MATHEMATICS OF FUZZY SETS

LOGIC, TOPOLOGY, AND MEASURE THEORY

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MATHEMATICS OF FUZZY SETS

LOGIC, TOPOLOGY, AND MEASURE THEORY

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Foreword

This book, Volume 3 of the *The Handbooks of Fuzzy Sets Series*, is intended to serve as a reference work for certain fundamental mathematical aspects of fuzzy sets, including mathematical logic, measure and probability theory, and especially general topology.

Its place in this *Handbook* series has a two-fold historical root: the need for standardization, both conceptually and notationally, in the mathematics of fuzzy sets; and the prominent role in the development of the mathematics of fuzzy sets played by the *International Seminar on Fuzzy Set Theory*—also known as the *Linz Seminar*—held annually since 1979 in Linz, Austria.

By the late 1980's, it was apparent to a broad spectrum of workers in fuzzy sets that the utility of the mathematics being rapidly developed in fuzzy sets was significantly restricted in every way by the overall incoherence of its literature, both conceptually and notationally. In keeping with the long-standing traditions of the Linz Seminar, many Seminar participants felt this problem should be addressed by a reference work which would standardize definitions, notations, and concepts, as well as make the mathematics of fuzzy sets more accessible and usable for the world-wide scientific community. This volume, in significant measure the work of Seminar participants, is an attempt to be such a reference work.

Given the role which this volume was intended to play, the editors kept always in mind the two critical criteria of a reference work: *standardization* and *state-of-the-art*. These criteria were often intertwined: in many of the chapters contained herein, significant new developments were sometimes necessary to facilitate standardization. It is the editors' hope that this volume so satisfies these two criteria that readers and workers from many backgrounds and interests will find it a useful reference tool as well as a platform for future research.

No work of this purpose and scope can hope for any degree of success without the support of many talented and committed individuals. The editors are grateful to the editors of the *Handbook* series, D. Dubois and H. Prade, for the opportunity to organize and edit this volume and for their supporting the inclusion of the mathematics of fuzzy sets in the *Handbook* series. Deepest thanks go to the chapter authors for their willingness to be a part of this important project and for their willingness to write, rewrite, rewrite yet again, polish, and enlarge their chapters to collectively produce this reference work. Appreciation is also expressed to the editors' respective universities for the use of facilities and secretarial and system staff help. Finally, gratitude is expressed to Mr. Gary Folven of Kluwer Academic Publishers for his cooperation, support, and infinite patience.

The Editors

Series Foreword

Fuzzy sets were introduced in 1965 by Lotfi Zadeh with a view to reconcile mathematical modeling and human knowledge in the engineering sciences. Since then, a considerable body of literature has blossomed around the concept of fuzzy sets in an incredibly wide range of areas, from mathematics and logics to traditional and advanced engineering methodologies (from civil engineering to computational intelligence). Applications are found in many contexts, from medicine to finance, from human factors to consumer products, from vehicle control to computational linguistics, and so on. Fuzzy logic is now currently used in the industrial practice of advanced information technology.

As a consequence of this trend, the number of conferences and publications on fuzzy logic has grown exponentially, and it becomes very difficult for students, newcomers, and even scientists already familiar with some aspects of fuzzy sets, to find their way in the maze of fuzzy papers. Notwithstanding circumstancial edited volumes, numerous fuzzy books have appeared, but, if we except very few comprehensive balanced textbooks, they are either very specialized monographs, or remain at a rather superficial level. Some are even misleading, conveying more ideology and unsustained claims than actual scientific contents.

What is missing is an organized set of detailed guidebooks to the relevant literature, that help the students and the newcoming scientist, having some preliminary knowledge of fuzzy sets, get deeper in the field without wasting time, by being guided right away in the heart of the literature relevant for her or his purpose. The ambition of the HANDBOOKS OF FUZZY SETS is to address this need. It will offer, in the compass of several volumes, a full picture of the current state of the art, in terms of the basic concepts, the mathematical developments, and the engineering methodologies that exploit the concept of fuzzy sets.

This collection will propose a series of volumes that aim at becoming a useful source of reference for all those, from graduate students to senior researchers, from pure mathematicians to industrial information engineers as well as life, human and social sciences scholars, interested in or working with fuzzy sets. The original feature of these volumes is that each chapter is written by one or several experts in the concerned topic. It provides introduction to the topic, outlines its development, presents the major results, and supplies an extensive bibliography for further reading.

The core set of volumes are respectively devoted to fundamentals of fuzzy set, mathematics of fuzzy sets, approximate reasoning and information systems, fuzzy models for pattern recognition and image processing, fuzzy sets in decision analysis, operations research and statistics, fuzzy systems modeling and control, and a guide to practical applications of fuzzy technologies.

Didier DUBOIS Henri PRADE
Toulouse

Introduction

The mathematics of fuzzy sets is the mathematics of lattice-valued maps. Lattice-valued maps play different roles and have different connotations in different areas of mathematics.

In mathematical logic, lattice-valued maps are *many-valued interpretations* of *predicate symbols*. This type of interpretation goes back to the work of A. Mostowski, who was the first to apply Heyting algebra valued maps to proof-theoretic problems of intuitionistic logic (cf. [11]). Seventeen years later, L.A. Zadeh attached a special philosophical meaning to lattice-valued maps; in the case of the real unit interval he interpreted $[0, 1]$-valued maps as *generalized characteristic functions* — i.e. as membership functions of a new kind of subsets, so-called fuzzy sets (cf. [12]).

In general topology, lattice-valued maps appear in the lattice-valued construction of the filter monad over **SET** — a concept on which, from the point of view of neighborhood systems, the notion of *topological space* can be based in a *strict* sense (cf. [3, 5, 6]). Here lattice-valued maps do not have immediately a special meaning or only that meaning which is intrinsically related to a special philosophical understanding of monads. If we follow E.G. Manes' non-deterministic interpretation of monads (cf. [10]), then lattice-valued filters can be understood as *non-standard points* of the respectively given *topological space*. On the other hand, if we identify neighborhood systems with collections of open subsets, then lattice-valued maps also appear in the formulation of open lattice-valued subsets — an approach originating with the seminal work of J.A. Goguen [4] in the early seventies, which in turn was a significant expansion of the initial paper of C.L. Chang [2] on $[0, 1]$-valued topologies.

In probability theory, especially in the field of measure-free conditioning, $[0, 1]$-valued maps appear in the representation of semisimple MV-algebras (cf. [1]) — a structure which is usually formed by the universe of conditional events. Here, in this context, lattice-valued maps can be understood as mathematical formulations of *conditional events*.

The purpose of this volume is to contribute to the standardization of the mathematics and notation of fuzzy sets in these three fields—mathematical logic and foundations, general topology, and probability and measure theory—and to do this by presenting the state of the art in a coordinated collection of carefully selected chapters. In one volume we are not able to treat all three fields with the same degree of completeness (e.g. in mathematical logic, the treatment of *fuzzy sets* as *separated presheaves* over a fixed GL-monoid is completely missing (cf. [7])). Instead, we focus special attention on the need for standardization in lattice-valued topology by giving for the first time a comprehensive overview on the most important lattice-valued, topological axioms together with canonical examples and non trivial applications to fields inside of mathematics.

This volume opens with two chapters on foundations. Chapter 1 presents various kinds of non-classical logics and their syntactic and semantic basis. Among other things this presentation pays special attention to *Gödel's* logic, *Hájek's* product logic, *Łukasiewicz'* logic, *Pawelka's* fuzzy logic, and their applications to fuzzy set theory. Chapter 2 investigates *image* and *pre-image operators* between fuzzy powersets, and establishes properties of powerset operators crucial to the lattice-theoretic foundations of any mathematical discipline in which image and pre-image play a critical role.

Chapter 3 and Chapter 4 make the extraordinary attempt to give a complete analysis of the axiomatic foundations of lattice-valued topologies in the case of *non completely distributive, complete lattices.* Both chapters treat in a different way the important problem of changing the underlying lattice. Chapter 3 is restricted to fixed-basis lattice-valued topologies and includes appropriate convergence theories which show that the basic axioms do have non-trivial consequences. Chapter 4 is devoted to the more general situation of variable-basis lattice-valued topologies and constructs a super-category unifying the theory of locales, fixed-basis lattice-valued topologies, and *Hutton* fuzzy topological spaces; that this more general setting has non-trivial consequences can be seen in Chapter 7. Both chapters contain sections on canonical examples related to the theory of frames (resp. locales), the theory of the fuzzy real line, and the theory of stochastic processes.

Chapter 5 fills the gap between probabilistic topological spaces and fuzzy neighborhood spaces and explains their relationship to lattice-valued topologies.

Chapter 6 and Chapter 7 investigate the important role of separation axioms within the theory of lattice-valued topological spaces with different perspectives. Chapter 6 puts special emphasize on space embedding and mapping extension problems, while Chapter 7 develops Stone-Čech-compactification and Stone-representation theorems even for the variable-basis case. Both chapters carefully present their results in light of the underlying lattice-theoretic bases. Comparing Chapters 3, 6, and 7, it is surprising to see there is not yet general agreement on the underlying lattice-theoretic structure or formulation of the Hausdorff and regularity separation axioms.

Chapter 8 and Chapter 9 present a brief introduction to the most important concepts of lattice-valued uniformities and their properties. In comparison to the topological situation, it is interesting to see that there exist three fundamentally different approaches to lattice-valued uniform spaces. In the case of the real unit interval as lattice-theoretic basis, Chapter 9 develops the basic theory of precompact or complete, $[0,1]$-valued uniform spaces.

This list of topological chapters is closed by Chapter 10 which is primarily devoted to the algebraic, topological, and uniform structures of the fuzzy real line and fuzzy unit interval. These spaces stem from the work of B. Hutton [8, 9] in the mid-seventies and play an important role in the study of higher separation axioms of lattice-valued topological spaces. An overlap between Chapter 6, Chapter 7, and Chapter 10 is therefore inevitable. A comprehensive study of these fundamental mathematical structures is important for future research work in the area of canonical examples in lattice-valued topology.

The remaining four chapters are dealing with measure-theoretic aspects of lattice-valued maps. Chapter 11 lays down the foundations of generalized measure theory, including its representation by Markov kernels. Subsequently, Chapter 12 develops the important theory of conditioning operators and its application to fundamental problems of measure-free conditioning. Further, Chapter 13 presents the elements of pseudo-analysis and their applications to the Hamilton-Jacobi equation and to optimization problems. Finally, Chapter 14 is a brief survey on the fundamentals of fuzzy random variables — a concept which can be understood as a $[0, 1]$-valued interpretation of the theory of random sets.

The Editors

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CHAPTER 1

Many–Valued Logic And Fuzzy Set Theory

S. GOTTWALD[1]

Introduction

Rather early in the (short) history of fuzzy sets it became clear that there is an intimate relationship between fuzzy set theory and many-valued logic. In the early days of fuzzy sets the main connection was given by fuzzy logic – in the understanding of this notion in those days: and this was as switching logic within a multiple-valued setting.

In parallel, soon it became obvious that fuzzy sets quite naturally could be discussed within the language of many-valued logic and, moreover, could be considered as sets of a particular (naive) set theory based on a suitable many-valued logic; cf. [29, 33].

Accordingly, the theory of fuzzy sets has given an important impetus for the recent revival of many-valued logic. And many-valued logic, on the other hand, retained its supporting role for the development of substantial results in fuzzy set theory. Some of the central topics which relate fuzzy set theory and many-valued logic shall be discussed in this chapter, essentially from the viewpoint of many-valued logic.

With the variety of set algebraic operations for fuzzy sets, like different types of unions and intersections, the fact is related that not only the larger sets of truth degrees of many-valued logics, as compared with classical two-valued logic, allow for "more" truth degree functions – the most common systems of many-valued logic, moreover, are not functionally complete.

Therefore in many-valued logic the suitable choice of basic connectives – which, as usual for crisp sets, also for fuzzy sets are in one-one correspondence with set algebraic operations – becomes much more intriguing. Accordingly the present chapter presents, after a short discussion of the basic ideas underlying many-valued logic, just this topic as the first one.

[1] Research for this paper was partly supported by the COST programs action no. 15.

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Having determined suitable families of connectives, particular systems of many-valued logic will be constituted by them. The most common ones among a large variety of possible systems shall be described.

It is well known from classical logic that logical equivalence as well as provable equivalence constitute congruence relations in the algebra of all well-formed formulas, and that the quotient structures – the so-called LINDENBAUM algebras – become BOOLEan algebras. Furthermore, logival validity in classical logic is just validity w.r.t. all BOOLEan algebras as value structures.

In many-valued logic one has corresponding relationships between logical systems and particular types of algebraic structures. These problems shall be discussed immediately following the presentation of particular systems of many-valued logic – on the level of propositional logics for simplicity.

The extension of many-valued propositional logics to first-order systems essentially is a routine matter and therefore treated only in passing. Nevertheless one has to take some care as infinitary inference rules may become necessary to get adequate axiomatisations for the class of logically valid formulas or of the entailment relation. And it proves to be an intriguing problem to extend first-order many-valued logics to logics with (nontrivial) identity relations.

First-order many-valued logics are suitable tools to develop fuzzy set theory within their language, as will be indicated with some few examples.

Fuzzy sets as a tool to treat vagueness phenomena have, however, influenced many-valued logic also in another important way – they supported the extension of many-valued logics to logics which allow for graded notions of entailment and of provability: to fuzzy logics for short. And these systems of generalised many-valued logics form the last main topic of the present chapter.

We finish this section with a brief outline:

§1 Basic ideas.
§2 Outline of the history.
§3 Characteristic connectives.
§4 Many–valued propositional logics.
§5 Algebraic structures for many–valued logics.
§6 Many–valued first–order logic.
§7 Fuzzy sets and many–valued logic.
§8 Fuzzy logic.

1 Basic ideas

1.1 From classical to many-valued logic

Logical systems in general are based on some formalised language which includes a notion of well formed formula, and then are determined either semantically or syntactically.

That a logical system is semantically determined means that one has a notion of *interpretation* or *model*[2] in the sense that w.r.t. each such interpretation every

[2]We prefer to use the word interpretation in general and to restrict the use of the word

well formed formula has some *(truth) value* or represents a function into the set of (truth) values. It furthermore means that one has a notion of validity for well formed formulas and based on it also a natural *entailment relation* between sets of well formed formulas and single formulas.

That a logical system is syntactically determined means that one has a notion of *proof* and of provable formula, i.e. of (formal) *theorem*, as well as a notion of *derivation from* a set of *premisses.*

From a philosophical, especially epistemological point of view the semantic aspect of (classical) logic is more basic than the syntactic one, because it are mainly the semantic ideas which determine what are suitable syntactic versions of the corresonding (system of) logic.

The most basic (semantic) assumptions of classical, i.e. two-valued – propositional as well as first order – logic are the principles of bivalence and of compositionality. Here the *principle of bivalence* is the assumption that each sentence[3] is either true or false under any one of the interpretations, i.e. has exactly one of the truth values $\top$ and $\bot$, usually numerically coded by 1 and 0. And the *principle of compositionality*[4] is the assumption that the value of each compound well formed formula is a function of the values of its (immediate) subformulas.

The most essential consequence of the principle of compositionality is the fact that each one of the propositional connectives as well as each one of the (first order) quantifiers is semantically determined by a function (of suitable arity) from the set of (truth) values into itself or by a function from its powerset into the set themselves.

Disregarding the quantifiers for a moment, i.e. restricting the considerations to the propositional case, the most essential (semantical) point is the determination of the truth value functions, i.e. of the operations in the truth value set which characterise the connectives. From an algebraic point of view, hence, the crucial point is to consider not only the *set* of truth values, but a whole *algebraic structure* with the truth value set as its support. And having in mind that all the classical connectives are definable from the connectives for conjunction, disjunction, and negation, this means to consider the set $\{0, 1\}$ of truth values together with the truth functions $\min, \max$ and $1 - \ldots$ of these connectives – and this is a particular BOOLEan algebra. The semantical notion of validity of a well formed formula H w.r.t. some interpretation now means that H has truth value 1 at this particular interpretation. (And universal validity of course means being valid for every interpretation.)

Generalising the notions of truth value, of interpretation and of validity in such a way that the truth value structure may be any (nontrivial) BOOLEan algebra $\mathcal{B} = \langle B, \sqcap, \sqcup, ^*, \mathbf{0}, \mathbf{1} \rangle$, that an interpretation is any mapping from the set of all propositional variables into B, the truth value functions for conjunction,

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model to particular interpretations which are tied in a specific way with sets of well formed formulas.

[3]By a sentence one means either any well formed formula of the corresponding formalised propositional language or any well formed formula of the corresponding formalised first order language which does not contain any free individual variable.

[4]Sometimes this principle is also named principle of extensionality.

disjunction, negation are chosen as $\sqcap, \sqcup, ^*$, respectively, and validity of a well formed formula H w.r.t. a given interpretation means that H has the BOOLEan value $\mathbf{1}$ at this interpretation.

This generalised type of interpretations can easily become extended to the first order case: one then has to consider only complete BOOLEan algebras and has to consider the operations of taking the infimum or supremum as the operations corresonding to the universal or existential quantifier.

It is well known that the class of universally valid formulas of classical logic is just the class of all formulas valid for all (nontrivial) – and complete (in the first order case) – BOOLEan algebras as truth value structures. This fact is refered to by saying that the class of (complete) BOOLEan algebras is characteristic for classical logic.

Many-valued logic deviates from these two basic principles only in that it neglects the principle of bivalence. Therefore, any system S of many-valued logic is characterised by a suitable *formalised language* $\mathcal{L}_\mathsf{S}$ which comprises

- its (nonempty) family $\mathcal{J}^\mathsf{S}$ of (basic) *propositional connectives*,
- its (possibly empty) family of *truth degree constants*,
- its set of *quantifiers*[5],

and adopts the usual way of defining the class of well formed formulas w.r.t. these syntactic primitives, and parallel to these syntactic data by the corresponding semantic data, i.e. by

- a (nonempty) set $\mathcal{W}^\mathsf{S}$ of *truth degrees*,
- a family of *truth degree functions* together with a correspondence between these truth degree functions and the propositional connectives of the (formal) language,
- a (possibly empty) family of nullary operations, i.e. of elements of the truth degree set together with a one-one correspondence between the members of this family and the truth degree constants of the (formal) language,
- a set of *quantifier interpreting functions* from $\mathbb{P}(\mathcal{W}^\mathsf{S})$ into $\mathcal{W}^\mathsf{S}$ together with a one-one correspondence between these functions and the quantifiers of the (formal) language.

Usually one additionally assumes that the classical truth values (or some "isomorphic" copies of them, also coded by 1 and 0) appear among the truth degrees of any suitable system S of many-valued logic:

$$\{0,1\} \subseteq \mathcal{W}^\mathsf{S}. \tag{1}$$

[5]Each quantifier for simplicity is supposed here to be a unary one. This means that we allow to have some kinds of generalised quantifiers in the sense of MOSTOWSKI [75] but we do not consider the possibility to have quantifiers with more than one scope, as is allowed e.g. in [90, 33].

As in the case of classical logic, any system of many-valued logic can either be semantically based or syntactically. Further on, we shall prefer to have the systems semantically based which we are going to consider, without however excluding syntactic foundations. This essentially also means that we usually start from a set $\mathcal{W}^{\mathsf{S}}$ of truth degrees together with some (finitary) operations in it, determining the truth degree functions of the basic connectives. Hence, also for (the semantic approach toward) many-valued logics some algebraic structures get a basic role. Unfortunately, however, there is (up to now) no single class of algebraic structures, like the BOOLEan algebras in the case of classical logic, which are characteristic for many-valued logic. Instead, different approaches toward many-valued logic rely to different algebraic structures.

1.2 Truth degrees

Formally, for the systems S of many-valued logic there is essentially no restriction concerning the set $\mathcal{W}^{\mathsf{S}}$ of truth degrees of S besides (1). Nevertheless the choice of $\mathcal{W}^{\mathsf{S}}$ as a set of numbers (either integers or rationals or even reels) is widely accepted use. At least as long as one is not interested to have an ordering of the truth degrees which allows for incomparable truth degrees[6]. The existence of incomparable truth degrees, however, may be crucial for certain particular applications, e.g. in a situation where the truth degrees are intended to code parallel evaluations of different points of view. To imagine such a situation assume to be interested, in image processing, to evaluate whether a certain point P belongs to a certain figure $\mathcal{F}$, i.e. to determine the truth value of the sentence "P is a point of $\mathcal{F}$". For pure black and white pictures, the evaluation yields one of the truth values $\top, \bot$. Being instead confronted with a graytone picture, the evaluation of this sentence may yield as truth degrees the values of some scale which characterises the different gray levels. And being, finally, confronted with a coloured picture, e.g. on the screen of some monitor, which consists of (coloured) pixels which themselves are generated by superposing pixels of the three basic colours, then it may be reasonable to evaluate the above mentioned sentence by a truth degree which is a triple of the levels of the basic colours which give point P.

Another widely accepted kind of approach is to assume that among the truth degrees there is a smallest one, usually interpreted as an equivalent for the truth value $\bot$, and a biggest one, usually interpreted as an equivalent for $\top$.

Based on these common assumptions it is usually at most a simple matter of isomorphic exchange of the structure of truth degrees to assume (1) together with

$$\mathcal{W}^{\mathsf{S}} \subseteq [0,1] \subseteq \mathbb{R}. \tag{2}$$

And this choice of the truth degree set shall be the standard one in the following discussions.

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[6] Of course, even in such a situation one can take a set of numbers for the set of truth degrees – and adjoin another ordering relation to these numbers than their natural ordering. But this is rather unusual.

For the case of infinitely many truth degrees it is common usage to consider either countably many or uncountably many truth degrees and furthermore to choose either one of the truth degree sets

$$\mathcal{W}_0 =_{def} \{x \in \mathbb{Q} \,|\, 0 \leq x \leq 1\} \quad \text{or} \quad \mathcal{W}_\infty =_{\text{def}} \{x \in \mathbb{R} \,|\, 0 \leq x \leq 1\}. \tag{3}$$

For the case of finite sets of truth degrees usually one additionally assumes that these truth degrees form a set of equidistant points of the real unit interval $[0, 1]$, i.e. are of the kind

$$\mathcal{W}_m =_{\text{def}} \{\frac{k}{m-1} \mid 0 \leqq k \leqq m-1\} \tag{4}$$

for some integer $m \geqq 2$.

1.3 Designated truth degrees

In classical logic there is a kind of superiority of the truth value $\top$ over the other one $\bot$: given some well formed formula (or some set of wff) one mainly is interested in those interpretations for the system of many-valued logic one is working in which make the given wff true.

With the set $\mathcal{W}^{\mathsf{S}}$ of truth degrees each system S of many-valued logic has its equivalent to the truth value set $\{\top, \bot\}$. However, even admitting condition (1), this does not mean that the truth degree 1 has to be *the* equivalent of $\top$. The rather vague philosophical idea that in many-valued logic one considers some kind of "splitting" of the classical truth values does not by itself determine which truth degrees "correspond" to $\top$. Therefore, to determine some system S of many-valued logic does not only mean to fix its set of truth degrees and its formal language, i.e. its connectives, quantifiers, predicate symbols and individual constants, together with their semantic interpretations, it also means to fix which truth degrees "correspond" to $\top$.

Formally this means, that with each system S of many-valued one connects not only his set $\mathcal{W}^{\mathsf{S}}$ of truth degrees but also some set $\mathcal{D}^{\mathsf{S}}$ of *designated* truth degrees. Of course, usually one supposes

$$1 \in \mathcal{D}^{\mathsf{S}} \subseteq \mathcal{W}^{\mathsf{S}} \quad \text{together with} \quad 0 \notin \mathcal{D}^{\mathsf{S}}. \tag{5}$$

Such a choice of designated truth degrees is of fundamental importance for the generalisation of the notions of logical validity and of logical consequence.

A further generalisation, although not so common as the use of a set of designated truth degrees, intends to have truth degrees which correspond to $\top$ as well as truth degrees which correspond to $\bot$. Accordingly, then, one has to distinguish *positively designated* truth degrees and *negatively designated* ones, i.e. one has to determine two disjoint sets $\mathcal{D}^{\mathsf{S}+}, \mathcal{D}^{\mathsf{S}-}$ such that

$$\mathcal{D}^{\mathsf{S}+} \cup \mathcal{D}^{\mathsf{S}-} \subseteq \mathcal{W}^{\mathsf{S}} \quad \text{and} \quad \mathcal{D}^{\mathsf{S}+} \cap \mathcal{D}^{\mathsf{S}-} = \emptyset$$

usually together with the additional assumptions

$$1 \in \mathcal{D}^{\mathsf{S}+} \quad \text{and} \quad 0 \in \mathcal{D}^{\mathsf{S}-}.$$

This idea of two types of designated truth degrees, mentioned e.g. in [90], seems not to be as important as the simpler idea to have only one type of designated truth degrees. This may depend on the fact that one usually in many-valued logic is interested in suitable generalisations of logically valid formulas and of logical consequences, and that the negatively designated truth degrees do not count for these generalisations. They would, instead, become crucial if one would intend to discuss the notion of contradiction (e.g. additionally in a context without negation connective). But this, up to now, has not been of much interest.

Therefore, we essentially shall restrict the considerations to the simpler case of only one set of designated truth degrees.

1.4 Logical validity and logical consequence

Based on the notion of designated truth degrees it is essentially a routine matter to generalise the notions of logical validity and of logical consequence to any system S of many-valued logic.

Adopting the standard usage to mean by a *sentence* either a well formed formula in the case of a propositional language $\mathcal{L}_\mathsf{S}$ or a well formed formula without free individual variables of a first order language $\mathcal{L}_\mathsf{S}$, one calls a sentence H *valid* in an interpretation iff it has a designated truth degree in that interpretation, and one calls a sentence H *logically valid* iff it is valid in every suitable interpretation.

As usual, furthermore, a formula H of a first order language $\mathcal{L}_\mathsf{S}$ is called valid in a given interpretation $\mathfrak{A}$, iff it has a designated truth degree w.r.t. any valuation of the individual variables of the language.

On the other hand, an interpretation $\mathfrak{A}$ is called a *model* of a well formed formula H iff this formula H is valid in $\mathfrak{A}$. And it is called a model of a set Σ of well formed formulae iff it is a model of any formula $H \in \Sigma$. The fact that $\mathfrak{A}$ is a model of H or Σ as usual is denoted by $\mathfrak{A} \models H$ or $\mathfrak{A} \models \Sigma$.

Sometimes, the notion of model is generalised a bit further in many-valued logic. Given some truth degree α and some sentence H, an interpretation $\mathfrak{A}$ is called an α*-model* of H iff the truth degree of H in the interpretation $\mathfrak{A}$ equals α or[7] iff it is greater or equal to α. We prefer here, to speak in the last mentioned case of a $(\geq \alpha)$-model.

Correspondingly the notion of α-model of a set of sentences is used. In this case, the most suitable way is to call an interpretation $\mathfrak{A}$ an $(\geq \alpha)$-model of a set Σ of sentences iff $\mathfrak{A}$ is an $(\geq \alpha)$-model of each sentence $H \in \Sigma$.

Based on these preliminaries, the notion of logical consequence is defined almost in the standard way, but again with mainly two slightly different basic intuitions. For simplicity, we restrict again the considerations to sentences only. The extension to all well formed formulas happens in many-valued logic exactly as in classical logic.

[7] Both these variants are in use. One has to check the use of the term α-model in the particular case to see which version applies.

One defines that a sentence H is a *logical consequence* of a set Σ of sentences, usually also written $\Sigma \models H$, iff

(*Version 1*): each model of Σ is also a model of H,

(*Version 2*): each $(\geq \alpha)$-model of Σ is also a $(\geq \alpha)$-model of H.

With $\models H$ as shorthand for $\emptyset \models H$ in any case $\models H$ means that H is logically valid.

The checking of well formed formulas of any propositional system of many-valued logic for being logically valid can be done in the same way as for classical logic by determining complete truth degree tables – and this is effective provided the set of truth degrees is finite.

Theorem 1.1 *For each finitely many-valued system* S *of propositional logic the property of being a logically valid sentence is decidable, and for each finite set* Σ *of sentences of* S *also the property of being a logical consequence of* Σ *is decidable.*

2 Outline of the history

Many-valued logic as a separate part of logic was created by the works of J. Łukasiewicz [59] and E.L. Post [87] in the beginning 1920th. Admittedly, both authors have not been the first ones which did not assume the principle of bivalence, but earlier attempts to do logic without this principle of bivalence did not prove to be influential[8].

The prehistory of many-valued logic, however, may be traced back up to Aristotle[9] who e.g. in his *De Interpretatione*, chapt. 9, discussed the problem of future contingencies, i.e. the problem which truth value a proposition should have today which asserts some future event. This problem was stimulating even for J. Łukasiewicz [60], and it is closely tied with the philosophical problem of determinism. The link is provided by the interpretation that the classification of some future event as (actually) "possible" or "undetermined" may well be seen as the acceptance of a third "truth value" besides $\top$ and $\bot$. Surely, this reading is not the necessary one. Nevertheless, the acient philosopical school of Epicureans which tended toward indeterminism refused the principle of bivalence, whereas the school of the Stoics did accept it – and strongly advocated determinism.

The same problem of *contingentia futura* was also the source for several extended discussions during the Middle Ages, cf. e.g. [61, 68, 1, 90], without getting resolved. And in the phase of the general revival of investigations into the field of logic during the second half of the 19th century the idea of neglecting the principle of bivalence appeared (partly without clear mentioning of this fact) to H. McColl [65], cf. also [57], and Ch.S. Peirce, cf. [85, vol. 4] as well as [21, 108].

[8]Even the previous paper [58] of J. Łukasiewicz which also admitted generalised truth "degrees" besides the traditional truth values $\top, \bot$ did not influence the development of logic toward many-valued logic in any perceivable manner.

[9]The interested reader may consult e.g. [60, 62, 90, 83].

The real starting phase of many-valued logic was the time interval from 1920 till about 1930, and the main force of development was the Polish school of logic under J. ŁUKASIEWICZ. The papers [63, 60] as well as the influential textbook [55], all published in 1930, explain the core ideas as well as the background of philosophical ideas and the main technical results proven up to this time[10]. They also stimulated further research into the topic.

In [60] Łukasiewicz intends to give a modal reading to his many-valued propositional logic, claiming that only the 3-valued and the infinite valued case (with the set of all rationals between 0 and 1 as truth degree set) are really of interest for applications. In [63] however all finitely many-valued propositional systems and the just mentioned infinitely many-valued one are discussed, always based on a negation and an implication connective as primitive ones characterised semantically by their truth degree functions.

Basic theoretical results for systems of many-valued logics which followed this initial phase of "Polish" many-valued logic have e.g. been

- M. WAJSBERG's [111] axiomatisation of the three valued (propositional) system L_3 of ŁUKASIEWICZ, i.e. of that one propositional system with ŁUKASIEWICZ's implication and negation connectives as primitive connectives,
- the extension of ŁUKASIEWICZ's system L_3 to a functionally complete one and its axiomatisation by J. SŁUPECKI [101],
- the work of K. GÖDEL [28] and S. JAŚKOWSKI [48] which clarified the mutual relations of intuitionistic and many-valued logic in the sense that it was proven that there does not exist a single (propositional) many-valued system whose set of logically valid formulas coincides with the set of logically valid formulas of intuitionistic (propositional) logic,
- the application of systems of three valued logic to the problems of logical antinomies by BOČVAR [5, 6] with the third truth value read as "senseless",
- the application of systems of three valued logic to problems of partially defined function by S. KLEENE [50, 51] with the third truth value read as "undefined".

Furthermore, during the 1940th basic approaches have been generalised and essential results were proven by B. ROSSER and A.R. TURQUETTE in a series of papers and later on most of this material collected in their monograph [94], which besides the ŁUKASIEWICZ papers of 1930 was the standard reference for years.

In the 1950th and 1960th then there was some decline in the interest in many-valued logics – and at the same time some shift in the focus of what

[10] As a side remark it has to be mentioned that P. BERNAYS [4] used more than the usual two truth values of classical logic to study independence problems for systems of axioms for systems of classical propositional calculus. But in his case these multiple values were only formal tools for his unprovability results.

were considered as the more interesting problems. The decline as well as the shift may have been caused by the same situation: the fact – mentioned in the ROSSER/TURQUETTE monograph [94] quite open – that up to this time no really convincing applications had been approached by the methods of many-valued logic. The shift in interest inside many-valued logic thus happened toward problems of definability of operations in $\{1, 2, \ldots, n\}$ from particular sets of such operations, i.e. toward problems connected with the functional incompleteness of most of the then "usual" systems of connectives of (finitely) many-valued systems of (propositional) logic. In the background, however, there was not only the theoretical problem of functional completeness, there was (and is) also the related problem in switching theory of sets of suitable elementary circuits which allow to generate all the (finitary) operations in $\{1, 2, \ldots, n\}$ by being combined into suitable circuits – or the related problem to determine the class of all operations which can be generated by some given ones.

3 Characteristic connectives

As customary in the language of classical logic, some types of "standard" logical connectives get main attendance, also because of their overwhelming use in colloquial language. These "basic" connectives usually are the connectives for conjunction, disjunction, negation, and implication.

It is, however, even in the field of classical logic and its subsystems a still unsolved problem to give structural criteria to characterise these different types of connectives, i.e. to determine by structural properties what e.g. is a negation connective or a conjunction connective. The more this is the case in many-valued logic.

Therefore the approach can not start with general definitions what are – in many-valued logic – e.g. conjunction connectives or negation connectives. Instead, one has to start with the consideration of typical, i.e. characteristic examples for these different types of connectives. Starting from such examples and from ideas on the ordinary use of the corresponding words in everyday communication, then of course typical properties of different types of connectives can be "inferred". This shall be done now for the most common types of connectives.

Additionally one usually assumes that any truth degree function τ for some connective which is intended to generalise some standard connective of classical logic gives back the truth value function of just this connective as its restriction $\tau \restriction \{0, 1\}$ to the argument set $\{0, 1\}$.

For general considerations it even is useful to distinguish among all truth degree functions those ones with a restriction to $\{0, 1\}$ which is an operation in $\{0, 1\}$. Therefore, a truth degree function τ is called *normal* iff one has $\mathsf{rg}\,(\tau \restriction \{0, 1\}) \subseteq \{0, 1\}$.

Any superposition of normal truth degree functions obviously is again a normal truth degree function. This gives immediately the

Proposition 3.1 *A necessary condition for the functional completeness of a system* S *of many-valued propositional logic is that at least one of the basic connectives from* $\mathcal{J}^{\mathsf{S}}$ *is interpreted by a non-normal truth degree function.*

It is an interesting problems for many-valued propositional logics then to find suitable sufficient conditions for functional completeness. This problem is, obviously, covered by the larger one to find results how to determine the class $\langle\mathfrak{F}\rangle$ of all truth degree functions (over a given set $\mathcal{W}_m$) generated by any given class $\mathfrak{F}$ of such functions, i.e. to determine the smallest closed class of m-valued functions containing $\mathfrak{F}$. And this itself is a large field of investigations, cf. e.g. [86].

Two of the simplest among those criteria are given in the following theorems.

Theorem 3.2 *A system* S *of finitely many-valued propositional logic with truth degree set* $\mathcal{W}_m$ *is functionally complete if the truth degree functions* vel_1 *and* non_2 *either are truth degree functions of some of the basic connectives of* S *or if they are definable within* S.

Theorem 3.3 *A system* S *of finitely many-valued propositional logic with truth degree set* $\mathcal{W}_m$ *is functionally complete if the truth degree functions* $\mathsf{vel}_1, \mathsf{et}_1$*, all the truth degrees*[11] $d \in \mathcal{W}_m$ *and all the unary functions*

$$\mathsf{j}^*_{m,k}(u) =_{\mathrm{def}} \begin{cases} 0 & \textit{if } u = \frac{k}{m-1} \\ 1 & \textit{otherwise} \end{cases} \tag{6}$$

either are truth degree functions of some of the basic connectives of S *or if they are definable within* S.

3.1 The J–connectives

The last mentioned Theorem 3.3 gives rise to a particular type of connectives, usually denoted J_s, $s \in \mathcal{W}$ any truth degree, and often considered in connection with systems of many-valued logic, which correspond to truth degree functions $\mathsf{j}_{m,k}$ defined in accordance with (6) by

$$\mathsf{j}_s(u) =_{\mathrm{def}} \begin{cases} 1 & \text{if } u = s \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

and which generalise the use of truth value constants $\top$ (true) and $\perp$ (false) in classical logic.

The interest in these connectives comes not only from their relationship with the problem of functional completeness[12] but, even more, from a rather uniform method of adequate axiomatisation of a whole class of finitely many-valued systems, cf. Section 4.6.

[11] This means, of course, the corresponding nullary functions over $\mathcal{W}_m$.

[12] Obviously Theorem 3.3 says that the availability of these connectives, either as primitives or as definable ones, almost suffices to have a functionally complete system.

3.2 Conjunction connectives

The most basic example for a truth degree function of a conjunction connective comes from the earlier papers of ŁUKASIEWICZ and is also used by GÖDEL. It is the (binary) function et_1 with the definition

$$\mathrm{et}_1(u,v) =_{\mathrm{def}} \min\{u,v\}. \tag{8}$$

Another example which also appeared already in the beginning papers by ŁUKASIEWICZ has the more complicated definition

$$\mathrm{et}_2(u,v) =_{\mathrm{def}} \max\{0, u+v-1\} \tag{9}$$

and is known as ŁUKASIEWICZ *(arithmetic) conjunction* or *bounded product*, but sometimes also called *bold(face) conjunction*[13]. Both these definitions are independent of the number of truth degrees of the particular system of many-valued logic they belong to. In the finitely many-valued systems with truth degree sets $\mathcal{W}_m$ one often prefers to describe these functions using (truth degree) tables instead of analytical formulas as done in (8), (9). In the present case for 5-valued systems this would mean to give the tables of Fig. 1.

Another type of truth degree function for a conjunction connective which can

et_1	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
0	0	0	0	0	0
$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{3}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
1	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1

et_2	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
0	0	0	0	0	0
$\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$
$\frac{1}{2}$	0	0	0	$\frac{1}{4}$	$\frac{1}{2}$
$\frac{3}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
1	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1

Figure 1: Truth table characterisations of conjunction connectives

only be used for some suitable truth degree sets is the function

$$\mathrm{et}_3(u,v) =_{\mathrm{def}} u \cdot v. \tag{10}$$

This truth degree function has seriously been considered only recently. That may, at least partly, be caused by the fact that among the typical truth degree sets, mentioned at the end of Section 1.2, besides the traditional truth value set $\mathcal{W}_2$ only the infinite ones are closed under et_3, i.e. under the product. And in the earlier periods of many-valued logic there was a tendency to prefer truth degree functions which can equally well be considered in the finitely as well as in the infinitely many-valued systems.

[13]The motivations for these names are historical in the first case, systematical in the second, and essentially accidental in the third one which refers to a notation sometimes used for the corresponding connective. The qualification "arithmetic" depends on the fact that in the ŁUKASIEWICZ systems as introduced in Section 4.1 also the connective (8) is present.

These three truth degree functions are, however, only particular cases of a much wider variety of possible candidates. Obviously, it is not of much interest to extend this list of examples, instead one likes to have some (few) leading principles which reasonable candidates for conjunction connectives should satisfy. Actually, there is a wide agreement that such principles should be the commutativity and associativity of the truth degree functions together with some suitable monotonicity and borderline condition. The type of truth degree functions thus determined are the t-norms.

Definition 3.1 *A binary operation* $\boldsymbol{t}$ *in the real unit interval* $[0,1]$ *is a* t-norm *iff it is*

(T1) *associative and commutative;*
(T2) *non-decreasing in the first – and hence in each – argument;*
(T3) *has* 1 *as neutral element, i.e. satisfies* $\boldsymbol{t}(u,1) = u$ *for each* $u \in [0,1]$.

As a corollary of this definition one immediately has that for any t-norm $\boldsymbol{t}$ there holds true

$$\boldsymbol{t}(u,0) = 0 \quad \text{for each } u \in [0,1]$$

because of $\boldsymbol{t}(u,0) = \boldsymbol{t}(0,u) \leqq \boldsymbol{t}(0,1) = 0$.

In algebraic terms, each t-norm represents a semigroup operation[14] in the unit interval $[0,1]$ with identity 1.

The notion of t-norm appeared in the context of probabilistic metric spaces in the work of MENGER about 1942, cf. his [67], as well as in the work of SCHWEIZER and SKLAR, cf. [98, 99], in the context of functional equations and in the work of FRANK [25]. A recent monograph devoted solely to t-norms and their applications is [54], other books which provide essential results on t-norms are [9, 24]. A short first introduction may also be [52].

The class of all t-norms is quite large. Therefore subclasses of it are taken into account for different reasons. One approach toward interesting subclasses of t-norms is to impose some continuity properties: from this point of view the most interesting subclasses are formed by the left continuous as well as by the continuous t-norms.

Another interesting subclass is formed by the *archimedean* t-norms which are characterised by the condition

$$\boldsymbol{t} \text{ archimedean} \quad \text{iff} \quad \begin{array}{l} \text{for all } u,v \in (0,1) \text{ there exists some} \\ n \in \mathbb{N} \text{ such that } u_{\boldsymbol{t}}^{(n)} < v \end{array} \tag{11}$$

with the power notation $u_{\boldsymbol{t}}^{(n)}$ defined by

$$u_{\boldsymbol{t}}^{(n)} =_{\text{def}} \begin{cases} u & \text{if } n = 1, \\ \boldsymbol{t}(\overbrace{u,u,\ldots,u}^{n \text{ times}}) & \text{if } n > 1. \end{cases} \tag{12}$$

[14]We prefer here, as later on with the t-conorms, the function notation $\boldsymbol{t}(u,v)$, but the infix notation $u\,\boldsymbol{t}\,v$ which is more common in the algebraic context is equally usual for t-norms.

For continuous t-norms there is a much simpler characterisation of archimedeanicity which for this class of t-norms often is even taken as the definition of that property.

Proposition 3.4 *A continuous t-norm* $\boldsymbol{t}$ *is archimedean iff* $\boldsymbol{t}(u,u) < u$ *holds true for all* $u \in (0,1)$.

A further important class of t-norms are the *strictly monotone* t-norms, i.e. those t-norms $\boldsymbol{t}$ which as binary functions over $[0,1]^2$ are stricly increasing functions over $(0,1)^2$, that means which satisfy:

$$u < v \quad \Rightarrow \quad \boldsymbol{t}(u,w) < \boldsymbol{t}(v,w). \tag{13}$$

One has to be careful with terminology in this case: being *strict* means for a t-norm to be continuous and stricly monotonous.

By Proposition 3.4, therefore, the strictness of a t-norm is sufficient for its archimedeanicity. However, there are stricly monotone and left continuous t-norms which are not archimedean. An example is the t-norm $\boldsymbol{t}^*$ which is for $x, y \in (0,1)$ given by

$$\boldsymbol{t}^*(x,y) =_{\mathsf{def}} \sum_{n=0}^{\infty} (2^{x_n+y_n-n})^{-1}$$

with reference to the binary expansions $x = \sum_{n=0}^{\infty} 2^{-x_n}$ and $y = \sum_{n=0}^{\infty} 2^{-y_n}$ with infinite, strictly monotone sequences[15] $(x_n)_{n\geq 0}, (y_n)_{n\geq 0}$, cf. [53].

Again another nice algebraic characterisation is available for the strictly monotone t-norms:

> a t-norms $\boldsymbol{t}$ is strictly monotone iff it satisfies the *cancellation law*,
> i.e. iff always $\boldsymbol{t}(u,w) = \boldsymbol{t}(v,w)$ for $w > 0$ implies $u = v$.

All the above mentioned truth degree functions et_k, $k = 1, \dots, 3$, are continuous t-norms. The t-norms et_2 and et_3 are even archimedean ones, but et_1 isn't archimedean. And et_3 is the only strict t-norm among them.

Two other intersting t-norms are the sometimes so-called *drastic product* $\boldsymbol{t}_0$ defined by

$$\boldsymbol{t}_0(u,v) =_{\mathsf{def}} \begin{cases} \min\{u,v\} & \text{if } \max\{u,v\} = 1 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

and the *nilpotent minimum* $\boldsymbol{t}_1$ of FODOR [23] defined by

$$\boldsymbol{t}_1(u,v) =_{\mathsf{def}} \begin{cases} \min\{u,v\} & \text{if } u+v > 1 \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

The drastic product $\boldsymbol{t}_0$ is archimedean but not continuous, and the nilpotent minimum $\boldsymbol{t}_1$ is neither archimedean nor continuous, but left continuous.

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[15]That means, that even rational numbers have to be described by infinite binary expansions.

Obviously, one has additionally for any t-norm $\boldsymbol{t}$ the inequality

$$\boldsymbol{t}_0 \leqslant \boldsymbol{t} \leqslant \mathsf{et}_1 = \min \tag{16}$$

with the ordering $\leqslant$ of the (binary) functions taken pointwise.

Further examples of t-norms are provided by a variety of – usually one-parametric – families. HAMACHER [41] e.g. considers the one-parametric family $\boldsymbol{t}_{H,\gamma}$ with parameter $\gamma \geq 0$ defined as

$$\boldsymbol{t}_{H,\gamma}(u,v) =_{\text{def}} \frac{uv}{\gamma + (1-\gamma)(u+v-uv)}. \tag{17}$$

These t-norms contain for $\gamma = 1$ the (algebraic) product et_3 and for $\gamma \to \infty$ the drastic product $\boldsymbol{t}_0$.

A problem on functional equations was the source of FRANK [25] to consider the family

$$\boldsymbol{t}_{F,s}(u,v) =_{\text{def}} \log_s \left(1 + \frac{(s^u-1)(s^v-1)}{s-1}\right) \tag{18}$$

of continuous archimedean t-norms with parameter $s < 0, s \neq 1$ and the particular cases

$$\begin{array}{rlcl} \boldsymbol{t}_{F,0}(u,v) =_{\text{def}} & \lim_{s\to 0} u\,\boldsymbol{t}_{F,s}\,v & = & \min\{u,v\}, \\ \boldsymbol{t}_{F,1}(u,v) =_{\text{def}} & \lim_{s\to 1} u\,\boldsymbol{t}_{F,s}\,v & = & u\cdot v, \\ \boldsymbol{t}_{F,\infty}(u,v) =_{\text{def}} & \lim_{s\to\infty} \boldsymbol{t}_{F,s}(u,v) & = & \max\{0, u+v-1\}. \end{array}$$

YAGER [113] introduced still another family $\boldsymbol{t}_{Y,p}$ with parameter $p > 0$ by

$$\boldsymbol{t}_{Y,p}(u,v) =_{\text{def}} 1 - \min\{1, ((1-u)^p + (1-v)^p)^{1/p}\}. \tag{19}$$

For $p \to 0$ these t-norms converge toward the drastic product $\boldsymbol{t}_0$, and for $p \to \infty$ they converge toward the minimum. Additionally, for $p = 1$ one has the t-norm et_2.

A fourth family $\boldsymbol{t}_{W,\lambda}$ with parameter $\lambda > -1$ was introduced by WEBER [112]. He defined

$$\boldsymbol{t}_{W,\lambda}(u,v) =_{\text{def}} \max\left\{0, \frac{u+v-1+\lambda uv}{1+\lambda}\right\}. \tag{20}$$

The parameter $\lambda = 0$ here gives the bounded product et_2. For $\lambda \to -1$ the t-norms $\boldsymbol{t}_{W,\lambda}$ converge toward the drastic product $\boldsymbol{t}_0$. And for $\lambda \to \infty$, finally, the t-norms $\boldsymbol{t}_{W,\lambda}$ converge toward the (algebraic) product et_3.

To define parametrised families of t-norms via (real) parameters is one approach toward subclasses of the class of t-norms. Another way is to refer to some method of generation of t-norms out of simpler ones or out of some type of generating functions. This idea turns out to work especially well for the classes of continuous and of archimedean t-norms.

To state such a reduction result for continuous t-norms we have to introduce the notion of ordinal sum of a family of t-norms.

Definition 3.2 *Suppose that* $([a_i, b_i])_{i\in I}$ *is a countable family of non-overlapping proper subintervals of the unit interval* $[0,1]$ *and let* $(\boldsymbol{t}_i)_{i\in I}$ *be a family of t-norms. Then the* ordinal sum *of the combined family* $(([a_i, b_i], \boldsymbol{t}_i))_{i\in I}$ *is the binary function* $T : [0,1]^2 \to [0,1]$ *characterised by*

$$T(u,v) = \begin{cases} a_k + (b_k - a_k)\boldsymbol{t}_k(\frac{u-a_k}{b_k-a_k}, \frac{v-a_k}{b_k-a_k}) & \text{if } (u,v) \in [a_k, b_k] \\ \min\{u,v\} & \text{otherwise.} \end{cases} \tag{21}$$

Then from results of MOSTERT and SHIELDS [74] one gets the following theorem.

Theorem 3.5 *For each continuous t-norm* $\boldsymbol{t}$ *one of the following cases appears:*

1. $\boldsymbol{t} = \mathrm{et}_1 = \min$,
2. $\boldsymbol{t}$ *is archimedean* ,
3. $\boldsymbol{t}$ *is the ordinal sum of a family* $(([a_i, b_i], \boldsymbol{t}_i))_{i\in I}$ *of continuous archimedean t-norms.*

For continuous archimedean t-norms one has even a further type of representation theorem. It refers to the *pseudoinverse* $f^{(-1)}$ of a function $f : [0,1] \to [0,\infty]$ which is for each non-increasing f defined for each $z \in [0,\infty]$ by

$$f^{(-1)}(z) =_{\mathrm{def}} \sup\{x \in [0,1] \mid f(x) > z\}$$

and which can for strictly decreasingand continuous functions f with $f(1) = 0$ be simpler characterised for each $z \in [0,\infty]$ by

$$f^{(-1)}(z) = \begin{cases} f^{-1}(z), & \text{if } z \in [0, f(0)] \\ 0, & \text{if } z \in (f(0), \infty]. \end{cases}$$

Theorem 3.6 *To each continuous and archimedean t-norm* $\boldsymbol{t}$ *there exists a continuous and monotonically decreasing function* $f : [0,1] \to [0,\infty]$ *with* $f(1) = 0$ *and for all* $u, v \in [0,1]$

$$\boldsymbol{t}(u,v) = f^{(-1)}(f(u) + f(v)). \tag{22}$$

And this generating function f *is uniquely determined up to a positive factor by the archimedean t-norm* $\boldsymbol{t}$.

Conversely, each binary function is a continuous archimedean t-norm which is given via (22) using a function f with these properties of continuity and monotonic decreasingness. However, already if $f : [0,1] \to [0,\infty]$ has the weaker properties to be non-increasing and to satisfy

$$f(u) + f(v) \in \mathrm{rg}\,(f) \cup (f(0), \infty]$$

for all $u, v \in [0,1]$, then by (22) an archimedean t-norm is determined.

The continuous archimedean t-norms themselves are determined by the strict t-norms in the sense of the following result of LING [56].

Theorem 3.7 *Each continuous and archimedean t-norm is the limit of a pointwise convergent sequence of continuous strict t-norms.*

So we may ask for another representation theorem for strict t-norms. To state such a result we need the notion of an *automorphism* of the unit interval. By such an automorphism we mean an continuous, strictly increasing surjection of $[0,1]$ onto itself, i.e. an automorphism of the lattice $\langle [0,1], \min, \max, 0, 1\rangle$.

Then one has the following theorem.

Theorem 3.8 *A continuous t-norm $\boldsymbol{t}$ is strict iff there exists an automorphism φ of the unit interval with*

$$\boldsymbol{t}(u,v) = \varphi^{-1}(\varphi(u)\cdot\varphi(v)) = \varphi^{-1}(\mathsf{et}_3(\varphi(u),\varphi(v))). \tag{23}$$

The t-norm et_3, the usual arithmetic product, hence is a kind of "prototype" for the continuous and strict t-norms.

There are more such representation results for different classes of t-norms, cf. e.g. [24, 54], which shall not be discussed here.

3.3 Negation connectives

Starting again from the historical origins, the papers of ŁUKASIEWICZ [59] and POST [87] gave two particular truth degree functions for negation connectives. ŁUKASIEWICZ used uniformly

$$\mathsf{non}_1(u) =_{\mathsf{def}} 1 - u \tag{24}$$

for any one of the standard truth degree sets, and POST restricting himself to the finite valued cases considered within the truth degree set $\mathcal{W}_m$ the truth degree function

$$\mathsf{non}_2(u) =_{\mathsf{def}} \begin{cases} 1 & \text{if } u = 0 \\ u - \frac{1}{m-1} & \text{if } u \neq 0. \end{cases} \tag{25}$$

Later on the work of GÖDEL added as another truth degree function for a negation connective the function

$$\mathsf{non}_0(u) =_{\mathsf{def}} \begin{cases} 1 & \text{if } u = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

The present discussion, however, considers POST's cyclic operation non_2 as a non-standard, even exotic example and tends to exclude it from more general considerations. Accordingly the following definition actually is prevalent.

Definition 3.3 *A function $\boldsymbol{n} : [0,1] \to [0,1]$ is called* negation function *iff $\boldsymbol{n}$ is nonincreasing and satisfies $\boldsymbol{n}(0) = 1$ and $\boldsymbol{n}(1) = 0$. A negation function $\boldsymbol{n}$ is called* strict *iff it is even stricly decreasing and continuous. And it is called* strong *iff it is strict and an* involution, *i.e. satisfies also the condition*

$$\boldsymbol{n}(\boldsymbol{n}(u)) = u \quad \textit{for all } u \in [0,1] \tag{27}$$

It is interesting to remark that the inverse function n^{-1} of each strict negation function n is again a strict negation function. And strong negation functions even coincide with their inverse functions.

Regarding the previous examples, non_2 is not a negation function at all assuming $m > 2$, but non_0 is a negation function, and non_1 is even a strong negation function. Another example of a negation function which is not a strict one is characterised by the equation

$$\mathsf{non}^*(u) =_{\mathrm{def}} \begin{cases} 1 & \text{if } u < 1 \\ 0 & \text{if} u = 1. \end{cases} \tag{28}$$

and thus obviously a kind of dual to non_0.

These two negation functions $\mathsf{non}_0, \mathsf{non}^*$ are extreme examples in the sense that for each negation function n it holds true

$$\mathsf{non}_0 \leqslant n \leqslant \mathsf{non}^*. \tag{29}$$

An example of a strict negation function which is not also a strong one is provided by the function non_3 with

$$\mathsf{non}_3(u) =_{\mathrm{def}} 1 - u^2. \tag{30}$$

Other examples of strong negation functions are all the functions of the family

$$n_{S,\lambda}(u) =_{\mathrm{def}} \frac{1-u}{1+\lambda u} \tag{31}$$

with parameter $\lambda > -1$ which M. Sugeno [103] introduced under the name of *λ-complement.*

For strict as well as for strong negations one has nice representation theorems, cf. [22, 106].

Theorem 3.9 **(i)** *A function $n : [0,1] \to [0,1]$ is a strong negation function iff there exists an automorphism φ of the unit interval such that*

$$n(u) = \varphi^{-1}(1 - \varphi(u)) \quad \textit{for all } u \in [0,1]. \tag{32}$$

(ii) *A function $n : [0,1] \to [0,1]$ is a strict negation function iff there exist automorphisms φ, ψ of the unit interval such that*

$$n(u) = \psi(1 - \varphi(u)) \quad \textit{for all } u \in [0,1]. \tag{33}$$

Strong as well as strict negation functions are hence "variants" of the negation function non_1.

Combining t-norms and negation functions allows to prove some further interesting representation theorem for a class of particular t-norms, cf. [81, 24].

Theorem 3.10 *Suppose that t is a continuous t-norm and n a strict negation function. Then one has*

$$t(u, n(u)) = 0 \quad \textit{for all } u \in [0,1]$$

iff there exists an automorphism φ *of the unit interval such that for all* $u, v \in [0,1]$ *one has*

$$t(u,v) = \varphi^{-1}(\mathsf{et}_2(\varphi(u), \varphi(v))) \quad \textit{and} \quad n(u) \leq \varphi^{-1}(1 - \varphi(u)).$$

Therefore, the pair $\mathsf{et}_2, \mathsf{non}_1$ represents in a suitable sense the "typical situation" of a continuous t-norm and a strict negation function which together satisfy a kind of generalised "law of contradiction".

3.4 Disjunction connectives

With the actual preliminaries two different approaches toward truth degree functions for disjunction connectives are suitable. One of them is to look for reasonable conditions such truth degree functions should satisfy – and thus to parallel the approach toward truth degree functions for conjunction connectives. The other one is to approach a truth degree function for a disjunction connective from a truth degree functions for a conjunction connective and another one for a negation connective – and to demand that some analogue of the usual DEMORGAN connection between these types of connectives should hold true.

There is no reasonable preference one of these approaches has over the other. Thus we should look at both of them.

The general properties one likes to have satisfied by disjunction connectives lead to the following class of t-conorms as truth degree functions for them.

Definition 3.4 *A binary operation* s *in* $[0,1]$ *is a* t-conorm *iff it is*

(S1) *associative and commutative;*
(S2) *non-decreasing in each argument;*
(S3) *has* 0 *as neutral element, i.e.* $s(u,0) = u$ *for each* $u \in [0,1]$.

As a corollary of this definition one has that for any t-conorm s it holds true

$$s(u,1) = 1 \quad \text{for each } u \in [0,1]$$

because of $s(u,1) = s(1,u) \geq s(1,0) = 1$.

In algebraic terms, each t-conorm represents a semigroup operation in the unit interval $[0,1]$ with identity 0.

The other approach toward truth degree functions via a DEMORGAN connection amounts to introduce some function $s : [0,1]^2 \to [0,1]$ by defining with reference to some negation function n and to some truth degree function t for a conjunction connective, e.g. thus by some t-norm t

$$s(u,v) =_{\text{def}} n(t(n(u), n(v))) \quad \text{for all } u, v \in [0,1] \tag{34}$$

or by defining

$$s(u,v) =_{\text{def}} n^{-1}(t(n(u), n(v))) \quad \text{for all } u, v \in [0,1] \tag{35}$$

or something like.

Both these main approaches, however, coincide as the following result shows.

Theorem 3.11 *Suppose that* $\boldsymbol{n}$ *is a strong negation function and that two binary oparations* $\boldsymbol{t}, \boldsymbol{s}$ *in* $[0,1]$ *are related such that*

$$\boldsymbol{s}(u,v) = \boldsymbol{n}(\boldsymbol{t}(\boldsymbol{n}(u),\boldsymbol{n}(v))) \quad \textit{for all } u,v \in [0,1]$$

holds true. Then $\boldsymbol{t}$ *is a t-norm iff* $\boldsymbol{s}$ *is a t-conorm.*

The fact that within this theorem a strong negation function, i.e. an involutive one, is taken into account causes that (34) and (35) become equivalent. Therefore, the particular form of the DEMORGAN connection between t-norms and t-conorms becomes inessential.

The standard choice of a strong negation function to connect t-norms and t-conorms is non_1. Starting from a t-norm $\boldsymbol{t}$ one then combines with it the t-conorm

$$\boldsymbol{s_t}(u,v) =_{\text{def}} 1 - \boldsymbol{t}(1-u, 1-v). \tag{36}$$

The main examples for truth degree functions of disjunction connectives which ever have been discussed fir into this schema. The most popular examples are the truth degree functions

$$\mathsf{vel}_1(u,v) =_{\text{def}} \max\{u,v\}, \tag{37}$$
$$\mathsf{vel}_2(u,v) =_{\text{def}} \min\{1, u+v\}, \tag{38}$$
$$\mathsf{vel}_3(u,v) =_{\text{def}} u + v - u \cdot v, \tag{39}$$

which are chosen in such a way that vel_i is related to et_i via (36) for $i = 1\ldots,3$. The usual names for these truth degree functions parallel the names for the conjunction connectives: vel_2 is the LUKASIEWICZ *disjunction* or *bounded sum*, and vel_3 is the *algebraic sum*. With the drastic product $\boldsymbol{t}_0$ and the nilpotent minimum $\boldsymbol{t}_1$ in the same manner the t-conorms

$$\boldsymbol{s}_0(u,v) =_{\text{def}} \begin{cases} \max\{u,v\}, & \text{if } \min\{u,v\} = 0 \\ 1 & \text{otherwise,} \end{cases} \tag{40}$$

$$\boldsymbol{s}_1(u,v) =_{\text{def}} \begin{cases} \max\{u,v\}, & \text{if } u+v < 1 \\ 1 & \text{otherwise} \end{cases} \tag{41}$$

are connected and correspondingly named *drastic sum* and *nilpotent maximum*.

But there are also other, extralogical aspects which relate sometimes t-norms and t-conorms like functional equations as we mentioned already in connection with the introduction of the FRANK family of t-norms $\boldsymbol{t}_{F,s}$. For them one has, using $\boldsymbol{s}_{F,s}$ to denote the t-conorm related with $\boldsymbol{t}_{F,s}$ via (36), the following result.

Theorem 3.12 *A continuous t-norm* $\boldsymbol{t}$ *and a continuous t-conorm* $\boldsymbol{s}$ *satisfy the functional equation*

$$\boldsymbol{t}(u,v) + \boldsymbol{s}(u,v) = u + v \quad \textit{for all } u,v \in [0,1] \tag{42}$$

iff one of the following conditions is satisfied:

(i) there exists some $s \in [0,\infty]$ *such that* $\boldsymbol{t} = \boldsymbol{t}_{F,s}$ *and* $\boldsymbol{s} = \boldsymbol{s}_{F,s}$*,*

(ii) t is an ordinal sum of t-norms $t_{F,s}$ from the FRANK *family with $s > 0$ and s is determined via equation (42).*

And all archimedean solutions of (42) have the form $(t, s) = (t_{F,s}, s_{F,s})$ of t-norms of the FRANK *family and their corresponding t-conorms.*

Via (36) also notions and results, like archimedeanicity or representation theorems, which hold true for t-norms can be transfered to t-conorms. For details we refer to [24].

3.5 Implication connectives

The last kind of truth degree functions which needs to be considered separately are the truth degree functions for implication connectives. For biimplication connectives, the last type of connective usually taken as (more or less) basic, then the standard approach also in many-valued logic is to take them as suitable conjunctions of implications.

Historically the first example of a truth degree function for an implication connective was given by ŁUKASIEWICZ [59]. This function seq_2 can be characterised[16] by the equation

$$\mathsf{seq}_2(u, v) =_{\text{def}} \min\{1, 1 - u + v\}. \tag{43}$$

This truth degree function as well as its corresponding connective are usually refered to as ŁUKASIEWICZ *implication.*

Another important example was introduced by GÖDEL [28] and may be characterised by

$$\mathsf{seq}_1(u, v) =_{\text{def}} \begin{cases} 1 & \text{if } u \leq v \\ v & \text{otherwise.} \end{cases} \tag{44}$$

As in the case of the previous example, this truth degree function as well as its corresponding implication connective are usually refered to as GÖDEL *implication.*

From a more general point of view, again one either looks for ways to reduce or relate implication connectives to other ones, or one asks for general properties any (suitable) implication connective should satisfy.

First examing ways to relate implications to other connectives, one of the standard methods in classical logic is to reduce implication to disjunction and negation or to conjunction and negation. And indeed the same reducibility property is satisfied for the ŁUKASIEWICZ implication seq_2 because one has for all $u, v \in [0, 1]$:

$$\mathsf{seq}_2(u, v) = \mathsf{vel}_2(\mathsf{non}_1(u), v) = \mathsf{non}_1(\mathsf{et}_2(u, \mathsf{non}_1(v))). \tag{45}$$

These results also give the motivation to call vel_2 the ŁUKASIEWICZ *(arithmetic) disjunction* and to call et_2 the ŁUKASIEWICZ *(arithmetic) conjunction.*

[16] The choice of the index "2" here has a systematic background which shall become clear soon, cf. (45).

The last type of reduction is impossible for the GÖDEL implication, even assuming that the negation function n involved in these formulas is a strong one. For in such a case $\mathsf{seq}_1(n(u), v)$ had to be a commutative binary operation in u, v which obviously is impossible. However, another type of reduction is possible and one has

$$\mathsf{seq}_1(u, v) = \sup\{w \mid \mathsf{et}_1(u, w) \leq v\}. \tag{46}$$

And this reduction is a quite reasonable one. This becomes clear if one has in mind that GÖDEL's paper [28], which introduced the truth degree function seq_1, discussed the possible relationships of many-valued and intuitionistic logic: in HEYTING algebras, the characteristic structures for intuitionistic logic, the intuitionistic implication is interpreted as the pseudocomplement and thus has a characterisation like (46) which can equivalently be written as

$$w \leq \mathsf{seq}_1(u, v) \quad \Leftrightarrow \quad \mathsf{et}_1(u, w) \leq v. \tag{47}$$

Algebraically, this construction amounts to having via this adjunction property (47) the GÖDEL implication as the relative pseudocomplement of the lattice $\langle [0,1], \min, \max \rangle$ resp. to having an adjoint pair $\mathsf{et}_1, \mathsf{seq}_1$.

It is interesting and important to notice that also for the ŁUKASIEWICZ implication (43) one has

$$\mathsf{seq}_2(u, v) = \sup\{w \mid \mathsf{et}_2(u, w) \leq v\}. \tag{48}$$

Therefore also $\mathsf{et}_2, \mathsf{seq}_2$ form an *adjoint pair.*

These examples (46), (48) indicate that the adjunction property

$$w \leq \mathsf{seq}(u, v) \quad \Leftrightarrow \quad \mathsf{et}(u, w) \leq v, \tag{49}$$

i.e. the fact that $\mathsf{et}, \mathsf{seq}$ form an adjoint pair, may be a characteristic connection between a conjunction connective et and an implication connective seq.

Straightforward calculations yield that from the adjunction property (49) one gets that the truth degree function seq is isotonic, i.e. nondecreasing in the second argument, and additionally satisfies the inequalities

$$\mathsf{et}(u, \mathsf{seq}(u, v)) \leq v \quad \text{and} \quad v \leq \mathsf{seq}(u, \mathsf{et}(u, v)) \tag{50}$$

which code[17] the soundness of the rule of detachment (for the implication connective based on the truth degree function seq) and of a kind of rule of introduction of conjunction. Particularly the availability of the inference schema of *modus ponens*, i.e. of the rule of detachment, for each adjoint pair $\mathsf{et}, \mathsf{seq}$ makes this approach highly valuable.

Together, the two types of reduction (45) on the one hand and (46), (48) resp. (49) on the other provide standard examples and motivate the following definition. Sometimes some further types of implication functions are discussed which shall also be introduced.

[17]This way of coding is the same as in classical propositional logic and based on the understanding that the higher truth degrees are the "better" ones.

Definition 3.5 *Let some t-norm t, some t-conorm s and some strong negation function n be given. Then the* R-implication function seq_t *determined by t is defined as*

$$\mathsf{seq}_t(u,v) =_{\mathrm{def}} \sup\{w \mid t(u,w) \leq v\}, \tag{51}$$

the S-implication function $\mathsf{seq}_{s,n}$ *determined by s and n is defined as*

$$\mathsf{seq}_{s,n}(u,v) =_{\mathrm{def}} s(n(u),v), \tag{52}$$

and the QL-implication function[18] $\mathsf{seq}_{t,s,n}$ *determined by t, s, and n is defined as*

$$\mathsf{seq}_{t,s,n}(u,v) =_{\mathrm{def}} s(n(u),t(u,v)). \tag{53}$$

Some care is necessary, however, with the definition of the R-implications: in the full generality as in (51) an adjoint pair t, seq_t results exactly in the cases where t is a left continuous t-norm. Therefore, this definition of R-implication is reasonable only for left-continuous t-norms.

R-implications as well as S-implications generalise well known relationships of classical and intuitionistic logic. From this point of view it is difficult to rank these generalisations. There is, nevertheless, an interesting point of difference: R-implications of left-continuous t-norms as implications coming from an adjoint pair satisfy the generalised *modus ponens principle* in the form that one always has

$$t(u, \mathsf{seq}_t(u,v)) \leq v, \tag{54}$$

the S-implications in general do not have this property.

As a side remark, it is interesting to notice that R-implications did not only appear in the context of intuitionistic logic, but also in connection with fuzzy relational equations and their solutions. There SANCHEZ [95, 96] interest in biggest solutions of fuzzy relational equations with max-min composition introduced a so-called α-operation[19] which was exactly our operation seq_1. Generalising the max-min composition to max-t composition, t some t-norm, W. PEDRYCZ [84] introduced the notion of *Φ-operator* φ_t (connected with t) for a binary operation that has to satisfy the isotonicity condition for the second argument together with further characterising conditions which exactly correspond to the properties mentioned in (50). Straightforward calculations yield also in this case that a t-norm and its corresponding Φ-operator form an adjoint pair. Therefore these Φ-operators φ_t are exactly the R-implications seq_t.

Quite another approach as giving these definitions is it to have some list of basic characteristic properties of implication operations i which collects basic properties all suitable implication operations should have. Unfortunately, however, for implication operators actually there does not exist a commonly agreed such list. Nevertheless, such a list was proposed by SMETS/MAGREZ [102] and extended by different authors, cf. [24]. A reasonably complete collection of such properties consists of the following ones:

[18] This name QL-implication reminds an implication connective used in quantum logic.

[19] This α-operation is discussed also in the chapter on fuzzy relational equations in vol. 1: *Fundamentals of Fuzzy Sets* of this set of handbooks.

1. *left antitonicity*: i is nonincreasing in the first argument.

2. *right isotonicity*: i is nondecreasing in the second argument.

3. *left boundary condition*: $i(0,v)=1$ for all $v \in [0,1]$.

4. *right boundary condition*: $i(u,1)=1$ for all $u \in [0,1]$.

5. *normality condition*: $i(1,0)=0$.

6. *degree ranking property*: $i(u,v)=1 \;\Leftrightarrow\; u \leq v$ for all $u,v \in [0,1]$.

7. *left neutrality*: $i(1,v)=v$ for all $u \in [0,1]$.

8. *exchange principle*: $i(u,i(v,w)) = i(v,i(u,w))$ for all $u,,wv \in [0,1]$.

9. *law of contraposition*: $i(u,v) = i(n(v),n(u))$ for all $u,v \in [0,1]$ w.r.t. some strict negation function n.

The first five of these conditions seem to represent a kind of minimal requirement for suitable truth degree functions for implication connectives. Hence, following [24], we give the

Definition 3.6 *A function $i : [0,1]^2 \to [0,1]$ is an* implication function *iff it satisfies the conditions of left antitonicity and right isotonicity together with the left and right boundary and the normality condition.*

Of course, not all the properties listed above are independent. One of the interesting results in this context is the following

Proposition 3.13 *Any function $i : [0,1]^2 \to [0,1]$ which satisfies the condition of right isotonicity, the exchange principle, and which has the degree ranking property is an implication function which satisfies also the left neutrality condition.*

Even more interesting is the fact that the R-implications as well as the S-implications have characterisations in terms of these properties, as the next two theorems show.

Theorem 3.14 *A function $i : [0,1]^2 \to [0,1]$ is an R-implication based on some suitable left continuous t-norm iff it satisfies the right isotonicity condition as well as the exchange principle, has the degree ranking property and is itself a left continuous function in the second argument for any first argument from $[0,1]$.*

And if additionally for each $u \in (0,1)$ the function $i_u : [0,1] \to [0,1]$ defined by $i_u(y) = i(u,y)$ is strictly increasing on $[0,u]$, then i is an R-implication which is based on a continuous t-norm.[20]

[20]The last part of this theorem was announced in [16].

Theorem 3.15 *An implication function is an S-implication function w.r.t. some strong negation function $\boldsymbol{n}$ and some suitable t-conorm iff it satisfies the exchange principle, the law of contraposition w.r.t. $\boldsymbol{n}$, and the left neutrality condition.*

For continuous R-implications, furthermore, the Łukasiewicz implication seq_2 again is a kind of "prototype" as the following representation theorem shows, cf. [24].

Theorem 3.16 *A function $\boldsymbol{i} : [0,1]^2 \to [0,1]$ is continuous and satisfies the right isotonicity condition as well as the exchange principle, and has the degree ranking property iff there exists an automorphism φ of the unit interval such that*

$$\boldsymbol{i}(u,v) = \varphi^{-1}(\mathsf{seq}_2(\varphi(u),\varphi(v))) \quad \textit{for all } u,v \in [0,1].$$

Despite the fact that we started from truth degree functions for conjunction and negation connectives, mainly because in these cases one widely agrees on the basic structural properties of such connectives, also (the truth degree functions of) implication connectives could form the starting point to derive other types of connectives.

This is most famous in the case of negations, again following the paradigm set by intuitionistic logic.

Theorem 3.17 *Suppose that a function $\boldsymbol{i} : [0,1]^2 \to [0,1]$ satisfies the right isotonicity condition and the exchange principle, and has the degree ranking property. Then the function $\boldsymbol{n} : [0,1] \to [0,1]$ which is defined by the equation*

$$\boldsymbol{n}(u) =_{\mathrm{def}} \boldsymbol{i}(u,0) \quad \textit{for all } u \in [0,1]. \tag{55}$$

has the following properties:

(i) $\boldsymbol{n}$ is a negation function.

(ii) $id \leqslant \boldsymbol{n} \circ \boldsymbol{n}$.

(iii) $\boldsymbol{n} = \boldsymbol{n} \circ \boldsymbol{n} \circ \boldsymbol{n}$.

(iv) If $\boldsymbol{n}$ is also continuous, then it is involutive and the law of contraposition is satisfied.

In the particular case of the Łukasiewicz negation, by (55) one gets the negation function non_1:

$$\mathsf{non}_1(u) = \mathsf{seq}_2(u,0) \quad \text{for all } u \in [0,1].$$

Furthermore, even the Łukasiewicz disjunction and conjunction functions are determined by the Łukasiewicz implication:

$$\begin{aligned} \mathsf{vel}_2(u,v) &= \mathsf{seq}_2(\mathsf{non}_1(u),v) \quad \text{for all } u,v \in [0,1], \\ \mathsf{et}_2(u,v) &= \mathsf{non}_1(\mathsf{seq}_2(u,\mathsf{non}_1(v))) \quad \text{for all } u,v \in [0,1]. \end{aligned}$$

4 Many-valued propositional logics

Based on the previously discussed varieties of truth degree functions of connectives for systems of many-valued propositional logic we now take a closer look at different such systems which have been considered for quite different reasons.

According to our remarks in Section 1.1 the languages $\mathcal{L}_\mathsf{S}$ of the following systems S of propositional logic are determined by their families of connectives and of truth degree constants. As set of *propositional variables* in any case the set

$$V = \{p', p'', p''', \dots\}$$

is chosen and the convention adopted to use as metavariables for propositional variables the symbols $p, q, r, \dots$ possibly with indices. The truth degree constants, however, play a rather marginal role in these systems, hence the sets of connectives $\mathcal{J}^\mathsf{S}$ are of main importance.

For the semantic interpretation of these systems one needs the sets $\mathcal{W}^\mathsf{S}$ of truth degrees and $\mathcal{D}^\mathsf{S}$ of designated truth degrees together with the families of truth degree functions for the connectives of $\mathcal{L}_\mathsf{S}$. These data commonly are collected into the *characteristic matrix*

$$\mathfrak{M}_\mathsf{S} = \langle \mathcal{W}^\mathsf{S}, \mathcal{D}^\mathsf{S}, \tau_1, \dots, \tau_n \rangle \tag{56}$$

in which $\tau_1, \dots, \tau_n$ is the family of the truth degree functions of the basic connectives of S.

Therefore the basic data for each of the following systems of many-valued propositional logic are the family of connectives and the characteristic matrix.

In most of these cases the set of truth degrees is not completely fixed. For simplicity of notation hence we additionally write $\mathcal{W}^*$ to indicate any one for the truth degree sets $\mathcal{W}_0, \mathcal{W}_\infty$ and all $\mathcal{W}_m, m \geq 2$.

Having in mind that the t-norms form a very basic class of candidates for truth degree functions of conjunction connectives and remembering that by Theorems 3.5 and 3.7 besides $\boldsymbol{t} = \mathrm{et}_1 = \min$ the continuous strict t-norms are of particular importance, then also $\boldsymbol{t} = \mathrm{et}_3$ becomes a kind of prototypical continuous t-norm. Another such prototypical t-norm, in presence of a suitable negation function, furthermore is $\boldsymbol{t} = \mathrm{et}_2$ according to Theorem 3.10. Therefore systems of many-valued logic with these conjunction operations form, independent of their historical importance, very interesting cases of such systems.

4.1 The Łukasiewicz systems

The Łukasiewicz propositional systems L_ν with $\nu = 0, 3, 4, \dots, \infty$ have in [59, 63] been originally formulated in negation and implication, i.e. their set of basic connectives $\mathcal{J}^{\mathsf{L}_\nu}$ is $\{\neg, \rightarrow_\mathsf{L}\}$. They have the characteristic matrices[21]

$$\mathfrak{M}_{\mathsf{L}_\nu} = \langle \mathcal{W}_\nu, \{1\}, \mathsf{non}_1, \mathsf{seq}_2 \rangle. \tag{57}$$

[21] Classical propositional logic (in implication and negation) fits well into this schema. Therefore sometimes L_2 is considered too and understood simply as classical logic. We shall not follow this use and restrict the considerations to the "proper" many-valued systems with $\mathcal{W}^* \neq \{0, 1\}$.

This choice of the truth degree functions is the reason that seq_2 is denoted as ŁUKASIEWICZ implication function.

According to Proposition 3.1 these ŁUKASIEWICZ systems are not functionally complete. This poses the problem of a characterisation of all those truth degree functions which can be generated by superposition of seq_2 and non_1, i.e. which can be defined in the ŁUKASIEWICZ systems. Obviously, it is enough to solve this problem for the case of the infinite valued system L_∞ because any truth degree function definable in any one of the systems L_m is a restriction of the truth degree function of L_∞ which is defined by the same defining formula. The following theorem of MCNAUGHTON [66] solves this problem.[22]

Theorem 4.1 *Let $f : [0,1]^n \to [0,1]$ be any n-ary function. The function f is a truth degree function determined by some sentence of the* ŁUKASIEWICZ *system L_∞ iff f is continuous and there exist a finite number of polynomials*

$$g_i(x_1, \ldots, x_n) = b_i + \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, \ldots, m$$

with integer coefficients a_{ij}, b_i such that for all $d_1, \ldots, d_n \in [0,1]$ there exists some $1 \le k \le m$ with

$$f(d_1, \ldots, d_n) = g_k(d_1, \ldots, d_n).$$

It is common usage to define further connectives for arbitrary well formed formulas H_1, H_2 by

$$H_1 \vee H_2 =_{\mathsf{def}} (H_1 \to_{\mathsf{L}} H_2) \to_{\mathsf{L}} H_2, \tag{58}$$
$$H_1 \wedge H_2 =_{\mathsf{def}} \neg(\neg H_1 \vee \neg H_2), \tag{59}$$
$$H_1 \otimes H_2 =_{\mathsf{def}} \neg(H_1 \to_{\mathsf{L}} \neg H_2), \tag{60}$$
$$H_1 \oplus H_2 =_{\mathsf{def}} \neg H_1 \to_{\mathsf{L}} H_2, \tag{61}$$
$$H_1 \leftrightarrow_{\mathsf{L}} H_2 =_{\mathsf{def}} (H_1 \to_{\mathsf{L}} H_2) \wedge (H_2 \to_{\mathsf{L}} H_1). \tag{62}$$

The first four of these connectives have well known truth degree functions:

connective	$\vee$	$\wedge$	$\otimes$	$\oplus$
truth degree function	vel_1	et_1	et_2	vel_2

And the truth degree function $\tau^{\mathsf{L}}_{\leftrightarrow}$ for the biimplication connective $\leftrightarrow_{\mathsf{L}}$ is characterised by the equation

$$\tau^{\mathsf{L}}_{\leftrightarrow}(u, v) = 1 - |u - v|. \tag{63}$$

[22] MCNAUGHTON's proof involved a lot of computetions and looked quite sophisticated. Recently, a more transparent proof which emphasises the geometric ideas was given in [77].

This has, by the way, the interesting consequence that the truth degree function determined by $\neg(H_1 \leftrightarrow_{\mathsf{L}} H_2)$ is a metric, i.e. measures the distance of the truth degrees of H_1 and H_2.

The logically valid sentences of different such ŁUKASIEWICZ systems are nicely related. Using the notation $\mathsf{taut}^{\mathsf{L}}_{\nu}$ for the set of all logically valid sentences of the system L_{ν} and additionally $\mathsf{taut}^{\mathsf{L}}_{2}$ for the set of all tautologies of classical two-valued propositional logic one has the following interesting results.

Theorem 4.2 *For all $m, n \in \mathbb{N}$ with $m, n \geq 2$ there hold true:*

$$\begin{array}{ll} (i) & \mathsf{taut}^{\mathsf{L}}_{m} \subseteq \mathsf{taut}^{\mathsf{L}}_{n} \Leftrightarrow \mathcal{W}_m \supseteq \mathcal{W}_n, \\ (ii) & \mathsf{taut}^{\mathsf{L}}_{m} \subseteq \mathsf{taut}^{\mathsf{L}}_{n} \Leftrightarrow (n-1) \mid (m-1), \\ (iii) & \mathsf{taut}^{\mathsf{L}}_{m} \not\subseteq \mathsf{taut}^{\mathsf{L}}_{m+1} \quad \textit{and} \quad \mathsf{taut}^{\mathsf{L}}_{m+2} \not\subseteq \mathsf{taut}^{\mathsf{L}}_{m+1}, \\ (iv) & \mathsf{taut}^{\mathsf{L}}_{\infty} = \displaystyle\bigcap_{m=3}^{\infty} \mathsf{taut}^{\mathsf{L}}_{m}, \\ (v) & \mathsf{taut}^{\mathsf{L}}_{\infty} = \mathsf{taut}^{\mathsf{L}}_{0}. \end{array}$$

Therefore, only one of the infinite valued systems needs consideration. Because of the quantifiers usually considered in first order logic it is preferable to prefer the system L_{∞}.

From the syntactical point of view, the problem of adequate axiomatisations for the ŁUKASIEWICZ systems is the most interesting one still to consider.

For the infinite valued system L_{∞} a very simple axiomatisation is available. It was conjectured as a adequate, i.e. sound and complete axiomatisation already in the 1930th. The proof of this fact, however, was given only in [92] in 1958.

Theorem 4.3 *A sound and complete axiomatisation of the system L_{∞} is given by the rule of detachment w.r.t. the* ŁUKASIEWICZ *implication $\rightarrow_{\mathsf{L}}$ together with the following list of axiom schemata:*

$$\begin{array}{ll} (i) & H_1 \rightarrow_{\mathsf{L}} (H_2 \rightarrow_{\mathsf{L}} H_1), \\ (ii) & (H_1 \rightarrow_{\mathsf{L}} H_2) \rightarrow_{\mathsf{L}} ((H_2 \rightarrow_{\mathsf{L}} H_3) \rightarrow_{\mathsf{L}} (H_1 \rightarrow_{\mathsf{L}} H_3)), \\ (iii) & (\neg H_2 \rightarrow_{\mathsf{L}} \neg H_1) \rightarrow_{\mathsf{L}} (H_1 \rightarrow_{\mathsf{L}} H_2), \\ (iv) & ((H_1 \rightarrow_{\mathsf{L}} H_2) \rightarrow_{\mathsf{L}} H_2) \rightarrow_{\mathsf{L}} ((H_2 \rightarrow_{\mathsf{L}} H_1) \rightarrow_{\mathsf{L}} H_1). \end{array}$$

A correspondingly simple axiomatisation exists for the other extreme case, the three-valued system L_3 and was given by WAJSBERG [111].

Theorem 4.4 *A sound and complete axiomatisation of the system L_3 is given by the rule of detachment w.r.t. the* ŁUKASIEWICZ *implication $\rightarrow_{\mathsf{L}}$ together with the following list of axiom schemata:*

$$\begin{array}{ll} (i) & H_1 \rightarrow_{\mathsf{L}} (H_2 \rightarrow_{\mathsf{L}} H_1), \\ (ii) & (H_1 \rightarrow_{\mathsf{L}} H_2) \rightarrow_{\mathsf{L}} ((H_2 \rightarrow_{\mathsf{L}} H_3) \rightarrow_{\mathsf{L}} (H_1 \rightarrow_{\mathsf{L}} H_3)), \\ (iii) & (\neg H_2 \rightarrow_{\mathsf{L}} \neg H_1) \rightarrow_{\mathsf{L}} (H_1 \rightarrow_{\mathsf{L}} H_2), \\ (iv) & ((H_1 \rightarrow_{\mathsf{L}} \neg H_1) \rightarrow_{\mathsf{L}} H_1) \rightarrow_{\mathsf{L}} H_1. \end{array}$$

Because for each set of logically valid sentences of L_m one has the inclusion $\mathsf{taut}^{\mathsf{L}}_\infty \subseteq \mathsf{taut}^{\mathsf{L}}_m$ it is quite natural to expect to get an adequate axiomatisation for L_m by extending a suitable axiom system for L_∞. The formulation of such an axiom systems needs, however, the finite iterations of the connectives $\otimes, \oplus$. Therefore we define recursively for any family $H_1, H_2, \ldots$ of well formed formulas of $\mathcal{L}_{\mathsf{L}}$

$$\prod_{i=1}^{n+1} H_i =_{\text{def}} \prod_{i=1}^{n} H_i \otimes H_{n+1} \quad \text{with} \quad \prod_{i=1}^{1} H_i =_{\text{def}} H_1, \tag{64}$$

$$\sum_{i=1}^{n+1} H_i =_{\text{def}} \sum_{i=1}^{n} H_i \oplus H_{n+1} \quad \text{with} \quad \sum_{i=1}^{1} H_i =_{\text{def}} H_1. \tag{65}$$

Theorem 4.5 *A sound and complete axiomatisation of the system L_m is given by the rule of detachment w.r.t. the* ŁUKASIEWICZ *implication $\rightarrow_{\mathsf{L}}$ together with the following list of axiom schemata:*

$$\begin{array}{ll}
(i) & H_1 \rightarrow_{\mathsf{L}} (H_2 \rightarrow_{\mathsf{L}} H_1), \\
(ii) & (H_1 \rightarrow_{\mathsf{L}} H_2) \rightarrow_{\mathsf{L}} ((H_2 \rightarrow_{\mathsf{L}} H_3) \rightarrow_{\mathsf{L}} (H_1 \rightarrow_{\mathsf{L}} H_3)), \\
(iii) & (\neg H_2 \rightarrow_{\mathsf{L}} \neg H_1) \rightarrow_{\mathsf{L}} (H_1 \rightarrow_{\mathsf{L}} H_2), \\
(iv) & ((H_1 \rightarrow_{\mathsf{L}} H_2) \rightarrow_{\mathsf{L}} H_2) \rightarrow_{\mathsf{L}} ((H_2 \rightarrow_{\mathsf{L}} H_1) \rightarrow_{\mathsf{L}} H_1), \\
(v) & \displaystyle\sum_{i=1}^{m} H \rightarrow_{\mathsf{L}} \sum_{i=1}^{m-1} H, \\
(vi) & \displaystyle\sum_{i=1}^{m-1} \left(\prod_{i=1}^{k} \oplus \left(\neg H \otimes \sum_{i=1}^{k-1} H \right) \right) \qquad \begin{array}{l} \textit{for all } 1 < k < m \\ \textit{with } (k-1) \not| (m-1). \end{array}
\end{array}$$

4.2 The GÖDEL systems

GÖDEL's family of many-valued propositional logics G_ν with $\nu = 0, 3, 4, \ldots, \infty$ was introduced in [28] in the context of investigations aimed to understand intuitionistic logic. Accordingly they have been formulated as systems in conjunction, disjunction, negation and, implication with a set of basic connectives $\mathcal{J}^{\mathsf{G}_\nu} = \{\wedge, \vee, \sim, \rightarrow_{\mathsf{G}}\}$. They have as their characteristic matrices[23] the structures

$$\mathfrak{M}_{\mathsf{G}_\nu} = \langle \mathcal{W}_\nu, \{1\}, \mathsf{et}_1, \mathsf{vel}_1, \mathsf{non}_0, \mathsf{seq}_1 \rangle. \tag{66}$$

This choice of the truth degree functions is the reason that seq_1 is called GÖDEL implication function.

According to Proposition 3.1 also these GÖDEL systems are not functionally complete.

[23]Classical propositional logic fits well into this schema. Therefore sometimes G_2 is considered too and understood simply as classical logic. We shall not follow this use and restrict the considerations to the "proper" many-valued systems with $\mathcal{W}^* \neq \{0, 1\}$.

For the sets $\mathsf{taut}^{\mathsf{G}}_{\nu}$ of all logically valid sentences of G_{ν} one has as for the ŁUKASIEWICZ systems the equality $\mathsf{taut}^{\mathsf{L}}_{\infty} = \mathsf{taut}^{\mathsf{L}}_{0}$ and therefore only one infinite valued GÖDEL system. Furthermore one has for all $2 \leq m \in \mathbb{N}$ the relations

$$\mathsf{taut}^{\mathsf{G}}_{\infty} \subset \mathsf{taut}^{\mathsf{G}}_{m+1} \subset \mathsf{taut}^{\mathsf{G}}_{m}, \tag{67}$$

$$\mathsf{taut}^{\mathsf{G}}_{\infty} = \bigcap_{m=2}^{\infty} \mathsf{taut}^{\mathsf{G}}_{m} \tag{68}$$

with again $m = 2$ indicating the case of classical logic.

The close relationship of the GÖDEL systems with intuitionistic logic proves to be essential for the axiomatisation of the GÖDEL systems as was shown by DUMMETT [17].

Theorem 4.6 *A sound and complete axiomatisation of the infinite valued* GÖDEL *system* G_{∞} *is provided by any adequate axiomatisation of intuitionistic propositional logic (with the connectives* $\wedge, \vee, \sim, \rightarrow_{\mathsf{G}}$ *of* $\mathcal{L}_{\mathsf{G}}$ *instead of the intuitionistic ones) enriched with the axiom schema*

$$(H_1 \rightarrow_{\mathsf{G}} H_2) \vee (H_2 \rightarrow_{\mathsf{G}} H_1). \tag{69}$$

An adequate axiomatisation for $\mathsf{taut}^{\mathsf{G}}_{\infty}$, hence, is provided by the propositional calculus LC which is constituted by the *rule of detachment* (w.r.t. the implication connective $\rightarrow_{\mathsf{G}}$) together with the following axiom schemata:

(LC1) $H_1 \rightarrow_{\mathsf{G}} (H_1 \wedge H_1)$,
(LC2) $(H_1 \wedge H_2) \rightarrow_{\mathsf{G}} (H_2 \wedge H_1)$,
(LC3) $(H_1 \rightarrow_{\mathsf{G}} H_2) \rightarrow_{\mathsf{G}} (H_1 \wedge H_3 \rightarrow_{\mathsf{G}} H_2 \wedge H_3)$,
(LC4) $((H_1 \rightarrow_{\mathsf{G}} H_2) \wedge (H_2 \rightarrow_{\mathsf{G}} H_3)) \rightarrow_{\mathsf{G}} (H_1 \rightarrow_{\mathsf{G}} H_3)$,
(LC5) $H_1 \rightarrow (H_2 \rightarrow_{\mathsf{G}} H_1)$,
(LC6) $H_1 \wedge (H_1 \rightarrow_{\mathsf{G}} H_2) \rightarrow_{\mathsf{G}} H_2$,
(LC7) $H_1 \rightarrow_{\mathsf{G}} H_1 \vee H_2$,
(LC8) $H_1 \vee H_2 \rightarrow_{\mathsf{G}} H_2 \vee H_1$,
(LC9) $(H_1 \rightarrow_{\mathsf{G}} H_3) \wedge (H_2 \rightarrow_{\mathsf{G}} H_3) \rightarrow_{\mathsf{G}} (H_1 \vee H_2 \rightarrow_{\mathsf{G}} H_3))$,
(LC10) $\sim H_1 \rightarrow_{\mathsf{G}} (H_1 \rightarrow_{\mathsf{G}} H_2)$,
(LC11) $(H_1 \rightarrow_{\mathsf{G}} H_2) \wedge (H_1 \rightarrow_{\mathsf{G}} \sim H_2) \rightarrow_{\mathsf{G}} \sim H_1$,
(LC12) $(H_1 \rightarrow_{\mathsf{G}} H_2) \vee (H_2 \rightarrow_{\mathsf{G}} H_1)$.

For a solution of the axiomatisation problem of the finitely many-valued systems $\mathsf{taut}^{\mathsf{G}}_{m}$ we shall look at extensions of this propositional calculus LC. And by a *proper extension* a propositional calculus shall be understood which proves more theorems than LC, cf. [18, 33].

Theorem 4.7 *If the propositional calculus* T *is a proper consistent extension of* LC *by some additional axiom schemata, then there exists some* $2 \leq m \in \mathbb{N}$ *such that the set of all theorems of* T *is the set* $\mathsf{taut}^{\mathsf{G}}_{m}$ *of all logically valid sentences of* G_m.

As a corollary one gets an adequate axiomatisation of each one of the systems $\mathsf{taut}^{\mathsf{G}}_m$ by extending LC by the axiom schema

$$\bigvee_{i=1}^{m} \bigvee_{k=i+1}^{m+1} ((H_i \to_{\mathsf{G}} H_k) \wedge (H_k \to_{\mathsf{G}} H_i)).$$

4.3 The product logic

A particular case of a many-valued propositional system with a t-norm based conjunction connective $\odot$ is provided by considering the case of the product $t = \mathsf{et}_3$ as corresponding truth degree function. This t-norm is continuous which allows to adjoin a corresponding R-implication connective $\to_P$ with truth degree function characterised by

$$\mathsf{seq}_3(u,v) = \begin{cases} 1 & \text{if } u \le v \\ \frac{v}{u} & \text{otherwise} \end{cases} \tag{70}$$

according to (51).

Based on this origin the *product logic* $\Pi\mathsf{L}$ has as its primitive connectives the set $\{\to_P, \odot, 0, 1\}$ and the structure

$$\langle \mathcal{W}_\infty, \{1\}, \mathsf{seq}_3, \mathsf{et}_3, 0, 1\rangle \tag{71}$$

of signature $\langle 2,2,0,0\rangle$ as its characteristic matrix.

As in the ŁUKASIEWICZ systems also here it is usual to introduce some additional connectives.

Definition 4.1 *For any well formed formulas H_1, H_2 of the language of product logic let be*

$$\begin{aligned} \neg H_1 &=_{\mathrm{def}} H_1 \to_P 0, \\ H_1 \wedge H_2 &=_{\mathrm{def}} H_1 \odot (H_1 \to_P H_2), \\ H_1 \vee H_2 &=_{\mathrm{def}} ((H_1 \to_P H_2) \to_P H_2) \wedge ((H_2 \to_P H_1) \to_P H_1), \\ H_1 \leftrightarrow H_2 &=_{\mathrm{def}} (H_1 \to_P H_2) \wedge (H_2 \to_P H_1). \end{aligned}$$

Obviously, the truth degree functions of $\wedge, \vee$ are $\mathsf{et}_1, \mathsf{vel}_1$, resprectively.

The problem of axiomatisation was solved for this system only recently in [40], cf. also [39].

Theorem 4.8 *A sound and complete axiomatisation of the infinite valued product logic* $\Pi\mathsf{L}$ *is given by the following axiom schemata*

ΠL1	$H_1 \to_P (H_2 \to_P H_1)$,
ΠL2	$(H_1 \to_P H_2) \to_P ((H_2 \to_P H_3) \to_P (H_1 \to_P H_3))$,
ΠL3	$H_1 \to_P 1$ *and* $0 \to_P H_1$,
ΠL4	$H_1 \odot H_2 \to_P H_2 \odot H_1$,

ΠL5	$(H_1 \odot (H_2 \odot H_3)) \leftrightarrow ((H_1 \odot H_2) \odot H_3)$,
ΠL6	$(H_1 \odot H_2 \to_P H_3) \leftrightarrow (H_1 \to_P (H_2 \to_P H_3))$,
ΠL7	$(H_1 \to_P H_2) \to_P (H_1 \odot H_3 \to_P H_2 \odot H_3)$,
ΠL8	$\neg\neg H_3 \to_P ((H_1 \odot H_3 \to_P H_2 \odot H_3) \to_P (H_1 \to_P H_2))$,
ΠL9	$(H_1 \to_P H_2) \to_P ((H_1 \to_P H_3) \to_P (H_1 \to_P H_2 \wedge H_3))$,
ΠL10	$(H_1 \to_P H_3) \to_P ((H_2 \to_P H_3) \to_P (H_1 \vee H_2 \to_P H_3))$,
ΠL11	$(H_1 \to_P H_2) \vee (H_2 \to_P H_1)$,
ΠL12	$(H_1 \wedge \neg H_1) \to_P 0$

together with the rule of detachment as the only inference rule.

4.4 The Post systems

The Post family of finitely many-valued propositional logics P_m with $m = 3, 4, \ldots$ was introduced in [87] in the context of investigations in classical logic, e.g. of investigations toward functional completeness. They have been formulated as systems in disjunction and negation with set of basic connectives $\mathcal{J}^{\mathsf{P}_m} = \{\vee, \sim_{\mathsf{P}}\}$. They have as their characteristic matrices[24] the structures

$$\mathfrak{M}_{\mathsf{P}_m} = \langle \mathcal{W}_m, \{1\}, \mathsf{vel}_1, \mathsf{non}_2 \rangle. \tag{72}$$

This choice of the truth degree functions is the reason that non_2 sometimes is called Post negation function.

According to Theorem 3.2 the Post systems are functionally complete. This is, as it seems, the most interesting property they have.

The set $\mathcal{D}^{\mathsf{P}}$ of designated truth degrees is, however, not completely fixed for these systems. The present version $\mathcal{D}^{\mathsf{P}} = \{1\}$ is mainly chosen, but alredy Post [87] discussed also other possibilities.

This (more or less open situation) is actually not an essential difficulty. But this is connected with the (somewhat astonishing) fact that from the point of view of formal logic these Post systems have not been the object of extended studies. So, for instance, there do not exist till now axiomatisations of these systems. They have, however, been the source for introducing and studying a particular class of algebraic structures – called Post algebras – which, after some reformulations which essentially used the functional completeness of the Post systems for being equivalent to the original Post matrix, motivated investigations on some essentially modified systems of many-valued logic formulated without reference to the Post negation.

The m-valued logical system based on these m-valued Post algebras has as set of basic connectives the set

$$\mathcal{J}_{\mathsf{P}} = \{\sim, \to, \vee, \wedge, \mathbf{D}_1, \ldots, \mathbf{D}_{m-1}\} \tag{73}$$

[24] Classical propositional logic (in disjunction and negation) fits well into this schema. Therefore sometimes P_2 is considered too and understood simply as classical logic. We shall not follow this use and restrict the considerations to the "proper" many-valued systems with $\mathcal{W}^* \neq \{0,1\}$.

of five binary and $m-1$ unary connectives together with a set

$$\{\mathbf{e}_1, \ldots, \mathbf{e}_{m-1}\} \tag{74}$$

of truth degree constants and is based on the axiom schemata (LC1) to (LC10) and for all $i = 1, \ldots, m-1$ the further schemata which additionally refer to the biimplication connective $\leftrightarrow$ defined as $H \leftrightarrow G =_{\text{def}} (H \to G) \land (G \to H)$:

(P11) $\mathbf{D}_i(H_1 \lor H_2) \leftrightarrow (\mathbf{D}_i H_1 \lor \mathbf{D}_i H_2)$,

(P12) $\mathbf{D}_i(H_1 \land H_2) \leftrightarrow (\mathbf{D}_i H_1 \land \mathbf{D}_i H_2)$,

(P13) $\mathbf{D}_i(H_1 \to H_2) \leftrightarrow ((\mathbf{D}_1 H_1 \to \mathbf{D}_1 H_2) \land (\mathbf{D}_2 H_1 \to \mathbf{D}_2 H_2) \land \ldots \land (\mathbf{D}_i H_1 \to \mathbf{D}_i H_2)$,

(P14) $\mathbf{D}_i(\sim H_1) \leftrightarrow \sim \mathbf{D}_1 H_1$,

(P15) $\mathbf{D}_i \mathbf{D}_j H_1 \leftrightarrow \mathbf{D}_j H_1$,

(P16) $\mathbf{D}_i \mathbf{e}_j$ for $i \leq j$ and $\sim \mathbf{D}_i \mathbf{e}_j$ for $i > j$,

(P17) $H_1 \leftrightarrow (\mathbf{D}_1 H_1 \land \mathbf{e}_1) \lor (\mathbf{D}_2 H_1 \land \mathbf{e}_2) \lor \ldots \lor (\mathbf{D}_{m-1} H_1 \land \mathbf{e}_{m-1})$,

(P18) $\mathbf{D}_1 H_1 \lor \sim \mathbf{D}_1 H_1$

together with the rule

$$\frac{H}{\mathbf{D}_{m-1} H}$$

and the rule of detachment (w.r.t. the implication $\to$) as inference rules.

4.5 A partial generalization: monoidal logic

To a large extend caused by developments in the field of fuzzy sets and their applications, in recent years a tendency goes toward systems of many-valued logic which are based on the lattice

$$\langle [0,1], \mathsf{et}_1, \mathsf{vel}_1 \rangle$$

with universal bounds $0, 1$ enriched with an adjoint pair $(\boldsymbol{t}, \mathsf{seq}_{\boldsymbol{t}})$ consisting of a (left) continuous t-norm $\boldsymbol{t}$ and their corresponding R-implication $\mathsf{seq}_{\boldsymbol{t}}$ together with a negation introduced in accordance with (55) such that for the lattice ordering $\leqq$ the structure

$$\langle [0,1], \leqq, \boldsymbol{t} \rangle$$

forms an integral lattice ordered commutative semigroup, i.e. an integral lattice ordered monoid.

There is no simple standard characteristic matrix in this case, the just mentioned type of monoid, considered in more detail in Section 5.6, gives a whole class of algebraic structures characterizing what U. HÖHLE [44, 47] calls *monoidal logic.*

This monoidal logic itself is not a (pure) many-valued logic but it is nicely related also to systems of many-valued logic and therefore should be mentioned here. In any case, it has as its set of designated truth degrees the singleton $\mathcal{D}^{\mathsf{M}} = \{\mathbf{1}\}$ of the universal bound.

Propositional monoidal logic $\mathcal{L}_{\mathrm{mon}}$ as a formal system has $\{\wedge, \vee, \neg, \otimes, \rightarrow\}$ as its set of basic connectives. Here $\wedge, \vee$ are to be read as conjunction and disjunction connectives based upon the lattice operations and $\otimes$ is another, usually non-idempotent conjunction connective, often called "context". The logical calculus characterizing the formal theorems of $\mathcal{L}_{\mathrm{mon}}$ may be determined by the *rule of detachment* (w.r.t. the implication connective $\rightarrow$) as its only primitive inference rule together with the following set of axiom schemes:

$$
\begin{array}{ll}
(M_1) & ((H_1 \rightarrow H_2) \rightarrow ((H_2 \rightarrow H_3) \rightarrow (H_1 \rightarrow H_3))), \\
(M_2) & (H_1 \rightarrow (H_1 \vee H_2)), \\
(M_3) & (H_2 \rightarrow (H_1 \vee H_2)), \\
(M_4) & ((H_1 \rightarrow H_3) \rightarrow ((H_2 \rightarrow H_3) \rightarrow ((H_1 \vee H_2) \rightarrow H_3))), \\
(M_5) & ((H_1 \wedge H_2) \rightarrow H_1), \\
(M_6) & ((H_1 \otimes H_2) \rightarrow H_1), \\
(M_7) & ((H_1 \wedge H_2) \rightarrow H_2), \\
(M_8) & ((H_1 \otimes H_2) \rightarrow (H_2 \otimes H_1)), \\
(M_9) & (((H_1 \otimes H_2) \otimes H_3) \rightarrow (H_1 \otimes (H_2 \otimes H_3))), \\
(M_{10}) & ((H_3 \rightarrow H_1) \rightarrow ((H_3 \rightarrow H_2) \rightarrow (H_3 \rightarrow (H_1 \wedge H_2)))), \\
(M_{11}) & ((H_1 \rightarrow (H_2 \rightarrow H_3)) \rightarrow ((H_1 \otimes H_2) \rightarrow H_3)), \\
(M_{12}) & (((H_1 \otimes H_2) \rightarrow H_3) \rightarrow (H_1 \rightarrow (H_2 \rightarrow H_3))), \\
(M_{13}) & ((H_1 \otimes \neg H_1) \rightarrow H_2), \\
(M_{14}) & ((H_1 \rightarrow (H_1 \otimes \neg H_1)) \rightarrow \neg H_1).
\end{array}
$$

It is a routine matter to prove that the provability-of-implication relation

$$H_1 \rhd H_2 \quad =_{\mathsf{def}} \quad \vdash (H_1 \rightarrow H_2)$$

is a preordering in the class of all well-formed formulas and hence determines an equivalence relation of "provable equivalence". The corresponding Lindenbaum algebra L_{mon} then is an integral lattice ordered monoid w.r.t. the ordering relation $\leq$ induced by the preordering $\rhd$ and the operation $*$ induced by the connective $\otimes$.

Furthermore, this operation $*$ together with the operation $\rightarrow_{[]}$ induced by the connective $\rightarrow$ form an adjoint pair, i.e. one has for all $\alpha, \beta, \gamma \in L_{\mathrm{mon}}$

$$\alpha * \beta \leq \gamma \quad \Leftrightarrow \quad \alpha \leq \beta \rightarrow_{[]} \gamma.$$

Extending the list of axiom schemes of $\mathcal{L}_{\mathrm{mon}}$ with the *law of idempotency*

$$(M_{15}) \qquad (H \rightarrow (H \otimes H))$$

makes the connectives $\wedge$ and $\otimes$ logically equivalent and transforms thus $\mathcal{L}_{\mathrm{mon}}$ into intuitionistic logic.

Extending instead the list of axiom schemes of $\mathcal{L}_{\mathrm{mon}}$ with the *law of double negation*

$$(M_{16}) \qquad (\neg\neg H \rightarrow H)$$

transforms $\mathcal{L}_{\mathrm{mon}}$ into GIRARD's commutative linear logic [26]. And adding additionally the *law of divisibility*

$$(M_{17}) \qquad ((H_1 \wedge H_2) \rightarrow (H_1 \otimes (H_1 \rightarrow H_2))$$

transforms then $\mathcal{L}_{\mathrm{mon}}$ into the ŁUKASIEWICZ infinite valued logic L_∞.[25]

4.6 A uniform method of axiomatisation

Up to now, we considered (classes of) quite particular systems of many-valued logic which have been discussed by various authors for various reasons. There was always a kind of uniformity in the definition of the systems of one class, there were, however, important differences in the particular approaches.

From the semantic point of view, i.e. from the choice of truth degree sets and truth degree functions constitutive for the particular systems, this is not a problem.

However, it would be welcome to have a kind of uniform syntactic approach toward systems of many-valued logic giving, e.g., adequate axiomatisations for semantically determined systems.

Such an approach was offered by J.B. ROSSER and A.R. TURQUETTE in [94] for a wide class of systems S of many-valued logic with finite truth degree sets of the form $\mathcal{W}^{\mathsf{S}} = \mathcal{W}_m, m \geq 2$.

They assume that the set $\mathcal{J}^{\mathsf{S}}$ of connectives of S shall contain a binary connective $\rightarrow$, denoting a kind of implication connective, and unary connectives J_s for each $s \in \mathcal{W}^{\mathsf{S}}$, or at least that such connectives are definable from the primitive connectives of the system S.

These connectives have to satisfy the following conditions:

RT_1 The truth degree function corresponding to the connective $\rightarrow$ assumes a non-designated truth degree just in the case that its first argument is a designated truth degree and its second argument is a non-designated one.

RT_2 The unary truth degree function corresponding to the connective $\mathsf{J}_s, s \in \mathcal{W}^{\mathsf{S}}$, assumes a designated truth degree just in the case that their argument value is s, and a non-designated truth degree in all other cases.

For a compact formulation of the axioms a sort of finite iteration of the connective $\rightarrow$ is needed. Therefore we define for any well-formed formulas

[25] As a side remark it should be mentioned that extending $\mathcal{L}_{\mathrm{mon}}$ with all the schemes $(M_{15}), (M_{16}), (M_{17})$ yields classical propositional logic.

$H_1, H_2, \ldots, G$ of the language $\mathcal{L}_\mathsf{S}$ recursively:

$$\bigominus_{i=1}^{0} (H_i, G) =_{\text{def}} G$$

$$\bigominus_{i=1}^{k+1} (H_i, G) =_{\text{def}} H_{k+1} \to \bigominus_{i=1}^{k} (H_i, G).$$

Additionally, for each connective $\varphi \in \mathcal{J}^\mathsf{S}$ its corresponding truth degree function shall be ver_φ.

Now let $\mathsf{Ax}_{\mathrm{RT}}$ be the set of all well-formed formulas falling under one of the following schemata:

$\mathsf{Ax}_{\mathrm{RT}}1: \quad A \to (B \to A)$,

$\mathsf{Ax}_{\mathrm{RT}}2: \quad (A \to (B \to C)) \to (B \to (A \to C))$,

$\mathsf{Ax}_{\mathrm{RT}}3: \quad (A \to B) \to ((B \to C) \to (A \to C))$,

$\mathsf{Ax}_{\mathrm{RT}}4: \quad (\mathsf{J}_s(A) \to (\mathsf{J}_s(A) \to B)) \to (\mathsf{J}_s(A) \to B)$ for each $s \in \mathcal{W}^\mathsf{S}$,

$\mathsf{Ax}_{\mathrm{RT}}5: \quad \bigominus_{i=1}^{m} (\mathsf{J}_{\frac{i-1}{m-1}}(A) \to B, B)$,

$\mathsf{Ax}_{\mathrm{RT}}6: \quad \mathsf{J}_s(\mathbf{s})$ for each truth degree s and each truth degree constant $\mathbf{s}$ denoting it,

$\mathsf{Ax}_{\mathrm{RT}}7: \quad \mathsf{J}_t(A) \to A$ for each designated truth degree t ,

$\mathsf{Ax}_{\mathrm{RT}}8: \quad \bigominus_{i=1}^{n} (\mathsf{J}_{s_i}(A_i), \mathsf{J}_t(\varphi(A1, \ldots, A_n)))$ for each n-ary connective $\varphi \in \mathcal{J}^\mathsf{S}$, for all $s_1, \ldots, s_n \in \mathcal{W}^\mathsf{S}$ and for the particular truth degree $t = \mathsf{ver}_\varphi(s_1, \ldots, s_n)$.

Particularly the last group ($\mathsf{Ax}_{\mathrm{RT}}8$) of axioms indicates, that the present approach follows some kind of "brute force" strategy: this group simply codes the truth degree behaviour of all the available connectives of the system S.

For this approach, then, one has the following axiomatisation theorem, proven in [94] and e.g. also in [33].

Theorem 4.9 *Suppose that* S *is a system of many-valued logic with* $\mathcal{W}^\mathsf{S} = \mathcal{W}_m$ *and with connectives satisfying conditions* RT_1 *and* RT_2*. Then this system* S *has as an adequate, i.e. sound and complete axiomatisation the set* $\mathsf{Ax}_{\mathrm{RT}}$ *of formulas together with the rule of detachment (w.r.t. the connective* $\to$*).*

The assumptions of this theorem can be weakened or changed suitably such that the theorem is also applicable to the finitely many-valued ŁUKASIEWICZ systems. But that is not of great importance because for them much simpler adequate axiomatisations are available.

5 Algebraic structures for many-valued logics

Corresponding to the diversity of systems of many-valued logics, also a wide variaty of algebraic systems has been connected with many-valued logic. In the following a survey is given which covers the most important ones, giving finally a unifying point of view for a lot of them – as was done for the systems of many-valued logic in the last section.[26]

5.1 MV-algebras

MV-algebras have been introduced by CHANG [10, 11] in investigations toward a completeness proof for the infinite valued LUKASIEWICZ system. They play, however, an important role in algebraic studies related to all the LUKASIEWICZ many-valued logics and proved to have interesting relationships to other structures too.[27] For simplicity we use for the operations in MV-algebras the same notation as for the corresponding connectives of the language $\mathcal{L}_{\text{L}}$.
The following definition is a simplification, essentially given by MANGANI [64], of the original definition of CHANG [10], and refers only to some of the fundamental operations one usually considers in the LUKASIEWICZ logics.

Definition 5.1 *An algebraic structure* $\mathcal{A} = \langle A, \oplus, \neg, 0\rangle$ *of signature* $\langle 2, 1, 0\rangle$ *is an* MV-algebra *iff* $\langle A, \oplus, 0\rangle$ *is an abelian monoid with neutral element* 0 *and if furthermore for all* $x, y \in A$ *there hold true*

$$\begin{array}{ll} (i) & \neg\neg(x) = x, \\ (ii) & x \oplus \neg 0 = \neg 0, \\ (iii) & \neg(\neg(x) \oplus y) \oplus y = \neg(\neg(y) \oplus x) \oplus x. \end{array}$$

Such an MV-algebra is nontrivial *iff it contains at least two elements.*

Obviously each structure $\langle \mathcal{W}^*, \mathsf{vel}_2, \mathsf{non}_1, 0\rangle$ is an MV-algebra. What is not explicitely taken into consideration here is e.g. the truth degree function of the LUKASIEWICZ implication. But an MV-algebraic equivalent of it and also equivalents of the other truth degree functions which are important for the LUKASIEWICZ systems are definable in MV-algebras.

Definition 5.2 *Suppose that* $\mathcal{A} = \langle A, \oplus, \neg, 0\rangle$ *is an MV-algebra. Then one defines for any* $x, y \in A$:

$$x \otimes y \;=_{\mathsf{def}}\; \neg(\neg(x) \oplus \neg(y)),$$

[26]It is, however, not intended to give – from an algebraic point of view – a fairly balanced survey. We concentrate on these structures which seem to be the most important ones from the perspective of fuzzy sets related many-valued logic.

[27]The theory of MV-algebras for quite a time seemed to be of particular interest only for the LUKASIEWICZ many-valued logics. Only more recently it became clear that this class of structures is much more important in mathematics. The first extended monograph devoted to MV-algebras is [15] and covers the majority of the important actual results in the field of MV-algebras.

$$\begin{aligned}
x \to y &=_{\text{def}} \neg(x) \oplus y, \\
x \vee y &=_{\text{def}} (x \otimes \neg(y)) \oplus y, \\
x \wedge y &=_{\text{def}} (x \oplus \neg(y)) \otimes y, \\
1 &=_{\text{def}} \neg(0).
\end{aligned}$$

Using these definitions, an MV-algebra is nontrivial iff $0 \neq 1$.

The original definition of CHANG [10] was based on the (primitive) operations $\oplus, \otimes, \neg$ and the constants $0, 1$, gave the definitions of $\vee, \wedge$ as in Definition 5.2 and consisted of the following much longer list of conditions:

$$\begin{array}{rl}
a \oplus b = b \oplus a & a \otimes b = b \otimes a, \\
a \oplus (b \oplus c) = (a \oplus b) \oplus c & a \otimes (b \otimes c) = (a \otimes b) \otimes c, \\
a \oplus \neg a = 1 & a \otimes \neg a = 0, \\
a \oplus 1 = 1 & a \otimes 0 = 0, \\
a \oplus 0 = a & a \otimes 1 = a, \\
\neg(a \oplus b) = \neg a \otimes \neg b & \neg(a \otimes b) = \neg a \oplus \neg b, \\
a \vee b = b \vee a & a \wedge b = b \wedge a, \\
a \vee (b \vee c) = (a \vee b) \vee c & a \wedge (b \wedge c) = (a \wedge b) \wedge c, \\
a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c) & a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c), \\
\neg\neg a = a, & \neg 0 = 1.
\end{array}$$

Of course, all these properties become provable with our simpler definition.

Each MV-algebra $\mathcal{A}$ is, in a natural way, equipped with an ordering relation $\leqq$ defined for all $a, b \in A$ by

$$a \leqq b \quad =_{\text{def}} \quad \text{there exists } c \in A \text{ with } a \oplus c = b \tag{75}$$

which is a partial ordering, i.e. reflexive, transitive, and antisymmetric. All the operations $\oplus, \otimes, \wedge, \vee$ are isotonic w.r.t. this ordering, and $\neg$ is order reversing and thus antitonic. Hence, the operation $\to$ is isotonic in its second argument, and antitonic in its first one.

For this ordering, also other quite natural characterisations are available, e.g. one always has:

$$a \leqq b \quad \Leftrightarrow \quad a \otimes \neg b = 0 \quad \Leftrightarrow \quad a \to b = 1. \tag{76}$$

And this ordering is even a *lattice ordering* w.r.t. the operations $\wedge, \vee$. Furthermore, the lattice $\langle A, \wedge, \vee \rangle$ with this ordering is a distributive lattice.

Therefore, each MV-algebra $\mathcal{A} = \langle A, \oplus, \neg, 0 \rangle$ yields in a natural way an abelian lattice ordered monoid $\langle A, \otimes, 0, 1, \leqq \rangle$ with universal lower and upper bounds.

Among the MV-algebras, the subclass of all the MV-*chains*, i.e. of all those MV-algebras for which their natural ordering $\leqq$ is total, is of particular importance. One of the main points for this is the following subdirect representation theorem.

Theorem 5.1 *Every nontrivial MV–algebra is a subdirect product of MV–chains.*

For finite MV-algebras an even stronger characterisation is possible.

Theorem 5.2 *An MV-algebra is finite iff it is isomorphic to a finite product of finite MV-chains.*

Moreover, given a finite MV-algebra, the MV-chains which appear as "factors" in its representation are uniquely determined (up to their order as factors).

For another interesting result which refers to MV-chains we need the notion of an MV-*equation*: this is any equation in the first-order language of MV-algebras, i.e. in the language with the same signature as the MV-algebras – and usually also with operation and relation symbols for all the notions introduced in Definition 5.2 – with its individual variables interpreted as elements of an MV-algebra under consideration..

The notion of satisfaction of such an MV-equation in an MV-algebra $\mathcal{A}$ (w.r.t. a given valuation, i.e. mapping of the set of individual variables into the support A of $\mathcal{A}$) is the usual model theoretical notion of satisfaction. The same holds true for the notion of validity (of an MV-equation in an MV-algebra). Then one has the following

Theorem 5.3 *An MV-equation is valid in all MV-algebras iff it is valid in all MV-chains.*

Any MV-equation for which there is an MV-counterexample hence is already not valid in a suitable MV-chain. In other words: *the equational theory of MV-algebras coincides with the equational theory of MV-chains.*

This result, however, can be further strenghtened to the following completeness theorem.

Theorem 5.4 *An MV-equation is valid in all MV-algebras iff it is valid in the particular infinite* ŁUKASIEWICZ *MV-algebra* $\langle \mathcal{W}_\infty, \mathsf{vel}_2, \mathsf{non}_1, 0\rangle$.

The MV-algebras, thus, have for the ŁUKASIEWICZ system of infinite-valued logic essentially the same importance as the BOOLEan algebras have for classical two-valued logic.

From a more algebraic point of view the notion of *ideal* is of crucial importance, i.e. of such subsets I of a MV-algebra $\mathcal{A}$ which contain the universal lower bound and are downward closed w.r.t. the natural ordering $\leqq$ and also closed under $\oplus$:

$$0 \in I, \qquad a \leqq b \in I \Rightarrow a \in I, \qquad a, b \in I \Rightarrow a \oplus b \in I. \tag{77}$$

Of course, A itself is an ideal of $\mathcal{A}$, the *trivial* one, all the other ideals are called *proper*. The *maximal* ideals are the $\subseteq$-maximal elements in the class of all proper ideals. And the intersection of all the maximal ideals of an MV-algebra $\mathcal{A}$ is the *radical* $\mathrm{Rad}\,(\mathcal{A})$ of $\mathcal{A}$.

Definition 5.3 *An MV-algebra* $\mathcal{A}$ *is* simple *iff it is nontrivial and has* $\{0\}$ *as its only proper ideal, and it is* semisimple *iff* $\mathrm{Rad}\,(\mathcal{A}) = \{0\}$.

There are characterisations of the simple and the semisimple MV-algebras which are interesting and important from the viewpoints of many-valued logic and of fuzzy set theory. For the simple MV-algebras one has the

Theorem 5.5 *An MV-algebra* $\mathcal{A}$ *is* simple *iff it is isomorphic to a subalgebra of the particular infinite* ŁUKASIEWICZ *MV-algebra* $\langle \mathcal{W}_\infty, \mathsf{vel}_2, \mathsf{non}_1, 0\rangle$.

This theorem immediately gives a characterisation of the truth degree structures of the finite many-valued ŁUKASIEWICZ logics.

Corollary 5.6 *The finite* ŁUKASIEWICZ *MV-algebras* $\langle \mathcal{W}_n, \mathsf{vel}_2, \mathsf{non}_1, 0\rangle$ *are, up to isomorphism, the only finite and simple MV-algebras.*

And for the semisimple MV-algebras one has the following two characterisations.

Theorem 5.7 (i) *An MV-algebra* $\mathcal{A}$ *is* semisimple *iff it is a subdirect product of subalgebras of the infinite* ŁUKASIEWICZ *MV-algebra* $\langle \mathcal{W}_\infty, \mathsf{vel}_2, \mathsf{non}_1, 0\rangle$.

(ii) *An MV-algebra* $\mathcal{A}$ *is* semisimple *iff it is isomorphic to an algebra of* $[0,1]$*-valued continuous functions, i.e. to an algebra of fuzzy subsets of some nonempty compact Hausdorff space.*

As algebraic structures, MV-algebras are closely related to some other types of structures. The most prominent examples are the bounded commutative BCK-algebras and the abelian lattice-ordered groups with a strong order unit.

Definition 5.4 *A* bounded commutative BCK-algebra *is an algebraic structure* $\mathcal{B} = \langle B, \circledast, \mathbf{0}, \mathbf{1}\rangle$ *of signature* $\langle 2,0,0\rangle$ *satisfying for all* $a,b,c \in B$ *the equations*

$$(a \circledast b) \circledast c = (a \circledast c) \circledast b, \qquad a \circledast (a \circledast b) = b \circledast (b \circledast a),$$
$$a \circledast a = 0, \qquad a \circledast \mathbf{0} = a, \qquad a \circledast \mathbf{1} = \mathbf{0}.$$

In this case one has the following relationship between both types of structures.

Theorem 5.8 (i) *If* $\langle A, \oplus, \neg, 0\rangle$ *is an MV-algebra, then the algebraic structure* $\langle A, \circledast, 0, \neg 0\rangle$ *with the operation* $\circledast$ *defined via* $a \circledast b =_{\mathrm{def}} a \oplus \neg b$ *is a bounded commutative BCK-algebra.*

(ii) *If* $\mathcal{B} = \langle B, \circledast, \mathbf{0}, \mathbf{1}\rangle$ *is a bounded commutative BCK-algebra, then the structure* $\langle B, \oplus, \neg, \mathbf{0}\rangle$ *with the operations* $\neg, \oplus$ *defined via* $\neg a =_{\mathrm{def}} \mathbf{1} \circledast a$ *and* $a \oplus b =_{\mathrm{def}} \mathbf{1} \circledast ((\mathbf{1} \circledast a) \circledast b)$ *is an MV-algebra.*

(iii) *These mappings between the classes of MV-algebras and of bounded commutative BCK-algebras are inverse to one another and hence bijections.*

Definition 5.5 *An abelian lattice-ordered group with a strong order unit is an algebraic structure* $\langle G, +, -, 0, u; \leqslant\rangle$ *of signature* $\langle 2, 1, 0; 2\rangle$ *such that* $\langle G, +, -, 0\rangle$ *is an abelian group endowed with a lattice ordering* $\leqslant$ *in* G*, with the corresponding lattice meet and join denoted by* $\sqcap, \sqcup$*, such that the group operation* $+$ *is monotonous w.r.t. the (reflexive partial) ordering* $\leqslant$*, and such that the strong order unit* $0 \leqslant u \in G$ *has the property that for each* $a \in G$ *there is an integer* $n \geq 0$ *with* $a \sqcup -a \leqslant nu$*.*

Starting from an abelian lattice-ordered group $\mathcal{G} = \langle G, +, -, 0, u; \leqslant\rangle$ with a strong order unit u and considering the interval $[0, u] = \{x \in G | 0 \leqslant x \leqslant u\}$ of G together with two operations defined as

$$a \oplus b =_{\mathsf{def}} u \sqcap (a + b), \qquad \neg a =_{\mathsf{def}} u - a, \tag{78}$$

one gets another algebraic structure as

$$\boldsymbol{\Gamma}(\mathcal{G}) =_{\mathsf{def}} \langle [0, u], \oplus, \neg, 0\rangle. \tag{79}$$

Theorem 5.9 *The operator* $\boldsymbol{\Gamma}$ *defined by (79) maps abelian lattice-ordered groups with strong order unit to MV-algebras and is even a natural equivalence between the categories of abelian lattice-ordered groups with a strong order unit and of MV-algebras (both with the respective natural homomorphisms as morphisms).*

Besides its algebraic content, this last result may may be read as stating that MV-algebras are not only important for ŁUKASIEWICZ's many-valued logics and interesting algebraic objects but also provide an equational formulation of the theory of magnitudes with an Archimedean unit.

This operator $\boldsymbol{\Gamma}$ is also of interest because of its relations to injective MV-algebras which form another algebraically interesting[28] class of MV-algebras.

Definition 5.6 *An MV-algebra* $\mathcal{A}$ *is* injective *iff each MV-homomorphism* $h : \mathcal{C} \to \mathcal{A}$ *from some MV-subalgebra* $\mathcal{C}$ *of an MV-algebra* $\mathcal{B}$ *into* $\mathcal{A}$ *can be extended to an MV-homomorphism from* $\mathcal{B}$ *into* $\mathcal{A}$*.*

For these injective MV-algebras the following interesting theorem was proven by GLUSCHANKOF [27].

Theorem 5.10 *An MV-algebra* $\mathcal{A}$ *is injective iff it is isomorphic to an MV-algebra of the form* $\boldsymbol{\Gamma}(\mathcal{G})$ *with* $\mathcal{G}$ *divisible abelian lattice-ordered group with strong order unit whose underlying lattice is complete. In the particular case* $|\mathcal{A}| \subseteq [0, 1]$ *one has even that* $\mathcal{A}$ *is injective iff* $|\mathcal{A}| = [0, 1]$*, and also iff the underlying lattice of* $\mathcal{A}$ *is complete.*

---

[28]The interest in this class of MV-algebras is not a purely algebraic one as indicates a compactness theorem for fuzzy logic mentioned later on in Section 8.4.1.

5.2 Łukasiewicz algebras

An algebraic approach toward the ŁUKASIEWICZ systems of many-valued logic which differs considerably from the MV-algebras was initiated by MOISIL [70, 71]. He startet from the 3-valued case and the observation that, notwithstanding the fact that it is impossible to define $\rightarrow_{\mathsf{L}}$ from $\wedge, \vee, \neg$, this becomes possible if one adds a unary connective m with truth degree function $u \mapsto \mathsf{vel}_2(u, u)$. The generalisation to the m-valued case resulted in the following definition.

Definition 5.7 *Given any $2 \leq m \in \mathbb{N}$, an m-valued* ŁUKASIEWICZ *algebra is an algebraic structure $\mathcal{A} = \langle A, \vee, \wedge, \neg, \mathsf{s}_1^m, \dots, \mathsf{s}_{m-1}^m, 0, 1\rangle$ of signature $\langle 2, 2, 1, 1, \dots, 1, 0, 0\rangle$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a distributive lattice with zero and unit and such that for all $x, y \in A$ and all $1 \leq i, j \leq m-1$ there hold true:*

$$
\begin{array}{ll}
(\mathsf{LA1}) & \neg(\neg(x)) = x, \\
(\mathsf{LA2}) & \neg(x \vee y) = \neg(x) \wedge \neg(y), \\
(\mathsf{LA3}) & \mathsf{s}_i^m(x \vee y) = \mathsf{s}_i^m(x) \vee \mathsf{s}_i^m(y), \\
(\mathsf{LA4}) & \mathsf{s}_i^m(x) \vee \neg \mathsf{s}_i^m(x) = 1, \\
(\mathsf{LA5}) & \mathsf{s}_j^m(\mathsf{s}_i^m(x)) = \mathsf{s}_i^m(x), \\
(\mathsf{LA6}) & \mathsf{s}_i^m(\neg(x)) = \neg \mathsf{s}_{m-i}^m(x), \\
(\mathsf{LA7}) & \mathsf{s}_i^m(x) \vee \mathsf{s}_{i+1}^m(x) = \mathsf{s}_{i+1}^m(x) \quad \textit{for } i < m-1, \\
(\mathsf{LA8}) & x \vee \mathsf{s}_{m-1}^m(x) = \mathsf{s}_{m-1}^m(x), \\
(\mathsf{LA9}) & (x \wedge \neg(\mathsf{s}_i^m(x)) \wedge \mathsf{s}_{i+1}^m(x)) \vee y = y \quad \textit{for } i < m-1.
\end{array}
$$

The theory of these structures is to a large extend developed in [14].

5.3 Product algebras

The completeness proof for the product logic $\Pi\mathsf{L}$ as given in [40], adopting the proof method which CHANG [10, 11] gave for the ŁUKASIEWICZ logics, is too based on suitable algebraic structures, called product algebras, which reflect the main structural properties of the matrix (71) and play for product logic a similar role as MV-algebras play for ŁUKASIEWICZ logic.

Definition 5.8 *An algebraic structure $\mathcal{A} = \langle A, \odot, \rightarrow, 0, 1\rangle$ is a* product algebra *if together with the additional operations*

$$
\begin{aligned}
x \wedge y &=_{\mathsf{def}} x \odot (x \rightarrow y), \\
x \vee y &=_{\mathsf{def}} ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x), \\
\neg(x) &=_{\mathsf{def}} x \rightarrow 0
\end{aligned}
$$

the algebraic structure $\langle A, \wedge, \vee, \odot, \rightarrow, 0, 1\rangle$ is a residuated lattice which furthermore satisfies the laws

$$
(i) \qquad ((x \rightarrow y) \vee (y \rightarrow x)) = 1,
$$

$$
\begin{array}{rl}
(ii) & \neg\neg(z) \odot ((x \odot z) \rightarrow (y \odot z)) \leq (x \rightarrow y), \\
(iii) & x \wedge \neg[x] = 0, \\
(iv) & x \odot (y \vee z) \leq (x \odot y) \vee (x \odot z), \\
(v) & x \odot (y \wedge z) \geq (x \odot y) \wedge (x \odot z).
\end{array}
$$

The LINDENBAUM algebra of the product logic is a product algebra. And the class of all product algebras is a characteristic class for the product logic, i.e. the class of all theorems of product logic (and hence the class of all logically valid sentences of this logic) is the class of all sentences satisfied in all product algebras.

The definition of product algebras can be rewritten as a sequence of (universally quantified) HORN formulas. Therefore the class of all product algebras is a variety. Thus the direct products and the subalgebras of product algebras are product algebras too. Furthermore, the only finite linearly ordered product algebra is the two element BOOLEan algebra $\{0, 1\}$ – and each BOOLEan algebra, with its lattice meet for the product $\odot$, is a product algebra since it is a subdirect product of copies of $\{0, 1\}$.

5.4 POST algebras

Algebraic counterparts of the POST systems of many-valued logic first have been introduced by ROSENBLOOM [93]. His POST algebras have been algebraic structures of signature $\langle 2, 1\rangle$ characterised by equations algebraically determining the truth degree functions $\mathsf{vel}_1, \mathsf{non}_2$. These characterising equations, however, had been rather difficult and not quite intuitive. Hence, POST algebras appeared as some kind of exotic structures for investigations. This situation changed quite to the opposite as EPSTEIN [20] redefined these structures within another signature. We shall explain here this (modified) notion of POST algebras[29] following a slightly different approach of DWINGER [19] who characterises POST algebras as particular lattices.

Some authors prefer, also based on the approach initiated by EPSTEIN, to define POST algebras as algebraic structures of a more complicated signature. One such possibility is to follow closely the formal structure of the language of the POST systems, which thus amounts to consider pseudo-BOOLEan algebras enriched with $n-1$ unary operations D_i and n distinguished elements e_i which together satisfy suitable algebraic translations of the basic axioms (P11) to (P18) of the POST systems. All these approaches, however, create classes of interdefinable structures, cf. [89, 91].

Definition 5.9 *A distributive lattice* $\mathcal{A} = \langle A, +, \cdot, 0, 1\rangle$ *with zero and unit element is a* POST algebra of order n, $n \geq 2$, *iff there exists a finite chain* $0 = \mathsf{e}_0 \leqq \mathsf{e}_1 \leqq \ldots \leqq \mathsf{e}_{n-1} = 1$ *w.r.t. the lattive ordering* $\leqq$ *such that for each* $a \in A$

[29]This modification is, however, an inessential one: starting from ROSENBLOOM's POST algebras one is able to define corresponding POST algebras in the sense of EPSTEIN, and vice versa.

there is a unique representation $a = \sum_{i=1}^{n-1} a_i \cdot \mathsf{e}_i$ *with* $a_1 \geqq a_2 \geqq \ldots \geqq a_{n-1}$ *and such that each of these coefficients* a_i *has a complement in the lattice* $\mathcal{A}$.

A POST algebra $\mathcal{A}$ of order n uniquely determines its finite chain of distinguished elements $\mathsf{e}_0, \mathsf{e}_1, \ldots, \mathsf{e}_{n-1} \in A$. Therefore every POST algebra has a unique order. Furthermore, POST algebras are relatively pseudo-complemented lattices.

A deeper understanding of the internal structure of algebras like the POST algebras often is provided by suitable representation theorems. The following results are particularly interesting for POST algebras.

Theorem 5.11 *A distributive lattice* $\mathcal{A} = \langle A, +, \cdot, 0, 1\rangle$ *with zero and unit element is a* POST *algebra of order* n, $n \geq 2$, *iff there exists a* BOOLE*an algebra* $\mathcal{B} = \langle B, \sqcup, \sqcap, {}^c, 0, 1\rangle$ *such that the lattice* $\mathcal{A}$ *is isomorphic to the lattice of all ordered* $(n-1)$*-tuples* $\mathsf{B} = \{(b_1, \ldots, b_{n-1}) \in B^n \mid b_1 \geqq b_2 \geqq \ldots \geqq b_{n-1}\}$ *with componentwise defined operations and with* $\mathbf{0} = (0, \ldots, 0)$ *as zero and* $\mathbf{1} = (1, \ldots, 1)$ *as unit element.*

Theorem 5.12 *A distributive lattice* $\mathcal{A} = \langle A, +, \cdot, 0, 1\rangle$ *with zero and unit element is a* POST *algebra iff it is the coproduct of a* BOOLE*an algebra and a finite chain.*

Theorem 5.13 *A distributive lattice* $\mathcal{A} = \langle A, +, \cdot, 0, 1\rangle$ *with zero and unit element is a* POST *algebra iff it is isomorphic to the lattice of continuous functions of a* BOOLE*an space* $\mathcal{X}$ *to a finite chain.*

5.5 DEMORGAN algebras

During the initial phase of the development of fuzzy set theory a lot of authors restricted their considerations to the union and intersection of fuzzy sets based on the maximum and minimum operations and to a complementation based on the (1-..)-operation. The corresponding system of many-valued logic thus was based on conjunction, disjunction, and negation connectives with truth degree functions $\mathsf{et}_1, \mathsf{vel}_1$ and non_1, resprectively.

The algebraic counterpart are the distributive DEMORGAN algebras. They have been introduced already by MOISIL [69] and studied later on also by RASIOWA, cf. [89], under the name of *quasi-*BOOLE*an algebras.*

Definition 5.10 *A* distributive DEMORGAN algebra $\mathcal{A} = \langle A, \wedge, \vee, \overline{}, 0, 1\rangle$ *is an algebraic structure of signature* $\langle 2, 2, 1, 0, 0\rangle$ *such that the reduct* $\langle A, \wedge, \vee, 0, 1\rangle$ *is a distributive lattice with zero and unit and* $\overline{}$ *is an involution which satisfies the* DEMORGAN *laws*

$$\begin{aligned} \overline{x \wedge y} &= \overline{x} \vee \overline{y}, \\ \overline{x \vee y} &= \overline{x} \wedge \overline{y}. \end{aligned}$$

These DEMORGAN algebras are taken into consideration occasionally. There seems, however, to be no really interesting theory of these algebras besides the fact that they are reducts of different types of algebras which themselves come into consideration as characteristic structures for the algebraic study of different systems of many-valued, and also of other types of non-classical logics.

5.6 Residuated ℓ-monoids

The discussion of truth degree functions for systems of many-valued logic which counts t-norms as basic candidates for truth degree functions and then aims to base further connectives to a large extend on the conjunction connective algebraically amounts to having in the truth degree set $\mathcal{W}$ a commutative semigroup operation $*$. But the monotonicity properties of the t-norms additionally forces to have an ordering relation $\leqslant$ for the set of truth degrees.

If one does not intend to restrict from the very beginning the truth degree sets to subsets of the real unit interval, then the most reasonable point of view is to have $\leqslant$ at least as a lattice ordering with universal upper and lower bounds within the set of truth degrees.

Hence the truth degree structure should at least be a lattice-ordered commutative monoid $\langle \mathcal{W}, \leqslant, * \rangle$ with universal upper and lower bounds for the lattice ordering, a *commutative ℓ-monoid* for short.

The approaches toward adding further connectives split in essentially two variants: either to first introduce a negation function or to first introduce an implication function. The approach toward residuated commutative ℓ-monoids prefers implication over negation and thus chooses R-implications as the suitable type of implication function. Algebraically this means to have a further binary operation $\rhd$ satisfying the adjointness condition

$$x * y \leqslant z \;\Leftrightarrow\; x \leqslant y \rhd z \tag{80}$$

for all $x, y, z \in \mathcal{W}$, which additionally characterises $\rhd$ uniquely.

Definition 5.11 *A* commutative residuated ℓ-monoid, *for short:* $r\ell$-monoid, *is an ordered algebraic structure $\mathcal{A} = \langle A, \leqslant, *, \rhd \rangle$ such that $\langle A, \leqslant, * \rangle$ is a commutative ℓ-monoid and $(*, \rhd)$ is an adjoint pair, i.e. satisfies condition (80).*

In any $r\ell$-monoid $\mathcal{A} = \langle A, \leqslant, *, \rhd \rangle$ one has by the lattice structure also the lattice operations $\sqcap, \sqcup$, and one has for all elements $x, y, z \in A$ satisfied the properties

$$\begin{aligned}
x * (x \rhd y) &\leqslant y, \\
x \rhd (y \rhd z) &= (x * y) \rhd z, \\
x * (y \sqcup z) &= (x * y) \sqcup (y * z), \\
x \rhd (y \sqcap z) &= (x \rhd y) \sqcap (x \rhd z), \\
(x \sqcup y) \rhd z &= (x \rhd z) \sqcap (y \rhd z).
\end{aligned}$$

At the present level of generality the unit element $\top$ of the semigroup operation $*$ needs not to coincide with the universal upper bound 1 of the lattice ordering. If this is the case, however, i.e. if $\top = 1$ holds true, the $r\ell$-monoid shall be called *integral.*

And, according to HÖHLE [45, 47], these integral $r\ell$-monoids are the appropriate algebraic structures for those many-valued logics which are determined by t-norm based conjunction connectives and which play a crucial role in a majority of theoretical papers related to fuzzy set theory.

Integrality of a $r\ell$-monoid may equivalently be characterised by demanding either one of the following two conditions:

$$\begin{aligned} x \leqslant y &\Leftrightarrow x \rhd y = 1, \\ x &= 1 \rhd x. \end{aligned}$$

And in each integral $r\ell$-monoid one has

$$x * y \leqslant x \sqcap y \quad \text{for all } x, y.$$

A further important requirement to be imposed on an integral $r\ell$-monoid $\mathcal{A} = \langle A, \leqslant, *, \rhd \rangle$ is its *divisibility*, i.e. the property that for any $x, y \in A$ which satisfy the condition $x \leqslant y$ there exists some $z \in A$ such that $x = y * z$. This divisibility can equivalently be characterised by the property that

$$x \sqcap y = x * (x \rhd y) \quad \text{for all } x, y \in A.$$

Integral divisible $r\ell$-monoids e.g. provide an interesting relationship to the HEYTING algebras characteristic for intuitionistic logic.

Theorem 5.14 *Let $\mathcal{A} = \langle A, \leqslant, *, \rhd \rangle$ be an integral $r\ell$-monoid which is divisible and satisfies the condition*

$$(x \rhd y) \sqcup (y \rhd x) = 1 \quad \textit{for all } x, y \in A. \tag{81}$$

Then the set $H_{\mathcal{A}} = \{x \in A \mid x * x = x\}$ of $*$-idempotent elements of $\mathcal{A}$ forms a HEYTING *algebra w.r.t. the underlying lattice of $\mathcal{A}$. And this* HEYTING *algebra has $\rhd$ as its pseudo-complementation.*

This relation to intuitionistic logic stimulates the idea to look w.r.t. any $r\ell$-monoid $\mathcal{A} = \langle A, \leqslant, *, \rhd \rangle$ at the unary operation – characterised by the equation

$$-x =_{\text{def}} x \rhd 0 \tag{82}$$

with 0 the universal lower bound of $\mathcal{A}$. This operation is related to a possibility to define negation in intuitionistic logic.

And indeed, also for integral $r\ell$-monoids one has

$$-0 = 1 \quad \text{and} \quad -1 = 0$$

and integrality together with the additional condition

$$x = (x \rhd 0) \rhd 0 = - - x \quad \text{for all } x \in A \tag{83}$$

suffice that the operation $-$ is antitonous in $\mathcal{A}$ and thus has all the properties a negation function in $[0,1]$ has to have. Therefore $-$ is termed *negation operation.*

Accepting this further assumption then leads into the area of well known algebraic counterparts of systems of many-valued logic, cf [45, 46].

Theorem 5.15 *An integral $r\ell$-monoid is a MV-algebra w.r.t. its semigroup and its negation operations iff it is divisible and satisfies condition (83).*

6 Many-valued first-order logic

6.1 Basic notions

The languages $\mathcal{L}_{\mathsf{S}}$ of systems S of first-order many-valued logic follow the same pattern as the languages of systems of classical first-order logic: they are based on some countable set $\mathcal{V} = \{v_0, v_1, v_2, \ldots\}$ of individual variables and are determined by

- their (nonempty) family $\mathcal{J}^{\mathsf{S}}$ of (basic) propositional connectives and, sometimes, their truth degree constants,
- their set $\mathcal{Q}^{\mathsf{S}}$ of quantifiers,
- their family $\mathcal{P}^{\mathsf{S}}$ of predicate symbols together with an arity function assigning each predicate symbol its arity,
- and possibly their set of individual constants and their set of function symbols.

It then is a routine matter to define the class of well formed formulas and to distinguish free and bounded occurences of individual variables within well formed formulas. All this are standard syntactic definitions with no differences[30] to the case of classical first-order logic.

The semantic parts of these systems, again, differ only slightly from the corresponding situation for classical first-order logic. Like there, one basically has

- a set $\mathcal{W}^{\mathsf{S}}$ of truth degrees, which again preferably shall be chosen as in (4) and (3), together with a subset $\mathcal{D}^{\mathsf{S}}$ of designated truth degrees,
- for each connective a truth degree function determining this connective,
- for each truth degree constant a corresponding truth degree, and

[30] Disregarding the inessential and minor difference which may be caused by the existence of truth degree constants for truth degrees different of 0 and 1.

- for each quantifier Q of arity k a corresponding truth degree mapping $\mathsf{ver}^{\mathsf{S}}_{\mathsf{Q}} : \mathbb{P}(\mathcal{W}^{\mathsf{S}})^k \to \mathcal{W}^{\mathsf{S}}$ determining this quantifier

as fixed data characterising the whole system S which have to be combined with particular *interpretations* $\mathfrak{A}$ collecting

- a nonempty set $A = |\mathfrak{A}|$ of individuals, the universe of the interpretation,
- for each individual constant a of $\mathcal{L}_{\mathsf{S}}$ a corresponding individual $a^{\mathfrak{A}} \in A$,
- for each predicate symbol P of $\mathcal{L}_{\mathsf{S}}$ of arity n a corresponding n-ary $\mathcal{W}^{\mathsf{S}}$-valued relation $P^{\mathfrak{A}}$ in A.

By an n-ary $\mathcal{W}^{\mathsf{S}}$-valued relation R we here understand a mapping $R : A^n \to \mathcal{W}^{\mathsf{S}}$, i.e. an n-ary fuzzy relation in the universe of discourse A (and with their membership degrees from $\mathcal{W}^{\mathsf{S}}$).

The satisfaction relation $(\mathfrak{A}, \alpha) \models H$, H any well formed formula and $\alpha : \mathcal{V} \to A$ any valuation, now becomes a graded relationship and hence preferably is defined as a function $\mathsf{Val}^{\mathsf{S}}{}_{\mathfrak{A}}(H, \alpha)$ mapping well formed formulas and valuations onto truth degrees. The details are standard as the quantifier case shows in which the definition reads for any unary quantifier Q binding one variable

$$\begin{aligned}\mathsf{Val}^{\mathsf{S}}{}_{\mathfrak{A}}(\mathsf{Q}xH, \alpha) \quad =_{\text{def}} \quad & \mathsf{ver}^{\mathsf{S}}_{\mathsf{Q}}(\{\mathsf{Val}^{\mathsf{S}}{}_{\mathfrak{A}}(H, \beta) \mid \beta : \mathcal{V} \to A \,\text{such that} \\ & \qquad \alpha(y) = \beta(y) \text{ for all } y \in \mathcal{V} \text{ with } y \neq x\}).\end{aligned}$$

Proceeding as usual now, a formula H is called S-*valid* (or simply *valid* if the reference to the particular system S is obvious) *in an interpretation* $\mathfrak{A}$ iff H has a designated truth degree for each suitable valuation in $\mathfrak{A}$. And H is called *logically* S-*valid* (or again simply *logically valid*) iff H is valid in any S-interpretation.

Therefore, all the usual basic model theoretic notions are available also for systems of many-valued first-order logic. It is, however, sometimes advisable to split classical notions into distinct ones for the many-valued case. To give an idea, how this can be done, we consider the notion of model.

As usual, an interpretation $\mathfrak{A}$ is a S-*model* of a set Σ of sentences iff each $H \in \Sigma$ is valid in $\mathfrak{A}$. By $\mathsf{Mod}^{\mathsf{S}}(\Sigma)$ the class of all S-models of Σ shall be denoted. It is, however, suitable to consider even for any truth degree t the notions of t-*model* and of $(\geq t)$-*model* $\mathfrak{A}$ of Σ by imposing the condition that $H \in \Sigma$ always has truth degree t or always has truth degree $\geq t$ in $\mathfrak{A}$, respectively.

Defining then for any sentence[31] H and any set Σ of sentences the consequence relation $\models_{\mathsf{S}}$ by

$$\Sigma \models_{\mathsf{S}} H =_{\text{def}} \mathsf{Mod}^{\mathsf{S}}(\Sigma) \subseteq \mathsf{Mod}^{\mathsf{S}}(H)$$

gives via

$$\mathsf{Cn}^{\mathsf{S}}(\Sigma) =_{\text{def}} \{H \mid \Sigma \models_{\mathsf{S}} H\}$$

[31]For simplicity we restrict here to the case of sentences. The extensions of these notions to well formed formulas in general proceed as in classical logic.

a consequence operator which is a closure operator within the set of all sentences.

More model theoretic constructions and results become available on this basis as e.g. the ultraproduct construction which works for any truth degree set together with a corresponding version of ŁOSs theorem which holds true for any finitely many-valued system, cf. [33], and can even be extended to suitable infinitely many-valued systems as done e.g. in [13].

The many-valued systems behave also w.r.t. other results much like classical logic, cf. [33]:

- each finitely many-valued one of them satisfies the compactness theorem,
- each one of them satisfies a suitable version of the downward LÖWENHEIM-SKOLEM theorem,
- and each finitely many-valued one of them satisfies a suitable version of the upward LÖWENHEIM-SKOLEM theorem.

and the restriction to the finitely many-valued case can again be deleted for suitable infinite truth degree sets, e.g. for compact HAUSDORFF spaces and truth degree functions of the connectives and the quantifiers which satisfy some suitable continuity conditions, cf. [13].

For particular systems we shall only look at the first-order ŁUKASIEWICZ logics because the type of transition from the propositional to the first-order case, standard for these systems, may be choosen in the case of other systems as well. For the monoidal logic this was done by U. HÖHLE e.g. in [47].

6.2 ŁUKASIEWICZ's first-order logics

The first-order systems of ŁUKASIEWICZ's many-valued logics are determined by the standard methods of transfer from propositional to first-order logic as explained in Secion 6.1 together with the choice of a universal quantifier $\forall$ and an existential quantifier $\exists$, both of them unary quantifiers, defined by the truth degree operations $\mathsf{ver}^{\mathsf{L}}_{\forall}, \mathsf{ver}^{\mathsf{L}}_{\exists}$ with characterising equations[32]

$$\mathsf{ver}^{\mathsf{L}}_{\forall}(X) =_{\mathsf{def}} \inf X \quad \text{and} \quad \mathsf{ver}^{\mathsf{L}}_{\exists}(X) =_{\mathsf{def}} \sup X \tag{84}$$

for all $X \subseteq \mathcal{W}^{\mathsf{L}}$.

Obviously, all the finite sets $\mathcal{W}_m$ as well as the infinite set $\mathcal{W}_\infty$ are closed under inf and sup. The truth degree set $\mathcal{W}_0$, however, is not closed neither under inf nor under sup. Hence one has also in the first-order case all the finitely many-valued ŁUKASIEWICZ systems as well as the infinitely one with truth degree set $\mathcal{W}_\infty$, there is, however, no reason to consider an infinitely many-valued ŁUKASIEWICZ system with truth degree set $\mathcal{W}_0$. As in the propositional case, by

[32]These two definitions are suitable ones in all the cases in which the truth degree set bears the structure of a complete lattice. And because this is satisfied for almost all of the important systems of many-valued logic, these definitions are the standard ones for these tow quantifiers. A rare exception is [105] where the author, instead of basing $\forall$ on min as its finite correlate, describes a method to combine a universal quantifier with any t-norm s its finite correlate.

L_ν the LUKASIEWICZ first-order logic with truth degree set $\mathcal{W}_\nu$, $\nu = 3, 4, \ldots, \infty$, is denoted[33].

The set of designated truth degrees is for first-order LUKASIEWICZ logic the same as in the propositional case. Therefore, semantic notions like validity in some interpretation, logical validity, the notion of model together with the notions of α- and $(\geq \alpha)$-model, as well as the semantic entailment relation can be defined in the standard ways predetermined by classical first-order and many-valued LUKASIEWICZ propositional logic.

Writing $\mathsf{lval}^{\mathsf{L}}{}_\nu$ for the set of all logically valid formulas of the system L_ν, one has

$$\mathcal{W}_\nu \subseteq \mathcal{W}_\rho \;\Leftrightarrow\; \mathsf{lval}^{\mathsf{L}}{}_\nu \supseteq \mathsf{lval}^{\mathsf{L}}{}_\rho \quad \text{for all } \nu, \rho \in \{3, 4, \ldots, \infty\} \tag{85}$$

as well as

$$\mathsf{lval}^{\mathsf{L}}{}_\infty = \bigcap_{n=3}^{\infty} \mathsf{lval}^{\mathsf{L}}{}_n. \tag{86}$$

Hence, each formula which is not L_∞-logically valid is already not logically valid in some of the finitely many-valued LUKASIEWICZ systems.

The situation is, however, more complicated with respect to satisfyability. There are sentences of the language $\mathcal{L}_{\mathsf{L}}$ of the LUKASIEWICZ systems which have a L_∞-model but do not have any L_n-model for any $2 \leq n \in \mathbb{N}$. An example is provided by the formula

$$\exists x P(x) \wedge \forall x \exists y (P(y) \ll P(x)),$$

where P is a unary predicate symbol and $\ll$ defined for any formulas H_1, H_2 by

$$H_1 \ll H_2 =_{\text{def}} \neg(H_1 \otimes H_1) \wedge (H_1 \oplus H_1 \;\leftrightarrow_{\mathsf{L}}\; H_2).$$

The essential idea here is that an interpretation $\mathfrak{A}$ is a model of a sentence $H_1 \ll H_2$ iff the truth degree of H_2 in $\mathfrak{A}$ is twice the truth degree of H_2 in $\mathfrak{A}$, cf. [33, 88].

For the finitely many-valued LUKASIEWICZ systems all the semantic results like the compactness theorem and LÖWENHEIM-SKOLEM theorems hold true which have been mentioned already in Section 6.1. Furthermore, also a suitable ommiting types theorem holds true for all finitely many-valued systems L_n, cf. [33].

But even the infinitely many-valued LUKASIEWICZ system L_∞ behaves semantically quite well:

- for L_∞ the compactness theorem holds true,
- for L_∞ hence suitable LÖWENHEIM-SKOLEM theorems hold true,
- for L_∞ the omitting types theorem holds true.

The problem of *axiomatisability* for the finitely many-valued LUKASIEWICZ systems L_n was solved by THIELE [104] in such a way as to add to any adequate system of axioms for the propositional system L_n some inference rules which

[33] Again $\nu = 2$ is possible and gives just classical first-order logic.

control the behaviour of the quantifiers and which allow some kinds of free and of bounded substitution. For details consult cf. [33, 104].

Much harder is the axiomatisability problem for the infinitely many-valued ŁUKASIEWICZ system L_∞. ROSSER/TURQUETTE [94] have proven the axiomatisability of its monadic fragment. But SCARPELLINI [97] proved that the set $\mathsf{lval}^{\mathsf{L}}{}_\infty$ of L_∞-logically valid formulas is not recursively enumerable[34] and hence not axiomatisable.

The axiomatisability situation changes however if one changes the set of designated truth degrees. If $r \in \mathbb{Q} \cap [0,1]$ is any rational number of the unit interval then

(i) for each set $\mathcal{D} = (r, 1]$ of designated truth degrees the corresponding set of logically valid formulas of the language $\mathcal{L}_\mathsf{L}$ of the ŁUKASIEWICZ systems is adequately axiomatisable, cf. [76, 2],

(ii) for each set $\mathcal{D} = [r, 1]$ of designated truth degrees the corresponding set of logically valid formulas of the language $\mathcal{L}_\mathsf{L}$ of the ŁUKASIEWICZ systems is not axiomatisable, cf. [2, 12].

Work of BELLUCE/CHANG [3] as well as of HAY [42] provides, however, a weaker axiomatisability result for L_∞. Both authors give particular axiom systems[35] and hence determine particular provability notions $\vdash_{\mathsf{BC}}$ and $\vdash_{\mathsf{H}}$ such that for each well formed formula H of $\mathcal{L}_\mathsf{L}$ it holds true

$$\begin{aligned} H \in \mathsf{lval}^{\mathsf{L}}{}_\infty \quad &\Leftrightarrow \quad \vdash_{\mathsf{BC}} H \oplus \prod_{i=1}^{n} H \quad \text{for each } n \geq 1 \\ &\Leftrightarrow \quad \vdash_{\mathsf{H}} H \oplus \prod_{i=1}^{n} H \quad \text{for each } n \geq 1. \end{aligned}$$

Therefore one gets adequate axiomatisations for the ŁUKASIEWICZ system L_∞ by enriching these axiom systems with the infinitary inference rule

$$\frac{\{H \oplus \prod_{i=1}^{n} H \mid 1 \leq n \in \mathbb{N}\}}{H} \tag{87}$$

or also, as shown in [44], by enriching them with the infinitary inference rule

$$\frac{\{\neg H \rightarrow_{\mathsf{L}} \prod_{i=1}^{n} H \mid 1 \leq n \in \mathbb{N}\}}{H}\,. \tag{88}$$

Denoting by LPC^* the infinitary ŁUKASIEWICZ predicate calculus, one has the following completeness theorem, cf. [44].

[34]The idea is to show that the recursive enumerability of the set $\mathsf{lval}^{\mathsf{L}}{}_\infty$ implies the recursive enumerability of the set of all finitely logically valid formulas of classical first-order logic, which however is known to be not recursively enumerable by results of TRAKHTENBROT [107], cf. e.g. [8].

[35]Both these axiom systems are only slight modifications of adequate axiom systems for classical first-order logic and shall not be given in detail here.

Theorem 6.1 *For each well formed formula H of $\mathcal{L}_{\mathsf{L}}$ there are equivalent*

(*i*) *H is LPC^*-provable,*

(*ii*) *H is valid in all interpretations whose truth degree structure is the standard matrix $\mathfrak{M}_{\mathsf{L}_\infty}$,*

(*iii*) *H is valid in all interpretations whose truth degree structure is any MV-algebra based on a complete lattice.*

This result also shows that MV-algebras play for L_∞ the same role as BOOLEan algebras play for classical first-order logic.

6.3 First-order monoidal logic

In this case there is the same kind of standard transfer from the propositional to the first-order case as for ŁUKASIEWICZ's many-valued logics. And the quantifiers $\forall, \exists$ again have to be understood via the equations

$$\mathsf{ver}_\forall^{\mathsf{M}}(X) =_{\mathsf{def}} \inf X \quad \text{and} \quad \mathsf{ver}_\exists^{\mathsf{M}}(X) =_{\mathsf{def}} \sup X$$

for all subsets X of the actual "truth degree", i.e. values structure, which now has to be a complete integral lattice ordered monoid. Of course, the infimum and supremum here have to be taken w.r.t. the lattice ordering in this monoid.

As in the propositional case, the set of designated truth degrees is the singleton $\mathcal{D}^{\mathsf{M}} = \{\mathbf{1}\}$ of the universal bound in each of these integral monoids.

An adequate axiomatisation of the class of universally valid formulas characterised by the class of all complete integral ℓ-monoids then is given by the axiom schemata $(M_1), \ldots, (M_{14})$ of Section 4.5 together with the first-order schemata

$$\begin{array}{ll} (M_{18}) & (\forall x H \to H[x/\tau]), \\ (M_{19}) & (H[x/\tau]) \to \exists x H \end{array}$$

for terms τ, and together with the inference rules of *detachment*, of *generalisation in the succedent*, and of *particularisation in the antecedent*:

$$\frac{H_1,\ H_1 \to H_2}{H_2}, \qquad \frac{G \to H}{G \to \forall x H}, \qquad \frac{H \to G}{\exists x H \to G} \tag{89}$$

with the crucial variable x not free in G.

Extending this system of first-order monoidal logic with the axiom schema (M_{15}) again yields an adequate axiomatisation of first-order intuitionistic logic, and extending it with the axiom (M_{16}) gives the first-order version of GIRARD's commutative linear logic.

The analogy with the propositional case and the particular logical systems brought up via extensions by additional axioms breaks down, however, in the case of the ŁUKASIEWICZ first-order system L_∞ because in this case one of the infinitary inference rules (87) or (88), or something equivalent, is necessary – and not available only via the rules (89) and additional axioms.

6.4 The uniform axiomatisation method of ROSSER/TURQUETTE

Concerning a unified approach toward formalisations of systems of first-order many-valued logics, the situation is as in the case of propositional logics: for the different types of first-order many-valued logics in each case a uniform approach exists, but up to now our considerations did not give a true unification which embraces different groups of first-order many-valued logics.

For a large class of finitely many-valued first-order logics there is, however, again a kind of "brute force" approach which extends the approach of Section 4.6 from the propositional to the first order case.

We do not intend to give all the details here which can be found e.g. in [94, 33], but sketch only the main ideas of how to get an adequate axiomatisation also in the first-order case.

Obviously, the problem is the treatment of quantifiers, having in mind the approach toward the connectives as realised for the propositional case in the group ($\mathsf{Ax}_{\mathrm{RT}}8$) of axioms, i.e. the coding of the value assignment of the truth degree functions by suitable axioms. The core idea again in the present case is to "code" also the truth degree behaviour of the quantifiers by appropriate axioms. To approach the problem in this way one has to assume that the truth degree behaviour of the quantifiers of the first-order finitely many-valued system S is describable in the language $\mathcal{L}_2$ of classical first-order logic. For this, we suppose to have in $\mathcal{L}_2$ negation, implication, and generalisation as its primitive logical constants, and to have for each predicate symbol P of the language $\mathcal{L}_\mathsf{S}$ and each truth degree $s \in \mathcal{W}^\mathsf{S}$ a predicate symbol $\pi_{P,s}$ of the same arity as P such that $\pi_{P,s}(\vec{\mathbf{a}})$, $\vec{\mathbf{a}}$ any assignment of individuals of a given interpretation $\mathfrak{A}$ to the individual variables of P, holds true just in case $P(\vec{\mathbf{a}})$ has truth degree s.

However, it is not enough to be able to discuss only quantified formulae, one has also to be able to discuss well-formed formulas with quantified parts. And this seems to be impossible in the previous elementary way of coding via a suitable choice of – almost obvious – axioms.

Denoting the truth degree of any formula H of $\mathcal{L}_\mathsf{S}$ w.r.t. an interpretation $\mathfrak{A}$ and a valuation $\vec{\mathbf{a}}$ again by $\mathsf{Val}^\mathsf{S}{}_\mathfrak{A}(H, \vec{\mathbf{a}})$, the *basic assumption* on S is that for all such H and all $s \in \mathcal{W}^\mathsf{S}$ the relation $\mathsf{Val}^\mathsf{S}{}_\mathfrak{A}(H, \vec{\mathbf{a}}) = s$ can adequately be expressed in $\mathcal{L}_2$ by a formula $\mathcal{V}_s(H)$. If this basic assumption is satisfied, we shall say that the first-order many-valued logic S *has truth degree conditions.*

Now we suppose additionally that in $\mathcal{L}_\mathsf{S}$ there are a binary connective $\rightarrow$, unary connectives $\sim$ and $\mathsf{J}_s, s \in \mathcal{W}^\mathsf{S}$, and a (unary) quantifier $\bigwedge$ for which we consider besides (RT_1), (RT_2) the additional conditions:

RT_3 The truth degree function corresponding to the connective $\sim$ assumes a non-designated truth degree just in the case that its argument is a designated truth degree.

RT_4 For each well-formed formula $H(x)$ of $\mathcal{L}_\mathsf{S}$ and each interpretation $\mathfrak{A}$ the following two conditions are equivalent:

(a) there is a designated truth degree $s \in \mathcal{D}^{\mathsf{S}}$ such that the truth degree condition $\mathcal{V}_s(\bigwedge x H(x))$ holds true,

(b) for each $a \in |\mathfrak{A}|$ there is a designated truth degree $s \in \mathcal{D}^{\mathsf{S}}$ such that the truth degree condition $\mathcal{V}_s(H(a))$ holds true.

Finally, to get the ROSSER/TURQUETTE axioms in the first-order case we have to consider S-*translations* $\mathcal{V}_s^{\mathsf{S}}(H)$ of the truth degree conditions $\mathcal{V}_s(H)$. The S-translation of $\mathcal{V}_s(H)$ is an $\mathcal{L}_{\mathsf{S}}$-formula which one gets from $\mathcal{V}_s(H)$ by writing always $\sim, \rightarrow, \bigwedge$ instead of the classical connectives $\neg, \Rightarrow, \forall$ and by substituting each subformula $\pi_{P,s}(\vec{x})$ of $\mathcal{V}_s(H)$ by $\mathsf{J}_s(P(\vec{x}))$.

The axiom schemata $(\mathsf{Ax}_{\mathrm{RT}}1), \ldots, (\mathsf{Ax}_{\mathrm{RT}}8)$ of the propositional case shall now be completed by the schemata:

$\mathsf{Ax}_{\mathrm{RT}}9:$ $\bigwedge x H(x) \rightarrow H(\mathtt{t})$ for all $\mathcal{L}_{\mathsf{S}}$-terms $\mathtt{t}$ suitable for this substitution,

$\mathsf{Ax}_{\mathrm{RT}}10:$ $\bigwedge x(G \rightarrow H(x)) \rightarrow (G \rightarrow \bigwedge x H(x))$ for all formulas G not containing x as a free variable,

$\mathsf{Ax}_{\mathrm{RT}}11:$ $\mathcal{V}_s^{\mathsf{S}}(H) \rightarrow \mathsf{J}_s(H)$ for all *quantified* formulas[36] H and all $s \in \mathcal{W}^{\mathsf{S}}$.

Again by $\mathsf{Ax}_{\mathrm{RT}}$ the set of all well-formed formulas falling under one of the schemata $(\mathsf{Ax}_{\mathrm{RT}}1), \ldots, (\mathsf{Ax}_{\mathrm{RT}}11)$ shall be denoted.

For the present approach, then, one has the following axiomatisation theorem, again proven in [94] and also in [33].

Theorem 6.2 *Suppose that* S *is a system of many-valued logic with* $\mathcal{W}^{\mathsf{S}} = \mathcal{W}_m$ *and with connectives* $\rightarrow, \sim$ *and* $\mathsf{J}_s, s \in \mathcal{W}^{\mathsf{S}}$ *and a quantifier* $\bigwedge$ *satisfying conditions* RT_1 *to* RT_4. *Suppose also that* S *has truth degree conditions. Then this system* S *has as an adequate, i.e. sound and complete axiomatisation the set* $\mathsf{Ax}_{\mathrm{RT}}$ *of formulas together with the rules of detachment (w.r.t. the connective* $\rightarrow$*) and of generalisation (w.r.t. the quantifier* $\bigwedge$*).*

As mentioned for the propositional case, also in the first-order case this result can be generalised in a way that makes it applicable also to the finitely many-valued ŁUKASIEWICZ systems. But, again, this is not very important because for these systems simpler axiomatisations are well known.

6.5 Adding identity

As in classical first-order logic, also in the many-valued case identity as a logical predicate can not be defined but has to be added explicitely with axioms governing its intended interpretation. For many-valued logic, however, the matter is even more complicated because there is no general agreement concerning the intuition underlying the intended understanding of identity.

The crucial point here is the problem whether the suitable understanding of identity should allow for truth degrees in between the "classical" degrees

[36] This means, for all those formulas which have an $\mathcal{L}_{\mathsf{S}}$-quantifier in front.

0, 1 to really appear as degrees of identity statements, or whether such identity statements have to have their truth degrees out of $\{0, 1\}$ only.

Therefore, we shall consider both variants: an "absolute" point of view allowing only two-valued identity relations in interpretations of many-valued first-order logic with identity, and the more "liberal" point of view compatible also with "truly" graded identity relations.

For the simplicity of formulations we introduce, previous to any formal definition of identity, the following notions.

Definition 6.1 *For any system* S *of many-valued first-order logic, an identity relation* $\mathsf{id} : |\mathfrak{A}| \to \mathcal{W}^{\mathsf{S}}$ *is called* crisp *iff one has* $\mathsf{rg}\,(\mathsf{id}) \subseteq \{0, 1\}$, *and it is called* many-valued *otherwise.*

6.5.1 Identity: the absolute point of view

The problem of many-valued identity logic was first considered explicitely by H. THIELE [104] for the finitely many-valued ŁUKASIEWICZ systems with the following restrictive result.

Theorem 6.3 *Let a suitable axiomatisation* Ax_{L} *of some* ŁUKASIEWICZ *first-order system* L_m *be given which satisfies the completeness theorem. Assume that the language* $\mathcal{L}_{\mathsf{L}}$ *is in the usual way extended with an equality sign* $\equiv$. *If one then extends* Ax_{L} *with the additional axiom schemata*[37]

$$\forall x(x \equiv x),$$
$$\forall x \forall y(x \equiv y \to_{\mathsf{L}} (H \to_{\mathsf{L}} H[x/y]))$$

for all wff H *of the extended language, then the extended first-order system* L_m *has only models with crisp identity relations.*

In some sence, therefore, the LEIBNIZ principles of identity trivialise the ŁUKASIEWICZ systems L_m with identity.

Without restriction to the ŁUKASIEWICZ systems L_m one is, however, able to axiomatize a whole class of many-valued logics with identity in the style of the ROSSER/TURQUETTE axiomatisation as explained in Sections 6.4 and 4.6.

Because of the absolute point of view to be considered here we call an S-interpretation $\equiv$*-absolute* iff it interprets the identity symbol $\equiv$ by a crisp identity relation.

Using the same assumptions for the language $\mathcal{L}_{\mathsf{S}}$ as in Section 6.4, and adding the further assumption that $\mathcal{L}_{\mathsf{S}}$ also shall have connectives $\sqcap$ for a conjunction and $\sqcup$ for a disjunction, the following additional axiom schemata provide a suitable axiomatisation:

$\mathsf{AxId^*1}$: $\bigwedge x \mathsf{J}_1(x \equiv x)$,

$\mathsf{AxId^*2}$: $\bigwedge x, y(\mathsf{J}_1(x \equiv y) \sqcup \mathsf{J}_0(x \equiv y))$,

[37] Here $H[x/y]$ is the usual substitution notation saying that the term y is substituted for the variable x.

$\mathsf{AxId^*3}$: for each function symbol F of $\mathcal{L}_\mathsf{S}$

$$\bigwedge x_1, \dots, x_n \bigwedge y_1, \dots, y_n \big(x_1 \equiv y_1 \sqcap \dots \sqcap x_n \equiv y_n \rightarrow \\ \rightarrow F(x_1, \dots, x_n) \equiv F(y_1, \dots, y_n)\big),$$

$\mathsf{AxId^*4}$: for each predicate symbol P of $\mathcal{L}_\mathsf{S}$ and each truth degree $s \in \mathcal{W}^\mathsf{S}$

$$\bigwedge x_1, \dots, x_n \bigwedge y_1, \dots, y_n \Big(x_1 \equiv y_1 \sqcap \dots \sqcap x_n \equiv y_n \sqcap \\ \sqcap \mathsf{J}_s(P(x_1, \dots, x_n)) \rightarrow \mathsf{J}_s(P(y_1, \dots, y_n))\Big).$$

The set of all formulae falling under one of these schemata ($\mathsf{AxId^*1}$), ..., ($\mathsf{AxId^*4}$) shall be denoted by $\mathsf{AxId^*}$.

The additional connectives of $\mathcal{L}_\mathsf{S}$ taken into account here force to consider also two further conditions:

RT_5 The truth degree function corresponding to the connective $\sqcap$ assumes a designated truth degree just in the case that both its arguments are designated truth degrees.

RT_6 The truth degree function corresponding to the connective $\sqcup$ assumes a non-designated truth degree just in the case that both its arguments are non-designated truth degrees.

For this absolute point of view, then, one has the following axiomatisation theorem of MORGAN [72], cf. also [33].

Theorem 6.4 *Suppose that* S *is a system of many-valued logic with* $\mathcal{W}^\mathsf{S} = \mathcal{W}_m$ *and with connectives* $\rightarrow, \sqcap, \sqcup, \sim$ *and* $\mathsf{J}_s, s \in \mathcal{W}^\mathsf{S}$, *and a quantifier* $\bigwedge$ *satisfying conditions* RT_1 *to* RT_6. *Suppose also that* S *has truth degree conditions. Then the class of all wff logically valid in all* $\equiv$*-absolute* S*-interpretations has as an adequate axiomatisation the set* $\mathsf{Ax}_{\mathrm{RT}} \cup \mathsf{AxId^*}$ *of formulas together with the rules of detachment (w.r.t. the connective* $\rightarrow$*) and of generalisation (w.r.t. the quantifier* $\bigwedge$*).*

6.5.2 Identity: the liberal point of view

Many-valued first-order identity logic which allows also many-valued identity relations was first concidered only in the mid 1970, was partly inspired by research on fuzzy set theory, and followed the lines of the ROSSER/TURQUETTE approach, cf. [73].

For a system S of finitely many-valued first-order logic the intended interpretations $\mathfrak{A}$ now shall be called $\equiv$*-normal.* They shall be characterised by the fact that the function $\mathsf{id} : |\mathfrak{A}|^2 \rightarrow \mathcal{W}^\mathsf{S} = \mathcal{W}_m$ which interprets in $\mathfrak{A}$ the identity symbol $\equiv$ has to satisfy for all $b_1, b_2 \in |\mathfrak{A}|$ the following three conditions:

$\mathsf{N}_\equiv 1$: $\mathsf{id}(b_1, b_2) = 1$,

$\mathsf{N}_\equiv 2$: $\mathsf{id}(b_1, b_2) = \mathsf{id}(b_2, b_1)$,

$\mathsf{N}_{\equiv}3:\ \mathsf{id}(b_1,b_2) \leq \inf\{1 - |P^{\mathfrak{A}}(\vec{a}) - P^{\mathfrak{A}}(\vec{c})| \mid P \in \mathcal{P}^{\mathsf{S}} \text{ and } (\vec{a},\vec{c}) \in \mathbf{C}(P;b_1,b_2)\}$
with $\mathbf{C}(P;b_1,b_2)$ the set of all pairs of n-tuples, n the arity of P, which coincide in all coordinates but one which is b_1 in $\vec{a}$ and b_2 in $\vec{c}$.

In the present case the same assumptions concerning $\mathcal{L}_{\mathsf{S}}$ as in the previous subsection shall apply. In particular, $\mathcal{L}_{\mathsf{S}}$ shall have the connectives $\sqcap, \sqcup$ as before, and we let $\bigsqcup$ denote the finite iteration of $\sqcup$. However, for simplicity we suppose that $\mathcal{L}_{\mathsf{S}}$ has only predicate symbols. Then the following schemata of axioms become important:

$\mathsf{AxId1}:\ \bigwedge x \mathsf{J}_1(x \equiv x),$

$\mathsf{AxId2}:\ \bigwedge x,y(\mathsf{J}_s(x \equiv y) \rightarrow \mathsf{J}_s(y \equiv x)) \quad \text{for all } s \in \mathcal{W}^{\mathsf{S}},$

$$\mathsf{AxId3}:\ \bigwedge x_i,y_i\big(\mathsf{J}_s(x_i \equiv y_i) \sqcap (\mathsf{J}_s(P(x_1 \dots x_i \dots x_n)) \rightarrow \\ \rightarrow \bigsqcup_{r=\mathsf{et}_2(s,t)}^{\mathsf{seq}_1(s,t)}(\mathsf{J}_r(P(x_1 \dots y_i \dots x_n)))\big)$$
for each predicate symbol P of $\mathcal{L}_{\mathsf{S}}$ and all $s,t \in \mathcal{W}^{\mathsf{S}}$.

The set of all formulae falling under one of these schemata (AxId1), ..., (AxId3) shall be denoted by AxId.

For this liberal point of view, then, one has the following axiomatisation theorem of MORGAN [73], cf. also [33].

Theorem 6.5 *Suppose that* S *is a system of many-valued logic with* $\mathcal{W}^{\mathsf{S}} = \mathcal{W}_m$ *and with connectives* $\rightarrow, \sqcap, \sqcup, \sim$ *and* $\mathsf{J}_s, s \in \mathcal{W}^{\mathsf{S}}$, *and a quantifier* $\bigwedge$ *satisfying conditions* RT_1 *to* RT_6. *Suppose also that* S *has truth degree conditions. Then the class of all wff logically valid in all* $\equiv$*-normal* S*-interpretations has as an adequate axiomatisation the set* $\mathsf{Ax}_{\mathrm{RT}} \cup \mathsf{AxId}$ *of formulas together with the rules of detachment (w.r.t. the connective* $\rightarrow$*) and of generalisation (w.r.t. the quantifier* $\bigwedge$*).*

Unfortunately, the axioms in AxId look rather complicated. Quite a bit simpler becomes the situation for the ŁUKASIEWICZ systems L_m. In this case it suffices to consider the axiom schemata:

$\mathsf{Id}_{\text{Ł}}1:\ \forall x(x \equiv x),$

$\mathsf{Id}_{\text{Ł}}2:\ \forall x,y(x \equiv y \rightarrow_{\mathsf{L}} y \equiv x) \quad ,$

$\mathsf{Id}_{\text{Ł}}3:\ \forall x_i,y_i\big(x_i \equiv y_i \otimes P(x_1 \dots x_i \dots x_n) \rightarrow_{\mathsf{L}} P(x_1 \dots y_i \dots x_n)\big)$
for each predicate symbol P of $\mathcal{L}_{\mathsf{S}}$.

As an immediate corollary one gets that an L_m-interpretation $\mathfrak{A}$ is $\equiv$-normal iff it is a model of the set $\mathsf{Id}_{\text{Ł}}$ of axioms determined by these schemata.

And the axiomatisation of L_m according to the slightly modified ROSSER/TURQUETTE approach suitable for L_m again gives that $\mathsf{Id}_{\text{Ł}}$ provides an adequate axiomatisation for the liberal finitely many-valued first-order ŁUKASIEWICZ identity logics, cf. [31, 32].

In the particular case of pure identity logic, i.e. in the case that the first-order language $\mathcal{L}_{\mathsf{L}}$ has $\equiv$ as its only predicate symbol, condition $\mathsf{Id}_{\mathsf{L}}3$ collapses to the requirement of $\otimes$-transitivity:

$$\forall x, y, z(x \equiv y \otimes y \equiv z \rightarrow_{\mathsf{L}} x \equiv z).$$

And this situaition, obviously, is easily generalised to the case of an arbitrary (left continuous) t-norm $\boldsymbol{t}$. In such a case, therefore, reflexivity, symmetry, and $\boldsymbol{t}$-transitivity form a suitable set of conditions[38] characterising a generalised, many-valued identity relation. In other words, in this case the fuzzy similarities[39] are just the suitable generalised identity relations.

6.5.3 Identity and degrees of existence

Still another, a bit more general approach toward many-valued identities, in the liberal understanding, is offered in the context of monoidal first-order logic by U. Höhle [43, 47]. The crucial point now is that even the assumption that for liberal identities $\equiv$ each formula $\mathtt{t} = \mathtt{t}$ has to have the truth degree 1, or at least a designated[40] truth degree, is abandoned.

Then the problem arises, how to intuitively interpret a situation that in an interpretation $\mathfrak{A}$ for some individual a the formula $a \equiv a$ has a truth degree different from the universal upper bound $\mathbf{1}$. The solution is to interpret each such truth degree of $a \equiv a$ as *degree of existence*[41] of the object a.

The intended *monoidal* $\equiv$*-interpretations* are in this case all the monoidal interpretations $\mathfrak{A}$ enriched with an M*-valued equality* $E : |\mathfrak{A}|^2 \rightarrow \mathsf{M}$ characterised by the properties:

E1: $E(x, y) \leq E(x, x) \wedge E(y, y)$,

E2: $E(x, y) = E(y, x)$,

E3: $E(x, y) * (E(y, y) \rightarrow E(y, z)) \leq E(x, z)$.

For the formal system of monoidal first-order identity logic M it is suitable to assume that the language $\mathcal{L}_{\mathsf{M}}$ besides the binary identity symbol $\equiv$ has a unary symbol e to denote the existence predicate. Given any monoidal $\equiv$-interpretation $\mathfrak{A}$ with M-valued equality E, the standard understanding is that the function $\epsilon : |\mathfrak{A}| \rightarrow \mathsf{M}$ with always $\epsilon(a) = E(a, a)$ is the meaning of the existence symbol e w.r.t. $\mathfrak{A}$.

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[38] This situation has, however, not yet been generally axiomatised.

[39] Often also called fuzzy equivalence relations, cf. e.g. [34].

[40] Actually, both these possibilities coincide because only the case that the universal bound of the monoidal truth structure is the single designated truth degree has been discussed.

[41] This is an adaption and generalisation of a corresponding approach of D. Scott [100] toward intuitionistic logic. Accordingly the following axiom schemata ($\mathsf{AxIE1}$) to ($\mathsf{AxIE7}$) reduce to his intuitionistic axioms of identity and existence. Intuitively, this degree of existence may be understood as indicating a kind of partial existence, e.g. a localised or locally restricted (in a suitable geometrical sense) one.

A suitable set AxIE of axioms for monoidal first-order identity logic, i.e. one which provides together with the other axioms of first-order monoidal logic an adequate axiomatisation, is then determined by the following axiom schemata, cf. [47]:

$\mathsf{AxIE1}:\ x \equiv y \to (\mathsf{e}(x) \wedge \mathsf{e}(y)),$

$\mathsf{AxIE2}:\ \mathsf{e}(x) \to x \equiv x,$

$\mathsf{AxIE3}:\ x \equiv y \to y \equiv x,$

$\mathsf{AxIE4}:\ P(x_1, \dots, x_n) \to \bigwedge_{i=1}^{n} \mathsf{e}(x_i)$ for each n-ary predicate symbol P,

$\mathsf{AxIE5}:\ (\mathsf{e}(y_i) \to y_i \equiv z_i) \to (P(x_1, \dots, y_i, \dots, x_n) \to P(x_1, \dots, z_i, \dots, x_n))$
for each n-ary predicate symbol P,

$\mathsf{AxIE6}:\ \mathsf{e}(F(x_1, \dots, x_n)) \to \bigwedge_{i=1}^{n} \mathsf{e}(x_i)$ for each n-ary function symbol F,

$\mathsf{AxIE7}:\ y_i \equiv z_i \to F(x_1, \dots, y_i, \dots, x_n) \equiv F(x_1, \dots, z_i, \dots, x_n)$
for each n-ary function symbol F.

7 Fuzzy Sets and Many-Valued Logic

7.1 Membership Degrees as Truth Degrees

A *fuzzy set* A is characterised by its generalised characteristic function $\mu_A : \mathcal{X} \to [0,1]$, called *membership function* of A and defined over some given *universe of discourse* $\mathcal{X}$, i.e. it is a *fuzzy subset* of $\mathcal{X}$.

The essential idea behind this approach was to have with the membership degree $\mu_A(a)$ for each point $a \in \mathcal{X}$ a graduation of its membership with respect to the fuzzy set A. And this degree obviously is nothing else as a degree to which the sentence *"a is a member of A"* holds true. Hence it is natural to interpret the membership degrees of fuzzy sets as truth degrees of the membership predicate in some (suitable system of) many-valued logic S.

To do this in a reasonable way one has to accept some minimal conditions concerning the language $\mathcal{L}_{\mathsf{S}}$.

Disregarding – for simplicity – fuzzy sets of type 2 and of every higher type as well (as is done in the overwhelming majority of fuzzy sets applications) one has, from the set theoretic point of view inspired by the usual idea of a cumulative hierarchy of sets, fuzzy sets as (generalized) sets of first level over a given set of urelements. Therefore the intended language needs besides a (generalized, i.e. graded) binary membership predicate ε e.g. two types of variables: (i) lower case latin letters $a, b, c, \dots, x, y, z$ for urelements, i.e. for points of the universe of discourse $\mathcal{X}$, and (ii) upper case latin letters $A, B, C, \dots$ for fuzzy subsets of $\mathcal{X}$. And of course it has some set of connectives and some quantifiers – and thus a suitable notion of well-formed formula.

Having in mind the standard fuzzy sets with membership degrees in the real unit interval $[0,1]$ thus forces to assume that S is an infinitely many-valued logic.

And the usual intuitive understanding of the membership degrees furthermore supports the assumption that $\mathcal{D}^{\mathsf{S}} = \{1\}$ is the set of designated truth degrees.

It is not necessary to fix all the details of the language $\mathcal{L}_{\mathsf{S}}$ in advance. We suppose, for simplicity of notation, that from the context it shall always be clear which objects the individual symbols are to denote.[42] Denoting the truth degree of a well-formed formula H by $[\![H]\!]$, to identify membership degrees with suitable truth degrees then means to put

$$\mu_A(x) = [\![\, x \,\varepsilon\, A \,]\!]. \tag{90}$$

This type of interpretation proves quite useful: it opens the doors to clarify far reaching analogies between notions and results related to fuzzy sets and those ones related to usual sets, cf. [33, 34].

7.2 Doing Fuzzy Set Theory with MVL Language

Based on the main idea to look at the membership degrees of fuzzy sets as truth degrees of a suitable membership predicate, it e.g. becomes quite natural to describe fuzzy sets by a (generalized) class term notation, adapting the corresponding notation $\{x \mid H(x)\}$ from traditional set theory and introducing a corresponding notation for fuzzy sets by

$$A = \{x \in \mathcal{X} \parallel H(x)\} \;\Leftrightarrow_{\mathrm{def}}\; \mu_A(x) = [\![H(x)]\!] \quad \text{for all } x \in \mathcal{X}, \tag{91}$$

with now H a well-formed formula of the language of our system S of many-valued logic for fuzzy set theory. As usual, the shorter notation $\{x \parallel H(x)\}$ is also used, and even preferred.

7.2.1 Fuzzy set algebra

With this notation the intersection and the cartesian product of fuzzy sets A, B are, even in their t-norm based version, determined as

$$\begin{aligned} A \cap_{\mathbf{t}} B &= \{x \parallel x \,\varepsilon\, A \wedge_{\mathbf{t}} x \,\varepsilon\, B\}, \\ A \times_{\mathbf{t}} B &= \{(x,y) \parallel x \,\varepsilon\, A \wedge_{\mathbf{t}} y \,\varepsilon\, B\}, \end{aligned}$$

and the standard, i.e. min-based form of the compositional rule of inference[43] applied w.r.t. a fuzzy relation R and a fuzzy set A becomes

$$A \circ R = \{y \parallel \exists x (x \,\varepsilon\, A \wedge (x,y) \,\varepsilon\, R)\}, \tag{92}$$

i.e. the analogue of a formula well known from elementary relation algebra.

[42] Which, formally, means that we assume that a valuation always is determined by the context and has not explicitely to be mentioned.

[43] This compositional rule of inference is of central importance for the applications of fuzzy sets to fuzzy control and to approximate reasoning, cf. Chapter 1 of the Handbook volume "Fuzzy Sets in Approximate Reasoning and Information Systems" edited by J. Bezdek, D. Dubois and H. Prade.

Also for the inclusion relation between fuzzy sets this approach works well. The standard definition of inclusion amounts to

$$A \subset B \quad \Leftrightarrow \quad \mu_A(x) \le \mu_B(x) \text{ for all } x \in \mathcal{X}$$

which in the language of many-valued logic is the same as

$$A \subset B \quad \Leftrightarrow \quad \models \forall x (x \,\varepsilon\, A \rightarrow_t x \,\varepsilon\, B) \tag{93}$$

w.r.t. any one R-implication connective based on a left continuous t-norm.

Obviously this version (93) of inclusion is easily generalised to a "fuzzified", i.e. (truly) many-valued inclusion relation defined as

$$A \subseteqq B =_{\mathsf{def}} \forall x (x \,\varepsilon\, A \rightarrow_t x \,\varepsilon\, B). \tag{94}$$

And this many-valued inclusion relation for fuzzy sets has still nice properties, e.g. it is t-transitive, i.e. one has:

$$\models \ (A \subseteqq B \wedge_t B \subseteqq C \rightarrow_t A \subseteqq C).$$

7.2.2 Fuzzy relation theory

This natural approach (92) toward the compositional rule of inference is almost the same as the usual definition of the relational product $R \circ S$ of two fuzzy relation, now – even related to some suitable t-norm – to be determined as

$$R \circ_t S = \{(x, y) \,\|\, \exists z \big((x, z) \,\varepsilon\, R \wedge_t (z, y) \,\varepsilon\, S\big)\}.$$

Relation properties become, in this context, again characterisations which formally read as the corresponding properties of crisp sets. Consider, as an example, transitivity of a fuzzy (binary) relation R in the universe of discourse $\mathcal{X}$ w.r.t. some given t-norm. The usual condition for all $x, y, z \in \mathcal{X}$

$$t(\mu_R(x, y), \mu_R(y, z)) \le \mu_R(x, z)$$

in the language $\mathcal{L}_S$ of the intended suitable $[0, 1]$-valued system for fuzzy set theory becomes the condition

$$\models R(x, y) \wedge_t R(y, z) \rightarrow_t R(x, z)$$

for all $x, y, z \in \mathcal{X}$ or, even better, becomes

$$\models \forall x, y, z \big(R(x, y) \wedge_t R(y, z) \rightarrow_t R(x, z)\big) \tag{95}$$

with the universal quantifier $\forall$ as in the ŁUKASIEWICZ systems.

This point of view not only opens the way for a treatment of fuzzy relations quite analogous to the usual discussion of properties of crisp relations, it also opens the way to consider *graded versions* of properties of fuzzy relations, cf. [34]. In the case of transitivity, a graded or "fuzzified" predicate Trans with the intended meaning "is transitive" may be defined as

$$\mathsf{Trans}(R) =_{\mathsf{def}} \forall x, y, z \big(R(x, y) \wedge_t R(y, z) \rightarrow_t R(x, z)\big). \tag{96}$$

7.2.3 Fuzzy sets and many-valued identities

Actually, there are essentially two interesting relationships between fuzzy sets and many-valued identity relations in the liberal sense.

Fuzzy sets determined by many-valued identities: Starting with a many-valued identity relation id in some given universe of discourse $\mathcal{X}$, having some t-norm $\boldsymbol{t}$ at hand, and assuming additionally that $\equiv$ is the only predicate symbol to be taken into account for the language under consideration, then – as mentioned in Section 6.5 – it is suitable to take id as a $\boldsymbol{t}$-transitive fuzzy similarity.

Extending the language then with the generalised membership predicate ε the most natural adoption of the identity axiom $\mathsf{Id}_{\text{L}}3$ to this situation is to assume that each fuzzy set A has to be characterised by a membership function μ_A satisfying

$$\boldsymbol{t}(\mu_A(a), \mathsf{id}(a,b)) \leq \mu_A(b)$$

for all $a, b \in \mathcal{X}$. In the language of many-valued logic this means to have satisfied

$$\models \forall x, y (x \,\varepsilon\, A \wedge_{\boldsymbol{t}} x \equiv y \rightarrow_{\boldsymbol{t}} y \,\varepsilon\, A). \tag{97}$$

This particular adoption of the identity axiom therefore $\mathsf{Id}_{\text{L}}3$ in the present situation becomes a condition of extensionality for fuzzy sets written down in the language of many-valued logic for fuzzy set theory.

The same type of approach also works in the case that one starts from an M-valued equality.

Extensionality in this sense was a basic idea behind the authors approach in [29, 30] toward a cumulative hierarchy of fuzzy sets, and is constitutive for the theory of M-valued sets of U. Höhle [45, 47].

As a particular case of (97) the fuzzy singleton $\{a\}_1$ of $a \in \mathcal{X}$ then may be defined by

$$\{a\}_1 =_{\text{def}} \{x \parallel x \equiv a\},$$

i.e. by the membership function $\mu_{(a)}$ characterised by the equation $\mu_{(a)}(x) = \mathsf{id}(x, a)$.

For $\mathcal{X} = \mathbb{R}$ and with a suitable choice of id such singletons become triangular fuzzy numbers and may be considered as the input and output values of fuzzy implications as used in systems of fuzzy control rules, cf. [49].

A many-valued identity for fuzzy sets: The previous discussions have related fuzzy sets and many-valued identities in such a way as to start from some many-valued identity and then to consider fuzzy sets as based on it.

There is also another possibility to connect fuzzy sets and many-valued identities: one can define a "fuzzified", i.e. many-valued identity between fuzzy sets. A quite natural basis for such an approach gives the many-valued inclusion relation (94), again w.r.t. some left continuous t-norm $\boldsymbol{t}$, via the definition

$$A \equiv_{\boldsymbol{t}} B =_{\text{def}} A \subseteqq B \wedge_{\boldsymbol{t}} B \subseteqq A. \tag{98}$$

This really gives a many-valued identity because one has e.g. the properties, cf. [34]:

$$
\begin{array}{rl}
(i) & \models A \equiv_t A, \\
(ii) & \models A \equiv_t B \rightarrow_t B \equiv_t A, \\
(iii) & \models A \equiv_t B \wedge_t B \equiv_t C \rightarrow_t A \equiv_t C, \\
(iv) & \models A \equiv_t B \rightarrow_t A \cap_t C \equiv_t B \cap_t C, \\
(v) & \models A \equiv_t B \rightarrow_t A \times_t C \equiv_t B \times_t C, \\
(vi) & \models A \equiv_t \emptyset \vee B \equiv_t \emptyset \rightarrow_t A \times_t B \equiv_t \emptyset.
\end{array}
$$

8 Fuzzy logic

The present state of development of the field of fuzzy logic forces to start with a terminological side remark concerning an essential ambiguity in the use of the terminus *fuzzy logic.* There is a widespread use, mainly originating from developments in the engineering fields of automatic control and of knowledge engineering, to understand by fuzzy logic any topic which involves fuzzy sets. Actually, this understanding of the terminus "fuzzy logic" which is not tied with the core ideas of formal logic often is refered to as: fuzzy logic in the wider sense. This understanding of the terminus "fuzzy logic" supersedes an older one which meant by fuzzy logic any system of many-valued logic related to fuzzy sets.

Contrary to this usage, *fuzzy logic in the narrower sense* of the word refers to a particular extension of many-valued logic, an extension which allows for fuzzy sets of premisses from which (graded) conclusions may be drawn.

It is this usage in the narrower sense which is our topic in the following parts of this paper. For more details, the reader should consult the quite recent book [39] of P. HÁJEK.

8.1 Adding partially sound inference rules to many-valued logic

Any well-formed formula of many-valued propositional logic as well as any closed formula of many-valued first order logic describes some propositions which may be only *partially* true (or partially false), i.e. which has a truth degree (instead of a traditional truth value).

Additionally, in many-valued first order logic formulas which contain free individual variables describe properties which hold for the objects, of which they are properties, only to some (truth) degree.

Finally, fuzzy logic combines the idea of a logical calculus as a formal system for inferring formulas from given (sets of) premises with the idea of fuzzy sets in the way that fuzzy logic allows for only *partially* "given" premises, i.e. for fuzzy sets of premises.

As systems of logic, both many-valued as well as fuzzy logic have a syntactic and a semantic aspect. The semantic considerations for many-valued logic are based (i) in the propositional case on valuations, i.e. mappings $v : V \to \mathcal{D}$ from the set V of propositional variables into the set $\mathcal{D}$ of truth degrees and completed by a consequence operation $\mathrm{Cn}_{\mathsf{S}} : I\!\!P(\mathcal{FOR}) \to I\!\!P(\mathcal{FOR})$ over subsets of the class $\mathcal{FOR}$ of all (well formed) formulas, and (ii) in the first order case on interpretations $\mathfrak{A}$ which map individual constants into a given universe of discourse $|\mathfrak{A}|$ and predicate letters (of arity n) to (n-ary) functions from $|\mathfrak{A}|$ into the set $\mathcal{D}$ of truth degrees and again completed by a consequence operation $\mathrm{Cn}_{\mathsf{S}} : I\!\!P(\mathcal{FOR}) \to I\!\!P(\mathcal{FOR})$.

The semantic considerations for fuzzy logic are based (i) on the valuations $v : V \to \mathcal{D}$ which map the set of propositional variables V into the set $\mathcal{D}$ of membership degrees and completed by a consequence operation $\mathrm{Cn}^{\star}_{\mathsf{S}} : I\!\!F(\mathcal{FOR}) \to I\!\!F(\mathcal{FOR})$ over the fuzzy subsets (with membership degrees in $\mathcal{D}$) of the class $\mathcal{FOR}$ of formulas, and (ii) in the first order case on interpretations $\mathfrak{A}$ which map individual constants into the universe of discourse $|\mathfrak{A}|$ and predicate letters (of arity n) to (n-ary) fuzzy relations in $|\mathfrak{A}|$ and completed again by a consequence operation $\mathrm{Cn}^{\star}_{\mathsf{S}} : I\!\!F(\mathcal{FOR}) \to I\!\!F(\mathcal{FOR})$ over the fuzzy subsets (with membership degrees in $\mathcal{D}$) of the class $\mathcal{FOR}$ of formulas.

The syntactic considerations for many-valued as well as fuzzy logic are based on suitable calculi fixed by their respective (crisp) sets of axioms and sets of inference rules. (Inferences in any case are finite sequences.) For fuzzy logic each inference rule R with k premises splits into two parts $\mathsf{R} = (\mathsf{R}^1; \mathsf{R}^2)$ such that R^1 is a (partial) k-ary mapping from $\mathcal{FOR}$ into $\mathcal{FOR}$ and R^2 is a k-ary mapping from $\mathcal{D}$ into $\mathcal{D}$. The idea is that R^2 associates with the degrees to which actual premises of R are given a degree to which the actual conclusion of R is given.

In any case, one of the main goals of those calculi is the axiomatization of the set of logically true formulas or of the logical part of elementary theories.

Let us first look at many-valued logic. Of course, not every inference schema is accepted as an inference rule: a necessary restriction is that to *sound* inference schemata, i.e. to inference schemata which led "from true premises only to a true conclusion". For many-valued logic this actually shall mean either that in case all premises have truth degree 1, which we suppose to be the only designated one, also the conclusion has truth degree 1 or that the conjunction of all premises has a truth degree not greater then the truth degree of the conclusion.

For fuzzy logic a comparable soundness condition is assumed, called "infallibility" in [7] instead of "soundness" as e.g. in [78], [80] or "Korrektheit" as in [33].

Now, looking back at the approaches we mentioned, one recognises that many-valued logic deals with partially true sentences, fuzzy logic deals with even partially given (sets of) premises – but in both cases soundness of the rules of inference is taken in some absolute sense.

What about partial soundness for inference schemata?

Is it possible to consistently discuss such a notion of partial soundness? And if possible: how such a notion can be integrated into many-valued as well as into fuzzy logic?

Regarding applications of partially sound rules of inference, it is quite obvious that an analysis of the heap paradox or the bold man paradox or other types of paradoxes may, instead of referring to the use of implications with a truth degree a little bit smaller then 1, become based upon the use of inference schemata which are "not completely sound".

8.2 Formalising the problem

As already in the preceding remarks we will restrict the considerations here to ŁUKASIEWICZ type many-valued systems: truth degrees a subset of the real interval $[0, 1]$, bigger truth degrees as the "better" ones, and 1 the only (positively) designated truth degree.

As the inference schema under discussion let us consider the schema

$$(\mathsf{R}) : \quad \frac{H_1, \dots, H_n}{H}$$

with n premises $H_1, \dots, H_n$ and the conclusion H. Furthermore, we use $[\![G]\!]$ to denote the truth degree of the formula G. The dependence of $[\![G]\!]$ from the supposed interpretation and a valuation of the (free) individual variables can be supposed to be clear from the context.

The usual soundness condition for (R) we shall consider is the inequality

$$[\![H_1 \& \dots \& H_n]\!] \leq [\![H]\!] \tag{99}$$

with & for a suitable conjunction operation, e.g. some t-norm. We prefer (99) because it is more general then demanding only

$$\text{IF} \quad [\![H_1]\!] = \dots = [\![H_n]\!] = 1 \quad \text{THEN} \quad [\![H]\!] = 1 \ . \tag{100}$$

A partially sound schema (R) of inference, which is not sound in the usual sense, has to deviate from (99) to some degree. And this degree of deviance we intend to use to "measure" the deviance of schema (R) from soundness, i.e. to define the "degree of soundness" of schema (R).

Referring to a measure of deviance of schema (R) from soundness condition (99) it is quite natural to look at the value $\delta(\mathsf{R})$ defined as

$$\delta(\mathsf{R}) = \sup \left(\max\{0, [\![H_1 \& \dots \& H_n]\!] - [\![H]\!]\} \right) \tag{101}$$

where the sup has to be taken with respect to all interpretations and all valuations of the individual variables. But other approaches as (101) can be discussed too.

Now, with respect to a suitable implication connective $\rightarrow$ one should have

$$[\![G_1]\!] \leq [\![G_2]\!] \quad \text{iff} \quad [\![G_1 \rightarrow G_2]\!] = 1 \tag{102}$$

for any formulas G_1, G_2 and hence

$$[\![H_1 \& \dots \& H_n \rightarrow H]\!] = 1 \tag{103}$$

as an equivalent soundness condition instead of (99). Using the symbol $\models$ for the consequence or satisfaction relation one has thus usually

$$\begin{aligned}(\mathsf{R}) \text{ sound} \quad &\Leftrightarrow \quad \text{always } [\![H_1\&\ldots\&H_n]\!] \leq [\![H]\!] \\ &\Leftrightarrow \quad \text{always } [\![H_1\&\ldots\&H_n \to H]\!] = 1 \\ &\Leftrightarrow \quad \models (H_1\&\ldots\&H_n \to H) \ .\end{aligned}$$

These equivalences give another way to approach partial soundness instead of (101).

Definition 8.1 *For a schema of inference* (R) *its degree of soundness shall be*

$$\kappa(\mathsf{R}) =_{\mathsf{def}} \inf([\![H_1\&\ldots\&H_n \to H]\!]) \tag{104}$$

with the infimum taken over all interpretations and all valuations of the individual variables.

Corollary 8.1 *With* $\to$ *the* ŁUKASIEWICZ *implication, characterised by the truth degree function* $\mathsf{seq}_L(u,v) = \min\{1, 1-u+v\}$, *one has*

$$\kappa(\mathsf{R}) = 1 - \delta(\mathsf{R})$$

and thus (104) as a suitable generalisation of the idea which led to (101).

For $\kappa(\mathsf{R})$ the degree of soundness of inference rule (R) one thus always has

$$\kappa(\mathsf{R}) \leq [\![H_1\&\ldots\&H_n \to H]\!]$$

or even, accepting $\underline{\kappa(\mathsf{R})}$ as a constant of the language to denote the degree $\kappa(\mathsf{R})$ and having (102) satisfied,

$$[\![\underline{\kappa(\mathsf{R})} \to (H_1\&\ldots\&H_n \to H)]\!] = 1$$

which via importation and exportation for the implication, and commutativity for the conjunction operations is equivalent to

$$[\![H_1\&\ldots\&H_n \to (\underline{\kappa(\mathsf{R})} \to H)]\!] = 1 \ .$$

This is a first way to "code" partially sound rules: it presupposes that one has to have each degree of soundness as a truth degree constant available within the language.

8.3 Partially sound rules in many-valued logic

The use of sound rules of inference within the process of inference of new propositions from given premises can be seen as a transfer of confidence in the premises to a confidence in the conclusion.

For classical logic, this type of interpretation looks completely unproblematic: "confidence" can be understood as the assumption of truth. A sound rule of inference does then rationally transfer this confidence from the premises to the conclusion. For many-valued logic, one way to interpret "confidence" in a formula or proposition H is to translate this into the statement $[\![H]\!] = 1$, i.e. again into the assumption of the (complete) truth of H. But it also seems natural to give "confidence in H" another reading meaning $[\![H]\!] \geq u$ for some truth degree u. In this second sense confidence itself is graded in some sense, and this is in very interesting coincidence with the basic idea of many-valued logic, i.e. with the graduation of truth.

Soundness condition (99) now can be read as allowing only such inference schemata as sound ones which transfer the "common confidence" in the premises, i.e. the confidence in the conjunction of the premises into a suitable confidence in the conclusion.

This reading of soundness shows that and why soundness condition (99) seems to be preferable over soundness condition (100). And this reading is based on a kind of identification of truth degrees with degrees of confidence (in the sense that degree u of confidence in H means $[\![H]\!] \geq u$) – and this does not seem completely unreasonable. Yet, here this interpretation of truth degrees as (lower bounds of) confidence degrees either refers to an identification of truth degrees with degrees of confidence – or it can simpler be seen as an addition of confidence degrees to classical logic.

Up to now the discussion was related simply to soundness. What about partially sound rules of inference? Of course, depending on their degree of soundness the confidence in the conclusion, given the confidences in the premises, should be smaller than in case of a completely sound rule of inference (with the same premises) - and becoming as smaller as the degree of soundness is becoming smaller.

Looking again at the schema (R) and first assuming full confidence in the premises, i.e. assuming $[\![H_i]\!] = 1$ for $1 \leq i \leq n$ or $\models H_i$ for $1 \leq i \leq n$, it seems reasonable to assume

$$\kappa(\mathsf{R}) \leq \text{confidence in } H \ ,$$

i.e. to assume

$$\text{IF} \quad [\![H_1]\!] = \ldots = [\![H_n]\!] = 1 \quad \text{THEN} \quad \kappa(\mathsf{R}) \leq [\![H]\!] \ . \tag{105}$$

This last condition in the present case also can be written as

$$\kappa(\mathsf{R}) \leq [\![H_1 \& \ldots \& H_n \to H]\!] \leq [\![H]\!] \tag{106}$$

simply assuming additionally that $[\![1 \to H]\!] = [\![H]\!]$ holds always true, which is only a mild and natural restriction concerning the implication connective $\to$.

Thus, from the intuitive point of view an application of a partially sound rule of inference can be understood via the additional idea of confidence degrees.

But, what about repeated applications of partially sound rules of inference? And is it really a convincing idea to identify degrees of confidence with (lower bounds of) the truth degrees?

At least for the second one of these questions a more cautious approach may help to separate confidence and truth degrees: a change from formulas to ordered pairs consisting of a formula and a degree of confidence. The consequence of this idea for inference schemata like (R) now is that premises and conclusions have to become ordered pairs, i.e. (R) changes into the modified schema

$$(\hat{\mathsf{R}}): \quad \frac{(H_1, \alpha_1), \dots, (H_n, \alpha_n)}{(H, \beta)} \ .$$

Choosing here confidence degrees, like membership degrees and truth degrees, from the real unit interval $[0,1]$ means that in $(\hat{\mathsf{R}})$ usually $\alpha_i > 0$ for $1 \leq i \leq n$ as well as $\beta > 0$; furthermore $\beta = \beta(\vec{\alpha})$ has to be a function of $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$.

A modified soundness condition for the inference schema $(\hat{\mathsf{R}})$ now is easily at hand.

Definition 8.2 *A modified inference schema of the form $(\hat{\mathsf{R}})$ is sound* iff it always holds true that in case one has $[\![H_i]\!] \geq \alpha_i$ for all $1 \leq i \leq n$, then one also has $[\![H]\!] \geq \beta$.*

The weaker form (100) of the usual soundness condition for (R) often is formulated in model theoretic terms as the condition that each model of $\{H_1, \dots, H_n\}$ also is a model of H. In its stronger form (99) this model theoretic version in many-valued logic simply reads: each interpretation is a model of

$$H_1 \& \dots \& H_n \ \rightarrow \ H \ .$$

Slightly modifying the notion of model, cf. [33], and calling for $\alpha \in [0,1]$ an interpretation A an α-model of a formula H iff $[\![H]\!] \geq \alpha$ in A (for all valuations of the individual variables) enables a model theoretic reformulation also of soundness*.

This formulation becomes even simpler if one calls A an $(\alpha_1, \dots, \alpha_n)$-model of a sequence $(H_1, \dots, H_n)$ of formulas iff A is an α_i-model of H_i for all $1 \leq i \leq n$.

Corollary 8.2 *A modified inference schema of the form $(\hat{\mathsf{R}})$ is sound* iff every $(\alpha_1, \dots, \alpha_n)$-model of $(H_1, \dots, H_n)$ is a β-model of H.*

Obviously, Definition 8.2 subsumes the traditional soundness condition (100), but also (99) and even condition (8.2) – simply depending on a suitable choice of β as a function of $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$.

Regarding condition (99) and assuming that (R) fulfils condition (99) one may take in $(\hat{\mathsf{R}})$ the degree $\beta = \mathrm{et}^{(n)}(\vec{\alpha})$ with $\mathrm{et}(u,v)$ the truth degree function characterising the conjunction & and $\mathrm{et}^{(n)}$ the n-ary generalisation of et, then of course this schema $(\hat{\mathsf{R}})$ is sound*. On the other hand, having the inference schema $(\hat{\mathsf{R}})$ sound* and using $\alpha_i = [\![H_i]\!]$ for all $1 \leq i \leq n$, so in case $\beta(\vec{\alpha}) = \mathrm{et}^{(n)}(\vec{\alpha})$ the corresponding "reduced" schema (R) is simply sound.

But if $(\hat{\mathsf{R}})$ is sound* and one does not always have $\beta(\vec{\alpha}) \geq \mathrm{et}^{(n)}(\vec{\alpha})$, then for $\delta^\star(\mathsf{R}) = \inf_{\alpha_1,\dots,\alpha_n}(\beta(\vec{\alpha}) - \mathrm{et}^{(n)}(\vec{\alpha}))$ one has always

$$1 + \delta^\star(\mathsf{R}) \leq 1 - \mathrm{et}^{(n)}(\vec{\alpha}) + \beta(\vec{\alpha}) \ . \tag{107}$$

With $\kappa^\star(\mathsf{R}) = 1 + \delta^\star(\mathsf{R})$ this can be written as

$$\kappa^\star(\mathsf{R}) \leq \mathsf{seq}_L(\mathrm{et}^{(n)}(\vec{\alpha}), \beta(\vec{\alpha})) \tag{108}$$

and obviously corresponds to (the first inequality of) condition (106).

Therefore, inference schemata of type $(\hat{\mathsf{R}})$ cover the usual sound inference rules as well as the partially sound rules of inference, and the soundness* for $(\hat{\mathsf{R}})$ covers usual soundness together with partial soundness.

Having a closer look at schema $(\hat{\mathsf{R}})$ one recognises that this schema can be split into two parts, viz. the old schema (R) together with a mapping R_{sem} : $(\vec{\alpha}) \mapsto \beta(\vec{\alpha})$. Taking $(\hat{\mathsf{R}}) = (\mathsf{R}_{syn}; \mathsf{R}_{sem})$ with now (R_{syn}) for (R), because (R) describes the "syntactic part" of $(\hat{\mathsf{R}})$, one has that $(\hat{\mathsf{R}})$ corresponds to

$$\left(\frac{H_1, \ldots, H_n}{H} ; \quad \frac{\alpha_1, \ldots, \alpha_n}{\beta(\alpha_1, \ldots, \alpha_n)} \right) . \tag{109}$$

Moreover, starting a deduction from a set Σ of "premises" using inference rules of type $(\hat{\mathsf{R}})$ needs premises which have the form of ordered pairs (formula; degree), i.e. one in such a case has to assume

$$\Sigma = \{(H_1, \alpha_1), (H_2, \alpha_2), \ldots\} . \tag{110}$$

But this exactly means to consider

$$\Sigma : \mathcal{FOR} \to \mathcal{D} \tag{111}$$

if one "completes" the set Σ of (110) with all the ordered pairs $(H, 0)$ for which $H \notin \{H_1, H_2, \ldots\}$.

With $\mathcal{D} = [0,1]$ or $\mathcal{D} \subseteq [0,1]$ as was always supposed up to now, (111) means that Σ is a *fuzzy subset* of the set $\mathcal{FOR}$ of all well-formed formulas.

8.4 Many-valued logic with graded consequences

In Section 8.1 our starting point was to extend the basic idea of many-valued logic, viz. that sentences need to be only partially true to the situation that rules of inference may be only partially sound.

Now we extend this idea of partiality in another direction assuming that the premisses a derivation is going to start from need to be given only partially, i.e. only to some degree. This degree, of course, should always be anyone of the truth degrees of the particular system of many-valued logic under consideration.

Having in mind the usual treatment of the situation that one derives formulas from some given set Σ of premisses, the common understanding in particular applications is that this set Σ should comprehend the (and only the) fundamental principles which are of basic importance for this application – and should not contain "redundant information" in the sense of (purely) logically valid sentences. To underline this situation, one says that by Σ some *(formalised)* *theory*[44] $\boldsymbol{T}$ is determined and that Σ is the (or: some possible) set of *nonlogical axioms* of this theory $\boldsymbol{T}$.

The extension we actually are interested in now amounts to consider elementary theories with *fuzzy sets of nonlogical axioms.* Such theories shall be called *fuzzy theories.* And the part of logic devoted to the examination of such fuzzy theories is the *fuzzy logic* (in the narrower sense).

In fuzzy logic, like in the standard approach toward many-valued logic and logical consequence, there is the duality of the semantic approach via interpretations (models), the satisfaction relation and the notion of semantical entailment, and there is the syntactic approach via rules of inference and a suitable notion of proof.

The formal language which we intend to use shall, as before, be some language of a suitable system of first-order many-valued logic. Their particular connectives and quantifiers as well as their truth degree constants shall become specified only if necessary. As we suppose that the membership degrees of the fuzzy sets which shall enter the discussion shall be truth degrees, there this language shall have enough of expressive power to handle the situation we have in mind.

8.4.1 The semantic approach

As in the case of (pure) many-valued logic, let us first consider the semantic approach. Because of the fact that the language for fuzzy logic is the same as for many-valued logic, the notion of interpretation is the same as before: based on fixed truth degree functions for the connectives and the quantifiers as well as on a fixed correspondence between the actual truth degree constants of the language and the truth degrees denoted by them, any interpretation has to provide the meanings of the predicate symbols and of the individual constants.

The essential point for fuzzy logic thus, regarding the semantic approach toward the entailment relation, is the definition of what shall be a model $\mathfrak{A}$ for a fuzzy set Σ of sentences. Intuitively, the crucial condition (for classical logic) reads

$$\text{if } H \text{ is a sentence of } \Sigma \text{ then } \mathfrak{A} \text{ is a model of } H. \tag{112}$$

Because now Σ is a fuzzy set of sentences the antecedent of this condition (112) either has to be read as the *graded* formula $H \varepsilon \Sigma$ – or as some statement involving the truth degree $[\![H \varepsilon \Sigma]\!]$ of it. The succedent of (112), however, is a classical statement, i.e. either true or false.

Thus, we either should transform $H \varepsilon \Sigma$ into a classical statement, or we should transform "$\mathfrak{A}$ is a model of H" into some *graded* formula.

We shall follow the second one of these ideas guided by the intuition that, classically, the truth value of "$\mathfrak{A}$ is a model of H" is the same as the truth value

[44] In the standard case that the system of logic one is refering to is the classical first-order logic, such formalised theories usually are called *elementary theories.* We shall extend this terminology and use this terminus "elementary theory" also in the case the actual logical system is some system of many-valued first-order logic.

of H if evaluated in the interpretation[45] $\mathfrak{A}$. Writing $\models^* (\mathfrak{A}, H)$ for the graded version of "$\mathfrak{A}$ is a model of H" it is natural to define

$$[\![\models^* (\mathfrak{A}, H)]\!] =_{\mathsf{def}} \mathsf{Val}^{\mathsf{S}}{}_{\mathfrak{A}}(H) \tag{113}$$

in accordance with the notation which was used in Section 6.1. Then (112) becomes the condition

$$H \,\varepsilon\, \Sigma \ \rightarrow \models^* (\mathfrak{A}, H) \tag{114}$$

which has to be "satisfied", i.e. which has to have truth degree 1, and in which $\rightarrow$ now has to be understood as a suitable implication connective of many-valued logic.

Having in mind that the degree ranking property is one of the basic properties of implication connectives for systems of many-valued logic, which is satisfied e.g. for all R-implications based upon left continuous t-norms, condition (114), and hence also (112), becomes

$$[\![H \,\varepsilon\, \Sigma]\!] \leq [\![\models^* (\mathfrak{A}, H)]\!]. \tag{115}$$

Therefore we give the

Definition 8.3 *An interpretation $\mathfrak{A}$ for some system S of many-valued logic is a* model *of a fuzzy set Σ of sentences of $\mathcal{L}_{\mathsf{S}}$, denoted: $\mathfrak{A} \models^* H$, iff condition (115) is satisfied for each $\mathcal{L}_{\mathsf{S}}$-sentence H.*

This definition opens the way to define the fuzzy set $\mathsf{Cn}^*(\Sigma)$ of all the sentences which are logically entailed by Σ. In classical logic the corresponding set $\mathsf{Cn}(\Sigma)$ may be characterised as

$$\mathsf{Cn}(\Sigma) = \{H \mid \mathfrak{A} \models H \text{ for all models } \mathfrak{A} \text{ of } \Sigma\} = \bigcap \{\mathsf{Th}(\mathfrak{A}) \mid \mathfrak{A} \models \Sigma\}$$

using the notion $\mathsf{Th}(\mathfrak{A}) = \{H \mid \mathfrak{A} \models H\}$ for the theory of a structure $\mathfrak{A}$.

For the present situation we only have to adapt this notion of the theory of a structure in such a way that it becomes a fuzzy set of formulas. And this is done simply by defining for any interpretation $\mathfrak{A}$

$$\mathsf{Th}^*(\mathfrak{A}) =_{\mathsf{def}} \{H \,\|\, \models^* (\mathfrak{A}, H)\}. \tag{116}$$

Then, as in the classical case, one defines furthermore the fuzzy set of all the sentences entailed by a fuzzy set Σ of sentences as

$$\mathsf{Cn}^*_{\models}(\Sigma) =_{\mathsf{def}} \bigcap \{\mathsf{Th}^*(\mathfrak{A}) \mid \mathfrak{A} \models^* \Sigma\} \tag{117}$$

with $\bigcap$ here for the intersection of a (crisp) set of fuzzy sets.

Of course, the degree to which some sentence H is entailed by Σ now is the membership degree $[\![H \,\varepsilon\, \mathsf{Cn}^*(\Sigma)]\!]$, and one obviously has

$$[\![H \,\varepsilon\, \mathsf{Cn}^*_{\models}(\Sigma)]\!] = \inf\{[\![\models^* (\mathfrak{A}, H)]\!] \mid \mathfrak{A} \models^* \Sigma\}. \tag{118}$$

[45]We remind the reader that we suppose that Σ is a (fuzzy) set of *sentences* which means that the truth value/degree of H in $\mathfrak{A}$ is indepent of any further valuation.

Further semantic notions, like e.g. the notion of satisfiability, then can be introduced along the lines well known from standard many-valued logic. Suitable generalisations of standard results become provable in this more general realm. As an example one has the following (propositional) compactness theoren, cf. [109], which holds true for all propositional fuzzy logics with an injective MV-algebra as set of truth degrees.

Theorem 8.3 *A fuzzy set Σ of well-formed (propositional) formulas is satisfiable iff each finite fuzzy subset*[46] *of Σ is satisfiable.*

8.4.2 The syntactic approach

The syntactic approach toward many-valued logic with graded consequences, i.e. toward fuzzy logic (in the narrower sense) has to refer to a suitable notion of derivation which allows to derive consequences from fuzzy sets of premisses, and which as usual has to be based on a suitable calculus, i.e. some algorithmic system with a rule base and some – possibly fuzzy – set Ax_{FL} of axioms, for fuzzy logic.

As is common usage, also in fuzzy logic a derivation shall be a finite sequence of well-formed formulas. But the intention to get the derived formula of every derivation with some degree forces to combine each derivation with some degree. Because of the usual understanding that each initial segment of a derivation again is a derivation (of its last formula), each initial segment of a derivation in fuzzy logic itself should come with a degree. And the most simple way to reach this goal seems to be a parallel treatment of formulas and degrees in derivations in fuzzy logic. Formally this means that the constituents of derivations in fuzzy logic should be ordered pairs of the form (formula, degree).

Fortunately this fits very well with the additional assumption that the sets of premisses now should be fuzzy sets of sentences, because fuzzy sets of formulas – or their membership functions – according to the usual set-theoretic understanding of functions are nothing but (crisp) sets of ordered pairs of just the type (formula, membership degree).

But then the inference rules should follow the same pattern: they either have to have as their premisses finite sequences of pairs of the form (formula, degree) and a consequent of the same form, or they have to consist itself of two parts acting in parallel – a first one that treats the formulas (in the standard way well known from calculi for classical logic) and a second one that treats the corresponding degrees. In any case, however, the result of an application of such a rule of inference has to be an ordered pair of the form (formula, degree).

That means, looking back at partial soundness of inference rules via their formulation in the form $(\hat{\mathsf{R}})$, the previous way to add partially sound rules of inference to many-valued logic was leading immediately into the field of fuzzy logic: the sets of premises to start a derivation from became in a very natural way read as fuzzy subsets of the set of all formulas, and the inference schemata proved to be prototypical for the form these schemata have in fuzzy logic.

[46]This means any fuzzy set $\Psi \subset \Sigma$ with a finite support.

However, for the moment we shall suppose that the (generalised) inference rules of the form ($\hat{\mathsf{R}}$) have to be sound – in one of the senses we discussed before.

A derivation in fuzzy logic of a sentence H from a fuzzy set Σ of premisses then is a finite sequence

$$(H_1, \alpha_1), (H_2, \alpha_2), \ldots, (H_n, \alpha_n), \tag{119}$$

of ordered pairs (H_i, α_i) of the form (formula, degree) such that

- either H_i comes from Σ and it is $\alpha_i = [\![H_i \,\varepsilon\, \Sigma]\!]$,
- or H_i comes from the fuzzy set of axioms Ax_{FL} and it is $\alpha_i = [\![H_i \,\varepsilon\, \mathsf{Ax}_{FL}]\!]$,
- or (H_i, α_i) results from the application of one of the rules of inference to pairs (H_k, α_k) with $1 \leq k < i$.

And the degree to which H_n is derived by the derivation (119) is α_n.

In fuzzy logic each derivation of a sentence H from a fuzzy set Σ of premisses provides a degree to which H is derived from Σ *by this derivation.* Unlike classical logic, however, different derivations of the same sentence H from the fuzzy set Σ may provide different degrees to which they are derivations of H. Therefore, a single derivation of H from Σ gives only a lower bound for what may be called the ***degree of derivability*** of the sentence H from the fuzzy set Σ. And this degree of derivability $\vdash^* (H, \Sigma)$ itself should be the supremum

$$\begin{aligned} \vdash^* (H, \Sigma) \quad =_{\mathsf{def}} \quad & \sup\{\alpha_n \mid (H_1, \alpha_1), (H_2, \alpha_2), \ldots, (H, \alpha_n) \\ & \qquad\qquad \text{is a derivation of } H \text{ from } \Sigma\}. \end{aligned} \tag{120}$$

As usual now, the (fuzzy) syntactic, i.e. derivability based consequence hull $\mathsf{Cn}^*_{\vdash}(\Sigma)$ of the fuzzy set Σ, i.e. the fuzzy set of theorems of the (fuzzy) elementary theory $\boldsymbol{T}$ with the fuzzy set Σ of nonlogical axioms is

$$\mathsf{Cn}^*_{\vdash}(\Sigma) =_{\mathsf{def}} \{H \,\|\, \vdash^* (H, \Sigma)\}. \tag{121}$$

8.4.3 Axiomatising fuzzy logic

As each system S^* of (propositional or first-order) fuzzy logic extends some system S of many-valued (propositional or first-order) logic, the problem of axiomatisability for this system S^* of fuzzy logic may be expected to be closely tied to the problem of axiomatisability of the "underlying" system S of many-valued logic. The crucial point here is that the semantic entailment operation $\mathsf{Cn}^*_{\models}$ of S^* extends the semantic entailment relation $\mathsf{Cn}_{\models}$ of S in such a way that for each *crisp*(!) set Σ of sentences of the (common) language $\mathcal{L}_{\mathsf{S}}$ of S^* and S one has $\mathsf{Cn}^*_{\models}(\Sigma) = \mathsf{Cn}_{\models}(\Sigma)$. Therefore, assuming that S^* and S have the same set of designated truth degrees, S^* has the same (crisp) set $\mathsf{Cn}^*_{\models}(\emptyset) = \mathsf{Cn}_{\models}(\emptyset)$ of logically valid sentences as S, and any adequate axiomatisation of of the logically valid sentences of S is at the same time an adequate axiomatisation for the logically valid sentences of S^*.

Even more holds true. Given an adequate axiomatisation of the semantic entailment operation $\mathsf{Cn}_{\models}$ of S, i.e. a notion of S-provability $\vdash_{\mathsf{S}}$ such that $H \in \mathsf{Cn}_{\models}(\Sigma)$ holds true iff $\Sigma \vdash_{\mathsf{S}} H$ for each sentence H and all sets Σ of sentences of S, then one already has an axiomatisation of a "fragment" of $\mathsf{Cn}^*_{\models}$, viz. of the restriction of $\mathsf{Cn}^*_{\models}$ to crisp argument sets. The only minor change which has to be made is a kind of rereading of the inference rules which, for fuzzy logic, have to treat in parallel formulas and degrees. As "nonfuzzy" many-valued logics can, from the point of view of fuzzy logic, be considered as treating all the formulas to degree one (in the set of axioms as well as in any set of premisses), the rereading of the inference rules simply amounts to combining with all the premisses of these rules and with their conclusions the truth degree 1.

At all thus, (adequate) axiomatisations of the semantic entailment operation of fuzzy logics may be chosen as extensions of (adequate) axiomatisations of the semantic entailment operation of the "underlying" many-valued logic in such a way that further axioms may be added (perhaps to some degree) and that the rules of inference need to become joined with a treatment of degrees to parallel the treatment of formulas they already regulate.

In accordance with the actual state of the art in fuzzy logic, however, we shall not intend to discuss the problem of axiomatisability for large classes of systems of many-valued logic as these remarks may seem to suggest. Instead, we restrict our considerations to the case that the "underlying" system of many-valued logic is the infinite valued ŁUKASIEWICZ logic L_∞ and take this case as a kind of prototype of how to approach the problem for other systems of fuzzy logic too.

For the ŁUKASIEWICZ propositional system L_∞ (in ŁUKASIEWICZ's implication and negation as primitive connectives) an adequate axiomatisation not only of the set of tautologies but even of the semantic entailment operation is provided by the axioms (i) to (iv) mentioned in Theorem 4.3 together with the rule of detachment w.r.t. $\rightarrow_{\mathsf{L}}$.

Before we extend this axiomatisation to the fuzzy ŁUKASIEWICZ propositional logic we should consider a possibility to "code" the membership degree $h = [\![H \,\varepsilon\, \mathsf{Cn}^*_{\models}(\Sigma)]\!]$ of H in the fuzzy consequence hull of Σ inside the language of this fuzzy logic. This goal can be reached because from (118) together with (113) one has that

$$h \leq \mathsf{Val}^{\mathsf{S}}{}_{\mathfrak{A}}(H) \quad \text{for all models } \mathfrak{A} \text{ of } \Sigma \tag{122}$$

and therefore that

$$\mathsf{Val}^{\mathsf{S}}{}_{\mathfrak{A}}(\boldsymbol{h} \rightarrow_{\mathsf{L}} H) = 1 \quad \text{for all models } \mathfrak{A} \text{ of } \Sigma \tag{123}$$

if one introduces the truth degree constant $\boldsymbol{h}$ into the language to denote the truth degree h. That means that from $h = [\![H \,\varepsilon\, \mathsf{Cn}^*_{\models}(\Sigma)]\!]$ one gets that $[\![(\boldsymbol{h} \rightarrow_{\mathsf{L}} H) \,\varepsilon\, \mathsf{Cn}^*_{\models}(\Sigma)]\!] = 1$. And then one can hope to find a suitable axiomatisation from this idea and a further strategy to have only minor changes in the sets of inference rules compared with axiomatisations of the "background" many-valued ŁUKASIEWICZ logic.

Guided by this observation we enrich the language of our fuzzy ŁUKASIEWICZ logic by truth degree constants for every truth degree, writing from now on $\underline{u}$ to denote the truth degree u.

Now we are in a position to give an adequate axiomatisation of the propositional fuzzy logic based upon the infinite valued ŁUKASIEWICZ logic L_∞. The *fuzzy set of axioms* $\mathsf{Ax}^{\mathsf{L}}_\infty$ for this system consists of:

- the axioms (i) to (iv) of Theorem 4.3 with membership degree 1,
- the truth degree constants $\underline{u}$ with membership degree u for every $u \in [0,1]$,
- the formulas $(\underline{u} \rightarrow_{\mathsf{L}} \underline{v}) \leftrightarrow_{\mathsf{L}} \underline{\mathsf{seq}_2(u,v)}$ with membership degree 1 for all $u, v \in [0,1]$,
- the formulas $\neg\underline{u} \leftrightarrow_{\mathsf{L}} \underline{\mathsf{non}_1(u)}$ with membership degree 1 for every $u \in [0,1]$.

And the *rules of inference*, written down as in (109) as rules which connect ordered pairs of the form (formula, degree) to finite lists of such pairs, are the following ones:

- the generalised *modus ponens* which allows to infer the ordered pair $(H_2, \mathsf{et}_2(u,v))$ from the ordered pairs (H_1, u) and $(H_1 \rightarrow_{\mathsf{L}} H_2, v)$, i.e.:

$$\left(\frac{H_1, H_1 \rightarrow_{\mathsf{L}} H_2}{H_2} \,;\quad \frac{u, v}{\mathsf{et}_2(u,v)} \right) ,$$

- the *constant introduction rule* which allows to infer the ordered pair $(\underline{u} \rightarrow_{\mathsf{L}} H, \mathsf{seq}_2(u,v))$ from the pair (H, v), i.e.:

$$\left(\frac{H}{\underline{u} \rightarrow_{\mathsf{L}} H} \,;\quad \frac{v}{\mathsf{seq}_2(u,v)} \right) .$$

Both the fuzzy set of axioms as well as the list of inference rules look, after some inspection, quite natural. The same still holds true for a lot of further generalised inference rules which prove to be sound in fuzzy logic, cf. [110]. There is, however, notwithstanding the fact that it is quite usual in set theoretically oriented mathematical logic to use such uncountable languages, a severe difficulty with this approach: it uses truth degree constants for all the real numbers of the unit interval $[0,1]$, i.e. for a non-denumerably infinite set of degrees – and therefore this extended language cannot be realised in the usual way as a language whose words are finite strings of symbols of a denumerable alphabet.

Fortunately there is a way to overcome this difficulty: for most applications it is sufficient to have as truth degree constants only symbols for the rational numbers of the real unit interval $[0,1]$. And because this is a denumerable set of numbers, the language of this *rational fuzzy* ŁUKASIEWICZ *logic* is itself a denumerable language.

What finally shall be discussed is the extension to the case of first-order fuzzy ŁUKASIEWICZ logic. Completely standard (as for ŁUKASIEWICZ logic in general) is the change in the language if changing from propositional to first-order logic. The only point for discussion is, again as usual, how to extend the (fuzzy) set of axioms and the list of inference rules. One possibility is to add as further axioms

- the formulas $\forall x H(x) \rightarrow_{\mathsf{L}} H(t)$, where t is any term free for the individual variable x in H, with membership degree 1,
- the formulas $\forall x(G \rightarrow_{\mathsf{L}} H(x)) \rightarrow_{\mathsf{L}} (G \rightarrow_{\mathsf{L}} \forall x H(x))$, where G does not contain the individual variable x free, with membership degree 1,
- the formulas $(G \rightarrow_{\mathsf{L}} \exists x H(x)) \rightarrow_{\mathsf{L}} \exists x(G \rightarrow_{\mathsf{L}} H(x))$, where G does not contain the individual variable x free, with membership degree 1,

and thus adding (essentially) just the same axioms which can also be used in extending an axiomatisation of classical propositional logic to an axiomatisation of classical first-order logic.

And the list of inference rules has e.g. to be extended by adding

- the suitable modified *generalisation rule* which allows to infer the ordered pair $(\forall x H, u)$ from the ordered pair (H, u), i.e.: i.e.:

$$\left(\frac{H}{\forall x H}; \ \frac{u}{u}\right).$$

8.4.4 Some theoretical results

Notwithstanding the fact, that the semantic as well as the syntactic approach toward graded consequences parallel the usual approaches in classical and in "standard" many-valued logic in a natural way, the syntactic approach has some particular pecularity. The crucial point is that, unlike the situation in classical logic, because of definition (120) the knowledge of a single derivation of a sentence H from Σ (with some degree α for this derivation of H) does not necessarily give the membership degree of H in $\mathsf{Cn}^*_{\mathsf{L}}(\Sigma)$. And in the case that the many-valued logic a particular system of fuzzy logic is based upon itself is an infinitely many-valued system, even a finite set of such derivations of H from Σ may not suffice to determine this membership degree. In this sense, despite the fact that all the inference rules are supposed to be finitary, i.e. to have each only a finite number of rule premisses, the present notion of provability in fuzzy logic is *not a finitary* notion of provability.

In this particular sense, provability in fuzzy logic is much more complicated a matter as is provability in classical first-order logic.

One of the most basic problems for graded consequences is the relation between the semantic and the syntactic approach. Here, again, the system of many-valued logic the approach toward graded consequences is based upon, becomes of importance. If suitably chosen, as done in the last section for propositional as well as first-order Łukasiewicz fuzzy logic, both these approaches give the same degrees of consequence.

Theorem 8.4 (Completeness Theorem)
For the approach toward graded consequences based on one of the (propositional or first-order) ŁUKASIEWICZ *systems* L_∞ *with truth degree constants for all the real numbers from* $[0,1]$ *one has for any fuzzy set* Σ *of sentences*

$$\mathsf{Cn}^*_{\models}(\Sigma) = \mathsf{Cn}^*_{\vdash}(\Sigma).$$

Therefore one has completeness in the strong sense of an adequate axiomatisation of the semantic entailment operation.

With respect to the problem of partially sound rules of inference, which was our starting point for looking at graded consequence operations, the essential results are given in the following two theorems.

Theorem 8.5 *Each many-valued elementary theory which one gets by enriching many-valued logic with some (crisp) set of nonlogical axioms and some partially sound rules of inference is equivalent to some elementary theory in the realm of fuzzy logic characterized by a fuzzy set of nonlogical axioms.*

Extending also fuzzy logic with partially sound rules of inference gives furthermore

Theorem 8.6 *Each many-valued elementary theory which one gets by enriching fuzzy logic with some (fuzzy) set of nonlogical axioms and some partially sound rules of inference is equivalent to some elementary theory in the realm of fuzzy logic characterized simply by a fuzzy set of nonlogical axioms and using only sound rules of inference.*

These last mentioned results have the disadvantage that the language of fuzzy logic becomes uncountable by the addition of all the truth degree constants. It is interesting that essentially the same result is already provable with only truth degree constants for the rationals from $[0,1]$, as was shown by P. Hájek [37]. He proves the following

Theorem 8.7 (Restricted Completeness Theorem) *Suppose that the approach toward graded consequences is based on one of the (propositional or first-order)* ŁUKASIEWICZ *systems* L_∞ *with truth degree constants for all the rational numbers from* $[0,1]$*, then one has for any "rational valued" fuzzy set* $\Sigma : \mathcal{FOR} \to [0,1] \cap \mathbb{Q}$ *of sentences*

$$\mathsf{Cn}^*_{\models}(\Sigma) = \mathsf{Cn}^*_{\vdash}(\Sigma).$$

Interestingly, the definitions (117), (121) of $\mathsf{Cn}^*_{\models}, \mathsf{Cn}^*_{\vdash}$ can remain unchanged in this new situation with the restricted language.

Even in this restricted form with truth degree constants only for the rationals the provability relation is not a simple one, and the theoremhood property is even a quite complicated one, as was proven also by P. H'AJEK [37, 38, 39] extending results of RAGAZ [88] and HÄHNLE [36].

Theorem 8.8 *Suppose that T is an axiomatizable rational fuzzy theory, i.e. that $T : \mathcal{FOR} \to [0,1] \cap \mathbb{Q}$ is a recursive function. Then*

(i) *the (crisp) satisfiability relation $\mathsf{Cn}^*_{\vdash}(T) = \{(H,r) \mid [\![H \,\varepsilon\, \mathsf{Cn}^*_{\vdash}(T)]\!] = r\}$ is a Π_2-relation,*

(ii) *and already for $T = \emptyset$ all the sets $\mathsf{Cn}^{[r]}_{\vdash} = \{H \mid [\![H \,\varepsilon\, \mathsf{Cn}^*_{\vdash}(\emptyset)]\!] \geq r\}$, $r \in \mathbb{Q}$, are Π_2-complete.*

In particular, this means that already the set of logically valid formulas of the first-order (ŁUKASIEWICZ style) fuzzy logic L_∞ with rational truth degree constants is Π_2-complete.[47]

8.4.5 Historical remarks

Fuzzy logic (in the narrower sense) was first studied in [82] by the Czech mathematician J. PAVELKA. He was concerned with fuzzy propositional logic and intended to be as general as possible regarding the truth degree sets. He presented axiomatisations for the fuzzy logics and succeded, however, with the proof of a completeness theorem only for the case that this truth degree set is (isomorphic to) the truth degree set of one of the ŁUKASIEWICZ systems.

The extension to the first-order case was mainly done in papers of another Czech mathematician V. NOVÁK and is reported e.g. in [78, 79]. He is using truth degree constants for all the reals of the unit interval.

The problem of partial soundness of inference rules was discussed by the present author in [35]. And the restriction to the rational fuzzy logic together with simplifications of the axiomatisation and with interesting further results on this fuzzy logic was quite recently given by a third Czech mathematician P. HÁJEK, e.g. in [37, 39].

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CHAPTER 2

Powerset Operator Foundations For Poslat Fuzzy Set Theories And Topologies

S. E. RODABAUGH

Introduction

This chapter summarizes those powerset operator foundations of all mathematical and fuzzy set disciplines in which the operations of taking the image and preimage of (fuzzy) subsets play a fundamental role; such disciplines include algebra, measure theory and analysis, and topology. We first outline such foundations for the fixed-basis case—where the lattice of membership values or basis is fixed for objects in a particular category, and then extend these foundations to the variable-basis case—where the basis is allowed to vary from object to object within a particular category. Such foundations underlie almost all chapters of this volume. Additional applications include justifications for the Zadeh Extension Principle [19] and characterizations of fuzzy associative memories in the sense of Kosko [9]. Full proofs of all results, along with additional material, are found in Rodabaugh [16]—no proofs are repeated even when a result below extends its counterpart of [16]; some results are also found in Manes [11] and Rodabaugh [14, 15].

1 History, motivation, and preliminaries

Traditional mathematics hinges disproportionately around the notion of powersets and powerset operators. More precisely, given $f : X \to Y$, the traditional powerset operators $f^{\to} : \mathcal{P}(X) \to \mathcal{P}(Y)$ and $f^{\leftarrow} : \mathcal{P}(X) \to \mathcal{P}(Y)$ given by

$$f^{\to}(A) = \{f(x) : x \in A\}, \;\; f^{\leftarrow}(B) = \{x : f(x) \in B\}$$

U. Höhle et al. (eds.), *Mathematics of Fuzzy Sets*

play a critical role in ordinary mathematics. While it took more than a century of mathematics to "empirically" confirm that these powerset operators are the "correct" liftings of f to the powersets of X and Y, it is first verified mathematically and directly by E. G. Manes [11]. The differently formatted, but equivalent proof of S. E. Rodabaugh [16] creates $f^{\rightarrow}$ by proving the adjunction between **SET** and **CSLAT** (the category of complete semilattices and arbitrary join-preserving maps), which is essentially the proof in P. T. Johnstone [7] that **CSLAT** is algebraic, and then creates $f^{\leftarrow}$ from $f^{\rightarrow}$ by the Adjoint Functor Theorem for order-preserving mappings between partially ordered sets. In fact, the powerset operator foundations of traditional mathematics may be viewed as entirely a consequence of the Adjoint Functor Theorem (AFT).

The development of the mathematics of fuzzy sets has seen a correspondingly disproportionate dependence on the powersets of fuzzy subsets and powerset operators. All chapters of this volume laying axiomatic and/or categorial foundations for topology, uniformities, measure theory, and algebra are themselves underpinned in an essential way by the powerset operator foundations summarized in this chapter; e.g. see U. Höhle / A. Šostak [5], Rodabaugh [17], etc.

The fundamental importance of powerset operators for fuzzy sets is recognized from the beginning in L. A. Zadeh's pioneering paper [19] introducing fuzzy sets. To state the Zadeh powerset operators and the associated Zadeh Extension Principle, let L be a complete lattice and $f : X \to Y$. Then the fuzzy powerset operators $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^X \longleftarrow L^Y$ are defined by

$$f_L^{\rightarrow}(a)(y) = \bigvee\{a(x) : f(x) = y\}, \quad f_L^{\leftarrow}(b) = b \circ f$$

While Zadeh's original definition is limited to the **base** $L = [0, 1]$, the needed properties of L are those of a complete (semi) lattice—cf. J. A. Goguen [4]. That Zadeh [19] puzzles over the definition of $f^{\rightarrow}$ (whether to use $\bigvee$ or $\bigwedge$) indicates it was not clear to Zadeh whether his powerset operators were the correct ones. But Manes [11] gives the first proof, using a monadic approach for a certain restricted class of lattices L, that the Zadeh operators were the right ones. Then in [16], Rodabaugh gives two different proofs for all complete lattices L vindicating Zadeh's definitions—first, using AFT to generate/lift the Zadeh operators from the traditional operators, and second, using classes of naturality diagrams indexed by L to generate the Zadeh operators directly from the original function. In this chapter we extend these results, along with several characterizations of each of the Zadeh operators, to lattices taken from **CIQML**, the largest category of lattices in which all the results of [16] can be derived; and then we stipulate these results as definitions of the Zadeh operators for ordered structures in **CQML**, the largest category of lattices appearing in this volume.

Starting with B. Hutton [6], Rodabaugh [13], P. Eklund [2], and others, fuzzy sets broadens to allow the change of base along with the change of set, begetting **variable-basis** fuzzy sets and **variable-basis** topology, the previously considered approaches being denoted **fixed-basis**. Thus the next natural question concerns the existence of variable-basis fuzzy powerset operators: given

$f : X \to Y$, how should the fuzzy powerset operators of f be defined from L^X to M^Y and from M^Y to L^X, where L, M are complete (quasi-monoidal) lattices with a known relationship (morphism) between them? Phrasing the question more precisely leads us to "ground" or "base" categories of the form **SET** $\times$ **LOQML** (defined below), which explicitly indicate our **point-set lattice-theoretic** or **poslat** approach to fuzzy sets, powerset operators, and the overlying topological, measure-theoretic, and algebraic theories. The AFT is used to generate variable-basis powerset operators from the Zadeh fixed-basis operators so that under appropriate conditions the variable-basis operators are adjunctive.

It is the purpose of this chapter to outline the above results and ideas, as well as their application to ground isomorphism characterizations, fuzzy systems, and fuzzy associative memories. All these results are taken or extended from [16], and the proofs are formally identical to those of [16] and shall not repeated here; it is left to the reader to check that these proofs work for the objects of **CQMIL**. Important related material is also found in [14–15].

The remainder of this section summarizes adjunctions, needed lattice-theoretic categories, the generation of the traditional powerset operators, and criteria appropriate for powerset operators both traditional and generalized. We begin with the definition of adjunction between categories as used in this chapter and adapted from S. Mac Lane [10].

1.1 Definition (Adjunction between categories). Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{C} \leftarrow \mathbf{D}$ be functors. We say F is **left-adjoint** to G iff the following two criteria are satisfied in the order stated:

(1) **Lifting/Continuity criterion**:

$$\begin{aligned} \forall A &\in |\mathbf{C}|, \exists \eta : A \to GF(A), \forall B \in |\mathbf{D}|, \forall f : A \to G(B), \\ \exists ! \overline{f} &: F(A) \to B,\ f = G(\overline{f}) \circ \eta \end{aligned}$$

(2) **Naturality criterion**:

$$\forall f : A_{\top} \to A_2 \text{ in } \mathbf{C},\ F(f) = (\text{or} \equiv)\ \overline{\eta_{A_2} \circ f}$$

where the "$\equiv$" option allows F to only be defined initially on objects, in which (1) and "$\equiv$" will stipulate an action for F on morphisms such that $F \dashv G$.

Major Diagram: Lifting/Continuity

$$\begin{array}{ccc} & \mathbf{C} & \mathbf{D} \end{array}$$

$$\begin{tikzcd}
 & \forall A\ (1) \arrow[dl, "\forall f\ (4)"'] \arrow[d, "\exists \eta_A\ (2)"] \arrow[dl, phantom, shift left=4, "\circ\ (6)"] & F(A) \arrow[d, "\exists ! \overline{f}\ (5)"] \\
G(B) & GF(A) \arrow[l, "G(\overline{f})"] & \forall B\ (3)
\end{tikzcd}$$

Minor/Naturality Diagram (Assuming Major Diagram)

$$\left[\begin{tikzcd}
A_1 \arrow[r, "f"] \arrow[d, "\eta_{A_1}"'] \arrow[dr, phantom, "\circ"] & A_2 \arrow[d, "\eta_{A_2}"] & F(A_1) \arrow[d, "! \overline{\eta_{A_2} \circ f}"] \\
GF(A_1) \arrow[r, "GF(\overline{\eta_{A_2} \circ f})"'] & GF(A_2) & F(A_2)
\end{tikzcd}\right]$$

$$\Longrightarrow \quad F(f) = (or \equiv)\ \overline{\eta_{A_2} \circ f}$$

We also say that G is **right-adjoint** to F or that (F, G) is an **adjunction**, or we may write $F \dashv G$. The map η is the **unit** of the adjunction; and the **D** morphism ε dual to η in the duals of the above statements is the **counit** of the adjunction. If both η and ε are isomorphisms in their respective categories, then (F, G) is an **equivalence (of categories)**.

It is well-known that if such a definition is instantiated with partially-ordered sets are taken as categories with order-preserving maps as functors between, then we obtain the following proposition:

1.2 Proposition. Let $f : L \to M$ and $g : L \leftarrow M$ be order-preserving maps between partially ordered sets L, M. Then $f \dashv g$ iff $[id_L \leq g \circ f$ and $f \circ g \leq id_M]$.

The inequalities of the proposition are dubbed the **Adjunction Inequalties** (AI) and will be so referenced *sequens*. The following result creating adjunctions in the partially-ordered case is well-known [7] and is critical to the study of powerset operators; its dual also holds, both are used *sequens*, and both are referred to as AFT.

1.3 Theorem (Adjoint functor theorem). Let L, M be partially-ordered sets, let $f : L \to M$ be a function between them, let L have arbitrary joins (or $\bigvee$), and let f preserve arbitrary joins. Then f is a functor from L to M; $\exists\, g$ a functor from M to L, given by

$$g(b) = \bigvee \{a \in L : f(a) \leq b\}$$

such that $f \dashv g$; g is the **unique** right adjoint of f; and g preserves all meets (or $\bigwedge$) in M.

The category **SET** comprises all sets and functions between sets. If $f : X \to Y$ in **SET** and P is a property identifying certain elements of Y, then

$$[f \text{ has } P] \equiv \{x \in X : f(x) \text{ has } P\}$$

If X is a set and $A \subset X$, then χ_A denotes the **characteristic function** from X to $\{\perp, \top\}$, where $\{\perp, \top\}$ usually denotes the set of bounds of some lattice L, i.e. $\chi_A : A \to \{\perp, \top\} \hookrightarrow L$. We shall notationally distinguish characteristic functions only by subset A, not by the various lattices L.

POSET is the category of partially-ordered sets and order-preserving maps, and **CSLAT** is the category of complete (join) semilattices and arbitrary $\bigvee$ preserving maps.

1.4 Definition (SFRM, FRM, SLOC, LOC). The category **SFRM** [17] comprises all complete lattices with arbitrary $\bigvee$ preserving and finite $\wedge$ preserving maps, and in the objects are called **semi-frames**. The full subcategory of objects which possess the first infinite distributive law is denoted **FRM**, and the objects are called **frames**. The dual **SFRM**^{op} of **SFRM** is denoted **SLOC**, and the objects are called **semilocales**; and the dual **FRM**^{op} of **FRM** is denoted **LOC**, and the objects are called **locales**.

1.5 Definition (CQML, LOQML). The objects of the category **CQML** comprise all **complete quasi-monoidal lattices**, i.e. $(L, \leq, \otimes)$ satisfying:

(1) $(L, \leq)$ is a complete lattice with upper bound $\top$ and lower bound $\perp$.

(2) $\otimes : L \times L \to L$ is a binary operation satisfying the following axioms:

(a) $\otimes$ is isotone in both arguments, i.e. $\alpha_1 \le \alpha_2$, $\beta_1 \le \beta_2 \Rightarrow \alpha_1 \otimes \beta_1 \le \alpha_2 \otimes \beta_2$;

(b) $\top$ is idempotent, i.e. $\top \otimes \top = \top$.

And the morphisms of **CQML** are mappings preserving $\otimes$, arbitrary $\bigvee$, and $\top$. The category **CQML** is studied extensively in [5] and used extensively in [17]. The dual $\mathbf{CQML}^{op}$ is denoted **LOQML**, its objects are called **localic quasi-monoids**, and it is used extensively in [17].

1.6 Definition (CQMIL). If an object $(L, \le, \otimes)$ of **CQML** additionally satisfies these axioms—

(1) $\top$ is a right-identity of $\otimes$,

(2) $\perp$ is a two-sided zero of $\otimes$

—then we call such an object a **complete quasi-monoidal integral lattice** and label the corresponding full subcategory of **CQML** by **CQMIL**. The dual of **CQMIL** is denoted **LOQMIL** and its objects are called **localic quasi-monoidal integral lattices**. Of course, $\otimes = \wedge$ satisfies this condition, as does $\otimes = \cdot$ for $L = [0,1]$.

1.7 Lemma. $\forall \phi \in \mathbf{LOQML}(A,B)$, $\exists\, [\phi] \in \mathbf{POSET}(A,B)$ given by

$$[\phi](a) = \bigwedge\{b : a \le \phi^{op}(b)\}$$

It is also the case that $[\phi](\perp) = \perp$.

The "ground category" of much of traditional mathematics is **SET**. This ground category may be rewritten as $\mathbf{SET} \times \mathbf{2}$, where **2** in this sense is the category whose sole object is the lattice $\{\perp, \top\}$ and sole morphism is $id_{\{\perp,\top\}}$. Similarly, the ground category of fixed-basis fuzzy set theory, for basis L, is $\mathbf{SET} \times \mathbf{L}$, where **L** is the category whose sole object is the localic quasi-monoid L and sole morphism is id_L. The definition of variable-basis ground categories, the largest of which we now define, hints at the extraordinary richness of morphisms in fuzzy theories based on such grounds.

1.8 Definition (Ground category $\mathbf{SET} \times \mathbf{LOQML}$ [17]). The **ground category $\mathbf{SET} \times \mathbf{LOQML}$** comprises the following data:

(1) **Objects.** (X, L), where $X \in |\mathbf{SET}|$, $L \in |\mathbf{LOQML}|$. The object (X, L) is a **(ground) set**.

(2) **Morphisms.** $(f, \phi) : (X, L) \to (Y, M)$, where $f : X \to Y$ in **SET** and $\phi : L \to M$ in **LOQML**, i.e. $\phi^{op} : L \leftarrow M$ in **CQML**.

(3) **Composition.** Component-wise in the respective categories.

(4) **Identities.** Component-wise in the respective categories $(id_{(X,L)} = (id_X, id_L))$.

1.9 Question. What sort of set theory must (or should) we have in each of **SET** $\times$ **2**, **SET** $\times$ **L**, or **SET** $\times$ **LOQML**? The answer lies in detailing the mathematical development of the classical powerset operators and seeing how that development can be carried over to the general case.

1.10 Criteria for traditional operators. Traditional set theory should be examined and found to be a poslat set theory in the sense that functions between sets give rise to two powerset operators between powersets, the latter two having important categorical and lattice-theoretic properties. More precisely:

(1) $\forall\, f : X \to Y$, there should be a unique lifting, the **forward powerset operator** $f^{\rightarrow} : \mathcal{P}(X) \to \mathcal{P}(Y)$ of f to the associated powersets which preserves arbitrary $\bigvee (= \bigcup)$;

(2) the $f^{\rightarrow}$ of (1) should uniquely generate the **backward powerset operator** $f^{\leftarrow} : \mathcal{P}(X) \leftarrow \mathcal{P}(Y)$ via adjunction between the pre-ordered categories $\mathcal{P}(X)$ and $\mathcal{P}(Y)$, so that $f^{\rightarrow} \dashv f^{\leftarrow}$.

We now state the result that the traditional set theory in fact satisfies these criteria.

1.11 Theorem (SET $\dashv$ CSLAT and generation of traditional powerset operators). Let $\mathbf{P} : |\mathbf{SET}| \to |\mathbf{CSLAT}|$ and $\mathbf{V} : \mathbf{SET} \leftarrow \mathbf{CSLAT}$ be defined by

$$\mathbf{P}(X) = \mathcal{P}(X), \ \mathbf{V}(L, \leq) = L, \ \mathbf{V}(f) = f$$

Then the following hold:

(1) The lifting criterion of 1.1(1) above is satisfied, i.e. $\forall X \in |\mathbf{SET}|, \exists\, \eta_X : X \to \mathbf{P}(X)$ defined by $\eta_X(x) = \{x\}$, $\forall (M, \leq) \in |\mathbf{CSLAT}|$, $\forall f : X \to M$ in **SET**, $\exists!\, \overline{f} : \mathbf{P}(X) \to M$ in **CSLAT**, $f = \mathbf{V}(\overline{f}) \circ \eta_X$.

$$
\begin{tikzcd}
& \textbf{SET} & & \textbf{CSLAT} \\
& \forall X\ (1) \arrow[dl, "\forall f\ (4)"'] \arrow[d, "\exists \eta_X\ (2)"] \arrow[dl, phantom, "\circ\ (6)", shift left=4] & & \mathcal{P}(X) \arrow[d, "\exists ! \overline{f}\ (5)"] \\
B & \mathcal{P}(X) \arrow[l, "\overline{f}"] & & \forall B\ (3)
\end{tikzcd}
$$

(2) If $f : X \to Y$ in **SET** is given and $\mathbf{P}(f)$ is stipulated to be $\overline{\eta_Y \circ f}$, then $\mathbf{P}$ is a functor and $\mathbf{P} \dashv \mathbf{V}$.

$$
\left[
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\circ"] & Y \arrow[d, "\eta_Y"] & \mathcal{P}(X) \arrow[d, "! \overline{\eta_Y \circ f}"] \\
\mathcal{P}(X) \arrow[r, "(\overline{\eta_Y \circ f})"'] & \mathcal{P}(Y) & \mathcal{P}(Y)
\end{tikzcd}
\right]
$$

$$\Longrightarrow \boldsymbol{P}(f) = (\mathit{or} \equiv) \ \overline{\eta_Y \circ f}$$

(3) $\overline{\eta_Y \circ f} : \mathbf{P}(X) \to \mathbf{P}(Y)$ is the traditional forward powerset operator $f^{\to}$, i.e.

$$\overline{\eta_Y \circ f}(A) = \{f(x) \in Y : x \in A\}$$

(4) Since $\overline{\eta_Y \circ f}$ is a **CSLAT** morphism, then $f^{\to}$ preserves arbitrary unions and so has a right adjoint (by AFT) which is the traditional backward

powerset operator $f^{\leftarrow}$, i.e. the action of this right adjoint at $B \in \mathbf{P}(Y)$ is

$$\{x \in X : f(x) \in B\}$$

1.12 Criteria for generalized operators. For any poslat set theory more "general" than the traditional set theory, the following properties should hold:

(1) The analogue of $f^{\rightarrow}[f^{\leftarrow}]$ should be a unique lifting of $f^{\rightarrow}[f^{\leftarrow}]$, i.e. the analogue of $f^{\rightarrow}[f^{\leftarrow}]$ should be a **generalization** of $f^{\rightarrow}[f^{\leftarrow}]$ in a natural way.

(2) The analogue of $f^{\rightarrow}$ should be left adjoint to the analogue of $f^{\leftarrow}$, i.e. the relationship $f^{\rightarrow} \dashv f^{\leftarrow}$ between $f^{\rightarrow}$ and $f^{\leftarrow}$ should hold for their analogues.

Both the fixed-basis fuzzy case and the change-of-basis fuzzy case satisfy Criteria 1.12—see §2 and §3 below, respectively. We also show in §2 that fixed-basis poslat set theories in fact satisfy Criteria 1.10 restated for L a complete quasi-monoidal integral lattice (Definition 1.6 above) with these substitutions: $\mathcal{P}(X), \mathcal{P}(Y)$ are replaced respectively by L^X, L^Y, and $f^{\rightarrow}$, $f^{\leftarrow}$ are replaced respectively by $f_L^{\rightarrow}$, $f_L^{\leftarrow}$.

2 Fuzzy powerset operators for fixed-basis poslat set theories

This section summarizes the results showing that fixed-basis poslat set theories — theories in which the **basis** or lattice L of membership values is held fixed as in [19], [4], [5], etc—satisfy Criteria 1.12 above. Restated, the results of this section establish poslat set theories which have grounds of the form $\mathbf{SET} \times \mathbf{L}$, where $L \in |\mathbf{LOQMIL}|$; we then stipulate these operators for all poslat set theories having grounds of the form $\mathbf{SET} \times \mathbf{L}$, where $L \in |\mathbf{LOQML}|$. As a consequence, this section gives a rigorous justification of the traditional Zadeh powerset operators and the so-called Zadeh Extension Principle, and completely characterizes the existence of such operators. These results also show these operators are uniquely determined with respect to Criteria 1.10 restated for the fixed-basis poslat case. We point out that the first justification of the Zadeh operators is given by [11]. The results of [16] summarized in this section rest on methods different from [11] and include many new justifications and characterizations of the Zadeh operators.

2.1 Discussion. Let $X \in |\mathbf{SET}|$, $L \in |\mathbf{LOQML}|$, and endow L^X point-wise with $\leq, \bigvee, \otimes$ from L. Then $L^X \in |\mathbf{LOQML}|$, and is the localic quasi-monoid of **L-subsets** of X because of

$$\mathcal{P}(X) \underset{\chi}{\rightarrow} \{\perp, \top\}^X \hookrightarrow L^X$$

where $\chi : \mathcal{P}(X) \to \{\bot, \top\}^X$ by $\chi(A) = \chi_A$ is an isomorphism in **CQML**, and $\hookrightarrow$ a monomorphism (embedding) in **CQML**. Let $\eta_X = \hookrightarrow \circ \chi$ in the sequel.

2.2 Criteria for fixed-basis operators. Let $X, Y \in |\mathbf{SET}|$ and fix $L \in |\mathbf{LOQML}|$. Then there should exist $f_L^{\to}$, $f_L^{\leftarrow}$ such that the following, precise restatements of the Criteria 1.12 for the fixed-basis case hold:

(1) $f_L^{\to}$ lifts $f^{\to}$ uniquely, i.e. $f_L^{\to}$ is the unique $g : L^X \to L^Y$ such that $g \circ \eta_X = \eta_Y \circ f^{\to}$.

(2) $f_L^{\leftarrow}$ lifts $f^{\leftarrow}$ uniquely, i.e. $f_L^{\leftarrow}$ is the unique $g : L^X \leftarrow L^Y$ such that $g \circ \eta_Y = \eta_X \circ f^{\leftarrow}$.

(3) $f_L^{\to} \dashv f_L^{\leftarrow}$, viewing L^X, L^Y as partially-ordered categories.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circ"] & Y \arrow[d] \\
\mathcal{P}(X) \arrow[r, "!\, f^{\to}"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\circ"] & \mathcal{P}(Y) \arrow[d, "\eta_Y"] \\
L^X \arrow[r, "?\,!\, f_L^{\to}"'] & L^Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P}(X) \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\circ"] & \mathcal{P}(Y) \arrow[l, "!\, f^{\leftarrow}"'] \arrow[d, "\eta_Y"] \\
L^X & L^Y \arrow[l, "?\,!\, f_L^{\leftarrow}"]
\end{tikzcd}$$

2.3 Discussion. These criteria are met in the sequel in each of two ways:

(1) $f_L^{\leftarrow}$ is determined **first** and **then** $f_L^{\to}$ is determined, the obverse of the classical case;

(2) $f_L^{\to}$ is determined **first** and **then** $f_L^{\leftarrow}$ is determined, as in the classical case.

2.4 Notation. $\forall X, \forall L, \forall \alpha \in L$, $\underline{\alpha}$ denotes the **constant map** in L^X having value α.

The next lemma is crucial to the proof of several of the characterizations of powerset operators given below and is similar to results in [12].

2.5 Lemma (Decomposition theorems). Let $X \in |\mathbf{SET}|$ and let $L \in |\mathbf{CQMIL}|$ (Definition 1.6 above). Then each $b \in L^X$ has each of the following forms:

$$b = \bigvee \{\underline{\alpha} \otimes \chi_{[b \geq \alpha]} : \alpha \in L\}, \quad b = \bigvee \left\{\underline{b(x)} \otimes \chi_{\{x\}} : x \in X\right\}$$

$$b = \bigwedge \{\underline{\alpha} \otimes \chi_{[b \leq \alpha]} : \alpha \in L\}, \quad b = \bigwedge \left\{\underline{b(x)} \vee \chi_{\{x\}} : x \in X\right\}$$

2.6A Theorem (Characterizations of existence of backward powerset operators). Let L be a $L \in |\mathbf{CQMIL}|$. Then both of the following statements hold:

I **(Existence and uniqueness of $f_L^{\leftarrow}$).** $\forall f \in \mathbf{SET}(X, Y)$, $\exists!\, g : L^X \leftarrow L^Y$ satisfying the following conditions:

(1) g preserves arbitrary $\bigvee$;

(2) $\forall \alpha \in L$, $g(\underline{\alpha}) = \underline{\alpha}$;

(3) $\forall \alpha \in L$, $\forall B \subset Y$, $g(\underline{\alpha} \otimes \chi_B) = \underline{\alpha} \otimes g(\chi_B)$; and

(4) g lifts $f^{\leftarrow}$, i.e. $g \circ \eta_Y = \eta_X \circ f^{\leftarrow}$ in 2.2(2) above.

Furthermore, this g is the traditional Zadeh backward powerset operator $f_L^{\leftarrow}$ given by:

$$\forall b \in L^Y, f_L^{\leftarrow}(b) = b \circ f$$

In particular, g is the **unique** choice of $f_L^{\leftarrow}$ with respect to (1–4) for each $L \in |\mathbf{CQMIL}|$ with $\otimes = \wedge$.

II **(Converse to I).** $g : L^X \leftarrow L^Y$ satisfies the following conditions——

(1) g preserves arbitrary $\bigvee$;

(2) g **preserves constants**, i.e. $\forall \alpha \in L$, $g(\underline{\alpha}) = \underline{\alpha}$;

(3) g **preserves crisp subsets**, i.e. $\forall \alpha \in L$, $\forall y \in Y$, $g(\underline{\alpha} \otimes \chi_{\{y\}}) = \underline{\alpha} \otimes g(\chi_{\{y\}})$;

(4) g **partitions** X in the sense that, given

$$B_y = \{x \in X : \chi_{\{x\}} \leq g(\chi_{\{y\}})\}$$

the collection $\{B_y : y \in Y\}$ partitions X

——iff $\exists!\, f : X \to Y$, $g = f_L^{\leftarrow}$.

2.6B Theorem (Characterizations dual to 2.6A). All assertions of 2.6A hold with the following respective replacements for I(1), I(3), II(1), II(3), II(5) of 2.6A:

I(1): g preserves arbitrary $\bigwedge$;

II(1): g preserves arbitrary $\bigwedge$;

I(3): $\forall \alpha \in L$, $\forall B \subset Y$, $g(\underline{\alpha} \vee \chi_B) = \underline{\alpha} \vee g(\chi_B)$;

II(3): $\forall \alpha \in L$, $\forall y \in Y$, $g(\underline{\alpha} \vee \chi_{Y-\{y\}}) = \underline{\alpha} \vee g(\chi_{Y-\{y\}})$;

II(5): g **partitions** X in the sense that, given

$$C_y = \left\{x \in X : \chi_{X-\{x\}} \geq g\left(\chi_{Y-\{y\}}\right)\right\}$$

the collection $\{C_y : y \in Y\}$ partitions X.

2.7 Remark. We **must** take $f_L^{\leftarrow}$ as the g in Theorem 2.6A/B, justifying p. 346 of [19]. But we now give even more characterizations confirming this definition of $f_L^{\leftarrow}$.

2.8 A/B Theorems (Characterizations alternative to 2.6 A/B). All the assertions of Theorem 2.6A [2.6B] hold with the following respective replacements for I(1), II(1), and II(5):

I(1): g preserves arbitrary $\bigvee$ [$\bigwedge$] and finite $\wedge$ [$\vee$];

II(1): g preserves arbitrary $\bigvee$ [$\bigwedge$] and finite $\wedge$ [$\vee$];

II(5): g **covers** X in the sense that $\{B_y : y \in Y\}$ [$\{C_y : y \in Y\}$] covers X.

The generation and characterizations of the forward powerset opeorator $f_L^{\rightarrow}$ hinge around the backward operator $f_L^{\leftarrow}$ and the AFT (dual to 1.3 above).

2.9 Theorem (Characterizations of existence of forward powerset operators). Let $L \in |\mathbf{CQMIL}|$. Then each of the following statements hold:

I. **(Existence and uniqueness of $f_L^{\rightarrow}$; Zadeh Extension Principle (ZEP)).** $\forall\, f \in \mathbf{SET}(X,Y)$, $\exists!\, h : L^X \to L^Y$ satisfying the following conditions:

(1) h preserves arbitrary $\bigvee$;

(2) h lifts the classical $f^{\rightarrow}$, i.e. $\eta_Y \circ f^{\rightarrow} = h \circ \eta_X$;

(3) h is left adjoint to $f_L^{\leftarrow}$, i.e. $h \dashv f_L^{\leftarrow}$.

This h is defined by AFT(1.3 dual) as follows:

$$\forall a \in L^X,\ h(a) = \bigwedge \{b \in L^Y : a \leq f_L^{\leftarrow}(b)\} \tag{2.9.1}$$

Furthermore, this h is the traditional Zadeh forward powerset operator $f_L^{\rightarrow}$ given by

$$\forall a \in L^X,\ \forall y \in Y,\ f_L^{\rightarrow}(a)(y) = \bigvee \{a(x) : f(x) = y\} \tag{2.9.2}$$

II. (**Converse to I**). $h : L^X \to L^Y$ satisfies the following conditions——

(1) h preserves arbitrary $\bigvee$; and

(2) h **preserves fuzzy singletons**, i.e.
$\forall x \in X,\ \exists y \in Y,\ \forall \alpha \in L - \{\perp\},\ h(\underline{\alpha} \otimes \chi_{\{x\}}) = \underline{\alpha} \otimes \chi_{\{y\}}$

——iff $\exists!\, f \in$ **SET** (X, Y) such $h = f_L^{\rightarrow}$.

2.10 Remark.

(1) We *must* take $f_L^{\rightarrow}$ as the h in Theorem 2.9, justifying p. 346 of [19]. Thus h, as defined by either (2.9.1) or (2.9.2), is the **correct** choice of $f_L^{\rightarrow}$.

(2) This gives a justification of the so-called **Zadeh Extension Principle** (**ZEP**) different from that of [11].

(3) The results of this section given so far show the criteria for a fixed-basis poslat set theory, as outlined in 1.12 and 2.2, are satisfied.

(4) Note that the building of the Zadeh operators from the original mapping f, even though it is the obverse of the classical powerset operator development summarized in Section 1, follows a well-known "lifting" pattern. Recall from elementary calculus that the map $1/x$ lifts by integration to $\ln(x)$; $\ln(x)$ then induces its adjoint $\exp(x)$ (by AFT), $\exp(x)$ lifts by well-known properties of exp to the general $\exp_a(x)$ for $a > 0$ and $a \neq 1$, and $\exp_a(x)$ then induces its adjoint $\log_a(x)$ (again by AFT) which lifts $\ln(x)$. This pattern characterizes and summarizes this section thus far: f lifts to $f^{\rightarrow}, f^{\rightarrow}$ induces $f^{\leftarrow}, f^{\leftarrow}$ lifts to $f_L^{\leftarrow}$, and $f_L^{\leftarrow}$ induces $f_L^{\rightarrow}$ which lifts $f^{\rightarrow}$, where the crucial steps have occurred by the AFT.

(5) But in calculus one can also lift directly from $1/x$ to $\log_a(x)$ by using standard limit constructions.We now proceed to give an alternate development of $f_L^{\rightarrow}$ and $f_L^{\leftarrow}$ in which we create $f_L^{\rightarrow}$ directly from f and then create $f_L^{\leftarrow}$ from $f_L^{\rightarrow}$. This will imitate the development of the powerset operators in the traditional set theory summarized in Section 1. In this way, the promise of 2.3(2) above is fulfilled.

2.11 Theorem (Characterization of direct lifting of $f_L^{\rightarrow}$ from f). Let $f : X \to Y$ and $L \in |\mathbf{CQMIL}|$ be given. $\forall \alpha \in L$, let $(\eta_\alpha)_X : X \to L^X$ by

$$(\eta_\alpha)_X(x) = \underline{\alpha} \wedge \chi_{\{x\}}$$

where η_α denotes both the arrow from X to L^X and the arrow from Y to L^Y. Then the following equivalence holds:

$$\left([\forall \alpha \in L,\ (\eta_\alpha)_Y \circ f = g \circ (\eta_\alpha)_X] \text{ and } g \text{ preserves arbitrary } \bigvee\right) \Leftrightarrow g = f_L^{\rightarrow}$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_\alpha"'] & Y \arrow[d, "\eta_\alpha"] \\ L^X \arrow[r, "g"'] & L^Y \end{tikzcd}$$

2.12 Theorem (Generation of $f_L^{\leftarrow}$ from $f_L^{\rightarrow}$). Let $f : X \to Y$ and let $g : L^X \leftarrow L^Y$ be the right adjoint of $f_L^{\rightarrow} : L^X \to L^Y$ generated via AFT(1.3) from the preservation of arbitrary $\bigvee$ by $f_L^{\rightarrow}$ (2.11). Then $g = f_L^{\leftarrow}$, i.e. the following are equivalent formulations of $f_L^{\leftarrow}$:

$$f_L^{\leftarrow}(b) = b \circ f \tag{2.12.1}$$

$$f_L^{\leftarrow}(b) = \bigvee \{a \in L^X : f_L^{\rightarrow}(a) \leq b\} \tag{2.12.2}$$

2.13 Remark. The classical pattern of f generating $f^{\rightarrow}$ which in turn generates $f^{\leftarrow}$ via AFT seems to be "duplicated" in 2.11 and 2.12 for the Zadeh powerset operators: f generates $f_L^{\rightarrow}$, which in turn generates $f_L^{\leftarrow}$ via AFT. Thus this section presents two basic schemes for generating the Zadeh powerset operators for fixed-basis fuzzy sets: first, generate these operators as liftings of the classical powerset operators; and second, generate these operators directly from the original function. Now the reader might have noted that this second approach, as presented in 2.11 and 2.12, is not quite an exact duplication of the classical pattern as given in 1.11: namely, in the classical case, f generates $f^{\rightarrow}$ via adjunction between the categories **SET** and **CSLAT**; but no adjunction between concrete categories appears overtly in 2.11, though clearly f generates $f_L^{\rightarrow}$ by means of families of "α-level naturality diagrams" indexed by L. The generation of $f_L^{\rightarrow}$ from f by means of adjunction between categories can be seen in [11] for a restricted subclass of complete lattices L.

The above results require that $\otimes$ satisfy the additional axiom that $\top$ is a right identity, i.e. that $L \in |\mathbf{CQMIL}|$. This is satisfied by many examples of L in which $\otimes \neq \wedge$, as well as satisfied by all L for which $\otimes = \wedge$. But what of L generally, i.e. what should be the powerset operators for the fixed-basis case when $L \in |\mathbf{CQML}|$? We handle this by stipulation in the following definition.

2.14 Definition (Powerset operators in general case). Let $f : X \to Y$ and $L \in |\mathbf{CQML}|$. Then $f_L^{\to} : L^X \to L^Y$ and $f_L^{\leftarrow} : L^X \leftarrow L^Y$ by

$$f_L^{\to}(a)(y) = \bigvee \{a(x) : f(x) = y\} \quad , \quad f_L^{\leftarrow}(b) = b \circ f$$

3 Fuzzy powerset operators for variable-basis poslat set theories

This section summarizes the extension of the foundations of the fixed-basis case of Section 2 to the variable-basis case. Restated, this section extends the foundations for poslat set theories having ground categories of the form $\mathbf{SET} \times \mathbf{L}$ to those poslat set theories having ground categories of the form $\mathbf{SET} \times \mathbf{C}$, where $\mathbf{C} \hookrightarrow \mathbf{LOQML} \equiv \mathbf{CQML}^{op}$; these latter theories are called **variable-basis poslat set theories**. For simplicity, we shall restrict our attention in our summary to the ground $\mathbf{SET} \times \mathbf{LOQML}$.

The question before us in this section has already been proposed in Question 1.9 of Section 1. Using Definition 1.8 of Section 1, we now reformulate this question precisely. Given a ground morphism $(f, \phi) : (X, L) \to (Y, M)$ in $\mathbf{SET} \times \mathbf{LOQML}$, how should the variable-basis powerset operators

$$(f, \phi)^{\to} : L^X \to M^Y, \quad (f, \phi)^{\leftarrow} : L^X \leftarrow M^Y$$

be defined so as to satisfy the Criteria for Generalized Operators 1.12 of Section 1 for variable-basis poslat set theories? We show that such definitions can be given which satisfy these criteria under rather mild conditions; when these mild conditions are not satisfied, these definitions are stipulated.

The set theory summarized below comes from [16] and is a significant modification of that proposed (but undeveloped and unjustified) in Rodabaugh [13], referenced in Rodabaugh [14–17] and Eklund [2], and greatly influenced by Goguen [4], Hutton [6], Eklund [2], and M. A. Erceg [3].

3.1 Construction (Arrows $\langle[\phi]\rangle$, $\langle\phi^{op}\rangle$). To describe the set theory in

$$\mathbf{SET} \times \mathbf{LOQML}$$

we need two more arrow constructions. First, recall the arrow $[\phi] : L \to M$ constructed in Lemma 1.7 from $\phi : L \to M$ in $\mathbf{LOQML}$. Now fix $X \in |\mathbf{SET}|$ and put

$$\langle[\phi]\rangle : L^X \to M^X, \quad \langle\phi^{op}\rangle : L^X \leftarrow M^X$$

by

$$\langle[\phi]\rangle(a) = [\phi]\circ a, \quad \langle\phi^{op}\rangle(b) = \phi^{op}\circ b$$

3.2 Lemma. $\langle[\phi]\rangle \in$ **POSET** (L^X, M^X) and $\langle\phi^{op}\rangle \in$ **CQML** (M^X, L^X).

3.3 Discussion. Starting with a morphism (f,ϕ) in **SET** $\times$ **LOQML** from (X, L) to (Y, M), we must determine the definitions of the forward powerset operator $(f,\phi)^{\rightarrow}$ and the backward powerset operator $(f,\phi)^{\leftarrow}$ so that Criteria 1.12(1,2) are satisfied. The most difficult matter is the determination of $(f,\phi)^{\rightarrow}$, i.e. the change-of-basis generalization of ZEP (2.9). While there are two possibly different definitions of $(f,\phi)^{\leftarrow}$ that happily turn out to coincide, there are two possibly different of $(f,\phi)^{\rightarrow}$ which in fact are really different, and the conditions under which these definitions of $(f,\phi)^{\rightarrow}$ coincide are precisely those which guarantee that $(f,\phi)^{\rightarrow} \dashv (f,\phi)^{\leftarrow}$ and that the other conditions of Criteria 1.12(1,2) are satisfied. Throughout this section use is made of the operators $f_L^{\rightarrow}$, $f_L^{\leftarrow}$, $f_M^{\rightarrow}$, $f_M^{\leftarrow}$ uniquely determined and characterized in Theorems 2.6A/B, 2.8A/B, 2.9, 2.11, and 2.12. Our development culminates below in Fundamental Theorem 3.9, which gives general conditions guaranteeing the uniqueness of $(f,\phi)^{\rightarrow}$ and $(f,\phi)^{\leftarrow}$ with respect to satisfying the Criteria 1.12(1,2) for variable-basis poslat set theories. As might be expected, the backward operator $(f,\phi)^{\leftarrow}$ proves much more tractable.

3.4 Definition (Two definitions of $(f,\phi)^{\rightarrow}$). Let $(f,\phi) : (X, L) \rightarrow (Y, M)$ in **SET** $\times$ **LOQML**.

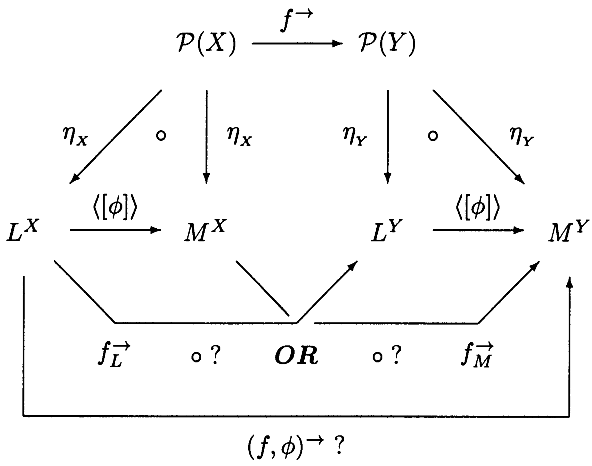
(1) **First definition of** $(f,\phi)^{\rightarrow} : L^X \rightarrow M^Y$.

$$(f,\phi)^{\rightarrow} \equiv \langle[\phi]\rangle \circ f_L^{\rightarrow} : L^X \rightarrow L^Y \rightarrow M^Y$$

(2) **Second definition of** $(f,\phi)^{\rightarrow} : L^X \rightarrow M^Y$.

$$(f,\phi)^{\rightarrow} \equiv f_M^{\rightarrow} \circ \langle[\phi]\rangle : L^X \rightarrow M^X \rightarrow M^Y$$

Diagram for $(f, \phi)^{\rightarrow}$



These definitions are distinct, but coincide under fairly general conditions given below in Fundamental Theorem 3.9.

3.5 Lemma (Characterizations of the two definitions of $(f, \phi)^{\rightarrow}$**).** Both definitions of 3.4 yield arrows in **POSET**. Furthermore, the first definition of $(f, \phi)^{\rightarrow} : L^X \rightarrow M^Y$ is equivalent to stipulating

$$(f, \phi)^{\rightarrow}(a) \equiv \bigwedge\{b : f_L^{\rightarrow}(a) \le \langle\phi^{op}\rangle(b)\}$$

And the second definition is equivalent to stipulating

$$(f, \phi)^{\rightarrow}(a) \equiv \bigwedge\{b : f_M^{\rightarrow}(\langle[\phi]\rangle(a)) \le b\}$$

3.6 Definition (Two definitions of $(f, \phi)^{\leftarrow}$**).** Let $(f, \phi) : (X, L) \rightarrow (Y, M)$ in **SET** $\times$ **LOQML**.

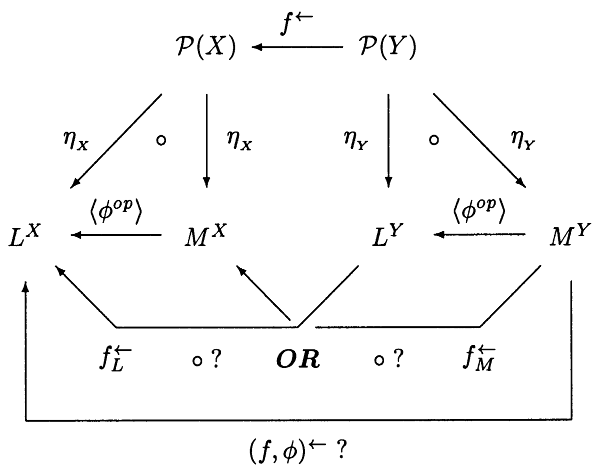
(1) **First definition of** $(f, \phi)^{\leftarrow} : L^X \leftarrow M^Y$.

$$(f, \phi)^{\leftarrow} \equiv f_L^{\leftarrow} \circ \langle\phi^{op}\rangle : L^X \leftarrow L^Y \leftarrow M^Y$$

(2) **Second definition of** $(f, \phi)^{\leftarrow} : L^X \leftarrow M^Y$.

$$(f, \phi)^{\leftarrow} \equiv \langle\phi^{op}\rangle \circ f_M^{\leftarrow} : L^X \leftarrow M^X \leftarrow M^Y$$

DIAGRAM FOR $(f, \phi)^{\leftarrow}$



3.7 Lemma (Resolution of the definitions of $(f, \phi)^{\leftarrow}$**).** Both definitions of 3.6 yield arrows in **CQML** which coincide, and the following are equivalent for $(f, \phi) : (X, L) \to (Y, M)$ in **SET** $\times$ **LOQML**:

(1) $(f, \phi)^{\leftarrow} \equiv f_L^{\leftarrow} \circ \langle\phi^{op}\rangle$;

(2) $(f, \phi)^{\leftarrow} \equiv \langle\phi^{op}\rangle \circ f_M^{\leftarrow}$;

(3) $(f, \phi)^{\leftarrow}(b) \equiv \phi^{op} \circ b \circ f$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "(f,\phi)^{\leftarrow}(b)"'] \arrow[dr, phantom, "\circ"] & Y \arrow[d, "b"] \\
L & M \arrow[l, "\phi^{op}"]
\end{tikzcd}$$

3.8 Convention. A statement in the literals 'X', 'Y', 'f' holds **universally** iff it holds $\forall X \in |\mathbf{SET}|, \forall Y \in |\mathbf{SET}|$, and $\forall f \in \mathbf{SET}(X, Y)$. This convention is used in the following Fundamental Theorem satisfying Criteria 1.12(1,2) for variable-basis poslat set theories.

3.9 Fundamental Theorem (General poslat set theory with unique liftings). Let $\phi \in \mathbf{LOQML}(L, M)$ and consider the following statements:

(1) ϕ^{op} preserves arbitrary $\bigwedge$;

(2) $[\phi]$ preserves arbitrary $\bigvee$;

(3) $[\phi] \dashv \phi^{op}$;

(4) $(f, \phi)^{\rightarrow} \dashv (f, \phi)^{\leftarrow}$ universally for at least one definition of $(f, \phi)^{\rightarrow}$;

(5) $(f, \phi)^{\rightarrow} \dashv (f, \phi)^{\rightarrow}$ universally for both definitions of $(f, \phi)^{\rightarrow}$;

(6) both definitions of $(f, \phi)^{\rightarrow}$ coincide universally;

(7) at least one definition of $(f, \phi^{\rightarrow}$ preserves arbitrary $\bigvee$ universally;

(8) both definitions of $(f, \phi)^{\rightarrow}$ preserve arbitrary $\bigvee$ universally;

(9) $(f, \phi)^{\leftarrow}$ preserves arbitrary $\bigwedge$ universally.

Then (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (9); and (1) implies each of (2), (6), (7), (8).

3.10 Corollary (Applications of fundamental theorem). Theorem 3.9(1–9) hold under the following conditions:

(1) L, M are equipped with order-reversing involutions and ϕ^{op} preserves such involutions; i.e. $L, M \in |\mathbf{DQML}|$ and $\phi^{op} \in \mathbf{DQML}$, where **DQML** [17] is the subcategory of **CQML** of objects equipped with order-reversing involutions and whose morphisms are **CQML** morphisms preserving such involutions. These conditions include all morphisms of **CBOOL**.

(2) ϕ^{op} is a backward (Zadeh) powerset operator; i.e. $\exists N \in |\mathbf{CQML}|$, $\exists W, Z \in |\mathbf{SET}|$, and $\exists g \in \mathbf{SET}(W, Z)$, $\phi^{op} \equiv g_N^{\leftarrow} : N^W \leftarrow N^Z$.

(3) ϕ is anyone of $\psi,^* \psi, \psi^*$ of 7.1.7.2 of [17].

(4) ϕ is anyone of ζ_l^* of 9.9(1)(b), $\zeta_l,^* \zeta_l$ of 9.9(2(b), and ζ_l of 9.9(3), all of [18].

(5) ϕ is an isomorphism in **LOQML**.

3.11 Remark. The factorization of $(f, \phi)^{\leftarrow}$ *à la* 3.7(1,2) has been studied at length in [2] under the restriction of 3.10(1).

3.12 Definition of $(f, \phi)^{\rightarrow}$. If (1) of 3.9 fails to hold, we stipulate $(f, \phi)^{\rightarrow}$ in accordance with 3.4(1).

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "(f,\phi)^{\rightarrow}(a)"] \arrow[dl, "f_L^{\rightarrow}(a)"'] \\
L \arrow[r, "{[\phi]}"'] \arrow[ur, phantom, shift right=3ex, "\circ"] & M
\end{tikzcd}$$

3.13 Remark. To sum up this section, it follows from Fundamental Theorem 3.9 that we have a generalization of the classical set theory appropriate for generalizing topology, the classical Stone machinery, measure theory, etc. This justifies a few terminologies and remarks. First, L^X is the (L-**fuzzy**) **powerset** of X, and each member of L^X is a(n) (L-**fuzzy**) **subset** of X (see [6]), which is **crisp** if its codomain is contained in $\{\perp, \top\}$ viewed as the subset of universal bounds of L. If $A \subset X$ in the usual sense, then its characteristic function χ_A is crisp; this induces the **CQML** isomorphism η_X, given previously, between the usual powerset and the complete sublattice $\{\perp, \top\}^X$ of the fuzzy powerset, and induces also a categorical isomorphism between **SET** and the full subcategory **SET** $\times \{\{\perp, \top\}\}$ of **SET** $\times$ **LOQML**. The members of L^X which are constant maps are called the **constant** (L-**fuzzy**) **subsets** of X and are denoted $\{\underline{\alpha} : \alpha \in L\}$ as stated in Section 2.

4 Powerset operator characterizations of ground isomorphisms

It is well known that in **SET**, each isomorphism f is characterized by the property: $\exists g$, $g^{\leftarrow} = f^{\rightarrow}$ and $g^{\rightarrow} = f^{\leftarrow}$. Because of Section 2 and Fundamental

Theorem 3.9 (and Corollary 3.10), this can be generalized to both fixed-basis and variable-basis grounds. The importance of such results is that they are the only means to a characterization of fuzzy homeomorphism for the fixed-basis topology of [5] and the variable-basis topology of [16, 17].

We first state the main result of the section, and then give the three lemmas needed to prove it which are of importance in their own right; the third lemma is a well-known schoolboy exercise. *En route* the classical case is greatly generalized not only with respect to the main result, but with respect to the first two supporting lemmas as well.

4.1 Theorem (Characterization of variable-basis ground isomorphisms). Let $(f,\phi) : (X,L) \to (Y,M)$ be a morphism in $\mathbf{SET} \times \mathbf{LOQML}$ and consider the following statements:

(1) (f,ϕ) is an isomorphism in $\mathbf{SET} \times \mathbf{LOQML}$;

(2) f and ϕ^{op} are both bijections;

(3) f has an inverse $f^{-1} \in \mathbf{SET}(Y,X)$, ϕ^{op} has an inverse $(\phi^{op})^{-1} \in \mathbf{CQML}$, $(\phi^{op})^{-1} = [\phi]$, $(f^{-1},[\phi]^{op})^{\leftarrow} = (f,\phi)^{\rightarrow}$, and $(f^{-1},[\phi]^{op})^{\rightarrow} = (f,\phi)^{\leftarrow}$.

(4) $\exists\,!\; g \in \mathbf{SET}(X,Y)$, $(g,[\phi]^{op})^{\leftarrow} = (f,\phi)^{\rightarrow}$, $(g,[\phi]^{op})^{\rightarrow} = (f,\phi)^{\leftarrow}$, and $(g,[\phi]^{op})^{\rightarrow} \dashv (g,[\phi]^{op})^{\leftarrow}$;

(5) $(f,\phi)^{\rightarrow}$ is a bijection (and hence a $\mathbf{CQML}$ isomorphism);

(6) $(f,\phi)^{\leftarrow}$ is a bijection (and hence a $\mathbf{CQML}$ isomorphism);

(7) $(f,\phi)^{\rightarrow}$ and $(f,\phi)^{\leftarrow}$ are bijections and inverses of each other.

Then the following statements hold:

I. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4), (5), (6), and (7) $\Rightarrow$ (3):

II. All of (1)–(7) are equivalent if ϕ^{op} preserves arbitrary $\bigwedge$.

4.2 Lemma (Fixed-basis characterization of SET isomorphisms). Let $f \in \mathbf{SET}(X,Y)$ and $L \in |\mathbf{LOQML}|$. The following are equivalent:

(1) f is an isomorphism;

(2) f has an inverse function f^{-1}, $(f^{-1})_L^{\leftarrow} = f_L^{\rightarrow}$, and $(f^{-1})_L^{\rightarrow} = f_L^{\leftarrow}$;

(3) $\exists g \in \mathbf{SET}(Y,X)$, $g_L^{\leftarrow} = f_L^{\rightarrow}, g_L^{\rightarrow} = f_L^{\leftarrow}$;

(4) $\exists ! g \in \mathbf{SET}(Y,X)$, $g_L^{\leftarrow} = f_L^{\leftarrow}$, $g_L^{\rightarrow} = f_L^{\leftarrow}$;

(5) $f_L^{\rightarrow}$ is a bijection (and hence an isomorphism in $\mathbf{CQML}$);

(6) $f_L^{\leftarrow}$ is a bijection (and hence an isomorphism in $\mathbf{CQML}$).

4.3 Lemma (Characterization of isomorphisms in LOQML). Let

$$\phi \in \mathbf{LOQML}(L, M)$$

and X be a set, and consider the following statements:

(1) ϕ is an isomorphism in **LOQML**.

(2) ϕ^{op} is a bijection (and hence an isomorphism in **LOQML**);

(3) $\langle \phi^{op} \rangle$ is a bijection (and hence an isomorphism in **LOQML**);

(4) $[\phi]$ and ϕ^{op} are function inverses of each other;

(5) $\langle [\phi] \rangle$ and $\langle \phi^{op} \rangle$ are function inverses of each other;

(6) $[\phi]$ is a bijection (and hence an isomorphism in **LOQML**);

(7) $\langle [\phi] \rangle$ is a bijection (and hence an isomorphism in **LOQML**).

Then the following statements hold:

I. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (6) $\Leftrightarrow$ (7).

II. All of (1)–(7) are equivalent if ϕ^{op} preserves arbitrary $\bigwedge$.

4.4 Lemma. Let $F \in \mathbf{SET}(Y, Z)$ and $G \in \mathbf{SET}(X, Y)$. Then the following hold:

(1) F and G are bijections $\Rightarrow F \circ G$ is a bijection.

(2) $F \circ G$ is a bijection and [F is an injection or G is a surjection] $\Rightarrow F$ and G are bijections.

5 Fuzzy powerset operators and fuzzy associative memories

This section borrows the concepts of fuzzy subsets as points in a cube, fuzzy systems, fuzzy associative memories, and the motivation for the latter, from B. Kosko [8, 9]. The purpose of this section is to summarize from [16] the characterization of how close the Zadeh powerset operators come to determining all fuzzy associative memories.

Throughout this section all underlying sets $X, Y, \ldots$ are finite, and we restrict all lattices $L, M, \ldots$ to be $\mathbb{I} \equiv [0, 1]$. Note that the powersets $\mathbb{I}^X$, $\mathbb{I}^Y, \ldots$ are order-theoretically isomorphic to finite-dimensional cubes, and may then be viewed as carrying the standard Euclidean topology. Possible generalizations, with the finiteness of sets removed and L, M taken as frames or continuous lattices, are not explored here; these are left for future work. For now, it is convenient to view fuzzy subsets as points in a finite-dimensional cube (in which the vertices are the crisp or classical subsets) so that closeness of fuzzy subsets can be measured using the Euclidean metric or distance formula. We view the Zadeh powerset operators as transferable back and forth between $\mathbb{I}^X$, $\mathbb{I}^Y$ and $\mathbb{I}^{|X|}$, $\mathbb{I}^{|Y|}$ by means of the lattice isomorphism mentioned above; we do not make this isomorphism explicit, nor do we make explicit how the powerset operators are thereby transferred, but leave this to the reader.

5.1 Definition (Fuzzy Systems and FAM's). Let X, Y be finite sets. Then a **fuzzy system with input space** X and **output space** Y is a mapping $F : \mathbb{I}^X \longrightarrow \mathbb{I}^Y$. And a **fuzzy associative memory (FAM)** is a fuzzy system which is continuous w.r.t. the Euclidean topologies on the powersets $\mathbb{I}^X$ and $\mathbb{I}^Y$.

5.2 Discussion (Motivation for fuzzy sets and FAM's). The powersets in 5.1 represent aggregates of attributes of the input and output spaces; i.e. $a \in \mathbb{I}^X$ represents an attribute of X. Fuzzy systems set up associations between attributes of X and attributes of Y. If X comprises data on traffic density and Y comprises data on traffic lights, then $a \in \mathbb{I}^X$ could be the attribute 'heavy' and $b \in \mathbb{I}^Y$ could be the attribute 'long duration of green'; and a fuzzy system could associate 'long duration of green' with 'heavy'—cf. some modern intersections in which traffic density controls both traffic light activation and duration. Further, it would seem desirable that traffic density attributes, which are close as measured by the Euclidean metric, should correspond to durations of the green traffic light, which are also close as measured by the Euclidean metric. This desirable trait is simply continuity w.r.t. the usual Euclidean topology on the fuzzy powersets viewed as cubes. Now it seems that this continuity would be a feature of a fuzzy system having "associative memory"; and so a fuzzy associative memory is then a fuzzy system for which continuity is formally stipulated.

5.3 Convention ($\vee$ and $\wedge$). The binary join and meet operators on $\mathbb{I}$ are mappings from $\mathbb{I} \times \mathbb{I}$ to $\mathbb{I}$, i.e. $\vee, \wedge : \mathbb{I}^2 \longrightarrow \mathbb{I}$. But we may view these as operators from $\mathbb{I}^n$ to $\mathbb{I}$ for $n \in \mathbb{N} \cup \{0\}$ as follows:

(1) for $n = 0$, define $\mathbb{I}^0$ to be $\emptyset$, in which case $\vee$ is the constant map $\underline{0}$, and $\wedge$ is the constant map $\underline{1}$;

(2) for $n = 1$, $\vee = \wedge = id_{\mathbb{I}}$;

(3) for $n = 2, \vee, \wedge$ are the usual binary operators;

(4) for $n \geq 3, \vee, \wedge$ are the usual binary operators extended via induction and the associative law.

Thus we have defined operators $\vee, \wedge : \mathbb{I}^n \longrightarrow \mathbb{I}$ for $n \in \mathbb{N} \cup \{0\}$.

5.4 Lemma. The join and meet operators $\vee, \wedge : \mathbb{I}^n \longrightarrow \mathbb{I}$ are continuous for $n \in \mathbb{N} \cup \{0\}$.

5.5 Theorem (Powerset operators as FAM's). Let $f : X \to Y$ be a function and $\phi \in \mathbf{LOQML}(\mathbb{I}, \mathbb{I})$.

(1) Each of $f_{\mathbb{I}}^{\rightarrow}$ and $f_{\mathbb{I}}^{\leftarrow}$ is a FAM.

(2) $(f, \phi)^{\rightarrow}$ is a FAM $\Leftrightarrow$ $[\phi] : \mathbb{I} \longrightarrow \mathbb{I}$ is continuous.

(3) $(f, \phi)^{\leftarrow}$ is a FAM $\Leftrightarrow$ $\phi^{op} : \mathbb{I} \longleftarrow \mathbb{I}$ is continuous, and the latter holds if ϕ^{op} preserves the order-reversing involution $(\alpha' \equiv 1 - \alpha)$ or arbitrary $\bigwedge$.

Recall from Theorems 2.6A/B(II), 2.8A/B(II), and 2.9II(2) the notions of a mapping between powersets preserving constants, fuzzy singletons, crisp subsets, partitioning the exponent of its codomain, and/or covering the exponent of its codomain. The following theorem and corollary are immediate consequences of Theorems 2.6A/B(II), 2.8A/B(II), 2.9II(2), and 5.5.

5.6 Theorem (FAM's as powerset operators). Let F be a fuzzy system. Then F preserves arbitrary joins and fuzzy singletons iff F is a forward Zadeh powerset operator, in which case it is also a FAM. Furthermore, F preserves arbitrary joins [meets], constants, crisp subsets, and partitions the exponent of its codomain [covers the exponent of its codomain, resp.] iff F is a backward Zadeh powerset operator, in which case it is also a FAM.

5.7 Corollary. A FAM is a forward Zadeh powerset operator iff it preserves arbitrary joins and fuzzy singletons.

5.8 Application. Strong doubts have been expressed in the literature, e.g. [9], about the relevance of the Zadeh powerset operators in fuzzy systems and fuzzy associative memories; but such doubts may be erroneously predicated upon the assumption that powerset operators require knowing *á priori* the input-output function underlying the powerset operators. The importance of Theorem 10.6 is that it **characterizes** those systems which can be **redescribed** as powerset operators of **induced** input-output functions. Thus Theorem 5.6 and Corollary 5.7 can be used in fuzzy logic to precisely identify those systems which cannot be *á posteriori* built up from input-output functions, and hence are not built upon any possible neutral network.

5.9 Question. Are there at least two or three important, **empirical** examples of FAMS which are **not** forward Zadeh powerset operators (as judged by Theorem 5.6 and Corollary 5.7)?

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Introductory Notes To Chapter 3

U. HÖHLE

The purpose of this section is to understand the conception of *space*[1] and the role of *lattice–valued maps* in topology. In particular, we answer the question to what extent topology is based on classical set theory.

If we choose the approach to topology given by neighborhood systems (cf. p. 213 in [9]), then it is clear that the *algebraic foundation* of topology is determined by the *filter monad* $\mathbf{F} = (F, \eta, \circ)$ (which is a submonad of the *double power set monad* [2] (cf. [5, 12, 15])). The situation is as follows: Let $\mathbf{SET_F}$ be the *Kleisli category* (cf. [12, 13]) associated with the filter monad $\mathbf{F}$ — i.e. *objects* are ordinary sets, but *morphisms* $\mathcal{A} : X \rightharpoondown Y$ are ordinary maps from X to the set $\mathbb{F}(Y)$ of all filters on Y. Further the *composition* in $\mathbf{SET_F}$ is determined by the clone-composition $\circ$ associated with the filter monad — i.e.

$$\begin{aligned} \mathcal{A} : X \longmapsto \mathbb{F}(Y) \quad &, \quad \mathcal{B} : Y \longmapsto \mathbb{F}(Z) \\ [\mathcal{B} \circ \mathcal{A}](x) \quad &= \quad \{ U \subseteq Z \mid \{y \in Y \mid U \in \mathcal{B}(y)\} \in \mathcal{A}(x) \} \;, \quad x \in X \quad . \end{aligned}$$

In particular, the *identity* of X in the sense of $\mathbf{SET_F}$ is the map $\eta_X : X \longmapsto \mathbb{F}(X)$ defined by

$$\eta_X(x) \quad = \quad \{A \subseteq X \,|\, x \in A\} \quad , \quad x \in X \; .$$

In this categorical framework, a *neighborhood system* on X is an endomorphism $\mathcal{U} : X \rightharpoondown X$ in the sense of $\mathbf{SET_F}$ satisfying the following axioms:

(i) $\quad \mathcal{U} \quad \subseteq \quad \eta_X(x) \quad , \quad \forall x \in X \quad ;$

(ii) $\quad \mathcal{U} \circ \mathcal{U} \quad = \quad \mathcal{U} \quad .$ $\qquad$ (*Idempotency*)

[1] An intuitive description of topological spaces is given by A. Appert; we quote from his book [3], page 10 : ...'Pour qu'un ensemble E d'objects de nature quelconque puisse être considéré comme répondant au concept intuitif de spatialité, il est nécessaire de s'être donné non seulement les éléments ou *points* de l'ensemble E, mais aussi d'avoir précisé certaines relations de proximité ou de situation de ces points les uns par rapport aux autres. C'est seulement quand ces relations auront été fixées que l'on pourra dire qu'une *topologie* est introduite dans l'ensemble E, ou encore que E est un *ensemble topologique*.

[2] In the case of $L = \{0, 1\}$, the monad $\mathbf{T}_L$ on page 120 in this Note coincides with the double power set monad.

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Moreover, a map $\varphi : X_1 \longmapsto X_2$ is *continuous* w.r.t. given neighborhood systems $\mathcal{U}_1$ on X_1 and $\mathcal{U}_2$ on X_2 iff the following axiom holds:

$$\text{(iii)} \quad (\mathcal{U}_2 \cdot \varphi)(x_1) \quad \subseteq \quad ((\eta_Y \cdot \varphi) \circ \mathcal{U}_1)(x_1) \qquad \forall\, x_1 \in X_1 \;. \qquad (\textit{Continuity})$$

Therefore the *algebraic foundations* of topology require only a partially ordered monad over **SET**, and do not require the full concept of classical set theory — e.g. in contrast to a wide-spread opinion the *explicit* formation of pre-images is *not necessary* for the definition of coninuity. As we have seen above it is sufficient to use the clone-compositon and the partial ordering of the underlying monad.

In order to obtain a deeper understanding of neighborhood systems, we refer to E.G. Manes' non-deterministic interpretation of monads over a given base category $\mathbf{K}$ (cf. Section 3 in Chapter 4 in [12]). We prepare this interpretation by recollecting the axioms of a monad $\mathbf{T} = (T, \eta, \circ)$ viewed as an algebraic theory in clone from (cf. p. 25 in [12]): $T : |\mathbf{K}| \longmapsto |\mathbf{K}|$ is an object function which assigns to each object X its 'extension' $T(X)$, the component η_X of η is a $\mathbf{K}$-morphism from X to $T(X)$ (so-called insertion-of-the-variables), and finally for each triple (X, Y, Z) of $\mathbf{K}$-objects X, Y and Z there exists a map $\circ : Hom(Y, T(Z)) \times Hom(X, T(Y)) \longmapsto Hom(X, T(Z))$ (so-called clone-composition). All these data are subjected to the following axioms:

$$\varphi_3 \circ (\varphi_2 \circ \varphi_1) \quad = \quad (\varphi_3 \circ \varphi_2) \circ \varphi_1 \;. \qquad \text{(Associativity)}$$

$$\varphi : X \longmapsto Y \quad , \quad \psi : Y \longmapsto T(Z) \quad , \quad \zeta : X \longmapsto T(Y)$$
$$\psi \circ (\eta_Y \cdot \varphi) \quad = \quad \psi \cdot \varphi \quad , \qquad \eta_Y \circ \zeta \quad = \zeta \quad .$$

The Kleisli category $\mathbf{K_T}$ associated with $\mathbf{T}$ has the same objects as $\mathbf{K}$, but morphisms from X to Y are $\mathbf{K}$-morphisms from X to the 'extension' $T(Y)$ of Y. In particular, the identity of X is η_X, and the composition is determined by the clone-composition. After this brief digression we return to the non-deterministic interpretation of monads.

Following closely Manes' arguments (cf. p. 309 in [12]) it is clear that elements of X are interpreted as *pure states* (or *standard points*) of X, and elements of $T(X)$ are interpreted as *fuzzy states* (or *non-standard points*) over X. Further, every morphism $\varphi : X \rightharpoondown Y$ of the associated Kleisli catgeory $\mathbf{K_T}$ can be understood as a *fuzzy morphism* assigning to every standard point of X a non-standard point over Y. Finally $\circ$ plays the role of the composition of fuzzy morphisms, and the component η_X of the natural transformation η can be viewed as the representation of standard points of X within the realm of non-standard points over X.

If we apply this philosophical understanding of monads to the filter monad, then *principal ultrafilters* on X serve as the representation of standard points of X, and *filters* on X play the role of non-standard points over X. In this context a neighborhood system $\mathcal{U}$ on X is an *idempotent, fuzzy endomorphism* of X provided with the additional property that every standard point x of X is 'inside of' its associated non-standard point $\mathcal{U}(x)$ over X (cf. (i)). In particular, it is not difficult to see that under the hypothesis of Axiom (i) the idempotentcy of $\mathcal{U}$ guarantees the **existence** of '*open subsets*' — i.e. $\mathcal{U}$ can be characterized by an appropriate *subframe* (cf. [11]) of the ordinary power set $\mathbb{P}(X)$ of X. We draw the following conclusions from these observations:

- A *topological space* is a pair $(X, \mathcal{U})$, where X is a set (more precisely an *object* in $\mathbf{SET}_{\mathbb{F}}$) and $\mathcal{U}$ is a *unary fuzzy operation* on X [3].
  - The axiom (i) reflects the topological idea of 'inside'.
  - The axiom (ii) opens the door to the important study of *covering properties* — concepts which are completely lattice-theoretic in nature.
- The object function $F : |\mathbf{SET}| \longmapsto |\mathbf{SET}|$ determines the *type of convergence theory* which can be associated with neighborhood systems (see also Section 5 and Section 6 in G. Choquet's paper [4]).
- 2-Valued maps (i.e. characteristic functions) appear exclusively in the construction of the filter space $\mathbb{F}(X)$ — the space of all non-standard points over X.

Since the axioms (i) – (iii) can be rephrased in any partially ordered monad, the previous conclusions motivate the following
Question 1: Can general topology be based on any partially ordered monad over a given base category $\mathbf{K}$?
A partial answer is given in [10]. Here we only ask the more restrictive question: Can we replace the filter monad by another partially ordered monad over $\mathbf{SET}$ such that the corresponding category of neighborhood systems (resp. topological spaces) form a topological category over $\mathbf{SET}$? An early attempt in this direction was performed by A. Appert in his work on $(\mathcal{V})$ spaces ("espaces à voisinages" [1, 2]) which originated in M. Fréchet's fundamental papers on neighborhood structures (cf. [6, 7]). In a more modern terminology we can describe A. Appert's approach as a replacement of the filter monad $\mathbf{F}$ by the *semi-filter monad* $\mathbf{SF}$ (which is also a submonad of the double power set monad). According to A. Appert, an idempotent endomorphism $\mathcal{V} : X \rightharpoondown X$ of the Kleisli category $\mathbf{SET}_{\mathbf{SF}}$ is called a *transitive (generalized) topology* on X iff $\mathcal{V}$ satisfies additionally Axiom (i) (cf. p. 12, p. 13 and p. 17 in [3]). By analogy to the previous considerations it is not difficult to see that transitive (generalized) topologies can be characterized by *complete join-subsemilattices* of the ordinary power set $\mathbb{P}(X)$ of X (cf. p. 19 in [3]); hence a non-trivial study of covering properties can always be performed in this framework (cf. [2], p. 19 in [3]). The situation changes drastically, if we look at the corresponding convergence theory. Since the supremum of all proper semi-filters on X is again a

[3] In contrast to A. Appert's mathematical approach to topological spaces (which is based on contiguity relations (cf. p.11 in [3]) and seems to be inconsistent with his own intuitive conception), we believe that only the monadic approach to topology (which requires neighborhood systems as a primitive concept) leads to a consistent mathematical model of Appert's intuitive idea of a topological space (cf. footnote 1).

proper semi-filter on X, every standard point is an *adherent point* of every non-standard point (i.e. proper semi-filter); hence the corresponding convergence theory is trivial. Therefore A. Appert's work on $(\mathcal{V})$ spaces or more general his book [3] on "Espaces Topologiques Intermédiares" co-authored by Ky Fan motivates the following

Question 2: Can we replace the filter monad by *another* partially ordered monad over **SET** such that

- the corresponding category of neighborhood systems (resp. topological spaces) form a **topological catgeory** over **SET**,
- neighborhood systems can be characterized by **complete join-semi-lattices**,
- a **non-trivial convergence theory** is available.

As the reader will see in *Chapter 3* of this volume fixed-basis topology give an affirmative answer to Question 2. The key idea consists in replacing 2-valued maps by L-valued maps where L is a fixed, but arbitrary complete lattice. This approach is accompanied by the insight that every set L induces a monad $\mathbf{T}_L = (T, \eta, \circ)$ over **SET** — an observation which is hidden in the double power set monad. In particular, T, η, $\circ$ are determined by the following formulas

$$\begin{aligned}
&T(X) \quad = \quad L^{(L^X)} \qquad \text{where} \quad X \in |\mathbf{SET}| \; ; \\
&\eta_X : X \longmapsto T(X) \qquad \text{by} \qquad [\eta_X(x)](f) \quad = \quad f(x) \quad , \; \forall\, x \in X \; ; \\
&\varphi : X \longmapsto L^{(L^Y)} \quad , \quad \psi : Y \longmapsto L^{(L^Z)} \\
&[(\psi \circ \varphi)(x)](g) \quad = \quad [\varphi(x)]\big((\psi(_)(g)\big) \qquad \text{where} \quad g \in L^Z \quad .
\end{aligned}$$

If we impose an isotone, commutative, but not necessarily associative, binary operation $\otimes$ on L, then we are in the position to define an interesting submonad $\mathbf{F}_L$ of the monad $\mathbf{T}_L$ such that Question 2 can be answered in the affirmative! Non-standard points over X w.r.t. $\mathbf{F}_L$ are L–filters on X (cf. Subsection 6.1 in *Chapter 3* of this volume). Moreover, idempotent endomorphisms $\mathcal{U} : X \rightharpoondown X$ in the sense of the Kleisli category $\mathbf{SET}_{\mathbf{F}_L}$ provided with the additional property

$$\text{(i')} \qquad [\mathcal{U}(x)](f) \;\; \leq \;\; [\eta_X(x)](f) \;\; = \;\; f(x) \quad , \quad \forall\, x \in X$$

form the new concept of neighborhood systems — so-called *L–neighborhood systems* on X. Further *continuity* of a map $\varphi : X_1 \longmapsto X_2$ can be defined by the following requirement

$$\text{(iii')} \qquad \big(\mathcal{U}_2 \cdot \varphi\big)(x_1) \;\; \leq \;\; \big((\eta_{X_2} \cdot \varphi) \circ \mathcal{U}_1\big)(x_1) \quad , \quad \forall\, x_1 \in X_1 \quad .$$

It is not difficult to see that L–neighborhood systems on X can be characterized by complete join-subsemilattices of L^X – so-called L-topologies on X (cf. Subsection 6.1 in *Chapter 3* of this volume). In particular, the study of *compactness* in terms of the *Heine-Borel* covering property can be performed in this framework (cf. Section 4 in *Chapter 7* of this volume). Moreover Subsection 6.4 and Subsection 7.3 in *Chapter 3* of this volume show that a non-trivial convergence theory is available w.r.t. L–neighborhood systems. All these observations confirm the fundamental cognition that the systematic use of lattice-valued maps opens new fields for general topology and leads to new, interesting, topological investigations (see e.g. Example 7.1.4 in *Chapter 3* of this volume). Hence from a monadic point of view *general topology* is **not** a closed theory. We try to illustrate this situation by a discussion of an important special case:

Let the underlying lattice L be given by the real unit interval $I = [0, 1]$ equipped with its usual ordering. Referring to the hypergraph functor (cf. Remark 7.1.5 in *Chapter 3* of this volume) it is not difficult to prove that every I–neighborhood system on X can be characterized by a complete join-subsemilattice of the ordinary power set $\mathbb{P}(X \times [0, 1[)$. In this sense we arrive again at A. Appert's concept of transitive (generalized) topologies. But, if we choose $\otimes \neq \min$ (e.g. if $\otimes$ is given by the *arithmetic mean* $\oplus$ on I), then it is easy to see that the corresponding convergence theory is fundamentally different from the usual one based on ordinary filters. Therefore, in the setting of $(L, \otimes) = (I, \oplus)$ the theory of I-neighborhood systems (i.e. of I–topological spaces) can be viewed as a *non-trivial continuation* of A. Appert's work on "Espaces Topologiques Intermédiares" arising from M. Fréchet's and W. Sierpinski's work in the late twenties of this century (cf. [8, 14]).

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CHAPTER 3

Axiomatic Foundations Of Fixed-Basis Fuzzy Topology

U. HÖHLE AND A.P. ŠOSTAK

Introduction

This paper gives the first comprehensive account on various systems of axioms of fixed–basis, L–fuzzy topological spaces and their corresponding convergence theory. In general we do not pursue the historical development, but it is our primary aim to present the state of the art of this field. We focus on the following problems:

- Which role is played by the idempotency, commutativity or associativity of the meet operation in L–fuzzy topology?
- Replacing $2 = \{0, 1\}$ by a fixed complete lattice L, what means *openness* (resp. *degree of openness*) of a given (L–fuzzy) subset? Does there exist a local characterization of openness – e.g. by neighborhood systems?
- Does there exist a *functorial* mechanism which permits a *change* of the fixed basis L leaving the topological axioms invariant?
- Do there exist non trivial, natural examples of L–fuzzy topological spaces which bridge to other fields of mathematics?

The first problem was already observed in [45] and leads to a careful analysis of the underlying lattice–theoretic structure (cf. Section 1). In this context examples range from complete orthomodular lattices to GL–quantales with square roots. As the reader will see below, the axioms of L–fuzzy topological (resp. L–topological) spaces only use a very simple lattice structure (so–called complete quasi–monoidal lattices $(L, \leq, \otimes)$) and are *not based* on commutativity or

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distributivity of the "meet operation" $\otimes$ over arbitrary joins, while the strength of the convergence theory seems to depend essentially on *idempotency*, *commutativity* and *distributivity* of $\otimes$ over arbitrary, non empty joins. Further, we emphasize that the entire development of the theory of L–fuzzy topological spaces (including convergence) does *not require* the *associativity* of $\otimes$. This insight underlines the importance of *monoidal mean operators* in fuzzy topology (e.g. arithmetic mean in the case of the real unit interval).

The second problem leads to the heart of this paper: We distinguish between openness and the degree of openness. Openness of a given map $g : X \longmapsto L$ is formulated as a crisp, binary topological information, while the degree of openness of g is represented by an element $\lambda \in L$. Both concepts are presented in this paper (cf. Section 2 and 3), and their various interrelationships are investigated extensively in sections 3 – 5 . The crisp notion "openness of g" leads to the category L–**TOP** of L–topological spaces (originally called (L–)fuzzy topological spaces by C.L. Chang [10] and J.A. Goguen [24], while now the notion L–fuzzy topological space is reserved for those topological structures which are capable to express the degree of openness of a given map g). In the case of GL–quantales it is surprising to see that strongly stratified L–topologies and strongly enriched L–fuzzy topologies come to the same thing (cf. Main Result II in Section 5). Moreover, in the general case of *not completely distributive* lattices we give for the first time a complete characterization of L–topological (resp. L–fuzzy topological) spaces by L–neighborhood (resp. L–fuzzy neighborhood) systems. This situation leads to a new convergence theory which intrinsically is based on the identification of "topologies" with "neighborhood systems" (cf. Sections 6, 8, 9). In the context of GL–monoids with square roots we obtain a new *Tychonoff–Theorem* for stratified, L–topological spaces.

The third problem of changing the underlying basis L is a long standing open problem. The work of B. Hutton [47] and S.E. Rodabaugh [92, 93] can be viewed as one possible solution which incorporates the effect of change of basis into the structure of continuous morphisms. In this paper the change of basis is given functorially and relies on certain *Galois connections* between the underlying lattices. In this context it is interesting to see that the axioms of L–fuzzy topological spaces are formulated in such a way that their *compatibility* with Galois connections is immediately apparent – i.e. having some Galois connection in mind the formations of infima or suprema appear in the right places.

In view of the fourth problem we present three fundametally different groups of *non trivial examples of L–topological spaces*: The first group of examples deals with the relationship between L–topologies, frames and the Booleanization of frames. (cf. Subsection 7.1). In particular we discuss the role of the *hypergraph functor* and show in the case of spatial locales L that L–topologies on X can be identified with ordinary topologies on $X \times pt(L)$ (cf. 7.1.5). The second group of examples uses complete MV–algebras (with square roots) as underlying lattice structure and explores the possibility of applying $[0,1]$–topologies to probability theory (cf. Subsections 7.2 and 7.3). A careful study of rigid, stratified L–topologies shows that the space of finitely additive probability measures on X is the *Čech–Stone compactification* of the discrete space X (cf. 7.3.2, 7.3.11). Finally the third group of examples is based on lower semicontinuous, lattice–valued maps which are intrinsically related to the concept of changing the underlying basis. It is interesting to see the way how L–topologies and M–topologies are related to each other– e.g. if $(L, \leq)$ is a complete Heyting algebra and $(\mathbb{B}(L), \leq)$ its Booleanization, then every stratified L–topology induces a *Boolean valued* stratified $\mathbb{B}(L)$–topology. Moreover, stratified and *co–stratified* L–topologies can be identified with their *Boolean valued* counterparts. A further application of lower semicontinuous maps appears in a special *relationship* linking general topology (or more general probabilistic topologies) with fuzzy topology (cf. Subsection 7.4).

Before we pass to comments and remarks addressed to each section separately, we would like to draw the reader's attention to the fact that all categories appearing in this paper are **topological** over **SET**. Therefore *fuzzy topology is* a part of *categorical topology.* Moreover, the whole body of the presented theory of fuzzy topological spaces is based at least on non–spatial locales and *a fortiori* on not completely distributive, complete lattices[1].

The first section presents the necessary lattice–theoretic foundations of the theory of fuzzy topological spaces. Here we compile some information on quantales with square roots, left–continuous t–norms, complete MV–algebras and the role of their idempotent elements. For the sake of completeness we also include the construction of the Booleanization of distributive lattices.

In sections 2 and 3 we lay down the fundamentals of the theory of L–fuzzy topological and L–topological spaces. We investigate the change of the underlying basis L and certain functorial relationships between L–topological and L–fuzzy topological spaces. Subsequently, we study various topological subcategories of fuzzy topological structures. We introduce strongly enriched (resp. strongly stratified), enriched (resp. stratified) and weakly enriched (resp. weakly stratified) L–fuzzy topological (resp. L–topological) spaces which require certain enrichments of the structure of the underlying lattice (cf. Sections 4 and 5).

Section 6 can be viewed as a *first* test demonstrating that the axioms of fixed-basis topology do really work. We develop an appropriate L–filter theory and show that there exists a *non trivial convergence theory* for L–topological spaces which is extremely beautiful on one hand and has non trivial applications on the other hand – e.g. every Hausdorff separated, S–compact, rigid, stratified, L–topological space is weakly star–regular (cf. 7.3.9), or pointwise π–almost everywhere converegnce is $\mathbb{A}$–topological (cf. 7.1.4) .

The convergence theory for L–fuzzy topological spaces is more complicated than the corresponding one for L–topological spaces (cf. Section 8 and 9). First of all we have to understand the *type* of fuzziness represented by openness operators $\mathcal{T} : L^X \longmapsto L$. In Section 8 we view $\mathcal{T}$ as a characteristic morphism on the "powerset" L^X and develop a convergence theory based on L–fuzzy

[1] With regard to the role of completely distributive lattices the reader is referred to Remark 7.1.5.

neighborhood systems which are generated by $\mathcal{T}$ by means of the corresponding L–fuzzy interior operator. In Section 9 we decide to view $\mathcal{T}$ as a ***generalized characteristic function***, and we construct its corresponding ***separated subpresheaf*** (cf. Remark 9.1.3, see also Section 6 in [40]). Starting from this local (i.e. sheaf–theoretic) understanding of lattice–valued maps, we develop a local convergence theory for weakly extensional L–fuzzy topological spaces which is not merely a level–wise repetition of convergence in L–topological spaces.

We close this paper with some brief comments on historical aspects related to this work (cf. Section 10).

Summing up the outline of this paper is as follows:

§1 Lattice–theortic foundations.
§1.1 Quantales.
§1.2 Enriched cqm–lattices.
§1.3 Complete MV–algebras with square roots and idempotent hulls.
§1.4 Booleanization of bounded distributive lattices.
§2. L–Fuzzy topological spaces.
§3. L–Topological spaces.
§4. Coreflective subcategories of L–**FTOP**.
§4.1 Extensional, L–fuzzy topologies.
§4.2 Enriched, L–fuzzy topologies.
§4.3 Strongly enriched, L–fuzzy topologies.
§5. Coreflective subcategories of L–**TOP**.
§5.1 Stratified L–topologies.
§5.2 Strongly stratified L–topologies.
§6. Convergence theory for L–topological spaces and its applications.
§6.1 L–Interior operators and L–neighborhood systems.
§6.2 L–Filter theory.
§6.3 The principle of L–continuous extension.
§6.4 Compactness and stratified L–topological spaces.
§6.5 A level–wise characterization of L–neighborhood axioms.
§7. Examples of L–topological spaces.
§7.1 Case of complete Heyting algebras.
§7.2 Case of complete MV–algebras.
§7.3 Case of complete MV–algebras with square roots and rigid L–topologies.
§7.4 Lower semicontinuous, lattice–valued maps.
§8. Convergence theory for L–fuzzy topological spaces.
§8.1 L–Fuzzy interior operators and L–fuzzy neighborhood systems.
§8.2 Convergence in L–fuzzy topological spaces.
§9. Local convergence theory for weakly extensional L–fuzzy topological spaces.
§9.1 Extensionality and singletons in L^X.
§9.2 A representation theory for weakly extensional L–fuzzy topologies.
§9.3 Local L–interior operators.
§9.4 Filter theory on local elements of L^X.
§9.5 Hausdorff separation axiom for weakly extensional, L–fuzzy topological spaces.
§10 Historical comments.

1 Lattice-theoretic foundations.

In this section we present the algebraic and lattice–theoretic foundations required by the axioms of fixed-basis fuzzy topologies. Familiarity with [6] and [86] will be helpful.

A *cqm–lattice* (short for complete quasi–monoidal lattice) is a triple $(L, \leq, \otimes)$ provided with the following properties

(I) $(L, \leq)$ is a complete lattice where $\top$ (resp. $\bot$) denotes the universal upper (resp. lower) bound.

(II) $(L, \leq, \otimes)$ is a partially ordered groupoid (cf. [6]) –i.e. $\otimes$ is a binary operation on L satisfying the *isotonicity* axiom
$$\alpha_1 \otimes \beta_1 \quad \leq \quad \alpha_2 \otimes \beta_2 \quad , \quad \text{whenever } \alpha_1 \leq \alpha_2 \text{ , } \beta_1 \leq \beta_2 \, .$$

(III) $\alpha \quad \leq \quad \alpha \otimes \top \quad , \quad \alpha \quad \leq \quad \top \otimes \alpha \qquad \forall \alpha \in L \, .$

Because of (III) the universal upper bound $\top$ is always idempotent w.r.t. $\otimes$ in any *cqm*–lattice. Further, every complete lattice $(L, \leq)$ carries an intrinsic structure of a *cqm*–lattice determined by the binary meet operation $\wedge$. As a matter of fact $(L, \leq, \wedge)$ is always a *cqm*–lattice.

On partially ordered groupoids there exist at least two fundamental different, but important conditions forcing Axiom (III):

Remark 1.1 Let $(L, \leq)$ be a complete lattice and $(L, \leq, \otimes)$ be a partially ordered groupoid.
(a) Let us assume that $(L, \leq, \otimes)$ has a *unit* (i.e. there exists $1 \in L$ with $\alpha \otimes 1 = 1 \otimes \alpha = \alpha$). Then Axiom (III) follows from the existence of the unit and the isotonicity axiom (II). In particular $(L, \leq, \otimes)$ is a *cqm*–lattice.
(b) If $\otimes$ is *idempotent* (i.e. $\alpha \otimes \alpha = \alpha$), then we again can invoke the isotonicity axiom and verify (III). Hence the idempotency is another sufficient condition guaranteeing that $(L, \leq, \otimes)$ is a *cqm*–lattice.

■

The category **CQML** of *cqm*–lattices consists of the following data:
Objects are *cqm*–lattices, and Morphisms $\varphi : (L_1, \leq_1, \otimes_1) \longmapsto (L_2, \leq_2, \otimes_2)$ are mappings $\varphi : L_1 \longmapsto L_2$ provided with the properties[2]

(M1) φ preserves arbitrary joins.

(M2) $\varphi(\alpha \otimes_1 \beta) = \varphi(\alpha) \otimes_2 \varphi(\beta)$.

(M3) φ preserves universal upper bounds - i.e. $\varphi(\top) = \top$.

The composition is the usual composition of maps, and the identity of $(L, \leq, \otimes)$ is the identity map of L. It is not difficult to show that **CQML** is finitely complete.

A partially ordered groupoid $(L, \leq, \otimes)$ is called a *cl*–groupoid (short for completely lattice–ordered groupoid (cf. [6])) iff $\otimes$ is distributive over *non empty* joins – i.e.

[2] Because of (M1) and (M3) the universal bounds are fixpoints of φ.

(IV) $$(\bigvee_{i\in I} \alpha_i) \otimes \beta \;=\; \bigvee_{i\in I} (\alpha_i \otimes \beta) \quad , \quad \beta \otimes (\bigvee_{i\in I} \alpha_i) \;=\; \bigvee_{i\in I} (\beta \otimes \alpha_i) \, .$$

A *cl*–quasi–monoid (short for completely lattice–ordered quasi–monoid) is a *cqm*–lattice which is also a *cl*–groupoid.

1.1 Quantales.

A triple $(L, \leq, \&)$ is called a *quantale* (cf. [80, 86]) iff $(L, \leq)$ is a complete lattice and

(V) $(L, \&)$ is a semigroup,

(VI) $\&$ is distributive over arbitrary joins –i.e.

$$(\bigvee_{i\in I} \alpha_i) \,\&\, \beta \;=\; \bigvee_{i\in I} (\alpha_i \,\&\, \beta) \quad , \quad \beta \,\&\, (\bigvee_{i\in I} \alpha_i) \;=\; \bigvee_{i\in I} (\beta \,\&\, \alpha_i) \; ,$$

where $\bigvee$ denotes the formation of arbitrary joins in $(L, \leq)$. Obviously the universal lower bound $\perp$ (viewed as the join of the empty set) is the zero element w.r.t. $\&$. Further every quantale is left– and right–residuated – i.e. there exist binary operations $\rightarrow_\ell$ and $\rightarrow_r$ on L satisfying the following axioms

$$\alpha \,\&\, \beta \;\leq\; \gamma \iff \beta \;\leq\; \alpha \rightarrow_r \gamma \quad , \quad \beta \,\&\, \alpha \;\leq\; \gamma \iff \beta \;\leq\; \alpha \rightarrow_\ell \gamma \, .$$

In particular $\rightarrow_r$ and $\rightarrow_\ell$ are determined by

$$\alpha \rightarrow_r \gamma \;=\; \bigvee\{\lambda \in L \mid \alpha \,\&\, \lambda \;\leq\; \gamma\}, \;\; \alpha \rightarrow_\ell \gamma \;=\; \bigvee\{\lambda \in L \mid \lambda \,\&\, \alpha \;\leq\; \gamma\}.$$

A quantale $(L, \leq, \&)$ is *commutative* iff the underlying semigroup $(L, \&)$ is commutative. In general quantales do not have units; therefore the following terminology is important: An element α is *right* (resp. *left*) *–sided* iff $\alpha \,\&\, \top \;\leq\; \alpha$ (resp. $\top \,\&\, \alpha \;\leq\; \alpha$). An element α is *strictly right* (resp. *left*) *–sided* iff $\alpha \,\&\, \top \;=\; \alpha$, (resp. $\top \,\&\, \alpha \;=\; \alpha$). An element α is *strictly two–sided* iff α is strictly right– *and* left–sided. A quantale is said to be *right* (resp. *left*) *–sided* iff every element $\alpha \in L$ is right (resp. left) –sided. A quantale is *strictly right* (resp. *left*) –sided iff every element in L is strictly right (resp. left) –sided. The universal upper bound $\top$ is the unit of $(L, \&)$ iff $(L, \leq, \&)$ is *strictly two–sided* – i.e. strictly right– and left–sided. A quantale is *right* (resp. *left*) *–divisible* iff the relation $\beta \;\leq\; \alpha$ is equivalent to the existence of an element $\gamma \in L$ satisfying the relation $\beta \;=\; \alpha \,\&\, \gamma$ (resp. $\beta \;=\; \gamma \,\&\, \alpha$). Obviously a right–sided (resp. left–sided) quantale is right (resp. left)–divisible iff for every inequality $\beta \;\leq\; \alpha$ there exists $\gamma \in L$ s.t. $\beta \;=\; \alpha \,\&\, \gamma$ (resp. $\beta \;=\; \gamma \,\&\, \alpha$). Further we note that in every right-divisible and right–sided quantale the meet operation can be expressed by the algebraic operations $\&$ and $\rightarrow_r$ on L

$$\alpha \wedge \beta \quad = \quad \alpha \,\&\, (\alpha \rightarrow_r \beta) \quad , \qquad \alpha, \beta \in L \; .$$

The analogous statement holds for left–divisible and left–sided quantales. A quantale $(L, \leq, \&)$ is called a *right Gelfand quantale* iff $(L, \leq, \&)$ is right–sided and $(L, \&)$ is an idempotent semigroup (cf. [80]). As an immediate consequence from the idempotency we obtain that every Gelfand quantale is strictly right–sided and *right–symmetric* – i.e.

$$\alpha \,\&\, (\beta \,\&\, \gamma) \quad = \quad \alpha \,\&\, (\gamma \,\&\, \beta) \quad \forall\, \alpha, \beta, \gamma \in L \qquad \text{(Right–Symmetry)}\ .$$

Moreover, it is not difficult to see that a Gelfand quantale $(L, \leq, \&)$ is commutative iff the universal upper bound $\top$ is the unit w.r.t. $\&$ iff $\& = \wedge$. Hence commutative Gelfand quantales and *complete Heyting algebras* ([49]) are the same things.

Example 1.1.1 (a) Let $(X, \mathcal{O})$ be an ordinary topological space. Then the lattice $(\mathcal{O}, \subseteq)$ of all *open* ordinary subsets of X forms a complete Heyting algebra.
(b) Let A be a *non–commutative* C^*–algebra, $R(A)$ be the lattice of all right–sided ideals of A, and let $\&$ be the usual multiplication of right–sided ideals. Then $(R(A), \subseteq, \&)$ is a right Gelfand quantale which is not a complete Heyting algebra (cf. [80]).
■

Let $Q = (L, \leq, \&)$ be a strictly right–sided, right–symmetric quantale; then the subquantale $S(Q) = \{\top \,\&\, \alpha \mid \alpha \in L\}$ of all strictly two–sided elements of Q is necessarily a commutative monoid. Further the *canonical quantale–homomorphism* j from Q onto $S(Q)$ is determined by:

$$j(\alpha) = \top \,\&\, \alpha \quad \forall\, \alpha \in L \quad .$$

Lemma 1.1.2 *In any strictly right–sided, right–symmetric quantale the following relations hold for all* $\alpha, \beta \in L$

(i) $\quad \alpha \rightarrow_\ell \beta \;=\; j(\alpha) \rightarrow_\ell \beta\ , \quad j(\alpha) \rightarrow_r j(\beta) \;=\; j(\alpha) \rightarrow_\ell j(\beta)\ .$

(ii) $\quad \alpha \rightarrow_r j(\beta) \;\leq\; j(\alpha) \rightarrow_r j(\beta)\ , \quad j(\alpha \rightarrow_r \beta) \;\leq\; \alpha \rightarrow_r \beta\ .$

(iii) $\quad \beta \rightarrow_\ell (\alpha \rightarrow_r \gamma) \;=\; ((\alpha \,\&\, \beta) \rightarrow_r \gamma\ .$

Proof. The relations follow immediately from the respective definitions.
■

A quantale $(L, \leq, \&)$ is called a *GL–monoid* iff $(L, \leq, \&)$ is a commutative, strictly two–sided, divisible quantale (cf. Section 5 in [39]). Typical examples of GL–monoids are complete Heyting algebras or *complete MV–algebras* (cf. [4]). For simplicity we recall here that complete MV–algebras are GL–monoids in which the law of double negation is valid (cf. Lemma 2.14 in [39]) – i.e.

$$(\alpha \rightarrow \bot) \rightarrow \bot \quad = \quad \alpha \quad , \qquad \forall\, \alpha \in L\ .$$

Example 1.1.3 Let $(G, \leq, +)$ be a (conditionally) complete ℓ–group – i.e. $(G, \leq, +)$ is a partially ordered commutative group in which for every bounded subset the supremum and infimum exist (cf. [6]). Further let u be an element of the positive cone $G^+ \setminus \{0\}$ of G. Then $L = \{g \in G \mid 0 \leq g \leq u\}$ is a complete lattice w.r.t. $\leq$. On L we consider the following binary operation defined by

$$g_1 * g_2 \quad = \quad (g_1 + g_2 - u) \vee 0 \qquad \forall g_1, g_2 \in L\,.$$

It is not difficult to verify that $(L, \leq, *)$ is a complete MV–algebra.
■

Further examples of GL–monoids are given by continuous semigroup structures on the real unit interval $[0,1]$ satisfying the following boundary conditions

$$\alpha \,\&\, 1 \;=\; 1 \,\&\, \alpha \;=\; 1 \quad , \quad \alpha \,\&\, 0 \;=\; 0 \,\&\, \alpha \;=\; 0\,.$$

In the context of probabilistic metric spaces continuous semigroup operations on $[0,1]$ satisfying the previous condition are also called continuous t-norms ([99]).

After these preliminaries we are in the position to define the structure of a right GL–quantale in the following way: A *right GL–quantale* is a strictly right–sided and right–symmetric quantale $Q = (L, \leq, \&)$ satisfying the additional axioms:

(GL1) $\quad (\alpha \rightarrow_\ell \alpha) \,\&\, \alpha \;=\; \alpha \qquad \forall \alpha \in L.$

(GL2) $\quad$ The subquantale $S(Q) = \{\top \,\&\, \alpha \mid \alpha \in L\}$ of all strictly two–sided elements of L is a GL–monoid (w.r.t. $\&$).

Obviously every GL–monoid is a right GL–quantale. Moreover, every right Gelfand quantale is a right GL–monoid. Indeed, the idempotency of $\&$ implies (GL1), and the fact that all two–sided elements of a right Gelfand quantale form a complete Heyting algebra (cf. [80]) guarantees the validity of (GL2).

Example 1.1.4 Let $M = (L, \leq, *)$ be a GL–monoid and U be a GL–submonoid of M – i.e. U is a submonoid of $(L, *)$ and U is closed w.r.t. arbitrary joins and meets performed in $(L, \leq)$. Then U is again a GL–monoid, and the residuation in U coincides with the restriction of the residuation in L. Moreover U induces a self–mapping $\bar{\ } : L \longmapsto L$ by

$$\overline{\alpha} \quad = \quad \bigwedge \{\lambda \in U \mid \alpha \,\leq\, \lambda\} \qquad\qquad \forall \alpha \in L\,.$$

Evidently, $\bar{\ }$ is a closure operator on L satisfying the following properties:

$$\overline{\alpha * \overline{\beta}} \;=\; \overline{\alpha} * \overline{\beta} \quad , \qquad \overline{\alpha \wedge \overline{\beta}} \quad = \quad \overline{\alpha} \wedge \overline{\beta}\,.$$

Indeed, we infer from

$$\alpha \quad \leq \quad \overline{\beta} \rightarrow (\alpha * \overline{\beta}) \quad \leq \quad \overline{\beta} \rightarrow (\overline{\alpha * \overline{\beta}})$$

that $\overline{\alpha} \le \overline{\beta} \to \overline{(\alpha * \overline{\beta})}$ holds; hence $\overline{\alpha} * \overline{\beta} \le \overline{\alpha * \overline{\beta}}$ follows. Further we obtain from the divisibility of M

$$\overline{\alpha} \wedge \overline{\beta} \quad \le \quad \overline{\alpha} * (\overline{\alpha} \to \overline{\beta}) \quad \le \quad \overline{\alpha * (\overline{\alpha} \to \overline{\beta})} \quad \le \quad \overline{\alpha \wedge \overline{\beta}} \, .$$

Now we are in the position to define an associative, binary operation on L as follows:

$$\alpha \,\&\, \beta \quad = \quad \alpha * \overline{\beta} \qquad\qquad \forall \alpha, \beta \in L \, .$$

Then it is not difficult to show that $\&$ induces the structure of a right GL–quantale on $(L, \le)$. In particular, the GL–quantale $Q = (L, \le, \&)$ fulfills the additional property

$$\text{(GL3)} \qquad \top \,\&\, (\alpha \wedge (\top \,\&\, \beta)) \quad = \quad (\top \,\&\, \alpha) \wedge (\top \,\&\, \beta) \, .$$

Finally, Q is non–commutative if and only if $U \neq L$. In this case there exists obviously an element $\alpha \in L$ with $\top \,\&\, \alpha \neq \alpha \,\&\, \top$.

■

Proposition 1.1.5 *Let Q be a right GL–quantale. Then the relation $\alpha \le \top \,\&\, \alpha$ holds for all $\alpha \in L$. Moreover, Q is* quasi–left–divisible – *i.e. for every pair $(\alpha, \beta) \in L \times L$ with $\beta \le \top \,\&\, \alpha$ there exists $\gamma \in L$ such that $\gamma \,\&\, \alpha = \beta$.*

Proof. Because of $\alpha \to_\ell \alpha \le \top$ the first assertion follows immediately from (GL1). In particular the canonical quantale homomorphism j is a closure operator on $(L, \le)$. Referring to Lemma 1.1.2 we observe that the residuation in the underlying GL–monoid $S(Q)$ of two–sided elements coincides with the restictions of $\to_r$ or $\to_\ell$. In the case of $\beta \le j(\alpha)$ we now apply Axiom (GL1) and the divisibility of $S(Q)$:

$$\beta \;=\; (\beta \to_\ell \beta) \,\&\, j(\beta) \;=\; (\beta \to_\ell \beta) \,\&\, (j(\alpha) \to_\ell j(\beta)) \,\&\, \top \,\&\, \alpha \, .$$

Hence the second assertion is established.

■

Corollary 1.1.6 *In any right GL–quantale Q the following relations hold:*

(i) $(\alpha \to_\ell \beta) \,\&\, \alpha \;=\; \beta$ *whenever* $\beta \le \top \,\&\, \alpha$.

(ii) $\gamma \,\&\, (\alpha \to_r (\alpha \,\&\, \beta)) \;=\; \gamma \,\&\, \beta$ *whenever* $\gamma \le \top \,\&\, \alpha$.

(iii) $\alpha \to_r (\top \,\&\, \beta) \;=\; (\top \,\&\, \alpha) \to_r (\top \,\&\, \beta)$.

(iv) $\alpha \to_r \beta \;=\; \top \,\&\, (\alpha \to_r \beta)$.

Proof. The relation (i) is equivalent to the quasi–left–divisibility of Q which is already verified in Proposition 1.1.5. Further the relation (ii) is an immediate consequence from the quasi–left–divisibility of Q. Finally (iii) and (iv) follow directly from Lemma 1.1.2 and Proposition 1.1.5.

■

1.2 Enriched *cqm*–lattices.

Sometimes we enrich *cqm*–lattices $(L, \leq, \otimes)$ by an additional, partially ordered, algebraic structure determined by quantales. For this purpose we fix a complete lattice $(L, \leq)$ and consider two binary operations $\otimes$ and $*$ on L such that the following conditions hold:

(CQM) $(L, \leq, \otimes)$ is a *cqm*–lattice.

(Q) $(L, \leq, *)$ is a quantale.

(VII) $*$ is *dominated* by $\otimes$ – i.e. for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in L$ the inequality

$$(\alpha_1 \otimes \beta_1) * (\alpha_2 \otimes \beta_2) \quad \leq \quad (\alpha_1 * \alpha_2) \otimes (\beta_1 * \beta_2) \quad \text{(Compatibility)}$$

holds.

Lemma 1.2.1 *Let $(L, \leq, *)$ be a quantale. Then the quadruple $(L \leq, \wedge, *)$ satisfies the axioms (I)–(III) and (V)–(VII). Moreover, if $(L, \leq, *)$ is a right–symmetric quantale satisfying the additional property*

$$\alpha \quad \leq \quad \alpha * \top \quad , \quad \alpha \quad \leq \quad \top * \alpha \qquad \forall \alpha \in L$$

then $(L, \leq, *, *)$ fulfills (I)–(VII).
■

Referring to a result due to L. Fuchs (cf. [18]) it is easy to see that the underlying lattice $(L, \leq)$ of a given quantale is *not* necessarily *distributive.* In this context Lemma 1.2.1 means that in the absence of distributivity w.r.t. $\wedge$ there exists a semigroup operation $*$ such that at least Axiom (VI) holds.

In the following consideration we describe a class of quadruples $(L, \leq, \otimes, *)$ satisfying (I)–(III) and (V)–(VII) which is not covered by Lemma 1.2.1. A natural source for this type of examples are quantales with square roots. We start with a simple definition.

A quantale $(L, \leq, *)$ has *square roots* (cf. [39]) iff there exists a unary operator $S : L \longmapsto L$ provided with the properties

(S1) $S(\alpha) * S(\alpha) \quad = \quad \alpha \qquad \forall \alpha \in L.$

(S2) $\beta * \beta \quad \leq \quad \alpha$ implies $\beta \leq S(\alpha)$.

Since the formation of square roots (i.e. the unary operator S) is uniquely determined by (S1) and (S2) we write also $\alpha^{1/2}$ instead of $S(\alpha)$.

Proposition 1.2.2 (Quantales with Square Roots)
Let $(L, \leq, *)$ be a quantale. Then the following assertions are equivalent

(i) $(L, \leq, *)$ has square roots.

(ii) $(L, \leq, *)$ satisfies the subsequent conditions

(1) $\forall \alpha \in L \ \exists \beta \in L : \ \alpha = \beta * \beta$.

(2) $\alpha * \beta \ \leq \ (\alpha * \alpha) \vee (\beta * \beta)$.

Proof. (a) ((i)$\Longrightarrow$(ii)) Property (1) follows immediately from (S1). Applying (S2) to $\gamma = (\alpha * \alpha) \vee (\beta * \beta)$ we obtain $\alpha \vee \beta \leq \gamma^{1/2}$. Now we take the square w.r.t. $*$ on both sides; hence Property (2) follows from (S1) and (VI).
(b) ((ii)$\Longrightarrow$(i)) We define a map $S : L \longmapsto L$ by

$$S(\alpha) \quad = \quad \bigvee \{ \lambda \in L | \quad \lambda * \lambda \leq \alpha \} \qquad .$$

Obviously S satisfies (S2). In order to verify (S1) we proceed as follows: First we infer from Property (2) and Axiom (VI) : $S(\alpha) * S(\alpha) \leq \alpha$. Then we invoke Property (1) and obtain: $S(\alpha) * S(\alpha) = \alpha$.
■

Example 1.2.3 (Quantales with Square Roots)
(a) Every idempotent quantale $Q = (L, \leq, \&)$ (cf. [86]) has obviously square roots. Namely, the formation of square roots is given by the identity id_L of L.
(b) Any continuous t–norm T (cf. [99]) induces on the real unit interval $[0,1]$ the structure of a (commutative) quantale with square roots. Significant, continuous t–norms are the following ones

$$\begin{array}{rcl} Min(\alpha,\beta) & = & \min(\alpha,\beta) \\ Prod(\alpha,\beta) & = & \alpha \cdot \beta \\ T_m(\alpha,\beta) & = & \max(\alpha + \beta - 1, 0) \end{array} \qquad .$$

The formation of square roots w.r.t. Min is given by the identity map of $[0,1]$, while square roots w.r.t. $Prod$ are the usual ones, and square roots w.r.t. T_m are determined by

$$\alpha^{1/2} \quad = \quad \frac{\alpha + 1}{2} \qquad \text{for all } \alpha \in [0,1] \qquad .$$

In this context we remark that every continuous t–norm can be written as an *ordinal sum* of Min, $Prod$ and T_m (cf. [60, 83]). Further we note that Min and T_m play a special role in the field of many–valued logics: Min is used by Gődel in his [0,1]–valued intuitionistic logic, while T_m is the arithmetic conjuction in Łukasiewicz' [0,1]–valued logic (cf. [17]). In particular, $([0,1], \leq, T_m)$ is a complete MV–algebra with square roots.
(c) The formation of square roots is preserved under arbitray products; i.e. products of quantales with square roots are again quantales with square roots.

■

Lemma 1.2.4 *Let $Q = (L, \leq, *)$ be a quantale with square roots. Further Q satisfies the additional property*

$$\text{(S3)} \quad (\alpha * \beta)^{1/2} \quad = \quad (\alpha^{1/2} * \beta^{1/2}) \vee \perp^{1/2} \qquad \forall \alpha, \beta \in L \qquad .$$

Then the formation of square roots preserves arbitrary, non empty joins – i.e. for any non empty subset $\{\alpha_i \,|\, i \in I\}$ *of* L *the relation*

$$(\bigvee_{i\in I} \alpha_i)^{1/2} \quad = \quad \bigvee_{i\in I} (\alpha_i)^{1/2}$$

holds.

Proof. Let $\mathcal{A} = \{\alpha_i \,|\, i \in I\}$ be a non empty subset of L. Referring to Proposition 1.2.2 we conclude from (VI) and (S1)

$$(\bigvee_{i\in I} (\alpha_i)^{1/2}) * (\bigvee_{i\in I} (\alpha_i)^{1/2}) \quad = \quad \bigvee_{i\in I} \alpha_i \qquad .$$

Now we take on both sides the square roots and apply (S1), (S3) :

$$\bot^{1/2} \vee (\bigvee_{i\in I} (\alpha_i)^{1/2}) \quad = \quad (\bigvee_{i\in I} \alpha_i)^{1/2} \qquad .$$

Since $\mathcal{A}$ is non empty, the inequality $\bot^{1/2} \;\leq\; \bigvee_{i\in I} (\alpha_i)^{1/2}$ holds; hence the assertion follows.

■

Example 1.2.5

(a) Let T be one of the fundamental, continuous t–norms – i.e. $T = Min$ or $T = Prod$ or $T = T_m$ (cf. Example 1.2.3(b)). Then $([0,1], \leq, T)$ is a quantale satisfying Axiom (S3). It is not difficult to see that (S3) is not preserved under ordinal sums (cf. [99]); e.g. let us consider the following continuous t–norm $T_{1/2}$ defined by

$$T_{1/2}(\alpha,\beta) \quad = \quad \begin{cases} \max(\alpha+\beta-1\,,\,1/2) & , \quad 1/2 \leq \alpha\;,\;1/2 \leq \beta \\ \max(\alpha+\beta-1/2\,,\,0) & , \quad \alpha \leq 1/2\;,\;\beta \leq 1/2 \\ \min(\alpha,\beta) & , \quad \text{elsewhere} \end{cases}$$

Then the map $S \,:\, [0,1] \longmapsto [0,1]$ determined by

$$S(\alpha) \quad = \quad \begin{cases} \frac{\alpha+1}{2} & , \quad 1/2 \leq \alpha \leq 1 \\ \frac{\alpha+1/2}{2} & , \quad 0 \leq \alpha \lneq 1/2 \end{cases}$$

coincides with the formation of square roots w.r.t. $T_{1/2}$. In the case of $\alpha = 5/8$ and $\beta = 3/4$ we obtain

$$\max(T_{1/2}(S(5/8), S(3/4))\,,\,1/4) \;\; = 11/16 \neq 3/4 = \quad S(T_{1/2}(5/8,3/4)) \;;$$

hence $([0,1], \leq, T_{1/2})$ does *not* satisfy (S3).

(b) The axiom (S3) is preserved under products of quantales with square roots. In particular, there exist non commutative quantales with square roots satisfying (S3) (cf. (a) and Example 1.2.3).

(c) Every complete MV–algebra with square roots satisfies Axiom (S3) (cf. Proposition 2.17 in [39]).

■

Remark 1.2.6 (Monoidal Mean Operator) Let $Q = (L, \leq, *)$ be a quantale with square roots. Then the *monoidal mean operator* $\circledast$ on L is given by

$$\alpha \circledast \beta \quad = \quad (\alpha^{1/2}) * (\beta^{1/2}) \qquad \forall\, \alpha, \beta \in L \quad .$$

If Q satisfies Axiom (S3), then Lemma 1.2.4 and Axiom (VI) imply that $(L, \leq, \circledast)$ is an *idempotent cl*–quasi–monoid. Moreover, if Q is a commutative quantale, then $(L, \leq, \circledast, *)$ fulfills the axioms (I)–(III) and (V)–(VII).

■

Motivated by the previous remark we ask the **question**: Let us assume that $\otimes$ is given by the monoidal mean operator. Does Axiom (VII) force the commutativity of the underlying quantale?

First we give an axiomatization of the monoidal mean operator:

Proposition 1.2.7 *Let $(L, \leq, \otimes, *)$ be a quadruple satisfying the axioms (I)–(III),(V)–(VII) and the additional axioms*

(VIII) $\quad (\top \otimes \alpha) * (\top \otimes \beta) \;=\; \alpha \otimes \beta$

(IX) $\quad (\top \otimes \alpha) * \top \;=\; \top \otimes \alpha$

(X) $\quad \alpha \otimes \alpha \;=\; \alpha$ (Idempotency)

Then $Q = (L, \leq, *)$ is a commutative, *strictly two–sided quantale. Furthermore Q has square roots and the monoidal mean operator $\circledast$ coincides with $\otimes$.*

Proof. (a) Let us consider an element $\alpha \in L$. Then (III), (VIII) and (IX) imply that $\top \otimes \alpha$ is strictly two–sided for all $\alpha \in L$. Further the relation $(\top \otimes \alpha) * (\top \otimes \alpha) = \alpha$ follows from (VIII) and (X); hence every element $\alpha \in L$ is strictly two–sided. In order to verify the commutativity of $*$ we proceed as follows: Because of (III), (VIII) and (IX) the relation

$$\alpha \otimes \top \quad = \quad (\top \otimes \alpha) * (\top \otimes \top) \quad = \quad \top \otimes \alpha$$

holds. Now we invoke (VII), (VIII), (X) and obtain:

$$\begin{aligned}
\alpha * \beta \quad &= \quad (\top \otimes \alpha) * (\top \otimes \alpha) * (\top \otimes \beta) * (\top \otimes \beta) \\
&= \quad (\top \otimes \alpha) * (\top \otimes \alpha) * (\beta \otimes \top) * (\top \otimes \beta) \\
&\leq \quad (\top \otimes \alpha) * (\beta \otimes \alpha) * (\top \otimes \beta) \\
&= \quad (\top \otimes \alpha) * (\top \otimes \beta) * (\top \otimes \alpha) * (\top \otimes \beta) \\
&\leq \quad (\beta \otimes \alpha) * (\beta \otimes \alpha) \\
&= \quad (\top \otimes \beta) * (\top \otimes \alpha) * (\top \otimes \beta) * (\top \otimes \alpha) \\
&\leq \quad (\top \otimes \beta) * (\beta \otimes \alpha) * (\top \otimes \alpha) \\
&= \quad (\top \otimes \beta) * (\top \otimes \beta) * (\top \otimes \alpha) * (\top \otimes \alpha) \\
&= \quad \beta * \alpha \qquad .
\end{aligned}$$

(b) In view of (VIII) it is sufficient to show that $\top \otimes \alpha$ is the square root of α w.r.t. $*$. The axiom (S1) follows immediately from (VIII) and (X) (cf. (a)). In order to verify (S2) we assume $\beta * \beta \leq \alpha$. Referring again to (VII) and (VIII) we obtain:

$$\beta \quad = \quad (\top \otimes \beta) * (\top \otimes \beta) \quad \leq \quad (\top * \top) \otimes (\beta * \beta) \quad \leq \quad \top \otimes \alpha \quad ;$$

hence the second assertion is verified.

■

Corollary 1.2.8 *Let $Q = (L, \leq, *)$ be a strictly two–sided quantale having square roots. Further let $\circledast$ be the monoidal mean operator on L w.r.t. $*$. Then the following assertions are equivalent:*

(i) Q is commutative.

(ii) The quadruple $(L, \leq, \circledast, *)$ satisfies Axiom (VII).

Proof. (a) Let us assume that Q is commutative. Then the inequality $\alpha^{1/2} * \beta^{1/2} \leq (\alpha * \beta)^{1/2}$ holds for all $\alpha, \beta \in L$. Hence Axiom (VII) follows from the commutativity of $*$.
(b) Because of $\alpha^{1/2} = \top \circledast \alpha$ the implication (ii)$\Longrightarrow$(i) is an immediate consequence from Proposition 1.2.7.

■

Referring to the previous corollary it is evident that we cannot omit the assumption of the commutativity of the underlying quantale in Remark 1.2.6.

Proposition 1.2.9 *Let $(L, \leq, *)$ be a strictly two–sided commutative quantale with square roots satisfying the axiom (S3). Further let ι_0 be the element of L determined by: $\iota_0 = (\bot^{1/2} \to \bot) * (\bot^{1/2} \to \bot)$. Then the following properties are valid*

(i) $(\bot^{1/2} \to \bot) \to \bot \quad = \quad \bot^{1/2}$.

(ii) $\iota_0 \to \bot \quad = \quad (\iota_0 \to \bot)^{1/2}$. In particular, $\iota_0 \to \bot$ is idempotent w.r.t. $*$.

Moreover, if $(L, \leq, *)$ satisfies the divisibility law (i.e. $(L, \leq, *)$ is a GL–monoid), then the following additional property holds

(iii) $\iota_0 \quad = \quad (\iota_0 \to \bot) \to \bot$.

Proof. The inequality $\alpha \leq (\alpha \to \bot) \to \bot$ holds for all $\alpha \in L$. In order to show the converse inequality in (i) we proceed as follows:
Because of $\bot^{1/2} \leq \bot^{1/2} \to \bot$ we obtain:

$$(\bot^{1/2} \to \bot) \to \bot \quad \leq \quad \bot^{1/2} \to \bot \quad ;$$

hence the relation

$$\begin{aligned} ((\bot^{1/2} \to \bot) \to \bot) * ((\bot^{1/2} \to \bot) \to \bot) \quad &\leq \\ \leq \quad (\bot^{1/2} \to \bot) * ((\bot^{1/2} \to \bot) \to \bot) \quad &= \quad \bot \end{aligned}$$

follows; i.e. $(\bot^{1/2} \to \bot) \to \bot \ \leq \ \bot^{1/2}$.
Since the formation of square roots preserves the "implication" (cf. Proposition 2.11 in [39]), we obtain from Assertion (i) and Axiom (S3):

$$\begin{aligned}[((\bot^{1/2} \to \bot) * (\bot^{1/2} \to \bot)) \to \bot]^{1/2} &= (\bot^{1/2} \to \bot) \to \bot^{1/2} \\ &= (\bot^{1/2} \to \bot) \to ((\bot^{1/2} \to \bot) \to \bot) \\ &= ((\bot^{1/2} \to \bot) * (\bot^{1/2} \to \bot)) \to \bot \, ;\end{aligned}$$

hence the assertion (ii) follows.
Referring again to Assertion (i) we derive the following relation from (S3) and the divisibility of $(L, \leq *)$:

$$\begin{aligned}\bot^{1/2} \to \bot &= [(\bot^{1/2} \to \bot) * ((\bot^{1/2} \to \bot) \to \bot^{1/2})] \to \bot \\ &= ((\bot^{1/2} \to \bot) \to \bot^{1/2}) \to ((\bot^{1/2} \to \bot) \to \bot) \\ &= [((\bot^{1/2} \to \bot) * (\bot^{1/2} \to \bot)) \to \bot]^{1/2} \to \bot^{1/2} \\ &= [(((\bot^{1/2} \to \bot) * (\bot^{1/2} \to \bot)) \to \bot) \to \bot]^{1/2} .\end{aligned}$$

Taking on both sides the squares then the assertion (iii) follows.
■

Definition 1.2.10 (Smoothness and strictness)
Let $(L, \leq, *)$ be a GL–monoid with square roots. $(L, \leq, *)$ is called *smooth* iff $\bot^{1/2} = \bot$. $(L, \leq, *)$ is said to be *strict* iff $\bot^{1/2} \to \bot = \bot^{1/2}$.
■

It is easy to see that the real unit interval provided with the usual multiplication *Prod* is a smooth GL–monoid (cf. Example 1.2.3). Further every complete Heyting algebra is a smooth GL–monoid.

Proposition 1.2.11 *Let $(L, \leq, *)$ be a smooth GL–monoid satisfying (S3). Further let ι be an idempotent element in L w.r.t. $*$. Then the following relations are valid:*

(i) $\quad \iota^{1/2} = \iota$.

(ii) $\quad \iota \to \bot$ is again idempotent w.r.t. $*$.

Proof. The assertion (i) follows from Axiom (S3) and the smoothness of $(L, \leq, *)$. Since the formation of square roots preserves the "implication", we conclude from assertion (i) and the smoothness of $(L, \leq, *)$:

$$(\iota \to \bot)^{1/2} \quad = \quad \iota^{1/2} \to \bot^{1/2} \quad = \quad \iota \to \bot \quad ;$$

hence the idempotency of $\iota \to \bot$ is verified.
■

Proposition 1.2.12 *Let $(L, \leq, *)$ be a strict GL–monoid. Then for all $\alpha \in L$ the following relation holds:*

$$(\alpha^{1/2} \to (\alpha^{1/2} * \bot^{1/2})) * (\alpha^{1/2} \to (\alpha^{1/2} * \bot^{1/2})) \quad = \quad \bot \quad .$$

Proof. By virtue of the divisibility of $(L, \leq, *)$ we obtain:

$$\begin{aligned} &\bot^{1/2} * (\alpha^{1/2} \to (\alpha^{1/2} * \bot^{1/2})) &&= \\ &(\alpha^{1/2} \to \bot^{1/2}) * \alpha^{1/2} * (\alpha^{1/2} \to (\alpha^{1/2} * \bot^{1/2})) &&= \\ &(\alpha^{1/2} \to \bot^{1/2}) * \alpha^{1/2} * \bot^{1/2} \quad = \quad \bot^{1/2} * \bot^{1/2} &&= \quad \bot \quad ; \end{aligned}$$

i.e. $(\alpha^{1/2} \to (\alpha^{1/2} * \bot^{1/2})) \leq \bot^{1/2} \to \bot$. Hence the assertion follows from the strictness of $(L, \leq, *)$.
∎

Under a mild hypothesis we can give a classification of GL–monoids with square roots as follows:

Theorem 1.2.13 (Classification of GL–monoids)
Let $(L, \leq, *)$ be a GL–monoids with square roots satisfying the additional condition

$$((\bot^{1/2} \to \bot) * (\bot^{1/2} \to \bot)) \vee (((\bot^{1/2} \to \bot) * (\bot^{1/2} \to \bot)) \to \bot) \quad = \quad \top .$$

Then $(L, \leq, *)$ is either a smooth GL–monoid or a strict GL–monoid or is isomorphic to a product of a smooth GL–monoid and a strict GL–monoid. Moreover, the factors of this product are uniquely deterimed up to an isomorphism.

Proof. We put $\iota_0 = (\bot^{1/2} \to \bot) * (\bot^{1/2} \to \bot)$. Since $\iota_0 \to \bot$ is idempotent (cf. Proposition 1.2.9 (ii)), we conclude from the hypothesis that ι_0 is also idempotent. Therefore we are in the position to repeat the arguments of Corollary 2.18, Remark 2.20 and Theorem 2.21 in [39] verbatim.
∎

The hypothesis of the previous theorem is satisfied by any GL–monoid with square roots fulfilling the *algebraic strong de Morgan law* (cf. Proposition 2.3 in [39]) – i.e.:

$$(\alpha \to \beta) \vee (\beta \to \alpha) \quad = \quad \top \qquad \forall \alpha, \beta \in L \quad .$$

At this moment we do not have any example of a strict GL–monoid which is not an MV–algebra. The following proposition throws some light on this problem:

Proposition 1.2.14 *Let $(L, \leq, *)$ be a strict GL–monoid. Then the following assertions are equivalent:*

(i) $(\alpha \to (\alpha * \beta)) \quad = \quad (\alpha \to \bot) \vee \beta \quad \forall \alpha, \beta \in L$.

(ii) $\alpha \quad = \quad (\bot^{1/2} \to (\bot^{1/2} * \alpha^{1/2})) * (\bot^{1/2} \to (\bot^{1/2} * \alpha^{1/2}))$.

(iii) $(L, \leq, *)$ is an MV–algebra.

Proof. The implication (i) $\Longrightarrow$ (ii) follows from Proposition 1.2.2 and the strictness of $(L, \leq, *)$. Since the formation of square roots preserves the "implication", we conclude from the divisibility law:

$$\begin{aligned} ((\alpha \to \perp) \to \perp)^{1/2} &= (\alpha \to \perp)^{1/2} \to ((\alpha \to \perp)^{1/2} * \alpha^{1/2}) \\ &\leq \perp^{1/2} \to (\perp^{1/2} * \alpha^{1/2}) \quad ; \end{aligned}$$

hence $(\alpha \to \perp) \to \perp = \alpha$ follows from Assertion (ii) – i.e. $(L, \leq, *)$ is an MV–algebra (cf. Subsection 1.1). Finally the assertion (i) holds in any MV–algebra (cf. Proposition 2.15 in [39]).
■

We close this subsection with the consideration of *pseudo–bisymmetric* subsets of the underlying lattice L.

Definition 1.2.15 Let $(L, \leq, \otimes, *)$ be a quadruple satisfying the axioms (I)–(III) and (V)–(VII). A non empty subset S of L is called *pseudo–bisymmetric* in $(L, \leq, \otimes, *)$ iff S satisfies the following condition

$$\text{(P)} \left\{ \begin{aligned} (\alpha * \beta) \otimes (\gamma * \delta) \;=\; & \Big((\alpha \otimes \gamma) * (\beta \otimes \delta)\Big) \bigvee \Big((\alpha \otimes \perp) * (\beta \otimes \top)\Big) \bigvee \\ & \bigvee \Big((\perp \otimes \gamma) * (\top \otimes \delta)\Big) \end{aligned} \right.$$

for all $\alpha, \gamma \in S$ and $\beta, \delta \in L$.

■

Proposition 1.2.16 *Let $(L, \leq, \otimes, *)$ be a quadruple satisfying the axioms (I)–(III), (V)–(IX) and the additional axiom*

$$\text{(XI)} \qquad \top \otimes (\alpha * \beta) = \Big((\top \otimes \alpha) * (\top \otimes \beta)\Big) \bigvee (\top \otimes \perp) \; .$$

Further let $\perp$ be idempotent w.r.t. $\otimes$ (i.e. $\perp \otimes \perp = \perp$), and let $(L, \leq, *)$ be right–symmetric. Then L is pseudo–bisymmetric in $(L, \leq, \otimes, *)$.

Proof. Because of (VIII) and (XI) the following relation holds:

$$\begin{aligned} &(\alpha * \beta) \otimes (\gamma * \delta) = \Big(\top \otimes (\alpha * \beta)\Big) * \Big(\top \otimes (\gamma * \delta)\Big) = \\ &\Big((\top \otimes \alpha) * (\top \otimes \beta) * (\top \otimes \gamma) * (\top \otimes \delta)\Big) \bigvee \Big((\top \otimes \alpha) * (\top \otimes \beta) * (\top \otimes \perp)\Big) \bigvee \\ &\bigvee \Big((\top \otimes \perp) * (\top \otimes \gamma) * (\top \otimes \delta)\Big) \bigvee \Big((\top \otimes \perp) * (\top \otimes \perp)\Big) \quad . \end{aligned}$$

Now, we apply the right–symmetricity of $(L, \leq, *)$, the axiom (VIII) and the idempotency of $\perp$ w.r.t. $\otimes$ and obtain:

$$\begin{aligned}
&(\alpha * \beta) \otimes (\gamma * \delta) \ = \\
&\Big((\top \otimes \alpha) * (\top \otimes \gamma) * (\top \otimes \beta) * (\top \otimes \delta)\Big) \bigvee \Big((\top \otimes \alpha) * (\top \otimes \perp) * (\top \otimes \beta)\Big) \bigvee \\
&\bigvee \Big((\top \otimes \perp) * (\top \otimes \gamma) * (\top \otimes \delta)\Big) \bigvee \Big((\top \otimes \perp) * (\top \otimes \perp)\Big) \ = \\
&= \Big((\alpha \otimes \gamma) * (\beta \otimes \delta)\Big) \bigvee \Big((\alpha \otimes \perp) * (\top \otimes \beta)\Big) \bigvee \Big((\perp \otimes \gamma) * (\top \otimes \delta)\Big) \ .
\end{aligned}$$

Since the axioms (VIII) and (IX) imply $\top \otimes \beta \ = \ \beta \otimes \top$, the assertion follows. ■

Example 1.2.17 (a) Let $(L, \leq, \otimes, *)$ be a quadruple provided with the properties (I)–(III) and (V)–(VII). Further let $(L, \leq, *)$ be strictly two–sided. If $\perp \otimes \top \ = \ \top \otimes \perp \ = \ \perp$, then $S_0 \ = \ \{\top, \perp\}$ is pseudo–bisymmetric in $(L, \leq, \otimes, *)$.
(b) Let $(L, \leq, *)$ be a right–symmetric quantale such that $\top$ satisfies the following condition

$$\alpha * \top \ \geq \ \alpha \quad , \quad \alpha \ \leq \ \top * \alpha \qquad \forall \alpha \in L \ .$$

Then $(L, \leq, *, *)$ satisfies (I)–(VII) (cf. Lemma 1.2.1). Further it is not difficult to see that L is pseudo–bisymmetric in $(L, \leq, *, *)$.
(c) Let $(L, \leq, *)$ be a strictly two–sided, commutative quantale. Further we assume that $(L, \leq, *)$ has square roots such that Axiom (S3) holds. Then in the case of the monoidal mean operator $\circledast$ the hypothesis of Proposition 1.2.16 is satisfied (cf. Proposition 1.2.7 and Corollary 1.2.8). Hence L is pseudo–bisymmetric in $(L, \leq, \circledast, *)$.

■

Example 1.2.18 (Conditional events)
Let $(L, \leq, *)$ be a complete MV–algebra and L_C be the set of all elements $(\alpha, \beta) \in L \times L$ with $\alpha \leq \beta$. Further we introduce two semigroup operations and a partial ordering on L_C as follows

$$\begin{aligned}
(\alpha_1, \beta_1) \ \preceq \ (\alpha_2, \beta_2) \quad &\iff \quad \alpha_1 \leq \alpha_2 \ \text{ and } \ \beta_1 \leq \beta_2 \quad . \\
(\alpha_1, \beta_1) \star (\alpha_2, \beta_2) \quad &= \quad (\alpha_1 * \alpha_2 \ , \ (\alpha_1 * \beta_2) \vee (\alpha_2 * \beta_1)) \quad . \\
(\alpha_1, \beta_1) \otimes (\alpha_2, \beta_2) \quad &= \quad (\alpha_1 * \alpha_2 \ , \ \beta_1 * \beta_2) \quad .
\end{aligned}$$

It is not difficult to show that $(L_C, \preceq, \star)$ and $(L_C, \preceq, \otimes)$ are commutative strictly two–sided quantales. Referring to [44] the quantale $(L_C, \preceq, , \star)$ can be understood as the algebra of conditional events determined by the unconditional events $\alpha \in L$. Further we observe

$$\begin{aligned}
&((\alpha_1, \beta_1) \otimes (\alpha_2, \beta_2)) \star ((\alpha_3, \beta_3) \otimes (\alpha_4, \beta_4)) \quad = \\
&((\alpha_1 * \alpha_2 * \alpha_3 * \alpha_4) \ , \ ((\alpha_1 * \alpha_2 * \beta_3 * \beta_4) \vee (\alpha_3 * \alpha_4 * \beta_1 * \beta_2))) \quad \leq \\
&((\alpha_1, \beta_1) \star (\alpha_3, \beta_3)) \otimes ((\alpha_2, \beta_2) \star (\alpha_4, \beta_4)) \qquad .
\end{aligned}$$

Hence $(L_C, \preceq, \otimes, \star)$ satisfies the axioms (I) – (VII). Now we consider the embedding $\mathcal{E} : L \longmapsto L_C$ determined by the diagonal – i.e. $\mathcal{E}(\alpha) = (\alpha, \alpha) \; \forall \, \alpha \in L$. Obviously the restrictions of $\otimes$ and $\star$ to $\mathcal{E}(L) \times \mathcal{E}(L)$ coincide. It is easy to show that $\mathcal{E}$ is a quantale morphism in the sense of $(L_C, \preceq, \star)$ and $(L_C, \preceq, \otimes)$. Moreover $\mathcal{E}$ preserves the residuations w.r.t. $*$, $\star$, $\otimes$. Finally we observe

$$\begin{aligned}
&((\alpha,\alpha) \otimes (\beta_1,\beta_2)) \star ((\gamma,\gamma) \otimes (\delta_1,\delta_2)) \quad = \\
&((\alpha * \gamma * \beta_1 * \delta_1) \, , \, ((\alpha * \gamma * \beta_1 * \delta_2) \vee (\alpha * \gamma * \beta_2 * \delta_1))) \quad = \\
&((\alpha,\alpha) \star (\gamma,\gamma)) \otimes ((\beta_1,\beta_2) \star (\delta_1,\delta_2)) \qquad ;
\end{aligned}$$

hence the range of $\mathcal{E}$ is pseudo–bisymmetric in $(L_C, \preceq, \otimes, \star)$.

■

1.3 Complete MV–algebras with square roots and idempotent hulls.

Let $M = (L, \leq, *)$ be a complete MV–algebra with square roots. Then

$$\iota_0 \quad := \quad (\bot^{1/2} \to \bot) * (\bot^{1/2} \to \bot)$$

is *idempotent* w.r.t. $*$ (see: Proposition 2.17 in [39]). We notice that smooth MV–algebras (cf. Definition 1.2.10) and complete Boolean algebras are synonymous notions (see: Proposition 2.19 in [39]). Further we recall: M is a *strict* MV–algebra iff $\bot^{1/2} \to \bot = \bot^{1/2}$ (iff $\iota_0 = \bot$). It is not difficult to see that Boolean algebras and strict MV–algebras are mutually exclusive concepts.
Since ι_0 is idempotent, every complete MV–algebra with square roots is either a Boolean algebra, or a strict MV–algebra, or a *product* of a Boolean and a strict MV–algebra (cf. Theorem 1.2.13, Theorem 2.21 in [39]).
Let M be a complete, strict MV–algebra containing at least three elements. Moreover, let us assume that $\bot$ and $\top$ are the only idempotent elements of L w.r.t. $*$. Then M is isomorphic to the real unit interval provided with Lukasiewicz' arithmetic conjunction T_m (cf. Corollary 6.12 in [39]). Referring to Example 1.2.3(b) we see immediately that the strict, complete MV–algebra $([0,1], \leq, T_m)$ has square roots.
The relation $\iota \wedge \alpha = \iota * \alpha$ holds for all $\alpha \in L$ and for all idempotent elements $\iota \in L$ (cf. Proposition 2.8 in [39]). In particular, $\iota \to \bot$ is idempotent, whenever ι is idempotent.
Moreover the semigroup operation $*$ is always distributive over arbitrary meets (cf. Theorem 5.2(c) in [39]). Since the "implication" can be expressed by $*$ and the negation $\_ \to \bot$ (cf. Proposition 2.8 in [39]), the following relation is valid in any complete MV–algebra:

$$\alpha \to \Big(\bigvee_{i \in I} \beta_i \Big) \quad = \quad \bigvee_{i \in I} (\alpha \to \beta_i) \qquad \text{where} \quad \{\beta_i \mid i \in I\} \subseteq L \, .$$

Finally in any (complete) MV–algebra the algebraic strong de Morgan's law holds (cf. Lemma 2.4 and Corollary 2.16 in [39]):

$$(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha) \quad = \quad \top \qquad \forall \alpha, \beta \in L \, .$$

Let $M = (L, \leq, *)$ be a complete MV–algebra. The *idempotent hull* $\overline{\alpha}$ of $\alpha \in L$ is defined by

$$\overline{\alpha} \quad := \quad \bigwedge \{ \iota \in L \mid \alpha \leq \iota \text{ and } \iota * \iota = \iota \} \qquad .$$

Similarly, we define the *idempotent kernel* $\alpha^{\smile}$ of $\alpha \in L$ by

$$\alpha^{\smile} \quad := \quad \bigvee \{ \iota \in L \mid \iota \leq \alpha \text{ and } \iota * \iota = \iota \} \qquad .$$

Since $*$ is distributive over arbitrary meets and joins, it can be shown that the idempotent hull (resp. kernel) $\overline{\alpha}$ (resp. $\alpha^{\smile}$) of α is again idempotent. In particular, the following relations are valid :[3]

$$\begin{aligned}
\bigvee_{i \in I} \overline{\alpha}_i &= \overline{\bigvee_{i \in I} \alpha_i} \\
\alpha^{\smile} &= \overline{(\alpha \rightarrow \bot)} \rightarrow \bot \\
\overline{\alpha} &= (\alpha \rightarrow \bot)^{\smile} \rightarrow \bot \\
\alpha^{\smile} &= \bigwedge_{n \in \mathbb{N}} \alpha^n
\end{aligned}$$

Proposition 1.3.1 (Properties of idempotent hulls)
Let $M = (L, \leq, *)$ be a complete MV–algebra with square roots. Then the following assertions hold :

(a) $\overline{\alpha \wedge \beta} \quad = \quad \overline{\alpha} \wedge \overline{\beta}$.

(b) If M is strict, then $\overline{\alpha^{1/2} * \bot^{1/2}} = \overline{\alpha}$ and $\overline{\alpha^{1/2} * \beta^{1/2}} = \overline{\alpha} \vee \overline{\beta}$.

(c) $\bot^{1/2} \wedge \overline{\alpha} \leq \overline{\alpha^{1/2} * \beta^{1/2}}$.

Proof. The inequality $\overline{\alpha \wedge \beta} \leq \overline{\alpha} \wedge \overline{\beta}$ is obvious. Now we consider an idempotent element ι with $\alpha \wedge \beta \leq \iota$. Then $\iota \rightarrow \bot$ is also idempotent, and the algebraic strong de Morgan law (cf. Lemma 2.4 and Corollary 2.16 in [39]) implies that the inequality

$$\iota \rightarrow \bot \quad \leq \quad (\alpha \rightarrow \bot)^{2^n} \vee (\beta \rightarrow \bot)^{2^n}$$

holds for all $n \in \mathbb{N}$. Taking into account that the underlying lattice of a complete MV–algebra is also a complete Brouwerian lattice (i.e. the dual lattice is a complete Heyting algebra (cf. Theorem 5.2 in [39])) we obtain

$$\iota \rightarrow \bot \quad \leq \quad (\alpha \rightarrow \bot)^{\smile} \vee (\beta \rightarrow \bot)^{\smile} \qquad ;$$

[3] α^n denotes the n–th power of α w.r.t. $*$.

hence $\overline{\alpha} \wedge \overline{\beta} \;\leq\; \iota$ follows.
In order to verify Assertion (b) we first observe : $((\alpha \to \bot)^{1/2})^{\smile} = (\alpha \to \bot)^{\smile}$; therefore we obtain

$$\overline{\alpha} \quad = \quad \overline{(\alpha \to \bot)^{1/2} \to \bot} \qquad .$$

Since M is strict, the relation

$$(\alpha \to \bot)^{1/2} \to \bot \;=\; (\alpha^{1/2} \to (\bot^{1/2} \to \bot)) \to \bot \;=\; \alpha^{1/2} * \bot^{1/2}$$

holds; i.e. $\overline{\alpha^{1/2} * \bot^{1/2}} = \overline{\alpha}$. Finally, we notice

$$\overline{\alpha} \vee \overline{\beta} \;=\; \overline{\alpha^{1/2} * \bot^{1/2}} \vee \overline{\beta^{1/2} * \bot^{1/2}} \;\leq\; \overline{\alpha^{1/2} * \beta^{1/2}} \;\leq\; \overline{\alpha} \vee \overline{\beta}\,.$$

Finally, since every complete MV–algebra with square roots can be uniquely decomposed into a product of a Boolean algebra and a strict MV–algebra, the assertion (c) follows from (b).
■

Corollary 1.3.2 *In any complete MV–algebra with square roots the following relations are valid*

(i) $\quad (\alpha \wedge \overline{\beta})^{1/2} * (\beta \wedge \overline{\alpha})^{1/2} \quad = \quad (\alpha^{1/2} * \beta^{1/2}) \wedge \overline{\alpha} \wedge \overline{\beta}\,.$

(ii) $\quad \overline{(\alpha \wedge (\overline{\beta} \to \bot))^{1/2} * \bot^{1/2}} \quad = \quad \overline{(\alpha^{1/2} * \beta^{1/2}) \wedge (\overline{\beta} \to \bot)}\,.$

(iii) $\quad \overline{\overline{\alpha}^{1/2} * \overline{\beta}^{1/2}} \quad = \quad \overline{\alpha^{1/2} * \beta^{1/2}}\,.$

Proof. Referring to the introductory remarks at the beginning of this subsection it is sufficient to verify the assertion for complete Boolean algebras or strict MV–algebras.
(a) Let M be a Boolean algebra (i.e. $* = \wedge$). Since every element of L is idempotent, the relations (i) – (iii) follow immediately.
(b) In the case of strict MV–algebras we apply Proposition 1.3.1 and obtain

$$\begin{aligned}
\overline{(\alpha \wedge (\overline{\beta} \to \bot))^{1/2} * \bot^{1/2}} \quad &= \quad \overline{\alpha \wedge (\overline{\beta} \to \bot)} \quad = \quad \overline{\alpha} \wedge (\overline{\beta} \to \bot) \\
&= \quad (\overline{\alpha} \vee \overline{\beta}) \wedge (\overline{\beta} \to \bot) \\
&= \quad \overline{(\alpha^{1/2} * \beta^{1/2}) \wedge (\overline{\beta} \to \bot)}\;;
\end{aligned}$$

$$\overline{(\overline{\alpha}^{1/2} * \overline{\beta}^{1/2})} \quad = \quad \overline{\overline{\alpha}} \vee \overline{\overline{\beta}} \quad = \quad \overline{\alpha} \vee \overline{\beta} \quad = \quad \overline{\alpha^{1/2} * \beta^{1/2}} \quad ;$$

hence (ii) and (iii) are verified. Further we observe

$$\overline{\beta}^{1/2} \quad = \quad \overline{\beta} \vee \bot^{1/2} \qquad , \qquad \beta \quad \leq \quad \beta^{1/2} \wedge \overline{\beta}$$

i.e. the relation

$$\beta^{1/2} * \overline{\beta}^{1/2} \quad = \quad (\beta^{1/2} * \overline{\beta}) \vee (\bot^{1/2} * \beta^{1/2}) \quad = \quad \beta^{1/2} \wedge \overline{\beta}$$

holds. Therefore we obtain from the distributivity of $*$ over finite meets

$$\begin{aligned}(\alpha \wedge \overline{\beta})^{1/2} * (\beta \wedge \overline{\alpha})^{1/2} &= (\alpha^{1/2} * \beta^{1/2}) \wedge (\alpha^{1/2} * \overline{\alpha}^{1/2}) \wedge (\overline{\beta}^{1/2} * \beta^{1/2}) \\ &= (\alpha^{1/2} * \beta^{1/2}) \wedge \overline{\alpha} \wedge \overline{\beta}\ .\end{aligned}$$

Thus the relation (i) is established.
■

1.4 Booleanization of bounded distributive lattices.

Let $(L, \leq, \top, \bot)$ be a bounded lattice with the universal upper (resp. lower) bound $\top$ (resp. $\bot$). A non empty subset $\mathbf{P}$ of L is said to be a *proper prime filter* in $(L, \leq)$ iff $\mathbf{P}$ satisfies the subsequent conditions:

- $\top \in \mathbf{P}$.
- $\alpha \leq \beta, \alpha \in \mathbf{P} \implies \beta \in \mathbf{P}$.
- $\alpha, \beta \in \mathbf{P} \implies \alpha \wedge \beta \in \mathbf{P}$.
- $\bot \notin \mathbf{P}$.
- $\alpha \vee \beta \in \mathbf{P} \implies \alpha \in \mathbf{P}\, or\, \beta \in \mathbf{P}$.

Referring to the Stone–representation of distributive lattices (cf. [49]) we quote the following

Lemma 1.4.1 *Let $(L, \leq, \top, \bot)$ be a bounded, distributive lattice. Then for every pair $(\alpha, \beta) \in L \times L$ with $\alpha \not\leq \beta$ there exists a proper prime filter $\mathbf{P}$ provided with the property $\alpha \in \mathbf{P}$, $\beta \notin \mathbf{P}$.*

Theorem 1.4.2 *Let $(L, \leq, \top, \bot)$ be a bounded, distributive lattice. Then there exists a complete Boolean algebra $(\mathbb{B}, \leq)$ and a map $j : L \longmapsto \mathbb{B}$ provided with the following properties:*

(i) $j(\alpha) \leq j(\beta) \iff \alpha \leq \beta$.

(ii) j is a lattice–homomorphism – i.e. $j(\top) = \top$, $j(\bot) = \bot$, $j(\alpha \vee \beta) = j(\alpha) \vee j(\beta)$, $j(\alpha \wedge \beta) = j(\alpha) \wedge j(\beta)$.

(iii) For all $\varkappa \in \mathbb{B}$ there exists a subset S of $L \times L$ such that

$$\varkappa = \bigvee \{ j(\alpha) \wedge (j(\beta) \to \bot) \mid (\alpha, \beta) \in S \} \quad .$$

Moreover the pair $(j, \mathbb{B})$ is uniquely determined by $(L, \leq)$ up to an isomorphism.

Proof. (a) (Existence) Let $Spec(L)$ be the *spectrum* of $(L, \leq)$ – i.e. the set of all *proper prime filters* $\mathbf{P}$ in $(L, \leq)$. We topologize $Spec(L)$ in the usual way – i.e. we consider the topology $\mathbb{T}_P$ on $Spec(L)$ which is determined by the following base:

$$\mathcal{B} \quad = \quad \{\, \{\mathbf{P} \in Spec(L) \,|\, \alpha \in \mathbf{P}, \beta \notin \mathbf{P}\} \quad | \quad (\alpha, \beta) \in L \times L\} \quad .$$

It is well known and not difficult to prove that $(Spec(L), \mathbb{T}_P)$ is a *totally disconnected, compact, topological space.*
Further let $\mathcal{A}(L)$ be the σ–field of all *Borel* subsets of $Spec(L)$ (w.r.t. $\mathbb{T}_P$), and let Δ be the σ–ideal of all Borel subsets of the *first category*. Then the quotient algebra $\mathbb{B} = \mathcal{A}(L)/\Delta$ is a complete Boolean algebra (cf. Example (C) in Section 21 in [100]). In particular, the equivalence classes of $\mathbb{B}$ are denoted by $[B]$ where $B \in \mathcal{A}(L)$. Now we are in the position to define a map $j : L \longmapsto \mathbb{B}$ by

$$j(\alpha) \quad = \quad [\{\mathbf{P} \in Spec(L) \mid \alpha \in \mathbf{P}\}] \qquad \forall \alpha \in L \quad .$$

In the case of $\alpha \leq \beta$ it is easy to see that $j(\alpha) \leq j(\beta)$ holds. On the other hand, if we assume $j(\alpha) \leq j(\beta)$, then by definition the set $B = \{\mathbf{P} \,|\, \alpha \in \mathbf{P},\ \beta \notin \mathbf{P}\}$ is a set of the first category. Since $(Spec(L), \mathbb{T}_P)$ is compact, the set B is empty; hence $\alpha \leq \beta$ follows from Lemma 1.4.1. Therewith Property (i) is established. Property (ii) is an immediate consequence from the prime filter axioms. In order to verify (iii) we proceed as follows: First we observe that for every element $[B] \in \mathbb{B}$ there exists a $\mathbb{T}_P$–open subset $U = \bigcup_{i \in I} \{\mathbf{P} \,|\, \alpha_i \in \mathbf{P},\ \beta_i \notin \mathbf{P}\}$ such that $[B] = [U]$. Now we refer to the special form of suprema in the *algebra of Borel subsets modulo sets of the first category* and obtain (cf. p. 75 in [100]):

$$[U] \quad = \quad \bigvee_{i \in I} [\{\mathbf{P} \,|\, \alpha_i \in \mathbf{P},\ \beta_i \notin \mathbf{P}\}] \quad = \quad \bigvee_{i \in I} (j(\alpha_i) \wedge (j(\beta_i) \to \bot)) \quad ;$$

hence (iii) is verified.
(b) (Unicity) Let us consider two complete Booelan algebras $\mathbb{B}_1$, $\mathbb{B}_2$ and two maps $j_1 : L \longmapsto \mathbb{B}_1$ and $j_2 : L \longmapsto \mathbb{B}_2$ such that $(j_1, \mathbb{B}_1)$ and $(j_2, \mathbb{B}_2)$ are satisfying (i)–(iii). We define a map $\Theta : \mathbb{B}_1 \to \mathbb{B}_2$ by ($\varkappa \in \mathbb{B}_1$):

$$\Theta(\varkappa) \;=\; \bigvee \{j_2(\alpha) \wedge (j_2(\beta) \to \bot) \,|\, \alpha, \beta \in L\,,\, j_1(\alpha) \wedge (j_1(\beta) \to \bot) \leq \varkappa\}\,.$$

It is not difficult to show that Θ is an isomorphism.
■

Remark 1.4.3 The *Booleanization* of a distributive, bounded lattice is a complete Boolean algebra $(\mathbb{B}, \leq)$ which satisfies the conditions (i)–(iii) in Theorem 1.4.2. Due to the unicity of $\mathbb{B}$ we also write $\mathbb{B}(L)$ instead of $\mathbb{B}$.
(a) If $(L, \leq)$ is a complete Heyting algebra, then the embedding of $(L, \leq)$ into its Booleanization preserves arbitrary joins. In fact, the subsequent set

$$\bigcap_{i \in I} \{\mathbf{P} \in Spec(L) \,|\, \bigvee_{i \in I} \alpha_i \in \mathbf{P},\ \alpha_i \notin \mathbf{P}\}$$

is a subset of the first category w.r.t. to the usual topology $\mathbb{T}_P$ on $Spec(L)$.
(b) The dual statement of (a) is also valid: If $(L, \leq)$ is a *complete co–Heyting algebra* (i.e. the dual lattice $(L, \leq)^{op}$ of $(L, \leq)$ is a complete Heyting algebra), then the embedding of $(L, \leq)$ into its Booleanization preserves arbitrary meets. Again the following set

$$\bigcap_{i \in I} \{\mathbf{P} \in Spec(L) \mid \bigwedge_{i \in I} \alpha_i \notin \mathbf{P},\, \alpha_i \in \mathbf{P}\}$$

is a subset of the first category in $Spec(L)$.
(c) The situation in (a) and (b) can be tied together to the following theorem (cf. [19]): A complete lattice $(L, \leq)$ can be embedded into a complete Boolean algebra by a *complete* lattice–homomorphism (i.e. by a map preserving arbitrary meets and arbitrary joins) iff $(L, \leq)$ is a complete Heyting algebra and co–Heyting algebra. ■

We close Section 1 with the observation that all algebraic operations on L can be extended pointwise to the set L^X of all maps from X to L – e.g.

- $f \leq g \Longleftrightarrow f(x) \leq g(x)$;
- $(f \otimes g)(x) = f(x) \otimes g(x)$;
- $(f * g)(x) = f(x) * g(x)$

for all $x \in X$.
Obviously, $(L^X, \leq, \otimes)$ is again a *cqm*–lattice, and the join of a subset $\{f_i \mid i \in I\}$ of L^X is given by

$$(\bigvee_{i \in I} f_i)(x) \quad = \quad \bigvee_{i \in I} f_i(x) \qquad \forall x \in X \quad .$$

Finally we use the following notations

$$1_X(x) = \top \quad , \quad \alpha \cdot 1_X(x) = \alpha \quad , \quad 1_\varnothing(x) = \bot \qquad \forall x \in X \quad .$$

2 L–fuzzy topological spaces.

In the following considerations we always assume that $(L, \leq, \otimes)$ is a *cqm*–lattice (cf. Section 1). If we need more structure, we will state these additional requirements explicitly.

Definition 2.1 (L–fuzzy topology, LF–continuity)
Let X be a non empty set. An *L–fuzzy topology* on X is a map $\mathcal{T} : L^X \to L$ satisfying the following axioms:

$$(\mathcal{O}1) \quad \mathcal{T}(1_X) \quad = \quad \top$$

$(\mathcal{O}2) \quad \mathcal{T}(f_1) \otimes \mathcal{T}(f_2) \;\leq\; \mathcal{T}(f_1 \otimes f_2) \qquad \forall f_1, f_2 \in L^X$

$(\mathcal{O}3)$ For every subset $\{f_i \mid i \in I\}$ of L^X the inequality

$$\bigwedge_{i \in I} \mathcal{T}(f_i) \;\leq\; \mathcal{T}(\bigvee_{i \in I} f_i)$$

holds.

An L*–fuzzy topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is an L–fuzzy topology on X. A map $\varphi : X \longmapsto Y$ is called LF*–continuous* w.r.t. the given L–fuzzy topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ iff φ satisfies the following inequality [4]

$$\mathcal{S}(g) \;\leq\; \mathcal{T}(g \circ \varphi)$$

for all $g \in L^Y$.

Addition. If X is the empty set, then $L^{\varnothing} = \{\varnothing\}$; in this context we regard the map $\mathcal{T} : L^{\varnothing} \to L$ defined by $\mathcal{T}(\varnothing) = \top$ as the (unique) L–fuzzy topology on $\varnothing$.

■

Remark 2.2
(a) If X is non-empty, then L^X consists at least of two elements. In particular, the universal lower bound in $(L^X, \leq)$ is given by $1_{\varnothing}$. Obviously Axiom $(\mathcal{O}3)$ implies in the case of the empty set

$$(\mathcal{O}1') \qquad \mathcal{T}(1_{\varnothing}) \;=\; \top.$$

(b) If we interpret the underlying lattice $(L, \leq)$ as the set of truth values, then the value $\mathcal{T}(f)$ of an L–fuzzy topology $\mathcal{T}$ on X expresses the degree to which "*f is open*". By virtue of $(\mathcal{O}1)$ and $(\mathcal{O}1')$ the universal bounds of $(L^X, \leq)$ are *always* "open".

■

On the set $\mathfrak{T}_L(X)$ of all L–fuzzy topologies on X we introduce a partial ordering by

$$\mathcal{T}_1 \preccurlyeq \mathcal{T}_2 \iff \mathcal{T}_1(f) \leq \mathcal{T}_2(f) \qquad \forall f \in L^X$$

In particular, $\mathcal{T}_1$ is called *coarser* than $\mathcal{T}_2$ (resp. $\mathcal{T}_2$ is said to be *finer* than $\mathcal{T}_1$) iff $\mathcal{T}_1 \preccurlyeq \mathcal{T}_2$.

Proposition 2.3 *$(\mathfrak{T}_L(X), \preccurlyeq)$ is a complete lattice.*

Proof. Obviously the map $\mathcal{T}_{dis} : L^X \longmapsto L$ defined by

$$\mathcal{T}_{dis}(f) \;=\; \top \qquad \forall f \in L^X$$

[4] $g \circ \varphi$ is denoted sometimes also by $\varphi_L^{\leftarrow}(g)$ (cf. Remark 1.1.5 in [93], see also [82]).

is an L–fuzzy topology and the universal upper bound in $\mathfrak{T}_L(X)$ w.r.t. $\preccurlyeq$. Therefore it is sufficient to verify that the infimum of any non empty family $\{\mathcal{T}_i \mid i \in I\}$ of L–fuzzy topologies $\mathcal{T}_i$ on X exists. Referring to the isotonicity of $\otimes$ the map $\mathcal{T}_0 : L^X \longmapsto L$ determined by

$$\mathcal{T}_0(f) \quad := \quad \bigwedge_{i \in I} \mathcal{T}_i(f) \qquad \forall f \in L^X$$

is an L–fuzzy topology on X. It is easy to show that $\mathcal{T}_0$ is the infimum of $\{\mathcal{T}_i \mid i \in I\}$.
■

Remark 2.4 (Generation of L–fuzzy topologies)
(a) Let $\mathcal{R} : L^X \longmapsto L$ be a map; then we consider the set $\mathfrak{Z}_{\mathcal{R}}$ of all L–fuzzy topologies $\mathcal{T}$ on X provided with the following property

$$\text{(G)} \qquad \mathcal{R}(f) \quad \leq \quad \mathcal{T}(f) \qquad \text{for all } f \in L^X \quad .$$

Obviously the infimum $\mathcal{T}_{\mathcal{R}}$ of $\mathfrak{Z}_{\mathcal{R}}$ w.r.t. $\preccurlyeq$ (cf. 2.3) is again an element of $\mathfrak{Z}_{\mathcal{R}}$. In particular $\mathcal{T}_{\mathcal{R}}$ is the coarsest L–fuzzy topology on X satisfying (G).
(b) We maintain the notation from (a). A map $\mathcal{R} : L^X \longmapsto L$ is called a *L-fuzzy subbase* of a L –fuzzy topology $\mathcal{T}$ iff $\mathcal{T} = \mathcal{T}_{\mathcal{R}}$.
■

Proposition 2.5 *Let $\varphi : X_1 \longmapsto X_2$ be a map, $\mathcal{T}_i$ be an L-fuzzy topology on X_i (i=1,2), and let $\mathcal{R}$ be a L-fuzzy subbase of $\mathcal{T}_2$. Then the following assertions are equivalent*

(i) φ is LF–continuous w.r.t. $\mathcal{T}_1$ and $\mathcal{T}_2$.

(ii) $\mathcal{R}(g) \leq \mathcal{T}_1(g \circ \varphi) \quad \forall g \in L^{X_2}$.

Proof. The implication $(i) \Longrightarrow (ii)$ is obvious. In order to show $(ii) \Longrightarrow (i)$ we consider an L–fuzzy topology $\bar{\mathcal{T}}_1$ on X_2 determined by

$$\bar{\mathcal{T}}_1(g) \quad = \quad \mathcal{T}_1(g \circ \varphi) \qquad \forall g \in L^{X_2} \quad .$$

In particular, $\bar{\mathcal{T}}_1$ is the finest L–fuzzy topology on X_2 making φ LF–continuous. Now we obtain from (ii)

$$\mathcal{R}(g) \quad \leq \quad \bar{\mathcal{T}}_1(g) \qquad \forall g \in L^{X_2} \quad .$$

Since $\mathcal{R}$ is a L–fuzzy subbase of $\mathcal{T}_2$, the assertion (i) follows.
■

The category L–**FTOP** consists of the following data: *Objects* are L–fuzzy topological spaces, and *morphisms* are LF–continuous maps. The *composition* is the usual composition of maps, and the *identity* of $(X, \mathcal{T})$ is the identity map id_X of X.

Theorem 2.6 *Let $\mathfrak{F}$: L-**FTOP** $\rightarrow$ **SET** be the forgetful functor. Then L–**FTOP** is a topological category over **SET** w.r.t. $\mathfrak{F}$.*

Proof. Because of Proposition 2.5 every co–structured map $\varphi : \mathfrak{F}(X,\tau) \longmapsto Y$ has a unique final lift $\varphi : (X,\tau) \longmapsto (Y,\sigma)$. Further, ($L$–**FTOP**, $\mathfrak{F}$) is fibre–complete (cf. Proposition 2.3). Hence it is sufficient to show that for every L–fuzzy topological space $(Y,\mathcal{S})$ a map $\varphi : X \longmapsto Y$ has a *unique initial lift* $\varphi : (X,\mathcal{T}) \longmapsto (Y,\mathcal{S})$ (cf. 21.C in [1]). Let us define a map $\mathcal{R} : L^X \longmapsto L$ by

$$\mathcal{R}(h) \quad = \quad \bigvee \{\mathcal{S}(g) \mid g \in L^Y \; , \; h = g \circ \varphi\} \qquad \forall\, h \in L^X$$

and consider the L–fuzzy topology $\mathcal{T}$ on X generated by $\mathcal{R}$ (i.e. the coarsest L–fuzzy topology "containing" $\mathcal{R}$ (cf. Remark 2.4)). Then φ is LF–continuous w.r.t. $\mathcal{T}$ and $\mathcal{S}$. In order to show that φ is an *initial morphism* we consider a further LF–continuous map $\psi : (Z,\mathcal{P}) \longmapsto (Y,\mathcal{S})$ and a map $\Theta : Z \longmapsto X$ s.t. the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] & Y \\ Z \arrow[u, "\Theta"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes. From the definition of $\mathcal{R}$ and the LF–continuity of ψ we derive the following relation

$$\mathcal{R}(h) \quad \leq \quad \mathcal{P}(h \circ \Theta) \qquad \forall\, h \in L^X \quad .$$

Since $\mathcal{R}$ is a L–fuzzy subbase of $\mathcal{T}$, we apply Proposition 2.5 and obtain that Θ is LF–continuous; i.e. φ is an initial arrow. The uniqueness of the lift follows from the antisymmetry of the partial ordering $\preccurlyeq$ "coarser".
■

Remark 2.7 (Change of basis L)

Let $(L_i, \leq_i, \otimes_i)$ $(i = 1,2)$ be a *cqm*–lattice and $\gamma : L_1 \rightarrow L_2$ be a **CQML**–morphism. In order to simplify the notation we omit the index attached to the inequality sign $\leq$. It will be clear from the context which inequality relation is meant. Since γ is join preserving the right adjoint map $\gamma^* : L_2 \rightarrow L_1$ exists and is given by

$$\gamma^*(\beta) := \vee\{\lambda \in L_1 \mid \gamma(\lambda) \leq \beta\}$$

Obviously γ^* preserves arbitrary meets and the following relations hold for all $\alpha \in L_1$ and $\beta, \beta' \in L_2$:

- $\gamma(\gamma^*(\beta)) \leq \beta, \qquad \alpha \leq \gamma^*(\gamma(\alpha));$
- $\gamma^*(\beta) \otimes_1 \gamma^*(\beta') \leq \gamma^*(\beta \otimes_2 \beta');$

- $\gamma(\gamma^*(\beta)) = \beta$ iff γ is surjective.

Moreover, if $(X, \mathcal{T})$ is an object of L_2–**FTOP**, then γ determines an L_1–fuzzy topology $\mathcal{H}(\mathcal{T})$ on X by

$$[\mathcal{H}(\mathcal{T})](f) = \gamma^*(\mathcal{T}(\gamma \circ f)) \qquad \forall f \in {L_1}^X.$$

It is not difficult to see that $\mathcal{H}$ gives rise to a functor

$\Gamma_\gamma : L_2$–**FTOP** $\longmapsto L_1$–**FTOP**, where $\Gamma_\gamma(X, \mathcal{T}) = (X, \mathcal{H}(\mathcal{T}))$, $\Gamma_\gamma(\varphi) = \varphi$.

We show that Γ_γ has a right adjoint functor. For this purpose we distinguish the following cases:

2.7.1 Let us assume that $\gamma : (L_1, \leq, \otimes_1) \longmapsto (L_2, \leq, \otimes_2)$ is a *surjective* **CQML**–morphism.

If $\mathcal{S}$ is a L_1–fuzzy topology on X, then $\mathcal{S}$ induces a map $\bar{\mathcal{S}} : L_2^X \longmapsto L_1$ by

$$\bar{\mathcal{S}}(g) \quad := \quad \bigvee \{\mathcal{S}(h) \mid \quad g = \gamma \circ h, \quad h \in {L_1}^X\} \quad , \qquad g \in {L_2}^X \; .$$

Further let $Z_\mathcal{S}$ be the set of all L_2–fuzzy topologies $\mathcal{P}$ on X provided with the property

$$\gamma \circ \bar{\mathcal{S}}(g) \quad \leq \quad \mathcal{P}(g) \qquad \forall g \in L_2^X \quad .$$

After these preparations we define a functor $\mathfrak{G}_\gamma$: L_1–**FTOP** $\longmapsto L_2$–**FTOP** as follows : Let $(X, \mathcal{S})$ be an object of L_1–**FTOP**, then

- $\mathfrak{G}_\gamma(X, \mathcal{S}) = (X, \mathcal{P}_\mathcal{S})$, where $\mathcal{P}_\mathcal{S}$ denotes the infimum of $Z_\mathcal{S}$ w.r.t. $\preccurlyeq$ (cf. 2.3).
- $\mathfrak{G}_\gamma(\varphi) = \varphi$.

In order to show that $\mathfrak{G}_\gamma$ preserves "continuity" we consider a morphism $\varphi : (X, \mathcal{T}) \to (Y, \mathcal{S})$ in L_1–**FTOP** ; then we obtain:

$$\bar{\mathcal{S}}(g) \leq \bar{\mathcal{T}}(g \circ \varphi),$$

$$\mathcal{P} \in Z_\mathcal{T} \Longrightarrow \mathcal{P}(\_ \circ \varphi) \in Z_\mathcal{S}$$

Thus $\mathcal{P}_\mathcal{S}(g) \leq \mathcal{P}_\mathcal{T}(g \circ \varphi)$ holds for all $g \in {L_2}^Y$, i.e, $\mathfrak{G}_\gamma(\varphi)$ is L_2F–continuous.

Further, we observe that for every L_2–fuzzy topology $\mathcal{T}$ on X the relation $\overline{\mathcal{H}(\mathcal{T})}(g) = \gamma^*(\mathcal{T}(g))$ holds for all $g \in {L_2}^X$. Since γ is surjective, we obtain :

$$\mathcal{T}(g) \quad = \quad \gamma \circ \gamma^* \circ \mathcal{T}(g) \quad = \quad \gamma \circ \overline{\mathcal{H}(\mathcal{T})}(g) \qquad ;$$

i.e. $\mathcal{P}_{\mathcal{H}(\mathcal{T})} = \mathcal{T}$. Hence $\mathfrak{G}_\gamma \circ \Gamma_\gamma = \mathrm{id}_{L_2\text{-FTOP}}$ is established.

In order to verify the *universal property* we proceed as follows: Let $\varphi : (X, \mathcal{T}) \to \mathfrak{G}_\gamma(Y, \mathcal{S})$ be L_2F–continuous; then we infer from the definition of $\mathcal{P}_\mathcal{S}$ that $\mathcal{P}_\mathcal{S} \in Z_\mathcal{S}$ (cf. 2.4) , i.e. the subsequent relation holds:

$$\gamma \circ \bar{\mathcal{S}}(g) \leq \mathcal{P}_\mathcal{S}(g) \leq \mathcal{T}(g \circ \varphi) \qquad \text{for all} \quad g \in {L_2}^Y.$$

Now we invoke the definition of $\bar{\mathcal{S}}$ and obtain for all $h \in {L_2}^Y$:

$$\mathcal{S}(h) \le \gamma^* \circ \gamma \circ \mathcal{S}(h) \le \gamma^* \circ \gamma \circ \bar{\mathcal{S}}(\gamma \circ h) \le \gamma^* \circ \mathcal{T}(\gamma \circ h \circ \varphi) = [\mathcal{H}(\mathcal{T})](h \circ \varphi),$$

i.e. $\varphi : \Gamma_\gamma(X,\mathcal{T}) \longmapsto (Y,\mathcal{S})$ is L_1F-continuous.

If we put $\ulcorner\varphi\urcorner = \varphi$, then we have verified the commutativity of the following diagram:

$$\begin{tikzcd}
(X,\mathcal{T}) \arrow[r, "id_X"] \arrow[dr, "\varphi"'] & \mathfrak{G}_\gamma(\Gamma_\gamma(X,\mathcal{T})) = (X,\mathcal{T}) \arrow[d, dotted, "\mathfrak{G}_\gamma(\ulcorner\varphi\urcorner)"] \\
 & \mathfrak{G}_\gamma(Y,\mathcal{S})
\end{tikzcd}$$

i.e. that $\mathfrak{G}_\gamma$ is right adjoint to Γ_γ.

2.7.2 Let $\gamma : (L_1, \le, \otimes_1) \to (L_2, \le, \otimes_2)$ be an embedding – i.e. $\alpha_1 \le \alpha_2$ if and only if $\gamma(\alpha_1) \le \gamma(\alpha_2)$.

Further let $\mathcal{S}$ be a L_1–fuzzy topology on X . We denote by $\Omega_\mathcal{S}$ the set of all L_2–fuzzy topologies $\mathcal{P}$ on X provided with the subsequent property

$$\gamma \circ \mathcal{S}(\gamma^{-1} \circ g) \quad \le \quad \mathcal{P}(g) \qquad \forall g \in \{g' \in {L_2}^X \,|\, g'(X) \subseteq \gamma(L_1)\}\,.$$

Then we define a functor $\mathfrak{E}_\gamma$: L_1–**FTOP** $\longmapsto$ L_2–**FTOP** as follows :

- $\mathfrak{E}_\gamma(X,\mathcal{S}) \;=\; (X,\varrho_\mathcal{S})$, where $\varrho_\mathcal{S}$ is the infimum of $\Omega_\mathcal{S}$ w.r.t. $\preccurlyeq$.
- $\mathfrak{E}_\gamma(\varphi) \;=\; \varphi$.

It is easy to see that $\mathfrak{E}_\gamma$ preserves "continuity". Indeed, let us consider an L_1F–continuous morphism $\varphi : (X,\mathcal{T}) \to (Y,\mathcal{S})$; then the implication

$$\mathcal{P} \in \Omega_\mathcal{T} \Longrightarrow \mathcal{P}(\_ \circ \varphi) \in \Omega_\mathcal{S}$$

follows from the inequality

$$\mathcal{S}(g) \le \mathcal{T}(g \circ \varphi) \qquad \forall g \in {L_1}^Y;$$

hence $\varphi : (X,\varrho_\mathcal{T}) \to (Y,\varrho_\mathcal{S})$ is L_2F–continuous. Moreover, since γ is injective, we obtain for all $g \in {L_2}^X$ with $g(X) \subseteq \gamma(L_1)$:

$$\gamma \circ \gamma^* \circ \mathcal{T}(\gamma \circ \gamma^{-1} \circ g) \quad = \quad \gamma \circ \gamma^* \circ \mathcal{T}(g) \quad \le \quad \mathcal{T}(g) \qquad ,$$

i.e. $\mathcal{T} \in \Omega_{\mathcal{H}(\mathcal{T})}$; hence $id_X : (X,\mathcal{T}) \to (X,\varrho_{\mathcal{H}(\mathcal{T})})$ is L_2F–continuous. Further, we consider an L_2F–continuous map $\varphi : (X,\mathcal{T}) \to \mathfrak{E}_\gamma(Y,\mathcal{S})$. Because of $\varrho_\mathcal{S} \in \Omega_\mathcal{S}$ we obtain for all $h \in {L_1}^Y$

$$\gamma \circ \mathcal{S}(h) = \gamma \circ \mathcal{S}(\gamma^{-1} \circ \gamma \circ h) \le \varrho_\mathcal{S}(\gamma \circ h) \le \mathcal{T}(\gamma \circ h \circ \varphi).$$

Now we invoke the fact that γ is an embedding:

$$\mathcal{S}(h) = \gamma^* \circ \gamma \circ \mathcal{S}(h) \le \gamma^* \circ \mathcal{T}(\gamma \circ h \circ \varphi) = [\mathcal{H}(\mathcal{T})](h \circ \varphi) \qquad \forall h \in {L_2}^Y \quad ,$$

i.e. $\varphi : (X, \mathcal{H}(\mathcal{T})) \to (Y, \mathcal{S})$ is L_1F–continuous. Putting $\varphi = \ulcorner \varphi \urcorner$ we establish the commutativity of the following diagram:

$$\begin{tikzcd}
(X, \mathcal{T}) \arrow[r, "id_X"] \arrow[dr, "\varphi"'] & \mathfrak{C}_\gamma(\Gamma_\gamma(X, \mathcal{T})) = (X, \varrho_{\mathcal{H}(\mathcal{T})}) \arrow[d, dotted, "\mathfrak{C}_\gamma(\ulcorner \varphi \urcorner)"] \\
 & \mathfrak{C}_\gamma(Y, \mathcal{S})
\end{tikzcd}$$

Hence the functor $\mathfrak{C}_\gamma$ is right adjoint to Γ_γ.

2.7.3 Since every **CQML**–morphism γ can be decomposed into a surjective **CQML**–morphism followed by an embedding, we conclude from **2.7.1** and **2.7.2** that Γ_γ has a right adjoint functor.
■

We close this section with a natural example of an L–fuzzy topology which indicates a close relationship between random topologies and fuzzy topologies.

Example 2.8 ([0,1]–fuzzy topologies generated by random bases)
Let X be a non empty set, $\mathcal{P}(X)$ be the ordinary power set of X, and let Π be a *random set* on $\mathcal{P}(X)$ – i.e. Π is a probability measure on $\mathcal{P}(\mathcal{P}(X))$ w.r.t. the σ–algebra generated by (cf. [79])

$$\Big\{ \{\mathcal{A} \in \mathcal{P}(\mathcal{P}(X)) \mid A \in \mathcal{A}\} \quad \Big| \quad A \in \mathcal{P}(X) \Big\} \qquad .$$

If we identify $\mathcal{P}(\mathcal{P}(X))$ with $\{0,1\}^{\mathcal{P}(X)}$ and consider the ordinary product topology $\mathbb{T}_p$ on $\mathcal{P}(\mathcal{P}(X))$ w.r.t. the discrete topology on $\{0,1\}$, then Π can be viewed as a regular Borel probability measure.
Further we observe that the set $\mathfrak{B}(X)$ of all topological bases on X (i.e. the set of all subsets $\mathcal{B}$ of $\mathcal{P}(X)$ which are closed under *finite* intersections) is a $\mathbb{T}_p$–closed subset of $\mathcal{P}(\mathcal{P}(X))$.
A random set Π on $\mathcal{P}(X)$ is called a *random base* on X iff the support of Π is contained in $\mathfrak{B}(X)$. Due to the additivity of probability measures every random base Π satisfies the condition

$$\Pi(\{\mathcal{A} \mid A_1 \in \mathcal{A}\}) \; + \; \Pi(\{\mathcal{A} \mid A_2 \in \mathcal{A}\}) \; - \; 1 \quad \le \quad \Pi(\{\mathcal{A} \mid A_1 \cap A_2 \in \mathcal{A}\}) \quad .$$

In the following considerations we view $[0,1]$ as a complete MV–algebra (cf. Example 1.1.3 in the case $G = \mathbb{R}$), and consider the case $\otimes = * = T_m$ (cf. Lemma 1.2.1) where T_m denotes Łukasiewicz arithmetic conjunction (cf. Example 1.2.3(b)). Then every random base on X induces a [0,1]–fuzzy topology $\mu : [0,1]^X \longmapsto [0,1]$ as follows

$$\mu(g) \quad = \quad \sup \{ \inf_{i \in I} T_m(\alpha_i, d_\Pi(A_i)) \mid \sup_{i \in I} \alpha_i \cdot \chi_{A_i} = g \}$$

where $d_\Pi(A_i) := \Pi(\{\mathcal{A} \mid A_i \in \mathcal{A}\})$. Because of

$$T_m\big(\inf_{i \in I} T_m(\alpha_i, d_\Pi(A_i)), \inf_{j \in J} T_m(\beta_j, d_\Pi(B_j))\big) \leq$$
$$\leq \inf_{(i,j) \in I \times J} T_m(T_m(\alpha_i, \beta_j), d_\Pi(A_i \cap B_j))$$

it is not difficult to show that in fact μ is a $[0,1]$–fuzzy topology.
■

3 L–Topological spaces

Let $(L, \leq, \otimes)$ be a *cqm*–lattice, and X be a non empty set. A subset τ of L^X is called a *L–topology* on X iff τ satisfies the following conditions

(o1) $1_X, 1_\varnothing \in \tau$.

(o2) $g_1, g_2 \in \tau \implies g_1 \otimes g_2 \in \tau$.

(o3) $\{g_i \mid i \in I\} \subseteq \tau \implies \bigvee_{i \in I} g_i \in \tau$.

A *L–topological space* is a pair (X, τ) where X is a set and τ an L–topology on X.

Example 3.1 (L–topology)
Let $(\mathcal{L}, \leqslant, \boxtimes)$ and $(L, \leq, \otimes)$ be *cqm*–lattices, and $\mathfrak{H}(\mathcal{L}, L)$ be the set of all **CQML**–morphisms $\varphi : (\mathcal{L}, \leqslant, \boxtimes) \longmapsto (L, \leq, \otimes)$. Then every element $a \in \mathcal{L}$ induces a map $f_a : \mathfrak{H}(\mathcal{L}, L) \longmapsto L$ by

$$f_a(\varphi) \quad = \quad \varphi(a) \qquad \text{(Evaluation)}$$

Then $\tau^\mathcal{L} := \{f_a \mid a \in \mathcal{L}\}$ is an L–topology on $\mathfrak{H}(\mathcal{L}, L)$.
■

Remark 3.2 (Generation of L–topologies)
(a) On the set $\mathbb{T}_L(X)$ of all L–topologies on X we introduce a partial ordering determined by the *set inclusion* $\subseteq$. It is easy to see that L^X is an element of $\mathbb{T}_L(X)$ and the universal upper bound in $(\mathbb{T}_L(X), \subseteq)$. In particular, $\tau_{dis} = L^X$ is called the *discrete* L–topology. Since arbitrary intersections of L–topologies are again L–topologies, we obtain that the pair $(\mathbb{T}_L(X), \subseteq)$ is a *complete* lattice. The universal lower bound τ_{ind} in $(\mathbb{T}_L(X), \subseteq)$ is said to be the *indiscrete* L–topology.

(b) Let ϱ be a non empty subset of L^X; then ϱ generates an L–topology τ_ϱ in the following sense

$$\tau_\varrho \quad := \quad \bigcap \{\tau \in \mathbb{T}_L(X) \mid \varrho \subseteq \tau\} \quad .$$

In particular τ_ϱ is the smallest L–topology containing ϱ.

(c) A non empty subset ϱ of L^X is called a *subbase* of an L–topology τ on X iff $\tau = \tau_\varrho$ (cf. (b)). Obviously $\{1_X, 1_\varnothing\}$ is a subbase of the indiscrete L–topology τ_{ind}. Hence τ_{ind} is contained in the L–topology $\{\alpha \cdot 1_X \mid \alpha \in L\}$.

(d) If X has at least two different elements, then the indiscrete and discrete L–topologies on X are always different (cf. (c)).

■

Let us consider L–topological spaces (X_1, τ_1) and (X_2, τ_2) ; a map $\varphi : X_1 \longmapsto X_2$ is called *L–continuous* iff φ fulfills the subsequent property

$$\{ g \circ \varphi \mid g \in \tau_2\} \subseteq \tau_1 \quad .$$

In an obvious way L–topological spaces and L–continuous maps form a category denoted by L–**TOP**.

Proposition 3.3 *Let (X_i, τ_i) be a L–topological space (i=1,2), ϱ be a subbase of τ_2 and let $\varphi : X_1 \longmapsto X_2$ be a map. Then the following assertions are equivalent*

(i) φ is L–continuous.

(ii) $\{g \circ \varphi \mid g \in \varrho\} \subseteq \tau_1$.

Proof. Since $\{g \in L^{X_2} \mid g \circ \varphi \in \tau_1 \}$ is an L–topology on X_2, the assertion follows from the definition of subbases (cf. 3.2).

■

Theorem 3.4 *Let $\mathfrak{F} : L$–**TOP** $\longmapsto$ **SET** be the forgetful functor. Then $(L$–**TOP**$, \mathfrak{F})$ is a topological category over* **SET**.

Proof. We can repeat the proof of Theorem 2.6 verbatim, provided we replace Proposition 2.3, Remark 2.4 and Proposition 2.5 by Remark 3.2 and Proposition 3.3 .

■

Since the universal upper bound $\top$ in L is idempotent w.r.t. $\otimes$ (cf. Axiom (III)), we observe that every L–fuzzy topology $\mathcal{T}$ on X induces an L–topology $\tau_\mathcal{T}$ by

$$\tau_\mathcal{T} \quad := \quad \{g \in L^X \mid \top = \mathcal{T}(g)\} \quad .$$

In particular, this situation gives rise to a functor $\mathfrak{K} : L$–**FTOP** $\longmapsto L$–**TOP** in the following sense

$$\mathfrak{K}(X, \mathcal{T}) \quad = \quad (X, \tau_\mathcal{T}) \quad , \quad \mathfrak{K}(\varphi) \;=\; \varphi.$$

Theorem 3.5 *The functor $\mathfrak{K}$ has a right adjoint functor.*

Proof. Let τ be an L–topology on X ; we denote by $\mathfrak{Z}_\tau$ the set of all L–fuzzy topologies $\mathcal{T}$ on X provided with the property

$$\mathcal{T}(g) \quad = \quad \top \qquad \forall\, g \in \tau \quad .$$

Further let $\mathcal{S}_\tau$ be the infimum of $\mathfrak{Z}_\tau$ w.r.t. $\preccurlyeq$ (cf. 2.3). Obviously $\mathcal{S}_\tau \in \mathfrak{Z}_\tau$, and the relation $\mathcal{S}_{\tau_\mathcal{T}} \preccurlyeq \mathcal{T}$ holds – i.e. $id_X \,:\, (X,\mathcal{T}) \longmapsto (X,\mathcal{S}_{\tau_\mathcal{T}})$ is LF–continuous. Further let $\varphi \,:\, (X_1,\tau_1) \longmapsto (X_2,\tau_2)$ be an L–**TOP**–morphism; then the L–continuity of φ implies

$$\mathcal{T}(g \circ \varphi) \quad = \quad \top \qquad \forall\, \mathcal{T} \in \mathfrak{Z}_{\tau_1}\,, \quad \forall\, g \in \tau_2 \quad ;$$

i.e. the implication

$$\mathcal{T} \in \mathfrak{Z}_{\tau_1} \quad \Longrightarrow \quad \mathcal{T}(\_ \circ \varphi) \in \mathfrak{Z}_{\tau_2}$$

holds. Hence $\varphi \,:\, (X_1,\mathcal{S}_{\tau_1}) \longmapsto (X_2,\mathcal{S}_{\tau_2})$ is LF–continuous.

Now we are in the position to define a functor $\mathfrak{G} \,:\, L$–**TOP** $\longmapsto L$–**FTOP** by

$$\mathfrak{G}(X,\tau) \quad = \quad (X,\mathcal{S}_\tau) \quad , \qquad \mathfrak{G}(\varphi) \quad = \quad \varphi \qquad .$$

Let us consider a L–**TOP**–morphism $\varphi \,:\, (X,\mathcal{T}) \longmapsto \mathfrak{G}(Y,\tau_0)$; then we infer from the LF–continuity of φ and the definition of $\mathcal{S}_{\tau_0}$:

$$\mathcal{T}(g \circ \varphi) \quad = \quad \top \qquad \forall\, g \in \tau_0 \ (\subseteq L^Y) \qquad ;$$

i.e. $\varphi \,:\, (X,\tau_\mathcal{T}) \longmapsto (Y,\tau_0)$ is L–continuous. Hence we have established the following universal property

$$\begin{tikzcd}
(X,\mathcal{T}) \arrow[r, "id_X"] \arrow[dr, "\varphi"'] & \mathfrak{G}(\mathfrak{K}(X,\mathcal{T})) \arrow[d, dotted, "\mathfrak{G}(\ulcorner\varphi\urcorner)"] \\
& \mathfrak{G}(Y,\tau_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{K}(X,\mathcal{T}) \arrow[d, "\ulcorner\varphi\urcorner=\varphi"] \\
(Y,\tau_0)
\end{tikzcd}
\quad ;$$

i.e. $\mathfrak{G}$ is right adjoint to $\mathfrak{K}$.
■.

Every **CQML**–morphism $\gamma \,:\, (L_1,\leq_1,\otimes_1) \longmapsto (L_2,\leq_2,\otimes_2)$ induces a functor $\Delta_\gamma \,:\, L_2$–**TOP** $\longmapsto L_1$–**TOP** by

- $\Delta_\gamma(X,\tau) \quad = \quad (X,\tau_\gamma) \quad$ where $\quad \tau_\gamma \quad = \quad \{f \in L_1^X \,|\, \gamma \circ f \in \tau\}$
- $\Delta_\gamma(\varphi) \quad = \quad \varphi \quad .$

In particular, Δ_γ fills in the following diagram

$$\begin{tikzcd} L_2\text{–}\mathbf{FTOP} \arrow[r, "\Gamma_\gamma"] \arrow[d, "\mathfrak{K}_1"'] & L_1\text{–}\mathbf{FTOP} \arrow[d, "\mathfrak{K}_2"] \\ L_2\text{–}\mathbf{TOP} \arrow[r, dotted, "\Delta_\gamma"'] & L_1\text{–}\mathbf{TOP} \end{tikzcd} \qquad \text{where} \quad \mathfrak{K}_i(X,\mathcal{T}) \;=\; (X,\tau_{\mathcal{T}}) \quad (i=1,2).$$

Theorem 3.6 (Change of basis L) Δ_γ *has a right adjoint functor.*

Proof. Let us consider a functor $\mathfrak{L}_\gamma \,:\, L_1\text{–}\mathbf{TOP} \longmapsto L_2\text{–}\mathbf{TOP}$ defined by

- $\mathfrak{L}_\gamma(X,\tau) \;=\; (X,\tau^{(\gamma)})$ where $\tau^{(\gamma)} \;=\; \{\gamma\circ f \mid f\in\tau\}$
- $\mathfrak{L}_\gamma(\varphi) \;=\; \varphi$.

We observe $(\tau_\gamma)^{(\gamma)} \subseteq \tau$; i.e. $id_X \,:\, (X,\tau) \longmapsto \mathfrak{L}_\gamma(\Delta_\gamma(X,\tau))$ is L_2–continuous. Let us now consider an L_2–**TOP**–object (X,τ), an L_1–**TOP**–object (Y,σ) and an L_2–continuous map $\varphi \,:\, (X,\tau) \longmapsto \mathfrak{L}_\gamma(Y,\sigma)$. In particular, φ satisfies the condition

$$\gamma\circ g\circ\varphi\in\tau \qquad \forall g\in\sigma \quad ;$$

i.e. $\ulcorner\varphi\urcorner = \varphi \,:\, \Delta_\gamma(X,\tau) \longmapsto (Y,\sigma)$ is L_1–continuous. Hence we have verified the following universal property

$$\begin{tikzcd} (X,\tau) \arrow[r, "id_X"] \arrow[dr, "\varphi"'] & \mathfrak{L}_\gamma(\Delta_\gamma(X,\tau)) \arrow[d, dotted, "\mathfrak{L}_\gamma(\ulcorner\varphi\urcorner)"] \\ & \mathfrak{L}_\gamma(Y,\sigma) \end{tikzcd} \qquad \begin{tikzcd} \Delta_\gamma(X,\tau) \arrow[d, "\ulcorner\varphi\urcorner=\varphi"] \\ (Y,\sigma) \end{tikzcd}$$

■.

Remark 3.7 (Relationship between TOP and L–TOP)
Let $(L,\leq,\otimes)$ be a *cqm*–lattice provided with the additonal property

$$\top\otimes\bot \;=\; \bot\otimes\top \;=\; \bot \quad .$$

Then $\{\top,\bot\}$ is a complete sub–*cqm*–lattice of $(L,\leq,\otimes)$. Further the category **TOP** of (ordinary) topological spaces is isomorphic to $\{\top,\bot\}$–**TOP**. Since the inclusion map $\iota \,:\, \{\top,\bot\} \longmapsto L$ is a **CQML** –morphism, we infer from Theorem 3.6 that L–**TOP** and **TOP** *are adjoint categories* – i.e. form an *adjoint situation w.r.t.* Δ_ι *and* $\mathfrak{L}_\iota$.
■

4 Coreflective subcategories of L–FTOP.

In this section we enrich the underlying *cqm*–lattice $(L, \leq, \otimes)$ by an additional semigroup operation $*$. More precisely,we assume that the quadruple $(L, \leq, \otimes, *)$ satisfies (I)–(III) and (V)–(VII) (cf. Subsection 1.2). Moreover, we require that $(L, \leq, *)$ is a strictly right–sided, right–symmetric quantale (cf. Subsection 1.1) provided with the additional property

$$\text{(XII)} \qquad \alpha \quad \leq \quad \top * \alpha \qquad \forall \alpha \in L .$$

Due to the monoidal closed structure we can construct a natural $L-$valued equality $[\![\ ,\ ]\!]$ on L^X as follows:

$$[\![f, g]\!] \quad = \quad \bigwedge_{x \in X} (f(x) \rightarrow_r g(x)) \wedge (g(x) \rightarrow_r f(x)) \quad , \qquad f, g \in L^X .$$

Because of (XII) we can apply Corollary 1.1.6(iv) and obtain $[\![f, g]\!] = \top * [\![f, g]\!]$ – i.e. $[\![f\,,\,g]\!]$ is strictly two–sided for all $f, g \in L^X$. Moreover, it is not difficult to verify that $[\![\ ,\ ]\!]$ satisfies the following properties:

- $[\![f, g]\!] \quad = \quad \top \quad \Longleftrightarrow \quad f = g$. *(Separation)*
- $[\![f, g]\!] \quad = \quad [\![g, f]\!]$. *(Symmetry)*
- $[\![f, g]\!] * [\![g, h]\!] \quad \leq \quad [\![f, h]\!]$. *(Transitivity)*
- $[\![f_1, f_2]\!] \otimes [\![g_1, g_2]\!] \quad \leq \quad [\![f_1 \otimes g_1\,,\, f_2 \otimes g_2]\!]$.
- $\bigwedge_{i \in I} [\![f_i, g_i]\!] \quad \leq \quad [\![\bigvee_{i \in I} f_i, \bigvee_{i \in I} g_i]\!]$.

A map $\xi : L^X \to L$ is called *extensional* iff ξ is compatible with the natural $L-$valued equality, i.e. ξ fulfills for all $f, g \in L^X$ the relation

$$(e) \qquad\qquad \xi(f) * [\![f, g]\!] \leq \xi(g).$$

For any map $\xi : L^X \to L$ we can compute its *extensional hull* $\bar{\xi}$ (i.e. the smallest extensional map "*containing* ξ") as follows:

$$\bar{\xi}(f) = \bigvee_{g \in L^X} \xi(g) * [\![g, f]\!] \qquad \forall f \in L^X \quad .$$

4.1 Extensional L–fuzzy topologies.

An L-fuzzy topological space $(X, \mathcal{T})$ is called *extensional* iff $\mathcal{T}$ is extensional w.r.t. $[\![\ ,\ ]\!]$. The class of all extensional, L–fuzzy topological spaces forms a full subcategory $\mathbb{E}L$–**FTOP** of L–**FTOP**.

Lemma 4.1.1 *Let $\varphi : X \longmapsto Y$ be a map and $\mathcal{T}$ be an extensional, L-fuzzy topology on X. Then $\mathcal{S} : L^Y \longmapsto L$ defined by*

$$\mathcal{S}(g) \quad = \quad \mathcal{T}(g \circ \varphi) \qquad \forall g \in L^Y$$

is an extensional, L-fuzzy topology on Y.

Proof. The assertion follows immediately from

$$[\![g_1, g_2]\!] \quad \leq \quad [\![g_1 \circ \varphi, g_2 \circ \varphi]\!] \qquad \forall\, g_1, g_2 \in L^Y \quad .$$

■

Remark 4.1.2 (Lattice of extensional L–fuzzy topologies)
The discrete, L–fuzzy topology $\mathcal{T}_{dis}$ (cf. 2.3) is extensional; i.e. $\mathcal{T}_{dis}$ is the universal upper bound in the set $\mathbb{E}\mathfrak{T}_L(X)$ of all extensional, L–fuzzy topologies on X. Moreover the infimum (w.r.t. $\preccurlyeq$) of a subset of $\mathbb{E}\mathfrak{T}_L(X)$ is again extensional. Hence $\mathbb{E}\mathfrak{T}_L(X)$ is a complete sublattice of $(\mathfrak{T}_L(X), \preccurlyeq)$ (cf. 2.3), and the corresponding embedding preserves arbitrary meets.
■

Theorem 4.1.3 *The category* $\mathbb{E}L$–**FTOP** *of all extensional, L–fuzzy topological spaces is a coreflective subcategory of* L–**FTOP**.

Proof. Let $\mathcal{S}$ be an L–fuzzy topology on X and $\mathbb{E}_{\mathcal{S}}$ be the set of all extensional, L–fuzzy topologies $\mathcal{T}$ on X with $\mathcal{S} \preccurlyeq \mathcal{T}$. Then the *extensional hull* $\bar{\mathcal{S}}$ of $\mathcal{S}$ is given by (cf. 4.1.2)

$$\bar{\mathcal{S}}(f) \quad = \quad \bigwedge \{\mathcal{T}(f) \mid \mathcal{T} \in \mathbb{E}_{\mathcal{S}}\} \qquad f \in L^X \quad .$$

If $\varphi : (X_1, \mathcal{S}_1) \longmapsto (X_2, \mathcal{S}_2)$ is LF–continuous, then we observe

$$\mathcal{S}_2(g) \quad \leq \quad \mathcal{S}_1(g \circ \varphi) \quad \leq \quad \bar{\mathcal{S}}_1(g \circ \varphi) \qquad g \in L^Y \quad .$$

From Lemma 4.1.1 and Remark 4.1.2 we conclude that $\bar{\mathcal{S}}_1(\_ \circ \varphi)$ is an element of $\mathbb{E}_{\mathcal{S}_2}$; i.e. φ is also LF–continuous w.r.t. $\bar{\mathcal{S}}_1$ and $\bar{\mathcal{S}}_2$.
Now we are in the position to define a functor $\mathfrak{F}_E : L$–**FTOP** $\longmapsto \mathbb{E}L$–**FTOP** by

$$\mathfrak{F}_E(X, \mathcal{S}) \quad = \quad (X, \bar{\mathcal{S}}) \qquad , \qquad \mathfrak{F}_E(\varphi) \quad = \quad \varphi \qquad .$$

It is not difficult to show that $\mathfrak{F}_E$ is a coreflection of the subcategory $\mathbb{E}L$–**FTOP**.
■.

Corollary 4.1.4 $\mathbb{E}L$–**FTOP** *is a topological subcategory*[5] *of* L–**FTOP**.

Proof. The assertion follows immediately from Theorem 2.6, Theorem 4.1.3 and from 21.35 in [1].
■

Theorem 4.1.5 (Change of basis L)

Let $(L_i, \leq_i, \otimes_i, *_i)$ *be a quadruple satisfying the properties stated at the beginning of Section 4, and let* $\mathfrak{J}_i : \mathbb{E}L_i$–**FTOP** $\longmapsto L_i$–**FTOP** *be the embedding of* $\mathbb{E}L_i$–**FTOP** *into* L_i–**FTOP** *(i=1,2). Further let* $\gamma : L_1 \longmapsto L_2$ *be a* **CQML**–*morphism provided with the additional property* [6]

[5] The definition of a topological subcatgeory is given in [1]

[6] i.e. $\gamma : (L_1, \leq_1 *_1) \longmapsto (L_2, \leq_2, *_2)$ is also a quantale–morphism (cf. [86])

$$\text{(QM)} \qquad \gamma(\alpha_1 *_1 \alpha_2) \quad = \quad \gamma(\alpha_1) *_2 \gamma(\alpha_2) \quad .$$

Then there exists a functor $\Gamma_\gamma^e : \mathbb{E}L_2\text{–}\mathbf{FTOP} \longmapsto \mathbb{E}L_1\text{–}\mathbf{FTOP}$ *satisfying the following properties*

(i) Γ_γ^e fills in the subsequent diagram

$$\begin{tikzcd} L_2\text{–}\mathbf{FTOP} \arrow[r, "\Gamma_\gamma"] & L_1\text{–}\mathbf{FTOP} \\ \mathbb{E}L_2\text{–}\mathbf{FTOP} \arrow[u, "\mathfrak{I}_2"] \arrow[r, dotted, "\Gamma_\gamma^e"'] & \mathbb{E}L_1\text{–}\mathbf{FTOP} \arrow[u, "\mathfrak{I}_1"'] \end{tikzcd}$$

(ii) Γ_γ^e has a right adjoint functor.

Proof. Referring to 4.1.3 and 2.7 it is sufficient to show that the functor Γ_γ preserves the *extensionality* of L–fuzzy topologies. We maintain the notations of 2.7. In particular let $\gamma^* : L_2 \to L_1$ be the right adjoint of γ. Then (QM) implies

- $\gamma(\alpha_1) *_2 \gamma(\alpha_1 \to_r \alpha_2) \quad \le \quad \gamma(\alpha_2) \qquad \forall \alpha_1, \alpha_2 \in L_1$.
- $\gamma^*(\beta_1) *_1 \gamma^*(\beta_2) \quad \le \quad \gamma^*(\beta_1 *_2 \beta_2) \qquad \forall \beta_1, \beta_2 \in L_2$.

Further let $\mathcal{T} : {L_2}^X \to L_2$ be an extensional, L–fuzzy topology and $\tilde{\mathcal{T}}$ be defined by

$$\tilde{\mathcal{T}}(f) \quad = \quad \gamma^* \circ \mathcal{T}(\gamma \circ f) \qquad \forall f \in {L_1}^X$$

Because of $\alpha \le \gamma^*(\gamma(\alpha))$ $(\alpha \in L_1)$ we obtain:

$$\begin{aligned} \tilde{\mathcal{T}}(f) *_1 [\![f, \bar{f}]\!] \quad &\le \quad \tilde{\mathcal{T}}(f) *_1 \gamma^*(\gamma([\![f, \bar{f}]\!])) \quad \le \quad \gamma^*(\mathcal{T}(\gamma \circ f) *_2 \gamma([\![f, \bar{f}]\!])) \\ &\le \quad \gamma^*(\mathcal{T}(\gamma \circ f) *_2 [\![\gamma \circ f, \gamma \circ \bar{f}]\!]) \quad \le \quad \gamma^*(\mathcal{T}(\gamma \circ \bar{f})) \\ &= \quad \tilde{\mathcal{T}}(\bar{f}) \qquad ; \end{aligned}$$

hence $\tilde{\mathcal{T}}$ is an extensional map from ${L_1}^X$ to L_1.
■

4.2 Enriched, L–fuzzy topologies.

An L–fuzzy topology is *weakly enriched* iff $\mathcal{T}(\alpha \cdot 1_X) = \top$ for all $\alpha \in L$. An L–fuzzy topology $\mathcal{T}$ on X is called *enriched* iff $\mathcal{T}$ satisfies the subsequent axiom

$$(\mathcal{R}) \qquad \mathcal{T}(f) \;\le\; \mathcal{T}(\alpha * f) \qquad \forall \alpha \in L, \quad \forall f \in L^X.$$

Since $(L, \le, *)$ is strictly right–sided, the axiom $(\mathcal{O}1)$ implies that every enriched L–fuzzy topology is weakly enriched.

An *(weakly) enriched L–fuzzy topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is an (weakly) enriched, L–fuzzy topology on X.

Example 4.2.1
(a) The discrete, L–fuzzy topology $\mathcal{T}_{dis}$ is obviously enriched. We show that there exist enriched, L–fuzzy topologies which are *not discrete*. For this purpose we consider a map $\mathcal{T} : L^X \longmapsto L$ defined by

$$\mathcal{T}(f) \quad = \quad (\bigvee_{x\in X} f(x)) \rightarrow_r (\bigwedge_{x\in X} f(x)) \qquad .$$

Evidently $\mathcal{T}$ fulfills $(\mathcal{O}1)$. In order to verify $(\mathcal{O}2)$ and $(\mathcal{O}3)$ we proceed as follows: First we derive from Axiom (VII)

$$\begin{aligned} (\bigvee_{x\in X} (f_1(x)\otimes f_2(x))) * (\mathcal{T}(f_1)\otimes\mathcal{T}(f_2)) \quad &\leq \\ ((\bigvee_{x\in X} f_1(x)) * \mathcal{T}(f_1)) \otimes ((\bigvee_{x\in X} f_2(x)) * \mathcal{T}(f_2)) \quad &\leq \\ (\bigwedge_{x\in X} f_1(x)) \otimes (\bigwedge_{x\in X} f_2(x)) \quad \leq \quad \bigwedge_{x\in X} (f_1(x)\otimes f_2(x)) \end{aligned}$$

– i.e. $\mathcal{T}(f_1)\otimes\mathcal{T}(f_2) \leq \mathcal{T}(f_1\otimes f_2)$. Further we observe :

$$(\bigvee_{i\in I}(\bigvee_{x\in X} f_i(x))) * (\bigwedge_{i\in I}\mathcal{T}(f_i)) \quad \leq \quad \bigvee_{i\in I}(\bigwedge_{x\in X} f_i(x)) \quad \leq \quad \bigwedge_{x\in X}(\bigvee_{i\in I} f_i(x)) \quad ;$$

i.e. $\bigwedge_{i\in I}\mathcal{T}(f_i) \quad \leq \quad \mathcal{T}(\bigvee_{i\in I} f_i)$. Since $(L, \leq, *)$ is strictly right–sided, the relation

$$\mathcal{T}(f) = \top \qquad \Longleftrightarrow \qquad \exists\, \alpha\in L : f = \alpha\cdot 1_X$$

holds; i.e. $\mathcal{T}$ is in general not discrete – e.g. $\mathcal{T} \neq \mathcal{T}_{dis}$, whenever X contains at least two different elements.
Finally Axiom $(\mathcal{R})$ follows from

$$(\bigvee_{x\in X}(\alpha * f(x))) * \mathcal{T}(f) \quad \leq \quad \alpha * (\bigwedge_{x\in X} f(x)) \quad \leq \quad \bigwedge_{x\in X}(\alpha * f(x)) \qquad ;$$

i.e. $\mathcal{T}$ is a non discrete, enriched L–fuzzy topology on X.

(b) Let $\mathbb{B}$ be a complete *Boolean algebra*, and let $* = \otimes = \wedge$ – i.e. the corresponding quadruple is given by $(\mathbb{B}, \leq, \wedge, \wedge)$. Then every *extensional* $\mathbb{B}$–fuzzy topology $\mathcal{T}$ is *enriched.* In fact, if we combine $(\mathcal{O}1)$ and $(\mathcal{O}1')$ with the extensionality of $\mathcal{T}$ (cf. section 4.1), we obtain immediately

$$\top \;=\; \alpha\vee(\alpha\rightarrow\bot) \;=\; [\![\alpha\cdot 1_X, 1_X]\!] \vee [\![\alpha\cdot 1_X, 1_\varnothing]\!] \quad \leq \quad \mathcal{T}(\alpha\cdot 1_X)$$

Since $*$ and $\otimes$ coincide, the axiom $(\mathcal{R})$ follows from $(\mathcal{O}2)$.
Moreover we observe that the indiscrete, extensional, $\mathbb{B}$–fuzzy topology $\mathcal{T}^e_{ind}$ on X is given by the following formulae (cf. (a)):

$$\mathcal{T}^e_{ind}(h) \quad = \quad (\bigwedge_{x\in X} h(x)) \vee ((\bigvee_{x\in X} h(x)) \rightarrow \bot) \quad = \quad (\bigvee_{x\in X} h(x)) \rightarrow (\bigwedge_{x\in X} h(x)).$$

■

Let **R**L–**FTOP** (resp. **WR**L–**FTOP**) be the full subcategory of L–**FTOP** formed by all enriched (resp. weakly enriched), L–fuzzy topological spaces. Since the infimum of a family $\mathfrak{A}$ of (weakly) enriched, L–fuzzy topologies is again (weakly) enriched (here we view $\mathfrak{A}$ as a subset of $\mathfrak{T}_L(X)$), we obtain immediately

Theorem 4.2.2 **R**L–**FTOP** *(resp.* **WR**L–**FTOP***) is a coreflective subcategory of* L–**FTOP***. In particular* **R**L–**FTOP** *(resp.* **WR**L–**FTOP***) is a topological subcategory of* L–**FTOP***.*

Proposition 4.2.3 *The terminal object* **1** *in* **WR**L–**FTOP** *is simultaneously an indiscrete and a discrete* **WR**L–**FTOP***-object. In particular, constant maps are always* **WR**L–**FTOP***-morphisms.*

Proof. Since the support set of **1** is given by a singleton $\{\cdot\}$, it is easy to see that the indiscrete, weakly enriched L–fuzzy topology and the discrete (enriched), L–fuzzy topology coincide on $\{\cdot\}$.
■

4.3 Strongly enriched L–fuzzy topologies

We start with the observation that every map $\xi : L^X \longmapsto L$ induces a further map $\underline{\cup}\xi : X \longmapsto L$ by

$$(\underline{\cup}\xi)(x) = \bigvee_{f \in L^X} \xi(f) * f(x) \qquad \forall x \in X.$$

If we understand ξ as a "fuzzy system of fuzzy subsets of X", then $\underline{\cup}\xi$ can be interpreted as the "join of ξ". The equation $\underline{\cup}\xi = \underline{\cup}\bar{\xi}$ follows immediately from the definition of extensional hulls.

An *$L-$fuzzy topology* $\mathcal{T}$ on X is called *strongly enriched* if and only if $\mathcal{T}$ is *extensional* and satisfies the additional axiom:

$$(\mathfrak{S}) \qquad \bigwedge_{f \in L^X} (\xi(f) \rightarrow_r \mathcal{T}(f)) \quad \leq \quad \mathcal{T}(\underline{\cup}\xi) \qquad \forall \xi \in L^{(L^X)}$$

A *strongly enriched, $L-$fuzzy topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a strongly enriched, $L-$fuzzy topology on X.

Theorem 4.3.1 *Let $\mathcal{T} : L^X \rightarrow L$ be an extensional, enriched $L-$fuzzy topology on X (cf. 4.1 and 4.2). Then the following assertions are equivalent:*

(i) $\mathcal{T}$ is a strongly enriched, $L-$fuzzy topology.

(ii) For all extensional maps $\xi : L^X \rightarrow L$ with $\xi \leq \mathcal{T}$ the relation $\mathcal{T}(\underline{\cup}\xi) = \top$ holds.

Proof. The implication (i) $\Longrightarrow$ (ii) is obvious. In order to verify $(ii) \Longrightarrow (i)$ we proceed as follows : Let $\xi : L^X \to L$ be given; then we put $\kappa_\xi := \bigwedge_{f \in L^X} (\xi(f) \to_r \mathcal{T}(f))$ and define $\zeta : L^X \to L$ by

$$\zeta(f) \quad = \quad \xi(f) * \kappa_\xi \qquad \forall f \in L^X$$

Obviously, $\zeta \leq \mathcal{T}$; hence $\bar{\zeta} \leq \mathcal{T}$ follows from the extensionality of $\mathcal{T}$. Now we apply assertion (ii) and obtain:

$$\top \quad = \quad \mathcal{T}(\underline{\cup}\bar{\zeta}) \quad = \quad \mathcal{T}(\underline{\cup}\zeta).$$

Because of the right–symmetry of $(L, \leq, *)$ the inequality $\kappa_\xi \leq [\![\underline{\cup}\zeta, \underline{\cup}\xi]\!]$ holds; hence once again we can apply Axiom (XII) and the extensionality of $\mathcal{T}$:

$$\kappa_\xi \quad \leq \quad \top * [\![\underline{\cup}\zeta, \underline{\cup}\xi]\!] \quad \leq \quad \mathcal{T}(\underline{\cup}\xi).$$

Therewith Assertion (i) is verified.
■

Proposition 4.3.2 *Let $\mathcal{T}$ be a strongly enriched, L–fuzzy topology on X. Then $\mathcal{T}$ fulfills the following properties*

(a) $\mathcal{T}(\mathcal{T}(f) * f) \quad = \quad \top \qquad \forall f \in L^X$.

(b) If $\mathcal{T}(f) = \top$, then $\mathcal{T}(\alpha * f) = \top$ for all $\alpha \in L$.

(c) If $\top * \mathcal{T}(f) = \mathcal{T}(f)$, then $\mathcal{T}(f * \mathcal{T}(f)) = \top$.

(d) $\mathcal{T}$ is an enriched L–fuzzy topology.

Proof. Every element $f \in L^X$ induces a map $\xi_1 : L^X \longmapsto L$ by

$$\xi_1(g) \quad = \quad \mathcal{T}(f) * [\![f, g]\!] \qquad \forall g \in L^X.$$

The extensionality of $\mathcal{T}$ implies $\xi_1 \leq \mathcal{T}$. Further we observe $(\underline{\cup}\xi_1)(x) = \alpha * \mathcal{T}(f) * f(x)\ \forall x \in X$; hence Property (a) follows from Theorem 4.3.1.
In order to verify Property (b) we fix $\alpha \in L$ and consider $f \in L^X$ with $\mathcal{T}(f) = \top$. Then we define a map $\xi_2 : L^X \longmapsto L$ by

$$\xi_2(g) \quad = \quad \alpha * [\![f, g]\!] \qquad \forall g \in L^X$$

and observe $(\underline{\cup}\xi_2)(x) = \alpha * f(x)$. Using again the extensionality of $\mathcal{T}$ we obtain

$$\xi_2(g) \quad \leq \quad \top * [\![f, g]\!] \quad = \quad \mathcal{T}(f) * [\![f, g]\!] \quad \leq \quad \mathcal{T}(g)\,;$$

hence Property (b) follows also from Theorem 4.3.1.
Now we assume $\top * \mathcal{T}(f) \leq \mathcal{T}(f)$ and consider a map $\xi_3 : L^X \longmapsto L$ determined by

$$\xi_3(g) \quad = \quad [\![f, g]\!] * \mathcal{T}(f) \qquad \forall g \in L^X.$$

Since $(L, \leq, *)$ is right–symmetric and Axiom(XII) holds, we obtain:

$$\xi_3(g) \quad \leq \quad \top * \llbracket f, g \rrbracket * \mathcal{T}(f) \quad = \quad \top * \mathcal{T}(f) * \llbracket f, g \rrbracket \quad \leq \quad \mathcal{T}(g) \ ;$$

hence the extensionality of $\mathcal{T}$ implies: $\xi_3 \ \leq \ \mathcal{T}$. Using again the right–symmetry of $\mathcal{T}$ we observe

$$(\underline{\sqcup}\xi_3)(x) \quad = \quad f(x) * \mathcal{T}(f) \qquad\qquad x \in X \ .$$

Therefore Property (c) follows from Theorem 4.3.1.
Further the inequality $\mathcal{T}(f) \ \leq \ \llbracket \alpha * \mathcal{T}(f) * f \, , \, \alpha * f \rrbracket$ is an immediate consequence from the right–symmetry of $(L, \leq, *)$. Now we combine the properties (a) and (b) and derive the following relation from Axiom (XII) and the extensionality of $\mathcal{T}$:

$$\begin{aligned} \mathcal{T}(f) \ &\leq \ \top * \llbracket \alpha * \mathcal{T}(f) * f \, , \alpha * f \rrbracket \ = \\ &= \ \mathcal{T}(\alpha * \mathcal{T}(f) * f) * \llbracket \alpha * \mathcal{T}(f) * f \, , \alpha * f \rrbracket \ \leq \\ &\leq \ \mathcal{T}(\alpha * f) \qquad ; \end{aligned}$$

hence the axiom $\mathcal{R}$ is verified.
■

The full subcategory of all strongly, enriched, L–fuzzy topological spaces is denoted by **S**$\mathbb{R}$$L$**–FTOP**. Since strong enrichedness is preserved under infima (of L–fuzzy topologies) we obtain immediately

Theorem 4.3.3 *The category* **SR**L**–FTOP** *is a coreflective subcategory of* **R**L**–FTOP** *and* $\mathbb{E}L$**–FTOP** *(resp.* L**–FTOP***). In particular* **SR**L**–FTOP** *is a topological subcategory of* **R**L**–FTOP** *and* $\mathbb{E}L$**–FTOP** *(resp.* L**–FTOP***).*
■

Hierarchy :

$$\begin{tikzcd} & \mathbf{SR}L\text{–}\mathbf{FTOP} \arrow[dl] \arrow[dr] & \\ \mathbb{E}L\text{–}\mathbf{FTOP} \arrow[dr] & & \mathbf{R}L\text{–}\mathbf{FTOP} \arrow[dl] \\ & L\text{–}\mathbf{FTOP} & \end{tikzcd}$$

Under the hypothesis that the underlying quantale is a right GL–quantale (cf. Subsection 1.1) we can give an important characterization of strongly enriched, L–fuzzy topological spaces.

Theorem 4.3.4 *Let $(L, \leq, *)$ be a right GL–quantale. Further let $\mathcal{T}$ be an extensional, L–fuzzy topology on X. Then the following assertions are equivalent:*

(i) $\mathcal{T}$ is a strongly enriched, $L-$fuzzy topology.

(ii) $\mathcal{T}$ is provided with the properties

(a) $\forall f \in L^X : \quad \mathcal{T}(\mathcal{T}(f) * f) \; = \; \top .$

(b) $\forall \alpha \in L : \quad \mathcal{T}(f) \; = \; \top \quad \Longrightarrow \quad \mathcal{T}(\alpha * f) \; = \; \top .$

Proof. The implication (i) $\Longrightarrow$ (ii) follows from Proposition 4.3.2(a) and (b). In order to verify the implication (ii) $\Longrightarrow$ (i) we proceed as follows:
If $\xi : L^X \to L$ is an extensional map with $\xi \leq \mathcal{T}$, then the relation

$$\xi(f) \quad = \quad (\mathcal{T}(f) \rightarrow_{\ell} \xi(f)) \, * \, \mathcal{T}(f) \qquad \forall f \in L^X$$

follows from Corollary 1.1.6(i). Now we invoke (ii) and obtain:

$$\mathcal{T}(\mathcal{T}(f) * f) \quad = \quad \top \qquad \text{and} \qquad \mathcal{T}(\xi(f) * f) \quad = \quad \top \qquad .$$

Referring to $(\mathcal{O}3)$ the relation $\mathcal{T}(\bigvee\limits_{f \in L^X} \xi(f) * f) = \top$ follows; hence by virtue of Theorem 4.3.1 the assertion (i) is verified.
■

Theorem 4.3.5 (Change of basis)
Let $(L_i, \leq_i, \otimes_i, *_i)$ be a quadruple provided with the axioms (I)–(III) and (V)–(VII) (i=1,2). Further let $\gamma \, : \, (L_1, \leq_1, \otimes_1) \longmapsto (L_2, \leq_2, \otimes_2)$ be a **CQML**–*morphism provided with Property (QM) (cf. 4.1.5). Then the following assertions are valid*

(a) The functor $\Gamma_\gamma : L_2$–**FTOP** $\to L_1$–**FTOP** (see Remark 2.7) preserves strongly enriched fuzzy topological spaces, i.e. the following diagram commutes

$$\begin{tikzcd} L_2\text{–}\mathbf{FTOP} \arrow[r, "\Gamma_\gamma"] & L_1\text{–}\mathbf{FTOP} \\ \mathbf{SR}L_2\text{–}\mathbf{FTOP} \arrow[u] \arrow[r, dotted] & \mathbf{SR}L_1\text{–}\mathbf{FTOP} \arrow[u] \end{tikzcd}$$

(b) $\Gamma_\gamma \Big|_{\mathbf{SR}L_2\text{–}\mathbf{TOP}}$ has a right adjoint functor.

Proof. (a) Let $\mathcal{T} \, : \, L_2^X \to L_2$ be a strongly enriched, L_2–fuzzy topology on X. We maintain the notation from 2.7 and 4.1.5. Referring to Theorem 4.1.5 it is sufficient to show that $\tilde{\mathcal{T}}$ fufills Axiom $(\mathfrak{S})$. Therefore let us consider a map $\xi \, : \, L_1^X \longmapsto L_1$ with $\xi \preccurlyeq \tilde{\mathcal{T}}$; then we define a map $\hat{\xi} \, : \, L_2^X \longmapsto L_2$ by

$$\hat{\xi}(g) \quad = \quad \bigvee \{ \gamma \circ \xi(f) \mid f \in L_1^X \text{ with } g \, = \, \gamma \circ f \}$$

and observe : $\underline{\sqcup}\hat{\xi} \;=\; \gamma\circ(\underline{\sqcup}\xi)$. Further we infer from $\gamma(\gamma^*(\beta)) \leq \beta \quad (\forall\, \beta \in L_2)$

$$\hat{\xi}(g) \quad \leq \quad \mathcal{T}(g) \qquad \forall\, g \in L_2^X \quad .$$

Since $\mathcal{T}$ satisfies $(\mathfrak{S})$, we obtain

$$\top \quad = \quad \mathcal{T}(\underline{\sqcup}\hat{\xi}) \quad = \quad \mathcal{T}(\gamma\circ(\underline{\sqcup}\xi)) \quad = \quad \tilde{\mathcal{T}}(\underline{\sqcup}\xi) \quad = \quad \top\,;$$

hence because of Theorem 4.3.1 the xiom $(\mathfrak{S})$ holds also for $\tilde{\mathcal{T}}$.

(b) The assertion (b) is a consequence of Assertion (a) and of the fact that infima of strongly enriched, fuzzy topologies can be computed pointwise.

■

5 Coreflective subcategories of L–TOP

In this section the quadruple $(L, \leq, \otimes, *)$ satisfies (I)–(III), (V)–(VII) (cf. Subsection 1.2). Further we assume that $(L, \leq, *)$ is a strictly right–sided, right–symmetric quantale provided with the additional axiom (XII) (cf. Section 4).

5.1 Stratified L–topologies

An L–topology τ on X is *weakly stratified* iff τ fulfills the so–called constants condition – i.e.

$(\Sigma 0) \quad \alpha \cdot 1_X \in \tau \qquad \forall\, \alpha \in L\,.$ (Constants Condition)

An L–topology on τ is said to be *stratified* iff τ is provided with the important property

$(\Sigma 1) \quad g \in \tau,\; \alpha \in L \quad \Longrightarrow \quad \alpha * g \in \tau\,.$ (Truncation Condition)

Since $(L, \leq, *)$ is strictly right–sided, Axiom (o1) implies that every stratified L–topology is also weakly stratified. More information about the relationship between weakly stratified and stratified L–topologies can be found in Theorem 5.2.7.

An L–topological space (X, τ) is said to be (*weakly*) *stratified* iff τ is (weakly) stratified. The full subcategory of all stratified (resp. weakly stratified), L–topological spaces is denoted by **S**L–**TOP** (resp. **WS**L–**TOP**). Since the *indiscrete*, weakly stratified L–topology τ^s_{ind} on X is given by

$$\tau^s_{ind} \quad = \quad \{\alpha \cdot 1_X \mid \alpha \in L\} \qquad ,$$

all *constant* morphisms are **WS**L–**TOP**–morphisms. Moreover we observe that the axioms (o1),(o2),(o3), $(\Sigma 0)$ and $(\Sigma 1)$ are preserved under arbitrary intersection. Therefore we obtain

Theorem 5.1.1 *The category* **S**L–**TOP** *(resp.* **WS**L–**TOP***) is a coreflective subcategory of* L–**TOP**. *In particular,* **S**L–**TOP** *(resp.* **WS**L–**TOP***) is a topological subcategory of* L–**TOP**. *Moreover* **S**L–**TOP** *is a reflective and coreflective subcategory of* **WS**L–**TOP**.

Proof. The first and third assertion are obvious. The second assertion follows from the first one, Theorem 3.4 and 21.35 in [1].
■

Referring to Section 4.2 we see immediately that the restriction $\mathfrak{K}_R$ of the functor $\mathfrak{K}$ to **R**L–**FTOP** (cf. Theorem 3.5) factors through **S**L–**TOP** :

$$\begin{array}{ccc} L\text{–}\mathbf{FTOP} & \xrightarrow{\ \mathfrak{K}\ } & L\text{–}\mathbf{TOP} \\ \uparrow & & \uparrow \\ \mathbf{R}L\text{–}\mathbf{FTOP} & \xrightarrow[\mathfrak{K}_R]{} & \mathbf{S}L\text{–}\mathbf{TOP} \end{array}$$

Theorem 5.1.2 $\mathfrak{K}_R$: **R**L–**FTOP** $\longmapsto$ **S**L–**TOP** *has a right adjoint functor.*

Proof. After some modifications the proof of Theorem 3.5 can be repeated verbatim.
■

An analogous version of Theorem 5.1.2 holds also for the categories **WR**L–**FTOP** of weakly enriched, L–fuzzy topological spaces and **WS**L–**TOP** of weakly stratified, L–topological spaces.

5.2 Strongly stratified L–topologies

A stratified L–topology τ on X (cf. Subsection 5.1) is called *strongly stratified* iff τ satisfies the additional property[7]

$$(\Sigma 2) \qquad h \in L^X, \quad g \in \tau \quad \Longrightarrow \quad [\![g, h]\!] * h \in \tau \quad \text{and} \quad h * [\![g, h]\!] \in \tau\ .$$

An L–topological space (X, τ) is said to be *strongly stratified* iff τ is strongly stratified.

Lemma 5.2.1 *If* $(L, \leq, *)$ *is commutative and has square roots (cf. Subsection 1.2, Proposition 1.2.7, Corollary 1.2.8), then* $(\Sigma 2)$ *implies* $(\Sigma 1)$.

Proof. Let $\alpha^{1/2}$ be the square root of $\alpha \in L$. Then the assertion follows from

$$[\![g\,,\,\alpha^{1/2} * g]\!] * \alpha^{1/2} * g \quad = \quad \alpha * g$$

■

Example 5.2.2 Let $([0,1], \leq, Prod)$ be the real unit interval provided with the ususal ordering $\leq$ and the usual multiplication $Prod$. Obviously the quadruple $([0,1], \leq, Min, Prod)$ satisfies the axioms (I) – (VII) (cf. Lemma 1.2.1, Example 1.2.3(b)). Because of

$$g * [\![g, k]\!] * [\![g, k]\!] \quad \leq \quad k * [\![g, k]\!]$$

---

[7] $[\![\,,\,]\!]$ denotes the natural L–valued equality on L^X defined at the beginning of Section 4.

we obtain that

$$\tau_0 = \{g \in [0,1]^X \mid g = 1_\varnothing \quad \text{or} \quad \inf_{x \in X} g(x) \neq 0\}.$$

is a strongly stratified $[0,1]$–fuzzy topology τ_0 on X which is *not discrete.* Referring to Lemma 7.3.12 we can show that τ_0 is the *indiscrete*, strongly stratified $[0,1]$–topology on X. ∎

The full subcategory of all strongly stratified, L–topological spaces is denoted by **S**$\mathbf{S}L$**–TOP**. Since the axioms (o1), (o2), (o3), (Σ1) and (Σ2) are preserved under arbitrary intersections, we obtain immediately:

Theorem 5.2.3 *The category* **SS**L**–TOP** *is a coreflective subcategory of* **S**L**–TOP** *and* L**–TOP**. *In particular* **SS**L**–TOP** *is a topological subcategory of* **S**L**–TOP** *and* L**–TOP**.

Lemma 5.2.4 *Let* $(X, \mathcal{T})$ *be a strongly enriched, L–fuzzy topological space, and let* $\mathfrak{K}$ *be the functor defined in Theorem 3.5. Then* $(X, \tau_{\mathcal{T}}) = \mathfrak{K}(X, \mathcal{T})$ *is a strongly stratified, L–topological space.*

Proof. Referring to 4.3.2 we only have to show that $\tau_{\mathcal{T}}$ satisfies (Σ2). For this purpose we fix $g \in \tau_{\mathcal{T}}$ and $f \in L^X$ and consider maps $\xi_1 : L^X \longmapsto L$, $\xi_2 : L^X \longmapsto L$ determined by

$$\xi_1(h) \;=\; [\![g, f]\!] * [\![f, h]\!] \quad , \quad \xi_2(h) \;=\; [\![f, h]\!] * [\![g, f]\!] \qquad \forall\, h \in L^X \,.$$

Since $\mathcal{T}$ is extensional, we obtain from the right–symmetry of $(L, \leq, *)$ and Axiom (XII) :

$$\begin{aligned} \xi_1(h) \;&\leq\; \top * \xi_1(h) \;\leq\; \mathcal{T}(h) \\ \xi_2(h) \;&\leq\; \top * \xi_2(h) \;=\; \top * [\![g, f]\!] * [\![f, h]\!] \;\leq\; \mathcal{T}(h)\,; \end{aligned}$$

hence (Σ2) follows from Axiom ($\mathfrak{S}$) and the observation

$$\underline{\sqcup}\xi \;=\; [\![g, f]\!] * f \quad , \quad \underline{\sqcup}\xi_2 \;=\; f * [\![g, f]\!] \quad .$$

∎

Because of the previous Lemma 5.2.4 the restriction $\mathfrak{K}_{SR}$ of $\mathfrak{K}$ to **SR**L**–FTOP** factors through **SS**L**–TOP** :

$$\begin{tikzcd} L\text{–}\mathbf{FTOP} \arrow[r, "\mathfrak{K}"] & L\text{–}\mathbf{TOP} \\ \mathbf{SR}L\text{–}\mathbf{FTOP} \arrow[u] \arrow[r, dotted, "\mathfrak{K}_{SR}"'] & \mathbf{SS}L\text{–}\mathbf{TOP} \arrow[u] \end{tikzcd}$$

Let **SR**L**–FTOP*** be the full subcategory of **SR**L**–FTOP** which consists of all strongly enriched, L–fuzzy topological spaces $(X, \mathcal{T})$ satisfying the additional axiom

$$(\Sigma 3) \qquad \top * \mathcal{T}(f) \;=\; \mathcal{T}(f) \qquad \forall f \in L^X .$$

In particular, for every $f \in L^X$ the element $\mathcal{T}(f)$ is strictly two–sided. Moreover **SR**L–**FTOP*** is a coreflective subcategory of **SR**L–**FTOP**. The categories **SR**L–**FTOP*** and **SR**L–**FTOP** coincide whenever the underlying quantale is strictly two–sided.

If $\mathfrak{K}^*_{SR}$ denotes the restriction of $\mathfrak{K}$ to **SR**L–**FTOP***, then we can complete the previous diagram as follows:

$$\begin{tikzcd}
L\text{–}\mathbf{FTOP} \arrow[r, "\mathfrak{K}"] & L\text{–}\mathbf{TOP} \\
\mathbf{SR}L\text{–}\mathbf{FTOP} \arrow[u] \arrow[r] & \mathbf{SS}L\text{–}\mathbf{TOP} \arrow[u] \\
\mathbf{SR}L\text{–}\mathbf{FTOP}^* \arrow[u] \arrow[ur, dotted, "\mathfrak{K}^*_{SR}"'] &
\end{tikzcd}$$

Theorem 5.2.5 (Main result I)
The category **SR**L–**FTOP*** *is isomorphic to a coreflective subcategory of* **SS**L–**TOP** *of all strongly stratified, L–topological spaces.*

Proof. (a) We show that $\mathfrak{K}^*_{SR}$ is an embedding. Let $\mathcal{T}$ be a strongly enriched, L–fuzzy topology on X provided with Property (Σ3); then we put $\mathfrak{K}(X, \mathcal{T}) = (X, \tau_{\mathcal{T}})$. Since $\mathcal{T}(f)$ is strictly two–sided for all $f \in L^X$, we deduce from Proposition 4.3.2(c):

$$\begin{aligned} \mathcal{T}(f) \;&\leq\; [\![f * \mathcal{T}(f) \,,\, f]\!] \;\leq\; \top * [\![f * \mathcal{T}(f) \,,\, f]\!] \;\leq \\ &\leq\; \bigvee_{g \in \tau_{\mathcal{T}}} \top * [\![g, f]\!] \;=\; \bigvee_{g \in \tau_{\mathcal{T}}} [\![g, f]\!] \;\leq\; \mathcal{T}(f) \,; \end{aligned}$$

hence the relation $\mathcal{T}(f) = \bigvee\limits_{g \in \tau_{\mathcal{T}}} [\![g, f]\!]$ follows – i.e. $\mathfrak{K}$ is injective on **SR**L–**FTOP***–objects and bijective on HOM–sets of **SR**L–**FTOP***. Therefore $\mathfrak{K}^*_{SR}$ is an embedding.

(b) In order to show that $\mathfrak{K}^*_{SR}$ has a right adjoint functor we can repeat the proof of Theorem 3.5 verbatim provided we make the obvious modifications (cf. 5.1.2).

■

In the next theorem we give a *sufficient* condition under which **SR**L–**FTOP*** and **SS**L–**TOP** are *isomorphic.*

Theorem 5.2.6 (Main result II)
Let $(L, \leq, \otimes)$ *be a cl–quasi–monoid (cf. Section 1) and* $(L, \leq, *)$ *be a GL–quantale such that* $(L, \leq, \otimes, *)$ *satisfies the compatibility axiom (VII) (cf. Subsections 1.1 and 1.2). Then* **SR**L–**FTOP*** *is isomorphic to* **SS**L–**TOP**.

Proof. Referring to the proof of the previous Theorem 5.2.5 it is sufficient to show that $\mathfrak{K}^*_{SR}$ is surjective on **SRL–FTOP***–objects. Therefore let τ be a strongly stratified L–topology on X; then τ induces a map $\mathcal{T}_\tau : L^X \longmapsto L$ by

$$\mathcal{T}_\tau(h) \quad = \quad \bigvee_{g\in\tau} [\![g,h]\!] \qquad \forall\, h \in L^X \quad .$$

(a) First we verify that $\mathcal{T}_\tau$ is a strongly enriched, L–fuzzy topology. $(\mathcal{O}1)$ follows immediately from $(o1)$. Further, since $[\![\ ,\ ]\!]$ is a L–valued congruence w.r.t. $\otimes$, we can deduce the subsequent relation from the axioms (IV) and $(o2)$:

$$\begin{aligned} \mathcal{T}_\tau(f_1)\otimes\mathcal{T}_\tau(f_2) \quad &= \quad \bigvee_{g_1,g_2\in\tau} ([\![f_1,g_1]\!]\otimes[\![f_2,g_2]\!]) \\ &\leq \quad \bigvee_{g_1,g_2\in\tau} [\![f_1\otimes f_2, g_1\otimes g_2]\!] \qquad ; \end{aligned}$$

hence $(o2)$ implies $(\mathcal{O}2)$.

(a1) Let us consider a map $\xi : L^X \to L$ with $\xi \leq \mathcal{T}_\tau$. Referring to Corollary 1.1.6(i), we obtain :

$$\begin{aligned} (\underline{\sqcup}\xi)(x) \quad &= \quad \bigvee_{f\in L^X} (\mathcal{T}_\tau(f) \to_\ell \xi(f)) * \mathcal{T}_\tau(f) * f(x) \\ &= \quad \bigvee_{f\in L^X, g\in\tau} (\mathcal{T}_\tau(f) \to_\ell \xi(f)) * [\![g,f]\!] * f(x) \qquad ; \end{aligned}$$

hence $\underline{\sqcup}\xi \in \tau$ follows from $(o3)$, $(\Sigma1)$ and $(\Sigma2)$; i.e. $\mathcal{T}_\tau$ satisfies the assertion (ii) in Theorem 4.3.1.

(a2) In order to verify $(\mathcal{O}3)$ we proceed as follows: Since $\mathcal{T}_\tau(f)$ is strictly two–sided for every $f \in L^X$, the extensionality of $\mathcal{T}_\tau$ implies:

$$\begin{aligned} \bigvee_{j\in I} [\![f_j,h]\!] * (\bigwedge_{i\in I} \mathcal{T}_\tau(f_i)) \quad &= \quad \top * \Big(\bigvee_{j\in I} (\bigwedge_{i\in I} \mathcal{T}_\tau(f_i)) * [\![f_j,h]\!]\Big) \quad \leq \\ &\leq \quad \top * \mathcal{T}_\tau(h) \quad = \quad \mathcal{T}(h) \quad . \end{aligned}$$

Now we use the result of the previous step (a1), the right–symmetry of $(L,\leq,*)$ and obtain that $\big(\bigvee_{j\in I} f_j\big) * \big(\bigwedge_{i\in I} \mathcal{T}(f_i)\big)$ is an element of τ. Applying again the extensionality of $\mathcal{T}_\tau$ and Axiom (XII) the relation

$$\bigwedge_{i\in I} \mathcal{T}_\tau(f_i) \quad \leq \quad \top * [\![(\bigvee_{j\in I} f_j) * (\bigwedge_{i\in I} \mathcal{T}_\tau(f_i)), \bigvee_{i\in I} f_i]\!] \quad \leq \quad \mathcal{T}_\tau(\bigvee_{i\in I} f_i) \qquad ,$$

follows; hence $(\mathcal{O}3)$ is verified.

Summing up we conclude from Theorem 4.3.1 that $\mathcal{T}_\tau$ is a strongly enriched, L–fuzzy topology.

(b) Let h be an element of L^X with $\mathcal{T}_\tau(h) = \top$. Then h admits the following representation

$$h(x) = h(x) * \top = \bigvee_{g \in \tau} h(x) * [\![g,h]\!] \quad , \quad x \in X \quad ;$$

hence we infer from $(o3)$ and $(\Sigma 2)$:

$$\mathcal{T}_\tau(h) = \top \iff h \in \tau \quad .$$

In particular, $\mathfrak{K}^*_{SR}$ is surjective on **SSL–TOP**–objects.
■

If the underlying GL–quantale is *commutative* (i.e. is a GL–monoid), then **SRL–FTOP** and **SRL–FTOP*** coincide , and **SRL–FTOP** is isomorphic to **SSL–TOP**. Moreover, if the underlying GL–monoid is idempotent – i.e. is determined by a *complete Heyting algebra* (cf. Subsection 1.1), then the following theorem shows that strongly stratified, stratified and weakly stratified L–topologies are *equivalent* concepts.

Theorem 5.2.7 *Let $(L, \leq)$ be a complete Heyting algebra and $\otimes = * = \wedge$ (i.e. $(L, \leq, \otimes, *) = (L, \leq, \wedge, \wedge)$). Further, let τ be an L–topology on X. Then the following assertions are equivalent:*

(i) τ is weakly stratified

(ii) τ is stratified.

(iii) τ is strongly stratified.

Proof. The assertion (i) together with $(o2)$ imply that τ is stratified. Further the idempotency of $\wedge$ implies

$$[\![g,h]\!] \wedge h = [\![g,h]\!] \wedge g \qquad \forall g,h \in L^X \quad ;$$

hence every stratified L–topology is also strongly stratified. Finally, Condition $(\Sigma 1)$ and Axiom $(o1)$ imply (i).
■.

In the case of *complete Heyting algebras* we conclude from Theorem 5.2.6 and Theorem 5.2.7 that L–topologies satisfying the Constants Condition (cf. (i) in 5.2.7) and strongly enriched, L–fuzzy topologies are the same things. From a historical point of view this observation was the fundamental message in [37].

In the case of GL–quantales we summarize the results of this section in the following commutative diagram:

$$
\begin{tikzcd}
& \mathbf{SR}L\text{–}\mathbf{FTOP}^* \simeq \mathbf{SS}L\text{–}\mathbf{TOP} \arrow[dl] \arrow[dr] & \\
\mathbf{R}L\text{–}\mathbf{FTOP} \arrow[rr, "\mathfrak{K}_R"] \arrow[d] & & \mathbf{S}L\text{–}\mathbf{TOP} \arrow[d] \\
L\text{–}\mathbf{FTOP} \arrow[rr, "\mathfrak{K}"'] & & L\text{–}\mathbf{TOP}
\end{tikzcd}
$$

6 Convergence theory for L–topological spaces and its applications

In this section the underlying quadruple satisfies the axioms (I)–(VII). In particular, $(L, \leq, \otimes)$ is a *cl*–quasi–monoid. Further, we will frequently make use of the monoidal mean operator (cf. Remark 1.2.6). Therfore, because of Proposition 1.2.7 we assume in addition that $(L, \leq, *)$ is a *commutative, strictly two–sided* quantale. In this context $\to_r$ and $\to_\ell$ coincide and are denoted by $\to$.

6.1 L–Interior operators and L– neighborhood systems

Let X be a set; a map $\mathcal{K} : L^X \to L^X$ is called an L*–interior operator* iff $\mathcal{K}$ satisfies the following conditions:

(K0) $\mathcal{K}(1_X) = 1_X$.

(K1) $f \leq g \implies \mathcal{K}(f) \leq \mathcal{K}(g)$. (Isotonicity)

(K2) $(\mathcal{K}(f)) \otimes (\mathcal{K}(g)) \leq \mathcal{K}(f \otimes g)$.

(K3) $\mathcal{K}(f) \leq f$.

(K4) $\mathcal{K}(f) \leq \mathcal{K}(\mathcal{K}(f))$.

Because of Axiom (II) it is easy to see that every L–topology τ on X induces an L–interior operator $\mathcal{K}_\tau$ by

$$\mathcal{K}_\tau(f) = \bigvee\{g \in \tau \mid g \leq f\}$$

and vice versa every L–interior operator $\mathcal{K}$ gives rise to an L–topology $\tau_\mathcal{K}$ on X defined by

$$\tau_\mathcal{K} = \{g \in L^X \mid g \leq \mathcal{K}(g)\} .$$

Obviously the relations $\mathcal{K}_{\tau_{\mathcal{K}}} = \mathcal{K}$ and $\tau_{\mathcal{K}_\tau} = \tau$ hold – i.e. L–interior operators and L–topologies are equivalent concepts.

An L–interior operator $\mathcal{K}$ is called *stratified* iff $\mathcal{K}$ satisfies the additional condition

(K5) $\quad \alpha * \mathcal{K}(f) \;\leq\; \mathcal{K}(\alpha * f) \qquad \forall\, \alpha \in L \quad \forall\, f \in L^X .$

Proposition 6.1.1 *Let τ be an L–topology on X and $\mathcal{K}$ be the corresponding L–interior operator. Then the following assertions are equivalent*

(i) τ is stratified.

(ii) $\mathcal{K}$ is stratified.

(iii) $\mathcal{K}$ is compatible with the natural equality $[\![\ ,\ ]\!]$ on L^X – i.e.

$$[\![f_1, f_2]\!] \;\leq\; [\![\mathcal{K}(f_1), \mathcal{K}(f_2)]\!] \qquad \forall\, f_1, f_2 \in L^X \quad .$$

Proof. It is not difficult to show that ($o3$) and ($\Sigma 1$) are equivalent to (K4) and (K5); hence the equivalence (i) $\Longleftrightarrow$ (ii) follows. In order to verify (ii) $\Longleftrightarrow$ (iii) we first observe

$$[\![f_1, f_2]\!] * f_1 \;\leq\; f_2 \quad , \quad [\![f_1, f_2]\!] * f_2 \;\leq\; f_1 \quad .$$

Then we infer from (K1) and (K5) :

$$[\![f_1, f_2]\!] * \mathcal{K}(f_1) \;\leq\; \mathcal{K}(f_2) \quad , \quad [\![f_1, f_2]\!] * \mathcal{K}(f_2) \;\leq\; \mathcal{K}(f_1) \quad ;$$

hence $[\![f_1, f_2]\!] \;\leq\; [\![\mathcal{K}(f_1), \mathcal{K}(f_2)]\!]$ follows – i.e. (ii) implies (iii). On the other hand, if we assume (iii), then we obtain from (K1):

$$\alpha \;\leq\; [\![(\alpha * f), f]\!] \;\leq\; [\![\mathcal{K}(\alpha * f), \mathcal{K}(f)]\!] \;=\; \mathcal{K}(f) \to \mathcal{K}(\alpha * f) \;;$$

hence the following relation holds

$$\alpha * \mathcal{K}(f) \;\leq\; (\mathcal{K}(f) \to \mathcal{K}(\alpha * f)) * \mathcal{K}(f) \;\leq\; \mathcal{K}(\alpha * f) \;\;;$$

i.e. $\mathcal{K}$ is stratified.
■

Let $\mathcal{U} : X \longmapsto L^{(L^X)}$ be a map; then for each $p \in X$ the image of p under $\mathcal{U}$ is denoted by μ_p. $\mathcal{U}$ is said to be an *L–neighborhood system* on X iff $\mathcal{U}$ satisfies the following axioms

(U0) $\quad \mu_p(1_X) \;=\; \top \;.$

(U1) $\quad f_1 \leq f_2 \;$ implies $\; \mu_p(f_1) \;\leq\; \mu_p(f_2) \;.$

(U2) $\quad (\mu_p(f_1)) \otimes (\mu_p(f_2)) \;\leq\; \mu_p(f_1 \otimes f_2) \;.$

(U3) $\quad \mu_p(f) \;\leq\; f(p) \;.$

(U4) $\quad \mu_p(f) \quad \leq \quad \bigvee\{\mu_p(h)) \mid h(q) \leq \mu_q(f) \; \forall q \in X\} \quad .$

Proposition 6.1.2 *Let $\mathcal{K}$ be an L–interior operator on X. Then $\mathcal{K}$ induces an L–neighborhood system $\mathcal{U}^{(\mathcal{K})} : X \longmapsto L^{(L^X)}$ by*

$$[\mathcal{U}^{(\mathcal{K})}(p)](f) \quad = \quad \mu_p^{(\mathcal{K})}(f) \quad = \quad [\mathcal{K}(f)](p) \quad .$$

Proof. The axioms (U0) – (U3) follow immediately from (K0) – (K3). Now we fix $f \in L^X$ and put $h = \mathcal{K}(f)$. In particular, $h(q) = \mu_q^{(\mathcal{K})}(f)$ holds for all $q \in X$. Referring to (K4) we obtain

$$\mu_p^{(\mathcal{K})}(f) \quad \leq \quad [\mathcal{K}(\mathcal{K}(f))](p) \quad = \quad \mu_p^{(\mathcal{K})}(h) \quad ;$$

hence (U4) follows.
■

Proposition 6.1.3 *Let $\mathcal{U} = (\mu_p)_{p \in X}$ be an L–neighborhood system on X. Then $\mathcal{U}$ induces an L–interior operator $\mathcal{K}_{\mathcal{U}} : L^X \longmapsto L^X$ by*

$$[\mathcal{K}_{\mathcal{U}}(f)](p) \quad = \quad \mu_p(f) \qquad p \in X \quad .$$

Proof. The axioms (K0) – (K3) are evident. Further we derive from (U1) and (U4) the following relation

$$\mu_p(f) \quad \leq \quad \mu_p(\mu_-(f)) \quad ;$$

i.e. $\mathcal{K}_{\mathcal{U}}(f) \quad \leq \quad \mathcal{K}_{\mathcal{U}}(\mathcal{K}_{\mathcal{U}}(f))$; hence (K4) is verified.
■

Referring to the previous propositions 6.1.2 and 6.1.3 it is not difficult to see that *L–interior operators* and *L–neighborhood systems* are *equivalent* concepts.

Definition 6.1.4 (L–Filters)
Let X be a set; a map $\nu : L^X \longmapsto L$ is called an *L–filter* on X iff ν is provided with the following properties

(F0) $\quad \nu(1_X) \quad = \quad \top$

(F1) $\quad f_1 \leq f_2 \quad \Longrightarrow \quad \nu(f_1) \leq \nu(f_2)$ (Isotonicity)

(F2) $\quad \nu(f_1) \otimes \nu(f_2) \quad \leq \quad \nu(f_1 \otimes f_2)$

(F3) $\quad \nu(1_\varnothing) \quad = \quad \bot \quad .$

An L–filter ν is *weakly stratified* iff $\bigwedge_{x \in X} f(x) \leq \nu(f)$ for all $f \in L^X$. An L–filter ν is said to be *stratified* iff ν fulfills the further axiom

(F4) $\quad \alpha * \nu(f) \quad \leq \quad \nu(\alpha * f) \qquad \forall \alpha \in L, f \in L^X \quad .$

■

Because of (F0) and (F1) every stratified L–filter is weakly stratified. Moreover, in the case of $\otimes = *$ the axiom (F2) implies that weak stratification and stratification of L–filters are equivalent concepts.

Let $\mathcal{U} = (\mu_p)_{p \in X}$ be an L–neighborhood system. Then every μ_p is an L–filter on X. In particular, μ_p is called the *L–neighborhood filter* at the point p.

Theorem 6.1.5 *An L–topology is (weakly) stratified iff each L–neighborhood filter is (weakly) stratified.*

Proof. 6.1.1 – 6.1.4
■

6.2 L–Filter theory

Lemma 6.2.1 *(a) Every stratified L–filter ν (cf. 6.1.4) fulfills the property:*

$$\text{(C)} \qquad \bigwedge_{x \in X} f(x) \quad \leq \quad \nu(f) \quad \leq \quad ((\bigvee_{x \in X} f(x)) \to \bot) \to \bot \quad .$$

(b) Let $(L, \leq, *)$ be a two–sided, commutative quantale with square roots satisfying Axiom (S3). Further, let $\otimes$ be given by the monoidal mean operator (cf. Remark 1.2.6). Then every stratified L–filter ν on X satisfies the following condition:

$$\text{(F3*)} \qquad \nu(\bot^{1/2} \cdot 1_X) \quad = \quad \bot^{1/2} \qquad .$$

Proof. If we combine (F4) with (F0), (F1) and (F3), then we obtain

$$\begin{aligned} \bigwedge_{x \in X} f(x) \quad \leq \quad \nu((\bigwedge_{x \in X} f(x)) \cdot 1_X) \quad &\leq \quad \nu(f) \\ ((\bigvee_{x \in X} f(x)) \to \bot) * \nu(f) \quad \leq \quad \nu(1_\varnothing) \quad &= \quad \bot \qquad ; \end{aligned}$$

hence (C) follows.
The assertion (b) is an immediate consequence from Condition (C) and Proposition 1.2.9(i).
■

Let $\mathfrak{F}_L(X)$ ($\mathfrak{F}_L^s(X)$) be the set of all (stratified) L–filters on X. On $\mathfrak{F}_L(X)$ we introduce a partial ordering $\preccurlyeq$ by

$$\nu_1 \quad \preccurlyeq \quad \nu_2 \qquad \Longleftrightarrow \qquad \nu_1(f) \quad \leq \quad \nu_2(f) \qquad \forall\, f \in L^X \quad .$$

In particular, we use on the set $\mathfrak{F}_L^s(X)$ of all stratified L–filters the restriction of $\preccurlyeq$.

Example 6.2.2 (Smallest stratified L–filter)
Let us consider a map $\nu_0 : L^X \to L$ defined by $\nu_0(f) = \bigwedge_{x \in X} f(x)$. Then it is easy to see that ν_0 is a stratified L–filter on X. Moreover, Lemma 6.2.1 implies that ν_0 is the smallest element in $(\mathfrak{F}_L^s(X), \preccurlyeq)$.

■

An L–filter ν is called *tight* iff ν satisfies the following important axiom

(F5) $\quad \nu(\alpha \cdot 1_X) = \alpha \qquad$ for all $\alpha \in L$.

Obviously (F5) implies (F0) and (F3). Moreover, if the underlying quantale is a complete MV–algebra (cf. Subsection 1.1, Subsection 1.3), then Lemma 6.2.1 implies that every stratified L–filter is tight. Moreover, if τ is a weakly stratified L–topology, then because of Axiom (U3) every L–neighborhood filter is tight; hence tightness of an L–filter is not a filter–theoretically primitive concept in its own right.

Remark 6.2.3 ($\top$–Filters)
(a) A $\top$–*filter* is an ordinary subset $\mathbf{F}$ of L^X provided with the following properties

($\mathbb{F}0$) $\quad 1_X \in \mathbf{F}$.

($\mathbb{F}1$) $\quad$ If $h \in L^X$ s.t. $\exists\, \varkappa : \mathbf{F} \longmapsto L$ with

$$\left\{ \begin{array}{lcl} \bigvee\{\varkappa(f) \mid f \in \mathbf{F}\} & = & \top \\ \varkappa(f) * f(x) & \le & h(x) \quad \forall f \in \mathbf{F}\,,\ \forall x \in X \end{array} \right\}, \quad \text{then} \qquad h \in \mathbf{F} \quad .$$

($\mathbb{F}2$) $\quad f_1, f_2 \in \mathbf{F} \implies f_1 \otimes f_2 \in \mathbf{F}$.

($\mathbb{F}3$) $\quad \bigvee_{x \in X} f(x) = \top \quad \forall f \in \mathbf{F}$.

Every $\top$–filter $\mathbf{F}$ induces a *stratified and tight* L–filter $\nu_{\mathbf{F}}$ by

$$(\mathcal{F}) \qquad \nu_{\mathbf{F}}(h) = \bigvee_{f \in \mathbf{F}} \Big(\bigwedge_{x \in X} (f(x) \to h(x))\Big) \qquad \forall\, h \in L^X \quad .$$

In fact, the axiom (F5) (which includes (F0) and (F3)) is an immediate consequence from ($\mathbb{F}0$) and ($\mathbb{F}3$). (F1) holds by definition, and (F2) follows from ($\mathbb{F}2$) and (VII). Finally, we derive the axiom (F4) from the inequality $\alpha * (\beta \to \gamma) \le \beta \to (\alpha * \gamma)$.
(b) Every tight L–filter ν induces a $\top$–filter $\mathbf{F}_\nu$ by $\mathbf{F}_\nu = \{f \in L^X \mid \nu(f) = \top\}$. On the other hand, if an L–filter ν is induced by a $\top$–filter $\mathbf{F}$ in the sense of (a) (i.e. $\nu = \nu_{\mathbf{F}}$), then the axiom ($\mathbb{F}1$) gurantees the validity of the following relation:

$$\mathbf{F}_{\nu_{\mathbf{F}}} = \{h \in L^X \mid \nu_{\mathbf{F}}(h) = \top\} = \mathbf{F} \quad .$$

In this sense $\top$–filters form a subclass of tight and stratified L–filters.

(c) Let $\mathbf{F}_0$ be the smallest $\top$–filter – i.e. $\mathbf{F}_0 = \{1_X\}$. Then it is easy to see that $\nu_{\mathbf{F}_0}$ coincides with the smallest (weakly) stratified L–filter (cf. 6.2.2). Therefore the smallest $\top$–filter is also the smallest (weakly) stratified L–filter.

■

Comment. It is not difficult to see that the hypothesis in Axiom ($\mathbb{F}1$) is an L–valued interpretation of the following formula $(\exists f)((f \in \mathbf{F}) \wedge (f \leq h))$. Moreover, if we view tight L–filters as characteristic morphisms, then Remark 6.2.3(b) shows that the corresponding $\top$–filters can be understood as the corresponding *subobjects* in an appropriate categorical framework.
■

Proposition 6.2.4 (Complete MV–algebras)
(a) Let $(L, \leq, *)$ be a complete MV-algebra and ν be a stratified L–filter on X. Then the following assertions are equivalent

(i) ν is induced by a $\top$–filter $\mathbf{F}$ in the sense of formula $(\mathcal{F})$.

(ii) $\alpha \rightarrow \nu(f) \quad \leq \quad \nu(\alpha \rightarrow f) \qquad \forall \alpha \in L, \forall f \in L^X$.

(b) Let $(\mathbb{B}, \leq)$ be a complete Boolean algebra and $\otimes = * = \wedge$. Then every stratified $\mathbb{B}$–filter is induced by a (unique) $\top$–filter. In particular, $\top$–filters and stratified $\mathbb{B}$–filters are equivalent concepts.

Proof (a) Since $*$ is distributive over arbitrary meets (cf. Theorem 5.2(c) in [39]), we obtain

$$\alpha \rightarrow (\bigvee_{i \in I} \beta_i) \quad = \quad \bigvee_{i \in I} (\alpha \rightarrow \beta_i) \qquad .$$

Hence the implication (i) $\Longrightarrow$ (ii) follows immeadiately from $(\mathcal{F})$. In order to verify (ii) $\Longrightarrow$ (i) we proceed as follows: Let ν be a stratified L–filter on X provided with Property (ii). Since the law of double negation is valid in any MV–algebra, the stratification implies the tightness of ν (cf. Lemma 6.2.1); hence $\mathbf{F}_\nu = \{f \in L^X \mid \nu(f) = \top\}$ is a $\top$–filter on X. Further we infer from Property (ii):

$$\nu(f) \rightarrow f \in \mathbf{F}_\nu \qquad \forall f \in L^X \quad .$$

Now we invoke the MV–property of MV–algebras (cf. p. 63 in [39]) and obtain

$$\begin{aligned} \bigwedge_{x \in X} ((\nu(f) \rightarrow f(x)) \rightarrow f(x)) \quad &= \quad \bigwedge_{x \in X} (\nu(f) \vee f(x)) \\ &= \quad \nu(f) \vee \Big(\bigwedge_{x \in X} f(x)\Big) \quad = \quad \nu(f) \quad ; \end{aligned}$$

hence ν is induced by $\mathbf{F}_\nu$. Therewith the assertion (a) is verified.
(b) Let $L = \mathbb{B}$ be a complete Boolean algebra and $* = \wedge$. Referring to Assertion (a) it is sufficient to show that every stratified $\mathbb{B}$–filter satisfies Property

(ii). Because of $\alpha \to \beta = (\alpha \to \perp) \vee \beta$ we are in the position to derive the following relation from the stratification:

$$\begin{aligned} \alpha \to \nu(f) &= (\alpha \to \perp) \vee \nu(f) = \nu((\alpha \to \perp) \cdot 1_X) \vee \nu(f) \\ &\leq \nu((\alpha \to \perp) \cdot 1_X \vee f) = \nu(\alpha \to f) \quad . \end{aligned}$$

■

Lemma 6.2.5 (Square roots)
Let $(L, \leq, *)$ be a commutative quantale with square roots such that Axiom (S3) is satisfied. Further let $\circledast$ be the monoidal mean operator (cf. Remark 1.2.6, Corollary 1.2.8). Then the quadruple $(L, \leq, \circledast, *)$ fulfills (I)–(VII), and every stratified L–filter ν on X satisfies the following property:

(F6) For every $n \in \mathbb{N}$ and for every n–tuple $(f_i)_{i=1}^n \in (L^X)^n$ the implication

$$f_1 * \ldots * f_n = 1_\varnothing \implies \nu(f_1) * \ldots * \nu(f_n) = \perp$$

holds.

Proof. Referring to Remark 1.2.6 the quadruple $(L, \leq, \circledast, *)$ fulfills (I)–(VII). Further, the recursive application of the formation of square roots leads to the subsequent notation

$$\alpha^{1/2^{n+1}} = (\alpha^{1/2^n})^{1/2} \qquad \forall \alpha \in L \,,\ \forall n \in \mathbb{N} \quad .$$

(a) We show : $(\perp^{1/2^n} \to \perp) \to \perp = \perp^{1/2^n}$ $\forall n \in \mathbb{N}$. Because of Proposition 1.2.9(i) the assertion holds for $n = 1$. If the assertion holds for an arbitrary $n \in \mathbb{N}$, then we observe:

$$\begin{aligned} \perp^{1/2^{n+1}} &= (\perp^{1/2^{n+1}} \to \perp^{1/2}) \to \perp^{1/2} \\ &= [\perp^{1/2^{n+1}} \to ((\perp^{1/2} \to \perp) \to \perp)] \to ((\perp^{1/2} \to \perp) \to \perp) \\ &= \Big([(\perp^{1/2} \to \perp) \to (\perp^{1/2^{n+1}} \to \perp)] * (\perp^{1/2} \to \perp)\Big) \to \perp \\ &\geq (\perp^{1/2^{n+1}} \to \perp) \to \perp \quad . \end{aligned}$$

Since the converse inequality is trivial, the assertion holds for $n + 1$.
(b) We verify by induction:

$$(\nu(f_1))^{1/2^n} * \ldots * (\nu(f_n))^{1/2^n} \leq \nu(\,(f_1 * \ldots * f_n)^{1/2^n}) \qquad (n \in \mathbb{N}) \quad .$$

In the case $n = 1$ the assertion follows from (F0) and (F2). Because of

$$(f_1 * \ldots * f_n)^{1/2^{n+1}} * (f_{n+1})^{1/2^{n+1}} \leq (f_1 * \ldots * f_n * f_{n+1})^{1/2^{n+1}}$$

we infer from (F1) and(F2)

$$[\nu((f_1 * \ldots * f_n)^{1/2^n}]^{1/2} * [\nu((f_{n+1})^{1/2^n})]^{1/2} \leq \nu((f_1 * \ldots * f_{n+1})^{1/2^{n+1}}) \;;$$

hence the induction hypothesis and the axioms (F0) and (F2) imply

$$(\nu(f_1))^{1/2^{n+1}} * \ldots * (\nu(f_n))^{1/2^{n+1}} * (\nu(f_{n+1}))^{1/2^{n+1}} \quad \leq \quad \nu((f_1 * \ldots * f_{n+1})^{1/2^{n+1}} .$$

(c) The assertion follows from the previous parts (a), (b) and Lemma 6.2.1(a) . ■

Theorem 6.2.6 *Let $(L, \leq, \otimes, *)$ be a quadruple satisfying (I)–(III), (V)–(X) and the additional axiom*

$$\text{(XI)} \qquad (\alpha * \beta) \otimes \top \quad = \quad ((\alpha \otimes \top) * (\beta \otimes \top)) \vee (\bot \otimes \top) \quad .$$

(a) Then $\otimes$ satisfies Axiom (IV).

(b) Further let $\mathcal{F} = \{\nu_i \mid i \in I\}$ be a family of (stratified) L–filters on X. If $\mathcal{F}$ is provided with the following property

$$\left\{ \begin{array}{l} \text{For every finite, non empty subset } \{i_1, \ldots, i_n\} \text{ of } I \text{ and for every} \\ n\text{–tuple } (f_{i_j})_{j=1}^n \in (L^X)^n \text{ the implication} \\ \quad f_{i_1} * \ldots * f_{i_n} = 1_\varnothing \quad \Longrightarrow \quad \nu_{i_1}(f_{i_1}) * \ldots * \nu_{i_n}(f_{i_n}) = \bot \\ \text{holds.} \end{array} \right\} ,$$

then $\mathcal{F}$ has an upper bound in $\mathfrak{F}_L(X)$ (resp. $\mathfrak{F}_L^s(X)$).

Proof. Referring to Proposition 1.2.7 we see immediately that $(L, \leq, *)$ has square roots and $\otimes$ coincides with the monoidal mean operator $\circledast$. Further the axiom (XI) guarantees that the formation of square roots satisfies Axiom (S3); i.e. (XI) forces the validity of (IV) (cf. Remark 1.2.6).
In order to verify the second assertion we define a map $\nu_\infty : L^X \longmapsto L$ by

$$\nu_\infty(h) \;=\; \bigvee\{\nu_{i_1}(f_{i_1}) * \ldots * \nu_{i_n}(f_{i_n}) \mid n \in \mathbb{N},\, f_{i_j} \in L^X,\, f_{i_1} * \ldots * f_{i_n} \leq h\}$$

and show that ν_∞ is a (stratified) L–filter on X. The axioms (F0) and (F1) (resp. (F4)) are evident. (F3) follows from the hypothesis on $\mathcal{F}$. In order to verify (F2) we proceed as follows: First let us consider the situation

$$f_{i_1} * \ldots * f_{i_n} \quad \leq \quad h_1 \qquad , \qquad g_{j_1} * \ldots * g_{j_m} \quad \leq \quad h_2 \qquad .$$

If $K = \{i_1, \ldots, i_n\} \cap \{j_1, \ldots, j_m\}$ is non empty, then we put

$$K = \{\ell_1, \ldots, \ell_p\} \quad , \qquad \begin{array}{l} \{i_1, \ldots, i_r\} = \{i_1, \ldots, i_n\} \cap \complement K \quad , \\ \{j_1, \ldots, j_s\} = \{j_1, \ldots, j_m\} \cap \complement K \quad ; \end{array}$$

where $\complement K$ means the complement of K. The axiom (S2) implies

$$\left. \begin{array}{c} (f_{i_1})^{1/2} * (\bot^{1/2} \cdot 1_X) * (f_{i_2})^{1/2} * \ldots * (f_{i_n})^{1/2} \leq (h_1)^{1/2} * (h_2)^{1/2} \\ (g_{j_1})^{1/2} * (\bot^{1/2} \cdot 1_X) * (g_{j_2})^{1/2} * \ldots * (g_{j_m})^{1/2} \leq (h_1)^{1/2} * (h_2)^{1/2} \\ (f_{i_1})^{1/2} * \ldots * (f_{i_r})^{1/2} * [((f_{\ell_1})^{1/2} * (g_{\ell_1})^{1/2}) * \ldots * ((f_{\ell_p})^{1/2} * \\ *(g_{\ell_p})^{1/2})] * (g_{j_1})^{1/2} * \ldots * (g_{j_s})^{1/2} \quad \leq \quad (h_1)^{1/2} * (h_2)^{1/2} \end{array} \right\} \quad (6.2.1)$$

Now we invoke (S3), (F0), (F2) and (F3) and obtain

$$
\begin{array}{l}
(\nu_{i_1}(f_{i_1}) * \ldots * \nu_{i_n}(f_{i_n}))^{1/2} \, * \, (\nu_{j_1}(g_{j_1}) * \ldots * \nu_{j_m}(g_{j_m}))^{1/2} \quad = \\
((\nu_{i_1}(f_{i_1}))^{1/2} * \bot^{1/2}) * (\nu_{i_2}(f_{i_2}))^{1/2} * \ldots * (\nu_{i_n}(f_{i_n}))^{1/2} \quad \bigvee \\
((\nu_{j_1}(g_{j_1}))^{1/2} * \bot^{1/2}) * (\nu_{j_2}(g_{j_2}))^{1/2} * \ldots * (\nu_{j_m}(g_{j_m}))^{1/2} \quad \bigvee \\
[(\nu_{i_1}(f_{i_1}))^{1/2} * \ldots * (\nu_{i_r}(f_{i_r}))^{1/2} \quad * \\
\quad *((\nu_{\ell_1}(f_{\ell_1}))^{1/2} * (\nu_{\ell_1}(g_{\ell_1}))^{1/2}) * \ldots * ((\nu_{\ell_p}(f_{\ell_p}))^{1/2} * (\nu_{\ell_p}(g_{\ell_p}))^{1/2}) \quad * \\
\qquad\qquad *(\nu_{j_1}(f_{j_1}))^{1/2} * \ldots * (\nu_{j_s}(f_{j_s}))^{1/2}] \quad \leq \\
\nu_{i_1}((f_{i_1})^{1/2} * \bot^{1/2} \cdot 1_X) * \nu_{i_2}((f_{i_2})^{1/2}) \ldots * \nu_{i_n}((f_{i_n})^{1/2}) \quad \bigvee \\
\nu_{j_1}((g_{j_1})^{1/2} * \bot^{1/2} \cdot 1_X) * \nu_{j_2}((g_{j_2})^{1/2}) * \ldots * \nu_{j_m}((g_{j_m})^{1/2}) \quad \bigvee \\
[\nu_{i_1}((f_{i_1})^{1/2}) * \ldots * \nu_{i_r}((f_{i_r})^{1/2}) \quad * \\
\quad *\nu_{\ell_1}((f_{\ell_1})^{1/2} * (g_{\ell_1})^{1/2}) * \ldots * \nu_{\ell_p}((f_{\ell_p})^{1/2} * (g_{\ell_p})^{1/2}) \\
\qquad\qquad *\nu_{j_1}((g_{j_1})^{1/2}) * \ldots * \nu_{j_s}((g_{j_s})^{1/2})] \quad \leq \\
\nu_\infty((h_1)^{1/2} * (h_2)^{1/2}) \quad .
\end{array}
$$

Because of (S3) the formation of square roots preserves arbitrary non empty joins; hence (F2) follows from formulae (6.2.1). Finally (F0) and the definition of ν_∞ shows that ν_∞ is an upper bound of $\mathcal{F}$ w.r.t. $\preccurlyeq$.

■

Corollary 6.2.7 *Let $(L, \leq, \otimes, *)$ be a quadruple satisfying (I)–(XI). A family $\{\nu_i \,|\, i \in I\}$ of* stratified *L–filters has an upper bound in $\mathfrak{F}^s_L(X)$ iff for every finite subset $\{i_1, \ldots, i_n\}$ of I and for every n–tuple $(f_{i_j})_{j=1}^n \in (L^X)^n$ the following implication holds*

$$f_{i_1} * \ldots * f_{i_n} \;=\; 1_\varnothing \;\Longrightarrow\; \nu_{i_1}(f_{i_1}) * \ldots * \nu_{i_n}(f_{i_n}) \;=\; \bot \quad .$$

Proof. Lemma 6.2.5 and Theorem 6.2.6.
■

Lemma 6.2.8 *Let $(L, \leq, \otimes, *)$ be a quadruple satisfying (I)–(XI). Then for every stratified L–filter ν there exists a stratified L–filter $\bar{\nu}$ provided with the subsequent properties*

(i) $\nu \;\preccurlyeq\; \bar{\nu}$

(ii) $\bar{\nu}(f_1) * \bar{\nu}(f_2) \;\leq\; \bar{\nu}(f_1 * f_2)$ (Submultiplicativity)

Proof. We consider the case $\nu_n = \nu$ for all $n \in \mathbb{N}$. Then the assertion follows from Lemma 6.2.5 and from the construction of ν_∞ in the proof of

Theorem 6.2.6.
■

In the next theorem we show that *Zorn's Lemma* and Axiom (IV) guarantee the existence of maximal (stratified) L–filters.

Theorem 6.2.9 (Existence of maximal (stratified) L–filters)
(a) The partially ordered set $(\mathfrak{F}_L(X), \preccurlyeq)$ has maximal elements.
(b) The partially ordered set $(\mathfrak{F}_L^s(X), \preccurlyeq)$ has maximal elements.

Proof. Referring to Zorn's Lemma it is sufficient to show that every chain $\mathcal{C}$ in $\mathfrak{F}_L(X)$ (resp. $\mathfrak{F}_L^s(X)$) has an upper bound in $\mathfrak{F}_L(X)$ (resp. $\mathfrak{F}_L^s(X)$). For this purpose let us consider a non empty chain $\mathcal{C} = \{\nu_i \mid i \in I\}$. We define a map $\nu_\infty : L^X \longmapsto L$ by

$$\nu_\infty(f) \quad = \quad \bigvee_{i \in I} \nu_i(f) \qquad .$$

It is easy to verify that ν_∞ fulfills (F0), (F1), (F3) (resp. (F4)). The axiom (F2) follows from (IV) and the linearity of the induced partial ordering on $\mathcal{C}$.
■

A maximal element in $(\mathfrak{F}_L(X), \preccurlyeq)$ (resp. $(\mathfrak{F}_L^s(X), \preccurlyeq)$) is also called an *$L$–ultrafilter* (resp. *stratified L–ultrafilter*).

Corollary 6.2.10 *For every (stratified) L–filter ν there exists a (stratified) L–ultrafilter μ with $\nu \preccurlyeq \mu$.*
■

In general it is *difficult* to give an algebraic characterization of the *maximality* of (stratified) L–ultrafilters. In the following considerations we present a solution of this problem in the important case of monoidal mean operators.

Theorem 6.2.11 (Case of complete Heyting algebras)
Let $H = (L, \leq)$ be a complete Heyting algebra, and let $\otimes = * = \wedge$. Further let ν be a (stratified) L–filter on X. Then the following assertions are equivalent

(i) ν is a (stratified) L–ultrafilter.

(ii) $\nu(f) \quad = \quad (\nu(f \to \bot)) \to \bot \qquad \forall f \in L^X$.

Proof. (a) Because of (F2) and (F3) every L–filter satisfies the condition

(F3') $\quad \nu(f) \quad \leq \quad ((\nu(f \to \bot)) \to \bot \qquad \forall f \in L^X$.

In order to verify (i) $\Longrightarrow$ (ii) it is sufficient to show that the maximality of ν implies

$$(\nu(f \to \bot)) \to \bot \quad \leq \quad \nu(f) \qquad \forall f \in L^X \quad .$$

For this purpose we fix $g \in L^X$ and define a map $\bar{\nu} : L^X \longmapsto L$ by

$$\bar{\nu}(f) = \nu(f) \vee (\nu(g \to f) \wedge ((\nu(g \to \perp)) \to \perp)) \qquad f \in L^X \quad .$$

Obviously $\bar{\nu}$ satisfies (F0), (F1) and (F3). The axiom (F2) follows from the distributivity of the underlying lattice $(L, \leq)$ and from the subsequent relation

$$\begin{aligned} &\bar{\nu}(f_1) \wedge \bar{\nu}(f_2) \quad \leq \\ &(\nu(f_1) \wedge \nu(f_2)) \vee (\nu(g \to f_1) \wedge \nu(g \to f_2) \wedge ((\nu(g \to \perp)) \to \perp)) \quad \leq \\ &\bar{\nu}(f_1 \wedge f_2) \qquad ; \end{aligned}$$

i.e. $\bar{\nu}$ is an L–filter. If ν is stratified, then we apply the inequality

$$\alpha \wedge (\beta \to \gamma) \quad \leq \quad \beta \to (\alpha \wedge \gamma)$$

and obtain immediately that $\bar{\nu}$ is also stratified.– Because of $\nu \preccurlyeq \bar{\nu}$ the maximality of ν implies $\nu = \bar{\nu}$; i.e.

$$(\nu(g \to \perp)) \to \perp \quad \leq \quad \bar{\nu}(g) \quad = \quad \nu(g) \qquad .$$

Hence (ii) is verified.

(b) Let $\tilde{\nu}$ be a (stratified) L–filter with $\nu \preccurlyeq \tilde{\nu}$; then we infer from (ii) and (F3'):

$$\tilde{\nu}(f) \quad \leq \quad (\tilde{\nu}(f \to \perp)) \to \perp \quad \leq \quad (\nu(f \to \perp)) \to \perp \quad = \quad \nu(f) \quad ;$$

i.e. $\tilde{\nu} = \nu$. Hence (ii) implies (i).

■

Remark 6.2.12 (L–Ultrafilters)

Let $H = (L, \leq)$ be a complete Heyting algebra.

(a) Let us assume that $(L, \leq)$ is a chain. Then an L–filter ν on X is an L–ultrafilter iff there exists a maximal filter $\mathbf{U}$ in the lattice $(L^X, \leq)$ with $\nu = \chi_{\mathbf{U}}$.

(b) If $L = \mathbb{B}$ is a complete Boolean algebra, then for every maximal filter $\mathbf{U}$ in the lattice $(\mathbb{B}^X, \leq)$ the characteristic function $\chi_{\mathbf{U}}$ is a $\mathbb{B}$–ultrafilter on X.

■

Lemma 6.2.13 (GL–Monoids with square roots)

Let $M = (L, \leq, *)$ be a GL–monoid with square roots satisfying the axiom (S3) and the following condition

$$\text{(S4)} \qquad ((\perp^{1/2} \to \perp)*(\perp^{1/2} \to \perp)) \vee (((\perp^{1/2} \to \perp)*(\perp^{1/2} \to \perp)) \to \perp) \;=\; \top.$$

Further let $\otimes = \circledast$ be the monoidal mean operator (cf. Remark 1.2.6). Then every stratified L–ultrafilter ν fulfills the property:

$$\text{(F7)} \qquad \nu(1_{\varnothing} \circledast f) \quad = \quad \perp \circledast \nu(f) \quad .$$

Proof. In order to simplify the notation we always denote $\alpha * \alpha$ by α^2. – Let ν be an L–ultrafilter on X. Because of the axiom (F2) the following relation holds:

$$\nu(f) \quad \leq \quad (\perp^{1/2} \to \nu(1_\varnothing \circledast f))^2 \quad \forall f \in L^X \quad .$$

Hence we obtain

$$\begin{aligned} &(\nu(f))^{1/2} * (\perp^{1/2} \to \nu(1_\varnothing \circledast g)) \leq (\perp^{1/2} \to \nu(1_\varnothing \circledast f)) * (\perp^{1/2} \to \nu(1_\varnothing \circledast g)) \\ = \quad & [(\perp^{1/2} \to \nu(1_\varnothing \circledast f))^{1/2} \; * \; (\perp^{1/2} \to \nu(1_\varnothing \circledast g))^{1/2}]^2 \leq \\ \leq \quad & [\perp^{1/2} \to (\nu(1_\varnothing \circledast f) \circledast \nu(1_\varnothing \circledast g))]^2 \quad \leq \quad (\perp^{1/2} \to \nu(1_\varnothing \circledast (f \circledast g)))^2 \quad . \end{aligned}$$

Therefore we can enlarge the L–filter ν as follows

$$\bar{\nu}(f) \quad := \quad \nu(f) \vee ((\iota_0 \to \perp) * (\perp^{1/2} \to \nu(1_\varnothing \circledast f))^2) \qquad \forall f \in L^X$$

where $\iota_0 = (\perp^{1/2} \to \perp)^2$.
We show that the map $\bar{\nu}$ is again a stratified L–filter on X. The axiom (S3) and Proposition 1.2.9(ii) imply

$$(\bar{\nu}(f))^{1/2} \quad = \quad (\nu(f))^{1/2} \vee ((\iota_0 \to \perp) * (\perp^{1/2} \to \nu(1_\varnothing \circledast f))) \vee \perp^{1/2} \quad ;$$

hence the previous considerations show that $\bar{\nu}$ satisfies (F2). The axioms (F0) and (F1) are evident, and (F3) holds by definition of $\bar{\nu}$. The stratification axiom (F4) follows from the subsequent relation:

$$\begin{aligned} \alpha * \bar{\nu}(f) \quad & \leq \quad \nu(\alpha * f) \vee \Big((\iota_0 \to \perp) * \Big(\perp^{1/2} \to (\alpha^{1/2} * \nu(1_\varnothing \circledast f))\Big)^2\Big) \\ & \leq \quad \bar{\nu}(\alpha * f) \quad . \end{aligned}$$

Now we invoke the maximality of ν and obtain:

$$(\iota_0 \to \perp) * (\perp^{1/2} \to \nu(1_\varnothing \circledast f))^2 \quad \leq \quad \nu(f) \qquad \forall f \in L^X . \qquad (\Diamond)$$

Since ν is stratified, we infer from (F1) and Lemma 6.2.1(b) : $\nu(1_\varnothing \circledast f) \leq \perp^{1/2}$; hence the relation

$$(\iota_0 \to \perp) * \nu(1_\varnothing \circledast f) \quad \leq \quad \perp \circledast \nu(f)$$

follows from $(\Diamond)$ and the divisibility of $(L, \leq, *)$. Now we observe $\iota_0 * \perp^{1/2} = \perp$; then we obtain from (S4):

$$\nu(f) \circledast \perp \quad \leq \quad \nu(1_\varnothing \circledast f) \quad = \quad (\iota_0 \to \perp) * \nu(1_\varnothing \circledast f) \quad \leq \quad \perp \circledast \nu(f) \; ;$$

hence (F7) is verified.
■

Theorem 6.2.14 (Characterization of stratified L–ultrafilters) *Let $(L, \leq, *)$ be a GL–monoid with square roots satisfying the axioms (S3) and (S4). Further let $\otimes$ be given by the monoidal mean operator $\circledast$. Then for every stratified L–filter ν on X the following assertions are equivalent*

(i) ν is an L–ultrafilter.

(ii) $\nu(f) \quad = \quad (\nu(f \to \perp)) \to \perp \qquad \forall f \in L^X \quad .$

Proof. Let ν be a stratified L–filter on X. In order to verify (i) $\Longrightarrow$ (ii) we distinguish the following cases:
(a) Let $(L, \leq, *)$ be a smooth GL–monoid (i.e. $\perp^{1/2} = \perp$). Then the formation of square roots is a quantale–automorphism of L. Hence the maximality of ν implies $\nu(f) = (\nu(f))^{1/2}$ for all $f \in L^X$. Then the map $\bar{\nu} : L^X \longmapsto L$ defined by

$$\bar{\nu}(f) \quad = \quad \nu(f) \vee (\nu(g \to f) * (\nu(g \to \perp) \to \perp)) \qquad \forall f \in L^X$$

is again a stratified L–filter on X (cf. Proof of Theorem 6.2.11). Applying again the maximality of ν we obtain $\nu(g \to \perp) \to \perp \leq \nu(g)$. The converse inequality follows from Lemma 6.2.8 and Axiom (F3). Hence the implication (i) $\Longrightarrow$ (ii) is verified in the case of smooth GL–monoids..
(b) Let $(L, \leq, *)$ be a strict GL–monoid (cf. Definition 1.2.10). Because of Axiom (F2) we can enlarge ν as follows:

$$\bar{\nu}(f) \quad := \quad ((\nu(g \to \perp))^{1/2} \to \nu(f \circledast (g \to \perp)))^2 \qquad \forall f \in L^X .$$

We show that the map $\bar{\nu}$ is a stratified L–filter on X. The axioms (F0) and (F1) are obvious. Further we observe

$$\perp^{1/2} \quad \leq \quad (\nu(f))^{1/2} \quad \leq \quad (\nu(g \to \perp))^{1/2} \to \nu(f \circledast (g \to \perp))$$

and conclude from (S3) and (F2):

$$\begin{aligned}
&\bar{\nu}(f_1) \circledast \bar{\nu}(f_2) \quad = \\
= \quad & ((\nu(g))^{1/2} \to \nu(f_1 \circledast (g \to \perp))) * ((\nu(g \to \perp))^{1/2} \to \nu(f_2 \circledast (g \to \perp))) = \\
= \quad & [((\nu(g \to \perp))^{1/2} \to \nu(f_1 \circledast (g \to \perp)))^{1/2} * \\
& \qquad * ((\nu(g \to \perp))^{1/2} \to \nu(f_2 \circledast (g \to \perp)))^{1/2}]^2 \quad \leq \\
\leq \quad & [(\nu(g \to \perp))^{1/2} \to (\nu(f_1 \circledast (g \to \perp)) \circledast \nu(f_2 \circledast (g \to \perp)))]^2 \quad \leq \\
\leq \quad & ((\nu(g \to \perp))^{1/2} \to \nu((f_1 \circledast f_2) \circledast (g \to \perp)))^2 \quad = \quad \bar{\nu}(f_1 \circledast f_2) \ ;
\end{aligned}$$

hence $\bar{\nu}$ fulfills (F2). The axiom (F3) follows from Axiom (F7) (cf. Lemma 6.2.13), the definition of $\bar{\nu}$ and Proposition 1.2.12. Since ν is stratified, we obtain:

$$\begin{aligned}
\alpha * \bar{\nu}(f) \quad &= \quad [\alpha^{1/2} * ((\nu(g \to \perp))^{1/2} \to \nu(f \circledast (g \to \perp)))]^2 \\
&\leq \quad [(\nu(g \to \perp))^{1/2} \to (\alpha^{1/2} * \nu(f \circledast (g \to \perp)))]^2 \\
&\leq \quad [(\nu(g \to \perp))^{1/2} \to \nu(\alpha^{1/2} * (f \circledast (g \to \perp)))]^2 \quad = \quad \bar{\nu}(\alpha * f) \ ;
\end{aligned}$$

hence $\bar{\nu}$ is also stratified.
Now we invoke the maximality of ν and conclude from the definition of $\bar{\nu}$ and Lemma 6.2.1(b):

$$\nu(g \to \bot) \to \bot \quad \leq \quad \nu(g) \quad .$$

Since the converse inequality is trivial (see Part (a)), ν satisfies the assertion (ii).
(c) The axiom (S4) guarantees the decomposition of $(L, \leq, *)$ into a smooth GL–monoid and a strict GL–monoid (cf. Theorem 1.2.13). Moreover, if we put $\iota_0 = (\bot^{1/2} \to \bot)^2$, then the axiom (S4) and Proposition 1.2.9(ii) imply the idempotency of ι_0. Therefore we can derive the following relation from the hypothesis (S4), the stratification axiom (F4), the idempotency of ι_0 and $\iota_0 \to \bot$:

$$\nu(f) \quad = \quad (\iota_0 * \nu(\iota_0 * f)) \vee ((\iota_0 \to \bot) * \nu((\iota_0 \to \bot) * f)) \ .$$

In particular, we observe: $\nu(\iota_0 \cdot 1_X) = \iota_0$, $\nu((\iota_0 \to \bot) \cdot 1_X) = \iota_0 \to \bot$ (cf. Proposition 1.2.9(iii), Lemma 6.2.1). Hence the assetion (ii) follows from the previous parts (a) and (b).
(d) In order to establish the implication (ii) $\Longrightarrow$ (i) let us consider the situation $\nu \preccurlyeq \tilde{\nu}$. Then Lemma 6.2.8, Axiom (F3) and the assertion (ii) imply:

$$\tilde{\nu}(f) \quad \leq \quad \tilde{\nu}(f \to \bot) \to \bot \quad \leq \quad \nu(f \to \bot) \to \bot \quad \leq \quad \nu(f) \ .$$

Hence ν is maximal.
■

Example 6.2.15 (Complete MV–algebras with square roots)
(a) For every element $p \in X$ the map $\delta_p : L^X \longmapsto L$ defined by

$$\delta_p(f) \quad = \quad f(p) \qquad \forall f \in L^X$$

is a stratified L–ultrafilter on X.
(b) In the case of $M = ([0,1], \leq, T_m)$ (cf. Example 1.2.3(b), Subsection 1.3) every finitely additive probability measure λ on X induces a stratified $[0,1]$–ultrafilter ν_λ on X by (cf. Example 6.7 in [41])

$$\nu_\lambda(f) \quad = \quad \int_X f \, d\lambda \qquad \forall f \in [0,1]^X \qquad .$$

and vice versa – i.e. finitely additive probability measures and stratified $[0,1]$–ultrafilters come to the same thing (cf. Example 6.7 in [41]).
■

Lemma 6.2.16 ($\mathbb{B}$–Filter extension) *Let $(\mathbb{B}, \leq)$ be a complete Boolean algebra and $\otimes = * = \wedge$. Further let ν be a (stratified) $\mathbb{B}$–filter on X. Then for every $g \in \mathbb{B}^X$ there exists a (stratified) $\mathbb{B}$–filter $\bar{\nu}$ on X provided with the following properties*

(i) $\nu \preccurlyeq \bar{\nu}$.

(ii) $\nu(g) = \bar{\nu}(g)$.

(iii) $(\bar{\nu}(g)) \to \bot = \bar{\nu}(g \to \bot)$.

Proof. We fix $g \in \mathbb{B}^X$ and define a map $\bar{\nu} : \mathbb{B}^X \to \mathbb{B}$ as follows:

$$\bar{\nu}(f) = \nu(f) \vee (\nu(g \vee f) \wedge ((\nu(g)) \to \bot)) \qquad \forall f \in \mathbb{B}^X .$$

It is not difficult to show that $\bar{\nu}$ fulfills the desired properties.
■

Theorem 6.2.17 (Boolean case) *Let* $(\mathbb{B}, \leq)$ *be a complete Boolean algebra and* $\otimes = * = \wedge$. *Then every (stratified)* $\mathbb{B}$*–filter can be written as an infimum of an appropriate family of (stratified)* $\mathbb{B}$*–ultrafilters.*

Proof. The assertion follows immediately from Lemma 6.2.16 and Corollary 6.2.10.
■

Proposition 6.2.18 (L–Filter extension w.r.t. strict MV–algebras) *Let* $(L, \leq, *)$ *be a strict* MV*–algebra and* $\otimes$ *be given by the monoidal mean operator* $\circledast$. *Then for every* $g \in L^X$ *and for every stratified* L*–filter* ν *on* X *satisfying the axiom*

(F7) $\quad \nu(1_\varnothing \circledast f) = \bot \circledast \nu(f) \qquad \forall f \in L^X$.

there exists a further stratified L*–filter* $\bar{\nu}$ *on* X *provided with the following properties:*

(i) $\nu \preccurlyeq \bar{\nu}$.

(ii) $\nu(g) = \bar{\nu}(g)$.

(iii) $(\bar{\nu}(g)) \to \bot = \bar{\nu}(g \to \bot)$.

Proof. Let ν be an L–filter provided with (F7). We fix $g \in L^X$. Referring to the part (b) of the proof of Theorem 6.2.14 we can construct a stratified L–filter $\bar{\nu}$ on X as follows:

$$\bar{\nu}(f) := ((\nu(g))^{1/2} \to \nu(f \circledast g))^2 \qquad \forall f \in L^X .$$

Oviously $\bar{\nu}$ satisfies (i) and (iii) (cf. Part (b) of the proof of Theorem 6.2.14). In order to verify (iii) we proceed as follows: Referring to Proposition 2.15 in [39] (see also Proposition 1.2.14) we conclude from the definition of $\bar{\nu}$:

$$\bar{\nu}(g) = ((\nu(g))^{1/2} \to \nu(g))^2 = [((\nu(g))^{1/2} \to \bot) \vee (\nu(g))^{1/2}]^2 .$$

Now we invoke the strictness of $(L, \leq, *)$ and obtain the desired property (iii).
■

If we combine Lemma 6.2.16 with Proposition 6.2.18, then in the case of stratified L–filters we can extend the assertion of Theorem 6.2.17 to the realm of complete MV–algebras with square roots (cf. Corollary 7.3. in [41]):

Theorem 6.2.19 (Complete MV–algebras with square roots)
Let $(L, \leq, *)$ be a complete MV–algebra with square roots and $\otimes = \circledast$ be the monoidal mean operator. Further let ν_0 be a stratified L–filter provided with the property

$$\text{(F7)} \qquad \nu_0(1_\varnothing \circledast f) \quad = \quad \bot \circledast \nu_0(f) \qquad \forall f \in L^X .$$

Then there exists a familiy $\{\nu_i \,|\, i \in I\}$ of stratified L–ultrafilters ν_i such that

$$\nu_0(f) \quad = \quad \bigwedge_{i \in I} \nu_i(f) \qquad \forall f \in L^X .$$

Proof. Since a complete MV–algebra can uniquely be decomposed into a complete Boolean algebra and a strict MV–algebra (cf. Theorem 2.21 in [39]; see also Theorem 1.2.13 and Subsection 1.3), the assertion follows immediately from Lemma 6.2.16 and Proposition 6.2.18.
■

Discussion. (a) In the Boolean case the axiom (F7) is redundant. Hence every stratified $\mathbb{B}$–filter can be written as an infimum of an appropriate family of stratified $\mathbb{B}$–ultrafilters (see also Theorem 6.2.17). Furthermore we remark that there exist $\mathbb{B}$–ultrafilters which are *not stratified* $\mathbb{B}$–ultrafilters (cf. Remark 6.2.12 and Proposition 6.2.4).
(b) Let $(L, \leq, *)$ be a complete MV–algebra with square roots. Then every stratified L–filter can be extended to a stratified L–filter satisfying (F7). We state again this result in Lemma 7.3.5 where we give its full proof.
■

Remark 6.2.20 (Images and inverse images of L–filters)
(a) Let $\varphi : X \longmapsto Y$ be a map and ν a (stratified) L–filter on X. Then the map $\varphi(\nu) : L^Y \longmapsto L$ defined by $[\varphi(\nu)](g) = \nu(g \circ \varphi)$ $(g \in L^Y)$ is a (stratified) L–filter on Y. In particular, $\varphi(\nu)$ is called the *image L–filter* of ν under φ. If the underlying quadruple $(L, \leq, \circledast, *)$ is given by a GL–monoid with square roots satisfying (S3) and (S4) , then we conclude from Theorem 6.2.14 that the maximality of *stratified* L–filters is preserved under the formation of images – i.e. if ν is a stratified L–ultrafilter on X, then $\varphi(\nu)$ is again a stratified L–ultrafilter on Y.

(b) Let $\varphi : X \longmapsto Y$ be a *surjective* map and ν be a (stratified) L–filter on X. Then $\varphi^{-1}(\nu) : L^X \longmapsto L$ defined by

$$[\varphi^{-1}(\nu)](f) \quad = \quad \bigvee \{\nu(g) \,|\, g \circ \varphi \leq f\} \qquad (f \in L^X)$$

is a (stratified) L–filter on X. In particular, the surjectivity of φ guarantees the validity of Axiom (F3). $\varphi^{-1}(\nu)$ is said to be the *inverse image L–filter* of ν under φ. Finally, the relation $\varphi(\varphi^{-1}(\nu)) = \nu$ holds for all surjective maps φ.
■

6.3 The principle of L-continuous extension

In this subsection we only refer to the triple $(L, \leq, \otimes)$ – i.e. all results of this subsection require only *cl*–quasi–monoids as an underlying lattice–theoretic structure.

If τ is an L–topology on X, then with every $g \in \tau$ we can associate a further element $g^* \in \tau$ defined as follows:

$$g^* \quad = \quad \bigvee \{ h \in \tau \mid \; h \otimes g \quad \leq \quad 1_{\varnothing} \otimes 1_X \} \quad .$$

Since $\otimes$ is distributive over arbitrary, non empty joins (see Axiom (IV)), we obtain the important relation

$$g^* \otimes g \quad \leq \quad 1_{\varnothing} \otimes 1_X \qquad \forall \, g \in \tau \quad .$$

Definition 6.3.1 (Separation Axioms)
Let (X, τ) be an L–topological space. (X, τ) is *Kolmogoroff separated* (i.e. fulfills the $\mathbf{T}_0$–axiom) iff τ separates points in X – i.e. for every pair $(p, q) \in X \times X$ with $p \neq q$ there exists $g \in \tau$ s.t. $g(p) \neq g(q)$. (X, τ) is *Hausdorff separated* (i.e. fulfills the $\mathbf{T}_2$–axiom) iff for every pair $(p, q) \in X \times X$ with $p \neq q$ there exists $g \in \tau$ s.t. $g^*(p) \otimes g(q) \not\leq \bot \otimes \top$.
■

Definition 6.3.2 (Density and Regularity)
Let (X, τ) be an L–topological space. An ordinary subset A of X is said to be *dense in X* iff every $g \in \tau$ satisfies the following implication[8]

$$g(a) \leq \bot \otimes \top \;\; \forall \, a \in A \quad \Longrightarrow \quad g(x) \leq \bot \otimes \top \;\; \forall \, x \in X \; .$$

An ordinary subset A of X is said to be *strictly dense in X* iff for all $g \in \tau$ the subsequent relation holds

$$g(x) \quad \leq \quad \bigvee \{ g(a) \mid a \in A \} \qquad \forall \, x \in X \quad .$$

(X, τ) is called *regular*[9] iff every $h \in \tau$ fulfils the following property

$$h \quad = \quad \bigvee \{ g \in \tau \mid g^* \vee h \; = \; 1_X \} \quad .$$

■

[8]In the case of $\otimes = \wedge$ the concept of density coincides with that one proposed by S.E. Rodabaugh in [?]. Examples of dense subspaces are given in Subsection 7.1.
[9]In the case of $\otimes = \wedge$ regularity is synonymous with *localic regularity* (cf. [92]).

Proposition 6.3.3 *Let* $(L, \leq, \otimes)$ *be a cl–quasi–monoid provided with the following property*

$$\text{(III*)} \qquad \top \otimes \alpha \leq \beta \otimes \top \quad \Longrightarrow \quad \alpha \leq \beta \ .$$

Further let (X, τ) *be a regular, L–topological space. Then the following assertions are equivalent:*

(i) (X, τ) is Hausdorff separated.

(ii) (X, τ) is Kolmogoroff separated.

Proof. (a) ((i) $\Longrightarrow$ (ii)) Let (p, q) be an element of $X \times X$ with $p \neq q$. Because of the $\mathbf{T}_2$–axiom there exists an element $g \in \tau$ provided with the property $g^*(p) \otimes g(q) \not\leq \bot \otimes \top$. Hence $g(p) \neq g(q)$ follows; i.e. (X, τ) is Kolmogoroff separated.
(b) ((ii) $\Longrightarrow$ (i)) Let (p, q) be a pair with $p \neq q$. Since (X, τ) satisfies the $\mathbf{T}_0$–axiom, there exists an element $h \in \tau$ with $h(p) \neq h(q)$. Without loss of generality we assume $h(p) \not\leq h(q)$. Further we assume that (X, τ) is not Hausdorff separated. Then we obtain for all $g \in \tau$ with $g^* \vee h = 1_X$ that the following estimation holds:

$$\begin{aligned} \top \otimes g(p) &= (g^*(q) \otimes g(p)) \vee (h(q) \otimes g(p)) \quad \leq \quad (\bot \otimes \top) \vee (h(q) \otimes \top) \\ &= h(q) \otimes \top \ ; \end{aligned}$$

hence $g(p) \leq h(q)$ follows from $(\text{III})^*$. Now we apply the regularity of (X, τ) and obtain $h(p) \leq h(q)$ which is a contradiction to the choice of h. Thus (X, τ) is Hausdorff separated.
■

Proposition 6.3.4 *Let* (X, τ) *be an L–topological space,* A *be a strictly dense subset of* X, *and let* $i_A : A \hookrightarrow X$ *be the inclusion map. Further let* (Y, σ) *be a Hausdorff separated, L–topological space, and let* $\varphi : X \longmapsto Y$ *and* $\psi : X \longmapsto Y$ *be an L–continuous map. Then the following assertions are equivalent:*

(i) $\varphi = \psi$.

(ii) $\varphi \circ i_A = \psi \circ i_A$.

Proof. The implication (i) $\Longrightarrow$ (ii) is obvious. In order to verify (ii) $\Longrightarrow$ (i) we proceed as follows: We fix $x \in X$ and choose an element $g \in \sigma$. Now we invoke the L–continuity of φ and ψ, the axiom (o2) (cf. Section 3) and the densitiy of A:

$$g^*(\varphi(x)) \otimes g(\psi(x)) \quad \leq \quad \bigvee_{a \in A} g^*(\varphi(a)) \otimes g(\psi(a)) \ ;$$

hence the definition of g^* and the assertion (ii) imply:

$$g^*(\varphi(x)) \otimes g(\psi(x)) \quad \leq \quad \bot \otimes \top \qquad \forall g \in \sigma \ .$$

Since (Y, σ) is Hausdorff separated, the relation $\varphi(x) = \psi(x)$ follows.
■

Theorem 6.3.5 (Principle of continuous extension)
Let $(L, \leq, \otimes)$ be a cl–quasi–monoid satisfying the axiom (III*). *Further (X, τ) be an L–topological space, (Y, σ) be a Hausdorff separated, regular, L–topological space, and let A be a strictly dense subset of X provided with the initial L–topology τ_A w.r.t. the inlusion map $i_A : A \hookrightarrow X$ (cf. Theorem 3.4). Finally let $\varphi : A \longmapsto Y$ be an L–continuous map. Then the following assertions are equivalent:*

(i) There exists an L–continuous map $\psi : X \longmapsto Y$ with $\psi \circ i_A = \varphi$; i.e. φ has an L–continuous extension.

(ii) For every $x \in X$ there exists $y \in Y$ satisfying the following condition

$$\forall h \in \sigma \; \exists g \in \tau : \quad (1) \quad h(y) \leq g(x) \quad , \quad (2) \quad g \circ i_A = h \circ \varphi \quad .$$

Proof. The implication (i) $\Longrightarrow$ (ii) follows immediately from the L–continuity of the extension ψ. In order to verify (ii) $\Longrightarrow$ (i) we proceed as follows:
(a) Let x be a given point of X. We show that the point y in Assertion (ii) is uniquely determined by the condition specified in Assertion (ii). Therefore let us consider the following situation: Let $h \in \sigma$, $g_1 \in \tau$, $g_2 \in \tau$ be such that

$$h^*(y_1) \leq g_1(x) \; , \quad h(y_2) \leq g_2(x) \; , \quad h^* \circ \varphi = g_1 \circ i_A \; , \quad h \circ \varphi = g_2 \circ i_A \quad .$$

Since A is strictly dense, we conclude from the axiom (o2) and the definition of h^*: $h^*(y_1) \otimes h(y_2) \leq g_1(x) \otimes g_2(x) \leq \bot \otimes \top$; hence the $\mathbf{T}_2$–axiom implies: $y_1 = y_2$.
(b) Because of the previous part (a) the assertion (ii) determines a map $\psi : X \longmapsto Y$. Since φ is L–continuous, the map ψ is obviously an extension of φ (see again Part (a)). We show that ψ is L–continuous. For this purpose we fix $h \in \sigma$ and consider $g \in \sigma$ with $g^* \vee h = 1_X$. The definition of ψ (cf. Assertion (ii)) guarantees that for every $x_0 \in X$ there exists an element $k_{x_0} \in \tau$ satisfying the following condition:

$$g(\psi(x_0)) \leq k_{x_0}(x_0) \quad , \quad k_{x_0} \circ i_A = g \circ \varphi \quad .$$

We show:

$$g^*(\psi(x)) \otimes k_{x_0}(x) \leq \bot \otimes \top \qquad \forall x \in X \quad . \qquad (\Diamond)$$

Referring again to the definition of ψ we observe that for every $x \in X$ there exists an element $\tilde{k} \in \tau$ with

$$g^*(\psi(x)) \leq \tilde{k}(x) \quad , \quad g^* \circ \varphi = \tilde{k} \circ i_A \quad .$$

Now we invoke the strict density of A, the axiom (o2) and obtain:

$$\begin{aligned} g^*(\psi(x)) \otimes k_{x_0}(x) \quad &\leq \quad \tilde{k}(x) \otimes k_{x_0}(x) \leq \bigvee_{a \in A} \tilde{k}(a) \otimes k_{x_0}(a) \\ &= \quad \bigvee_{a \in A} g^*(\varphi(a)) \otimes g(\varphi(a)) \leq \bot \otimes \top \quad . \end{aligned}$$

Because of $g^* \vee h = 1_Y$ we conclude from $(\diamondsuit)$

$$\top \otimes k_{x_0}(x) \quad \leq \quad h(\psi(x)) \otimes \top \qquad \forall x \in X \quad .$$

In particular, the axiom (III*) implies : $k_{x_0}(x) \leq h(\psi(x))$. Thus we obtain

$$g(\psi(x)) \quad \leq \quad \bigvee_{x_0 \in X} k_{x_0}(x) \quad \leq \quad h(\psi(x)) \qquad \forall x \in X \quad .$$

Since τ is closed w.r.t. arbitrary joins, we conclude from the regularity of (Y, σ) that $h \circ \psi$ is an element of τ; hence ψ is L–continuous.
■

In the remaining part of this subsection we consider the case $\otimes = \circledast$ where $\circledast$ denotes the monoidal mean operator w.r.t. to a given quantale. Therefore we assume that a quadruple $(L, \leq, \otimes, *)$ is given and satisfies the axioms (I) – (XI). In particular, $(L, \leq, *)$ is a strictly two–sided, commutative quantale, $\otimes$ coincides with the monoidal mean operator $\circledast$, and the triple $(L, \leq, \circledast)$ is a *cl*–quasi–monoid satisfying the additional axiom (III*) (cf. Remark 1.2.6, Proposition 1.2.7, Corollary 1.2.8). In this context it is interesting to see that for any L–topological space (X, τ) the following relation holds

$$g^* \quad = \quad \bigvee \{ h \in \tau \mid h * g = 1_\varnothing \} \quad .$$

Theorem 6.3.6 (Characterization of the T_2–axiom)
Let (X, τ) be an L–topological space with the corresponding L–neighborhood system $(\mu_p)_{p \in X}$. Then the following assertions are equivalent:

(i) (X, τ) is Hausdorff separated .

(ii) For every pair $(p, q) \in X \times X$ with $p \neq q$ there exist elements $g_1, g_2 \in \tau$ provided with the following properties:

$$g_1(p) * g_2(q) \quad \neq \quad \bot \quad , \qquad g_1 * g_2 \quad = \quad 1_\varnothing \quad .$$

If in addition (X, τ) is stratified, then the following assertions are equivalent

(i) (X, τ) is Hausdorff separated .

(iii) For every pair $(p, q) \in X$ with $p \neq q$ the L–neighborhood filters μ_p and μ_q do not have a common upper bound in the set of all stratified L–filters on X.

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) follows from the commutativity of $\otimes$ and the axioms (III), (VII), (VIII) and (X) – i.e.

$$\alpha \otimes \beta \;=\; (\alpha \otimes \top) * (\beta \otimes \top) \quad \leq \quad (\alpha * \beta) \otimes \top \quad , \qquad (\alpha \otimes \top) * (\alpha \otimes \top) \;=\; \alpha \;.$$

The equivalence (i) $\Longleftrightarrow$ (iii) is an immediate consequence from Corollary 6.2.7 and the L–neighborhood axioms (U3), (U4).
■

Definition 6.3.7 Let $(L, \leq, *)$ be a commutative, strictly two-sided quantale with square roots satisfying the condition (S3) (cf. Subsection 1.2). Further let $\otimes$ be given by the monoidal mean operator $\circledast$. An L-topological space is called *star-regular* iff every element $h \in \tau$ fulfills the following condition:

$$h \quad = \quad \bigvee \{ g \in \tau \mid (g^*) \to \perp \ \leq \ h \} \quad .$$

■

Evidently every regular, L-topological space is star-regular. Morever, Theorem 6.3.5 remains valid, if we replace regularity by star-regularity. For the sake of completeness we formulate the following

Theorem 6.3.8 (Quantales with square roots)
Let $(L, \leq, *)$ be a commutative, strictly two-sided quantale with square roots such that Condition (S3) is valid. Further, let (X, τ) be an L-topological space, (Y, σ) be a Hausdorff separated, star-regular, L-topological space, and let A be a strictly dense subset of X provided with the initial L-topology τ_A w.r.t. the inclusion map $i_A : A \hookrightarrow X$. Finally let $\varphi : A \longmapsto Y$ be an L-continuous map. Then the following assertions are equivalent:

(i) There exists an L-continuous map $\psi : X \longmapsto Y$ with $\psi \circ i_A = \varphi$; i.e. φ has an L-continuous extension.

(ii) For every $x \in X$ there exists $y \in Y$ satisfying the following condition

$$\forall h \in \sigma \ \exists g \in \tau : \quad (1) \quad h(y) \ \leq \ g(x) \quad , \quad (2) \quad g \circ i_A \ = \ h \circ \varphi \quad .$$

Proof. In principle we can repeat the proof of Theorem 6.3.5 with obvious modifications; e.g. the part (a) remains valid, if we replace $\circledast$ by $*$. In order to verify the L-continuity of the extension ψ of φ we can argue as follows: Let us fix $h \in \sigma$ and choose an element $g \in \sigma$ with $(g^*) \to \perp \ \leq \ h$. Then for every $x_0 \in X$ there exists an element $k_{x_0} \in \tau$ provided with the following properties

$$g(\psi(x_0)) \ \leq \ k_{x_0}(x_0) \quad , \quad g \circ \varphi \ = \ k_{x_0} \circ i_A \quad .$$

We show:

$$k_{x_0} * g^* \quad = \quad 1_\varnothing \quad . \tag{$\Diamond'$}$$

Referring to Assertion (ii) we choose an element $\tilde{k} \in \tau$ depending on $x \in X$ such that

$$g^*(\psi(x)) \ \leq \ \tilde{k}(x) \quad , \quad g^* \circ \varphi \ = \ \tilde{k} \circ i_A$$

Now we apply the axiom (o2), the strict density of A :

$$k_{x_0}(x) \circledast g^*(\psi(x)) \quad \leq \quad k_{x_0}(x) \circledast \tilde{k}(x) \quad \leq \quad \bigvee_{a \in A} g(\varphi(a)) \circledast g^*(\varphi(a))$$

and obtain from Proposition 1.2.2

$$k_{x_0}(x) * g^*(\psi(x)) \quad \leq \quad k_{x_0}(x) * \tilde{k}(x) \quad \leq \quad \bigvee_{a \in A} g(\varphi(a)) * g^*(\varphi(a)) \quad = \quad \bot \; ;$$

hence the relation $(\Diamond')$ is verified. Further the choice of g and the relation $(\Diamond')$ imply:

$$g(\psi(x)) \quad \leq \quad \bigvee_{x_0 \in X} k_{x_0}(x) \quad \leq \quad h(\psi(x)) \qquad \forall x \in X \quad .$$

We invoke the star–regularity of (Y, σ) and conclude from (o3) that $h \circ \psi$ is an element of τ; hence ψ is L–continuous.

■

Remark 6.3.9 If we replace strict density by density in 6.3.4, 6.3.5 and 6.3.8, then an immediate inspection of the respective proofs shows that the assertions of Proposition 6.3.4, Theorem 6.3.5 and Theorem 6.3.8 remain valid.

■

Remark 6.3.10 Let $(L, \leq, *)$ be a complete MV–algebra with square roots. Then the "negation" in L (i.e. $\alpha' \; = \; \alpha \rightarrow \bot$) determines an order reversing involution on L. In this context star–regularity and Hutton–Reilly–regularity (cf. [58, 92], [59, 94]) are equivalent concepts. Further the Hausdorff separation axiom specified in Definition 6.3.1 implies the Hausdorff separation axiom formulated by T. Kubiak (cf. [58, 59]). Finally we remark that in the case of *stratified*, L–topological spaces (X, τ) a subset A is *strictly dense* in X iff A is dense in X iff the L–interior of $X \cap \complement A$ is *empty* – i.e. $\mathcal{K}(1_{X \cap \complement A}) \; = \; 1_{\varnothing}$ where $\mathcal{K}$ denotes the L–interior operator corresponding to τ.

■

6.4 Compactness and stratified L–topological spaces

Definition 6.4.1 (Adherent point, limit point)
Let (X, τ) be a L–topological space and $(\mu_p)_{p \in X}$ be the corresponding L–neighborhood system (cf. Subsection 6.1). Further let ν be an L–filter on X.
(a) A point $p \in X$ is called *adherent point* of ν iff there exists a further L-filter $\bar{\nu}$ on X provided with the following properties

(i) $\quad \bar{\nu}((\bot \otimes \top) \cdot 1_X) \quad \leq \quad \bot \otimes \top \quad .$

(ii) $\quad \mu_p \quad \preccurlyeq \quad \bar{\nu} \quad , \quad \nu \quad \preccurlyeq \quad \bar{\nu} \quad .$

(b) A point $p \in X$ is said to be a *limit point* of ν iff $\mu_p \preccurlyeq \nu$.

■

If an L-filter ν has adherent points, then ν satisfies necessarily the following condition

$$\text{(F8)} \qquad \nu((\bot \otimes \top) \cdot 1_X) \quad \leq \quad \bot \otimes \top \ .$$

Since in the case of monoidal mean operators (i.e. $\otimes = \circledast$) the axiom (F3*) implies (F8), we conclude from Lemma 6.2.1(b) that every *stratified* L-filter fulfills (F8). Moreover under the hypothesis of (F8) every limit point is an adherent point.

Remark 6.4.2 (Adherence map, limit map)
Let $(\mu_p)_{p\in X}$ be a L-neighborhood system and $\mathfrak{F}_L(X)$ be the set of all L-filters on X. Then $(\mu_p)_{p\in X}$ induces two maps $adh : \mathfrak{F}_L(X) \longmapsto L^X$, $lim : \mathfrak{F}_L(X) \longmapsto L^X$ in the following way

$$\begin{aligned} [adh(\nu)](p) &= \bigwedge_{h\in L^X} (\mu_p(h) \to ((\nu(h \to \bot)) \to \bot)) \quad , \\ [lim(\nu)](p) &= \bigwedge_{h\in L^X} (\mu_p(h) \to \nu(h)) \quad . \end{aligned}$$

If $(L, \leq, \circledast, *)$ is given by a GL-monoid with square roots satisfying (S3) and (S4), then we conclude from Theorem 6.2.14 that for every stratified L-ultrafilter ν the adherence map $adh(\nu)$ and the limit map $lim(\nu)$ coincide.
∎

A L-topological space (X, τ) is called *compact* iff every L-filter provided with (F8) has at least one adherent point. A stratified L-topological space (X, τ) is said to be S-*compact* iff every stratified L-filter has at least one adherent point.

Proposition 6.4.3 (Images of compact L-topological spaces)
Let (X, τ) be a compact (S-compact, stratified) L-topological space, (Y, σ) be a (stratified) L-topological space, and let $\varphi : X \longmapsto Y$ be a surjective, L-continuous map. Then (Y, σ) is compact (S-compact).

Proof. Let ν be a (stratified) L-filter on Y, and $\varphi^{-1}(\nu)$ be the inverse image of ν under φ (cf. 6.2.20(b)). Then $\varphi^{-1}(\nu)$ has an adherent point $p \in X$ – i.e. there exists a (stratified) L-filter $\bar{\nu}$ on X with $\mu_p \preccurlyeq \bar{\nu}$ and $\varphi^{-1}(\nu) \preccurlyeq \bar{\nu}$. Now we form the corresponding image L-filters (cf. 6.2.20(a)) and obtain from the L-continuity of φ the following relation

$$\mu_{\varphi(p)} \quad \preccurlyeq \quad \varphi(\mu_p) \quad \preccurlyeq \quad \varphi(\bar{\nu}) \quad , \quad \nu \quad = \quad \varphi(\varphi^{-1}(\nu)) \quad \preccurlyeq \quad \varphi(\bar{\nu}) \quad ;$$

hence $\varphi(p)$ is an adherent point of ν.
∎

In the following considerations we do not persue the development of the general theory of compact L-topological spaces[10]; rather we restrict our interest to stratified L-topological spaces and S-compactness. Further we make the following **General Assumptions**

[10]In the case of complete Heyting algebras we refer to [42].

(A1) $(L, \leq, *)$ is a GL–monoid with square roots satisfying the axioms (S3) and (S4) (cf. Lemma 1.2.4 and Lemma 6.2.13).

(A2) $\otimes$ is determined by the monoidal mean operator $\circledast$.

Lemma 6.4.4 *Let ν be a stratified L–filter on X and p be a point of X. Further let (X, τ) be a stratified L–topological space. Then the following assertions are equivalent*

(i) p is an adherent point of ν.

(ii) $adh(\nu)(p) \quad = \quad \top$.

Proof. If p is an adherent point of ν, then we conclude from the L–filter axiom (F2) and Property (i) in Definition 6.4.1:

$$\mu_p(h) * \nu(h \to \bot) \quad \leq \quad [\bar{\nu}(\bot^{1/2} \cdot 1_X)]^2 \quad \leq \quad \bot \qquad \forall\, h \in L^X .$$

Hence $adh(\nu)(p) \; = \; \top$ follows. The implication (ii) $\Longrightarrow$ (i) is an immediate consequence of Corollary 6.2.7.
■

Lemma 6.4.5 ($\mathbf{T_2}$–Axiom and S–compactness) *Let (X, τ) be a stratified L–topological space.*
(a) (X, τ) is Hausdorff separated iff every stratified L–filter on X has at most one limit point.
(b) (X, τ) is S–compact and Hausdorff separated iff every stratified L–ultrafilter has a unique limit point.

Proof. The assertion (a) follows immediately from Theorem 6.3.6 and Corollary 6.2.7. Further it is not difficult to derive the assertion (b) from Remark 6.4.2, Lemma 6.4.4 and the previous assertion (a).
■

Theorem 6.4.6 (Compactness in the case of Boolean algebras) [11]
Let $(L, \leq, \otimes, *)$ be determined by a complete Boolean algebra – i.e. $* = \otimes = \wedge$ and the law of double negation is valid in $\mathbb{B} = (L, \leq)$. A stratified L–topological space (X, τ) is S–compact iff the following version of the so-called Heine–Borel property *holds*

$$(\mathrm{HB}) \quad \left\{ \begin{array}{l} \text{For very subfamily } \{g_i \,|\, i \in I\} \text{ of } \tau \text{ with } \bigvee\limits_{i \in I} g_i(x) \;\neq\; \bot \quad \forall\, x \in X \\ \text{there exists a finite subfamily } \{g_i \,|\, i \in H\} \text{ of } \{g_i \,|\, i \in I\} \text{ with} \\ \bigwedge\limits_{x \in X} (\bigvee\limits_{i \in H} g_i(x)) \quad \neq \quad \bot \ . \end{array} \right.$$

■

[11] In the more general case of complete Heyting algebras we refer to Theorem 3.3 in [42].

Proof. (a) Let ν be a stratified L–filter, μ_p be the stratified L–neighborhood filter of p, and let $\mathbf{F}_\nu$, $\mathbf{F}_{\mu_p}$ be the associated $\top$–filters (cf. 6.2.4(b)). We invoke Property $(\mathcal{F})$ in 6.2.3 and obtain for all $h \in L^X$:

$$\begin{aligned}\mu_p(h) \wedge \nu(h \to \bot) \quad &= \\ \bigvee_{(f,d_p)\in \mathbf{F}_\nu \times \mathbf{F}_{\mu_p}} \Big(\bigwedge_{x\in X} (d_p(x) \to h(x)) \wedge (f(x) \to (h(x) \to \bot))\Big) \quad &.\end{aligned}$$

Hence p is an adherent point of ν iff for all $f \in \mathbf{F}_\nu$ and for all $d_p \in \mathbf{F}_{\mu_p}$ the relation

$$\bigvee_{x\in X} (d_p(x) \wedge f(x)) \quad = \quad \top$$

holds.

(b) Let $(\mu_p)_{p\in X}$ be the L–neighborhood system corresponding to τ, and let $\mathbf{F}_{\mu_p}$ be the associated $\top$–filter of μ_p ($p \in X$). Then an element $f \in L^X$ (i.e. a L–subset of X) is τ*–closed*

$$\text{iff} \quad \mu_p(f \to \bot) \quad = \quad f(p) \to \bot \qquad \forall p \in X$$

$$\text{iff} \quad \bigvee_{d_p\in \mathbf{F}_{\mu_p}} \Big(\bigwedge_{x\in X} (d_p(x) \to (f(x) \to \bot))\Big) \quad = \quad f(p) \to \bot \qquad \forall p \in X$$

$$\text{iff} \quad \bigwedge_{d_p\in \mathbf{F}_{\mu_p}} \Big(\bigvee_{x\in X} d_p(x) \wedge f(x)\Big) \quad = \quad f(p) \qquad \forall p \in X \quad .$$

(c) We show that the condition (HB) is necessary. For this purpose we consider a subfamily $\{g_i \,|\, i \in I\}$ of τ with

$$\bigvee_{i\in I} g_i(x) \quad \neq \quad \bot \qquad \forall x \in X \qquad .$$

We denote by $J(I)$ the set of all finite, non empty , ordinary subsets of I and assume

$$\bigwedge_{x\in X} \Big(\bigvee_{i\in H} g_i(x)\Big) \quad = \quad \bot \qquad \forall H \in J(I) \qquad .$$

Then $\mathcal{G} \;=\; \{g_i \to \bot \,|\, i \in I\}$ generates a $\top$–filter $\mathbf{F}_\mathcal{G}$. In particular the map $\nu : L^X \longmapsto L$ defined by

$$\nu(h) \quad = \quad \bigvee_{f\in \mathbf{F}_\mathcal{G}} \Big(\bigwedge_{x\in X} f(x) \to h(x)\Big)$$

is a stratified L–filter on X. Because of Axiom (F1) the associated $\top$–filter of ν coincides with $\mathbf{F}_\mathcal{G}$ (cf. 6.2.3(b)). Since (X,τ) is S–compact, ν has an adherent point p_0. Referring to the previous parts (a) and (b) we obtain for all $H \in J(I)$:

$$\begin{aligned}\top \quad &= \quad \bigwedge_{d_{p_0}\in \mathbf{F}_{\mu_{p_0}}} \Big(\bigvee_{x\in X} \big(\big(\bigwedge_{i\in H} (g_i(x) \to \bot)\big) \wedge d_{p_0}(x)\big)\Big) \quad = \\ &= \quad \bigwedge_{i\in H} (g_i(p_0) \to \bot)\end{aligned}$$

–i.e. $\bigvee_{i\in I} g_i(p_0) = \bot$ which is a contradiction to the hypothesis on $\{g_i \mid i \in I\}$.

(d) We show that the condition (HB) is sufficient. Therefore let us consider a stratified L–filter ν and its associated $\top$–filter $\mathbf{F}_\nu$. Referring to part (b) the *τ–closure* of $f \in \mathbf{F}_\nu$ is given by

$$\mathsf{cl}f(p) = \bigwedge_{d_p \in \mathbf{F}_{\mu_p}} \big(\bigvee_{x\in X} (f(x) \wedge d_p(x)) \big) \qquad \forall p \in X \quad .$$

Obviously $\{\mathsf{cl}f \to \bot \mid f \in \mathbf{F}_\nu\}$ is a subset of τ. Further we obtain from ($\mathbb{F}$3)

$$\bigwedge_{x\in X} \mathsf{cl}f(x) \to \bot = \bot \qquad \forall f \in \mathbf{F}_\nu \quad .$$

Since $\mathbf{F}_\nu$ is directed downwards (cf. ($\mathbb{F}$2)), the condition (HB) implies the existence of a point $p_0 \in X$ with $\bigvee_{f\in\mathbf{F}_\nu} (\mathsf{cl}f(p_0) \to \bot) = \bot$; hence we obtain $\bigwedge_{f\in\mathbf{F}_\nu} \mathsf{cl}f(p_0) = \top$. Referring to the parts (a) and (b) this means that p_0 is an adherent point of ν.

■

Corollary 6.4.7 *Let $(L, \leq, \otimes, *)$ be determined by a complete Boolean algebra (cf. 6.4.6). Every S–compact, stratified L–topological space (X, τ) satisfies the following condition*

$$\text{(HB')} \quad \begin{cases} \text{Every subfamily } \{g_i \mid i \in I\} \text{ of } \tau \text{ with } \bigvee_{i\in I} g_i(x) = \top \quad \forall x \in X \\ \text{has the property} \quad \bigvee_{H\in J(I)} \big(\bigwedge_{x\in X} \big(\bigvee_{i\in H} g_i(x) \big)\big) = \top \\ \text{where } J(I) \text{ denotes all finite, ordinary subsets of } I\ . \end{cases}$$

Proof. Let us assume that there exists a subfamily $\{g_i \mid i \in I\}$ of τ provided with the subsequent properties

(i) $\quad \bigvee_{i\in I} g_i(x) = \top \qquad \forall x \in X$

(ii) $\quad \bigvee_{H\in J(I)} \big(\bigwedge_{x\in X} \big(\bigvee_{i\in H} g_i(x) \big)\big) \neq \top$.

Then we put $\varkappa = \bigwedge_{H\in J(I)} \big(\bigvee_{x\in X} \big(\bigwedge_{i\in H} (g_i(x) \to \bot)\big)\big) \neq \bot$, and the system $\mathcal{G} = \{(g_i \to \bot) \vee (\varkappa \to \bot) \cdot 1_X \mid i \in I\}$ generates a $\top$–filter $\mathbf{F}_\mathcal{G}$. Let $\nu_\mathcal{G}$ be the associated stratified L–filter of $\mathbf{F}_\mathcal{G}$ (cf. 6.2.3(a)). Since (X, τ) is S–compact, $\nu_\mathcal{G}$ has an adherent point p_0. Further we use the stratification property of τ and observe that the set $\mathcal{G}$ consists of τ–closed, L–fuzzy subsets. Hence we obtain (cf. part (a) and (b) of the proof of 6.4.6)

$$\bigwedge_{i\in I} (g_i(p_0) \to \bot) \vee (\varkappa \to \bot) = \top \qquad ;$$

i.e. $\bigvee_{i\in I} g_i(p_0) \quad \leq \quad \varkappa \to \perp \quad \neq \quad \top$ which is a contradiction to (i).

■

One of the main goals of any theory of compactness is to investigate the problem to which extent the formulation of a Tychonoff–Theorem is possible. Here we present a version of *Tychonoff's Theorem* within the category of stratified L–topological spaces. As underlying lattice–theoretical structure we only assume a GL–monoid with square roots satisfying the axioms (S3) and (S4). In particular the "conjunction–operator" $\otimes$ is determined by the monoidal mean operator $\circledast$ (cf. (A1) and (A2)).

Theorem 6.4.8 (Tychonoff Theorem)
Let $\mathfrak{F} = \{(X_i, \tau_i) \,|\, i \in I\}$ be a non empty family of stratified L–topological spaces. Then the following assertions are equivalent

(i) (X_i, τ_i) is S–compact for all $i \in I$.

(ii) The L–topological product (X_0, τ_0) of $\mathfrak{F}$ in the sense of **SL–TOP** is S–compact.

Proof. Let X_0 be the set–theoretical product of $\{X_i \,|\, i \in I\}$, and let Π_j be the projection from X_0 onto X_j $(j \in I)$. Further let $(\mu_{p_i})_{p_i \in X_i}$ be the L–neighborhood system corresponding to τ_i $(i \in I)$.
(a) For every $\vec{p} = (p_i)_{i\in I} \in X_0$ we define a map $\mu_{\vec{p}} : L^{X_0} \longmapsto L$ by

$$\mu_{\vec{p}}(h) \quad =$$
$$= \quad \bigvee\{\mu_{p_{i_1}}(f_{i_1}) * \ldots * \mu_{p_{i_n}}(f_{i_n}) \,|\, n \in \mathbb{N},\, (f_{i_1} \circ \Pi_{i_1}) * \ldots * (f_{i_n} \circ \Pi_{i_n}) \;\leq\; h\}$$

and show that $(\mu_{\vec{p}})_{\vec{p}\in X_0}$ is a L–neighborhood system on X_0. The axioms (U0), (U1) and (U3) are evident. The proof of (U2) is analogous to the verification of (F2) in the proof of 6.2.6. In order to verify (U4) we proceed as follows: Let us consider finitely many $f_{i_j} \in L^{X_{i_j}}$ with $f_{i_1} \circ \Pi_{i_1} * \ldots * f_{i_n} \circ \Pi_{i_n} \;\leq\; h$; then we infer from (U4)

$$\mu_{p_{i_1}}(f_{i_1}) * \ldots * \mu_{p_{i_n}}(f_{i_n}) \quad \leq$$
$$\bigvee\{\mu_{p_{i_1}}(k_{i_1}) * \ldots * \mu_{p_{i_n}}(k_{i_n}) \,|\, k_{i_j}(x_{i_j}) \,\leq\, \mu_{x_{i_j}}(f_{i_j}) \quad \forall x_{i_j} \in X_{i_j}\} \;\leq$$
$$\bigvee\{\mu_{\vec{p}}((k_{i_1} \circ \Pi_{i_j}) * \ldots * (k_{i_n} \circ \Pi_{i_n})) \,|\, (k_{i_1} \circ \Pi_{i_1}) * \ldots * (k_{i_n} \circ \Pi_{i_n}) \;\leq\; \mu_{-}(h)\,\};$$

hence (U4) follows also for $(\mu_{\vec{p}})_{\vec{p}\in X_0}$.
(b) We show (i) $\Longrightarrow$ (ii) . For this purpose let ν be a stratified L–ultrafilter on X_0. Since the image L–filter $\Pi_j(\nu)$ is also a stratified L–ultrafilter on X_j (cf. 6.2.20(a)), there exists a point $p_j \in X_j$ with $\mu_{p_j} \preccurlyeq \Pi_j(\nu)$. We put $\vec{p}_0 = (p_j)_{j\in I}$. By virtue of the maximality all stratified L–ultrafilter are submultiplicative (cf. 6.2.8); hence the definition of $\mu_{\vec{p}_0}$ implies $\mu_{\vec{p}_0} \preccurlyeq \nu$ – i.e. $\vec{p}_0$ is a limit point of ν. Since the stratified product L–topology τ_0 is coarser than

the topology determined by $(\mu_{\vec{p}})_{\vec{p}\in X_0}$, the assertion (ii) follows.
(c) Since all projections are surjective and L–continuous, the implication (ii) $\Longrightarrow$ (i) is an immediate consequence from Proposition 6.4.3.
■.

Theorem 6.4.9 *Let $\mathfrak{F} = \{(X_i, \tau_i) \mid i \in I\}$ be a non empty family of stratified L–topological spaces. Then the following assertions are equivalent*

(i) (X_i, τ_i) is Hausdorff separated for all $i \in I$.

(ii) The L–topological product (X_0, τ_0) of $\mathfrak{F}$ in the sense of **SL–TOP** is Hausdorff separated.

Proof. Since the family $\{\Pi_j \mid j \in I\}$ of projections is a family of L–continuous maps separating points, the implication (i) $\Longrightarrow$ (ii) is an immediate consequence from Theorem 6.3.6. In order to verify (ii) $\Longrightarrow$ (i) we proceed as follows: Let $(\mu_{\vec{p}})_{\vec{p}\in X_0}$ be the L–neighborhood system constructed in part (a) of the proof of 6.4.8. Since the stratified product L–topology is Hausdorff separated, we obtain that the stratified L–topology determined by $(\mu_{\vec{p}})_{\vec{p}\in X_0}$ is also Hausdorff separated. We choose $i_0 \in I$ and a pair $(p_{i_0}, q_{i_0}) \in X_{i_0} \times X_{i_0}$ with $p_{i_0} \neq q_{i_0}$. Then there exists a pair $(\vec{p}, \vec{q}) \in X_0 \times X_0$ satisfying the following condition

$$\Pi_{i_0}(\vec{p}) = p_{i_0} \quad , \quad \Pi_{i_0}(\vec{q}) = q_{i_0} \quad , \quad \Pi_j(\vec{p}) = \Pi_j(\vec{q}) \qquad \forall\, j \neq i_0 \qquad (\spadesuit)$$

Because of $\vec{p} \neq \vec{q}$ the Hausdorff separation axiom guarantees the existence of $h \in L^{X_0}$ provided with the property (cf. Corollary 6.2.7)

$$\mu_{\vec{p}}(h) * \mu_{\vec{q}}(h \to \bot) \quad \neq \quad \bot \quad .$$

Referring to the construction of $\mu_{\vec{p}}$ and $\mu_{\vec{q}}$ we can find finite index sets $\{i_1, \ldots, i_n\}$, $\{j_1, \ldots, j_m\}$ and elements $f_{i_k} \in L^{X_{i_k}}$, $g_{j_\ell} \in L^{X_{j_\ell}}$ such that the following relations hold

$$\begin{aligned}
&f_{i_1} \circ \Pi_{i_1} * \ldots * f_{i_n} \circ \Pi_{i_n} \quad \leq \quad h \quad , \\
&g_{j_1} \circ \Pi_{j_1} * \ldots * g_{j_m} \circ \Pi_{j_m} \quad \leq \quad h \to \bot \quad , \\
&\mu_{p_{i_1}}(f_{i_1}) * \ldots * \mu_{p_{i_n}}(f_{i_n}) * \mu_{q_{j_1}}(g_{j_1}) * \ldots * \mu_{q_{j_m}}(g_{j_m}) \quad = \varkappa \quad \neq \quad \bot \, .
\end{aligned}$$

If $i_0 \notin \{i_1, \ldots, i_n\} \cap \{j_1, \ldots j_m\}$, then we deduce from (U3) and $(\spadesuit)$ that $\varkappa = \bot$ which is impossible; hence $i_0 \in \{i_1, \ldots, i_m\} \cap \{j_1, \ldots, j_m\}$. Now we are in the position to define $\bar{f}, \bar{g} \in L^{X_{i_0}}$ as follows

$$\left.\begin{aligned}
\bar{f}(x_{i_0}) \quad &= \quad \underbrace{f_{i_1}(p_{i_1}) * \ldots * f_{i_n}(p_{i_n})}_{i_k \neq i_0} * f_{i_0}(x_{i_0}) \\
\bar{g}(x_{i_0}) \quad &= \quad \underbrace{g_{j_1}(q_{j_1}) * \ldots * g_{j_m}(q_{j_m})}_{j_\ell \neq i_0} * g_{i_0}(x_{i_0})
\end{aligned}\right. \qquad x_{i_0} \in X_{i_0} \quad .$$

Because of (♠) the relation

$$\bar{f}(x_{i_0}) * \bar{g}(x_{i_0}) \quad \leq \quad h(\vec{p}) * (h(\vec{p}) \to \perp) \quad = \quad \perp$$

follows for all $\vec{p}$ with $\Pi_{i_0}(\vec{p}) \ = \ x_{i_0}$, $\Pi_j(\vec{p}) \ = \ \Pi_j(\vec{q}) \ = \ p_j$ for all $j \neq i_0$. Now we apply (U3), (F1), (F4) and obtain

$$\perp \quad \neq \quad \kappa \quad \leq \quad \mu_{p_{i_0}}(\bar{f}) * \mu_{q_{i_0}}(\bar{g}) \quad \leq \quad \mu_{p_{i_0}}(\bar{f}) * \mu_{q_{i_0}}(\bar{f} \to \perp) \quad .$$

By virtue of Theorem 6.3.6 (see also Corollary 6.2.7) the L–topological space (X_{i_0}, τ_{i_0}) is Hausdorff separated. Hence assertion (i) is verified.
■

As immediate consequence of the previous Theorems 6.4.8 and 6.4.9 we obtain that in the category of stratified L–topological spaces an arbitrary product of stratified L–topological spaces is S–compact and Hausdorff separated if and only if each factor is S–compact and Hausdorff separated.

We close this subsection with a result which can be viewed as a first step towards a clarification of the relationship between S–compactness and some type of regularity.

Theorem 6.4.10 (Complete MV–algebras with square roots)
Let $(L, \leq, *)$ be a complete MV-algebra with square roots, and let (X, τ) be a S-compact, stratified L-topological space. Further let ν be a stratified L-filter on X satisfying the axiom (F7). If ν has a **unique** *adherent point, then ν is convergent.*

Proof. Let q be the unique adherent point of ν. If μ is a stratified L–ultrafilter on X with $\nu \preccurlyeq \mu$, then the S–compactness implies that μ has a limit point p. Because of $\nu \preccurlyeq \mu$ the point p is also an adherent point of ν; hence $p = q$. Now we embark on Theorem 6.2.19 and obtain that q is also a limit point of ν.
■

An important application of Theorem 6.4.10 appears in the proof of Theorem 7.3.9.

6.5 A level–wise characterization of L–neighborhood axioms

Let X be a non empty set, $(L, \leq)$ a complete lattice and α be an element of $L^0 := L \setminus \{\perp\}$. The *$\alpha$–level set* A_α of a given *L–valued map* $f : X \longmapsto L$ is determined by

$$A_\alpha \quad = \quad \{x \in X \mid \alpha \leq f(x)\}$$

The characterization of L–valued maps by their corresponding system of α–level sets goes back to the early work of C.V. Negoita and D. Ralescu on **Fuzzy Set Theory** ([81]). Here we summarize this result as follows:

Lemma 6.5.1 (Negoita–Ralescu)
(a) Let X be a non empty set, f be a L–valued map defined on X, and let $(A_\alpha)_{\alpha \in L^0}$ be the system of α–level sets associated with f. Then $(A_\alpha)_{\alpha \in L^0}$ satisfies the following condition:

$$(\mathbb{L}1) \qquad \bigcap_{\beta \in I} A_\beta \;=\; A_{\vee I} \qquad \forall I \subseteq L, \quad I \neq \varnothing \qquad \text{(Left Continuity)}$$

(b) For every system $(A_\alpha)_{\alpha \in L^0}$ of subsets A_α of X satisfying Axiom $(\mathbb{L}1)$ there exists a *unique* L–valued map f such that for each α the subset A_α coincides with the α–level set of f. In particular the following relation holds:

$$(\mathbb{L}2) \qquad f(x) \;=\; \bigvee \{\alpha \in L^0 \mid x \in A_\alpha\} \qquad \forall x \in X \qquad .$$

■

The previous lemma shows that L–valued maps and L^0–indexed systems of subsets provided with $(\mathbb{L}1)$ are equivalent concepts. In the case of $L = \{0,1\}$ this equivalence means the identification of *ordinary subsets* with their *characteristic functions.* Therefore we introduce the following terminology: An L–valued map defined on X is called a *L–subset of X.*

In the following considerations of this subsection we assume that $(L, \leq, \otimes)$ is a cl–quasi–monoid (cf. Section 1).

Lemma 6.5.2 (Characterization of L–filters by their α–levels)
Let $\nu : L^X \longmapsto L$ be a map (i.e. a L–subset of L^X) and $(\mathbb{F}_\alpha)_{\alpha \in L^0}$ be its corresponding system of α–level sets. Then the following assertions are equivalent:

(i) ν is an L–filter on X.

(ii) $(\mathbb{F}_\alpha)_{\alpha \in L^0}$ fulfills the subsequent properties:

$$(\mathfrak{f}0) \qquad 1_X \in \mathbb{F}_\alpha \quad .$$

$$(\mathfrak{f}1) \qquad f \leq h,\, f \in \mathbb{F}_\alpha \quad \Longrightarrow \quad h \in \mathbb{F}_\alpha \quad .$$

$$(\mathfrak{f}2) \qquad \begin{cases} \alpha \otimes \beta \neq \bot\,,\ f_1 \in \mathbb{F}_\alpha\,,\ f_2 \in \mathbb{F}_\beta & \Longrightarrow \quad f_1 \otimes f_2 \in \mathbb{F}_{\alpha \otimes \beta}\,, \\ \alpha \otimes \bot \neq \bot\,,\ f \in \mathbb{F}_\alpha & \Longrightarrow \quad f \otimes 1_\varnothing \in \mathbb{F}_{\alpha \otimes \bot} \quad . \end{cases}$$

$$(\mathfrak{f}3) \qquad 1_\varnothing \notin \mathbb{F}_\alpha \quad .$$

Proof. Referring to Lemma 6.5.1 it is easy to verify the equivalence between the L–filter axioms (F0), (F1), (F3) (cf. 6.1.4) and the axioms $(\mathfrak{f}0)$, $(\mathfrak{f}1)$ and $(\mathfrak{f}3)$. The equivalence of (F2) and $(\mathfrak{f}2)$ is based on Axiom (IV).
■

Theorem 6.5.3 *Let X be a non empty set, $(\mu_x)_{x \in X}$ be a system of maps $\mu_x : L^X \longmapsto L$. Further let $(\mathbb{U}_{(x,\alpha)})_{\alpha \in L^0}$ be the system of α–level sets corresponding to μ_x. Then the following assertions are equivalent:*

(i) $\mathcal{U} = (\mu_x)_{x \in X}$ is a L–neigborhood system on X.

(ii) $(\mathbb{U}_{(x,\alpha)})_{(x,\alpha) \in X \times L^0}$ fulfills the subsequent properties:

(u0) $1_X \in \mathbb{U}_{(x,\alpha)}$.

(u1) $u \leq v,\ u \in \mathbb{U}_{(x,\alpha)} \implies v \in \mathbb{U}_{(x,\alpha)}$.

(u2) $\begin{cases} \alpha \otimes \beta \neq \perp,\ u_1 \in \mathbb{U}_{(x,\alpha)},\ u_2 \in \mathbb{U}_{(x,\beta)} \implies u_1 \otimes u_2 \in \mathbb{U}_{(x,\alpha\otimes\beta)}, \\ \alpha \otimes \perp \neq \perp,\ u \in \mathbb{U}_{(x,\alpha)} \implies u \otimes 1_\varnothing \in \mathbb{U}_{(x,\alpha\otimes\beta)} \end{cases}$.

(u3) $u \in \mathbb{U}_{(x,\alpha)} \implies \alpha \leq u(x)$.

(u4) $\forall\, u \in \mathbb{U}_{(x,\alpha)}\ \exists\, v \in \mathbb{U}_{(x,\alpha)}\ :\ u \in \mathbb{U}_{(y,v(y))}\quad \forall\, y$ with $v(y) \neq \perp$.

Proof. In view of Lemma 6.5.2 the conditions (U0) – (U3) are equivalent to (u0) – (u3). In order to verify also the equivalence between (U4) and (u4) we proceed as follows:
(a) ((U4)$\Longrightarrow$(u4)) Let us consider $u \in \mathbb{U}_{(x,\alpha)}$ – i.e. $\alpha \leq \mu_x(u)$; then we define a map $v : X \longmapsto L$ by

$$v(y) = \mu_y(u) \qquad \forall\, y \in X \quad .$$

Now we apply (U1) and (U4) and obtain: $\mu_x(u) \leq \mu_x(v)$ – i.e. $v \in \mathbb{U}_{(x,\alpha)}$. Further

$$u \in \mathbb{U}_{(y,v(y))} \qquad \text{whenever} \quad v(y) \neq \perp$$

holds by definition of v; hence (u4) follows.
(b)((u4)$\Longrightarrow$(U4)) According to ($\mathbb{L}$2) the relation

$$\mu_z(u) = \bigvee \{ \alpha \in L^0 \mid u \in \mathbb{U}_{(z,\alpha)} \} \tag{$\blacklozenge$}$$

holds for all $z \in X$. Now we fix $x \in X$ and denote by M_x the set of all $\alpha \in L^0$ with $u \in \mathbb{U}_{(x,\alpha)}$. By virtue of (u4) we choose for every $\alpha \in M_x$ a map $v \in \mathbb{U}_{(x,\alpha)}$ such that $u \in \mathbb{U}_{(y,v(y))}$ whenever $v(y) \neq \perp$; then the relation ($\blacklozenge$) implies:

$$\alpha \leq \mu_x(v) \ , \qquad v(y) \leq \mu_y(u), \quad \forall\, y \in X \quad .$$

Now we apply again ($\blacklozenge$) and obtain:

$$\mu_x(u) = \bigvee M_x \leq \bigvee \{ \mu_x(v) \mid v(y) \leq \mu_y(u)\ \forall\, y \in X \} \qquad ;$$

hence (U4) is verified.
■

Remark 6.5.4 (Complete Heyting algebras)
Let $(L, \leq)$ be a complete Heyting algebra, and let us consider the special case $\otimes = \wedge$. Then the axioms $(\mathfrak{f}0)$ – $(\mathfrak{f}3)$ simply mean that $\mathbb{F}_\alpha$ is a *filter* in the sense of the lattice $(L^X, \preccurlyeq)$, where $\preccurlyeq$ is defined pointwise w.r.t. $\leq$. In this context we can restate the result of Lemma 6.5.2 as follows: L–filters and L^0–index systems of filters of L^X satisfying the left continuity condition $(\mathbb{L}1)$ are equivalent concepts. Moreover, if we extend these considerations to L–neighborhood systems, then we obtain that L–neighborhood systems can be characterized by systems $(\mathbb{U}_{(x,\alpha)})_{(x,\alpha) \in X \times L^0}$ provided with the following properties:

- $(\mathbb{U}_{(x,\alpha)})_{\alpha \in L^0}$ satisfies $(\mathbb{L}1)$ for all $x \in X$.
- $\mathbb{U}_{(x,\alpha)}$ is a filter of L^X for all $\alpha \in L^0$.
- $(\mathbb{U}_{(x,\alpha)})_{(x,\alpha) \in X \times L^0}$ satisfies $(\mathfrak{u}3)$ and $(\mathfrak{u}4)$.

In particular $(\mathbb{U}_{(x,\alpha)})_{(x,\alpha) \in X \times L^0}$ is called a *neighborhood filter system of L–subsets of X*. Finally, for every L–subset a of X its *neighborhood filter* $\mathbb{V}_a$ *of L–subsets of X* is determined by:

$$\mathbb{V}_a = \bigcap \{ \mathbb{U}_{(x,a(x))} \mid a(x) \neq \perp \} .$$

Then it is not difficult to show that the axioms $(\mathbb{L}1)$, $(\mathfrak{u}0)$ – $(\mathfrak{u}4)$ are equivalent to those axioms of fuzzy neighborhood systems proposed by S.E. Rodabaugh in [91] (see also Section 4 in [29]).

■

7 Examples of L–topological spaces

7.1 Case of complete Heyting algebras

Let $(L, \leq)$ be a complete Heyting algebra and let us consider the case $\otimes = * = \wedge$. Further, let $\mathcal{L}$ be a complete lattice and $\mathrm{H}(\mathcal{L}, L)$ be the set of all frame morphisms $\varphi : \mathcal{L} \to L$ (i.e. φ preserves arbitrary joins and finite meets).Referring to the evaluation map every element $a \in \mathcal{L}$ induces a map $f_a : \mathrm{H}(\mathcal{L}, L) \longmapsto L$ by $f_a(\varphi) = \varphi(a)$ Then $\tau^{\mathcal{L}} := \{f_a \mid a \in \mathcal{L}\}$ is a L–topology on $\mathrm{H}(\mathcal{L}, L)$, and the pair $(\mathrm{H}(\mathcal{L}, L), \tau^{\mathcal{L}})$ is a L–topological space (cf. Example 3.1).

Proposition 7.1.1 *Let $(L, \leq)$ be a complete Heyting algebra. Then the following assertions are equivalent*

(i) For all complete lattices $\mathcal{L}$ with $\mathrm{H}(\mathcal{L}, L) \neq \varnothing$ the L–topological space $(\mathrm{H}(\mathcal{L}, L), \tau^{\mathcal{L}})$ is **not stratified.**

(ii) The L-topological space $(\mathrm{H}(L, L), \tau^L)$ is not stratified.

(iii) There exist $\alpha \in L \setminus \{\top, \perp\}$ and $\varphi \in \mathrm{H}(L, L)$ such that $\varphi(\alpha) \neq \alpha$.

Proof. The implications (i)$\Longrightarrow$(ii) and (iii)$\Longrightarrow$(i) are obvious. In order to verify (ii)$\Longrightarrow$(iii) we proceed as follows: Since $(\mathrm{H}(L,L),\tau^L)$ is not stratified, there exists $\alpha \in L \setminus \{\top,\bot\}$ such that

$$\alpha \cdot 1_{\mathrm{H}(L,L)} \quad \neq \quad f_\beta \qquad \forall\, \beta \in L \qquad ;$$

hence there exists $\varphi \in \mathrm{H}(L,L)$ with $\varphi(\alpha) \neq \alpha$.
■

If $(L,\leq)$ contains a prime element α with $\bot \neq \alpha \neq \top$, then $(L,\leq)$ satisfies condition (iii) in 7.1.1. In particular, any complete chain contains enough prime element. Therefore we *emphasize* that in contrast to the ideological statement 3.13 in [70] there exist natural, *non–stratified* L–topological spaces arising quite canonically in the study of frames (resp. locales; cf. [49]). Obviously, every subframe $\mathcal{L}$ of L^I (where I denotes an appropriate, non empty index set) can be identified with an L–topological space $(\mathrm{H}(\mathcal{L},L),\tau^{\mathcal{L}})$ (for more details see Section 5 in [93]).

The Booleanization of complete Heyting algebras leads to a further important class of examples of L–topological spaces.

Example 7.1.2 (Frames and $\mathbb{B}$–topologies)
Let $(L,\leq)$ be a complete Heyting algebra and $\mathbb{B}(L,\leq) = (\mathbb{B}(L),\leq)$ its Booleanization (cf. Theorem 1.4.2). Then the embedding $j : L \longmapsto \mathbb{B}(L)$ is a frame–morphism (cf. Remark 1.4.3(a)). In particular, $\mathrm{H}(L,\mathbb{B}(L))$ separates points in L – i.e. L is frame–isomorphic to the $\mathbb{B}(L)$–topology $\tau^L = \{f_a \,|\, a \in L\}$. Hence every *frame* can be identified with a *Boolean valued topology.* Since in the Boolean case a non trivial convergence theory is available (cf. Theorem 6.2.17), the reader is invited to pursue the consequences of this fact for the theory of frames (see also [42, 49]).

■

Remark 7.1.3 (Dense L–topological subspaces)
(a) Let $\mathcal{A}$ be a dense sublocale of $\mathcal{B}$ – i.e. there exists a *surjective* framemorphism $\Theta : \mathcal{B} \longmapsto \mathcal{A}$ provided with the property

$$\Theta(b) \quad = \quad \bot \quad \Longrightarrow \quad b = \bot \quad .$$

Further let $(L,\leq)$ be a complete Heyting algebra, and let us consider the case $\otimes = \wedge$. Then, according to the above–mentioned construction the locales $\mathcal{A}$ and $\mathcal{B}$ generate L–topological spaces of the following form, namely $(\mathrm{H}(\mathcal{A},L),\tau^{\mathcal{A}})$ and $(\mathrm{H}(\mathcal{B},L),\tau^{\mathcal{B}})$. Moreover, the surjective framemorphism Θ induces an injective, L–continuous map $\varphi_\Theta : \mathrm{H}(\mathcal{A},L) \longmapsto \mathrm{H}(\mathcal{B},L)$ by[12]

$$[\varphi_\Theta(p)](b) \quad = \quad p \circ \Theta(b) \qquad \forall\, b \in B \quad .$$

[12] See also Theorem 2.11 in [95].

If $\mathcal{A}$ is L–spatial (i.e. the set of all L–valued framemorphisms separates points in $\mathcal{A}$), then the density of $\mathcal{A}$ implies that the range of φ_Θ is a dense, L–topological subspace of $\mathrm{H}(\mathcal{B}, L)$ in the sense of Definition 6.3.2.
(b) The situation presented in (a) is realized by the following standard example: Let $\mathcal{B}$ be a locale and $\mathcal{B}_{\neg\neg}$ be the sublocale of all regular elements of $\mathcal{B}$ (i.e. of all elements of $\mathcal{B}$ satisfying the condition: $(b \to \bot) \to \bot \;=\; b$). It is well known that $\mathcal{B}_{\neg\neg}$ is a *complete Boolean algebra* and the smallest *dense* sublocale of $\mathcal{B}$ (cf. [49, 1.13,2.4]). Further we choose $\mathcal{B}_{\neg\neg}$ as an underlying complete Boolean algebra $\mathbb{B}$ and consider the $\mathbb{B}$–topological space $(\mathrm{H}(\mathcal{B},\mathbb{B}), \tau^{\mathcal{B}})$ generated by $\mathcal{B}$. Due to the special relationship between $\mathcal{B}$ and $\mathcal{B}_{\neg\neg} = \mathbb{B}$ we can replace the $\mathbb{B}$–topological space $(\mathrm{H}(\mathcal{B}_{\neg\neg},\mathbb{B}), \tau^{\mathcal{B}_{\neg\neg}})$ by the *singleton space* $(\{\cdot\},\mathbb{B})$ and consider the framemorphism $\varphi_{\neg\neg} : \mathcal{B} \longmapsto \mathcal{B}_{\neg\neg}$ defined by

$$\varphi_{\neg\neg}(b) \quad = \quad (b \to \bot) \to \bot \qquad \forall\, b \in \mathcal{B} \quad .$$

Then it is not difficult to show that $\{\varphi_{\neg\neg}\}$ is a dense (but in general *not strictly dense*) $\mathbb{B}$–topological subspace of $\mathrm{H}(\mathcal{B},\mathbb{B})$.
■

Example 7.1.4 (Pointwise almost everywhere convergence)
Let $(\Omega, \mathcal{M}, \pi)$ be an atomless probability space and $L^0(\Omega,\mathcal{M},\pi)$ be the vector space of all π–almost everywhere defined, real–valued, random variables. Further let $\mathcal{I}_0$ be the σ–ideal of all $\mathcal{M}$–measurable π–null sets. Then the quotient algebra $\mathbb{A} := \mathcal{M}/\mathcal{I}_0$ is a complete, atomless, weakly σ–distribitive Boolean algebra of countable type (see [100, 50]). In particular, the quotient map from $\mathcal{M}$ to $\mathbb{A}$ is denoted by q.

It is well known that there does *not exist* an ordinary (i.e. 2–)topology τ on $L^0(\Omega,\mathcal{M},\pi)$ such that the pointwise π–almost everywhere convergence is *τ–topological* – i.e. a sequence $(\zeta_n)_{n\in\mathbb{N}}$ of π–almost everywhere defined, real–valued random variables converges π–almost everywhere to ξ iff $(\zeta_n)_{n\in\mathbb{N}}$ is topologically convergent to ξ w.r.t. a given (2–)topology τ. This deficiency plays an important role in the theory of *linear stochastic processes* (cf. [3, 38]). The aim of this example is to show that there exists a stratified $\mathbb{A}$–topology on $L^0(\Omega,\mathcal{M},\pi)$ such that pointwise π–almost everywhere convergence is actually *$\mathbb{A}$–topological.*

Let τ_R be the usual, ordinary (2–)topology on the real line $\mathbb{R}$. Since τ_R satisfies the second countability axiom, we deduce from the Stone representation of $\mathbb{A}$ that every framemorphism $p : \tau_R \longmapsto \mathbb{A}$ can be identified with a π–almost everywhere defined, real–valued random variable ξ, and vice versa, every π–almost everywhere defined random variable determines a framemorphism by: $p(U) = q(\xi^{-1}(U)) \quad \forall\, U \in \tau_R$. According to the above–mentioned construction it is clear that τ_R generates an $\mathbb{A}$–topology τ^{τ_R} on $L^0(\Omega,\mathcal{M},\pi)$ as follows:

$$\tau^{\tau_R} \quad = \quad \{f_U \mid U \in \tau_R\} \qquad \text{where } f_U(\xi) \;=\; q(\xi^{-1}(U)) \; .$$

In particular, $(L^0(\Omega,\mathcal{M},\pi)\,,\, \tau^{\tau_R})$ can be viewed as the $\mathbb{A}$–sobrification of the usual real line $(\mathbb{R},\tau_R)$ (cf. [95]). Now we add all constant, $\mathbb{A}$–valued maps to τ^{τ_R} and arrive at the stratification τ_{RS} of τ^{τ_R} – i.e. τ_{RS} is the smallest $\mathbb{A}$–topology on $L^0(\Omega, \mathcal{M}, \pi)$ which contains τ^{τ_R} and all constant maps (see also Theorem 5.1.1 and Theorem 5.2.7). Referring to Subsection 6.1 and Proposition 6.2.4(b) it is clear that the $\mathbb{A}$–neighborhood system corresponding to τ_{RS} can be identified with a system $(\mathbb{U}_\xi)_{\xi \in L^0(\Omega,\mathcal{M},\pi)}$ of $\top$–(neighborhood)filters $\mathbb{U}_\xi$. Referring to Lemma 3.2 in [38] it is not difficult to show that every $\mathbb{A}$–valued map $d_{(\xi,\frac{1}{n})} : L^0(\Omega, \mathcal{M}, \pi) \longmapsto \mathbb{A}$ of the following form

$$d_{(\xi,\frac{1}{n})}(\zeta) \quad = \quad q((\xi - \zeta)^{-1}(] - \tfrac{1}{n}, +\tfrac{1}{n}[)) \qquad \forall \zeta \in L^0(\Omega, \mathcal{M}, \pi)$$

is a $\mathbb{A}$–valued neighborhood of ξ (i.e. $d_{(\xi,\frac{1}{n})} \in \mathbb{U}_\xi$), and vice versa for every $\mathbb{A}$–valued neighborhood $d_\xi \in \mathbb{U}_\xi$ there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\bigvee\limits_{n\in\mathbb{N}} \alpha_n = \top$ such that the following relation holds:

$$\alpha_n \wedge d_{(\xi,\frac{1}{n})}(\zeta) \quad \leq \quad d(\zeta) \qquad \forall \zeta \in L^0(\Omega, \mathcal{M}, \pi) \ .$$

A sequence $(\zeta_n)_{n\in\mathbb{N}}$ is $\mathbb{A}$–topological convergent iff the corresponding elementary filter $\mathbf{E}((\zeta_n)_{n\in\mathbb{N}}) = \{d \in \mathbb{A}^{L^0(\Omega,\mathcal{M},\pi)} \mid \bigvee\limits_{m=1}^{\infty} (\bigwedge\limits_{n=m}^{\infty} d(\zeta_n)) = \top\}$ is convergent w.r.t a given $\mathbb{A}$–topology in the sense of Definition 6.4.1(b) where $\top$–filters are identified with stratified $\mathbb{A}$–filters (cf. Proposition 6.2.4(b)). Referring to the previous construction it is now clear that a sequence $(\zeta_k)_{k\in\mathbb{N}}$ in $L^0(\Omega, \mathcal{M}, \pi)$ is $\mathbb{A}$–topological convergent to ξ w.r.t. the stratified $\mathbb{A}$–topology τ_{RS} iff

$$\bigvee_{m=1}^{\infty} \bigwedge_{k=m}^{\infty} q((\xi - \zeta_k)^{-1}(] - \tfrac{1}{n}, +\tfrac{1}{n}[)) \quad = \quad \top \qquad \forall n \in \mathbb{N}$$

iff $(\zeta_k)_{k\in\mathbb{N}}$ converges π–almost everywhere to ξ. For more details on the relationship between pointwise π–almost everywhere convergence and $\mathbb{A}$–topological convergence we refer the reader to [38] – e.g. a map T from a metric space X to $L^0(\Omega, \mathcal{M}, \pi)$ (i.e. a stochastic process T with parameter set X) is $\mathbb{A}$–*continuous*[13] iff T transfers convergent sequences in X into pointwise π–almost everywhere convergent sequences (in $L^0(\Omega, \mathcal{M}, \pi)$).
■

Remark 7.1.5 (Spatial locales and L–stable subsets)
(a) Let $(L, \leq)$ be a complete Heyting algebra, $pt(L)$ be the set of all frame morphisms $p : L \longmapsto \{\perp, \top\}$, and let X be a non empty set. A subset G of $X \times pt(L)$ is called *L–stable* iff G is provided with the following property

$$\begin{aligned} &\forall (x,p) \in G \quad \exists \alpha_0 \in L \quad \text{s.t.} \\ &p(\alpha_0) = \top \quad , \quad q(\alpha_0) = \perp \quad \forall q \in pt(L) \text{ with } (x,q) \notin G \ . \end{aligned}$$

Further every subset G of $X \times pt(L)$ induces a map $f_G : X \longmapsto L$ by

$$f_G(x) \quad = \quad \bigvee \{\alpha \in L \mid \forall q \in pt(L) \text{ with } (x,q) \notin G : q(\alpha) = \perp\} \quad .$$

[13]See Section 3 and Remark 7.4.13.

If G is L–stable, then we obtain:

$$G \;=\; \{(x,p) \in X \times pt(L) \mid p(f_G(x)) \;=\; \top\} \quad .$$

On the other hand, every map $f : X \longmapsto L$ determines an L–stable subset G_f of $X \times pt(L)$ by

$$G_f \;=\; \{ (x,p) \in X \times pt(L) \mid p(f(x)) \;=\; \top \} \quad .$$

If L is a *spatial* locale (i.e. $pt(L)$ separates points in L (cf. [49])), then $f_{G_f}(x) = f(x)$ holds for all $x \in X$. Hence in the case of spatial locales L–valued maps $f : X \longmapsto L$ and L–stable subsets of $X \times pt(L)$ are equivalent concepts. Finally, an ordinary topology $\mathbb{T}$ on $X \times pt(L)$ is called ***L–stable*** iff every element $G \in \mathbb{T}$ is L–stable.

(b) Let $(L, \leq)$ be a spatial locale and $\otimes = * = \wedge$. Then L–topologies on X and L–stable, ordinary topologies on $X \times pt(L)$ are *equivalent concepts.* In particular, we can introduce a functor $\mathfrak{T} : L\text{–}\mathbf{TOP} \longmapsto \mathbf{TOP}$ which is determined on objects and morphisms as follows (see also [54]):

$$\begin{aligned}
\mathfrak{T}(X,\tau) \quad &= \quad (X \times pt(L)\,,\, \mathbb{T}_\tau) \qquad \text{where} \\
\mathbb{T}_\tau \quad &= \quad \{\, \{(x,p) \in X \times pt(L) \mid p(g(x)) \;=\; \top\} \mid \quad g \in \tau \,\} \quad , \\
\mathfrak{T}(\varphi) \quad &= \quad \varphi \times id_{pt(L)} \qquad .
\end{aligned}$$

It is not difficult to show that $\mathfrak{T}$ is an embedding functor[14] (cf. [54]). Moreover, an L–topology τ on X is weakly stratified (resp. stratified or strongly stratified (cf. Theorem 5.2.7)) iff the corresponding ordinary topology $\mathbb{T}_\tau$ on $X \times pt(L)$ satisfies the following condition

$$(\Sigma^*0) \qquad \{(x,p) \in X \times pt(L) \mid p(\alpha) \;=\; \top\} \;\in\; \mathbb{T}_\tau \qquad\qquad \forall\, \alpha \in L\,.$$

(c) Since the real unit interval $[0,1]$ is a spatial locale, we conclude from *Raney's Theorem* (cf. p. 204/205 in [22]) that every **completely distributive lattice** is spatial. Hence, if we restrict exclusively our interest to purely lattice–theoretic operations (i.e. $\otimes = * = \wedge$), then the previous part (b) shows that the theory of L–topological spaces can be embedded into the theory of ordinary topological spaces. We can draw two conclusions from this situation:

- If we insist on $\otimes = \wedge$, then the non trivial part of the theory of L–topological spaces is based on **non–spatial locales** L. In order to obtain a tentative conception of such a theory the reader is referred to Example 7.1.2, Remark 7.1.3 and Example 7.1.4.

- If we actually prefer completely distributive lattices (or more general spatial locales), then a non trivial theory of L–topological spaces requires binary operations $\otimes \neq \wedge$. The direction of such a theory is illustrated by the role of the **monoidal mean operator** and its application to the theory of rigid L–topological spaces (cf. Subsection 7.3).

■

[14] In the case of $L = [0,1]$ the functor $\mathfrak{T}$ coincides with the *hypergraph functor* (cf. [88]).

7.2 Case of complete MV–algebras

The underlying quadruple $(L, \leq, \otimes, *)$ satisfies the axioms (I) – (VII). Further we require that $(L, \leq, *)$ is a complete MV–algebra.

A stratified L–topology τ on X is called a ***probabilistic L–topology*** iff τ is provided with the additional property (cf. [33])

$$\text{(LP)} \quad g \in \tau \implies \alpha \to g \in \tau \quad \forall \alpha \in L \ .$$

If τ is a probabilistic L–topology on X, then the pair (X, τ) is said to be a ***probabilistic L–topological space.*** A simple characterization of probabilistic L–topologies is given in:

Lemma 7.2.1 *Let τ be a stratified L–topology on X and $(\mu_p)_{p\in X}$ be the corresponding L–neighborhood system. Then the following assertions are equivalent:*

(i) τ is a probabilistic L–topology .

(ii) $\alpha \to \mu_p(f) \ \leq \ \mu_p(\alpha \to f) \quad \forall p \in X\,,\ \forall \alpha \in L\,,\ \forall f \in L^X$.

■

Combining Proposition 6.2.4(a) with Lemma 7.2.1 we obtain that every probabilistic L–topology can be characterized by systems of $\top$–filters (cf. Remark 6.2.3). The precise situation is as follows:
Let X be a non empty set, and for every $p \in X$ let $\mathbf{U}_p$ be a non empty subset of L^X. $(\mathbf{U}_p)_{p\in X}$ is called a *system of crisp sets of L–valued neighborhoods* iff $(\mathbf{U}_p)_{p\in X}$ satisfies the following conditions (cf. [33])

$$(\mathbb{U}1/2) \quad \mathbf{U}_p \ \text{ is a } \top\text{–filter for all } p \in X \ .$$

$$(\mathbb{U}3) \quad d \in \mathbf{U}_p \implies d(p) = \top \ .$$

$$(\mathbb{U}4) \quad \forall\, d_p \in \mathbf{U}_p \ \exists\, d^* \in \mathbf{U}_p \ \forall\, q \in X \ \exists\, d_q \in \mathbf{U}_q \ : \quad d_q(x) * d^*(q) \ \leq \ d_p(x) \quad \forall\, x \in X \ .$$

Let $(\mathbf{U}_p)_{p\in X}$ be a system of crisp sets of L–valued neighborhoods. Then $(\mathbf{U}_p)_{p\in X}$ induces an L–neighborhood system $(\mu_p)_{p\in X}$ of a probabilistic L–topology by

$$\mu_p(f) \ = \ \bigvee_{d_p\in \mathbf{U}_p} \Big(\bigwedge_{x\in X} d_p(x) \to f(x) \Big) \quad .$$

Obviously, $(\mu_p)_{p\in X}$ satisfies the axioms (U1) – (U3). In order to verify (U4) we first deduce the following relation from Axiom $(\mathbb{U}4)$

$$\bigwedge_{x\in X} \Big(d_p(x) \to f(x) \Big) \ \leq \ d^*(q) \to \Big(\bigvee_{d_q\in \mathbf{U}_q} \Big(d_q(x) \to f(x) \Big) \Big) \quad ;$$

hence the relation $\mu_p(f) \leq \mu_p(\mu\_(f))$ follows – i.e. (U4) holds.
On the other hand, every L–neighborhood system $(\mu_p)_{p\in X}$ of a probabilistic L–topology determines a system of crisp sets of L–valued neighborhoods $(\mathbf{U}_p)_{p\in X}$

by

$$\mathbf{U}_p = \{ d \in L^X \mid \mu_p(d) = \top \} \quad .$$

In fact, the axioms ($\mathbb{U}1$) – ($\mathbb{U}3$) are evident. In order to prove ($\mathbb{U}4$) we choose an element $d_p \in \mathbf{U}_p$ and define a map $d^* : X \longmapsto L$ by $d^*(q) = \mu_q(d_p)$. Because of (U4) the map d^* is again an element of $\mathbf{U}_p$. Now we fix $q \in X$ and conclude from Lemma 7.2.1 that the map $d_q : X \longmapsto L$ defined by

$$d_q(x) = d^*(q) \to d_p(x) \qquad \forall x \in X$$

is an element of $\mathbf{U}_q$. Hence the axiom ($\mathbb{U}4$) follows from the subsequent relation

$$d^*(q) * d_q(x) \leq d_p(x) \qquad \forall x \in X \quad .$$

Suming up the previous considerations we state the important fact that probabilistic L–topologies and crisp systems of L–valued neighborhoods are *equivalent* concepts. In particular, a map $g : X \longmapsto L$ is τ–open (i.e. $g \in \tau$) iff for every $p \in X$ there exists an L–valued neighborhood d_p of p such that

$$g(p) * d_p(x) \leq g(x) \qquad \forall x \in X \quad .$$

Example 7.2.2 (Random Variables) Let $(\Omega, \mathfrak{M}, \Pi)$ be a probability space and $L^0(\Omega, \mathfrak{M}, \Pi)$ be the space of all almost Π–everywhere defined, real valued random variables. Further we view the real unit interval as a MV–algebra – i.e. we consider the complete MV–algebra $([0,1], \leq, T_m)$ (cf. Example 1.2.3(b)) and put $\otimes = \min$. Then we can introduce on $X = L^0(\Omega, \mathfrak{M}, \Pi)$ a system $(\mathbf{U}_p)_{p \in X}$ of crisp sets of $[0,1]$–valued neighborhoods by

$$\mathbf{U}_p = \Big\{ d \in [0,1]^X \mid \forall n \in \mathbb{N}\, \exists m_n \in \mathbb{N} \text{ s.t. } : \\ d(x) \geq \Pi(\{\omega \in \Omega | \, |p(\omega) - x(\omega)| \leq \frac{1}{m_n}\}) - \frac{1}{n} \ \forall x \in X \Big\}.$$

The [0,1]–probabilistic topology τ_0 associated with $(\mathbf{U}_p)_{p \in X}$ is called the *canonical probabilistic topology* on $L^0(\Omega, \mathfrak{M}, \Pi)$.

7.3 Case of complete MV–algebras with square roots and rigid L–topologies

Let $(L, \leq, *)$ be a complete MV–algebra with square roots (cf. Subsection 1.1, Example 1.2.5(c), Subsection 1.3), and $\circledast$ be the monoidal mean operator – i.e.

$$\alpha \circledast \beta = \alpha^{1/2} * \beta^{1/2}$$

Because of Example 1.2.5(c) and Remark 1.2.6 the quadruple $(L, \leq, \circledast, *)$ satisfies the axioms (I) – (VII). Moreover the axioms (VIII)–(XI) (cf. Proposition 1.2.7 and Proposition 1.2.16) are also valid.

Lemma 7.3.1 *Let $\mathcal{B}$ be an L–*topology base *on X i.e.*

(i) $1_X, 1_\varnothing \in \mathcal{B}$

(ii) $f_1, f_2 \in \mathcal{B} \Longrightarrow f_1 \circledast f_2 \in \mathcal{B}$

Then $\tau := \{\bigvee_{i\in I}(\alpha_i * g_i) \mid \alpha_i \in L, g_i \in \mathcal{B}, i \in I\}$ *is a stratified L–topology on X.*

Proof. Obviously τ fulfills $(o1), (o3)$ and $(\Sigma 1)$. Since square roots preserve arbitrary joins and the relation $(\alpha * \beta)^{1/2} = (\alpha^{1/2} * \beta^{1/2}) \vee \bot^{1/2}$ holds, we obtain

$$\begin{aligned}
&(\textstyle\bigvee_{i\in I}(\alpha_i * g_i)) \circledast (\bigvee_{j\in J}(\beta_j * f_j)) &=\\
&\textstyle\bigvee_{(i,j)\in I\times J}(({\alpha_i}^{1/2} * {g_i}^{1/2}) \vee \bot^{1/2}) * (({\beta_j}^{1/2} * {f_j}^{1/2}) \vee \bot^{1/2}) &=\\
&(\textstyle\bigvee_{(i,j)\in I\times J}({\alpha_i}^{1/2} * {\beta_j}^{1/2} * {g_i}^{1/2} * {f_j}^{1/2})) \bigvee &\\
&\textstyle\bigvee (\bigvee_{i\in I}({\alpha_i}^{1/2} * {g_i}^{1/2} * \bot^{1/2})) \bigvee (\bigvee_{j\in J}({\beta_j}^{1/2} * {f_j}^{1/2} * \bot^{1/2})) &=\\
&(\textstyle\bigvee_{(i,j)\in I\times J}({\alpha_i}^{1/2} * {\beta_j}^{1/2}) * (g_i \circledast f_j)) \bigvee &\\
&\textstyle\bigvee (\bigvee_{i\in I}({\alpha_i}^{1/2} * (g_i \circledast 1_\varnothing)) \vee (\bigvee_{j\in J}({\beta_j}^{1/2} * (f_j \circledast 1_\varnothing))) &,
\end{aligned}$$

i.e. τ fulfills also $(o2)$.
■

Example 7.3.2 Let us consider the case $(L, \leq, *) = ([0,1], \leq, \mathrm{T}_m)$ (cf. Example 1.2.3(b)); then the monoidal mean operator $\circledast$ is given by $\alpha \circledast \beta = \frac{\alpha+\beta}{2}$ (cf. Remark 1.2.6). Further we denote by $\mathbb{P}(X)$ the set of all *finitely additive probability measures* μ on X (in particular μ is defined on the ordinary power set of X). Then every map $f \in [0,1]^X$ induces a map $\bar{f} : \mathbb{P}(X) \to [0,1]$ by

$$\bar{f}(\mu) := \int_X f\, d\mu, \qquad \mu \in \mathbb{P}(X)$$

Obviously $\mathcal{B}_\mathbb{P} := \{\bar{f} \mid f \in [0,1]^X\}$ is a $[0,1]$–topology base on $\mathbb{P}(X)$ and determines according to 7.3.1 a stratified $[0,1]$–topology $\tau_\mathbb{P}$ on $\mathbb{P}(X)$. We remark that $(\mathbb{P}(X), \tau_\mathbb{P})$ is a stratified, *Hausdorff separated, S–compact,* $[0,1]$–topological space. In fact, if ν is a stratified $[0,1]$–ultrafilter on $\mathbb{P}(X)$, then ν determines a finitely additive, probability measure λ_0 on X by (cf. Example 6.2.15(b)):

$$\nu(\bar{f}) = \int_X f\, d\lambda_0 \qquad f \in [0,1]^X \quad .$$

Further, let $(\mu_p)_{p\in\mathbb{P}(X)}$ be the stratified [0,1]–neighborhood system corresponding to $\tau_\mathbb{P}$. Then we obtain for any $p \in \mathbb{P}(X)$ (cf. 6.1) :

$$\begin{aligned}
\bar{f}(p) &= \mu_p(\bar{f}) && \forall\, f \in [0,1]^X\\
\mu_p(F) &= \textstyle\bigvee\{\alpha * \bar{f}(p) \mid \alpha * \bar{f} \leq F\} && \forall\, F \in [0,1]^{\mathbb{P}(X)}
\end{aligned}$$

We show that $\lambda_0 = p_0$ is the unique limit point of ν. By virtue of the definition of p_0 we immediately deduce from the previous relations

$$\begin{aligned}
\nu(\bar{f}) &= \mu_{p_0}(\bar{f}) && \forall f \in [0,1]^X\\
\mu_{p_0}(F) &\leq \nu(F) && \forall F \in [0,1]^{\mathbb{P}(X)} \quad .
\end{aligned}$$

In particular p_0 is a limit point of ν. In order to verify the uniqueness of p_0 we consider a further limit point q_0 of ν – i.e.

$$\mu_{q_0}(\bar{f}) \quad \leq \quad \nu(\bar{f}) \quad = \quad \mu_{p_0}(\bar{f}) \qquad\qquad \forall f \in [0,1]^X \quad .$$

Because of $\overline{1-f} \;=\; 1 - \bar{f}$ we conclude

$$\bar{f}(q_0) \quad = \quad \bar{f}(p_0) \qquad\qquad \forall f \in [0,1]^X \qquad ;$$

hence $q_0 \;=\; p_0$ follows.
■

In the following considerations we show to which extent the previous example can be viewed as the **Čech–Stone compactification** of the discrete [0,1]–toplogical space $(X,[0,1]^X)$. For this purpose we present a link between the Hausdorff separation axiom (cf. Subsection 6.3), the S–compactness axiom (cf. Subsection 6.4) and the *weak star–regularity* axiom which will be defined in Definition 7.3.8. First we introduce a subclass of stratified, L–topological spaces: A stratified L–topology is called *rigid* iff τ satisfies the additional axiom

$$(\Sigma 3) \qquad g \in \tau \quad \text{with} \quad g \;\leq\; \bot^{1/2} \cdot 1_X \qquad \Longrightarrow \qquad [(\bot^{1/2} \cdot 1_X) \rightarrow g]^2 \;\in\; \tau\,.$$

A stratified, L–topological spaces (X,τ) is said to be *rigid* iff τ is rigid. Since the axiom $(\Sigma 3)$ is preserved under arbitrary intersections, the class of rigid, L–topological spaces forms a coreflexive subcategory of the category **SL–TOP** of stratified L–topological spaces.
Referring to Subsection 1.3 we know that every complete MV–algebra $(L,\leq,*)$ contains a prominent, **idempotent** element ι_0 where ι_0 is given as follows:

$$\iota_0 \quad = \quad (\bot^{1/2} \rightarrow \bot)^2 \quad .$$

Further there exists a *Galois connection* on $(L,\leq)$ determined by

$$\frac{1}{2} \cdot \alpha \quad := \quad \bot \circledast \alpha \qquad , \qquad 2 \cdot \alpha \quad := \quad (\bot^{1/2} \rightarrow \alpha)^2 \qquad \forall \alpha \in L\,.$$

Because of $2 \cdot g \;=\; \iota_0 \cdot 1_X \,\vee\, \Big((\iota_0 \rightarrow \bot) \cdot 1_X \;*\; 2 \cdot g\Big)$ we can simplify the axiom $(\Sigma 3)$ as follows:

$$(\Sigma 3') \qquad g \in \tau \quad \text{with} \quad g \;\leq\; \tfrac{1}{2} \cdot (1_X) \quad \Longrightarrow \quad ((\iota_0 \rightarrow \bot) \cdot 1_X) \;*\; (2 \cdot g) \;\in\; \tau\,.$$

In the Boolean case (i.e. $\bot^{1/2} \;=\; \bot$) the axiom $(\Sigma 3)$ is redundant. On the other hand, if the underlying MV–algebra is strict (i.e. $\bot^{1/2} \;=\; \bot^{1/2} \rightarrow \bot$), then there exist non trivial rigid L–topologies which form an important subclass of stratified L–topologies.
A characterization of rigid, stratified L–topologies is given in the following

Lemma 7.3.3 *Let τ be a stratified L–topology on X and $(\mu_p)_{p \in X}$ be the corresponding L–neighborhood system. Then the following assertions are equivalent*

(i) τ is rigid.

(ii) For all $p \in X$ the L–neighborhood filter μ_p satisfies (F7) .

(iii) Every L–neighborhood filter μ_p satisfies the condition

$$(\mathrm{F7}') \qquad (\iota_0 \to \bot) * \mu_p(1_\varnothing \circledast f) \quad \le \quad \bot \circledast \mu_p(f) \qquad \forall f \in L^X .$$

Proof. (a) In oder to show (i) $\Longrightarrow$ (ii) we fix an element $f \in L^X$ and define a map $g : X \longmapsto L$ by $g(p) = \mu_p(1_\varnothing \circledast f)$. Then we conclude from (U3) and (U4) that g is an element of τ satisfying the condition $g \le \bot^{1/2} \cdot 1_X$. Referring again to (U3) we infer from Proposition 2.15 in [39] (cf. Proposition 1.2.14):

$$\bot^{1/2} \to g \quad \le \quad f^{1/2} \vee \Big((\bot^{1/2} \to \bot) \cdot 1_X \Big) \quad .$$

Hence $\Big((\iota_0 \to \bot) \cdot 1_X \Big) * 2 \cdot g \quad \le \quad f$ follows from Proposition 1.2.2 and the definition of ι_0 . Because of (Σ1), (Σ3) the relation $(\iota_0 \to \bot) * (2 \cdot g(p)) \le \mu_p(f)$ holds. Further we observe

$$\tfrac{1}{2} \cdot \top \quad = \quad \bot^{1/2} \qquad , \qquad \tfrac{1}{2} \cdot (2 \cdot \alpha) \quad = \quad \alpha \wedge \tfrac{1}{2} \cdot \top \quad . \qquad (\Diamond)$$

Hence Proposition 1.2.9(ii) implies: $(\iota_0 \to \bot) * g \quad \le \quad \bot^{1/2} * (\mu_p(f))^{1/2}$. On the other hand we infer from Lemma 6.2.1(b):

$$\iota_0 * \mu_p(1_\varnothing \circledast f) \quad \le \quad \iota_0 * \mu(\bot^{1/2} \cdot 1_X) \quad \le \quad \iota_0 * \bot^{1/2} \quad \le \quad \bot \quad ;$$

i.e.

$$\mu_p(1_\varnothing \circledast f) \quad \le \quad (\iota_0 \to \bot) * \mu_p(1_\varnothing \circledast f) \quad \le \quad \bot \circledast \mu_p(f) \quad .$$

Because of (F2) and (F3) the converse inequality is always valid; hence the assertion (ii) is verified.
(b) The implication (ii) $\Longrightarrow$ (iii) is trivial. In order to verify (iii) $\Longrightarrow$ (i) we proceed as follows: We choose an element $g \in \tau$ with $g \le \frac{1}{2} \cdot (1_X)$ and conclude from ($\Diamond$) and Assertion (iii) that the relation

$$(\iota_0 \to \bot) * g(p) \;=\; (\iota_0 \to \bot) * \mu_p(g) \quad \le \quad \mu_p(2 \cdot g) \circledast \bot$$

holds for all $p \in X$. Now we apply the idempotency of $\iota_0 \to \bot$ and obtain:

$$(\iota_0 \to \bot) * 2 \cdot g(p) \quad \le \quad (\iota_0 \to \bot) * [\bot^{1/2} \to ((\iota_0 \to \bot) * g(p))]^2 \le \quad \mu_p(2 \cdot g) \; .$$

Because of $\iota_0 \cdot 1_X \le 2 \cdot g$ the stratification of μ_p implies (cf. Lemma 6.2.1(a)): $\iota_0 \le \mu_p(2 \cdot g)$; hence the relation

$$2 \cdot g(p) \quad \le \quad \iota_0 \vee (\iota_0 \to \bot) * (2 \cdot g(p)) \quad \le \quad \mu_p(2 \cdot g)$$

follows for all $p \in X$; i.e. the assertion (i) is verified.
∎

Corollary 7.3.4 *Every rigid and stratified L–topology τ fulfills the following property*

$$(\Sigma 4) \qquad g \in \tau \quad \Longrightarrow \quad g * g \in \tau \quad .$$

Proof. Let $(\mu_p)_{p \in X}$ be the L–neighborhood system corresponding to τ, and let g be an element of τ. Since τ is stratified and rigid, we conclude from Lemma 7.3.3

$$\begin{aligned} g(p) &= \mu_p(g) \leq \mu_p((g^2)^{1/2}) \leq \bot^{1/2} \to (\bot^{1/2} * \mu_p((g^2)^{1/2})) \\ &\leq \bot^{1/2} \to \mu_p(\bot^{1/2} * (g^2)^{1/2}) = \bot^{1/2} \to (\bot \circledast \mu_p(g^2)) \\ &= (\bot^{1/2} \to \bot) \vee (\mu_p(g^2))^{1/2} \quad ; \end{aligned}$$

hence the inequality $(\iota_0 \to \bot) * (g(p))^2 \leq \mu_p(g^2)$ holds for all $p \in X$. Further the relation $\iota_0 * \alpha = \iota_0 * \alpha^2$ follows from

$$\iota_0 * \alpha * (\alpha^2 \to \bot) \quad \leq \quad \iota_0 * \alpha * (\alpha \to \bot^{1/2}) \quad = \quad \bot \quad .$$

Therefore we obtain for all $p \in X$:

$$\begin{aligned} (g(p))^2 &= (\iota_0 * g(p)) \vee ((\iota_0 \to \bot) * \mu_p(g^2)) \\ &\leq (\iota_0 * \mu_p((\iota_0 \cdot 1_X) * g^2)) \vee ((\iota_0 \to \bot) * \mu_p(g^2)) \leq \mu_p(g^2) \; ; \end{aligned}$$

hence $g^2 \in \tau$.
■

Since $\otimes$ coincides with the monoidal mean operator $\circledast$, we infer from (o2) and (Σ4) that every rigid, stratified L–topology τ satisfies the following axiom

$$(\Sigma 4') \qquad g_1, g_2 \in \tau \quad \Longrightarrow \quad g_1 * g_2 \in \tau \quad .$$

Further a probabilistic L–topology (cf. Subsection 7.2) τ is rigid iff τ fulfills (Σ4).
The coreflection $(X, \tilde{\tau})$ of a stratified, L–topological space (X, τ) in the category of all rigid, stratified, L–topological spaces is called the *rigid hull* of (X, τ). If $\mathfrak{RS}(X)$ denotes the set of all rigid, stratified L–topologies on X, then $\tilde{\tau}$ is given by

$$\tilde{\tau} \quad = \quad \bigcap \{ \tau' \mid \tau \subseteq \tau' \, , \, \tau' \in \mathfrak{RS}(X) \} \quad .$$

The goal of the following considerations is to give an explicit description of the L–neighborhood system corresponding to $\tilde{\tau}$. As a preparation we verify the following Lemma 7.3.5 : First we define recursively the following operations:

$$\begin{aligned} 2^{-(n+1)} \cdot \alpha &= \tfrac{1}{2} \cdot (2^{-n} \cdot \alpha) \qquad \text{where} \qquad 2^{-1} \cdot \alpha = \tfrac{1}{2} \cdot \alpha \\ 2^{n+1} \cdot \alpha &= 2 \cdot (2^n \cdot \alpha) \qquad \text{where} \qquad 2^1 \cdot \alpha = 2 \cdot \alpha \end{aligned}$$

Lemma 7.3.5 (cf. Lemma 6.2 in [41])
Let ν be a stratified L–filter on X. Then the map $\hat{\nu} : L^X \longmapsto L$ defined by

$$\hat{\nu}(f) \quad = \quad \nu(f) \vee \Big((\iota_0 \to \bot) * \Big(\bigvee_{n \in \mathbb{N}} 2^n \cdot (\nu(2^{-n} \cdot f)) \Big) \Big) \qquad f \in L^X$$

is a stratified L–filter satisfying the axiom (F7). In particular, $\hat{\nu}$ is the smallest L–filter which contains ν (i.e. $\nu \preccurlyeq \hat{\nu}$) and fulfills (F7).

Proof. (a) First we quote a list of formulas which are valid for the Galois connection $(\frac{1}{2} \cdot \_\, , \; 2 \cdot \_)$:

$$\begin{aligned}(\iota_0 \to \bot) * (2 \cdot (2^{-1} \cdot \alpha)) &= (\iota_0 \to \bot) * \alpha \\ (\iota_0 \to \bot) * (2 \cdot \alpha) &= (\iota_0 \to \bot) * (2 \cdot ((\iota_0 \to \bot) * \alpha) \qquad (\diamond\diamond) \\ \alpha * (2^{-n} \cdot \beta) &= 2^{-n} \cdot (\alpha^{2^n} * \beta)\end{aligned}$$

Further we verify by induction:

$$(\iota_0 \to \bot) * \alpha * (2^n \cdot \beta) \;=\; (\iota_0 \to \bot) * (2^n \cdot (\alpha^{1/2^n} * \beta)) \text{ whenever } \beta \;\leq\; 2^{-n} \cdot \top.$$

In the case of $n = 1$ we observe:

$$\begin{aligned}(\iota \to \bot) * \alpha * (2 \cdot \beta) &= (\iota_0 \to \bot) * \alpha * (\bot^{1/2} \to \beta)^2 \\ &= (\iota_0 \to \bot) * \Big((\bot^{1/2} \to \bot) \vee (\alpha^{1/2} * (\bot^{1/2} \to \beta))\Big)^2 \\ &= (\iota_0 \to \bot) * \Big(\bot^{1/2} \to (\alpha^{1/2} * \bot^{1/2} * (\bot^{1/2} \to \beta))\Big) \\ &= (\iota_0 \to \bot) * (2 \cdot (\alpha^{1/2} * \beta)) \; .\end{aligned}$$

In order to perform the step from n to $n + 1$ we proceed as follows:

$$\begin{aligned}(\iota_0 \to \bot) * (2^{n+1} \cdot (\alpha^{1/2^{n+1}} * \beta)) &= (\iota_0 \to \bot) * \Big(2^n \cdot \Big(2 \cdot ((\alpha^{1/2^n})^{1/2} * \beta)\Big)\Big) \\ &= (\iota_0 \to \bot) * \Big(2^n \cdot \Big((\iota_0 \to \bot) * \Big(2 \cdot ((\alpha^{1/2^n})^{1/2} * \beta)\Big)\Big)\Big) \\ &= (\iota_0 \to \bot) * \Big(2^n \cdot \Big((\iota_0 \to \bot) * \alpha^{1/2^n} * (2 \cdot \beta)\Big)\Big) \\ &= (\iota_0 \to \bot) * \alpha * \Big(2^n \cdot ((\iota_0 \to \bot) * (2 \cdot \beta))\Big) \\ &= (\iota_0 \to \bot) * \alpha * (2^{n+1} \cdot \beta) \; .\end{aligned}$$

Further we obtain:

$$\begin{aligned}(2 \cdot \alpha) \circledast (2 \cdot \beta) &= (\bot^{1/2} \to \alpha) * (\bot^{1/2} \to \beta) \\ &= \Big((\bot^{1/4} \to \alpha^{1/2}) * (\bot^{1/4} \to \beta^{1/2})\Big)^2 \\ &\leq 2 \cdot (\alpha \circledast \beta) \; ;\end{aligned}$$

hence $(2^n \cdot \alpha) \circledast (2^n \cdot \beta) \;\leq\; 2^n \cdot (\alpha \circledast \beta)$ follows by induction. Finally we note:

$$(2^{-n} \cdot \alpha) \circledast (2^{-n} \cdot \beta) \;=\; 2^{-n} \cdot (\alpha \circledast \beta) \quad .$$

(b) Let ν be a stratified L–filter on X. Because of $\otimes = \circledast$ the relation $2^{-1} \cdot \nu(f) \;\leq\; \nu(2^{-1} \cdot f)$ follows immediately from the filter axiom (F2); hence we obtain (cf. Part (a)):

$$(\iota_0 \to \bot) * \nu(f) \quad \leq \quad (\iota_0 \to \bot) * (2 \cdot \nu(2^{-1} \cdot f)) \qquad \forall f \in L^X \, . \quad (\blacklozenge)$$

We show that $\hat{\nu}$ is a stratified L–filter. The axioms (F0) and (F1) are obvious. Since ν is stratified, the axioms (F3) follow immediately from Lemma 6.2.1(a) and the first formula in $(\diamond\diamond)$ (cf. Part (a)). Referring to the formulas quoted at the end of Part (a) we conclude from $(\blacklozenge)$ that $\hat{\nu}$ also fulfills the filter axiom (F2). In order to verify the stratification axiom (F4) we proceed as follows (cf. Part(a)):

$$\begin{aligned}
\alpha * \hat{\nu}(f) &= (\alpha * \nu(f)) \vee (\iota_0 \to \bot) * \Big(\bigvee_{n \in \mathbb{N}} \alpha * (2^n \cdot \nu(2^{-n} \cdot f))\Big) \\
&= (\alpha * \nu(f)) \vee (\iota_0 \to \bot) * \Big(\bigvee_{n \in \mathbb{N}} 2^n \cdot \big(\alpha^{1/2^n} * \nu(2^{-n} \cdot f)\big)\Big) \\
&\leq \nu(\alpha * f) \vee (\iota_0 \to \bot) * \Big(\bigvee_{n \in \mathbb{N}} 2^n \cdot \nu(\alpha^{1/2^n} * (2^{-n} \cdot f))\Big) \\
&= \nu(\alpha * f) \vee (\iota_0 \to \bot) * (2^n \cdot \nu(2^{-n} \cdot (\alpha * f))) \quad = \quad \hat{\nu}(\alpha * f) \, .
\end{aligned}$$

Further the construction of $\hat{\nu}$ implies: $(\iota_0 \to \bot) * (2 \cdot \hat{\nu}(2^{-1} \cdot f)) \leq \hat{\nu}(f)$; hence $\hat{\nu}$ fulfills the axiom (F7). Referring again to $(\blacklozenge)$ it is easy to verify that $\hat{\nu}$ is the smallest stratified L–filter which contains ν and satisfies (F7).
■

Proposition 7.3.6 *Let (X, τ) be a stratified, L–topological space and $(\mu_p)_{p \in X}$ be the L–neighborhood system corresponding to τ. Further let $(X, \tilde{\tau})$ be the rigid hull of (X, τ). Then the L–neighborhood system $(\tilde{\mu}_p)_{p \in X}$ is given by*

$$\tilde{\mu}_p(f) \quad = \quad \mu_p(f) \vee (\iota_0 \to \bot) * \Big(\bigvee_{n \in \mathbb{N}} 2^n \cdot \mu_p(2^{-n} \cdot f)\Big) \qquad \forall f \in L^X \, .$$

Proof. Referring to Lemma 7.3.3 and Lemma 7.3.5 it is sufficient to show that $(\tilde{\mu}_p)_{p \in X}$ satisfies the L–neighborhood axioms (U3) and (U4). The axiom (U3) is obvious. In order to verify (U4) we proceed as follows: If $h \leq 2^{-n} \cdot 1_X$, then we conclude from the stratification axiom (F4) and the formulas quoted in Part(a) of the proof of Lemma 7.3.5:

$$\begin{aligned}
(\iota_0 \to \bot) * (2^n \cdot \mu_p(h)) &= (\iota_0 \to \bot) * (2^n \cdot \mu_p\Big((\iota_0 \to \bot) * (2^{-n} \cdot (2^n \cdot h))\Big) \\
&= (\iota_0 \to \bot) * \Big(2^n \cdot \mu_p(2^{-n} \cdot (2^n \cdot h))\Big) \qquad ;
\end{aligned}$$

hence we obtain

$$\begin{aligned}
\tilde{\mu}_p(f) &= \mu_p(\mu_(f)) \vee (\iota_0 \to \bot) * \Big(\bigvee_{n \in \mathbb{N}} 2^n \cdot \mu_p(\mu_(2^{-n} \cdot f))\Big) \\
&= \mu_p(\mu_(f)) \vee (\iota_0 \to \bot) * \Big(\bigvee_{n \in \mathbb{N}} 2^n \cdot \mu_p(2^{-n} \cdot (2^n \cdot \mu_(2^{-n} \cdot f)))\Big) \\
&\leq \mu_p(\tilde{\mu}_(f)) \vee (\iota_0 \to \bot) * \Big(\bigvee_{n \in \mathbb{N}} 2^n \cdot \mu_p(2^{-n} \cdot \tilde{\mu}_(f))\Big) \\
&= \tilde{\mu}_p(\tilde{\mu}_(f)) \qquad .
\end{aligned}$$

■

Corollary 7.3.7 *Let (X,τ) be a S–compact, stratified, L–topological space. Then the rigid hull of (X,τ) is also S–compact.*

Proof. Since every stratified L–ultrafilter fulfills Axiom (F7) (cf. Lemma 6.2.13), the assertion follows from Lemma 7.3.5 and Proposition 7.3.6.
■

Before we define the axiom of *weak star–regularity* we recall and introduce some notations: If τ is an L–topology on X, then for every $f \in L^X$ we define an element $f^* \in L^X$ by (cf. Subsection 6.3):

$$f^* = \bigvee \{ g \in \tau \mid g * f = 1_\varnothing \} \quad .$$

Obviously f^* is an element of τ. In particular, we put $\overline{f} := f^* \to \perp$ and view $\overline{f}$ as the τ–closed hull of f.

Definition 7.3.8 (Weak star–regularity)
A rigid, stratified, L–topological space is called *weak star–regular* iff every element $h \in \tau$ satisfies the following relation

$$\begin{aligned} h = & \Big(\bigvee \{ \alpha * g \mid g \in \tau, \alpha \in L, \alpha * \overline{g} \le h \}\Big) \vee \\ & \vee (\iota_0 \to \perp) * \Big(\bigvee \{ 2^n \cdot (\alpha * g) \mid n \in \mathbb{N}, g \in \tau, \alpha \in L, \alpha * \overline{g} \le 2^{-n} \cdot h \}\Big). \end{aligned}$$

■

It is clear that every star–regular, rigid, stratified L–topological space is also weak star–regular. Moreover we are now in the position to prove the important third main result:

Theorem 7.3.9 (Main result III)
Every Hausdorff separated, S–compact, rigid, stratified, L–topological space (X,τ) is weak star–regular.

Proof. We fix a point $p \in X$ and consider the L–neighborhood filter μ_p at p (w.r.t. τ). Further, we define a map $\nu_p : L^X \longmapsto L$ by

$$\nu_p(h) = \bigvee \{ \alpha * \mu_p(f) \mid \alpha \in L, \alpha * \overline{f}(p) \le h(p) \} \quad h \in L^X \,.$$

(a) We show that ν_p is a stratified L–filter. The axioms (F0), F(1), (F3) and (F4) are evident. In order to verify (F2) we proceed as follows: First we show

$$\overline{f_1 \circledast f_2} \le \overline{f_1} \circledast \overline{f_2} \qquad \forall f_1, f_2 \in L^X \quad . \tag{$\diamond$}$$

If $f \in L^X$, then the relation $\iota_0 * (\iota_0 * f)^* \le f^*$ follows from the stratification axiom $(\Sigma 1)$; hence the idempotency of ι_0 implies:

$$\iota_0 * \overline{f} \le \iota_0 * \overline{\iota_0 * f} \le \iota_0 * \overline{f} \,.$$

Further we observe: $\iota_0 \wedge \alpha = \iota_0 * \alpha = \iota_0 * \alpha^2 \ \forall \alpha \in L$ (cf. Proof of Corollary 7.3.4); then we obtain:

$$\iota_0 * \overline{f_1 \circledast f_2} = \iota_0 * \overline{(\iota_0 * f_1) * (\iota_0 * f_2)} \leq \iota_0 * (\overline{f_1} \wedge \overline{f_2}) = \iota_0 * (\overline{f_1} \circledast \overline{f_2}) .$$

Therefore, in order to verify $(\diamond)$ it is sufficient to show

$$(\iota_0 \to \bot) * \overline{f_1 \circledast f_2} \leq (\iota_0 \to \bot) * (\overline{f_1} \circledast \overline{f_2}) . \qquad (\diamond')$$

First we observe:

$$\begin{aligned}
(\iota_0 \to \bot) * (\bot^{1/2} \to \bot) &= \bot^{1/2} = (\iota_0 \to \bot) * \bot^{1/2} , \\
(\iota_0 \to \bot) * (\beta \to \bot)^{1/2} &= (\iota_0 \to \bot) * ((\bot \circledast \beta) \to \bot) , \\
(\iota_0 \to \bot) * ((\alpha^{1/2} * \beta^{1/2}) \to \bot) &\leq (\iota_0 \to \bot) * (\alpha^{1/2} \to (\bot^{1/2} \to \bot)) \\
&= (\iota_0 \to \bot) * (\alpha^{1/2} \to \bot^{1/2}) ;
\end{aligned}$$

then we obtain from Proposition 2.15 in [39] (cf. Proposition 1.2.14) and the divisibility law (cf. Lemma 2.5 in [39]):

$$\begin{aligned}
&(\iota_0 \to \bot) * ((\alpha \to \bot) \circledast (\beta \to \bot)) = \\
&= (\iota_0 \to \bot) * [((\alpha^{1/2} * \beta^{1/2}) \to \bot) \vee (\alpha^{1/2} \to \bot^{1/2})] * ((\bot \circledast \beta) \to \bot) \\
&= (\iota_0 \to \bot) * ((\alpha \circledast \beta) \to (\bot \circledast \beta)) * ((\bot \circledast \beta) \to \bot) \\
&= (\iota_0 \to \bot) * ((\alpha \circledast \beta) \to \bot) .
\end{aligned}$$

Because of $f_1^* \circledast f_2^* \leq (f_1 \circledast f_2)^*$ we conclude from the preceding considerations

$$\begin{aligned}
(\iota_0 \to \bot) * (\overline{f_1 \circledast f_2}) &\leq (\iota_0 \to \bot) * ((f_1^* \circledast f_2^*) \to \bot) \\
&= (\iota_0 \to \bot) * (\overline{f_1} \circledast \overline{f_2}) ;
\end{aligned}$$

hence the relation $(\diamond')$ is verified.
Let us now consider the following situation: $\alpha_i * \overline{f_i} \leq h_i \ (i = 1, 2)$. Since square roots in MV-algebras satisfy the axiom (S3), we obtain from $(\diamond)$:

$$\begin{aligned}
(\alpha_1 \circledast \alpha_2) * \overline{f_1 \circledast f_2} &\leq (\alpha_1 \circledast \alpha_2) * (\overline{f_1} \circledast \overline{f_2}) \leq h_1 \circledast h_2 \\
(\alpha_1 \circledast \bot) * (\overline{f_2 \circledast 1_X}) &\leq (\alpha_1 \circledast \bot) * (\overline{f_2} \circledast 1_X) \leq h_1 \circledast h_2 \\
(\alpha_2 \circledast \bot) * (\overline{f_1 \circledast 1_X}) &\leq (\alpha_2 \circledast \bot) * (\overline{f_1} \circledast 1_X) \leq h_1 \circledast h_2 ;
\end{aligned}$$

hence the inequality $(\alpha_1 * \mu_p(f_1)) \circledast (\alpha_2 * \mu_p(f_2)) \leq \nu_p(h_1 \circledast h_2)$ follows – i.e. ν_p satisfies the axiom (F2).
(b) We show that ν_p has a unique adherent point $q = p$. Since μ_p satisfies the axioms (F1) and (F4), the construction of ν_p implies that p is an adherent point of ν_p (cf. Defintion 6.4.1(a)). Let q be a further adherent point of ν_p. Since the "negation" in $(L, \leq, *)$ is an involution, we obtain for every $g \in \tau$:

$\overline{g \to \perp} = g \to \perp$. We put $g^* = \bigvee\{g' \in \tau \mid g' * g = 1_\varnothing\}$ and conclude from Definition 6.4.1(a) (see also Corollary 6.2.7):

$$\begin{aligned} g^*(p) * g(q) \quad &\leq \quad \mu_p(g^*) * \mu_q(g) \quad \leq \quad \mu_p(g \to \perp) * \mu_q(g) \\ &\leq \quad \nu_p(g \to \perp) * \mu_q(g) \quad \leq \quad \perp \ . \end{aligned}$$

Since (X, τ) is Hausdorff separated, the points p and q coincide (cf. Theorem 6.3.6).
(c) Let $\hat{\nu}_p$ be the smallest stratified L–filter which contains ν_p and satisfies the axiom (F7) (cf. Lemma 7.3.5). Since τ is a rigid L–topology (cf. Lemma 7.3.3), we conclude from Part (b) that the point p is the unique adherent point of $\hat{\nu}_p$. Hence the Theorem 6.4.10 implies that p is also the limit point of $\hat{\nu}_p$.
(d) The previous argumentation holds for all $p \in X$. Since the unary operation $2 \cdot \_$ preserves arbitrary joins (cf. Subsection 1.3[15]), we conclude from the construction of $\hat{\nu}_p$ that (X, τ) is weak star–regular.
■

If we replace star–regularity by weak regularity, then we can generalize Theorem 6.3.8 as follows:

Theorem 7.3.10 (Principle of L–continuous extension)
Let (X, τ) be a rigid, stratified, L–topological space, (Y, σ) be a Hausdorff separated, weak–star–regular, L–topological space, and let A be a dense subset of X provided with the initial L–topology τ_A w.r.t. the inclusion map $i_A : A \hookrightarrow X$. Finally let $\varphi : A \longmapsto Y$ be an L–continuous map. Then the following assertions are equivalent:

(i) There exists an L–continuous map $\psi : X \longmapsto Y$ with $\psi \circ i_A = \varphi$; i.e. φ has an L–continuous extension.

(ii) For every $x \in X$ there exists $y \in Y$ satisfying the following condition

$$\forall\, h \in \sigma\ \exists\, g \in \tau : \quad (1) \quad h(y) \leq g(x) \quad , \quad (2) \quad g \circ i_A = h \circ \varphi \quad .$$

Proof. In principle we can repeat the proof of Theorem 6.3.8 with obvious modifications. In order to give an example we restrict our interest to the verification of the L–continuity of the extension ψ; i.e. we show (ii) $\Longrightarrow$ (i): Let us fix $h \in \sigma$ and choose $n \in \mathbb{N}_0$, $\alpha \in L$ and $g \in \sigma$. If $n = 0$ we use the following convention: $2^{-0} \cdot \varkappa = 2^0 \cdot \varkappa = \varkappa$ for all $\varkappa$ in L. Further we assume: $\alpha * \overline{g} \leq 2^{-n} \cdot h$ where $\overline{g} = g^* \to \perp$ is the τ–closed hull of g. Then for every $x_0 \in X$ there exists an element $k_{x_0} \in \tau$ provided with the following properties

$$g(\psi(x_0)) \leq k_{x_0}(x_0) \quad , \quad g \circ \varphi = k_{x_0} \circ i_A \quad .$$

We show:

$$k_{x_0} \quad \leq \quad \overline{g} \circ \psi \quad = \quad 1_\varnothing \quad . \tag{$\Diamond\Diamond$}$$

[15]Here we use Proposition 2.8, Lemma 2.14, Theorem 5.2(c) and Proposition 2.11 in [39].

Referring to Assertion (ii) we choose an element $\tilde{k} \in \tau$ depending on $x \in X$ such that

$$g^*(\psi(x)) \le \tilde{k}(x) \quad , \quad g^* \circ \varphi = \tilde{k} \circ i_A$$

We apply the axiom ($\Sigma 4'$) (resp. (o2)), the density of A and obtain:

$$k_{x_0}(x) * g^*(\psi(x)) \quad \le k_{x_0}(x) * \tilde{k}(x) \quad \le \quad \bigvee_{a \in A} g(\varphi(a)) * g^*(\varphi(a)) \quad = \quad \perp \quad ;$$

hence the relation ($\Diamond\Diamond$) follows. Now we take into account the choice of the triple (n, α, g) and distinguish the following cases:
In the case $n = 0$ we conclude from $\alpha * \overline{g} \le h$ and the previous relation ($\Diamond\Diamond$):

$$\alpha * g(\psi(x)) \quad \le \quad \bigvee_{x_0 \in X} \alpha * k_{x_0}(x) \quad \le \quad h(\psi(x)) \qquad \forall x \in X \quad .$$

In the case of $1 \le n$ we conclude from $\alpha * \overline{g} \le 2^{-n} \cdot h$:

$$(\iota_0 \to \perp) * \Big(2^n \cdot (\alpha * \overline{g})\Big) \quad \le \quad h \quad .$$

Further the relation ($\Diamond\Diamond$) implies: $\alpha * k_{x_0} \le 2^{-n} \cdot 1_X \quad \forall\, x_0 \in X$. Since τ is stratified and rigid, $(\iota_0 \to \perp) * (2^n \cdot (\alpha * k_{x_0}))$ is an element of τ for all $x_0 \in X$. Now we apply again ($\Diamond\Diamond$) and obtain for all $x \in X$:

$$\begin{aligned} (\iota_0 \to \perp) * \Big(2^n \cdot (\alpha * g(\psi(x)))\Big) \quad &\le \quad \bigvee_{x_0 \in X} (\iota_0 \to \perp) * \Big(2^n \cdot (\alpha * k_{x_0}(x))\Big) \\ &\le \quad h(\psi(x)) \quad . \end{aligned}$$

Finally we invoke the weak star–regularity of (Y, σ) and conclude from (o3) that $h \circ \psi$ is an element of τ ; hence ψ is L–continuous.
■

Corollary 7.3.11 *Let* $(L, \le, *) = ([0,1], \le, T_m)$ *be the canonical MV–algebra structure on* $[0,1]$ *(cf. Example 1.2.3(b)). Further let* X *be a non empty set,* $\mathbb{P}(X)$ *be the set of all finitely additive probability measures on* X, *and let* $(\mathbb{P}(X), \tau_{\mathbb{P}})$ *be the Hausdorff separated, S–compact L–topological space constructed in Example 7.3.2. If* (Y, σ) *is an arbitrary, Hausdorff separated, S–compact, rigid, stratified* $[0,1]$*–topological space, then every map* $\varphi : X \longmapsto Y$ *has a unique* $[0,1]$*–continuous extension to the rigid hull of* $(\mathbb{P}(X), \tau_{\mathbb{P}})$.

Proof. (a) We identify every element $x \in X$ with the Dirac measure whose support coincides with $\{x\}$ and show that X is dense in $\mathbb{P}(X)$ w.r.t. the $[0,1]$–topology $\tilde{\tau}_{\mathbb{P}}$ of the rigid hull $(\mathbb{P}(X), \tilde{\tau}_{\mathbb{P}})$ of $(\mathbb{P}(X), \tau_{\mathbb{P}})$. Since every integral can be estimated as follows:

$$\int_X f \, d\lambda \quad \le \quad \sup\{f(x) \mid x \in X\} \qquad \forall \lambda \in \mathbb{P}(X) \quad , \quad \forall f \in [0,1]^X \quad ,$$

we obtain from the definition of $\tau_{\mathbb{P}}$ that X is dense in $\mathbb{P}(X)$ w.r.t. $\tau_{\mathbb{P}}$. Hence Proposition 7.3.6 implies that X is also dense in $\mathbb{P}(X)$ w.r.t. $\tilde{\tau}_{\mathbb{P}}$.
(b) By virtue of Theorem 7.3.9 the $[0,1]$–topological space (Y,σ) is weak star–regular; hence it is sufficient to verify the assertion (ii) in Theorem 7.3.10. Since finitely additive probability measures on X and stratified $[0,1]$–ultrafilters on X are the same things, every finitely additive probability measure λ_0 induces a stratified ultrafilter on Y by

$$\nu(f) \quad = \quad \int_X f \circ \varphi \, d\lambda_0 \qquad \forall f \in [0,1]^Y \quad .$$

In particular ν is the image $[0,1]$–filter of λ under φ (cf. Remark 6.2.20(a)). Since (Y,σ) is S–compact, there exists a point $y_0 \in Y$ with $\mu_{y_0}(f) \leq \nu(f)$ where μ_{y_0} denotes the $[0,1]$–neighborhood filter at y_0. For every $g \in \sigma$ we define a map $k : \mathbb{P}(X) \longmapsto [0,1]$ by

$$k(\lambda) \quad = \quad \int_X g \circ \varphi \, d\lambda \qquad \forall \lambda \in \mathbb{P}(X) \quad .$$

Then $k \in \tau_{\mathbb{P}} \subseteq \tilde{\tau}_{\mathbb{P}}$ and the following relations are valid:

$$g(y_0) \quad \leq \quad k(\lambda_0) \qquad , \qquad k(x) \quad = \quad g \circ \varphi(x) \qquad \forall x \in X \quad .$$

Hence the assertion (ii) of Theorem 7.3.10 is verified.
■

The previous Corollary 7.3.11 shows that the set $(\mathbb{P}(X), \tau_{\mathbb{P}})$ of all finitely additive probability measures on X is the **Čech–Stone compactification** of the discrete space X in the category of Hausdorff separated, S–compact, rigid, stratified, $[0,1]$–topological spaces. In this sense the real unit interval $[0,1]$ is the Čech–Stone compactification of the set $\{0,1\}$ (consisting of two elements).

We finish this subsection with a study of indiscrete, strongly stratified L–topological spaces. We start from a lemma which is also valid, when the underlying algebraic framework only satisfies (I)–(III) and (V)–(VI).

Lemma 7.3.12 *Let X be a non-empty set, $x \in X, \kappa \in L$ and let τ be a subset of L^X provided with (o1) and (Σ2). Then the map $g_n : X \to L$ defined by*

$$g_n(y) \quad = \quad \left\{ \begin{array}{ccc} \kappa & : & x = y \\ \kappa^{2n} & : & x \neq y \end{array} \right\} \quad \text{where } \kappa^{2n} \text{ is the } 2n\text{–th power w.r.t. } * ,$$

is an element of τ for all $n \in \mathbb{N}$

Proof. We verify the assertion by induction over n. First we consider a map $k_1 : X \to L$ determined by

$$k_1(y) \quad = \quad \left\{ \begin{array}{ccc} \top & : & y = x \\ \kappa & : & y \neq x \end{array} \right\} \quad , \qquad y \in X \quad ;$$

then we infer from $(o1)$ and $(\Sigma 2)$ that

$$g_1 = k_1 * [\![k_1, 1_X]\!]$$

is an element of τ. Further we assume that g_n is an element of τ. Now we consider a map $k_{n+1} : X \to L$:

$$k_{n+1}(y) \quad = \quad \left\{ \begin{array}{lcc} \top & : & y = x \\ \kappa^{2n+1} & : & y \neq x \end{array} \right\} \quad , \qquad y \in X$$

and obtain: $[\![g_n, k_{n+1}]\!] = \kappa$. By virtue of $(\Sigma 2)$ $\quad g_{n+1} = [\![g_n, k_{n+1}]\!] * k_{n+1}$ is an element of τ.

■

Proposition 7.3.13 *Let $M = (L, \leq, *)$ be a complete MV-algebra with square roots and $\circledast$ be the monoidal mean operator. Then the following assertions are equivalent*

(i) M is a strict MV-algebra – i.e. $\perp^{1/2} \to \perp \; = \; \perp^{1/2}$ (cf. 1.3).

(ii) The indiscrete and discrete strongly stratified L–topology coincide.

Proof (a) ((i)$\Longrightarrow$(ii)) If M is strict, then we conclude from Corollary 6.10(ii) in [39] that for every element $\kappa \in L$ the following relation holds:

$$\kappa \quad = \quad \bigvee_{n \in \mathbb{N}} \kappa \wedge \perp^{1/2^n} \qquad .$$

Now we invoke Lemma 7.3.12 and obtain from (o1), (o3) and $(\Sigma 2)$ that for any strongly stratified L-topology τ and for any $\kappa \in L$ the map $g : X \longmapsto L$ defined by

$$g^{(x_0)}(y) \quad = \quad \left\{ \begin{array}{lcc} \kappa & : & y = x_0 \\ \perp & : & y \neq x_0 \end{array} \right\} \qquad y \in X$$

is an element of τ. By virtue of (o3) τ is necessarily the discrete L–topology on X.

(b) ((ii) $\Longrightarrow$(i)) If M is *not* a strict MV–algebra (cf. Subsection 1.3), then

$$\iota_0 \quad = \quad (\perp^{1/2} \to \perp) * (\perp^{1/2} \to \perp)$$

is *idempotent* w.r.t. $*$ and different from $\perp$. It is not difficult to show that

$$\tau^{ss}_{ind} = \{\alpha \vee f \mid \alpha \in L \quad \text{with} \quad \alpha \leq \iota_0, \quad f(x) \leq \iota_0 \to \perp \quad \forall x \in X\}$$

is the indiscrete, strongly stratified L–topology on X which is not *discrete.*

■.

The unique **Boolean part** of a complete MV–algebra $M = (L, \leq, *)$ with square roots is given (up to an isomorphism) by (cf. Section 1.3, Theorem 2.21 in [39])

$$L_b \quad = \quad \{\lambda \in L \mid \ \lambda \ \leq \ (\bot^{1/2} \to \bot) * (\bot^{1/2} \to \bot) \ \} \qquad .$$

As an immediate consequence from Propostion 7.3.13 we quote:

Corollary 7.3.14 *Let $M = (L, \leq, *)$ be a complete MV–algebra with square roots. Then the categories* **S**SL**–TOP** *and* **S**L_b**–TOP** *are isomorphic.*

■

7.4 Lower semicontinuous, lattice–valued maps

In this subsection we fix quadruples $(L, \leq, \otimes_1, *_1)$ and $(M, \leq, \otimes_2, *_2)$ satisfying (I)–(VII). Moreover, we assume that $(L, \leq, *_1)$ and $(M, \leq, *_2)$ are strictly two–sided, commutative quantales. Now we fix a map $\gamma : L \longmapsto M$ provided with the subsequent properties:

(E1) $\gamma : (L, \leq, \otimes_1) \longmapsto (M, \leq, \otimes_2)$ is a **CQML**–morphism.

(E2) $\gamma : (L, \leq, *_1) \longmapsto (M, \leq, *_2)$ is a quantale–morphism.

(E3) The range $\gamma(L)$ of γ is a pseudo–bisymmetric subset of M in $(M, \leq, \otimes_2, *_2)$ (cf. Definition 1.2.15).

Then we conclude from Theorem 3.6 and Theorem 5.1.1 that the category L–**TOP** of L–topological spaces and the category **S**M–**TOP** of stratified M–topological spaces are adjoint to each other. The aim of the following considerations is to give an explicit presentation of this adjoint situation. As the reader will see below, *lower semicontinuous maps* play a dominant role in this context.

Definition 7.4.1 Let (X, τ) be an L–topological space, and $(\mu_p)_{p \in X}$ be the L–neighborhood system corresponding to (X, τ). A map $g : X \to M$ is called *lower (τ, γ)–semicontinuous* iff g satisfies the following condition

$$g(p) \quad \leq \quad \bigvee_{f \in L^X} \gamma(\mu_p(f)) *_2 \big(\bigwedge_{x \in X} (\gamma(f(x)) \to g(x))\big)$$

where $\to$ denotes the "implication" w.r.t. $*_2$.

■

Lemma 7.4.2 (Lower semicontinuous regularization)
Let (X, τ) be an L–topological space and $h : X \to M$ be an arbitrary map. The lower (τ, γ)–semicontinuous regularization $\check{h}$ *of h (i.e. the largest lower (τ, γ)–semicontinuous map g with $g \leq h$) is given by*

$$\check{h}(p) = \bigvee_{f \in L^X} \gamma(\mu_p(f)) *_2 \big(\bigwedge_{x \in X} (\gamma(f(x)) \to h(x))\big) \qquad \forall p \in X \quad .$$

Proof. Obviously $\check{h} \leq h$, and every lower semicontinuous map g with $g \leq h$ is smaller than or equal to $\check{h}$. Therefore it is sufficient to show that $\check{h}$ is lower (τ, γ)–semicontinuous. First we observe:

$$\begin{aligned} &\bigwedge_{x\in X} (\gamma(f(x)) \to h(x)) \quad \leq \\ &\leq \quad \bigwedge_{z\in X} \Big(\gamma(\mu_z(f)) \to \Big[\gamma(\mu_z(f)) *_2 \big(\bigwedge_{x\in X} (\gamma(f(x)) \to h(x))\big)\Big]\Big) \quad \leq \\ &\leq \bigwedge_{z\in X} \gamma(\mu_z(f)) \to \check{h}(z) \quad . \end{aligned}$$

Now we invoke the L–neighborhood axiom (U4) (cf. Subsection 6.1) and obtain:

$$\begin{aligned} \check{h}(p) \quad &= \quad \bigvee_{f\in L^X} \gamma(\mu_p(f)) *_2 \Big(\bigwedge_{x\in X} (\gamma(f(x)) \to h(x))\Big) \\ &\leq \quad \bigvee_{f\in L^X} \gamma(\mu_p(\mu_-(f))) *_2 \bigwedge_{z\in X} \big(\gamma(\mu_z(f)) \to \check{h}(z)\big) \\ &\leq \quad \bigvee_{k\in L^X} \gamma(\mu_p(k)) *_2 \Big(\bigwedge_{z\in X} (\gamma(k(z)) \to \check{h}(z))\Big) \quad ; \end{aligned}$$

hence the lower (τ, γ)–semicontinuity of $\check{h}$ is verified.

■

Proposition 7.4.3 *Let (X, τ) be an $L-$topological space, $\ell(\tau, \gamma)$ be the set of all lower (τ, γ)–semicontinuous maps $g : X \to M$, and $\mathcal{U} = (\mu_p)_{p\in X}$ be the $L-$neighborhood system corresponding to (X, τ). Then $\ell(\tau, \gamma)$ is a stratified M–topology and the corresponding M–neighborhood system $(\nu_p)_{p\in X}$ is given by formulae*

$$\nu_p(h) \quad = \quad \check{h}(p) \quad = \quad \bigvee_{f\in L^X} \Big(\gamma(\mu_p(f)) *_2 \big(\bigwedge_{x\in X} \gamma(f(x)) \to h(x)\big)\Big) .$$

Proof. The axioms (Σ1),(o1) and (o3) are obvious. Further, let us consider two lower (τ, γ)–semicontinuous maps g_1, g_2. Since the range of γ is pseudo–bisymmetric in $(M, \leq, \otimes_2, *_2)$, we can derive the following relation from (IV), (VII) and the L–neighborhood axioms (U2), (U3) (cf. Subsection 6.1):

$$\begin{aligned} &g_1(p) \otimes_2 g_2(p) \quad = \quad \bigvee_{f_1, f_2\in L^X} \Big(\\ &\bigvee \Big[\big(\gamma(\mu_p(f_1)) \otimes_2 \gamma(\mu_p(f_2))\big) *_2 \\ &\quad *_2 \big(\big(\bigwedge_{x\in X} \gamma(f_1(x)) \to g_1(x)\big) \otimes_2 \big(\bigwedge_{x\in X} \gamma(f_2(x)) \to g_2(x)\big)\big)\Big] \\ &\bigvee \Big[\big(\gamma(\mu_p(f_1)) \otimes_2 \gamma(\mu_p(1_\varnothing))\big) *_2 \\ &\quad *_2 \big(\big(\bigwedge_{x\in X} \gamma(f_1(x)) \to g_1(x)\big) \otimes_2 \big(\bigwedge_{x\in X} \gamma(1_\varnothing(x)) \to g_2(x)\big)\big)\Big] \\ &\bigvee \Big[\big(\gamma(\mu_p(1_\varnothing)) \otimes_2 \gamma(\mu_p(f_2))\big) *_2 \\ &\quad *_2 \big(\big(\bigwedge_{x\in X} \gamma(1_\varnothing(x)) \to g_1(x)\big) \otimes_2 \big(\bigwedge_{x\in X} \gamma(f_2(x)) \to g_2(x)\big)\big)\Big]\Big) \\ &\leq \quad \bigvee_{f\in L^X} \Big(\gamma(\mu_p(f)) *_2 \big(\bigwedge_{x\in X} \gamma(f(x)) \to (g_1(x) \otimes_2 g_2(x))\big)\Big) \quad ; \end{aligned}$$

hence $g_1 \otimes_2 g_2$ is also lower (τ, γ)–semicontinuous – i.e. (o2) is verified. Finally Section 6.1 and Lemma 7.4.2 show that the M–neighborhood filter ν_p at the point $p \in X$ is determined by

$$\nu_p(h) \quad = \quad \check{h}(p) \quad = \quad \bigvee_{f \in L^X} \Big(\gamma(\mu_p(f)) *_2 \big(\bigwedge_{x \in X} \gamma(f(x)) \to h(x)\big)\Big) \quad .$$

■

Let $\Delta_\gamma : M\text{–}\mathbf{TOP} \longmapsto L\text{–}\mathbf{TOP}$ and $\mathfrak{L}_\gamma : L\text{–}\mathbf{TOP} \longmapsto M\text{–}\mathbf{TOP}$ be the functors determined by γ according to Section 3 (cf. Theorem 3.6). Further, let $\mathfrak{E} : \mathbf{S}M\text{–}\mathbf{TOP} \longmapsto M\text{–}\mathbf{TOP}$ be the embedding of $\mathbf{S}M\text{–}\mathbf{TOP}$ into $M\text{–}\mathbf{TOP}$ and $\mathfrak{F}_S : M\text{–}\mathbf{TOP} \longmapsto \mathbf{S}M\text{–}\mathbf{TOP}$ be the coreflector of the category of M–topological spaces in $\mathbf{S}M\text{–}\mathbf{TOP}$ (cf. Theorem 5.1.1). In particular the action of $\mathfrak{F}_S$ on objects is given by

$$\begin{aligned} \mathfrak{F}_S(X, \tau) \quad &= \quad (X, \tau_S) \\ \text{where} \quad \tau_S \quad &= \quad \bigcap \{\sigma \mid \sigma \text{ stratified } M\text{–topology on } X\,,\ \tau \subseteq \sigma\,\} \end{aligned}$$

Now we consider the composition $\omega_\gamma = \mathfrak{F}_S \circ \mathfrak{L}_\gamma$ and infer from Theorem 3.6 and Theorem 5.1.1 that

$$\omega_\gamma : L\text{–}\mathbf{TOP} \longmapsto \mathbf{S}M\text{–}\mathbf{TOP}$$

has a left adjoint which is determined by $\delta_\gamma = \Delta_\gamma \circ \mathfrak{E} : \mathbf{S}M\text{–}\mathbf{TOP} \longmapsto L\text{–}\mathbf{TOP}$. An explicit description of the range of ω_γ is given in the following proposition.

Proposition 7.4.4 (Lower semicontinuous, M–valued maps)
Let (X, τ) be an L–topological space and $\ell(\tau, \gamma)$ be the set of all lower (τ, γ)–semicontinuous maps $g : X \longmapsto M$. Then

$$\omega_\gamma(X, \tau) \quad = \quad (X, \ell(\tau, \gamma)) \quad .$$

Proof.
(a) Let $h : X \longmapsto M$ be a lower (τ, γ)–semicontinuous map; we show that h has the following representation

$$h(p) \quad = \quad \bigvee_{g \in \tau} \Big(\gamma(g(p)) *_2 \big(\bigwedge_{x \in X} \gamma(g(x)) \to h(x)\big)\Big) \qquad (\blacklozenge)$$

The inequality $\bigvee_{g \in \tau} (\gamma \circ g) *_2 \big(\bigwedge_{x \in X} \gamma(g(x)) \to h(x)\big) \leq h$ is obvious. Because of the L–neighborhood axioms (U3) and (U4) the map $g : X \longmapsto L$ defined by $g(z) = \mu_z(f)$ is τ–open and fulfills the inequality $g \leq f$. Now we invoke the lower (τ, γ)–semicontinuity of h and obtain:

$$h(p) \quad \leq \quad \bigvee_{g \in \tau} \gamma(g(p)) *_2 \big(\bigwedge_{x \in X} \gamma(g(x)) \to h(x)\big) \quad ;$$

hence (♦) follows.
(b) Since $\gamma \circ g$ is lower (τ, γ)–semicontinuous for all τ–open maps g, the stratified M–topology $\ell(\tau, \gamma)$ (cf. Proposition 7.4.3) is always finer than $\tau^\gamma = \{\gamma \circ g \mid g \in \tau\}$ (cf. proof of Theorem 3.6). On the other hand, referring to (♦), it is easy to see that every stratified M–topology σ with $\tau^\gamma \subseteq \sigma$ contains $\ell(\tau, \gamma)$; hence the assertion is verified.
■

It is easy to see that the functor δ_γ : **S**M**–TOP** $\longmapsto$ L**–TOP** factors through **S**L**–TOP** (cf. 3.6). Hence the restriction ω_γ^s of ω_γ to **S**L**–TOP** has a left adjoint functor δ_γ^s : **S**M**–TOP** $\longmapsto$ **S**L**–TOP**. In the following considerations we describe the range of δ_γ^s.

Definition 7.4.5 (γ–saturation) A stratified L–topology τ on X is said to be γ*–saturated* iff the implication

$$\gamma \circ h \;=\; \bigvee_{g \in \tau} (\gamma \circ g) *_2 \big(\bigwedge_{x \in X} \gamma(g(x)) \to \gamma(h(x))\big) \quad \Longrightarrow \quad h \in \tau$$

holds for all $h \in L^X$.

Proposition 7.4.6 *If $\gamma : L \longmapsto M$ satisfies the additional properties*

(i) γ is injective and arbitrary meet preserving ,

(ii) $\gamma(\alpha) \to \gamma(\beta) \;=\; \gamma(\alpha \to \beta) \quad \forall\, \alpha, \beta \in L$,

then every stratified L–topology is γ–saturated.

Proof. Let τ be a stratified L–topology and h be an element of L^X provided with the subsequent property

$$\gamma \circ h \;=\; \bigvee_{g \in \tau} \Big((\gamma \circ g) *_2 \big(\bigwedge_{x \in X} \gamma(g(x)) \to \gamma(h(x))\big)\Big) \quad ;$$

then we apply (i) and (ii) and obtain:

$$\gamma(h(p)) \;=\; \gamma\big(\bigvee_{g \in \tau} (g(p) *_2 (\bigwedge_{x \in X} g(x) \to h(x)))\big) \qquad \forall\, p \in X\,.$$

Since γ is injective, we conclude from (o3) and (Σ1) that h is an element of τ – i.e. τ is γ–saturated.
■

Proposition 7.4.7 *The range of δ_γ^s consists of all γ–saturated, stratified L–topologies.*

Proof. (a) Let (X,τ) be a stratified L–topological space such that there exists a stratified M–topological space (X,σ) with $\tau = \{h \in L^X \,|\, \gamma \circ h \in \sigma\}$ – i.e. $\delta^s_\gamma(X,\sigma) = (X,\tau)$. Further let $k \in L^X$ be provided with the following property

$$\gamma \circ k \quad = \quad \bigvee_{h \in \tau} \Big((\gamma \circ h) *_2 \big(\bigwedge_{x \in X} \gamma(h(x)) \to \gamma(k(x)) \big) \Big) \qquad ;$$

then the relationship between τ and σ and the application of $(\Sigma 1)$ to σ imply: $\gamma \circ k \in \sigma$ – i.e. $k \in \tau$; hence τ is γ–saturated.

(b) Let (X,τ) be a γ–saturated, stratified L–topological space. We show: $\delta^s_\gamma(\omega^s_\gamma(X,\tau)) = (X,\tau)$. Let us consider $k \in L^X$ such that $\gamma \circ k$ is lower (τ,γ)–semicontinuous. Since τ is γ–saturated, k is an element of τ; hence the assertion follows.

■

Corollary 7.4.8 *The subcategory* $\mathbf{S}_\gamma L$–$\mathbf{TOP}$ *of all γ–saturated, stratified, L–topological spaces is isomorphic to a coreflective subcategory of* $\mathbf{S}M$–$\mathbf{TOP}$.

Proof. Let $\breve{\omega}^s_\gamma$ be the restriction of ω^s_γ to $\mathbf{S}_\gamma L$–$\mathbf{TOP}$. It is not difficult to show that $\breve{\omega}^s_\gamma$ is an embedding from $\mathbf{S}_\gamma L$–$\mathbf{TOP}$ into $\mathbf{S}M$–$\mathbf{TOP}$ (cf. Proof of 7.4.7). Since the range of δ^s_γ coincides with $\mathbf{S}_\gamma L$–$\mathbf{TOP}$ (cf. Proposition 7.4.7), δ^s_γ is left adjoint to $\breve{\omega}^s_\gamma$; hence the assertion follows.

■

We close this subsection with a presentation of four important special cases.

Case A (Probabilistic L–topologies versus M–topologies)

We assume that the underlying quadruple $(M,\leq,\otimes,*)$ satisfies the axioms (I) – (VII) . Further let $(L,\leq,*)$ be a subquantale of $(M,\leq,*)$ provided with the following properties (see also Example 1.2.18):

(i) The canonical embedding $\iota : L \longmapsto M$ preserves the residuation – i.e. $\iota(\alpha) \to \iota(\beta) = \iota(\alpha \to \beta)$.

(ii) The restriction of $\otimes$ to $L \times L$ coincides with $*$.

(iii) $(L,\leq,*)$ is a complete MV–algebra.

(iv) The range $\iota(L)$ of L is pseudo–bisymmetric in $(M,\leq,\otimes,*)$.

We conclude from Proposition 7.4.6 and Corollary 7.4.8 that $\mathbf{S}L$–$\mathbf{TOP}$ is isomorphic to a coreflective subcategory of $\mathbf{S}M$–$\mathbf{TOP}$. Moreover is not difficult to show that the category $\mathbf{P}L$–$\mathbf{TOP}$ of probabilistic L–topological spaces is a coreflective subcategory of $\mathbf{S}L$–$\mathbf{TOP}$. Hence $\mathbf{P}L$–$\mathbf{TOP}$ is isomorphic to a coreflective subcategory of $\mathbf{S}M$–$\mathbf{TOP}$. In particular, every probabilistic L–topology τ on X can be identified with the set $\ell(\tau,\iota)$ of all lower (τ,ι)–semicontinuous maps – i.e. with the set of all those mappings $g : X \longmapsto M$ satisfying the following axiom

$$g(p) \quad \leq \quad \bigvee_{d \in \mathbf{U}_p} \Big(\bigwedge_{x \in X} \iota(d(x)) \to g(x) \Big) \qquad \forall p \in X$$

where $\mathbf{U}_p$ denotes the set of all L–valued neighborhoods at p w.r.t. τ (cf. Subsection 7.2).

If we assume additionally that $(M, \leq)$ is *completely distributive*, then we can show that $\omega_\iota \circ \mathcal{E} : \mathbf{P}L\textbf{–TOP} \longmapsto \mathbf{S}M\textbf{–TOP}$ has a right adjoint functor where $\mathcal{E}$ denotes the embedding functor from $\mathbf{P}L\textbf{–TOP}$ to $\mathbf{S}L\textbf{–TOP}$.
We prepare the proof by three lemmata. First we need some notations: Every $\alpha \in M$ determines a map $h_\alpha : M \longmapsto L$ by

$$h_\alpha(\beta) \quad = \quad \bigvee \{\lambda \in L \mid \iota(\lambda) * \beta \leq \alpha\} \qquad \forall \beta \in M \quad .$$

Since ι preserves arbitrary joins, we obtain: $\iota(h_\alpha(\beta)) * \beta \leq \alpha$. Moreover, in the case of $L = \{\top, \bot\}$ the map h_α coincides with the ordinary characteristic function of the interval $[0, \alpha]$.

Lemma 7.4.9 (Charcterization of lower semicontinuity)
Let $(M, \leq)$ be a completely distributive, complete lattice and τ be an L–probabilistic topology on X. Then for every map $g : X \longmapsto M$ the following assertions are equivalent

(i) g is lower (τ, ι)–semicontinuous.

(ii) $(h_\alpha \circ g) \to \bot \in \tau \qquad \forall \alpha \in M$.

Proof. (a) ((*i*) $\Longrightarrow$ (*ii*)) Since $(L, \leq, *)$ is a MV–algebra, it is sufficient to show that $h_\alpha \circ g$ is τ–closed – i.e.

$$\bigwedge_{d_p \in \mathbf{U}_p} \Big(\bigvee_{x \in X} h_\alpha(g(x)) * d_p(x) \Big) \quad \leq \quad h_\alpha(g(p)) \qquad \forall p \in X \quad .$$

Since g is lower (τ, ι)–semicontinuous, we obtain

$$\iota\Big(\bigwedge_{d_p \in \mathbf{U}_p} \Big(\bigvee_{x \in X} h_\alpha(g(x)) * d_p(x) \Big)\Big) * g(p) \quad \leq$$

$$\Big(\bigwedge_{d_p \in \mathbf{U}_p} \Big(\bigvee_{x \in X} \iota(h_\alpha(g(x))) * \iota(d_p(x)) \Big)\Big) \; * \; \Big(\bigvee_{d_p \in \mathbf{U}_p} \Big(\bigwedge_{x \in X} \iota(d_p(x)) \to g(x) \Big)\Big) \quad \leq$$

$$\bigvee_{d_p \in \mathbf{U}_p} \bigvee_{x \in X} \Big(\iota(h_\alpha(g(x)) * g(x) \Big) \quad \leq \quad \alpha \quad ;$$

hence the τ–closedness of $h_\alpha \circ g$ follows from the definition of h_α.
(b) ((*ii*) $\Longrightarrow$ (*i*)) For every map $\xi : \mathbf{U}_p \longmapsto X$ we define an element $\alpha_\xi \in M$ by

$$\alpha_\xi \quad = \quad \bigvee_{d_p \in \mathbf{U}_p} \iota(d_p(\xi(d_p))) \to g(\xi(d_p)) \qquad .$$

Because of $\delta \leq \beta \to \delta$ the relation $h_{\alpha_\xi}(g(\xi(d_p))) = \top$ holds for all $d_p \in \mathbf{U}_p$. Since $h_{\alpha_\xi} \circ g$ is τ–closed (cf. (ii)), we obtain:

$$\bigwedge_{d_p \in \mathbf{U}_p} \Big(\bigvee_{x \in X} h_{\alpha_\xi}(g(x)) * d_p(x) \Big) = \top \leq h_{\alpha_\xi}(g(p)) \quad ;$$

i.e. $g(p) \leq \alpha_\xi$ for all $\xi \in X^{\mathbf{U}_p}$. Hence the complete distributivity of $(M, \leq)$ implies

$$g(p) \leq \bigwedge_{\xi \in X^{\mathbf{U}_p}} \alpha_\xi = \bigvee_{d_p \in \mathbf{U}_p} \Big(\bigwedge_{x \in X} \iota(d_p(x)) \to g(x) \Big) \quad ;$$

i.e. g is lower (τ, ι)–semicontinuous.
■

As an immediate consequence of the previous lemma we obtain

Lemma 7.4.10 *Let $(M, \leq)$ be a completely, distributive lattice. Then for every non empty family $\{\tau_i \mid i \in I\}$ of probabilistic L–topologies on X the following relation holds:*

$$\ell((\bigcap_{i \in I} \tau_i), \iota) = \bigcap_{i \in I} \ell(\tau_i, \iota) \quad .$$

■

Now we are in the position to define an object function $\vartheta : |\mathbf{S}M\text{–}\mathbf{TOP}| \longmapsto |\mathbf{P}L\text{–}\mathbf{TOP}|$ as follows:

$$\vartheta(X, \tau) = (X, \Upsilon_\tau)$$

$$\text{where} \quad \Upsilon_\tau = \bigcap \{ \tau_0 \mid \tau_0 \ \text{ probabilistic } L\text{–topology on } X, \ \tau \subseteq \ell(\tau_0, \iota) \}$$

Lemma 7.4.11 *Let $(M, \leq)$ be completely distributive. If $\varphi : (X, \tau) \longmapsto (Y, \sigma)$ is a $\mathbf{S}M$–$\mathbf{TOP}$–morphism (in particular φ is an M–continuous map), then $\varphi : \vartheta(X, \tau) \longmapsto \vartheta(Y, \sigma)$ is L–continuous.*

Proof. (a) Let τ_0 be a probabilistic L–topology on X and $(\mathbf{U}_p)_{p \in X}$ be the system of crisp sets of L–valued neighborhoods corresponding to τ_0. Further let $\varphi(\tau_0)$ be the final L–topology on Y w.r.t. φ –i.e.

$$\varphi(\tau_0) = \{ g \in L^Y \mid g \circ \varphi \in \tau_0 \} \quad .$$

Then $\varphi(\tau_0)$ is again a probabilistic L–topology on Y and the corresponding system $(\mathbf{V}_q)_{q \in Y}$ of crisp sets of L–valued neighborhoods is given by[16]

$$\mathbf{V}_{\varphi(p)} = \{ f \in L^Y \mid f \circ \varphi \in \mathbf{U}_p \} = \{ f \in L^Y \mid \exists\, d \in \mathbf{U}_p : \varphi[d] \leq f \}$$

$$\mathbf{V}_q = \{ f \in L^Y \mid f(q) = \top \} \quad \text{where} \quad q \notin \varphi(X) \quad .$$

[16] $\varphi[g]$ is the L–valued image of the map g under φ – i.e. $(\varphi[g])(y) = \vee\{g(x) \mid \varphi(x) = y\}$.

Because of

$$\begin{aligned}\bigvee_{f \in \mathbf{V}_{\varphi(p)}} \Big(\bigwedge_{y \in Y} \iota(f(y)) \to g(y)\Big) &= \bigvee_{d_p \in \mathbf{U}_p} \Big(\bigwedge_{x \in X} \iota(\varphi[d_p](\varphi(x))) \to g(\varphi(x))\Big)\\ &= \bigvee_{d_p \in \mathbf{U}_p} \Big(\bigwedge_{x \in X} \iota(d_p(x)) \to g \circ \varphi(x)\Big)\end{aligned}$$

we obtain the important relation $\ell(\varphi(\tau_0), \iota) = \varphi(\ell(\tau_0, \iota))$.
(b) Let φ be a M–continuous map from (X, τ) to (Y, σ). We put $\tau_0 = \Upsilon_\tau$; then $\tau \subseteq \ell(\tau_0, \iota)$ follows from Lemma 7.4.10. Referring to the consideration of Step (a) we conclude from the M–continuity of φ that φ is L–continuous from $\vartheta(X, \tau)$ to $\vartheta(Y, \sigma)$.
■

With regard to the previous lemma the object function ϑ can be completed to a functor from $\mathbf{S}M$–$\mathbf{TOP}$ to $\mathbf{P}L$–$\mathbf{TOP}$ in an obvious way.

Theorem 7.4.12 *Let $(M, \leq)$ be a completely distributive lattice. Then the functor ϑ is right adjoint to $\omega_\iota \circ \mathcal{E}$.*

Proof. Obviously $\vartheta \circ \omega_\iota \circ \mathcal{E} = id_{\mathbf{P}L\text{–}\mathbf{TOP}}$; we show that the following universal property holds:

$$\begin{tikzcd} (X,\tau_0) \arrow[r, "id_X"] \arrow[dr, "\varphi"'] & \vartheta(\omega_\iota \circ \mathcal{E}(X,\tau_0)) \arrow[d, "\vartheta(\ulcorner\varphi\urcorner)"] \\ & \vartheta(Y,\tau)) \end{tikzcd} \qquad \begin{tikzcd} \omega_\iota \circ \mathcal{E}(X,\tau_0) \arrow[d, dotted, "\varphi = \ulcorner\varphi\urcorner"] \\ (Y,\tau) \end{tikzcd}$$

Since $\varphi : (X, \tau_0) \longmapsto \vartheta(Y, \tau)$ is L–continuous, we obtain that $\varphi : \omega_\iota \circ \mathcal{E}(X, \tau_0) \longmapsto \omega_\iota \circ \mathcal{E} \circ \vartheta(Y, \tau)$ is M–continuous. Now we invoke Lemma 7.4.10 and observe that $id_Y : \omega_\iota \circ \mathcal{E} \circ \vartheta(Y, \tau) \longmapsto (Y, \tau)$ is also M–continuous; hence we can view φ as a M–continuous morphism from $\omega_\iota \circ \mathcal{E}(X, \tau_0)$ to (Y, τ). Therewith the assertion is established.
■

We can summarize the preceding considerations as follows:

If $(M, \leq)$ is completely distributive, then $\mathbf{P}L$–$\mathbf{TOP}$ is isomorphic to a coreflective and reflective subcategory of $\mathbf{S}M$–$\mathbf{TOP}$. In this context a probabilistic L-topology τ_0 on X is identified with the set of all lower (τ_0, ι)-semicontinuous maps $g : X \longmapsto M$.

Case B (Ordinary topologies versus M–topologies)

We assume that the underlying quadruple $(M, \leq, \otimes, *)$ satisfies the additional axiom

$$\bot \;=\bot \otimes \top \;=\; \top \otimes \bot \;.$$

Then we put $L = 2 = \{\bot, \top\}$; obviously the canonical embedding $\iota : 2 \longmapsto M$ fulfills the conditions (i) – (iv) in **Case A** (see also Example 1.2.17(a)). Further, every 2–topology is stratified and is also a probabilistic 2–topology; hence the catgeories **TOP** and **P2–TOP** = **S2–TOP** (cf. 3.7) are isomorphic. We conclude from **Case A** that **TOP** is isomorphic to a coreflective subcategory of **S**M**–TOP**. In particular every ordinary topology $\mathbb{T}$ on X can be identified with the set of all lower $\mathbb{T}$–semicontinuous maps $g : X \longmapsto M$ – i.e. with the set of all those mappings $g : X \longmapsto M$ satisfying the following axiom

$$g(p) \quad \leq \quad \bigvee_{U \in \mathbb{U}_p} \Big(\bigwedge_{x \in U} g(x) \Big) \qquad \forall\, p \in X$$

where $\mathbb{U}_p$ denotes the set of all neighborhoods at p w.r.t. $\mathbb{T}$.

Remark 7.4.13 If we assume in addition that $(M, \leq, *)$ is a (complete) MV–algebra, then the set $\ell(\mathbb{T})$ of all lower semicontinuous maps $g : X \longmapsto M$ forms obviously a *probabilistic M-topology* on X (cf. Subsection 7.2). In fact, if g is lower $\mathbb{T}$–semicontinuous, then for all $\alpha \in M$ the map $\alpha \to g$ is again lower $\mathbb{T}$–semicontinuous. Moreover, the corresponding $\top$–neighborhood filter $\mathbf{U}_p$ at the point $p \in X$ is given by

$$\mathbf{U}_p \quad = \quad \{ d \in L^X \mid \check{d}(p) \,=\, \top \} \qquad\qquad \text{(cf. 7.4.2)} \quad .$$

In particular $\mathbf{U}_p$ can be viewed as the $\top$–filter which is determined by

$$\{ \chi_U \mid U \in \mathbb{U}_p \}$$

and the saturation process w.r.t. the κ–condition (cf. Axiom ($\mathbb{F}$1) in Remark 6.2.3) where $\mathbb{U}_p$ is the ordinary neighborhood filter at p w.r.t. $\mathbb{T}$.
■

If in addition $(M, \leq)$ is a completely distributive lattice, then we conclude from Theorem 7.4.12 (see **Case A**) that **Top** is isomorphic to a coreflective and reflective subcategory of **S**M**–TOP**. In this context, an ordinary topology $\mathbb{T}$ on X is identified with the set of all lower $\mathbb{T}$–semicontinuous maps $g : X \longmapsto M$.

Case C (Booleanization of L–topologies)

Let $(L, \leq)$ be a complete Heyting algebra, $\mathbb{B}(L, \leq) = (\mathbb{B}(L), \leq)$ be its *Booleanization* and $j : L \longmapsto \mathbb{B}(L)$ be the canonical embedding (cf. Theorem 1.4.2). Further, we consider the setting $(L, \leq, \wedge, \wedge)$ and $(\mathbb{B}(L), \wedge, \wedge)$ – i.e. the case $\otimes = * = \wedge$. Since j is a frame–morphism (cf. Remark 1.4.3(a)), we see immediately (cf. Example 1.2.17(b)) that j satisfies Condition (E3) (see the beginning of Subsection 7.4). Because of Corollary 7.4.8 the category $\mathbf{S}_j L$**–TOP** of j–saturated, stratified, L–topological spaces is isomorphic to a coreflective subcategory of stratified, $\mathbb{B}(L)$–topological spaces. In the following considerations we assume additionally that the binary join operator is distributive over arbitrary meets– i.e. $(L,\leq)$ is a *complete Heyting and co–Heyting algebra* (cf. Remark 1.4.3(b)). In particular, $(L,\leq)$ carries a further binary operation $\rhd$ determined by

$$\alpha \rhd \beta \quad = \quad \bigwedge\{\,\lambda \in L \mid \alpha \leq \lambda \vee \beta\,\} \qquad \forall\, \alpha,\beta \in L \quad .$$

Definition 7.4.14 (Co-stratification)
Let $(L,\leq)$ be a complete Heyting and co–Heyting algebra. An L–topology τ on X is said to be *co–stratified* iff τ satisfies the following condition

$(\Sigma 3)\quad g \rhd \alpha \,\in\, \tau \qquad \forall\, g \in \tau,\ \forall\, \alpha \in L\quad .$

■

Proposition 7.4.15 *Let $(L,\leq)$ be a complete Heyting and co–Heyting algebra. Further let $j : L \longmapsto \mathbb{B}(L)$ be the canonical embedding from L into its Booleanization $\mathbb{B}(L)$. Then every stratified and co–stratified L–topology τ is j–saturated.*

Proof. Since $(L,\leq)$ is a complete co–Heyting algebra, the canonical embedding $j : L \longmapsto \mathbb{B}(L)$ is meet preserving (cf. Remark 1.4.3(b)). Hence the left adjoint map j_* of j exists and is given by:

$$j_*(\kappa) \quad = \quad \bigwedge\{\lambda \in L \mid \kappa \,\leq\, j(\lambda)\} \qquad \forall\, \kappa \in \mathbb{B}(L) \quad .$$

In particular, the following relation

$$(f \,\rhd\, \beta)(x) \quad = \quad f(x) \,\rhd\, \beta \quad = \quad j_*(j(f(x)) \wedge (j(\beta) \to \bot)) \qquad (\blacklozenge)$$

holds for all $f \in L^X$, $x \in X$, $\beta \in L$.
Now let τ be a stratified and co–stratified L–topology on X. In order to show that τ is j–saturated we consider an element $h \in L^X$ with

$$j \circ h \quad = \quad \bigvee_{g \in \tau} (j \circ g) \wedge \Big(\bigwedge_{x \in X} j(g(x)) \to j(h(x)) \Big) \quad .$$

Referring to Property (iii) in Theorem 1.4.2 there exists an index set I provided with the following properties

$$\{g_i \,|\, i \in I\} \;\subseteq\; \tau \quad , \quad \{(\alpha_i,\beta_i) \,|\, i \in I\} \;\subseteq\; L \times L\,,$$
$$j \circ h \quad = \quad \bigvee_{i \in I} \,(j \circ g_i) \,\wedge\, j(\alpha_i) \,\wedge\, (j(\beta_i) \to \bot) \quad .$$

Since τ is stratified, for each $i \in I$ there exists a τ–open map $\tilde{g}_i \in \tau$ such that

$$j \circ h \quad = \quad \bigvee_{i \in I} \,(j \circ \tilde{g}_i) \,\wedge\, (j(\beta_i) \to \bot) \qquad (\blacklozenge\blacklozenge)$$

Now we invoke the fact that j is injective (cf. Property (i) in Theorem 1.4.2) and j_* is join preserving; hence the relation

$$h \quad = \quad \bigvee_{i \in I} (\tilde{g}_i \rhd \beta_i)$$

follows from (♦) and (♦♦). Since τ is co–stratified, h is an element of τ.
■

As an immediate consequence from the previous considerations we obtain that every *co–stratified* and stratified L–topology can be identified with a stratified $\mathbb{B}(L)$–topology. Since in the Boolean case stratified $\mathbb{B}$–topologies can be described by neighborhood systems of $\top$–filters (cf. Theorem 6.1.5, Proposition 6.2.4(b)), the previous statement means that co–stratified and stratified L–topologies can be characterized by neighborhood systems consisting of crisp sets of $\mathbb{B}(L)$–valued neighborhoods only.

Example 7.4.16 (Complete chains) Let $(L, \leq)$ be a complete chain; then $(L, \leq)$ is a complete Heyting and co–Heyting algebra. In particular, the subsequent relations are valid:

$$\alpha \rhd \beta \quad = \quad \begin{cases} \bot \,, & \alpha \leq \beta \\ \alpha \,, & \beta < \alpha \end{cases} \quad , \qquad g \rhd \beta \; = \; g \wedge \chi_{\{\beta < g\}} \quad \text{where } g \in L^X \,.$$

Further we consider the case $\otimes = * = \wedge$ and choose an element $\beta_0 \in L$ with $\bot \neq \beta_0 \neq \top$. Then

$$\tau \quad = \quad \{\alpha \cdot 1_X \mid \alpha \in L\} \cup \{ f \in L^X \mid f \leq \beta_0 \cdot 1_X \}$$

is a co–stratifed and stratified L–topology on X. Moreover there does not exist an ordinary topology $\mathbb{T}$ on X such that $\omega_\iota(X, \mathbb{T}) = (X, \tau)$ – i.e. τ is not topologically generated in the sense of **Case B**. With regard to the previous considerations τ has systems of neighborhoods which are **not** 2–valued, but $\mathbb{B}(L)$–valued.

■

A more interesting example is the following one

Example 7.4.17 Let $(L, \leq, *)$ be a strict MV–algebra (cf. Section 1.3). Then it is well known that the underlying lattice $(L, \leq)$ is a complete Heyting and co–Heyting algebra. Besides the natural implication $\rightarrow$ based on the semigroup operation $*$ we can consider a further implication $\rightharpoondown$ determined by the binary meet operation $\wedge$

$$\alpha \rightharpoondown \beta \quad = \quad \bigvee \{\lambda \in L \mid \alpha \wedge \lambda \leq \beta\} \quad , \qquad \alpha, \beta \in L \quad .$$

Now we define a binary operation E on L by $E(\alpha, \beta) = (\alpha \rightharpoondown \beta) \wedge (\beta \rightharpoondown \alpha)$. In particular, E can be viewed as an *L–valued equality* on L. Further, let τ_L

be the subset of all maps $g : L \longmapsto L$ satisfying the following property

$$g(\alpha) \leq \bigwedge_{\beta \in L} \Big((E(\alpha,\beta) \to \bot) \vee g(\beta)\Big) \qquad \forall \alpha \in L \quad .$$

It is not difficult to show that τ_L is a stratified L–topology on L in the sense of the quadruple $(L, \leq, \wedge, *)$. Moreover, τ_L is co–stratified. In fact, for every $g \in \tau_L$ and every $\lambda \in L$ the relation

$$(g \rhd \lambda)(\alpha) \leq \bigwedge_{\beta \in L} \Big((E(\alpha,\beta) \to \bot) \vee (g \rhd \lambda)(\beta)\Big)$$

follows from

$$\begin{aligned} g(\alpha) &\leq \bigwedge_{\beta \in L} \Big((E(\alpha,\beta) \to \bot) \vee g(\beta)\Big) \\ &\leq \Big(\bigwedge_{\beta \in L} \Big((E(\alpha,\beta) \to \bot) \vee (g \rhd \lambda)(\beta)\Big)\Big) \vee \lambda \quad . \end{aligned}$$

We show that τ_L is *not* a probabilistic L–topology. The proof is divided into two steps:
Step 1: Let g be an element of τ_L and $\alpha \in L$ with $g(\alpha) = \top$. Because of the formula $(\gamma \to \bot) \vee \beta = \gamma \to (\gamma * \beta)$ (cf. Proposition 2.15 in [39]) we obtain

$$E(\alpha,\beta) \to (E(\alpha,\beta) * g(\beta)) = \top \qquad \forall \beta \in L \quad ;$$

i.e. $E(\alpha,\beta) = E(\alpha,\beta) * g(\beta) \quad \forall \beta \in L$. Hence the idempotent kernel (cf. Subsection 1.3) $(g(\beta))\check{}$ of $g(\beta)$ w.r.t. $*$ fulfills the following property

$$E(\alpha,\beta) \leq (g(\beta))\check{} \qquad \forall \beta \in L \quad .$$

Since $(L, \leq, *)$ is a strict MV–algebra (cf. Subsection 1.3), we obtain that $\top$ is the only idempotent element of L which is larger than $\bot^{1/2}$. Hence we conclude from the previous considerations:

$$g(\top) = \top \iff g(\bot^{1/2}) = \top \quad . \qquad (\Diamond)$$

Step 2: Because of $E(\alpha,\beta) \wedge \beta \leq \alpha$ we obtain that the map $g_0 : L \longmapsto L$ defined by $g_0(\beta) = \beta \to \bot$ is τ_L–open – i.e. $g_0 \in \tau_L$. Let us assume that τ_L is a probabilistic L–topology; then for every point $\alpha \in L$ there exists an open L–valued neighborhood $d_\alpha \in \mathbf{U}_\alpha$ such that (cf. Subsection 7.2)

$$g_0(\alpha) * d_\alpha(\beta) \leq g_0(\beta) \qquad \forall \beta \in L$$

In the case of $\alpha = \bot^{1/2}$ we obtain from the relation $(\Diamond)$ and the strictness of $(L, \leq, *)$:

$$\bot^{1/2} = g_0(\bot^{1/2}) * \top = g_0(\bot^{1/2}) * d_{\bot^{1/2}}(\top) \leq g_0(\top)$$

which is a contradiction to $g_0(\top) = \bot$. Hence the assumption is false.

We can summarize the previous considerations as follows: Even though τ_L *cannot* be described by L–valued neighborhoods in the sense of Subsection 7.2, we can identify τ_L with its Booleanization which offers the possibility to characterize τ_L by $\mathbb{B}(L)$–valued neighborhoods.

■

Case D (Strict MV–algebras)

Let $(L, \leq, *)$ be a complete, strict MV–algebra (cf. Subsection 1.3). Since $([0,1], \leq, T_m)$ is the initial object in the category of complete, strict MV–algebras (cf. Theorem 6.9 in [39]), there exists an MV–algebra–homomorphism γ : $([0,1], \leq, T_m) \longmapsto (L, \leq, *)$. Further, we consider the corresponding *monoidal mean operators* on $[0,1]$ and L (i.e. *the arithmetic mean* $\boxplus$ on $[0,1]$ and the monoidal mean operator $\circledast$ on L determined by $*$). Then $\gamma : [0,1] \longmapsto L$ fulfills (E1)–(E3) in the setting given by $([0,1], \leq, \boxplus, T_m)$ and $(L, \leq, \circledast, *)$ (cf. Example 1.2.17(c)). Since γ is an MV–algebra–homomorphism, γ preserves the implication; hence γ satisfies the conditions (i) and (ii) in Proposition 7.4.6. Therefore we conclude from Proposition 7.4.6, Proposition 7.4.7 and Corollary 7.4.8 that the category $\mathbf{S}[0,1]$–$\mathbf{TOP}$ is isomorphic to a coreflective subcategory of $\mathbf{S}L$–$\mathbf{TOP}$.

8 Convergence theory for L–fuzzy topological spaces

In this section the underlying quadruple $(L, \leq, \otimes, *)$ satisfies the axioms (I)–(VII). Moreover we assume that $(L, \leq, *)$ is a strictly two–sided, commutative quantale.

8.1 L–fuzzy interior operators and L–fuzzy neighborhood systems

Definition 8.1.1 A mapping $\mathcal{I} : L^X \times L \to L^X$ is called an L–fuzzy interior operator on X iff $\mathcal{I}$ satisfies the following conditions:

$(I0)$ $\quad \mathcal{I}(1_X, \alpha) \;=\; 1_X \qquad \forall \alpha \in L$.

$(I1)$ $\quad \mathcal{I}(g, \beta) \;\leq\; \mathcal{I}(f, \alpha) \quad$ whenever $\quad g \leq f$, $\alpha \leq \beta$.

$(I2)$ $\quad \mathcal{I}(f, \alpha) \otimes \mathcal{I}(g, \beta) \;\leq\; \mathcal{I}(f \otimes g, \alpha \otimes \beta)$.

$(I3)$ $\quad \mathcal{I}(f, \alpha) \;\leq\; f$.

$(I4)$ $\quad \mathcal{I}(f, \alpha) \;\leq\; \mathcal{I}(\mathcal{I}(f, \alpha), \alpha)$.

$(I5)$ $\quad \mathcal{I}(f, \bot) \;=\; f$.

$(I6)$ $\quad$ If $\varnothing \neq K \subseteq L$, $\mathcal{I}(f, \alpha) = f^o \;\; \forall \alpha \in K$, then $\mathcal{I}(f, \vee K) \;=\; f^o$.

■

Theorem 8.1.2 *Given an $L-$fuzzy topology $\mathcal{T} : L^X \to L$, the mapping $\mathcal{I}_\mathcal{T} : L^X \times L \to L^X$ defined by the equality*

$$\mathcal{I}_\mathcal{T}(f,\alpha) \quad = \quad \bigvee \{u \in L^X \mid u \leq f , \mathcal{T}(u) \geq \alpha\} \quad , \quad \forall f \in L^X, \alpha \in L$$

is an $L-$fuzzy interior operator on X. Conversely, given an $L-$fuzzy interior operator $\mathcal{I} : L^X \times L \to L^X$, the formula

$$\mathcal{T}_\mathcal{I}(f) \quad = \quad \bigvee \{\alpha \in L \mid \mathcal{I}(f,\alpha) \geq f\} \quad , \quad f \in L^X$$

defines an $L-$fuzzy topology $\mathcal{T}_\mathcal{I} : L^X \to L$. Moreover, the equalities $\mathcal{T}_{\mathcal{I}_\mathcal{T}} = \mathcal{T}$ and $\mathcal{I}_{\mathcal{T}_\mathcal{I}} = \mathcal{I}$ hold. Thus $L-$fuzzy topologies and $L-$fuzzy interior operators come to the same thing.

The proof of Theorem 8.1.2 is based on the following sequence of lemmata.

Lemma 8.1.3 *Let $\mathcal{T} : L^X \to L$ be a map satisfying Axiom ($\mathcal{O}3$). Then the map $\mathcal{I}_\mathcal{T} : L^X \times L \to L^X$ defined by*

$$\mathcal{I}_\mathcal{T}(f,\alpha) \quad = \quad \bigvee \{u \in L^X \mid u \leq f, \alpha \leq \mathcal{T}(u)\}$$

fulfills the following properties:

(8.1) $\alpha \leq \mathcal{T}(\mathcal{I}_\mathcal{T}(f,\alpha)) \quad \forall \alpha \in L$.

($I1$) $\mathcal{I}_\mathcal{T}(f,\alpha) \leq \mathcal{I}_\mathcal{T}(g,\beta)$ *whenever* $f \leq g$ *and* $\beta \leq \alpha$.

($I3$) $\mathcal{I}_\mathcal{T}(f,\alpha) \leq f$.

($I4$) $\mathcal{I}_\mathcal{T}(f,\alpha) \leq \mathcal{I}_\mathcal{T}(\mathcal{I}_\mathcal{T}(f,\alpha),\alpha)$.

($I5$) $\mathcal{I}_\mathcal{T}(f,\bot) = f$.

($I6$) *If* $\varnothing \neq K \subseteq L$, $\mathcal{I}_\mathcal{T}(f,\alpha) = f^0 \quad \forall \alpha \in K$, *then* $\mathcal{I}_\mathcal{T}(f,\vee K) = f^0$.

(8.1$'$) $\mathcal{T}(f) = \bigvee\{\alpha \in L \mid f \leq \mathcal{I}_\mathcal{T}(f,\alpha)\}$.

Proof. The relation (8.1) follows immediately from Axiom ($\mathcal{O}3$), while ($I1$), ($I3$) and ($I5$) hold by definition of $\mathcal{I}_\mathcal{T}$. Further, (8.1) and the definition of $\mathcal{I}_\mathcal{T}$ imply ($I4$). In order to verify ($I6$) we proceed as follows. Let K be a non-empty subset of L such that the relation

$$\mathcal{I}_\mathcal{T}(f,\alpha) \quad = \quad f^0$$

holds for all $\alpha \in K$. Referring again to (8.1) we obtain: $\vee K \leq \mathcal{T}(f^0)$. Now we invoke again the definition of $\mathcal{I}_\mathcal{T}$ and the properties ($I1$) and ($I3$):

$$f^0 \quad = \quad \mathcal{I}_\mathcal{T}(f^0,\vee K) \quad \leq \quad \mathcal{I}_\mathcal{T}(f,\vee K).$$

The converse inequality $\mathcal{I}_{\mathcal{T}}(f, \vee K) \leq f^0 (= \mathcal{I}_{\mathcal{T}}(f,\alpha))$ follows again from $(I1)$; hence the equality $f^0 = \mathcal{I}_{\mathcal{T}}(f, \vee K)$ has been established. Finally, we infer from (8.1) and $(I3)$:

$$\bigvee \{\alpha \in L \mid f \leq \mathcal{I}_{\mathcal{T}}(f,\alpha)\} \leq \mathcal{T}(f) \quad .$$

On the other hand, the definition of $\mathcal{I}_{\mathcal{T}}$ implies: $f = \mathcal{I}_{\mathcal{T}}(f, \mathcal{T}(f))$; hence the converse inequality

$$\mathcal{T}(f) \leq \bigvee \{\alpha \in L \mid f \leq \mathcal{I}_{\mathcal{T}}(f,\alpha)\}$$

follows. Therewith the relation (8.1$'$) is verified.
■

Lemma 8.1.4 *Let $\mathcal{I} : L^X \times L \to L^X$ be a map provided with the properties $(I1)$, $(I3)$, $(I4)$, $(I5)$ and $(I6)$. Then the map $\mathcal{T}_{\mathcal{I}} : L^X \to L$ defined by*

$$\mathcal{T}_{\mathcal{I}}(f) = \bigvee \{\alpha \in L \mid f \leq \mathcal{I}(f,\alpha)\}$$

satisfies the following conditions:

(8.2) $\quad \beta \leq \mathcal{T}_{\mathcal{I}}(f) \iff f \leq \mathcal{I}(f,\beta) \quad \forall \beta \in L \quad .$

$(\mathcal{O}3)$ $\quad \bigwedge_{i\in I} \mathcal{T}_{\mathcal{I}}(f_i) \leq \mathcal{T}_{\mathcal{I}}(\bigvee_{i\in I} f_i) \quad .$

(8.2$'$) $\quad \mathcal{I}(f,\alpha) = \bigvee \{u \in L^X \mid u \leq f \,,\ \alpha \leq \mathcal{T}_{\mathcal{I}}(u)\} \quad .$

Proof. The implication $f \leq \mathcal{I}(f,\beta) \Longrightarrow \beta \leq \mathcal{T}_{\mathcal{I}}(f)$ holds by definition. Because of $(I5)$ the set $\{\alpha \in L \mid f \leq \mathcal{I}(f,\alpha)\}$ is non empty; hence $(I6)$ and $(I3)$ imply $f = \mathcal{I}(f, \mathcal{T}_{\mathcal{I}}(f)) \quad \forall f \in L^X$. Let us now assume $\beta \leq \mathcal{T}_{\mathcal{I}}(f)$, then $f \leq \mathcal{I}(f,\beta)$ follows from $(I1)$; hence the relation (8.2) is established. In order to verify $(\mathcal{O}3)$ we put $\beta := \bigwedge_{i\in I} \mathcal{T}_{\mathcal{I}}(f_i)$. Now we invoke (8.2) and $(I1)$:

$$\bigvee_{i\in I} f_i \leq \bigvee_{i\in I} \mathcal{I}(f_i,\beta) \leq \mathcal{I}(\bigvee_{i\in I} f_i, \beta) \quad ;$$

hence $\beta \leq \mathcal{T}_{\mathcal{I}}(\vee_{i\in I} f_i)$ follows from the definition of $\mathcal{T}_{\mathcal{I}}$. Finally, we fix $\alpha \in L$ and put

$$g = \bigvee \{u \in L^X \mid u \leq f \,,\ \alpha \leq \mathcal{T}_{\mathcal{I}}(u)\} \quad .$$

With regard to (8.2), $(I1)$ and $(I4)$ we obtain:

$$g \leq \mathcal{I}(g,\alpha) \leq \mathcal{I}(f,\alpha) \quad , \quad \alpha \leq \mathcal{T}_{\mathcal{I}}(\mathcal{I}(f,\alpha)) \quad ;$$

hence the relation $g = \mathcal{I}(f,\alpha)$ follows; and (8.2$'$) is verified.
■

Proof of Theorem 8.1.2
Referring to the definition of $\mathcal{I}_{\mathcal{T}}$ and $\mathcal{T}_{\mathcal{I}}$ we conclude from Lemmata 8.1.3 and 8.1.4 that there exists a bijection between the set of all maps $\mathcal{T} : L^X \to L$ provided with $(\mathcal{O}3)$ and the set of all maps $\mathcal{I} : L^X \times L \to L^X$ satisfying $(I1), (I3), (I4), (I5)$ and $(I6)$. Therefore Theorem 8.1.2 will be established if we can verify the equivalences $(\mathcal{O}1) \iff (I0)$ and $(\mathcal{O}2) \iff (I2)$.

The first equivalence is obvious. In order to verify the second equivalence we proceed as follows:
(a) Let us assume that $\mathcal{I}$ satisfies $(I2)$; then we apply Axiom (IV) and obtain

$$\begin{aligned} \mathcal{T}(f) \otimes \mathcal{T}(g) \quad &= \quad \bigvee \{\alpha \otimes \beta \mid f \leq \mathcal{I}(f, \alpha) \, , \, g \leq \mathcal{I}(g, \beta)\} \\ &\leq \quad \bigvee \{\gamma \mid f \otimes g \leq \mathcal{I}(f \otimes g, \gamma)\} \quad = \quad \mathcal{T}(f \otimes g) \, ; \end{aligned}$$

hence the L–fuzzy topology corresponding to $\mathcal{I}$ fulfills $(\mathcal{O}2)$.
(b) We assume that $\mathcal{T}$ is provided with $(\mathcal{O}2)$; then Axiom (IV) implies:

$$\begin{aligned} &\mathcal{I}(f, \alpha) \otimes \mathcal{I}(g, \beta) \quad = \\ &\bigvee \{u \otimes v \mid u, v \in L^X, u \leq f, v \leq g, \ \alpha \leq \mathcal{T}(u) \, , \, \beta \leq \mathcal{T}(v)\} \quad \leq \\ &\bigvee \{w \in L^X \mid w \leq f \otimes g, \ \alpha \otimes \beta \leq \mathcal{T}(w)\} \quad = \quad \mathcal{I}(f \otimes g, \alpha \otimes \beta) \, ; \end{aligned}$$

hence the L–fuzzy interior operator corresponding to $\mathcal{T}$ satisfies $(I2)$.
■

Definition 8.1.5 An L–fuzzy interior operator $\mathcal{I} : L^X \times L \to L^X$ is called *enriched* if it satisfies the additional axiom

$$(IE) \qquad \mathcal{I}(f * \lambda, \alpha) \quad \geq \quad \mathcal{I}(f, \alpha) * \lambda \qquad \forall f \in L^X, \quad \forall \lambda \in L, \quad \forall \alpha \in L \, .$$

■

Proposition 8.1.6 *Let $\mathcal{T}$ be an L-fuzzy topology and $\mathcal{I}$ be the corresponding L–fuzzy interior operator. Then $\mathcal{T}$ is enriched iff $\mathcal{I}$ is enriched.*

Proof. Assume that $\mathcal{I}$ is enriched and let $\mathcal{T}(u) \geq \alpha$. Then $\mathcal{I}(u, \alpha) = u$ and hence

$$\mathcal{I}(u * \lambda, \alpha) \quad \geq \quad \mathcal{I}(u, \alpha) * \lambda \quad = \quad u * \lambda \quad ,$$

i.e. $\mathcal{T}(u * \lambda) \geq \alpha$. Conversely, assume that $\mathcal{T}$ is enriched. Then

$$\begin{aligned} \mathcal{I}(f * \lambda, \alpha) \quad &= \quad \bigvee \{u \in L^X \mid u \leq f * \lambda \, , \, \mathcal{T}(u) \geq \alpha\} \\ &\geq \quad \bigvee \{v * \lambda \mid v \in L^X \, , \, v \leq f \, , \, \mathcal{T}(v) \geq \alpha\} \\ &= \quad \lambda * \bigvee \{v \mid v \in L^X \, , \, v \leq f \, , \, \mathcal{T}(v) \geq \alpha\} \quad = \quad \lambda * \mathcal{I}(f, \alpha) \, . \end{aligned}$$

■

Proposition 8.1.7 *$\mathcal{I}$ is enriched iff $\mathcal{I}$ is compatible with the natural equality on L^X, i.e. if*

$$[\![f,g]\!] \quad \leq \quad [\![\mathcal{I}(f,\alpha),\mathcal{I}(g,\alpha)]\!] \qquad \forall\, f,g \in L^X, \quad \forall\, \alpha \in L \quad .$$

Proof. If $\mathcal{I}$ is enriched, then because of $[\![f,g]\!] * f \leq g$ we get

$$[\![f,g]\!] * \mathcal{I}(f,\alpha) \quad \leq \quad \mathcal{I}(f * [\![f,g]\!],\alpha) \quad \leq \quad \mathcal{I}(g,\alpha) \quad .$$

In a similar way, we conclude that $[\![f,g]\!] * \mathcal{I}(g,\alpha) \leq \mathcal{I}(f,\alpha)$; and hence

$$[\![f,g]\!] \quad \leq \quad [\![\mathcal{I}(f,\alpha)\,,\,\mathcal{I}(g,\alpha)]\!] \quad .$$

Conversely, if $\mathcal{I}$ is compatible with the natural equality, then

$$\lambda \quad \leq \quad [\![f,\lambda * f]\!] \quad \leq \quad [\![\mathcal{I}(f,\alpha),\,\mathcal{I}(\lambda * f,\alpha)]\!] \quad = \quad \mathcal{I}(f,\alpha) \to \mathcal{I}(\lambda * f,\alpha) \quad ;$$

hence $\mathcal{I}(\lambda * f,\alpha) \geq \mathcal{I}(f,\alpha) * \lambda$ follows.

■

Definition 8.1.8 (L–fuzzy neighborhood system)
Let X be a set. A map $\mathcal{N} : X \times L^X \times L \to L$ is called *an L–fuzzy neighborhood system* iff $\mathcal{N}$ satisfies the following axioms:

($N0$) $\quad \mathcal{N}(x,1_X,\alpha) \quad = \quad \top \quad .$

($N1$) $\quad u \leq v\,,\ \beta \leq \alpha \quad \Longrightarrow \quad \mathcal{N}(x,u,\alpha) \leq \mathcal{N}(x,v,\beta) \quad .$

($N2$) $\quad \mathcal{N}(x,u_1,\alpha) \otimes \mathcal{N}(x,u_2,\beta) \quad \leq \quad \mathcal{N}(x,u_1 \otimes u_2,\alpha \otimes \beta) \quad .$

($N3$) $\quad \mathcal{N}(x,u,\alpha) \quad \leq \quad u(x) \quad .$

($N4$) $\quad \mathcal{N}(x,u,\alpha) \quad \leq \quad \bigvee \{\mathcal{N}(x,v,\alpha) \mid v(y) \leq \mathcal{N}(y,u,\alpha) \quad \forall\, y \in X\} \quad .$

($N5$) $\quad \mathcal{N}(x,u,\bot) \quad = \quad u(x) \quad .$

($N6$) $\quad$ *For any non empty subset $\{\alpha_i \mid i \in I\}$ of L the implication holds:*

$$\begin{aligned} u(x) \quad \leq \quad \mathcal{N}(x,u,\alpha_i) \quad \forall\, i \in I,\ \forall\, x \in X \quad \Longrightarrow \\ \Longrightarrow \quad u(x) \quad \leq \quad \mathcal{N}(x,u,\textstyle\bigvee_{i\in I}\alpha_i) \quad \forall\, x \in X \end{aligned} \quad .$$

■

Proposition 8.1.9 *(a) Every L–fuzzy interior operator $\mathcal{I} : L^X \times L \to L^X$ induces an L–fuzzy neighborhood system $\mathcal{N}_{\mathcal{I}}$ by*

$$\mathcal{N}_{\mathcal{I}}(x,u,\alpha) \quad = \quad [\mathcal{I}(u,\alpha)](x) \quad \forall x \in X \quad .$$

(b) Every L–fuzzy neighborhood system $\mathcal{N} : X \times L^X \times L \to L$ determines an L–fuzzy interior operator $\mathcal{I}_{\mathcal{N}}$ by

$$[\mathcal{I}_{\mathcal{N}}(u,\alpha)](x) \quad = \quad \mathcal{N}(x,u,\alpha)$$

Proof. In order to verify assertion (a) we see immediately that $(I0)$ – $(I3)$ and $(I5)$ imply $(N0)$ – $(N3)$ and $(N5)$. Further we put:

$$v(y) \quad = \quad [\mathcal{I}(u,\alpha)](y) \quad (=\mathcal{N}_{\mathcal{I}}(y,u,\alpha)) \qquad \forall\, y \in X \quad .$$

Referring to $(I4)$ we obtain:

$$\mathcal{N}_{\mathcal{I}}(x,u,\alpha) \quad = \quad [\mathcal{I}(u,\alpha)](x) \quad \leq \quad [\mathcal{I}(v,\alpha)](x) \quad = \quad \mathcal{N}_{\mathcal{I}}(x,v,\alpha) \quad ;$$

hence Axiom $(N4)$ follows. Finally, $(N6)$ is an immediate consequence of $(I6)$ and $(N3)$. Now we assume that $\mathcal{N}$ is an L–fuzzy neighborhood system and show that $\mathcal{I}_{\mathcal{N}}$ is an L–fuzzy interior operator. The axioms $(I0)$ – $(I3)$ and $(I5)$, $(I6)$ follow immediately from properties $(N0)$ – $(N3)$ and $(N5)$, $(N6)$. Further we put $w(y) = \mathcal{N}(y,u,\alpha) \quad \forall y \in X$. Then we infer from $(N1)$ and $(N4)$: $\mathcal{N}(x,u,\alpha) \leq \mathcal{N}(x,w,\alpha)$, hence the inequality $\mathcal{I}_{\mathcal{N}}(u,\alpha) \leq \mathcal{I}_{\mathcal{N}}(\mathcal{I}_{\mathcal{N}}(u,\alpha),\alpha)$ follows; i.e. Axiom $(I4)$ holds.
■

We conclude from Proposition 8.1.9 that L–fuzzy interior operators and L–fuzzy neighborhood systems are equivalent concepts. In particular, an L–fuzzy neighborhood system $\mathcal{N}$ is called *enriched* iff the corresponding L–fuzzy interior operator is enriched, – i.e. $\mathcal{N}$ satisfies the additional axiom

$$(NE) \qquad \mathcal{N}(x,u,\alpha) * \lambda \leq \mathcal{N}(x,u*\lambda,\alpha) \qquad \forall x \in X, \forall u \in L^X, \forall \alpha \in L \quad .$$

Finally, it is obvious that the concept of L–fuzzy neighborhood systems offers the possibility to define "continuity" at each point $x \in X$. Details of this approach are left to the reader.

We close this subsection with a remark dealing with the impact of the idempotency of $\otimes$ on the L–fuzzy interior (resp. neighborhood) axioms.

Remark 8.1.10 (Idempontency of $\otimes$)
(a) We assume that $\otimes$ fulfills Axiom (X), - i.e. $\otimes$ is idempotent, but not necessarily associative. Then every L–fuzzy interior operator $\mathcal{I}$ determines an L–interior operator $\mathcal{K}_\alpha$ for each $\alpha \in L$ as follows:

$$\mathcal{K}_\alpha(f) \quad = \quad \mathcal{I}(f,\alpha) \qquad \forall\, f \in L^X.$$

Moreover, every L–fuzzy neighborhood system $\mathcal{N}$ induces an L–neighborhood system for all $\alpha \in L$ by

$$\mathcal{U}_\alpha \quad = \quad (\mu_p^{(\alpha)})_{p\in X} \qquad \text{where} \quad \mu_p^{(\alpha)} : L^X \longmapsto L \quad , \quad \mu_p^{(\alpha)}(f) \;=\; \mathcal{N}(p,f,\alpha)\ .$$

(b) Now we assume that $\otimes$ is not only idempotent, but also $\top$ acts as a unity w.r.t. $\otimes$; then we obtain

$$\alpha \wedge \beta \quad = \quad (\alpha \wedge \beta) \otimes (\alpha \wedge \beta) \quad \leq \quad \alpha \otimes \beta \quad \leq \quad \alpha \wedge \beta \quad ,$$

i.e. $\otimes$ and $\wedge$ coincide. Under the hypothesis of $(I1)$ (resp. $(N1)$) we can make the following observations:

- Axiom $(I2)$ is equivalent to

$$(I2^*) \qquad \mathcal{I}(u,\alpha) \wedge \mathcal{I}(v,\alpha) \quad \leq \quad \mathcal{I}(u \wedge v,\alpha) \qquad \forall \alpha \in L \quad ;$$

- Axiom $(N2)$ is equivalent to

$$(N2^*) \qquad \mathcal{N}(x,u,\alpha) \wedge \mathcal{N}(x,v,\alpha) \quad \leq \quad \mathcal{N}(x,u \wedge v,\alpha) \qquad \forall \alpha \in L \quad .$$

Further, if $\mathcal{K}$ (resp. $\mathcal{U} = (\mu_p)_{p\in X}$) is an $L-$interior operator (resp. $L-$neighborhood system), then $\mathcal{I}_{\mathcal{K}}$ (resp. $\mathcal{N}_{\mathcal{U}}$) defined by $\mathcal{I}_{\mathcal{K}}(f,\alpha) = \mathcal{K}(f)$ (resp. $\mathcal{N}_{\mathcal{U}}(x,f,\alpha) = \mu_x(f)$) $\forall \alpha \in L$, is an $L-$fuzzy interior operator (resp. $L-$fuzzy neighborhood system).

■

8.2 Convergence in L–fuzzy topological spaces

Definition 8.2.1 (L–Fuzzy filter)
A map $\mathcal{F} : L^X \times L \to L$ is called an *L–fuzzy filter on X* iff $\mathcal{F}$ is provided with the following properties:

$$\text{(FF0)} \quad \mathcal{F}(1_X,\alpha) \quad = \quad \top \quad .$$

$$\text{(FF1)} \quad u \leq v,\, \beta \leq \alpha \quad \Longrightarrow \quad \mathcal{F}(u,\alpha) \;\leq\; \mathcal{F}(v,\beta) \quad .$$

$$\text{(FF2)} \quad \mathcal{F}(u,\alpha) \otimes \mathcal{F}(v,\beta) \quad \leq \quad \mathcal{F}(u \otimes v, \alpha \otimes \beta) \quad .$$

$$\text{(FF3)} \quad \mathcal{F}(1_\varnothing,\alpha) \quad = \quad \bot \quad .$$

■

Remark 8.2.2
(a) Let $\mathcal{F}$ be a L–fuzzy filter on X. Then the map $\nu : L^X \longmapsto L$ defined by $\nu(f) = \mathcal{F}(f,\bot)$ is a L–filter on X.
(b) Every L–filter ν on X induces an L–fuzzy filter $\mathcal{F}_\nu$ by $\mathcal{F}_\nu(f,\alpha) = \nu(f)$ for all $\alpha \in L$.
(c) Let $\mathcal{N}$ be a L–fuzzy neighborhood system on X. Then the map $\mathcal{N}_x : L^X \times L \longmapsto L$ defined by $\mathcal{N}_x(u,\alpha) = \mathcal{N}(x,u,\alpha)$ is a L–fuzzy filter on X. In particular, $\mathcal{N}_x$ is called the *L–fuzzy neighborhood filter at* x.

■

On the set $\mathfrak{F}\mathfrak{F}_L(X)$ of all L–fuzzy filters on X we introduce a partial ordering $\preceq$ as follows:

$$\mathcal{F}_1 \preceq \mathcal{F}_2 \quad \Longleftrightarrow \quad \mathcal{F}_1(u,\alpha) \;\leq\; \mathcal{F}_2(u,\alpha) \quad \forall\, \alpha \in L\,,\; \forall\, u \in L^X \quad .$$

Referring to Remark 8.2.2(a) and (b) it is easy to see that every L–fuzzy filter is dominated by an L–filter in the sense of $\preceq$. Moreover, by virtue of

Zorn's Lemma $(\mathfrak{F}\mathfrak{F}_L(X), \preceq)$ has maximal elements. It is not difficult to show that every maximal L–fuzzy filter can be identified with an L–ultrafilter – i.e. for every maximal L–fuzzy filter $\mathcal{U}$ there exists an L–ultrafilter μ such that $\mathcal{U}(f,\alpha) = \mu(f) \quad \forall f \in L^X$.

Definition 8.2.3 (Limit point) Let $\mathcal{N}$ be an L–fuzzy neighborhood system and $\mathcal{F}$ be an L–fuzzy filter on X. An element x_0 is called a *limit point* of $\mathcal{F}$ iff $\mathcal{N}_{x_0} \preceq \mathcal{F}$ where $\mathcal{N}_{x_0}$ is the L–fuzzy neighborhood filter at x_0.

■

In the sense of the previous definition the convergence theory of L–topological spaces can be extended to L–fuzzy topological spaces. Details of this theory are left to the reader.

9 Local convergence theory for weakly exstensional L–fuzzy topological spaces

The aim of this section is to investigate the role of idempotent elements in the convergence theory for L–fuzzy topological spaces[17]. The key idea is to understand L–fuzzy topologies $\mathcal{T} : L^X \longmapsto L$ as *generalized characteristic functions.* Hence this approach requires that $\mathcal{T}$ satisfies a certain type of extensionality condition. In order to give a preparation and motivation for the general setting we start with a brief overview on the extensionality and classification of L–valued maps defined on L^X.

9.1 Extensionality and singletons in L^X

Lemma 9.1.1 *Let $(L,\leq)$ be a complete Heyting algebra, and let us consider the case $\otimes = * = \wedge$. Further let $\mathcal{T} : L^X \longmapsto L$ be a map. Then the following assertions are equivalent*

(i) $\mathcal{T}$ *is extensional w.r.t. the natural equality* $[\![\ ,\ ]\!]$ *on* L^X.

(ii) $\alpha \leq \bigwedge_{f \in L^X} (\mathcal{T}(f) \to \mathcal{T}(\alpha \wedge f)) \wedge (\mathcal{T}(\alpha \wedge f) \to \mathcal{T}(f)) \quad \forall (\alpha, f) \in L \times L^X$.

Proof. In view of Section 4 the implication (i) $\Longrightarrow$ (ii) is obvious. The converse implication (ii) $\Longrightarrow$ (i) follows from the subsequent relation

$$f \wedge [\![f, g]\!] = g \wedge [\![f, g]\!] \qquad \forall f, g \in L^X .$$

■

Let $(L, \leq, *)$ be a GL–monoid – i.e. a commutative strictly two–sided quantale provided with the additional property (cf. Subsection 1.1):

[17]With regard to general aspects the reader is referred to Section 8.

$$\forall (\alpha, \beta) \in L \text{ with } \beta \leq \alpha \ \exists \gamma \in L \text{ s.t. } \beta = \alpha * \gamma \ . \qquad (\textit{Divisibility})$$

The divisibility implies that for all $\alpha \in L$ and for all idempotent elements $\iota \in L$ (w.r.t. $*$) the relation $\iota * \alpha = \iota \wedge \alpha$ holds (cf. 1.3).

An extentional map $s : L^X \longmapsto L$ is called a *singleton in* L^X iff s is provided with the additional property

$$s(f) * (\mathbb{E}(s) \rightarrow s(g)) \quad \leq \quad [\![f , g]\!] \qquad (\textit{Singleton Condition})$$

where $\mathbb{E}(s) := \bigvee\{s(x) \mid x \in X\}$ is interpreted as the *extent* to which s *exists*. The type of the following singletons is of special interest:

$$s_{(f,\alpha)}(g) \quad := \quad \alpha * [\![f , g]\!] \qquad \forall g \in L^X \ .$$

It is not difficult to show that $s_{(f,\alpha)}$ is a singleton in L^X. In particular, we view $s_{(f,\alpha)}$ as the *restriction* of f to the *domain* represented by α. In this sense we understand $s_{(f,\alpha)}$ as a *local element of* L^X induced by (f, α), and we denote $s_{(f,\alpha)}$ by $f \upharpoonright \alpha$. Obviously, $f \upharpoonright \alpha$ is not uniquely determined by the pair (f, α) as the following lemma demonstrates:

Lemma 9.1.2 *Let $*$ be distributive over arbitary meets in $(L, \leq)$. Then for all $(f, \alpha), (g, \beta) \in L^X \times L$ the following equivalence holds*

$$f \upharpoonright \alpha \ = \ g \upharpoonright \beta \quad \Longleftrightarrow \quad \alpha = \beta \, , \ \overline{\alpha} \wedge f \ = \overline{\alpha} \wedge g$$

where $\overline{\alpha}$ is the idempotent hull of α w.r.t $*$ (cf. 1.3). In particular, for every local element $f \upharpoonright \alpha$ of L^X there exists a unique (ordinary) element $f_0 \in L^X$ provided with the following conditions

$$f_0(x) \ \leq \ \overline{\alpha} \quad \forall x \in X \qquad , \qquad f \upharpoonright \alpha \ = \ f_0 \upharpoonright \alpha \qquad .$$

Proof. Let us consider the situation $f \upharpoonright \alpha \ = \ g \upharpoonright \beta$; then for all $h \in L^X$ the relation

$$\alpha * [\![f , h]\!] \quad = \quad \beta * [\![g , h]\!]$$

holds. Hence $\alpha = \beta$ and $\alpha = \alpha * [\![f , g]\!]$ follows. Since $*$ is distributive over arbitrary meets, we obtain that

$$\bigwedge_{n \in \mathbb{N}} \underbrace{[\![f , g]\!] * \ldots * [\![f , g]\!]}_{n \text{ times}}$$

is idempotent; hence α is smaller than or equal to the idempotent kernel of $[\![f , g]\!]$ – or the other way around, the idempontent hull $\overline{\alpha}$ of α is smaller than or equal to $[\![f , g]\!]$. Now we invoke the definition of $[\![\ , \ ]\!]$ and obtain $\overline{\alpha} * f \ \leq \ g$, $\overline{\alpha} * g \ \leq \ f$; hence $\overline{\alpha} \wedge f \ = \ \overline{\alpha} \wedge g$ follows from the idempotency of $\overline{\alpha}$. On the other hand, if $\overline{\alpha} \wedge f \ = \ \overline{\alpha} \wedge g$, then it is not difficult to show that $\overline{\alpha} \ \leq \ [\![f , g]\!]$; hence the divisibility of $(L, \leq, *)$ implies $\alpha \ \leq \ \alpha \ * \ [\![f , g]\!]$; and therewith the first assertion is verified. In order to establish the second assertion we put $f_0 \ = \overline{\alpha} \wedge f$ and observe $\overline{\alpha} \ \leq \ [\![f_0 , f]\!]$. Further details of this proof are left to the reader.

∎

Remark 9.1.3 (Classification of L–valued maps in L^X) [18]
Let $\mathcal{T} : L^X \longmapsto L$ be an extensional map and $\mathfrak{S}(\mathcal{T})$ be the set of all singletons of type $s_{(f,\alpha)}$ with $s_{(f,\alpha)}(h) \leq \mathcal{T}(h) \quad \forall\, h \in L^X$. Then the extensionality of $\mathcal{T}$ implies

$$\mathcal{T}(h) \quad = \quad \bigvee\{\, s_{(f,\alpha)}(h) \mid s_{(f,\alpha)} \in \mathfrak{S}(\mathcal{T})\,\} \qquad .$$

Now we apply Lemma 9.1.2 and obtain that $\mathcal{T}$ can be identified with the set of all pairs $(f,\alpha) \in L^X \times L$ with $\alpha \leq \mathcal{T}(f) \quad , \quad f \leq \overline{\alpha}$. In particular, the value α is interpreted as the *domain* of the local element (f,α) (cf. Lemma 9.2.1). Finally, the set

$$\sigma_\alpha \quad = \quad \{f \in L^X \mid \alpha \leq \mathcal{T}(f) \,,\, f \leq \overline{\alpha}\}$$

is called the *α–section of* $\mathcal{T}$.

■

In the following considerations of Section 9 we restrict the structure of the underlying lattice L and make the following

General Assumptions. Let $(L, \leq, \otimes, *)$ always be a quadruple satisfying the axioms (I) – (VII) . We assume the divisibility of $(L, \leq, *)$ and the existence of idempotent hulls (i.e. for every element $\alpha \in L$ there exists (w.r.t. $*$) a smallest idempotent element $\overline{\alpha}$ which is greater than or equal to α). Moreover, we require the following compatibility axioms between the formation of idempotent hulls, the meet operation $\wedge$ and the given, binary operation $\otimes$:

(XIII) $\quad \overline{\alpha \wedge \overline{\beta}} \quad = \quad \overline{\alpha} \wedge \overline{\beta}$

(XIV) $\quad \overline{\alpha} \rightarrow \perp$ is idempotent w.r.t. $*$ for all $\alpha \in L$.

(XV) $\quad (\alpha \wedge \overline{\beta}) \otimes (\beta \wedge \overline{\alpha}) \quad = \quad (\alpha \otimes \beta) \wedge \overline{\alpha} \wedge \overline{\beta}$

(XVI) $\quad \overline{\overline{\alpha} \otimes \overline{\beta}} \quad = \quad \overline{\alpha \otimes \beta}$.

(XVII) $\quad \alpha \otimes \beta \quad \leq \quad (\overline{\alpha} \wedge \overline{\beta}) \vee (\overline{\beta} \rightarrow \perp)$

In the case of $\otimes = * = \wedge$ every complete Heyting algebra $(L, \leq)$ fulfills the above axioms (cf. Lemma 1.2.1). Moreover, if $(L, \leq, *)$ is a complete MV–algebra with square roots, then we conclude from Subsection 1.3 that either $(L, \leq, \wedge, *)$ or $(L, \leq, \circledast, *)$ satisfies the axioms (I) – (VII), (X), (XIII) – (XVII)

[18] Further details on the classification of extensional lattice–valued maps can be found in [40]

where $\circledast$ denotes the monoidal mean operator (cf. Remark 1.2.6 and Corollary 1.2.8). In particular, in the later case the axioms (XIII), (XV) – (XVII) follow immediately from Proposition 1.3.1 and Corollary 1.3.2.

Lemma 9.1.4 *Let ι be an idempotent element of L w.r.t. $*$. Then the relation*

$$\text{(XVIII)} \qquad \iota \wedge (\alpha \otimes \beta) \;=\; (\iota \wedge \alpha) \otimes (\iota \wedge \beta)$$

holds for all $\alpha, \beta \in L$.

Proof. Let ι be idempotent w.r.t $*$; then we deduce from (II), (VII), (X) and the divisibility axiom:

$$\begin{aligned} \iota \wedge (\alpha \otimes \beta) &= (\iota \otimes \iota) * (\alpha \otimes \beta) \leq (\iota * \alpha) \otimes (\iota * \beta) \\ &= (\iota \wedge \alpha) \otimes (\iota \wedge \beta) \leq (\alpha \otimes \beta) \wedge \iota \quad . \end{aligned}$$

Hence the assertion follows.
■

As a preparation for the concept of local L–interior operators we first specify a representation theory for L–fuzzy topologies.

9.2 A representation theory for weakly extensional L–fuzzy topologies

Let $\mathcal{T} : L^X \longmapsto L$ be a map. We fix $\alpha \in L$; in accordance with Remark 9.1.3 the so–called α–section of $\mathcal{T}$ is given by

$$\tau_\alpha \quad = \quad \{ f \in L^X \mid \alpha \leq \mathcal{T}(f), \quad f(x) \leq \overline{\alpha} \quad \forall x \in X \} \quad .$$

Further, $\mathcal{T}$ is called *weakly exentional* iff $\mathcal{T}$ satisfies the following condition

$$\left.\begin{array}{l} \text{The inequality} \\ \iota \;\leq\; \bigwedge\limits_{f \in L^X} (\mathcal{T}(f) \rightarrow \mathcal{T}(\iota \wedge f)) \wedge (\mathcal{T}(\iota \wedge f) \rightarrow \mathcal{T}(f)) \\ \text{holds for all idempotent elements } \iota \in L \quad (\text{w.r.t.} *) \, . \end{array}\right\} \qquad \text{(WE)}$$

If $\bot$ and $\top$ are the only idempotent elements in L (w.r.t. $*$), then the condition (WE) is *redundant* – i.e. every map $\mathcal{T}$ is weakly extensional. In this context α–sections and α–cuts of $\mathcal{T}$ coincide for all $\alpha \in L \setminus \{\bot\}$. On the other hand, if every element of L is idempotent (i.e. $\otimes = * = \wedge$), then weak extensionality and extensionality are equivalent concepts (cf. Lemma 9.1.1). In particular, $\bigcup\limits_{\alpha \in L} \tau_\alpha$ is the support of the "subpresheaf" corresponding to $\mathcal{T}$ (cf. Remark 9.1.3, Example 6.6 in [40]).

Lemma 9.2.1 *Let $\mathcal{T} : L^X \longmapsto L$ be a weakly extensional map. Then the corresponding system $(\tau_\alpha)_{\alpha \in L}$ of α–sections is provided with the following properties*

$$\text{(i)} \quad \alpha, \beta \in L \text{ with } \beta \leq \alpha,\ f \in \tau_\alpha \quad \Longrightarrow \quad f \wedge \overline{\beta} \in \tau_\beta \qquad \text{(WR)}$$

$$\text{(ii)} \quad \left.\begin{array}{l} \text{If } h \in L^X,\ \{\alpha_i \,|\, i \in I\} \subseteq L \text{ with } h \wedge \overline{\alpha}_i \in \tau_{\alpha_i}\ \forall i \in I, \\ \text{then } h \wedge \big(\overline{\bigvee_{i \in I} \alpha_i}\big) \in \tau_{\bigvee_{i \in I} \alpha_i} \end{array}\right\} \qquad \text{(WC)}$$

$$\text{(iii)} \quad \mathcal{T}(f) \;=\; \bigvee \{\alpha \in L \,|\, f \wedge \overline{\alpha} \in \tau_\alpha\}$$

Proof. In the following considerations we frequently use the fact

$$\alpha * \overline{\alpha} \;=\; \alpha \wedge \overline{\alpha} \;=\; \alpha \quad .$$

Further we observe that the equivalence

$$h \wedge \overline{\alpha} \in \tau_\alpha \iff \alpha \leq \mathcal{T}(h) \quad \forall h \in L^X$$

follows from (WE). Therefore (i) and (ii) are a direct consequence of the definition of α–sections and the weak extensionality of $\mathcal{T}$. In order to verify (iii) we first infer from (WC) :

$$\bigvee \{\alpha \in L \,|\, f \wedge \overline{\alpha} \in \tau_\alpha\} \;\leq\; \mathcal{T}(f) \quad ;$$

on the other hand $f \wedge \overline{\mathcal{T}(f)} \in \tau_{\mathcal{T}(f)}$ holds; hence (iii) is established.
■

Comment. The condition (WR) describes the operation of the restiction map from the α–section to the β–section. In contrast to (WR) the condition (WC) can be understood as a kind of continuity condition.
■

Lemma 9.2.2 *Let $(\tau_\alpha)_{\alpha \in L}$ be a system of subsets $\tau_\alpha \subseteq \{f \in L^X | f \leq \overline{\alpha}\}$ satisfying (WR) and (WC). Then the map $\hat{\mathcal{T}} : L^X \longmapsto L$ defined by*

$$\hat{\mathcal{T}}(h) \;=\; \bigvee \{\alpha \in L \,|\, \overline{\alpha} \wedge h \in \tau_\alpha\} \qquad \forall h \in L^X$$

is weakly extensional, and for every $\alpha \in L$ the α–section of $\hat{\mathcal{T}}$ coincides with τ_α.

Proof. An immediate application of (WC) leads to the relation

$$(\overline{\hat{\mathcal{T}}(h)}) \wedge h \in \tau_{\hat{\mathcal{T}}(h)} \qquad \forall h \in L^X \qquad (\spadesuit)$$

Further let ι be an idempotent element of L (w.r.t. $*$).
(a) Referring to (WR) and (XIII) we obtain:

$$\overline{\iota \wedge \hat{\mathcal{T}}(h)} \wedge (\iota \wedge h) \;=\; (\iota \wedge \overline{\hat{\mathcal{T}}(h)}) \wedge h \in \tau_{\iota \wedge \hat{\mathcal{T}}(h)} \quad ;$$

hence the inequality $\iota \wedge \hat{\mathcal{T}}(h) \leq \hat{\mathcal{T}}(\iota \wedge h)$ follows from the definition of $\hat{\mathcal{T}}$.
(b) Referring again to (WR) and (XIII) we make the observation:

$$\overline{(\iota \wedge \hat{\mathcal{T}}(\iota \wedge h))} \wedge h \quad = \quad (\overline{\hat{\mathcal{T}}(\iota \wedge h)} \wedge h \wedge \iota \in \tau_{\iota \wedge \hat{\mathcal{T}}(\iota \wedge h)} \qquad ;$$

hence we obtain $\iota \wedge \hat{\mathcal{T}}(\iota \wedge h) \leq \hat{\mathcal{T}}(h)$.
(c) From (a) and (b) we conclude that $\hat{\mathcal{T}}$ is weakly extensional. Moreover, by definition of $\hat{\mathcal{T}}$ the subset τ_α is contained in the α–section corresponding to $\hat{\mathcal{T}}$. On the other hand, if f is an element of an α–section corresponding to $\mathcal{T}$, then we infer from (WR) and (♠):

$$f \quad = \quad \overline{\alpha} \wedge f \quad = \quad \overline{\alpha} \wedge (\overline{\hat{\mathcal{T}}(f)} \wedge f) \in \tau_\alpha \qquad ;$$

hence every element of an α–section is also an element of τ_α. Therewith the second part of the assertion is established.
■

From Lemma 9.2.1 and Lemma 9.2.2 we conclude that weakly extensional maps $\mathcal{T} : L^X \longmapsto L$ and systems $(\tau_\alpha)_{\alpha \in L}$ of subsets τ_α of $\{f \in L^X \mid f \leq \overline{\alpha}\}$ provided with (WR) and (WC) come to the same thing.

Proposition 9.2.3 (Weakly extensional, L–fuzzy topologies)
Let $\mathcal{T} : L^X \longmapsto L$ be a weakly extensional map and $(\tau_\alpha)_{\alpha \in L}$ be the system of α–sections corresponding to $\mathcal{T}$.

(a) $\mathcal{T}$ *satisfies* $(\mathcal{O}1)$ *iff*

$(\mathcal{O}1^*)$ $\qquad 1_X \in \tau_\top \qquad .$

(b) $\mathcal{T}$ *satisfies* $(\mathcal{O}2)$ *iff* $(\tau_\alpha)_{\alpha \in L}$ *fulfills the condition*

$(\mathcal{O}2^*)$ $\qquad h \wedge \overline{\alpha} \in \tau_\alpha\,, \quad k \wedge \overline{\beta} \in \tau_\beta \quad \Longrightarrow \quad (\overline{\alpha \otimes \beta}) \wedge (h \otimes k) \in \tau_{\alpha \otimes \beta} \qquad .$

(c) $\mathcal{T}$ *satisfies* $(\mathcal{O}3)$ *iff* $(\tau_\alpha)_{\alpha \in L}$ *fulfills the condition*

$(\mathcal{O}3^*)$ $\qquad \{g_i \mid i \in I\} \subseteq \tau_\alpha \quad \Longrightarrow \quad \bigvee_{i \in I} g_i \in \tau_\alpha \qquad .$

Proof. Referring to 9.2.1 and 9.2.2 it is easy to verify (a) – (c).
■

Lemma 9.2.4 *Let $\mathcal{T}$ be an L–fuzzy topology on X and $\mathcal{I}_{\mathcal{T}}$ be the L–fuzzy interior corresponding to $\mathcal{T}$. Then the following assertions are equivalent*

(i) $\mathcal{T}$ is weakly extensional.

(ii) For all $\alpha \in L$ the equivalence

$$h = \mathcal{I}_{\mathcal{T}}(h,\alpha) \iff h \wedge \overline{\alpha} = \mathcal{I}_{\mathcal{T}}(h \wedge \overline{\alpha}, \alpha)$$

holds.

Proof. By the definition of $\mathcal{I}_{\mathcal{T}}$ the subsequent relations hold

$$\begin{array}{ccccccc} h & = & \mathcal{I}_{\mathcal{T}}(h,\alpha) & \iff & \alpha & \leq & \mathcal{T}(h) \\ h \wedge \overline{\alpha} & = & \mathcal{I}_{\mathcal{T}}(h \wedge \overline{\alpha}, \alpha) & \iff & \alpha & \leq & \mathcal{T}(\overline{\alpha} \wedge h) \end{array} \quad ;$$

hence (i) implies (ii).
In order to verify (ii)$\Longrightarrow$(i) we proceed as follows: Referring to Lemma 9.2.1 it is sufficient to verify (WR) and (WC). Let us assume

$$\alpha \leq \mathcal{T}(f) \;, \quad f \leq \overline{\alpha} \;, \quad \beta \leq \alpha \quad ;$$

then the relations

$$\mathcal{I}_{\mathcal{T}}(f,\alpha) = f \;, \quad \mathcal{I}_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}(f,\alpha),\beta) = \mathcal{I}_{\mathcal{T}}(f,\alpha) \quad .$$

follow from the definition of $\mathcal{I}_{\mathcal{T}}$ and $(I1), (I3)$ and $(I4)$. Now we apply (ii):

$$\begin{array}{rcl} \overline{\beta} \wedge f & = & \mathcal{I}_{\mathcal{T}}(f,\alpha) \wedge \overline{\beta} \; = \; \mathcal{I}_{\mathcal{T}}(\overline{\beta} \wedge \mathcal{I}_{\mathcal{T}}(f,\alpha), \beta) \\ & \leq & \mathcal{I}_{\mathcal{T}}(\overline{\beta} \wedge f, \beta) \end{array}$$

–i.e. $\beta \leq \mathcal{T}(\overline{\beta} \wedge f)$. Therewith (WR) is verified.
Let us now assume

$$\overline{\alpha}_i \wedge h = \mathcal{I}_{\mathcal{T}}(\overline{\alpha}_i \wedge h, \alpha_i) \quad \forall i \in I \quad .$$

We invoke (ii) and $(I6)$ and obtain the following relations

$$h = \mathcal{I}_{\mathcal{T}}(h,\alpha_i) \;\; \forall i \in I \;, \quad h = \mathcal{I}_{\mathcal{T}}(h, \bigvee_{i\in I} \alpha_i)$$

$$\overline{(\bigvee_{i\in I} \alpha_i)} \wedge h = \mathcal{I}_{\mathcal{T}}(\overline{(\bigvee_{i\in I} \alpha_i)} \wedge h, \bigvee_{i\in I} \alpha_i) \quad ;$$

hence (WC) is established.
■

9.3 Local L–interior operators

Referring to the subsection 9.1 we understand a pair $(f, \alpha) \in L^X \times L$ as a local element of L^X with domain α. Keeping this interpretation in mind we are now going to introduce interior operators as self–mappings of the set of all local elements of L^X.

Definition 9.3.1 (Local L–interior operator)
A map $\mathcal{I}nt : L^X \times L \longmapsto L^X$ is called a *local L–interior operator* on X if and only if $\mathcal{I}nt$ satisfies the following axioms:

(ℑ0) $\quad \mathcal{I}nt(1_X, \top) \; = \; 1_X \; .$

(ℑ1) $\quad \beta \leq \alpha,\ f \leq h \quad \Longrightarrow \quad \overline{\beta} \wedge \mathcal{I}nt(f, \alpha) \; \leq \; \mathcal{I}nt(h, \beta) \; .$

(ℑ2) $\quad \mathcal{I}nt(f, \alpha) \otimes \mathcal{I}nt(h, \beta) \; \leq \; \mathcal{I}nt(f \otimes h\, , \alpha \otimes \beta) \; .$

(ℑ3) $\quad \mathcal{I}nt(f, \alpha) \; \leq \; \overline{\alpha} \wedge f \; .$

(ℑ4) $\quad \mathcal{I}nt(f, \alpha) \; \leq \; \mathcal{I}nt(\mathcal{I}nt(f, \alpha), \alpha) \; .$

(ℑ5) $\quad (\overline{\alpha \otimes \bot} \wedge \mathcal{I}nt(f, \alpha)) \otimes (\overline{\alpha \otimes \bot} \wedge h) \; \leq \; \mathcal{I}nt(f \otimes h\, , \alpha \otimes \bot) \; .$

(ℑ6) $\quad \overline{\alpha}_i \wedge h = \mathcal{I}nt(h, \alpha_i) \;\; \forall\, i \in I \quad \Longrightarrow \overline{\bigvee_{i \in I} \alpha_i} \wedge h = \mathcal{I}nt(h, \bigvee_{i \in I} \alpha_i) \; .$

■

An obvious consequence of (ℑ0),(ℑ1) and (ℑ3) is the property

(ℑ0′) $\quad \mathcal{I}nt(1_X, \alpha) \; = \; \overline{\alpha} \cdot 1_X \; .$

Further we infer from (ℑ1),(ℑ3) and (ℑ4)

(ℑ4′) $\quad h \wedge \overline{\alpha} = \mathcal{I}nt(f, \alpha) \quad \Longrightarrow \quad h \wedge \overline{\alpha} = \mathcal{I}nt(h, \alpha) = \mathcal{I}nt(\overline{\alpha} \wedge h\, , \alpha) \; .$

Proposition 9.3.2 *Let $\mathcal{T} : L^X \longmapsto L$ be a weakly extensional, L–fuzzy topology on X and $(\tau_\alpha)_{\alpha \in L}$ be the corresponding system of α–sections. Then the map $\mathcal{I}nt_{\mathcal{T}} : L^X \times L \longmapsto L^X$ defined by*

$$\mathcal{I}nt_{\mathcal{T}}(f, \alpha) \; = \; \bigvee \{ g \in \tau_\alpha \mid g \leq f \} \qquad f \in L^X, \quad \alpha \in L$$

is a local L-interior operator. In particular, the relation

$$\tau_\alpha \; = \; \{ h \in L^X \mid h \leq \mathcal{I}nt_{\mathcal{T}}(h, \alpha) \} \qquad (\spadesuit)$$

holds for all $\alpha \in L$.

Proof. (a) As an immediate consequence from ($\mathcal{O}3$) (cf. Proposition 9.2.3(c)) we obtain $\mathcal{I}nt_{\mathcal{T}}(f,\alpha) \in \tau_\alpha$; hence relation (♠) holds. In particular, the axiom ($\mathfrak{I}4$) follows.
(b) Because of 9.2.3(a) Axiom ($\mathfrak{I}0$) is valid. Further ($\mathfrak{I}1$) follows from (WR) (cf. 9.2.1(i)) and the definition of $\mathcal{I}nt_{\mathcal{T}}$. Further, ($\mathfrak{I}3$) holds again by definition. Now we invoke Axiom (XVI) and obtain:

$$\mathcal{I}nt_{\mathcal{T}}(f,\alpha) \otimes \mathcal{I}nt_{\mathcal{T}}(h,\beta) \quad \leq \quad \overline{\overline{\alpha} \otimes \overline{\beta}} \quad = \quad \overline{\alpha \otimes \beta} \qquad ;$$

i.e. ($\mathfrak{I}2$) follows from ($\mathcal{O}2$) (cf. 9.2.3(b)). In order to verify ($\mathfrak{I}5$) we proceed as follows: First we put $g = \mathcal{I}nt_{\mathcal{T}}(f,\alpha)$; then $g = \overline{\alpha} \wedge g \in \tau_\alpha$. Since $h \wedge \bot = 1_\varnothing \in \tau_\bot$, Proposition 9.2.3(b) implies $\overline{\alpha \otimes \bot} \wedge (g \otimes h) \in \tau_{\alpha\otimes\bot}$; hence we obtain from Lemma 9.1.4 and the definition of $\mathcal{I}nt_{\mathcal{T}}$:

$$\begin{aligned}(\overline{\alpha\otimes\bot} \wedge \mathcal{I}nt_{\mathcal{T}}(f,\alpha)) \otimes (\overline{\alpha\otimes\bot} \wedge h) \quad &\leq \quad \mathcal{I}nt_{\mathcal{T}}(g\otimes h\,,\,\alpha\otimes\bot) \\ &\leq \quad \mathcal{I}nt_{\mathcal{T}}(f\otimes h\,,\,\alpha\otimes\bot) \quad .\end{aligned}$$

Therewith ($\mathfrak{I}5$) is established.
Finally, it is easy to see that ($\mathfrak{I}6$) is equivalent to (WC).
■

Proposition 9.3.3 *Let $\mathcal{I}nt : L^X \times L \longmapsto L^X$ be a local L–interior operator on X. Then the system $(\tau_\alpha)_{\alpha\in L}$ defined by*

$$\tau_\alpha \quad = \quad \{\, f \in L^X \mid f \leq \mathcal{I}nt(f,\alpha)\,\}$$

is a system of α–sections of a weakly extensional, L–fuzzy topology $\mathcal{T}_{\mathcal{I}nt}$.

Proof. (a) First we show that $(\tau_\alpha)_{\alpha\in L}$ fulfills (WR) and (WC). Let us assume $f \in \tau_\alpha$; then ($\mathfrak{I}1$) implies:

$$\overline{\beta} \wedge f \quad \leq \quad \overline{\beta} \wedge \mathcal{I}nt(f,\alpha) \quad \leq \quad \mathcal{I}nt(f,\beta) \qquad \text{whenever} \quad \beta \leq \alpha \qquad ;$$

i.e. $\overline{\beta} \wedge f \in \tau_\beta$; hence (WR) is verified. Further, we consider $h \in L^X$ and a subset $\{\alpha_i | i \in I\}$ of L with $h \wedge \overline{\alpha_i} \in \tau_{\alpha_i}$ $\forall i \in I$. Then the definition of τ_{α_i} and ($\mathfrak{I}3$) imply

$$h \wedge \overline{\alpha_i} \quad = \quad \mathcal{I}nt(h,\alpha_i) \qquad \forall i \in I \qquad .$$

Now we invoke ($\mathfrak{I}6$) and obtain: $h \wedge \overline{\bigvee_{i\in I} \alpha_i} \in \tau_{\bigvee_{i\in I}\alpha_i}$; hence (WC) follows.
(b) Referring to Proposition 9.2.3 it is sufficient to prove ($\mathcal{O}1^*$) – ($\mathcal{O}3^*$).
Obviously ($\mathfrak{I}0$) implies ($\mathcal{O}1^*$). In order to verify ($\mathcal{O}2^*$) we proceed as follows: Let us assume $\overline{\alpha} \wedge h \in \tau_\alpha$ $\overline{\beta} \wedge k \in \tau_\beta$ – i.e.

$$\overline{\alpha} \wedge h \quad = \quad \mathcal{I}nt(h,\alpha) \quad , \quad \overline{\beta} \wedge k \quad = \quad \mathcal{I}nt(k,\beta) \qquad .$$

Step 1 : From ($\mathfrak{I}1$) we infer:

$$\overline{\alpha} \wedge \overline{\beta} \wedge h \quad \leq \quad \mathcal{I}nt(h,\alpha\wedge\overline{\beta}) \quad , \quad \overline{\alpha} \wedge \overline{\beta} \wedge k \quad \leq \quad \mathcal{I}nt(k,\beta\wedge\overline{\alpha}) \quad .$$

Now we invoke (XV), (XVIII), ($\mathfrak{J}$2) and obtain:

$$\begin{aligned}
\overline{\alpha} \wedge \overline{\beta} \,\wedge\, (h \otimes k) \quad &\leq \quad \mathcal{I}nt(h, \alpha \wedge \overline{\beta}) \,\otimes\, \mathcal{I}nt(k, \beta \wedge \overline{\alpha}) \\
&\leq \quad \mathcal{I}nt((h \otimes k)\,,\, (\alpha \wedge \overline{\beta}) \otimes (\beta \wedge \overline{\alpha})) \\
&= \quad \mathcal{I}nt((h \otimes k)\,,\, (\alpha \otimes \beta) \wedge (\overline{\alpha} \wedge \overline{\beta})) \quad .
\end{aligned}$$

Referring to (X), (XIII), (XV) and (XVI) we note

$$\overline{(\alpha \otimes \beta) \,\wedge\, (\overline{\alpha} \wedge \overline{\beta})} \quad = \quad \overline{((\overline{\alpha \wedge \overline{\beta}}) \otimes (\overline{\beta \wedge \overline{\alpha}})} \quad = \quad \overline{\alpha} \,\wedge\, \overline{\beta} \qquad ;$$

hence the relation

$$\overline{(\alpha \otimes \beta) \wedge \overline{\alpha} \wedge \overline{\beta}} \,\wedge\, (h \otimes k) \quad = \quad \mathcal{I}nt((h \otimes k)\,,\, ((\alpha \otimes \beta) \wedge (\overline{\alpha} \wedge \overline{\beta}))) \qquad .$$

holds.

Step 2 : We apply again ($\mathfrak{J}$1) and obtain from the idempotency of $\overline{\beta} \to \perp$ (cf. Axiom (XIV)):

$$\overline{\alpha} \wedge (\overline{\beta} \to \perp) \,\wedge\, h \quad \leq \quad \mathcal{I}nt(h\,,\, \alpha \wedge (\overline{\beta} \to \perp)) \qquad .$$

Further the axioms (II) and (X) imply $(\alpha \wedge (\overline{\beta} \to \perp)) \otimes \perp \;\leq\; \alpha \wedge (\overline{\beta} \to \perp)$; hence we infer from ($\mathfrak{J}$5) :

$$\begin{aligned}
&\overline{(\alpha \wedge (\overline{\beta} \to \perp)) \otimes \perp} \,\wedge\, (h \otimes k) \quad = \\
&((\overline{(\alpha \wedge (\overline{\beta} \to \perp)) \otimes \perp} \,\wedge\, \overline{\alpha} \wedge (\overline{\beta} \to \perp) \wedge h) \,\otimes\, ((\overline{(\alpha \wedge (\overline{\beta} \to \perp)) \otimes \perp} \,\wedge\, k) \quad \leq \\
&((\overline{(\alpha \wedge (\overline{\beta} \to \perp)) \otimes \perp} \,\wedge\, \mathcal{I}nt(h\,,\, \alpha \wedge (\overline{\beta} \to \perp))) \,\otimes\, ((\overline{(\alpha \wedge (\overline{\beta} \to \perp)) \otimes \perp} \,\wedge\, k \;\leq \\
&\mathcal{I}nt(h \otimes k\,,\, (\alpha \wedge (\overline{\beta} \to \perp)) \otimes \perp)
\end{aligned}$$

Now we apply again (XIV), (XVIII) and observe:

$$(\alpha \otimes \beta) \wedge (\overline{\beta} \to \perp) \quad = \quad (\alpha \wedge (\overline{\beta} \to \perp)) \otimes \perp \qquad ;$$

hence the following inequality holds:

$$\overline{(\alpha \otimes \beta) \wedge (\overline{\beta} \to \perp)} \,\wedge\, (h \otimes k) \quad \leq \quad \mathcal{I}nt(h \otimes k\,,\, (\alpha \otimes \beta) \wedge (\overline{\beta} \to \perp)) \quad .$$

Step 3 : Now we combine the results from Step 1 and Step 2 and apply the axioms ($\mathfrak{J}$3), ($\mathfrak{J}$6) and (XVII) :

$$\begin{aligned}
\overline{\alpha \otimes \beta} \,\wedge\, (h \otimes k) \quad &= \quad \overline{\alpha \otimes \beta} \wedge (\overline{\alpha} \vee \overline{\beta}) \,\wedge\, (h \otimes k) \\
&= \quad \overline{(\alpha \otimes \beta) \wedge ((\overline{\alpha} \wedge \overline{\beta}) \,\vee\, (\overline{\beta} \to \perp))} \,\wedge\, (h \otimes k) \\
&= \quad \mathcal{I}nt((h \otimes k)\,,\, ((\alpha \otimes \beta) \wedge ((\overline{\alpha} \wedge \overline{\beta}) \,\vee\, (\overline{\beta} \to \perp)))) \\
&= \quad \mathcal{I}nt(h \otimes k\,,\, \alpha \otimes \beta) \qquad ;
\end{aligned}$$

i.e. $\overline{\alpha \otimes \beta} \wedge (h \otimes k) \in \tau_{\alpha\otimes\beta}$. Therewith $(\mathcal{O}2^*)$ is verified.
(c) Let $\{g_i \mid i \in I\}$ be a subset of τ_α; then $g_i = \mathcal{I}nt(g_i, \alpha)\ \forall\, i \in I$. Because of $(\mathfrak{I}1)$ the inequality

$$\bigvee_{i\in I} g_i \quad \leq \quad \mathcal{I}nt(\bigvee_{i\in I} g_i\,,\, \alpha)$$

follows – i.e. $\bigvee_{i\in I} g_i \in \tau_\alpha$; hence $(\mathcal{O}3^*)$ is verified.
■

Remark 9.3.4 (Role of the axiom $(\mathfrak{I}5)$)
(a) In the case of $\otimes = \wedge$ the axiom $(\mathfrak{I}5)$ is redundant.
(b) Let us consider the MV–algebra $([0,1], \leq, T_m)$ (cf. Example 1.2.3(b)) and the monoidal mean operator $\circledast$ determined by T_m (cf. Remark 1.2.6). Obviously, $\circledast$ coincides with the arithmetic mean (cf. Example 1.2.3(b)). Then the axiom $(\mathfrak{I}5)$ is equivalent to

$$\mathcal{I}nt(f,\alpha) \circledast k \quad \leq \quad \mathcal{I}nt(f \circledast k\,,\, \frac{\alpha}{2}) \qquad \forall\, \alpha \neq 0 \quad .$$

■

In the subsequent propositions 9.3.5 and 9.3.6 we investigate some aspects of the relationship between L–fuzzy interior operators and local L-interior operators.

Proposition 9.3.5 *Let $\mathcal{I}$ be an L–fuzzy interior operator on X satisfying the additional axiom*

$$(I7) \qquad h = \mathcal{I}(h,\alpha) \iff h \wedge \overline{\alpha} = \mathcal{I}(h \wedge \overline{\alpha}, \alpha) \qquad \forall\, \alpha \in L \quad .$$

Then $\mathcal{I}nt : L^X \times L \longmapsto L^X$ defined by

$$\mathcal{I}nt(f,\alpha) \quad = \quad \overline{\alpha} \wedge \mathcal{I}(f,\alpha) \quad , \quad \alpha \in L$$

is a local L–interior operator on X.

Proof. The axioms $(\mathfrak{I}0)$ and $(\mathfrak{I}1)$ follow immediately from $(I0)$ and $(I1)$. Further we infer from (II), (XVI) and $(I2)$

$$\begin{aligned}(\overline{\alpha} \wedge \mathcal{I}(f,\alpha)) \otimes (\overline{\beta} \wedge \mathcal{I}(g,\beta)) \quad &\leq \quad (\overline{\alpha} \otimes \overline{\beta}) \wedge \big(\mathcal{I}(f,\alpha) \otimes \mathcal{I}(g,\beta)\big) \\ &\leq \quad (\overline{\alpha \otimes \beta}) \wedge \mathcal{I}(f \otimes g, \alpha \otimes \beta) \quad ;\end{aligned}$$

hence $(\mathfrak{I}2)$ is verified. The axiom $(\mathfrak{I}3)$ is a direct consequence from $(I3)$. Because of $(I3)$, $(I4)$ and $(I7)$ we obtain:

$$\begin{aligned}\mathcal{I}nt(f,\alpha) \quad &= \quad \overline{\alpha} \wedge \mathcal{I}(f,\alpha) \quad = \quad \overline{\alpha} \wedge \mathcal{I}(\mathcal{I}(f,\alpha)\,,\,\alpha) \\ &= \quad \overline{\alpha} \wedge \mathcal{I}(\overline{\alpha} \wedge \mathcal{I}(f,\alpha)\,,\,\alpha\,) \quad = \quad \mathcal{I}nt(\mathcal{I}nt(f,\alpha)) \quad ;\end{aligned}$$

hence $(\mathfrak{I}4)$ is also verified. Further the axiom $(\mathfrak{I}5)$ follows immediately from $(I5)$, $(I2)$ and (XVIII). Finally $(\mathfrak{I}6)$ is a direct consequence of $(I6)$ and $(I7)$.
■

Proposition 9.3.6 *Let $\mathcal{I}nt$ be a local L–interior operator. Then the map $\mathcal{I} : L^X \times L \longmapsto L$ defined by*

$$\mathcal{I}(f,\alpha) \quad = \quad (\overline{\alpha} \to \mathcal{I}nt(f,\alpha)) \wedge f$$

is an L–fuzzy interior operator provided with Property $(I7)$.

Proof. We show that $\mathcal{I}$ satisfies $(I0)$ – $(I7)$. The axiom $(I0)$ is obvious. Referring to $(\mathcal{I}1)$ we obtain in the case of $\beta \le \alpha$ and $f \le g$:

$$\begin{aligned} \overline{\beta} * \mathcal{I}(f,\alpha) \quad &= \quad (\overline{\beta} * (\overline{\alpha} \to \mathcal{I}nt(f,\alpha))) \wedge f \\ &\le \quad \overline{\beta} \wedge \mathcal{I}nt(f,\alpha) \quad \le \quad \mathcal{I}nt(g,\beta) \quad ; \end{aligned}$$

hence $(I1)$ follows. The verification of $(I2)$ is divided into three steps:
Step 1: We show:

$$\begin{aligned} &\overline{(\alpha \otimes \beta) \wedge (\overline{\alpha} \wedge \overline{\beta})} * (\mathcal{I}(f,\alpha) \otimes \mathcal{I}(g,\beta)) \quad \le \\ &\qquad \le \quad \mathcal{I}nt((\mathcal{I}(f,\alpha) \otimes \mathcal{I}(g,\beta)), ((\alpha \otimes \beta) \wedge \overline{\alpha} \wedge \overline{\beta})) \quad . \end{aligned}$$

Referring to (X), (XIII), (XV), (XVI) and $(\mathfrak{J}1)$ – $(\mathfrak{J}4)$ we obtain:

$$\begin{aligned} &\overline{(\alpha \otimes \beta) \wedge (\overline{\alpha} \wedge \overline{\beta})} * (\mathcal{I}(f,\alpha) \otimes \mathcal{I}(g,\beta)) \quad = \\ &(\overline{\alpha} \wedge \overline{\beta}) * (\mathcal{I}(f,\alpha) \otimes \mathcal{I}(g,\beta)) \quad \le \\ &(\overline{\beta} \wedge \mathcal{I}nt(\mathcal{I}nt(f,\alpha),\alpha)) \otimes (\overline{\alpha} \wedge \mathcal{I}nt(\mathcal{I}nt(g,\beta),\beta)) \quad \le \\ &\mathcal{I}nt(\mathcal{I}(f,\alpha), \alpha \wedge \overline{\beta}) \otimes \mathcal{I}nt(\mathcal{I}(g,\beta), \beta \wedge \overline{\alpha}) \quad \le \\ &\mathcal{I}nt(\mathcal{I}(f,\alpha) \otimes \mathcal{I}(g,\beta), ((\alpha \wedge \overline{\beta}) \otimes (\beta \wedge \overline{\alpha}))) \quad = \\ &\mathcal{I}nt(\mathcal{I}(f,\alpha) \otimes \mathcal{I}(g,\beta), ((\alpha \otimes \beta) \wedge \overline{\alpha} \wedge \overline{\beta})) \quad . \end{aligned}$$

Step 2: We show

$$\begin{aligned} &(\overline{(\alpha \otimes \beta) \wedge (\overline{\beta} \to \bot)}) * (\mathcal{I}(f,\alpha) \otimes \mathcal{I}(g,\beta)) \quad \le \\ &\qquad \le \quad \mathcal{I}nt(\mathcal{I}(f,\alpha) \otimes \mathcal{I}(g,\beta), ((\alpha \otimes \beta) \wedge (\overline{\beta} \to \bot))) \quad . \end{aligned}$$

Because of (X), (XIV) and (XVIII) the relation

$$\overline{(\alpha \otimes \beta) \wedge (\overline{\beta} \to \bot)} \quad = \quad \overline{(\alpha \wedge (\overline{\beta} \to \bot)) \otimes \bot} \quad \le \quad \overline{\alpha} \wedge (\overline{\beta} \to \bot)$$

holds. Then we infer from $(\mathfrak{J}1)$, $(\mathfrak{J}3)$ – $(\mathfrak{J}5)$:

$$\begin{aligned} &\overline{(\alpha \otimes \beta) \wedge (\overline{\beta} \to \bot)} * (\mathcal{I}(f,\alpha) \otimes \mathcal{I}(g,\beta)) \quad \le \\ &((\overline{(\alpha \wedge (\overline{\beta} \to \bot)) \otimes \bot}) \wedge \mathcal{I}nt(\mathcal{I}nt(f,\alpha)\,\alpha)) \otimes \\ &\qquad \otimes \; ((\overline{(\alpha \wedge (\overline{\beta} \to \bot)) \otimes \bot}) \wedge \mathcal{I}(g,\beta)) \quad \le \\ &((\overline{(\alpha \wedge (\overline{\beta} \to \bot)) \otimes \bot}) \wedge \mathcal{I}nt(\mathcal{I}(f,\alpha), (\alpha \wedge (\overline{\beta} \to \bot)))) \otimes \\ &\qquad \otimes \; ((\overline{(\alpha \wedge (\overline{\beta} \to \bot)) \otimes \bot}) \wedge \mathcal{I}(g,\beta)) \quad \le \\ &\mathcal{I}nt(\mathcal{I}(f,\alpha) \otimes \mathcal{I}(g,\beta), ((\alpha \otimes \beta) \wedge (\overline{\beta} \to \bot)) \quad . \end{aligned}$$

Step 3: Since Axiom (XVII) implies

$$\begin{aligned}\overline{\alpha \otimes \beta} &= \overline{(\alpha \otimes \beta) \wedge ((\overline{\alpha} \wedge \overline{\beta}) \vee (\overline{\beta} \to \bot))} \\ &= (\overline{\alpha} \wedge \overline{\beta}) \vee \overline{(\alpha \wedge (\overline{\beta} \to \bot)) \otimes \bot} \quad ,\end{aligned}$$

we conclude from *Step* 1, *Step* 2, ($\mathfrak{J}1$) and ($\mathfrak{J}6$):

$$\begin{aligned}&\overline{\alpha \otimes \beta} * (\mathcal{I}(f,\alpha) \otimes \mathcal{I}(g,\beta)) \\ &\qquad \leq \mathcal{I}nt(\mathcal{I}(f,\alpha) \otimes \mathcal{I}(g,\beta), \alpha \otimes \beta) \\ &\qquad \leq \mathcal{I}nt(f \otimes g, \alpha \otimes \beta) \quad ;\end{aligned}$$

hence ($I2$) follows.
Further ($I3$) and ($I5$) hold by the definition of $\mathcal{I}$. Now we apply ($\mathfrak{J}1$), ($\mathfrak{J}3$) and ($\mathfrak{J}4$) and obtain:

$$\begin{aligned}\mathcal{I}(f,\alpha) &= (\overline{\alpha} \to \mathcal{I}nt(f,\alpha)) \wedge f \\ &\leq (\overline{\alpha} \to \mathcal{I}nt(\mathcal{I}nt(f,\alpha),\alpha)) \wedge f \\ &\leq (\overline{\alpha} \to \mathcal{I}nt(\mathcal{I}(f,\alpha),\alpha)) \wedge (\overline{\alpha} \to \mathcal{I}nt(\mathcal{I}nt(f,\alpha),\alpha)) \wedge f \\ &= (\overline{\alpha} \to \mathcal{I}nt(\mathcal{I}(f,\alpha),\alpha)) \wedge \mathcal{I}(f,\alpha) \\ &= \mathcal{I}(\mathcal{I}(f,\alpha),\alpha) \quad ;\end{aligned}$$

hence ($I4$) is established.
In order to verify ($I6$) let us consider $f, f^o \in L^X$ with the following property:

$$(\overline{\alpha}_i \to \mathcal{I}nt(f,\alpha_i)) \wedge f = f^o \qquad \forall i \in I \quad .$$

Then we obtain:

$$f^o \leq f \,, \qquad \mathcal{I}nt(f,\alpha_i) = \overline{\alpha}_i \wedge f^o \quad \forall i \in I \quad .$$

Now we apply ($\mathfrak{J}4'$) and ($\mathfrak{J}6$) :

$$\mathcal{I}nt(f^o, \bigvee_{i \in I} \alpha_i) = (\overline{\bigvee_{i \in I} \alpha_i}) \wedge f^o \quad ;$$

i.e.

$$f^o \leq \mathcal{I}(f^o, \bigvee_{i \in I} \alpha_i) \leq f^o \quad ;$$

hence ($I6$) is verified.
Because of the definition of $\mathcal{I}$ we infer from ($\mathfrak{J}3$) and ($\mathfrak{J}4$) :

$$\begin{aligned}&\mathcal{I}(f,\alpha) = f \iff \mathcal{I}nt(f,\alpha) = \overline{\alpha} \wedge f \iff \\ &\iff \mathcal{I}nt(\overline{\alpha} \wedge f, \alpha) = \overline{\alpha} \wedge f \iff \mathcal{I}(\overline{\alpha} \wedge f, \alpha) = \overline{\alpha} \wedge f \quad ;\end{aligned}$$

hence ($I7$) follows.
■

Referring to 9.3.2 and 9.3.3 it is easy to see that weakly extensional, L–fuzzy topologies and local L–interior operators come to the same thing. Moreover, a weakly extensional, L–fuzzy topology $\mathcal{T}$ is enriched if and only if the corresponding local L–interior operator $\mathcal{I}nt_{\mathcal{T}}$ is enriched – i.e. $\mathcal{I}nt_{\mathcal{T}}$ satisfies the axiom (IE) for all $\alpha \in L$.
By analogy to subsection 8.1 local L–interior operators admit a characterization by local L–neighborhood systems in a sense which is explained below.

Definition 9.3.7 (Local L–fuzzy neighborhood system)
Let X be a set. A map $\mathfrak{N} : X \times L^X \times L \to L$ is called *a local L–neighborhood system* iff $\mathfrak{N}$ satisfies the following axioms:

$$(\mathfrak{N}0) \qquad \mathfrak{N}(x, 1_X, \top) \;=\; \top, \quad \forall x \in X \quad .$$

$$(\mathfrak{N}1) \qquad u \le v\,,\; \beta \le \alpha \;\Longrightarrow\; \overline{\beta} \wedge \mathfrak{N}(x,u,\alpha) \;\le\; \mathfrak{N}(x,v,\beta) \quad .$$

$$(\mathfrak{N}2) \qquad \mathfrak{N}(x,u_1,\alpha) \otimes \mathfrak{N}(x,u_2,\beta) \;\le\; \mathfrak{N}(x, u_1 \otimes u_2, \alpha \otimes \beta) \quad .$$

$$(\mathfrak{N}3) \qquad \mathfrak{N}(x,u,\alpha) \;\le\; u(x) \wedge \overline{\alpha} \quad .$$

$$(\mathfrak{N}4) \qquad \mathfrak{N}(x,u,\alpha) \;\le\; \bigvee \{\mathfrak{N}(x,v,\alpha) \mid v(y) \le \mathfrak{N}(y,u,\alpha) \quad \forall y \in X\} \quad .$$

$$(\mathfrak{N}5) \qquad (\overline{\alpha \otimes \bot} \wedge \mathfrak{N}(x,u,\alpha)) \otimes (\overline{\alpha \otimes \bot} \wedge v(x)) \;\le\; \mathfrak{N}(x, u \otimes v, \alpha \otimes \bot) \quad .$$

$$(\mathfrak{N}6) \qquad \overline{\alpha}_i \wedge u(x) \;\le\; \mathfrak{N}(x,u,\alpha_i)\; \forall i \in I\,, \forall x \in X \;\Longrightarrow$$
$$\Longrightarrow\; u(x) \wedge \overline{\bigvee_{i \in I} \alpha_i} \;\le\; \mathfrak{N}(x, u\,, \bigvee_{i\in I} \alpha_i) \quad \forall x \in X \quad .$$

■

An obvious modification of 8.1.9 leads to the following

Proposition 9.3.8 *Every local, L–interior operator $\mathcal{I}nt$ can be identified with a local, L–neighborhood system $\mathfrak{N}$ and vice versa. In particular the relationship between $\mathcal{I}nt$ and $\mathfrak{N}$ is determined by*

$$\mathfrak{N}(x,u,\alpha) \;=\; [\mathcal{I}nt(f,\alpha)](x) \quad .$$

9.4 Filter theory on local elements of L^X

In this subsection we are going to adjust the concept of L–fuzzy filters to the situation given in the previous subsection.

Definition 9.4.1 (Localized, L–fuzzy filters) A map $\mathcal{F} : L^X \times L \to L$ is called a *localized, L–fuzzy filter* on X iff $\mathcal{F}$ has the following properties:

$$(\mathcal{F}0) \qquad \mathcal{F}(1_X, \alpha) = \overline{\alpha} \quad \forall \alpha \in L \quad .$$

$$(\mathcal{F}1) \qquad u \le v\,,\; \beta \le \alpha \;\Longrightarrow\; \mathcal{F}(u,\alpha) \wedge \overline{\beta} \;\le\; \mathcal{F}(v,\beta) \quad .$$

$$(\mathcal{F}2) \quad \mathcal{F}(u,\alpha) \otimes \mathcal{F}(v,\beta) \quad \leq \quad \mathcal{F}(u \otimes v, \alpha \otimes \beta) \quad .$$

$$(\mathcal{F}3) \quad \mathcal{F}(1_{\emptyset},\alpha) \quad = \quad \perp \quad .$$

A localized, L–fuzzy filter $\mathcal{F}$ is called enriched if $\mathcal{F}$ satisfies the additional axiom

$$(\mathcal{F}4) \quad (\mathcal{F}(u,\alpha)) * \lambda \quad \leq \quad \mathcal{F}(u * \lambda, \alpha) \qquad \forall \alpha \in L, \quad \forall \lambda \in L \quad .$$

■

Caution. In general a localized, L–fuzzy filter is *not* an L–fuzzy filter. This is only true, if $\top$ and $\perp$ are the only idempotent elements in L (w.r.t. $*$).
■

Remark 9.4.2 (a) Let ν be an L–filter on X; then $\mathcal{F}_\nu : L^X \times L \to L$ defined by

$$\mathcal{F}_\nu(f,\alpha) \quad = \quad \overline{\alpha} \wedge \nu(f) \qquad \forall f \in L^X, \forall \alpha \in L$$

is a localized, L–fuzzy filter on X. Moreover, if $\mathfrak{N}$ is a local L–neighborhood system on X, then for every $x \in X$ the map $\mathfrak{N}_x : L^X \times L \to L$ determined by

$$\mathfrak{N}_x(f,\alpha) \quad = \quad \mathfrak{N}(x,f,\alpha)$$

is a localized, L–fuzzy filter on X. In particular, $\mathfrak{N}_x$ is called the *localized, L-fuzzy neighborhood filter at the point x.*
(c) Let us consider the case of $\otimes = \wedge$. Then under the hypothesis of $(\mathcal{F}1)$ the axiom $(\mathcal{F}2)$ can be replaced by

$$(\mathcal{F}2^*) \quad \mathcal{F}(u,\alpha) \wedge \mathcal{F}(v,\alpha) \quad \leq \quad \mathcal{F}(u \wedge v, \alpha) \qquad \forall \alpha \in L.$$

(d) As an immediate corollary of the previous considerations we obtain that a weakly extensional, L–fuzzy topology $\mathcal{T}$ is enriched iff all localized, L–fuzzy neighborhood filters are enriched.

■

Lemma 9.4.3 *Every enriched localized, L–fuzzy filter $\mathcal{F}$ fulfills the following property*

$$(\mathcal{C}) \quad \left\{ \begin{array}{lll} \overline{\alpha} \wedge (\bigwedge\limits_{x \in X} f(x)) \leq & \mathcal{F}(f,\alpha) \leq & \\ & \leq \quad \overline{\alpha} \wedge \Big(\Big((\bigvee\limits_{x \in X} f(x)) \to \perp\Big) \to \perp\Big) & \end{array} \right\} \quad \forall \alpha \in L$$

The arguments of the proof of 6.2.1 can be repeated more or less verbatim.
■

Let $\mathcal{FF}_L(X)$ (resp. $\mathcal{FF}_L^E(X)$) be the set of all (resp. enriched) localized, L–fuzzy filters on X. On $\mathcal{FF}_L(X)$ (resp. $\mathcal{FF}_L^E(X)$) we can introduce a partial ordering $\preceq$ as follows:

$$\mathcal{F}_1 \preceq \mathcal{F}_2 \quad \Longleftrightarrow \quad \mathcal{F}_1(u,\alpha) \leq \mathcal{F}_2(u,\alpha) \qquad \forall \alpha \in L, \forall u \in L^X.$$

As a direct consequence of Lemma 9.4.3 we obtain that the smallest enriched localized, L–fuzzy filter in $\mathcal{FF}_L^E(X)$ coincides with the smallest stratified $L-$filter (cf. Remark 9.4.2 (a)).

Proposition 9.4.4 (Strict $MV-$algebras)
Let $M = (L, \leq, *)$ be a strict (complete) $MV-$algebra with square roots and $\circledast$ be the monoidal mean operator (cf. 1.2.6). Further let $\mathcal{F}$ be an enriched, localized, $L-$fuzzy filter on X. Then $\mathcal{F}$ satisfies the following property:

$$(\mathcal{F}5)\quad \begin{cases} \forall n \in \mathbb{N},\ (f_i)_{i=1}^n \in (L^X)^n,\ (\alpha_i)_{i=1}^n \in L^n \quad \textit{the implication} \\ f_1 * \ldots * f_n = 1_\varnothing \implies \mathcal{F}(f_1, \alpha_1) * \ldots * \mathcal{F}(f_n, \alpha_n) = \perp \ \textit{holds.} \end{cases}$$

Proof . (a) Let α be an element of L. We recall that the m–th power and 2^n–th root of α are defined recursively as follows:

$$\alpha^m = \alpha^{m-1} * \alpha \quad , \quad \alpha^{1/2^{n+1}} = (\alpha^{1/2^n})^{1/2} \quad ;$$

hence we obtain

$$(\alpha^{1/2^{n+1}})^{2^n} = [(\alpha^{1/2^{n+1}})^2]^{2^{n-1}} = (\alpha^{1/2^n})^{2^{n-1}} \quad .$$

In particular the relation $(\alpha^{1/2^n})^{2^{n-1}} = \alpha^{1/2}$ holds for all $n \in \mathbb{N}$. Therefore we can derive from $(S3)$ the following important relation

$$((\perp^{1/2^n})^{2^n-1})^{1/2} = (\perp^{1/2^{n+1}})^{2^n-1} \quad .$$

Now we are in the position to verify by induction upon n :

$$(\perp^{1/2^n})^{2^n-1} \to \perp = \perp^{1/2^n} \tag{$\bigstar$}$$

Because of the strictness of $(L, \leq, *)$ the previous formula holds for $n = 1$. We assume the validity of ($\bigstar$) for n, and take square roots on both sides:

$$((\perp^{1/2^n})^{2^n-1})^{1/2} \to \perp^{1/2} = \perp^{1/2^{n+1}} \quad .$$

Applying again the strictness we substitute $\perp^{1/2}$ by $\perp^{1/2} \to \perp$ and obtain:

$$\begin{aligned} \perp^{1/2^{n+1}} &= [(\perp^{1/2^{n+1}})^{2^n-1} * (\perp^{1/2^{n+1}})^{2^n}] \to \perp \\ &= (\perp^{1/2^{n+1}})^{2^{n+1}-1} \quad ; \end{aligned}$$

hence ($\bigstar$) holds also for $n + 1$.

(b) Since the idempotent kernel $(\perp^{1/2^n})\check{}$ of $\perp^{1/2^n}$ coincides with $\perp$, we conclude from Subsection 1.3 : $\overline{\perp^{1/2^n} \to \perp} = \top$. Hence the relation

$$\overline{(\perp^{1/2^n})^n} = \top \qquad \forall n \in \mathbb{N} \tag{$\bigstar\bigstar$}$$

follows from ($\bigstar$) .

(c) Let $\mathcal{F}$ be an enriched, localized, L–fuzzy filter. We verify by induction upon n:

$$(\Diamond\Diamond) \quad \begin{cases} (\mathcal{F}(f_1,\alpha_1))^{1/2^n} * \ldots * (\mathcal{F}(f_n,\alpha_n))^{1/2^n} \quad \leq \\ \quad \leq \quad \mathcal{F}((f_1 * \ldots * f_n)^{1/2^n}, (\alpha_1^{1/2^n} * \ldots * \alpha_n^{1/2^n})) \quad . \end{cases}$$

Case 1: $(n = 1)$ The relation

$$(\mathcal{F}(f_1,\alpha_1))^{1/2} = (\mathcal{F}(f_1,\alpha_1))^{1/2} * (\mathcal{F}(1_X,\top))^{1/2} \leq \mathcal{F}({f_1}^{1/2},{\alpha_1}^{1/2})$$

follows from $(\mathcal{F}0)$ and $(\mathcal{F}2)$.

Case 2: (Induction step) Now we assume that $(\Diamond\Diamond)$ holds for some $n \in \mathbb{N}$. Because of

$$(\alpha_1)^{1/2^{n+1}} * \ldots * (\alpha_n)^{1/2^{n+1}} \leq ((\alpha_1)^{1/2^n} * \ldots * (\alpha_n)^{1/2^n})^{1/2}$$

and

$$((f_1 * \ldots * f_n)^{1/2^n})^{1/2} * (f_{n+1})^{1/2^{n+1}} \leq (f_1 * \ldots * f_n * f_{n+1})^{1/2^{n+1}}$$

we infer from $(\mathcal{F}1)$, $(\mathcal{F}2)$ and $(\bigstar\bigstar)$:

$$\begin{aligned} &\mathcal{F}((f_1 * \ldots * f_{n+1})^{1/2^{n+1}}, (\alpha_1)^{1/2^{n+1}} * \ldots * (\alpha_{n+1})^{1/2^{n+1}}) \quad \geq \\ &\mathcal{F}(((f_1 * \ldots * f_n)^{1/2^n})^{1/2} * ((f_{n+1})^{1/2^n})^{1/2}, \\ &\qquad ((\alpha_1)^{1/2^n} * \ldots * (\alpha_n)^{1/2^n})^{1/2} * ((\alpha_{n+1})^{1/2^n})^{1/2}) \quad \geq \\ &[\mathcal{F}((f_1 * \ldots * f_n)^{1/2^n}, (\alpha_1)^{1/2^n} * \ldots * (\alpha_n)^{1/2^n})]^{1/2} * \\ &\qquad * [\mathcal{F}((f_{n+1})^{1/2^n}, (\alpha_{n+1})^{1/2^n})]^{1/2} \quad . \end{aligned}$$

Now we invoke the induction hypothesis and obtain:

$$\begin{aligned} &\mathcal{F}((f_1 * \ldots * f_{n+1})^{1/2^{n+1}}, (\alpha_1)^{1/2^{n+1}} * \ldots * (\alpha_{n+1})^{1/2^{n+1}}) \quad \geq \\ &((\mathcal{F}(f_1,\alpha_1))^{1/2^n} * \ldots * (\mathcal{F}(f_n,\alpha_n))^{1/2^n})^{1/2} * ((\mathcal{F}(f_{n+1},\alpha_{n+1}))^{1/2^n})^{1/2} \geq \\ &(\mathcal{F}(f_1,\alpha_1))^{1/2^{n+1}} * \ldots * (\mathcal{F}(f_n,\alpha_n))^{1/2^{n+1}} * (\mathcal{F}(f_{n+1},\alpha_{n+1}))^{1/2^{n+1}} \quad ; \end{aligned}$$

hence relation $(\Diamond\Diamond)$ is established.
(d) Let us consider $(f_i)_{i=1}^n \in (L^X)^n$ with $f_1 * \ldots * f_n = 1_\varnothing$. Then we infer from formulae $(\Diamond\Diamond)$ (cf. Step (c)):

$$(\mathcal{F}(f_1,\alpha_1))^{1/2^n} * \ldots * (\mathcal{F}(f_n,\alpha_n))^{1/2^n} \leq \mathcal{F}(0^{1/2^n} \cdot 1_X, (\alpha_1)^{1/2^n} * \ldots * (\alpha_n)^{1/2^n}) \quad .$$

Since $\mathcal{F}$ is enriched, the relation

$$\mathcal{F}(\perp^{1/2^n} \cdot 1_X, (\alpha_1)^{1/2^n} * \ldots * (\alpha_n)^{1/2^n}) \quad = \quad \perp^{1/2^n} \quad ;$$

follows from (★★) and Lemma 9.4.3. Hence we obtain:

$$(\mathcal{F}(f_1,\alpha_1))^{1/2^n} * \ldots * (\mathcal{F}(f_n,\alpha_n))^{1/2^n} \quad \leq \quad \perp^{1/2^n} \quad .$$

Taking the 2^n–th power on both sides, $\mathcal{F}(f_1,\alpha_1) * \ldots * \mathcal{F}(f_n,\alpha_n) \quad = \quad \perp$ follows.
■

Corollary 9.4.5 *Let $(L,\leq,*)$ be a complete MV–algebra with square roots, and $\circledast$ be the corresponding, monoidal mean operator. Then every enriched, localized, L–fuzzy filter satisfies $(\mathcal{F}5)$.*

Proof . In the Boolean case $L = \mathbb{B}$ it is easy to see that every (enriched) local, $\mathbb{B}$–fuzzy filter fulfills $(\mathcal{F}5)$. Since every MV–algebra with square roots can be uniquely decomposed into a Boolean algebra and a strict MV–algebra (cf. Subsection 1.3), the assertion follows from Proposition 9.4.4.
■

Corollary 9.4.6 *Let $(L,\leq,*)$ be a strict (complete) $MV-$algebra without non-trivial idempotent elements (i.e. $L = [0,1]$ and $* = T_m$ (see Example 1.2.3)). Further let $\circledast$ be the monoidal (i.e. arithmetic) mean operator. Then for every enriched, localized $L-$fuzzy filter $\mathcal{F}$ there exists a stratified $L-$filter $\hat{\nu}$ on X such that*

$$\mathcal{F}(f,\alpha) \leq \hat{\nu}(f) \quad \text{for all} \quad \alpha \in L \quad .$$

Proof . Since $\top$ is the only idempotent element in L being different from $\perp$, we obtain that any enriched, localized L–fuzzy filter $\mathcal{F}$ induces a family $\mathbf{F} = \{\,\nu_\alpha \mid \alpha \in L \setminus \{\perp\}\,\}$ of stratified, L–filters as follows:

$$\nu_\alpha(f) \quad = \quad \mathcal{F}(f,\alpha) \qquad \text{whenever} \quad \alpha \neq \perp \quad .$$

Because of Proposition 9.4.4 (see $(\mathcal{F}5)$) $\mathbf{F}$ satisfies the hypothesis of Theorem 6.2.6(b). Hence there exists a stratified L–filter which dominates $\mathcal{F}$.
■

Theorem 9.4.7 *Let $(L,\leq,*)$ be a complete MV–algebra with square roots, and $\circledast$ be the corresponding monoidal mean operator. Further let $\mathcal{FF} =$ $= \{\mathcal{F}_i \mid i \in I\}$ be a family of enriched, localized, $L-$fuzzy filters on X provided with the following property*

$$(\mathfrak{F}) \quad \begin{cases} \text{For every finite, non empty subset } \{i_1,\ldots,i_n\} \text{ of } I \text{ and} \\ \text{for all } (f_{i_j})_{j=1}^n \in (L^X)^n\,,\ (\alpha_{i_j})_{j=1}^n \in L^n \text{ the implication} \\ f_{i_1} * \ldots * f_{i_n} = 1_\varnothing \quad \Longrightarrow \quad \mathcal{F}_{i_1}(f_{i_1},\alpha_{i_1}) * \ldots * \mathcal{F}_{i_n}(f_{i_n},\alpha_{i_n}) \quad = \perp \\ \text{holds} \quad . \end{cases}$$

Then $\mathcal{FF}$ has an upper bound in $\mathcal{FF}_L^E(X)$.

Proof. We define a map $\mathcal{F}_\infty : L^X \times L \longmapsto L$ by

$$\begin{aligned}\mathcal{F}_\infty(h,\varkappa) \quad &= \quad \bigvee \{\overline{\varkappa} \wedge (\mathcal{F}_{i_1}(f_{i_1},\alpha_{i_1}) * \ldots * \mathcal{F}_{i_n}(f_{i_n},\alpha_{i_n}) \quad \mid \\ &\exists\, n \in \mathbb{N},\ \alpha_{i_j} \in L,\ f_{i_j} \in L^X,\ \alpha_{i_1} * \ldots * \alpha_{i_n} \ \le\ \varkappa,\ f_{i_1} * \ldots * f_{i_n} \ \le\ h\}\end{aligned}$$

(a) We show that $\mathcal{F}_\infty$ is an enriched, localized, L–fuzzy filter on X. The axioms ($\mathcal{F}0$) and ($\mathcal{F}4$) are evident. The hypothesis guarantees the validity of ($\mathcal{F}3$). The axiom ($\mathcal{F}1$) follows from

$$\overline{\beta} \wedge \mathcal{F}_{i_j}(f_{i_j},\alpha_{i_j}) \quad \le \quad \mathcal{F}_{i_j}(f_{i_j},\alpha_{i_j} \wedge \beta) \qquad .$$

In order to verify ($\mathcal{F}2$) we first observe:

$$\begin{aligned}&(\overline{\varkappa} \wedge (\mathcal{F}_{i_1}(f_{i_1},\alpha_{i_1}) * \ldots * \mathcal{F}_{i_n}(f_{i_n},\alpha_{i_n})))^{1/2} \quad = \\ &= \quad [\overline{\varkappa} \wedge ((\mathcal{F}_{i_1}(f_{i_1},\alpha_{i_1}))^{1/2} * \ldots * (\mathcal{F}_{i_n}(f_{i_n},\alpha_{i_n}))^{1/2}] \vee \bot^{1/2} \quad ;\end{aligned}$$

$$\begin{aligned}\bot^{1/2} * (\mathcal{F}_{j_1}(g_{j_1},\beta_{j_1}))^{1/2} \quad &= \quad (\mathcal{F}_{j_1}(1_\varnothing,\varkappa_1))^{1/2} * (\mathcal{F}_{j_1}(g_{j_1},\beta_{j_1}))^{1/2} \quad \le \\ &\le \quad \mathcal{F}_{j_1}((g_{j_1})^{1/2} * (\bot^{1/2} \cdot 1_X),\, \varkappa_1^{1/2} * \beta_{j_1}^{1/2}) \quad ;\end{aligned}$$

$$\bot^{1/2} \wedge \overline{\varkappa_2} \quad \le \quad \overline{\varkappa_1 \circledast \varkappa_2} \qquad \text{(see: Proposition 1.3.1(c))} \qquad .$$

Therefore we are in the position to transfer that strategy developed for the verification of (F2) in 6.2.6 to the present case. Since this proof is extremely technical, we leave the details to the reader.

(b) Because of ($\mathcal{F}0$) the enriched, localized, L–fuzzy filter $\mathcal{F}_\infty$ is an upper bound of $\mathcal{FF}$.

■

An immediate consequence from Corollary 9.4.5 and Theorem 9.4.7 is the following

Proposition 9.4.8 *Let $(L,\le,*)$ be a complete MV–algebra with square roots and $\circledast$ be the corresponding monoidal mean operator. Then a family $\mathcal{FF}$ of enriched, localized, L–fuzzy filters on X has an upper bound in $\mathcal{FF}_L^E(X)$ if and only if $\mathcal{FF}$ satisfies Condition ($\mathfrak{F}$) in 9.4.7.*

9.5 Hausdorff separation axiom for weakly extensional, L–fuzzy topological spaces

Let $\mathcal{T}$ be an (enriched) weakly extensional, L–fuzzy topology on X and $\mathfrak{N}$ be the local L–neighborhood system corresponding to $\mathcal{T}$. An (enriched) localized, L–fuzzy filter $\mathcal{F}$ on X converges to $x_0 \in X$ (i.e. x_0 is a *limit point* of $\mathcal{F}$) w.r.t. $\mathcal{T}$ if and only if the following relation holds:

$$\mathfrak{N}(x_0,f,\alpha) \quad \le \quad \mathcal{F}(f,\alpha) \qquad \forall\,(f,\alpha) \in L^X \times L \qquad .$$

An (enriched) weakly extensional, L–fuzzy topological space $(X,\mathcal{T})$ is called *Hausdorff separated* if and only if every (enriched) localized, L–fuzzy filter has at most one limit point.

Proposition 9.5.1 (Complete MV–algebras with square roots)
Let $(L, \leq, *)$ be a complete MV-algebra with square roots and $\circledast$ be the corresponding monoidal mean operator. Further let $\mathcal{T}$ be a weakly extensional, enriched L-fuzzy topology on X and $\mathfrak{N}$ be the local L-neighborhood system corresponding to $\mathcal{T}$. Then $(X, \mathcal{T})$ is Hausdorff separated if and only if for every pair $(x_1, x_2) \in X \times X$ with $x_1 \neq x_2$ the set $\{\mathfrak{N}_{x_1}, \mathfrak{N}_{x_2}\}$ consisting of localized, L-fuzzy neighborhood filters at x_1 and x_2 **does not** *satisfy the condition $(\mathfrak{F})$ in 9.4.7.*

Proof. The assertion follows immediately from Lemma 9.2.4, Proposition 9.3.5, Proposition 9.3.8, Remark 9.4.2(a) and Proposition 9.4.8.
■

We emphasize that in the spirit of subsection 6.4 a general development of a local convergence theory for weakly extensional, L–fuzzy topological spaces is possible. Details of this theory are left to the reader. In particular we do not touch the links between convergence in L–topological spaces and weakly extensional, L–fuzzy topological spaces. What is more important is the observation that due to the characterization of weakly extensional, L–fuzzy topologies by local L–neighborhood systems (cf. 9.2 and 9.3) an *intrinsic local convergence theory* is always available.

10 Historical comments[19]

10.1 L-topologies, stratified L-topologies

In the case of a complete Heyting algebras L (i.e. $\otimes = * = \wedge$) the *axioms of L-topological spaces* and the concept of *L-continuity* trace back to the works of C.L. Chang [10] and J.A.Goguen [24]. Actually, C.L. Chang is using the real unit interval (i.e. $L = I = [0,1]$) while J.A. Goguen, referring to *cl*-monoids indeed relies on complete Heyting algebras. The fact that the constant maps between L-topological spaces are not necessarily continuous induces R. Lowen [64] to modify the axioms of L-topologies: he replaces Axiom $(o1)$ by a stronger requirement that all constant maps $\alpha : X \to L$ are always *open*. Thus Lowen's work is placed in the category of *(weakly) stratified* I-topological spaces where $I = [0,1]$.

[19] The aim of this section is to present some information about investigations done in Fuzzy Topology, more or less directly related to the topic of our work. Having neither the possibility, nor the intention to give here a comprehensive survey and analysis of the subject on the whole we proceed from the following two criteria when mentioning a paper in these *Comments*: its importance for and closedness to the main trend of our work, and the priority of the corresponding publication. Note also that some essential comments of historical nature have been already made at appropriate places in the main text.

10.2 L-interior operators, L-neighborhood structure

L-interior operators (in case of completely distributive, complete lattices) first appear in B.Hutton's paper [46] on normality in L-topological spaces.

In the case of the Heyting algebra $L = [0,1]$ early concepts of various *L-neighborhood structures* of L-topological spaces are contained in papers written by C.K. Wong [117], R.H. Warren [114, 115], Pu Paoming and Liu Yingming [84], S. Gottwald [25], C. De Mitri and E. Pascali [12], E.E. Kerre and P.L. Ottoy [52]. In the more general setting of completely distributive, complete lattices L-neighborhood structures of L-topological spaces are discussed by S.E. Rodabaugh [91], Wang Guojun [112], Zhao Xiadong [119] and some other (mainly Chinese) authors. Note also that under certain assumptions *Chapter 5 in this volume* ([43]) is specially devoted to the characterization of L-topologies by L-valued neighborhoods.

10.3 Convergence in L-topologies

In the papers on *Fuzzy Topology* one can find several different approaches to the study of convergence in L-topological spaces.

In the case of $L = [0,1]$ the research of convergence structures in the framework of L-topologies by means of *L-nets* has been initiated by Pu Paoming and Liu Yingming [84, 61]. Later on, M.Macho Stadler and M.A. de Prada Vicente (see e.g. [74, 75]) and some other researchers have worked in this direction. The L-net convergence structure in the case of completely distributive, complete lattices has been studied by Liu Yingming and Luo Maokang [63].

Convergence theory on the basis of *prefilters* (i.e. certain subsets of L^X, cf. e.g. [7]) is initiated by R. Lowen [67] in case of stratified $[0,1]$-topologies and later extended by R. Warren [116] to arbitrary $[0,1]$-topologies. Certain contributions to the investigation of prefilter convergence in $[0,1]$-topologies have been done by A. Katsaras [51], M.Macho Stadler and M.A.de Prada Vicente [76] and some other authors. In [68] the relations between net convergence and prefilter convergence in $[0,1]$-topologies have been studied. B. Hutton [47] is the first one to apply prefilter convergence to the study of L-topologies in case of *arbitrary completely distributive, complete lattices L*. In the *absence of complete distributivity* of L certain prefilters (actually $\mathbf{1}$-filters, cf. also the definition of $\top$-filters in 6.2.3) have been used by U. Höhle [30] (cf. also [36]) to develop a convergence theory for Boolean valued topologies ($\mathbb{B}$-topologies). Among other things, this approach has important applications to *Probability Theory*.

Under different lattice-theoretic assumptions on the lattice L convergence on the basis of *L-filters* (cf. Definition 6.1.4.) is studied in a series of papers by P. Eklund and W. Gähler (see e.g. [13, 14, 15]). In the case of complete MV–algebras the theory of L–filters is initiated by U. Höhle [41]. Recently this theory has been continued by J. Gutiérrez García, I. Mardones Pérez and M. Burton (see e.g. [26]). In particular, in the case of GL–monoids they investigate carefully various relations between different kinds of lattice-valued filters (see also Section 2 in *Chapter 5* of this volume).

10.4 Topological properties of L-topological spaces

10.4.1 Compactness in L-topological spaces

The compactness property being one of the central topological concepts, naturally drew attention of almost everyone interested in Fuzzy Topology. Of the numerous papers considering analogues of compactness in the context of L-topological spaces (usually under the assumption of complete distributivity of the lattice L) we shall mention here the papers by C.L. Chang [10] ($L = [0,1]$), R. Lowen [64] (stratified $[0,1]$-topologies), J.A. Goguen [24], T.E. Gantner, R.C. Steinlage and R.H. Warren [20], B. Hutton [47], S.E. Rodabaugh [92], J.J. Chadwick [8] ($L = [0,1]$), T. Kubiak [57], Wang Guojun [113] and Zhao Dongsheng [120] (L-net approach to compactness in L-topological spaces). A concept of $\mathcal{T}$-compactness for $[0,1]$-topologies, where $\mathcal{T}$ is a fixed t-norm on $[0,1]$, is considered in [111]. A. Šostak [103, 107] in his study of compactness properties although referring to L-fuzzy topological spaces, actually reduces the topic to the case of L-topologies by considering corresponding α-cuts. Compactness in probabilistic topological spaces and related problems are studied in [31]. A notion of compactness for *non-completely distributive* Boolean valued topologies has been developed by U. Höhle [35].

For more information on different approaches to the concept of compactness in L-topological spaces the reader is referred for instance to the following papers [57, 92, 94, 104].

10.4.2 Separation in L-topological spaces

Many authors devote themselves to the problem of finding appropriate definitions of separation properties (in particular, Hausdorff-type separation axioms) for L-topological spaces. Among the early works dealing with this problem we shall mention here papers by Pu Paoming and Liu Yingming [84, 85], R.Srivastava, S.N.Lal and A.K. Srivastava [108, 109], W. Kotzé ("separation" of *disjoint* $[0,1]$-points — i.e. $[0,1]$-subsets with one-point supports, by disjoint open [0,1]-subsets), U. Höhle [36] (a very weak Hausdorff-type separation property in Boolean-valued topologies), M. Sarkar [97, 98], D. Adnadjević [2] ("separation" of *distinct* L-points by L-neighborhoods (in [97] and in [2] $L = [0,1]$)), B. Hutton and I. Reilly [48] (a purely lattice-theoretic approach), S.E. Rodabaugh [87, 88] (an α-level type approach), R. Lowen and P. Wuyts [118] (an approach based on prefilters), A. Šostak [104] (a spectral approach, $L = [0,1]$), Liu Yingming and Luo Maokang [62] (an approach using so called *remote neighborhoods* [112]).

Separation properties of regularity type have been studied by M. Sarkar [97, 98], D. Adnadjević [2], B. Hutton and I. Reilly [48], S.R. Malghan and S.S. Benchalli [77].

For more information concerning different schemes of separation axioms, in particular, concerning different definitions of Hausdorffness and regularity in L-topological spaces the reader is referred for instance to the following papers [48, 58, 59, 89, 92, 94, 104].

10.4.3 Extending continuous functions

The problem of extending continuous functions from a subset A in an L-topological space X to the whole space X has two essentially different settings: the existence of a continuous extension under assumption that A is a closed subset of X and the existence of a continuous extension under assumption that A is a dense subset of X (the second one is the so called *Extension Principle*). *Chapter 6 in this Volume* is devoted to the first one of these problems (see [59]); this paper contains also the relevant and quite extensive bibliography on this subject.

On the other hand, as far as we know, the second problem in its general setting is considered in our paper (see Section 6.3) for the first time. In the special case when the function is defined on a dense subset of a Boolean valued topological space and takes its values in a (probabilistic) Boolean valued uniform space, the *Extension Principle* has been established by U. Höhle [33] (see also [36]).

10.5 Probabilistic L-topologies

The axioms of probabilistic L-topologies are introduced by U. Höhle [30, 31, 33, 34] in order to provide probabilistic metric spaces [99] with appropriate L-topological structures (cf. also [43]). In the same papers mentioned above foundations of the theory of probabilistic L-topologies have been developed. In particular, characterizations of probabilistic L-topologies by means of closure operators and by means of systems of crisp sets of L–valued neighborhoods have been obtained here.

10.6 [0,1]-topologies on spaces of probability measures

[0,1]-topologies on spaces of probability measures on the Borel σ-algebra of a separable metric space have been considered by R. Lowen [69] — see also the construction of the L-real interval by Hutton [46] and the construction of the L-real line by T.E. Gantner, R.C. Steinlage and R.H. Warren [20].

10.7 L-fuzzy topological spaces

The axioms of L-fuzzy topological spaces and the corresponding concept of LF-continuity (in case of a complete Heyting algebra L) go back to A. Šostak [101] ($L = [0,1]$), [103, 104]. Similar ideas appear also in U. Höhle [32], G. Gerla [21] and T. Kubiak [55].

Characterization of L-fuzzy topologies by means of L-fuzzy interior operators and L-fuzzy closure operators is given in [11] (in case $L = [0,1]$) and in [107] (in case of completely distributive, complete lattices L endowed with an order reversing involution). A characterization of $[0,1]$-fuzzy topologies by means of certain $[0,1]$-fuzzy neighborhood systems is obtained in [105]. In [106] the ($[0,1]$-net) convergence structure of $[0,1]$-fuzzy topologies has been studied.

10.8 The use of "enriched" lattices

J.A. Goguen [24] was, probably, the first one who felt the necessity to enrich the structure of the underlying lattice L with an additional, binary operation $*$ in order to get a deeper insight into the subject of *Fuzzy Topology* and to develop the corresponding adequate theory of L-topological spaces. However, proceeding from a *cl*-monoid $(L, \leq, \wedge, \vee, *)$ J.A. Goguen's theory actually relies only on the structure of the underlying complete Heyting algebra and does not derive great benefit from those possibilities provided by the additionally given monoidal operation $*$.

On the other hand, already in the late seventies, U. Höhle's investigations in the field of *Fuzzy Topology* rely essentially on complete lattices enriched with additional, binary operation(s). In particular, complete lattices L equipped with two additional commutative semigroup operations $*$ and $\otimes$ form the "environment" where the theory of (fuzzy) probabilistic topologies has been developed (see [30, 31, 33, 34]).

In order to understand the *importance* of enriching the underlying lattice with additional, algebraic structure(s) the reader is also referred to Remark 7.1.5 and the next subsection.

10.9 Role of algebraic properties of the meet operator in the theory of L-topologies

The investigation of the role of idempotency, commutativity and associativity played by the meet operator is first conducted in the authors' paper [45] and it leads to a replacement of complete Heyting algebras by GL-monoids with square roots. In this context the equivalence between strongly enriched L-fuzzy topologies and strongly stratified L-topologies (cf. Theorem 5.2.6) has already been proved in [45] (which uses a different terminology).

10.10 Categorical aspects of Fuzzy Topology. Functors between categories of L-(fuzzy) topological spaces

Efforts of many authors have been directed to the problem to establish essential categorical relations between different categories of L-(fuzzy) topology as well as between categories of L-(fuzzy) topological spaces and the category **TOP** of topological spaces: see e.g. [88] and [104] which give a survey on some earlier attempts in this direction. Here we shall linger on some aspects of this problem which are most closely related to our work.

10.10.1 In the special case $L = [0, 1]$ the functor $\omega_\iota \ (= \omega) : \mathbf{TOP} \to L\text{-}\mathbf{TOP}$ appears for the first time in R. Lowen' s paper [64]. In particular, ω is defined to be the functor which assigns to each topological space (X, T) the $[0, 1]$-topological space $(X, \ell(T))$ where $\ell(T)$ denotes the set of all lower T-semicontinuous mappings from X to $[0, 1]$ (cf. Case B in Subsection 7.4). Generalizing the concept of lower T-semicontinuity to the case of L-valued maps,

T. Kubiak [57] extends the functor ω to the general setting given by complete lattices L. In the case of completely distributive lattices T. Kubiak's definition becomes equivalent to the one accepted in Subsection 7.4. If L is a complete MV-algebra, then the range of the functor ω_ι is contained in the category of probabilistic L-topological spaces (cf. Remark 7.4.13).

10.10.2 The fact that ω has a right adjoint is first established by R. Lowen, P. Wuyts and E. Lowen in the case of the real unit interval (cf. [73]). Later on, T. Kubiak [57] extends this result to the scope of completely distributive lattices. A careful analysis of T. Kubiak's arguments (cf. Remark 3.7 in [57]) shows that by virtue of Kubiak's definition of ω this result holds even true in a more general lattice-theoretic context.

10.10.3 The functor Δ_ι used in the study of the adjoint situation between **TOP** and L-**TOP** (see Subsection 3.7) essentially traces back to H.W. Martin's paper [78] (in case $L = [0,1]$). Later analogous functors have been used by various authors studying different problems in *Fuzzy Topology* — see e.g. the functor G_χ in S.E. Rodabaugh's paper [90] which investigates topological and algebraic properties of L-fuzzy real lines.

10.10.4 The hypergraph functor $\mathcal{G}$ from L-**TOP** to **TOP** has been introduced (independently) by E.S. Santos [96] and R. Lowen [66] in the case of $L = [0,1]$, and later it has been systematically used in the study of relations between these categories — see e.g. [54, 87, 88]. It is an interesting and remakable fact that the hypergraph functor is even definable in the case in which the underlying complete lattice L is given by a *spatial locale* (see e.g. [54] or Subsection 7.1.5).

10.10.5 Investigation of reflectivity and coreflectivity of some subcategories of the category L-**TOP** has been conducted in a series of papers by H. Herrlich, R. Lowen and P. Wuyts; see e.g. [27, 71, 72].

10.10.6 Coreflective subcategories of the category $[0,1]$-**FTOP** of $[0,1]$-fuzzy topological spaces are studied by A. Šostak [102]. In particular, the coreflectivity of the subcategory $[0,1]$-**TOP** in $[0,1]$-**FTOP** has been established here.

10.10.7 In the case of complete Heyting algebras U. Höhle [37] proved the equivalence between strongly stratified L-topologies and weakly stratified L-topologies (cf. Theorem 5.2.7) as a by-product of the identification of stratified L-topological spaces with topological space objects in the topos of L-valued sets (cf. [110]).

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CHAPTER 4

Categorical Foundations Of Variable-Basis Fuzzy Topology

S. E. Rodabaugh

Introduction

This chapter lays categorical foundations for topology and fuzzy topology in which the basis of a space—the lattice of membership values—is allowed to change from one object to another *within the same category* (the basis of a space being distinguished from the basis of the topology of a space). It is the goal of this chapter to create foundations which answer all the following questions in the affirmative:

(1) Are there categories for variable-basis topology or variable-basis fuzzy topology which are topological over their ground (or base) categories?

(2) Are there categories for variable-basis topology or variable-basis fuzzy topology which cohere or unite all known canonical examples of point-set lattice-theoretic (or **poslat**) spaces, even when they are based on different lattices of membership values? For example, is there a *single category* containing all the fuzzy real lines $\{\mathbb{R}(L) : L \in |\mathbf{DQML}|\}$ [12, 24, 39, 62, 69] and in which fuzzy real lines with different underlying bases may be "compared" by "homeomorphisms" or "non-homeomorphic continuous morphisms" (where **DQML** is defined in 1.3.2 below)?

(3) Are there categories for variable-basis topology or variable-basis fuzzy topology which cohere or unite known, important fixed-basis categories for topology or fuzzy topology as subcategories within a *single category*?

(4) Are there categories for variable-basis topology or variable-basis fuzzy topology which make no essential use of algebraic notions such as associativity, commutivity, and idempotency of the traditional meet operation? (Cf. introduction of [23].)

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The question of change-of-basis between *fixed-basis categories* by functors is solved in [23]; and that solution becomes a special case of our solution once their fixed-basis categories are mapped into the supercategories studied in this chapter (*à la* (3) above). But the above questions concern the *internalizing* of change-of-basis within a *single category*.

Restated, the four questions above may be thought of as comprising three **boundary conditions** for the theory constructed in this chapter: (1) is the **topological** condition, (2,3) are the **unification/coherence** conditions, and (4) is the **non-algebraic** condition.

This chapter and the companion chapter [23] of U. Höhle and A. Šostak together give a definitive, conclusive answer to the question posed to this author many years ago by M. E. Rudin [71]: *is fuzzy topology really topological or is it part of algebra?* Categorically, fuzzy topology is certainly topological—each category for fuzzy topology in this *Handbook* is topological; with respect to the algebraic properties of associativity, commutivity, and idempotency of "intersection", fuzzy topology is non-algebraic; and hence fuzzy topology is more topological, and less algebraic, than traditional Hausdorff spaces and algebraic topology! (Indeed, it is fuzzy topology which reveals that the topological character of ordinary topology is non-algebraic w.r.t. intersection).

The outline of this chapter is as follows:

1 Preliminary discussion and motivation

It is the specific purpose of this section to motivate variable-basis foundations—and the questions posed above in the chapter introduction—from the historical and categorical perspectives, and then give needed mathematical preliminaries.

1.1 History of variable-basis thinking

There have historically been two distinct streams of thought which have promoted variable-basis thinking in fuzzy sets, one stream internal to fuzzy sets and the other external to fuzzy sets, and early on neither stream was aware of the other. In this subsection we summarize these two streams and their eventual merger in the theory being developed in this chapter.

In 1965, the famous paper [81] of L.A. Zadeh based fuzzy sets on the unit interval $\mathbb{I}$, with the associated powerset and powerset operators as follows: $a : X \to \mathbb{I}$ is an $\mathbb{I}$-subset of X, $\mathbb{I}^X$ is the $\mathbb{I}$-powerset of X, and $f : X \to Y$ induces powerset operators of the form:

$$f_{\mathbb{I}}^{\to} : \mathbb{I}^X \to \mathbb{I}^Y \text{ by } f_{\mathbb{I}}^{\to}(a)(y) = \bigvee_{f(x)=y} a(x)$$

$$f_{\mathbb{I}}^{\leftarrow} : \mathbb{I}^X \leftarrow \mathbb{I}^Y \text{ by } f_{\mathbb{I}}^{\leftarrow}(b) = b \circ f$$

(See the author's earlier chapter in this volume concerning powerset operators [67]). And in 1968, C. L. Chang [2] proposed an "$\mathbb{I}$-topology" based on Zadeh's set theory in which an object of the shape (X, τ) is an **$\mathbb{I}$-topological space** with τ closed under arbitrary $\bigvee$ and finite $\wedge$, and a morphism $f : (X, \tau) \to (Y, \sigma)$ is a mapping $f : X \to Y$ satisfying $(f_{\mathbb{I}}^{\leftarrow})_{|\sigma} : \tau \leftarrow \sigma$. Such a topological theory is called a **fixed-basis theory**: the lattice-theoretic basis, the unit interval, is held fixed for all objects of Chang's (implicitly defined) category; and the implicit endomorphism of $\mathbb{I}$ in each morphism is fixed at the identity mapping $id_{\mathbb{I}}$ of $\mathbb{I}$ (cf. Subsection 6.2 below).

Almost simultaneous with Chang's historic paper was the seminal paper [15] of J. A. Goguen in 1967 which proposed a "multi-basis" approach to fuzzy sets, replacing the unit interval $\mathbb{I}$ with a complete lattice (ordered semigroup) L; in fact, Goguen proposed a family of separate, fixed-basis fuzzy set theories generally related only by the fact that their bases came from the same category of lattices. If L was such a lattice, then an L-subset of X was a mapping $a : X \to L$ and the **L-powerset** was L^X. (In this context we should mention that Goguen pioneered the use of the term "base" or "basis" for the underlying lattice L, a usage that continues today.) Now in the last section [15] Goguen sketched some functorial relationships between separate fuzzy set theories with different bases, further developing his multi-basis approach, but did not reconcile these theories into one category or variable-basis framework; and a similar multi-basis approach obtains in [23] for fixed-basis fuzzy topology [23].

Thus it was inevitable that Goguen's own proposal for fuzzy topology [16], based on lattices other than the unit interval, created a family of separate, fixed-basis topological theories whose bases came from the same category of lattices. Given such a lattice L, an **L-topological space** is a pair (X, τ) with $\tau \subset L^X$ closed under arbitrary $\bigvee$ and finite $\wedge$, and a(n) **(L-)continuous morphism** $f : (X, \tau) \to (Y, \sigma)$ satisfying $(f_L^{\leftarrow})_{|\sigma} : \tau \leftarrow \sigma$, where $f_L^{\leftarrow}$ is defined formally as in the Chang case. Such spaces were also called **(L-)Chang-Goguen topological spaces**. In modern usage, the terms **L-subset**, **L-powerset**, **L-topological space** or **L-Chang-Goguen topological space**, and **L-continuous morphism** only require L to be a complete lattice or complete quasi-monoidal lattice (see [23] or Subsection 1.3.2 below) leading to a family of categories each of the form L-**TOP**.

It is this author's conjecture that Goguen's multi-basis approach did not progress at that time to a variable-basis category because the needed powerset operators had not yet been developed, i.e. there was no way to determine continuiuty (using backward operator) and homeomorphisms (using forward and backward operators). So the variable-basis potential of [15] remained unfulfilled until the 1980's, a fulfillment we will describe after considering the parallel development of localic theory.

A **frame** is a complete lattice satisfying the first infinite distributive law of finite meets over arbitrary joins, a frame morphism is a mapping between frames preserving arbitrary joins and finite meets, and the ensuing category is denoted **FRM**. The category **LOC** of **locales** is the opposite (or dual) category of **FRM**, i.e. **LOC** = $\mathbf{FRM}^{op}$; which means that a locale is a frame, and the morphisms of **LOC** and **FRM** are in a bijection such that $f \in \mathbf{LOC}(A, B) \Leftrightarrow f^{op} \in \mathbf{FRM}(B, A)$. Part of the motivation behind locales is that if $(X, T), (Y, S)$ are topological spaces and $f : (X, T) \to (Y, S)$ is continuous, then T, S are locales and $\left(f^{\leftarrow}_{|S}\right)^{op} : T \to S$ is a localic morphism from T to S.

It turns out that many locales may be viewed simply as classical topologies without underlying sets of points. But for reasons that will be clear in Section 6 below, the category **LOC** is in reality *a variable-basis category of singleton topological spaces* in which a given locale "is" the basis of membership values of the overlying topological space and a given localic morphism "is" a backward powerset operator.

If fuzzy theorists had known of locales, then the variable-basis approach could have had an earlier resolution. Localic theory was being developed in the 1960's and 1970's—e.g. see [3, 4, 10, 14, 27, 28, 51]—and was given a coherent statement by P. T. Johnstone in [28]; but this development was largely unknown to fuzzy theorists until Johnstone's book.

Turning back to the fuzzy community, the first attempt to describe a topological theory, which could be said to be variable-basis, was B. Hutton's fundamental 1980 paper [25] in which a formally point-free theory was developed that is remarkably similar to localic theory. Noting that L-topological spaces $(X, \tau), (Y, \sigma)$ and a continuous morphism $f : (X, \tau) \to (Y, \sigma)$ induce objects and a morphism of the form $(f^{\leftarrow}_L)^{op} : (L^X, \tau) \to (L^Y, \sigma)$ with $f^{\leftarrow}_L : L^X \leftarrow L^Y$ and $\tau \supset (f^{\leftarrow}_L)^{\rightarrow}(\sigma)$, and observing that L^X need not be isomorphic to L^Y, Hutton introduced a category **HTOP** (our designation) having objects of the form (L, τ) and morphisms of the form $\phi : (L, \tau) \to (M, \sigma)$, where $\tau \subset L$ is closed under arbitrary $\bigvee$ and finite $\bigwedge$ and $\phi^{op} : L \leftarrow M$ has the property that $\tau \supset (\phi^{op})^{\rightarrow}(\sigma)$. As in the localic case, Section 6 shows that **HTOP** is really *a variable-basis category of singleton topological spaces.*

What is missing from our synopsis in the preceding paragraph is an explicit description of the properties of the lattices being used for bases and of the morphisms between them. Another sense in which Hutton was a pioneer is that he was the first worker in fuzzy sets to separate out the description of the **ground category** over which he was building his topological theory. Let **HUT** (our designation) have as objects those lattices (called **Hutton algebras**) which are complete, completely distributive (in the strong sense), and equipped with an order-reversing involution, and have as morphisms those mappings which preserve arbitrary joins and the involution (and hence arbitrary meets as well); and let **FUZLAT** (Hutton's designation) be $\mathbf{HUT}^{op}$. Then in the preceding paragraph the bases (i.e. the "L", "M") are objects of **FUZLAT** (called fuzzy lattices), and the morphisms between them are **FUZLAT** morphisms. See the formal definition of Hutton's approach in Subsection 6.1 below.

There are two important threads left to weave. Soon after [25] there appeared [53] in which this author constructed the first variable-basis category for topology in which the underlying sets of the spaces are non-singletons, i.e. have unrestricted cardinality: objects are of the form (X, L, τ), where τ is a **topol-**

ogy on (X, L)—i.e. τ is an L-topology on X, and morphisms are of the form (f, ϕ), where f is a mapping between sets and ϕ is a dual morphism between bases—see Section 3 below. Thus, in such a theory, both the underlying set was allowed to change (as in a fixed-basis theory) and the basis was allowed to change (as in the localic or Hutton theory). In [53], this author gave the needed backward powerset operator for continuity, but only described his ground category implicitly. Further, [53] formally required complete distributivity and order-reversing involutions of the underlying lattices, the latter being retained in hopes that at this abstract level closed fuzzy subsets would somehow play an important role.

The first explicit description of this author's ground was given by P. Eklund in papers [5, 6] taken from his Ph.D. dissertation [7], in which Eklund also began the first study of the categorical properties of variable-basis topology, thus initiating categorical fuzzy topology. While Hutton was to the first to separately describe a ground category vis-a-vis the overlying topological framework, it was Eklund who first formalized the notion of ground category in fuzzy topology (e.g. see [5]).

Influenced by Eklund and by localic theory, especially by the book of Johnstone (*op. cit.*), this author began to refine the ground categories for variable-basis topology in two basic ways: *firstly* by removing restrictions—relaxing the requirements of order-reversing involution (for locales do not have such involutions, and neither will the bases of their fuzzy sober space representations), complete distributivity (locales generate singleton spaces with underlying bases which have only the first infinite distributive law), and then any distributivity whatsoever (the classical representation theory for locales makes no essential use of distributivity)—so that by the end of [53, 56, 61, 62, 64, 65], the only requirements were completeness (of the base lattice) and preservation of arbitrary joins and finite meets (by the base morphisms); and *secondly* by giving rigorous developments and justifications of forward and backward powerset operators for both fixed-basis and variable-basis ground categories ([62, 64], especially [66]). Out of these refinements come the new categories **SFRM** and **SLOC** of semiframes and semilocales (see Subsection 1.3 below).

The second and last thread is the notion of fuzzy topology as defined by A. Šostak in [75, 78] in which the topology is a fuzzy subset of the L-powerset having certain properties. This notion was motivated by an early paper of U. Höhle [19] in which the topology was a fuzzy subset of the classical powerset. All these theories are fixed-basis (L-)fuzzy topological theories. A refinement of this approach, with bases taken from **CQML** (complete quasi-monoidal lattices—Subsubsection 1.3.2 below), is thoroughly developed elsewhere in this volume [23] leading to fixed-basis categories of the form L-**FTOP**, and is also distinguished by the non-algebraic character of the multiplication or "intersection" endued from the underlying objects of **CQML** (see also [22] in this regard).

¿From the above paragraphs comes a three-fold historical motivation for this chapter:

(1) Construct ground categories for variable-basis topology in the manner of

[64] but with bases from **CQML**, i.e. make "non-algebraic" grounds for variable-basis topology.

(2) Construct categorical frameworks for variable-basis fuzzy topology which incorporates the fixed-basis categories of the extant literature, especially [23] of this volume, thus marrying this author's variable-basis approach to fuzzy topologies in the sense of Höhle/Šostak.

(3) Explicitly place the constructions of (1) and (2) upon the powerset operator foundation detailed in [66] and summarized in [67].

1.2 Categorical motivation for variable-basis theories

The categorical motivation for variable-basis theories in topology is three-fold:

(1) For each $\mathbf{C} \hookrightarrow \mathbf{LOQML} \equiv \mathbf{CQML}^{op}$, provide a single categorical framework **C-TOP** or **C-FTOP** (Subsection 3.1) for all canonical examples residing in fixed-basis frameworks with bases from $|\mathbf{C}|$. As an example, let $\mathbf{C} = \mathbf{DQML}^{op}$ (Subsubsection 1.3.2 below); then for each $L \in |\mathbf{DQML}^{op}|$, there is the fuzzy real line $\mathbb{R}(L)$ [12, 62] and fuzzy unit interval $\mathbb{I}(L)$ [24, 39] as canonical objects in the fixed-basis category L-**TOP**. A variable-basis category for topology of the form $\mathbf{DQML}^{op}$**-TOP** should allow a comparison of two fuzzy real lines with different bases in the sense of morphisms between them. An expected result should be: $\mathbb{R}(L_1)$ is categorically isomorphic to $\mathbb{R}(L_2)$ iff L_1 is order-isomorphic to L_2. We note that *only* in a variable-basis category is the question of such comparisons *well-posed.*

(2) For each $\mathbf{C} \hookrightarrow \mathbf{LOQML}$, provide a single categorical framework **C-TOP** or **C-FTOP** for all fixed-basis frameworks with bases from $|\mathbf{C}|$. As an example, let $\mathbf{C} = \mathbf{LOC}$; then $\forall L \in \mathbf{LOC}$, L-**TOP** should be embeddable as a subcategory of **LOC-TOP**, and L-**FTOP** should be embeddable as a subcategory of **LOC-FTOP**. An even more strategic "subcategory" example: a variable-basis category of the form **LOC-TOP** should be a supercategory (up to functorial embeddings) of both classical categories **TOP** and **LOC**, and as such **LOC-TOP** would be the first such cohering of **TOP** with **LOC**.

(3) For each $\mathbf{C} \hookrightarrow \mathbf{LOQML}$, provide categorical frameworks **C-TOP** and **C-FTOP** for variable-basis theories which are *topological over their ground categories* in the sense of [1]. This will not only assure that these frameworks have good properties whenever their grounds have good properties, but that they behave categorically like classical topological frameworks such as **TOP**, **UNIF**, etc; and this means, from a categorical point of view, one is doing topology whenever one is doing variable-basis topology and fuzzy topology.

It follows that the categorical justification of variable-basis theories for topology and fuzzy topology will therefore also be three-fold:

(1) **Internal justification**—that from within resting on the cohering of similar canonical examples having different bases. See Section 7 below, based partly on Section 5 below.

(2) **External justification**—that from without resting on the framework in question cohering fixed-basis frameworks and dissimiliar categories (such as **TOP** and **LOC**). See Section 6 below.

(3) **Categorical justification**—that from without resting on the framework in question being topological over its ground category. See Sections 3 and 4 below.

When variable-basis theories are constructed and justified (Sections 3, 6, 7), and fixed-basis theories are embedded in these variable-basis theories (Section 6), a new understanding of fixed-basis theories emerges: not only does a fixed-basis approach fix a base lattice, but it also fixes the identity mapping as the endomorphism for that lattice. This raises the question of fixing a base lattice and fixing a base endomorphism other than the identity mapping. And such a question leads unavoidably to categories of **endomorphism-saturated spaces** and their distinctive grounds, categories best studied by adapting techniques from the pure variable-basis setting. Categories of endomorphism-saturated spaces are defined, justified by examples, and justified as topological over their grounds in Section 4.

Finally, this chapter complements the companion chapter [23]: the fixed-basis categories of that chapter embed into the variable-basis categories of this chapter, and the proofs of topological in that chapter can be recovered by applying these embeddings to the proofs of topological in this chapter.

1.3 Mathematical preliminaries and foundations

It is the purpose of this subsection to give those mathematical preliminaries underlying the preceding two subsections and needed in subsequent sections.

1.3.0. Adjunction of categories. Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{C} \leftarrow \mathbf{D}$ be functors. We say F is **left-adjoint** to G iff the following two criteria are satisfied:

(1) **Lifting/Continuity criterion**:

$$\forall A \in |\mathbf{C}|, \exists \eta : A \to GF(A), \forall B \in |\mathbf{D}|, \forall f : A \to G(B),$$

$$\exists ! \overline{f} : F(A) \to B,\ f = G(\overline{f}) \circ \eta$$

In diagram form, we have the following "Major Diagram":

MAJOR DIAGRAM: LIFTING/CONTINUITY

$$\begin{tikzcd}
& \mathbf{C} & \mathbf{D} \\
& \forall A\ (1) \arrow[dl, "\forall f\ (4)"'] \arrow[d, "\exists \eta_A\ (2)"] \arrow[dl, phantom, shift left=3ex, "\circ\ (6)"] & F(A) \arrow[d, "\exists ! \overline{f}\ (5)"] \\
G(B) & GF(A) \arrow[l, "G(\overline{f})"] & \forall B\ (3)
\end{tikzcd}$$

(2) **Naturality criterion**:

$$\forall f : A_1 \to A_2 \text{ in } \mathbf{C},\ F(f) = (\text{or} \equiv)\ \overline{\eta_{A_2} \circ f}$$

where the "$\equiv$" option allows F to only be defined initially on objects, in which (1) and "$\equiv$" will stipulate an action of F on morphisms such that $F \dashv G$. In daigram form, we have the following "Minor Diagram":

MINOR DIAGRAM (ASSUMING MAJOR DIAGRAM)

$$\left[\begin{tikzcd}
A_1 \arrow[r, "f"] \arrow[d, "\eta_{A_1}"'] \arrow[dr, phantom, "\circ"] & A_2 \arrow[d, "\eta_{A_2}"] & F(A_1) \arrow[d, "! \overline{\eta_{A_2} \circ f}"] \\
GF(A_1) \arrow[r, "GF(\overline{\eta_{A_2} \circ f})"'] & GF(A_2) & F(A_2)
\end{tikzcd}\right]$$

$$\Longrightarrow F(f) = (\mathit{or} \equiv)\ \overline{\eta_{A_2} \circ f}$$

We also say that G is **right-adjoint** to F or that (F, G) is an **adjunction**, or we may write $F \dashv G$. The map η is the **unit** of the adjunction; and the dual

D morphism ε of η in the duals of the above statements is the **counit** of the adjunction. If both η and ε are isomorphisms in their respective categories, then (F, G) is an **equivalence (of categories)**.

The foregoing definition is reworded from [48] and is used repeatedly in Section 6. If it is instantiated with two posets viewed as categories and two order-preserving maps viewed as functors, then this definition reduces to saying that $id_{\mathbf{C}} \leq G \circ F$ and $F \circ G \leq id_{\mathbf{D}}$.

We also need the following statement of the Adjoint Functor Theorem referred to in Lemma 1.3.2 below and used repeatedly throughout this chapter.

Theorem (Adjoint Functor Functor (AFT)). Let $f \in \mathbf{SET}(L, M)$, where $L, M \in |\mathbf{POSET}|$, L has arbitrary $\bigvee$, and f preserves $\bigvee$. Then $f \in \mathbf{POSET}(L, M)$ and $\exists\,!\, g \in \mathbf{POSET}(M, L)$ such that $f \dashv g$. This g preserves all arbitrary $\bigwedge$ existing in M and is given by

$$g(b) = \bigvee \{a \in L : f(a) \leq b\}$$

The proof may be found in [28]; and we note the dual of this statement holds and is also designated Adjoint Functor Theorem (AFT).

1.3.1 Discussion (Definition of a topological category). We first give a short and concise definition from [1].

Short Definition. Category $\mathbf{A}$ is **topological w.r.t. category X and functor** $V : \mathbf{A} \to \mathbf{X}$ iff each V-structured source in $\mathbf{X}$ has a unique, initial V-lift in $\mathbf{A}$. We may also say that $\mathbf{A}$ is **topological over X w.r.t. functor** V. If the category $\mathbf{X}$ and the functor V are understood, we say $\mathbf{A}$ is (a) **topological (category)**.

We now midrash this short definition into a longer definition to make it absolutely clear what is the intention of [1] in this definition w.r.t. each of the terms "V-structured source", "V-lift", "initial", and "unique".

Long Definition. Let categories $\mathbf{A}$ and $\mathbf{X}$ be given, as well as a functor $V : \mathbf{A} \to \mathbf{X}$, and let $X \in |X|$, $A_j \in |A|$, and $f_j : X \to V(A_j)$ in $\mathbf{X}$, where $j \in J$ for some indexing set J. Then $(X,\ f_j : X \to V(A_j))_J$ is a V-**structured source**. Category $\mathbf{A}$ is **topological w.r.t. category X and functor** V iff for each such V-structured source all the following hold:

(1) V-**lift**: $\exists\, \hat{X} \in |\mathbf{A}|$, $\exists \left(\hat{f}_j : \hat{X} \to A_j\right)_J$ in $\mathbf{A}$, $V(\hat{X}) = X$, $V\left(\hat{f}_j\right) = f_j$.

(2) **Initial V-lift**: Given $\left(\hat{X},\ \hat{f}_j : \hat{X} \to A_j\right)_J$ in **A** from (1), then $\forall \left(\hat{Y},\ \hat{g}_j : \hat{Y} \to A_j\right)$ in **A** with $Y \equiv V\left(\hat{Y}\right)$ and $g_j \equiv V(\hat{g}_j)$, $\forall h : Y \to X$ in **X** with $g_j = f_j \circ h$ in **X**, $\exists!\hat{h} : \hat{Y} \to \hat{X}$ in **A** with $h = V(\hat{h})$ and $\hat{g}_j = \hat{f}_j \circ h$ in **A**.

(3) **Unique initial V-lift**. Given $\left(\hat{X},\ \hat{f}_j : \hat{X} \to A_j\right)_J$ satisfying (1) and (2), and given $\left(\bar{X},\ \bar{f}_j : \bar{X} \to A_j\right)_J$ satisfying (1) and (2), $\hat{X} = \bar{X}$ and $\hat{f}_j = \bar{f}_j$.

Interpretation of Long Definition. Let categories **A** and **X** and functor V be given such that the following hold:

(1) Each object of **A** is of the form (X, S), where $X \in |\mathbf{X}|$ and S is a **structure on** X, where $(X,S) = (\overline{X}, \overline{S}) \Leftrightarrow X = \overline{X},\ S = \overline{S}$.

(2) Each morphism of **A** is a morphism of **X** satisfying some stipulated property P in this sense: $f \in \mathbf{A}\left((X_1, S_1), (X_2, S_2)\right)$ means $f \in \mathbf{X}(X_1, X_2)$ and f satisfies property P.

(3) $V : \mathbf{A} \to \mathbf{X}$ is a forgetful functor in the usual sense: $V(X,S) = X$, $V(f) = f$. This implies that the composition of **A** is formally that of **X**.

Then **A** is topological w.r.t. **X** and functor V iff the relevant parts of the Long Definition read as follows (modulo labeling):

- **V-Structured source**: $(X,\ f_j : X \to V(X_j, S_j))_J$.
- **V-Lift**: $\exists S$ on X, $(X,S) \in |\mathbf{A}|$ and each f_j satisfies P.
- **Initial V-lift**: $\forall ((Y,T),\ g_j : (Y,T) \to (X_j, S_j))_J$ in $|\mathbf{A}|$, $\forall h : Y \to X$ in **X**,
$$(\forall j \in J,\ g_j = f_j \circ h \text{ in X}) \Rightarrow (h \text{ satisfies P})$$
- **Unique, initial V-lift**: $\forall \bar{S}$ on X, $(X, \bar{S}) \in |\mathbf{A}|$,
$$\forall \left(\bar{f}_j : (X, \bar{S}) \to (X_j, S_j)\right)_J$$
in **A**,
$$\left((X,\bar{S}),\ \left(\bar{f}_j : (X,\bar{S}) \to (X_j, S_j)\right)_J\right)$$
is another initial V-lift for the original V-source, $\bar{S} = S$.

If **A** is topological over **SET** w.r.t. the usual forgetful functor V, then **A** is called a **topological construct** [1].

Proposition. If $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{X} \to \mathbf{Y}$ are isomorphisms of categories and $V : \mathbf{A} \to \mathbf{X}$ is a functor, then **A** is topological over **X** w.r.t. V iff **B** is topological over **Y** w.r.t. $G \circ V \circ F^{-1}$.

1.3.2. Lattice-theoretic foundations. Each of the following categories defined below is either concrete over **SET** or $\mathbf{SET}^{op}$, in which case the composition and identities of each of the following categories is from **SET** or $\mathbf{SET}^{op}$. The categories **FRM**, **LOC**, and **HUT** were defined in Subsection 1.1 above. Throughout this chapter, the lower [upper] bounds of a complete lattice are denoted $\perp$ [$\top$, respectively].

Definition (Semiframes and semilocales). The category **SFRM** of **semiframes** comprises complete lattices and morphisms preserving arbitrary joins and finite meets; and the category **SLOC** of **semilocales** is $\mathbf{SFRM}^{op}$. Note that **FRM** [**LOC**] is a full, isomorphism closed category of **SFRM** [**SLOC**].

Definition (DeMorgan lattices/algebras). The category **DMRG** comprises all complete lattices having an order-reversing involution or **deMorgan complementation**, called **complete deMorgan lattices** (or **algebras**), and morphisms preserving arbitrary joins and all involutes (and hence arbitrary meets)—said objects have the minimum axioms needed to guarantee that both deMorgan laws hold, justifying their name. The lack of any required distributivity is justified by **DMRG** including many natural and important examples. The lattice of closed linear subspaces of Hilbert space is a complete deMorgan algebra in our sense, but is non-distributive for $dim \geq 2$ (cf. Open Question X of [72]); actually, such lattices are objects in **ORTHODMRG**, the full subcategory of **DMRG** in which the deMorgan complementation is also an orthocomplementation. Easier examples of non-distributive deMorgan lattices include the symmetric, five-point diamond (which is in **DMRG** – **ORTHODMRG**) and the symmetric, six-point diamond (which is in **ORTHODMRG**).

DMRG [$\mathbf{DMRG}^{op}$] is an isomorphism closed subcategory of **SFRM** [**SLOC**], as is **HUT** [**FUZLAT**] defined in Subsection 1.1 above of **DMRG** [$\mathbf{DMRG}^{op}$]; analogous statements hold for the category **CBOOL** [$\mathbf{CBOOL}^{op}$] of complete Boolean algebras and [duals] with arbitrary $\bigvee$ and arbitrary $\bigwedge$ preserving maps.

Definition (Complete quasi-monoidal lattices). The category **CQML** of **complete quasi-monoidal lattices** comprises the following data (cf. [23]), where composition and identities are taken from **SET**:

(1) **Objects**: Complete lattices L equipped with a binary operation or **tensor product** $\otimes : L \times L \to L$ satisfying:

 (a) $\otimes$ is isotone in both arguments,

 (b) $\top \otimes \top = \top$, where $\top$ is the upper bound of the lattice L.

(2) **Morphisms**: All **SET** morphisms, between the above objects, which preserve $\otimes$ and $\top$ and arbitrary $\bigvee$.

CQML$_\perp$ is the full subcategory of **CQML** for which in each object the identities $\perp \otimes \perp = \perp$, $\perp \otimes \top = \top$, $\top \otimes \perp = \perp$ hold, where $\perp$ is the lower bound of the lattice.

Definition (Localic quasi-monoidal lattices). The category **LOQML** is the dual of **CQML**, i.e. $\mathbf{LOQML} = \mathbf{CQML}^{op}$; and $\mathbf{LOQML}_\perp = \mathbf{CQML}_\perp^{op}$. The prefix "localic" is used to indicated the dual category consistent with the usage of **LOC** and **SLOC**. The objects of **LOQML**, **SLOC**, and **LOC** all "are" singleton spaces according to Subsection 6.2 and topologies according to Subsection 7.4.

Definition (DeMorgan quasi-monoidal lattices). The category **DQML** is the subcategory of **CQML** in which each object is additionally equipped with an order-reversing involution and each morphism additionally preserves the involution.

Definition (Categories L, $\mathbf{L}_\phi$, and $\mathbf{L}_\mathbf{C}$). Given $L \in |\mathbf{LOQML}|$, the category **L** has L as its sole object and id_L from **LOQML** as its sole morphism; and the category $\mathbf{L}_\phi$ has L as its sole object, $\phi \in \mathbf{LOQML}(L, L)$ as its sole morphism, composition as $\phi \circ \phi = \phi$, and identity morphism for L is ϕ. Note that $\mathbf{L}_\phi$ is a subcategory of **LOQML** iff $\phi^{op} = id_L$, in which case $\mathbf{L}_\phi = \mathbf{L}$. The reasons for $\mathbf{L}_\phi$ and for using duals will emerge in Sections 2–4. If $\mathbf{C} \hookrightarrow \mathbf{LOQML}$, then $\mathbf{L}_\mathbf{C}$ is the category having L as its sole object and $\mathbf{C}(L, L)$ as its morphisms.

Proposition. Each of **SFRM** and **FRM** embeds into **CQML** as full isomorphism closed subcategories by assigning to each $(L, \leq, \vee, \wedge, \perp, \top)$ the quasi-monoidal lattice $(L, \leq, \vee, \wedge, \perp, \top, \otimes)$, where $\otimes$ is defined by restricting $\wedge$ to two arguments. Each of **DMRG**, **CBOOL**, and **HUT** embeds similarly into **CQML**, as well as **SLOC**, **LOC**, $\mathbf{CBOOL}^{op}$, $\mathbf{DMRG}^{op}$, and **FUZLAT** into **LOQML**. Similar results hold for **DMRG**, **CBOOL**, and **HUT** w.r.t. **DQML** and for $\mathbf{CBOOL}^{op}$, $\mathbf{DMRG}^{op}$, and **FUZLAT** w.r.t. $\mathbf{DQML}^{op}$.

The following lemmas about lattice and **CQML** morphisms are extremely useful throughout this chapter and make reference to the adjoint functor theorem which, together with other adjunction issues, is given in 1.3.0 above. Lemma 7.A of Section 7 uses, and gives extensions of, these lemmas for **DQML** morphisms.

Lemma (Order-embeddings). Let L, M be lattices and $f : L \to M$ be a function. Then f is an order-isomorphism of L with $f^{\to}(L)$ (i.e. $\alpha \leq \beta$ iff $f(\alpha) \leq f(\beta)$) if either of the following hold:

(1) f is injective and preserves finite $\vee$;

(2) f is injective and preserves finite $\wedge$.

Lemma (Properties of right adjoints). Let $\psi : L \leftarrow M$ be given in **CQML**. Then there exists by the adjoint functor theorem for partial-order preserving maps (AFT) a right-adjoint $\psi^* : L \rightarrow M$ having the following properties:

(1) $\psi^*(\alpha) = \bigvee\{\beta \in M : \psi(\beta) \leq \alpha\}$.

(2) $\psi(\psi^*(\alpha)) \leq \alpha,\ \psi^*(\psi(\beta)) \geq \beta$.

(3) ψ^* is a right [left] inverse of ψ iff ψ is surjective [injective], i.e.

$$\psi\left(\psi^*\left(\alpha\right)\right) = \alpha \Leftrightarrow \psi \text{ surjective}, \quad \psi^*\left(\psi\left(\alpha\right)\right) = \alpha \Leftrightarrow \psi \text{ injective}$$

(4) ψ^* preserves arbitrary $\bigwedge$, and hence is order-preserving;

(5) ψ^* is submultiplicative, i.e. $\psi^*(\alpha) \otimes \psi^*(\beta) \leq \psi^*(\alpha \otimes \beta)$.

Proof. (1,2,4) are immediate consequences of AFT or its proof. Now (3) follows from (2), and (5) follows from (1) and the properties of ψ. We note (3,5) are observed in the companion chapter [23]. □

1.3.3 Powerset operator foundations. We give the powerset operators, developed and justified in detail in [66] and extended in [67], which are specifically needed in this chapter. Let $f \in \mathbf{SET}(X,Y)$, $L, M \in |\mathbf{CQML}|$, $\phi \in \mathbf{LOQML}(L,M)$, and $\mathcal{P}(X)$, $\mathcal{P}(Y)$, L^X, M^Y be the classical powerset of X, the classical powerset of Y, the L-powerset of X, and the M-powerset of Y, respectively. Then the following powerset operators are defined:

$$f^{\rightarrow} : P(X) \rightarrow P(Y) \text{ by } f^{\rightarrow}(A) = \{f(x) \in Y : x \in A\}$$

$$f^{\leftarrow} : P(X) \leftarrow P(Y) \text{ by } f^{\leftarrow}(B) = \{x \in X : f(x) \in B\}$$

$$f_L^{\rightarrow} : L^X \rightarrow L^Y \text{ by } f_L^{\rightarrow}(a)(y) = \bigvee_{f(x)=y} a(x)$$

$$f_L^{\leftarrow} : L^X \leftarrow L^Y \text{ by } f_L^{\leftarrow}(b) = b \circ f$$

$${}^*\phi : L \rightarrow M \text{ in } \mathbf{POSET} \text{ by } {}^*\phi(\alpha) = \bigwedge\{\beta \in M : \alpha \leq \phi^{op}(\beta)\}$$

$$\langle{}^*\phi\rangle : L^X \rightarrow M^X,\ \langle{}^*\phi\rangle : L^Y \rightarrow M^Y \text{ by } \langle{}^*\phi\rangle(a) = {}^*\phi \circ a$$

$$\langle\phi^{op}\rangle : L^X \leftarrow M^X,\ \langle\phi^{op}\rangle : L^Y \leftarrow M^Y \text{ by } \langle\phi^{op}\rangle(b) = \phi^{op} \circ b$$

$$(f,\phi)^{\rightarrow} : L^X \rightarrow M^Y \text{ by } (f,\phi)^{\rightarrow} = \langle{}^*\phi\rangle \circ f_L^{\rightarrow}$$

$$(f,\phi)^{\leftarrow} : L^X \leftarrow M^Y \text{ by } (f,\phi)^{\leftarrow} = f_L^{\leftarrow} \circ \langle\phi^{op}\rangle$$

These powerset operators link the ground categories constructed in Section 2 with the topological categories constructed in Sections 3 and 4.

Notation (Constant fuzzy subsets). Constant members of L^X are denoted $\underline{\alpha}$, where $\alpha \in L$ is the sole value taken by this fuzzy subset. E.g. $\underline{\perp}$ is that fuzzy subset which takes only the value $\perp$.

The following lemma is proved in [67] for $\otimes = \wedge$ and is used repeatedly throughout this chapter.

Lemma. $(f, \phi)^{\leftarrow}$ preserves $\otimes$, $\underline{\top}$, and arbitrary $\bigvee$.

2 Ground categories SET $\times$ C and SET $\times$ $\mathbf{L}_\phi$

Concrete categories, such as found in classical algebra and topology, rest on what [5, 7] calls "ground categories" or what [1] calls "base categories". The topological theories being created in fuzzy sets are determined by certain data, including the following: the ground (or base) category for a set theory, the powerset operators of the ground category, the forgetful functor from the category for "topology" into the ground category, and the particular structure carried by the objects of the category for "topology" and preserved by the morphisms of that category.

Variable-basis topology and fuzzy topology require grounds which are products of two categories: the first category, which in this chapter will always be **SET**, carries "point-set" information, and the second category, which in this chapter will always be a subcategory **C** of the category **LOQML**, carries "lattice-theoretic" information. These grounds, and the topology built on such grounds, are said to be **point-set lattice-theoretic** or **poslat** (see Section 3 below).

For clarity of terminology, we shall not use the term "base category", but rather the term **ground category** or simply **ground**. The avoidance of the term "base category" stems from our need to have a label for the objects of category **C** carrying the lattice-theoretic information—an object L of **C** is called a **base** since the L(-fuzzy) powerset of a set X is of the form L^X—and our desire to avoid using the word "base" in too many different ways, since it will also be used of the base of a topology or fuzzy topology.

As shown in [1], if a category can be shown "topological" over its ground w.r.t. a forgetful functor, then that category essentially behaves (relative to its ground) as **TOP** (relative to **SET**) and inherits many nice properties from the ground (providing that ground itself has nice properties *ab initio*). In the next section we show that all categories for variable-basis topology and and variable-basis fuzzy topology are topological over their grounds with the expected forgetful functor.

Therefore, it is the primary goal of this section is to define grounds of the form $\mathbf{SET} \times \mathbf{C}$, where $\mathbf{C}$ is a subcategory of **LOQML**, and discuss the categorical properties of these grounds. Since **SET** is well-known, discussion of ground categorical properties essentially reduces to understanding the categorical properties of $\mathbf{C}$.

2.1 Definitions of $\mathbf{SET} \times \mathbf{C}$ and $\mathbf{SET} \times \mathbf{L}_\phi$

2.1.1 Definition. Let $\mathbf{C}$ be a subcategory of **LOQML**. The **ground category** $\mathbf{SET} \times \mathbf{C}$ comprises the following data:

(1) **Objects.** (X, L), where $X \in |\mathbf{SET}|$ and $L \in |\mathbf{C}|$. The object (X, L) is a **(ground) set.**

(2) **Morphisms.** $(f, \phi) : (X, L) \to (Y, M)$, where $f : X \to Y$ in **SET**, $\phi : L \to M$ in $\mathbf{C}$. The morphism (f, ϕ) is **(ground) function**.

(3) **Composition**. Component-wise in the respective categories.

(4) **Identities.** Component-wise in the respective categories, i.e. $id_{(X,L)} = (id_X, id_L)$.

2.1.2 Definition. Let $\mathbf{C}$ be a subcategory of **LOQML**, $L \in |\mathbf{C}|$, and $\phi \in \mathbf{C}(L, L)$. The **ground category** $\mathbf{SET} \times \mathbf{L}_\phi$ (1.3.2) comprises the following data:

(1) **Objects**. (X, L), where $X \in |\mathbf{SET}|$. The object (X, L) is a **(ground) set.**

(2) **Morphisms**. $(f, \phi) : (X, L) \to (Y, L)$, where $f : X \to Y$ in **SET**. The morphism (f, ϕ) is **(ground) function.**

(3) **Composition**. $(f, \phi) \circ (g, \phi) = (f \circ g, \phi)$, where the composition in the first component is in **SET**.

(4) **Identities.** $id_{(X,L)} = (id_X, \phi)$, where the identity in the first component is in **SET**.

2.1.3 Definition (Ground equalities). In $\mathbf{SET} \times \mathbf{C}$, $(X, L) = (Y, M)$ iff $X = Y$ and $L = M$, and $(f, \phi) = (g, \psi)$ iff they have the same domain and codomain, $f = g$ and $\phi = \psi$. Equality of objects and morphisms is defined similarly in $\mathbf{SET} \times \mathbf{L}_\phi$ by requiring equality in **SET** of the first components.

2.1.4 Remark (Powerset operators). For $(f,\phi) : (X,L) \to (Y,M)$ in $\mathbf{SET} \times \mathbf{C}$ or $\mathbf{SET} \times \mathbf{L}_\phi$, the forward powerset operator $(f,\phi)^\to : L^X \to M^Y$ of 1.3.3 can be rewritten

$$(f,\phi)^\to(a) = \bigwedge\{b : f_L^\to(a) \le \langle\phi^{op}\rangle(b)\}$$

and the backward powerset operator $(f,\phi)^\leftarrow : L^X \leftarrow M^Y$ of 1.3.3 can be rewritten

$$(f,\phi)^\leftarrow(b) = \phi^{op} \circ b \circ f$$

2.1.5 Notation. If a set [lattice] is clear in context, then its identity mapping in $\mathbf{SET}$ [$\mathbf{C}$] may be simply written as *id*. In particular we use the following notation: $\mathbf{SET} \times \mathbf{L} = \mathbf{SET} \times \mathbf{L}_{id}$. (See the sixth definition of 1.3.2 above.)

2.1.6 Theorem. Let $\mathbf{C}$ be a subcategory of $\mathbf{LOQML}$, $L \in |\mathbf{C}|$, and $\phi \in \mathbf{C}(L,L)$. The following hold:

(1) $\mathbf{SET} \times \mathbf{C}$ is a category.

(2) $\mathbf{SET} \times \mathbf{L}_\phi$ is a category.

(3) The objects and morphisms of $\mathbf{SET} \times \mathbf{L}_\phi$ are objects and morphisms, respectively, of $\mathbf{SET} \times \mathbf{C}$.

(4) $\mathbf{SET} \times \mathbf{L}_\phi$ is a subcategory of $\mathbf{SET} \times \mathbf{C} \Leftrightarrow \phi = id \Leftrightarrow \mathbf{SET} \times \mathbf{L}_\phi = \mathbf{SET} \times \mathbf{L}$.

(5) If ϕ^{op} preserves arbitrary $\bigwedge$, then $^*\phi \dashv \phi^{op}$ and $(f,\phi)^\to \dashv (f,\phi)^\leftarrow$, in which case $(f,\phi)^\to$ preserves arbitrary $\bigvee$ and $(f,\phi)^\leftarrow$ preserves arbitrary $\bigwedge$. It is also the case that $(f,\phi)^\leftarrow$ preserves arbitrary $\bigvee$.

(6) (f,ϕ) is an isomorphism in $\mathbf{SET} \times \mathbf{C} \Leftrightarrow f$, ϕ are isomorphisms in $\mathbf{SET}$, $\mathbf{C}$, respectively $\Leftrightarrow f$, ϕ are bijections.

(7) (f,ϕ) is an isomorphism in $\mathbf{SET} \times \mathbf{L}_\phi \Leftrightarrow f$ is a bijection. Hence (f, id) is an isomorphism in $\mathbf{SET} \times \mathbf{L} \Leftrightarrow f$ is a bijection.

Proof. Statements (1–3, 6–7) are trivial; statement (4) is straightforward; and see [66] for details of (5). □

2.1.7 Remark.

(1) Categories of the form $\mathbf{SET} \times \mathbf{C}$ provide a ground for variable-basis topology and variable-basis fuzzy topology. See Section 3.

(2) As an application of (1), let $\mathbf{L}_{\mathbf{LOQML}}$ be the subcategory of **LOQML** whose single object is L in $|\mathbf{LOQML}|$ and whose morphisms comprise $\mathbf{LOQML}(L, L)$. Then the ground $\mathbf{SET} \times \mathbf{L}_{\mathbf{LOQML}}$ underlies variable-basis fuzzy topology in which the basis is fixed even though the basis morphisms are **not** fixed. More generally we can define $\mathbf{SET} \times \mathbf{L}_{\mathbf{C}}$ for $\mathbf{C} \hookrightarrow \mathbf{LOQML}$. See 1.3.2 and Section 6.

(3) Categories of the form $\mathbf{SET} \times \mathbf{L}_{\phi}$ provide ground categories for internalized change-of-basis topology and fuzzy topology, i.e. topologies invariant under a fixed endomorphism ϕ of L. See 1.3.2 and Section 4.

(4) Categories of the form $\mathbf{SET} \times \mathbf{L}$ provide ground categories for both fixed-basis topology and fixed-basis fuzzy topology in the sense of [23], in which case the functions, or ground morphisms, are often written f instead of (f, ϕ). See 1.3.2 and Section 6 below.

There are categorical isomorphisms between internalized change-of-basis set theories on one hand and fixed-basis set theories on the other; and these isomorphisms bridge the formal breach of 2.1.6(4) between variable-basis and internalized change-of-basis for the cases $\phi \neq id$. In fact, internalized change-of-basis set theories may always be viewed (up to functorial embeddings or isomorphisms) as both classical theories and subcategories of variable-basis set theories, as is seen in the next result.

2.1.8 Proposition. Let $\mathbf{C}$ be a subcategory of **LOQML**, $L \in |\mathbf{C}|$, and $\phi \in \mathbf{C}(L, L)$.

(1) The functor $F : \mathbf{SET} \to \mathbf{SET} \times \mathbf{L}$, given by $F(X) = (X, L)$, $F(f) = (f, id)$, is an isomorphism.

(2) The functor $F : \mathbf{SET} \times \mathbf{L}_{\phi} \to \mathbf{SET} \times \mathbf{L}$, given by $F(X, L) = (X, L)$, $F(f, \phi) = (f, id)$, is an isomorphism embedding $\mathbf{SET} \times \mathbf{L}_{\phi}$ onto a subcategory of $\mathbf{SET} \times \mathbf{C}$.

Proof. Straightforward. □

2.1.9 Remark.

(1) Even though **SET**, $\mathbf{SET} \times \mathbf{L}$, and $\mathbf{SET} \times \mathbf{L}_{\phi}$ are all isomorphic, we maintain these distinct notations for the following reasons:

   (a) The syntax of the morphisms is different—f *vis-a-vis* (f, id) *vis-a-vis* (f, ϕ).

   (b) The *associated powerset operators*, i.e. the underlying monadic structure, are mathematically different— $f^{\to}$, $f^{\leftarrow}$ *vis-a-vis* $(f, id)^{\to}$, $(f, id)^{\leftarrow}$ (essentially $f_L^{\to}$, $f_L^{\leftarrow}$) *vis-a-vis* $(f, \phi)^{\to}$, $(f, \phi)^{\leftarrow}$ (see 1.3.3).

(c) The topological theories based on these ground categories are different due to (1,2) above—see (2–4) below and Sections 3, 4, 6 of this chapter.

(2) The F of 2.1.8(1) is an isomorphism between **SET** and **SET** $\times$ **L** which weakens with each known lifting to **TOP** and an overlying topological category in Sections 3, 4, and 6. More precisely, using the functorial embeddings of Subsection 6.2, we have:

(a) when F is lifted to G_χ, it remains an isomorphism iff $L = \{\bot, \top\}$; otherwise it need only have a right adjoint;

(b) when F is lifted to the functor ω_L (for L a continuous lattice [34, 40]), it becomes an epicoreflection; and

(c) when F is lifted to the α-level functor F_α, only under certain lattice-theoretic conditions does it have a right adjoint.

(3) The F of 2.1.8,(2) is an isomorphism between **SET** $\times$ $\mathbf{L}_\phi$ and **SET** $\times$ **L** which weakens when lifted to overlying topological categories—more precisely, when topological categories are built in Sections 3, 4, and 6 upon **SET** $\times$ $\mathbf{L}_\phi$ and **SET** $\times$ **L**, the isomorphism lifts via the forgetful functor to an isocoreflective adjunction which is not a categorical equivalence.

(4) We make two conclusions from (1–3). First, categorical properties of **SET** $\times$ $\mathbf{L}_\phi$ are the same as **SET**, and this has pleasant consequences for topological theories which are either internalized change-of-basis or fixed-basis (see Subsection 6.3). Second, topological theories based on these grounds are categorically significant and different *vis-a-vis* **TOP**.

2.1.10 Discussion (Ground categories and disjoint sum categories). It is possible to interpret internalized change-of-basis ground categories with the same base as summands in the sum category of **SET** with itself in the sense of [17, III.4.10]. To make this precise, let **C** be a subcategory of **LOQML** and $L \in |\mathbf{C}|$. Then we construct the sum category of **SET** with itself over the indexing set $\mathbf{C}(L, L)$ as follows: put

$$\mathbf{SET}_\phi \equiv \mathbf{SET} \times (L, \phi) \cong \mathbf{SET}$$

and then take the disjoint sum category

$$\coprod_{\phi \in \mathbf{C}(L,L)} \mathbf{SET}_\phi$$

in which composition is given by

$$(f, \phi) \circ (g, \psi) = (f \circ g, \phi) \Leftrightarrow \phi = \psi$$

Then it follows that this sum contains each internalized change-of-basis ground as a subcategory; and so this sum provides a common ground for the endomorphism-saturated topological categories of Section 4 and Subsection 6.2. On the other hand, the sum category is rather meagre in good properties.

2.2 Categorical properties of $\mathbf{SET} \times \mathbf{C}$ and $\mathbf{SET} \times \mathbf{L}_\phi$

As stated in the Section introduction,the properties of the variable-basis grounds of the previous Subsection are essentially determined by the properties of the category $\mathbf{C}$; and since $\mathbf{C}$ is a (full) subcategory of **LOQML**, its properties will be dual to that of the concrete (over **SET**) category $\mathbf{C}^{op}$. Furthermore, as a consequence of Remark 2.1.9(3), the properties of each internalized change-of-basis ground $\mathbf{SET} \times \mathbf{L}_\phi$ is completely known—they are exactly the properties of **SET**. Therefore this Subsection shall focus primarily on the categorical properties of variable-basis grounds.

The categorical properties of greatest interest are those which lift from a ground to an overlying topological category (cf. Theorem 21.16 of [1]). Such properties include both the following and some of the associated limits and colimits important in their own right (e.g. products, co-equalizers):

(co)completeness
(regular, extremally) (co-)well-poweredness
(regular, (Epi, Mono-Source)) factorizability
(co)separators

To avoid repetition below, we emphasize that $\mathbf{C}$ is a subcategory of **LOQML**. The gist of the next two results is the rough intuition that some categorical properties based on factoring and equality (e.g. products) are both preserved and reflected by the operation of taking a product category.

2.2.1 Lemma. Let $\prod_{j \in J} E_j$ be a product of categories $(E_j)_{j \in J}$, and let P be any of the following categorical properties: products, coproducts, equalizers, coequalizers, initial obejects, terminal objects. Then $\prod_{j \in J} E_j$ has P iff $\forall j \in J$, E_j has P.

Proof. The proof is quite straightforward. For illustration we prove a particular property for the case $|J| = 2$, leaving the general case and other properties to the reader. We prove:

$$(e, \nu) \text{ equalizes } (f, \phi) \text{ and } (g, \psi) \text{ iff } e \text{ equalizes } f, g \text{ and } v \text{ equalizes } \phi, \psi$$

For necessity, note that since $(f, \phi) \circ (e, \nu) = (g, \psi) \circ (e, \nu)$, then $f \circ e = g \circ e$ and $\phi \circ \nu = \psi \circ \nu$. Now if e' equalizes f, g, and v' equalizes ϕ, ψ, then (e', v') equalizes $(f, \phi), (g, \psi)$; in which case there is a unique (h, ρ) factoring (e', ν')

through (e, ν). But then h factors e' through e, and ρ factors v' through v. And if $\overline{h}$ also factors e' through e, and $\overline{\rho}$ also factors v' through v, then it follows that $(\overline{h}, \overline{\rho})$factors (e', n') through (e, ν) ; in which case it follows that $(\overline{h}, \overline{\rho}) = (h, \rho)$, i.e. $\overline{h} = h,\ \overline{\rho} = \rho$. This completes the proof of necessity for equalizers.

Sufficiency for equalizers follows by reversing the above argumentation. □

2.2.2 Corollary. $\prod_{j\in J} E_j$ is [co]complete iff $\forall\, j \in J,\ E_j$ is [co]complete.

2.2.3 Lemma.

(1) $\prod_{j\in J} E_j$ has [co]separators $\Rightarrow \forall\, j \in J,\ E_j$ has [co]separators.

(2) $\forall\, j \in J,\ E_j$ is connected and has [co]separators $\Rightarrow \prod_{j\in J} E_j$ has [co]separators.

Proof. We give the separator proof for the case $|J| = 2$; the general case and coseparator proof are left to the reader. For (1), let (Z, N) be a separator in the product category, WLOG let $f, g : X \to Y$ be a pair of distinct morphisms in the first category, let L be an object in the second category, and ϕ let be a morphism from L to L in the second category (e.g. $\phi = id$). Then $(f, \phi), (g, \phi) : (X, L) \to (Y, L)$ is a pair of distinct morphisms in the product category. So $\exists\, (h, \rho) : (Z, N) \to (X, L),\ (f, \phi) \circ (h, \rho) \neq (g, \phi) \circ (h, \rho)$, which means $(f \circ h, \phi \circ \rho) \neq (g \circ h, \phi \circ \rho)$, i.e. $f \circ h \neq g \circ h$. It follows that Z is a separator in the first category. A similar argument shows N is a separator in the second category.

For sufficiency, we use the notation of necessity and let Z be a separator in the first category, N be a separator in the second category, and let $(f, \phi), (g, \psi) : (X, L) \to (Y, M)$ be distinct morphisms in the product category. Suppose $f \neq g$ in the first category. Invoke connectedness to obtain $\rho : N \to L$ in the second category, and invoke the existence of separators to obtain a separation $h : Z \to X$ in the first category. Then $f \circ h \neq g \circ h$, and so $(f, \phi) \circ (h, \rho) \neq (g, \psi) \circ (h, \rho)$. This shows that (Z, N) is a separator in the product category in this case. The case when $\phi \neq \psi$ proceeds analogously by appealing to connectedness in the first category and separators in the second category. The case when $f \neq g, \phi \neq \psi$ appeals only to separators in the factor categories. □

2.2.4 Example. Concerning separators, the condition of connectedness in 2.2.3(2) is nonsuperfluous as can be seen by a standard example: letting $\mathbf{E}_1 = \mathbf{SET} = \mathbf{E}_2$, each factor category has separators and coseparators, but is not connected; yet $\mathbf{SET} \times \mathbf{SET}$ does not have separators [1, 7.15(3)]. On the other hand, letting $\mathbf{E}_1 =$ four element Boolean algebra $= \mathbf{E}_2$, each factor as a category has separators [1, 7.18(6)] and is not connected (because of the two incomparable elements); yet the product category $\mathbf{E}_1 \times \mathbf{E}_2$ as a partially-ordered set has separators.

2.2.5 Lemma. The following hold:

(1) **LOC**, **FUZLAT**, and **L** (1.3.2) are complete and cocomplete.

(2) **LOQML**, **SLOC**, $\mathbf{CLAT}^{op}$, $\mathbf{DMRG}^{op}$, and $\mathbf{CBOOL}^{op}$ are cocomplete.

Proof. For (1), see [28] for **LOC** and [25] for **FUZLAT**; cocompleteness in (2) follows from the completeness of the concrete duals, which is done coordinatewise. □

2.2.6 Theorem (Categorical properties of $\mathbf{SET} \times \mathbf{C}$ and $\mathbf{SET} \times \mathbf{L}_\phi$). Let **C** be a subcategory of **LOQML**.

(1) The ground $\mathbf{SET} \times \mathbf{C}$ is:

  (a) Complete and cocomplete if $\mathbf{C} = \mathbf{LOC}$, **FUZLAT**, or **L**.

  (b) Cocomplete if $\mathbf{C} = \mathbf{LOQML}$, **SLOC**, $\mathbf{CLAT}^{op}$, $\mathbf{DMRG}^{op}$, or $\mathbf{CBOOL}^{op}$.

(2) If $L \in |\mathbf{C}|$, and $\phi \in \mathbf{C}(L, L)$, then the ground $\mathbf{SET} \times \mathbf{L}_\phi$ has the following properties: completeness, cocompleteness, well-poweredness, co-well-poweredness, (Epi, Mono-Source) factorizability, separators, and coseparators.

Proof. For (1), Lemmas 2.2.1, 2.2.3, 2.2.5; and for (2), Remark 2.1.9(3). □

3 Topological categories for variable-basis topology and fuzzy topology

For each **C** a subcategory of **LOQML**, we define categorical frameworks for variable-basis topology and fuzzy topology over a ground category of the form $\mathbf{SET} \times \mathbf{C}$. These frameworks are justified externally in that they are shown to be topological over their ground w.r.t. the expected forgetful functor. En route we define notions of subbase and prove the crucial equivalence of subbasic continuity with continuity. We conclude the section by indicating some of the categorical consequences resulting from these frameworks being topological and fibre-small.

From a philosophical point of view, this section generalizes previous developments of variable-basis topology in that bases are now supplied from the richer and more general category **CQML**, and it extends the fuzzy topology approach of [23] by synthesizing it with the variable-basis approach of this author. In this connection, Prof. Höhle is especially thanked both for his encouragement to make this synthesis and for his suggestions thereto.

What both variable-basis topology and variable-basis fuzzy topology have in common is that they share the same ground categories. In particular, this means we put a topology or fuzzy topology on a ground object of the form (X, L) which is literally a **point-set lattice-theoretic object**. Thus we may *sequens* call these approaches to topology **point-set lattice-theoretic,** or **poslat** for short [62], and speak of **poslat topology**, **poslat fuzzy topology**, etc. For convenience, we sometimes distinguish the topology case in this chapter from the fuzzy topology case by calling one **poslat** and the other **fuzzy**, though the latter is actually **poslat fuzzy**.

Finally, we remind the reader that the internal justification of these frameworks via unification of canonical examples is taken up in Section 7.

3.1 Definitions of C-TOP and C-FTOP with examples of objects

The categorical frameworks detailed below rest on a base or ground category of the form $\mathbf{SET} \times \mathbf{C}$, where $\mathbf{C}$ is a subcategory of the category **LOQML**. Often $\mathbf{C}$ is taken to be **SLOC** or **LOC** (Proposition 1.3.2). While fixed-basis topology and fuzzy topology allow the change of both the underlying set and the topology, the frameworks below additionally permit the change of the underlying basis or lattice/monoid of membership values.

3.1.1. Definition (The category C-TOP). Let **C** be a subcategory of **LOQML**. The category **C-TOP** comprises the following data:

(1) **Objects.** Objects are ordered triples (X, L, τ) satisfying the following axioms:

  (a) **Ground axiom.** $(X, L) \in |\mathbf{SET} \times \mathbf{C}|$.

  (b) **Topological axiom.** $\tau \subset L^X$ is closed under $\otimes$ and arbitrary $\bigvee$ and contains $\underline{\top}$, i.e. the inclusion map $\tau \hookrightarrow L^X$ is a **CQML** morphism. (X, L, τ) is a **(poslat) topological space**, τ is a **(poslat) topology** on the (ground) set (X, L), and $v \in \tau$ is a **(fuzzy) open subset** of (X, L).

  (c) **Equality of objects.** $(X, L, \tau) = (Y, M, \sigma) \Leftrightarrow (X, L) = (Y, M)$ in $\mathbf{SET} \times \mathbf{C}$ (2.1.3) and $\tau = \sigma$ as subsets of $L^X \equiv M^Y$.

(2) **Morphisms.** Morphisms are ordered pairs

$$(f, \phi) : (X, L, \tau) \to (Y, M, \sigma)$$

called **continuous (morphisms),** satisfying the following axioms:

  (a) **Ground axiom.**

$$(f, \phi) : (X, L) \to (Y, M)$$

  is a morphism in $\mathbf{SET} \times \mathbf{C}$.

(b) **Continuity axiom.** $(f,\phi)^{\leftarrow} : L^X \leftarrow M^Y$ maps σ into τ, i.e. $(f,\phi)^{\leftarrow}_{|\sigma} : \tau \leftarrow \sigma$.

(c) **Equality of morphisms.** As in **SET** $\times$ **C** (2.1.3).

(3) **Composition.** As in **SET** $\times$ **C**.

(4) **Identity morphisms.** Given (X, L, τ), $id_{(X,L,\tau)} = (id_X, id_L)$ (or (id, id) for short).

3.1.2. Proposition. Let **C** be a subcategory of **LOQML**. Then **C-TOP** is a concrete category over **SET** $\times$ **C**.

Proof. The main point to be checked is composition, and the proof is substantially no different than that of [53]; but we give it a modern statement using powerset operators. The reader can check (using 1.3.3 or [62]) that

$$((f,\phi)\circ(g,\psi))^{\leftarrow} = (f\circ g, \phi\circ\psi)^{\leftarrow} = (g,\psi)^{\leftarrow}\circ(f,\phi)^{\leftarrow}$$

from which the preservation of continuity by composition immediately follows. □

3.1.3. Definition (The category C-FTOP). Let **C** be a subcategory of **LOQML**. The category **C-FTOP** comprises the following data:

(1) **Objects.** Objects are ordered triples $(X, L, \mathcal{T})$ satisfying the following axioms:

(a) **Ground axiom.** $(X, L) \in |\mathbf{SET}\times\mathbf{C}|$.

(b) **Fuzzy topological axiom.** $\mathcal{T} : L^X \to L$ is a mapping satisfying:

(i) $\forall$ indexing set J, $\forall\{u_j : j \in J\} \subset L^X$,

$$\bigwedge_{j\in J} \mathcal{T}(u_j) \leq \mathcal{T}\left(\bigvee_{j\in J} u_j\right)$$

(ii) $\forall$ indexing set J with $|J| = 2$, $\forall\{u_j : j \in J\} \subset L^X$,

$$\bigotimes_{j\in J} \mathcal{T}(u_j) \leq \mathcal{T}\left(\bigotimes_{j\in J} u_j\right)$$

(iii) $\tau(\underline{\top}) = \top$.

(c) **Equality of objects.** $(X, L, \mathcal{T}) = (Y, M, \mathcal{S}) \Leftrightarrow (X, L) = (Y, M)$ in **SET** $\times$ **C** (2.1.1) and $\mathcal{T} = \mathcal{S}$ as **SET** mappings from $L^X \equiv M^Y$ to $L \equiv M$.

Such objects are **(poslat) fuzzy topological spaces** and $\mathcal{T}$ is a **(poslat) fuzzy topology** on the (ground) set (X, L).

(2) **Morphisms**. Morphisms are ordered pairs

$$(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$$

called **fuzzy continuous (morphisms)**, satisfying the following axioms:

(a) **Ground axiom.**

$$(f, \phi) : (X, L) \to (Y, M)$$

is a morphism in **SET** $\times$ **C**.

(b) **Fuzzy continuity axiom.** $\mathcal{T} \circ (f, \phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S}$ on M^Y.

(c) **Equality of morphisms**. As in **SET** $\times$ **C** (2.1.1).

(3) **Composition**. As in **SET** $\times$ **C**.

(4) **Identities**. As in **SET** $\times$ **C**.

3.1.3.1. Proposition (Alternate fuzzy continuity axiom). On M^Y the following holds:

$$\mathcal{T} \circ (f, \phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S} \Leftrightarrow \phi^* \circ \mathcal{T} \circ (f, \phi)^{\leftarrow} \geq \mathcal{S}$$

where ϕ^* comes from Lemma 1.3.2 and is a shorthand for $(\phi^{op})^*$.

Proof. For necessity, apply ϕ^* to both sides, invoking its preservation of order (Lemma 1.3.2(4)) and the fact that $\phi^* \circ \phi^{op} \geq id_M$ (Lemma 1.3.2(2)); and for suficiency, apply ϕ^{op} to both sides, invoking its preservation of order (since it is a **CQML** morphism) and the fact that $\phi^{op} \circ \phi^* \leq id_L$. □

3.1.3.2. Remark. The form of the fuzzy continuity axiom given in 3.1.3.1 is closer to the intuitive meaning of fuzzy continuity—the degree of openness of the "preimage" of a fuzzy subset of the codomain is as great as the degree of openness of that fuzzy subset; the intuition behind the axiom as given in 3.1.3(2)(b) involves comparing degrees of openness of a fuzzy subset in each of two "secret" fuzzy topological spaces, namely $(Y, L, \mathcal{T} \circ (f, \phi)^{\leftarrow})$ and $(Y, L, \phi^{op} \circ \mathcal{S})$. What recommends both forms is technical convenience; either one shall be used in this chapter without necessarily being referenced. The reader should note the left side $\phi^* \circ \mathcal{T} \circ (f, \phi)^{\leftarrow}$ of 3.1.3.1 in fundamental to 3.2.12 of Subsection 3.2, a result that ultimately lays the foundation for **C-FTOP** being topological over **SET** $\times$ **C** in Subsection 3.3.

3.1.4. Proposition. Let **C** be a subcategory of **LOQML**. Then **C-FTOP** is a concrete category over **SET** $\times$ **C**.

Proof. We only check the composition of morphisms. Let the following morphisms be given:

$$(f,\phi):(X,L,\mathcal{T}_1)\to(Y,M,\mathcal{T}_2),\quad (g,\psi):(Y,M,\mathcal{T}_2)\to(Z,N,\mathcal{T}_3)$$

Then the following holds, using the displayed line of 3.1.2 above, and justifies that $(g\circ f,\psi\circ\phi)$ is a (continuous) morphism:

$$\begin{aligned}\mathcal{T}_1\circ(g\circ f,\psi\circ\phi)^{\leftarrow} &= \mathcal{T}_1\circ(f,\phi)^{\leftarrow}\circ(g,\psi)^{\leftarrow}\\ &\geq \phi^{op}\circ\mathcal{T}_2\circ(g.\psi)^{\leftarrow}\\ &\geq \phi^{op}\circ\psi^{op}\circ\mathcal{T}_3\\ &\geq (\psi\circ\phi)^{op}\circ\mathcal{T}_3 \quad \square\end{aligned}$$

3.1.5. Remarks and Examples. Let $L\in|\mathbf{LOQML}|$.

(1) Each L-topological space (X,τ), as discussed in Subsection 1.1, when written in the form (X,L,τ), is an object in **LOQML-TOP**. And each L-continuous morphism f can be rewritten as the morphism (f,id) in **LOQML-TOP**. Restated, each category L-**TOP** of [23] can be rewritten as the subcategory **L-TOP** of **LOQML-TOP**, since it is the subcategory **C-TOP** with $\mathbf{C}=\mathbf{L}$ (Subsection 1.3.2 above). See Subsection 6.2 other functorial embeddings into **LOQML-TOP**. By these remarks, a treasure trove of specific examples sit inside **LOQML-TOP** which are discussed at greater length in Section 7. These examples include:

(a) The fuzzy real lines $\mathbb{R}(L)$ [12] and fuzzy unit intervals $\mathbb{I}(L)$ [24], where $L\in|\mathbf{DMQL}|$, can be rewritten as $(\mathbb{R}(L),L,\tau(L))$ and $(\mathbb{I}(L),L,\tau(L)(\mathbb{I}(L)))$, respectively; and their co-fuzzy duals [59] can be written as $(\mathbb{R},L,co\text{-}\tau(L))$ and $(\mathbb{I},L,co\text{-}\tau(L))$. See Subsections 7.1 and 7.3 below.

(b) The L-sober topological spaces of the form $(Lpt(A),(\Phi_L)^{\rightarrow}(A))$, where

$$A,L\in|\mathbf{CQML}|,\quad Lpt(A)=\mathbf{CQML}(A,L)$$

and

$$\Phi_L:A\to L^{Lpt(A)}\quad\text{by}\quad \Phi_L(a)(p)=p(a)$$

may rewritten in the form

$$(Lpt(A),L,(\Phi_L)^{\rightarrow}(A))$$

Such spaces play a fundamental role in the representation and compactification theories for fixed-basis topology when the lattice-theoretic base is an object in **FRM**, as well as a fundamental role in compactification theories for variable-basis topology for all objects of **LOC-TOP**. It should be noted that for *all* presently known L, including $L = \mathbb{I}$, *none* of these spaces have any constant maps in their topologies other than $\underline{\perp}$ and $\underline{\top}$. This guarantees that *no* general framework for fixed-basis topology can be a c-category over its ground category; and this confirms that the definition of topological category in [1] is "correct". As an application, we conclude that **FTS,** contrary to its extensive promotion in [42–44] and also in E. Lowen and R. Lowen of [70], *cannot be a general framework for* $\mathbb{I}$-*topology.* Results and detailed proofs and applications are given in [56–57, 64–65, 68] and Subsection 7.4 below.

(c) The generated stratified L-topological spaces of the form $(X, \omega_L(\Im))$ —where $(X, \Im) \in |\mathbf{TOP}|$ and $\omega_L(\Im) = \mathbf{TOP}((X, \Im), (L, SUP(L)))$ —may be rewritten $(X, L, \omega_L(\Im))$. The restricitive case $L = \mathbb{I}$ is found in [42], but the general case for $L \in |\mathbf{SFRM}|$ is found in [39].

(2) Each L-fuzzy topological space $(X, \mathcal{T})$, as defined in [23] and written in the form $(X, L, \mathcal{T})$, is an object in **LOQML-FTOP**. And each L-fuzzy continuous morphism f can be rewritten as the morphism (f, id) in **LOQML-FTOP**. Restated, each category L-**FTOP** of [23] can be rewritten as the subcategory **L-FTOP** of **LOQML-TOP** since it is the subcategory **C-FTOP** with $\mathbf{C} = \mathbf{L}$ (Subsection 1.3.2 above). See Section 6.2 for other functorial embeddings into **LOQML-FTOP**.

(3) We have expressed the axioms of variable-basis fuzzy topology explicitly in terms of functional inequalities and powerset operators, thus rooting this theory directly into the foundations of [66–67]. This will be our approach throughout this chapter.

(4) Note a fourth condition obtains from 3.1.3(1)(b): $\mathcal{T}(\underline{\perp}) = \top$. This follows from choosing the indexing set J to be empty in (i), so that $\top \le \mathcal{T}(\underline{\perp})$.

(5) If $\otimes$ is chosen as $\wedge$, then 3.1.3(1)(b)(ii) reads as

$$\forall \text{ finite indexing set } J,\ \forall \{u_j : j \in J\},\ \bigwedge_{j\in J} \mathcal{T}(u_j) \le \mathcal{T}\left(\bigwedge_{j\in J} u_j\right)$$

and implies $\mathcal{T}(\underline{\top}) = \top$ (by choosing J empty), which then obviates (iii). Thus in a semi-locale, only 3.1.3(1)(b)(i,ii) are needed.

3.2 Subbasic continuities and final structures in C-TOP and C-FTOP

We develop subbases and subbasic continuity first for the variable-basis topology case and then extend this development to the variable-basis fuzzy topology case. En route we show the existence of final structures and conclude with some canonical examples of continuous morphisms.

The importance of this subsection is that it is the foundation stone of much of this chapter, especially the proofs in the next subsection that **C-TOP** and **C-FTOP** are topological over the ground **SET** $\times$ **C**.

3.2.1. Proposition. Let **C** be a subcategory of **LOQML** and let $(X, L) \in |\mathbf{SET} \times \mathbf{C}|$. Then the following hold:

(1) The collection

$$T = \{\tau \subset L^X : \tau \text{ is a topology on } (X, L)\}$$

is a complete lattice ordered by inclusion.

(2) $\forall \sigma \subset L^X$, $\exists$ a smallest topology on (X, L) containing σ, namely

$$\bigcap \{\tau \in T : \sigma \subset \tau\}$$

Proof. (1) follows from the fact that T is small and complete with respect to arbitrary intersections; and (2) follows from (the proof of) (1). $\square$

3.2.2. Definition (Subbase and base). Let $(X, L, \tau) \in |\mathbf{C\text{-}TOP}|$. Then σ is a **subbase** [β is a **base**] of τ if τ is the smallest topology on (X, L) containing σ [if each member of τ is a join of some subcollection of β, resp.], in which case we write $\tau = \langle\langle\sigma\rangle\rangle$ [$\tau = \langle\beta\rangle$, resp]. The existence of $\langle\langle\sigma\rangle\rangle$ is guaranteed by 3.2.1(2) above. These notions originally stem from [79] for the fixed-basis case with $L = \mathbb{I}$.

Note that to have a subbase σ induce a topology τ via a base β *á la* classical topology, the underlying lattice L must satisfy the first infinite distributive law of $\otimes$ over arbitrary $\bigvee$; e.g. this will be the case if $L \in |\mathbf{LOC}|$ and $\otimes = \wedge$ (restricted to finitely many arguments), in which case an open subset can be represented as a suprema of finite infima of subbasic open sets. If is σ a subbase for τ, and β is a base for τ induced from by taking finite products (via $\otimes$), then we write $\tau = \langle\beta\rangle$ and $\beta = \langle\sigma\rangle$. If one bypasses a base and goes "directly" from a subbase to a topology, infinite distributivity may not be needed—as seen in the following definitions and results, in which case we write $\tau = \langle\langle\sigma\rangle\rangle$ without presuming there exists a base β with either $\tau = \langle\beta\rangle$ or $\beta = \langle\sigma\rangle$.

3.2.3. Definition (Join of topologies). If $\{\tau_i\}_{i \in I}$ is a collection of topologies on $(X, L) \in |\mathbf{SET} \times \mathbf{C}|$, then $\bigvee i \in I \tau_i$ is the topology $\langle\langle \bigcup_{i \in I} \tau_i \rangle\rangle$ on (X, L) having $\bigcup_{i \in I} \tau_i$ as its subbase.

3.2.4. Definition (Subbasic and co-subbasic continuity). Let $(f, \phi) : (X, L) \to (Y, M)$ in **SET** $\times$ **C**, and let τ be a topology on (X, L) and $\langle\langle\sigma\rangle\rangle$ [$\langle\beta\rangle$] be a topology on (Y, M) with subbase σ [base β, resp.]. We say (f, ϕ)

is **subbasic continuous** [**basic continuous**] from (X, L, τ) to $(Y, M, \langle\langle\sigma\rangle\rangle)$ [$(Y, M, \langle\beta\rangle)$, resp.] iff

$$(f, \phi)^{\leftarrow}_{|\sigma} : \tau \leftarrow \sigma \;[(f, \phi)^{\leftarrow}_{|\beta} : \tau \leftarrow \beta, \text{ resp.}]$$

and (f, ϕ) is **co-subbasic continuous** [**co-basic continuous**] from $(Y, M, \langle\langle\sigma\rangle\rangle)$ [$(Y, M, \langle\beta\rangle$, resp.] to (X, L, τ) iff

$$(f, \phi)^{\leftarrow}_{|\tau} : \sigma \leftarrow \tau \;[(f, \phi)^{\leftarrow}_{|\tau} : \beta \leftarrow \tau, \text{ resp.}]$$

3.2.5. Theorem (Final structures and morphisms for topology). Let $\mathbf{C}$ be a subcategory of **LOQML**, let $(f, \phi) : (X, L) \to (Y, M)$ in $\mathbf{SET} \times \mathbf{C}$, and let τ be a topology on (X, L). Then the following hold:

(1) $\tau_{(f,\phi)} \equiv \{v \in M^Y : (f, \phi)^{\leftarrow}(v) \in \tau\}$ is a topology on (Y, M);

(2) $(f, \phi) : (X, L, \tau) \to \left(Y, M, \tau_{(f,\phi)}\right)$ is continuous;

(3) $(f, \phi) : (X, L, \tau) \to (Y, M, \hat{\tau})$ is continuous $\Leftrightarrow \hat{\tau} \subset \tau_{(f,\phi)}$;

(4) $\tau_{(f,\phi)} =$
$$\bigvee \left\{ \begin{array}{c} \hat{\tau} \subset M^Y : (Y, M, \hat{\tau}) \in |\mathbf{C\text{-}TOP}|, \\ (f, \phi) \in \mathbf{C\text{-}TOP}\,((X, L, \tau), (Y, M, \hat{\tau})) \end{array} \right\}$$

(5) $(f, \phi) : (X, L, \tau) \to \left(Y, M, \tau_{(f,\phi)}\right)$ is a final morphism in **C-TOP**;

(6) $(f, \phi) : (X, L, \tau) \to (Y, M, \hat{\tau})$ is a final morphism in **C-TOP** $\Leftrightarrow \hat{\tau} = \tau_{(f,\phi)}$.

Proof. *Ad (1).* The reader can verify that Lemma 1.3.3 and the fact τ is a topology on (X, L) imply that $\tau_{(f,\phi)}$ is a topology on (Y, M).

Ad (2). This is immediate from the definition of $\tau_{(f,\phi)}$.

Ad (3). $(\Leftarrow)$ is obvious from (2) and the definiiton of continuity applied to (f, ϕ) w.r.t. $\tau_{(f,\phi)}$. For $(\Rightarrow)$, let a continuous $(f, \phi) : (X, L, \tau) \to (Y, M, \hat{\tau})$ be given. Then $\forall v \in \hat{\tau}$, $(f, \phi)^{\leftarrow}(v) \in \tau$. This implies $v \in \tau_{(f,\phi)}$, so $\hat{\tau} \subset \tau_{(f,\phi)}$.

Ad (4). This is immediate from (3) and (4) using 3.2.3 (cf. Proposition 3.2.1(1)).

Ad (5). To show (f, ϕ) is final, we must verify the following:

$$\forall (Z, N, \tau^*) \in |\mathbf{C\text{-}TOP}|,\ \forall (g, \psi) \in (\mathbf{SET} \times \mathbf{C})\,((Y, M), (Z, N)),$$

$$(g, \psi) \circ (f, \phi) \in \mathbf{C\text{-}TOP}\,((X, L, \tau), (Z, N, \tau^*)) \Rightarrow$$
$$(g, \psi) \in \mathbf{C\text{-}TOP}\,\left((Y, M, \tau_{(f,\phi)}), (Z, N, \tau^*)\right)$$

So let all antecedents be instantiated and let $v \in \tau^*$. Then

$$(f, \phi)^{\leftarrow}\left((g, \psi)^{\leftarrow}(v)\right) = (g \circ f, \psi \circ \phi)^{\leftarrow}(v) \in \tau$$

by the continuity of $(g,\psi)\circ(f,\phi)$. But by (1) we have $(g,\psi)^{\leftarrow}(v) \in \tau_{(f,\phi)}$. Thus (g,ψ) is continuous and (f,ϕ) is final as claimed.

Ad (6). Sufficiency comes from (5). As for necessity, let (f,ϕ) be a final morphism in **C-TOP**. Since (f,ϕ) is assumed in $\mathbf{C\text{-}TOP}((X,L,\tau),(Y,M,\hat{\tau}))$, it is continuous; and so by (3) we have $\hat{\tau} \subset \tau_{(f,\phi)}$. To apply the finality of (f,ϕ) using the notation of the proof of (5), put $Z = Y$, $N = M$, $\hat{\tau} = \tau_{(f,\phi)}$, and $(g,\psi) = (id,id)$. Note $(g,\psi)\circ(f,\phi) = (f,\phi)$ is continuous from (X,L,τ) to $(Y,M,\tau_{(f,\phi)})$ by (2). Now finality applies to say that $(id,id) : (Y,M,\hat{\tau}) \to (Y,M,\tau_{(f,\phi)})$ is continuous. This implies that $\tau_{(f,\phi)} \subset \hat{\tau}$. So $\hat{\tau} = \tau_{(f,\phi)}$. □

3.2.6. Theorem (Equivalence of subbasic continuity with continuity). Let **C** be a subcategory of **LOQML**, let $(f,\phi) : (X,L) \to (Y,M)$ in $\mathbf{SET} \times \mathbf{C}$, let τ_1 be a topology on (X,L) with $\tau_1 = \langle\beta_1\rangle$ and $\tau_1 = \langle\langle\sigma_1\rangle\rangle$, and let τ_2 be a topology on (Y,M) with $\tau_2 = \langle\beta_2\rangle$ and $\tau_2 = \langle\langle\sigma_2\rangle\rangle$ (where it is not assumed that $\beta_i = \langle\sigma_i\rangle$). Then the following are equivalent:

(1) $(f,\phi) : (X,L,\tau_1) \to (Y,M,\tau_2)$ is continuous.

(2) $(f,\phi) : (X,L,\tau_1) \to (Y,M,\tau_2)$ is basic continuous.

(3) $(f,\phi) : (X,L,\tau_1) \to (Y,M,\tau_2)$ is subbasic continuous.

And (1–3) are implied by each of the following:

(4) $(f,\phi) : (X,L,\tau_1) \to (Y,M,\tau_2)$ is co-basic continuous.

(5) $(f,\phi) : (X,L,\tau_1) \to (Y,M,\tau_2)$ is co-subbasic continuous.

Proof. (1) $\Rightarrow$ (2) $\wedge$ (3) is immediate, (4) $\vee$ (5) $\Rightarrow$ (1) is trivial, and (1) $\Leftarrow$ (2) follows from $(f,\phi)^{\leftarrow}$ preserving arbitrary $\bigvee$. As for (1) $\Leftarrow$ (3), the assumption of (3), 3.2.4, and 3.2.5(1) imply that $\sigma_2 \subset \tau_{(f,\phi)}$. Now the definition of subbase (3.2.2) implies that $\tau_2 \subset \tau_{(f,\phi)}$. Apply 3.2.5(3) to conclude that (f,ϕ) is continuous. □

3.2.7. Remark. Theorem 3.2.6(1–3) stems from the special case for $\mathbf{C} = \mathbf{SLOC}$ given in [56, 66] and proved in detail in [39].

We turn now to the variable-basis fuzzy topology case. While there are parallels to the topology case, subbasic and final structures in the fuzzy topology case require deeper work and rest ultimately on the Adjoint Functor Theorem (AFT); and this changes the presentation significantly from that for the topology case.

3.2.8. Proposition. Let **C** be a subcategory of **LOQML** and let $(X,L) \in |\mathbf{SET} \times \mathbf{C}|$. Then the following hold:

(1) The collection

$$T = \left\{ \mathcal{T} \in L^{(L^X)} : \mathcal{T} \text{ is a fuzzy topology on } (X, L) \right\}$$

is a complete lattice with the ordering $\leq$ of $L^{(L^X)}$; in fact, it is a complete sub-meet-semilattice of $L^{(L^X)}$, i.e. closed under the $\bigwedge$ of $L^{(L^X)}$.

(2) $\forall \mathcal{S} \in L^{(L^X)}$, $\exists$ a smallest fuzzy topology on (X, L) dominating $\mathcal{S}$, namely

$$\bigwedge \{ \mathcal{T} \in T : \mathcal{S} \leq \mathcal{T} \}$$

Proof. (1) is proved in detail in the companion chapter [23]; and (2) follows from (1). □

3.2.9. Definition (Fuzzy subbases and fuzzy bases). Let $\mathcal{S}$ be a **fuzzy subcollection** in L^X, i.e. $\mathcal{S} : L^X \to L$ is a function with no other presumed properties. Then the **fuzzy topology $\mathcal{T}_\mathcal{S}$ generated from fuzzy subbasis $\mathcal{S}$** is given by:

$$\mathcal{T}_\mathcal{S} = \bigwedge \{ \mathcal{T} : L^X \to L \mid \mathcal{T} \text{ is a fuzzy topology on } (X, L),\ \mathcal{S} \leq \mathcal{T} \}$$

We are assured by 3.2.6 that this topology is well-defined. Alternatively, we write $\mathcal{T}_\mathcal{S} = \langle\langle \mathcal{S} \rangle\rangle$. If the only topology under consideration is the one generated by $\mathcal{S}$, then we drop the subscript and write $\mathcal{T} = \langle\langle \mathcal{S} \rangle\rangle$ in analogy with the topology case above. If a fuzzy topology $\mathcal{T} = \langle\langle \mathcal{B} \rangle\rangle$ has subbase $\mathcal{B}$, and $\mathcal{B}$ satisfies fuzzy topological axiom 3.1.3(1)(b)(i), i.e.

$$\forall \text{ indexing set } J,\ \forall \{u_j : j \in J\} \subset L^X,\ \bigwedge_{j \in J} \mathcal{B}(u_j) \leq \mathcal{B}\left(\bigvee_{j \in J} u_j \right)$$

then we say $\mathcal{B}$ is a (fuzzy) base for $\mathcal{T}$ and write $\mathcal{T} = \langle \mathcal{B} \rangle$. The reader should compare this with 3.2.2.

3.2.10. Definition (Join of fuzzy topologies). Let $\{\mathcal{T}_{j \in J}\}$ be a collection of fuzzy topologies on a set (X, L). Then $\bigvee j \in J \mathcal{T}_j$ is the fuzzy toplogy $\langle\langle \bigvee j \in J \mathcal{T}_j \rangle\rangle$ on (X, L), where the left join symbol is part of the notation being defined, and the right join symbol refers to the join of the subset $\{\mathcal{T}_{j \in J}\}$ of mappings in the lattice $L^{(L^X)}$. Because of 3.2.8 above, there is no harm from this abuse of notation.

3.2.11. Definition (Fuzzy subbasic and co-subbasic continuity). These definitions are analogous to 3.2.4, except the defining conditions now read:

- **fuzzy subbasic continuity**: $\mathcal{T} \circ (f, \phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S}$ on M^Y ($\mathcal{S}$ is subbase of codomain of (f, ϕ)),
- **fuzzy basic continuity**: $\mathcal{T} \circ (f, \phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{B}$ on M^Y ($\mathcal{B}$ is base of codomain of (f, ϕ)),
- **fuzzy co-subbasic continuity**: $\mathcal{S} \circ (f, \phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{T}$ on M^Y ($\mathcal{S}$ is subbase of domain of (f, ϕ)),
- **fuzzy co-basic continuity**: $\mathcal{B} \circ (f, \phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{T}$ on M^Y ($\mathcal{B}$ is base of domain of (f, ϕ)).

Each of these can be reformulated *á la* 3.1.3.1 above.

3.2.12. Theorem (Final structures and morphisms for fuzzy topology). Let **C** be a subcategory of **LOQML**, let $(f, \phi) : (X, L) \to (Y, M)$ in **SET** $\times$ **C**, let $\phi^* \equiv (\phi^{op})$ be the right adjoint of $\phi^{op} : L \leftarrow M$ guaranteed by Lemma 1.3.2, and let $\mathcal{T}$ be a fuzzy topology on (X, L). Then the following hold:

(1) $\mathcal{T}_{(f,\phi)} \equiv \phi^* \circ \mathcal{T} \circ (f, \phi)^{\leftarrow}$ is a fuzzy topology on (Y, M);

(2) $(f, \phi) : (X, L, \mathcal{T}) \to \left(Y, M, \mathcal{T}_{(f,\phi)}\right)$ is fuzzy continuous;

(3) $(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{T}^\wedge)$ is fuzzy continuous $\Leftrightarrow \mathcal{T}^\wedge \leq \mathcal{T}_{(f,\phi)}$;

(4) $\mathcal{T}_{(f,\phi)} =$

$$\bigvee \left\{ \begin{array}{c} \mathcal{T}^\wedge \in M^{\left(M^Y\right)} : (Y, M, \mathcal{T}^\wedge) \in |\textbf{C-FTOP}|\,, \\ (f, \phi) \in \textbf{C-FTOP}\left((X, L, \mathcal{T}), (Y, M, \mathcal{T}^\wedge)\right) \end{array} \right\}$$

(5) $(f, \phi) : (X, L, \mathcal{T}) \to \left(Y, M, \mathcal{T}_{(f,\phi)}\right)$ is a final morphism in **C-FTOP**;

(6) $(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{T}^\wedge)$ is a final morphism in **C-FTOP** $\Leftrightarrow \mathcal{T}^\wedge = \mathcal{T}_{(f,\phi)}$.

Proof. *Ad (1).* Using 1.3.3 and Lemma 1.3.2 (4,5), we check the fuzzy topological axioms of 3.1.3(b) as follows, where the $\otimes$-axiom requires $|J| = 2$:

$$\begin{aligned} \bigwedge_{j \in J} \mathcal{T}_{(f,\phi)}(u_j) &= \bigwedge_{j \in J} \phi^*\left(\mathcal{T}\left(f, \phi\right)^{\leftarrow}(u_j)\right) \\ &= \phi^*\left(\bigwedge_{j \in J} \mathcal{T}\left(f, \phi\right)^{\leftarrow}(u_j)\right) \\ &\leq \phi^*\left(\mathcal{T}\left(\bigvee_{j \in J} (f, \phi)^{\leftarrow}(u_j)\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \phi^*\left(\mathcal{T}\,(f,\phi)^{\leftarrow}\left(\bigvee_{j\in J} u_j\right)\right)\\
&= \mathcal{T}_{(f,\phi)}\left(\bigvee_{j\in J} u_j\right)
\end{aligned}
$$

$$
\begin{aligned}
\bigotimes_{j\in J}\mathcal{T}_{(f,\phi)}\,(u_j) &= \bigotimes_{j\in J}\phi^*\,(\mathcal{T}\,(f,\phi)^{\leftarrow}\,(u_j))\\
&\le \phi^*\left(\bigotimes_{j\in J}\mathcal{T}\,(f,\phi))^{\leftarrow}\,(u_j)\right)\\
&\le \phi^*\left(\mathcal{T}\left(\bigotimes_{j\in J}(f,\phi)^{\leftarrow}\,(u_j)\right)\right)\\
&= \phi^*\left(\mathcal{T}\,(f,\phi)^{\leftarrow}\left(\bigotimes_{j\in J} u_j\right)\right)\\
&= \mathcal{T}_{(f,\phi)}\left(\bigotimes_{j\in J} u_j\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_{(f,\phi)}\,(\underline{\top}) &= \phi^*\,(\mathcal{T}\,(f,\phi)^{\leftarrow}\,(\underline{\top}))\\
&= \underline{\top}
\end{aligned}
$$

Note the last computation rests on $\underline{\top}$ being the meet over the empty set, ϕ^* preserving arbitrary $\bigwedge$ and hence $\underline{\top}$, and $(f,\phi)^{\leftarrow}$ preserving $\underline{\top}$ (1.3.3).

Ad (2). From Lemma 1.3.2 (2) we have that

$$\phi^{op}\circ\phi^*\le id_L$$

which inplies that

$$
\begin{aligned}
\mathcal{T}\circ(f,\phi)^{\leftarrow} &\ge \phi^{op}\circ\phi^*\circ\mathcal{T}\circ(f,\phi)^{\leftarrow}\\
&= \phi^{op}\circ\mathcal{T}_{(f,\phi)}
\end{aligned}
$$

This verifies that $(f,\phi):(X,L,\mathcal{T})\to\left(Y,M,\mathcal{T}_{(f,\phi)}\right)$ is fuzzy continuous.

Ad (3). For ($\Rightarrow$), let $(f,\phi):(X,L,\mathcal{T})\to(Y,M,\mathcal{T}^{\wedge})$ be fuzzy continuous. Then we have

$$\mathcal{T}\circ(f,\phi)^{\leftarrow}\ge\phi^{op}\circ\mathcal{T}^{\wedge}$$

Whence Lemma 1.3.2 (2,4) it follows that

$$
\begin{aligned}
\mathcal{T}_{(f,\phi)} &= \phi^*\circ\mathcal{T}\circ(f,\phi)^{\leftarrow}\\
&\ge \phi^*\circ\phi^{op}\circ\mathcal{T}^{\wedge}\\
&\ge \mathcal{T}^{\wedge}
\end{aligned}
$$

For $(\Leftarrow)$, let $\mathcal{T}^\wedge \le \mathcal{T}_{(f,\phi)}$. Since $(f,\phi) : (X, L, \mathcal{T}) \to \left(Y, M, \mathcal{T}_{(f,\phi)}\right)$ is fuzzy continuous from (2), we have

$$\begin{aligned} \mathcal{T} \circ (f,\phi)^{\leftarrow} &\ge \phi^{op} \circ \mathcal{T}_{(f,\phi)} \\ &\ge \phi^{op} \circ \mathcal{T}^\wedge \end{aligned}$$

Ad (4). This follows from (3) and (4) together with 3.2.10.

Ad(5). To show $(f,\phi) : (X, L, \mathcal{T}) \to \left(Y, M, \mathcal{T}_{(f,\phi)}\right)$ is final, we must verify the following:

$$\forall (Z, N, \mathcal{T}^*) \in |\mathbf{C\text{-}FTOP}|, \ \forall (g,\psi) \in (\mathbf{SET} \times \mathbf{C})\left((Y, M), (Z, N)\right),$$

$$(g,\psi) \circ (f,\phi) \in \mathbf{C\text{-}FTOP}\left((X, L, \mathcal{T}), (Z, N, \mathcal{T}^*)\right) \Rightarrow$$

$$(g,\psi) \in \mathbf{C\text{-}FTOP}\left((Y, M, \mathcal{T}_{(f,\phi)}), (Z, N, \mathcal{T}^*)\right)$$

Letting all antecedents be instantiated, we have from the continuity of $(g,\psi) \circ (f,\phi)$ and the facts $(g \circ f, \psi \circ \phi)^{\leftarrow} = (f,\phi)^{\leftarrow} \circ (g,\psi)^{\leftarrow}$ and $(\psi \circ \phi)^{op} = \phi^{op} \circ \psi^{op}$ that

$$\mathcal{T} \circ (f,\phi)^{\leftarrow} \circ (g,\psi)^{\leftarrow} \ge \phi^{op} \circ \psi^{op} \circ \mathcal{T}^*$$

Now from the definition of $\mathcal{T}_{(f,\phi)}$, Lemma 1.3.2 (4), and Lemma 1.3.2 (2), in that order, it follows

$$\begin{aligned} \mathcal{T}_{(f,\phi)} \circ (g,\psi)^{\leftarrow} &= \phi^* \circ \mathcal{T} \circ (f,\phi)^{\leftarrow} \circ (g,\psi)^{\leftarrow} \\ &\ge \phi^* \circ \phi^{op} \circ \psi^{op} \circ \mathcal{T}^* \\ &\ge \psi^{op} \circ \mathcal{T}^* \end{aligned}$$

Ad (6). Sufficiency comes from (5). As for necessity, let $(f,\phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{T}^\wedge)$ be final in $\mathbf{C\text{-}FTOP}$. Since (f,ϕ) is presumed fuzzy continuous, we have from (3) that $\mathcal{T}^\wedge \le \mathcal{T}_{(f,\phi)}$. To apply the finality of (f,ϕ) using the notation of the proof of (5), put $Z = Y$, $N = M$, $\mathcal{T}^* = \mathcal{T}_{(f,\phi)}$, and $(g,\psi) = (id, id)$. Note $(g,\psi) \circ (f,\phi) = (f,\phi)$ is fuzzy continuous from $(X, L, \mathcal{T})$ to $(Y, M, \mathcal{T}_{(f,\phi)})$ by (2). Now finality applies to say that $(id, id) : (Y, M, \mathcal{T}^\wedge) \to (Y, M, \mathcal{T}_{(f,\phi)})$ is fuzzy continuous, and this implies

$$\begin{aligned} \mathcal{T}^\wedge &= \mathcal{T}^\wedge \circ (id, id)^{\leftarrow} \\ &\ge id^{op} \circ \mathcal{T}_{(f,\phi)} \\ &= \mathcal{T}_{(f,\phi)} \end{aligned}$$

Thus $\mathcal{T}^\wedge = \mathcal{T}_{(f,\phi)}$. □

3.2.13. Theorem (Equivalence of fuzzy subbasic continuity with fuzzy continuity). Let $\mathbf{C}$ be a subcategory of $\mathbf{LOQML}$, let $(f,\phi) : (X, L) \to (Y, M)$ in $\mathbf{SET} \times \mathbf{C}$, let $\mathcal{T}_1$ be a fuzzy topology on (X, L) with $\mathcal{T}_1 = \langle \mathcal{B}_1 \rangle$ and $\mathcal{T}_1 = \langle\langle \mathcal{S}_1 \rangle\rangle$, and let $\mathcal{T}_2$ be a fuzzy topology on (Y, M) with $\mathcal{T}_2 = \langle \mathcal{B}_2 \rangle$ and $\mathcal{T}_2 = \langle\langle \mathcal{S}_2 \rangle\rangle$. Then the following are equivalent:

(1) $(f, \phi) : (X, L, \mathcal{T}_1) \to (Y, M, \mathcal{T}_2)$ is fuzzy continuous.

(2) $(f, \phi) : (X, L, \mathcal{T}_1) \to (Y, M, \mathcal{T}_2)$ is fuzzy basic continuous.

(3) $(f, \phi) : (X, L, \mathcal{T}_1) \to (Y, M, \mathcal{T}_2)$ is fuzzy subbasic continuous.

And (1–3) are implied by each of the following:

(4) $(f, \phi) : (X, L, \mathcal{T}_1) \to (Y, M, \mathcal{T}_2)$ is fuzzy co-basic continuous.

(5) $(f, \phi) : (X, L, \mathcal{T}_1) \to (Y, M, \mathcal{T}_2)$ is fuzzy co-subbasic continuous.

Proof. (1) $\Rightarrow$ (3). $\mathcal{T}_1 \circ (f, \phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{T}_2 \geq \phi^{op} \circ \mathcal{S}_2$ since $\mathcal{S}_2 \leq \mathcal{T}_2$.

(1) $\Leftarrow$ (3). The fuzzy subbasic continuity of (f, ϕ) gives

$$\mathcal{T}_1 \circ (f, \phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S}_2$$

And applying the definition of $(\mathcal{T}_1)_{(f,\phi)}$ from 3.2.12(1), Lemma 1.3.2(4), and Lemma 1.3.2(2), in that order, yields

$$\begin{aligned} (\mathcal{T}_1)_{(f,\phi)} &= \phi^* \circ \mathcal{T}_1 \circ (f, \phi)^{\leftarrow} \\ &\geq \phi^* \circ \phi^{op} \circ \mathcal{S}_2 \\ &\geq \mathcal{S}_2 \end{aligned}$$

Now the definition of fuzzy subbase (3.2.9) implies that $(\mathcal{T}_1)_{(f,\phi)} \geq \mathcal{T}_2$. We now apply 3.2.12(3) to conclude that (f, ϕ) is fuzzy continuous.

(1) $\Leftarrow$ (5). $\mathcal{T}_1 \circ (f, \phi)^{\leftarrow} \geq \mathcal{S}_1 \circ (f, \phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{T}_2$ since $\mathcal{T}_1 \geq \mathcal{S}_1$.

(1) $\Rightarrow$ (2), (1) $\Leftarrow$ (2), (1) $\Leftarrow$ (4). Respective consequences of (1) $\Rightarrow$ (3), (1) $\Leftarrow$ (3), (1) $\Leftarrow$ (5) since each fuzzy base is a fuzzy subbase. □

3.2.14. Canonical examples of morphisms. Let $L \in |\mathbf{DMRG}|$ be totally ordered.

(1) Let the fuzzy real line $\mathbb{R}(L)$ be equipped both with the standard L-topology $\tau(L)$ [12] and with fuzzy addition $\oplus$, and let $\mathbb{R}(L) \times \mathbb{R}(L)$ be equipped with the standard (Goguen-Wong) L-product topology $\tau(L) \times \tau(L)$ (see [16, 80]). Then each proof that

$$(\oplus, id) : (\mathbb{R}(L) \times \mathbb{R}(L), L, \tau(L) \times \tau(L)) \to (\mathbb{R}(L), L, \tau(L))$$

is jointly (poslat) continuous is based on showing $(\oplus, id)$ is subbasic continuous, and the proof that

$$(\otimes, id) : (\mathbb{R}(L) \times \mathbb{R}(L), L, \tau(L) \times \tau(L)) \to (\mathbb{R}(L), L, \tau(L))$$

is jointly (poslat) continuous is based on showing $(\otimes, id)$ is subbasic continuous. See [55] or its references for the hypergraph proofs for $(\oplus, id)$ and $(\otimes, id)$, respectively.

(2) Let the classical real line $\mathbb{R}$ be equipped with the standard L-topology $co\text{-}\tau(L)$—the co-fuzzy dual topology [59] induced on $\mathbb{R}$ from $\mathbb{R}(L)$—and let $\mathbb{R} \times \mathbb{R}$ be equipped with the standard (Goguen-Wong) L-product topology $co\text{-}\tau(L) \times co\text{-}\tau(L)$. Then three of the four known proofs that classical addition

$$(+, id) : (\mathbb{R} \times \mathbb{R}, L, co\text{-}\tau(L) \times co\text{-}\tau(L)) \to (\mathbb{R}, L, co\text{-}\tau(L))$$

is jointly (poslat) continuous are based on showing $(+, id)$ is subbasic continuous, and two of the three known proofs that classical multiplication

$$(\bullet, id) : (\mathbb{R} \times \mathbb{R}, L, co\tau_L \times co\tau_L) \to (\mathbb{R}, L, co\text{-}\tau(L))$$

is jointly (poslat) continuous are based on showing $(\bullet, id)$ is subbasic continuous. See [59–60] for details.

(3) If $\phi \in \mathbf{DMRG}^{op}(L, L)$ and the G_ϕ functor of Subsection 4.1 is applied to each object of the previous examples, then each of $(\oplus, \phi), (\otimes, \phi), (+, \phi), (\bullet, \phi)$ is respectively jointly (poslat) continuous. If $\phi \in Auto(L)$, then each of $(+, \phi), (\bullet, \phi)$ is respectively jointly (poslat) continuous between the objects as originally given in (3)—this follows from Proposition 9.25 of [66].

3.3 C-TOP and C-FTOP are topological over SET × C

This subsection builds directly on Subsections 1.3, 3.1, and especially 3.2. We use the Interpretation of the Long Definition of 1.3.1. In particular, the forgetful functors

$$V : \mathbf{C-TOP} \to \mathbf{SET} \times \mathbf{C},\ V : \mathbf{C-FTOP} \to \mathbf{SET} \times \mathbf{C}$$

in this subsection have the following shape:

$$V(X, L, \tau) = (X, L) \text{ or } V(X, L, \mathcal{T}) = (X, L),\ \ V(f, \phi) = (f, \phi)$$

Then a V-structured source from variable-basis topology looks like

$$((X, L),\ (f_i, \phi_i) : (X, L) \to V(X_i, L_i, \tau_i)))_I$$

and a V-structured source from variable-basis fuzzy topology looks like

$$((X, L),\ (f_i, \phi_i) : (X, L) \to V(X_i, L_i, \mathcal{T}_i)))_I$$

It is our task to show that each V-structured source from one of these categories has a unique initial V-lift from that category. We first prove the **C-TOP** case, and then extend the argument to the deeper **C-FTOP** case.

3.3.1. Lemma (V**-Lift**). Let $((X, L), (f_i, \phi_i) : (X, L) \to (X_i, L_i, \tau_i)))_I$ be a V-structured source in **SET** $\times$ **C** from **C-TOP**. There is a topology τ on (X, L) such that each $(f_i, \phi_i) : (X, L, \tau) \to (X_i, L_i, \tau_i)$ is continuous.

Proof. From 3.2.3, put

$$\tau = \bigvee_{i \in I} ((f_i, \phi_i)^{\leftarrow})^{\rightarrow} (\tau_i) = \langle\langle \bigcup_{i \in I} ((f_i, \phi_i)^{\leftarrow})^{\rightarrow} (\tau_i) \rangle\rangle$$

where $((f_i, \phi_i)^{\leftarrow})^{\rightarrow}$ is the classical forward powerset operator of the fuzzy backward operator $(f_i, \phi_i)^{\leftarrow}$. Then by 3.2.1 and 3.2.2, τ is a topology on (X, L). Each (f_i, ϕ_i) is co-subbasic continuous by 3.2.4, and hence by 3.2.6(5) is continuous. □

3.3.2. Lemma (Initial V**-lift).** $((X, L, \tau), (f_i, \phi_i) : (X, L, \tau) \to (X_i, L_i, \tau_i)))_I$ is an initial V-lift of the V-structured source of 3.3.1, where τ is given in the proof of 3.3.1.

Proof. Let $((X, L, \upsilon), (g_i, \psi_i) : (X, L, \upsilon) \to (X_i, L_i, \tau_i)))_I$ be another V-lift of the V-structured source of 3.3.1, and let $(h, \phi) : (X, L) \to (X, L)$ be a ground morphism such that

$$\forall i \in I, \ (g_i, \psi_i) = (f_i, \phi_i) \circ (h, \phi)$$

Then $(h, \phi)^{\leftarrow} \circ (f_i, \phi_i)^{\leftarrow} = (g_i, \psi_i)^{\leftarrow}$. For the subbasic continuity of

$$(h, \phi) : (X, L, \upsilon) \to (X, L, \tau)$$

let

$$v \in \bigcup_{i \in I} ((f_i, \phi_i)^{\leftarrow})^{\rightarrow} (\tau_i)$$

Then $\exists\, i \in I, \exists\, u \in \tau_i, \ (f_i, \phi_i)^{\leftarrow}(u) = v$; and so

$$(h, \phi)^{\leftarrow}(v) = (h, \phi)^{\leftarrow}((f_i, \phi_i)^{\leftarrow}(u)) = (g_i, \psi_i)^{\leftarrow}(u) \in \upsilon$$

Hence (h, ϕ) is subbasic continuous, and so is continuous by 3.2.6. □

3.3.3. Lemma (Unique initial V**-lift).**

$$((X, L, \tau), (f_i, \phi_i) : (X, L, \tau) \to (X_i, L_i, \tau_i)))_I$$

is the unique V-initial lift of the V-structured source of 3.3.1, where τ is given in the proof of 3.3.1.

Proof. Let $((X, L, \upsilon), (g_i, \psi_i) : (X, L, \upsilon) \to (X_i, L_i, \tau_i)))_I$ be another V-initial lift. Then we must show that $\upsilon = \tau$. Since both constructions are lifts, it follows immediately that

$$(f_i, \phi_i) = V(f_i, \phi_i) = V(g_i, \psi_i) = (g_i, \psi_i)$$

Let $(h, \phi) : (X, L) \to (X, L)$ by $h = id_X, \psi = id_L$. Then using the initial properties of each lift, we have (h, ϕ) must be continuous, both from (X, L, v) to (X, L, τ) , and from (X, L, τ) to (X, L, v); the former implies $\tau \subset v$, and the latter implies $v \subset \tau$. So $v = \tau$. □

3.3.4. Theorem. For each subcategory **C** of **LOQML**, the category **C-TOP** is topological over **SET** × **C** with respect to the forgetful functor V.

Proof. Conjoin 3.3.1, 3.3.2, 3.3.3, and Interpretation of Definition 1.3.1. □

We turn now to the **C-FTOP** case.

3.3.5. Lemma (V**-Lift**). Let $((X, L), (f_i, \phi_i) : (X, L) \to (X_i, L_i, \mathcal{T}_i)))_I$ be a V-structured source in **SET** × **C** from **C-FTOP**. There is a fuzzy topology $\mathcal{T}$ on (X, L) such that each $(f_i, \phi_i) : (X, L, \mathcal{T}) \to (X_i, L_i, \mathcal{T}_i)$ is fuzzy continuous.

Proof. Construct a map $\mathcal{S} : L^X \to L$ by defining $\mathcal{S}$ at $a \in L^X$ as follows:

$$\mathcal{S}(a) = \begin{cases} \bigvee\limits_{i \in I,\ b \in ((f_i, \phi_i)^{\leftarrow})^{\leftarrow}(\{a\})} \phi_i^{op}(\mathcal{T}_i(b)) & , \ \exists i \in I,\ ((f_i, \phi_i)^{\leftarrow})^{\leftarrow}(\{a\}) \neq \emptyset \\ \perp & , \ \forall i \in I,\ ((f_i, \phi_i)^{\leftarrow})^{\leftarrow}(\{a\}) = \emptyset \end{cases}$$

where $b \in ((f_i, \phi_i)^{\leftarrow})^{\leftarrow}(\{a\})$ means $(f_i, \phi_i)^{\leftarrow}(b) = a$, i.e. $((f_i, \phi_i)^{\leftarrow})^{\leftarrow}$ is the classsical backward powerset operator of the fuzzy backward powerset operator —see 1.3.3. Put $\mathcal{T} = \langle\langle \mathcal{S} \rangle\rangle$.

We now verify that each (f_i, ϕ_i) is fuzzy co-subbasic continuous; it will then follow by 3.2.13(5) that (f_i, ϕ_i) is fuzzy continuous. And to do this, we must verify that $\mathcal{S} \circ (f_i, \phi_i) \geq \phi_i^{op} \circ \mathcal{T}_i$ on $L_i^{X_i}$. Let $b \in L_i^{X_i}$ and put $a = (f_i, \phi_i)^{\leftarrow}(b)$. Then $a \in L^X$ and it follows that $((f_i, \phi_i)^{\leftarrow})^{\leftarrow}(a) \neq \emptyset$ (since this particular b is in $((f_i, \phi_i)^{\leftarrow})^{\leftarrow}(a)$). We therefore have that

$$\begin{aligned} \mathcal{S}((f_i, \phi_i)^{\leftarrow}(b)) &= \mathcal{S}(a) \\ &= \bigvee_{i \in I,\ b \in ((f_i, \phi_i)^{\leftarrow}(a))^{\leftarrow}} \phi_i^{op}(\mathcal{T}_i(b)) \\ &\geq \phi_i^{op}(\tau_i(b)) \end{aligned}$$

from which it follows that $\mathcal{S} \circ (f_i, \phi_i) \geq \phi_i^{op} \circ \mathcal{T}_i$ on $L_i^{X_i}$. This concludes the proof. □

3.3.6. Lemma (Initial V**-lift).**

$$((X, L, \mathcal{T}), (f_i, \phi_i) : (X, L, \mathcal{T}) \to (X_i, L_i, \mathcal{T}_i)))_I$$

is a V-initial lift of the V-structured source of 3.3.5, where $\mathcal{T}$ is given in the proof of 3.3.5.

Proof. Let $\mathcal{U}$ be a topology on (X, L); $\forall i \in I$, let (g_i, ψ_i) fuzzy continuous from $(X, L, \mathcal{U})$ to (X_i, L_i, τ_i), and let $(h, \psi) : (X, L) \to (X, L)$ be a ground morphism such that

$$\forall i \in I,\ (f_i, \phi_i) \circ (h, \psi) = (g_i, \psi_i) \text{ in } \mathbf{SET} \times \mathbf{C} \qquad (*)$$

The goal is to show that $(h, \psi) : (X, L, \upsilon) \to (X, L, \tau)$ is fuzzy continuous by showing (h, ψ) is fuzzy subbasic continuous and then invoking 3.2.13(3) of Subsection 3.2 to conclude (h, ψ) is fuzzy continuous. Restated, our goal is to verify on L^X the inequality

$$\mathcal{U} \circ (h, \psi)^{\leftarrow} \geq \psi^{op} \circ \mathcal{S}$$

where we recall $\mathcal{S}$ is the subbase for $\mathcal{T}$ created in the proof of 3.3.5.

First, we observe from the fuzzy continuity of each (g_i, ψ_i) that on $L_i^{X_i}$ there obtains the inequality

$$\mathcal{U} \circ (g_i, \psi_i)^{\leftarrow} \geq \psi_i^{op} \circ \mathcal{T}_i \qquad (**)$$

And we observe that the identities of (*) above imply

$$\forall i \in I,\ \phi_i \circ \psi = \psi_i,\ \psi^{op} \circ \phi_i^{op} = \psi_i^{op},\ (h, \psi)^{\leftarrow} \circ (f_i, \phi_i)^{\leftarrow} = (g_i, \psi_i)^{\leftarrow} \qquad (***)$$

Second, the map $\psi^{op} : L \leftarrow L$ has a right adjoint $\psi^* : L \to L$ with the properties listed in Lemma 1.3.2. Our intermediate goal is to verify on L^X the following inequality:

$$\psi^* \circ \mathcal{U} \circ (h, \psi)^{\leftarrow} \geq \mathcal{S} \qquad (****)$$

To verify (****), we write L^X as a disjoint union $A \cup_{\emptyset} B$, where A and B are defined as follows:

$$a \in A \Leftrightarrow \exists i \in I,\ ((f_i, \phi_i)^{\leftarrow})^{\leftarrow}(a) \neq \emptyset$$

$$B = L^X - A$$

Note on B the inequality (****) holds trivially since $\mathcal{S}_{|B} = \perp_{|B}$. To show that (****) holds on A, let $a \in A$ and let $i \in I$ such that $((f_i, \phi_i)^{\leftarrow})^{\leftarrow}(a) \neq \emptyset$. Then $\exists b \in L_i^{X_i}$, $b \in ((f_i, \phi_i)^{\leftarrow})^{\leftarrow}(a)$. Using Lemma 1.3.2(2), the definition of $\mathcal{S}$, (**), and (***) yields this sequence of cogitations:

$$\mathcal{U} \circ (h, \psi)^{\leftarrow} \circ (f_i, \phi_i)^{\leftarrow} \geq \psi^{op} \circ \phi_i^{op} \circ \mathcal{T}_i,$$

$$\psi^* \circ \mathcal{U} \circ (h, \psi)^{\leftarrow} \circ (f_i, \phi_i)^{\leftarrow} \geq \phi_i^{op} \circ \mathcal{T}_i,$$

$$\psi^* \left(\mathcal{U}\left((h, \psi)^{\leftarrow}\left((f_i, \phi_i)^{\leftarrow}(b)\right)\right)\right) \geq \phi_i^{op}\left(\mathcal{T}_i(b)\right),$$

$$\begin{aligned} \psi^* \left(\mathcal{U}\,(h, \psi)^{\leftarrow}(a)\right) &= \psi^* \left(\mathcal{U}\left((h, \psi)^{\leftarrow}\left((f_i, \phi_i)^{\leftarrow}(b)\right)\right)\right) \\ &\geq \bigvee_{i \in I,\ b \in ((f_i, \phi_i)^{\leftarrow})^{\leftarrow}(a)} \phi_i^{op}\left(\mathcal{T}_i(b)\right) \\ &= \mathcal{S}(a) \end{aligned}$$

This completes the proof of (****) on L^x.

To finish the proof of the lemma, we invoke the last sentence of 3.2.10 (*á la* 3.1.3.1) to obtain:

$$\mathcal{U} \circ (h, \psi)^{\leftarrow} \geq \psi^{op} \circ \mathcal{S}$$

This concludes the proof of the fuzzy subbasic continuity of (h, ψ), the fuzzy continuity of (h, ψ) (3.2.13), and hence the lemma. □

3.3.7. Remark (Alternate finish to proof of 3.3.6). Given that (****) is established in the proof of 3.3.6, we have from 3.2.12(1) that (****) may be rewritten as

$$\mathcal{T}_{(h,\psi)} \geq \mathcal{S}$$

Then it follows from 3.2.8 and 3.2.9 that $\mathcal{T}_{(h,\psi)} \geq \mathcal{T}$, which implies from 3.2.12(3) that (h, ψ) is fuzzy continuous, finishing the proof of 3.3.6. Of course, this is really the proof of 3.2.13((1) $\Leftarrow$ (3)) in disguise.

3.3.8. Lemma (Unique initial V-lift).

$$((X, L, \mathcal{T}),\ (f_i, \phi_i) : (X, L, \mathcal{T}) \to (X_i, L_i, \mathcal{T}_i)))_I$$

is the unique initial V-lift of the V-structured source of 3.3.5, where $\mathcal{T}$ is given in the proof of 3.3.5.

Proof. Let $((X, L, \mathcal{U}),\ (g_i, \psi_i) : (X, L, \mathcal{U}) \to (X_i, L_i, \mathcal{T}_i)))_I$ be another V-initial lift. Then we must show that $\mathcal{U} = \mathcal{T}$ and $\forall i \in I,\ (g_i, \psi_i) = (f_i, \phi_i)$. Since both constructions are lifts, it follows immediately that

$$(f_i, \phi_i) = V(f_i, \phi_i) = V(g_i, \psi_i) = (g_i, \psi_i)$$

Let $(h, \psi) : (X, L) \to (X, L)$ by $h = id_X, \psi = id_L$. Then using the initial properties of each lift, we have (h, ψ) must be fuzzy continuous, both from $((X, L, \mathcal{U})$ to $(X, L, \mathcal{T})$ and from $(X, L, \mathcal{T})$ to $(X, L, \mathcal{U})$. These two fuzzy continuities yield on L^X the following inequalities:

$$\mathcal{U} \circ (h, \psi)^{\leftarrow} \geq \psi^{op} \circ \mathcal{T}, \quad \mathcal{T} \circ (h, \psi)^{\leftarrow} \geq \psi^{op} \circ \mathcal{U}$$

But $(h, \psi)^{\leftarrow} = id_{L^X},\ \psi^{op} = id_L$. So these inequalities on L^X reduce to

$$\mathcal{U} \geq \mathcal{T}, \quad \mathcal{T} \geq \mathcal{U}$$

Hence $\mathcal{U} = \mathcal{T}$, and the initial V-lift of 3.3.5 is unique. □

3.3.9. Theorem. For each subcategory **C** of **LOQML**, the category **C-FTOP** is topological over **SET** $\times$ **C** with respect to the forgetful functor V.

Proof. Conjoin 3.3.5, 3.3.6, 3.3.8, and Interpretation of Definition 1.3.1. □

3.3.10 Corollary (Applications to fixed-basis topology and fuzzy topology). For each $L \in |\mathbf{CQML}|$, each of **L-TOP** and **L-FTOP** is topological over $\mathbf{SET} \times \mathbf{L}$.

Proof. Apply 3.3.4 and 3.3.9 with $\mathbf{C} = \mathbf{L}$. □

This corollary, filtered through Proposition 1.3.1 and the functorial embeddings of Section 6 below, gives an alternate proof that the categories L-**TOP** and L-**FTOP** studied in [23] are topological over **SET**.

3.4 Categorical consequences of topological and fibre-smallness

3.4.1. Lemma. Let **C** a subcategory of **LOQML**. Then each object of **C-TOP** and **C-FTOP** is fibre-small over $\mathbf{SET} \times \mathbf{C}$; in fact, each fibre is a complete lattice.

Proof. Given a ground object (X, L), its fibre comprises all spaces of the form (X, L, τ) or the form $(X, L, \mathcal{T})$, and so is in a bijection with all topologies $\tau \subset L^X$ or all fuzzy topologies $\mathcal{T} : L^X \to L$. Hence this fibre may be regarded as a subclass of the set $\mathcal{P}\left(L^X\right)$ or the set $L^{\left(L^X\right)}$, and so is a set, the condition for fibre-smallness. That it is a complete lattice follows from 3.2.1 or 3.2.8 (by dualizing the ordering of 3.2.1/3.2.8). □

3.4.2. Theorem. Let **C** be a subcategory of **LOQML**. Then the following properties or statements hold for **C-TOP** and **C-FTOP** over $\mathbf{SET} \times \mathbf{C}$ w.r.t. the appropriate forgetful functor V.

(1) Initial completeness—each V-structured source has a unique initial V-lift.

(2) Final completeness—each V-structured sink has a unique final V-lift.

(3) Discrete structures—each object in $\mathbf{SET} \times \mathbf{C}$ has a discrete V-lift.

(4) Indiscrete structures—each object in $\mathbf{SET} \times \mathbf{C}$ has an indiscrete V-lift.

(5) V is faithful and possesses both a left adjoint—the discrete functor—and a right adjoint—the indiscrete functor; and both adjoints are full embeddings.

Proof. Conjoin Section 21 of [1] with Theorems 3.3.4 and 3.3.9. □

3.4.3. Theorem. The following properties or statements hold for **C-TOP** and **C-FTOP** over **SET** $\times$ **C** w.r.t. the appropriate forgetful functor V and the specified subcategory **C** of **LOQML**.

(1) Completeness and cocompleteness for **C** = **LOC** and **FUZLAT**.

(2) Cocompleteness for

$$\mathbf{C} = \mathbf{LOQML},\ \mathbf{SLOC},\ \mathbf{CLAT}^{op},\ \mathbf{DMRG}^{op},\ \mathbf{CBOOL}^{op}$$

(3) Completeness, cocompleteness, well-poweredness, co-well-poweredness, (Epi, Extremal Mono-Source)-category, regular factorizations, separators, coseparators for **C** = **L** (for $L \in |\mathbf{CQML}|$).

Proof. Conjoin Theorems 2.2.6, 3.3.4, and 3.3.9 with Section 21 of [1]. □

4 Topological categories for internalized change-of-basis

4.1 $\mathbf{L}_\phi$-TOP and endomorphism-saturated spaces

The variable-basis or change-of-basis approach enriches topology both respect to objects and morphisms; in particular, even the morphisms are enriched when the base of objects is fixed, providing the base has endomorphisms other than the identity—see the subcategory **C-TOP**(L) in Subsection 6.1 below. In this subsection, we show how a fixed base with endomorphisms other than the identity can be used to enrich objects, specifically the topologies of objects. This enrichment of topologies differs from stratification (Subsection 6.2), generates "free" objects from old objects, yields a "saturation" functor, and embeds into fixed-basis topology of the same base.

This approach is called **internalized change-of-basis topology**, and it rests on the ground category **SET** $\times$ $\mathbf{L}_\phi$ (2.1.2). The reader should note the contrast with fixed-basis topology: in that setting both the base is fixed **and** the endomorphism on that base is fixed at the identity. Further related ideas are found in [56, 62, 66]. We close this subsection with a substantial list of natural and canonical examples.

4.1.1. Definition (Endomorphism-saturated spaces). Let $L \in |\mathbf{CQML}|$ and $\phi \in \mathbf{LOQML}(L, L)$. The notation $\mathbf{L}_\phi$**-TOP** comprises the following data:

(1) **Objects**: Each object is of the form (X, L, τ), with L fixed, satisfying:

(a) **Ground axiom.** $(X, L) \in |\mathbf{SET} \times \mathbf{L}_\phi|$;

(b) **Topological axiom.** $(X, L, \tau) \in |\mathbf{L\text{-}TOP}|$ (see 3.1.5(6)) for **L-TOP**);

(c) **ϕ-Saturation Axiom.** $\forall b \in L^X$, $b \in \tau \Leftrightarrow \langle \phi^{op} \rangle (b) \in \tau$ (see 1.3.4 for $\langle \phi^{op} \rangle$).

Each such object is called a **ϕ-saturated topological space** (see 4.1.5 below).

(2) **Morphisms**: Each morphism is of the form (f, ϕ), with ϕ fixed, satisfying:

(a) **Ground axiom.** $(f, \phi) \in \mathbf{SET} \times \mathbf{L}_\phi$;

(b) **Continuity axiom.** $(f, \phi) \in \mathbf{LOQML\text{-}TOP}$.

(3) **Composition**: as in $\mathbf{SET} \times \mathbf{L}_\phi$.

(4) **Identities**: as in $\mathbf{SET} \times \mathbf{L}_\phi$.

4.1.2. Proposition. $\mathbf{L}_\phi$**-TOP** is a concrete category over the ground category $\mathbf{SET} \times \mathbf{L}_\phi$.

4.1.3. Remark. Note that $\mathbf{L}_\phi$**-TOP** is a subcategory of **LOQML-TOP** if $\phi = id_L$, in which case it is **L-TOP** and isomorphic to the category L-**TOP** discussed in [23]. The precise relationship between the identity case and the non-identity case, i.e. *id_L vis-a-vis ϕ*, is given in the adjunction of the next result.

4.1.4. Theorem (ϕ-Saturation adjunction $F \dashv G_\phi$). Let $L \in |\mathbf{CQML}|$ and $\phi \in \mathbf{LOQML}(L, L)$. Then the following hold:

(1) $\forall (X, L, \tau) \in |\mathbf{LOQML\text{-}TOP}|$, $\exists$ a smallest topology $\tau_\phi \supset \tau$, $(X, L, \tau_\phi) \in |\mathbf{L}_\phi\text{-}\mathbf{TOP}|$.

(2) There is a morphism-invertible [53], categorical isomorphism F of $\mathbf{L}_\phi$-**TOP** onto a full subcategory of **L-FTOP**, namely

$$F(X, L, \tau) = (X, L, \tau), \; F(f, \phi) = (f, id)$$

(3) There is a full, object-onto (hence dense) functor G_ϕ of **L-FTOP** onto $\mathbf{L}_\phi$-**TOP** defined by:

$$G_\phi(\tau) = \tau_\phi, \; G_\phi(X, L, \tau) = (X, L, \tau_\phi), \; G_\phi(f, id) = (f, \phi)$$

(4) (F, G_ϕ) is an adjunction; and $F \dashv G_\phi$ is an isocoreflection, but need not be an adjunction even if $L = \mathbb{I} \equiv [0, 1]$, and so $F \dashv G_\phi$ need not be an equivalence.

Proof. See Theorem 9.22 of [66] for the case $L \in |\mathbf{SFRM}|$ and $\phi \in \mathbf{SLOC}(L, L)$ —all details of proof carry over without alteration; cf. 2.1.8 above. □

4.1.5. Remark. The functor G_ϕ saturates or enriches a topological space by adding to its topology τ all the change-of-basis "shifts" or "mutations" by ϕ^{op} of the open sets in τ; this justifies the label "saturated" in 4.1.1 for the objects of $\mathbf{L}_\phi$-$\mathbf{TOP}$. Since the functor F of 4.1.4(2) is essentially an inclusion, the adjunction $F \dashv G_\phi$ of 4.1.4(4) suggests calling G_ϕ the **ϕ-saturation functor**. There is a well-established tradition of saturation operations in fuzzy topology: the stratification functor G_k [56, 62] of Subsection 6.2 is motivated by [42]; the translation-closed saturations of [18, 20]; and the soberification functors for both fixed and variable basis topology [56, 57, 62, 64, 65]. It is important to understand the relationship between saturation operations: every translation-closed space is stratified, no stratified space is sober if the underlying base has an endomorphism other than the identity [62, 64], and the relationship between stratification and ϕ-saturation is partly understood [66] and summarized below.

4.1.6. Question.

(1) Are there natural or even canonical objects in $\mathbf{L}_\phi$-$\mathbf{TOP}$?

(2) What is the relationship between ϕ-saturation and soberification?

4.1.7. Theorem (Catalogue of examples / Answer to 4.1.6). Let $L \in |\mathbf{CQML}|$ and $\phi \in \mathbf{LOQML}(L, L)$. Then the following statements hold:

(1) Each space of the form (X, L, L^X) is in $\mathbf{L}_\phi$-$\mathbf{TOP}$.

(2) Each space of the form $(X, L, \{\underline{\alpha} : \alpha \in L\})$ is in $\mathbf{L}_\phi$-$\mathbf{TOP}$.

(3) For each space (X, L, τ), $G_\phi(X, L, \tau)$ is in $\mathbf{L}_\phi$-$\mathbf{TOP}$. In particular, if $L \in |\mathbf{DMRG}|$ and $\phi \in \mathbf{DMRG}^{op}(L, L)$, then $G_\phi(\mathbb{R}(L))$, $G_\phi(\mathbb{R}, co\text{-}\tau(L))$, $G_\phi(\mathbb{I}(L))$, and $G_\phi(\mathbb{I}, co\text{-}\tau(L))$ are in $\mathbf{L}_\phi$-$\mathbf{TOP}$. (Cf. [12, 24, 39, 62] and Subsections 7.1 and 7.3 below).

(4) Let $L \in |\mathbf{DMRG} \cap \mathbf{FRM}|$ and $\phi \in Auto(L)$. Then $(\mathbb{R}, co\text{-}\tau(L))$ and $(\mathbb{I}, co\text{-}\tau(L))$ are in $\mathbf{L}_\phi$-$\mathbf{TOP}$.

(5) If (X, L, τ) is S_0 and ϕ-saturated, then its soberification

$$(Lpt(\tau), L, (\Phi_L)^{\rightarrow}(\tau))$$

is in $\mathbf{L}_\phi$-$\mathbf{TOP}$ and if (X, L, τ) is sober and ϕ-saturated, then

$$(\Psi_L, \phi) : (X, L, \tau) \to (Lpt(\tau), L, (\Phi_L)^{\rightarrow}(\tau))$$

is a morphism in $\mathbf{L}_\phi$-$\mathbf{TOP}$. (Cf. Subsection 7.4).

Proof. See [66] for the case $L \in |\mathbf{SFRM}|$ and $\phi \in \mathbf{SLOC}(L, L)$—all details of proof carry over without alteration. □

4.1.8. Remark. From 4.1.7(3), we have other natural forms of the fuzzy real line, the dual real line, the fuzzy unit interval, and the dual unit interval—namely their endo-saturated forms. Also see Subsection 5.1 for results concerning categorical isomorphisms in $\mathbf{L}_\phi$-**TOP**.

4.2 $\mathbf{L}_\phi$-FTOP and endomorphism-saturated fuzzy topologies

This Subsection brings the notion of internalized change-of-basis over from Subsection 4.1 to the objects and morphisms of **C-FTOP**, and in so doing makes reconciliation between the internalized change-of-basis concept and the notion of fixed-basis fuzzy topology as developed in [23]. The ground category in this Subsection is of the same form as that for Subsection 4.1, namely $\mathbf{SET} \times \mathbf{L}_\phi$, where $L \in |\mathbf{CQML}|$ and $\phi \in \mathbf{LOQML}(L, L)$.

4.2.1. Definition. (Endomorphism-saturated fuzzy spaces). Let $L \in |\mathbf{CQML}|$ and $\phi \in \mathbf{LOQML}(L, L)$. The category $\mathbf{L}_\phi$-**FTOP** comprises the following data:

(1) **Objects.** Each object is of the form $(X, L, \mathcal{T})$, with L fixed, satisfying:

   (a) **Ground axiom.** $(X, L) \in |\mathbf{SET} \times \mathbf{L}_\phi|$.

   (b) **Fuzzy topological axiom.** $(X, L, \mathcal{T}) \in |\mathbf{L}\text{-}\mathbf{FTOP}|$.

   (c) **ϕ-Saturation axiom.** $\forall$ object $(X, L, \mathcal{T})$, $\mathcal{T} = \mathcal{T} \circ \langle \phi^{op} \rangle$ on L^X.

   Each such object is called a **ϕ-saturated fuzzy topological spaces** (see 4.2.7 below).

(2) **Morphisms.** Each morphism is of the form

$$(f, \phi) : (X, L, \mathcal{T}) \to (Y, L, \mathcal{S})$$

   with ϕ fixed, satisfying:

   (a) **Ground axiom.** $(f, \phi) \in \mathbf{SET} \times \mathbf{L}_\phi$.

   (b) **Fuzzy continuity axiom.** $\mathcal{T} \circ (f, \phi)^{\leftarrow} \geq \mathcal{S}$ on L^Y.

   Each such morphism is said to be **fuzzy continuous**.

(3) **Composition.** As in $\mathbf{SET} \times \mathbf{L}_\phi$.

(4) **Identities.** As in $\mathbf{SET} \times \mathbf{L}_\phi$.

4.2.2. Proposition. $\mathbf{L}_\phi$**-FTOP** is a concrete category over the ground category $\mathbf{SET} \times \mathbf{L}_\phi$ (2.1.2).

Proof. That the identity morphism (id, ϕ) is fuzzy continuous follows immediately since

$$\mathcal{T} \circ (id, \phi)^{\leftarrow} = \mathcal{T} \circ \langle \phi^{op} \rangle = \mathcal{T}$$

(In fact, this shows the identity morphism to be a categorical isomorphism.) Now let continuous morphisms

$$(X_1, L, \mathcal{T}_1) \xrightarrow{(f,\phi)} (X_2, L, \mathcal{T}_2) \xrightarrow{(g,\phi)} (X_3, L, \mathcal{T}_3)$$

be given. The proof of the continuity of $(g \circ f, \phi)$ divides into two remarks. First, we claim the identity

$$\mathcal{T}_1 \circ (g \circ f, \phi)^{\leftarrow} = \mathcal{T}_1 \circ (f, \phi)^{\leftarrow} \circ (g, \phi)^{\leftarrow}$$

holds on L^X: for if $b \in L^X$, then

$$\mathcal{T}_1 \left(\phi^{op} \circ b \circ g \circ f \right) = \mathcal{T}_1 \left(\phi^{op} \circ (\phi^{op} \circ b \circ g \circ f) \right)$$

follows from 4.2.1(1)(c) and implies the claimed identity. Second, it follows from the continuities of $(f, \phi), (g, \phi)$ and this identity that

$$\mathcal{T}_1 \circ (g \circ f, \phi)^{\leftarrow} = \mathcal{T}_1 \circ (f, \phi)^{\leftarrow} \circ (g, \phi)^{\leftarrow} \geq \mathcal{T}_2 \circ (g, \phi)^{\leftarrow} \geq \mathcal{T}_3$$

where the last equality comes from 4.2.1(1)(c). Thus $(g \circ f, \phi)$ is continuous. □

4.2.3. Remark. Note that $\mathbf{L}_\phi$**-FTOP** is a subcategory of **LOQML-FTOP** if $\phi = id_L$, in which case it is precisely the subcategory $\mathbf{L}_{id}$**-FTOP** $\equiv$ **L-FTOP** and is isomorphic to L-**FTOP** studied in [23]. The precise relationship between the identity case and the non-identity case, i.e. id_L *vis-a-vis* ϕ, is given in the adjunction of 4.2.6 below. We return to this question after first answering an easier question.

4.2.4. Question. What is the precise relationship between the $\mathbf{L}_\phi$**-TOP** and $\mathbf{L}_\phi$**-FTOP**?

4.2.5. Proposition (Answer to 4.2.4). Let $(X, L, \tau) \in |\mathbf{C\text{-}TOP}|$ and let $(f, \phi) \in \mathbf{C\text{-}TOP}$. Then (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4), where:

(1) $(X, L, \tau) \in |\mathbf{L}_\phi\text{-}\mathbf{TOP}|$;

(2) $(X, L, \chi_\tau) \in |\mathbf{L}_\phi\text{-}\mathbf{FTOP}|$, where $\chi_\tau : L^X \to L$ with $coker\,(\chi_\tau) = \tau$;

(3) $(f, \phi) \in \mathbf{L}_\phi\text{-}\mathbf{TOP}\,((X, L, \tau), (Y, L, \sigma))$;

(4) $(f, \phi) \in \mathbf{L}_\phi\textbf{-FTOP}\left((X, L, \chi_\tau), (Y, L, \chi_\sigma)\right)$.

Proof. (1) $\Rightarrow$ (2). $\chi_\tau(b) = \top \Leftrightarrow b \in \tau \Leftrightarrow \langle\phi^{op}\rangle(b) \in \tau \Leftrightarrow \chi_\tau(\langle\phi^{op}\rangle(b)) = \top$.

(3) $\Rightarrow$ (4). Let $(f, \phi) : (X, L, \tau) \to (Y, L, \sigma)$ in $\mathbf{L}_\phi\textbf{-TOP}$. Then we note

$$\chi_\sigma(v) = \top \;\Rightarrow\; v \in \sigma \;\Rightarrow\; (f,\phi)^{\leftarrow}(v) \in \tau \;\Rightarrow\; \chi_\tau\left((f,\phi)^{\leftarrow}(v)\right) = \top$$

which implies $\chi_\tau \circ (f, \phi)^{\leftarrow} \geq \chi_\sigma$ on L^X, i.e. $(f, \phi) \in \mathbf{L}_\phi\textbf{-FTOP}$.

(1) $\Leftarrow$ (2), (3) $\Leftarrow$ (4). Reorder the steps of the proofs above. $\square$

4.2.6. Theorem (Extension of $F \dashv G_\phi$ to fuzzy topology). Let

$$\phi \in \mathbf{LOQML}(L, L)$$

Then the following hold:

(1) For each (X, L), the set of ϕ-saturated fuzzy topologies on (X, L) forms a complete lattice under the partial order of $L^{\left(L^X\right)}$. Hence, for each $(X, L, \mathcal{T})$ in **LOQML-FTOP**, there is a smallest fuzzy topology $\mathcal{T}_\phi \geq \mathcal{T}$ such that $(X, L, \mathcal{T})$ is an object in $\mathbf{L}_\phi\textbf{-FTOP}$.

(2) There is a morphism-invertible categorical isomorphism F of $\mathbf{L}_\phi\textbf{-FTOP}$ onto a full subcategory of **L-FTOP** namely

$$F(X, L, \mathcal{T}) = (X, L, \mathcal{T}), \; F(f, \phi) = (f, id)$$

(3) There is a full, object-onto (hence dense) functor G_ϕ of **L-FTOP** onto $\mathbf{L}_\phi\textbf{-FTOP}$ defined by:

$$G_\phi(\mathcal{T}) = \mathcal{T}_\phi, \; G_\phi(X, L, \mathcal{T}) = (X, L, G_\phi(\mathcal{T})), \; G_\phi(f, id) = (f, \phi)$$

(4) (F, G_ϕ) is an adjunction, and $F \dashv G_\phi$ is an isocoreflection; but (G_ϕ, F) need not be an adjunction even if $L = \mathbb{I} \equiv [0,1]$ and ϕ $\in$Auto(L), and so $F \dashv G_\phi$ need not be an equivalence.

Proof. *Ad (1).* Consider a subcollection $(\mathcal{T}_j : j \in J\}$ of ϕ-saturated fuzzy topologies on (X, L). We verify that $\bigwedge_j \mathcal{T}_j$ is a ϕ-saturated fuzzy topology on (X, L). That $\bigwedge_j \mathcal{T}_j$ is a fuzzy topology on (X, L) is already established (3.2.8(1) above). It remains to show that it is ϕ-saturated. Since each $\mathcal{T}_j$ is ϕ-saturated, it follows

$$\mathcal{T}_j = \mathcal{T}_j \circ \langle\phi^{op}\rangle$$

whence follows

$$\bigwedge_j \mathcal{T}_j = \bigwedge_j (\mathcal{T}_j \circ \langle \phi^{op} \rangle) = \bigwedge_e dge_j \mathcal{T}_j \ \circ \ \langle \phi^{op} \rangle$$

Ad (2). All the claims boil down to showing that $(f, \phi) : (X, L, \mathcal{T}) \to (Y, L, \sigma)$ is fuzzy continuous iff $(f, id) : (X, L, \mathcal{T}) \to (Y, L, \sigma)$ is fuzzy continuous, i.e.

$$\mathcal{T} \circ (f, \phi)^{\leftarrow} \geq \sigma \Leftrightarrow \mathcal{T} \circ (f, id)^{\leftarrow} \geq \sigma$$

Now from the identity of 4.1.1(1)(c) comes the following:

$$\mathcal{T} \circ (\phi^{op} \circ b \circ f) = \mathcal{T} \circ ((\phi^{op} \circ (b \circ f)) = \mathcal{T} \circ (b \circ f)$$

which implies the identity

$$\mathcal{T} \circ (f, id)^{\leftarrow} = \mathcal{T} \circ (f, \phi)^{\leftarrow}$$

and hence our claims.

Ad(3). It follows immediately from *(1)* that G_ϕ sends objects to ϕ-saturated objects. So the main issue is whether the fuzzy continuity of (f, id) implies that of (f, ϕ). We proceed in three steps, letting (f, id) be fuzzy continuous.

Step 1. $(\mathcal{T}_1)_\phi \circ (f, \phi)^{\leftarrow} = (\mathcal{T}_1)_\phi \circ (f, id)^{\leftarrow}$.
Proof. Since $(\mathcal{T}_1)_\phi$ is ϕ-saturated, the proof is immediate from the identity of 4.1.1(1)(c), as was done in *Ad (2)* above.

Step 2. $(\mathcal{T}_1)_\phi \circ (f, id)^{\leftarrow}$ is a ϕ-saturated fuzzy topology on (X_2, L).
Proof. Using the ϕ-saturation of $(\mathcal{T}_1)_\phi$, we note:

$$\begin{aligned}
\left((\mathcal{T}_1)_\phi \circ (f, id)^{\leftarrow} \right) (\phi^{op} \circ b) &= (\mathcal{T}_1)_\phi (id \circ (\phi^{op} \circ b) \circ f) \\
&= (\mathcal{T}_1)_\phi (\phi^{op} \circ (b \circ f)) \\
&= (\mathcal{T}_1)_\phi (b \circ f) \\
&= (\mathcal{T}_1)_\phi (id \circ b \circ f) \\
&= \left((\mathcal{T}_1)_\phi \circ (f, id)^{\leftarrow} \right) (b)
\end{aligned}$$

Step 3. $(\mathcal{T}_1)_\phi \circ (f, \phi)^{\leftarrow} \geq (\mathcal{T}_2)_\phi$; hence (f, ϕ) is fuzzy continuous.
Proof. Using the properties of ϕ-saturated fuzzy topologies, the previous two steps, the fuzzy continuity of (f, id), and the definition of ϕ-saturated fuzzy topologies, we make the following two arguments which prove this step and complete the proof of (3):

$$\begin{aligned}
(\mathcal{T}_1)_\phi \circ (f, \phi)^{\leftarrow} &= (\mathcal{T}_1)_\phi \circ (f, id)^{\leftarrow} \\
&\geq \mathcal{T}_1 \circ (f, id)^{\leftarrow} \\
&\geq \mathcal{T}_2,
\end{aligned}$$

$$(\mathcal{T}_1)_\phi \circ (f,\phi)^{\leftarrow} \geq (\mathcal{T}_2)_\phi$$

Ad (4). The negative claims of (4) follow from 4.1.4(4) followed by 4.2.5. Thus we need only verify $F \dashv G_\phi$. As indicated in 1.3.4, we divide the work into verifying continuity and naturality criteria.

Continuity Criterion. We must verify the following statement:

$$\forall (X,L,\mathcal{T}) \in |\mathbf{L}_\phi\text{-}\mathbf{FTOP}|,\ \exists (\eta,\phi) \in \mathbf{L}_\phi\text{-}\mathbf{FTOP}((X,L,\mathcal{T}), G_\phi F(X,L,\mathcal{T})),$$

$$\forall (Y,L,\mathcal{S}) \in |\mathbf{L}\text{-}\mathbf{FTOP}|,\ \forall (f,\phi) \in \mathbf{L}_\phi\text{-}\mathbf{FTOP}((X,L,\mathcal{T}), G_\phi(Y,L,\mathcal{S})),$$

$$\exists!\, \overline{(f,\phi)} \in \mathbf{L}\text{-}\mathbf{FTOP}(F(X,L,\mathcal{T}) \to (Y,L,\mathcal{S})),\ (f,\phi) = G_\phi\overline{(f,\phi)} \circ (\eta,\phi).$$

Let $(X,L,\mathcal{T})$ be given. Then we note $\mathcal{T}_\phi = \mathcal{T}$, so that $G_\phi F(X,L,\mathcal{T}) = (X,L,\mathcal{T})$. Choose $\eta = id$; then (id,ϕ) is fuzzy continuous (4.2.2). Now given (f,ϕ) as specified above, choose $\overline{(f,\phi)} = (f,id)$. Clearly $G_\phi\overline{(f,\phi)} = (f,\phi)$ is the unique ground morphism giving the required factorization. One issue remains: given the fuzzy continuity of $(f,\phi) : (X,L,\mathcal{T}) \to (Y,M,\mathcal{S}_\phi)$, verify the fuzzy continuity of $(f,id) : (X,L,\mathcal{T}) \to (Y,M,\mathcal{S})$. But this follows immediately using the identity from *Ad (2)* above:

$$\mathcal{T} \circ (f,id)^{\leftarrow} = \mathcal{T} \circ (f,\phi)^{\leftarrow} \geq \mathcal{S}_\phi \geq \mathcal{S}$$

Naturality Criterion. The issue is whether $\overline{(id,\phi) \circ (f,\phi)} = F(f,\phi)$, where the bar operator is taken from the previous paragraph. But each side is trivially (f,id).

This completes the proof of 4.2.6. □

4.2.7. Remark. The justifications for the label "ϕ-saturated" and calling G_ϕ the ϕ-saturation functor are analogous to those given in 4.1.5 and will not be repeated. The intuition for ϕ-saturated fuzzy topological spaces is rather different from that for ϕ-saturated topological spaces—the latter enriches the topology with translations by ϕ^{op} of all open sets, while the former enriches the fuzzy topology with commutative triangular diagrams; of course, in the setting of Subsection 4.1, these two notions coincide via 4.2.5. There is no relationship between this type of saturation and the stratification of [23]—this is a consequence of 4.1.5 and 4.2.5. Also note the examples of 4.1.7 carry over immediately to the fuzzy topological frameworks of this subsection if we apply 4.2.5 together with the answer to the following question.

4.2.8. Question. What is the precise relationship between the G_ϕ of 4.1.4(3) and the G_ϕ of this 4.2.6(3)?

4.2.9. Theorem (Answer to 4.2.8). Let $L \in |\mathbf{CQML}|$, $\phi \in \mathbf{LOQML}(L, L)$, and $(X, L, \tau) \in |\mathbf{L\text{-}TOP}|$. Then on L^X the following holds:

$$(\chi_\tau)_\phi = \chi_{(\tau_\phi)}$$

where the left-hand side uses the ϕ-saturation operation of this Subsection, and the right-hand side uses that of Subsection 4.1.

Proof. For the direction, the proof of 4.2.5(1,2) shows that $(\chi_\tau)_\phi$ is ϕ-saturated in the sense of this Subsection, and so $(\chi_\tau)_\phi \le \chi_{(\tau_\phi)}$ obtains by definition of G_ϕ in the fuzzy topology case. The reverse direction requires more work.

Step 1. Let $\mathcal{S}$ be a ϕ-saturated fuzzy topology on (X, L) in the sense of this Subsection. Then *coker* $(\mathcal{S})$ is a ϕ-saturated topology on (X, L) in the sense of Subsection 4.1. It will then follow that *coker* $(\chi_\tau)_\phi$ is a ϕ-saturated topology on (X, L) in the sense of Subsection 4.1.
Proof. To see *coker* $(\mathcal{S})$ is closed under arbitrary joins and binary products, let $\{u_\alpha\}_\alpha \subset$ *coker* $(\mathcal{S})$; note each $\mathcal{S}(u_\alpha) = 1$ and

$$\mathcal{S}\left(\bigvee_\alpha u_\alpha\right) \ge \bigwedge_\alpha \mathcal{S}(u_\alpha) = \top,$$

$$\mathcal{S}\left(u_\alpha \bigotimes u_\beta\right) \ge \mathcal{S}(u_\alpha) \bigotimes \mathcal{S}(u_\beta) \ge \top \bigotimes \top = 1$$

As for being ϕ-saturated, the steps are essentially those of 4.2.5(1,2).

Step 2. coker $(\chi_\tau)_\phi \supset \tau$.
Proof. $(\chi_\tau)_\phi \ge \chi_\tau \Rightarrow$ *coker* $\left((\chi_\tau)_\phi\right) \supset$ *coker* $(\chi_\tau) = \tau$.

Step 3. $(\chi_\tau)_\phi \ge \chi_{(\tau_\phi)}$.
Proof. From the previous steps and the definition of τ_ϕ from Subsection 4.1 follows

$$\mathit{coker}\left((\chi_\tau)_\phi\right) \supset \tau_\phi$$

which implies that $(\chi_\tau)_\phi \ge \chi_{(\tau_\phi)}$.

This completes the proof of the theorem. □

4.3 $\mathbf{L}_\phi$-TOP and $\mathbf{L}_\phi$-FTOP are topological over SET × $\mathbf{L}_\phi$

4.3.1. Theorem. For each $L \in |\mathbf{CQML}|$ and each endomorphism $\phi \in \mathbf{LOQML}(L, L)$, the category $\mathbf{L}_\phi$**-TOP** is topological over **SET** $\times\, \mathbf{L}_\phi$ with respect to the forgetful functor $V : \mathbf{L}_\phi\text{-}\mathbf{TOP} \to \mathbf{SET} \times \mathbf{L}_\phi$ given by

$$V(X, L, \tau) = (X, L), \; V(f, \phi) = (f, \phi)$$

Proof. It suffices to indicate modifications to the proofs of the analogues of 3.3.1–3.3.3 needed for this theorem—the statements of these analogues and the details of their proofs not addressed below are left to the reader.
In the proof of 3.3.1, given the V-structured source $((X, L), (f_i, \phi) : (X, L) \to (X_i, L, \tau_i)))_I$, put τ on (X, L) by

$$\tau = G_\phi \left(\bigvee_{i \in I} ((f_i, id_L)^{\leftarrow})^{\rightarrow} (\tau_i) \right)$$

Note that each (f_i, id_L) is continuous w.r.t. the topology

$$\bigvee i \in I \; ((f_i, id_L)^{\leftarrow})^{\rightarrow} (\tau_i)$$

which the proof of 3.3.1 would construct if L were held fixed and the endomorphism ϕ of L fixed at id_L. It now follows from 4.1.4(3), and the fact that $(f_i, \phi) = G_\phi(f_i, id_L)$, that each (f_i, ϕ) is continuous w.r.t. τ.

To modify the proof of 3.3.2, let v be a ϕ-saturated topology on (X, L) making each (g_i, ϕ) continuous, and let $(h, \phi) : (X, L) \to (X, L)$ be a ground morphism in $\mathbf{SET} \times \mathbf{L}_\phi$ such that

$$(f_i, \phi) \circ (h, \phi) = (g_i, \phi)$$

Using 4.1.4(2), we have that

$$(f_i, id_L) = F(f_i, \phi), \; (g_i, id_L) = F(g_i, \phi)$$

and hence that each of $(f_i, id_L), (g_i, id_L)$ is continuous. Since in $\mathbf{SET} \times \mathbf{L}$ we have

$$(f_i, id_L) \circ (h, id_L) = (g_i, id_L)$$

we apply the argument of 3.3.2 with $\mathbf{C} = \mathbf{L}$ to conclude that (h, id_L) is continuous w.r.t. the topology $\bigvee i \in I \; ((f_i, \phi)^{\leftarrow})^{\rightarrow} (\tau_i)$. Since $(h, \phi) = G_\phi(h, id_L)$, we now apply 4.1.4(3) to conclude that (h, ϕ) is continuous w.r.t. τ.

The proof of 3.3.3 needs no modification other than to assume all spaces are ϕ-saturated. □

4.3.2. Theorem. For each $L \in |\mathbf{CQML}|$ and each endomorphism $\phi \in \mathbf{LOQML}(L, L)$, the category $\mathbf{L}_\phi$-$\mathbf{FTOP}$ is topological over $\mathbf{SET} \times \mathbf{L}_\phi$ with respect to the forgetful functor $V : \mathbf{L}_\phi\text{-}\mathbf{FTOP} \to \mathbf{SET} \times \mathbf{L}_\phi$ given by

$$V(X, L, \mathcal{T}) = (X, L), \; V(f, \phi) = (f, \phi)$$

Proof. Analogous to the proof of 4.3.1, we only give modifications of the proofs of 3.3.5, 3.3.6, and 3.3.8.
In the proof of 3.3.5, given the V-structured source $((X, L), (f_i, \phi) : (X, L) \to (X_i, L, \mathcal{T}_i)))_I$, put $\mathcal{S}$ on L^X by defining $\mathcal{S}$ at each $a \in L^X$ as follows:

$$\mathcal{S}(a) = \begin{cases} \displaystyle\bigvee_{i \in I, \; b \in ((f_i, id_L)^{\leftarrow})^{\leftarrow}(\{a\})} (\mathcal{T}_i(b)), & \exists\, i \in I, \; ((f_i, id_L)^{\leftarrow})^{\leftarrow} (\{a\}) \neq \emptyset \\ \perp & , \; \forall\, i \in I, \; ((f_i, id_L)^{\leftarrow})^{\leftarrow} (\{a\}) = \emptyset \end{cases}$$

Apply G_ϕ of 4.2.6(3) and set

$$\mathcal{T} = G_\phi\left(\langle\langle\mathcal{S}\rangle\rangle\right)$$

By the argument of 3.3.5, each (f_i, id_L) is fuzzy continuous w.r.t. $\langle\langle\mathcal{S}\rangle\rangle$. Since $(f_i, \phi) = G_\phi(f_i, id_L)$, it follows from 4.2.6(3) that (f_i, ϕ) is fuzzy continuous w.r.t. the fuzzy topology $\mathcal{T}$.

To modify the proof of 3.3.6, let $\mathcal{U}$ be a ϕ-saturated fuzzy topology on (X, L) making each (g_i, ϕ) fuzzy continuous, and let $(h, \phi) : (X, L) \to (X, L)$ be a ground morphism in $\mathbf{SET} \times \mathbf{L}_\phi$ such that

$$(f_i, \phi) \circ (h, \phi) = (g_i, \phi)$$

Using 4.2.6(2), we have that

$$(f_i, id_L) = F(f_i, \phi),\ (g_i, id_L) = F(g_i, \phi)$$

and hence that each of $(f_i, id_L), (g_i, id_L)$ is continuous. Since in $\mathbf{SET} \times \mathbf{L}$ we have

$$(f_i, id_L) \circ (h, id_L) = (g_i, id_L)$$

we apply the argument of 3.3.6 with $\mathbf{C} = \mathbf{L}$ to conclude that (h, id_L) is fuzzy continuous w.r.t. the fuzzy topology $\langle\langle\mathcal{S}\rangle\rangle$. Since $(h, \phi) = G_\phi(h, id_L)$, we now apply 4.2.6(3) to conclude that (h, ϕ) is fuzzy continuous w.r.t. $\mathcal{T} \equiv G_\phi\left(\langle\langle\mathcal{S}\rangle\rangle\right)$.

The proof of 3.3.8 needs no modification other than to assume all spaces are ϕ-saturated. □

4.3.3. Corollary. $\forall L \in |\mathbf{CQML}|$, L-**TOP** and L-**FTOP** of [23] are topological over **SET**.

Proof. Note from Subsection 6.2 that L-**TOP** $\cong$ **L-TOP** $\cong$ $\mathbf{L}_{id}$-**TOP** and L-**FTOP** $\cong$ **L-FTOP** $\cong$ $\mathbf{L}_{id}$-**FTOP**. Now apply Proposition 1.3.1, 2.1.9(1), 4.3.1, and 4.3.2 above. □

4.4 Categorical consequences of topological and fibre-smallness

4.4.1. Lemma. Let $L \in |\mathbf{CQML}|$ and $\phi \in \mathbf{LOQML}(L, L)$. Then $\mathbf{L}_\phi$-**TOP** and $\mathbf{L}_\phi$-**FTOP** are fibre-small over $\mathbf{SET} \times \mathbf{L}_\phi$.

Proof. See proof of 3.4.1. □

4.4.2. Theorem. Let $L \in |\mathbf{CQML}|$ and $\phi \in \mathbf{LOQML}(L, L)$. Then the following properties or statements hold for $\mathbf{L}_\phi$-**TOP** and $\mathbf{L}_\phi$-**FTOP** over $\mathbf{SET} \times \mathbf{L}_\phi$ w.r.t. the appropriate forgetful functor V:

(1) All properties and statements listed in 3.4.2(1–5) above, including initial and final completeness and discrete and indiscrete structures.

(2) All the properties listed in 3.4.3(3) above: completeness, co-completeness, well-poweredness, co-well-poweredness, (Epi, Extremal Mono-Source)-category, regular factorizations, separators, and co-separators.

Proof. Modify the proofs of 3.4.2 and 3.4.3 using 2.1.9(1), 4.3.1, and 4.3.2. □

5 Categorical isomorphisms and embeddings

This section defines and characterizes isomorphisms and embeddings in **C-TOP**, **C-FTOP**, $\mathbf{L}_\phi$**-TOP**, and $\mathbf{L}_\phi$**-FTOP.** The applications of these results are widespread: some have already appeared in 4.1.4(4) and 4.2.6(4) above; and they will be appealed to repeatedly throughout Section 7 and show this section is foundational to the internal justification of the categories presented in this chapter. Further applications include the following: this section charaterizes the categorical embeddings of L-**TOP** and L-**FTOP** (5.3.9); and these characterizations are fundamental for the fixed-basis compactification reflectors [65]. Throughout this section **C** is a subcategory of **LOQML**, $L \in |\mathbf{CQML}|$, and $\phi \in \mathbf{LOQML}(L, L)$.

5.1 Homeomorphisms, fuzzy homeomorphisms, and categorical isomorphisms

5.1.1. Definition (Homeomorphisms and open maps). Let $(f, \phi) : (X, L) \to (Y, M)$ in **SET** × **C**. Then (f, ϕ) is a **homeomorphism** in **C-TOP** from (X, L, τ) to (Y, M, σ) if each of f and ϕ^{op} is bijective and each of (f, ϕ) and $(f^{-1}, ({}^*\phi)^{op})$ is continuous; and (f, ϕ) is **open** in **C-TOP** from (X, L, τ) to (Y, M, σ) if $(f, \phi)^{\rightarrow}_{|\tau} : \tau \to \sigma$. (See 1.3.3 for powerset operators, including ${}^*\phi$ and $(f, \phi)^{\rightarrow}$.)

5.1.2. Theorem (Topological isomorphisms). Let $(f, \phi) : (X, L) \to (Y, M)$ be a ground morphism in **SET** × **C**. Then the following are equivalent:

(1) $(f, \phi) : (X, L, \tau) \to (Y, M, \sigma)$ is a homeomorphism in **C-TOP**.

(2) $(f, \phi) : (X, L, \tau) \to (Y, M, \sigma)$ is a categorical isomorphism in **C-TOP**.

(3) $(f, \phi) : (X, L, \tau) \to (Y, M, \sigma)$ is both open and continuous in **C-TOP**.

(4) Both $(f, \phi) : (X, L, \tau) \to (Y, M, \sigma)$ and

$$\left(f^{-1}, \left((\phi^{op})^{-1}\right)^{op}\right) : (X, L, \tau) \leftarrow (Y, M, \sigma)$$

are continuous.

Proof. This result occurs in [66]; its proof is a straightforward consequence of the definitions of composition of ground morphisms, continuity, and powerset operators (see 1.3.3). □

5.1.3. Definition (Fuzzy homeomorphisms and fuzzy open maps). Let $(f, \phi) : (X, L) \to (Y, M)$ in **SET** × **C**. Then (f, ϕ) is a **fuzzy homeomorphism** in **C-FTOP** from $(X, L, \mathcal{T})$ to $(Y, M, \mathcal{S})$ if each of f and ϕ^{op} is bijective and each of (f, ϕ) and $(f^{-1}, ({}^*\phi)^{op})$ is fuzzy continuous; and (f, ϕ) is **fuzzy open** in **C-FTOP** from $(X, L, \mathcal{T})$ to $(Y, M, \mathcal{S})$ if ${}^*\phi \circ \mathcal{T} \leq \mathcal{S} \circ (f, \phi)^{\to}$ on L^X.

5.1.4. Theorem (Fuzzy topological isomorphisms). Let

$$(f, \phi) : (X, L) \to (Y, M)$$

be a ground morphism in **SET** × **C**. Then the following are equivalent:

(1) $(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$ is a fuzzy homeomorphism in **C-FTOP**.

(2) $(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$ is a categorical isomorphism in **C-FTOP**.

(3) $(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$ is both fuzzy open and fuzzy continuous in **C-FTOP**.

(4) $(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$ and $\left(f^{-1}, \left((\phi^{op})^{-1}\right)^{op}\right) : (X, L, \mathcal{T}) \leftarrow (Y, M, \mathcal{S})$ are continuous.

(5) $(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$ satisfies

$$\mathcal{T} \circ (f, \phi)^{\leftarrow} = \phi^{op} \circ \mathcal{S} \text{ on } M^Y$$

(6) $(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$ satisfies

$${}^*\phi \circ \mathcal{T} = \mathcal{S} \circ (f, \phi)^{\to} \text{ on } L^X$$

Proof. We first show the equivlence of (5) with (6), and then indicate how these are equivalent to each of (1)–(4).

(5) ⇔ (6). First note that for a ground isomorphism (f, ϕ), we have ${}^*\phi = (\phi^{op})^{-1}$ and $(f, \phi)^{\to} = ((f, \phi)^{\leftarrow})^{-1}$(see [66, 67]). Second, observe now that the equations of (5) and (6) are equivalent using these inverses.

$((5) \vee (6)) \Leftrightarrow (1)$. For necessity, to show (f, ϕ) is a fuzzy homeomorphism, we must show both (f, ϕ) and $(f^{-1}, (^*\phi)^{op})$ are fuzzy continuous, i.e. show both the following inequalities hold:

$$\mathcal{T} \circ (f, \phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S} \text{ on } M^Y, \mathcal{S} \circ (f^{-1}, (^*\phi)^{op})^{\leftarrow} \geq {}^*\phi \circ \mathcal{T} \text{ on } L^X$$

The first is immediate from (5). Now, note that (f, ϕ) being a ground isomorphism implies that $(f^{-1}, (^*\phi)^{op})^{\leftarrow} = (f, \phi)^{\rightarrow}$ (see [66, 67]), so that (5) via (6) implies the second inequality. For sufficiency, we assume the two inequalities displayed above. We also assume f, ϕ^{op} are bijections. Using $^*\phi = (\phi^{op})^{-1}$ and $(f, \phi)^{\rightarrow} = ((f, \phi)^{\leftarrow})^{-1}$, we obtain from the first assumed inequality

$$^*\phi \circ \mathcal{T} \geq \mathcal{S} \circ (f, \phi)^{\rightarrow}$$

and from the second assumed inequality

$$\mathcal{S} \circ (f, \phi)^{\rightarrow} \geq {}^*\phi \circ \mathcal{T}$$

the both of which imply (6). Hence (5) is also obtained (by the equivalence of (5) with (6)).

$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$. These are similar and left to the reader. □

5.1.5. Definition (Homeomorphisms and fuzzy homeomorphisms in $\mathbf{L}_\phi$-TOP and $\mathbf{L}_\phi$-FTOP). Let $(f, \phi) : (X, L) \to (Y, M)$ in **SET** $\times$ $\mathbf{L}_\phi$. Then (f, ϕ) is a **homeomorphism** in $\mathbf{L}_\phi$**-TOP** from (X, L, τ) to (Y, M, σ) if f is bijective and each of (f, ϕ) and (f^{-1}, ϕ) is continuous; (f, ϕ) is a **fuzzy homeomorphism** in $\mathbf{L}_\phi$**-FTOP** from $(X, L, \mathcal{T})$ to $(Y, M, \mathcal{S})$ if f is bijective and each of (f, ϕ) and (f^{-1}, ϕ) is fuzzy continuous; and (f, ϕ) is **open** [**fuzzy open**] if (f^{-1}, ϕ) is continuous [fuzzy continuous].

5.1.6. Theorem (Topological isomorphisms in $\mathbf{L}_\phi$-TOP). Let $(f, \phi) : (X, L) \to (Y, L)$ be a ground morphism in **SET** $\times$ $\mathbf{L}_\phi$. Then the following are equivalent:

(1) $(f, \phi) : (X, L, \tau) \to (Y, L, \sigma)$ is a homeomorphism in $\mathbf{L}_\phi$**-TOP**.

(2) $(f, \phi) : (X, L, \tau) \to (Y, L, \sigma)$ is a categorical isomorphism in $\mathbf{L}_\phi$**-TOP**.

(3) $(f, \phi) : (X, L, \tau) \to (Y, L, \sigma)$ is both open and continuous in $\mathbf{L}_\phi$**-TOP**.

Proof. Similar to that of 5.1.2. □

5.1.7. Theorem (Fuzzy topological isomorphisms in $\mathbf{L}_\phi$-FTOP). Let $(f, \phi) : (X, L) \to (Y, L)$ be a ground morphism in **SET** $\times$ $\mathbf{L}_\phi$. Then the following are equivalent:

(1) $(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$ is a fuzzy homeomorphism in $\mathbf{L}_\phi$**-FTOP**.

(2) $(f,\phi):(X,L,\mathcal{T})\to(Y,M,\mathcal{S})$ is a categorical isomorphism in $\mathbf{L}_\phi$**-FTOP.**

(3) $(f,\phi):(X,L,\mathcal{T})\to(Y,M,\mathcal{S})$ is both fuzzy open and fuzzy continuous in $\mathbf{L}_\phi$**-FTOP.**

(4) $(f,\phi):(X,L,\mathcal{T})\to(Y,M,\mathcal{S})$ satisfies

$$\mathcal{T}\circ(f,\phi)^{\leftarrow}=\mathcal{S}\text{ on }L^Y$$

(5) $(f,\phi):(X,L,\mathcal{T})\to(Y,M,\mathcal{S})$ satisfies

$$\mathcal{T}=\mathcal{S}\circ\left(f^{-1},\phi\right)^{\leftarrow}\text{ on }L^X$$

Proof. (1) $\Leftrightarrow$ (3) is trivial, and (1) $\Leftrightarrow$ (2) is straightforward using the composition of morphisms in $\mathbf{L}_\phi$**-FTOP.** We turn to the other implications, first noting from [66, 67] that for a bijection f, we have

$$\left(f^{-1}\right)_L^{\leftarrow}=\left(f_L^{\leftarrow}\right)^{-1}$$

and second noting that $\mathcal{T},\mathcal{S}$ are ϕ-saturated fuzzy topologies, so that

$$\mathcal{T}=\mathcal{T}\circ\langle\phi^{op}\rangle\text{ on }L^X,\;\;\mathcal{S}=\mathcal{S}\circ\langle\phi^{op}\rangle\text{ on }L^Y$$

(4) $\Leftrightarrow$ (5). We prove necessity; sufficiency essentially reverses the steps of necessity. Assuming (4), we obtain (5) in this way:

$$\mathcal{T}\circ(f,\phi)^{\leftarrow}=\mathcal{S},$$
$$\mathcal{T}\circ\langle\phi^{op}\rangle\circ f_L^{\leftarrow}=\mathcal{S},$$
$$\mathcal{T}\circ\langle\phi^{op}\rangle=\mathcal{S}\circ\left(f^{-1}\right)_L^{\leftarrow},$$
$$\mathcal{T}=\mathcal{S}\circ\langle\phi^{op}\rangle\circ\left(f^{-1}\right)_L^{\leftarrow},$$
$$\mathcal{T}=\mathcal{S}\circ\left(f^{-1},\phi\right)^{\leftarrow}$$

(1) $\Leftrightarrow$ (4). We prove necessity, leaving sufficiency to the reader. Assuming (1) is equivalent to assuming the following two inequalities:

$$\mathcal{T}\circ(f,\phi)^{\leftarrow}\geq\mathcal{S},\;\;\mathcal{S}\circ\left(f^{-1},\phi\right)^{\leftarrow}\geq\mathcal{T}$$

The second inequality leads to this sequence of steps:

$$\mathcal{S}\circ\langle\phi^{op}\rangle\circ\left(f^{-1}\right)_L^{\leftarrow}\geq\mathcal{T},$$
$$\mathcal{S}\circ\langle\phi^{op}\rangle\geq\mathcal{T}\circ f_L^{\leftarrow}.$$
$$\mathcal{S}\geq\mathcal{T}\circ\langle\phi^{op}\rangle\circ f_L^{\leftarrow},$$
$$\mathcal{S}\geq\mathcal{T}\circ(f,\phi)^{\leftarrow}$$

Now this last inequality together with the first inequality from above yield (4).

(1) $\Leftrightarrow$ (5). Analogous proof to that for (1) $\Leftrightarrow$ (4). □

5.2 Subspaces, initial structures, and initial morphisms

Throughout this subsection **C** is a subcategory of **LOQML**. In this subsection we determine the initial structures and initial morphisms for **C-TOP**, **C-FTOP**, $\mathbf{L}_\phi$**-TOP**, and $\mathbf{L}_\phi$**-FTOP**. Of course, initial structures in these categories are guaranteed by their being topological over their respective grounds (Subsections 3.3, 3.4, 4.3); but our present purpose is to apply this theory via this subsection to characterize embeddings in the next subsection.

5.2.1. Theorem (Initial structures and initial morphisms of C-TOP and $\mathbf{L}_\phi$-TOP). Let $(f,\phi) : (X,L) \to (Y,M)$ be a ground morphism in $\mathbf{SET} \times \mathbf{C}$, σ be a topology on (Y,M),

$$\Im \equiv \{\{\tau \mid (f,\phi) : (X,L,\tau) \to (Y,M,\sigma) \text{ is continuous in } \mathbf{C\text{-}TOP}\}\}$$

and

$$\tau_\wedge \equiv ((f,\phi)^{\leftarrow})^{\rightarrow}(\sigma)$$

where $((f,\phi)^{\leftarrow})^{\rightarrow}$ is the classical forward operator of the fuzzy backward operator $(f,\phi)^{\leftarrow}$ (1.3.3). Then the following hold:

(1) $\tau_\wedge$ is a topology on (X,L);

(2) $\tau_\wedge \in \Im$;

(3) $\tau \in \Im \Leftrightarrow \tau \geq \tau_\wedge$;

(4) $\tau_\wedge = \bigwedge \Im$;

(5) $((X,L,\tau_\wedge),\ (f,\phi) : (X,L,\tau_\wedge) \to (Y,M,\sigma))$ is the unique initial V-lift of $((X,L),\ (f,\phi) : (X,L) \to V(Y,M,\sigma))$, where $V : \mathbf{C\text{-}TOP} \to \mathbf{SET} \times \mathbf{C}$ is the forgetful functor of Subsection 3.3.

(6) $(f,\phi) : (X,L,\tau_\wedge) \to (Y,M,\sigma)$ is an initial morphism in **C-TOP**.

(7) $(f,\phi) : (X,L,\tau) \to (Y,M,\sigma)$ is an initial morphism in **C-TOP** $\Leftrightarrow \tau = \tau_\wedge$

All of (1–7) hold if the following replacements are made: $\mathbf{SET} \times \mathbf{C}$ by $\mathbf{SET} \times \mathbf{L}_\phi$, M by L, **C-TOP** by $\mathbf{L}_\phi$**-TOP**, $\tau_\wedge$ by $G_\phi(\tau_\wedge)$ (where G_ϕ is given in Subsection 4.1), "topology" by "ϕ-saturated topology", and "Subsection 3.3" by "Subsection 4.3".

Proof. *Ad (1–4).* (1) is a straightforward consequence of Lemma 1.3.3, (2) is immediate, (3) is a restatement of the definition of continuity, and (4) is clear from (2,3).

Ad (5). It follows from the proof of Theorem 3.3.4 that the topology on (X,L) giving the unique initial lift is $\langle\langle((f,\phi)^{\leftarrow})^{\rightarrow}(\sigma)\rangle\rangle$. But this topology must be $\tau_\wedge$ by (1) and the definition of subbase.

Ad (6). Note that $(f,\phi) : (X,L,\tau_\wedge) \to (Y,M,\sigma)$ is a morphism by (2). For initiality, let $(g,\psi) : (Z,N,\upsilon) \to (X,L,\tau_\wedge)$ such that $(f,\phi)\circ(g,\psi)$ is continuous, and let $u \in \tau_\wedge$. Then $\exists\, v \in \tau$, $u = (f,\phi)^{\leftarrow}(v)$ and

$$(g,\psi)^{\leftarrow}(u) = (g,\psi)^{\leftarrow}\left((f,\phi)^{\leftarrow}(v)\right) = \left((f,\phi)\circ(g,\psi)\right)^{\leftarrow}(v) \in \upsilon$$

Ad (7). Sufficiency is (6). For necessity, suppose that $(f,\phi) : (X,L,\tau) \to (Y,M,\sigma)$ is initial. Choose $(Z,N,\upsilon) = (X,L,\tau_\wedge)$ and $(g,\psi) = (id,id)$. Then $(f,\phi)\circ(g,\psi) = (f,\phi)$ as ground morphisms, and the latter is continuous by (2) above. Now initiality applies to say that (g,ψ) is continuous, which forces $\tau \subset \tau_\wedge$. For the reverse containment, we note that (f,ϕ) being initial in **C-TOP** includes its being a morphism in **C-TOP**. So (f,ϕ) is continuous, which implies by (3) that $\tau \supset \tau_\wedge$. Hence $\tau = \tau_\wedge$.

The proofs of (1–7) in the endo-saturated case are analogous to the proofs given above. □

5.2.2. Definition (Subspaces in C-TOP). Let $(X,L,\tau) \in |\mathbf{C\text{-}TOP}|$ and $A \subset X$. Then the **subspace** $(A,L,\tau_{|A})$ is defined by putting

$$\tau_{|A} = \{u_{|A} : u \in \tau\}.$$

This notion stems from [79] and is necessary for compactification theories in fixed-basis topology [65] as well as for determining embeddings.

5.2.3. Proposition. Let $(X,L) \in |\mathbf{SET} \times \mathbf{C}|$ and let $A \subset X$. Then the powerset operator $(\hookrightarrow,\phi)^{\leftarrow} : L^A \leftarrow L^X$ is the fuzzy subset restriction operator, i.e.

$$(\hookrightarrow,\phi)^{\leftarrow}(b) = \phi^{op} \circ b_{|A}$$

For $\phi^{op} = id$, $(\hookrightarrow, id)^{\leftarrow}(b) = b_{|A}$.

Proof. The point-wise computation is straightforward from 1.3.3. □

5.2.4. Corollary (Subspaces and initial morphisms in C-TOP and $\mathbf{L}_\phi$-TOP). Let $(X,L,\tau) \in |\mathbf{C\text{-}TOP}|$ and $A \subset X$. Then the following hold:

(1) $\tau_{|A}$ is a topology on (A,L);

(2) $(\hookrightarrow, id) : (A,L,\upsilon) \to (X,L,\tau)$ is continuous in **C-TOP** $\Leftrightarrow \upsilon \supset \tau_{|A}$;

(3) $\tau_{|A} = \bigwedge \{\upsilon \mid (\hookrightarrow, id) : (A,L,\upsilon) \to (X,L,\tau)$ is continuous in $\mathbf{C\text{-}TOP}\}$;

(4) $\left((A,L,\tau_{|A}),\ (\hookrightarrow, id) : (A,L,\tau_{|A}) \to (X,L,\tau)\right)$ is the unique initial V-lift of $((A,L),\ (\hookrightarrow, id) : (A,L) \to V(X,L,\tau))$, where V is the appropriate forgetful functor.

(5) $(\hookrightarrow, id) : (A,L,\upsilon) \to (X,L,\tau)$ is an initial morphism in **C-TOP** $\Leftrightarrow \upsilon = \tau_{|A}$.

All of (1–5) hold if the following replacements are made: **C-TOP** by $\mathbf{L}_\phi$**-TOP**, $\tau|_A$ by $G_\phi\left(\tau|_A\right)$, "topology" by "$\phi$-saturated topology", and $(\hookrightarrow, id)$ by $(\hookrightarrow, \phi)$.

Proof. Apply 5.2.1–5.2.3. □

Initial morphisms and subspaces comprise a more delicate issue for variable-basis fuzzy topology than for variable-basis topology. In the next theorem, we retain the format of 5.2.1 in order to facilitate comparisons between these theorems.

5.2.5. Theorem (Initial structures and initial morphisms of C-FTOP and $\mathbf{L}_\phi$-FTOP). Let $(f, \phi) : (X, L) \to (Y, M)$ be a ground morphism in $\mathbf{SET} \times \mathbf{C}$, $\mathcal{S}$ be a topology on (Y, M),

$$\Im \equiv \{\mathcal{T} \mid (f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S}) \text{ is fuzzy continuous in } \mathbf{C\text{-}FTOP}\}$$

$\mathcal{S}^* : L^X \to L$ by

$$\mathcal{S}^*(a) = \bigvee_{(f,\phi)^{\leftarrow}(b)=a} \phi^{op}(\mathcal{S}(b))$$

and

$$\mathcal{T}_\wedge \equiv \langle\langle \mathcal{S}^* \rangle\rangle$$

Then the following hold:

(1) $\mathcal{T}_\wedge$ is a fuzzy topology on (X, L);

(2) $\mathcal{T}_\wedge \in \Im$;

(3) $\mathcal{T} \in \Im \Leftrightarrow \mathcal{T} \geq \mathcal{T}_\wedge$;

(4) $\mathcal{T}_\wedge = \bigwedge \Im$;

(5) $((X, L, \mathcal{T}_\wedge),\ (f, \phi) : (X, L, \mathcal{T}_\wedge) \to (Y, M, \mathcal{S}))$ is the unique initial V-lift of $((X, L),\ (f, \phi) : (X, L) \to V(Y, M, \mathcal{S}))$, where

$$V : \mathbf{C - FTOP} \to \mathbf{SET} \times \mathbf{C}$$

is the forgetful functor of Subsection 3.3;

(6) $(f, \phi) : (X, L, \mathcal{T}_\wedge) \to (Y, M, \mathcal{S})$ is an initial morphism in **C-TOP**;

(7) $(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$ is an initial morphism in **C-TOP** $\Leftrightarrow \mathcal{T} = \mathcal{T}_\wedge$.

All of (1–7) hold if the following replacements are made: $\mathbf{SET} \times \mathbf{C}$ by $\mathbf{SET} \times \mathbf{L}_\phi$, M by L, **C-FTOP** by $\mathbf{L}_\phi$**-FTOP**, $\langle\langle \mathcal{S}^* \rangle\rangle$ by $G_\phi\left(\langle\langle \mathcal{S}^* \rangle\rangle\right)$ (where G_ϕ is given in Subsection 4.1), "fuzzy topology" by "ϕ-saturated fuzzy topology", and "Subsection 3.3" by "Subsection 4.3".

Proof. *Ad (1).* Immediate from 3.2.8 and 3.2.9.

Ad (2). The proof that $\mathcal{S}^* \circ (f,\phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S}$ is essentially that of 3.3.5 above. Hence (f,ϕ) is fuzzy co-subbasic continuous. Now 3.2.13(4) applies to say (f,ϕ) is fuzzy continuous. It follows $\mathcal{T}_\wedge \in \Im$.

Ad (3). From (2) it follows that $\mathcal{T}_\wedge \circ (f,\phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S}$. For sufficiency, suppose $\mathcal{T} \geq \mathcal{T}_\wedge$. Then $\mathcal{T} \circ (f,\phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S}$, so $\mathcal{T} \in \Im$. Now for necessity, suppose $\mathcal{T} \in \Im$, let $a \in L^X$, and fix $b \in M^Y$ such that $a = (f,\phi)^{\leftarrow}(b)$. Since $\mathcal{T} \circ (f,\phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S}$, we have

$$\mathcal{T}(a) = \mathcal{T}\left((f,\phi)^{\leftarrow}(b)\right) \geq \phi^{op}\left(\mathcal{S}(b)\right)$$

Since this is true for each b such that $a = (f,\phi)^{\leftarrow}(b)$, we have

$$\mathcal{T}(a) \geq \bigvee_{a=(f,\phi)^{\leftarrow}(b)} \phi^{op}\left(\mathcal{S}(b)\right) = \mathcal{S}^*(a)$$

It follows that $\mathcal{T} \geq \mathcal{T}_\wedge$.

Ad (4). This is immediate from (2,3).

Ad (5). This is a corollary of the proofs of 3.3.5–3.3.9: in this case there is only one arrow in our V-structured source, and this simplifies the subbase of the proof of 3.3.5 to $\mathcal{S}^*$ as defined in the statement of the theorem.

Ad (6). Note that $(f,\phi) : (X,L,\mathcal{T}_\wedge) \to (Y,M,\mathcal{S})$ is a morphism by (2). For initiality, let $(g,\psi) : (Z,N,\mathcal{U}) \to (X,L,\mathcal{T}_\wedge)$ such that $(f,\phi) \circ (g,\psi)$ is fuzzy continuous, and recall the alternate fuzzy continuity axiom of Lemma 3.1.3.1. From the fuzzy continuiuty of the composition we have

$$\psi^* \circ \mathcal{U} \circ (g,\psi)^{\leftarrow} \circ (f,\phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S}$$

which implies that

$$\psi^* \circ \mathcal{U} \circ (g,\psi)^{\leftarrow} \in \Im$$

and hence that

$$\psi^* \circ \mathcal{U} \circ (g,\psi)^{\leftarrow} \geq \mathcal{T}_\wedge$$

Invoking 3.1.3.1 now yields the continuity of (g,ψ).

Ad (7). Sufficiency is immediate from (6). For necessity, we first note that the fuzzy continuity of (f,ϕ) from initiality implies via (3) that $\mathcal{T} \geq \mathcal{T}_\wedge$. Now to further apply initiality, let $(g,\psi) : (Z,N,\mathcal{U}) \to (X,L,\mathcal{T}_\wedge)$ be given with $(Z,N,\mathcal{U}) = (X,L,\mathcal{T}_\wedge)$ and $(g,\psi) = (id,id)$. Then $(f,\phi) \circ (g,\psi)$ is fuzzy continuous, so initiality implies the fuzzy continuity of (id,id), which gives

$$\mathcal{T}_\wedge \circ (id,id)^{\leftarrow} \geq id \circ \mathcal{T}$$

This forces $\mathcal{T}_\wedge \geq \mathcal{T}$. So $\mathcal{T}_\wedge = \mathcal{T}$.

The proofs of (1–7) in the endo-saturated case are analogous to the proofs given above. □

5.2.6. Definition (Fuzzy subspaces in C-FTOP). Let

$$(X, L, \mathcal{T}) \in |\mathbf{C\text{-}FTOP}|$$

and $A \subset X$. Then the **fuzzy subspace** $(A, L, \mathcal{T}_{|A})$ is defined by putting

$$\mathcal{T}_{|A} = \langle\langle \mathcal{S} \rangle\rangle$$

where $\mathcal{S} : L^A \to L$ by

$$\mathcal{S}(a) = \bigvee_{b_{|A} = a,\ b \in L^X} \mathcal{T}(b)$$

This is the fuzzy topological counterpart to 5.2.2 above.

5.2.7. Corollary (Subspaces and initial morphisms in C-FTOP and $\mathbf{L}_\phi$-FTOP). Let $(X, L, \mathcal{T}) \in |\mathbf{C\text{-}FTOP}|$ and $A \subset X$. Then the following hold:

(1) $\mathcal{T}_{|A}$ is a fuzzy topology on (A, L);

(2) $(\hookrightarrow, id) : (A, L, \mathcal{U}) \to (X, L, \mathcal{T})$ is fuzzy continuous in **C-FTOP** $\Leftrightarrow$ $\mathcal{U} \geq \mathcal{T}_{|A}$;

(3) $\mathcal{T}_{|A} =$

$$\bigwedge \left\{ \begin{array}{c} \mathcal{U} \mid (\hookrightarrow, id) : (A, L, \mathcal{U}) \to (X, L, \mathcal{T}) \\ \text{is fuzzy continuous in } \mathbf{C\text{-}FTOP} \end{array} \right\}$$

(4) $\left((A, L, \mathcal{T}_{|A}),\ (\hookrightarrow, id) : (A, L, \mathcal{T}_{|A}) \to (X, L, \mathcal{T})\right)$ is the unique initial V-lift of $((A, L),\ (\hookrightarrow, id) : (A, L) \to V(X, L, \mathcal{T}))$, where V is the appropriate forgetful functor.

(5) $(\hookrightarrow, id) : (A, L, \mathcal{U}) \to (X, L, \mathcal{T})$ is an initial morphism in **C-FTOP** $\Leftrightarrow$ $\mathcal{U} = \mathcal{T}_{|A}$.

All of (1–5) hold if the following replacements are made: **C-FTOP** by $\mathbf{L}_\phi$**-FTOP**, $\mathcal{T}_{|A}$ by $G_\phi(\mathcal{T}_{|A})$, "fuzzy topology" by "ϕ-saturated fuzzy topology", and $(\hookrightarrow, id)$ by $(\hookrightarrow, \phi)$.

Proof. Apply 5.2.3, 5.2.5, and 5.2.6. □

5.3 Topological, fuzzy topological, and categorical embeddings

In this subsection we bring together Subsections 5.1 and 5.2 to characterize the embeddings in **C-TOP** and **C-FTOP** (and hence $\mathbf{L}_\phi$**-TOP** and $\mathbf{L}_\phi$**-FTOP**). Throughout, **C** is a subcategory of **LOQML** and $\mathbf{L}_\phi$ is the category determined from $L \in |\mathbf{CQML}|$ and $\phi \in \mathbf{LOQML}(L, L)$.

5.3.1. Definition (Restricted functions). Let $f : X \to Y$ be a function. Then $f_{|f^{\to}(X)} : X \to f^{\to}(X)$ is the function given by $f_{|f^{\to}(X)}(x) = f(x)$. It is the **restriction of f to its range**.

5.3.2. Definition (Topological and fuzzy topological embeddings). Let $(f, \phi) : (X, L) \to (Y, M)$ be a ground morphism in **SET** $\times$ **C**, τ be a topology, and $\mathcal{T}$ be a fuzzy topology on (X, L), and σ be a topology and $\mathcal{S}$ be a fuzzy topology on (Y, M).

(1) $(f, \phi) : (X, L, \tau) \to (X, M, \sigma)$ is a **topological embedding** in **C-TOP** if

$$\left(f_{|\, f^{\to}(X)}, \phi\right) : (X, L, \tau) \to \left(f^{\to}(X), M, \sigma_{|\, f^{\to}(X)}\right)$$

is a homeomorphism in **C-TOP** in the sense of 5.1.1, where $\sigma_{|\, f^{\to}(X)}$ is the subspace topology in the sense of 5.2.2.

(2) $(f, \phi) : (X, L, \mathcal{T}) \to (X, M, \mathcal{S})$ is a **fuzzy topological embedding** in **C-FTOP** if

$$\left(f_{|\, f^{\to}(X)}, \phi\right) : (X, L, \mathcal{T}) \to \left(f^{\to}(X), M, \mathcal{S}_{|\, f^{\to}(X)}\right)$$

is a homeomorphism in **C-FTOP** in the sense of 5.1.3, where $\mathcal{S}_{|\, f^{\to}(X)}$ is the subspace topology in the sense of 5.2.6.

A **topological embedding** of $\mathbf{L}_{\phi}$**-TOP** is a topological embedding of **C-TOP** contained in $\mathbf{L}_{\phi}$**-TOP**; and a **fuzzy topological embedding** of $\mathbf{L}_{\phi}$**-FTOP** is a topological embedding of **C-FTOP** contained in $\mathbf{L}_{\phi}$**-FTOP**.

5.3.3. Definition (Categorical embeddings [1]**).** Let **A** and **X** be categories and $V : \mathbf{A} \to \mathbf{X}$ a functor. Then $f \in \mathbf{A}$ is a **categorical embedding over X w.r.t.** V iff f is an initial morphism in **A**—see [1] and the proof of 5.2.1(6)—and a monomorphism in **X**. In the case of the categories **C-TOP**, **C-FTOP**, $\mathbf{L}_{\phi}$-**TOP**, and $\mathbf{L}_{\phi}$**-FTOP**, we simply say (f, ϕ) is a **categorical embedding** if the previous sentence is satisfied over the appropriate ground w.r.t. the appropriate forgetful functor.

5.3.4. Lemma (Backward powerset operators of restricted morphisms). Let $(f, \phi) : (X, L) \to (Y, M)$ be a ground morphism and let $b \in M^Y$. Then

$$(f, \phi)^{\leftarrow}(b) = \left(f_{|\, f^{\to}(X)}, \phi\right)^{\leftarrow} \left(b_{|\, f^{\to}(X)}\right)$$

Proof. Given $y \in Y$, the computation is straightforward:

$$\begin{aligned} \left(f_{|\, f^{\to}(X)}, \phi\right)^{\leftarrow} \left(b_{|\, f^{\to}(X)}\right)(y) &= \phi^{op}\left(b_{|\, f^{\to}(X)}\left(f_{|\, f^{\to}(X)}(y)\right)\right) \\ &= \phi^{op}(b(f(y))) \\ &= (f, \phi)^{\leftarrow}(b)(y) \quad \square \end{aligned}$$

5.3.5. Theorem. Each topological embedding is a morphism in **C-TOP**, and each fuzzy topological embedding is a morphism in **C-FTOP**. More generally, a ground morphism (f, ϕ) is [fuzzy] continuous iff the restricted ground morphism $\left(f_{|\, f^{\rightarrow}(X)}, \phi\right)$ is [fuzzy] continuous.

Proof. If $(f, \phi) : (X, L, \tau) \to (Y, M, \sigma)$ be a topological embedding in **C-TOP**. Then $\left(f_{|\, f^{\rightarrow}(X)}, \phi\right) : (X, L, \tau) \to \left(Y, M, \sigma_{|\, f^{\rightarrow}(X)}\right)$ is continuous; and now 5.3.4 yields immediately that

$$v \in \sigma \Rightarrow (f, \phi)^{\leftarrow}(v) = \left(f_{|\, f^{\rightarrow}(X)}, \phi\right)^{\leftarrow} \left(v_{|\, f^{\rightarrow}(X)}\right) \in \tau$$

This establishes that continuity of $\left(f_{|\, f^{\rightarrow}(X)}, \phi\right)$ implies continuity of (f, ϕ). The converse direction follows similarly from 5.3.4.

As for the fuzzy topological case, let $(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$ be a fuzzy topological embedding in **C-FTOP**, which implies that $\left(f_{|\, f^{\rightarrow}(X)}, \phi\right) : (X, L, \mathcal{T}) \to \left(Y, M, \mathcal{S}_{|\, f^{\rightarrow}(X)}\right)$ is fuzzy continuous, i.e. that

$$\mathcal{T} \circ \left(f_{|\, f^{\rightarrow}(X)}, \phi\right)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S}_{|\, f^{\rightarrow}(X)}$$

It is our purpose to show that $\mathcal{T} \circ (f, \phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S}$ on M^Y. Letting $b \in M^Y$, recalling the definition of the fuzzy subspace topology (5.2.6), and applying 5.3.4 yield:

$$\begin{aligned}
\mathcal{T}\left((f, \phi)^{\leftarrow}(b)\right) &= \mathcal{T}\left(\left(f_{|\, f^{\rightarrow}(X)}, \phi\right)^{\leftarrow} \left(b_{|\, f^{\rightarrow}(X)}\right)\right) \\
&\geq \phi^{op}\left(\mathcal{S}_{|\, f^{\rightarrow}(X)}\left(b_{|\, f^{\rightarrow}(X)}\right)\right) \\
&\geq \phi^{op}\left(\bigvee_{c_{|\, f^{\rightarrow}(X)} = b_{|\, f^{\rightarrow}(X)}} \mathcal{S}(c)\right) \\
&\geq \phi^{op}\left(\mathcal{S}(b)\right)
\end{aligned}$$

This establishes that continuity of $\left(f_{|\, f^{\rightarrow}(X)}, \phi\right)$ implies that of (f, ϕ). For the converse direction, fix $b \in M^Y$ and let $c \in M^Y$ such that $c_{|\, f^{\rightarrow}(X)} = b_{|\, f^{\rightarrow}(X)}$. Then

$$\begin{aligned}
\mathcal{T}\left(\left(f_{|\, f^{\rightarrow}(X)}, \phi\right)^{\leftarrow} \left(b_{|\, f^{\rightarrow}(X)}\right)\right) &= \mathcal{T}\left(\left(f_{|\, f^{\rightarrow}(X)}, \phi\right)^{\leftarrow} \left(c_{|\, f^{\rightarrow}(X)}\right)\right) \\
&= \mathcal{T}\left((f, \phi)^{\leftarrow}(c)\right) \\
&\geq \phi^{op}\left(\mathcal{S}(c)\right)
\end{aligned}$$

¿From this it follows that

$$\begin{aligned}
\mathcal{T}\left(\left(f_{|\, f^{\rightarrow}(X)}, \phi\right)^{\leftarrow} \left(b_{|\, f^{\rightarrow}(X)}\right)\right) &\geq \bigvee_{c_{|\, f^{\rightarrow}(X)} = b_{|\, f^{\rightarrow}(X)}} \phi^{op}\left(\mathcal{S}(c)\right) \\
&= \phi^{op}\left(\bigvee_{c_{|\, f^{\rightarrow}(X)} = b_{|\, f^{\rightarrow}(X)}} \mathcal{S}(c)\right)
\end{aligned}$$

To finish the proof, we observe from the foregoing and 1.3.2(2) that

$$\phi^* \circ \mathcal{T} \circ \left(f_{|\, f^{\to}(X)}, \phi\right)^{\leftarrow} \geq \text{Subbase of } \mathcal{S}_{|\, f^{\to}(X)}$$

that $\phi^* \circ \mathcal{T} \circ \left(f_{|\, f^{\to}(X)}, \phi\right)^{\leftarrow}$ is a fuzzy topology on $(f^{\to}(X), M)$, and hence from 5.2.6 that

$$\phi^* \circ \mathcal{T} \circ \left(f_{|\, f^{\to}(X)}, \phi\right)^{\leftarrow} \geq \mathcal{S}_{|\, f^{\to}(X)}$$

We now invoke 3.1.3.1. □

5.3.6. Theorem (Embeddings in C-TOP). The following are equivalent in **C-TOP**:

(1) $(f, \phi) : (X, L, \tau) \to (Y, M, \sigma)$ is a topological embedding;

(2) $(f, \phi) : (X, L, \tau) \to (Y, M, \sigma)$ is a categorical embedding and ϕ is a retraction in **C**;

(3) $(f, \phi) : (X, L, \tau) \to (Y, M, \sigma)$ is a categorical embedding and ϕ is a regular epi in **C**.

Proof. (1) $\Rightarrow$ (2). Given 5.3.5, we must show (f, ϕ) is both mono in **SET** $\times$ **C** and initial in **C-TOP**. Combining 5.1.1 and 5.3.2, each of $f_{|\, f^{\to}(X)}$ and ϕ^{op} are bijections and hence monos in **SET**. It then follows, using the compositions of **C**, $\mathbf{C}^{op}$, and **SET** $\times$ **C**, that (f, ϕ) is a mono in **SET** $\times$ **C**. To show initiality of (f, ϕ), let (Z, N, υ) be a topological space and $(g, \psi) : (Z, N) \to (X, L)$ be a morphism in **SET** $\times$ **C** such that $(f, \phi) \circ (g, \psi) : (Z, N, \upsilon) \to (Y, M, \sigma)$ is continuous; we must now show that $(g, \psi) : (Z, N, \upsilon) \to (X, L, \tau)$ is continuous. Let $v \in \tau$; we show that $(g, \psi)^{\leftarrow}(v) \in \upsilon$. Using the fact that $\left(f_{|\, f^{\to}(X)}, \phi\right)$ is a homeomorphism, $\exists\, u \in \sigma$ such that $\left(f_{|\, f^{\to}(X)}, \phi\right)^{\to}(v) = u_{|\, f^{\to}(X)}$. Applying 5.3.4 yields

$$v = \left(f_{|\, f^{\to}(X)}, \phi\right)^{\leftarrow} \left(f_{|\, f^{\to}(X)}, \phi\right)^{\to}(v) = \left(f_{|\, f^{\to}(X)}, \phi\right)^{\leftarrow}\left(u_{|\, f^{\to}(X)}\right) = (f, \phi)^{\leftarrow}(u)$$

and applying continuity of $(f, \phi) \circ (g, \psi)$ yields

$$(g, \psi)^{\leftarrow}(v) = (g, \psi)^{\leftarrow}(f, \phi)^{\leftarrow}(u) \in \upsilon$$

Finally, ϕ^{op} is a bijection in **SET** $\Rightarrow$ ϕ^{op} is an isomorphism in $\mathbf{C}^{op}$ $\Rightarrow$ ϕ is an isomorphism in **C**, and so ϕ is a retraction in **C**.

(1) $\Leftarrow$ (2). Since (f, ϕ) is a monomorphism in **SET** $\times$ **C**, it follows that f is a mono in **SET** and ϕ is a mono in **C**. Hence f is injective; and since ϕ is assumed a retraction in **C**, we have ϕ is an iso in **C** and hence ϕ^{op} is a bijection To apply the initiality of (f, ϕ) , we make the following choices for a space (Z, N, υ) and a ground morphism $(g, \psi) : (Z, N) \to (X, L)$:

$$Z = f^{\to}(X),\ N = M,\ \upsilon = \sigma_{|f^{\to}(X)},\ g = \left(f_{|\, f^{\to}(X)}\right)^{-1},\ \psi = \left(\left(\phi^{op}\right)^{-1}\right)^{op}$$

To prove $(f,\phi)\circ(g,\psi):(Z,N,v)\to(Y,M,\sigma)$ is continuous, let $v\in\sigma$ and $z\in Z$. Observe

$$\psi^{op}\left(\phi^{op}\left(v\left(f\left(g\left(z\right)\right)\right)\right)\right) = \left(\phi^{op}\right)^{-1}\left(\phi^{op}\left(v\left(f\left(f^{-1}\left(z\right)\right)\right)\right)\right) = v(z)$$

so that

$$(g,\psi)^{\leftarrow}(f,\phi)^{\leftarrow}(v) = v_{|f^{\rightarrow}(X)} = v\in\sigma_{|f^{\rightarrow}(X)}$$

This shows $(f,\phi)\circ(g,\psi)$ is continuous. Initiality of (f,ϕ) now implies that (g,ψ) is continuous, which means that $\left(f_{|f^{\rightarrow}(X)},\phi\right)$ is an open morphism in **C-TOP** (5.1.1). Since (f,ϕ) is assumed a morphism in **C-TOP**, i.e. is continuous, it follows from 5.3.5 that $\left(f_{|f^{\rightarrow}(X)},\phi\right)$ is continuous. We conclude from 5.1.1 that $\left(f_{|f^{\rightarrow}(X)},\phi\right)$ is a homeomorphism and from 5.3.2 that (f,ϕ) is a topological embedding.

(2) $\Leftrightarrow$ (3). This is immediate since ϕ is a mono in **C** for each of (2) and (3). □

5.3.7. Theorem (Embeddings in C-FTOP). The following are equivalent in **C-FTOP**:

(1) $(f,\phi):(X,L,\mathcal{T})\to(Y,M,\mathcal{S})$ is a fuzzy topological embedding;

(2) $(f,\phi):(X,L,\mathcal{T})\to(Y,M,\mathcal{S})$ is a categorical embedding and ϕ is a retraction in **C**;

(3) $(f,\phi):(X,L,\mathcal{T})\to(Y,M,\mathcal{S})$ is a categorical embedding and ϕ is a regular epi in **C**.

Proof. The proof of (2) $\Leftrightarrow$ (3) is just as for 5.3.6; and the ground issues of monomorphism in (1) $\Rightarrow$ (2) and (1) $\Leftarrow$ (2) are also just as for 5.3.6. It remains to show initiality of (f,ϕ) in (1) $\Rightarrow$ (2) and that (f,ϕ) is a topological embedding in (1) $\Leftarrow$ (2).

(1) $\Rightarrow$ (2). To show initiality of (f,ϕ), let $(Z,N,\mathcal{U})$ be a fuzzy topological space and $(g,\psi):(Z,N)\to(X,L)$ be a morphism in $\mathbf{SET}\times\mathbf{C}$ such that $(f,\phi)\circ(g,\psi):(Z,N,\mathcal{U})\to(Y,M,\mathcal{S})$ is fuzzy continuous; we must now show that $(g,\psi):(Z,N,\mathcal{U})\to(X,L,\mathcal{T})$ is continuous, i.e. on L^X we must show

$$\mathcal{U}\circ(g,\psi)^{\leftarrow}\geq\psi^{op}\circ\mathcal{T}$$

Let $b\in L^X$; then using the fact that $\left(f_{|f^{\rightarrow}(X)},\phi\right)$ is a fuzzy homeomorphism, let $c\in M^Y$ such that

$$\left(f_{|f^{\rightarrow}(X)},\phi\right)^{\rightarrow}(b) = c_{|f^{\rightarrow}(X)}$$

First, using 5.3.4 and the fuzzy continuity of $(f,\phi)\circ(g,\psi)$, we observe

$$\begin{aligned}
\left(\mathcal{U}\circ(g,\psi)^{\leftarrow}\right)(b) &= \left(\mathcal{U}\circ(g,\psi)^{\leftarrow}\circ\left(f_{|f^{\rightarrow}(X)},\phi\right)^{\leftarrow}\circ\left(f_{|f^{\rightarrow}(X)},\phi\right)^{\rightarrow}\right)(b)\\
&= \left(\mathcal{U}\circ(g,\psi)^{\leftarrow}\circ\left(f_{|f^{\rightarrow}(X)},\phi\right)^{\leftarrow}\right)\left(c_{|f^{\rightarrow}(X)}\right)\\
&= \left(\mathcal{U}\circ(g,\psi)^{\leftarrow}\circ(f,\phi)^{\leftarrow}\right)(c)\\
&\geq \left(\psi^{op}\circ\phi^{op}\circ\mathcal{S}\right)(c)
\end{aligned}$$

Second, using the preceding observation and letting $c \in M^Y$ range freely such that $\left(f_{|f^{\rightarrow}(X)}, \phi\right)^{\rightarrow}(b) = c_{|f^{\rightarrow}(X)}$, we observe

$$\begin{aligned}(\mathcal{U} \circ (g,\psi)^{\leftarrow})(b) &\geq \bigvee_{c_{|f^{\rightarrow}(X)} = \left(f_{|f^{\rightarrow}(X)},\phi\right)^{\rightarrow}(b)} (\psi^{op} \circ \phi^{op} \circ \mathcal{S})(c) \\ &= (\psi^{op} \circ \phi^{op}) \left(\bigvee_{c_{|f^{\rightarrow}(X)} = \left(f_{|f^{\rightarrow}(X)},\phi\right)^{\rightarrow}(b)} \mathcal{S}(c) \right)\end{aligned}$$

which implies via Lemma 1.3.2.1 that

$$(\psi^* \circ \phi^* \circ \mathcal{U} \circ (g,\psi)^{\leftarrow})(b) \geq \bigvee_{c_{|f^{\rightarrow}(X)} = \left(f_{|f^{\rightarrow}(X)},\phi\right)^{\rightarrow}(b)} \mathcal{S}(c)$$

Third, using the preceding observation, 3.2.12(1), and 5.2.6, and letting b range freely, we observe

$$\psi^* \circ \phi^* \circ \mathcal{U} \circ (g,\psi)^{\leftarrow} \geq \mathcal{S}_{|f^{\rightarrow}(X)} \circ \left(f_{|f^{\rightarrow}(X)}, \phi\right)^{\rightarrow}$$

which implies via Lemma 1.3.2.1 that

$$\mathcal{U} \circ (g,\psi)^{\leftarrow} \geq \psi^{op} \circ \phi^{op} \circ \mathcal{S}_{|f^{\rightarrow}(X)} \circ \left(f_{|f^{\rightarrow}(X)}, \phi\right)^{\rightarrow}$$

Fourth, using the preceding observation, invoking that $\left(f_{|f^{\rightarrow}(X)}, \phi\right)$ is a fuzzy homeomorphism, and applying 5.1.4(5), we observe

$$\mathcal{U} \circ (g,\psi)^{\leftarrow} \geq \psi^{op} \circ \mathcal{T} \circ (f,\phi)^{\leftarrow} \circ \left(f_{|f^{\rightarrow}(X)}, \phi\right)^{\rightarrow}$$

Fifth, applying (the proof) of 5.1.2((3) $\Leftrightarrow$ (4)), we observe

$$\begin{aligned}\left((f,\phi)^{\leftarrow} \circ \left(f_{|f^{\rightarrow}(X)}, \phi\right)^{\rightarrow}\right)(b) &= \\ \left((f,\phi)^{\leftarrow} \circ \left(\left(f_{|f^{\rightarrow}(X)}\right)^{-1}, \left((\phi^{op})^{-1}\right)^{op}\right)^{\rightarrow}\right)(b) &= \\ \phi^{op} \circ (\phi^{op})^{-1} \circ b \circ \left(f_{|f^{\rightarrow}(X)}\right)^{-1} \circ f &= b\end{aligned}$$

¿From the preceding two observations, we have on L^X that

$$\mathcal{U} \circ (g,\psi)^{\leftarrow} \geq \psi^{op} \circ \mathcal{T}$$

This completes the proof of the fuzzy continuity of (g,ψ).

(1) $\Leftarrow$ (2). We are to prove (f,ϕ) is a topological embedding. Note (f,ϕ) is assumed a morphism in **C-FTOP**, i.e. continuous; so by 5.3.5, $\left(f_{|f^{\rightarrow}(X)}, \phi\right)$ is continuous. To show $\left(f_{|f^{\rightarrow}(X)}, \phi\right)$ is an open morphism in **C-FTOP**, we apply the initiality of (f,ϕ) by making the following choices for a space $(Z, N, \mathcal{U})$ and a ground morphism $(g,\psi) : (Z,N) \to (X,L)$:

$$Z = f^{\rightarrow}(X),\ N = M,\ \mathcal{U} = \mathcal{S}_{|f^{\rightarrow}(X)},\ g = \left(f_{|f^{\rightarrow}(X)}\right)^{-1},\ \psi = \left((\phi^{op})^{-1}\right)^{op}$$

To prove $(f,\phi)\circ(g,\psi):(Z,N,\mathcal{U})\to(Y,M,\mathcal{S})$ is continuous, we must show that on M^Y that the following inequality holds:

$$\mathcal{U}\circ(g,\psi)^{\leftarrow}\circ(f,\phi)^{\leftarrow}\geq\psi^{op}\circ\phi^{op}\circ\mathcal{S}$$

The right-hand side reduces to $\mathcal{S}$. Fixing $b\in M^Y$ and letting $z\in Z$, we have $f\left(\left(f_{|f^{\rightarrow}(X)}\right)^{-1}(z)\right)=z$. This implies

$$\begin{aligned}((g,\psi)^{\leftarrow}\circ(f,\phi)^{\leftarrow})(b) &= (\phi^{op})^{-1}\circ\phi^{op}\circ b\circ f\circ\left(f_{|f^{\rightarrow}(X)}\right)^{-1}\\ &= b\end{aligned}$$

which in turn implies via 5.2.6 that

$$\begin{aligned}(\mathcal{U}\circ(g,\psi)^{\leftarrow}\circ(f,\phi)^{\leftarrow})(b) &= \mathcal{U}(b)\\ &= \mathcal{S}_{|f^{\rightarrow}(X)}(b)\\ &\geq \bigvee_{c_{|f^{\rightarrow}(X)}=b}\mathcal{S}(c)\\ &\geq \mathcal{S}(b)\end{aligned}$$

It follows that $(f,\phi)\circ(g,\psi)$ is fuzzy continuous. Applying the initiality of (f,ϕ) yields the fuzzy continuity of (g,ψ), which implies $\left(\left(f_{|f^{\rightarrow}(X)}\right)^{-1},\left((\phi^{op})^{-1}\right)^{op}\right)$ is fuzzy continuous. We conclude from 5.1.3 that $\left(f_{|f^{\rightarrow}(X)},\phi\right)$ is a fuzzy homeomorphism, and hence from 5.3.2 that (f,ϕ) is a fuzzy topological embedding. □

5.3.8A. Theorem (Embeddings in $\mathbf{L}_\phi$-TOP). The following are equivalent in $\mathbf{L}_\phi$-**TOP**:

(1) $(f,\phi):(X,L,\tau)\to(Y,L,\sigma)$ is a topological embedding;

(2) $(f,\phi):(X,L,\tau)\to(Y,L,\sigma)$ is a categorical embedding;

5.3.8B. Theorem (Embeddings in $\mathbf{L}_\phi$-FTOP). The following are equivalent in $\mathbf{L}_\phi$-**FTOP**:

(1) $(f,\phi):(X,L,\mathcal{T})\to(Y,L,\mathcal{S})$ is a fuzzy topological embedding;

(2) $(f,\phi):(X,L,\mathcal{T})\to(Y,L,\mathcal{S})$ is a categorical embedding;

Proof. The proof of 5.3.8A is essentially contained that of 5.3.6 and is left to the reader. And the proof of 5.3.8B is essentially contained in 5.3.7, except that at the appropriate step, appeal is not made to 3.2.12(1), but rather to its extension to the endo-saturated implicit in the proof of 4.3.2 via the G_ϕ functor; and this extension and the proof of 5.3.8B is al;so left to the reader. □

5.3.9. Corollary (Embeddings in L-TOP and L-FTOP). In L-**TOP** and L-**FTOP** the topological embeddings and the categorical embeddings are the same if "topological embedding" means "homeomorphism onto subspace of the range".

Proof. Apply 5.3.8 and the functorial embeddings of Section 6.2. □

5.3.10 Remark. The characterizations of 5.3.9 for fixed-basis topology provide a link between categorical and topological compactifications, and therefore they play a fundamental role in the development of compactification reflectors for fixed-basis topology when the lattice-theoretic base is a frame [65].

6 Unification of topology and fuzzy topology by C-TOP and C-FTOP

For $\mathbf{C} \hookrightarrow$ **LOQML**, each of **C-TOP** and **C-FTOP** has a rich infrastructure of subcategories. Detailing this infrastructure provides an important justification of variable-basis theories as pointed out in Subsection 1.2. There are four important aspects of this justification, all related to the poslat construction of these variable-basis categories (see the term **poslat** at the beginning of Sections 2 and 3):

(1) A poslat variable-basis theory brings coherence to fixed-basis theories. For example, **C-TOP** provides, up to functorial embedding, a single supercategory unifying all the following categories:

  (a) the well-known category **TOP**;

  (b) $\forall L \in \mathbf{C}$, the fixed-basis category L-**TOP**, i.e. all the "Chang-Goguen" categories;

  (c) $\forall L \in \mathbf{C}$, the fixed-basis category L-$\mathbf{TOP}_k$, i.e. all the "Lowen" categories;

  (d) $\forall L \in \mathbf{C}, \forall \phi \in \mathbf{C}(L, L)$, the endo-saturated category $\mathbf{L}_\phi$-**TOP**, i.e. all the internalized change-of-basis categories.

In a similar vein, **C-FTOP** provides, up to functorial embeddings, a single supercategory for **TOP**, **C-TOP**, and each L-**FTOP** (for $L \in |\mathbf{C}|$), i.e. all the Höhle-Šostak categories of [23].

(2) A poslat variable-basis theory re-incorporates lattice-based theories of topology back into a point-set context, thus mitigating traditional criticisms of these approaches from the point-set topology community (see

[63]). For example, if **LOC** $\hookrightarrow$ **C**, then **C-TOP** provides, up to functorial embedding, a single supercategory unifying all the following categories:

(a) the classical category **LOC** for point-free topology;

(b) the category **C-HTOP**, the "Hutton" category for point-free (fuzzy) topology (6.1.4 *infra*);

(c) the category **C** as a generalized categrory for "point-free" (fuzzy) topology.

As a particular illustration, it appears that **LOC-TOP** (choosing **C** = **LOC**) is the "smallest" supercategory, up to functorial embeddings, of both **TOP** and **LOC**; i.e. **LOC-TOP** = "**TOP** $\vee$ **LOC**".

(3) A poslat variable-basis theory enriches topology and fuzzy topology especially with respect to morphisms, even when the lattice-theoretic base is fixed. In particular, it enriches **TOP** with respect to both objects and morphisms (see Remark 6.2.2(1)). As for the enrichment of morphisms for base larger than $\{\perp, \top\}$, say $L = \mathbb{I}$, let $\mathbb{R}(\mathbb{I})$ be the fuzzy real line based on the unit interval [12, 24, 39, 69]; then the category **C-TOP** (with **C** = **LOQML**) provides at least $|\mathbb{R}|$ more (continuous) morphisms from $\mathbb{R}(\mathbb{I})$ to $\mathbb{R}(\mathbb{I})$ than can exist in $\mathbb{I}$-**TOP** (or any L-**TOP** with $\mathbb{I} \hookrightarrow L$). This will follow from Subsections 6.2 and 7.1 since $\mathbb{I}$ has (at least) $|\mathbb{R}|$ many non-identity endomorphisms in **LOQML** each of which induces a morphism from $\mathbb{R}(\mathbb{I})$ to $\mathbb{R}(\mathbb{I})$; while in $\mathbb{I}$-**TOP** as embedded in **LOQML-TOP**, the only permissible morphisms from $\mathbb{R}(\mathbb{I})$ to $\mathbb{R}(\mathbb{I})$ require the identity morphism on $\mathbb{I}$.

(4) A variable-basis theory, if topological over its ground, immediately generates more topological frameworks from those subcategories which are defined by properties preserved by the meet operation of the fibre over a ground object (6.3.1). For example, **C-TOP** is topological w.r.t. **SET** $\times$ **C** and V : **C-TOP** $\to$ **SET** $\times$ **C** (3.3); and if **WIDGET** be a subcategory of **C-TOP** defined by the topologies of its objects being "widgets", and if in $\mathcal{P}\left(L^X\right)$, the powerset of the fibre of an object, "widgetness" is preserved under arbitrary $\bigcap$ or $\bigwedge$, then **WIDGET** would be a topological category w.r.t. **SET** $\times$ **C** and the restricted forgetful functor $V_{|\mathbf{WIDGET}}$. Furthermore, any category which would embed into **C-TOP** onto **WIDGET** would also be topological w.r.t. its ground and the appropriate modification of the restricted forgetful functor by the embedding using (an extension of) Proposition 1.3.1. All subcategories in 6.1, as well as those embedding into **C-TOP** or **C-FTOP** in 6.2, are in this way topological over their grounds—these include L-**TOP** and L-**FTOP** for each $L \in |\mathbf{CQML}|$, thus implying as corollaries all proofs of topological given in the companion chapter [23].

It is the purpose of this section to make explicit the justification of **C-TOP** and **C-FTOP** outlined above. In the first subsection, we catalog the important subcategories of **C-TOP** and **C-FTOP**, define explicitly the Hutton approach to topology, and then define our extension of the Hutton approach to fuzzy topology. The second subsection provides the embeddings and adjunctions onto subcategories of **C-TOP** and **C-FTOP**, and the third section makes explicit the argument for these subcategories being topological. In many cases, we cite previous papers along these lines; and we also cite the analysis of subcategories of fixed-basis theories in [23].

Throughout this section $\mathbf{C} \hookrightarrow \mathbf{LOQML}$ and $L \in |\mathbf{C}|$, unless indicated otherwise. Finally, we take **subcategory** in the strict sense of **subclasses** (*á la* [1]).

6.1 Roster of subcategories

6.1.1 Definition (Subcategories of C-TOP and C-FTOP).

(1) **C-TOP**(L) [**C-FTOP**(L)] is the full subcategory of **C-TOP** [**C-FTOP**] each of whose objects has the same lattice L of membership values. See Remark 6.1.2 below.

(2) **C-TOP**(L, id) [**C-FTOP**(L, id)] is the subcategory of **C-TOP** [**C-FTOP**] each of whose morphisms has second component identity id_L. A morphism (f, id_L) is also written (f, id) or f if L is understood.

(3) **C-TOP**$_k$ [**C-FTOP**$_k$] is the full subcategory of **C-TOP** [**C-FTOP**] for which each space satisfies $\tau \supset \{\underline{\alpha} : \alpha \in L\}$ [$coker(\mathcal{T}) \supset \{\underline{\alpha} : \alpha \in L\}$]. Such spaces were first defined in [42] and are called **stratified** if $\otimes = \wedge$ (e.g. if $\mathbf{C} = \mathbf{SLOC}$)—the history of "stratified" can be traced in the references of [46] back to [52]. But to make the connection with [23], we generally call such spaces **weakly enriched** or **weakly stratified.**

(4) **C-TOP**$_s$ [**C-FTOP**$_s$] is the full subcategory of **C-TOP** [**C-FTOP**] of "singleton spaces", i.e. the full subcategory for which the ground object of each space has a singleton set in the first component.

(5) **C-SFTOP** [**C-TFTOP**, **C-RFTOP**] is the full subcategory of **C-FTOP** of all spaces having strong [continuous, enriched] fuzzy topologies in the sense of [23].

Other subcategories are defined by combining symbols; e.g. **C-TOP**$_k$ (L) is the full subcategory of **C-TOP**(L) satisfying the condition of (3), or equivalently, is the full subcategory of **C-TOP**$_k$ satisfying the condition of (1).

6.1.1.1 Theorem (Justification of C-TOP$_k$ and C-FTOP$_k$). Let

$$(f, \phi) : (X, L) \to (Y, M)$$

in $\mathbf{SET} \times \mathbf{C}$, with $\phi^{op} : L \leftarrow M$ a surjection, and let τ, σ [$\mathcal{T}$, $\mathcal{S}$] be respective poslat [fuzzy] topologies on (X, L), (Y, M) [respectively]. Consider the following statements:

I. (X, L, τ) [$(X, L, \mathcal{T})$] is weakly stratified.

II. $f : X \to Y$ is a constant map.

III. (f, ϕ) is [fuzzy] continuous from (X, L, τ) [$(X, L, \mathcal{T})$] to (Y, M, σ) [$(Y, M, \mathcal{S})$].

IV. (Y, M, σ) [$(Y, M, \mathcal{S})$] is weakly stratified.

Then the following hold:

(1) $(\mathrm{I} \wedge \mathrm{II}) \Rightarrow \mathrm{III}$;

(2) $(\mathrm{IV} \wedge \mathrm{III}) \Rightarrow \mathrm{I}$.

Hence the following holds:

(3) $(\mathrm{II} \wedge \mathrm{IV}) \Rightarrow (\mathrm{I} \Leftrightarrow \mathrm{III})$.

Proof. We leave the poslat case to the reader and consider only the fuzzy poslat case.

Ad (1). If f is a constant map, it follows that $\forall b \in M^Y$, $(f, \phi)^{\leftarrow}(b)$ is a constant map; in which case it then follows that

$$\mathcal{T}\left((f, \phi)^{\leftarrow}(b)\right) = \top \geq \phi^{op}\left(\mathcal{S}(b)\right)$$

i.e. (f, ϕ) is fuzzy continuous.

Ad (2). Let $\alpha \in L$; then $\exists\, \beta \in M$, $\phi^{op}(\beta) = \alpha$. It follows that

$$\underline{\alpha} = (f, \phi)^{\leftarrow}\left(\underline{\beta}\right)$$

Hence, the fuzzy continuity of (f, ϕ) implies

$$\mathcal{T}(\underline{\alpha}) = \mathcal{T}\left((f, \phi)^{\leftarrow}\left(\underline{\beta}\right)\right) \geq \phi^{op}\left(\mathcal{S}\left(\underline{\beta}\right)\right) = \phi^{op}(\top) = \top$$

So $(X, L, \mathcal{T})$ is weakly stratified.

Ad (3). Immediate from (1) and (2). □

6.1.2 Remark (Examples).

(1) Note $\mathbf{C}$-$\mathbf{TOP}(L) = \mathbf{L_C}$-$\mathbf{TOP}$ and $\mathbf{C}$-$\mathbf{FTOP}(L) = \mathbf{L_C}$-$\mathbf{FTOP}$; i.e., these are instantiations of $\mathbf{D}$-$\mathbf{TOP}$ and $\mathbf{D}$-$\mathbf{FTOP}$ with $\mathbf{D} = \mathbf{L_C}$ (recall 1.3.2 and 2.1.7(2)) and have their own ground $\mathbf{SET} \times \mathbf{L_C} \hookrightarrow \mathbf{SET} \times \mathbf{C}$.

(2) Note **C-TOP**(L, id) = **L-TOP** and **C-FTOP**(L, id) = **L-FTOP**; i.e., these are instantiations of **D-TOP** and **D-TOP** with **D** = **L** (recall 1.3.2 and 2.1.7(4)) and have their own ground **SET** × **L** $\hookrightarrow$ **SET** × **C**.

(3) The richness of morphisms in **C-TOP**(L) and **C-FTOP**(L) is directly related to $|\mathbf{C}(L, L)|$, the cardinality of the endomorphisms of L in **C**. For example, if **C** = **SLOC** and L is a semiframe (i.e. with $\otimes = \wedge$) and L admits a prime element, then $|\mathbf{C}(L, L)| \geq 2$—if α is the prime element, then put $p : L \to L$ by

$$p = \chi_{coker(\downarrow(\alpha))}$$

where $coker(\downarrow(\alpha))$ is the complement in L of the principal ideal of α (cf. [28, 56–57, 62–65]). However, the existence of a prime element is not necessary, since $reg(\mathfrak{T}(\mathbb{R}))$, the pointless locale of regular open sets of the usual real line, has at least $|\mathbb{R}|$ endomorphisms other than the identity—e.g. put $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x + r, \quad r \neq \bot$$

Then each of $f^{\to}_{reg(\mathfrak{T}(\mathbb{R}))}$, $f^{\leftarrow}_{reg(\mathfrak{T}(\mathbb{R}))}$, the powerset operators of f on $\mathcal{P}(\mathbb{R})$ restricted to $reg(\mathfrak{T}(\mathbb{R}))$, is a non-identity endomorphism in

$$\mathbf{C}(reg(\mathfrak{T}(\mathbb{R})), reg(\mathfrak{T}(\mathbb{R})))$$

By restricting these morphisms to the symmetric regular open sets of $\mathbb{R}$, we again have these same conclusions holding for this pointless locales as well. The foregoing comments are modified from results of [47].

6.1.3 Definition (The category C-HTOP). The category **C-HTOP** comprises the following data:

(1) **Objects**. Objects are ordered pairs (L, τ) satisfying the following axioms:

  (a) **Ground axiom.** $L \in |\mathbf{C}|$;

  (b) **Topological axiom.** $\tau \subset L$ is closed under $\otimes$ and arbitrary $\bigvee$.

(2) **Morphisms**. Morphisms are of the form $\phi : (L, \tau) \to (M, \sigma)$, are called **continuous**, and satisfy the following axioms:

  (a) **Ground axiom.** $\phi : L \to M$ in **C**.

  (b) **Continuity axiom.** $\phi^{op}|_{\sigma} : \tau \leftarrow \sigma$, i.e. ϕ^{op} maps σ into τ.

(3) **Composition**. As in **C**.

(4) **Identities**. Given (L, τ), $id_{(L,\tau)} = id_L$ in **C**.

If **C** = **FUZLAT** (recall paragraph 11 of 1.1 above), then we write

$$\mathbf{C\text{-}HTOP} = \mathbf{HTOP}$$

Note that **C** is the ground category for **C-HTOP**. The "**H**" is these categorical names refers to [25] (and see the ninth and tenth paragraphs of 1.1 above).

6.1.4 Definition (The category C-HFTOP). The category **C-HFTOP** comprises the following data:

(1) **Objects**. Objects are ordered pairs $(L, \mathcal{T})$ satisfying the following axioms:

   (a) **Ground axiom.** $L \in |\mathbf{C}|$;

   (b) **Topological axiom.** $\mathcal{T} : L \to L$ is a mapping satisfying:

      (i) $\forall$ indexing set J, $\forall \{u_j : j \in J\} \subset L$,

$$\bigwedge_{j \in J} \mathcal{T}(u_j) \leq \mathcal{T}\left(\bigvee_{j \in J} u_j\right)$$

      (ii) For $|J| = 2$, $\forall \{u_j : j \in J\} \subset L$,

$$\bigotimes_{j \in J} \mathcal{T}(u_j) \leq \mathcal{T}\left(\bigotimes_{j \in J} u_j\right)$$

      (iii) $\mathcal{T}(\top) = \top$.

(2) **Morphisms**. Morphisms are of the form $\phi : (L, \mathcal{T}) \to (M, \mathcal{S})$, are called **continuous**, and satisfy the following axioms:

   (a) **Ground axiom.** $\phi : L \to M$ in **C**.

   (b) **Continuity axiom.** $\mathcal{T} \circ \langle \phi^{op} \rangle \geq \phi^{op} \circ \mathcal{S}$, or equivalently (*á la* Proposition 3.1.3.1), $\phi^* \circ \mathcal{T} \circ \langle \phi^{op} \rangle \geq \mathcal{S}$ on M.

(3) **Composition**. As in **C**.

(4) **Identities**. Given (L, τ), $id_{(L,\tau)} = id_L$ in **C**.

In keeping with the previous definition, if $\mathbf{C} = \mathbf{FUZLAT}$ (recall paragraph 11 of 1.1 above), then we write **C-HFTOP** = **HFTOP**. As above, **C** is the ground category for **C-HFTOP** and "**H**" refers to Hutton. These categories marry the Hutton "pointless" approach to the Höhle-Šostak approach.

6.2 Functorial embeddings onto subcategories

6.2.1 Theorem (Functors and embeddings into C-TOP and stratification functors). The following hold:

(1) **TOP** maps into $\mathbf{C\text{-}TOP}(L, id) \hookrightarrow \mathbf{C\text{-}TOP}(L)$ via the functor G_χ given by the following sequence of definitions, where $(X, \mathfrak{T}) \in |\mathbf{TOP}|$:

$$G_\chi(\mathfrak{T}) = \langle\langle\{\chi_U : U \in \mathfrak{T}\}\rangle\rangle$$
$$G_\chi(X, \mathfrak{T}) = (X, L, G_\chi(\mathfrak{T}))$$
$$G_\chi(f) = (f, id)$$

And $\mathbf{C\text{-}TOP}(L, id)$ maps back into **TOP** via the functor M_χ given by the following sequence of definitions, where $(X, L, \tau) \in |\mathbf{C\text{-}TOP}(L)|$:

$$M_\chi(\tau) = \langle\langle\{U \subset X : \chi_U \in \tau\}\rangle\rangle$$
$$M_\chi(X, L, \tau) = (X, M_\chi(\tau))$$
$$M_\chi(f, id) = f$$

If $\mathbf{C} \hookrightarrow \mathbf{LOQML}_\perp$, then

$$G_\chi(\mathfrak{T}) = \{\chi_U : U \in \mathfrak{T}\}, \quad M_\chi(\tau) = \{U \subset X : \chi_U \in \tau\}$$

and the following hold: G_χ is an embedding of **TOP** into **C-TOP**; $M_\chi \dashv G_\chi$; the adjunction $M_\chi \dashv G_\chi$ is a monoreflection, but generally not an equivalence of categories since $M_\chi \vdash G_\chi$ need not be the case; and $M_\chi \dashv G_\chi$ is an equivalence iff $L = \{\perp, \top\}$, in which case G_χ and M_χ are isomorphisms; i.e. $\mathbf{TOP} \approx \mathbf{C\text{-}TOP}(L, id) \approx \{\perp, \top\}\text{-}\mathbf{TOP}$ (see (4) below).

(2) If $\mathbf{C} \hookrightarrow \mathbf{SLOC}$, then **TOP** embeds into $\mathbf{C\text{-}TOP}_k(L, id) \hookrightarrow \mathbf{C\text{-}TOP}_k(L)$ via ω_L given by the following sequence of definitions, where $(X, \mathfrak{T}) \in |\mathbf{TOP}|$:

$$SUP(L) = \langle\langle\{L - \downarrow(a) : a \in L\}\rangle\rangle,$$
$$\omega_L(\mathfrak{T}) = \mathbf{TOP}((X, \mathfrak{T}), (L, SUP(L))),$$
$$\omega_L(X, \mathfrak{T}) = (X, L, \omega_L(\mathfrak{T})),$$
$$\omega_L(f) = (f, id)$$

The right adjoint ι_L of ω_L on $\mathbf{C\text{-}TOP}_k(L, id)$ is given by the following sequence of definitions, where $(X, L, \tau) \in |\mathbf{C\text{-}TOP}_k(L)|$:

$$\iota_L(\tau) = \langle\langle\{[u \not\leq \alpha] : u \in \tau,\ \alpha \in L\}\rangle\rangle,$$
$$\iota_L(X, L, \tau) = (X, \iota_L(\tau)),$$
$$\iota_L(f, id) = f$$

This adjunction is an epicoreflection if L is a hypercontinuous [34, 40].

(3) $\forall\, \alpha \in L - \{\top\}$, **TOP** maps functorially into **C-TOP**(L, id) via F_α given by the following sequence of definitions, where $(X, \mathfrak{T}) \in |\mathbf{TOP}|$:

$$F_\alpha(\mathfrak{T}) = \langle\langle\{u \in L^X : [u \not\leq \alpha] \in \mathfrak{T}\}\rangle\rangle$$

$$F_\alpha(X, \mathfrak{T}) = (X, L, F_\alpha(\mathfrak{T}))$$

$$F_\alpha(f) = (f, id)$$

The functor S_α maps in the opposite direction and is given by the following sequence of definitions, where $(X, L, \tau) \in |\mathbf{C\text{-}TOP}(L, id)|$:

$$S_\alpha(\tau) = \langle\langle\{[u \not\leq \alpha] : u \in \tau\}\rangle\rangle$$

$$S_\alpha(X, L, \tau) = (X, S_\alpha(\tau))$$

$$S_\alpha(f, id) = f$$

If $\otimes = \wedge$ and $\alpha \in L^a$—α is comparable to each element of L and $[\alpha < \beta,\ \alpha < \gamma \Rightarrow \alpha < \beta \wedge \gamma]$, then $F_\alpha \dashv S_\alpha$.

(4) L-**TOP** is isomorphic to **C-TOP**$(L, id) \equiv$ **L-TOP**, and hence embeds into **C-TOP**, via

$$(X, \tau) \leftrightarrow (X, L, \tau), \quad f \leftrightarrow (f, id)$$

Hence all subcategories of L-**TOP**, e.g. L-$\mathbf{TOP}_k$, embed into **C-TOP**(L, id).

(5) $\forall L \in |\mathbf{C}|, \forall \phi \in \mathbf{C}(L, L)$, $\mathbf{L}_\phi$-**TOP** embeds into **C-TOP**(L, id) as detailed in 4.1.4 by a non-equivalence isocoreflection.

(6) $\mathbf{C\text{-}TOP}_k$ embeds into **C-TOP** via the inclusion $\hookrightarrow$ which has a faithful right adjoint G_k given by the following sequence of definitions, where $(X, L, \tau) \in |\mathbf{C\text{-}TOP}|$:

$$G_k(\tau) = \tau \vee \{\underline{\alpha} : \alpha \in L\} \text{ (see 3.2.2)}$$

$$G_k(X, L, \tau) = (X, L, G_k(\tau))$$

$$G_k(f, \phi) = (f, \phi)$$

Furthermore, $\hookrightarrow \dashv G_k$ is an isocoreflection, even when restricted to pairs of corresponding subcategories (e.g. **C-TOP**(L, id) and $\mathbf{C\text{-}TOP}_k(L, id)$), but it is generally not an equivalence since $\hookrightarrow \vdash G_k$ need not be the case.

(7) **C-HTOP** embeds into $\mathbf{C\text{-}TOP}_s$ via the **singleton functor** S as follows:

$$S(L, \tau) = \left(\mathbf{1}, L, \tau^{\mathbf{1}}\right)$$

$$S(\phi) = (id, \phi)$$

where $\mathbf{1}$ is a fixed one-point set. The **Hutton functor** $H : \mathbf{C\text{-}HTOP} \leftarrow C\text{-}\mathbf{TOP}$ is defined by the following:

$$H(X, L, \tau) = \left(L^X, \tau\right)$$

$$H(f, \phi) = ((f, \phi)^{\leftarrow})^{op}$$

If H is restricted to the full subcategory $\mathbf{C\text{-}TOP}_s$, then $S \dashv H$ is an equivalence of categories; i.e. $\mathbf{C\text{-}HTOP} \approx \mathbf{C\text{-}TOP}_s$.

(8) $\mathbf{C}$ embeds into $\mathbf{C\text{-}TOP}$ via the **singleton functor**, also denoted S, as follows:

$$S(A) = (\mathbf{1}, A, A^{\mathbf{1}})$$

$$S(\phi) = (id, \phi)$$

where $\mathbf{1}$ is a fixed one-point set. The **localic functor** $\Omega : \mathbf{C} \leftarrow \mathbf{C\text{-}TOP}$ is defined by the following:

$$\Omega(X, L, \tau) = \tau$$

$$\Omega(f, \phi) = ((f, \phi)^{\leftarrow}|_{\sigma})^{op}$$

where σ is the topology of the codomain of (f, ϕ). If Ω is restricted to the full subcategory $\mathbf{C\text{-}TOP}_s$, then $S \dashv \Omega$ is an equivalence of categories; i.e. $\mathbf{C} \approx \mathbf{C\text{-}TOP}_s$. These statements hold in particular for $\mathbf{C} = \mathbf{LOQML}$, $\mathbf{SLOC}$, and $\mathbf{LOC}$.

Proof. These well-known results are found in [28, 39, 42, 49, 53, 56, 62], or are modifications of same: the only change in most cases is from $\otimes = \wedge$ to general $\otimes$. Details are left to the reader. $\square$

6.2.2 Remark (Consequences of 6.2.1 and its proof).

(1) The ramification of 6.2.1(1–3) is that poslat topology enriches ordinary topology both respect to objects and morphisms. Poslat topologies in this sense contain, via characteristic mappings, an ordinary topology *plus more*; and so poslat topologies are *richer* and contain *more* topological information. It follows poslat continuity in the sense of fixed-basis and variable-basis topology is *stronger* than ordinary continuity since the former requires the preservation (by the backward powerset operator) of *all* open sets, not just the "characteristic" open sets. The reader who works out the details of 6.2.1(1) will see that the greater strength of poslat continuity guarantees $M_\chi \dashv G_\chi$, and the relative weakness of ordinary continuity guarantees that $M_\chi \nvdash G_\chi$ iff $L \neq \{\bot, \top\}$. Similarly, the details of (2) show that stratified poslat topologies contain the classical topological information *plus* a copy of the base lattice, and poslat continuity

preserves this information; and the details of (3) show that a poslat topology contains classical topological information *at every level*, and poslat continuity preserves *every* level of information. But not only does poslat topology give richer morphisms than ordinary topology, but in its variable-basis form gives a richer variety of morphisms, as is commented on in (2) below.

(2) The ramification of 6.2.1(4–5) is that variable-basis topology makes fixed-basis topology categorically coherent. This means that questions of how spaces with different lattice-theoretic bases relate can now be well-posed. The enrichment of the fixed-basis settings by variable-basis topology with respect to morphisms is taken up in detail in Section 7 for various canonical examples from fixed-basis topology. It follows from these functors and Section 7 that variable-basis topology also enriches ordinary topology w.r.t. variety of morphisms.

(3) The ramification of 6.2.1(6) is that variable-basis topology is naturally related to the stratified setting, but is strictly more general, a generality justified indetail using fuzzy sobriety in [56, 62, 64] and summarized in 3.1.5(1)(b) above and Subsection 7.4 below. Since G_k constructs the free weakly-stratified space over a topological space, it is called the **weak stratification functor** [56, 62, 64]. Note that for $\mathbf{C}$ the category of completely distributive lattices and $L \in |\mathbf{C}|$, the functor ω_L coincides with $G_\omega^L \equiv G_k \circ G_\chi^L$; see Proposition 3.8 of [39], compare [59–60], and see Subsection 7.2 below.

(4) The ramification of 6.2.1(7–8) is that every Hutton space and every complete quasi-monoidal lattice (including every locale) is a *singleton (poslat) topological space.* Several consequences:

   (a) Locales are topology *if* one is willing to say that poslat topology is topology.
   (b) A purely lattice-theoretic approach to topology, uniformities, subsapces is important because it points the way for doing these things in a variable-basis, hence categorically coherent, way.
   (c) On the other hand, contrary to what many workers in fuzzy sets claim, purely lattice-theoretic methods are *not* more general than point-set methods since the former are dealing with only *singleton* topological spaces, *singleton* uniform spaces, etc.

In the next result, the analogues to most of the functors of 6.2.1 are constructed for the fuzzy topology case, and we shall consistently use the same notation for analogous functors since their roles are essentially the same and their contexts are different and not easily confused.

6.2.3 Theorem (Functors and embeddings into C-FTOP). The following hold:

(1) **C-TOP** embeds into **C-FTOP** via a functor G_χ given by the following sequence of definitions, where $(X, L, \tau) \in |\mathbf{C\text{-}TOP}|$:

$$G_\chi(\tau) = \bigwedge_{\mathcal{T} \geq \chi_\tau} \mathcal{T} \equiv \langle\langle\{\chi_\tau\}\rangle\rangle$$

$$G_\chi(X, L, \tau) = \left(X, L, G_\chi^L(\tau)\right)$$

$$G_\chi(f, \phi) = (f, \phi)$$

The left adjoint M_χ of G_χ on **C-FTOP** is given by the following sequence of definitions, where $(X, L, \mathcal{T}) \in |\mathbf{C\text{-}FTOP}|$:

$$M_\chi(\mathcal{T}) = coker(\mathcal{T}) \equiv [\mathcal{T} = \top]$$

$$M_\chi(X, L, \mathcal{T}) = (X, L, M_\chi(\mathcal{T}))$$

$$M_\chi(f, \phi) = (f, \phi)$$

The adjunction $M_\chi \dashv G_\chi$ is a bireflection, but generally not an equivalence of categories since $M_\chi \vdash G_\chi$ need not be the case.

(2) Fix $\alpha \in L$. $\mathbf{L}_\phi$**-TOP** maps functorially into $\mathbf{L}_\phi$**-FTOP** via $F_{\alpha\phi}$ given by the following sequence of definitions, where $(X, L, \tau) \in |\mathbf{L}_\phi\text{-}\mathbf{TOP}|$ and G_ϕ is the functor developed in 4.2.6 above:

$$F_{\alpha\phi}(\tau) = G_\phi\left(\bigwedge_{[\mathcal{T} \geq \alpha] \supset \tau} \mathcal{T}\right)$$

$$F_{\alpha\phi}(X, L, \tau) = (X, L, F_{\alpha\phi}(\tau))$$

$$F_{\alpha\phi}(f, \phi) = (f, \phi)$$

The functor $S_{\alpha\phi}$, mapping in the opposite direction, is given by the following sequence of defintions, where $(X, L, \mathcal{T}) \in |\mathbf{L}_\phi\text{-}\mathbf{FTOP}|$:

$$S_{\alpha\phi}(\mathcal{T}) = [\mathcal{T} \geq \alpha]$$

$$S_{\alpha\phi}(X, L, \mathcal{T}) = (X, L, S_{\alpha\phi}(\mathcal{T}))$$

$$S_{\alpha\phi}(f, \phi) = (f, \phi)$$

Then the following hold, where G_ϕ is the functor recalled in 4.1.4 above, and the instantiations of $F_{\alpha\phi}$ and $S_{\alpha\phi}$ when $\phi = id_L$ are resepctively denoted F_α and S_α:

(a) $S_{\alpha\phi} = G_\phi \circ S_\alpha$;

(b) $S_{\alpha\phi} \dashv F_{\alpha\phi}$;

(c) $S_{\alpha\phi} \dashv F_{\alpha\phi}$ is a bireflection, but generally not an equivalence of categories.

(3) L-**FTOP** is isomorphic to **C-FTOP**$(L, id) \equiv$ **L-FTOP**, and hence embeds into **C-FTOP**, via

$$(X, \mathcal{T}) \leftrightarrow (X, L, \mathcal{T}), \quad f \leftrightarrow (f, id)$$

Hence all subcategories of L-**FTOP**, e.g. L-**SFTOP**, L-**TFTOP**, L-**RFTOP**, **C-FTOP**$_k$, embed into **C-FTOP**(L, id).

(4) $\forall L \in |\mathbf{C}|$, $\forall \phi \in \mathbf{C}(L, L)$, $\mathbf{L}_\phi$-**FTOP** embeds into **C-FTOP**(L, id) as detailed in 4.2.6 by a non-equivalence isocoreflection.

(5) **C-FTOP**$_k$ embeds into **C-FTOP** via the inclusion $\hookrightarrow$ which has a faithful right adjoint given by the following sequence of definitions, where $(X, L, \mathcal{T}) \in |\mathbf{C\text{-}FTOP}|$:

$$G_k(\mathcal{T}) = \bigwedge \{\mathcal{T}^* : \mathcal{T}^* \geq \mathcal{T},\ coker(\mathcal{T}^*) \supset \{\underline{\alpha} : \alpha \in L\}\}$$

$$G_k(X, L, \mathcal{T}) = (X, L, G_k(\mathcal{T}))$$

$$G_k(f, \phi) = (f, \phi)$$

The following hold:

(a) $\hookrightarrow \dashv G_k$ is an isocoreflection, even when restricted to pairs of corresponding subcategories, but is not generally an equivalence since $\hookrightarrow \vdash G_k$ need not hold.

(b) $G_\chi^{6.2.3(1)} \circ G_k^{6.2.1(6)} = G_k^{6.2.3(5)} \circ G_\chi^{6.2.3(1)}$, where the superscript indicates whence in this chapter a functor is taken, and the functor $G_k^{6.2.3(5)}$ from **C-TOP** to **C-FTOP**$_k$ has a left adjoint, namely $M_\chi^{6.2.3(1)} \circ \hookrightarrow^{6.2.3(5)} = \hookrightarrow^{6.2.1(6)} \circ M^{6.2.3(1)}$.

(6) **C-HFTOP** embeds into **C-FTOP**$_s$ via the **singleton functor** S as follows, where $\mathbf{1}$ is a fixed one-point set and $j : L^{\mathbf{1}} \to L$ is the unique isomorphism:

$$\mathcal{T}^{\mathbf{1}} \equiv \mathcal{T} \circ j$$

$$S(L, \mathcal{T}) = \left(\mathbf{1}, L, \mathcal{T}^{\mathbf{1}}\right)$$

$$S(\phi) = (id, \phi)$$

The **Hutton functor** H : **C-HFTOP** $\leftarrow$ **C-FTOP** is defined by the following:

$$H(X, L, \mathcal{T}) = \left(L^X, \mathcal{T}\right)$$

$$H(f, \phi) = ((f, \phi)^{\leftarrow})^{op}$$

$G_{\chi s}$, the point-free version of $G_\chi^{6.2.3(1)}$, is defined from **C-HTOP** to **C-HFTOP** as follows:

$$G_{\chi s}(\tau) = \bigwedge_{\mathcal{T} \geq \chi_\tau} \mathcal{T}$$

$$G_{\chi s}(L,\tau) = (L, G_{\chi s}(\tau))$$
$$G_{\chi s}(\phi) = \phi$$

And $M_{\chi s}$, the point-free version of $M^{6.2.3(1)}$, is then defined from **C-HFTOP** to **C-HTOP** in the expected way. The following hold:

(a) If H is restricted to the full subcategory $\mathbf{C\text{-}TOP}_s$, then $S \dashv H$ holds and is an equivalence of categories; i.e. $\mathbf{C\text{-}HFTOP} \approx \mathbf{C\text{-}FTOP}_s$.

(b) $G_\chi^{6.2.3(1)} \circ S^{6.2.1(7)} = S^{6.2.3(6)} \circ G_{\chi s}$.

(c) $H^{6.2.1(7)} \circ M_\chi^{6.2.3(1)} = M_{\chi s} \circ H^{6.2.3(6)}$.

(7) **C** embeds into **C-FTOP** via the **singleton functor**, also denoted S, as follows, where $\mathbf{1}$ is a fixed one-point set:

$$A_\chi^{\mathbf{1}} \equiv \chi_{(A^{\mathbf{1}})} : A^{\mathbf{1}} \to A$$

$$S(A) = (\mathbf{1}, A, A_\chi^{\mathbf{1}})$$
$$S(\phi) = (id, \phi)$$

The **localic functor** $\Omega : \mathbf{C} \leftarrow \mathbf{C\text{-}FTOP}$ is defined by the following:

$$\Omega(X, L, \mathcal{T}) = coker(\mathcal{T})$$

$$\Omega(f,\phi) = \left((f,\phi)^{\leftarrow}|_{coker(\mathcal{S})}\right)^{op}$$

where $\mathcal{S}$ is the fuzzy topology of the codomain of (f,ϕ). The following hold:

(a) If Ω is restricted to the full subcategory $\mathbf{C\text{-}TFOP}_s$, then $S \dashv \Omega$ holds and is an equivalence of categories; i.e. $\mathbf{C} \approx \mathbf{C\text{-}FTOP}_s$.

(b) $G_\chi^{6.2.3(1)} \circ S^{6.2.1(8)} = S^{6.2.3(7)}$.

(c) $\Omega^{6.2.1(8)} \circ M_\chi^{6.2.1(1)} = \Omega^{6.2.3(7)}$.

Proof. *Ad* (1). Our first task is to show that G_χ is well-defined and functorial. The well-definedness of G_χ on objects follows from 3.2.8(1), so now let $(f,\phi) : (X,L,\tau) \to (Y,M,\sigma)$ in **C-TOP** be given. That $(f,\phi) : (X,L,\mathcal{T}_\tau) \to (Y,M,\mathcal{S}_\sigma)$ is a morphism in **C-FTOP** is immediate from the following sequence of steps.

First, $\phi^* \circ \mathcal{T}_\tau \circ (f,\phi)^{\leftarrow}$ is a fuzzy topology on (Y,L) by 3.2.12(1). Second, we observe that if $\chi_\sigma(v) = \top$, i.e. $v \in \sigma$, then $(f,\phi)^{\leftarrow}(v) \in \tau$; and hence

$$\chi_\tau((f,\phi)^{\leftarrow}(v)) = \top = \phi^{op}(\top) = \phi^{op}(\chi_\sigma(v))$$

which means that on M^Y we have

$$\chi_\tau \circ (f,\phi)^{\leftarrow} \geq \phi^{op} \circ \chi_\sigma$$

Third, we apply the definition of $\mathcal{T}_\tau$ and the fact that ϕ^* preserves order and $\top$ to conclude that on M^Y we have

$$\phi^* \circ \mathcal{T}_\tau \circ (f,\phi)^{\leftarrow} \geq \phi^* \circ \chi_\tau \circ (f,\phi)^{\leftarrow} \geq \chi_\sigma$$

Finally, we apply the definition of $\mathcal{T}_\sigma$ to conclude that

$$\phi^* \circ \mathcal{T}_\tau \circ (f,\phi)^{\leftarrow} \geq \mathcal{T}_\sigma$$

and that $(f,\phi) : (X, L, \mathcal{T}_\tau) \to (Y, M, \mathcal{S}_\sigma)$ is a morphism in **C-FTOP**.

As for M_χ, the reader can easily check that $[\mathcal{T} = \top]$ is a topology on (X, L) if $(X, L, \mathcal{T})$ is a fuzzy topological space. Given $(f,\phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$ a morphism in **C-FTOP**, it is straightforward to see that $v \in [\mathcal{S} = \top]$ and

$$\mathcal{T} \circ (f,\phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S}$$

imply that $(f,\phi)^{\leftarrow}(v) \in [\mathcal{T} = \top]$. It follows that M_χ is functorial.

As for $M_\chi \dashv G_\chi$, the unit is (id, id) which is clearly continuous—for each $(X, L, \mathcal{T}) \in |\mathbf{C\text{-}FTOP}|$, we have $\mathcal{T} \geq \chi_{[\mathcal{T}=\top]}$, which implies $\mathcal{T} \geq \mathcal{T}_{[\mathcal{T}=\top]}$. The reader can verify from 1.3.4 that the proof of the lifting criterion for $M_\chi \dashv G_\chi$ is equivalent to showing the following: $\forall (X, L, \mathcal{T}) \in |\mathbf{C\text{-}FTOP}|$, $\forall (Y, M, \sigma) \in |\mathbf{C\text{-}TOP}|$, and

$$\forall (f,\phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S}_\sigma)$$

fuzzy continuous in **C-FTOP**,

$$(f,\phi) : (X, L, [\mathcal{T} = \top]) \to (Y, M, \sigma)$$

is continuous in **C-TOP**. And this statement is easily verified: if $v \in \sigma$, then $\phi^{op}(\mathcal{S}_\sigma(v)) = \top$, so the the fuzzy continuity of (f,ϕ) implies $\mathcal{T}((f,\phi)^{\leftarrow}(v)) = \top$, i.e. $(f,\phi)^{\leftarrow}(v) \in [\mathcal{T} = \top]$, verifying (f,ϕ) is continuous.

The other assertions of (1) are left to the reader. Note the failure of $G_\chi \dashv M_\chi$ follows from 6.2.1(1) if $\mathbf{C} \hookrightarrow \mathbf{LOQML}_\perp$, since in such cases $G_\chi(\tau) = \chi_\tau$.

Ad (2). To verify that $F_{\alpha\phi}$ is a functor, it suffices to show that if

$$(f,\phi) : (X, L, \tau) \to (Y, L, \sigma)$$

is continuous in $\mathbf{L}_\phi$**-TOP**, then

$$(f, id) : \left(X, L, \bigwedge_{[\mathcal{T} \geq \alpha] \supset \tau} \mathcal{T}\right) \to \left(Y, L, \bigwedge_{[\mathcal{S} \geq \alpha] \supset \sigma} \mathcal{S}\right)$$

is fuzzy continuous in **L-FTOP**—since G_ϕ of 4.2.6 is a functor, it would then follow that $(f,\phi) : (X, L, F_{\alpha\phi}(\tau)) \to (Y, L, F_{\alpha\phi}(\tau))$ is fuzzy continuous in $\mathbf{L}_\phi$-**FTOP**. Letting

$$\mathcal{T}_\alpha \equiv \bigwedge_{[\mathcal{T} \geq \alpha] \supset \tau} \mathcal{T}, \quad \mathcal{S}_\alpha \equiv \bigwedge_{[\mathcal{S} \geq \alpha] \supset \sigma} \mathcal{S}$$

we note if $v \in \sigma$, then applying the functor F of 4.2.6 to the continuity of (f, ϕ) yields that $(f, id)^{\leftarrow}(v) \in \tau$. Now if $\mathcal{T}$ is a fuzzy topology on (X, L) with $[\mathcal{T} \geq \alpha] \supset \tau$, then

$$\mathcal{T}\left((f, id)^{\leftarrow}(v)\right) \geq \alpha$$

It follows that

$$\mathcal{T}_{\alpha}\left((f, id)^{\leftarrow}(v)\right) \geq \alpha$$

We now invoke 3.2.12 to conclude that $\mathcal{T}_{\alpha} \circ (f, id)^{\leftarrow}$ is a fuzzy topology on (Y, L) with

$$[\mathcal{T}_{\alpha} \circ (f, id)^{\leftarrow} \geq \alpha] \supset \sigma$$

It now follows from the definition of $\mathcal{S}_{\alpha}$ that $\mathcal{T}_{\alpha} \circ (f, id)^{\leftarrow} \geq \mathcal{S}_{\alpha}$, from which the functoriality of $F_{\alpha\phi}$ follows as indicated above.

It is straightforward to check that $S_{\alpha\phi}(\mathcal{T})$ is ϕ-saturated and that $S_{\alpha\phi}$ is functorial, i.e. $S_{\alpha\phi}$ changes fuzzy continuity to continuity. It follows that (a) is true. Now to verify $S_{\alpha\phi} \dashv F_{\alpha\phi}$, we first check that $(id, \phi) : (X, L, \mathcal{T}) \to (X, L, F_{\alpha\phi}[\mathcal{T} \geq \alpha])$ is fuzzy continuous as follows, using the ϕ-saturation of $\mathcal{T}$:

$$\begin{aligned}
\mathcal{T} \circ (id, \phi)^{\leftarrow} &= \mathcal{T} \circ \langle \phi^{op} \rangle \\
&= \mathcal{T} \\
&\geq F_{\alpha\phi}[\mathcal{T} \geq \alpha]
\end{aligned}$$

The remainder of the adjunction reduces to checking that if

$$(f, \phi) : (X, L, \mathcal{T}) \to (Y, L, F_{\alpha\phi}(\sigma))$$

is fuzzy continuous, where $(Y, L, \sigma) \in |\mathbf{L}_{\phi}\text{-}\mathbf{TOP}|$, then

$$(f, \phi) : (X, L, [\mathcal{T} \geq \alpha]) \to (Y, L, \sigma)$$

is continuous. To that end, let $v \in \sigma$. Then *á la* the proof of the functoriality of $F_{\alpha\phi}$ above, we have

$$F_{\alpha\phi}(\sigma)(v) \geq \alpha$$

Fuzzy continuity now implies

$$\left(\mathcal{T}\left((f, \phi)^{\leftarrow}(v)\right)\right) \geq \alpha$$

Hence $(f, \phi)^{\leftarrow}(v) \in [\mathcal{T} \geq \alpha]$, and this proves (b). Finally, the bireflectivity of the adjunction follows trivially from the unit being of the form (id, ϕ), from which (c) follows.

Ad (3). This is straightforward and left to the reader.

Ad (5). *À la* previous proofs, we begin by checking the functoriality of G_k. First note that

$$\{\underline{\alpha} : \alpha \in L\} \subset coker\,(G_k(\mathcal{T}))$$

Second, given fuzzy continuous

$$(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$$

the definition of $G_k(\mathcal{T})$ yields

$$\mathcal{T} \circ (f,\phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S} \Rightarrow$$
$$G_k(\mathcal{T}) \circ (f,\phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S} \Rightarrow$$
$$\phi^* \circ G_k(\mathcal{T}) \circ (f,\phi)^{\leftarrow} \geq \mathcal{S}$$

Third, we note that $\phi^* \circ G_k(\mathcal{T}) \circ (f,\phi)^{\leftarrow}$ is a fuzzy topology on (Y, M) (3.2.12) which is weakly stratified—for $\beta \in M$, it is easy to show that $(f,\phi)^{\leftarrow}(\underline{\beta}) \in \{\underline{\alpha} : \alpha \in L\}$, hence $G_k(\mathcal{T})\left((f,\phi)^{\leftarrow}(\underline{\beta})\right) = \top$, and the claim of weakly stratified follows since $\phi^*(\top) = \top$ (ϕ^* is a right adjoint and hence preserves $\bigwedge$). Fourth, the definition of $G_k(\mathcal{S})$ implies that

$$\phi^* \circ G_k(\mathcal{T}) \circ (f,\phi)^{\leftarrow} \geq G_k(\mathcal{S})$$

The adjunction $\hookrightarrow \dashv G_k$ has unit (id, id) and hinges around this detail—given

$$(X, L, \mathcal{T}) \in |\mathbf{C\text{-}FTOP}_k|, \quad (Y, M, \mathcal{S}) \in |\mathbf{C\text{-}FTOP}|$$

and fuzzy continuous

$$(f,\phi) : (X, L, \mathcal{T}) \to (Y, M, G_k(\mathcal{S}))$$

then

$$(f,\phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$$

is immediately fuzzy continuous since $\phi^* \circ \mathcal{T} \circ (f,\phi)^{\leftarrow} \geq G_k(\mathcal{S}) \geq \mathcal{S}$. That $\hookrightarrow \vdash G_k$ fails follows immediately from 6.2.1(6) and (b). Thus all of (a) holds once (b) is confirmed. The second claim of (b) is a corollary of the composition of adjunctions and the first claim of (b). So the proof of (5) is complete once the first claim of (b) is verified, and this reduces to proving

$$\bigwedge A = \bigwedge B$$

where

$$A \equiv \left\{\mathcal{T}_* : \mathcal{T}_* \geq \chi_{\mathcal{T}} \vee \{\underline{\alpha} : \alpha \in L\}\right\}$$
$$B \equiv \left\{\mathcal{T}_* : \mathcal{T}_* \geq \left(\bigwedge_{\mathcal{T} \geq \chi_{\mathcal{T}}} \mathcal{T}\right) \vee \left(\chi_{\{\underline{\alpha} : \alpha \in L\}}\right)\right\}$$

Clearly $A \supset B$, so that $\bigwedge A \leq \bigwedge B$. But clearly $\bigwedge A \in B$, so that $\bigwedge A \geq \bigwedge B$. This completes the proof of (5).

Ad (6). The functoriality of S and H are left to the reader, the functoriality proof of $M_{\chi s}$ mimics that of M_χ of (1), and the functoriality of $G_{\chi s}$ follows from 6.3.2 of the next section and the proof of the functoriality of G_χ of (1). As for (a), it suffices to show the adjunction—the other claims are left to the reader. The unit of $S \dashv H$, given $(L, \mathcal{T})$, is

$$j^{op} : (L, \mathcal{T}) \to \left(L^1, \mathcal{T}^1\right)$$

And given $(\mathbf{1}, M, \mathcal{S})$ and $\phi : (L, \mathcal{T}) \to (M^{\mathbf{1}}, \mathcal{S})$, the unique arrow from $(\mathbf{1}, L, \mathcal{T}^{\mathbf{1}})$ to $(\mathbf{1}, M, \mathcal{S})$ whose H-image factors ϕ through j^{op} is

$$(id, \psi) : (\mathbf{1}, L, \mathcal{T}^{\mathbf{1}}) \to (\mathbf{1}, M, \mathcal{S})$$

defined by taking $\psi : L \to M$ to be the unique (non-concrete) morphism satisfying

$$\langle \psi^{op} \rangle = j \circ \phi^{op}$$

The reader can now verify the adjunction and (a).

We verify (b), i.e.

$$G_{\chi}^{6.2.3(1)} \circ S^{6.2.1(7)} = S^{6.2.3(6)} \circ G_{\chi s}$$

The claimed identity clearly holds on morphisms. As for objects, the claim reduces, given (L, τ), to showing that the following agree on $L^{\mathbf{1}}$:

$$\mathcal{A} \equiv \bigwedge_{\mathcal{T} \geq \chi_{\tau^{\mathbf{1}}}} \mathcal{T}, \quad \mathcal{B} \equiv \left(\bigwedge_{\mathcal{T} \geq \chi_{\tau}} \mathcal{T} \right) \circ j$$

It is straightforward to verify that $\mathcal{B}$ is a fuzzy topology dominating $\chi_{\tau^{\mathbf{1}}}$, from which it immediately follows that $\mathcal{A} \leq \mathcal{B}$; and it is also straightforward to verify that $\mathcal{A} \circ j^{-1}$ is a fuzzy topology dominating χ_{τ}, from which it follows that $\mathcal{A} \geq \mathcal{B}$. The proof of (c) is left to the reader. This completes the proof of (6).

Ad (7). The proof generally is a modification of that for (6) and is left to the reader. One detail common to the proof of the functoriality of S, the adjunction $S \dashv \Omega$, and the identity of (b) is the identity

$$A_{\chi}^{\mathbf{1}} = \bigwedge_{\mathcal{T} \geq \chi_{(A^{\mathbf{1}})}} (\mathcal{T} : A^{\mathbf{1}} \to A)$$

which holds trivially. This concludes the proof of (7) and of the theorem. □

6.2.4 Remark (Consequences of 6.2.3 and its proof).

(1) The appeal to 6.3.5 in the proof of the functoriality of $G_{\chi s}$ in 6.2.3(6) is more efficient than a *de novo* proof that **C-HFTOP** is topological over its ground and hence has the final structures needed in 6.2.3(2). The proof of 6.3.x does not depend on the functoriality of $G_{\chi s}$; rather, it depends on the embedding $S^{6.2.3(6)}$, the proof of which is independent of both $G_{\chi s}$ and 6.3.5.

(2) Many of the comments of 6.2.2, giving the philosophical ramifications of 6.2.1, can be repeated almost verbatim to give the philosophical ramifications of 6.2.3. Though we now summarize these ramifications briefly (pointing out a few differences as well), it is left to the reader to restate each ramification of 6.2.1 as a ramification of 6.2.3 whenever possible.

(a) Fuzzy topological structures are richer than poslat topological structures, and fuzzy continuity is stronger than poslat continuity. This follows from 6.2.3(1).

(b) The categorical relationship between fuzzy topology and its related "levels" of poslat topologies is precisely the obverse of the categorical relationship between a poslat topology and its related levels of ordinary topologies; namely, in the latter we have $F_\alpha \dashv S_\alpha$, while in the former we have $S_{\alpha\phi} \dashv F_{\alpha\phi}$ and hence $S_\alpha \dashv F_\alpha$. While the results of 6.2.1 consistently give the interpretation that poslat topology is a generalization of ordinary topology (6.2.1(1–3)), the results of 6.2.3(1–2) do not consistently give the interpretation that fuzzy topology is a generalization of poslat topology. However, since $G_{\mathcal{X}}^{6.2.3(1)}$ has stronger properties than $S_{\alpha\phi}$ and is not limited by fixing a base, we adopt the interpretation of 6.2.3(1) that fuzzy topology is a generalization of poslat topology.

(c) Variable-basis fuzzy topology makes fixed-basis fuzzy topology categorically coherent. This follows from 6.2.3(3–4).

(d) The functor $G_k^{6.2.3(5)}$ is the **weak-stratification functor**. This, and the conclusion that weakly stratified spaces are an appropriate special setting, follows from 6.2.3(5).

(e) Every fuzzy Hutton space and every complete quasi-monoidal lattice is a singleton fuzzy topological space. This comes from 6.2.3(6–7). But combining 6.2.1(8) and 6.2.3(7) yields a small surprise, namely

$$\mathbf{C} \approx \mathbf{C\text{-}TOP}_s,\ \mathbf{C} \approx \mathbf{C\text{-}FTOP}_s \quad \Rightarrow \quad \mathbf{C\text{-}TOP}_s \approx \mathbf{C\text{-}FTOP}_s$$

Thus the singleton spaces representing complete quasi-monoidal lattices, semilocales, locales, etc may be categorically regarded as either poslat or fuzzy.

6.3 Topological subcategories

6.3.1 Proposition (Topological subcategories of topological categories). Let $\mathbf{B}$ be a subcategory of $\mathbf{A}$ and suppose $\mathbf{A}$ is topological w.r.t. $\mathbf{X}$ and $V : \mathbf{A} \to \mathbf{X}$. If each V-structured source $(X,\ f_j : X \to V(B_j))_J$ with $\{B_j\}_J \subset |\mathbf{B}|$ and $\{f_j\}_J \subset \mathbf{B}$ has its unique, initial V-lift in $\mathbf{B}$, then $\mathbf{B}$ is topological w.r.t. $\mathbf{X}$ and $V_{|\mathbf{B}} : \mathbf{B} \to \mathbf{X}$.

Proof. Immediate. □

6.3.2 Proposition (Meet criterion for topological subcategories). Let **A** be a subcategory of **C-TOP** [**C-FTOP**] satisfying the following "meet criterion":

$$\forall \{(X, L, \tau_\gamma)\}_\gamma \subset |\mathbf{C\text{-}TOP}|, \ \left(X, L, \bigcap_\gamma \tau_\gamma\right) \in |\mathbf{C\text{-}TOP}|$$

$$\left[\forall \{(X, L, \mathcal{T}_\gamma)\}_\gamma \subset |\mathbf{C\text{-}FTOP}|, \ \left(X, L, \bigwedge_\gamma \mathcal{T}_\gamma\right) \in |\mathbf{C\text{-}FTOP}|\right]$$

Then **A** is topological w.r.t. **SET** $\times$ **C** and $V_{|\mathbf{A}}$, where

$$V : \mathbf{C\text{-}TOP} \ [\mathbf{C\text{-}FTOP}] \to \mathbf{SET} \times \mathbf{C}$$

is the forgetful functor of Section 3.

Proof. This can be justified from 6.3.1 and the proofs of 3.3.4 and 3.3.9—specifically the proofs of 3.3.1 and 3.3.5 as coupled with the notions of [fuzzy] subbase in 3.2.2/3.2.9 and join of [fuzzy] topologies in 3.2.3/3.2.10. □

6.3.3 Theorem (Topological subcategories of C-TOP and C-FTOP).

(1) Each of the following subcategories of **C-TOP** is topological w.r.t. **SET** $\times$ **C** and the appropriate restriction of the forgetful functor:

$$\mathbf{C\text{-}TOP}(L), \ \mathbf{C\text{-}TOP}(L, id), \ \mathbf{C\text{-}TOP}_k, \ \mathbf{C\text{-}TOP}_s, \ \mathbf{C\text{-}TOP}(L)_k$$

(2) Each of the following subcategories of **C-FTOP** is topological w.r.t. **SET** $\times$ **C** and the appropriate restriction of the forgetful functor:

$$\mathbf{C\text{-}FTOP}(L), \ \mathbf{C\text{-}FTOP}(L, id), \ \mathbf{C\text{-}FTOP}_k, \ \mathbf{C\text{-}FTOP}_s$$

$$\mathbf{C\text{-}FTOP}(L)_k, \ \mathbf{C\text{-}SFTOP}, \ \mathbf{C\text{-}TFTOP}, \ \mathbf{C\text{-}RFTOP}$$

Proof. Apply the meet criterion (6.3.2) to each listed subcategory. □

6.3.4 Proposition (Embedded Topological Categories). Let **A** be topological w.r.t. **X** and $V : \mathbf{A} \to \mathbf{X}$. If **B** is a subcategory of **A** topological w.r.t. **X** and $V_{|\mathbf{B}} : \mathbf{B} \to \mathbf{X}$, and $U : \mathbf{D} \to \mathbf{B}$ is a categorical isomorphism, then **D** is topological w.r.t. **X** and $V \circ U$.

Proof. Use Proposition 1.3.1. □

6.3.5 Theorem (Topological categories embedded into C-TOP and C-FTOP).

(1) Each of the following categories is topological w.r.t. **SET** $\times$ **C** and the indicated modification of the forgetful functor by functors of 6.2.1:

 (a) **TOP**, $V \circ G_\chi$ or $V \circ \omega_L$.
 (b) L-**TOP**, $V \circ J$, where J is the embedding of 6.2.1(4).
 (c) **C-HTOP**, $V \circ S$.
 (d) **C**, $V \circ S$.

(2) Each of the following categories is topological w.r.t. **SET** $\times$ **C** and the indicated modification of the forgetful functor by functors of 6.2.3:

 (a) L-**FTOP**, $V \circ J$, where J is the embedding of 6.2.3(3).
 (b) L-**SFTOP** [L-**TFTOP**, L-**RFTOP**], $V \circ J$, where J is the appropriate restriction of the J of (a).
 (c) **C-HFTOP**, $V \circ S$.
 (d) **C**, $V \circ S$.

Proof. Apply 6.3.4 by noting, directly for (1)(c,d) and (2)(c,d) and by 6.3.3 for (1)(a,b) and (2)(a,b), that the image of each embedded category is topological . □

7 Unification of canonical examples by C-TOP and C-FTOP

This section is dedicated to canonical examples of poslat topology, examples which internally justify variable-basis frameworks in two related ways:

(1) **C-TOP** and **C-FTOP** provide necessary, coherent frameworks for these canonical examples. Only in these frameworks can certain questions concerning these examples be well-formed, let alone resolved. For example, when are two fuzzy real lines homeomorphic?

(2) **C-TOP** and **C-FTOP** enrich topology and fuzzy topology w.r.t. morphisms, both with respect to richness of continuity and variety of morphisms. This section will provide many examples of non-trivial morphisms between canonical examples not possible in fixed-basis topology and fixed-basis fuzzy topology, including traditional point-set topology. More precisely, none of these morphisms are in the ranges (or images) of the categorical embeddings catalogued in the previous section. For example, **C-TOP** provides many non-homeomorphic continuous morphisms between certain real lines with different underlying bases.

The results implicit in (1,2) above explicitly require virtually all the results of Sections 3—6: the results of Sections 3–4 constructing initial structures from subbases; the results from Sections 5 dealing with subbases, subspaces, embeddings, and isomorphisms in **C-TOP** and **C-FTOP**; and the results from Section 6 dealing with embeddings into **C-TOP** and **C-FTOP**.

The canonical examples considered here are not exhaustive, but are examples with extensive literatures or closely related to same. These by order include: fuzzy real lines and unit intervals; generated and weakly generated spaces; co-fuzzy dual real lines and unit intervals; and sober L-topological spaces.

Each subsection below is quadpartite and has the same organization—construction of a particular class of canonical examples, characterization of continuous morphisms between members of that class having different bases, characterization of homemorphisms between same, construction of specific classes of continuous morphisms between same; and examples of non-homeomorphic continuous morphisms between same.

Frequent appeal is made throughout the subsections below to the following extension of Lemmas 1.3.2.

7.A. Lemma (Adjoints of CQML morphisms).

(1) Let $\psi : L_1 \to L_2$ be a **CQML** morphism. The following hold:

(a) ψ has a right adjoint $\psi^* : L_1 \leftarrow L_2$ given by

$$\psi^*(\beta) = \bigvee_{\psi(\alpha) \le \beta} \alpha$$

(b) $\psi^* \circ \psi = id_{L_1}$ iff ψ is an **CQML** embedding.

(2) Let $\psi : L_1 \to L_2$ be a **CQML** morphism preserving arbitrary $\bigwedge$ (e.g. $\psi \in$ **DMQL**). The following hold:

(a) ψ has a left adjoint $^*\psi : L_1 \leftarrow L_2$ given by

$$^*\psi(\beta) = \bigwedge_{\beta \le \psi(\alpha)} \alpha$$

(b) $^*\psi \circ \psi = id_{L_1}$ iff ψ is an **CQML** embedding.

(c) ψ is an isomorphism $\Leftrightarrow$ $^*\psi = \psi^*$.

(d) If ψ is a **DQML** morphism, then $\psi^*(\beta') = [^*\psi(\beta)]'$, $^*\psi(\beta') = [\psi^*(\beta)]'$ (deMorgan Laws for Adjoints).

7.B. Corollary. The category **DQML** is not "adjoint-closed"—there are deMorgan morphisms whose adjoints are not deMorgan morphisms (by (2)(d)). If a non-isomorphism embedding in **DQML** has an adjoint in **SFRM**, it must be outside **DQML** (by (2)(c,d)). It follows these remarks also hold w.r.t. **DMRG**.

Proof of 7.A. (1)(a) and (2)(a) come from AFT; (1)(b) are (2)(b) are exercises for the reader; and (2)(c) is an immediate consequence of the inequalities defining adjunction for order-preserving maps between posets. We prove one of the deMorgan laws of (2)(d), leaving the other to the reader.

$$\begin{aligned}
\psi^*(\beta') &= \bigvee_{\psi(\alpha)\le\beta'} \alpha \\
&= \bigvee_{\psi(\alpha)\le\beta'} \alpha'' \\
&= \left[\bigwedge_{\psi(\alpha)\le\beta'} \alpha'\right]' \\
&= \left[\bigwedge_{[\psi(\alpha)]'\ge(\beta')'} \alpha'\right]' \\
&= \left[\bigwedge_{\psi(\alpha')\ge\beta} \alpha'\right]' \\
&= [{}^*\psi(\beta)]' \quad \square
\end{aligned}$$

7.1 Fuzzy real lines and unit intervals

The real lines and unit intervals are referred to in 3.1.5(1)(a) and 4.1.7(3) and references to the literature are given there. We begin with the definitions of these fundamental examples.

7.1.1. Discussion (Construction of fuzzy real lines and unit intervals). Let $L \in |\mathbf{DQML}|$, and let $\lambda : \mathbb{R} \to L$ be a map or an L-(fuzzy) subset of $\mathbb{R}$ satisfying the following conditions:

- λ is antitone;
- $\bigvee_{t\in\mathbb{R}} \lambda(t) = \top$, $\bigwedge_{t\in\mathbb{R}} \lambda(t) = \bot$.

Putting $\lambda(t+) = \bigvee_{s>t} \lambda(s)$, $\lambda(t-) = \bigwedge_{s<t} \lambda(s)$, the following are equivalent for two such mappings λ, μ:

- $\forall t \in \mathbb{R}$, $\lambda(t+) = \mu(t+)$;

- $\forall t \in \mathbb{R},\ \lambda(t-) = \mu(t-)$;
- $\forall t \in \mathbb{R},\ \lambda(t+) = \mu(t+)$ and $\lambda(t-) = \mu(t-)$.

A relation $\equiv$ is defined by stipulating $\lambda \equiv \mu$ iff any of these three conditions hold. The L-**(fuzzy) real line** or L-**(poslat) real line** $\mathbb{R}(L)$ is the set of all equivalence classes, i.e.

$$\mathbb{R}(L) = \left\{ [\lambda] \;\middle|\; \lambda : \mathbb{R} \to L,\ \lambda \text{ antitone},\ \bigvee_{t \in \mathbb{R}} \lambda(t) = \top,\ \bigwedge_{t \in \mathbb{R}} \lambda(t) = \bot \right\}$$

Each $[\lambda]$ is an L-**number**.

The subbase for the standard topology on $\mathbb{R}(L)$ comprises all maps L_t, R_t defined as follows, where $t \in \mathbb{R}$:

$$\mathfrak{L}_t : \mathbb{R}(L) \to L \text{ by } \mathfrak{L}_t(\lambda) = (\lambda'(t-))$$

$$\mathfrak{R}_t : \mathbb{R}(L) \to L \text{ by } \mathfrak{R}_t(\lambda) = \lambda(t+)$$

These maps are L-(fuzzy) subsets of $\mathbb{R}(L)$. And the standard poslat topology $\tau(L)$ is defined by

$$\tau(L) = \langle\langle \{\mathfrak{L}_t, \mathfrak{R}_t : t \in \mathbb{R}\} \rangle\rangle$$

The "$\mathfrak{L}$" in "$\mathfrak{L}_t$" means "left" and the "$\mathfrak{R}$" in "$\mathfrak{R}_t$" means "right" since these subbasic sets are the generalizations of the subbasic left and right open rays of $\mathbb{R}$ (this is made explicit below in the proof of 7.1.3). The space $(\mathbb{R}(L), \tau(L))$ is also called the L-**(fuzzy) real line** and the space $(\mathbb{R}(L), L, \tau(L))$ is a **(fuzzy) real line** or **(poslat) real line**, either one often denoted $\mathbb{R}(L)$.

The L-**(fuzzy) unit interval** or L-**(poslat) unit interval** $\mathbb{I}(L)$ is the subset of all L-numbers $[\lambda]$ satisfying

$$\lambda(1+) = \bot, \quad \lambda(0-) = \top$$

and is equipped with the subspace topology $\tau(L)_{|\mathbb{I}(L)}$ of 5.2.2, more conveniently written as $\tau(\mathbb{I}(L))$, namely

$$\tau(\mathbb{I}(L)) = \left\{ u_{|\mathbb{I}(L)} : u \in \tau(L) \right\}$$

The space $(\mathbb{I}(L), \tau(\mathbb{I}(L)))$ is also called the L-**(fuzzy) unit interval** and the space $(\mathbb{I}(L), L, \tau(\mathbb{I}(L)))$ is a **(fuzzy) unit interval** or **(poslat) unit interval**, either one often denoted $\mathbb{I}(L)$.

7.1.2. Remark (Canonicity of $\mathbb{R}(L)$ and $\mathbb{I}(L)$). The following hold:

(1) $G_\chi(\mathbb{R}, \mathfrak{T})$ is homeomorphic to $(\mathbb{R}\{\bot, \top\}, L, \tau\{\bot, \top\})$, and $G_\chi(\mathbb{I}, \mathfrak{T})$ is homeomorphic to $(\mathbb{I}\{\bot, \top\}, L, \tau(\mathbb{I}\{\bot, \top\}))$—see 7.1.3 below—where $\mathfrak{T}$ is the usual topology on $\mathbb{R}$ or $\mathbb{I}$. Both 7.1.3(1,2) below are part of folklore (but see [12, 24, 55]).

(2) For the normality axiom of [24] and for $L \in |\mathbf{DMRG}|$, there are Urysohn Lemmas [24] with codomain $\mathbb{I}(L)$ and Tiezte-Extension Theorems [36, 37, 41] with codomains $\mathbb{I}(L)$ and $\mathbb{R}(L)$. And $\mathbb{I}(L)$ and $\mathbb{R}(L)$ for $L \in |\mathbf{DMRG} \cap \mathbf{SFRM}|$ have all the middle-level and upper-level separation axioms of [26]—see [40, 54].

(3) For each linear deMorgan L (with $\otimes = \wedge$) and for each classical T_1 topological space $(X, \mathfrak{T})$ with at most finitely many components, there is an L-topological space $(X(L), \mathfrak{T}(L))$ such that $(X(L), \mathfrak{T}(L))$ is homeomorphic to:

   (a) $G_\chi(X, \mathfrak{T})$ if $L = \{\perp, \top\}$.

   (b) $\mathbb{R}(L)$ if $X = \mathbb{R}$ (with $\phi = id$);

   (c) $\mathbb{I}(L)$ if $X = \mathbb{I}$ (with $\phi = id$).

   See [30–32]; and cf. [73–74] and [59, 62].

(4) For each linear deMorgan L (with $\otimes = \wedge$), $\exists \oplus, \odot$ on $\mathbb{R}(L)$ such that the following hold:

   (a) $(\oplus, id)$ and $(\odot, id)$ are continuous on $(\mathbb{R}(L)) \times \mathbb{R}(L)$ equipped with the categorical product of L-**TOP** carried isomorphically to **L-TOP**;

   (b) $(\oplus, id)$ is (quasi-)uniformly continuous with respect to the appropropriate product (quasi-)uniformity;

   (c) $\oplus$ and $\odot$ are the unique extensions (in certain senses) of $+$ and $\cdot$ from $\mathbb{R}$ to $\mathbb{R}(L)$ and they possess many algebraic properties—$(\mathbb{R}(L), \oplus)$ is a cancellation abelian semigroup with identity.

   See [55, 58, 69] and the references given there.

7.1.3. Proposition. Let $L \in |\mathbf{DQML}|$.

(1) $G_\chi(\mathbb{R}, \mathfrak{T}) \equiv (\mathbb{R}, L, \{\chi_U : U \in \mathfrak{T}\})$ embeds into $(\mathbb{R}(L), L, \tau(L))$.

(2) The embedding of (1) is a homeomorphism iff $L = \{\perp, \top\}$.

Proof. Put $\phi = id$, and define $f : \mathbb{R} \to \mathbb{R}(L)$ by $f(r) = [\lambda_r]$, where $\lambda_r : \mathbb{R} \to L$ by

$$\lambda_r(t) = \begin{cases} \top, & t < r \\ \perp, & t > r \end{cases}$$

where the definition at r is non-essential because of the equivalence relation. It follows that (f, id) is a ground morphism. To see that it is a topological embedding in the sense of 5.3.2(1), it suffices to show that it is a homeomorphism from $(\mathbb{R}, L, \{\chi_U : U \in \mathfrak{T}\})$ to $(\mathbb{R}\{\perp, \top\}, L, \tau\{\perp, \top\})$, the latter clearly being the subspace of $(\mathbb{R}(L), L, \tau(L))$ required by 5.3.2(1). Now to show that (f, id) is a homeomorphism onto $(\mathbb{R}\{\bot, \top\}, L, \tau\{\bot, \top\})$, it suffices by 5.1.1 to show that each of (f, id) and (f^{-1}, id) are continuous; and it therefore suffices by 3.2.6 to show that each is subbasic continuous. These proofs follow from the following identities in the variable r:

$$\chi_{(r,+\infty)} = \mathfrak{L}_t|_{f^{\to}(\mathbb{R})}$$

$$\chi_{(-\infty,r)} = \mathfrak{R}_t|_{f^{\to}(\mathbb{R})}$$

These identities establish (1), (2), and the notation "$\mathfrak{L}_t, \mathfrak{R}_t$", and are left to the reader to verify. □

We finish this subsection by tackling the broader question of relating real lines with different underlying bases. Our intermediate goal is a collection of characterizations of continuous morphisms which help us achieve our ultimate goal of specific examples of non-homeomorphic continuous morphisms between real lines with different underlying bases. We begin be defining restrictive types of subbasic continuity relevant to the fuzzy real lines and fuzzy unit intervals (and indeed generalizable to any class of spaces whose subbases have a "natural" or "useful" partition).

7.1.4 Subbasic notation. Let $\mathcal{L}$ and $\mathcal{R}$ respectively denote the collections of left and right subbasic open sets of a fuzzy real line or fuzzy unit interval. If two real lines $\mathbb{R}(L_1)$ and $\mathbb{R}(L_2)$ are given, then the collections of left and right subbasic sets are respectively denoted ${}^1\mathcal{L}, {}^1\mathcal{R}, {}^2\mathcal{L}, {}^2\mathcal{R}$, and the member subbasic sets are respectively denoted ${}^1\mathfrak{L}_t, {}^1\mathfrak{R}_t, {}^2\mathfrak{L}_t, {}^2\mathfrak{R}_t$, with similar conventions holding if two unit intervals, or a unit interval and a real line, are given.

7.1.5 Definition (Special subbasic continuities).

(1) We say (f, ϕ) is a **ground morphism between $\mathbb{R}(L_1)$ and $\mathbb{R}(L_2)$** iff $L_1, L_2 \in |\mathbf{DMQL}|$ and $(f, \phi) : \mathbb{R}(L_1) \to \mathbb{R}(L_2)$ is a ground morphism in **SET × LOQML**. Similar definitions can be given for ground morphisms between $\mathbb{I}(L_1)$ and $\mathbb{I}(L_2)$, between $\mathbb{R}(L_1)$ and $\mathbb{I}(L_2)$, and between $\mathbb{I}(L_1)$ and $\mathbb{R}(L_2)$. All these cases are jointly described by saying that (f, ϕ) is a **ground morphism between fuzzy real lines and/or fuzzy unit intervals.**

(2) A ground morphism (f, ϕ) between fuzzy real lines and/or fuzzy unit intervals is **left-subbasic continuous (ls-continuous)**, **right-subbasic continuous (rs-continuous)**, or **symmetrically-subbasic continuous (ss-continuous)** iff the following respectively hold:

(ls) $((f, \phi)^{\leftarrow})^{\to}({}^2\mathcal{L}) \subset {}^1\mathcal{L}$;

(rs) $((f, \phi)^{\leftarrow})^{\to}({}^2\mathcal{R}) \subset {}^1\mathcal{R}$;

(ss) both (ls) and (rs) hold.

Note ss-continuity $\Rightarrow$ subbasic-continuity $\Rightarrow$ continuity (3.2.6).

7.1.6 Lemma (Characterization of ss-continuous maps). Let

$$(f, \phi) : (\mathbb{R}(L_1), L_1) \to (\mathbb{R}(L_2), L_2)$$

be a ground morphism in **SET** $\times$ **LOQML**. Then (f, ϕ) is ss-continuous iff both the following statements hold:

(1) $\forall\, t \in \mathbb{R},\ \exists\, u \in \mathbb{R},\ \forall\, [\lambda] \in \mathbb{R}(L_1),\ \forall\, \mu \in f[\lambda]$,

$$\phi^{op}(\mu'(t-)) = \lambda'(u-)$$

(2) $\forall\, t \in \mathbb{R},\ \exists\, u \in \mathbb{R},\ \forall\, [\lambda] \in \mathbb{R}(L_1),\ \forall\, \mu \in f[\lambda]$,

$$\phi^{op}(\mu(t+)) = \lambda(u+)$$

In the latter case, (f, ϕ) is continuous. This characterization, appropriately modified, is valid for ground morphisms between any combination of fuzzy real lines and/or fuzzy unit intervals.

Proof. It suffices to show (1) $\Leftrightarrow$ ls-continuity and (2) $\Leftrightarrow$ rs-continuity. We prove only the former; the latter is similar and simpler. Now the former bi-implication is equivalent to saying:

$$((f,\phi)^{\leftarrow})^{\rightarrow}({}^{2}\mathcal{L}) \subset {}^{1}\mathcal{L} \Leftrightarrow \forall\, t \in \mathbb{R},\ \exists\, u \in \mathbb{R},\ (f,\phi)^{\leftarrow}({}^{2}\mathfrak{L}_t) = {}^{1}\mathfrak{L}_u \qquad \text{(I)}$$

Note the condition of the right-hand side of I is equivalent to

$$\forall\, [\lambda] \in \mathbb{R}(L_1),\ (f,\phi)^{\leftarrow}({}^{2}\mathfrak{L}_t)[\lambda] = {}^{1}\mathfrak{L}_u[\lambda] \qquad \text{(II)}$$

Since $f[\lambda] \in \mathbb{R}(L_2)$, then $\forall\, \mu \in f[\lambda]$, $[\mu] = f[\lambda]$, and so the left-hand side of II reduces as follows:

$$\begin{aligned} (f,\phi)^{\leftarrow}({}^{2}\mathfrak{L}_t)[\lambda] &= \phi^{op}({}^{2}\mathfrak{L}_t(f[\lambda])) \\ &= \phi^{op}(\mu'(t-)) \end{aligned}$$

Since the right hand side of II is $\lambda'(u-)$, this implies that

$$\phi^{op}(\mu'(t-)) = \lambda'(u-)$$

It now follows that (1) holds iff (f, ϕ) is ls-continuous. $\square$

We now apply 7.1.6 via Lemma 7.A.

7.1.7 Theorem (Continuous morphisms between real lines).

(1) Let $\psi : L_1 \to L_2$ be an embedding in **DQML** and let $f : \mathbb{R}(L_1) \to \mathbb{R}(L_2)$ by
$$f[\lambda] = [\psi \circ \lambda]$$
Then $(f, ({}^*\psi)^{op}) : (\mathbb{R}(L_1), L_1, \tau(L_1)) \to (\mathbb{R}(L_2), L_2, \tau(L_2))$ is a continuous morphism in **LOQML-TOP** iff ${}^*\psi$ preserves $\otimes$ and $\top$; in which case, if ψ is not an isomorphism, then $(f, ({}^*\psi)^{op})$ is not in $\mathbf{DQML}^{op}$**-TOP**.

(2) Let $\psi : L_1 \to L_2$ be an embedding in **DMRG** and let $f : \mathbb{R}(L_1) \to \mathbb{R}(L_2)$ by
$$f[\lambda] = [\psi \circ \lambda]$$
Then $(f, (\psi^*)^{op}) : (\mathbb{R}(L_1), L_1, \tau(L_1)) \to (\mathbb{R}(L_2), L_2, \tau(L_2))$ is a continuous morphism in **SLOC-TOP** iff ψ^* preserves arbitrary $\bigvee$; in which case, if ψ is not an isomorphism, then $(f, (\psi^*)^{op})$ is not in $\mathbf{DMRG}^{op}$**-TOP**.

7.1.7.1. Corollary. $\mathbb{R}(L_1) \cong \mathbb{R}(L_2) \Leftrightarrow L_1 \cong L_2$.

Proof. This follows from 5.1.1, 7.1(5), and [7.1.7(1) or 7.1.7(2)]. □

7.1.7.2. Applications of 7.1.7. Let $L_1 = \{0, \alpha, 1\}$ with $\alpha' = \alpha$,
$$L_2 = \{0, 1/4, 1/2, 3/4, 1\}$$
with $(1/4)' = 3/4$ and $(1/2)' = 1/2$, $\otimes$ in both cases be $\wedge$, and $\psi : L_1 \to L_2$ by
$$\psi(\gamma) = \begin{cases} 0, & \gamma = 0 \\ 1/2, & \gamma = \alpha \\ 1, & \gamma = 1 \end{cases}$$
Then it follows that ${}^*\psi$ and ψ^* are defined as follows:
$$\psi^*(\beta) = \begin{cases} 1, & \beta = 1 \\ \alpha, & \beta = 1/2, 3/4 \\ 0, & \beta = 0, 1/4 \end{cases}$$
$${}^*\psi(\beta) = \begin{cases} 1, & \beta = 3/4, 1 \\ \alpha, & \beta = 1/4, 1/2 \\ 0, & \beta = 0 \end{cases}$$
And it also follows that each of ${}^*\psi$ and ψ^* preserves both arbitrary $\bigvee$ and arbitrary $\bigwedge$. The theorem therefore yields that both of
$$(f, ({}^*\psi)^{op}) : (\mathbb{R}(L_1), L_1, \tau(L_1)) \to (\mathbb{R}(L_2), L_2, \tau(L_2))$$
and
$$(f, (\psi^*)^{op}) : (\mathbb{R}(L_1), L_1, \tau(L_1)) \to (\mathbb{R}(L_2), L_2, \tau(L_2))$$
are continuous morphisms in **SLOC-TOP.** Further, by Corollaries 7.B and 7.1.7.1 above, neither of these continuous functions is a $\mathbf{DMRG}^{op}$**-TOP** morphism and neither is a homeomorphism in **LOQML-TOP**.

Proof of 7.1.7. We let the reader prove (2). For (1), necessity follows from the fact that

$$\begin{aligned}((f,({}^*\psi)^{op}) \in \mathbf{LOQML\text{-}TOP}) &\Rightarrow ((f,({}^*\psi)^{op}) \in \mathbf{SET} \times \mathbf{LOQML}) \\ &\Rightarrow ({}^*\psi)^{op} \in \mathbf{LOQML} \\ &\Rightarrow {}^*\psi \in \mathbf{CQML} \\ &\Rightarrow {}^*\psi \text{ preserves } \otimes \text{ and} \top\end{aligned}$$

Now for sufficiency in (1), we let the reader verify that f as defined is well-defined from $\mathbb{R}(L_1)$ to $\mathbb{R}(L_2)$. And to finish sufficiency, it suffices to satisfy the conditions of Lemma 7.1.6(1,2); we show 7.1.6(1) is satisfied and leave (2) to the reader. To that end, put $\phi = ({}^*\psi)^{op}$, let $t \in \mathbb{R}$, and put $u = t$. Note that for each $\mu \in f[\lambda]$,

$$\mu(t-) = (\psi \circ \lambda)(t-)$$

It therefore follows from ψ preserving deMorgan complements and from 7.A(3) that

$$\begin{aligned}\phi^{op}(\mu'(t-)) &= {}^*\psi\left((\psi \circ \lambda)'(t-)\right) \\ &= {}^*\psi\left(((\psi \circ \lambda)(t-))'\right) \\ &= {}^*\psi\left(\left(\bigwedge_{s<t} (\psi \circ \lambda)(s)\right)'\right) \\ &= {}^*\psi\left(\left(\psi\left(\bigwedge_{s<t} \lambda(s)\right)\right)'\right) \\ &= {}^*\psi\left(\psi\left(\left(\bigwedge_{s<t} \lambda(s)\right)'\right)\right) \\ &= \left(\bigwedge_{s<t} \lambda(s)\right)' \\ &= (\lambda(t-))' \\ &= \lambda'(t-)\end{aligned}$$

This completes the proof that 7.1.6(1) is satisfied. Finally, the last statement of (1) now follows from Corollary 7.B. □

7.1.8 Remark. Note 7.1.3 and 7.1.7 fulfill the claims of both (1) and (2) of the opening paragraph of this section.

7.1.9 Remark. Note 7.1.7, 7.1.7.1, and the example of 7.1.7.2 remain valid if real lines are replaced by unit intervals; and appropriate modifications remain valid if real lines and unit intervals are mixed.

7.2 Generated (weakly stratified) poslat spaces

This subsection examines the morphisms between generated, stratified spaces with different underlying bases. The importance of this section is two-fold: we build non-homeomorphic (as well as homemorphic) variable-basis morphisms between spaces generated from ordinary topology via certain functors of Section 6; and these results apply to build variable-basis morphisms between the co-fuzzy duals of the fuzzy real lines and fuzzy unit intervals in the next subsection.

7.2.1. Definition (Functor G_ω^L). Given $\mathbf{C} \hookrightarrow \mathbf{LOQML}$ and $L \in |\mathbf{C}|$, the functor

$$G_\omega^L : \mathbf{TOP} \to \mathbf{C\text{-}TOP}_k$$

is defined by

$$G_\omega^L = G_\chi^L \circ G_k$$

See [60] and 6.2.2(3) of the preceding subsection for motivation behind this notation.

7.2.2. Remark. To make G_ω^L explicit, let $(X, \mathfrak{T}) \in |\mathbf{TOP}|$ and $f \in \mathbf{TOP}$, and put:

$$G_\omega^L(\mathfrak{T}) = \langle\langle \{\chi_U : U \in \mathfrak{T}\} \cup \{\underline{\alpha} : \alpha \in L\} \rangle\rangle$$
$$G_\omega^L(X, \mathfrak{T}) = \left(X, L, G_\omega^L(\mathfrak{T})\right)$$
$$G_\omega^L(f) = f$$

7.2.3. Theorem (Characterization of variable-basis mappings between stratified spaces). Let

$$\mathbf{C} \hookrightarrow \mathbf{LOQML}, (f, \phi) \in (\mathbf{SET} \times \mathbf{C})\left((X, L), (Y, M)\right)$$

and $\mathfrak{T}, \mathfrak{S}$ be ordinary topologies on X, Y, respectively. Then

$$f : (X, \mathfrak{T}) \to (Y, \mathfrak{S})$$

is continuous in **TOP** if and only if

$$(f, \phi) : \left(X, L, G_\omega^L(\mathfrak{T})\right) \to \left(Y, M, G_\omega^M(\mathfrak{S})\right)$$

is continuous in $\mathbf{C\text{-}TOP}_k$.

Proof. *Necessity.* A subbasis of $G_\omega^M(\mathfrak{S})$ is

$$\{\chi_V : V \in \mathfrak{S}\} \cup \{\underline{\beta} : \beta \in M\}$$

By Lemma 1.3.3, $(f,\phi)^{\leftarrow}$ takes the constants of this subbase to the constants of the subbase of $G_\omega^L(\mathfrak{T})$, i.e.

$$(f,\phi)^{\leftarrow}(\underline{\beta}) = \underline{\phi^{op}(\beta)}$$

and invoking the continuity of f in **TOP**, $(f,\phi)^{\leftarrow}$ takes the characteristics of this subbase to the characteristics of the subbase of $G_\omega^L(\mathfrak{T})$, i.e.

$$(f,\phi)^{\leftarrow}(\chi_V) = \chi_{f^{\leftarrow}(V)}$$

where $f^{\leftarrow}$ is the classical backward powerset operator. Thus $(f,\phi)^{\leftarrow}$ maps the subbase of $G_\omega^M(\mathfrak{S})$ to that of $G_\omega^L(\mathfrak{T})$, making (f,ϕ) subbasic continuous, and hence continuous in **C-TOP**$_k$ by 3.2.6.

Sufficiency. Given (f,ϕ) is continuous, then for $V \in \mathfrak{S}$, we have

$$(f,\phi)^{\leftarrow}(\chi_V) = \chi_{f^{\leftarrow}(V)} \in G_\omega^L(\mathfrak{T})$$

Next, noting that the first infinite distributive law

$$a \wedge B = \bigvee \{a \wedge b : b \in B\}$$

holds when $B \subset \{\perp, \top\}$ (or as in this setting, $B \subset \{\underline{\perp}, \underline{\top}\}$), we then obtain the expected base for $G_\omega^L(\mathfrak{T})$ from its subbase, i.e.

$$G_\omega^L(\mathfrak{T}) = \langle\{\underline{\alpha} \wedge \chi_U : \alpha \in L,\, U \in \mathfrak{T}\}\rangle$$

Thus

$$\chi_{f^{\leftarrow}(V)} = \bigvee_i (\underline{\alpha}_i \wedge \chi_{U_i})$$

Now if $x \notin f^{\leftarrow}(V)$, then

$$\forall i,\ (\underline{\alpha}_i \wedge \chi_{U_i})(x) = \perp$$

And if $x \in f^{\leftarrow}(V)$, then

$$\exists j,\ (\underline{\alpha}_j \wedge \chi_{U_j})(x) = \top$$

So that

$$\chi_{f^{\leftarrow}(V)} = \chi_{U_j}$$

It follows that $f^{\leftarrow}(V) = U_j \in \mathfrak{T}$, making f continuous in **TOP**. $\square$

7.2.4. Remark. We make the following comments in regard to 7.2.3:

(1) The condition of the theorem being fully sufficient, without the imposition of the first infinite distributive law, is due to helpful criticisms of T. Kubiak.

(2) Each ordinary continuous map generates a class of variable-basis morphisms indexed by the union of the morphism classes of $\mathbf{C} \hookrightarrow \mathbf{LOQML}$, which is generally a very large class indeed. This shows the rich variety of morphisms in variable-basis topology, but only partly.

7.2.5. Corollary. Let $\mathbf{C}$ be a subcategory of completely distributive lattices,

$$(f, \phi) \in (\mathbf{SET} \times \mathbf{C})\left((X, L), (Y, M)\right)$$

and $\mathfrak{T}, \mathfrak{S}$ be ordinary topologies on X, Y, respectively. Then the $f : (X, \mathfrak{T}) \to (Y, \mathfrak{S})$ is continuous in $\mathbf{TOP}$ iff

$$(f, \phi) : (X, L, \omega_L(\mathfrak{T})) \to (Y, M, \omega_M(\mathfrak{S}))$$

is continuous in $\mathbf{C}\text{-}\mathbf{TOP}_k$, where ω_L, ω_M are defined in 6.2.1(2).

Proof. Combine 7.2.3 with Proposition 3.8 of [39]. $\square$

7.2.6. Definition (Alternate extension of ω). For $\mathbf{C} \hookrightarrow \mathbf{LOQML}$ and $L \in |\mathbf{C}|$, put $\omega_L^* : \mathbf{TOP} \to \mathbf{C}\text{-}\mathbf{TOP}_k(L, id)$ by the following steps, where $(X, \mathfrak{T}) \in |\mathbf{TOP}|$:

$$\omega_L^*(\mathfrak{T}) = \langle\langle \{u \in L^X : \forall \alpha \in L,\ [u > \alpha] \in \mathfrak{T}\} \rangle\rangle$$
$$\omega_L^*(X, \mathfrak{T}) = (X, \omega_L^*(\mathfrak{T}))$$
$$\omega_L^*(f) = f$$

This functor is given in [60] with slightly different notation. If L is linear, then $\omega_L^*(\mathfrak{T})$ is the indicated subbasis.

7.2.7. Corollary. Let M be a complete chain (which holds if $\mathbf{C}$ is a subcategory of complete chains),

$$(f, \phi) \in (\mathbf{SET} \times \mathbf{C})\left((X, L), (Y, M)\right)$$

and $\mathfrak{T}, \mathfrak{S}$ be ordinary topologies on X, Y, respectively. Then $f : (X, \mathfrak{T}) \to (Y, \mathfrak{S})$ is continuous in $\mathbf{TOP}$ implies

$$(f, \phi) : (X, L, \omega_L^*(\mathfrak{T})) \to (Y, M, \omega_M^*(\mathfrak{S}))$$

is continuous in $\mathbf{C}\text{-}\mathbf{TOP}_k$, where ω_L^*, ω_M^* are defined in 7.2.6; and the converse holds if L is also a complete chain.

Proof. From Theorem 3.3(1,2) of [60] we have $\omega_L^*(\mathfrak{T}) \supset G_\omega^L(\mathfrak{T})$ and $\omega_M^*(\mathfrak{S}) = G_\omega^M(\mathfrak{S})$, and this is enough to guarantee continuity of (f, ϕ) from 7.2.3. $\square$

7.3 (Co-fuzzy) Dual real lines and unit intervals

The purpose of this subsection is twofpld: summarize the construction of the duals of the fuzzy real lines and fuzzy unit intervals studied in [59–60]; characterize morphisms between duals based on different underlying lattices; and construct examples of non-homeomorphic morphisms between such duals.

These duals turn out to be $\mathbb{R}$ and $\mathbb{I}$, respectively, equipped with a rich inventory of canonical poslat topologies. It is the three-fold extension of the ω functor, resulting from combining [39] with [58–59], which allows us to apply the preceding subsection in a surprising way to generate a rich store of variable-basis morphisms between these duals, morphisms not possible in ordinary or fixed-basis topology.

The key to constructing duals lies in building, on a product of two sets, topologies which are not (usually) product topologies. Then we section such a topology onto each set to create a topology on that set. The two sets, so respectively equipped, become dual topological spaces. If a space is given, then constructing its dual requires finding the second set, and the topology on the product of the two sets, in such a way that when we section this topology back to the first set, we recover the original topology on the first set. In this section, we summarize the application of this method to $\mathbb{R}(L)$ and $\mathbb{I}(L)$: this means in the case for $\mathbb{R}(L)$, we choose $\mathbb{R}$ for the second set, construct a topology on $\mathbb{R}(L) \times \mathbb{R}$ which then sections back to $\mathbb{R}(L)$ to recover the canonical topology $\tau(L)$; and it follows that the poslat topology induced on $\mathbb{R}$, denoted *co*-$\tau(L)$, is that topology making $\mathbb{R}$ into the dual space of $\mathbb{R}(L)$, i.e. $(\mathbb{R}(L), L, \tau(L))$ and $(\mathbb{R}, L, co\text{-}\tau(L))$ are dual spaces, the latter being the **(co-fuzzy) dual real line**. It has already been shown in [59–60] that the duals of $\mathbb{R}(L)$ and $\mathbb{I}(L)$ possess such a wealth of important properties as to merit canonical status alongside $\mathbb{R}(L)$ and $\mathbb{I}(L)$. Finally, we remark that duals can be constructed for spaces besides the fuzzy real lines and fuzzy unit intervals.

7.3.1. Construction of poslat duals. Fix $L \in |\mathbf{LOQML}|$. Let T, X, $\mathcal{J} = \{j_\gamma\}_\gamma$ be non-empty sets with $X \subset L^X$ and $\mathcal{J} \subset \left(L^T\right)^X$. Now $\forall\, x \in X$, $\forall\, t \in T$, put $u_t^\gamma(x) = j_\gamma(x)(t)$. Then $\{u_t^\gamma\}_{(t,\gamma)} \subset L^X$. Put

$$\tau_{\mathcal{J}} = \langle\langle \{u_t^\gamma\}_{(t,\gamma)} \rangle\rangle$$

i.e. $\tau_{\mathcal{J}}$ is the L-topology on X with subbasis $\{u_t^\gamma\}_{(t,\gamma)}$, making $(X, L, \tau_{\mathcal{J}})$ an L-topological space called the (L-)**poslat dual** of T and write $X = T(L)$. If $(T, \mathfrak{T}) \in |\mathbf{TOP}|$, then $(X \equiv T(L), L, \tau_{\mathcal{J}})$ is the (L-)**poslat topological dual** of $(T, \mathfrak{T})$ iff $\exists$ injection $h : T \to X$ such that

$$(\tau_{\mathcal{J}})_{|h^{\to}(T)} = G_\chi(\mathfrak{T})$$

where $(\tau_{\mathcal{J}})_{|h^{\to}(T)}$ is the subspace topology on the image of h (5.2.2), and G_χ is the functor from 6.2.1(1). More generally, one can define topological duals w.r.t. other functors from **TOP** to **L-TOP**.

7.3.2. Construction of dynamic duals. Let $(X, L, \tau_{\mathcal{J}})$ be a poslat topological dual of T as in 7.2.1, let $(x,t) \in X \times T$, put $U_\gamma(x,t) = u_t^\gamma(x)$, and put

$$\tau_d^{\mathcal{J}}(T) = \langle\langle \{U_\gamma\}_\gamma \rangle\rangle$$

Then $\tau_d^{\mathcal{J}}(T)$ is the L-topology on $X \times T$ with subbasis $\{U_\gamma\}_\gamma$, and we speak of $\left(X \times T, L, \tau_d^{\mathcal{J}}(T)\right)$ as the **dynamic dual** of $(X, L, \tau_{\mathcal{J}})$: the members of $\tau_d^{\mathcal{J}}(T)$ are fuzzy subsets of $X \times T$ which can be interpreted as "time-dependent"—or "dynamic"—fuzzy subsets of X if T (or $t \in T$) represents time.

7.3.3. Construction of (co-fuzzy) duals. Let $(X, L, \tau_{\mathcal{J}})$ be a poslat topological dual of T as in 7.2.1, and $\left(X \times T, L, \tau_d^{\mathcal{J}}(T)\right)$ be the dynamic dual of $(X, L, \tau_{\mathcal{J}})$ as in 7.2.2. Fix $x \in X$; then $\forall u \in \tau_d^{\mathcal{J}}(T)$, put $u_x : T \to L$ by

$$u_x(t) = u(x,t)$$

It follows that

$$\left[\tau_d^{\mathcal{J}}(T)\right]_x \equiv \left\{u_x : u \in \tau_d^{\mathcal{J}}(T)\right\}$$

is an L-topology on T. Put

$$co\text{-}\tau_{\mathcal{J}} = \langle\langle \bigcup \left\{ \left[\tau_d^{\mathcal{J}}(T)\right]_x : x \in X \right\} \rangle\rangle$$

The space $(T, L, co\text{-}\tau_{\mathcal{J}})$ is the **(co-fuzzy) dual** of $(X, L, \tau_{\mathcal{J}})$.

7.3.4. Interpretation of fuzzy real lines and construction of (co-fuzzy) dual real lines. We now give a particular instantiation of 7.3.1–7.3.3 to give an alternate description of the fuzzy real lines and thereby generate the co-fuzzy real lines as the co-fuzzy duals of the fuzzy real lines. Fix $L \in |\mathbf{DQML}|$, and stipulate $T = \mathbb{R}$, $X = \mathbb{R}(L)$—re-interpreted, if necessary, as a subset of $L^{\mathbb{R}}$ by injecting $\mathbb{R}(L)$ onto a subset by mapping each equivalence class to its, say, left-continuous member:

$$\mathcal{J} = \{j_1, j_2\}$$

where

$$j_1[\lambda](t) = \lambda'(t-),\ j_2[\lambda](t) = \lambda(t+)$$

and $\mathfrak{T}$ to be the usual topology on $\mathbb{R}$. It can be shown that the subbasic members of $\tau_{\mathcal{J}}$ are as follows:

$$u_t^1 = \mathfrak{L}_t,\ u_t^2 = \mathfrak{R}_t$$

This implies that $\tau_{\mathcal{J}}$ has the same subbasis as $\tau(L)$, i.e. the fuzzy real line

$$(\mathbb{R}(L), L, \tau(L)) \equiv (X, L, \tau_{\mathcal{J}})$$

is the poslat dual of $T \equiv \mathbb{R}$, and because of 7.1.3(1), the poslat topological dual of $(\mathbb{R}, \mathfrak{T})$. Now the fuzzy real line induces the dynamic dual

$$\left(\mathbb{R}(L) \times \mathbb{R}, L, \tau_d^{\mathcal{J}}(\mathbb{R})\right)$$

as in 7.3.2 as follows:

$$\tau_d^{\mathcal{J}}(\mathbb{R}) = \langle\langle\{\mathfrak{L}, \mathfrak{R}\}\rangle\rangle$$

where

$$\mathfrak{L}, \mathfrak{R} : \mathbb{R}(L) \times \mathbb{R} \to L$$

by

$$\mathfrak{L}([\lambda], t) = \mathfrak{L}_t[\lambda] = \lambda'(t-), \quad \mathfrak{R}([\lambda], t) = \mathfrak{R}_t[\lambda] = \lambda(t+)$$

It follows that

$$\tau_d^{\mathcal{J}}(\mathbb{R}) = \{\underline{\perp}, \mathfrak{L}, \mathfrak{R}, \mathfrak{L} \wedge \mathfrak{R}, \mathfrak{L} \vee \mathfrak{R}, \underline{\top}\}$$

This dynamic dual induces the (co-fuzzy) dual topology $co\text{-}\tau(L) \equiv co\text{-}\tau_{\mathcal{J}}$ on $T \equiv \mathbb{R}$; and it can be shown that

$$co\text{-}\tau(L) = \langle\langle\{\mathfrak{L}_\lambda, \mathfrak{R}_\lambda\}\rangle\rangle_{[\lambda]\in\mathbb{R}(L)}$$

precisely the obverse—or "dual"—of

$$\tau(L) = \langle\langle\{\mathfrak{L}_t, \mathfrak{R}_t\}_{t\in\mathbb{R}}\rangle\rangle$$

It can be shown that $(\mathbb{R}, L, co\text{-}\tau(L))$ has virtually every "good" property proposed for L-topological spaces providing the underlying L has appropriate properties [59]: e.g. if $L \in |\mathbf{HUT}|$, then this space is Erceg-metrizable with a metric extending the Euclidean metric. Such "canonical" topologies on $\mathbb{R}$ (and $\mathbb{I}$) justify the constructions of 7.3.1–7.3.3.

7.3.4.1. Proposition (Basic properties of (co-fuzzy) duals). Let $L \in |\mathbf{DQML}|$ and consider the usual real line $(\mathbb{R}, \mathfrak{T})$ and co-fuzzy dual real line

$$(\mathbb{R}, L, co\text{-}\tau(L))$$

The following hold:

(1) $G_\omega^L(\mathfrak{T}) \subset co\text{-}\tau(L)$ (see 7.2.2);

(2) $G_\omega^L(\mathfrak{T}) = co\text{-}\tau(L)$ if L is a complete chain (see 7.2.2).

(3) $(\mathbb{R}, L, co\text{-}\tau(L))$ is (weakly-)stratified.

(4) $G_\chi(\mathfrak{T}) \subset co\text{-}\tau(L)$, and "=" holds iff $L = \{\perp, \top\}$ (see 6.2.1(1)).

(5) $(\mathbb{R}, L, co\text{-}\tau(L))$ maps onto $G_\chi(\mathbb{R}, \mathfrak{T})$, and this morphism is a (poslat) homeomorphism iff $L = \{\perp, \top\}$.

(6) For $L = \{\perp, \top\}$, $(\mathbb{R}, \mathfrak{T})$ is self-dual in the sense that $G_\chi(\mathbb{R}, \mathfrak{T})$ is poslat homeomorphic to each of the fuzzy real line $(\mathbb{R}\{\perp, \top\}, \{\perp, \top\}, \tau\{\perp, \top\})$ and the co-fuzzy dual real line $(\mathbb{R}, \{\perp, \top\}, co\text{-}\tau\{\perp, \top\})$.

Proof. For (1), Theorem 3.1.6 of [59]; for (2), Theorem 4.1(2) of [60]; and (3) is immediate from (1). The first assertion of (4) holds since

$$\forall r \in \mathbb{R}, \mathfrak{L}_{[\lambda_r]} = \chi_{(r,+\infty)}, \ \mathfrak{R}_{[\lambda_r]} = \chi_{(-\infty,r)}$$

where $[\lambda_r]$ is defined in the proof of 7.1.3 above. For necessity of the second assertion of (4), suppose $G_\chi(\mathfrak{T}) \supset co\text{-}\tau(L)$. Then invoking (1), we have

$$\{\underline{\alpha} : \alpha \in L\} \subset \{\underline{\perp}, \underline{\top}\}$$

which forces $L = \{\perp, \top\}$. Now sufficiency of the second assertion of (4) follows immediately from (2) given L is a chain. The morphism for (5) is (id, id) : $(\mathbb{R}, L, co\text{-}\tau(L)) \to G_\chi(\mathbb{R}, \mathfrak{T})$—its continuity comes from the first assertion of (4), and the rest of (5) comes from the second assertion of (4). Finally, (6) is a consequence of 7.1.3(2) and (5). □

7.3.5. Definition (Subbasic notation and symmetrically-subbasic continuities for duals). The subbasic notation (7.1.4) and ground morphisms and symmetric-subbasic continuities (7.1.5) for the fuzzy real lines and unit intervals can be analogously defined for their duals. For example, we let ${}^1\mathfrak{L}_\lambda$, ${}^1\mathfrak{R}_\lambda$, ${}^2\mathfrak{L}_\lambda$, ${}^2\mathfrak{R}_\lambda$ denote left-subbasic [right-subbasic, left-subbasic, right-subbasic] fuzzy sets of the domain [co-domain, domain, co-domain, respectively] of a ground morphism between dual spaces, and we let the respective collections of subbasic fuzzy sets be denoted ${}^1\mathcal{L}_{co}$, ${}^1\mathcal{R}_{co}$, ${}^2\mathcal{L}_{co}$, ${}^2\mathcal{R}_{co}$. As in Subsection 7.1, ss-continuity implies continuity (by 3.2.6). In the sequel, it is also natural to consider **left-right [right-left, mixed-symmetrically-] subbasic continuity**, respectively denoted **lrs- [rls-, mss-] continuity**, respectively defined by the conditions $((f,\phi)^{\leftarrow})^{\rightarrow}({}^2\mathcal{L}) \subset {}^1\mathcal{R}$, $((f,\phi)^{\leftarrow})^{\rightarrow}({}^2\mathcal{R}) \subset {}^1\mathcal{L}$, both the preceding; and of course, mss-continuity also implies continuity.

7.3.6. Lemma (Characterization of ss-continuous morphisms between duals). Let

$$(f, \phi) : (\mathbb{R}, L_1) \to (\mathbb{R}, L_2)$$

be a ground morphism in **SET** $\times$ **LOQML**. Then (f, ϕ) is ss-continuous iff both the following statements hold:

(1) $\forall\, [\mu] \in \mathbb{R}(L_2)\,, \exists\, [\lambda] \in \mathbb{R}(L_1)\,, \forall t \in \mathbb{R}$,

$$\phi^{op}\left(\mu'\left((f(t))-\right)\right) = \lambda'(t-)$$

(2) $\forall\, [\mu] \in \mathbb{R}(L_2)\,, \exists\, [\lambda] \in \mathbb{R}(L_1)\,, \forall t \in \mathbb{R}$,

$$\phi^{op}\left(\mu\left((f(t))+\right)\right) = \lambda(t+)$$

And (f, ϕ) is mss-continuous iff both the following statements hold:

(3) $\forall\, [\mu] \in \mathbb{R}(L_2)\,,\, \exists\, [\lambda] \in \mathbb{R}(L_1)\,,\, \forall t \in \mathbb{R},$

$$\phi^{op}\left(\mu'\left(\left(f\left(t\right)\right)-\right)\right) = \lambda\left(t+\right)$$

(4) $\forall\, [\mu] \in \mathbb{R}(L_2)\,,\, \exists\, [\lambda] \in \mathbb{R}(L_1)\,,\, \forall t \in \mathbb{R},$

$$\phi^{op}\left(\mu\left(\left(f\left(t\right)\right)+\right)\right) = \lambda'\left(t-\right)$$

Proof. As in the proof of 7.1.6, it suffices to show (1) $\Leftrightarrow$ ls-continuity, (2) $\Leftrightarrow$ rs-continuity, (3) $\Leftrightarrow$ lrs-continuity, (4) $\Leftrightarrow$ rls-continuity; and we prove only the first, the others being similar and/or simpler. Now the former bi-implication is equivalent to saying:

$$\begin{array}{rcl} \left(\left(f,\phi\right)^{\leftarrow}\right)^{\rightarrow}\left({}^2\mathcal{L}_{co}\right) & \subset & {}^1\mathcal{L}_{co} \quad \Leftrightarrow \\ \forall\,[\mu] & \in & \mathbb{R}(L_2)\,,\, \exists[\lambda] \in \mathbb{R}(L_1)\,,\, (f,\phi)^{\leftarrow}\left({}^2\mathfrak{L}_{[\mu]}\right) = {}^1\mathfrak{L}_{[\lambda]} \quad \text{(I)} \end{array}$$

and the condition of the right-hand side of I is equivalent to

$$\forall\, t \in \mathbb{R},\ (f,\phi)^{\leftarrow}\left({}^2\mathfrak{L}_{[\mu]}\right)(t) = {}^1\mathfrak{L}_{[\lambda]}\,(t) \qquad \text{(II)}$$

The left-hand side of (II) reduces as follows:

$$\begin{array}{rcl} (f,\phi)^{\leftarrow}\left({}^2\mathfrak{L}_{[\mu]}\right)(t) & = & \phi^{op}\left({}^2\mathfrak{L}_{[\mu]}(f\left(t\right))\right) \\ & = & \phi^{op}\left(\mu'\left(\left(f\left(t\right)\right)-\right)\right) \end{array}$$

Since the right-hand side of II is $\lambda'\,(t-)$, this implies that the condition of II is equivalent to

$$\phi^{op}\left(\mu'\left(\left(f\left(t\right)\right)-\right)\right) = \lambda'\left(t-\right)$$

It now follows that (1) holds iff (f,ϕ) is ls-continuous. □

7.3.7. Theorem (Classes of continuous morphisms between duals). Let $(f,\phi) : (\mathbb{R}, L_1) \to (\mathbb{R}, L_2)$ be a ground morphism in **SET** $\times$ **LOQML**. Then the following hold:

(1) If $f : \mathbb{R} \to \mathbb{R}$ is continuous w.r.t. the usual topology on $\mathbb{R}$ and order-preserving with

$$f\left(\left(+\infty\right)-\right) = +\infty,\ f\left(\left(-\infty\right)+\right) = -\infty$$

then $(f,\phi) : (\mathbb{R}, L_1, co\text{-}\tau(L_1)) \to (\mathbb{R}, L_2, co\text{-}\tau(L_2))$ is a continuous morphism.

(2) If $f : \mathbb{R} \to \mathbb{R}$ is continuous w.r.t. the usual topology on $\mathbb{R}$ and order-reversing with

$$f((+\infty)-) = -\infty, \ f((-\infty)+) = +\infty$$

then $(f, \phi) : (\mathbb{R}, L_1, co\text{-}\tau(L_1)) \to (\mathbb{R}, L_2, co\text{-}\tau(L_2))$ is a continuous morphism.

Proof. We first note for the f of (1) and each $[\nu] \in \mathbb{R}(L_2)$ that

$$\nu((f(t))-) = \nu(f(t-)), \ \nu((f(t))+) = \nu(f(t+))$$

and similarly for the f of (2) and each $[\nu] \in \mathbb{R}(L_2)$ that

$$\nu((f(t))-) = \nu(f(t+)), \ \nu((f(t))+) = \nu(f(t-))$$

$$\nu'((f(t))-) = \nu'(f(t+)), \ \nu'((f(t))+) = \nu'(f(t-))$$

Now the proof of (1) is similar and simpler than that of (2), so we prove (2); and for (2), it suffices to satisfy 7.3.6(3,4). To that end let $[\mu] \in \mathbb{R}(L_2)$, and for (3) choose

$$\lambda = \phi^{op} \circ \mu' \circ f$$

It can be checked that $\lambda : \mathbb{R} \to L_1$ satisfies the required conditions of 7.1.1 above, so that $[\lambda] \in \mathbb{R}(L_1)$. Now letting $t \in \mathbb{R}$, we obtain the left-hand side of the condition of 7.3.6(3):

$$\begin{aligned} \phi^{op}(\mu'((f(t))-)) &= \phi^{op}(\mu'(f(t+))) \\ &= \lambda(t+) \end{aligned}$$

As for (4), the proof is analogous to that of (3), but choosing

$$\lambda = (\phi^{op} \circ \mu \circ f)'$$

The details are left to the reader. □

7.3.7.1. Corollary. $(\mathbb{R}, L_1, co\text{-}\tau(L_1))$ is homeomorphic to $(\mathbb{R}, L_2, co\text{-}\tau(L_2))$ in **C-TOP**$_k$ iff $L_1 \cong L_2$ in **DQML** (5.1.1 above).

7.3.8. Theorem (Another class of continuous morphisms between duals). Let $\mathfrak{T}$ be the usual topology on $\mathbb{R}$, let $f : (\mathbb{R}, \mathfrak{T}) \to (\mathbb{R}, \mathfrak{T})$ be continuous in **TOP**, and let $(f, \phi) : (\mathbb{R}, L_1) \to (\mathbb{R}, L_2)$ be a ground morphism in **SET** $\times$ **LOQML** with L_1 arbitrary in **DQML** and L_2 a complete chain. Then

$$(f, \phi) : (\mathbb{R}, L_1, co\text{-}\tau(L_1)) \to (\mathbb{R}, L_2, co\text{-}\tau(L_2))$$

is a continuous morphism in **C-TOP**$_k$.

Proof. Note from 7.3.4.1 (1) that $G_{\omega}^{L_1}(\mathfrak{T}) \subset co\text{-}\tau(L_1)$ and from 7.3.4.1(2) that $G_{\omega}^{L_2}(\mathfrak{T}) = co\text{-}\tau(L_2)$. This is enough to guarantee the continuity of (f, ϕ) from 7.2.3. (Cf. proof of 7.2.7.) □

7.3.9. Specific examples of continuous morphisms between (co-fuzzy) duals.

$$(f, \phi) : (\mathbb{R}, L, co\text{-}\tau(L)) \to (\mathbb{R}, M, co\text{-}\tau(M))$$

is continuous in $\mathbf{C\text{-}TOP}_k$ if we make the following instantiations of 7.3.7 and/or 7.3.8 above:

(1) Let $L \equiv L_2$, $M \equiv L_1$, $\phi^{op} \equiv \psi : L \leftarrow M$ as constructed in 7.1.7.2 above with any $f : (\mathbb{R}, \mathfrak{T}) \to (\mathbb{R}, \mathfrak{T})$ continuous in **TOP**.

(2) Let L, M be arbitrary from **DQML** with $f(x) = x^3$ or $f(x) = 1 - x^5$.

(3) Put $M = [0, 1]$ with the usual ordering, involution, and tensor product $(\wedge)$. Construct L as a **two-sided continuous diamond** (cf. [60]) by identifying two disjoint copies of [0,1], denoted $\mathbb{I}_l$ and $\mathbb{I}_r$ (for "left copy" and "right copy") precisely at 0 and 1: $\mathbb{I}_l$ and $\mathbb{I}_r$ are the **sides**, the partial order is defined by saying $\alpha \le \beta$ in L iff α, β are on the same side and $\alpha \le \beta$ in $\mathbb{I}$, and the involution is defined by saying

$$\alpha' = \begin{cases} \alpha_r' \equiv 1 - \alpha_r, & \alpha = \alpha_r \in \mathbb{I}_r \\ \alpha_l' \equiv 1 - \alpha_l, & \alpha = \alpha_l \in \mathbb{I}_l \end{cases}$$

Choosing $\otimes = \wedge$ makes L into an object of **DQML.** Put $\phi^{op} : L \leftarrow M$ by $\phi^{op}(\beta) = \beta_r$. Then $\phi \in \mathbf{DQML}^{op}(L, M)$. Now let $f : (\mathbb{R}, \mathfrak{T}) \to (\mathbb{R}, \mathfrak{T})$ be any continuous map in **TOP**, e.g. $f(x) = \sin x$.

(4) For a discrete version of (2), let $M = \{0, 1/2, 1\}$ with the usual ordering, involution, and tensor product $(\wedge)$. And let $L = \{0, \alpha, \beta, 1\}$, where $\{0, \alpha, 1\}$ is the left-side of L isomorphic to M, and $\{0, \beta, 1\}$ is the right-side of L isomorphic to M. Proceed now as in (2).

(5) Let $M = \{\bot, \top\}$ and $L = \{\bot, \alpha, \beta, \top\}$ be Boolean algebras, with L equipped with the Boolean complementation as its involution. In each case $\otimes = \wedge$. Put $\phi^{op} : L \leftarrow M$ by $\phi^{op}(\bot) = \bot$ and $\phi^{op}(\top) = \top$. Now proceed as in (2).

7.3.10. Remark (Other specific examples). Tying poslat topologies on $\mathbb{R}$ to those of the fuzzy real lines yields big dividends in terms of richness of canonical properties. But it exacts a price in terms of the lattice-theoretic components of the morphisms in $\mathbf{C\text{-}TOP}_k$ preserving the involution. One way out is to replace $co\text{-}\tau(L)$, which requires the involution, with $G_{\omega}^{L}(\mathfrak{T})$ which makes no special requirements of the lattice—the motivation being that the two coincide for chains in **DQML**. This generates from 7.2.3 a whole class of examples not contained in the classes considered above. Consider the following application of 7.2.3: let L, M both be [0,1] with the usual ordering and tensor product. Put

$$\phi^{op} : L \leftarrow M \quad \text{by} \quad \phi^{op}(\alpha) = \alpha^2$$

and let $f : (\mathbb{R}, \mathfrak{T}) \to (\mathbb{R}, \mathfrak{T})$ be any continuous map in **TOP**:

(1) Then $(f, \phi) : \left(\mathbb{R}, L, G_\omega^L(\mathfrak{T})\right) \to \left(\mathbb{R}, M, G_\omega^M(\mathfrak{T})\right)$ is continuous in **C-TOP**$_k$ whenever $f : (\mathbb{R}, \mathfrak{T}) \to (\mathbb{R}, \mathfrak{T})$ is a continuous map in **TOP**.

(2) $(f, \phi) : \left(\mathbb{R}, L, G_\omega^L(\mathfrak{T})\right) \to \left(\mathbb{R}, M, G_\omega^M(\mathfrak{T})\right)$ is a **C-TOP**$_k$ isomorphism (or homeomorphism) whenever $f : (\mathbb{R}, \mathfrak{T}) \to (\mathbb{R}, \mathfrak{T})$ is a homeomorphism in **TOP** (since ϕ^{op} is a **CQML** isomorphism).

7.3.11. Remark. There are analogous classes of examples when one considers the duals of the fuzzy unit intervals or combinations of the duals of fuzzy real lines with duals of fuzzy unit intervals.

7.3.12. Remark. To sum up, all continuous, real-valued functions of a real variable are included in the above classes of examples, all fixed-basis morphisms on the real line as a poslat dual space, plus a variety of **classes** of variable-basis morphisms considerably richer than those currently known on the fuzzy real lines. In particular, the claims of both (1) and (2) of the opening paragraphs of this section have been fulfilled in the case of the real line and unit interval as co-fuzzy duals of the fuzzy real line and fuzzy unit interval. The point is that poslat topology and variable-basis topology significantly enrich traditional topology w.r.t. morphims.

7.4 Soberifications of spaces with different underlying bases

The purpose of this section is twofold: adapt the L-sober topological spaces of [56, 62–65,68] for $L \in |\mathbf{SFRM}|$ to the more general case for $L \in |\mathbf{CQML}|$ using ideas from [57]; and then characterize and show the existence of continuous, non-homeomorphic morphisms in **LOQML-TOP** between soberifications having different underlying bases. The key ideas from [57] are the following, stated in rough terms:

(1) If a certain property is required of a "point" on a lattice-theoretic structure, then an analogous property passes through to the evaluation map defined on that lattice-theoretic structure.

(2) If the evaluation map on a lattice-theoretic structure has a certain property, then the structure induced on the range (or image) of that map inherits an analogous property.

In the sequel, we let $\mathbf{C}$ be a subcategory of **LOQML**. Technical details of proofs for 7.4.1–7.4.3 *et sequens* are as in [64] and are therefore omitted.

7.4.1. Construction of C-TOP objects from C objects. Let $A \in |\mathbf{C}|$ and fix $L \in |\mathbf{CQML}|$. Put

$$Lpt(A) = \mathbf{CQML}\,(A, L)$$

These are called the *L-points of A* and comprise all maps from A to L preserving $\otimes$, arbitrary $\bigvee$, and $\top$. On the L-powerset of *Lpt(A)* we define the "Stone" separation map Φ_L as follows:

$$\Phi_L : A \to L^{Lpt(A)} \text{ by } \Phi_L(p)(a) = p(a)$$

Then it can be shown that Φ_L preserves $\otimes$, arbitrary $\bigvee$, and $\top$, where these are inhertied by the codomain of Φ_L from L. It can now be shown that $\Phi_L^{\rightarrow}(A)$ is closed under these operations and hence is an L-topology on *Lpt(A).* Thus we have

$$A \mapsto (Lpt(A), L, \Phi_L^{\rightarrow}(A))$$

where the latter is an L-topological space; so we put

$$\mathbf{LPT}\,(A) = (Lpt(A), L, \Phi_L^{\rightarrow}(A))$$

7.4.2. Construction of C-TOP morphisms from C morphisms. Continuing from 7.4.1, let $f : A \to B$ in $\mathbf{C}$, i.e. $f^{op} : A \leftarrow B$ in $\mathbf{C}^{op}$. We define

$$Lpt\,(f) : Lpt(A) \to Lpt(B)$$

by

$$Lpt(f)(p) = p \circ f^{op}$$

Then setting

$$\mathbf{LPT}\,(f) = (Lpt\,(f)\,, id)$$

it can be proved that $\mathbf{LPT}\,(f) : \mathbf{LPT}\,(A) \to \mathbf{LPT}\,(B)$ is continuous, i.e. a morphism in **C-TOP**. Then together with 7.4.1 we have the functor $\mathbf{LPT} :$ **C-TOP** $(L, id) \leftarrow \mathbf{C}$.

7.4.3. Construction of the adjunctions $\mathbf{L\Omega} \dashv \mathbf{LPT}$. Put

$$\mathbf{L\Omega} : \textbf{C-TOP}(L, id) \to \mathbf{C}$$

as follows:

$$\mathbf{L\Omega}\,(X, L, \tau) = \tau$$

$$\mathbf{L\Omega}\,(f : (X_1, L_1, \tau_1) \to (X_2, L_2, \tau_2)) \;=\; ((f, id)^{\leftarrow})^{op} : \tau_1 \to \tau_2$$

Then it can be proved that $\mathbf{L\Omega} \dashv \mathbf{LPT}$ with counit $(\Phi_L)^{op}$ and unit (Ψ_L, id), where the latter is an evaluation map defined by

$$\Psi_L : (X, L, \tau) \to \mathbf{LPT}(\tau) \equiv (Lpt(\tau), L, \Phi_L^{\to}(\tau)), \quad \Psi_L(x)(u) = u(x)$$

The unit is continuous, open w.r.t. its range viewed as a subobject (Section 5), an embedding iff (X, L, τ) is L-T_0, and a categorical isomorphism (Section 5) iff (X, L, τ) is L-sober; the counit is an isomorphism iff it is injective iff its domain is L-spatial. (Cf. [64, 65, 68].) It follows that this adjunction restricts to the equivalence of categories

$$\mathbf{SOB}\text{:}\mathbf{C}\text{-}\mathbf{TOP}(L, id) \approx \mathbf{LSPAT}\text{:}\mathbf{C}$$

It is now our purpose to construct continuous, non-homeomorphisms between spaces of the form $\mathbf{LPT}(A)$ and $\mathbf{MPT}(B)$ for possibly different L and M from $\mathbf{C}$ to indicate the richness of morphisms in $\mathbf{C}$-$\mathbf{TOP}$. We begin with the following definition and characterization.

7.4.4. Defnition. Let Let A, B be objects in $\mathbf{C}$ and let

$$(f, \phi) : (Lpt(A), L) \to (Mpt(B), M)$$

be a ground morphism in $\mathbf{SET} \times \mathbf{C}$. Then for $g : A \leftarrow B$ a mapping in $\mathbf{SET}$, (f, ϕ) is **g-indexed-continuous** iff $\forall b \in B, (f, \phi)^{\leftarrow}(\Phi_M(b)) = \Phi_L(g(b))$; and (f, ϕ) is **indexed-continuous** if it is g-indexed continuous when $A = B$ and $g = id_A$.

7.4.4.1. Remark. Note that g-indexed-continuity, and hence indexed-continuity, imply continuity.

7.4.5. Lemma (Characterization of C-TOP morphisms between soberifications). Let A, B be objects in $\mathbf{C}$ and let

$$(f, \phi) : (Lpt(A), L) \to (Mpt(B), M)$$

be a ground morphism in $\mathbf{SET} \times \mathbf{C}$. Then the following hold:

(1) $(f, \phi) : \mathbf{LPT}(A) \to \mathbf{MPT}(B)$ is continuous iff

$$\forall b \in B,\ \exists a \in A,\ \forall p \in Lpt(A),\ (\phi^{op} \circ f(p))(b) = p(a)$$

(2) If $g : A \leftarrow B$ is a mapping in $\mathbf{SET}$, then $(f, \phi) : \mathbf{LPT}(A) \to \mathbf{MPT}(B)$ is g-indexed-continuous iff

$$\forall p \in Lpt(A),\ \phi^{op} \circ f(p) = p \circ g$$

(3) If $A = B$, then $(f, \phi) : \mathbf{LPT}(A) \to \mathbf{MPT}(B)$ is indexed-continuous iff

$$\forall p \in Lpt(A),\ \phi^{op} \circ f(p) = p$$

Proof. Since (3) is a corollary of (2), and the proof of (2) can be obtained from the proof of (1), we prove only (1). For (1), let $b \in B$ and $p \in Lpt(A)$, and note

$$\begin{aligned} (f,\phi)^{\leftarrow}(\Phi_M(b))(p) &= (\phi^{op} \circ \Phi_M(b) \circ f)(p) \\ &= \phi^{op}(\Phi_M(b)(f(p))) \\ &= \phi^{op}(f(p))(b) \end{aligned}$$

Now (f,ϕ) is continuous iff $\forall v \in \Phi_M^{\rightarrow}(B)$, $\exists u \in \Phi_L^{\rightarrow}(A)$, $(f,\phi)^{\leftarrow}(v) = u$, which is the case iff $\forall b \in B$, $\exists a \in A$, $\forall p \in Lpt(A)$, $(f,\phi)^{\leftarrow}(\Phi_M(b))(p) = (\Phi_L(a))(p)$. But this latter condition holds iff $\phi^{op}(f(p))(b) = p(a)$. The claim of (1) follows. □

7.4.6 Theorem (Continuous morphisms between soberifications). Let $g : A \to B$ in **LOQML.**

(1) Let $\psi : L \to M$ be an embedding in **CQML** preserving arbitrary $\bigwedge$, and $f : Lpt(A) \to Mpt(B)$ by

$$f(p) = \psi \circ p \circ g^{op}$$

Then $(f, ({}^*\psi)^{op}) : \mathbf{LPT}(A) \to \mathbf{MPT}(B)$ is a continuous morphism in **LOQML-TOP** iff ${}^*\psi$ preserves $\otimes$ and $\top$.

(2) Let $\psi : L \to M$ be an embedding in **CQML**, and $f : Lpt(A) \to Mpt(B)$ by

$$f(p) = \psi \circ p \circ g^{op}$$

Then $(f, (\psi^*)^{op}) : \mathbf{LPT}(A) \to \mathbf{MPT}(B)$ is a continuous morphism in **SLOC-TOP** iff ψ^* preserves $\bigvee$.

7.4.6.1. Corollary.

(1) Both of 7.4.6(1,2) hold with $A = B$ and $g = id$.

(2) Given $A \in |\mathbf{CQML}|$, then $\mathbf{LPT}(A)$ is homeomorphic to $\mathbf{MPT}(A)$ iff $L \cong M$.

Proof. Note (2) follows from 5.1.1, 7.1(5), and [7.4.6(1) or 7.4.6(2)]. □

7.4.6.2. Applications of 7.4.6.

(1) Choose $A = B$ and $g = id$, and let L, M and ψ be the lattices and morphism constructed in 7.1.7.2. Then $(f, ({}^*\psi)^{op}) : \mathbf{LPT}(A) \to \mathbf{MPT}(A)$ and $(f, (\psi^*)^{op}) : \mathbf{LPT}(A) \to \mathbf{MPT}(A)$ are morphisms in **SLOC-TOP**, neither of which are homeomorphisms.

(2) By Meßner's theorem [50], $\mathbb{R}(L)$ is homeomorphic to $\mathbf{LPT}(\tau(L))$ iff L is a complete Boolean algebra. So for the lattices of 7.1.7.2, we have $\mathbb{R}(L)$ is not homeomorphic to $\mathbf{LPT}(\tau(L))$ and $\mathbb{R}(M)$ is not homeomorphic to $\mathbf{MPT}(\tau(M))$. Thus the point spaces $\mathbf{LPT}(\tau(L))$, $\mathbf{MPT}(\tau(L))$, $\mathbf{LPT}(\tau(M))$, $\mathbf{MPT}(\tau(M))$ are of interest. And it follows that there are non-homeomorphic morphisms between $\mathbf{LPT}(\tau(L))$ and $\mathbf{MPT}(\tau(L))$, as well as non-homeomorphic morphisms between $\mathbf{LPT}(\tau(M))$ and $\mathbf{MPT}(\tau(M))$. Furthermore, $\mathbf{LPT}(\tau(L))$ and $\mathbf{MPT}(\tau(M))$ are non-homeomorphic, as are $\mathbf{LPT}(\tau(M))$ and $\mathbf{MPT}(\tau(L))$.

(3) For each $\psi : L \to M$ in **CQML**, with L non-Boolean and M Boolean and ψ satisfying the conditions of 7.4.6(1) [7.4.6(2)], there is a non-homeomorphic continuous morphism from $\mathbb{R}(L)$ to $\mathbb{R}(M)$ of the following shape:

$$(\Psi_M, id)^{-1} \circ (f, ({}^*\psi)^{op}) \circ (\Psi_L, id)$$
$$\left[(\Psi_M, id)^{-1} \circ (f, (\psi^*)^{op}) \circ (\Psi_L, id)\right]$$

respectively, where f is defined as in 7.4.6(1) [7.4.6(2)]. This is a consequence of 7.4.6 and (2) above.

Proof of 7.4.6. We prove (1), leaving (2) to the reader. Now necessity is identical to that of the proof of 7.1.7(1) and is not given. For sufficiency, the reader can verify that $f : Lpt\,(A) \to Mpt\,(B)$ is well-defined. We are to show that

$$(f, ({}^*\psi)^{op}) : \mathbf{LPT}\,(A) \to \mathbf{MPT}\,(A)$$

is g^{op}-indexed-continuous by satisfying the condition of 7.4.5(2); then by 7.4.4.1 we will be able to conclude that $(f, ({}^*\psi)^{op})$ is continuous. To satisfy the condition of 7.4.5(2), let $p \in Lpt\,(A)$. Our goal is to verify

$${}^*\psi \circ f\,(p) = p \circ g^{op}$$

Let $b \in B$. Applying Lemma 7.1A(2)(b), we have

$$\begin{aligned} {}^*\psi\,(f\,(p)\,(b)) &= {}^*\psi\,(\psi\,(p\,(g^{op}\,(b)))) \\ &= p\,(g^{op}\,(b)) \quad \square \end{aligned}$$

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CHAPTER 5

Characterization Of L–Topologies By L–Valued Neighborhoods

U. Höhle

Introduction

It is well known that L–topologies can be characterized by L–neighborhood systems (cf. Subsection 6.1 in [14]). The aim of this paper is to give a characterization of a subclass of L–topologies by crisp systems of L–valued neighborhoods. This subclass consists of stratified and transitive L–topologies and covers simultaneously probabilistic L–topologies and $[0,1]$–topologies determined by fuzzy neighborhood spaces. We present this characterization depending on the structure of the underlying lattice L. In the case of probabilistic L–topologies it is remarkable to see that the structure of complete MV–algebras (cf. [2, 12]) is sufficient, while in all other case the complete distributivity of the underlying lattice L seems to be essential. Further, if L is given by the real unit interval $[0,1]$, then the Booleanization of $[0,1]$–topologies corresponding to fuzzy neighborhood spaces exists. Hence fuzzy neighborhood spaces can be chararcterized by two different types of many valued neighborhoods – namely by Booelan valued neighborhoods or by $[0,1]$–valued neighborhoods as the name "fuzzy neighborhood space" suggests (cf. Remark 3.17, Proposition 5.1). Moreover, $[0,1]$–fuzzifying topologies and $[0,1]$–topologies of fuzzy neighborhood spaces are equivalent concepts. Finally, we underline the interesting fact that a special class of stratified and transitive, $[0,1]$–topological spaces is induced by *Menger spaces* which form an important subclass of *probabilistic metric spaces* (cf. Example 5.6).

The paper is organized as follows: After some lattice–theoretic prerequisites we present an L–valued filter theory (cf. Section 2) which plays a significant role in the characterization of stratified and transitive L–topologies by crisp systems of L–valued neighborhoods (cf. Theorem 3.8 and Theorem 3.9). In Section 4

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we discuss some categorical properties of stratified and transitive, L–topological spaces – e.g. the fact that in the case of completely distributive lattices stratified and transitive L–topological spaces form a category which is isomorphic to a bireflective subcategory of the category of probabilistic L–topological spaces. Finally, in Section 5 we collect some special $[0,1]$–topological results.

1 Lattice–theoretic prerequisites

1.1 A triple $(L, \leq, *)$ is called a strictly two–sided, commutative quantale (cf. [25]) iff $(L, \leq, *)$ satifies the following conditions

(L1) $(L, \leq)$ is a complete lattice.

(L2) $(L, *)$ is a commutative semigroup.

(L3) The universal upper bound $\top$ is the unit element w.r.t. $*$.

(L4) $*$ is distributive over arbitrary joins –i.e. $\alpha * (\bigvee_{i \in I} \beta_i) = \bigvee_{i \in I} (\alpha * \beta_i)$.

In any (commutative) quantale the self–mapping $\lambda \rightsquigarrow \alpha * \lambda$ has a right adjoint map $\alpha \to \_$. In particular $\alpha \to \beta$ is given by

$$\alpha \to \beta \quad = \quad \bigvee \{\lambda \mid \alpha * \lambda \leq \beta\} \quad .$$

A strictly two–sided, commutative quantale is called a *MV–algebra* iff $(L, \leq, *)$ satisfies the important additional property

(MV) $\quad (\alpha \to \beta) \to \beta \quad = \quad \alpha \vee \beta \qquad\qquad \forall \alpha, \beta \in L \quad .$

1.2 Let $(L, \leq)$ be a partially ordered set. A *semi–ideal* in $(L, \leq)$ is a non empty subset I of L provided with the property

$$\alpha \leq \beta \,,\, \beta \in I \quad \Longrightarrow \quad \alpha \in I \quad .$$

A *semi–filter* in $(L, \leq)$ is a semi–ideal in $(L, \leq^{op})$ where $\leq^{op}$ denotes the dual (resp. opposite) partial ordering on L. If $(L, \leq)$ is a completely distributive lattice, then for every $\varkappa \in L$ there exists a unique semi–ideal $I(\varkappa)$ satisfying the following conditions (cf. [23]):

(I1) $\quad \varkappa \leq \bigvee I(\varkappa)$.

(I2) If I is a semi–ideal with $\varkappa \leq \bigvee I$, then $I(\varkappa) \subseteq I$.

By virtue of the principle of duality the complete distributivity implies also that for every $\varkappa \in L$ there exists a unique semi–filter $F(\varkappa)$ provided with the subsequent properties

(F1) $\quad \bigwedge F(\varkappa) \leq \varkappa$.

(F2) If F is a semi–filter with $\bigwedge F \leq \varkappa$, then $F(\varkappa) \subseteq F$.

Lemma 1.1 *Let $(L, \leq)$ be a completely distributive lattice and $(L, \leq, *)$ be a complete MV–algebra. Further let $\varkappa \in L$ and $\bot$ be the universal lower bound in $(L, \leq)$. Then the following relations are valid*

$$\text{(i)} \quad F(\varkappa) \;=\; \{(\epsilon \to \bot) \to \varkappa \mid \epsilon \in F(\bot)\} \quad .$$

$$\text{(ii)} \quad I(\varkappa) \;=\; \{(\epsilon \to \bot) * \varkappa \mid \epsilon \in F(\bot)\} \quad .$$

Proof. (a) We show that $F_\varkappa := \{(\epsilon \to \bot) \to \varkappa \mid \epsilon \in F(\bot)\}$ is a semi–filter. Let us consider $\epsilon \in F(\bot)$ and the situation $(\epsilon \to \bot) \to \varkappa \leq \lambda$. Then ϵ' defined by

$$\epsilon' \;=\; ((\epsilon \to \bot) \to (\lambda \to \varkappa)) \to \epsilon$$

is an element of $F(\bot)$. Now we invoke the property (MV), the divisibility and the algebraic strong de Morgan law of MV–algebras (cf. Proposition 2.3, Lemma 2.5 in [12]) and obtain

$$\begin{aligned}
(\epsilon' \to \bot) \to \varkappa \;&=\; \Big((\epsilon \to \bot) * ((\epsilon \to \bot) \to (\lambda \to \varkappa))\Big) \to \varkappa \\
&=\; \Big((\epsilon \to \bot) \wedge (\lambda \to \varkappa)\Big) \to \varkappa \\
&=\; ((\epsilon \to \bot) \to \varkappa) \vee ((\lambda \to \varkappa) \to \varkappa) \\
&=\; (\epsilon \to \bot) \to \varkappa) \vee \lambda \vee \varkappa \;=\; \lambda \,.
\end{aligned}$$

(b) Because of

$$\bigwedge_{\epsilon \in F(\bot)} (\epsilon \to \bot) \to \varkappa \;=\; \Big(\bigvee_{\epsilon \in F(\bot)} (\epsilon \to \bot)\Big) \to \varkappa \;=\; \varkappa$$

we infer from (F2) that $F(\varkappa)$ is a subset of $F_\varkappa$. On the other hand, since in any complete MV–algebra the semigroup operation $*$ is distributive over arbitrary meets (cf. Theorem 5.2 in [12]), the relation $\bot = \bigwedge\{\lambda * (\varkappa \to \bot) \mid \lambda \in F(\varkappa)\}$ holds. Now we apply again (F2) and obtain

$$\forall \epsilon \in F(\bot) \;\; \exists \lambda \in F(\varkappa) : \quad \lambda * (\varkappa \to \bot) \leq \epsilon \quad .$$

Referring again to (MV) it is easy to see that the equivalence

$$\lambda * (\varkappa \to \bot) \leq \epsilon \iff \lambda \;(= (\lambda \to \varkappa) \to \varkappa) \leq (\epsilon \to \bot) \to \varkappa$$

holds; hence $F_\varkappa$ is contained in $F(\varkappa)$.
(c) By virtue of the principle of duality the assertion (ii) follows from(i).

□

Lemma 1.2 *Let $(L, \leq)$ be a completely distributive lattice and $(L, \leq, *)$ be a complete MV–algebra. Further let $\tau_\leq$ be the interval topology on L (i.e. the coarsest (ordinary) topology on L s.t. each order interval $[\alpha, \beta]$ is closed). Then the following assertions hold:*

(a) $(L, \tau_{\leq})$ is a compact Hausdorff space.

(b) For every $\epsilon \in F(\bot)$ the order interval $[\varkappa * (\epsilon \to \bot)\ ,\ (\epsilon \to \bot) \to \varkappa]$ is a neighborhood of $\varkappa$ w.r.t. $\tau_{\leq}$.

Proof. The interval topology is compact on any complete lattice (cf. p.250 in [3]). Further we fix $\varkappa \in L$. For each $\alpha \in L$ with $\varkappa \not\leq \alpha$ we choose an element $\beta_\alpha \in L$ with $\beta_\alpha \not\leq \alpha$. Then we obtain: $\varkappa \leq \bigvee\{\beta_\alpha \mid \varkappa \not\leq \alpha\}$; hence the complete distributivity of $(L, \leq)$ implies

$$\varkappa \quad = \quad \bigvee_{\{\alpha,\, \varkappa \not\leq \alpha\}} \Big(\bigwedge\{\beta \mid \beta \not\leq \alpha\}\Big) \quad . \qquad (\Diamond)$$

By virtue of the principle of duality we obtain also

$$\varkappa \quad = \quad \bigwedge_{\{\alpha,\, \alpha \not\leq \varkappa\}} \Big(\bigvee\{\beta \mid \alpha \not\leq \beta\}\Big) \quad . \qquad (\Diamond')$$

Because of $(\Diamond)$ and $(\Diamond')$ the interval topology is Hausdorff separated[1]. Moreover, we combine Lemma 1.1 with $(\Diamond)$ and $(\Diamond')$ and obtain that for every $\epsilon \in F(\bot)$ there exist elements $\delta_1,\ \delta_2 \in L$ provided with the following property

$$\varkappa \in \ (\complement[\bot, \delta_1]) \cap (\complement[\delta_2, \top]) \quad \subseteq \quad [\varkappa * (\epsilon \to \bot), (\epsilon \to \bot) \to \varkappa] \qquad ;$$

hence the assertion (b) is also verified.

□

Theorem 1.3 *Let $(L, \leq)$ be a completely distributive lattice, $(L, \leq, *)$ be a complete MV–algebra, and let X be a non empty set. Then for every map $f : X \longmapsto L$ and for every $\epsilon \in F(\bot)$ there exists a map $g_\epsilon : X \longmapsto L$ satisfying the following conditions*

(i) The range $g_\epsilon(X)$ of g_ϵ is a finite subset of L.

(ii) $g_\epsilon(x) * (\epsilon \to \bot) \quad \leq \quad f(x) \quad \leq \quad g_\epsilon(x) \qquad \forall\, x \in X$.

Proof. We infer from Axiom (I2) and the algebraic strong de Morgan law (cf. Lemma 2.4 in [12]) that for every element $\epsilon \in F(\bot)$ there exists $\delta \in F(\bot)$ such that $\epsilon \to \bot \ \leq\ (\delta \to \bot) * (\delta \to \bot)$. Since $(L, \tau_{\leq})$ is a compact topological space, we are in the position to choose a finite subset $\{\varkappa_1, \ldots, \varkappa_n\}$ of L such that the following relation holds (cf. Assertion (b) in Lemma 1.2):

$$L \quad = \quad \bigcup_{i=1}^{n} [\varkappa_i * (\delta \to \bot), (\delta \to \bot) \to \varkappa_i] \qquad .$$

Further let $f : X \to L$ be a map; then we define a partition $\{A_i \,|\, i = 1, \ldots, n\}$ in X as follows

[1]The Hausdorff separation axiom holds already in hypercontinuous lattices (cf. p. 167 in [6]).

$$\begin{aligned} A_1 &= f^{-1}([\varkappa_1 * (\delta \to \perp), (\delta \to \perp) \to \varkappa_1] \\ A_i &= f^{-1}([\varkappa_i * (\delta \to \perp), (\delta \to \perp) \to \varkappa_i] \bigcap \complement\Big(\\ &\qquad \bigcup_{j=1}^{i-1} [\varkappa_j * (\delta \to \perp), (\delta \to \perp) \to \varkappa_j]\Big)) \quad , \quad i \geq 2 . \end{aligned}$$

Now we introduce a map $g_\epsilon : X \longmapsto L$ by:

$$g_\epsilon(x) = (\delta \to \perp) \to \varkappa_i \quad , \quad \text{whenever} \quad x \in A_i \quad (1 \leq i \leq n) \quad .$$

Obviously $f \leq g_\epsilon$. Further, for every $x \in A_i$ we observe

$$\begin{aligned} g_\epsilon(x) * (f(x) \to \perp) &\leq ((\delta \to \perp) \to \varkappa_i) * (\varkappa_i \to \delta) \\ &\leq (\delta \to \perp) \to \delta) \leq \epsilon ; \end{aligned}$$

hence the inequality $g_\epsilon * (\epsilon \to \perp) \leq f(x)$ follows.

□

2 L–Filters in the case of $(L, \leq, \wedge, *)$

Let $(L, \leq, *)$ be a complete MV–algebra and X be a non empty set. Then the operations in L can be extended pointwise to L^X. In particular, L^X can also be viewed as a complete MV–algebra. Further we use the following notation: 1_A is always the ordinary characteristic function of a subset A of X; the constant map from X to L determined by $\alpha \in L$ is denoted by $\alpha \cdot 1_X$.

In the case of $\otimes = \wedge$ we recall the notion of a stratified L–filter (cf. Definition 6.1.4 in [14]): A map $\nu : L^X \longmapsto L$ is called a *stratified L–filter on X* iff ν fulfills the following axioms:

(F0) $\nu(1_X) = \top$.

(F1) $f_1 \leq f_2 \implies \nu(f_1) \leq \nu(f_2)$.

(F2) $\nu(f_1) \wedge \nu(f_2) \leq \nu(f_1 \wedge f_2)$.

(F3) $\nu(1_\varnothing) = \perp$.

(F4) $\alpha * \nu(f) \leq \nu(\alpha * f)$.

Because of (F1) the equality holds in (F2). Moreover, since the underlying quantale is a complete MV–algebra, we conclude from (F4) that every stratified L–filter is *tight* (cf. Lemma 6.2.1 in [14]) – i.e. the following axiom holds

(F5) $\nu(\alpha \cdot 1_X) = \alpha \qquad \forall \alpha \in L$.

Referring again to (F1) and (F4) it is not difficult to show that every stratified L–filters fulfills the property

(F4′) $\nu(\alpha \rightarrow f) \;\leq\; \alpha \rightarrow \nu(f) \qquad \forall \alpha \in L,\, \forall f \in L^X$.

Theorem 2.1 (Uniqueness) *Let $(L, \leq)$ be a completely distributive, complete lattice, X be a non empty set , and let $\mathcal{P}(X)$ be the ordinary power set of X. If ν_1 and ν_2 are stratified L–filters on X, then the following assertions are equivalent*

(i) $\nu_1 \;=\; \nu_2$.

(ii) $\nu_1(1_A \vee \alpha \cdot 1_X) \;=\; \nu_2(1_A \vee \alpha \cdot 1_X) \qquad \forall A \in \mathcal{P}(X), \forall \alpha \in L$.

Proof. The implication (i) $\Longrightarrow$ (ii) is obvious. On the other hand we infer from Assertion (ii) and the axioms (F1) and (F2) that $\nu_1(g)$ and $\nu_2(g)$ coincide for all $g \in L^X$ with *finite* range. Now we invoke Theorem 1.3 and Axiom (F4) and obtain for all $f \in L^X$:

$$\nu_1(f) * (\epsilon \rightarrow \bot) \;\leq\; \nu_2(f) \quad , \quad \nu_2(f) * (\epsilon \rightarrow \bot) \;\leq\; \nu_1(f) \qquad \forall \epsilon \in F(\bot)\,;$$

hence the assertion (i) follows.

□

Definition 2.2 (L–filters of ordinary subsets) Let X be a non empty set and $\mathcal{P}(X)$ be the ordinary power set of X. A map $\varphi : \mathcal{P}(X) \rightarrow L$ is an *L–filter of ordinary subsets of X* iff φ satisfies the following axioms

(f1) $\varphi(X) \;=\; \top \;,\; \varphi(\varnothing) \;=\; \bot$. (Boundary Conditions)

(f2) $\varphi(A \cap B) \;=\; \varphi(A) \wedge \varphi(B)$. (Intersection Property)

□

If φ is an L–filter of ordinary subsets, then the value $\varphi(A)$ can be interpreted as the *degree* to which A *is contained in* φ. In the case of the real unit interval (i.e. $L = [0,1]$) [0,1]–filters of ordinary subsets and necessity measures are the same things (cf. [22]).

Since the restriction of L–filters to $\mathcal{P}(X) \cong \{\top, \bot\}^X$ is an L–filter of crisp subsets of X, the following lemma is of principle interest:

Lemma 2.3 *Let φ be an L–filter of crisp subsets of X. Then the map ν_φ : $L^X \rightarrow L$ defined by*

$$\nu_\varphi(f) \;=\; \bigvee_{A \in \mathcal{P}(X)} \varphi(A) \wedge \Big(\bigwedge_{x \in A} f(x) \Big)$$

is a stratified L–filter on X which is an extension of φ and satisfies the additional property

(F6) $\nu_\varphi(f \vee \alpha) \;=\; (\nu_\varphi(f)) \vee \alpha$.

Proof. It is not difficult to show that ν_φ is an extension of φ. The axioms (F1) – (F3) are obvious. The axiom (F4) follows from:

$$\begin{aligned} \alpha * (\varphi(A) \wedge \Big(\bigwedge_{x\in A} f(x)\Big) &\leq \varphi(A) \wedge (\alpha * \Big(\bigwedge_{x\in A} f(x)\Big)) \\ &= \varphi(A) \wedge \Big(\bigwedge_{x\in A} \alpha * f(x)\Big) \ . \end{aligned}$$

□

An immediate consequence from Theorem 2.1 and Lemma 2.3 is

Corollary 2.4 *Let $(L, \leq)$ be completely distributive. Then there exists a bijective map from the set of all L–filters of crisp subsets of X and the set of all stratified L–filters satisfying the additional axiom (F6).*

□

After a moment's reflection it is clear that there exist various extensions from L–filters of crisp subsets to stratified L–filters. The aim of the following considerations is to present a construction which is based on an additional semigroup operation $\odot$. For this purpose we agree upon the following

General Assumptions: We assume always that $(L, \leq)$ is a completely distributive lattice and $(L, \leq, *)$ is a complete MV–algebra. Further we consider a semigroup operation $\odot$ provided with the properties

(L5) $(L, \leq, \odot)$ is a quantale (cf. [25]) .

(L6) $\alpha = \alpha \odot \top = \top \odot \alpha$ – i.e. $(L, \leq, \odot)$ is strictly two–sided.

(L7) $\odot$ is distributive over arbitrary meets .

(L8) $(\alpha \odot \beta) * (\gamma \odot \delta) \leq (\alpha * \gamma) \odot (\beta * \delta)$

The axioms (L6) and (L8) imply immediately: $\alpha * \beta \leq \alpha \odot \beta$; hence $*$ is always dominated by $\odot$ (see also p. 209 in [27]). Further it is easy to see that in the case of $\odot \in \{\wedge, *\}$ the quadruple $(L, \leq, *, \odot)$ satisfies the axioms (L5) – (L8) (cf. Lemma 1.2.1 in [14]).

Theorem 2.5 *Let φ be an L–filter of crisp subsets of X. Then the map $\nu_\varphi : L^X \longmapsto L$ defined by*

$$\nu_\varphi(f) = \bigwedge \{((\varphi(A) \to \bot) \odot (\varkappa \to \bot)) \to \bot \mid f \leq 1_A \vee \varkappa \cdot 1_X\}$$

is a stratified L–filter which extends φ and satisfies the following condition:

$$(\mathrm{F6})_\odot \qquad \begin{aligned} &\nu_\varphi(((f \to \bot) \odot (\alpha \to \bot)) \to \bot) = \\ &= (((\nu_\varphi(f)) \to \bot) \odot (\alpha \to \bot)) \to \bot \end{aligned} \ .$$

Proof. Because of (L6) the map ν_φ is an extension of φ. The axiom $(F1)$ is obvious. In order to verify (F2), (F4) and (F6$_\odot$) we proceed as follows:
(a) In the case of $f_1 \wedge f_2 \leq 1_C \vee \varkappa \cdot 1_X$ we define two sets for $\epsilon \in F(\bot)$:

$$\begin{aligned} A_\epsilon^{(1)} &= \{x \in X \cap \complement C \mid f_1(x) \not\leq (\epsilon \to \bot) \to \varkappa\} \quad , \\ A_\epsilon^{(2)} &= \{x \in X \cap \complement C \mid f_2(x) \not\leq (\epsilon \to \bot) \to \varkappa\} \quad . \end{aligned}$$

Then $A_\epsilon^{(1)}$ and $A_\epsilon^{(2)}$ are disjoint, and the following relation holds:

$$f_i \leq 1_{C \cup A_\epsilon^{(i)}} \vee (\epsilon \to \bot) \to \varkappa) \cdot 1_X \qquad (i = 1, 2) \quad .$$

Now we invoke $(\mathfrak{f}2)$:

$$\nu_\varphi(f_1) \wedge \nu_\varphi(f_2) \leq ((\varphi(C) \to \bot) \odot ((\epsilon \to \bot) * (\varkappa \to \bot))) \to \bot \quad .$$

Since $\epsilon \in F(\bot)$ is arbitrary, the axiom (F2) follows.
(b) If $\alpha \cdot 1_X * f \leq 1_A \vee \varkappa \cdot 1_X$, then $f \leq 1_A \vee (\alpha \to \varkappa) \cdot 1_X$. We apply (L8):

$$\begin{aligned} &\alpha * \Big(((\varphi(A) \to \bot) \odot ((\varkappa \to \bot) * \alpha)) \to \bot\Big) * ((\varphi(A) \to \bot) \odot (\varkappa \to \bot)) \leq \\ &\Big(((\varphi(A) \to \bot) \odot ((\varkappa \to \bot) * \alpha)) \to \bot\Big) * \Big((\varphi(A) \to \bot) \odot ((\varkappa \to \bot) * \alpha)\Big) \\ &\leq \bot \quad ; \end{aligned}$$

hence $\alpha * \nu_\varphi(f) \leq ((\varphi(A) \to \bot) \odot (\varkappa \to \bot)) \to \bot$ follows. Therewith the axiom (F4) is verified.
(c) Since the relation $f \leq 1_A \vee \varkappa \cdot 1_X$ implies

$$\begin{aligned} &((f \to \bot) \odot ((\alpha \to \bot) \cdot 1_X)) \to \bot \leq \\ &\qquad \leq 1_A \vee (((\varkappa \to \bot) \odot (\alpha \to \bot)) \to \bot) \cdot 1_X \quad , \end{aligned}$$

we obtain

$$\begin{aligned} &\nu_\varphi(((f \to \bot) \odot ((\alpha \to \bot) \cdot 1_X)) \to \bot) \leq \\ &\qquad \leq ((\varphi(A) \to \bot) \odot (\varkappa \to \bot) \odot (\alpha \to \bot)) \to \bot \quad ; \end{aligned}$$

hence the inequality

$$\nu_\varphi(((f \to \bot) \odot ((\alpha \to \bot) \cdot 1_X)) \to \bot) \leq ((\nu_\varphi(f) \to \bot) \odot (\alpha \to \bot)) \to \bot$$

follows. On the other hand, for every pair $(\alpha, \varkappa) \in L \times L$ we define an element $\beta \in L$ by

$$\beta = \bigwedge\{\lambda \in L \mid \varkappa \to \bot \leq \lambda \odot (\alpha \to \bot)\} \quad .$$

Because of $(L7)$ we obtain: $\varkappa \to \bot \leq \beta \odot (\alpha \to \bot)$. Hence we conclude from

$$((f \to \bot) \odot ((\alpha \to \bot) \cdot 1_X)) \to \bot \leq 1_A \vee \varkappa \cdot 1_X$$

that the relation $f \leq 1_A \vee (\beta \to \perp) \cdot 1_X$ holds. In particular we obtain:

$$\begin{aligned} ((\nu_\varphi(f) \to \perp) \odot (\alpha \to \perp)) \to \perp &\leq ((\varphi(A) \to \perp) \odot \beta \odot (\alpha \to \perp)) \to \perp \\ &\leq ((\varphi(A) \to \perp) \odot (\varkappa \to \perp)) \to \perp \quad ; \end{aligned}$$

i.e.

$$((\nu_\varphi(f) \to \perp) \odot (\alpha \to \perp)) \to \perp \quad \leq \quad \nu_\varphi(((f \to \perp) \odot ((\alpha \to \perp) \cdot 1_X)) \to \perp) \; .$$

Therewith the axiom $(F6)_\odot$ is verified.

□

In the case of the real unit interval we can consider a family $\{T_p \mid p \in [-1, +\infty]\}$ of associative, continuous, binary operations defined by

$$\begin{array}{lcll} T_\infty(\alpha,\beta) & = & \min(\alpha,\beta) & p = +\infty . \\ T_p(\alpha,\beta) & = & (\alpha^{-p} + \beta^{-p} - 1)^{-1/p} & 0 < p < +\infty . \\ T_0(\alpha,\beta) & = & \alpha \cdot \beta & p = 0 . \\ T_p(\alpha,\beta) & = & (\max(\alpha^{-p} + \beta^{-p} - 1, 0))^{-1/p} & -1 \leq p < 0 . \end{array}$$

It is easy to see that $([0,1], \leq, T_{-1})$ is an MV–algebra; in particular T_{-1} coincides with *Lukasiewicz' arithmetic conjunction* (cf. [5]). Moreover, we obtain:

Proposition 2.6 *The quadruple* $([0,1], \leq T_{-1}, T_p)$ *satisfies the axioms (L5) – (L8) for all* $p \in [-1, +\infty]$.

Proof. The axioms (L5) and (L7) follow from the continuity of T_p. Since 1 is the unity of T_p, the axiom (L6) is evident. In order to verify (L8) we proceed as follows: In the case of $p \in \{-1, +\infty\}$ Axiom (L8) is obvious. Since T_0 is the pointwise limit of binary operations T_p with $p \neq 0$, it is sufficient to verify (L8) in the case of $p \in]-1, 0[\cup]0, +\infty[$.

Step 1. We fix $\beta \in]0,1]$ and consider a map $e_\beta : [0,1] \longmapsto [0,1]$ determined by $e_\beta(\alpha) = T_p(\beta, \alpha)$. Obviously, e_β is continuous on $[0,1]$ and differentiable on $]0,1[$. In particular, the derivative of e_β in α is given by

$$e'_\beta(\alpha) \quad = \quad \left(\frac{T_p(\beta,\alpha)}{\alpha}\right)^{p+1} \quad .$$

Since e'_β is bounded by 1 on $]0,1[$, the function $f_\beta : [0,1] \to [0,1]$ defined by $f_\beta(\alpha) = e_\beta(\alpha) - \alpha - \beta + 1$ is non increasing. Hence f_β is non negative.

Step 2. We fix $(\alpha, \beta) \in]0,1] \times]0,1]$ and introduce a map $g_{(\alpha,\beta)} : [0,1] \longmapsto [0,1]$ by

$$g_{(\alpha,\beta)}(\gamma) \quad = \quad e_\beta(T_{-1}(\alpha,\gamma)) - \alpha - e_\beta(\gamma) + 1 \quad .$$

We show the non–negativity of $g_{(\alpha,\beta)}$. In the case of $\gamma \in [0, 1-\alpha]$ it is easy to see that the relation

$$0 \quad \leq \quad 1 - \gamma - \alpha \quad \leq \quad g_{(\alpha,\beta)}(\gamma)$$

holds. Further we observe

$$\left(\frac{e_\beta(\gamma)}{\gamma}\right)' = \frac{e_\beta(\gamma)}{\gamma^2}\cdot\left(\left(\frac{e_\beta(\gamma)}{\gamma}\right)^p - 1\right) \quad ;$$

i.e. the function $\gamma \rightsquigarrow \frac{e_\beta(\gamma)}{\gamma}$ is non decreasing (resp. non increasing) for all $p \in]-1,0[$ (resp. $p \in]0,+\infty[$). Hence the restriction of $g_{(\alpha,\beta)}$ to $[1-\alpha,1]$ is monotone. Because of

$$g_{(\alpha,\beta)}(1-\alpha) = 0 \quad , \quad g_{(\alpha,\beta)}(1) = f_\beta(\alpha)$$

we obtain from *Step 1* that $g_{(\alpha,\beta)}$ is also non negative on $[1-\alpha,1]$.

Step 3. In order to verify (L8) we fix $(\alpha,\beta,\gamma) \in [0,1]^3$ and define a map $h_{(\alpha,\beta,\gamma)} : [0,1] \longmapsto [0,1]$ by

$$h_{(\alpha,\beta,\gamma)}(\epsilon) = T_p(T_{-1}(\alpha,\gamma),T_{-1}(\epsilon,\beta)) - T_p(\alpha,\epsilon) - T_p(\gamma,\beta) + 1 \quad .$$

It is sufficient to show that $h_{(\alpha,\beta,\gamma)}$ is non negative. In the case of $(\alpha,\beta,\gamma) \notin]0,1]^3$ we obtain immediately $0 \le h_{(\alpha,\beta,\gamma)}(\epsilon)$. Further, if $T_{-1}(\epsilon,\beta) = 0$ or $T_{-1}(\alpha,\gamma) = 0$, then the non–negativity of $h_{(\alpha,\beta,\gamma)}$ follows from

$$0 \le 1 - \max(T_p(\alpha,1-\beta)\,,\,T_p(1-\gamma,\epsilon)) - T_p(\gamma,\beta) \le h_{(\alpha,\beta,\gamma)}(\epsilon)\,.$$

Hence it is sufficient to consider $(\alpha,\beta,\gamma) \in]0,1]^3$, $0 < T_{-1}(\alpha,\gamma)$ and to assume that the global minimum of $h_{(\alpha,\beta,\gamma)}$ is attained at $\epsilon_0 \in]1-\beta,1]$. We distinguish the following cases:

Case 1 : $\epsilon_0 = 1$. Then $h_{(\alpha,\beta,\gamma)}(1) = g_{(\alpha,\beta)}(\gamma)$; thus the non–negativity of $h_{(\alpha,\beta,\gamma)}$ follows from *Step 2*.

Case 2 : $\epsilon_0 \in]1-\beta,1[$. Then the first derivative of $h_{(\alpha,\beta,\gamma)}$ vanishes in ϵ_0 ; i.e.

$$\left(\frac{T_p(T_{-1}(\alpha,\gamma)\,,\,T_{-1}(\epsilon_0,\beta))}{T_{-1}(\epsilon_0,\beta)}\right)^{p+1} - \left(\frac{T_p(\alpha,\epsilon_0)}{\epsilon_0}\right)^{p+1} = 0 \quad .$$

Hence we obtain:

$$\begin{aligned} h_{(\alpha,\beta,\gamma)}(\epsilon_0) &= \frac{T_p(\alpha,\epsilon_0)}{\epsilon_0}\cdot(T_{-1}(\epsilon_0,\beta) - \epsilon_0) - T_p(\gamma,\beta) + 1 \\ &\ge (T_{-1}(\epsilon_0,\beta) - \epsilon_0) - \beta + 1 \\ &= 0 \quad . \end{aligned}$$

In particular, the non–negativity of $h_{(\alpha,\beta,\gamma)}$ on $]1-\beta,1[$ follows.

□

Combining Proposition 2.6 with Theorem 2.5 we see that in the case of the real unit interval there exist continuously many extensions of a $[0,1]$–filter of crisp subsets to a stratified [0,1]–filter. In this situation, the additional binary operations $\odot = T_p$ exhaust the interval determined by $* = T_{-1}$ and $\wedge = \min$.

Corollary 2.7 *Let $(L, \leq, *, \odot)$ be a quadruple provided with (L5) – (L8). Then there exists a bijective map between the set of all L-filters of crisp subsets of X and the set of all stratified L-filters on X satisfying Axiom $(F6)_\odot$.*

Proof. The assertion follows immediately from Theorem 2.1 and Theorem 2.5. □

In the following definition we recall the notion of $\top$–filters (cf. Remark 6.2.3 in [14]) which we need for an important characterization of L–filters of crisp subsets.

Definition 2.8 ($\top$–Filters) Let $(L, \leq, *)$ be a complete MV–algebra. Further let the algebraic operation $\otimes$ be determined by the binary meet operation $\wedge$. A subset $\mathbf{F}$ of L^X is called a $\top$*–filter on* X iff $\mathbf{F}$ satisfies the following conditions (cf. Definition 1.5 in [11]):

(𝔽0) $\quad 1_X \in \mathbf{F}$.

(𝔽1) $\quad h \in L^X$ with $\exists \varkappa : \mathbf{F} \longmapsto L$ s.t. $\left\{ \begin{array}{rcl} \bigvee\{\varkappa(f) \mid f \in \mathbf{F}\} & = & \top \\ \varkappa(f) * f(x) & \leq & h(x) \ \forall f \in \mathbf{F} \end{array} \right\}$

$\qquad \Longrightarrow \quad h \in \mathbf{F}$.

(𝔽2) $\quad f_1, f_2 \in \mathbf{F} \quad \Longrightarrow \quad f_1 \wedge f_2 \in \mathbf{F}$.

(𝔽3) $\quad \bigvee\limits_{x \in X} f(x) \quad = \quad \top \quad \forall f \in \mathbf{F}$.

□

If the underlying lattice L is completely distributive, then the condition (𝔽1) is equivalent to

(𝔽1′) If $h \in L^X$ with $\forall \epsilon \in F(\bot)\ \exists f \in \mathbf{F} : f(x) * (\epsilon \to \bot) \leq h(x)\ \forall x \in X$, then $h \in \mathbf{F}$.

In the case of $(L \ \leq, *) = ([0,1], \leq, T_{-1})$ the condition (𝔽1′) coincides with the so-called saturation condition of pre–filters introduced by R. Lowen in the context of fuzzy neighborhood spaces (cf. [18]).

Lemma 2.9 *Let $(L, \leq)$ be a completely distributive lattice, X be a non empty set, and let $\mathcal{P}(X)$ be the ordinary power set of X. If $\mathbf{F}_1$ and $\mathbf{F}_2$ are $\top$-filters, then the following assertions are equivalent*

(i) $\quad \mathbf{F}_1 \quad = \quad \mathbf{F}_2$.

(ii) $\quad \forall \varkappa \in L,\ \forall A \in \mathcal{P}(X) : 1_A \vee \varkappa \cdot 1_X \in \mathbf{F}_1 \quad \Longleftrightarrow \quad 1_A \vee \varkappa \cdot 1_X \in \mathbf{F}_2$.

Proof. The assertion follows immediately from (𝔽1′), (𝔽2) and Theorem 1.3. □

It is easy to see that every $\top$–filter $\mathbf{F}$ on X induces an L–filter $\xi(\mathbf{F})$ of crisp subsets of X by

$$\xi(\mathbf{F})(A) \quad = \quad \bigvee_{f\in\mathbf{F}} \Big(\bigwedge_{x\notin A} (f(x) \to \bot) \Big) \qquad .$$

Lemma 2.10 *Let $(L,\leq)$ be a completely distributive lattice and X be a non empty set. Further let $\mathbf{F}$ be a $\top$-filter on X and $\xi(\mathbf{F})$ be the associated L–filter of crisp subsets of X. If $\varkappa \in L$ and $A \subseteq X$, then the following assertions are equivalent:*

(i) $\quad 1_A \vee \varkappa \cdot 1_X \in \mathbf{F} \quad .$

(ii) $\quad \varkappa \to \bot \quad \leq \quad \xi(\mathbf{F})(A) \quad .$

Proof. The implication (i) $\Longrightarrow$ (ii) is obvious. On the other hand, if $\varkappa \to \bot \ \leq \ \xi(\mathbf{F})(A)$, then we conclude from Lemma 1.1 that for every $\epsilon \in F(\bot)$ there exists $f_\epsilon \in \mathbf{F}$ s.t.

$$(\varkappa \to \bot) * (\epsilon \to \bot) \quad \leq \quad \bigwedge_{x\notin A} (f_\epsilon(x) \to \bot) \quad ;$$

hence the relation

$$f_\epsilon(x) \, * \, (\epsilon \to \bot) \quad \leq \quad 1_A(x) \, \vee \, \varkappa$$

holds for all $\epsilon \in F(\bot)$. Now we apply $(\mathbb{F}1')$ and obtain that $1_A \vee \varkappa \cdot 1_X$ is an element of $\mathbf{F}$.

□

Theorem 2.11 *Let $(L,\leq)$ be a completely distributive lattice. Then the map $\mathbf{F} \rightsquigarrow \xi(\mathbf{F})$ is a bijective map from the set of all $\top$-filters on X onto the set of all L–filters of crisp subsets of X.*

Proof. The injectivity of ξ follows from Lemma 2.9 and Lemma 2.10. In order to verify the surjectivity of ξ we proceed as follows: Let φ be an L–filter of crisp subsets; then we show that

$$\mathbf{F}_\varphi \quad = \quad \{ \, f \in L^X \mid \bigwedge_{x\notin A} (f(x) \to \bot) \ \leq \ \varphi(A) \ \forall \, A \subseteq X \, \}$$

is a $\top$–filter. Since the relation $f(x) * \varkappa(f) \ \leq \ h(x)$ is equivalent to $(h(x) \to \bot) * \varkappa(f) \ \leq \ f(x) \to \bot$, $\mathbf{F}_\varphi$ fulfills Axiom $(\mathbb{F}1)$. Now we invoke the complete distributivity of $(L,\leq)$ and obtain:

$$\bigwedge_{x\notin A} ((f_1(x) \wedge f_2(x)) \to \bot) \quad \leq$$

$$\leq \quad \bigvee_{E_1 \cap E_2 = A} \Big(\Big(\bigwedge_{x\notin E_1} (f_1(x) \to \bot) \Big) \wedge \Big(\bigwedge_{x\notin E_2} (f_2(x) \to \bot) \Big) \Big) \ ;$$

hence $(\mathbb{F}2)$ follows from $(\mathfrak{f}2)$. The axiom $(\mathbb{F}3)$ is evident. Finally, the map $1_A \vee (\varphi(A) \to \bot) \cdot 1_X$ is always an element of $\mathbf{F}_\varphi$; hence $\xi(\mathbf{F}_\varphi)(A) = \varphi(A)$ holds for all ordinary subsets A of X.

□

Because of Corollary 2.7 and Theorem 2.11 there exists a bijective map $\Psi_\odot$ from the set of all $\top$–filters on X onto the set of all stratified L–filters on X satisfying $(\mathrm{F6})_\odot$. In the following considerations we give an explicit description of $\Psi_\odot$. First we prove:

Lemma 2.12 *Let* $\mathbf{F}$ *be a* $\top$*–filter on* X. *Then the map* $\nu_{\mathbf{F}} : L^X \longmapsto L$ *defined by*

$$\nu_{\mathbf{F}}(h) \quad = \quad \bigvee_{f \in \mathbf{F}} \Big(\bigwedge_{x \in X} ((f(x) \odot (h(x) \to \bot)) \to \bot) \Big) \qquad \forall\, h \in L^X$$

is a stratified L*–filter provided with Property* $(\mathrm{F6})_\odot$.

Proof. It is easy to see that $\nu_{\mathbf{F}}$ satisfies the axioms (F0), (F1) and (F3). Moreover, (F2) follows from $(\mathbb{F}2)$. The verification of (F4) is based on Axiom (L8). First we observe:

$$\begin{array}{lc} \beta * ((\alpha \odot (\gamma \to \bot)) \to \bot) \; * \; (\alpha \odot ((\beta * \gamma) \to \bot)) & \leq \\ ((\alpha \odot (\gamma \to \bot)) \to \bot) \; * \; (\alpha \odot (\beta * (\beta \to (\gamma \to \bot)))) & \leq \\ ((\alpha \odot (\gamma \to \bot)) \to \bot) \; * \; (\alpha \odot (\gamma \to \bot)) \quad = \quad \bot & ; \end{array}$$

i.e.

$$\beta * ((\alpha \odot (\gamma \to \bot)) \to \bot) \quad \leq \quad (\alpha \odot ((\beta * \gamma) \to \bot)) \to \bot \quad .$$

Hence we obtain: $\beta * \nu_{\mathbf{F}}(h) \;\leq\; \nu_{\mathbf{F}}(\beta * h)$. The axiom $(\mathrm{F6})_\odot$ follows immediately from (L5) and (L7).

□

Proposition 2.13 *The bijective map* $\Psi_\odot$ *from the set of all* $\top$*–filters on* X *onto the set of all stratified* L*–filters satisfying* $(\mathrm{F6})_\odot$ *is determined by (cf. Corollary 2.7, Theorem 2.11):*

$$\Psi_\odot(\mathbf{F})(h) \quad = \quad \nu_{\mathbf{F}}(h) \; = \quad \bigvee_{f \in \mathbf{F}} \Big(\bigwedge_{x \in X} ((f(x) \odot (h(x) \to \bot)) \to \bot) \Big) \quad \forall\, h \in L^X .$$

Proof. Let ν be a stratified L–filter satisfying $(\mathrm{F6})_\odot$, and let φ be the L–filter of crisp subsets corresponding to ν (cf. Theorem 2.5, Corollary 2.7). Further let $\mathbf{F}_\varphi$ be the $\top$–filter associated with φ (cf. Proof of Theorem 2.11). Because of Lemma 2.12 and Theorem 2.1 it is sufficient to prove:

$$\nu(1_A \vee \varkappa \cdot 1_X) \quad = \quad \nu_{\mathbf{F}_\varphi}(1_A \vee \varkappa \cdot 1_X) \qquad \forall \varkappa \in L\,,\; \forall\, A \subseteq X\,.$$

We apply $(F6)_\odot$, (L5) – (L7) and obtain:

$$\begin{aligned}\nu(1_A \vee \varkappa\cdot 1_X) &= ((\varphi(A)\to\bot)\odot(\varkappa\to\bot))\to\bot\\ &= (((\bigvee_{f\in \mathbf{F}_\varphi}\Big(\bigwedge_{x\notin A}(f(x)\to\bot)\Big)\Big)\to\bot)\odot(\varkappa\to\bot))\to\bot\\ &= \bigvee_{f\in\mathbf{F}_\varphi}\Big(\bigwedge_{x\notin A}\Big((f(x)\odot(\varkappa\to\bot))\to\bot\Big)\Big)\\ &= \nu_{\mathbf{F}_\varphi}(1_A\vee\varkappa\cdot 1_X)\quad .\end{aligned}$$

□

Remark 2.14 (a) Let $(L,\leq)$ be a completely distributive lattice. Then in the case of $\odot = \wedge$ it is easy to see that $\Psi_\wedge$ is also the composition of ξ (cf. Theorem 2.11) with the map specified in Lemma 2.3. Indeed, the relation

$$\Psi_\wedge(\mathbf{F})(h) = \bigvee_{A\subseteq X}\Big(\bigvee_{f\in\mathbf{F}}\Big(\Big(\bigwedge_{x\notin A}(f(x)\to\bot)\Big)\wedge\Big(\bigwedge_{x\in A}h(x)\Big)\Big)\Big)$$

follows from the complete distributivity of $(L,\leq)$.
(b) In the case of $\odot = *$ the assertion of Proposition 2.13 remains valid without assuming the complete distributivity of the underlying lattice $(L,\leq)$ (cf. Remark 6.2.3, Proposition 6.2.4 in [14]). This observation motivates the following *Question*:
Let $(L,\leq,*)$ be a complete MV–algebra and $(L,\leq,*,\odot)$ be a quadruple satisfying (L5) – (L8) . To which extent depends the validity of Proposition 2.13 on the complete distributivity of the underlying lattice $(L,\leq)$?

□

3 Stratified and transitive L–topologies

Let $(L,\leq,*)$ be a complete MV–algebra and X be an arbitrary, non empty set. Obviously, the algebraic and lattice–theoretic operations can be extended pointwise to L^X. Then L^X is again a complete MV–algebra. Since the underlying lattice of a complete MV–algebra is always a *frame* (resp. complete Heyting algebra (cf. [15], Theorem 5.2 in [12])), we can also view L^X as a frame, if we confine ourselves to the exclusive application of lattice–theoretic operations on L^X.

Referring to Section 3 in [14] we only consider the case $\otimes = \wedge$ in this paper; in particular an *L–topology τ on X* is always a *subframe* of L^X – i.e. τ is a subset of L^X which is closed under finite meets and arbitrary joins performed in L^X. An L–topology τ on X is called *stratified* iff τ satisfies the additional axiom (cf. Subsection 5.1 in [14])

$(\Sigma 1)\quad g\in\tau,\ \alpha\in L \implies (\alpha\cdot 1_X)*g\in\tau\,.$ (*Truncation Condition*)

In order to formulate the transivity axiom of a stratified L–topology we are forced to enrich the underlying MV–algebra $(L, \leq, *)$ by a further binary operation $\odot$ such that $(L \leq, \odot)$ is a *commutative quantale.* Based on the pair $(*, \odot)$ we are now in the position to introduce a commutative semigroup operation $\boxdot$ on L as follows

$$\alpha \boxdot \beta \quad = \quad ((\alpha \to \bot) \odot (\beta \to \bot)) \to \bot \qquad \forall \alpha, \beta \in L$$

where $\alpha \to \_$ denotes the right adjoint map to $\alpha * \_$. Obviously, $\boxdot$ is distributive over arbitrary meets in $(L, \leq)$; hence the self–mapping $\lambda \rightsquigarrow \lambda \boxdot \beta$ has a left adjoint map $\_ \rhd \beta$. In particular $\alpha \rhd \beta$ is given by

$$\alpha \rhd \beta \quad = \quad \bigwedge \{ \lambda \in L \mid \alpha \leq \lambda \boxdot \beta \} \qquad ;$$

and the following relations hold: $\alpha \leq (\alpha \rhd \beta) \boxdot \beta \quad , \quad (\alpha \boxdot \beta) \rhd \beta \leq \alpha$.

Remark 3.1 (Special cases) (a) Let us consider the case $\odot = \wedge$. Then the semigroup operation $\boxdot$ and the binary join operation $\vee$ coincide; in particular, $\rhd$ is just the *co–implication* in $(L, \leq)$.
(b) Let $\alpha \rightharpoondown \_$ be the right adjoint map to $\alpha \odot \_$. Further we assume that the relation

$$\alpha \to \bot \quad = \quad \alpha \rightharpoondown \bot \qquad (:= \quad \bigvee \{ \lambda \mid \alpha \odot \lambda \; = \; \bot \} \;) \qquad\qquad (\lozenge)$$

holds for all $\alpha \in L$. Then we obtain

$$\begin{aligned} \alpha \quad \leq \quad \lambda \boxdot \beta \qquad & \Longleftrightarrow \qquad (\lambda \to \bot) \odot (\beta \to \bot) \quad \leq \quad \alpha \to \bot \\ & \Longleftrightarrow \qquad \lambda \to \bot \quad \leq \quad (\beta \to \bot) \rightharpoondown (\alpha \to \bot) \\ & \Longleftrightarrow \qquad \lambda \to \bot \quad \leq \quad (\alpha \odot (\beta \to \bot)) \to \bot \\ & \Longleftrightarrow \qquad \alpha \odot (\beta \to \bot) \quad \leq \quad \lambda \qquad ; \end{aligned}$$

hence the relation $\alpha \rhd \beta \; = \; \alpha \odot (\beta \to \bot)$ follows.
(c) In the case of $* = \odot$ the property $(\lozenge)$ (cf. (b)) is trivial. A more interesting example can be given in the case of the real unit interval $[0,1]$: Let T_{-1} be the Łukasiewicz' arithmetic conjunction (cf. Section 2) and $T_{\min}$ Fodor's nilpotent minimum (cf. [4])

$$T_{\min}(\alpha, \beta) \quad = \quad \left\{ \begin{array}{ccc} \min(\alpha, \beta) & , & \alpha + \beta > 1 \\ 0 & , & \alpha + \beta \leq 1 \end{array} \right\} \quad .$$

It is easy to see that $([0,1], \leq, T_{\min})$ is a commutative quantale provided with the property

$$\bigvee \{ \lambda \in [0,1] \mid T_{\min}(\alpha, \lambda) \; = \; 0 \} \quad = \quad 1 - \alpha \qquad \forall \alpha \in [0,1] \, .$$

i.e. the quadruple $([0,1] \leq, T_{-1}, T_{\min})$ fulfills Property $(\lozenge)$ in (b). Finally, in view of the *General Assumptions* in Section 2 we note that $([0,1], \leq, T_{-1}, T_{\min})$ does not satisfy the axioms (L7) and (L8) .

□

A stratified L–topology τ on X is called *transitive* iff τ satisfies the additional axioms

(T1) $\quad g \in \tau\,,\ \alpha \in L \quad \Longrightarrow \quad (\alpha \cdot 1_X) \boxdot g \,\in\, \tau\,.$ (*Translation–invariance*)

(T2) $\quad g \in \tau\,,\ \alpha \in L \quad \Longrightarrow \quad g \rhd (\alpha \cdot 1_X) \,\in\, \tau\,.$ (*Co–Stratification*)

Remark 3.2 (a) In the case of $\odot = \wedge$ the axiom (T1) is redundant (cf. Remark 3.1(a)). Thus a stratified L–topology τ is transitive iff τ is co–stratified – i.e. τ fulfills (T2).
(b) Let us consider the case $\odot = *$ (cf. Remark 3.1(b)). Then the axioms ($\Sigma 1$) and (T2) are equivalent; hence a stratified L–topology τ is transitive iff τ is translation–invariant – i.e. τ satisfies (T1). Finally, referring to Subsection 7.2 in [14] it is easy to see that probabilistic L–topologies and stratified and translation–invariant L–topologies are equivalent concepts.
(c) Let us assume that the underlying lattice $(L, \leq)$ is completely distributive. Then *Raney's Theorem* implies that L is a spatial locale (cf. p. 204/205 in [6], [15]); hence every L–topology on X can be identified with an ordinary topology on $X \times pt(L)$ where $pt(L)$ denotes the set of all frame–morphisms from L to $2 = \{\bot, \top\}$ (cf. Subsection 7.1 in [14]). In this context the stratification, co–stratification and the translation–invariance can be viewed as special 'topological' axioms. We return to this point in Section 5.

□

In the following considerations we identify always L–topologies τ with their corresponding L–interior operators $\mathcal{K}$, respectively with their corresponding L–neighborhood systems $(\mu_p)_{p\in X}$ (cf. Subsection 6.1 in [14]). In particular $\mathcal{K}$ and $(\mu_p)_{p\in X}$ are given by

$$\mathcal{K}(f) \;=\; \bigvee\{g \in \tau \,|\, g \leq f\} \quad , \quad \mu_p(f) \;=\; (K(f))(p) \qquad \forall\, f \in L^X\,.$$

Proposition 3.3 *Let τ be an L–topology on X and $\mathcal{K}$ be the L–interior operator corresponding to τ. Then the following assertions are equivalent:*

(i) τ is a stratified and transitive L–topology.

(ii) $\mathcal{K}$ satisfies the subsequent properties

(K5) $\quad (\alpha \cdot 1_X) * \mathcal{K}(f) \;\leq\; \mathcal{K}((\alpha \cdot 1_X) * f)\quad.$

(K6) $\quad (\alpha \cdot 1_X) \boxdot \mathcal{K}(f) \;=\; \mathcal{K}((\alpha \cdot 1_X) \boxdot f)\quad.$

Proof. It is not difficult to show the equivalence between ($\Sigma 1$) and (K5) (cf. Proposition 6.1.1 in [14]). In order to verify the equivalence between (K6) and (T1),(T2) we proceed as follows: Let us assume that τ is transitive. Then (T1) implies

$$(\alpha \cdot 1_X) \boxdot \mathcal{K}(f) \;\leq\; \mathcal{K}((\alpha \cdot 1_X) \boxdot f)\quad.$$

On the other hand we use $(\alpha \boxdot \beta) \rhd \alpha \leq \beta$ and deduce from (T2)

$$\mathcal{K}((\alpha \cdot 1_X) \boxdot f) \rhd (\alpha \cdot 1_X) \quad \leq \quad \mathcal{K}(f) \qquad \forall f \in L^X .$$

Hence we obtain:

$$\begin{aligned} \mathcal{K}((\alpha \cdot 1_X) \boxdot f) \quad &\leq \quad (\mathcal{K}((\alpha \cdot 1_X) \boxdot f) \rhd (\alpha \cdot 1_X)) \boxdot (\alpha \cdot 1_X) \\ &\leq \quad (\mathcal{K}(f)) \boxdot (\alpha \cdot 1_X) \quad . \end{aligned}$$

Therewith (K6) is established. Further the implication (K6) $\Longrightarrow$ (T1) is evident. In order to verify (K6) $\Longrightarrow$ (T2) we first observe

$$\mathcal{K}(g) \quad \leq \quad \mathcal{K}((g \rhd (\alpha \cdot 1_X)) \boxdot (\alpha \cdot 1_X)) \quad = \quad \big(\mathcal{K}(g \rhd (\alpha \cdot 1_X))\big) \boxdot (\alpha \cdot 1_X);$$

hence (T2) follows from $\mathcal{K}(g) \rhd (\alpha \cdot 1_X) \leq \mathcal{K}(g \rhd (\alpha \cdot 1_X))$.

□

Since probabilistic L–topologies (cf. Remark 3.2(b)) can be identified with crisp systems of L–valued neighborhoods (cf. Subsection 7.2 in [14]), the aim of the following considerations is to extend this result to the case of arbitrary stratified and transitive L–topologies. As the reader will see below the complete distributivity of the underlying lattice plays an important role. First we start with a definition:

Definition 3.4 (L–valued neighborhoods)
Let $(L, \leq, *)$ be a complete MV–algebra and $(L, \leq, \odot)$ be a commutative quantale such that the quadruple $(L, \leq, *, \odot)$ satisfies the axioms (L6) – (L8). Further let X be a non empty set and $\mathbf{U}_p$ be a $\top$–filter on X for all $p \in X$ (cf. Definition 2.8). $(\mathbf{U}_p)_{p \in X}$ is called a *crisp system of L–valued neighborhoods* iff $(\mathbf{U}_p)_{p \in X}$ satisfies the following axioms (cf. Subsection 7.2 in [14], Definition 2.7 in [11])

(𝕌3) $\quad d \in \mathbf{U}_p \quad \Longrightarrow \quad d(p) \;=\; \top$.

(𝕌4)$_\odot$ $\quad \forall p \in X \; \forall d_p \in \mathbf{U}_p \quad \exists \quad \varkappa : \bigcup\limits_{q \in X} \mathbf{U}_p \times \mathbf{U}_q \longmapsto L \quad$ s.t.

(i) $$\bigvee_{\hat{d} \in \mathbf{U}_p} \Big(\bigwedge_{q \in X} \Big(\bigvee_{d_q \in \mathbf{U}_q} \varkappa(\hat{d}, d_q) \Big) \Big) \quad = \quad \top \; .$$

(ii) $$(\hat{d}(q) \odot d_q(x)) * \varkappa(\hat{d}, d_q) \quad \leq \quad d_p(x) \quad \forall x \in X \; .$$

In this context every element $d_p \in \mathbf{U}_p$ is called an *L–valued neighborhood of p*.

□

At a first glance the axiom (𝕌4)$_\odot$ looks a little bit monstrous. Therefore we first make a digression from the principle train of thoughts and give a characterization of (𝕌4)$_\odot$ in two important special cases:

Lemma 3.5 *Let $(L, \leq, *)$ be a complete MV–algebra and X be a non empty set. Further let $(\mathbf{U}_p)_{p \in X}$ be a crisp system of $\top$–filters on X provided with Property $(\mathbb{U}3)$. If $\odot$ and $*$ coincide, then the following assertions are equivalent*

(i) $(\mathbf{U}_p)_{p\in X}$ satisfies $(\mathbb{U}4)_*$.

(ii) $\forall\, p \in X \;\forall\, d_p \in \mathbf{U}_p \;\exists\, d^* \in \mathbf{U}_p \;\forall\, q \in X \;\exists\, d_q \in \mathbf{U}_p$ s.t.
$d^*(q) * d_q(x) \;\leq\; d_p(x) \quad \forall\, x \in X$ (cf. $(\mathbb{U}4)$ in 7.2 in [14]) .

Proof. The implication (ii) $\Longrightarrow$ (i) is obvious. In order to verify (i) $\Longrightarrow$ (ii) we proceed as follows: We define a map $\mathcal{K} : L^X \longmapsto L^X$ by

$$(\mathcal{K}(f))(x) \;=\; \bigvee_{d\in\mathbf{U}_x} \Big(\bigwedge_{z\in X} (d(z) \to f(z)) \Big) \qquad \forall\, f \in L^X .$$

The axiom $(\mathbb{U}3)$ implies $K(f) \leq f$. We show that $\mathcal{K}(f) \leq \mathcal{K}(\mathcal{K}(f))$ – i.e. $\mathcal{K}$ is an L–interior operator on X. For this purpose we fix $x \in X$, $d_x \in \mathbf{U}_x$ and consider a map $\varkappa_x : \bigcup_{q\in X} \mathbf{U}_x \times \mathbf{U}_q \longmapsto L$ provided with Property (i) in $(\mathbb{U}4)_*$. Then we define a map $\bar{\varkappa}_x : \mathbf{U}_{\text{,}} \longrightarrow L$ by

$$\bar{\varkappa}_x(\hat{d}) \;=\; \bigwedge_{q\in X} \Big(\bigvee_{d_q\in\mathbf{U}_q} \varkappa(\hat{d}, d_q) \Big) \qquad \forall\, \hat{d} \in \mathbf{U}_x \quad .$$

Now we are in the position to introduce a further map $\tilde{d} : X \longmapsto L$ as follows

$$\tilde{d}(q) \;=\; \bigvee_{\hat{d}\in\mathbf{U}_x} \bar{\varkappa}_x(\hat{d}) * \hat{d}(q) \qquad \forall\, q \in X \quad .$$

We conclude from Axiom $(\mathbb{F}1)$ and Property (i) in $(\mathbb{U}4)_*$ that $\tilde{d}$ is a L–valued neighborhood of x. Further we infer from Property (ii) in $(\mathbb{U}4)_*$:

$$\begin{aligned}
\bigwedge_{z\in X} (d_x(z) \to f(z)) \;&\leq\; \bigwedge_{d_q\in\mathbf{U}_q} \Big(\bigwedge_{z\in X} \Big(\big(\varkappa(\hat{d}, d_q) * \hat{d}(q) * d_q(z)\big) \to f(z) \Big) \Big) \\
&=\; \bigwedge_{d_q\in\mathbf{U}_q} \Big((\varkappa(\hat{d}, d_q) * \hat{d}(q)) \to \Big(\bigwedge_{z\in X} (d_q(z) \to f(z)) \Big) \Big) \\
&\leq\; \bigwedge_{d_q\in\mathbf{U}_q} \Big((\varkappa(\hat{d}, d_q) * \hat{d}(q)) \to (\mathcal{K}(f))(q) \Big) \\
&=\; \tilde{d}(q) \to (\mathcal{K}(f))(q) \qquad ;
\end{aligned}$$

hence the relation $(\mathcal{K}(f))(x) \;\leq\; (\mathcal{K}(\mathcal{K}(f)))(x)$ follows. Now we turn to the verification of assertion (ii). If d_p is a L–valued neighborhood of p, then we define a map $d^* : X \longmapsto L$ and for all $q \in X$ a map $d_q : X \longmapsto L$ as follows

$$d^*(q) \;=\; (\mathcal{K}(d_p))(q) \quad , \quad d_q(x) \;=\; d^*(q) \to d^*(x) \qquad , \qquad q, x \in X \quad .$$

Evidently the relation $d^*(q) * d_q(x) \le (\mathcal{K}(d_p))(x) \le d_p(x)$ holds. Hence it is sufficient to verify that d^* is a L–valued neighborhood of p and d_q is a L–valued neighborhood of q. Since $\mathcal{K}$ is *idempotent*, we obtain

$$(\mathcal{K}(d^*))(p) = d^*(p) = \top \quad ;$$

hence the map $\varkappa^* : \mathbf{U}_p \longmapsto L$ defined by $\varkappa^*(\hat{d}) = \bigwedge_{x \in X} (\hat{d}(x) \to d^*(x))$ satisfies the following properties

$$\bigvee_{\hat{d} \in \mathbf{U}_p} \varkappa^*(\hat{d}) = \top \quad , \qquad \varkappa^*(\hat{d}) * \hat{d}(x) \le d^*(x) \quad .$$

Because of (F1) the map d^* is an element of $\mathbf{U}_p$. Now we invoke again the idempotence of $\mathcal{K}$ and obtain

$$\bigvee_{\hat{d}_q \in \mathbf{U}_q} \Big(d^*(q) \to \Big(\bigwedge_{x \in X} (\hat{d}_q(x) \to d^*(x))\Big)\Big) = d^*(q) \to (\mathcal{K}(d^*))(q) = \top \quad .$$

Hence the map $\varkappa_q : \mathbf{U}_q \longmapsto L$ defined by

$$\varkappa_q(\hat{d}_q) = d^*(q) \to \Big(\bigwedge_{x \in X} (\hat{d}_q(x) \to d^*(x))\Big)$$

has the property $\bigvee\{\varkappa_q(\hat{d}_q) \mid \hat{d}_q \in \mathbf{U}_q\} = \top$. Further we observe:

$$d^*(q) * \varkappa_q(\hat{d}_q) * \hat{d}_q(x) \le d^*(x) \quad ;$$

i.e. $\varkappa_q(\hat{d}_q) * \hat{d}(x) \le d_q(x)$. Finally, we conclude from (**F**1) that d_q is contained in $\mathbf{U}_q$.

□

Lemma 3.6 *Let $(L, \le, *)$ be a complete MV–algebra and X be a non empty set. Further let $(\mathbf{U}_p)_{p \in X}$ be a crisp system of $\top$–filter on X. If the underlying lattice $(L, \le)$ is completely distributive, then the following assertions are equivalent*

(i) $(\mathbf{U}_p)_{p \in X}$ satisfies $(\mathbb{U}4)_{\odot}$.

(ii) $\forall p \in X \; \forall d_p \in \mathbf{U}_p \; \forall \epsilon \in F(\bot)^2 \; \exists d^* \in \mathbf{U}_p \; \forall q \in X \; \exists d_q \in \mathbf{U}_q$ s.t. $(d^*(q) \odot d_q(x)) * (\epsilon \to \bot) \le d_p(x) \quad \forall x \in X$.

Proof. In any complete MV–algebra, the relation

$$\bigvee_{\epsilon \in F(\bot)} \epsilon \to \bot = \bigvee_{\delta \in F(\bot)} (\delta \to \bot) * (\delta \to \bot)$$

[2]cf. 1.2 in Section 1.

holds. Referring to Lemma 1.1 we observe that for every $\epsilon \in F(\perp)$ there exists $\delta \in F(\perp)$ such that $\epsilon \to \perp \;\leq\; (\delta \to \perp) * (\delta \to \perp)$. Hence the implication (i) $\Longrightarrow$ (ii) follows from the previous formula and Lemma 1.1. On the other hand, let us consider $d_p \in \mathbf{U}_p$ and fix $\epsilon \in F(\perp)$. According to assertion (ii) we determine an element $d_\epsilon^* \in \mathbf{U}_p$ and $d_q \in \mathbf{U}_q$ depending on $q \in X$ such that $(d_\epsilon^* \odot d_q(x)) * (\epsilon \to \perp) \leq d_p(x)$ holds. Now we are in the position to define a map $\varkappa : \bigcup_{q \in X} \mathbf{U}_p \times \mathbf{U}_q \longmapsto L$ as follows:

$$\varkappa(\hat{d}, \hat{d}_q) \quad = \quad \begin{cases} \perp & : \quad \hat{d} \neq d_\epsilon^* \\ \perp & : \quad \hat{d} = d_\epsilon^* \;,\; \hat{d}_q \neq d_q \\ \epsilon \to \perp & : \quad \hat{d} = d_\epsilon^* \;,\; \hat{d}_q = d_q \end{cases} .$$

It is easy to see that $\varkappa$ fulfills the properties (i) and (ii) in $(\mathbb{U}4)_\odot$; hence the implication (ii) $\Longrightarrow$ (i) is verified.

□

Remark 3.7 (a) The axioms of L–valued neighborhoods appear for the first time in the author's paper on *Probabilistic Topologies Induced by L–Fuzzy Uniformities* (cf. [11]). The simplification of Axiom $(\mathbb{U}4)_\odot$ in the case $\odot = *$ (cf. Lemma 3.5) is known since the mid–eighties. Unfortunately, this result remained unpublished for many years.
(b) Let us consider the special case of the real unit interval – i.e. $(L, \leq, *, \odot) = ([0,1], \leq, T_{-1}, \min)$ where T_{-1} denotes the Łukasiewicz arithmetic conjunction (cf. Proposition 2.6). In this context Lemma 3.6 shows that the axioms of [0,1]–valued neighborhoods coincide with the axioms of *Fuzzy Neighborhood Systems* introduced by R. Lowen in 1982 (cf. [18]).

□

After this historical remark we return to our main subject of investigation.

Theorem 3.8 *Let $(\mathbf{U}_p)_{p \in X}$ be a crisp system of L–valued neighborhoods. Further let τ be the set of all $g \in L^X$ satisfying the following property*

$$g(p) \quad \leq \quad \bigvee_{d \in \mathbf{U}_p} \Big(\bigwedge_{x \in X} (d(x) \odot (g(x) \to \perp)) \to \perp \Big) \qquad \forall p \in X \quad .$$

Then τ is a stratified and transitive L–topology on X.

Proof. Referring to Proposition 3.3 it is sufficient to prove that the self–mapping $\mathcal{K} : L^X \to L^X$ defined by

$$(\mathcal{K}(f))(p) \quad = \quad \bigvee_{d \in \mathbf{U}_p} \Big(\bigwedge_{x \in X} (d(x) \odot (f(x) \to \perp)) \to \perp \Big)$$

is an L–interior operator provided with the properties (K5) and (K6).
(a) The isotonocity of $\mathcal{K}$ follows from the construction. Further the axioms

(F2) and (U3) imply: $\mathcal{K}(f_1) \wedge \mathcal{K}(f_2) \leq \mathcal{K}(f_1 \wedge f_2)$, $\mathcal{K}(f) \leq f$. In order to verify the idempotency of $\mathcal{K}$ it is sufficient to show $\mathcal{K}(f) \leq \mathcal{K}(\mathcal{K}(f))$. For this purpose we fix $p \in X$ and $d_p \in \mathbf{U}_p$. Because of $(\mathbb{U}4)_{\odot}$ there exists a map $\varkappa : \bigcup_{q \in X} \mathbf{U}_p \times \mathbf{U}_q \longmapsto L$ provided with the following properties

$$\bigvee_{\hat{d} \in \mathbf{U}_p} \Big(\bigwedge_{q \in X} \Big(\bigvee_{d_q \in \mathbf{U}_q} \varkappa(\hat{d}, d_q) \Big) \Big) \; = \; \top \; . \tag{$\Diamond$}$$

$$(\hat{d}(q) \odot d_q(x)) * \varkappa(\hat{d}, d_q) \;\leq\; d_p(x) \; . \tag{$\Diamond\Diamond$}$$

First we deduce the following relation from (L6), (L8) and ($\Diamond\Diamond$):

$$\begin{aligned} d_p(x) \odot (f(x) \to \bot) \;&\geq\; \Big((\hat{d}(q) \odot d_q(x)) * \varkappa(\hat{d}, d_q) \Big) \odot (f(x) \to \bot) \\ &\geq\; \Big(\hat{d}(q) \odot d_q(x) \odot (f(x) \to \bot) \Big) * \Big(\varkappa(\hat{d}, d_q) \odot \top \Big) \\ &\geq\; \Big(\hat{d}(q) \odot d_q(x) \odot (f(x) \to \bot) \Big) * \varkappa(\hat{d}, d_q) \; . \end{aligned}$$

Because of

$$\begin{aligned} &\Big(\hat{d}(q) \odot \Big(\bigvee_{x \in X} (d_q(x) \odot (f(x) \to \bot)) \Big) \Big) * \varkappa(\hat{d}, d_q) \;\; \geq \\ &\Big(\hat{d}(q) \odot \Big(\bigwedge_{\hat{d}_q \in \mathbf{U}_q} \Big(\bigvee_{x \in X} (\hat{d}_q(x) \odot (f(x) \to \bot)) \Big) \Big) \Big) * \varkappa(\hat{d}, d_q) \end{aligned}$$

we obtain

$$\bigvee_{x \in X} (d_p(x) \odot (f(x) \to \bot)) \;\;\geq\;\; \Big(\hat{d}(q) \odot ((\mathcal{K}(f))(q) \to \bot) \Big) * \Big(\bigvee_{d_q \in \mathbf{U}_q} \varkappa(\hat{d}, d_q) \Big).$$

Since the left side of the previous relation is independent of $q \in X$, we can continue the estimation as follows:

$$\begin{aligned} &\bigvee_{x \in X} (d_p(x) \odot (f(x) \to \bot)) \;\;\geq \\ &\Big(\bigvee_{q \in X} (\hat{d}(q) \odot ((\mathcal{K}(f))(q) \to \bot)) \Big) * \Big(\bigwedge_{q \in X} \Big(\bigvee_{d_q \in \mathbf{U}_q} \varkappa(\hat{d}, d_q) \Big) \Big) \;\;\geq \\ &\Big(\bigwedge_{\tilde{d} \in \mathbf{U}_p} \Big(\bigvee_{q \in X} (\tilde{d}(q) \odot ((\mathcal{K}(f))(q) \to \bot)) \Big) \Big) * \Big(\bigwedge_{q \in X} \Big(\bigvee_{d_q \in \mathbf{U}_q} \varkappa(\hat{d}, d_q) \Big) \Big) \;\;= \\ &\Big((\mathcal{K}(\mathcal{K}(f)))(q) \to \bot \Big) * \Big(\bigwedge_{q \in X} \Big(\bigvee_{d_q \in \mathbf{U}_q} \varkappa(\hat{d}, d_q) \Big) \Big) \; . \end{aligned}$$

Finally, we invoke ($\Diamond$) and obtain:

$$\bigwedge_{x \in X} \Big(\Big(d_p(x) \odot (f(x) \to \bot) \Big) \to \bot \Big) \;\;\leq\;\; (\mathcal{K}(\mathcal{K}(f)))(p) \;\; ;$$

hence the relation $\mathcal{K} \leq \mathcal{K} \circ \mathcal{K}$ is verified.

(b) Since $*$ is distributive over arbitary joins and meets, the axiom (K5) follows from the subsequent relation (cf. Proof of Lemma 2.12):

$$\beta * ((\gamma_1 \odot (\gamma_2 \to \bot)) \to \bot) \;\;\leq\;\; (\gamma_1 \odot ((\beta * \gamma_2) \to \bot)) \to \bot$$

which requires the axioms (L6) and (L8). Further the axioms (L5) and (L7) imply that the operation $\boxdot$ (associated with $\odot$) is also distributive over arbitary joins and meets; hence we obtain:

$$\begin{aligned} \alpha \boxdot (\mathcal{K}(f))(p) &= \alpha \boxdot \Big(\bigvee_{d \in \mathbf{U}_p} \Big(\bigwedge_{x \in X} (d(x) \to \bot) \boxdot f(x) \Big) \Big) \\ &= \bigvee_{d \in \mathbf{U}_p} \Big(\bigwedge_{x \in X} ((d(x) \to \bot) \boxdot (f(x) \boxdot \alpha)) \Big) \quad ; \end{aligned}$$

i.e. (K6) is verified.

□

Motivated by the previous theorem we rise the following *question*: Is every stratified and transitive L–topology induced by a unique crisp system of L–valued neighborhoods in the sense of Theorem 3.8 ? In the case of $\odot = *$ we can answer this question in the affirmative (cf. Remark 3.2(b), Subsection 7.2 in [14]). A further sufficient condition forcing an affirmative answer is the complete distributivity of the underlying lattice as the following theorem demonstrates:

Theorem 3.9 *Let τ be a stratified and transitive L–topology on X, and let $\mathcal{K}$ be the corresponding L–interior operator. If $(L, \leq)$ is completely distributive, then there exists a unique crisp system $(\mathbf{U}_p)_{p \in X}$ of L–valued neighborhoods satisfying the following condition*

$$(\mathfrak{U}) \qquad (\mathcal{K}(f))(p) = \bigvee_{d \in \mathbf{U}_p} \Big(\bigwedge_{x \in X} ((d(x) \odot (f(x) \to \bot)) \to \bot) \Big)$$
$$\forall p \in X \ , \ \forall f \in L^X \ .$$

The proof requires the following lemmata:

Lemma 3.10 *Let $(L, \leq)$ be a completely distributive lattice, X be an arbitrary non empty set, and let $\mathcal{P}(X)$ be the ordinary power set of X. Further, for every $p \in X$ let μ_p be a stratified L–filter satisfying* $(\mathrm{F6})_\odot$ *and let φ_p be the L–filter of crisp subsets corresponding to μ_p. Then the following assertions are equivalent:*

(i) $(\mu_p)_{p \in X}$ is provided with the subsequent properties

$$\text{(U3)} \qquad \mu_p(f) \leq f(p) \qquad \forall f \in L^X .$$
$$\text{(U4)} \qquad \mu_p(f) \leq \bigvee \{ \mu_p(h) \mid h(q) \leq \mu_q(f) \, \forall q \in X \} .$$

(ii) $(\varphi_p)_{p \in X}$ is provided with the subsequent properties

$$\text{(u3)} \qquad \varphi_p(A) = \bot \qquad \text{whenever} \qquad p \notin A \ .$$
$$\text{(u4)} \qquad \varphi_p(A) \leq \bigvee_{q \notin B} \Big(((\varphi_q(A) \to \bot) \odot (\varphi_p(B) \to \bot)) \to \bot \Big)$$
$$\forall B \in \mathcal{P}(X) \ .$$

Proof. (a) ((i) $\Longrightarrow$ (ii)) Because of $\mu_p(1_A) = \varphi_p(A)$ the property (u3) follows immediately from (U3). In order to verify the implication (U4) $\Longrightarrow$ (u4) we fix $A, B \in \mathcal{P}(X)$ and define a map $h : X \longmapsto L$ by

$$h(x) \quad = \quad 1_B(x) \vee \Big(\bigvee_{q \notin B} \varphi_q(A)\Big) \qquad \forall x \in X \quad .$$

Then $\varphi_x(A) \leq h(x)$ holds for all $x \in X$. Since φ_x is the restriction of μ_x to $\mathcal{P}(X) \cong \{\perp, \top\}^X$, we infer from (U4) and (F1):

$$\varphi_p(A) \quad = \quad \mu_p(\mu\_(1_A)) \quad \leq \quad \mu_p(h) \quad .$$

Now we apply $(\mathrm{F6})_\odot$, (L7) and obtain:

$$\begin{aligned} \varphi_p(A) \quad &\leq \quad ((\Big(\bigvee_{q \notin B} \varphi_q(A)\Big) \to \perp) \odot (\varphi_p(B) \to \perp)) \to \perp \\ &= \quad \bigvee_{q \notin B} \Big(((\varphi_q(A) \to \perp) \odot (\varphi_p(B) \to \perp)) \to \perp\Big) \quad . \end{aligned}$$

(b) ((ii) $\Longrightarrow$ (i)) Referring to Theorem 2.5 and Corollary 2.7 the stratified L–filter μ_p is determined by

$$\mu_p(h) \quad = \quad \bigwedge\{\, ((\varphi_p(A) \to \perp) \odot (\alpha \to \perp)) \to \perp \;\mid\; h \leq 1_A \vee \alpha \cdot 1_X \,\} \quad . \quad (\Diamond)$$

Obviously (U3) is an immediate consequence of (u3). Further we fix $h \in L^X$ and define a map $f_0 : X \longmapsto L$ as follows

$$f_0(x) \quad = \quad \mu_x(h) \qquad\qquad \forall x \in X \, .$$

In order to verify (U4) it is sufficient to prove: $\mu_p(h) \leq \mu_p(f_0)$. For this purpose we fix $\epsilon \in F(\perp)$ (cf. Lemma 1.1). According to Theorem 1.3 there exist finitely many $A_i \in \mathcal{P}(X)$ and $\alpha_i \in L$ $(i = 1, \dots, n)$ such that the following relation holds:

$$(\epsilon \to \perp) * \Big(\bigwedge_{i=1}^{n} (1_{A_i} \vee \alpha_i \cdot 1_X)\Big) \quad \leq \quad h \quad \leq \quad \bigwedge_{i=1}^{n} (1_{A_i} \vee \alpha_i \cdot 1_X) \qquad .$$

Step 1: Let $f_i : X \longmapsto L$ be a map defined by $f_i(x) = \mu_x(1_{A_i} \vee \alpha_i \cdot 1_X)$ for all $x \in X$ $(i = 1, \dots, n)$. We show: $\mu_p(h) \leq \mu_p(f_i)$. For this purpose we choose $\varkappa \in L$ and $B \in \mathcal{P}(X)$ with $f_i \leq 1_B \vee \varkappa \cdot 1_X$. Now we apply $(\mathrm{F6})_\odot$ and obtain

$$\Big(((\bigvee_{x \notin B} \varphi_x(A_i)) \to \perp) \odot (\alpha_i \to \perp)\Big) \to \perp \quad \leq \quad \varkappa \qquad ;$$

hence the relation

$$\begin{aligned} ((\varphi_p(B) \to \perp) \odot (\varkappa \to \perp)) \to \perp \quad &\geq \quad ((\varphi_p(A_i) \to \perp) \odot (\alpha_i \to \perp)) \to \perp \\ &\geq \quad \mu_p(h) \end{aligned}$$

follows from (F1) and (u4). Finally, we use Formula $(\lozenge)$ and obtain the desired inequality: $\mu_p(h) \le \mu_p(f_i)$.
Step 2 : It is easy to see that for all $x \in X$ the relation

$$(\epsilon \to \bot) * \bigwedge_{i=1}^{n} f_i(x) \quad \le \quad (\epsilon \to \bot) * \mu_x(\bigwedge_{i=1}^{n}(1_{A_i} \vee \alpha_i \cdot 1_X)) \quad \le \quad f_0(x)$$

is an immediate consequence from (F2) and (F4). Now we apply again (F2), (F4) and obtain from *Step 1*:

$$(\epsilon \to \bot) * \mu_p(h) \quad \le \quad (\epsilon \to \bot) * \Big(\bigwedge_{i=1}^{n} \mu_p(f_i)\Big) \quad \le \quad \mu_p(f_0) \quad .$$

Since $\epsilon \in F(\bot)$ is arbitrary, the inequality $\mu_p(h) \le \mu_p(f_0)$ follows.

□

Lemma 3.11 *Let $(L, \le)$ be a completely distrbutive lattice, X be a non empty set, and let $\mathcal{P}(X)$ be the ordinary power set of X. Further, for every p in X let φ_p be an L-filter of crisp subsets of X and let $\mathbf{U}_p$ be the $\top$-filter corresponding to φ_p (cf. Theorem 2.11). If $(\varphi_p)_{p\in X}$ satisfies Axiom $(\mathrm{u4})_\odot$, then $(\mathbf{U}_p)_{p\in X}$ fulfills the following property*

$$(\mathfrak{P}) \qquad \left\{ \begin{array}{l} \forall f \in \mathbf{U}_p \ \forall (\gamma,\delta,\epsilon) \in L^3 \ \textit{with} \ \gamma \odot \epsilon \not\le \delta \ \exists \hat{f} \in \mathbf{U}_p \\ \forall q \in \{x \in X \mid \gamma \le \hat{f}(x)\} \ \exists d_q \in \mathbf{U}_q \ \textit{s.t.} \\ \{x \in X \mid \epsilon \le d_q(x)\} \quad \subseteq \quad \{x \in X \mid f(x) \not\le \delta\} \quad . \end{array} \right\}$$

Proof. We assume that Property $(\mathfrak{P})$ is false – i.e. there exists an element $f_0 \in \mathbf{U}_p$ and a triple $(\gamma,\delta,\epsilon) \in L^3$ with $\gamma \odot \epsilon \not\le \delta$ such that *for all* $\hat{f} \in \mathbf{U}_p$ we can choose an element $q_{\hat{f}} \in X$ provided with the following properties:

$$\gamma \quad \le \quad \hat{f}(q_{\hat{f}}) \quad ,$$

$$\{x \in X \mid \epsilon \le d_{q_{\hat{f}}}(x)\} \cap \{x \in X \mid f_0(x) \le \delta\} \quad \neq \quad \varnothing \qquad \forall d_{q_{\hat{f}}} \in \mathbf{U}_{q_{\hat{f}}} \ .$$

Now we are in the position to introduce the following crisp subsets of X

$$E \quad = \quad \{x \in X \mid f_0(x) \le \delta\} \quad , \qquad F \quad = \quad \{q_{\hat{f}} \mid \hat{f} \in \mathbf{U}_p\} \quad .$$

Referring to the map ξ defined in Section 2 (cf. Lemma 2.10, Theorem 2.11) we obtain:

$$\delta \to \bot \le \varphi_p(\complement E) \quad , \quad \varphi_p(\complement F) \le \gamma \to \bot \quad , \quad \varphi_q(\complement E) \le \epsilon \to \bot \quad \forall q \in F \, ;$$

hence Axiom (u4) implies the estimation

$$\begin{aligned} \delta \to \bot \quad &\le \quad \varphi_p(\complement E) \le \bigvee_{q \in F} \Big(((\varphi_q(\complement E) \to \bot) \odot (\varphi_p(\complement F) \to \bot)) \to \bot \\ &\le \quad (\epsilon \odot \gamma) \to \bot\Big) \end{aligned}$$

which is inconsistent with the choice of $(\gamma, \delta, \epsilon)$.

$\square$

Proof of Theorem 3.9. Let τ be a stratified and transitive L–topology on X, and let $\mathcal{K}$ (resp. $(\mu_p)_{p\in X}$) be the corresponding L–interior operator (resp. L–neighborhood system) on X. Since τ is stratified and transitive, we conclude from Proposition 2.13 and Proposition 3.3 that for every L–neighborhood filter μ_p (at p) there exists a unique $\top$–filter $\mathbf{U}_p$ on X satisfying the following condition

$$(\mathcal{K}(h))(p) \quad = \quad \mu_p(h) \quad = \quad \bigvee_{d\in\mathbf{U}_p} \Big(\bigwedge_{x\in X} (d(x) \odot (h(x) \to \bot)) \to \bot \Big) \quad \forall\, h \in L^X.$$

The aim of the following considerations is to show that $(\mathbf{U}_p)_{p\in X}$ is a crisp system of L–valued neighborhoods. For this purpose let φ_p be the L–filter of crisp subsets corresponding to μ_p (resp. $\mathbf{U}_p$). Since $(\mu_p)_{p\in X}$ is an L–neighborhood system, $(\mu_p)_{p\in X}$ fulfills the conditions (U3) and (U4) in Lemma 3.10 (cf. Subsection 6.1 in [14]). By the construction of the map ξ (cf. Proof of Theorem 2.11) the following relation

$$d(p) \to \bot \quad \leq \quad (\xi(\mathbf{U}_p))(X \setminus \{p\}) \quad = \quad \varphi_p(X \setminus \{p\})$$

holds for all $d \in \mathbf{U}_p$. Now we invoke Lemma 3.10 and obtain

$$d(p) \to \bot \quad = \quad \bot \qquad \forall\, d \in \mathbf{U}_p \qquad ;$$

i.e. $(\mathbf{U}_p)_{p\in X}$ satisfies Axiom $(\mathbb{U}3)$. In order to establish Axiom $(\mathbb{U}4)_\odot$ it is sufficient to verify the assertion (ii) in Lemma 3.6. We divide this proof into two parts:

(a) We fix $p \in X$. For every element $\varkappa \in L \setminus \{\top\}$ and every element $\check{d} \in \mathbf{U}_p$ we introduce a map $\check{d}_\varkappa : X \longmapsto L$ by

$$\check{d}_\varkappa(x) \quad = \quad \left\{ \begin{array}{ccc} \top & : & \check{d}(x) \not\leq \varkappa \\ \varkappa & : & \check{d}(x) \leq \varkappa \end{array} \right\} \quad ,$$

and we prove the following

Assertion.
$$\begin{gathered} \forall\, \epsilon \in F(\bot)\ \exists\, d_\epsilon \in \mathbf{U}_p\ \forall\, q \in X\ \exists\, d_q \in \mathbf{U}_q \text{ s.t.} \\ (d_q(x) \odot d_\epsilon(q)) * (\epsilon \to \bot) \quad \leq \quad \check{d}_\varkappa(x) \qquad \forall\, x \in X\,. \end{gathered}$$

Let us consider an element $\epsilon \in F(\bot)$ with $(\epsilon \to \bot) \to \varkappa \neq \top$. Then there exists a further element $\tilde{\epsilon} \in F(\bot)$ with $(\tilde{\epsilon} \to \bot) \to \tilde{\epsilon} \leq \epsilon$ (cf. Proof of Lemma 3.6). For every $\beta \in L$ with $\beta \not\leq (\tilde{\epsilon} \to \bot) \to \varkappa$ we define an element $\bar{\beta} \in L$ by

$$\bar{\beta} \quad = \quad \bigwedge \{\lambda \in L \mid \ \beta \odot \lambda \not\leq (\tilde{\epsilon} \to \bot) \to \varkappa\} \qquad .$$

Then Lemma 1.1 implies $\beta \odot \bar{\beta} \not\leq \varkappa$. Further we invoke Lemma 3.11 and obtain in the case of $f = \check{d}_\varkappa$ that there exists $\hat{d}_\beta \in \mathbf{U}_p$ provided with the property

$$\begin{array}{lll} \forall q \in V_\beta & := \{x \in X \mid \beta \le \hat{d}_\beta(x)\} & \exists\, d_q \in \mathbf{U}_q \text{ s.t.} \\ \{x \in X \mid \bar{\beta} \le d_q(x)\} & \subseteq & \{x \in X \mid \check{d}_\varkappa(x) \not\le \varkappa\} \end{array} \qquad (\Diamond)$$

Now we introduce a map $d^*_\varkappa : X \longmapsto L$ as follows:

$$d^*_\varkappa(q) \;=\; ((\tilde{\epsilon} \to \bot) \to \varkappa) \vee \left(\bigvee\{\beta \in L \mid q \in V_\beta\,,\ \beta \not\le (\tilde{\epsilon} \to \bot) \to \varkappa\}\right).$$

Step 1: We show $d^*_\varkappa \in \mathbf{U}_p$. For every $\gamma \in F(\bot)$ we choose $g_\gamma \in L^X$ provided with the following properties (cf. Theorem1.3):

$$g_\gamma(x) * (\gamma \to \bot) \;\le\; d^*_\varkappa(x) \;\le\; g_\gamma(x) \qquad \forall x \in X\,.$$
$$g_\gamma(X) \;=\; \{\delta_1, \delta_2, \dots, \delta_n\}\,.$$

Now we define a map $g_i : X \longmapsto L$ by $(i = 1, \dots, n)$:

$$g_i(x) \;=\; \left\{\begin{array}{lcl} \top & : & g_\gamma(x) \not\le \delta_i \\ \delta_i & : & g_\gamma(x) \le \delta_i \end{array}\right\} \qquad \forall x \in X\,.$$

Since $g_\gamma = \bigwedge_{i=1}^{n} g_i$, we conclude from (𝔽1) and (𝔽2) that it is sufficient to show $g_i \in \mathbf{U}_p$ for all $i = 1, \dots, n$. We assume the contrary – i.e. there exists $i_0 \in \{1, \dots, m\}$ s.t. $g_{i_0} \notin \mathbf{U}_p$. Referring to the relationship between φ_p and $\mathbf{U}_p$ (determined by the map ξ, cf. Theorem 2.11) we observe that there exists a subset E of X with the following property

$$\bigwedge_{x \in E} (g_{i_0}(x) \to \bot) \;\not\le\; \varphi(\complement E)\,.$$

If we put $\chi = \varphi_p(\complement E)$, then we obtain:

$$\chi \neq \top\,, \qquad E \subseteq \{x \in X \mid g_{i_0}(x) \le \delta_{i_0}\} \qquad (\Diamond\Diamond)$$

In particular, E is non empty (cf. Axiom ($\mathfrak{f}1$)), and the relation $\delta_{i_0} \to \bot \not\le \chi$ holds. Further we choose an arbitrary element $x \in E$ and conclude from the definition of $d^*_\varkappa$ and the choice of g_γ:

$$(\tilde{\epsilon} \to \bot) \to \varkappa \;\le\; d^*_\varkappa(x) \;\le\; \delta_{i_0}\,;$$

hence because of Lemma 1.1 there exists an element $\iota \in F(\bot)$ such that the following relations are valid

$$(\chi \to \bot) * (\iota \to \bot) \;\not\le\; (\tilde{\epsilon} \to \bot) \to \varkappa\,, \qquad (\chi \to \bot) * (\iota \to \bot) \not\le \delta_{i_0}\,.$$

Now we put $\beta = (\chi \to \bot) * (\iota \to \bot)$. Since $\hat{d}_\beta$ is an element of $\mathbf{U}_p$, we use again the special relationship between $\mathbf{U}_p$ and φ_p and obtain:

$$\chi \to \bot \;=\; \varphi_p(\complement E) \to \bot \;\le\; \bigvee_{x \in E} \hat{d}_\beta(x)\,;$$

hence there exists an element $x_0 \in E$ with $\beta \leq \hat{d}_\beta(x_0)$ (cf. Lemma 1.1). Because of $\beta \not\leq (\tilde{\epsilon} \to \bot) \to \varkappa$ we use again the definition of $d^*_\varkappa$ and obtain: $\beta \leq d^*_\varkappa(x_0)$. Since $\beta \not\leq \delta_{i_0}$ and $d^*_\varkappa \leq g_{i_0}$, the relation $g_{i_0}(x_0) \not\leq \delta_{i_0}$ follows which is a contradiction to $g_{i_0}(x_0) \leq \delta_{i_0}$ (cf. Relation $(\Diamond\Diamond)$). Therefore the assumption is false, and g_{i_0} is an element of $\mathbf{U}_p$.

Step 2 : We show

Assertion'.
$$\forall q \in X \quad \exists\, d_q \in \mathbf{U}_q \quad \text{s.t.} \quad (d_q(x) \odot d^*_\varkappa(q)) * (\epsilon \to \bot) \quad \leq \quad \check{d}_\varkappa(x) \qquad \forall x \in X \quad .$$

In the case of $d^*_\varkappa(q) \leq (\tilde{\epsilon} \to \bot) \to \varkappa$ the *Assertion'* is evident. Therefore we assume $d^*_\varkappa(q) \not\leq (\tilde{\epsilon} \to \bot) \to \varkappa$ and conclude from Lemma 1.1 that there exists an element $\beta \in L$ provided with the following properties:

$$d^*_\varkappa(q) * (\tilde{\epsilon} \to \bot) \quad \leq \quad \beta \vee ((\tilde{\epsilon} \to \bot) \to \varkappa) \quad , \quad \beta \quad \not\leq \quad (\tilde{\epsilon} \to \bot) \to \varkappa$$
$$q \in V_\beta \quad = \quad \{x \in X \mid \beta \leq \hat{d}_\beta(x)\} \quad .$$

In particular, because of $(\Diamond)$ there exists $d_q \in \mathbf{U}_q$ such that the relation

$$\{x \in X \mid \bar{\beta} \leq d_q(x)\} \quad \subseteq \quad \{x \in X \mid \check{d}_\varkappa(x) \not\leq \varkappa\} \qquad (\blacklozenge)$$

holds. In the case of $\check{d}_\varkappa(x_0) = \top$ there is nothing to show. Therefore we assume $\check{d}_\varkappa(x_0) \neq \top$ – i.e. $\check{d}_\varkappa(x_0) = \varkappa$. Then the relation $(\blacklozenge)$ implies: $\bar{\beta} \not\leq d_q(x_0)$; hence $d_q(x_0) \odot \beta \leq (\tilde{\epsilon} \to \bot) \to \varkappa$ follows from the definition of $\bar{\beta}$. Now we observe $\epsilon \to \bot \leq (\tilde{\epsilon} \to \bot) * (\tilde{\epsilon} \to \bot)$ and use again (L6) and (L8):

$$\begin{aligned}
(d_q(x_0) \odot d^*_\varkappa(x_0)) * (\epsilon \to \bot) \quad &\leq \quad \Big(d_q(x_0) \odot (d^*_\varkappa(x_0) * (\tilde{\epsilon} \to \bot))\Big) * (\tilde{\epsilon} \to \bot) \\
&\leq \quad \Big(d_q(x_0) \odot (((\tilde{\epsilon} \to \bot) \to \varkappa) \vee \beta)\Big) * (\tilde{\epsilon} \to \bot) \\
&\leq \quad ((\tilde{\epsilon} \to \bot) \to \varkappa) * (\tilde{\epsilon} \to \bot) \quad \leq \quad \varkappa \quad .
\end{aligned}$$

Therewith the *Assertion'* is verified. Finally the *Assertion* follows from *Step 1* and *Step 2*.

(b) Let us consider $d \in \mathbf{U}_p$ and $\epsilon \in F(\bot)$. Then we choose $\delta \in F(\bot)$ with $(\delta \to \bot) \to \delta \leq \epsilon$. By virtue of Theorem 1.3 there exists a map $\check{d} \in L^X$ equipped with the subsequent properties:

$$\begin{aligned}
\check{d}(x) * (\delta \to \bot) \quad &\leq \quad d(x) \quad \leq \quad \check{d}(x) \qquad \forall x \in X \\
\check{d}(X) \quad &= \quad \{\varkappa_1, \varkappa_2, \dots, \varkappa_n\} \quad .
\end{aligned}$$

For each $i \in \{1, 2, \dots, n\}$ we define a map $\check{d}_{\varkappa_i} \longmapsto L$ by

$$\check{d}_{\varkappa_i} \quad = \quad \begin{Bmatrix} \top & : & \check{d}(x) \not\leq \varkappa_i \\ \varkappa_i & : & \check{d}(x) \leq \varkappa_i \end{Bmatrix} \quad .$$

We conclude from the *Assertion* in part (a) that there exists an element $d^*_{\varkappa_i} \in \mathbf{U}_p$ such that the following relation holds:

$$\forall q \in X \ \exists d^{(i)}_q \in \mathbf{U}_q \ \text{ s.t. } \ (d^{(i)}_q(x) \odot d^*_{\varkappa_i}(q)) * (\delta \to \bot) \quad \leq \quad \check{d}_{\varkappa_i}(x) \quad ;$$

hence we obtain for all $x \in X$:

$$\begin{array}{l}\Big(\Big(\bigwedge\limits_{i=1}^{n} d_q^*(x)\Big) \odot \Big(\bigwedge\limits_{i=1}^{n} d_{\varkappa_i}^*(q)\Big)\Big) * (\epsilon \to \bot) \quad \leq \\ \Big(\bigwedge\limits_{i=1}^{n} \check{d}_{\varkappa_i}(x)\Big) * (\delta \to \bot) \quad = \quad \check{d}(x) * (\delta \to \bot) \quad \leq \quad d(x)\end{array} \quad . \qquad (\blacklozenge\blacklozenge)$$

Because of ($\mathbb{F}2$) the map $\bigwedge\limits_{i=1}^{n} d_q^{(i)}$ (resp. $\bigwedge\limits_{i=1}^{n} d_{\varkappa_i}^*$) is an element of $\mathbf{U}_q$ (resp. $\mathbf{U}_p$) ; hence the assertion (ii) in Lemma 3.6 follows from $(\blacklozenge\blacklozenge)$.

□

Remark 3.12 (L–fuzzy contiguity relations) Let $(L, \leq, \odot)$ be a strictly two–sided, commutative quantale. Further let X be an arbitrary non empty set and $\mathcal{P}(X)$ be the (ordinary) power set of X. A map $c : X \times \mathcal{P}(X) \longmapsto L$ is called an *L-fuzzy contiguity relation on the space* X iff c fulfills the following axioms (cf. [13]):

(C1) $\quad c(p, E) \;=\; \top \quad$ whenever $\quad p \in E$.

(C2) $\quad c(p, \varnothing) \;=\; \bot \quad$ for all $\quad p \in X$.

(C3) $\quad c(p, E \cup F) \;=\; c(p, E) \vee c(p, F)$. (*Distributivity*)

(C4) $\quad \Big(\bigwedge\limits_{q \in F} c(q, E)\Big) \odot c(p, F) \;\leq\; c(p, E)$. (*Transitivity*)

If c is an L–fuzzy contiguity relation on X, then the value $c(p, E)$ can be interpreted as the *degree* to which the (crisp) point p is *contiguous* to the (crisp) subset E of X. In this sense the axioms (C1) – (C4) form an immediate fuzzification of the usual contiguity axioms (cf. [1]).
Now we assume a further binary operation $*$ on L such that $(L, \leq, *)$ is a complete MV–algebra and $(L, \leq, *, \odot)$ satisfies the conditions (L5) – (L8). If the underlying lattice $(L, \leq)$ is completely distributive, then we conclude from Lemma 3.10 and Theorem 3.9 that *stratified and transitive L–topologies*, *L–fuzzy contiguity relations* and crisp systems of *L–valued neighborhoods* satisfying ($\mathbb{F}1$), ($\mathbb{F}2$), ($\mathbb{U}3$), $(\mathbb{U}4)_\odot$ are respectively equivalent concepts. In particular, the list of their mutual relationsships looks like as follows:

$$\begin{array}{rcl}(\mathcal{K}(f))(p) & = & \bigvee\{ g(p) \mid g \in \tau \,,\, g \leq f \} \\ (\mathcal{K}(f))(p) & = & \bigvee\limits_{d \in \mathbf{U}_p} \Big(\bigwedge\limits_{x \in X} (d(x) \odot (f(x) \to \bot)) \to \bot\Big) \\ (\mathcal{K}(1_A))(p) & = & c(p, \complement A) \to \bot \\ \mathbf{U}_p & = & \{ d \in L^X \mid c(p, A) \leq \bigvee\limits_{x \in A} d(x) \quad \forall A \in \mathcal{P}(X) \} \quad .\end{array}$$

Finally, the axiom $(C4)$ justifies to understand the *conjunction* of translation–invariance and co–stratification as the *transitivity axiom* of L–topologies.

□

We close this section with the presentation of some results in the *case* of $\odot = \wedge$. First we recall the concept of fuzzifying topologies[3] which appeared in [10] under the name 'L–fuzzy topology' (cf. Definition 4.6, Proposition 4.11 in [10]): Let $\mathcal{P}(X)$ be the ordinary power set of a given set X; a map $\mu : \mathcal{P}(X) \longmapsto L$ is called an *L–fuzzifying topology on X* iff μ satisfies the following conditions:

$$(\mathcal{O}1) \qquad \mu(X) \;=\; \mu(\varnothing) \;=\; \perp \quad .$$

$$(\mathcal{O}2) \qquad \mu(A) \wedge \mu(B) \;\leq\; \mu(A \cap B) \quad .$$

$$(\mathcal{O}3) \qquad \bigwedge_{i \in I} \mu(A_i) \;\leq\; \mu(\bigcup_{i \in I} A_i) \qquad \forall \, \{A_i \mid i \in I\} \subseteq \mathcal{P}(X) \quad .$$

Proposition 3.13 *Let $(L, \leq)$ be a completely distributive lattice. Then every L–fuzzifying topology induces a system $(\varphi_p)_{p \in X}$ of L–filters φ_p of crisp subsets of X by*

$$\varphi_p(A) \;=\; \bigvee \{\, \mu(G) \mid p \in G \subseteq A \} \qquad \forall p \in X \, , \, \forall A \in \mathcal{P}(X) \quad .$$

Moreover $(\varphi_p)_{p \in X}$ satisfies the axioms (u3), (u4) *(cf. Lemma 3.10), and the following relation holds*

$$\mu(A) \;=\; \bigwedge_{p \in A} \varphi_p(A) \qquad \forall A \in \mathcal{P}(X) \quad . \qquad (\blacklozenge)$$

Proof. The axioms (f1) , (f2) and (u3) follow immediately from $(\mathcal{O}1)$, $(\mathcal{O}2)$ and the construction of φ_p . Further we fix $p \in X$ and consider subsets A, B, G of X with $p \in G \subseteq A$. We distinguish the following cases

$$\left. \begin{array}{rcl} \bigvee_{q \notin B} \varphi_q(A) \;\geq\; \mu(G) & : & G \cap \complement B \;\neq\; \varnothing \\ \varphi_p(B) \;\geq\; \mu(G) & : & G \cap \complement B \;=\; \varnothing \end{array} \right\} ;$$

hence the axiom (u4) follows. The inequality $\mu(A) \;\leq\; \bigwedge_{p \in A} \varphi_p(A)$ is trivial. On the the other hand, we apply $(\mathcal{O}3)$, the complete distributivity of the underlying lattice $(L, \leq)$ and obtain

$$\begin{aligned} \bigwedge_{p \in A} \varphi_p(A) \;&=\; \bigwedge_{p \in A} \Big(\bigvee \{\, \mu(G) \mid p \in G \subseteq A \,\} \Big) \\ &=\; \bigvee \{\, \bigwedge_{p \in A} \mu(G_p) \;\mid\; (G_p)_{p \in A} \text{ with } p \in G_p \subseteq A \,\} \\ &\leq\; \bigvee \{\, \mu(\bigcup_{p \in A} G_p) \;\mid\; (G_p)_{p \in A} \text{ with } p \in G_p \subseteq A \,\} \\ &=\; \mu(A) \qquad . \end{aligned}$$

Hence the formula $(\blacklozenge)$ is verified.

$\square$

[3]In the case of $L = [0,1]$ this terminology traces back to Mingsheng Ying (cf. [30])

Proposition 3.14 *Let $(L, \leq)$ be a completely distributive lattice. Further let $(\varphi_p)_{p \in X}$ be a system of L–filters φ_p of crisp subsets of X satisfying* (u3) *and* (u4). *Then $(\varphi_p)_{p \in X}$ induces an L–fuzzifying topology μ on X by*

$$\mu(A) \quad = \quad \bigwedge_{p \in A} \varphi_p(A) \qquad\qquad \forall A \in \mathcal{P}(X) \quad .$$

Moreover the following formula holds

$$\varphi_p(A) \quad = \quad \bigvee \{ \mu(G) \mid p \in G \subseteq A \} \qquad \forall p \in X \,,\, \forall A \in \mathcal{P}(X) \quad . \qquad (\blacklozenge\blacklozenge)$$

Proof. The axioms $(\mathcal{O}1)$ and $(\mathcal{O}2)$ follow immediately from $(\mathfrak{f}1)$ and $(\mathfrak{f}2)$. Further let $\{A_i \mid i \in I\}$ be a family of subsets of X; then the isotonicity of L–filters of crisp subsets implies

$$\bigwedge_{i \in I} \mu(A_i) \;=\; \bigwedge_{i \in I} \Big(\bigwedge_{p \in A_i} \varphi_p(A_i) \Big) \;\leq\; \bigwedge_{i \in I} \Big(\bigwedge_{p \in A_i} \varphi_p(\bigcup_{i \in I} A_i) \Big) \;=\; \mu(\bigcup_{i \in I} A_i) \;;$$

hence the axiom $(\mathcal{O}3)$ is verified. The inequality

$$\bigvee \{ \mu(G) \mid p \in G \subseteq A \} \quad \leq \quad \varphi_p(A)$$

is trivial. In the case of $p \notin A$ the axiom (u3) shows that in the previous relation the equality sign holds. Therefore we assume $p \in A$ and consider the set Υ of all maps $\zeta : \{G \mid p \in G \subseteq A\} \longmapsto X$ with $\zeta(G) \in G$. Now we fix $\zeta \in \Upsilon$ and put $\varkappa_\zeta = \bigvee \{ \varphi_{\zeta(G)}(G) \mid p \in G \subseteq A \}$. Further let F be the set of all $q \in X$ with $\varphi_q(A) \leq \varkappa_\zeta$. We show that F contains p. Let us assume the contrary; then $p \in G_0 = A \cap \complement F$. In particular, we put $s = \zeta(G_0)$. Then the axioms (u3) and (u4) imply:

$$\varphi_s(A) \quad \leq \quad \varphi_s(G_0) \vee \Big(\bigvee_{q \notin G_0} \varphi_q(A) \Big) \quad \leq \quad \varkappa_\zeta \quad ;$$

hence $s \in F$ which is a contradiction to $s = \zeta(G_0) \in G_0$. Therefore the assumption is false; hence the relation

$$\varphi_p(A) \qquad \leq \qquad \varkappa_\zeta \qquad\qquad \forall \zeta \in \Upsilon \quad . \qquad (\lozenge)$$

holds. Now we invoke the complete distributivity of $(L, \leq)$ and derive the following relation from the definition of Υ and Formula $(\lozenge)$:

$$\begin{aligned} \varphi_p(A) \;\leq\; \bigwedge_{\zeta \in \Upsilon} \varkappa_\zeta \;&=\; \bigwedge_{\zeta \in \Upsilon} \Big(\bigvee \{ \varphi_{\zeta(G)}(G) \mid p \in G \subseteq A \} \Big) \\ &=\; \bigvee \{ \bigwedge_{q \in G} \varphi_q(G) \mid p \in G \subseteq A \} \;=\; \bigvee \{ \mu(G) \mid p \in G \subseteq A \} ; \end{aligned}$$

hence the relation $(\blacklozenge\blacklozenge)$ is verified.

$\square$

Corollary 3.15 *Let $(L, \leq, *)$ be a complete MV–algebra and $\odot = \wedge$. Further let $(L, \leq)$ be a completely distributive lattice. Then L–fuzzifying topologies, L–fuzzy contiguity relations and stratified and transitive L–topologies are equivalent concepts.*

Proof. The assertion follows immediately from Proposition 3.3, Lemma 3.10, Remark 3.12, Proposition 3.13 and Proposition 3.14.

□

Remark 3.16 (a) We assume the hypothesis of Corollary 3.15 , and consider a stratified, transitive L–topology τ on X. Further, let μ be the L–fuzzifying topology on X and c be the L–fuzzy contiguity relation on X corresponding to $\tau$and let τ (cf. Corollary 3.15). Then the mutual relationships between τ, c, μ are given by (cf. (♦) and (♦♦) in 3.13 and 3.14):

$$\begin{array}{ccccc} \bigvee\{\, \mu(G) \mid p \in G \subseteq A\} & = & c(p, \complement A) \rightarrow \bot & = & (\mathcal{K}_\tau(1_A))(p) \\ \bigwedge\limits_{p \in A} (\mathcal{K}_\tau(1_A))(p) & = & \mu(A) & = & \bigwedge\limits_{p \in A} \Big(c(p, \complement A) \rightarrow \bot\Big) \end{array}$$

where $\mathcal{K}_\tau$ denotes the L–interior operator associated with τ. Moreover, we conclude from Theorem 3.9 that every L–fuzzifying topology on X can be characterized by a *unique* crisp system of L–valued neighborhoods.
(b) If the underlying lattice $(L, \leq)$ is given by the real unit interval $([0,1], \leq)$, then Proposition 3.13 and Proposition 3.14 are closely related to Theorem 3.1 and Theorem 3.2 in [30]. Even though the formulas (♦) and (♦♦) appear in Mingsheng Ying's work, the important transitivity axiom (u4) (cf. (C4) in Remark 3.12) is missing.
(c) In the case of arbitrary complete MV–algebras (without any additional assumption on the underlying lattice) the equivalence between L–fuzzifying topologies and L–fuzzy contiguity relations remains valid, if we use appropriate modifications of the axioms (C4) and ($\mathcal{O}$3). For details the reader is referred to Section 4 in [10].

□

Remark 3.17 (Case $\odot = \wedge$)
Let $(L, \leq, *)$ be a complete MV–algebra, and $\mathbb{B}(L)$ be the *Booleanization* of the underlying lattice $(L, \leq)$ (cf. Subsection 1.4 in [14]). Every stratified L–topology is weakly stratified (cf. Subsection 5.1 in [14]) – i.e. every stratified L–topology τ fulfills the so–called *constants conditions*

$$(\Sigma 0) \qquad (\alpha \cdot 1_X) * g \in\in \tau \qquad \forall \alpha \in L \quad .$$

Further let us consider the important case $\odot = \wedge$. Then, under the hypothesis of the stratification axiom transitivity and co–stratification of L–topologies are equivalent concepts (cf. Remark 3.2(a)). Hence we conclude from Proposition 7.4.15 in [14] that every stratified and transitive L–topology can be identified with its Booleanziation. An important consequence of this fact is the possibility to describe stratified and transitive L–topologies by crisp systems of $\mathbb{B}(L)$–valued neighborhoods. Details of this construction can be found in Subsection 7.4 in [14].

On the other hand, if the underlying lattice is *completely distributive*, then we conclude from Theorem 3.8 and Theorem 3.9 that every stratified and transitive L–topology can also be identified with a crisp system of L–valued neighborhoods. Hence in the case of completely distributive lattices there exist at least *two different* characterizations of stratified and transitive L–topologies by *many–valued* neighborhoods.

We finish this remark with a simple example in which we compute explicitly the $\mathbb{B}(L)$–valued and respectively L–valued neighborhoods of a given stratified and transitive L–topology:

Example. We emphasize that $(L, \leq)$ is a completely distributive lattice, and we denote the implication in the Boolean algebra $\mathbb{B}(L)$ by $\Rightarrow$. Further let X be a non empty set. Then

$$\tau \quad = \quad \{ (\alpha \cdot 1_X) \vee f \mid \alpha \in L \, , \, f \in L^X \text{ with } f \; \leq \; \beta \cdot 1_X \} \qquad\qquad (\beta \in L)$$

is a stratified and co–stratified (resp. transitive) L–topology on X. Indeed, the stratification and co–stratification axiom follow from the subsequent relations:

$$\begin{array}{rclcrclc} & (\alpha_1 \vee \alpha_2) \rhd \gamma & = & (\alpha_1 \rhd \gamma) \vee (\alpha_2 \rhd \gamma) & & & & \\ (\alpha \cdot 1_X) * h & \leq & h \quad , & h \rhd \alpha & \leq & h & \forall\, h \in L^X & . \end{array}$$

Obviously the L–interior operator $\mathcal{K}$ corresponding to τ is given by

$$(\mathcal{K}(h))(p) \quad = \quad \Big(\bigwedge_{x \in X} h(x) \Big) \vee (h(p) \wedge \beta) \qquad\qquad \forall\, p \in X \quad .$$

Then it is not difficult to show that the system $\mathbf{U}_p$ (resp. $\mathbf{V}_p$) of L–valued (resp. $\mathbb{B}(L)$–valued) neighborhoods of p are given by

$$\begin{array}{rcl} \mathbf{U}_p & = & \{ d \in L^X \mid d(p) = \top \, , \; \beta \to \bot \; \leq \; d(x) \;\; \forall\, x \in X \} \quad , \\ \mathbf{V}_p & = & \{ f \in \mathbb{B}(L)^X \mid f(p) = \top \, , \; \beta \Rightarrow \bot \; \leq \; f(x) \;\; \forall\, x \in X \} \quad . \end{array}$$

□

4 Categorical properties of stratified and transitive L–topological spaces

Let $(L, \leq, *)$ be a complete MV–algebra. A pair (X, τ) is a *stratified L–topological space* iff X is a set and τ is a stratified L–topology on X. Let (X, τ) and (Y, σ) be stratified L–topological spaces; a map $\Phi : X \longmapsto Y$ is *L–continuous* (cf. Section 3 in [14]) iff Φ satisfies the following condition

$$\{ g \circ \Phi \mid g \in \sigma \} \quad \subseteq \quad \tau \quad .$$

In an obvious way stratified L–topological spaces and L–continuous maps form a category denoted by $\mathbf{S}L_{\wedge}$**–TOP**. Here we emphasize again that in contrast to the general concept of stratified L–topological spaces (cf. Section 3 and Subsection 5 in [14]) we *restrict* ourselves to the *case* $\otimes = \wedge$ – i.e. stratified L–topologies on X form always subframes of L^X.

Further let $\odot$ be a commutative semigroup operation on L such that the quadruple $(L, \leq, *, \odot)$ satisfies the axioms (L5) – (L8). A pair (X, τ) is called a *stratified and transitive L–topological space* iff X is a set and τ is a stratified and transitive L–topology on X. Again stratified and transitive L–topological spaces and L–continuous maps form a category denoted by $\mathbf{ST}_{\odot}$**–TOP**. In the special case $\odot = *$ we observe that $\mathbf{ST}_{*}$**–TOP** and the category $\mathbf{P}L$**–TOP** of probabilistic L–topological spaces coincide (cf. Remark 3.2(b)).

Theorem 4.1 *The category* $\mathbf{ST}_{\odot}$**–TOP** *is a coreflective subcategory of* $\mathbf{S}L_{\wedge}$**–TOP**. *If in addition* $(L \leq, \odot)$ *satisfies the algebraic strong De Morgan law, then* $\mathbf{ST}_{\odot}$**–TOP** *is also a bireflective subcategory of* $\mathbf{S}L_{\wedge}$**–TOP**.

Proof. It is sufficient to show that for any stratified L–topology σ on X there exists the coarsest, stratified and transitive L–topology τ_0 on X with $\sigma \subseteq \tau_0$ and the finest, stratified and transitive L–topology τ_∞ with $\tau_\infty \subseteq \sigma$. Since the axioms of stratified and transitive L–topologies on a given set X are preserved under arbitrary intersections, the existence of τ_0 is trivial. In order to show the existence of τ_∞ we proceed as follows: Let $\mathcal{K}_\sigma$ be the L–interior operator corresponding to σ; then we consider the set $\mathfrak{K}$ of all L–interior operators provided with the subsequent properties

- $\mathcal{K}(f) \;\leq\; \mathcal{K}_\sigma(f) \qquad \forall f \in L^X$.
- $\mathcal{K}$ satisfies (K5) and (K6) .

Since the coarsest stratified L–topology is also transitive, the set $\mathfrak{K}$ is non empty. Further let $\mathfrak{J}_0(\mathfrak{K})$ be the set of all non empty, finite subsets $\mathbf{H}$ of $\mathfrak{K}$. Then the map $\mathcal{K}_\infty : L^X \longmapsto L^X$ defined by

$$\mathcal{K}_\infty(h) \;=\; \bigvee\{ \bigwedge_{\mathcal{K}\in\mathbf{H}} \mathcal{K}(f_{\mathcal{K}}) \;\mid\; \mathbf{H} \in \mathfrak{J}_0(\mathfrak{K}) \,,\; \bigwedge_{\mathcal{K}\in\mathbf{H}} f_{\mathcal{K}} \;\leq\; h\}$$

is again an L–interior operator satisfying (K5) . In order to verify (K6) we proceed as follows: We fix $\varkappa \in L$, $h \in L^X$, $\mathbf{H} \in \mathfrak{J}_0(\mathfrak{K})$ and choose $f_{\mathcal{K}} \in L^X$ with $\bigwedge\limits_{\mathcal{K}\in\mathbf{H}} f_{\mathcal{K}} \leq h \boxdot \varkappa \cdot 1_X$. Then we define a map $\bar{f}_{\mathcal{K}} : X \longmapsto L$ by

$$\bar{f}_{\mathcal{K}}(x) \;=\; \Big((\varkappa \rightarrow \bot) \rightharpoondown (f_{\mathcal{K}}(x) \rightarrow \bot)\Big) \rightarrow \bot \qquad \forall x \in X$$

where $\alpha \rightharpoondown \_$ is the right adjoint map to $\alpha \odot \_$. Now we apply the algebraic strong De Morgan law (cf. Proposition 2.3. in [12]) and obtain:

$$\bigvee_{\mathcal{K}\in\mathbf{H}} (\bar{f}_{\mathcal{K}}(x) \rightarrow \bot) \;=\; \bigvee_{\mathcal{K}\in\mathbf{H}} (\varkappa \rightarrow \bot) \rightharpoondown (f_{\mathcal{K}}(x) \rightarrow \bot)$$

$$
\begin{aligned}
&= (\varkappa \to \bot) \rightharpoondown \Big(\bigvee_{\mathcal{K}\in \mathbf{H}} (f_{\mathcal{K}}(x) \to \bot) \Big) \\
&\geq (\varkappa \to \bot) \rightharpoondown ((\varkappa \to \bot) \odot (h(x) \to \bot)) \\
&\geq h(x) \to \bot \qquad ;
\end{aligned}
$$

i.e. $\bigwedge_{\mathcal{K}\in \mathbf{H}} \bar{f}_{\mathcal{K}} \leq h$. Further it is not difficult to verify $f_{\mathcal{K}} \leq \varkappa \cdot 1_X \boxdot \bar{f}_{\mathcal{K}}$. Summing up we get

$$
\begin{aligned}
\bigwedge_{\mathcal{K}\in \mathbf{H}} \mathcal{K}(f_{\mathcal{K}}) &\leq \bigwedge_{\mathcal{K}\in \mathbf{H}} \mathcal{K}(\varkappa \cdot 1_X \boxdot \bar{f}_{\mathcal{K}}) = \varkappa \cdot 1_X \boxdot \Big(\bigwedge_{\mathcal{K}\in \mathbf{H}} \mathcal{K}(\bar{f}_{\mathcal{K}}) \Big) \\
&\leq \varkappa \cdot 1_X \boxdot \mathcal{K}_\infty(h) \qquad ;
\end{aligned}
$$

hence the inequality $\mathcal{K}_\infty(\varkappa \cdot 1_X \boxdot h) \leq \varkappa \cdot 1_X \boxdot \mathcal{K}_\infty(\varkappa \cdot 1_X \boxdot h)$ follows. Since the converse inequality is trivial, the axiom (K6) is established. Finally, the subsequent relation

$$
\mathcal{K} \leq \mathcal{K}_\infty \leq \mathcal{K}_\sigma \qquad \forall \mathcal{K} \in \mathfrak{K}
$$

follows from the construction of $\mathcal{K}_\infty$ and the usual L–interior operator axioms (K0) – (K2) (cf. case $\otimes = \wedge$ in Subsection 6.1 in [14]); hence the L–topology τ_∞ corresponding to $\mathcal{K}_\infty$ fulfills the desired properties.

□

Sometimes it is not easy to compute explicitly the coreflection of a stratified, L–topological space in $\mathbf{ST}_\odot L$–$\mathbf{TOP}$. In the case of $\odot = *$ there exists a simple solution as the following proposition demonstrates:

Proposition 4.2 *Let (X, τ) be a stratified, L–topological space, and let (X, τ_0) be the coreflection of (X, τ) in $\mathbf{ST}_* L$–$\mathbf{TOP}$ (= $\mathbf{P}L$–$\mathbf{TOP}$). Further let σ_0 be the subframe of L^X generated given by $\{\alpha \to g \mid g \in \tau,\ \alpha \in L\}$. Then τ_0 and σ_0 coincide.*

Proof. We note that in any complete MV–algebra the following relations are valid

$$
\text{(i)} \quad \alpha \to (\beta_1 \wedge \beta_2) = (\alpha \to \beta_1) \wedge (\alpha \to \beta_2) \quad .
$$

$$
\text{(ii)} \quad \alpha \to \Big(\bigvee_{i\in I} \beta_i \Big) = \bigvee_{i\in I} (\alpha \to \beta_i) \quad .
$$

$$
\text{(iii)} \quad \alpha * (\beta \to \gamma) = (\alpha \to \beta) \to ((\beta \to \alpha) * (\beta \wedge \gamma)) \quad .
$$

The relations (i) and (ii) are well known (cf. Proposition 2.1, Proposition 2.8, Proposition 5.2(c) in [14]). Further the divisibility axiom of MV–algebras implies (cf. Lemma 2.5, Lemma 2.14 in [14]):

$$
\begin{aligned}
(\alpha \to \beta) * \alpha * (\beta \to \gamma) &= (\alpha \wedge \beta) * (\beta \to \gamma) = (\beta \to \alpha) * \beta * (\beta \to \gamma) \\
&= (\beta \to \alpha) * (\beta \wedge \gamma) \quad .
\end{aligned}
$$

Since $(\alpha \to \beta) \to \bot \;=\; \alpha * (\beta \to \bot) \;\leq\; \alpha * (\beta \to \gamma)$, we obtain from Proposition 2.15 in [14]:

$$\begin{aligned} \alpha * (\beta \to \gamma) &= (\alpha \to \beta) \to \Big((\alpha \to \beta) * \alpha * (\beta \to \gamma)\Big) \\ &= (\alpha \to \beta) \to ((\beta \to \alpha) * (\beta \wedge \gamma)) \qquad ; \end{aligned}$$

therewith the relation (iii) is verified. Now we return to a stratified L–topology on X; then the previous relations (i) – (iii) show that the subframe σ_0 of L^X generated by

$$\{\alpha \to g \mid g \in \tau,\; \alpha \in L\}$$

is a probabilistic (i.e. stratified and translation–invariant) L–topology on X (see also Remark 3.2(b)). In particular τ_0 is the coarsest probabilistic L–topology containing τ.

□

The next lemma gives a characterization of L–continuity in the case of stratified and transitive L–topological spaces.

Lemma 4.3 (Characterization of L–continuity)
Let $(L, \leq)$ be a completely distributive lattice and $(L, \leq, *, \odot)$ be a quadruple satisfying (L5) – (L8). Further, let (X, τ) and (Y, σ) be stratified and transitive L–topological spaces, and let $(\mathbf{U}_p)_{p\in X}$ and $(\mathbf{V}_q)_{q\in Y}$ be their corresponding crisp system of L–valued neighborhoods. Then for every $\Phi : X \longmapsto Y$ the following assertions are equivalent

(i) $\Phi : (X, \tau) \longmapsto (Y, \sigma)$ is L–continuous .

(ii) $d \circ \Phi \in \mathbf{U}_p \quad \forall\, d \in \mathbf{V}_{\Phi(p)} \qquad \forall\, p \in X$.

Proof. The L–interior operator corresponding to τ (resp. σ) is denoted by $\mathcal{K}_\tau$ (resp. $\mathcal{K}_\sigma$). Then the L–continuity of Φ is equivalent to the following relation

$$(\mathcal{K}_\sigma(f)) \circ \Phi \quad \leq \quad \mathcal{K}_\tau(f \circ \Phi) \qquad \forall\, f \in L^Y \quad . \tag{$\Diamond$}$$

Because of Theorem 3.8 and Theorem 3.9 it is easy to see that the implication (ii) $\Longrightarrow$ (i) holds. In order to verify (i) $\Longrightarrow$ (ii) we proceed as follows: Because of Theorem 1.3 and the axioms ($\mathbb{F}$1) and ($\mathbb{F}$2) it is sufficient to establish (ii) in the case of $d = 1_B \vee \varkappa \cdot 1_X$ where $B \subseteq Y$ and $d \in \mathbf{V}_{\Phi(p)}$. Referring to the relationship between L–interior operators, L–fuzzy contiguity relations and crisp systems of L–valued neighborhoods (cf. Remark 3.12) we have to show

$$\bigwedge_{x \notin A} (1_B(\Phi(x)) \vee \varkappa) \to \bot \quad \leq \quad (\mathcal{K}_\tau(1_A))(p) \qquad \forall\, A \subseteq X \quad . \tag{$\Diamond\Diamond$}$$

In the case of $\complement A \cap \Phi^{-1}(B) \neq \varnothing$ there is nothing to prove. Therefore we assume: $\Phi^{-1}(B) \subseteq A$. Now we invoke ($\Diamond$) and obtain

$$\begin{aligned} \varkappa = \bigwedge_{y \notin B} 1_B(y) \vee \varkappa) \;&\leq\; (\mathcal{K}_\sigma(1_B))(\Phi(p)) \;\leq\; (\mathcal{K}_\tau(1_{\Phi^{-1}(B)}))(p) \\ &\leq\; (\mathcal{K}_\tau(1_A))(p) \qquad ; \end{aligned}$$

hence $(\Diamond\Diamond)$ follows.

$\square$

In the following considerations we always consider a completely distributive lattice $(L,\leq)$ and two quadruples $(L,\leq,*,\odot_1)$ and $(L,\leq,*,\odot_2)$ satisfying (L5) – (L8) (see *General Assumptions* in Section 2). Further we assume that $\odot_2 \leq \odot_1$ – i.e. $\alpha \odot_2 \beta \leq \alpha \odot_1 \beta \;\; \forall \alpha, \beta \in L$. Then there exists a functor $\mathcal{E}_{12} : \mathbf{ST}_{\odot_1}L\text{–}\mathbf{TOP} \longmapsto \mathbf{ST}_{\odot_2}L\text{–}\mathbf{TOP}$ defined as follows: Let (X,τ) be an object of $\mathbf{ST}_{\odot_1}\text{–}\mathbf{TOP}$, and $(\mathbf{U}_p)_{p\in X}$ be the corresponding crisp system of L–valued neighborhoods. Since the axiom $(\mathbb{U}4)_{\odot_1}$ implies always $(\mathbb{U}4)_{\odot_2}$, we infer from Theorem 3.8 that $(\mathbf{U}_p)_{p\in X}$ induces w.r.t $\odot_2$ an L–interior operator $\mathcal{K}_{12}$ by

$$(\mathcal{K}_{12}(f))(p) \quad = \quad \bigvee_{d\in \mathbf{U}_p} \Big(\bigwedge_{x\in X} ((d(x) \odot_2 (f(x) \to \bot)) \to \bot)\Big) \qquad \forall f \in L^X \quad .$$

If we denote the corresponding L–topology by τ_{12}, then the actions of $\mathcal{E}_{12}$ on objects and morphisms can be specified as follows:

$$\mathcal{E}_{12}(X,\tau) \quad = \quad (X,\tau_{12}) \qquad , \qquad \mathcal{E}_{12}(\Phi) \quad = \quad \Phi \qquad .$$

Because of Theorem 3.9 it is easy to see that $\mathcal{E}_{12}$ is even an *embedding functor.*

Theorem 4.4 *The embedding functor* $\mathcal{E}_{12} : \mathbf{ST}_{\odot_1}L\text{–}\mathbf{TOP} \longmapsto \mathbf{ST}_{\odot_2}L\text{–}\mathbf{TOP}$ *has a left adjoint. In particular, the unit of this adjoint situation is a natural bimorphism.*

Proof. In order prove that $\mathcal{E}_{12}$ has a left adjoint it is sufficient to show that for any object $(X,\sigma) \in \mathbf{ST}_{\odot_2}L\text{–TOP}$ there exists w.r.t. $\odot_1$ the finest stratified and transitive L–topology τ_∞ on X such that $id_X : (X,\sigma) \longmapsto \mathcal{E}_{12}(X,\tau_\infty)$ is L–continuous. For this purpose let $\mathfrak{T}$ be the family of all stratified L–topologies τ on X provided with the following properties:

- τ is transitive w.r.t. $\odot_1$.
- $id_X : (X,\sigma) \longmapsto \mathcal{E}_{12}(X,\tau)$ is L–continuous .

Further, let $(\mathbf{U}_p^{(\tau)})_{p\in X}$ be the crisp system of L–valued neighborhoods corresponding to τ, and let $(\mathbf{V}_p)_{p\in X}$ be again the crisp system of L–valued neighborhoods corresponding to σ. Because of Lemma 4.3 the L–continuity of the identity map $id_X : (X,\sigma) \longmapsto \mathcal{E}_{12}(X,\tau)$ is equivalent to $\mathbf{U}_p^{(\tau)} \subseteq \mathbf{V}_p \quad \forall p \in X$. Since $\mathbf{V}_p$ satisfies the axiom $(\mathbb{F}2)$, the $\top$–filter $\mathbf{U}_p^\infty$ generated by $\cup\{\mathbf{U}_p^{(\tau)} \mid \tau \in \mathfrak{T}\}$ exists and is contained in $\mathbf{V}_p$. It is not difficult to show that $(\mathbf{U}_p^\infty)_{p\in X}$ is w.r.t. $\odot_1$ a crisp system of L–valued neighborhoods. We denote the corresponding stratified and transitive L–topology by τ_∞. Referring again to Lemma 4.3 we obtain that $id_X : (X,\sigma) \longmapsto \mathcal{E}_{12}(X,\tau_\infty)$ is L–continuous; hence τ_∞ fulfills the desired properties. In particular the unit of this adjoint situation is a natural bimorphism.

$\square$

Corollary 4.5 *The category* $\mathbf{ST}_{\odot_1}L\text{–}\mathbf{TOP}$ *is isomorphic to a bireflectice subcategory of* $\mathbf{ST}_{\odot_2}L\text{–}\mathbf{TOP}$. *In particular, every category* $\mathbf{ST}_{\odot}L\text{–}\mathbf{TOP}$ *is isomorphic to a bireflective subcategory of the category of probabilistic L–topological spaces.*

Proof. The assertion follows immeadiately from Theorem 4.4.

□

5 Case of the real unit interval

Various authors use the real unit interval [0,1] as the underlying lattice for investigations in fuzzy topology. Therefore it seems to be reasonable to devote an entire section to this special case.

In order to have the usual operations $+$ and $-$ at our disposal we always view [0,1] as a complete MV–algebra – i.e. the operation $*$ on $[0,1]$ is given by the Łukasiewicz' arithmetic conjunction:

$$\alpha * \beta \quad = \quad T_{-1}(\alpha,\beta) \quad = \quad \max(\alpha+\beta-1\,,0) \qquad \forall\, \alpha,\beta \in [0,1] \quad .$$

Proposition 5.1 *Let* $\odot = \min$, *and let* τ *be a* $[0,1]$*–topology on* X. *Then the following assertions are equivalent:*

(i) τ is stratified and transitive .

(ii) τ is weakly stratified and co–stratified – i.e. τ fulfills the following properties:

(Σ0) $\quad \alpha \cdot 1_X \in \tau \qquad \forall\, \alpha \in [0,1]$. (*Constant Conditions*)

(T2) $\quad g \in \tau \implies g \rhd (\alpha \cdot 1_X) \in \tau \quad \forall\, \alpha \in [0,1]$. (*Co–Stratification*)

Proof. The implication (i) $\implies$ (ii) is obvious. In order to verify (ii) $\implies$ (i) it is sufficient to show that τ is stratified. Because of

$$\begin{aligned} (g \rhd \alpha)(x_0) \quad &= \quad \inf\{\lambda \in [0,1] \mid g(x_0) \le \max(\lambda,\alpha)\} \\ &= \quad \min\Big(g(x_0)\ ,\ 1_{\{x\,:\,\alpha<g(x)\}}(x_0)\Big) \end{aligned}$$

we obtain

$$\begin{aligned} T_{-1}(\alpha, g(x_0)) \quad &= \quad \sup_{\beta\in[0,1[} \min\Big(\beta\ ,\ 1_{\{x\,:\,\beta<T_{-1}(\alpha,g(x))\}}(x_0)\Big) \\ &= \quad \sup_{\beta\in[0,1[} \min\Big(\beta\ ,\ 1_{\{x\,:\,1-\alpha+\beta<g(x)\}}(x_0)\,,\ g(x_0)\Big) \\ &= \quad \sup_{\beta\in[0,1[} \min\Big(\beta\ ,\ (g \rhd (1-\alpha+\beta))(x_0)\Big) \qquad ; \end{aligned}$$

hence the stratification axiom follows from ($\Sigma 0$) and (T2) .

□

Because of Proposition 5.1 we can conclude from Theorem 3.8 and Theorem 3.9 that in the case of $\odot = \min$ weakly stratified and co-stratified, $[0,1]$-topological spaces and *fuzzy neighborhood spaces* (introduced by Lowen in [18]) are equivalent concepts. This result was first established by P. Wuyts, R. Lowen and E. Lowen [29] (see also Theorem 2.6 in [16]). The characterization of weakly stratified and co-stratified $[0,1]$-topologies by *Boolean valued* (resp. $\mathbb{B}([0,1])$-valued) *neighborhoods* is new (cf. Remark 3.17). Further it follows from Proposition 3.3 and Proposition 5.1 that $[0,1]$-interior operators corresponding to fuzzy neighborhood spaces are characterized by the simple condition

$$\text{(K6)} \quad \max(\alpha\,,\,(\mathcal{K}(f))(x)) \quad = \quad (\mathcal{K}(\max(f\,,\,\alpha\cdot 1_X))(x) \qquad \forall x \in X$$

which is dual to

$$\text{(F)} \quad \min(\alpha\,,\,(\mathcal{I}(f))(x)) \quad = \quad (\mathcal{I}(\min(f\,,\,\alpha\cdot 1_X))(x) \qquad \forall x \in X$$

where $\mathcal{I}$ denotes the corresponding *closure operator.* The condition (F) appeared independently in [20, 21] and in [19] and is an important simplification of the condition (C′) given originally by R. Lowen in [18]. Moreover, we conclude from Corollary 3.15 (cf. Remark 3.16(a)) that $[0,1]$-fuzzifying topologies (cf. [30]) and $[0,1]$-topologies corresponding to fuzzy neighborhood spaces are equivalent concepts. In this context we now exhibit a close relationship between α-cuts of $[0,1]$-fuzzifying topologies on one hand and α-level topologies associated with fuzzy neighborhood spaces on the other hand.

Let τ be a $[0,1]$-topology on X. Then for every $\alpha \in [0,1[$ the set

$$\iota_\alpha(\tau) \quad = \quad \{g^{-1}(]\alpha,1]) \quad | \quad g \in \tau\}$$

is an ordinary topology on X. Following the terminology proposed by R. Lowen and P. Wuyts (cf. [17, 28]) we call $\iota_\alpha(\tau)$ the *α-level topology associated with* τ. If τ is co-stratified w.r.t. $\odot = \min$, then it is not difficult to show that the corresponding system $(\iota_\alpha(\tau))_{\alpha\in[0,1[}$ is nested – i.e.

$$\beta \;\leq\; \alpha \qquad \Longrightarrow \qquad \iota_\alpha(\tau) \;\subseteq\; \iota_\beta(\tau) \qquad .$$

On the other hand, if μ is a $[0,1]$-fuzzifying topology on X, then the *α-cuts μ_α of μ*

$$\mu_\alpha \quad = \quad \{G \in \mathcal{P}(X) \mid \alpha \leq \mu(G)\} \qquad\qquad (\alpha \in]0,1])$$

form also ordinary topologies on X (cf. Corollary 2.1 in [30]).

Proposition 5.2 *Let μ be a $[0,1]$-fuzzifying topology on X and let τ be the corresponding weakly stratified and co-stratified $[0,1]$-topology associated with a given fuzzy neighborhood system. Then for all $\alpha \in]0,1]$ the following relation holds*[4]

$$\mu_\alpha \quad = \quad \bigcap_{\beta\in[0,\alpha[} \iota_\alpha(\tau) \qquad .$$

[4]More details on α-cuts and $\mathcal{V}_D$-closure operators (cf. [1]) can be found in [13].

Proof. Let $\mathcal{K}_\tau$ be the $[0,1]$–interior operator corresponding to τ, and let G be an element of $\iota_\beta(\tau)$. We choose $g \in \tau$ with $G = \{x \in X \mid \beta < g(x)\}$. Since τ is co–stratified w.r.t. $\odot = \min$, we obtain:

$$g \rhd \beta \quad = \quad g \wedge 1_G \quad = \quad g \wedge \mathcal{K}_\tau(1_G) \qquad ;$$

i.e. $\beta \leq \inf_{q \in G} (\mathcal{K}_\tau(1_G))(q)$; hence $\beta \leq \mu(G)$ follows from Remark 3.16(a). Therefore $\bigcap_{\beta \in [0,\alpha[} \iota_\beta(\tau)$ is contained in μ_α. On the other hand, if $A \in \mu_\alpha$, then we obtain (cf. Remark 3.16(a))

$$\alpha \quad \leq \quad (\mathcal{K}_\tau(1_A))(q) \qquad \forall q \in A \qquad , \qquad (\mathcal{K}_\tau(1_A))(q) \quad = \quad 0 \qquad \forall q \notin A\,;$$

hence $A = \{x \in X \mid \beta < (\mathcal{K}_\tau(1_A))(x)\}$ for all $\beta < \alpha$ – i.e. μ_α is also a subset of $\bigcap_{\beta \in [0,\alpha[} \iota_\beta(\tau)$.

$\square$

By virtue of Corollary 4.5 and Proposition 5.1 the category of fuzzy neighborhood spaces is isomorphic to a bireflective subcategory of the category of probabilistic [0,1]–topological spaces. In the following considerations we clarify the problem to which extend fuzzy neighborhood spaces and probabilistic $[0,1]$–topological spaces can be considered as ordinary topological spaces. Since $[0,1]$ is a spatial locale (cf. [15]) – in particular every element $\alpha \in [0,1[$ is prime, it is well known that $[0,1]$– topological spaces (X,τ) can be identified with ordinary topological spaces $(X \times [0,1[\,, \mathbb{T}_\tau)$ where $\mathbb{T}_\tau$ is given by (cf. Remark 3.2(c), p. 453 in [17], p.249 in [24], Remark 7.1.5 in [14]):

$$\mathbb{T}_\tau \quad = \quad \{\,\{(x,\alpha) \mid \alpha < g(x)\} \;\mid\; g \in \tau\} \qquad .$$

A subset G of $X \times [0,1[$ is called $[0,1[$*–stable* iff G satisfies the following conditions

(S1) $\quad (x,\alpha) \in G$ and $\beta \leq \alpha \quad \Longrightarrow \quad (x,\beta) \in G$.

(S2) $\quad \forall\, (x,\alpha) \in G \;\exists\, \gamma > \alpha$ s.t. $(x,\gamma) \in G$.

It is easy to see that every map $g : X \longmapsto [0,1]$ determines a $[0,1[$–stable subset G_g of $X \times [0,1[$ by $G_g = \{(x,\alpha) \mid \alpha < g(x)\}$, and vice versa every $[0,1[$–stable subset G of $X \times [0,1[$ induces a map $g_G : X \longmapsto [0,1]$ by

$$g_G(x) \quad = \quad \sup\{\alpha \in [0,1[\mid (x,\alpha) \in G\} \qquad .$$

In particular, the following relations are valid: $G_{g_G} = G$ and $g_{G_g} = g$; i.e. $[0,1[$–stable subsets of $X \times [0,1[$ and $[0,1]$–valued maps with domain X are equivalent concepts.

As immediate consequence of the foregoing considerations we can state the following

Lemma 5.3 *Let $\mathbb{T}$ be an ordinary topology on $X \times [0,1[$. Then the following assertions are equivalent:*

(i) Every set $G \in \mathbb{T}$ is $[0,1[$–stable .

(ii) There exists a *unique* $[0,1]$–topology τ on X s.t. $\mathbb{T} = \mathbb{T}_\tau$.

□

Proposition 5.4 *Let $\mathbb{T}$ be an ordinary topology on $X \times [0,1[$ provided with the property that every element $G \in \mathbb{T}$ is $[0,1[$–stable. Further let τ be the $[0,1]$–topology corresponding to $\mathbb{T}$. Then the following assertions are equivalent:*

(i) τ is weakly stratified and co–stratified w.r.t. $\odot = \min$ (i.e τ is a $[0,1]$–topology of a fuzzy neighborhood space).

(ii) $\mathbb{T}$ fulfills the additonal properties

$$\text{(ws)} \qquad X \times [0,\alpha[\in \mathbb{T} \qquad \forall \alpha \in]0,1] \ .$$

$$\text{(cos)} \quad \begin{cases} G \in \mathbb{T} \quad \Longrightarrow \\ \{(x,\alpha) \in X \times [0,1[\mid (x, \max(\alpha,\beta)) \in G\} \in \mathbb{T} \quad \forall \beta \in [0,1[\ . \end{cases}$$

Proof. The equivalence between (ws) and the weak stratification axiom is evident. The equivalence between (cos) and the co–stratification axiom follows from

$$(g \rhd \beta)(x) \quad = \quad \sup\{\alpha \in [0,1[\mid \alpha < g(x) \ , \ \beta \leq \alpha\} \qquad \forall x \in X \quad .$$

□

Proposition 5.5 *Let $\mathbb{T}$ be an ordinary topology on $X \times [0,1[$ provided with the property that every element $G \in \mathbb{T}$ is $[0,1[$–stable. Further let τ be the $[0,1]$–topology on X corresponding to $\mathbb{T}$. Then the following assertions are equivalent:*

(i) τ is a probabilistic $[0,1]$–topology on X

(ii) $\mathbb{T}$ fulfills the additonal properties

$$\text{(s)} \quad G \in \mathbb{T} \quad \Longrightarrow \quad \{(x, \alpha+\beta-1) \mid (x,\beta) \in G \ , \ 1-\alpha \leq \beta\} \in \mathbb{T} \quad \forall \alpha \in]0,1] \ .$$

$$\text{(t)} \quad \begin{cases} G \in \mathbb{T} \quad \Longrightarrow \\ \{(x, 1-\alpha+\beta) \mid (x,\beta) \in G, \ \beta < \alpha\} \cup (X \times [0,1-\alpha[) \in \mathbb{T} \\ \forall \alpha \in]0,1] \ . \end{cases}$$

Proof. The equivalence of (i) and(ii) follows from the subsequent relations:

$$\begin{aligned} \max(g(x)+\alpha-1, 0) &= \sup\{\alpha+\beta-1 \mid \beta < g(x) \ , 1-\alpha \leq \beta\} \\ \min(1-\alpha+g(x), 1) &= \max(\sup\{1-\alpha+\beta \mid \beta < g(x), \ \beta < \alpha\}, 1-\alpha) \end{aligned}$$

where g denotes a $[0,1]$–valued map with domain X.

$\square$

We close this section with an important class of examples of stratified and transitive $[0,1]$–topological spaces generated by probabilistic metric spaces.

Example 5.6 Let T be a continuous t-norm which dominates the Łukasiewicz' arithmetic conjunction T_{-1} – i.e. the quadruple $([0,1],\leq,T_{-1},T)$ satisfies the axioms (L5) – (L8) (cf. Proposition 2.6). Further let $\mathcal{D}^+$ be the set of all nonnegative probability distribution functions – i.e.

$$\mathcal{D}^+ \quad = \quad \{F \in [0,1]^{\mathbb{R}} \mid F(0) = 0 \,,\; \sup_{r'<r} F(r') = F(r) \,,\; \sup_{n\in\mathbb{N}} F(n) = 1\}$$

A map $\mathfrak{F} : X \times X \longmapsto \mathcal{D}^+$ is called a *probabilistic metric* on X w.r.t. T iff $\mathfrak{F}$ satisfies the following axioms

(PM1) $\quad (\mathfrak{F}(x,y))(\frac{1}{n}) \;=\; 1 \;\; \forall n \in \mathbb{N} \iff x = y\,.$ (*Identity*)

(PM2) $\quad \mathfrak{F}(x,y) \;=\; \mathfrak{F}(y,x)\,.$ (*Symmetry*)

(PM3) $\quad T((\mathfrak{F}(x,y))(r_1)\,,\,(\mathfrak{F}(y,z))(r_2)) \;\leq\; (\mathfrak{F}(x,z))(r_1+r_2)\,.$ (*Triangle–Inequality*)

A *Menger space*[5] is a triple $(X,\mathfrak{F},T)$ where X is a set , and $\mathfrak{F}$ is a probabilistic metric on X w.r.t. T. In the case of $\odot = T$ every probabilistic metric $\mathfrak{F}$ on X induces a crisp system $(\mathbf{U}_p)_{p\in X}$ of $[0,1]$–valued neighborhoods by (cf. Theorem 4.8 and Proposition 6.3 in [11]):

$$\mathbf{U}_p \;=\; \{f \in [0,1]^X \mid \forall n \in \mathbb{N}\; \exists m_n \in \mathbb{N} : (\mathfrak{F}(p,q))(\tfrac{1}{m_n}) - \tfrac{1}{n} \;\leq\; f(q) \quad \forall q \in X\}.$$

It is not difficult to show that $(\mathbf{U}_p)_{p\in U}$ fulfills the axioms $(\mathbb{F}1)$, $(\mathbb{F}2)$, $(\mathbb{U}3)$ and $(\mathbb{U}4)_T$ (cf. Lemma 3.6). Hence $(\mathbb{U}_p)_{p\in X}$ can be identified with a stratified and transitive $[0,1]$–topology τ (cf. Theorem 3.8 and Theorem 3.9). Summing up every Menger space $(X,\mathfrak{F},T)$ (provided T dominates T_{-1}) determines a stratified and transitive $[0,1]$–topological space. Concrete examples of Menger spaces can be found in [9] and in Subsection 7.2. in [14].

$\square$

6 Concluding Remark

Probabilistic L–topological spaces play a dominant role in the theory of those L–topological spaces which can be characterized by crisp systems of L–valued neighborhoods (cf. Corollary 4.5). It is remarkable to see that the theory of

[5]Details of the theory of probabilistic metric spaces can be found in [27].

probabilistic L–topological spaces including its characterization by systems of L–valued neighborhoods is only based on complete MV–algebras $(L, \leq, *)$ and does not require the complete distributivity of the underlying lattice $(L, \leq)$ (cf. Subsection 7.2 in [14]). If $\odot$ and $\wedge$ coincide, then the transitivity is equivalent to the co–stratification axiom, and the Booleanization of stratified and transitive L–topologies exists. In this case stratified and transitive L–topologies can be characterized by Boolean–valued neighborhoods (cf. Case C in Subsection 7.4 in [14]). If in addition the underlying complete lattice $(L, \leq)$ is completely distributive, then in the case of $\odot \neq *$ a characterization of stratified and transitive L–topologies by L–valued neighborhoods is also available (cf. Theorem 3.9, Remark 3.17). Moreover, in the case of $\odot = \wedge$ the complete distributivity of $(L, \leq)$ guarantees the equivalence between L–fuzzifying topologies, L–fuzzy contiguity relations, crisp systems of L–valued neighborhoods and stratified and transitive L–topologies (cf. Theorem 3.9 and Corollary 3.15).

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CHAPTER 6

Separation Axioms: Extension Of Mappings And Embedding Of Spaces*

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Introduction

The present chapter is intended to give a self-contained development of some of the topics of fuzzy topology (54A40) that involve L-real-valued continuous functions on L-topological spaces. The four major themes include separation of L-sets by continuous L-real functions, insertion of continuous L-real functions between two comparable L-real functions, extension of continuous L-real functions from a subspace to the entire space and embedding L-topological spaces into products. (In actual fact, it is the embedding of L-Tychonoff spaces into L-cubes which falls into that category of results.)

In general topology (i.e. when the lattice L is the two-point chain) these ideas are among the most important aspects, and related results, such as e.g. Urysohn lemma, Tietze extension theorem or Tychonoff embedding theorem, rank among the fundamental theorems.

The fuzzy topology we shall deal with is that which in this Handbook is named the fixed-basis fuzzy topology. This means that throughout this article the lattice L is fixed (modulo some aditional assumptions) as in the chapter by Höhle and Šostak [13]. The assumptions about L have been minimalized and, consequently, our main results are complete lattice results. We recall that most of the separation axioms that will concern us as well as the concept of an L-real-valued function require L to be isomorphic to its dual lattice via an involution. Thus, a complete lattice with an order-reversing involution is the minimal assumption for this chapter. We notice that, unlike [13], we classically use the meet operation in the definition of an L-topology by letting $\otimes = \wedge$ (see [13] for details).

*Dedicated to Professor Kiyoshi Iseki on his 80th birthday.

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In some instances, in particular when the poset of all continuous L-real functions is required to be a lattice, we shall assume that L is meet-continuous. The distributivity (in a form of the infinite distribution law) is mainly needed when making statements about universal spaces.

We note that our results truly generalize their topological counterparts. This means that not only statements of the latter but also their proofs are obtainable by letting L to be a two-point lattice. We recall that this is rarely the case in those papers in which L is the real unit interval $[0,1]$. These papers freely use the order-density of $[0,1]$ while the two-point lattice (general topology) is not order-dense.

The outline of this paper is as follows:

§1 First L–topological concepts.
Lattices. L–sets. L–topological spaces. Continuity, subspaces and products. Stratification.
§2 L–real–valued functions.
Monotone L–subsets of the real line. L–real line and L–intervals. Semicontinuous and continuous L–real functions. Characteristic functions of L–sets. Separating L–sets by continuous L–real functions. Generating continuous L–real functions.
§3 Separation axioms.
Completely L–regular spaces and L–regular spaces. L–normal spaces. Separation properties of L–reals.
§4 Insertion and extension of mappings.
Two insertion lemmas. Insertion and extension theorems for L–normal spaces. General L–topological insertion and extension theorems. Extending $\mathbb{R}(L)$–valued functions.
§5 Embedding of L–topological spaces.
Completely regular families of functions. Embedding theorems. Universal L–topological spaces.

1 First L-topological concepts

Concerning the general background, we refer the reader to the foundation chapters [13] and [47]. For the reader's convenience we compile here those L-topological concepts (and only those) that we shall need in the sequel. These are all standard and come from [4, 8, 9, 34, 53]. Some of those notions can essentially be found in [13] and [47], but not all of them and not always in the form we need herein. Our references for lattices are [6] and [2].

Lattices

Let L be a complete lattice. Its universal bounds are denoted by $\perp$ and $\top$. Thus $\perp \leq \alpha \leq \top$ for all $\alpha \in L$. We set $\bigvee \emptyset = \perp$ and $\bigwedge \emptyset = \top$. The two-point lattice $\{\perp, \top\}$ is denoted by $\mathbf{2}$.

A unary operation $'$ on L is a *quasi-complementation* if it is an involution (i.e. $\alpha'' = \alpha$ for all $\alpha \in L$) that inverts the ordering (i.e. $\alpha \leq \beta$ implies $\beta' \leq \alpha'$). In $(L, ')$ the de Morgan laws hold:

$$\left(\bigvee A\right)' = \bigwedge\{\alpha' : \alpha \in A\} \quad \text{and} \quad \left(\bigwedge A\right)' = \bigvee\{\alpha' : \alpha \in A\}$$

for every $A \subset L$. In particular, $\perp' = \top$ and $\top' = \perp$.

In each complete lattice L one has the two generalized associative laws:

$$\bigvee_{j \in J} \bigvee A_j = \bigvee\Big(\bigcup_{j \in J} A_j\Big) \quad \text{and} \quad \bigwedge_{j \in J} \bigwedge A_j = \bigwedge\Big(\bigcup_{j \in J} A_j\Big)$$

for arbitrary $\{A_j \subset L : j \in J\}$.

A complete lattice L is *meet-continuous* if

$$\text{(MC)} \qquad \alpha \wedge \bigvee D = \bigvee\{\alpha \wedge \gamma : \gamma \in D\}$$

for all $\alpha \in L$ and all directed sets $D \subset L$ (D is directed if, whenever $\gamma_1, \gamma_2 \in D$, there is $\gamma \in D$ such that $\gamma_1 \leq \gamma$ and $\gamma_2 \leq \gamma$). It is easy to show that (MC) is equivalent to

$$\text{(MC)}^\star \qquad \left(\bigvee D_1\right) \wedge \left(\bigvee D_2\right) = \bigvee\{\gamma_1 \wedge \gamma_2 : \gamma_1 \in D_1, \gamma_2 \in D_2\}$$

where D_1, $D_2 \subset L$ are directed.

We remark that the set D in (MC) (likewise those of (MC*)) can without loss of generality be assumed to be a chain (cf. [10, p. 122]). This actually will always be the case in this chapter.

A complete lattice L is *infinitely distributive* (or a *frame* or a *complete Heyting algebra*) if

$$\text{(ID)} \qquad \alpha \wedge \bigvee B = \bigvee\{\alpha \wedge \beta : \beta \in B\}$$

for all $\alpha \in L$ and $B \subset L$. We note, using the generalized associative law, that (ID) is equivalent to

$$\text{(ID)}^\star \qquad \bigwedge_{j \in J} \bigvee A_j = \bigvee\left\{\bigwedge_{j \in J} \varphi(j) : \varphi \in \prod_{j \in J} A_j\right\}$$

for arbitrary $A_j \subset L$ with a finite index set J.

Clearly, in $(L, ')$ we also have the dual versions of (MC), (ID), etc., i.e directed infs distribute over finite sups in L, etc.

If the index set J is arbitrary, (ID*) becomes the complete distributive law (CD). We shall never use (CD) in this chapter, but may refer to it when making historical remarks.

1.1 Example. The following provide some examples[1] of a complete lattice L with a quasi-complementation (= order-reversing involution) $'$:

(1) $L = [0,1]$ with $\alpha' = \varphi^{-1}(1-\varphi(\alpha))$, $\alpha \in L$, where φ is an order-preserving bijection from L to L. In the standard example one has $\psi = \mathrm{id}_L$.

(2) $L = \{\frac{i}{n-1} : \quad i = 1, \ldots, n-1\}$ with the natural ordering and $(\frac{i}{n-1})' = \frac{n-i-1}{n-1}$.

(3) $L = \{\bot, \alpha, \beta, \top\}$ (where $\alpha \wedge \beta = \bot$ and $\alpha \vee \beta = \top$) with $\alpha' = \alpha$ and $\beta' = \beta$ (cf. [41]).

(4) $L = \mathcal{P}(X)$ (the power set of a set X) with $A' = X \backslash g(A)$ for every $A \subset X$, where g is an involution of X onto itself (this is an example of a quasi-Boolean algebra, see [41] for more details and for a representation theorem).

(5) Let $(L, *)$ be a complete MV-algebra. Then $\alpha' = \alpha \to \bot = \bigvee\{\lambda \in L : \alpha * \lambda = \bot\}$, $\alpha \in L$, is a quasi-complementation (see [12] for details; also see [13]). Note that such L does not need to be completely distributive.

(6) Among nondistributive lattices with a quasi-complementation $'$ are also those in which $'$ is an orthocomplementation (i.e., $\alpha \wedge \alpha' = \bot$ and, thus, $\alpha \vee \alpha' = \top$ for all $\alpha \in L$). A particular class of such lattices is provided by weakly orthomodular lattices. These are complete lattices with an orthocomplementation $'$ satisfying the following weak modularity condition: if α is orthogonal to β (i.e. $\alpha \le \beta'$) and $\beta \le \gamma$, then $(\alpha \vee \beta) \wedge \gamma = (\alpha \wedge \gamma) \vee \beta$. Typical examples are the lattices of closed subspaces of a Hilbert space. For more information see [52] (also cf. [2]).

(7) Finally, we mention complete Boolean algebras (which are special cases of both (5) and (6)).

L–sets

Let $(L, ')$ be a complete lattice. For X a set, L^X is the complete lattice of all maps from X into L, called *L-sets* or *L-subsets* of X, under pointwise ordering:

$$a \le b \text{ in } L^X \quad \text{if and only if} \quad a(x) \le b(x) \text{ in } L$$

for all $x \in X$. In particular, given $\mathcal{A} \subset L^X$ we have $(\bigvee \mathcal{A})(x) = \bigvee\{a(x) : a \in \mathcal{A}\}$ for all $x \in X$, and the same for infs.

For $Y \subset X$ and $a \in L^X$, $a|Y \in L^Y$ is the restriction of a to Y. If $A \subset X$, $1_A \in \mathbf{2}^X \subset L^X$ is the characteristic function of A. The constant member of L^X with value α is denoted α too. We write $\alpha 1_A$ for $\alpha \wedge 1_A$. The cardinality of a set X is denoted $|X|$.

Clearly, L^X has a quasi-complementation $'$ defined pointwisely: $a'(x) = a(x)'$ for all $a \in L^X$ and $x \in X$. Thus, the de Morgan laws are inherited by $(L^X, ')$.

[1] Examples (5) and (6) have been suggested by U. Höhle.

Images and inverse images of L-sets

Let $f\colon X \to Y$ be an arbitrary mapping. We define

$$f^{\rightarrow}\colon L^X \to L^Y \quad \text{and} \quad f^{\leftarrow}\colon L^Y \to L^X$$

by

$$f^{\rightarrow}(a) = \bigvee_{x\in X} a(x)1_{\{f(x)\}} \quad \text{and} \quad f^{\leftarrow}(b) = b\circ f$$

for all $a\in L^X$ and $b\in L^Y$ (where $b\circ f$, the composition of f and b, will later be denoted by just bf). Clearly, $f^{\rightarrow}(a)(y) = \bigvee\{a(x) : f(x) = y\}$ for all $y\in Y$. Note that $f^{\rightarrow}(1_A) = 1_{f(A)}$ and $f^{\leftarrow}(1_B) = 1_{f^{-1}(B)}$.

All the properties of images and inverse images of sets are inherited by the two operators just defined. We shall need a number of them, viz.:

1.2 Properties. *Let $(L,')$ be complete. Let $f\colon X \to Y$, $g\colon Y \to Z$, $a\in L^X$, $b\in L^Y$, $c\in L^Z$. Then:*

(1) *$f^{\leftarrow}$ preserves arbitrary sups and arbitrary infs; $f^{\rightarrow}$ preserves arbitrary sups (in particular both are order-preserving),*

(2) $a \le f^{\leftarrow}(f^{\rightarrow}(a))$,

(3) $f^{\rightarrow}(f^{\leftarrow}(b)) \le b$, *and* $f^{\rightarrow}(f^{\leftarrow}(b)) = b$ *if and only if* f *is onto,*

(4) $(g\circ f)^{\rightarrow}(a) = g^{\rightarrow}(f^{\rightarrow}(a))$,

(5) $(g\circ f)^{\leftarrow}(c) = f^{\leftarrow}(g^{\leftarrow}(c))$,

(6) $f^{\leftarrow}(b') = f^{\leftarrow}(b)'$.

Proof. Let us check (4). We calculate

$$\begin{aligned}
g^{\rightarrow}(f^{\rightarrow}(a)) &= \bigvee_{y\in Y} f^{\rightarrow}(a)(y)\, 1_{\{g(y)\}} \\
&= \bigvee_{y\in Y} \left(\bigvee_{x\in X} a(x)\, 1_{\{f(x)\}}(y)\right) 1_{\{g(y)\}} \\
&= \bigvee_{x\in X} \left(\bigvee_{x\in X} a(x)\, 1_{\{f(x)\}}(f(x))\right) 1_{\{g(f(x))\}} \\
&= \bigvee_{x\in X} a(x)\, 1_{\{g(f(x))\}} \\
&= (g\circ f)^{\rightarrow}(a). \quad \square
\end{aligned}$$

We shall distinguish notationally between images and inverse images of L-sets generated by $f\colon X \to Y$ and its restriction $\tilde{f}\colon X \to f(X)$ where $\tilde{f}(x) = f(x)$ for all $x\in X$. For $a\in L^X$ and $b\in L^Y$ we trivially have:

$$\tilde{f}^{\rightarrow}(a) = f^{\rightarrow}(a)|f(X) \quad \text{and} \quad f^{\leftarrow}(b) = \tilde{f}^{\leftarrow}(b|f(X)).$$

Notation. If $\mathcal{A} \subset L^Y$, then $f^{\leftarrow}(\mathcal{A}) = \{f^{\leftarrow}(a) : a\in \mathcal{A}\}$.

L-topological spaces

Let $(L,')$ be a complete lattice. A subfamily $\mathcal{T} \subset L^X$ which is closed under the formation of any sups and finite infs (both formed in L^X) is called an *L-topology* on X and its members are called *open* L-sets (thus, $1_\emptyset$, $1_X \in \mathcal{T}$)[2]. The *L-topological space* $(X, \mathcal{T})$ will often be denoted just by X and its L-topology by $o(X)$. Members of $\kappa(X) = \{k \in L^X : k' \text{ is open}\}$ are called *closed.* Clearly, $\kappa(X)$ is closed under any infs and finite sups.

Many of the point-free topological concepts easily generalize to an L-topological setting. An $\mathcal{S} \subset L^X$ is said to *generate* $o(X)$ if

$$o(X) = \bigcap\{\mathcal{T} \supset \mathcal{S} : \mathcal{T} \text{ is an } L\text{-topology on } X\},$$

and we write $o(X) = \langle\!\langle \mathcal{S} \rangle\!\rangle$. If $\mathcal{B} \subset L^X$ and

$$o(X) = \left\{\bigvee B : B \subset \mathcal{B}\right\},$$

then $\mathcal{B}$ is a *base* for X. A family $\mathcal{K}$ of closed L-sets is a *closed base* if $\mathcal{K}' = \{k' : k \in \mathcal{K}\}$ is a base. The *weight* $w(X)$ of X is the smallest infinite cardinal such that X has a base of cardinality $\leq w(X)$, i.e.

$$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\} + \aleph_0.$$

1.3 Lemma. *Let L be a complete lattice, X be an L-topological space, and let $\mathcal{U}$ be a family of open L-sets. Then there exists a subfamily $\mathcal{U}_0 \subset \mathcal{U}$ such that $\bigvee \mathcal{U}_0 = \bigvee \mathcal{U}$ and $|\mathcal{U}_0| \leq w(X)$.*

Proof. Let $\mathcal{B}$ be a base for X with $|\mathcal{B}| \leq w(X)$. Let

$$\mathcal{V} = \{v \in \mathcal{B} : v \leq u \text{ for some } u \in \mathcal{U}\}.$$

For $v \in \mathcal{V}$ choose $u_v \in \mathcal{U}$ such that $v \leq u_v$. We set

$$\mathcal{U}_0 = \{u_v \in \mathcal{U} : v \in \mathcal{V}\}.$$

Clearly, $|\mathcal{U}_0| \leq w(X)$. Since $\mathcal{B}$ is a base, for every $u \in \mathcal{U}$ there is $\mathcal{V}_u \subset \mathcal{V}$ such that $\bigvee \mathcal{V}_u = u$. Also, for every $v \in \mathcal{V}_u$ one has $v \leq u_v \leq \bigvee \mathcal{U}_0$. Thus $\bigvee \mathcal{V}_u \leq \bigvee \mathcal{U}_0$ for all $u \in \mathcal{U}$. Therefore,

$$\bigvee \mathcal{U} = \bigvee_{u \in \mathcal{U}} \bigvee \mathcal{V}_u \leq \bigvee \mathcal{U}_0.$$

This proves the nontrivial inequality. □

We note that if L is a frame and $\mathcal{T} = \langle\!\langle \mathcal{S} \rangle\!\rangle$, then $\mathcal{B} = \{\bigwedge S : S \in \mathcal{S}, S \text{ is finite}\}$ is a base.

[2]We note that this is a special case of the concept investigated in [13] (see Introduction).

Let $a \in L^X$. Then

$$\mathrm{Cl}_X\, a = \bigwedge\{k \in \kappa(X) : a \leq k\}$$

is the *closure* of a (also denoted by $\overline{a}$), and

$$\mathrm{Int}_X\, a = \bigvee\{u \in o(X) : u \leq a\}$$

is the *interior* of a (also denoted by Int a). We note that Cl_X satisfies all the four Kuratowski's axioms and Int_X has the well-known dual properties. Also notice that $\mathrm{Int}_X\, a = (\mathrm{Cl}_X(a'))'$.

Continuity, subspaces and products

For L-topological spaces X and Y, $f\colon X \to Y$ is called *continuous* if, whenever u is open in Y, uf is open in X. Clearly, the composition of two continuous functions is continuous by 1.2(5). The collection of all continuous functions from X to Y is denoted $C(X,Y)$.

1.4 Proposition. *Let $(L,')$ be complete. For $f\colon X \to Y$, the following statements are equivalent:*

(1) *f is continuous,*

(2) $f^{\leftarrow}(\kappa(Y)) \subset \kappa(X)$,

(3) *$f^{\leftarrow}(\mathcal{S}) \subset o(X)$ for any $\mathcal{S} \subset L^Y$ with $o(Y) = \langle\!\langle \mathcal{S} \rangle\!\rangle$,*

(4) *$f^{\rightarrow}(\mathrm{Cl}_X\, a) \leq \mathrm{Cl}_Y f^{\rightarrow}(a)$ for all $a \in L^X$,*

(5) *$\mathrm{Cl}_X f^{\leftarrow}(b) \leq f^{\leftarrow}(\mathrm{Cl}_Y\, b)$ for all $b \in L^Y$.*

Proof. (1) $\Leftrightarrow$ (2) is trivial by 1.2(6); (1) $\Leftrightarrow$ (3) is proved in *Chapter 3* or *Chapter 4* of this volume (see [13, 3.3] or [47, 3.2.6]); (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2) follows by standard argument by using 1.2. $\square$

A continuous bijection $f\colon X \to Y$ is a *homeomorphism* if f^{-1} is continuous. This is equivalent to the statement that f is *closed* continuous bijection, i.e. $f^{\rightarrow}(k)$ is closed in Y for every k closed in X. Indeed, $(f^{-1})^{\leftarrow} = f^{\rightarrow}$.

For $(Y, \mathcal{T})$ an L-topological space and a map $f\colon X \to Y$, $f^{\leftarrow}(\mathcal{T})$ is an L-topology on X by 1.2(1), and is clearly the smallest L-topology on X making f continuous from X to $(Y, \mathcal{T})$. When $X \subset Y$ and f is the identity embedding, then

$$\mathcal{T}_X = f^{\leftarrow}(\mathcal{T}) = \{u|X : u \in \mathcal{T}\}$$

is called the *subspace L-topology*.

If $f\colon X \to Y$ is continuous, then so is the restriction $f|A\colon A \to Y$ for every subspace A of X (as $f|A$ is the composition of j_A, the identity embedding of A into X, and f). Continuous is also $\tilde{f}\colon X \to f(X)$, because for every open u in Y one has $\tilde{f}^{\leftarrow}(u|f(X)) = f^{\leftarrow}(u)$.

More generally, if $B \subset Y$, then $g\colon X \to B$ is continuous if and only if $f = j_B \circ g\colon X \to Y$ is continuous. Indeed,

$$g^{\leftarrow}(o(B)) = g^{\leftarrow}(j_B{}^{\leftarrow}(o(Y)) = (j_B \circ g)^{\leftarrow}(o(Y)) = f^{\leftarrow}(o(Y)).$$

We say that $f\colon X \to Y$ is an *embedding* if $\tilde{f}\colon X \to f(X)$ is a homeomorphism.

For future use we include the following:

1.5 Lemma. *Let $(L,')$ be a complete lattice. If $f\colon X \to Y$ is continuous and $a \in L^X$, then*

$$\mathrm{Cl}_{f(X)}\, \tilde{f}^{\rightarrow}(a) = (\mathrm{Cl}_Y f^{\rightarrow}(a)) \mid f(X).$$

Proof. Since $\tilde{f}^{\rightarrow}(a) = f^{\rightarrow}(a)$ on $f(X)$ and $f^{\rightarrow}(a) = \perp$ on $Y \setminus f(X)$, hence

$$\begin{aligned}\mathrm{Cl}_{f(X)}\, \tilde{f}^{\rightarrow}(a) &= \bigwedge\{k|f(X) : \tilde{f}^{\rightarrow}(a) \le k|f(X),\ k\in\kappa(Y)\}\\ &= \bigwedge\{k|f(X) : f^{\rightarrow}(a) \le k,\ k\in\kappa(Y)\}\\ &= (\mathrm{Cl}_Y\, f^{\rightarrow}(a)) \mid f(X). \quad \square\end{aligned}$$

1.6 Lemma. *Let L be complete. If $\mathcal{S} \subset L^Y$ and f maps a set X into Y, then $f^{\leftarrow}(\langle\!\langle \mathcal{S} \rangle\!\rangle) = \langle\!\langle f^{\leftarrow}(\mathcal{S}) \rangle\!\rangle$.*

Proof. The inclusion $\supset$ is obvious. Since $f^{\leftarrow}(\mathcal{S}) \subset \langle\!\langle f^{\leftarrow}(\mathcal{S}) \rangle\!\rangle$, hence f is continuous from $(X, \langle\!\langle f^{\leftarrow}(\mathcal{S}) \rangle\!\rangle)$ to $(Y, \langle\!\langle \mathcal{S} \rangle\!\rangle)$ by 1.4(3). This yields the non-trivial inclusion. $\square$

1.7 Corollary. *Let L be complete. If $\mathcal{T} = \langle\!\langle \mathcal{S} \rangle\!\rangle$ on X and $A \subset X$, then $\mathcal{T}_A = \langle\!\langle \mathcal{S}_A \rangle\!\rangle$ where $\mathcal{S}_A = \{u|A : u \in \mathcal{S}\}$.* $\square$

Now, let $(\mathcal{T}_j)_{j\in J}$ be a family of L-topologies on a set X. The *supremum L-topology* $\bigvee_{j\in J} \mathcal{T}_j$ is one which is generated by $\bigcup_{j\in J} \mathcal{T}_j$.

Let $(X_j)_{j\in J}$ be a family of L-topological spaces. The *product* of the X_j is the usual Cartesian product $\prod_{j\in J} X_j$ L-topologized by the L-topology

$$\bigvee_{j\in J} \pi_j^{\leftarrow}\,(o(X_j)),$$

where π_j is the j th projection.

As a consequence of the definition and 1.4(3) (the latter is the most important statement in this section, because it enables us to deal with continuity in a distributivity-free environment!) we have the following:

1.8 Proposition. *Let L be a complete lattice. Then $f\colon Y \to \prod_{j\in J} X_j$ is continuous if and only if $\pi_j \circ f\colon Y \to X_j$ is continuous for every $j \in J$.* $\square$

Stratification

An L-topological space X is called, in this Handbook, *weakly stratified*[3] if all the constant L-sets of X are open (cf. [34]).

That we deal here with (weak) stratification serves essentially only one purpose, mainly in section 5, viz. to be able to embed sub-products into products (see below).

Let $\mathcal{C}_X$ denote all the constant L-sets of X. For $\mathcal{T}$ an L-topology on X, we let

$$\mathcal{T}^c = \mathcal{T} \vee \mathcal{C}_X.$$

Then X with $\mathcal{T}^c$ is called *weakly stratified* and is denoted by X^c. If $X = X^c$, we obviously have

$$C(X,Y) = C(X,Y^c).$$

1.9 Proposition. *Let L be a complete lattice. Let $f\colon X \to Y$ (a mapping from a set to a set). Let $A \subset Y$ and let $\mathcal{T}$ and $(\mathcal{T}_j)_{j\in J}$ be L-topologies on Y. Then:*

(1) $f^{\leftarrow}(\mathcal{C}_Y) = \mathcal{C}_X$,

(2) $f^{\leftarrow}(\mathcal{T}^c) = (f^{\leftarrow}(\mathcal{T}))^c$,

(3) $(\bigvee_{j\in J} \mathcal{T}_j)^c = \bigvee_{j\in J} \mathcal{T}_j^c$,

(4) $(\mathcal{T}_A)^c = (\mathcal{T}^c)_A$.

Proof. (1) is obvious. For (2), since $\mathcal{T}^c$ is generated by $\mathcal{T} \cup \mathcal{C}_Y$, hence $f^{\leftarrow}(\mathcal{T} \cup \mathcal{C}_Y) = f^{\leftarrow}(\mathcal{T}) \cup \mathcal{C}_X$ generates $f^{\leftarrow}(\mathcal{T}^c)$ by 1.6. Statement (3) is obvious and (4) follows from (2) with f the identity embedding of A into Y. □

As a consequence of (2) and (3) above, we obtain:

1.10 Proposition. *Let L be complete. Let $(X_j)_{j\in J}$ be a family of L-topological spaces. Then*

$$(\prod_{j\in J} X_j)^c = \prod_{j\in J} X_j^c. \quad \square$$

We observe that even if only a factor, X_j say, is weakly stratified then so is the whole product $X = \prod_{j\in J} X_j$. Indeed, we then clearly have

$$\mathcal{C}_X \subset \pi_j^{\leftarrow}(o(X_j)).$$

This is the reason that only those factors (or products of those factors), that are weakly stratified can standartly be embedded into the product (cf. [39]). For, the usual bijective copy of X_{j_0} in X, viz.

$$Z = \{x \in X : x_j = z_j \text{ whenever } j \neq j_0\}$$

[3]For what is stratified and strongly stratified, we refer to [13, 5.1 and particularly 5.2.7].

(where z_j is fixed in X_j for $j \neq j_0$) always has constant L-sets when considered as the subspace of the product. Namely, L-sets $u_j = \pi_j^{\leftarrow}(u)|Z$ (u open in X_j) generate the L-topology on Z (cf. 1.7) and for every $x \in Z$

$$u_j(x) = u(z_j) \quad \text{whenever} \quad j \neq j_0,$$

i.e. $u_j \in \mathcal{C}_Z$ if $j \neq j_0$.

Thus, while $\varphi = \pi_{j_0}|Z$ is a continuous bijection from Z to X_{j_0}, its inverse φ^{-1} is continuous provided X_{j_0} has enough constant L-sets (precisely, when every constant L-set with values in $\{u(z_j) : u \text{ is open in } X_j \text{ and } j \neq j_0\}$ is open in X_{j_0}). This is guaranteed by the case when X_{j_0} is weakly stratified. This discussion is summarized by the following:

1.11 Proposition. *Let L be a complete lattice. For $(X_j)_{j\in J}$ a family of L-topological spaces, the following are equivalent for any $j \in J$:*

(1) *$\prod_{j\in J} X_j$ is weakly stratified and X_j can be embedded into the product.*
(2) *X_j is weakly stratified.* □

2 L-real-valued functions

The usual symbols $\mathbb{R}$ and $\mathbb{Q}$ stand for real and rational numbers, and $I = [0,1] \subset \mathbb{R}$. We shall deal with the L-reals of [14] and [5]. (See [11] and [35] for other fuzzy reals.) The details of the presentation which follows come from [22, 24, 27]. In most cases, we do not make specific references to these papers.

Monotone L–subsets of the real line

Let L be a complete lattice and $\mathbb{R}_L$ be the set of all order–reversing maps $\lambda : \mathbb{R} \to L$ provided with the following properties

$$\bigvee \lambda(\mathbb{R}) = \top \quad , \qquad \bigwedge \lambda(\mathbb{R}) = \bot \quad .$$

For $\lambda \in \mathbb{R}_L$ and $t \in \mathbb{R}$, we let

$$\lambda^+(t) = \bigvee \lambda(t,\infty) \quad \text{and} \quad \lambda^-(t) = \bigwedge \lambda(-\infty,t).$$

Clearly $\lambda^+ \leq \lambda \leq \lambda^-$.

2.1 Lemma. *Let L be a complete lattice. For $\lambda, \mu \in \mathbb{R}_L$, the following hold:*

(1) *$(\cdot)^+$ and $(\cdot)^-$ are order-preserving self-mappings of $\mathbb{R}_L$,*
(2) *$\lambda^{++} = \lambda^+ = \lambda^{-+}$ and $\lambda^{--} = \lambda^- = \lambda^{+-}$,*
(3) *$\lambda^+ \leq \mu^+$ iff $\lambda^- \leq \mu^-$ iff $\lambda^+ \leq \mu^-$,*
(4) *$\lambda^+ = \mu^+$ iff $\lambda^- = \mu^-$,*

(5) $(\cdot)^+$ *preserves all the existing sups in* $\mathbb{R}_L$ *and* $(\cdot)^-$ *preserves all the existing infs* (*both formed in* $\mathbb{R}_L$),

(6) *If* $\phi\colon \mathbb{R} \to \mathbb{R}$ *is an increasing bijection, then* $\lambda \circ \phi \in \mathbb{R}_L$, $(\lambda \circ \phi)^+ = \lambda^+ \circ \phi$ *and* $(\lambda \circ \phi)^- = \lambda^- \circ \phi$.

Proof. (1) We have $\bigwedge \lambda^+(\mathbb{R}) \leq \bigwedge \lambda(\mathbb{R}) = \perp$ and

$$\bigvee \lambda^+(\mathbb{R}) = \bigvee_{t\in\mathbb{R}} \bigvee \lambda(t,\infty) = \bigvee\left(\bigcup_{t\in\mathbb{R}} \lambda(t,\infty)\right) = \bigvee \lambda(\mathbb{R}) = \top;$$

similarly for $(\cdot)^-$.

(2) As above,

$$\lambda^+(t) = \bigvee\left(\bigcup_{s>t} \lambda(s,\infty)\right) = \bigvee_{s>t} \bigvee \lambda(s,\infty) = \bigvee_{s>t} \lambda^+(s) = \lambda^{++}(t).$$

If $s > t$, then $\lambda^+(t) \geq \lambda^-(s)$, hence $\lambda^+(t) \geq \bigvee_{s>t} \lambda^-(s) = \lambda^{-+}(t)$ and $\lambda^{-+} \geq \lambda^+$ is obvious. This proves the first assertion of (2). The proof of the second one is similar.

(3) Let $\lambda^+ \leq \mu^+$. By monotonicity of $(\cdot)^-$ and (2), we have $\lambda^- = \lambda^{+-} \leq \mu^{+-} = \mu^-$ which clearly implies $\lambda^+ \leq \mu^-$. Now, the latter inequality and (2) yield $\lambda^+ = \lambda^{++} \leq \mu^{-+} = \mu^+$.

(4) is a consequence of (3).

(5) is obvious.

(6) Clearly $\lambda \circ \phi \in \mathbb{R}_L$ and

$$(\lambda \circ \phi)^+(t) = \bigvee \lambda(\phi(t,\infty)) = \bigvee \lambda(\phi(t),\infty) = \lambda^+(\phi(t));$$

the same for $(\cdot)^-$. □

2.2 Example. In general, $\mathbb{R}_L$ need not be a sublattice of $L^{\mathbb{R}}$, i.e. need not be closed under $\wedge$ and $\vee$ formed in $L^{\mathbb{R}}$. To see this, consider the complete lattice L obtained by identifying the top and bottom elements of two disjoint copies I_1 and I_2 of the real unit interval with the natural ordering (if $\alpha_i \in I_i \setminus \{0,1\}$, then α_1 and α_2 are incomparable). Now let $\lambda_1, \lambda_2 \in \mathbb{R}_L$ be such that $\lambda_i(\mathbb{R}) \subset I_i \setminus \{1\}$. Then $(\lambda_1 \wedge \lambda_2)(\mathbb{R}) = \{0\}$, hence $\lambda_1 \wedge \lambda_2 \notin \mathbb{R}_L$.

We also note that L has a quasi-complementation, viz. $\alpha' = 1 - \alpha \in I_i$ whenever $\alpha \in I_i$.

The point of this example is that L is not meet-continuous (for if $D = [0,1) \subset I_1$ and $\alpha \in I_2 \setminus \{0,1\}$, then $\alpha \wedge \bigvee D = \alpha \neq 0 = \bigvee\{\alpha \wedge \gamma : \gamma \in D\}$).

2.3 Lemma. *Let* $(L,')$ *be meet-continuous. Then* $\mathbb{R}_L$ *is a sublattice of* $L^{\mathbb{R}}$ *and both* $(\cdot)^+$ *and* $(\cdot)^-$ *are lattice homomorphism, i.e. preserve* $\wedge$ *and* $\vee$.

Proof. To avoid repetitions we first observe that for all $\lambda, \mu \in \mathbb{R}_L$ and all $A \subset \mathbb{R}$ one has the following:

(a) $\bigvee(\lambda \wedge \mu)(A) = \bigvee \lambda(A) \wedge \bigvee \mu(A)$,

(b) $\bigwedge(\lambda \vee \mu)(A) = \bigwedge \lambda(A) \vee \bigwedge \mu(A)$.

Indeed, since both $\lambda(A)$ and $\mu(A)$ are chains in L, we have using (MC*)

$$\begin{aligned} \bigvee \lambda(A) \wedge \bigvee \mu(A) &= \bigvee\{\lambda(t) \wedge \mu(s) : t, s \in A\} \\ &\leq \bigvee\{\lambda(t \wedge s) \wedge \mu(t \wedge s) : t, s \in A\} \\ &= \bigvee(\lambda \wedge \mu)(A), \end{aligned}$$

with the reverse inequality being obvious. This proves (a). Now (b) follows similarly by using the dual of (MC*) which holds on account of $'$.

With $A = \mathbb{R}$ we have $\bigvee(\lambda \wedge \mu)(\mathbb{R}) = \top$ and $\bigwedge(\lambda \vee \mu)(\mathbb{R}) = \bot$ by (a) and (b), respectively. Thus $\mathbb{R}_L$ is a sublattice (the other details for this are trivial).

Furthermore, using (a) with $A = (t, \infty)$ yields $(\lambda \wedge \mu)^+(t) = \lambda^+(t) \wedge \mu^+(t)$, and by (b) with $A = (-\infty, t)$ we obtain $(\lambda \vee \mu)^-(t) = \lambda^-(t) \vee \mu^-(t)$. The two remaining cases are trivial. □

L-real line and L-intervals

Let $(L, ')$ be a complete lattice. Given $\lambda, \mu \in \mathbb{R}_L$, let

$$\lambda \sim \mu \quad \text{iff} \quad \lambda^+ = \mu^+.$$

This is an equivalence relation and the set $\mathbb{R}(L)$ of all the equivalence classes $[\lambda]$ is called the *L-real line* [14], [5]. It becomes a poset with

$$[\lambda] \leq [\mu] \quad \text{iff} \quad \lambda^+ \leq \mu^+.$$

The *natural* L-topology on $\mathbb{R}(L)$ is generated by the L-sets $R_t, L_t \in L^{\mathbb{R}(L)}$ ($t \in \mathbb{R}$) defined by

$$R_t[\lambda] = \lambda^+(t) \quad \text{and} \quad L_t[\lambda] = \lambda^-(t)'.$$

2.4 Remark. *For each $t \in \mathbb{R}$ the following hold:*

(1) $R_t = \bigvee_{s>t} R_s = \bigvee_{s>t} L'_s$,

(2) $L_t = \bigvee_{s<t} L_s = \bigvee_{s<t} R'_s$,

where all the s can be assumed to be rational.

Proof. This is a restatement of 2.1(2). □

Clearly, $R_t \wedge R_s = R_{t \vee s}$ and $L_t \wedge L_s = L_{t \wedge s}$. We thus have another two L-topologies on $\mathbb{R}(L)$, viz.

$$\mathcal{R}_L = \{R_t : t \in \mathbb{R}\} \cup \{1_\emptyset, 1_{\mathbb{R}(L)}\}$$

and

$$\mathcal{L}_L = \{L_t : t \in \mathbb{R}\} \cup \{1_\emptyset, 1_{\mathbb{R}(L)}\},$$

whose supremum is the natural L-topology of $\mathbb{R}(L)$.

If not otherwise stated, $\mathbb{R}(L)$ carries its natural L-topology and so does any subspace of $\mathbb{R}(L)$.

Two important subspaces of $\mathbb{R}(L)$ are the following:

$$I(L) = \{[\lambda] \in \mathbb{R}(L) : \lambda^-(0)' = \lambda^+(1) = \bot\}$$

and

$$(0,1)(L) = \{[\lambda] \in I(L) : \lambda^+(0)' = \lambda^-(1) = \bot\}.$$

They are called the *closed* and *open unit L-intervals*, respectively. The terms "closed" and "open" should be understood in an order-theoretic sense and not in a topological one (for, we do not know any $L \neq \mathbf{2}$ with which $1_{I(L)}$ would be closed in $\mathbb{R}(L)$ and $1_{(0,1)(L)}$ would be open in $I(L)$).

Both these L-intervals are equipped with the ordering induced from $\mathbb{R}(L)$. Of course, pertinent are only the restrictions of L_t and R_t with $t \in I$, and these will also be written as L_t and R_t.

2.5 Proposition. (1) *If $(L, ')$ is a complete lattice, then $(I(L), {}^\bullet)$ is a complete lattice (in which $\bigvee_{j\in J}[\lambda_j] = [\bigvee_{j\in J}\lambda_j]$ and similarly for infs) with a quasi-complementation $[\lambda]^\bullet = [\lambda^\bullet]$ where $\lambda^\bullet(t) = \lambda(1-t)'$ for all $t \in \mathbb{R}$.*

(2) *If $(L, ')$ is meet-continuous, then $\mathbb{R}(L)$ and $\mathbb{R}(L)^c$ are L-topological lattices (in which $[\lambda] \vee [\mu] = [\lambda \vee \mu]$ and similarly for infs).*

(3) *If $(L, ')$ is meet-continuous, then $(I(L), {}^\bullet)$ and $(I(L)^c, {}^\bullet)$ are L-topological lattices with a continuous quasi-complementation.*

Proof. (1) We observe that $I_L = \{\lambda \in L^{\mathbb{R}} : [\lambda] \in I(L)\}$ is a complete sublattice of $L^{\mathbb{R}}$. The relation $\sim$ on I_L has the substitution property for arbitrary $\vee$ and $\wedge$, i.e. if $\lambda_j \sim \mu_j$ for all j, then $\bigvee_j \lambda_j \sim \bigvee_j \mu_j$ and $\bigwedge_j \lambda_j \sim \bigwedge_j \mu_j$ by (5) and (4) of 2.1. Since $I(L) = I_L \backslash \sim$, it is a complete lattice.

It is easy to verify that

$$(\blacklozenge) \qquad (\lambda^+)^\bullet = (\lambda^\bullet)^- \quad \text{and} \quad (\lambda^-)^\bullet = (\lambda^\bullet)^+$$

for all $\lambda \in I_L$, hence $\lambda^\bullet \in I_L$ and $\lambda \sim \mu$ iff $\lambda^\bullet \sim \mu^\bullet$. Thus $[\lambda]^\bullet$ is well defined and is clearly a quasi-complementation.

(2) $\mathbb{R}(L)$ is a lattice by the same argument as that for (1), using 2.3. Let π_1 and π_2 be the projections of $\mathbb{R}(L) \times \mathbb{R}(L)$ onto $\mathbb{R}(L)$. Since $(\cdot)^+$ and $(\cdot)^-$ are lattice-homomorphisms (2.3), we have

$$\wedge^{\leftarrow}(R_t) = \pi_1^{\leftarrow}(R_t) \wedge \pi_2^{\leftarrow}(R_t), \qquad \wedge^{\leftarrow}(L_t) = \pi_1^{\leftarrow}(L_t) \vee \pi_2^{\leftarrow}(L_t)$$

and dually for the two remaining cases. Thus, $\wedge$ and $\vee$ are continuous from $\mathbb{R}(L) \times \mathbb{R}(L)$ into $\mathbb{R}(L)$ by 1.4(3). The case of $\mathbb{R}(L)^c$ follows by 1.10.

(3) The same argument as in (2). Alternatively, it follows by (2), (1), and the easily proven fact that

$$o(I(L) \times I(L)) = o(\mathbb{R}(L) \times \mathbb{R}(L))_{I(L)\times I(L)}.$$

The quasi-complementation is continuous on account of ($\blacklozenge$). Specifically, $L_t \circ (\cdot)^\bullet = R_{1-t}$ and $R_t \circ (\cdot)^\bullet = L_{1-t}$ for every $0 \le t \le 1$. $\square$

2.6 Remark. *For any complete lattice $(L,')$, $\mathbb{R}(L)$ and $(0,1)(L)$ are homeomorphic.*

Proof. Let h: $(0,1) \to \mathbb{R}$ be an increasing homeomorphism. Define

$$H\colon \mathbb{R}(L) \to (0,1)(L) \quad \text{by} \quad H([\lambda]) = [\lambda_h]$$

where

$$\lambda_h(t) = \begin{cases} \top & \text{if} \quad t \le 0, \\ \lambda(h(t)) & \text{if} \quad 0 < t < 1, \\ \perp & \text{if} \quad t \ge 1. \end{cases}$$

It is left to the reader to check that H is a well-defined continuous bijection and its inverse is continuous too. □

2.7 Remark. The subspaces R$(\mathbf{2})$, $I(\mathbf{2})$ and $(0,1)$ $(\mathbf{2})$ of $\mathbb{R}(L)$ are homeomorphic to $\mathbb{R}$, I and $(0,1)$ equipped with $\mathbf{2}$-topologies consisting of characteristic functions of their open sets. They embed into $\mathbb{R}(L)$ via e: $t \to [1_{(-\infty,t)}]$, where $R_t e = 1_{(t,\infty)}$ and $L_t e = 1_{(-\infty,t)}$. Clearly, e is also a lattice embedding.

Notation. Given $t \in \mathbb{R}$, $[t]$ is the member of $\mathbb{R}(L)$ generated by $1_{(-\infty,t)}$. We write (α) for the member of $I(L)$ generated by an element of $\mathbb{R}_L$ which is constant on I with the value α.

Semicontinuous and continuous L-real functions

Let $(L,')$ be a complete lattice. The partial ordering on $\mathbb{R}(L)^X$ is defined by

$$f \le g \quad \text{iff} \quad f(x) \le g(x)$$

for all $x \in X$. Of course, $f \le g$ iff $R_t f \le R_t g$ iff $L_t' f \le L_t' g$ for all $t \in \mathbb{R}$. When f and g take values in a sublattice of $\mathbb{R}(L)$ (or when $\mathbb{R}(L)$ itself is a lattice), $f \wedge g$ and $f \vee g$ exist, and $(f \wedge g)(x) = f(x) \wedge g(x)$ and $(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in X$.

Let X be an L-topological space. We distinguish the following:

$$LSC(X) = \{f \in \mathbb{R}(L)^X : f^{\leftarrow}(\mathcal{R}_L) \subset o(X)\},$$

$$USC(X) = \{f \in \mathbb{R}(L)^X : f^{\leftarrow}(\mathcal{L}_L) \subset o(X)\},$$

$$LSC^*(X) = I(L)^X \cap LSC(X),$$

$$USC^*(X) = I(L)^X \cap USC(X).$$

We now have

$$C(X) = C(X, \mathbb{R}(L)) = LSC(X) \cap USC(X),$$

$$C^*(X) = C(X, I(L)) = LSC^*(X) \cap USC^*(X).$$

Members of $LSC(X)$ ($USC(X)$) are called *lower* (*upper*) *semicontinuous.*

Notation. For $t \in \mathbb{R}$, $\mathbf{t}$ denotes the constant function of X into $\mathbb{R}(L)$ with the value $[t]$. Clearly, $\mathbf{t} \in C(X)$.

Given an $f\colon X \to \mathbb{R}(L)$, we write f_x for a representative of $f(x)$, i.e.

$$f(x) = [f_x].$$

For $f \in C^*(X)$, $f^\bullet \in C^*(X)$ is the composition of f and $(\cdot)^\bullet$.

2.8 Lemma. *Let $(L, ')$ be a complete lattice and let X be an L-topological space. Then:*

(1) $\mathbf{t} \wedge f$, $\mathbf{t} \vee f \in C(X)$ *for all* $t \in \mathbb{R}$ *and* $f \in C(X)$.

(2) *If* $\mathcal{F} \subset LSC^*(X)$, *then* $\mathbf{t} \wedge \bigvee \mathcal{F} = \bigvee\{\mathbf{t} \wedge f : f \in \mathcal{F}\} \in LSC^*(X)$ *for every* $t \in \mathbb{R}$.

Proof. (1) As $R_s\mathbf{t} = 1_\emptyset$ if $s \geq t$, and $R_s\mathbf{t} = 1_X$ if $s < t$, we have $R_s(\mathbf{t} \wedge f) = R_s\mathbf{t} \wedge R_s f$, open L-set. Similarly for L_s and for the case of $\mathbf{t} \vee f$.

(2) $LSC^*(X)$ is clearly closed under arbitrary pointwise sups. Condition (ID) is not needed to have the equality, for we clearly have for all $x \in X$

$$1_{(-\infty,t)} \wedge \bigvee_{f \in \mathcal{F}} f_x = \bigvee_{f \in \mathcal{F}} (1_{(-\infty,t)} \wedge f_x),$$

and these two functions are representative of $(\mathbf{t} \wedge \bigvee \mathcal{F})(x)$ (the left one) and $\bigvee_{f \in \mathcal{F}} (\mathbf{t} \wedge f)(x)$ (the right one). $\square$

2.9 Remark. *For $(L, ')$ a meet-continuous lattice, $C(X)$ and $C^*(X)$ are closed under finite $\wedge$ and $\vee$. The same hold for $C(X, \mathbb{R}(L)^c)$ and $C(X, I(L)^c)$ provided X is weakly stratified.*

Proof. This is now obvious. E.g., $f \wedge g$ is the composition of two continuous functions; the first is $X \to \mathbb{R}(L) \times \mathbb{R}(L)$ via $x \mapsto (f(x), g(x))$ (continuous by 1.8) and the second is $\wedge$ on $\mathbb{R}(L) \times \mathbb{R}(L)$ (continuous by 2.5). $\square$

Characteristic functions of L-sets

Let $(L, ')$ be a complete lattice. A *characteristic function* of an $a \in L^X$ is a map $\chi_a\colon X \to I(L)$ defined by

$$\chi_a(x) = (a(x))$$

for all $x \in X$. These functions have many properties analogous to those of characteristic functions of sets (cf. [22] and [24]).

2.10 Remark. *If $\mathcal{A} \subset L^X$ and $a \in L^X$, then:*

(1) $\chi_{\vee\mathcal{A}} = \bigvee_{a \in \mathcal{A}} \chi_a$,

(2) $\chi_{\wedge\mathcal{A}} = \bigwedge_{a \in \mathcal{A}} \chi_a$.

(3) $(\chi_a)^\bullet = \chi_{a'}$. $\square$

2.11 Remark. *Let X be an L-topological space and $a \in L^X$. Then:*

(1) *a is open if and only if $\chi_a \in LSC^*(X)$,*

(2) *a is closed if and only if $\chi_a \in USC^*(X)$.* □

The following decomposition lemma [21] generalizes the representation $a = \bigvee_{t\in I}(t \wedge 1_{a^{-1}(t,1]})$ holding for each $a \in I^X$.

2.12 Lemma. *For L a complete lattice and X a set we have*

$$a = \bigvee_{t\in I}(\mathbf{t} \wedge \chi_{R_t a})$$

for all $a \in I(L)^X$.

Proof. Let $x \in X$ and $s < 1$. Since $R_s\mathbf{t} = 1_\emptyset$ if $s \geq t$, we have (cf. the proof of (1) of 2.8)

$$\begin{aligned} R_s\left(\bigvee_{t\in I}(\mathbf{t} \wedge \chi_{R_t a})\right)(x) &= \bigvee_{t\in I}(R_s\mathbf{t} \wedge R_s\chi_{R_t a})(x) = \bigvee_{t>s} R_s\chi_{R_t a}(x) \\ &= \bigvee_{t>s} R_t a(x) = R_s a(x), \end{aligned}$$

which yields the required equality. □

Separating L-sets by continuous L-real functions

Let $(L, ')$ be a complete lattice and X an L-topological space. Given $a, b \in L^X$, we write

$$a \prec b$$

if and only if

$$\text{(S)} \qquad a \ \leq \ L_t{}'f \ \leq \ R_s f \ \leq \ b$$

for some $f \in C(X)$ and $s < t$ in $\mathbb{R}$.[4] (Note that the point is in the first and third inequalitites, because the second one always holds.)

If necessary, we shall write

$$a \prec_f b$$

to indicate the function involved in (S).

Note. Let us have a look into what does (S) mean when $L = \mathbf{2}$. Then $a = 1_A$, $b = 1_B$ and, using the isomorphism e: R $\to \mathbb{R}(\mathbf{2})$, $e(t) = [t]$, we have with

[4] Referring to the terminology introduced by S.E. Rodabaugh ([48, §3]) a is called *completely inside* b iff $a \prec b$ (see 2.17 infra).

$g = e^{-1} \circ f \colon X \to \mathbb{R}$ the following: $L'_t f = L'_t \circ e \circ e^{-1} \circ f = 1_{[t,\infty)} g$ and, similarly, $R_s f = 1_{(s,\infty)} g$. Thus, (S) becomes

$$1_A \le 1_{[t,\infty)} g \le 1_{(s,\infty)} g \le 1_B,$$

and this says that $g(A) \subset [t, \infty)$ and $g(X \backslash B) \subset (-\infty, s]$. In particular, if $f \colon X \to I(\mathbf{2})$, $s = 0, t = 1$, then $g(A) = \{1\}$ and $g(X \backslash B) = \{0\}$.

2.13 Lemma. *Let $(L, ')$ be complete. For $a, b \in L^X$ the following are equivalent:*

(1) $a \prec b$,

(2) $a \le L_1{}'g \le R_0 g \le b$ *for some* $g \in C^*(X)$,

(3) $a \le R_0{}'g \le L_1 g \le b$ *for some* $g \in C^*(X)$,

(4) $\chi_a \le g \le \chi_b$ *for some* $g \in C^*(X)$.

Proof. (1) $\Rightarrow$ (2): Let $a \prec_f b$ (as in (S)). Let φ be an increasing bijection of $\mathbb{R}$ onto itself such that $\varphi(0) = s$ and $\varphi(1) = t$. For each $x \in X$ we set $f_\varphi(x) = [f_x \circ \varphi]$. By 2.1(6), $f_\varphi \in C(X)$. By 2.8(1) $g = (\mathbf{0} \vee f_\varphi) \wedge \mathbf{1} \in C^*(X)$ and we have (as in proving 2.8(1)):

$$L_1{}'g = L_1{}'\mathbf{0} \vee L_1{}'f_\varphi = L_1{}'f_\varphi = L_{\varphi(1)}{}'f = L_t{}'f$$

and, similarly, $R_0 g = R_s f$. Thus, g is as required.

(2) $\Rightarrow$ (1): Obvious

(2) $\Leftrightarrow$ (3): For each $g \in C^*(X)$ we have $R_0 g^\bullet = L_1 g$, and since $(\cdot)^\bullet$ is an involution, we also have $R_0 g = L_1 g^\bullet$.

(2) $\Rightarrow$ (4): We must have $R_t \chi_a \le R_t g$ $(0 \le t < 1)$ and $L_t{}'g \le L_t{}'\chi_b$ $(0 < t \le 1)$. Indeed, $R_t \chi_a = a \le L_1{}'g \le R_t g$ and $L_t{}'g \le R_0 g \le b = L_t{}'\chi_b$.

(4) $\Rightarrow$ (2): $a = L_1{}'\chi_a \le L_1{}'g \le R_0 g \le R_0 \chi_b = b$. □

2.14 Lemma. *Let X be an L-topological space. If $(L, ')$ is complete and a, b, c, $d \in L^X$, then:*

(1) $a \prec b \Rightarrow a \le b$ (*in fact,* $a \prec b \Rightarrow \overline{a} \prec \operatorname{Int} b$).

(2) $c \le a \prec b \le d \Rightarrow c \prec d$.

(3) $a \prec b \Rightarrow b' \prec a'$.

(4) $a \prec b \Rightarrow a \prec u \prec b$ *for some* $u \in o(X)$.

(5) *If* $g \in C(Y, X)$, *and* $a \prec b$, *then* $ag \prec bg$ (*in* L^Y).

Furthermore, for $(L, ')$ a meet-continuous lattice we have:

(6) $a \prec c$ *and* $b \prec c \quad \Rightarrow \quad a \vee b \prec c$.

(7) $a \prec b$ *and* $a \prec c \quad \Rightarrow \quad a \prec b \wedge c$.

Proof. (1) and (2) are trivial. For (3), if $a \prec_f b$ with $f \in C^*(X)$, then $b' \prec_{f^\bullet} a'$. To prove (4), observe that

$$a \le L_1{}'f \le R_{\frac{1}{2}} f = u \le L_{\frac{1}{2}}{}'f \le R_0 f \le b,$$

i.e. $a \prec u \prec b$ and u is open. For (5), if $a \prec_f b$, then $ag \prec_{f\circ g} bg$.

Now assume $(L,')$ is meet-continuous. For (6), if $a \prec_f c$ and $b \prec_g c$, then $a \vee b \prec_{f \vee g} c$ where $f \vee g \in C(X)$ by 2.9. Similarly for (7). □

Definition. If $a \prec b'$, then a and b are said to be *completely separated.* An $a \in L^X$ is called an *L-zero-set* if

$$a = R_0{}'f$$

for some $f \in C(X)$ and $b \in L^X$ is called an *L-cozero-set* if b' is an L-zero-set.

2.15 Lemma. *Let X be an L-topological space with $(L,')$ a complete lattice. For $a \in L^X$, the following are equivalent:*

(1) *a is an L-zero-set.*

(2) *There exists $f \in C^*(X)$ such that $a = R_0{}'f$.*

(3) *There exists $f \in C^*(X)$ and $t \in I$ such that $a = R_t{}'f$.*

(4) *There exists $f \in C^*(X)$ and $t \in I$ such that $a = L_t{}'f$.*

Proof. (1) $\Rightarrow$ (2): If $a = R_0{}'g$ with $g \in C(X)$, then $a = R_0{}'f$ with $f = (\mathbf{0} \vee g) \wedge \mathbf{1} \in C^*(X)$.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Leftrightarrow$ (4): $a = R_t{}'f = L_{1-t}{}'f^\bullet$. Similarly, $a = L_t{}'f = R_{1-t}{}'f^\bullet$.

(3) $\Rightarrow$ (1): $a = R_t{}'f = R_0{}'h$ for $h \in C(X)$ such that $h_x(s) = f_x(s+t)$ for all $s \in \mathbb{R}$. □

Generating continuous L-real functions

We develop here a general procedure of constructing a continuous L-real function using monotone families of L-sets, cf. [14, 20, 23]. In our presentation we follow [28].

Let $(L,')$ be a complete lattice and let X be a set. Each non-decreasing family $\mathcal{C} = \{c_r : r \in \mathbb{Q}\} \subset L^X$ such that $\bigvee \mathcal{C} = 1_X$ and $\bigwedge \mathcal{C} = 1_\emptyset$ is called a *scale.*

Let for every $x \in X$ and $t \in \mathbb{R}$

$$f_x(t) = \bigwedge_{r<t} c_r{}'(x).$$

Notice that $f_x \in \mathbb{R}_L$. Indeed it is order-reversing, $\bigwedge f_x(\mathbb{R}) = (\bigvee \mathcal{C})'(x) = \bot$, and $\bigvee f_x(\mathbb{R}) \geq \bigvee_{t\in\mathbb{R}} \bigvee_{r>t} c_r'(x) = (\bigwedge \mathcal{C})'(x) = \top$.

Also, it is easy to see that $f_x = f_x^-$.

The function $f \colon X \to \mathbb{R}(L)$, defined by

$$f(x) = [f_x]$$

is said to be *generated* by the scale $\mathcal{C}$.

In what follows p, q, r stand for rationals.

2.16 Lemma. *Let $(L, ')$ be complete and let $f, g\colon X \to \mathbb{R}(L)$ be generated by the scales $\{c_r : r \in \mathbb{Q}\}$ and $\{d_r : r \in \mathbb{Q}\}$. Then:*

(1) $L_t f = \bigvee_{r<t} c_r$ *for every* $t \in \mathbb{R}$.

(2) $R_t' f = \bigwedge_{r>t} c_r$ *for every* $t \in \mathbb{R}$.

(3) $L_r f \le c_r \le R_r' f$ *for every* $r \in \mathbb{Q}$.

(4) $f \le g$ *if and only if* $d_r \le c_q$ *whenever* $r < q$.

(5) f *is generated by both* $\{L_r f : r \in \mathbb{Q}\}$ *and* $\{R_r' f : r \in \mathbb{Q}\}$.

For X an L-topological space we have:

(6) $f \in C(X)$ *if and only if* $\overline{c_r} \le \mathrm{Int}\, c_q$ *whenever* $r < q$.

(7) $f \in C^*(X)$ *if and only if* $f \in C(X)$, *and* $c_r = 1_\emptyset$ *if* $r < 0$, *and* $c_r = 1_X$ *if* $r > 1$.

Proof. (1) For each $x \in X$ we have $L_t' f(x) = f_x^-(t) = f_x(t) = \bigwedge_{r<t} c_r'(x)$. Hence $L_t f = \bigvee_{r<t} c_r$.

(2) Since $R_t' f = \bigwedge_{s>t} L_s f$, hence by (1) we obtain

$$R_t' f = \bigwedge_{s>t} \bigvee_{r<s} c_r \le \bigwedge_{s>t} \bigwedge_{r>s} c_r = \bigwedge_{r>t} c_r.$$

Since $\bigwedge_{r>t} c_r \le \bigvee_{r<s} c_r$ for every $s > t$, hence

$$\bigwedge_{r>t} c_r \le \bigwedge_{s>t} \bigvee_{r<s} c_r = R_t' f.$$

(3) Obvious by (1) and (2).

(4) If $r < q$ then $d_r \le R_r' g \le R_r' f \le L_q f \le c_q$. Conversely, for every $q < t$, $d_q \le \bigvee_{r<t} c_r = L_t f$. Thus, $L_t g = \bigvee_{q<t} d_q \le L_t f$, i.e. $f \le g$.

(5) Clearly, both $\{L_r f\}$ and $\{R_r' f\}$ are scales by 2.1(1). Since $R_p' f \le L_q f \le R_r' f$ whenever $p < q < r$, they generate the same function by (4). Let h be the function generated by $\{L_r f\}$. Then for all $x \in X$ and $t \in \mathbb{R}$, $L_t' h(x) = \bigwedge_{s<t} \bigwedge_{r<s} L_r' f(x) = L_t' f(x)$.

(6) $\Rightarrow$: For $r < q$ we have by (1) and (2)

$$c_r \le \bigwedge_{p>r} c_p = R_r' f \le L_q f = \bigvee_{p<q} c_p \le c_q.$$

Since $R_r' f$ is closed and $L_q f$ is open, we obtain $\overline{c_r} \le \mathrm{Int}\, c_q$.

$\Leftarrow$: Since $\bigvee_{r<t} c_r = \bigvee_{r<t} \mathrm{Int}\, c_r$, hence $L_t f$ is open by (1). Similarly, since $\bigwedge_{r>t} c_r = \bigwedge_{r>t} \overline{c_r}$, hence $R_t' f$ is closed by (2). By 1.4(3), $f \in C(X)$.

(7) f is $I(L)$-valued if and only if

$$\bigvee_{r<0} c_r = L_0 f = 1_\emptyset = R_1 f = \bigvee_{r>1} c_r'. \quad \square$$

2.17 Lemma. *Let X be an L-topological space with $(L,')$ a complete lattice. For $a, b \in L^X$, the following are equivalent:*

(1) $a \prec b$.

(2) *There exists a family $\{u_r \in o(X) : r \in \mathbb{Q} \cap (0,1)\}$ such that $a \leq u_r \leq b$ for every $r \in \mathbb{Q} \cap (0,1)$, and $\overline{u_r} \leq u_q$ whenever $r < q$.*

Proof. (1) $\Rightarrow$ (2): Referring to 2.13, assume $a \leq R_0{}'f \leq L_1 f \leq b$ with $f \in C^*(X)$. For every $r \in \mathbb{Q} \cap (0,1)$ we have $R_0{}'f \leq L_r f \leq L_1 f$. Also, $L_r f \leq R_p{}'f \leq L_q f$ if $0 < r < p < q < 1$. Hence $\overline{L_r f} \leq L_q f$ if $r < q$. Therefore, $\{L_r f : r \in \mathbb{Q} \cap (0,1)\}$ is as required.

(2) $\Rightarrow$ (1): Let $\{u_r \in o(X) : r \in \mathbb{Q} \cap (0,1)\}$ be given. Put $u_r = 1_\emptyset$ for $r \leq 0$ and $u_r = 1_X$ for $r \geq 1$. Then $\mathcal{U} = \{u_r \in o(X) : r \in \mathbb{Q}\}$ is a scale of open L-sets such that $\overline{u_r} \leq u_q$ if $r < q$ in $\mathbb{Q}$. Let f be the function generated by $\mathcal{U}$. By 2.16(7), $f \in C^*(X)$. Moreover, since $a \leq u_r \leq b$ if $0 < r < 1$, we obtain by (1) and (2) of 2.16:

$$a \leq \bigwedge_{r>0} u_r = R_0{}'f \leq L_1 f = \bigvee_{r<1} u_r \leq b. \quad \square$$

3 Separation axioms

In this section we shall consider only those separation axioms which are relevant to the main subject of this article, viz. extending continuous L-real functions and embedding L-topological spaces. Those axioms include: L-T_0, L-regularity, L-tychonoffness, L-normality and perfect L-normality. All of them but L-T_0 were introduced by Hutton [14, 15] and Hutton and Reilly [17]. The L-T_0 axiom has been introduced independently by a number of authors: Liu [32], Rodabaugh [45], Wuyts and Lowen [54], Šostak [49]. In actual fact, Liu has a different and much more sophisticated, yet equivalent, formulation (see [27] for details).

Completely L-regular spaces and L-regular spaces

In what follows we adopt the following notation: given $\mathcal{A} \subset L^X$ and $b \in L^X$, we write

$$\mathcal{A} \prec b$$

if and only if $a \prec b$ for every $a \in \mathcal{A}$.

Definition. Let $(L,')$ be a complete lattice. An L-topological space is *completely L-regular* if, whenever $u \in L^X$ is open, there exists a family $\mathcal{A} \subset L^X$ such that

$$\mathcal{A} \prec u \quad \text{and} \quad \bigvee \mathcal{A} = u.$$

We shall say that $\mathcal{A}$ is a *$\prec$-decomposition* of u. If $\mathcal{A}$ consists of open L-sets, we say it is an *open $\prec$-decomposition.*

The following are taken from [27].

3.1 Theorem. *Let $(L,')$ be a complete lattice. For X an L-topological space, the following are equivalent:*

(1) *X is completely L-regular.*

(2) *For every basis $\mathcal{B}$ and each $u \in \mathcal{B}$, u has an open $\prec$-decomposition.*

(3) *$u = \bigvee\{v \in o(X) : v \prec u\}$ for every open $u \in L^X$.*

(4) *$\mathrm{Int}_X\, a = \bigvee\{v \in o(X) : v \prec a\}$ for every $a \in L^X$.*

(5) *For every $g \in LSC^*(X)$ there exists a family $\mathcal{F} \subset C^*(X)$ such that $g = \bigvee \mathcal{F}$, a pointwise sup, i.e. $g(x) = \bigvee_{f\in\mathcal{F}} f(x)$ for all $x \in X$.*

(6) *The family of all L-cozero-sets is a base for X.*

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{A}$ be a $\prec$-decomposition of u. Then $a \prec u$ for every $a\in\mathcal{A}$ and, using 2.14(4), there exists an open v_a such that $a \prec v_a \prec u$. Clearly, $\bigvee_{a\in\mathcal{A}} v_a = u$. Thus $\{v_a : a\in\mathcal{A}\}$ is an open $\prec$-decomposition of u.

(2) $\Rightarrow$ (1): Obvious

(2) $\Rightarrow$ (3): Let u be open (it is a member of the basis $o(X)$). For $\mathcal{V}$ an open $\prec$-decomposition of u we have $u = \bigvee \mathcal{V} \le \bigvee\{w \in o(X) : w \prec u\} \le u$.

(3) $\Rightarrow$ (4): For every $a \in L^X$ we have

$$\begin{aligned}\mathrm{Int}_X\, a &= \bigvee\{u\in o(X) : u \le a\} = \bigvee_{u\le a} \bigvee\{v\in o(X) : v \prec u\}\\ &\le \bigvee\{v\in o(X) : v \prec a\} \le \mathrm{Int}_X\, a.\end{aligned}$$

(4) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (5): Let $f\in LSC^*(X)$. For every $t \in I$ we let $\mathcal{V}_t = \{v\in o(X) : v \prec R_t f\}$. By 2.13, $\chi_v \le f_v \le \chi_{R_t f}$ for some $f_v\in C^*(X)$. By 2.10 we thus have

$$\chi_{R_t f} = \bigvee_{v\in\mathcal{V}_t} f_v.$$

Using 2.8 and 2.12 we obtain

$$f = \bigvee_{t\in I}(\mathbf{t} \wedge \bigvee_{v\in\mathcal{V}_t} f_v) = \bigvee_{t\in I}\bigvee_{v\in\mathcal{V}_t} \mathbf{t}\wedge f_v,$$

a pointwise sup of members of $C^*(X)$.

(5) $\Rightarrow$ (6): Let u be open in X. Then $\chi_u\in LSC^*(X)$ and, thus, there exists an $\mathcal{F} \subset C^*(X)$ with $\bigvee \mathcal{F} = \chi_u$. Clearly, $u = R_0\chi_u = \bigvee_{f\in\mathcal{F}} R_0 f$.

(6) $\Rightarrow$ (1): A basic open L-set $R_0 f$, where $f\in C^*(X)$, has $\{L_s{}'f : 0 < s \le 1\}$ as a $\prec$-decomposition. Indeed, by 2.4, $R_0 f = \bigvee_{s>0} L_s{}'f$ and $L_s{}'f \prec_f R_0 f$ for all $s \in (0,1]$. $\square$

3.2 Lemma. *Let $(L,')$ be a frame. Let $\mathcal{S} \subset L^X$ generate the L-topology of X. Then X is completely L-regular if and only if every member of $\mathcal{S}$ has an open $\prec$-decomposition.*

Proof. To prove the non-trivial implication, let J be a finite index set. Let $\{v_j : j \in J\} \subset \mathcal{S}$ and let $v = \bigwedge_{j\in J} v_j$. We show that v has an open $\prec$-decomposition. Let $\mathcal{V}_j$ be an open $\prec$-decomposition of v_j. Let $\varphi \in \Phi = \prod_{j\in J} \mathcal{V}_j$ and let $v_\varphi = \bigwedge_{j\in J} \varphi(j)$. Since $\varphi(j) \in \mathcal{V}_j$, hence $\varphi(j) \prec v_j$. Thus $v_\varphi \prec v_j$ for every $j \in J$. Hence, by 2.14(7) we have

$$v_\varphi \prec \bigwedge_{j\in J} v_j = v.$$

By (ID*) we get

$$v = \bigwedge_{j\in J} \bigvee \mathcal{V}_j = \bigvee_{\varphi\in\Phi} \bigwedge_{j\in J} \varphi(j) = \bigvee_{\varphi\in\Phi} v_\varphi.$$

Thus, every basic open L-set has an open $\prec$-decomposition, and X is completely L-regular by 3.1(2). □

3.3 Remark. *Let $(L,')$ be a frame. The following statements hold:*

(1) *If $f\colon X \to Y$ and $(Y,\mathcal{T})$ is completely L-regular, then $(X, f^{\leftarrow}(\mathcal{T}))$ is completeley L-regular.*

(2) *Every subspace of a completely L-regular space is completely L-regular.*

(3) *A supremum of completely L-regular L-topologies is completely L-regular.*

(4) *A product of completely L-regular spaces is completely L-regular.*

Proof. By 2.14(5) we get (1). Clearly (2) is a consequence of (1) (note that these hold for L a complete lattice). Also, (3) follows from 3.2, and (4) is a consequence of (1) and (3). □

3.4 Remark. *If $(L,')$ is a frame, then $\mathbb{R}(L)$, its subspaces and products of those subspaces, are all completely L-regular.*

Proof. After 3.3, it suffices to show $\mathbb{R}(L)$ is completely L-regular. We have $R_t = \bigvee_{s>t} L_s{}'$ and $L_s{}' \prec_{id} R_t$, id being the identity map of $\mathbb{R}(L)$ into itself, as well as $L_t = \bigvee_{s<t} R_s{}'$ with $R_s{}' \prec_{id} L_t$. By 3.2 we get the assertion. □

3.5 Theorem. *Let $(L,')$ be a complete lattice and let X be an L-topological space. Consider the following statements:*

(1) *X is completely L-regular.*

(2) *$k = \bigwedge\{f^{\leftarrow}(\overline{f^{\rightarrow}(k)}) : f \in C^*(X)\}$ for every $k \in \kappa(X)$.*

(3) *$\mathrm{Cl}_X\, a = \bigwedge\{f^{\leftarrow}(\overline{f^{\rightarrow}(a)}) : f \in C^*(X)\}$ for every $a \in L^X$.*

(4) *$\bigcup\{f^{\leftarrow}(o(I(L)) : f \in C^*(X)\}$ is a base for X.*

(5) $o(X) = \bigvee\{f^{\leftarrow}(o(I(L)) : f \in C^*(X)\}$. *i.e. X has the smallest L-topology with respect to which all members of $C^*(X)$ are continuous.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).

If L is a frame, then (5) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2): By 3.1(6), for every closed L-set k there exists $\mathcal{F} \subset C^*(X)$ such that

$$k = \bigwedge_{f \in \mathcal{F}} f^{\leftarrow}(R_0{}').$$

Therefore $f^{\rightarrow}(k) \leq f^{\rightarrow}(f^{\leftarrow}(R_0{}')) \leq R_0{}'$, a closed L-set. Hence $\overline{f^{\rightarrow}(k))} \leq R_0{}'$ for all $f \in \mathcal{F}$. Therefore

$$\bigwedge_{f \in C^*(X)} f^{\leftarrow}(\overline{f^{\rightarrow}(k)}) \leq \bigwedge_{f \in \mathcal{F}} f^{\leftarrow}(R_0{}') = k.$$

Clearly, $k \leq \bigwedge\{f^{\leftarrow}(\overline{f^{\rightarrow}(k)}) : f \in C^*(X)\}$ always.

(2) $\Rightarrow$ (3): Let $\mathcal{F} = C^*(X)$. For every $a \in L^X$ we have

$$\begin{aligned} \mathrm{Cl}_X\, a &= \bigwedge\{k \in \kappa(X) : a \leq k\} \\ &= \bigwedge_{a \leq k} \bigwedge_{f \in \mathcal{F}} f^{\leftarrow}(\overline{f^{\rightarrow}(k)}) \\ &\geq \bigwedge_{f \in \mathcal{F}} \bigwedge\{f^{\leftarrow}(\overline{f^{\rightarrow}(k)}) : f^{\leftarrow}(\overline{f^{\rightarrow}(a)}) \leq f^{\leftarrow}(\overline{f^{\rightarrow}(k)})\} \\ &\geq \bigwedge_{f \in \mathcal{F}} f^{\leftarrow}(\overline{f^{\rightarrow}(a)}). \end{aligned}$$

By continuity of f we have $f^{\rightarrow}(\overline{a}) \leq \overline{f^{\rightarrow}(a)}$, hence

$$\mathrm{Cl}_X\, a \leq \bigwedge_{f \in \mathcal{F}} f^{\leftarrow}(\overline{f^{\rightarrow}(a)}).$$

(3) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (4): With $u = k'$ we obtain from (2)

$$u = \bigvee_{f \in C^*(X)} f^{\leftarrow}(\,\mathrm{Int}(f^{\rightarrow}(u')')),$$

and the assertion follows.

(4) $\Rightarrow$ (2): Let k be closed. Let $\mathcal{F} \subset C^*(X)$ and let $\{k_f : f \in \mathcal{F}\} \subset \kappa(I(L))$ be such that $k = \bigwedge\{f^{\leftarrow}(k_f) : f \in \mathcal{F}\}$. Then the proof of (1) implies (2) goes unchanged with k_f playing the role of $R_0{}'$.

(4) $\Rightarrow$ (5): Obvious.

Now assume L is a frame.

(5) $\Rightarrow$ (1): $o(X)$ is a supremum of completely L-regular L-topologies by 3.4 and 3.3(1), hence completely L-regular by 3.3(3). $\square$

3.6 Proposition. *Let $(L,')$ be a complete lattice and let X be an L-topological space such that $\alpha \in o(X)$ if and only $\alpha' \in o(X)$. Then the following are equivalent:*

(1) $o(X) \subset \mathcal{C}_X$.

(2) *X is completely L-regular and each member of $C^*(X)$ is constant.*

Proof. (1) $\Rightarrow$ (2): X is completely L-regular, because we have $\alpha = L_1{}'\mathcal{X}_\alpha \leq R_0\mathcal{X}_\alpha \leq \alpha$ and $\mathcal{X}_\alpha \in C^*(X)$. Thus $\alpha \prec \alpha$. Also, $f(x) \not\leq f(y)$ with $x \neq y$ implies $R_t f(x) \not\leq R_t f(y)$ for some $0 \leq t < 1$, a contradiction.

(2) $\Rightarrow$ (1): Assume there is a non-constant open L-set. Then there is a closed L-set k with $k(x) \neq k(y)$ for some $x \neq y$. By (1) $\Rightarrow$ (2) of 3.5, there is an $f \in C^*(X)$ such that $\overline{f^{\rightarrow}(k)}(f(x)) \neq \overline{f^{\rightarrow}(k)}(f(y))$, i.e. $f(x) \neq f(y)$, a contradiction. □

3.7 Proposition. *Let $(L,')$ be a frame. If X is a completely L-regular space, then so is X^c.*

Proof. 3.6 and 3.3(3). □

Definition. Let $(L,')$ be a complete lattice. An L-topological space X is called *L-regular* if

$$u = \bigvee \{v \in o(X) : \overline{v} \leq u\}$$

for every $u \in o(X)$.

Clearly every completely L-regular space is L-regular (cf. 3.1(3)).

We observe, following [29], that with[5]

$$v \sqsubset u \quad \text{iff} \quad \overline{v} \leq u$$

an L-regular space is one in which every open L-set has an open $\sqsubset$-decomposition. Also note that $\sqsubset$ has all the properties of the relation $\prec$ stated in 2.14, except of the property (4) of 2.14. Thus the proof of 3.1 applies unchanged to the case of $\sqsubset$. Consequently for $(L,')$ a frame, if $\mathcal{S} \subset L^X$ generates the L-topology of X, then X is L-regular if and only if $u = \bigvee\{v \in o(X) : \overline{v} \leq u\}$ for every $u \in \mathcal{S}$. Thus, we also have a counterpart of 3.3 and 3.7. In particular, we have:

3.8 Remark. *Let $(L,')$ be a frame. Every subspace of an L-regular space is L-regular. A product of L-regular spaces is L-regular. If X is L-regular, so is X^c.* □

Definition. Let $(L,')$ be a complete lattice. An L-topological space X is called L-T_0 if, whenever $x \neq y$ in X, there exists an open L-set u such that $u(x) \neq u(y)$.

[5] Referring again to the terminology porposed by S.E. Rodabaugh ([48, §3]) v is *quite–inside* u iff $v \sqsubset u$.

3.9 Remark. (1) *If $(L,')$ is complete, then every subspace of an L-T_0 space is L-T_0 and a product of L-T_0 spaces is L-T_0.*

(2) *If $(L,')$ is a frame and $\prod_{j\in J} X_j$ is L-T_0, then so is every X_j.*

Proof. (1) is obvious. For (2), let $\prod_{j\in J} X_j$ be L-T_0. Let $x_j \neq y_j$ in X_j. Select x and y in the product such that $\pi_i(x) = \pi_i(y)$ for all $i \neq j$. We observe that the open L-set in the definition of L-T_0 can without loss of generality be assumed to be a basic open L-set. Let $u = \bigwedge_{i\in J_0} u_i\pi_i$ ($J_0 \subset J$ is finite) be the basic open L-set of the product which separates x and y. Clearly, there is $i_0 \in J_0$ such that $u_{i_0}\pi_{i_0}(x) \neq u_{i_0}\pi_{i_0}(y)$. The way we have chosen x and y clearly implies that $i_0 = j$ and we are done. □

Definition. Let $(L,')$ be a complete lattice. An L-topological space X is called *L-Tychonoff* if it is completely L-regular and L-T_0.

3.10 Proposition. *Let $(L,')$ be a frame. The following statements hold:*

(1) *Every subspace of an L-Tychonoff space is L-Tychonoff.*

(2) *A supremum of L-Tychonoff L-topologies is L-Tychonoff.*

(3) *A product of L-Tychonoff spaces is L-Tychonoff.*

(4) *If X is L-Tychonoff, then so is X^c.*

Proof. By 3.3 and 3.9 we get (1)–(3). By 3.7 we have (4). □

3.11 Remark. *Let $(L,')$ be a frame and let X_j be weakly stratified for every $j \in J$. Then, if $\prod_{j\in J} X_j$ is L-T_0 (resp., L-regular, completely L-regular, L-Tychonoff), then so is every X_j.*

Proof. By 1.11 and 3.9 (resp., 3.8, 3.3, 3.10). □

It is of interest to note that L-Tychonoff spaces (in fact, L-regular L-T_0 spaces) admit certain pointwise separation property which when $L = \mathbf{2}$ becomes the Hausdorff separation property. Namely (cf. [27]):

3.12 Proposition. *Let $(L,')$ be a complete lattice. Let X be an L-regular space. Then X is L-T_0 if and only if it satisfies*

L-T_2: If $x \neq y$ in X, there exist $u, v \in o(X)$ such that $u(x) \not\leq u(y)$, $v(y) \not\leq v(x)$, and $u \leq v'$.

Proof. Clearly, always L-$T_2 \Rightarrow L$-T_0. Assume X is L-T_0 and L-regular. Let $x \neq y$ in X. Let $u(x) \neq u(y)$ for some $u \in o(X)$. Then either $u(x) \not\leq u(y)$ or $u(y) \not\leq u(x)$. We without loss of generality assume that the first inequality holds. Since X is L-regular, there is a family $\mathcal{V} \subset o(X)$ such that $u = \bigvee \mathcal{V} = \bigvee \overline{\mathcal{V}}$ where $\overline{\mathcal{V}} = \{\overline{v} : v \in \mathcal{V}\}$. Thus $(\bigvee \mathcal{V})(x) \not\leq (\bigvee \overline{\mathcal{V}})(y)$, so that there exist $v \in \mathcal{V}$ such that $v(x) \not\leq \overline{v}(y)$. Consequently, $v(x) \not\leq v(y)$, and $\overline{v}(x) \not\leq \overline{v}(y)$. Also $w(y) \not\leq w(x)$ where $w = (\overline{v})'$. Finally note that $w = \mathrm{Int}(v') \leq v'$. Thus X satisfies L-T_2. □

For more about L-T_2 we refer to [27].

L-normal spaces

Definition. Let $(L,')$ be a complete lattice. An L-topological space X is called *L-normal* if whenever $k \le u$ (k is closed and u is open), there exists an open v such that

$$k \le v \le \overline{v} \le u.$$

A *perfectly L-normal* space is an L-normal space in which every open L-set is F_σ, i.e a countable sup of closed L-sets. An $a \in L^X$ is G_δ if a' is F_σ.

The following comes from [27]:

3.13 Theorem. *Let $(L,')$ be a meet-continuous lattice. Every L-regular space X with a countable base is perfectly L-normal.*

Proof. By L-regularity, every open L-set u can be written as $u = \bigvee \mathcal{V} = \bigvee \overline{\mathcal{V}}$, where $\mathcal{V} \subset o(X)$ and $\overline{\mathcal{V}} = \{\overline{v} : v \in \mathcal{V}\}$. By 1.3, there is a countable subfamily $\{v_n : n \in \mathrm{N}\}$ of $\mathcal{V}$ such that

$$(*) \qquad u = \bigvee_n v_n \le \bigvee_n \overline{v_n} \le u\,.$$

Thus, every open L-set is an F_σ.

To show L-normality, let $k \le u$ where k is closed and u is open. By $(*)$ (and its dual version for closed L-sets), there exist countable families $\{v_n : n \in \mathrm{N}\}$ and $\{w_n : n \in \mathrm{N}\}$ consisting of open L-sets such that

$$k \le \bigvee_n v_n = \bigvee_n \overline{v_n} = u$$

and

$$k = \bigwedge_n w_n = \bigwedge_n \overline{w_n} \le u.$$

Clearly, we can assume both these families are non-decreasing. Define

$$h_1 = v_1,$$

$$h_n = v_n \wedge \bigwedge_{i<n} w_i \quad \text{for} \quad n \ge 2.$$

We observe that if $m \le n$, then $h_m \le v_m \le \bigvee_{i \le n} \overline{v_i}$, and if $m > n$, then $h_m \le w_n \le \overline{w_n}$. Thus $h_m \le \overline{w_n} \vee \bigvee_{i \le n} \overline{v_i}$ for all m and n, and consequently

$$h = \bigvee_{m \ge 1} h_m \le \bigwedge_{n \ge 1} (\overline{w_n} \vee \bigvee_{i \le n} \overline{v_i}) = f,$$

where, thus, h is open and f is closed. Furthermore, we have

$$\begin{aligned} k &\le \bigvee_n v_n \wedge \bigwedge_n w_n = \bigvee_n (v_n \wedge \bigwedge_n w_n) \quad \text{(by (MC))} \\ &\le v_1 \vee \bigvee_{n \ge 2} (v_n \wedge \bigwedge_{i<n} w_i) = \bigvee_n h_n = h \end{aligned}$$

and

$$\begin{aligned} f &\leq \bigwedge_n(\overline{w_n} \vee \bigvee_n \overline{v_n}) = \bigwedge_n \overline{w_n} \vee \bigvee_n \overline{v_n} \quad \text{(by the dual of (MC))} \\ &\leq u. \quad \square \end{aligned}$$

The next theorem is due to Hutton [14]. It will also later be obtained as a simple corollary from an insertion theorem for L-normal spaces.

3.14 Theorem (Urysohn's lemma). *For X an L-topological space with $(L,')$ a complete lattice, the following are equivalent:*

(1) *X is L-normal,*

(2) *If k is closed, u is open, and $k \leq u$, then there exists an $f \in C^*(X)$ such that $k \leq L_1{}'f \leq R_0 f \leq u$.*

Proof. (1) $\Rightarrow$ (2): Let $\{q_n\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$ with $q_1 = 0$ and $q_2 = 1$. For every $n \geq 2$ we inductively construct a family $\{u_{q_i} : i < n\}$ of open L-sets such that

$$(I_n) \qquad k \leq u_{q_i} \leq \overline{u_{q_i}} \leq u_{q_j} \leq u \text{ if } q_i < q_j \quad (i,j < n).$$

There is u_{q_1} with $k \leq u_{q_1} \leq \overline{u_{q_1}} \leq u$. With $u_{q_2} = u$ this is (I_2). Assume that u_{q_i} are already defined for $i < n$ and satisfy (I_n). Let $q_l = \max\{q_i : q_i < q_n, i < n\}$ and $q_r = \min\{q_i : q_i > q_n, i < n\}$. Then $q_l \leq q_r$ and thus there exists u_{q_n} with $\overline{u_{q_l}} \leq u_{q_n} \leq \overline{u_{q_n}} \leq u_{q_r}$. Now u_{q_i} $(i < n+1)$ satisfy (I_{n+1}). We have constructed a family $\{u_r : r \in \mathbb{Q} \cap (0,1)\}$ of open L-sets such that $k \leq u_r \leq u$ and $\overline{u_r} \leq u_q$ if $r < q$. By 2.17 we conclude that $k \prec u$.
(2) $\Rightarrow$ (1): If $k \prec_f u$ with $f \in C^*(X)$, then $k \leq v \leq \overline{v} \leq u$ with $v = R_{\frac{1}{2}} f$. $\square$

3.15 Corollary. *Let $(L,')$ be a complete lattice. Every L-normal and L-regular space is completely L-regular.* $\square$

Separation properties of L-reals

We summarize here separation properties of $\mathbb{R}(L)$, $I(L)$, their products as well as stratifications of them. These results are easily deduced from some of the above propositions, see also [27, 30]. We remark that we do not require complete distributivity of L (cf. [45]).

3.16 Proposition. *Let $(L,')$ be a frame. The following hold:*

(1) *$\mathbb{R}(L)^J$, $(\mathbb{R}(L)^c)^J$, $I(L)^J$ and $(I(L)^c)^J$ are L-Tychonoff for any J.*

(2) *If $|J| \leq \aleph_0$, then $\mathbb{R}(L)^J$ and $I(L)^J$ are perfectly L-normal.*

(3) *If $|J| \leq \aleph_0$ and $|L| \leq \aleph_0$, then $(\mathbb{R}(L)^c)^J$ and $(I(L)^c)^J$ are perfectly L-normal.*

Proof. (1) First note that $\mathbb{R}(L)$ and $I(L)$ are L-T_0 (in fact, they are L-T_2 for any L; see [27]), and so are their weakly stratified products by 3.9. By 3.4, 3.7 and 1.10 all the spaces of (1) are completely L-regular, hence L-Tychonoff.

(2) We first observe that with L a frame and with $|J| \leq \aleph_0$

$$\left\{ \bigwedge_{j\in K} \pi_j^{\leftarrow}(R_s \wedge L_t) : s,t\in\mathbb{Q}, \quad K\subset J \text{ is finite} \right\}$$

is a countable base for $\mathbb{R}(L)^J$, an L-regular space by 3.4 and 3.3(4). Hence $\mathbb{R}(L)^J$ is perfectly L-normal by 3.13. The same argument applies to $(I(L))^J$.

(3) As in (2) by noticing that $\{R_s \wedge L_t \wedge \alpha : s,t\in\mathbb{Q}, \alpha\in L\}$ is a countable base of $\mathbb{R}(L)$ whenever $|L| \leq \aleph_0$. $\square$

3.17 Remark. The author does not know if $I(L)^J$ is L-normal for an arbitrary J (this is a long-standing open question of [44]) and the same for $(I(L)^c)^J$. In fact, I do not know if X^c must necessarily be normal whenever X is (see also [30]).

4 Insertion and extension of mappings

All the results of this section are given or were announced in [22-25]. We begin with two abstract insertion lemmas which will be used latter to prove insertion and extension theorems for L-normal spaces and extremally disconnected L-topological spaces with $(L,')$ a complete lattice. These include the Katětov-Tong insertion theorem for $\mathbb{R}(L)$-valued functions and Tietze-Urysohn extension theorem for $I(L)$-valued functions. When $(L,')$ is meet-continuous, those insertion lemmas yield general L-topological insertion, extension and separation theorems which generalize the classical results of Blair [3] and Lane [31], Mrówka [37], and Gillman and Jerison [7]. A particular case is an extension theorem for $I(L)$-valued functions of Hutton-compact subspaces of completely L-regular spaces [27]. Finally, we discuss the possibility of extending $\mathbb{R}(L)$-valued functions.

Two insertion lemmas

Definition. Let L be a complete lattice. A binary relation $\Subset$ in L^X is called a *subordination* if for all $a,b,c,d\in L$ the following hold:

(P1) $a \Subset b \Rightarrow a \leq b$,

(P2) $a \leq b \Subset c \leq d \Rightarrow a \Subset d$,

(P3) $a \Subset c$ and $b \Subset c \Rightarrow a \vee b \Subset c$,

(P4) $a \Subset b$ and $a \Subset c \Rightarrow a \Subset b \wedge c$, (IP) (Insertion Property) $a \Subset b \Rightarrow a \Subset c \Subset b$ for some $c \in L^X$.

Such a relation has various names in the literature. It is sometimes called a quasi-proximity relation.

4.1 Example. The two examples that will concern us are the following:

(1) Let $(L,')$ be a complete lattice. For X an L-topological space and $a, b \in L^X$,

$$a \Subset b \quad \Leftrightarrow \quad \overline{a} \leq \operatorname{Int} b$$

satisfies (P1)–(P4), and X is L-normal iff (IP) holds.

(2) Let $(L,')$ be a meet-continuous lattice. Let X be an arbitrary L-topological space. If $a, b \in L^X$, then

$$a \Subset b \quad \Leftrightarrow \quad a \prec_f b$$

for some $f \in C^*(X)$ satisfies (P1)–P(4), and (IP) (cf. 2.14).

One can show [22] that a subordination $\Subset$ is equivalent to the relation ρ of Katětov [18]. Thus, the lemmas which follow are essentially the two insertion lemmas of [18]. The proofs are written down as in [23] where $\Subset$ is the relation of 4.1(1) (see also [26]). Actually, the proof of Lemma 4.2 is essentially the proof that separated F_σ-sets in a normal topological space have disjoint open neighborhoods (cf. 4.3).

4.2 Lemma. *Let L be a complete lattice and let $\Subset$ a subordination in L^X. Let $\mathcal{A}, \mathcal{B} \subset L^X$ be two countable subfamilies such that*

$$\bigvee \mathcal{A} \Subset b \quad \textit{and} \quad a \Subset \bigwedge \mathcal{B}$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then there exists a $c \in L^X$ such that $a \Subset c \Subset b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Proof. (as in [26]) Let $\{a_n\}$ and $\{b_n\}$ be enumerations of $\mathcal{A}$ and $\mathcal{B}$. We set $a_0 = \bigvee \mathcal{A}$ and $b_0 = \bigwedge \mathcal{B}$. Then

$$a_n \leq a_0 \Subset b_m \quad \text{and} \quad a_n \Subset b_0 \leq b_m$$

for all n and m. We construct inductively two countable collections $\{c_n : n \in \mathrm{N}\}$ and $\{d_n : n \in \mathrm{N}\}$ of elements of L^X such that

(I_n) $a_i \Subset c_i \Subset d_j \Subset b_j$, $a_0 \Subset d_j$ and $c_i \Subset b_0$ for each $n \geq 2$ and $i, j < n$.

To begin with, select c_1 with $a_1 \Subset c_1 \Subset b_0$. Since $b_0 \leq b_1$, hence $c_1 \Subset b_1$. Also, $a_0 \Subset b_1$, hence $a_0 \vee c_1 \Subset b_1$. Thus there exists d_1 with $a_0 \vee c_1 \Subset d_1 \Subset b_1$. We now have

(I_2) $a_1 \Subset c_1 \Subset d_1 \Subset b_1$, $a_0 \Subset d_1$ and $c_1 \Subset b_0$.

Now assume that for some $n \geq 2$ we have already constructed $c_i, d_i \in L^X$ $(i < n)$ such that (I_n) holds. Since

$$a_n \leq a_0 \Subset d_j \quad (j < n) \quad \text{and} \quad a_n \Subset b_0,$$

hence there exists c_n such that

$$a_n \Subset c_n \Subset b_0 \wedge \bigwedge_{j<n} d_j.$$

Similarly, since $c_i \Subset b_0 \leq b_n \quad (i \leq n)$ and $a_0 \Subset b_n$, hence there is d_n such that

$$a_0 \vee \bigvee_{i \leq n} c_i \Subset d_n \Subset b_n.$$

This proves (I_{n+1}). Now, we set

$$c = \bigvee_{n \geq 1} c_n.$$

Then $a_n \Subset c_n \leq c$ for all n. Also, since $c_n \Subset d_m$ for all n and m, hence $c \leq d_m \Subset b_m$ for all m. Thus $a_n \Subset c \Subset b_m$ for all n, m. $\square$

4.3 Corollary. *Let $(L, ')$ be a complete lattice and let X be L-normal. Let $a \in L^X$ be F_σ and $b \in L^X$ be G_δ. Then, if $\overline{a} \leq b$ and $a \leq \operatorname{Int} b$, then there exists an open u such that $a \leq u \leq \overline{u} \leq b$.*

Proof. 4.2 with $\Subset$ of 4.1(1). $\square$

4.4 Lemma. *Let L be a complete lattice and let $\Subset$ be a subordination in L^X. Let $\{a_r : r \in \mathbb{Q}\}$ and $\{b_r : r \in \mathbb{Q}\}$ be two non-decreasing subfamilies of L^X such that*

$$a_r \Subset b_s \quad \textit{whenever} \quad r < s.$$

Then there exists a family $\{c_r : r \in \mathbb{Q}\}$ of L^X such that

$$a_r \Subset c_p \Subset c_q \Subset b_s \quad \textit{whenever} \quad r < p < q < s.$$

Proof. (as in [26]) Let $\{r_n\}$ enumerate $\mathbb{Q}$. We will define a family $\{c_{r_i} : i \in \mathrm{N}\} \subset L^X$ such that

$$a_r \Subset c_{r_i} \Subset c_{r_j} \Subset b_s \quad \text{whenever} \quad r < r_i < r_j < s.$$

We proceed inductively . If $r < r_1 < s$, then $a_r \leq a_{r_1} \Subset b_s$ and $a_r \Subset b_{r_1} \leq b_s$. By 4.2, there is c_{r_1} such that $a_r \Subset c_{r_1} \Subset b_s \quad (r < r_1 < s)$.

Suppose we have constructed everything for all r_i with $i < n$. We set with $i < n$

$$\mathcal{C}_0 = \{c_{r_i} : r_i < r_n\}, \quad \mathcal{C}_1 = \{c_{r_i} : r_i > r_n\},$$
$$\mathcal{A} = \{a_r : r < r_n\}, \quad \mathcal{B} = \{b_r : r > r_n\}.$$

Also, let $\mathcal{D} = \mathcal{A} \cup \mathcal{C}_0$ and $\mathcal{E} = \mathcal{B} \cup \mathcal{C}_1$. We then have

$$\bigvee \mathcal{D} \leq a_{r_n} \vee \bigvee \mathcal{C}_0 \Subset e \quad \text{and} \quad d \Subset b_{r_n} \wedge \bigwedge \mathcal{C}_1 \leq \bigwedge \mathcal{E}$$

for all $d \in \mathcal{D}$ and $e \in \mathcal{E}$. Thus 4.2 applies, and there exists c_{r_n} such that

$$d \Subset c_{r_n} \Subset e$$

for every $d \in \mathcal{D}$ and $e \in \mathcal{E}$. This means that

$$\begin{aligned} a_r &\Subset c_{r_n} && \text{if} \quad r < r_n, \\ c_{r_i} &\Subset c_{r_n} && \text{if} \quad r_i < r_n, \\ c_{r_n} &\Subset c_{r_i} && \text{if} \quad r_n < r_i, \\ c_{r_n} &\Subset b_r && \text{if} \quad r_n < r, \end{aligned}$$

and we are done. □

4.5 Remark. These two lemmas yield more than what we are going to deduce from them. These yield also insertion theorems for (quasi-)uniformly and (quasi-)proximally continuous functions and some others; see [26].

Insertion and extension theorems for L-normal spaces

The following, given in [22, 23], extends the classical Katětov-Tong insertion theorem [18, 51].

4.6 Theorem (Insertion theorem). *Let $(L, ')$ be a complete lattice. For X an L-topological space, the following statements are equivalent:*

(1) *X is L-normal.*

(2) *If $g \in USC(X)$, $h \in LSC(X)$ and $g \leq h$, then there exists an $f \in C(X)$ such that $g \leq f \leq h$.*

(3) (Urysohn's lemma) *If $k', u \in o(X)$ and $k \leq u$, then there exists an $f \in C^*(X)$ such that $k \leq L_1{}'f \leq R_0 f \leq u$.*

Proof. (1) $\Rightarrow$ (2): For every $r \in \mathbb{Q}$ let

$$h_r = R_r{}'h \quad \text{and} \quad g_r = L_r g.$$

Then $\{h_r : r \in \mathbb{Q}\}$ and $\{g_r : r \in \mathbb{Q}\}$ are scales generating h and g, respectively, by 2.16(5). Since h_r is closed, g_r is open, and $h_r \leq g_s (r < s)$ by 2.16(4), hence

$$h_r \Subset g_s \quad \text{whenever} \quad r < s \text{ in } \mathbb{Q},$$

where $\Subset$ is the subordination relation of 4.1(1). By 4.4 there exists $\mathcal{F} = \{f_r : r \in \mathbb{Q}\} \subset L^X$ such that

$$(\bigstar) \qquad h_r \Subset f_p \Subset f_q \Subset g_s \quad \text{whenever} \quad r < p < q < s.$$

Clearly, $\mathcal{F}$ is a scale (because $\{h_r\}$ and $\{g_r\}$ are) such that

$$\overline{f_p} \le \mathrm{Int} f_q \quad \text{if} \quad p < q.$$

By 2.16(6), $\mathcal{F}$ generates an $f \in C(X)$ which – on account of ($\bigstar$) and 2.16(4) – is such that $g \le f \le h$.

(2) $\Rightarrow$ (3): If k is closed, u is open and $k \le u$, then $\chi_k \in USC^*(X)$, $\chi_u \in LSC^*(X)$ and $\chi_k \le \chi_u$ (cf. 2.11). For $f \in C^*(X)$ with

$$\chi_k \le f \le \chi_u$$

we obtain $k \prec u$ by 2.13.

(3) $\Rightarrow$ (1): See the proof of 3.14. □

Theorem 4.6 suggests to characterize those L-topological spaces that admit the property of inserting a continuous function between lower and upper semicontinuous functions. For $L = \mathbf{2}$ these are the extremally disconnected spaces; see Lane [31] who deduced it from the **2**-version of Theorem 4.16. There is however a short direct proof given below.

Definition. Let $(L, ')$ be a complete lattice. An L-topological space X is called *extremally disconnected* if $\overline{u}$ is open whenever $u \in L^X$ is open.

4.7 Remark. *For a complete lattice $(L, ')$, X is extremally disconnected iff $\overline{u} \le \mathrm{Int}\ k$ whenever $u \le k$ and $u, k' \in o(X)$ iff $\mathrm{Int}\ k$ is closed for every $k \in \kappa(X)$.*

Proof. If X is extremally disconnected and $u \le k$ with $u, k' \in o(X)$, then $\overline{u}$ is open and $\overline{u} \le k$. Thus $\overline{u} \le \mathrm{Int}\ k$. Now, if k is closed, then $\mathrm{Cl}(\mathrm{Int}\ k) \le \mathrm{Int}\ k$ since $\mathrm{Int}\ k \le k$. Finally, if u is open, then $\overline{u} = (\mathrm{Int}\ (u'))'$ is open since $\mathrm{Int}(u')$ is closed. □

4.8 Theorem. *For an L-topological space X with $(L, ')$ a complete lattice the following are equivalent:*

(1) *X is extremally disconnected.*

(2) *If $g \in LSC(X)$, $h \in USC(X)$ and $g \le h$, then there exists an $f \in C(X)$ such that $g \le f \le h$.*

(3) *If $u, k' \in o(X)$ and $u \le k$, then there exists an $f \in C^*(X)$ such that $u \le L_1{}'f \le R_0 f \le k$.*

Proof. (1) $\Rightarrow$ (2): Since $L_r h \le R_r{}'g$ and $L_r h$ is open for all $r \in \mathbb{Q}$, hence $L_r h \le \mathrm{Int}\ (R_r{}'g)$. Let $f_r = \mathrm{Int}\ (R_r{}'g)$. Let $\mathcal{F} = \{f_r : r \in \mathbb{Q}\}$. Then $\mathcal{F}$ is a scale, for it is non-decreasing, and $\bigwedge \mathcal{F} \le \bigwedge_{r \in \mathbb{Q}} R_r{}'g = 1_\emptyset$ and $\bigvee \mathcal{F} \ge \bigvee_{r \in \mathbb{Q}} L_r h = 1_X$. Since $R_r{}'g$ is closed, hence f_r is closed and open by 4.7. Thus $\mathcal{F}$ generates an $f \in C(X)$. Since $L_r h \le f_p \le R_q{}'g$ whenever $r < p < q$, hence $g \le f \le h$ by 2.16 (4).

(2) $\Rightarrow$ (3): As in the proof of (2) $\Rightarrow$ (3) of 4.6.

(3) $\Rightarrow$ (1): Let $u \leq k$ with $u, k' \in o(X)$. Since $u \prec_f k$ for some $f \in C^*(X)$, hence $\overline{u} \leq \operatorname{Int} k$. By 4.7, X is extremally disconnected. $\square$

We now proceed to applications of 4.6 and 4.8 to extending continuous $I(L)$-valued functions.

4.9 Lemma. *Let $(L, ')$ be complete. Let X be an L-topological space, $A \subset X$, and let $f \in C^*(A)$. Let $g, h\colon X \to I(L)$ be defined by $f = g = h$ on A, $g = \mathbf{0}$, $h = \mathbf{1}$ on $X \setminus A$. Then:*

(1) *If 1_A is open, then $g \in LSC^*(X)$ and $h \in USC^*(X)$.*

(2) *If 1_A is closed, then $g \in USC^*(X)$ and $h \in LSC^*(X)$.*

Proof. The following equalities hold:

$$(1_g) \qquad R_t g = \begin{cases} u_t \wedge 1_A & \text{if } \ t \geq 0, \\ 1_X & \text{if } \ t < 0. \end{cases}$$

$$(2_h) \qquad R_t h = \begin{cases} u_t \vee 1_{X\setminus A} & \text{if } \ t < 1, \\ 1_\emptyset & \text{if } \ t \geq 1. \end{cases}$$

where $u_t \in o(X)$ is such that $u_t|A = R_t f$ for all $t \in \mathbb{R}$. Similarly, we have

$$(1_h) \qquad L_t h = \begin{cases} v_t \wedge 1_A & \text{if } \ t \leq 1, \\ 1_X & \text{if } \ t > 1. \end{cases}$$

$$(2_g) \qquad L_t g = \begin{cases} v_t \vee 1_{X\setminus A} & \text{if } \ t > 0, \\ 1_\emptyset & \text{if } \ t \leq 0. \end{cases}$$

where $v_t \in o(X)$ is such that $v_t|A = L_t f$ for all $t \in \mathbb{R}$.

For $i = 1, 2$, property (i) for g (resp., h) follows from (i_g) (resp., (i_h)). $\square$

For convenience we shall use the notion of suitability [for extending functions] introduced in [42].

Definition. An L-topological space is called *suitable* if there exists a nonempty proper subset A of X such that 1_A is open. Then A is called *suitable open.* Also, A is called *suitable closed* if 1_A is closed.

Definition. Let X and Y be L-topological spaces, $A \subset X$ and $f\colon A \to Y$ be continuous. A continuous $g\colon X \to Y$ is called a *continuous extension* of f over X if $g|A = f$ (i.e. $g(x) = f(x)$ for all $x \in A$).

4.10 Theorem (Tietze-Urysohn extension theorem). *Let $(L, ')$ be a complete lattice. Let X be L-normal and $A \subset X$ be suitable closed. Then every continuous function $f\colon A \to I(L)$ has a continuous extension over X.*

Proof 1. Let $g, h\colon X \to I(L)$ be such that $g = h = f$ on A, $g = \mathbf{0}$ and $h = \mathbf{1}$ on $X \setminus A$. Then $g \leq h$, $g \in USC^*(X)$ and $h \in LSC^*(X)$ by 4.9(2). Hence, there is $F \in C^*(X)$ such that $g \leq F \leq h$ by 4.6. Clearly $F|A = f$. $\square$

Proof 2. For every $r \in \mathbb{Q} \cap (0,1)$ there exist $v_r, w_r \in o(X)$ such that $R_r f = v_r|A$ and $L_r f = w_r|A$. Let

$$k_r = v_r' \wedge 1_A \quad \text{and} \quad u_r = w_r \vee 1_{X\setminus A}.$$

Clearly, $k_r \leq u_s$ if $r < s$ in $\mathbb{Q} \cap (0,1)$, and since k_r is closed and u_r is open, we have

$$k_r \Subset u_s \quad \text{if} \quad r < s,$$

where $\Subset$ is the relation of 4.1(1). Thus, by 4.4, there exists a family $\{c_r : r \in \mathbb{Q} \cap (0,1)\} \subset L^X$ such that $k_r \Subset c_p \Subset c_q \Subset u_s$ whenever $r < p < q < s$. Let $c_r = 1_\emptyset$ if $r \leq 0$ and $c_r = 1_X$ if $r \geq 1$. Then $\mathcal{C} = \{c_r : r \in \mathbb{Q}\}$ is a scale, and since $c_p \Subset c_q$ iff $\overline{c_p} \leq \operatorname{Int} c_q$, hence $\mathcal{C}$ generates an $F \in C^*(X)$ by 2.16(7).

Let us now check that $F = f$ on A. Indeed, for every $t \in (0,1]$ and every $x \in A$ we have

$$\begin{aligned} L_t' f(x) &= \bigwedge_{r<t} L_r' f(x) = \bigwedge_{r<t} (w_r|A)'(x) \\ &= \bigwedge_{r<t} u_r'(x) \leq \bigwedge_{r<t} c_r'(x) \\ &= L_t' F(x) \end{aligned}$$

by 2.16(1). Similarly,

$$\begin{aligned} L_t' F(x) &= \bigwedge_{r<t} c_r'(x) \leq \bigwedge_{r<t} k_r'(x) \\ &= \bigwedge_{r<t} R_r f(x) = L_t' f(x) \end{aligned}$$

for all $x \in A$. Thus $f = F$ on A. $\square$

We note that the Tietze-Urysohn extension theorem does not characterize L-normality (if $|L| > 2$) just because there are non-suitable L-normal spaces. It is also not a characterization of suitable L-normal spaces as the following simple example of [22] shows.

4.11 Example. Let $L = I = [0,1]$ with $\alpha' = 1 - \alpha$ for all $\alpha \in I$. Let $X = [0, \infty)$. For each $t \in I$ and $i = 0, 1, 2$, let

$$u_{t,i} = t\, 1_I \vee 2^{-i} 1_{(1,\infty)}.$$

Then $\mathcal{T} = \{u_{t,i} : t \in I,\ i = 0,1,2\} \cup \{1_\emptyset\}$ is an L-topology on X. It is easy to see that $(X, \mathcal{T})$ has a unique suitable closed set, namely the interval I ($1_I = u_{0,0}'$).

Also, $\mathcal{T}_I = \mathcal{C}_I$ and $f \in C^*(I)$ iff f is constant. Let $f \in C^*(I)$ with $f(x) = [\lambda] \in I(L)$ for all $x \in X$. Then $F\colon X \to I(L)$ defined by

$$F(X) = \begin{cases} f(x) & \text{if } x \in I, \\ (\frac{1}{2}) & \text{if } x \in X \setminus I \end{cases}$$

(recall $(\frac{1}{2})$ is a member of $I(L)$ generated by a member of $\mathbb{R}_L$ which is constant on I with value $\frac{1}{2}$) is continuous. Clearly, $F|I = f$ (actually, F is the unique continuous extension of f).

Finally observe that $(X, \mathcal{T})$ fails to be L-normal. Indeed, let $k = u'_{\frac{1}{4},2}$ and $u = u_{\frac{3}{4},0}$. Then $k \leq u$. If $a \in \mathcal{T}$ (resp., $a' \in \mathcal{T}$) and $k \leq a \leq u$, then $a = u$ (resp., $a = k$). Since $k \neq u$, k is not open, u is not closed, we infer that $(X, \mathcal{T})$ is not L-normal. $\square$

4.12 Question. Under what conditions is L-normality equivalent to the extension property with respect to suitable closed sets (in the class of all suitable L-topological spaces)?

The space (X, L^X) shows that $L = \mathbf{2}$ is not a necessary condition.

4.13 Corollary. *Let $(L, ')$ be a complete lattice. Let X be an L-normal space with a suitable closed subset A. The following hold:*

(1) *Each continuous $f\colon A \to I(L)^J$ has a continuous extension over X.*

(2) *If X is weakly stratified, each continuous $f\colon A \to (I(L)^c)^J$ has a continuous extension over X.*

Proof. (1) For every $j \in J$, $\pi_j \circ f\colon A \to I(L)$ has a continuous extension $F_j\colon X \to I(L)$. Then $F\colon A \to I(L)^J$ such that $\pi_j \circ F = F_j$ is a continuous extension of f (cf. 1.8).

(2) This follows from (1). For if X is weakly stratified, then so is A by 1.9(4). Also, $C(A, I(L)^J) = C(A, (I(L)^c)^J)$ because $(I(L)^c)^J = (I(L)^J)^c$ by 1.10. $\square$

There are counterparts of 4.10-4.13 for extremally disconnected L-topological spaces. In particular, we have the following result whose proof is the same as that of 4.10 (proof 1) except that the appeal to 4.9(2) and (4.6) is replaced by an application of 4.9(1) and (4.8).

4.14 Theorem. *Let $(L, ')$ be complete. Let X be an extremally disconnected L-topological space and $A \subset X$ be suitable open. Then every continuous function $f\colon A \to I(L)$ has a continuous extension over X.* $\square$

4.15 Remark. For $L = \mathbf{2}$, 4.14 is a characterization of extremal disconnectedness (see [7, 1.H.6]). If $|L| > 2$, as in the case of L-normality, suitable L-topological spaces with the extension property with respect to suitable open sets need not be extremally disconnected. This can be shown by a modification of Example 4.11. Namely, for the space $(X, \mathcal{T})$ of 4.11, $(X, \mathcal{U})$ is also an

L-topological space where $\mathcal{U} = \{u' : u \in \mathcal{T}\}$. In $(X,\mathcal{U})$, the interval I is a unique suitable open set, $\mathcal{U}_I = \mathcal{C}_I$, and $C^*(I)$ consists of constant functions. Also, the function F of 4.11 is a continuous extension of f over $(X,\mathcal{U})$ too. Finally, $(X,\mathcal{U})$ is not extremally disconnected, because $\mathrm{Cl}(u_{t,0}') = u_{1-t,2} \notin \mathcal{U}$ for all $0 \leq t < 1$. Therefore, one can formulate a counterpart of Question 4.12 for the case of extremally disconnected spaces.

General L-topological insertion and extension theorems

There are remarkable generalizations of the insertion and extension theorems we have already proved. Those general results provide: (1) necessary and sufficient conditions for the insertion of a continuous $\mathbb{R}(L)$-valued function between two comparable $\mathbb{R}(L)$-valued functions (Blair [3], Lane [31] for $L = \mathbf{2}$), (2) necessary and sufficient conditions for extending a given continuous $I(L)$-valued function from a given subspace to the whole space (Mrówka [38] for $L = \mathbf{2}$), and (3) necessary and sufficient conditions for extending an arbitrary continuous $I(L)$-valued function from a given subspace to the whole space (if $L = \mathbf{2}$, this is the well-known Urysohn extension theorem due to Gillman and Jerison [7, 1.17]).

We note that the insertion and extension theorems of the preceding subsection can be deduced from the general results of this subsection. We however have included direct proofs because these are complete lattice proofs while those which follow require $(L,')$ to be a meet-continuous lattice.

Results of this subsection were announced in [24] and stated (without proofs) in [25].

Notation. If $A \subset X$ and $a \in L^A$, then $(a)_0 \in L^X$ is defined as follows:

$$(a)_0(x) = \begin{cases} a(x) & \text{if } x \in A, \\ \perp & \text{if } x \in X \setminus A. \end{cases}$$

4.16 Theorem (general insertion theorem). *Let $(L,')$ be a meet-continuous lattice. Let X be an L-topological space and let g, h: $X \to \mathbb{R}(L)$ be two arbitrary functions. The following are equivalent:*

(1) *There exists $f \in C(X)$ such that $g \leq f \leq h$.*

(2) *If $s < t$ in $\mathbb{R}$, then $R_s'h$ and $L_t'g$ are completely separated .*

Proof. (1) $\Rightarrow$ (2): Trivial, for if $s < r < t$, then $L_t'g \leq L_t'f \leq R_r f \leq R_s h$, hence $L_t'g \prec_f R_s h$.

(2) $\Rightarrow$ (1): By 2.16(5), $\{L_r g : r \in \mathbb{Q}\}$ and $\{R_r'h : r \in \mathbb{Q}\}$ are scales that generate g and h, respectively. Now, (2) says that $R_r'h \prec L_s g$ whenever $r < s$ in $\mathbb{Q}$. Since L is meet-continuous, hence $\Subset = \prec$ is a subordination (cf. 4.1(2)), and by 4.4 there exists $\mathcal{C} = \{c_r : r \in \mathbb{Q}\} \subset L^X$ such that $R_r'h \prec c_p \prec c_q \prec L_s g$ if $r < p < q < s$. Clearly, $\bigwedge \mathcal{C} = 1_\emptyset$, $\bigvee \mathcal{C} = 1_X$, and the continuous function f generated by $\mathcal{C}$ satisfies $g \leq f \leq h$ on account of 2.16(4). □

4.17 Theorem (general extension theorem). *Let $(L,')$ be a meet-continuous lattice. Let X be an L-topological space, $A \subset X$, and let $f \in C^*(A)$. The following are equivalent:*

(1) *f extends continuously over X.*

(2) *$(R_s'f)_0$ and $(L_t'f)_0$ are completely separated in X for every $s < t$ in* $[0,1]$.

Proof. (1) $\Rightarrow$ (2): Let $g \in C^*(X)$ be a continuous extension of f. Clearly, then $(R_s'f)_0 \leq R_s'g$ and $(L_t'f)_0 \leq L_t'g$, and this provides the required complete separation.

(2) $\Rightarrow$ (1): Define $g, h\colon X \to I(L)$ by $g = f = h$ on A, $g = \mathbf{0}$ on $X\backslash A$ and $h = \mathbf{1}$ on $X\backslash A$. Then $R_s'h = (R_s'f)_0$ and $L_t'g = (L_t'f)_0$ $(s < t)$ are completely separated in X. Since g and h satisfy the assumption of 4.16, there is $F \in C^*(X)$ between g and h which is the extension of f. □

4.18 Theorem (Urysohn extension theorem). *Let $(L,')$ be meet-continuous. For an L-topological space X and $A \subset X$, the following are equivalent:*

(1) *Every $f \in C^*(A)$ has a continuous extension over X.*

(2) *For all $a, b \in L^A$, if a and b are completely separated in A, then $(a)_0$ and $(b)_0$ are completely separated in X.*

Proof. (1) $\Rightarrow$ (2): If $a \prec_f b'$ via $f \in C^*(A)$ and $g \in C^*(X)$ extends f, then $(a)_0 \prec_g (b)_0'$

(2) $\Rightarrow$ (1): Let $f \in C^*(A)$. If $0 \leq s < t \leq 1$, then $R_s'f, L_t'f \in L^A$ are completely separated in A by f itself, hence $(R_s'f)_0$ and $(L_t'f)_0$ are completely separated in X. By 4.17, f has a continuous extension over X. □

As an application of 4.18 we shall have an extension theorem for H-compact [16] suitable closed subsets of completely L-regular spaces (proved directly in [27]).

Definition. An L-topological space is *H-compact* if every $k \in \kappa(X)$ is compact, i.e. whenever $k \leq \bigvee \mathcal{U}$ with $\mathcal{U} \subset o(X)$, there is a finite $\mathcal{U}_0 \subset \mathcal{U}$ such that $k \leq \bigvee \mathcal{U}_0$.

4.19 Proposition (Urysohn's type lemma). *Let $(L,')$ be meet-continuous. If X is a completely L-regular space, $k \in L^X$ is compact, $u \in L^X$ is open, and $k \leq u$, then $k \leq L_1'f \leq R_0 f \leq u$ for some $f \in C^*(X)$.*

Proof. Let $\mathcal{V}$ be an open $\prec$-decomposition of u (cf. 3.1). Then $k \leq \bigvee \mathcal{V}$ and there exists a finite subfamily $\mathcal{V}_0 \subset \mathcal{V}$ with $k \leq \bigvee \mathcal{V}_0$. By 2.14(6) we have $\bigvee \mathcal{V}_0 \prec u$, hence $k \prec u$. □

4.20 Lemma. *Let X be an L-topological space with $(L,')$ a complete lattice and let $A \subset X$. Then A is an H-compact subspace of X if and only if $(k)_0$ is compact in X for every $k \in \kappa(A)$.*

Proof. Let A be H-compact, $k \in \kappa(A)$, and let $(k)_0 \leq \bigvee \mathcal{U}$ with $\mathcal{U} \subset o(X)$. Then $k \leq (\bigvee \mathcal{U})|A = \bigvee_{u \in \mathcal{U}} (u|A)$ and, thus, there exists a finite $\mathcal{U}_0 \subset \mathcal{U}$ such that $k \leq \bigvee_{u \in \mathcal{U}_0} (u|A)$. Hence $(k)_0 \leq \bigvee \mathcal{U}_0$.

Conversely, let $k \in \kappa(A)$ and $k \leq \bigvee \mathcal{V}$ with $\mathcal{V} \subset o(A)$. Then $(k)_0 \leq (\bigvee \mathcal{V})_0 = \bigvee_{v \in \mathcal{V}} (v)_0 \leq \bigvee_{v \in \mathcal{V}} w_v$, where $w_v|A = v$ with $w_v \in o(X)$. Therefore, there is a finite $\mathcal{V}_0 \subset \mathcal{V}$ such that $(k)_0 \leq \bigvee_{v \in \mathcal{V}_0} w_v$. Consequently, $k = (k)_0|A \leq \bigvee_{v \in \mathcal{V}_0} (w_v|A) = \bigvee \mathcal{V}_0$. □

4.21 Theorem. *Let $(L, ')$ be a meet-continuous lattice. Let A be an H-compact subspace of a completely L-regular space in which A is suitable closed. Then every continuous $f\colon A \to I(L)$ has a continuous extension over X.*

Proof. Let $a, b \in L^A$ be completely separated, i.e. $a \prec b'$ in A. Then

$$\mathrm{Cl}_A a \leq \mathrm{Int}_A(b') = (\mathrm{Cl}_A b)'.$$

Let $\mathrm{Cl}_A a = k_1|A$ and $\mathrm{Cl}_A b = k_2|A$ where $k_1, k_2 \in \kappa(X)$. Since $1_A \in \kappa(X)$, hence

$$(\mathrm{Cl}_A a)_0 = k_1 \wedge 1_A \quad \text{and} \quad (\mathrm{Cl}_A b)_0 = k_2 \wedge 1_A$$

are closed in X. Furthermore, ($\mathrm{Cl}_A a)_0$ is compact in X by 4.20, and

$$(\mathrm{Cl}_A a)_0 \leq ((\mathrm{Cl}_A b)')_0 \leq (\mathrm{Cl}_A b)_0'$$

Now, by 4.19 we have

$$(a)_0 \leq (\mathrm{Cl}_A a)_0 \prec (\mathrm{Cl}_A b)_0' \leq (b)_0',$$

i.e. $(a)_0$ and $(b)_0$ are completeley separated in X. By 4.18, f has a continuous extension over X. □

4.22 Remark. All the results of this subsection are valid for weakly stratified L-topological spaces and $I(L)^c$ instead of $I(L)$.

Extending $\mathbb{R}(L)$–valued functions

Very little is known about the possibility of continuous extending $\mathbb{R}(L)$-valued functions. For instance, we do not know any $L \neq \mathbf{2}$ for which the Tietze-Urysohn theorem is valid for $\mathbb{R}(L)$-valued functions. The reason is that (with L meet-continuous; see 2.9) $C(X, \mathbb{R}(L))$ is merely a lattice and not a ring as is the case when $L = \mathbf{2}$. However, the following holds:

4.23 Lemma. *Let X be an L-topological space with $(L, ')$ a meet-continuous lattice. Let $A \subset X$ be such that 1_A is an L-zero-set and let every function of $C^*(A)$ has a continuous extension over X. Then every function of $C(A)$ has a continuous extension over X.*

Proof. Referring to 2.6. we let $f\colon A \to (0,1)(L) \subset I(L)$ be continuous. Let $F\colon X \to I(L)$ be its continuous extension. Let $1_A = R_0{}'g$ with $g \in C^*(X)$. Then $1_A = L_1{}'g^\bullet$. Next, define

$$g_1 = g \wedge \frac{\mathbf{1}}{\mathbf{2}} \quad \text{and} \quad g_2 = g^\bullet \vee \frac{\mathbf{1}}{\mathbf{2}}.$$

Then $R_0 g_1 = R_0 g = 1_{X\setminus A} = L_1 g^\bullet = L_1 g_2$ and $L_0 g_1 = R_0 g_2 = 1_X$. Also, $g_1|A = \mathbf{0}$ and $g_2|A = \mathbf{1}$. Therefore by 2.9

$$F_1 = (F \vee g_1) \wedge g_2 \in C^*(X),$$

and $F_1|A = F|A = f$. We also have

$$F_1(X) \subset (0,1)(L).$$

Indeed, for every $x \in X\setminus A$

$$R_0 F_1(x) = (R_0 F(x) \vee R_0 g_1(x)) \wedge R_0 g_2(x) = (R_0 F(x) \vee \top) \wedge \top = \top$$

and, similarly,

$$L_1{}'F_1(x) = (L_1{}'F(x) \vee \bot) \wedge \bot = \bot,$$

i.e. $F_1(X \setminus A) \subset (0,1)(L)$, and $F_1(A) \subset (0,1)(L)$ since $F_1|A = f$. $\square$

4.24 Remark. By 4.23, if the set A of 4.10, 4.13, 4.14 and 4.21 is such that 1_A is an L-zero-set, then in all of these results $I(L)$ can be replaced by $\mathbb{R}(L)$. The proof of 4.23 remains valid (without any change) for functions taking values in $(0,1)(L) \cup \{[0]\}$, $(0,1)(L) \cup \{[1]\}$ and $(0,1)(L) \cup \{[0],[1]\}$ (cf. related questions of [43, Question 2.5] and [44, Question 8.7]).

In particular, since in a perfectly L-normal space every closed L-set is an L-zero-set (see [14]), we have the following [24]:

4.25 Theorem. *Let $(L,')$ be a meet-continuous lattice. Let X be a perfectly L-normal [weakly stratified] space and A be a suitable closed. Then every continuous $f\colon A \to \mathbb{R}(L)$ $[\mathbb{R}(L)^c]$ has a continuous extension over X.* $\square$

5 Embedding of L-topological spaces

In this section we treat the problems of embedding L-topological spaces into products. From the very begining we embed spaces of prescribed weight, i.e. the amount of factors of the range space is controlled according to the weight of the embedable space. Crucial for this is a pointfree setting for the concept of separating points from closed sets introduced in [27]. Two other results are included: an embedding theorem for zero-dimensional L-topological spaces [40] and an L-version of the Urysohn embedding theorem.

Historically, the first I-topological Tychonoff embedding theorem was given by Katsaras [19] for completely I-regular[6] and Katsaras-T_1 spaces (note that $I(L)$ is not Katsaras-T_1). For $(L,')$ a completely distributive lattice, Liu [32] has embedded an L-Tychonoff space into an L-cube with $|C^*(X)|$ factors by using Liu's pointwise characterizations of complete L-regularity and L-uniformities. Later, Šostak [49] introduced and studied the notion of E-regularity in a I-topological setting.

Completely regular families of functions

Definition. Let $(L,')$ be a complete lattice. Let $f\colon X \to Y_f$ be continuos for every $f \in \mathcal{F}$. The family $\mathcal{F}$ is called *completely regular* if for every $k \in \kappa(X)$

(CR) $$k = \textstyle\bigwedge_{f\in\mathcal{F}} f^{\leftarrow}(\mathrm{Cl}_{Y_f} f^{\rightarrow}(k)).$$

In a topological setting, $\mathcal{F}$ satisfies (CR) iff $\mathcal{F}$ separates points from closed sets, i.e. whenever $x \notin K \subset X$(a closed subset), there is an $f \in \mathcal{F}$ such that $f(x) \notin \mathrm{Cl}_{Y_f} f(K)$. Condition (CR) is equivalent (see [27]) to the condition of separating "fuzzy points" from closed L-sets introduced in [32]. For $L = I$, another equivalent pointed condition is given in [49].

We note that we could write $\geq$ in (CR) in place of $=$, since $\leq$ always holds. Thus, if $\mathcal{G} \subset \mathcal{F}$ is completely regular, then so is $\mathcal{F}$. In what follows we write $\overline{f^{\rightarrow}(k)}$ rather than $\mathrm{Cl}_{Y_f} f^{\rightarrow}(k)$.

5.1 Remark. *For any complete $(L,')$ the following are equivalent:*

(1) *$\mathcal{F}$ is completely regular.*

(2) $\mathrm{Cl}_X\, a = \bigvee\{f^{\leftarrow}(\overline{f^{\rightarrow}(a)}) : f\in\mathcal{F}\}$ *for all* $a \in L^X$.

(3) $\bigcup\{(o(Y_f)) : f\in\mathcal{F}\}$ *is a base for* X.

Proof. Replace $I(L)$ by Y_f and $C^*(X)$ by $\mathcal{F}$ in the proof of (2) $\Leftrightarrow$ (3) and (2) $\Leftrightarrow$ (4) of 3.5. $\square$

With our new terminology, a part of 3.5 can now be stated as follows:

5.2 Remark. (1) *If $(L,')$ is complete and X is completely L-regular, then $C^*(X)$ is a completely regular family.*

(2) *If $(L,')$ is a frame, then $C^*(X)$ is completely regular iff X is completely L-regular.* $\square$

5.3 Fundamental Lemma. *Let $(L,')$ be a complete lattice. Let X and Y_j $(j \in J)$ be arbitrary L-topological spaces. Every completely regular family*

$$\mathcal{F} \subset \bigcup_{j\in J} C(X, Y_j)$$

[6] Actually, Katsaras has a different, but equivalent [22], [49], definition (see also [27, Sect. 4]).

has a completely regular subfamily

$$\mathcal{F}_0 \subset \mathcal{F} \quad \textit{such that} \quad |\mathcal{F}_0| \leq w(X).$$

Proof. Let $\mathcal{K}$ be a closed base of X such that $|\mathcal{K}| \leq w(X)$. By (a closed L-set version of) 1.3, for every $k \in \mathcal{K}$ there exists $\mathcal{F}_k \subset \mathcal{F}$ such that $|\mathcal{F}_k| \leq w(X)$ and

$$k = \bigwedge_{f \in \mathcal{F}_k} f^{\leftarrow}(\overline{f^{\rightarrow}(k)})$$

Let

$$\mathcal{F}_0 = \bigcup_{k \in \mathcal{K}} \mathcal{F}_k.$$

Since $w(X)$ is, by definition, an infinite cardinal hence $|\mathcal{F}_0| \leq w(X)$. We claim $\mathcal{F}_0$ is completely regular. Indeed, given an arbitrary closed h, we choose $\mathcal{H} \subset \mathcal{K}$ with $h = \bigwedge \mathcal{H}$. Since $k \geq h$ for all $k \in \mathcal{H}$, we have

$$\begin{aligned} h &= \bigwedge_{k \in \mathcal{H}} \bigwedge_{f \in \mathcal{F}_k} f^{\leftarrow}(\overline{f^{\rightarrow}(k)}) \\ &\geq \bigwedge_{k \in \mathcal{H}} \bigwedge_{f \in \mathcal{F}_k} f^{\leftarrow}(\overline{f^{\rightarrow}(h)}) \\ &\geq \bigwedge_{f \in \mathcal{F}_0} f^{\leftarrow}(\overline{f^{\rightarrow}(h)}). \quad \square \end{aligned}$$

5.4 Remark. *Let $(L, ')$ be a complete lattice. Let X be an L-T_0 space and let $\mathcal{F}$ be a completely regular family. Then $\mathcal{F}$ separates points of X, i.e whenever $x \neq y$ in X, then there is an $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.*

Proof. Cf. the argument for (2) $\Rightarrow$ (1) of 3.6. $\square$

Embedding theorems

The following, proved in [27], is a fundamental theorem in this section, which enables us to deduce all the embedding theorems for spaces of prescribed weight.

5.5 Embedding Theorem. *Let $(L, ')$ be a complete lattice and let X be L-T_0. Let $f \colon X \to Y_f$ be continuous for every $f \in \mathcal{F}$. If $\mathcal{F}$ is completely regular, then X can be embedded into $\prod_{g \in \mathcal{G}} Y_g$ where $\mathcal{G} \subset \mathcal{F}$ is such that $|\mathcal{G}| \leq w(X)$.*

Proof. By 5.3, there is a completely regular family $\mathcal{G} \subset \mathcal{F}$ such that $|\mathcal{G}| \leq w(X)$. By 5.4, $\mathcal{G}$ separates points of X. Let $e_{\mathcal{G}} \colon X \to \prod_{g \in \mathcal{G}} Y_g$ be the diagonal map, i.e. $\pi_g \circ e_{\mathcal{G}} = g$ for every $g \in \mathcal{G}$. Thus $e_{\mathcal{G}}$ is continuous by 1.8 and injective by 5.4. We now show that $\tilde{e}_{\mathcal{G}} \colon X \to e_{\mathcal{G}}(X)$ is a closed mapping. Using continuity of projections, 1.2, and 1.4, we have for every $k \in \kappa(X)$ and $g \in \mathcal{G}$ (we let $E = e_{\mathcal{G}}(X)$ and $e = \tilde{e}_{\mathcal{G}}$):

$$\begin{aligned} g^{\leftarrow}(\overline{g^{\rightarrow}(k)}) &= e_{\mathcal{G}}^{\leftarrow}(\pi_g^{\leftarrow}(\overline{\pi_g^{\rightarrow}(e_{\mathcal{G}}^{\rightarrow}(k))})) \\ &\geq e_{\mathcal{G}}^{\leftarrow}(\pi_g^{\leftarrow}(\pi_g^{\rightarrow}(\overline{e_{\mathcal{G}}^{\rightarrow}(k)}))) \\ &\geq e_{\mathcal{G}}^{\leftarrow}(\overline{e_{\mathcal{G}}^{\rightarrow}(k)}) \\ &= e^{\leftarrow}(\overline{e_{\mathcal{G}}^{\rightarrow}(k)}|E) \\ &= e^{\leftarrow}(\mathrm{Cl}_E\, e^{\rightarrow}(k)) \end{aligned}$$

by 1.5. Therefore

$$k = \bigwedge_{g \in \mathcal{G}} g^{\leftarrow}(\overline{g^{\rightarrow}(k)}) \geq e^{\leftarrow}(\mathrm{Cl}_E\, e^{\rightarrow}(k))$$

and $e^{\leftarrow}(k) \geq e^{\rightarrow}(e^{\leftarrow}(\mathrm{Cl}_E\, e^{\rightarrow}(k)) = \mathrm{Cl}_E\, e^{\rightarrow}(k)$ since e is onto. Thus $e = \tilde{e}_{\mathcal{G}}$ is a closed mapping. □

As an important corollary we have:

5.6 Theorem (embedding lemma). *Let $(L, ')$ be complete and let X be L-T_0. For every L-topological space Y, if $C(X, Y)$ is completely regular, then there is a cardinal $\mathbf{m} \leq w(X)$ such that X can be embedded into the cube $Y^{\mathbf{m}}$.* □

For weakly stratified X and Y, one can simply say that X embeds into $Y^{w(X)}$ (cf. 1.11). For such spaces we formulate the following corollary of 5.5 (Mrówka [37] for $L = \mathbf{2}$):

5.7 Theorem. *Let $(L, ')$ be complete, let X and Y be weakly stratified and let X be L-T_0. If $\bigcup\{C(X, Y^n) : n \in \mathbb{N}\}$ is completely regular, then X can be embedded into $Y^{w(X)}$.* □

5.8 Theorem (Tychonoff embedding theorem). *For every complete $(L, ')$, each L-Tychonoff [weakly stratified] space X can be embedded into a cube $I(L)^{\mathbf{m}}$ for some $\mathbf{m} \leq w(X)$ [into the cube $(I(L)^c)^{w(X)}$].*

Proof. By 5.2 and 5.6. □

5.9 Theorem (Urysohn embedding theorem). *Let $(L, ')$ be a meet-continuous lattice. Every L-regular second countable [weakly stratified] L-T_0 space can be embedded into $I(L)^{\aleph_0}$ [into $(I(L)^c)^{\aleph_0}$].*

Proof. By 3.13, 3.15 and 5.8. □

5.10 Corollary. *Let $(L, ')$ be a frame. The following hold:*

(1) *A [weakly stratified] space is L-Tychonoff if and only if it is homeomorphic to a subspace of a [weakly stratified] L-Tychonoff cube.*

(2) *A [weakly stratified] space is L-regular, second countable and L-T_0 if and only if it is homeomorphic to a subspace of a [weakly stratified] L-Hilbert cube.*

Proof. By 5.8, 5.9, 3.3, 3.4 and 3.8. □

The following L-topological versions of the Sierpiński space and of the two-point discrete space were introduced in [50] and [1] respectively.

Definition. For $(L,')$ a complete lattice, $S(L)$ (resp., $D(L)$) is the L-topological space with L as the underlying set and with the L-topology $\{1_\emptyset, id_L, 1_L\}$ (with the L-topology generated by $\{id_L, id'_L\}$, resp.).

We note that $S(\mathbf{2}) = \chi S$ and $D(\mathbf{2}) = \chi D$ where $S = (\mathbf{2}, \{\emptyset, \{\top\}, \mathbf{2}\})$ (the Sierpiński space), $D = \mathbf{2}$ with the discrete topology, and χS consists of characteristic functions of members of S.

The following two embedding theorems were first proved in [36] and [40], respectively.

5.11 Theorem. *Let $(L,')$ be a complete lattice. Every [weakly stratified] L-T_0 space can be embedded into $S(L)^{\mathbf{m}}$ for some $\mathbf{m} \leq w(X)$ [into $(S(L)^c)^{w(X)}$].*

Proof. Let $\mathcal{K}$ be a closed base of X with $|\mathcal{K}| \leq w(X)$. For each $k \in \mathcal{K}$, let $f_k\colon \underline{X \to S(L)}$, $f_k(x) = k'(x)$ for all $x \in X$. Clearly, f_k is continuous and $f_k^{\leftarrow}(\overline{f_k^{\rightarrow}(k)}) = k$. Therefore $k \geq \bigwedge_{f\in\mathcal{F}} f^{\leftarrow}(f^{\rightarrow}(k))$ where $\mathcal{F} = \{f_k : k \in \mathcal{K}\}$. Hence $\mathcal{F}$ is completely regular and 5.6 applies. □

Definition. An L-topological space is called *zero-dimensional* if it has a base consisting of closed and open L-sets (clopen for short).

Of course, subspaces of zero-dimensional spaces are zero-dimensional. It is also clear that if $(L,')$ is a frame, then zero-dimensionality is productive (cf. [40]).

5.12 Theorem. *Let $(L,')$ be complete. Every [weakly stratified] zero-dimensional L-T_0 space X can be embedded into $D(L)^{\mathbf{m}}$ for some $\mathbf{m} \leq w(X)$ [into $(D(L)^c)^{w(X)}$].*

Proof. Let $\mathcal{B}$ be a base consisting of clopen L-sets such that $|\mathcal{B}| \leq w(X)$. We have $\mathcal{B} \subset C(X, D(L))$. Indeed $u^{\leftarrow}(id_L) = \underline{u}$ and $u^{\leftarrow}(id'_L) = \underline{u'}$ are open for every $u \in \mathcal{B}$. Also for every $u \in \mathcal{B}$, $u = u^{\leftarrow}(\overline{u^{\rightarrow}(u)}) \geq \bigwedge_{v\in\mathcal{B}} v^{\leftarrow}(\overline{v^{\rightarrow}(u)})$. Thus $\mathcal{B}$ is completely regular and 5.6 applies. □

Universal L-topological spaces

A universal L-topological space for a given class of L-topological spaces is a space in this class in which every space belonging to the class can be embedded. In formulating universal space theorems we shall restrict ourselves to weakly stratified spaces.

5.13 Theorem. *Let $(L,')$ be a frame. Then $(I(L)^c)^{max\{\mathbf{m},|L|\}}$ is universal for all weakly stratified L-Tychonoff spaces of weight $\le max\{\mathbf{m},|L|\}$ where $\mathbf{m} \ge \aleph_0$.*

Proof. Since $I(L)$ is second countable, hence $w((I(L)^c)^{\mathbf{m}}) \le \max\{\mathbf{m},|L|\}$. Therefore $(I(L)^c)^{\max\{\mathbf{m},|L|\}}$ is weakly stratified (by 1.10), L-Tychonoff (by 3.16) and has weight $\le \max\{\mathbf{m},|L|\}$. By 5.8, an L-Tychonoff X embeds into $(I(L)^c)^{w(X)}$, and since we are in the class of weakly stratified spaces, hence X embeds into $(I(L)^c)^{\max\{\mathbf{m},|L|\}}$. □

Similarly one proves the two results which follow:

5.14 Theorem. *Let $(L,')$ be a complete lattice. Then $(S(L)^c)^{max\{\mathbf{m},|L|\}}$ is universal for all weakly stratified L-T_0 spaces of weight $\le max\{\mathbf{m},|L|\}$ where $\mathbf{m} \ge \aleph_0$.* □

5.15 Theorem. *Let $(L,')$ be a frame. Then $(D(L)^c)^{max\{\mathbf{m},|L|\}}$ is universal for all weakly stratified zero-dimensional L-T_0 spaces of weight $\le max\{\mathbf{m},|L|\}$ where $\mathbf{m} \ge \aleph_0$.* □

Another universal space for weakly stratified zero-dimensional L-T_0 spaces is given in [40]. We refer to [33] for further results related to 5.9 (viz. Urysohn metrization theorem).

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CHAPTER 7

Separation Axioms: Representation Theorems, Compactness, And Compactifications

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Introduction

In the historical development of general topology, the searches for appropriate compactness axioms and appropriate separation axioms are closely intertwined with each other. That such intertwining is important is proven by both the Alexandrov and Stone-Čech compactifications; that such intertwining is to be expected follows from the duality between compactness and separation—the former restricts, and the latter increases, the number of open sets; and that such intertwining is categorically necessary is proven by the categorical nature of the Stone-Čech compactification and its relationship to the Stone representation theorems. Furthermore, these compactifications and many other well-known results justify the compact Hausdorff spaces of traditional mathematics.

As mathematicians sought to develop and extend general topology in various ways using the concept of a fuzzy subset, it is not surprising that searches for the appropriate axioms of compactness and separation axioms were intertwined with varying degrees of success. Some of these attempts include the following:

(1) The Hutton axioms of normality and compactness (denoted H-compactness (by us)) [12–13], together with the Hutton-Reilly (HR) separation axioms schemum [14], give the H-compact, HR-Hausdorff spaces.

(2) The α-compactness axiom of Gantner, Steinlage, and Warren [7] and the α-separation axiom schemum of Rodabaugh [32] together yield α-compact α-Hausdorff spaces.

(3) Stratified compact Hausdorff spaces in the sense of Lowen may be found in reference [27] of [42].

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(4) The compactness axiom of Chang [3], denoted C-compactness in [39, 42], is joined by Rodabaugh [35, 42] to various separation axioms for poslat topological spaces, including those motivated from locales.

(5) Compact Hausdorff-separated spaces as defined in Subsections 6.3–6.4 of [11], with related and extensive developments in references [20] and [25] of [11].

We make a preliminary comment on notation. Based both on the results of [35, 42] and on the additional results contributed by this chapter, it is this author's view that this chapter should simplify the notation of [35, 42] and rename C-compactness by **compactness** and localic regularity by **regularity**, and make other corresponding simplifications, e.g. **C-COMP-LOC-REG-SOB-L-TOP** should be renamed **K-REG-SOB-L-TOP**.

With regard to content, this chapter summarizes both the compactification reflectors of fixed-basis topology w.r.t. compactness given in [35, 42] as well as the development in [35, 42] of closely related low-order separation axioms—the L-T_0 and L-sobriety axioms, which together with certain regularity axioms and compactness establish the fixed-basis categorical equivalent of compact Hausdorff spaces and generate *en route* various needed classes of generalizations of Stone representation theorems. But this chapter also adds many new results to those of [35, 42]:

(1) extensions of the fixed-basis compactifications of [42] to variable-basis topology in the sense of [43], giving the first compactification reflectors for variable-basis topology;

(2) detailed object-level and category-level comparisons of traditional and fixed-basis sobriety, including applications to the fuzzy real lines, the general fuzzification problem, and the construction of new fuzzy real lines and fuzzy arithmetic operations;

(3) detailed object-level and category-level comparisons of compact Hausdorff with its fixed-basis categorical equivalent of compact, regular, sober L-spaces, including applications to the fuzzy unit interval;

(4) detailed morphism-level comparison between continuous mappings on compact Hausdorff spaces and L-continuous mappings on compact, regular, sober L-spaces; and

(5) connections to the compact Hausdorff-separated spaces of Subsections 6.3–6.4 of [11].

All proofs not given in this chapter are found in detail in [42], a paper on which this chapter is primarily based. Much of the new material in (2–4) was in response to, or arose out of, extensive discussions with U. Höhle; and the material in (5) is contributed by U. Höhle.

It might be helpful to the reader to point out certain fundamental philosophies underlying this chapter and its predecessor [42] (see also [39]):

(1) Tihonov theorems are not more important than compactification reflectors. The justification for this is that the latter considers compactness and separation together. Furthermore, compactification reflectors reveal the proper role played by morphisms in compactifications. In this sense, categorical compactifications play a more fundamental role than topological compactifications (see Section 1 below).

(2) Traditional mathematics has much to teach fuzzy sets. In particular, the theory of locales informs fuzzy sets of the following: compactness is really localic compactness, and both are intimately related to H-compactness; localic separation axioms give rise to separation axioms for fixed-basis spaces which are intimately related to corresponding HR-separation axioms; and sobriety, approporiately generalized, has as fundamental role to play in fuzzy sets as in general topology. An important corollary is that the categorical equivalent of compact Hausdorff spaces for L-topological spaces is the compact, regular, L-sober topological spaces.

(3) Fuzzy sets has much to teach traditional mathematics. In particular, the general classes of Stone-Čech compactifications constructed in fixed-basis topology, along with the associated general classes of Stone representation theorems, enable fuzzy sets to enrich traditional spaces with many new compactifications and representations, as well as teach traditional mathematics several lessons: the Stone representation theorem for distributive lattices rests on $\mathbf{2} \equiv \{\bot, \top\}$ being a distributive semiframe; the Stone representation theorem for Boolean algebras rests on $\mathbf{2} \equiv \{\bot, \top\}$ being a complete Boolean algebra; and the Stone-Čech compactification for general topology rests on $\mathbf{2} \equiv \{\bot, \top\}$ being a frame.

In order to further clarify the philosophy and methods of this chapter, it is needful that we discuss in more detail various approaches to compactifications, state with more specificity the goals of this chapter, as well as give a few lattice-theoretic preliminaries; this is done in Section 1 below. We close these opening remarks with an outline of this chapter.

1 Views and approaches to compactification

In traditional topology, there have historically been two fundamental views (or metaphysics) of compactification, together with the two associated methodologies (or epistomologies). It is necessary to understand these views and approaches in order to understand what has been done in recent years in **point-set lattice-theoretic (poslat)** topology with regard to compactification, and what therefore should be done with respect to compactness, separation axioms, and representation theorems.

1.1 Discussion (Two definitions of compactification in traditional topology).

(1) **Topological definition.** Let $(X, \mathfrak{T}) \in \mathbf{TOP}$. Then $((X^*, \mathfrak{T}^*), h)$ is a **compactification** of $(X, \mathfrak{T})$ iff $(X^*, \mathfrak{T}^*)$ is compact and:

(a) $h : X \to X^*$ is a homeomorphic embedding (i.e. regular monomorphism);

(b) $h^{\to}(X)$ is dense in X^*.

(2) **Categorical definition.** $\exists$ a subcategory $\mathbf{C}$ of $\mathbf{TOP}$ of compact spaces, $\exists$ a functor $K : \mathbf{TOP} \to \mathbf{C}$, $K \dashv \hookrightarrow$. Applying the diagrams for adjunction given in [35, 39, 40, 45], based in turn on [29], we have the following restatement of the foregoing condition:

$$\forall\, (X, \mathfrak{T}) \in |\mathbf{TOP}|\,,\ \exists \eta : (X, \mathfrak{T}) \to K\,(X, \mathfrak{T})\,,$$

$$\forall\, (Y, \mathfrak{S}) \in |\mathbf{C}|\,,\ \forall f : (X, \mathfrak{T}) \to (Y, \mathfrak{S})\,,$$

$$\exists!\, \overline{f} : K\,(X, \mathfrak{T}) \to (Y, \mathfrak{S})\,,\ f = \overline{f} \circ \eta.$$

For each $(X, \mathfrak{T})$, the pair $(\mathbf{K}\,(X, \mathfrak{T})\,, \eta)$ is the **compactification** of $(X, \mathfrak{T})$.

$$\begin{tikzcd}
& \textbf{TOP} & \textbf{C} \\
& \forall X\ (1) \arrow[dl, "\forall f\ (4)"'] \arrow[d, "\exists \eta_X\ (2)"] \arrow[dl, phantom, shift left=4, "\circ\ (6)"] & \mathbf{K}(X) \arrow[d, "\exists ! \bar{f}\ (5)"] \\
Y & \mathbf{K}(X) \arrow[l, "\bar{f}"] & \forall Y\ (3)
\end{tikzcd}$$

The topological definition may be found in [6], and the categorical definition may be found in [10]. Note the categorical definition says a compactification is a compact space together with a universal arrow from the space to the compact space, i.e. a continuous map through which all liftings must uniquely factor; equivalently, we may say **C** is a reflector in **TOP**.

There need be no relationship between these two definitions: the Alexandrov (one-point) compactification is a topological compactification but generally yields not a universal arrow; and the Stone-Čech compactification is a categorical compactification which satisfies the embedding and density criteria only when the original space is Tihonov. What the two viewpoints have in common is the stipulation of an object (the compact space) and an arrow (the map), and where they differ is in the criteria which the map must satisfy in categorical terms: the topological approach requires the map be a regular monomorphism (plus density), and the categorical approach requires the map be a universal arrow.

As indicated above, topological and categorical viewpoints coincide for Stone-Čech compactifications of Tihonov (= completely regular T_0) spaces. (It is a non-trivial exercise to derive for such spaces the topological criteria from the categorical criteria.) This coincidence shows the canonicity of the Stone-Čech approach. This in turn illustrates that the categorical criteria are more fundamental than the topological criteria in this sense: every topological space has a Stone-Čech compactification, so that "unique liftings of maps" or "categorical reflection" applies more broadly (or "universally") than "existence of regular monomorphisms". Furthermore, the categorical criteria (in the case of Stone-Čech) are based on the machinery of the Stone representation theorems for Boolean algebras and distributive lattices; they are in fact based ultimately on the relationship between locales and topological spaces, a topic explored more fully below.

1.2 Discussion (Two approaches/methodologies to compactification). There are two fundamentally distinct approaches to contructing the compactification $(X^*, \mathfrak{T}^*)$ of a space $(X, \mathfrak{T})$, and these two approaches reflect the two different definitions discussed above.

(1) **Topological/point-set approach.** Given a space $(X, \mathfrak{T})$, the first (and more traditional) approach first builds the set X^* and then furnishes the topology $\mathfrak{T}^*$ to make the space $(X^*, \mathfrak{T}^*)$; in the case of Stone-Čech, the set βX is a subset of the appropriate cube formed using evaluation, and the topology $\beta\mathfrak{T}$ is inherited as a subspace topology. The topological or point-set definition of compactification is reflected in this traditional approach since it builds the point-set first. A typical reference is [6].

(2) **Categorical/localic approach.** The second approach first builds the topology $\mathfrak{T}^*$ and then assigns a suitable point-set X^* as the underlying set. Because this approach in its first step is "point-free", it tends to rely heavily on categorical methods and so reflects the categorical viewpoint. In the case of Stone-Čech, the topology $\beta\mathfrak{T}$ is first constructed, and then a suitable point-set βX is attached as the underlying set. The construction of $\beta\mathfrak{T}$ uses the construction of the Stone-Čech compactification A for any locale A; and then the machinery of the Stone representation theorems, actually the adjunction between **TOP** and **LOC**, builds the underlying set X. See [16] for details.

In the case of Stone-Čech, both approaches coincide (or "commute") and yield the same compactification, even for a general space. In particular, the first approach

$$(X, \mathfrak{T}) \longmapsto X \longmapsto \beta X \longmapsto (\beta X, \beta\mathfrak{T})$$

uses the forgetful functor V: **TOP** $\to$ **SET**, and then builds the compactification as indicated above; while the second approach

$$(X, \mathfrak{T}) \longmapsto \mathfrak{T} \longmapsto \beta\mathfrak{T} \longmapsto (\beta X, \beta\mathfrak{T})$$

uses, in succession, the functors

$$\Omega : \mathbf{TOP} \to \mathbf{LOC},\ \beta\colon \mathbf{LOC} \to \mathbf{K\text{-}REG\text{-}LOC},\ PT : \mathbf{LOC} \to \mathbf{TOP}$$

1.3 Discussion (Categories L-TOP and C-TOP). To motivate these questions for poslat topology, we need to review some terminology from [11, 43]; and the reader should review the powerset operators of [44–45]—these are constantly used throughout this chapter. The category **SFRM** comprises all complete lattices, together with morphisms preserving arbitrary $\bigvee$ and finite $\wedge$, and taken with the usual composition and identities. For $X \in |\mathbf{SET}|$ and $L \in |\mathbf{SFRM}|$, we form the L-**powerset** L^X comprising all L-(**fuzzy**) **subsets** $a : X \to L$. Now for $L \in |\mathbf{SFRM}|$, the category L-**TOP** comprises all spaces (X, τ) in which the L-topology τ is closed under all arbitrary $\bigvee$ and finite $\wedge$ of L^X—the members of τ are (L-(**fuzzy**)) **open subsets**—together with functions f whose backward powerset operators $f_L^{\leftarrow}$ preserve L-topologies (i.e. $f_L^{\leftarrow}(v) \equiv v \circ f$ is an L-open subset of the domain whenever v is an L-open subset of the codomain), and taken with the usual composition and identities. The categories L-**TOP** are the categorical models for **fixed-basis topology** in the sense of Chang-Goguen, where L is the lattice-theoretic base fixed for the entire topological category; and in this context we also speak of **fixed-basis (poslat) topology**. We point out that the restriction from the general lattice-theoretic category **CQML** of [11, 43] to **SFRM** in this chapter is appropriate since the compactifications given below use $\otimes = \wedge$.

The categorical models for variable-basis topology are determined by first stipulating a subcategory **C** of lattice-theoretic bases. For this chapter **C** is taken as the category $\mathbf{SLOC} \equiv \mathbf{SFRM}^{op}$. The category **C-TOP** comprises all spaces (X, L, τ) such that $L \in |\mathbf{C}|$ and $(X, \tau) \in L$-**TOP**, together with all morphisms $(f, \phi) \in \mathbf{SET} \times \mathbf{C}$ whose backward powerset operators $(f, \phi)^{\leftarrow}$ preserves the poslat topologies (i.e. $(f, \phi)^{\leftarrow}(v) \equiv \phi^{op} \circ v \circ f$ is an open subset of the domain whenever v is an open subset of the codomain), and taken with the composition and identities of $\mathbf{SET} \times \mathbf{C}$. The category **C-TOP** is the categorical model for **variable-basis topology** since it unifies as subcategories all the L-**TOP**'s for $L \in |\mathbf{C}|$; and in this context we also speak of **variable-basis point-set lattice-theoretic (poslat) topology**. Again, we point out that **C** is restricted to be **SLOC** rather than a more general subcategory of $\mathbf{LOQML} \equiv \mathbf{CQML}^{op}$ since the compactifications of objects in $|\mathbf{C\text{-}TOP}|$ use $\otimes = \wedge$.

1.4 Discussion (**"Topological definition/approach" to compactification in fixed-basis topology**). The topological approach to compactification in traditional topology finds many approximate counterparts in fixed-basis topology [1, 2, 25–28, 30]. Each approach of these sources was analyzed in [42] and none were found to have all of the following desirable properties:

(1) truly generalize the classical β, i.e. without using it;

(2) be valid for a broad category of lattices L (instead of restricted to just $L = [0, 1] \subset \mathbb{R}$);

(3) be categorical, i.e. generate a compactification reflector; and

(4) be a reflector on *all* of L-**TOP**, instead of reflecting on a restrictive subcategory of L-**TOP**.

1.5 Discussion ("Categorical definition/approach" to compactification in fixed-basis topology). In [42] we see the emergence of a canonical approach satisfying all the conditions of 1.4(1–4), as well as satisfying topological criteria for spaces with appropriate separation properties, for all categories L-**TOP** for $L \in |\mathbf{FRM}|$. From that work, using the criteria of [39], is affirmed: a strong vindication of Chang's definition of compactness [3] for poslat topology, a definition ignored in the literature; the establishment of L-sobriety, and its included L-T_0 axiom, as the lowest-level separation axioms for L-topology; and the establishment of both the localic and the Hutton-Reilly axioms of regularity, complete regularity, and normality as the middle-level separation axioms for L-topology. It turns out that the categorical equivalent of the compact Hausdorff spaces of classical mathematics is the compact [3], localic or Hutton-Reilly regular, sober L-topological spaces. (For a discussion of other separation axioms, see [20] and [22].)

1.6 Question. What are the precise relationships, object-level and category-level, between traditional sobriety and L-sobriety, when are these sobrieties actually equivalent, and what role does this relationship play in giving a general solution to the fuzzification—lattice-dependent extension—problem in topology?

1.7 Question. What are the precise relationships, object-level, morphism-level, and category-level, between classical compact Hausdorff spaces and the compact, localic or Hutton-Reilly regular, sober L-topological spaces?

1.8 Question. Is there a counterpart in *variable-basis* topology of the categorical/localic approaches to compactification in traditional and fixed-basis topology?

The first goal of this chapter is to summarize for the reader, *sans* proofs, the categorical/localic development in [42] for fixed-basis topology; the second goal is to settle Questions 1.6–1.7, thereby further clarifying the roles of L-sobriety and density in the developments of [42]; and the third goal is to answer Question 1.8 affirmatively, thereby transferring to variable-basis topology the development of the categorical approach given in [42] for fixed-basis topology. Achieving these latter goals completes the contribution of [42] to the debate over compactness and separation axioms.

In Section 2, we summarize the fuzzification of spatiality and sobriety previously given in [35, 39–42]—this allows us to generalize the adjunction between **TOP** and **LOC**, and the resulting equivalence between **SOB-TOP** and **SPAT-LOC**; and the relationship between traditional sobriety and its generalizations is fully detailed, answering Question 1.6. We then discuss in Section 3 localic-like separation axioms and their relationship to the axioms of [14]. Then localic compactness, compactness [3], and H-compactness [13] are discussed and related in Section 4. After these preliminary sections and using the above-mentioned ideas, we present results in which can be found applications to both multi-valued and traditional topology:

(1) General classes of Stone representation theorems for distributive lattices (Section 5). It follows in the fixed-basis context that compactness with sobriety are a categorically equivalent replacement to traditional compactness with traditional sobriety.

(2) Compact, regular, sober, fixed-basis topological space representations of compact regular locales (Section 6). It follows that the compact, regular, sober, fixed-basis topological spaces are categorically equivalent to the traditional compact Hausdorff topological spaces. Furthermore, the former spaces may be viewed as a type of soberification of the latter spaces, answering part of Question 1.7. Further, if the basis of the spaces is a complete Boolean algebra, then H-compact, HR-regular, sober spaces may replace the compact, regular, sober spaces in the preceding sentence. In this context we summarize results of U. Höhle connecting compact, regular, sober, fixed-basis topological spaces to the compact, Hausdorff-separated spaces of Subsections 6.3–6.4 of [11], including the result that when the basis is a complete Boolean algebra, these two approaches coincide.

(3) General classes of Stone representation theorems for Boolean algebras (Section 7). It follows in the fixed-basis context that sobriety with any pairing of either compactness or H-compactness with either of regularity or HR-regularity is also categorically equivalent to pairing traditional compactness with traditional Hausdorff.

(4) Stone-Čech compactifications for all poslat topological spaces in the fixed-basis category L-**TOP** with $L \in |\mathbf{FRM}|$ (Section 8). In this context compactness plays the same role as traditional compactness, the completely regular, T_0, L-topological spaces play the same role as the traditional completely regular T_0 spaces, and the compact, regular, sober, L-topological spaces play the same role as the traditional compact Hausdorff spaces. For some frames L, including the complete Boolean algebras, the previous sentence holds with H-compactness replacing compactness, HR-complete regularity replacing complete regularity, and HR-regularity replacing regularity. Finally, the relationship between classical compactifications and L-compactifications is detailed using notions of soberifications and density, thereby completing the answer to Question 1.7.

(5) Stone-Čech compactifications for all poslat topological spaces in certain subcategories of the variable-basis category **LOC-TOP** (Section 9). As in (4) above, these use compactness, sobriety, and complete regularity. These compactifications are equivalently generated for all spaces in corresponding subcategories in the variable-basis category $\mathbf{BOOL}^{op}$-**TOP** using H-compactness, sobriety, and HR-complete regularity. These results answer Question 1.8.

We finish this section with a few lattice-theoretic preliminaries. All lattice-theoretic categories not defined in this chapter are found in [42–4]; the prefix "**D**" or "**C**" or "**ORTHO**" in a categorical name indicates the attribute of distributivity, completeness, or orthocomplemented is stipulated for its objects. Unless specified otherwise, all base lattices L are taken from the category **SFRM** or its dual **SLOC**. We maintain the practice of [11, 43] of denoting the lower and upper bounds of a lattice by $\perp$ and $\top$, respectively; and the constant maps from a set X into a lattice L with values $\perp$ and $\top$ are denoted $\underline{\perp}$ and $\underline{\top}$, respectively. We use the symbols "$\{\perp, \top\}$" and "2" interchangeably.

2 Fixed-basis sobriety and spatiality

The ideas of sobriety and spatiality comprise the machinery of compactification as developed below. This section summarizes such ideas as carried over to poslat or Chang-Goguen fixed-basis topology. Proofs not given below, as well as philosophical and mathematical justification of the ideas constructed and the methods employed, can be seen (in more general form) in [35, 39–41]. Toward the end of the section we cite the two-fold justification from [41] of fuzzy sets which these ideas give us; cf. 5.14, 6.8, 7.11 below. At the end of the section we add a detailed analysis with proofs comparing traditional sobriety with the sobriety described below. Classical notation and ideas are taken from [16]. Each fixed base lattice is taken from **SFRM** or **SLOC**.

2.1 Lemma (Lattice-dependent extension of Ω : TOP $\to$ LOC). $L\Omega : L$-**TOP** $\to$ **SLOC** is a functor, where

$$L\Omega(X, \tau) = \tau, \quad L\Omega(f : (X, \tau) \to (Y, \sigma)) = ([(f_L^{\leftarrow})|_\sigma]^{op} : \tau \to \sigma)$$

2.2 Lemma (Lattice-dependent extension of PT: TOP $\leftarrow$ LOC). LPT : L-**TOP** $\leftarrow$ **SLOC** is a functor, where

$$Lpt(A) = \{p : A \to L \mid p \in \mathbf{SFRM}\},$$

$$\Phi_L : A \to L^{Lpt(A)} \text{ by } \Phi_L(a)(p) = p(a),$$

$$LPT(A) = (Lpt(A), \Phi_L^{\to}(A)),$$

$$LPT[f : A \to B] = [f^{op}]_L^{\leftarrow}, \text{ i.e. } LPT(f)(p) = p \circ f^{op}$$

where $f^{op} : A \leftarrow B$ is a concrete map in **SFRM**. Also, $\Phi_L \in$ **SFRM**.

2.3 Lemma (Lattice-dependent extension of Ψ : TOP $\to$ |PT$^{\to}$(LOC)|). Put

$$\Psi_L : |L\text{-}\mathbf{TOP}| \to |LPT^{\to}(\mathbf{SLOC})|$$

as follows: given $(X, \leq) \in |L\text{-}\mathbf{TOP}|$, put

$$\Psi_L : X \to Lpt(\tau) \quad \text{by} \quad \Psi_L(x) : \tau \to L \quad \text{by} \quad \Psi_L(x)(u) = u(x)$$

Then $\Psi_L : (X, \tau) \to (Lpt(\tau), \Phi_L^{\to}(\tau))$ is L-continuous, and L-open w.r.t. its range in $(Lpt(\tau), \Phi_L^{\to}(\tau))$ (see [43] for notions of openness of morphisms and the underlying subspace topology and subobject.)

2.4 Theorem (Lattice-dependent extension of $\Omega \dashv PT$). $L\Omega \dashv LPT$ with unit Ψ_L and counit Φ_L. As special cases, $L \in |\mathbf{FRM}| \Rightarrow L\text{-}\mathbf{TOP}$ is adjunctive with $\mathbf{LOC}$ via $L\Omega \dashv LPT$; and $L = \{\perp, \top\} \Rightarrow \Omega \dashv PT$.

2.5 Definition (Lattice-dependent extension of sobriety and T_0). A space $(X, \leq) \in |L\text{-}\mathbf{TOP}|$ is **$(L\text{-})$sober** $[(L\text{-})T_0\ ]$ iff Ψ_L is bijective [injective]. **SOB-L-TOP** is the full subcategory of L-**TOP** of all sober objects.

2.6 Remark. Given the G_χ: **TOP** $\to$ L-**TOP** functor—see Definition 4.1.1 of [39], references [11, 94, 98] of [39], Theorems 6.2.1(1) and 6.2.3(1) of [43], and Remark 3.7 of [11], a classical space $(X, \mathfrak{T})$ is sober $[T_0] \Leftrightarrow G_\chi(X, \mathfrak{T})$ is $\{\perp, \top\}$-sober $[\{\perp, \top\}\text{-}T_0]$. We also denote "$\{\perp, \top\}$-sober" by "2-sober". And from [16] for **TOP**, Hausdorff $\Rightarrow$ sobriety $\Rightarrow$ T_0, with no implications reversible; and there is no relationship between sobriety and T_1. But in **TOP** under regularity, sobriety is equivalent to each of T_0, T_1, and T_2; and hence in **TOP**, compact Hausdorff spaces are equivalent to compact regular sober spaces. To "fuzzify" compact Hausdorff spaces, it suffices to combine the "appropriate" fuzzy notions of compactness and regularity with the fuzzy notion of (L-)sobriety defined in 2.5. In [42] we determined these "appropriate" notions to be compactness [3] or H-compactness [13] together with Hutton-Reilly regularity [14] or "localic" regularity [35, 39] (cf. [4, 16]).

The concepts of L-subspace topology, L-open morphism, L-embedding, and L-homeomorphism are fundamental to the compactification theory summarized in this chapter, used repeatedly in the sequel, and found with detailed categorical analysis in [43].

2.7 Definition (Lattice-dependent extension of spatiality). The lattice $A \in \mathbf{SLOC}$ is **L-spatial** iff $\Phi_L : A \to L^{Lpt(A)}$ is injective. L-**SPAT-SLOC** is the full subcategory of **SLOC** of all L-spatial semilocales. If $L = \{\perp, \top\}$, then we speak of both $\{\perp, \top\}$-spatiality or 2-spatiality. We point out that 2-spatiality is precisely the "crisp" or "classical" spatiality of [16].

2.8 Lemma. The following hold:

(1) $\forall (X, \tau) \in |L\text{-}\mathbf{TOP}|$, (X, τ) is L-sober $[L\text{-}T_0\ ] \Leftrightarrow \Psi_L$ is an L-homeomorphism [L-embedding] (see [43]).

(2) $\forall A \in |\mathbf{SLOC}|$, A is L-spatial $\Leftrightarrow \Phi_L : A \to \Phi_L^{\to}(A)$ is an isomorphism.

(3) If $L_1 \hookrightarrow L_2$ in **SFRM**, then L_1-spatiality $\Rightarrow$ L_2-spatiality. Thus $\{\bot, \top\}$-spatiality, or 2-spatiality, implies L-spatiality for each L in **SFRM**.

2.9 Lemma. $\forall (X, \tau) \in |L\text{-}\mathbf{TOP}|$, $L\Omega(X, \tau) = \tau$ is L-spatial; and $\forall A \in |\mathbf{SLOC}|$,

$$LPT(A) = (Lpt(A), \Phi_L^{\rightarrow}(A))$$

is L-sober. Thus

$$L\Omega : L\text{-}\mathbf{TOP} \rightarrow L\text{-}\mathbf{SPAT\text{-}SLOC}$$

and

$$LPT : \mathbf{SOB}\text{-}L\text{-}\mathbf{TOP} \leftarrow \mathbf{SLOC}$$

where the full subcategories **SOB-L-TOP** and **L-SPAT-SLOC** are defined in the obvious way. Furthermore, the functors

$$L\Omega \circ LPT : \mathbf{SLOC} \rightarrow L\text{-}\mathbf{SPAT\text{-}SLOC}$$

$$LPT \circ L\Omega : L\text{-}\mathbf{TOP} \rightarrow \mathbf{SOB}\text{-}L\text{-}\mathbf{TOP}$$

are valid, called the **L-spatialization** and **L-soberification functors**, respectively.

2.10 Theorem (Lattice-dependent extension of SOB-TOP $\approx$ SPAT-LOC representation theorem). $\forall L \in |\mathbf{SFRM}|$,

$$\mathbf{SOB}\text{-}L\text{-}\mathbf{TOP} \approx L\text{-}\mathbf{SPAT\text{-}SLOC}$$

2.11 Theorem. Let $L \in |\mathbf{SFRM}|$, $(X, \tau) \in |L\text{-}\mathbf{TOP}|$, $A \in |\mathbf{SLOC}|$, and $h : A \rightarrow \tau$ in **SFRM** be given; and put $H : X \rightarrow Lpt(A)$ by $H(x) = \Psi_L(x) \circ h$. Then:

(1) $H : X \rightarrow Lpt(A)$ is L-continuous (i.e. in L-**TOP**).

(2) If h is an epimorphism in **SET** (i.e. surjective), then H is a L-open morphism w.r.t. the L-subspace topology on $H^{\rightarrow}(X)$ inherited from $LPT(A)$.

(3) If H is injective, then Ψ_L is injective. And conversely, if h is an epimorphism in **SET** and Ψ_L is injective, then H is an L-embedding into $LPT(A)$, i.e. an L-homeomorphism onto $H^{\rightarrow}(X)$ with the L-subspace topology inherited from $LPT(A)$.

2.12 Remark. Lemma 2.8(1) is a corollary of Theorem 2.11: set $A = \tau$ and set $h = id_\tau$.

2.13 Remark. The full generality of 2.11 is fundamental to Stone-Čech compactification as developed in this paper.

2.14 Remark (Necessity of fuzzy subsets). Many justifications of fuzzy or lattice-valued sets and the ideas of this section appear in Sections 5–8. But for now we cite a two-fold justification proved in [35, 39–41] and derived in part from 2.10 above. First, there are many locales which cannot be represented by any classical sober topological space; but every locale, indeed every complete lattice, has at least one representing sober fixed-basis topological space. Second, many localic products are not (isomorphic to) any product topology; but every localic product is some L-product topology, where L-product topology is taken in the sense of [9, 55]. The point is that only with L-subsets (Discussion 1.3) do we have the option of choosing the lattice L of membership values; whereas the restriction $L = \{\bot, \top\}$ in classical mathematics insures that classical topology has the above two gaps which can only be filled by fuzzy or lattice-valued subsets (the word "only" being justified because of the categorical nature of these gaps and their lattice-valued "solutions").

2.15 Discussion (Comparison between L-sobriety and classical sobriety). The purpose of this discussion is to establish categorically the relationship between L-sobriety and classical sobriety by introducing two new functors: the L-2-soberification and 2-L-soberification functors. Motivated from and implicitly contained in Remark 6.3 of [42], these functors will help us, in Discussion 6.3 below, to formalize that remark. Furthermore, these functors find important applications in this Section and Sections 6 and 8. These applications include the general fuzzification problem in topology (Discussion 2.16), the construction of new fuzzy arithmetic operations (Discussion 2.16), the relationship between traditional compact Hausdorff spaces and compact, regular, sober L-spaces (Discussion 6.3), and the relationship between traditional compactifications and L-compactifications via both density and the L-continuous extension of continuous maps (Discussions 8.14–8.15).

2.15.1 Definition (L-2-Soberification). The L-2-**soberification functor** is the functor

$$LPT \circ \Omega : \mathbf{TOP} \to \mathbf{SOB}\text{-}L\text{-}\mathbf{TOP}$$

Object-wise $(X, \mathfrak{T}) \mapsto (Lpt(\mathfrak{T}), \Phi_L^{\rightarrow}(\mathfrak{T}))$, with the latter called the L-2-**soberification** of the former. (Cf. Lemma 2.9.)

2.15.2 Remark. $LPT \circ L\Omega \circ G_\chi : \mathbf{TOP} \to \mathbf{SOB}\text{-}L\text{-}\mathbf{TOP}$ is naturally isomorphic to the L-2-soberification functor (where G_χ is given and studied in [43]).

Proof. Note $\mathfrak{T} \approx G_\chi(\mathfrak{T})$. And it is proved in [35]: $A \approx B \Rightarrow LPT(A) \approx LPT(B)$ (using the universality of $L\Omega \dashv LPT$). $\square$

2.15.3 Definition (2-L-**Soberification**). The 2-L-**soberification functor** is the functor

$$PT \circ L\Omega : L\text{-}\mathbf{TOP} \to \mathbf{SOB\text{-}TOP}$$

Object-wise $(X, \tau) \mapsto (Pt(\tau), \Phi_2^{\rightarrow}(\tau))$ with the latter called the 2-L-**soberification** of the former. (Cf. Lemma 2.9.)

2.15.4 Remark. There are other functors which are candidates for 2-L-soberification, namely $PT \circ \Omega \circ S_\perp$ and $PT \circ \Omega \circ M_\chi$ (where $S_\perp$ ($\perp$-level functor) and M_χ ("Martin" functor) may be found in [43]). But these are apparently different, and neither of these "fits" as nicely with the L-2-soberification functor defined above.

2.15.5 Theorem (Adjunction between sobrieties). Let $L \in |\mathbf{SFRM}|$. Then **SOB-TOP** $\dashv$ **SOB-L-TOP.** More precisely, the following hold:

(1) $LPT \circ \Omega \dashv PT \circ L\Omega$.

(2) The unit of this adjunction is a homeomorphism, i.e. this adjunction is an iso-reflection.

Proof. *Ad (1).* The basic tool is the fact that the composition of the adjunctions is again an adjunction (see Lemma 9.17.1 below for a precise statement). To use this tool, we put together a few facts. First, each ordinary topology is 2-spatial (2.9), and 2-spatiality implies L-spatiality (2.8(3)); and this means the adjunction $\Omega \dashv PT$ restricts to the categories **SOB-TOP** and L-**SPAT-SLOC**. Second, the adjunction $L\Omega \dashv LPT$ restricts to a categorical equivalence when restricted to the categories **SOB-L-TOP** and L-**SPAT-SLOC** (2.10); and this means on these categories we have the reverse adjunction $LPT \dashv L\Omega$. Now composing the adjunctions $\Omega \dashv PT$ and $LPT \dashv L\Omega$ yields the claimed adjunction $LPT \circ \Omega \dashv PT \circ L\Omega$.

Ad (2). The verification of (2) is implicit in the verification of (1). To be more explicit, let $(X, \mathfrak{T})$ be an ordinary sober topological space. Then the image of this space under $PT \circ L\Omega \circ LPT \circ \Omega$ is the ordinary sober topological space

$$(Pt(\Phi_L^{\rightarrow}(\mathfrak{T})), \Phi_2^{\rightarrow}(\Phi_L^{\rightarrow}(\mathfrak{T})))$$

Since $\mathfrak{T}$ is L-spatial (see proof of (1)), then $\mathfrak{T} \approx \Phi_L^{\rightarrow}(\mathfrak{T})$. Now the universal properties of $\Omega \dashv PT$ imply that

$$(Pt(\mathfrak{T}), \Phi_2^{\rightarrow}(\mathfrak{T})) \approx (Pt(\Phi_L^{\rightarrow}(\mathfrak{T})), \Phi_2^{\rightarrow}(\Phi_L^{\rightarrow}(\mathfrak{T})))$$

Since $(X, \mathfrak{T}) \approx (Pt(\mathfrak{T}), \Phi_2^{\rightarrow}(\mathfrak{T}))$ (by the sobriety of $(X, \mathfrak{T})$), it follows that the unit

$$\eta : (X, \mathfrak{T}) \to (Pt(\Phi_L^{\rightarrow}(\mathfrak{T})), \Phi_2^{\rightarrow}(\Phi_L^{\rightarrow}(\mathfrak{T})))$$

of the adjunction $LPT \circ \Omega \dashv PT \circ L\Omega$ is a homeomorphism. $\square$

2.15.6 Definition (Topological L-topological spaces). Let $L \in |\mathbf{SFRM}|$. An L-topological space (X, τ) is **topological** or **2-topological** iff τ is 2-spatial. This definition was first given in [35], and was later shown by Meßner [31] to be independent of previous definitions of "topological" L-spaces; and other references include [39, 40]. We denote (full) subcategories of 2-topological fixed-basis spaces by the prefix **2-TOP**.

2.15.7 Theorem (Sobriety Representation Theorem). Let $L \in |\mathbf{SFRM}|$.

(1) The L-2-soberification of each ordinary topological space is a 2-topological sober L-space.

(2) **SOB-TOP** $\approx$ **2-TOP-SOB-L-TOP** via the restriction of $LPT \circ \Omega \dashv PT \circ L\Omega$.

Proof. *Ad (1).* The details are similar to those of 2.15.5 above. Given an ordinary topology $\mathfrak{T}$, it follows that $\Phi_2^{\rightarrow}(\mathfrak{T})$ is also an ordinary topology, and so both $\mathfrak{T}$ and $\Phi_2^{\rightarrow}(\mathfrak{T})$ are both 2-spatial and L-spatial, i.e. $\mathfrak{T} \approx \Phi_2^{\rightarrow}(\mathfrak{T})$, $\mathfrak{T} \approx \Phi_L^{\rightarrow}(\mathfrak{T})$, and $\Phi_2^{\rightarrow}(\mathfrak{T}) \approx \Phi_L^{\rightarrow}(\Phi_2^{\rightarrow}(\mathfrak{T}))$. Now it follows that

$$\Phi_L^{\rightarrow}(\mathfrak{T}) \approx \Phi_L^{\rightarrow}(\Phi_2^{\rightarrow}(\mathfrak{T})) \approx \Phi_2^{\rightarrow}(\mathfrak{T}) \approx \Phi_2^{\rightarrow}(\Phi_L^{\rightarrow}(\mathfrak{T}))$$

This implies $\Phi_L^{\rightarrow}(\mathfrak{T})$ is 2-spatial, and so $(Lpt(\mathfrak{T}), \Phi_L^{\rightarrow}(\mathfrak{T}))$ is a 2-topological sober L-space.

Ad (2). The adjunction $LPT \circ \Omega \dashv PT \circ L\Omega$ is now a composition of categorical equivalences. Namely, we now view $\Omega \dashv PT$ as the categorical equivalence between **SOB-TOP** and **SPAT-SLOC,** and we view the categorical equivalence $LPT \dashv L\Omega$ between **SOB-L-TOP** and **L-SPAT-SLOC** as restricting to an equivalence between **2-TOP-SOB-L-TOP** and **SPAT-SLOC.** □

The following result stems from discussions with U. Höhle concerning L-2-soberifications.

2.15.8 Application (Fuzzy Real Lines and Unit Intervals). If L is a complete Boolean algebra, then $\mathbb{R}(L)$ is L-homeomorphic to the L-2-soberification of $\mathbb{R}$ and $\mathbb{I}(L)$ is L-homeomorphic to the L-2-soberification of $\mathbb{I}$.

Proof. We prove the fuzzy unit interval case; the other case is identical. From Theorem 3 of [12] comes the fact that if $L \in |\mathbf{CBOOL}|$, then the L-topology $\tau(L)$ of $\mathbb{I}(L)$ is lattice-isomorphic to the usual topology $\tau(2)$ of the real line interval $\mathbb{I}$. Now from [31], we have $L \in |\mathbf{CBOOL}| \Leftrightarrow (\mathbb{I}(L), \tau(L)) \approx LPT(\tau(L))$. Finally, we have $\tau(2) \approx \tau(L) \Rightarrow LPT(\tau(L)) \approx LPT(\tau(2))$ by the universality of $L\Omega \dashv LPT$ [35]. The claim now follows. □

2.15.9 Remark. Theorems 2.15.5 and 2.15.7 in essence say that L-sobriety is closely related to, yet richer than, traditional sobriety, i.e. L-2-soberification is richer than 2-L-soberification. Restated, ordinary sober topological spaces may be categorically regarded as sober L-spaces which are "topological". Thus L-sobriety provides the traditional sober spaces, plus possibly many more depending on the lattice L. We state the following problem and related conjecture:

2.15.10 Problem. Find a condition on L which is necessary and sufficient to have **SOB-TOP** $\approx$ **SOB-L-TOP**; and find all such conditions on L.

2.15.11 Conjecture. Let $L \in |\mathbf{SFRM}|$. Then **SOB-TOP** $\approx$ **SOB-L-TOP** $\Leftrightarrow$ L is 2-spatial.

2.15.12 Remark. If the conjecture be true, then for the non-spatial locales L given in [16], L-sobriety would be strictly richer categorically than traditional sobriety.

2.16 Discussion (Dual solution to the general fuzzification problem). One of the unanticipated benefits of the L-2soberification functor is that it resolves the general fuzzification problem in topology in a way dual to the Klein fuzzification.

2.16.1 Question (Fuzzification problem). In reference [97] of [39], the author essentially posed the following **fuzzification problem**. Fix L a complete deMorgan chain. $\forall\ (X, \mathfrak{T}) \in |\mathbf{TOP}|$, $\exists$ (?) $(X(L), \mathfrak{T}(L)) \in |L\text{-}\mathbf{TOP}|$, the following boundary conditions are satisfied:

(1) For $L = 2 \equiv \{\perp, \top\}$, $\exists\, h : X \to X(L)$ a bijection, $\mathfrak{T}$ is order-isomorphic to $\mathfrak{T}(L)$ via $h^{\leftarrow} \circ S_\perp$, where $S_\perp(u) = [u > \perp]$ (see the S_α functor of 6.2.1(3) of [43] with $\alpha = \perp$);

(2) For $(X, \mathfrak{T}) = \mathbb{R}$, $(X(L), \mathfrak{T}(L))$ is L-homeomorphic to the L-fuzzy real line $\mathbb{R}(L)$ (reference [69] of [43]).

(3) For $(X, \mathfrak{T}) = \mathbb{I} \equiv [0, 1]$, $(X(L), \mathfrak{T}(L))$ is L-homeomorphic to the L-fuzzy unit interval $\mathbb{I}(L)$ (reference [69] of [43]).

The following summarizes the substantial partial solution by Klein [17–19]:

2.16.2 Theorem (Klein fuzzification). Let $(X, \mathfrak{T})$ be an ordinary topological space with at most finitely many components, and let $L \in |\mathbf{HUT}|$ such that $\perp$ is meet-irreducible. Then $\exists!\ (X(L), \mathfrak{T}(L)) \in |L\text{-}\mathbf{TOP}|$. Further, this fuzzification satisfies the following conditions *à la* those of 2.16.1:

(1) For $L = 2 \equiv \{\bot, \top\}$, 2.16.1(1) holds providing it is further assumed that $(X, \mathfrak{T})$ is T_1 and $[\forall\, x \in X,\ X - \{x\}$ is connected$]$.

(2) For $(X, \mathfrak{T}) = \mathbb{R}$, 2.16.1(2) holds providing L is assumed a complete deMorgan chain.

(3) For $(X, \mathfrak{T}) = \mathbb{I} \equiv [0, 1]$, 2.16.1(2) providing L is assumed a complete deMorgan chain.

The Klein fuzzification leads to some additional open questions.

2.16.3 Question (Fuzzification of complex plane). Given $\mathbb{R} \times \mathbb{R}$, then $(\mathbb{R} \times \mathbb{R})(L)$ exists by 2.16.2 and $\mathbb{R}(L) \times \mathbb{R}(L)$ exists by the Goguen-Wong product [9,55], which is the categorical product on L-**TOP**. We repeat the question first posed as an open question in reference [52] of [39]:

- Is there an unambiguous model for the L-complex plane, i.e. is $(\mathbb{R} \times \mathbb{R})(L)$ L-homeomorphic to $\mathbb{R}(L) \times \mathbb{R}(L)$?

2.16.4 Question (Klein fuzzification functorial).

(1) Is the Klein fuzzification functorial, i.e. can it be made into a functor K : **CONN**-T_1-**TOP** $\to$ L-**TOP**?

(2) If the answer to (1) is affirmative, does K have a left-adjoint or at least preserve products?

We now give partial answers to 2.16.1, 2.16.3, and 2.16.4 using the L-2-soberification functor for L a semiframe. The conclusions of the theorem are stated so as to facilitate comparison with 2.16.1 above and 2.6.2 above.

2.16.5 Theorem (L-2-Soberification and fuzzification). Let $(X, \mathfrak{T}) \in |\mathbf{TOP}|$ and $L \in |\mathbf{SFRM}|$. Then its L-2-soberification

$$(X^*(L), \mathfrak{T}^*(L)) \equiv LPT(\Omega(X, \mathfrak{T})) = (Lpt(\mathfrak{T}), \Phi_L^{\rightarrow}(\mathfrak{T}))$$

satisfies the following conditions:

(1) For $L = 2$, $\mathfrak{T}$ is order-isomorphic to $\mathfrak{T}^*(L)$; and X is bijective with $X(L)$ iff $(X, \mathfrak{T})$ is sober, in which case 2.1.16(1) holds as stated.

(2) For $(X, \mathfrak{T}) = \mathbb{R}$, $(X^*(L), \mathfrak{T}^*(L)) \equiv \mathbb{R}^*(L)$ is L-homeomorphic to the L-fuzzy real line $\mathbb{R}(L)$ providing $L \in |\mathbf{CBOOL}|$.

(3) For $(X, \mathfrak{T}) = \mathbb{I} \equiv [0, 1]$, $(X^*(L), \mathfrak{T}^*(L)) \equiv \mathbb{I}^*(L)$ is L-homeomorphic to the L-fuzzy unit interval $\mathbb{I}(L)$ providing $L \in |\mathbf{CBOOL}|$.

(4) $(\mathbb{R} \times \mathbb{R})^* (L)$ is L-homeomorphic to $\mathbb{R}^* (L) \times \mathbb{R}^* (L)$.

(5) The following are all L-homeomorphic

$$(\mathbb{R} \times \mathbb{R})^* (L) \approx \mathbb{R}^* (L) \times \mathbb{R}^* (L) \approx \mathbb{R}(L) \times \mathbb{R}(L)$$

providing providing $L \in |\mathbf{CBOOL}|$.

(6) Results analogous to (4,5) hold for the L-fuzzy unit square; i.e. (4,5) holds with "$\mathbb{R}$" replaced by "$\mathbb{I}$".

Proof. For (1), $\mathfrak{T}$ is 2-spatial; the second clause is the defintion of sobriety; and the third clause holds since $\Psi^{\leftarrow} \circ S_{\perp} = \Phi^{-1} \circ S_{\perp} = \Phi_2^{-1}$. The proofs of (2,3) are given in 2.15.8. As for (4), we have $\mathbb{R} \times \mathbb{R}$ is Hausdorff, hence sober. So using this fact and the 2-spatiality of $\mathfrak{T}$, we may restriction our attention to **SOB-TOP** and **2-TOP-SOB-L-TOP**. By 2.15.7(2), the adjunction

$$LPT \circ \Omega \dashv PT \circ L\Omega$$

may be regarded as a categorical equivalence, which implies the reverse adjunction

$$PT \circ L\Omega \dashv LPT \circ \Omega$$

holds, i.e. the L-2-soberification functor has a left-adjoint. This implies that this functor preserves all products existing in **SOB-TOP**. Since **SOB-L-TOP** and **SOB-TOP** are closed under all products (Theorem 5.2.1 and Corollary 5.2.2 of [35]), we have

$$(\mathbb{R} \times \mathbb{R})^* (L) \approx \mathbb{R}^* (L) \times \mathbb{R}^* (L)$$

in **SOB-L-TOP**. To finish (4), we note

$$(\mathbb{R} \times \mathbb{R})^* (L) \in |\mathbf{2\text{-}TOP\text{-}SOB\text{-}}L\mathbf{\text{-}TOP}|$$

θthat L-homeomorphs of 2-topological sober L-spaces are again 2-topological sober L-spaces (the 2-topological claim follows from the universality of $\Omega \dashv PT$ and the L-sobriety claim follows from the universality of $L\Omega \dashv LPT$), and hence that $\mathbb{R}^* (L) \times \mathbb{R}^* (L) \in |\mathbf{2\text{-}TOP\text{-}SOB\text{-}}L\mathbf{\text{-}TOP}|$. Now (5) follows from (2) and (4) using the universality of the Goguen-Wong product. And the proof of (6) is analogous to that of (4,5). □

2.16.6 Remark (Fuzzification and new class of fuzzy real lines and unit intervals).

(1) It would seem that the L-2-soberification functor should be considered as a new fuzzification functor. Note that this functor provides an L-fuzzification for every topological space for every $L \in |\mathbf{SFRM}|$ and does

so functorially—this latter property has implications for a new fuzzy addition and fuzzy multiplication detailed in 2.16.7 below. In particular, this means we have new models of the fuzzy real line and fuzzy unit interval for ***arbitrary semiframes***, namely $\mathbb{R}^*(L)$ and $\mathbb{I}^*(L)$, which coincide with the Hutton models if L is a complete Boolean algebra. There are other fuzzification procedures in the literature which should in the future be related whenever possible to the L-2-soberification: the L(-fuzzy) duals and L(-fuzzy) topological duals of the author in [36]; the Lowen fuzzification in reference [24] of [36] in which each separable metric space has an L-fuzzification for $L = [0,1]$; the Lowen fuzzification in reference [25] of [36] in which each linearly-ordered topological space has an L-fuzzification for $L = [0,1]$; the Šostak fuzzification in reference [130] of [39] in which each linearly-ordered space has has an L-fuzzification for $L = [0,1]$; etc. The most general of the now known fuzzification machines is that provided by the L-2-soberification functor.

(2) It is instructive philosophically to consider the different paths taken by the Klein fuzzification and the alternatiive provided by L-2-soberification. These show the fuzzification offered here to be dual to the Klein fuzzification.

   (a) The Klein fuzzification first builds the set $X(L)$ and then the L-topology $\mathfrak{T}(L)$, while L-2-soberification builds the L-topology $\mathfrak{T}^*(L)$ (up to isomorphism) first and then the set $X^*(L)$. This should be compared with the two approaches to compactification outlined in Discussion 1.2 above.

   (b) There are two paths from T_0 to Hausdorff: through T_1 and through sobriety. The Klein fuzzification is related to T_1 and our alternative is related to sobriety—see 2.16.2(1) and compare with 2.16.5(1).

   (c) There are two basic ways in which to view the traditional lattice 2 of membership values: as a chain and as a Boolean algebra. The Klein fuzzification is partly related to chains and our alternative is partly related to Boolean algebras—see 2.16.2(2,3) and compare with 2.16.5(2,3).

2.16.7 Application (Fuzzification and new fuzzy arithemetic operations). Jointly L-continuous fuzzy addition and fuzzy multiplication on the L-fuzzy real line with many algebraic properties were constructed by the author in references [96, 100] of [39] under the restriction that L is a complete deMorgan chain. If we consider the L-2-soberification of the real line, namely the new fuzzy real line $\mathbb{R}^*(L)$ from 2.16.6(1), then using 2.16.5(4) and the fact $LPT \circ \Omega$ is functorial, we get as a corollary the generation of new fuzzy arithmetic operations on our new L-fuzzy real lines for L only a *semiframe*:

$$\begin{aligned}[+ : \mathbb{R}\times\mathbb{R}\to\mathbb{R}] \quad &\mapsto \quad [LPT(\Omega(+)) : (\mathbb{R}\times\mathbb{R})^*(L) \to \mathbb{R}^*(L)] \\ &\equiv \quad [(LPT(\Omega(+)))' : \mathbb{R}^*(L)\times\mathbb{R}^*(L) \to \mathbb{R}^*(L)]\end{aligned}$$

$$\begin{aligned}[\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}] \quad &\mapsto \quad \left[LPT\left(\Omega\left(\cdot\right)\right) : \left(\mathbb{R} \times \mathbb{R}\right)^{*}\left(L\right) \to \mathbb{R}^{*}\left(L\right)\right] \\ &\equiv \quad \left[\left(LPT\left(\Omega\left(\cdot\right)\right)\right)' : \mathbb{R}^{*}\left(L\right) \times \mathbb{R}^{*}\left(L\right) \to \mathbb{R}^{*}\left(L\right)\right]\end{aligned}$$

where $LPT\left(\Omega\left(+\right)\right)$ and $LPT\left(\Omega\left(+\right)\right)'$, and $LPT\left(\Omega\left(\cdot\right)\right)$ and $LPT\left(\Omega\left(\cdot\right)\right)'$, are in the same morphism class of L-**TOP** in the sense defined in 8.15 below.

3 Separation axioms for representation and compactification

This section catalogues the separation axioms for L-spaces important for representation and compactification theorems. For this purpose, the appropriate low-level axioms are L-sobriety and axioms subsumed by L-sobriety, and the approriate middle-level axioms are either the regularity, complete regularity, and normality motivated by locales, or the Hutton-Reilly regularity, complete regularity, and normality axioms. It is well-established for some time that the Hutton-Reilly high-level axioms are appropriate if taken without the Hutton-Reilly low-level axioms; we show in [42] that these high-level axioms should be used with the L-T_0 axiom subsumed in L-sobriety. See also [20, 21–22, 37, 39]: these references include comparisons of separation axiom schemes, relationships to uniform structures, and studies of separation properties of the fuzzy intervals and fuzzy real lines. Proofs of the results of this section are found in [42].

3.1 Discussion (Low-level separation axioms). The low-level axioms suitable for compactification in poslat (or L-) topological spaces are those defined in the preceding section, namely $(L\text{-})T_0$ and $(L\text{-})$sobriety of Definition 2.5 above. For convenience, we also now name an axiom implicit in Definition 2.5: (X, τ) is $(L\text{-})S_0$ iff is surjective. For L-topology we have that

$$(L\text{-})\text{sober} \Leftrightarrow T_0 + S_0.$$

Furthermore, it is shown for L-topology [35]:

(1) (X, τ) is $T_0 \Leftrightarrow \forall\, x \in X,\ \forall y \in X,\ x \neq y, \exists\, u \in \tau,\ u(x) \neq u(y)$.

(2) (X, τ) is $S_0 \Leftrightarrow \forall\, p \in Lpt(\tau),\ \exists\, x \in X,\ \forall\, u \in \tau,\ p(u) = u(x)$.

3.1.1 Proposition. $\forall\, L \in |\mathbf{DMRG}|$, $\mathbb{R}(L)$ and each $\mathbb{J}\left(L\right)$, for J an interval of $\mathbb{R}$, are L-T_0 in the sense of Definition 2.5 above (see [12], [7], [34, 43] for definitions and discussion of $\mathbb{R}(L)$ and $\mathbb{J}\left(L\right)$).

A far more sophisticated result is due to [31]: given $L \in |\mathbf{DMRG}|$, $\mathbb{R}(L)$ (and in fact $\mathbb{J}\left(L\right)$ for $\mathbb{J}$ a non-degenerate interval of $\mathbb{R}$) is sober $L \in |\mathbf{CBOOL}|$.

3.1.2 Corollary. Every fuzzy real line in the literature is L-T_0.

Because of 3.1.1 and 3.1.2, together with the fact no other T_0-like axiom in the literature is satisfied for $\mathbb{R}(L)$, the L-T_0 axiom of Definition 2.5 should be regarded as canonical. Interestingly enough, our L-T_0 axiom was independently discovered in reference [55] of [42], justified by categorical methods for $L = \mathbb{I}$ in references [29–30] of [42], and is apparently the same as the sub-T_0 axiom of [23]. Note that the HR-T_0 axiom is deemed unacceptable since it is not satisfied by the various fuzzy real lines and intervals [34].

3.2 Discussion (Middle-level separation axioms). The middle-level axioms of importance for compactification for fixed-basis poslat topology include the Hutton-Reilly axioms of regularity, complete regularity, and normality [14] denoted HR-regularity, HR-completely regularity, and HR-normality (but without the HR-T_0 axiom, so e.g. we want HR-normality but not HR-T_0), as well as the "localic" axioms of regularity, complete regularity, and normality given in [35] denoted loc-regularity, loc-completely regularity, and loc-normality in [35, 42]. The latter axioms are motivated by the separation axioms for locales, and in this chapter the prefix "loc" is dropped for the regular and completely regular case. All these axioms are defined below; but also see [22] regarding these axioms.

3.3 Discussion (High-level separation axioms). These include the Hutton-Reilly axioms [14] above normality, but without the HR-T_0 axiom of Hutton-Reilly; e.g. we want HR-pseudometrizability, but not HR-metrizability (the former plus HR-T_0). In keeping with [35], we define a space to be a **metric space** if it is HR-pseudometric and L-T_0 as in Definition 2.5; and we also define a space to be a **sober metric space** if it is HR-pseudometric and L-sober as in Definition 2.5. Using [34, 35], $\mathbb{R}(L)$ is a metric space for $L \in |\mathbf{HUT}|$; and using [31,34], $\mathbb{R}(L)$ is a sober metric space iff $L \in |\mathbf{ORTHOHUT}|$. Also see [20] regarding metric spaces.

3.4 Discussion (Refinements of the partial order for regularity axioms). "Refinements" or strengthenings of a partial order (e.g. the way-below relation $\ll$ of [8]) have been studied for many years; and applications to the fuzzy context are developed in [21–22, 42] and their references. Middle-level separation axioms important to compactification involve various refinements of the partial order either in the base lattice L or in a poslat powerset L^X. We now define and discuss those refinements relevant to regularity.

3.4.1 Definition. Let $L \in |\mathbf{SFRM}|$, $M \subset L$, and $a, b \in M$.

(1) a is **well-inside** b (w.r.t. M), denoted $a \lessdot b$, if

$$\exists c \in M,\ a \wedge c = \bot,\ b \vee c = \top$$

This comes from [16].

(2) a is **quite-inside** b (w.r.t. M), denoted $a \Subset b$, if

$$\exists c \in M, \ a \leq c' \leq b$$

where L is assumed in **DMRG** and c' is the involute of c in L. This comes from [35, 42].

3.4.2 Remark. Trivially $\Subset$ is a refinement, for $a \Subset b \Rightarrow a \leq b$. Generally $\lessdot$ and $\leq$ are not related; but under distributivity, $a \lessdot b \Rightarrow a \leq b$. The notions of $\lessdot$, $\Subset$, and $\ll$ appear to be pair-wise distinct. The first two are related by following "Insertion Axioms", the first one of which guarantees that $\lessdot$ refines $\leq$.

3.4.3 Insertion Axioms (IA). Let $L \in |\mathbf{DMRG}|$ and $M \subset L$.

(IA1) $\forall a, b \in M, a \lessdot b$ (w.r.t. M) $\Rightarrow$ $a \Subset b$ (w.r.t. M).

(IA2) $\forall a, b \in M$, $a \Subset b$ (w.r.t. M) $\Rightarrow$ $a \lessdot b$ (w.r.t. M).

3.4.4 Proposition (Sufficient conditions for the Insertion Axioms). Let $L \in |\mathbf{DMRG}|$, $X \in |\mathbf{SET}|$, and $M \subset L$, and consider the following conditions:

(1) $\perp$ is meet-irreducible in L (or $\perp \in L^b$—see [32, 33]);

(2) L is a vertical sequence of at least two generalized diamonds, each with at least two facets [37];

(3) $L \in |\mathbf{CBOOL}|$, or more generally, L is a complete MV-algebra;

(4) L is a generalized diamond with two facets [37];

(5) $L \in |\mathbf{ORTHODMRG}|$, or more generally, L is a weakly orthomodular deMorgan algebra.

Then the following statements hold:

(I) each of (1,2,3,4) $\Rightarrow$ (IA1) holds for $M \subset L^X$;

(II) each of (3,4,5) $\Rightarrow$ (IA2) holds for $M \subset L^X$.

3.4.5 Remark. None of the conditions of 3.4.4 are superfluous. The generalized conditions in 3.4.4 (3,5) are suggestions of U. Höhle.

3.5 Definition (Regularity axioms). Let $A, L \in |\mathbf{SFRM}|$ and $(X, \tau) \in |L\text{-}\mathbf{TOP}|$.

(1) A is **localic regular**, or **regular**, iff

$$\forall\, a \in A,\ \exists\, \{b_\gamma\}_\gamma \subset \{b \in A : b \lessdot a \text{ w.r.t. } A\},\ a = \bigvee_\gamma b_\gamma$$

This is a modification of [5, 16]; see 3.6(1) below. **REG-SLOC** is the full subcategory of **SLOC** of regular semilocales.

(2) (X, τ) is **localic regular**, or **regular**, iff τ is regular *à la* (1) above. This is a modification of [35]. **REG-L-TOP** denotes the full subcategory of L-**TOP** of regular spaces.

(3) (X, τ) is HR-**regular** iff

$$\forall\, u \in \tau,\ \exists\, \{v_\gamma\}_\gamma \subset \{v : \overline{v} \le u\},\ u = \bigvee_\gamma v_\gamma$$

where L is further assumed to be in **DMRG**, and $\overline{v}$ denotes the (L-)**closure** $\bigwedge \{k : v \le k,\ k' \in \tau\}$ of v [14]. **HR-REG-L-TOP** is the full subcategory of L-**TOP** of HR-regular objects.

The following proposition links regularity of spaces with regularity of locales, shows that HR-regularity can be redefined *à la* the definition of localic regularity using $\Subset$, and shows how these regularities are related to each other and the $L\Omega$ and LPT functors. *Sequens*, **DSFRM** [**DSLOC**] denotes the full subcategory of **SFRM** [**SLOC**] of disdtributive objects.

3.6 Proposition. Let $A, L \in |\mathbf{SFRM}|$ and $(X, \tau) \in |L\text{-}\mathbf{TOP}|$.

(1) If $A \in |\mathbf{DSLOC}|$, then A is regular $\Leftrightarrow$

$$\forall\, a \in A,\ a = \bigvee \{b \in A : b \lessdot a \text{ w.r.t. } A\}$$

Hence for $A \in |\mathbf{LOC}|$, A is regular $\Leftrightarrow$ A is a regular locale in the sense of [5, 16]. Hence **REG-LOC** = **REG-LOC** of [16].

(2) If $L \in |\mathbf{DSFRM}|$, (X, τ) is regular $\Leftrightarrow$

$$\forall\, u \in \tau,\ u = \bigvee \{v : v \lessdot u \text{ w.r.t. } \tau\}$$

Hence if $L \in |\mathbf{FRM}|$, then (X, τ) is regular $\Leftrightarrow$ $L\Omega(X, \tau)$ is a regular locale; and if $L = \{\perp, \top\}$, then $(X, \mathfrak{T}) \in |\mathbf{TOP}|$ is (classically) regular $\Leftrightarrow$ $G_\chi(X, \mathfrak{T})$ is regular, where G_χ can be found in [39, 42].

(3) If $L \in |\mathbf{DMRG}|$, then the following are equivalent:

(i) (X, τ) is HR-regular;

(ii) $\forall\, u \in \tau,\ \exists\, \{v_\gamma\}_\gamma \subset \{v \in \tau : v \Subset u \text{ w.r.t. } \tau\},\ u = \bigvee_\gamma v_\gamma$;

(iii) $\forall u \in \tau$, $u = \bigvee\{v \in \tau : v \Subset u\}$.

(4) If $L \in |\mathbf{DMRG}|$ and (IA1) holds for w.r.t. L^X, then regularity $\Rightarrow$ HR-regularity. This implication holds under 3.4.4(1-3).

(5) If $L \in |\mathbf{DMRG}|$ and (IA2) holds for w.r.t. L^X, then HR-regularity $\Rightarrow$ regularity. This implication holds under 3.4.4(3-5).

(6) A is regular $\Rightarrow$ $LPT(A)$ is regular and L-sober; the converse holds if A is L-spatial.

(7) If $L \in |\mathbf{DMRG}|$ and (IA1) holds for $\Phi_L^{\rightarrow}(A)$ w.r.t. $L^{Lpt(A)}$, A is regular $\Rightarrow$ $LPT(A)$ is HR-regular and L-sober; the converse is valid if also A is L-spatial and (IA2) holds for $\Phi_L^{\rightarrow}(A)$ w.r.t. $L^{Lpt(A)}$.

(8) The following hold (cf. 2.9 above):

$$L\Omega : \textbf{REG-}L\textbf{-TOP} \rightarrow \textbf{REG-SLOC}$$

$$LPT : \textbf{REG-}L\textbf{-TOP} \leftarrow \textbf{REG-SLOC}$$

$$L\Omega : \textbf{HR-REG-}L\textbf{-TOP} \rightarrow \textbf{REG-SLOC} \quad \text{(under (IA2))}$$

$$LPT : \textbf{HR-REG-}L\textbf{-TOP} \leftarrow \textbf{REG-SLOC} \quad \text{(under (IA1)}$$

3.7 Corollary. Let $L \in |\mathbf{CBOOL}|$, or more generally, L be a complete MV-algebra, and $(X, \tau) \in |L\textbf{-TOP}|$. Then (X, τ) is regular $\Leftrightarrow$ (X, τ) is HR-regular.

3.8 Remark.

(1) It is known that $\mathbb{R}(L)$ and $\mathbb{J}\,(L)$, for $\mathbb{J} \subset \mathbb{R}$, are HR-regular (in fact, HR-pseudometrizable) if $L \in |\mathbf{HUT}|$: see [34]. It follows that if (IA2) also holds, $\mathbb{R}(L)$ and $\mathbb{J}\,(L)$ are regular. For example, if $L \in |\mathbf{CBOOL}|$, then $\mathbb{R}(L)$ is regular; of course, for such an L it is well-known that the Hutton topology $\tau(L)$ on $\mathbb{R}(L)$ is isomorphic to the standard topology $\mathfrak{T}$ on $\mathbb{R}$, which yields the same result.

(2) Under certain restrictions, Sarkar-regularity [46] may be substituted for HR-regularity. See 5.1(2,3) of [34] for the conditions.

(3) From [35] comes the notion of a L-topological space being loc-Hausdorff: if $L \in |\mathbf{FRM}|$, then (X, τ) is **loc-Hausdorff** iff τ is strongly Hausdorff as a locale (for which notion see [15, 16]). Under the conditions of (1) guaranteeing regularity, the fuzzy intervals are localic Hausdorff as well. They also satisfy the R_0 and R_1 axioms of [14]. From above, they satisfy for $L \in |\mathbf{DMRG}|$ the L-T_0 axiom.

3.9 Discussion (Refinements of the partial order for complete regularity axioms). As in the case with regularity, complete regularity requires refinements of the partial order, refinements related to those given in 3.4 and its sequel, which we now define and discuss.

3.9.1 Definition. Let $L \in |\mathbf{SFRM}|$, $M \subset L$, and $\forall\, a, b \in M$. And let $\mathbb{D}$ be the dyadic rationals in $\mathbb{I} \equiv [0, 1]$.

(1) a is **really inside** b (w.r.t. M), denoted $a \overline{\overline{\lessdot}}\, b$, iff $\exists\ \{a_q : q \in \mathbb{D}\} \subset M$,

   (i) $a \leq a_0 \lessdot a_1 \leq b$; and

   (ii) $p < q \Rightarrow a_p \lessdot a_q$.

   This definition comes from [16]. We shall call $\{a_q : q \in \mathbb{D}\}$ a **loc-scale between a and b.**

(2) a is **completely inside** b (w.r.t. M), denoted $a \Subset\!\Subset b$, iff $\exists\ \{a_r : r \in \mathbb{I}\} \subset M$,

   (i) $a \leq a_0 \Subset\!\Subset a_1 \leq b$; and

   (ii) $r < s \Rightarrow a_r \Subset\!\Subset a_s$.

   This definition comes from [35,43]; a similar refinement, the $\prec$-relation, has since appeared in [21,22]. We shall call $\{a_r : r \in \mathbb{I}\}$ an **HR-scale between a and b.**.

3.9.2 Remark. The way in which $\overline{\overline{\lessdot}}$ and $\Subset\!\Subset$ are derived from $\lessdot$ and $\Subset$, respectively, guarantees that the new notions are related by the same Insertion Axioms used above to relate the previous refinements of $\leq$, and that the above development concerning $\lessdot$, $\Subset$, regularity, and HR-regularity has an analogous development for $\overline{\overline{\lessdot}}$, $\Subset\!\Subset$, and the definitions of complete regularity given below.

3.9.3 Lemma (Sufficiency of the Insertion Axioms). Let $L \in |\mathbf{DMRG}|$ and $M \subset L$.

(1) If (IA1) holds for M w.r.t. L and M is a complete join subsemilattice of L, then $\forall\, a, b \in M$, $a \overline{\overline{\lessdot}}\, b$ (w.r.t. M) $\Rightarrow a \Subset\!\Subset b$ (w.r.t. M).

(2) If (IA2) holds for M w.r.t. L, then $\forall\, a, b \in M$, $a \Subset\!\Subset b$ (w.r.t. M) $\Rightarrow a \overline{\overline{\lessdot}}\, b$ (w.r.t. M).

3.9.4 Proposition. Let $L \in |\mathbf{DMRG}|$, $X \in |\mathbf{SET}|$, and $M \subset L$. Then:

(I) If M is a complete join semisublattice of L, then each of 3.4.4(1,2,3,4) $\Rightarrow$ $[\forall\, a, b \in M,\ a \overline{\overline{<}} b$ (w.r.t. M) $\Rightarrow a \Subset b$ (w.r.t. M)$]$.

(II) Each of 3.4.4(3,4,5) $\Rightarrow$ $[\forall\, a, b \in M,\ a \Subset b$ (w.r.t. M) $\Rightarrow a \overline{\overline{<}} b$ (w.r.t. M)$]$.

3.10 Definition (**Complete regularity axioms**). Let $A, L \in |\mathbf{SFRM}|$ and $(X, \tau) \in |L\text{-}\mathbf{TOP}|$.

(1) A is **localic completely regular**, or **completely regular**, abbreviated **c-reg**, iff

$$\forall\, a \in A,\ \exists\ \{b_\gamma\}_\gamma \subset \left\{b \in A : b \overline{\overline{<}} a \text{ w.r.t. } A\right\},\ a = \bigvee_\gamma b_\gamma$$

This is a modification of [5, 16]; see 3.11(1) below. **CREG-SLOC** is the full subcategory of **SLOC** of c-reg semilocales.

(2) (X, τ) is **localic completely regular**, or **completely regular**, abbreviated **c-reg**, iff τ is c-reg *à la* (1) above. This is a modification of [35]. **CREG-L-TOP** denotes the full subcategory of L-**TOP** of c-reg spaces.

(3) (X, τ) is **HR-completely regular**, abbreviated **HR-cr**, iff

$$\forall\, u \in \tau,\ \exists \mathcal{I} \in |\mathbf{SET}|,\ \exists\ \{u_{i\,r} : (i, r) \in \mathcal{I} \times \mathbb{I}\},$$

$$u = \bigvee \{u_{i\,0} : i \in \mathcal{I}\} \text{ and } [\forall\, i \in \mathcal{I},\ r < s \Rightarrow \bar{u}_{i\,r} \le u_{i\,s} \le u]$$

where L is further assumed to be in $|\mathbf{DMRG}|$, and $\bar{u}$ denotes the L-closure of u (3.5(3) above). This definition comes from [14]. **HR-CR-L-TOP** is the full subcategory of L-**TOP** of HR-cr spaces.

3.10.1 Remark. Loc-cr $\Rightarrow$ regularity, and HR-cr $\Rightarrow$ HR-regularity.

Analogous to Proposition 3.6, the following proposition links complete regularity of spaces with complete regularity of locales, shows that HR-complete regularity can be redefined *à la* the definition of loc-complete regularity using $\Subset$, and shows how these complete regularities are related to each other and the $L\Omega$ and LPT functors.

3.11 Proposition. Let $A, L \in |\mathbf{SFRM}|$ and $(X, \tau) \in |L\text{-}\mathbf{TOP}|$.

(1) If $A \in |\mathbf{DCSLF}|$, then A is c-reg $\Leftrightarrow \forall\, a \in A,\ a = \bigvee \left\{b : b \overline{\overline{<}} a \text{ w.r.t. } A\right\}$. Hence for $A \in |\mathbf{LOC}|$, A is c-reg $\Leftrightarrow$ A is a completely regular locale in the sense of [5, 16]. Restated, **CREG-LOC** = **COMPL-REG-LOC** of [16].

(2) If $L \in |\mathbf{DCSLF}|$, (X,τ) is c-reg $\Leftrightarrow \forall\, u \in \tau,\ u = \bigvee \left\{v : v \overline{\overline{<}} u \text{ w.r.t. } \tau\right\}$; if $L \in |\mathbf{FRM}|$, then (X,τ) is c-reg $\Leftrightarrow L\Omega(X,\tau)$ is a regular locale; and if $L = \{\perp, \top\}$, then $(X,T) \in |\mathbf{TOP}|$ is (classically) completely regular $\Leftrightarrow G_\chi(X,\mathfrak{T})$ is c-reg, where G_χ can be found in [39, 43].

(3) If $L \in |\mathbf{DMRG}|$, then the following are equivalent:

(i) (X,τ) is HR-cr;

(ii) $\forall\, u \in \tau,\ \exists\, \{v_\gamma\}_\gamma \subset \{v \in \tau : v \Subset\!\!\Subset u \text{ w.r.t. } \tau\},\ u = \bigvee_\gamma v_\gamma$;

(iii) $\forall\, u \in \tau,\ u = \bigvee \{v \in \tau : v \Subset\!\!\Subset u\}$.

(4) If $L \in |\mathbf{DMRG}|$ and (IA1) holds for w.r.t. L , then c-reg $\Rightarrow$ HR-cr. This implication holds under 3.4.4(1-3).

(5) If $L \in |\mathbf{DMRG}|$ and (IA2) holds for w.r.t. L , then c-reg $\Leftarrow$ HR-cr. This implication holds under 3.4.4(3-5).

(6) A is c-reg $\Rightarrow LPT(A)$ is c-reg and L-sober; the converse holds if A is L-spatial.

(7) If $L \in |\mathbf{DMRG}|$ and (IA1) holds for $\Phi_L^{\rightarrow}(A)$ w.r.t. $L^{Lpt(A)}$, A is c-reg $\Rightarrow LPT(A)$ is HR-cr and L-sober; the converse is also valid if A is L-spatial and (IA2) holds for $\Phi_L^{\rightarrow}(A)$ w.r.t. $L^{Lpt(A)}$.

(8) The following hold (cf. 2.9 and 3.6(8)) above:

$$L\Omega : \mathbf{CREG}\text{-}L\text{-}\mathbf{TOP} \to \mathbf{CREG}\text{-}\mathbf{SLOC}$$

$$LPT : \mathbf{CREG}\text{-}L\text{-}\mathbf{TOP} \leftarrow \mathbf{CREG}\text{-}\mathbf{SLOC}$$

$$L\Omega : \mathbf{HR}\text{-}\mathbf{CR}\text{-}L\text{-}\mathbf{TOP} \to \mathbf{CREG}\text{-}\mathbf{SLOC} \quad \text{(under (IA2))}$$

$$LPT : \mathbf{HR}\text{-}\mathbf{CR}\text{-}L\text{-}\mathbf{TOP} \leftarrow \mathbf{CREG}\text{-}\mathbf{SLOC} \quad \text{(under (IA1)}$$

3.12 Corollary. Corollary 3.7 and Remark 3.8 hold with "regular" ["HR-regular"] replaced by "c-reg" ["HR-cr"].

3.13 Remark. To the statements of 3.12 can be added some of the additional consequences of c-reg; e.g. if L is a frame, then (X,τ) is c-reg $\Rightarrow \tau$ is isomorphic to a (not necessarily closed) sublocale of a localic product of the localic unit interval. The reader can peruse IV: 1.7, 2.1-2.11 of [16] for further implications of c-reg, and of HR-cr through c-reg and Propostion 3.11.

3.14 Discussion (Refinements of the partial order for normality axioms). As with regularity, normality requires refinements of the partial order *à la* those given in 3.4 and *sequens*, which we now define and discuss.

3.14.1 Definition. Let $L \in |\mathbf{SFRM}|$, $M \subset L$, and $a, b \in M$.

(1) $a \lll b$ **via** c means a, b, c satisfy the condition of 3.4.1(1) above.

(2) $a \Subset b$ **via** c means a, b, c satisfy the condition of 3.4.1(2) above, where L is assumed in **DMRG**.

3.14.2 Modified Insertion Axioms (**MIA**). Let $L \in |\mathbf{DMRG}|$ and $M \subset L$.

(1) $\forall\, a, b, c \in M$, $a \lll b$ via $c \Rightarrow a \Subset b$ via c'.

(2) $\forall\, a, b, c \in M$, $a \Subset b$ via $c \Rightarrow a \lll b$ via c'.

3.14.3 Remark. Clearly MIA(1) [(2)] is strictly stronger than IA(1) [(2)].

3.14.4 Proposition. Let $L \in |\mathbf{DMRG}|$, $X \in |\mathbf{SET}|$, and $M \subset L$.

(1) Each of 3.4.4(1,2,3,4) implies MIA(1) holds for M w.r.t. L^X.

(2) Each of 3.4.4(3,4,5) implies MIA(2) holds for M w.r.t. L^X.

3.15 Definition (**Normality axioms**). Let $A, L \in |\mathbf{SFRM}|$ and $(X, \tau) \in |L\text{-}\mathbf{TOP}|$.

(1) A is **localic normal**, or **loc-normal**, iff

$$\forall\, a, b \in A,\ a \vee b = \top,\ \exists\, c, d \in A,\ c \wedge d = \bot,\ c \vee b = \top,\ a \vee d = \top$$

This comes from [5, 16].

(2) (X, τ) is **localic normal**, or **loc-normal**, iff τ is loc-normal *à la* (1) above; cf. [35].

(3) If $L \in |\mathbf{DMRG}|$, then (X, τ) is **H-normal** iff $\forall$ **closed** k (i.e. $k' \in \tau$), $\forall\, u \in \tau$, $k \le u$, $\exists\, v$, $k \le v \le \overline{v} \le u$. This comes from [12].

3.15.1 Remark. Loc-normality $\Rightarrow$ c-reg, and H-normality $\Rightarrow$ HR-cr. The former implication is the "localic Urysohn Lemma" [4, 16] and the latter implication is the "Hutton-Urysohn Lemma" [12].

Analogous to 3.6 and 3.11, the following proposition links loc-normality with normality of locales, shows loc-normality and H-normality can be redefined using the refinements of the partial order given in 3.14.1, and relates these normalities to each other and the $L\Omega$ and LPT functors.

3.16 Proposition. Let $A, L \in |\mathbf{SFRM}|$ and $(X, \tau) \in |L\text{-}\mathbf{TOP}|$.

(1) A is loc-normal $\Leftrightarrow \forall\, a, b \in A,\ a \vee b = \top,\ \exists\, c, d \in A,$

$$c \lessdot a \text{ via } d,\ d \lessdot b \text{ via } c$$

For $A \in |\mathbf{LOC}|$, A is loc-normal $\Leftrightarrow$ A is a normal locale in the sense of [5, 16].

(2) (X, τ) is loc-normal $\Leftrightarrow \forall u, v \in \tau,\ u \vee v = \underline{\top},\ \exists\, w, z \in \tau,\ w \lessdot u$ via z, $z \lessdot v$ via w. For $L \in |\mathbf{FRM}|$, (X, τ) is loc-normal $\Leftrightarrow L\Omega(X, \tau)$ is a normal locale; and if $L = \{\bot, \top\}$, $(X, \mathfrak{T}) \in |\mathbf{TOP}|$ is (classically) normal $\Leftrightarrow G_\chi(X, \mathfrak{T})$ is loc-normal.

(3) If $L \in |\mathbf{DMRG}|$, then (X, τ) is H-normal $\Leftrightarrow \forall u, v \in,\ v' \leq u,\ \exists\, w, z \in \tau,\ v' \lessdot w'$ via z, $z \lessdot u$ via w'.

(4) If $L \in |\mathbf{DMRG}|$, (MIA1) holds for τ w.r.t. L^X, and τ has the property that

$$\forall\, u, v \in \tau,\ u' \leq v \Rightarrow u \vee v = \underline{\top}$$

then (X, τ) is loc-normal $\Rightarrow$ (X, τ) is H-normal.

(5) If $L \in |\mathbf{DMRG}|$, (MIA2) holds for τ w.r.t. L^X, and τ has the property that

$$\forall\, u, v \in \tau,\ u \vee v = \underline{\top} \Rightarrow u' \leq v$$

then (X, τ) is H-normal $\Rightarrow$ (X, τ) is loc-normal.

(6) If $L \in |\mathbf{DMRG}|$ and (MIA1,2) both hold for τ w.r.t. L^X, then (X, τ) is H-normal $\Leftrightarrow$ (X, τ) is loc-normal. These hypotheses hold if $L \in |\mathbf{CBOOL}|$ or is a complete MV-algebra.

(7) A is loc-normal $\Rightarrow LPT(A)$ is loc-normal and sober; the converse holds if the kernel $\Phi_L^{\leftarrow}\{\underline{\bot}\}$ and the co-kernel $\Phi_L^{\leftarrow}\{\underline{\top}\}$ of Φ_L are singleton sets.

(8) If in (7) it is also assumed that $L \in |\mathbf{DMRG}|$ and (MIA1,2) both hold for $\Phi_L^{\rightarrow}(A)$ w.r.t. $L^{Lpt(A)}$, then A is loc-normal $\Rightarrow LPT(A)$ is H-normal and sober.

3.17 Corollary. Corollary 3.7 and Remark 3.8 hold with "regular" ["HR-regular"] replaced by "loc-normal" ["H-normal", respectively].

4 Compactness axioms for representation and compactification

The purpose of this section is to define and relate the concept of localic compactness to the L-topological notions of compactness [3] and Hutton compactness

[13]. Many other concepts of compactness for fuzzy sets and fuzzy topology have from the poslat perspective been given—α-compactness, strong compactness, fuzzy compactness, ultra-compactness, N-compactness, etc; but none of these are known to admit a truly general compactification in either sense of 1.1 above. Localic compactness and compactness [3] do admit a general compactification in both senses of Section 1 (see Section 8 below); and for certain lattices (or "bases"), Hutton compactness [13] is equivalent to compactness. Hence these three compactness axioms will be considered in this section. Our reasons for the term "compactness" designating the definition due to Chang [3] emerge later in Sections 5–9. See Section 5.3 of [39] for detailed comments on the proper criteria for compactness in poslat fuzzy sets; see Sections 5–8 below and [14, 16] for relations between these compactness axioms and separation axioms, including connections to the compact spaces in the sense of [11]; and see [42] for proofs.

4.1 Definition. Let $A, L \in |\mathbf{SFRM}|$ and $(X, \tau) \in |L\text{-}\mathbf{TOP}|$.

(1) An element $a \in A$ is **finite** or **compact** iff

$$\forall S \subset A, \bigvee S \geq a, \exists F\,(\text{finite}) \subset S, \bigvee F \geq a$$

and the subset of A of compact elements is denoted $K(A)$. If A is a locale, then finiteness is equivalently defined by replacing "$\geq$" by "$=$"; see [16].

(2) A is **localic compact** or **compact** iff $\top$ in A is finite *à la* (1), i.e.

$$\forall S \subset A, \bigvee S = \top, \exists F\,(\text{finite}) \subset S, \bigvee F = \top$$

See [16]. **K-SFRM** [**K-SLOC**, **K-LOC**] is the full subcategory of **SFRM** [resp. **SLOC**, **LOC**] of compact objects.

(3) (X, τ) is **compact** [3] or **localic compact** [35] iff $\underline{\top}$ ($\equiv \chi_X$) in τ is finite *à la* (1), i.e. $\forall$open covering $\mathcal{U} \subset \tau, \bigvee \mathcal{U} = \underline{\top}, \exists$ finite open subcovering $\mathcal{V} \subset \mathcal{U}$, $\bigvee \mathcal{V} = \underline{\top}$. **K-$L$-TOP** is the full subcategory of L-**TOP** of compact objects.

(4) If $L \in |\mathbf{DMRG}|$, then (X, τ) is **H-compact** [13] iff $\forall$closed $k \in L^X$ (3.15(3)), $\forall$ **open covering** $\mathcal{U} \subset \tau$ **of** k, $k \leq \bigvee \mathcal{U}$), $\exists$ **finite open subcovering** $\mathcal{V} \subset \mathcal{U}$ **of** k, $k \leq \bigvee \mathcal{V}$. **H-COMP-$L$-TOP** is the full subcategory of L-**TOP** of H-compact objects.

4.2 Definition. The subsets A^c [7] and A_d (cf. [33]) of a lattice are defined as follows:

$$A^c = \{a \in A : \forall b \in A, \, a \leq b \text{ or } a \geq b\}$$

$$A_d = \{b \in A : \exists a \in A, \, a < b \text{ and } [a, b] \subset A^c\}$$

Also $\top$ is **nonsup in** A iff $\nexists$ non-empty $B \subset L - \{\top\}$, $\bigvee B = \top$ [32].

4.3 Theorem. Let $A, L \in |\mathbf{SFRM}|$ and $(X, \tau) \in |L\text{-}\mathbf{TOP}|$.

(1) (X, τ) is compact $\Leftrightarrow$ $L\Omega(X, \tau) \equiv \tau$ is compact.

(2) If A is L-spatial or A, L are finite, then A is compact $\Leftrightarrow$ $LPT(A)$ is compact.

(3) If A is a generalized diamond of two facets with $\top$ nonsup on each facet [37] or $\top \in A_d$, then A is compact $\Rightarrow$ $LPT(A)$ is compact.

(4) If Φ_L is finite-to-one, and the hypotheses of (3) hold, then $LPT(A)$ is compact $\Rightarrow$ A is compact.

4.4 Remark. The importance of 4.3 cannot be overstated: compactness, maligned in the literature, is precisely the space equivalent of localic compactness (for complete lattices, and particularly for locales). The wholesale abandonment of compactness is due to fuzzy theorists being both unaware of locale theory and preoccupied with "unrestricted" Tihonov Theorems, rather than focusing on categorical reflectors (i.e. categorical compactifications) and Tihonov Theorems which truly generalize the classical Tihonov Theorem. Compactness excels on both issues—see the detailed analyses in [39, 42] as well as the next three sections. And our next results show there is an intimate connection between H-compactness and localic compactness and compactness.

4.5 Theorem. Let $L \in |\mathbf{DMRG}|$ and $(X, \tau) \in |L\text{-}\mathbf{TOP}|$.

(1) H-compactnes $\Rightarrow$ compactness.

(2) If $L \in |\mathbf{CBOOL}|$, then compactness $\Rightarrow$ H-compactness.

4.6 Corollary. Let $A \in |\mathbf{SFRM}|$, $L \in |\mathbf{DMRG}|$, and $(X, \tau) \in |L\text{-}\mathbf{TOP}|$.

(1) (X, τ) is H-compact $\Rightarrow$ $L\Omega(X, \tau) \equiv \tau$ is compact, and the converse holds if $L \in |\mathbf{CBOOL}|$.

(2) If A is L-spatial or A, L are finite, then A is compact $\Leftarrow$ $LPT(A)$ is H-compact, and the converse of "$\Leftarrow$" holds if $L \in |\mathbf{CBOOL}|$.

(3) If $L \in |\mathbf{CBOOL}|$, and if A is a generalized diamond of two facets with $\top$ nonsup on each facet [37] or $\top \in A$, then A is compact $\Rightarrow$ $LPT(A)$ is H-compact.

(4) If Φ_L is finite-to-one, and the hypotheses of (3) hold, then $LPT(A)$ is H-compact $\Rightarrow$ A is compact.

5 Fixed-basis Stone representation theorems for distributive lattices

This section shows how distributive lattices can be categorically represented by certain compact, L-sober, poslat topological spaces. Use is made of the representation of distributive lattices by coherent locales given in [16]. The classical Stone representation theorem for distributive lattices is generalized in poslat topology in the sense that it is reformulated and incorporated into a schemum or class of representation theorems, each theorem indexed by the base lattice L of membership values, the classical theorem being recovered by stipulating $L = \{\bot, \top\}$. This implies that compactness and L-sobriety, in the context of coherence, are the counterpart to traditional compactness and sobriety.

From [16] comes the modern form of the classical theory, the material concerning poslat spaces is modified from [35], while Stone's original theorem is in [51]. Proofs for this section are in [42].

5.1 Definition (Coherent Lattices). Let $A \in |\mathbf{SLOC}|$. Then A is **coherent** iff both of the following hold:

(1) Each element of A is a join of finite elements of A.

(2) The set $K(A)$ of all finite elements is a sublattice of A; equivalently, $K(A)$ is closed under finite meets (so that $\top \in K(A)$). (The equivalence follows since $K(A)$ is always closed under finite joins).

5.1.1 Remark. Note that a coherent object of **SLOC** is compact (since $\top$ is finite).

5.1.2 Examples. Every finite, non-distributive lattice is coherent (and complete), and there are easily obtained infinite, non-distributive, complete, coherent lattices; e.g., let A be the set-theoretic union $\mathbb{I}_l \cup \mathbb{I}_r$ of two copies of $\mathbb{I} \equiv [0, 1]$, say $\mathbb{I}_l$ for the "left" copy and $\mathbb{I}_r$ for the "right" copy, but with 1_l and 1_r identified as $\top$, 0_l and 0_r identified as $\bot$, and the obvious order with only $\top$ and $\bot$ comparable to all the elements of A. So there are many non-trivial examples in

$$|\mathbf{COH\text{-}SLOC}| - |\mathbf{COH\text{-}LOC}|$$

5.2 Definition (The category COH-SLOC) . The category **COH-SLOC** is the subcategory of **SLOC** consisting of all coherent objects (5.1) together with all **coherent morphisms**: given $A, B \in |\mathbf{SLOC}|$, $g : A \to B$ in **SLOC** is **coherent** iff $g^{op}|_{K(B)} : K(A) \leftarrow K(B)$ in **SFRM**. **COH-LOC** is the subcategory of **LOC** of all coherent objects and coherent morphisms.

5.3 Definition (The category COH-TOP). The category **COH-TOP** is the subcategory of **TOP** comprising the following data:

(1) **Coherent spaces** $(X, \mathfrak{T})$: a space $(X, \mathfrak{T})$ is **coherent** iff it is sober and $K(\mathfrak{T})$, the collection of compact open subsets of $\mathfrak{T}$, is closed under finite intersections and is a basis for $\mathfrak{T}$.

(2) **Coherent morphisms:** a set mapping $f : X \to Y$ is a **coherent morphism** or **coherent mapping** $f : (X, \mathfrak{T}) \to (Y, \mathfrak{S})$ iff

$$f^{\leftarrow}|_{\mathfrak{S}} : \mathfrak{T} \leftarrow \mathfrak{S} \quad \text{and} \quad f^{\leftarrow}|_{K(\mathfrak{S})} : K(\mathfrak{T}) \leftarrow K(\mathfrak{S})$$

i.e. f is continuous and $[f^{\leftarrow}]^{op}$ is a coherent lattice morphism (in **LOC**).

5.4 Proposition. $(X, \mathfrak{T}) \in |\mathbf{TOP}|$ is coherent iff it is sober and $\mathfrak{T}$ is a coherent locale.

5.5 Definition (The category COH-L-TOP). Let $L \in |\mathbf{SFRM}|$. The category **COH-L-TOP** is the subcategory of **L-TOP** comprising the following data:

(1) (L)-**Coherent spaces**: (X, τ) is (L)-**coherent** iff it is L-sober and τ is a coherent lattice.

(2) (L)-**Coherent morphisms**: a set mapping $f : X \to Y$ is a(n) (L)-**coherent morphism** or (L)-**coherent mapping** $f : (X, \tau) \to (Y, \sigma)$ iff

$$f_L^{\leftarrow}|_{\sigma} : \tau \leftarrow \sigma \quad \text{and} \quad f_L^{\leftarrow}|_{K(\sigma)} : K(\tau) \leftarrow K(\sigma)$$

i.e. f is L-continuous and $[f_L^{\leftarrow}]^{op}$ is a coherent lattice morphism (in **SLOC**).

5.6 Remark. Clearly **COH-$\{\perp, \top\}$-TOP** is isomorphic to **COH-TOP**. Note that the notion of coherence in poslat topology hinges on the notions of compactness (and its equivalence with the finiteness of $\underline{\top}$ in τ, or $\underline{\top}$ being in $K(\tau)$) and fixed-basis sobriety.

We begin to list the tools used in this and the next two sections. The main idea is to relate the concept of ideals to the concept of finite (or compact) elements: J is an **ideal** of B iff J is a **lower set** ($\forall\, a, b \in B$, $[a \le b,\, b \in J] \Rightarrow a \in J$) and J is closed under finite joins (implying $\perp$ of B is in J). We let $Idl\,(B)$ denote the set of all ideals of B. Much of what follows is taken from the classical representation of distributive lattices by coherent locales and coherent spaces as presented in [16].

5.7 Lemma [16: cf. Proposition II.3.2]. A complete distributive lattice is coherent iff it is isomorphic to the locale of ideals of some distributive lattice. Hence, every coherent, complete, distributive lattice is a frame.

5.8 Remark. From the proof of 5.7 in [16] comes the following important fact: $\forall B \in |\mathbf{DLAT}|$, $B \approx K(Idl(B))$, where **DLAT** is the category of distributive lattices and lattice morphisms.

5.9 Lemma (Representation of distributive lattices by coherent locales: [16: Corollary II.3.3]). $\mathbf{DLAT}^{op} \approx$ **COH-LOC**, i.e. $\mathbf{DLAT}^{op}$ is categorically equivalent to **COH-LOC** via the functors FIN and IDL induced respectively by the following correspondences: $A \in |\mathbf{COH\text{-}LOC}|$ determines $K(A) \in \mathbf{DLAT}$, and $B \in \mathbf{DLAT}$ determines $Idl(B) \in |\mathbf{COH\text{-}LOC}|$.

5.10 Lemma [16: Theorem II.3.4]. Each coherent locale is (classically) spatial.

5.11 Lemma. $\forall L \in |\mathbf{SFRM}|$, each coherent locale is L-spatial.

5.12 Lemma. Let $L \in |\mathbf{SFRM}|$. Then the following hold:

(1) $L\Omega$ maps **COH-L-TOP** into **COH-SLOC**.

(2) $L\Omega$ maps **COH-L-TOP** into **COH-LOC** iff $L \in |\mathbf{DSFRM}|$, where **DSFRM** is the full subcategory of **SFRM** of distributive objects.

(3) LPT, restricted to **COH-LOC**, maps **COH-LOC** into **COH-L-TOP**.

The following theorem synergizes these results: 2.10, 5.5, 5.9, 5.11, and 5.12.

5.13 Theorem (Schemum of Stone representation theorems for distributive lattices). The following hold:

(1) $\forall L \in |\mathbf{DSFRM}|$, **COH-$L$-TOP** $\approx$ **COH-LOC**.

(2) $\forall L \in |\mathbf{DSFRM}|$, **COH-$L$-TOP** $\approx \mathbf{DLAT}^{op}$.

(3) **COH-TOP** $\approx$ **DLAT** [16: Corollary II.3.4].

(4) $\forall L \in |\mathbf{DSFRM}|$, **COH-$L$-TOP** $\approx$ **COH-TOP**.

5.14 Remark (Justification of fuzzy sets to traditional mathematics). To traditional mathematics we can say that fuzzy sets, i.e. lattice-valued subsets, give a much richer representation of distributive lattices than traditional topology can alone—for each complete distributive lattice L, there is a category of representing L-topological spaces (5.13(2)). Furthermore, fuzzy sets explain the classical Stone representation theorem for distributive lattices: the membership lattice implicit in Stone's theorem is $\{\perp, \top\}$, and Stone's theorem hinges on this lattice being complete and distributive; and in this way fuzzy sets as lattice-valued sets pinpoint the properties of $\{\perp, \top\}$ crucial to, yet hidden in, Stone's theorem.

5.15 Remark (Justification of compactness and L-sobriety to fuzzy sets). To poslat topology we can say that compact, L-sober spaces are a proper generalization of traditionally compact sober spaces (5.13(4))—indeed, fixing a complete semiframe L, each distributive lattice B generates a compact, L-sober space which is additionally L-coherent (cf. 4.3(2-4) above and see 5.16 below).

5.16 Discussion. Analogously to the traditional case in [16], the equivalence of 5.13(2) sends an L-coherent space (X, τ) to the distributive lattice $K(\tau)$, and sends a distributive lattice B to the L-coherent space $LPT(Idl(B))$—this comes by composing the $L\Omega$, LPT functors appropriately with the FIN, IDL functors of 5.9. Now by Lemma II.3.4 of [16], the prime elements of $Idl(B)$ are precisely the prime ideals of B; and from Proposition 5.4.1(3) of [35] and Proposition 2 of [37], we have that for all $L \in |\mathbf{SFRM}|$, the prime elements of $Idl(B)$ are adjunctive with $Lpt(Idl(B))$ (viewing both as pre-ordered categories) iff $\perp$ is meet-irreducible in L. Hence, analogous to the traditional case, we may regard $LPT(Idl(B))$ as the "**L-prime spectrum**" of B when $\perp$ is meet-irreducible in L, and otherwise as the "**L-spectrum**" of B. We therefore state:

5.17 Definition (L-(Prime) spectrum of lattice). Let $B \in |\mathbf{LAT}|$ and $L \in |\mathbf{SFRM}|$. Then $LPT(Idl(B))$ is the **L-spectrum** of B, denoted L-$Spec(B)$. If $\perp$ is meet-irreducible in L, then L-$Spec(B)$ is called the **L-prime spectrum** of B. The L-spectrum is essential for the representation of Boolean algebras; see 7.8 below.

6 Fixed-basis space representations of compact regular locales and compact Hausdorff spaces

In this section we show that categorically, the classical compact Hausdorff spaces, the compact, regular, distributive objects of **SLOC**, and the compact, regular, L-sober spaces are all categorically equivalent, the latter equivalence requiring L a distributive object of **SFRM**. If L is also a Boolean algebra, then H-compactness, HR-regularity, and L-sobriety are another categorical equivalent, as are the compact, Hausdorff-separated spaces of Subsections 6.3–6.4 of [11]. We point out that these categorical equivalences are obtained in two different ways: using the representation theorems of 5.13 above; and using the L-2 and 2-L soberification functors of 2.15 (*et sequens*) above. Indicated below are some philosophical ramifications. Proofs not given below are found in [42].

6.1 Lemma. **K-REG-DSLOC** $\hookrightarrow$ **2-SPAT-SLOC**, where **DSLOC** denotes $\mathbf{DSFRM}^{op}$.

6.2 Theorem (Representations of compact Hausdorff spaces). The following statements hold:

(1) $\forall L \in |\mathbf{DSFRM}|$, **K-REG-SOB-$L$-TOP** $\approx$ **K-REG-DSLOC**.

(2) $\forall L \in |\mathbf{DSFRM}|$, **K-REG-DSLOC** $\approx$ **K-REG-LOC**.

(3) $\forall L \in |\mathbf{DSFRM}|$, **K-REG-SOB-$L$-TOP** $\approx$ **K-REG-LOC**.

(4) **K-HAUS-TOP** $\approx$ **K-REG-LOC** [16: Corollary III.1.10].

(5) $\forall L \in |\mathbf{DSFRM}|$, **K-REG-SOB-$L$-TOP** $\approx$ **K-HAUS-TOP**.

(6) $\forall L \in |\mathbf{CBOOL}|$,

H-COMP-HR-REG-SOB-L-TOP $\approx$ **K-REG-SOB-L-TOP**

and so statements (1,3,5) hold with "**H-COMP-HR-REG**" replacing "**K-REG**".

6.3 Discussion. The equivalences of (5),(6)(5) above send an L-topological space (X, τ) to the traditional or "crisp" point space $(Pt(\tau), \Phi^{\rightarrow}(\tau))$, and send a traditional topological space $(X, \mathfrak{T})$ to the L-point space $(Lpt(\mathfrak{T}), \Phi_L^{\rightarrow}(\mathfrak{T}))$—the reader will note that these correspondences are object-level descriptions of the L-2-soberification and 2-L-soberification functors (2.15 *et sequens*). Discussions with U. Höhle have motivated the author to use these functors to express an alternative categorical equivalence between **K-REG-SOB-L-TOP** and **K-HAUS-TOP**, i.e. one not factoring through the Stone representation theorems of Section 5. This will prove instructive both here and in Section 8 below. In particular, we are able to provide a partial converse to 6.2(5) above and 6.3.2(1) below—under certain conditions, the distributivity of L is also necessary.

6.3.1 Proposition. The following hold:

(1) $(X, \mathfrak{T})$ compact Hausdorff $\Rightarrow$ $LPT(\mathfrak{T})$ (the L-2-soberification of $(X, \mathfrak{T})$) is compact, regular, and L-sober.

(2) (X, τ) is compact, regular, and L-sober space $\Rightarrow$ $PT(\tau)$ (the 2-L-soberification of (X, τ)) is compact Hausdorff.

(3) Every compact, regular, and L-sober space arises (up to L-homeomorphism) as the L-2-soberification of $(X, \mathfrak{T})$ of some compact Hausdorff space, providing L is a distributive semiframe.

Proof. (1,2) follow from 4.3(1,2), 3.6(6) above. As for (3), we have the following correspondences determined by applying first $PT \circ L\Omega$ and then $LPT \circ \Omega$:

$$(X, \tau) \mapsto (Pt(\tau), \Phi_2^{\rightarrow}(\tau)) \mapsto (Lpt(\Phi_2^{\rightarrow}(\tau)), \Phi_L^{\rightarrow}(\Phi_2^{\rightarrow}(\tau)))$$

Since L is distributive, we have τ is 2-spatial (Lemma 6.1 above), which implies that $\tau \approx \Phi_2^{\rightarrow}(\tau)$. Using L-sobriety followed by the universality of $L\Omega \dashv LPT$, the following L-homeomorphisms set up:

$$(X, \tau) \approx (Lpt(\tau), \Phi_L^{\rightarrow}(\tau)) \approx (Lpt(\Phi_2^{\rightarrow}(\tau)), \Phi_L^{\rightarrow}(\Phi_2^{\rightarrow}(\tau)))$$

This says that every compact, regular, and L-sober space can be recovered as the L-2-soberification of the compact Hausdorff space $(Pt(\tau), \Phi_2^{\rightarrow}(\tau))$, which in turn is the 2-L-soberification of the original compact, regular, and L-sober space. □

6.3.2 Theorem (Direct equivalence of K-REG-SOB-L-TOP to K-HAUS-TOP). Let $L \in |\mathbf{SFRM}|$. The following hold:

(1) If $L \in |\mathbf{DSFRM}|$, then the adjunction $LPT \circ \Omega \dashv PT \circ L\Omega$ of 2.15.5 restricts to give an equivalence of the categories **K-REG-SOB-L-TOP** and **K-HAUS-TOP**.

(2) The converse of (2) holds if L is also assumed to be compact and regular; in fact, under these additional assumptions, if the adjunction $LPT \circ \Omega \dashv PT \circ L\Omega$ restricts to give an equivalence of the categories **K-REG-SOB-L-TOP** and **K-HAUS-TOP**, then $L \in |\mathbf{FRM}|$.

Proof. *Ad (1).* From 6.3.1 above, we have that the adjunction $LPT \circ \Omega \dashv PT \circ L\Omega$ of 2.15.5 restricts to give an adjunction of the categories **K-REG-SOB-L-TOP** and **K-HAUS-TOP**. Note **K-HAUS-TOP** is a subcategory of **SOB-TOP**. Now by Lemma 6.1 we have **K-REG-SOB-L-TOP** is a subcategory of **2-TOP-SOB-L-TOP**. Thus the restriction of the adjunction $LPT \circ \Omega \dashv PT \circ L\Omega$ of 2.15.5 in this case turns to be a restriction of the equivalence $LPT \circ \Omega \dashv PT \circ L\Omega$ of 2.15.7.

Ad (2). Assume that $LPT \circ \Omega \dashv PT \circ L\Omega$ restricts to give an equivalence of the categories **K-REG-SOB-L-TOP** and **K-HAUS-TOP**. Then given (X, τ) in **K-REG-SOB-L-TOP**, we have

$$\tau \approx \Phi_L^{\rightarrow}(\tau)$$

by Lemma 2.9 above,

$$\Phi_L^{\rightarrow}(\tau) \approx \Phi_L^{\rightarrow}(\Phi_2^{\rightarrow}(\tau))$$

via the assumed restriction of $LPT \circ \Omega \dashv PT \circ L\Omega$ and the subsequent L-homeomorphism

$$(X, \tau) \approx (Lpt(\Phi_2^{\rightarrow}(\tau)), \Phi_L^{\rightarrow}(\Phi_2^{\rightarrow}(\tau)))$$

and

$$\Phi_L^{\rightarrow}\left(\Phi_2^{\rightarrow}(\tau)\right) \approx \Phi_2^{\rightarrow}(\tau)$$

by Lemma 2.8(3) followed by Lemma 2.9. Composing isomorphisms, we have that

$$\tau \approx \Phi_2^{\rightarrow}(\tau)$$

Now note that $\Phi_2^{\rightarrow}(\tau)$ is a 2-topology (i.e. a traditional topology) and hence is a frame; and so τ is a frame. Since all of the foregoing is true for each space in **K-REG-SOB-L-TOP**, it must be true of the L-soberification

$$\left(Lpt\left(L^{\mathbf{1}}\right), \Phi_L^{\rightarrow}\left(L^{\mathbf{1}}\right)\right)$$

of the particular singleton compact, regular space $\left(\mathbf{1}, L^{\mathbf{1}}\right)$: the compactness and regularity of $\left(\mathbf{1}, L^{\mathbf{1}}\right)$ are immediate from the localic compactness and regularity of L as a semiframe; and the compactness and regularity of $\left(Lpt\left(L^{\mathbf{1}}\right), \Phi_L^{\rightarrow}\left(L^{\mathbf{1}}\right)\right)$ now follow from 3.11(6) and 4.3(2), with the sobriety trivially true. It follows from the earlier remarks that that $\Phi_L^{\rightarrow}\left(L^{\mathbf{1}}\right)$, hence $L^{\mathbf{1}}$, and hence L, is a frame. So we obtain L is distributive and the converse of (1) holds under the conditions of L being compact and regular. □

6.4 Remark (Role of 2-topological spaces). Recall the definition of 2-topological l-spaces given in 2.15.6. We note that given (X, τ) in L-**TOP** with L in **DSFRM**, (X, τ) is compact, regular, and L-sober $\Rightarrow$ (X, τ) is topological, with an analogous result holding for the Hutton/Hutton-Reilly conditions if L is a complete Boolean algebra: this follows immediately from 2.15.7(1) coupled with 6.3.1(3). Indeed, such spaces have many other important properties, as summarized in the following theorem:

6.5 Theorem. Let $L \in |\mathbf{DSFRM}|$ [$|\mathbf{CBOOL}|$] and $(X, \tau) \in |L\text{-}\mathbf{TOP}|$. Then (X, τ) is compact, regular [H-compact, HR-regular], and L-sober $\Rightarrow$ τ is isomorphic to the topology of some compact Hausdorff object in **TOP**, in which case (X, τ) and τ have these additional properties:

(1) (X, τ) is topological in the sense defined in 2.15.6;

(2) (X, τ) is loc-normal [H-normal] (3.15 above), c-reg [HR-cr] (3.10 above), loc-Hausdorff (3.8(3) above), and loc-T_U (i.e. τ is T_U in the sense of [16]);

(3) localic maps from flat sublocales of locales into have Joyal extensions;

(4) τ is a retract in **LOC** of some coherent locale;

(5) τ is isomorphic to a closed sublocale of the localic Tihonov cube;

(6) τ is a stably continuous lattice.

6.6 Corollary. If $L \in |\mathbf{CBOOL}|$, then all the properties and conclusions of 6.5 hold for the fuzzy unit interval $\mathbb{L}(L)$.

6.7 Remark. Corollary 6.6 should be compared with 2.15.8 above, since both are related to the fact that $\mathbb{I}(L)$ is a 2-topological space for $L \in |\mathbf{CBOOL}|$.

6.8 Remark.

(1) Philosophically, depending on the lattice-theoretic assumptions involved, some combination of compactnes/H-compactness, regularity/HR-regularity, and L-sobriety must be viewed as an appropriate generalization of classical compactness and Hausdorff: our mandate for the word "must" stems from our use of categorical equivalences. The reader might ask: given that the lattice-theoretic assumptions are more restrictive for the Hutton/Hutton-Reilly approach than the localic approach (the restrictions for the latter allowing the important lattice $L = [0,1]$), why not work only with the latter? The answer is that not only are these approaches often logically (and categorically) equivalent, but in some contexts the Hutton/Hutton-Reilly approach is more convenient; indeed, as seen in the next section, the localic approach must factor through the Hutton/Hutton-Reilly approach. In any case, some combination of these two approaches is an appropriate generalization of classical compactness and Hausdorff, a point repeatedly reinforced in Section 7.

(2) In this section we have presented the categorical equivalence of **K-REG-SOB-L-TOP** with **K-HAUS-TOP**. This means that from a categorical point of view, compact, regular, sober, L-spaces behave like compact Hausdorff spaces. But what kind of relationship is there between a specific compact, regular, sober, L-space and compact Hausdorff spaces? This question for its resolution requires notions of subspaces and density and will be presented at the close of Section 8.

6.9 Remark (Connections to compact Hausdorff-separated spaces of [11]). We now summarize some comments communicated to the author from U. Höhle concerning the connections between compact, regular, sober L-spaces and the compact Hausdorff-separated L-spaces of Subsections 6.3–6.4 of [11] and extensively developed in references [20, 25] of [11]. The following comparisons can be made:

(1) Let L be a bi-frame, i.e. $L, L^{op} \in |\mathbf{FRM}|$. Then each compact, sober, L-space is compact in the sense of Subsection 6.4 of [11].

(2) If $L \in |\mathbf{CBOOL}|$, then an L-space is compact, regular, and sober $\Leftrightarrow$ it is compact and Hausdorff separated in the sense of Subsections 6.3–6.4 of [11].

The proof of (1) uses Theorem 6.2.6 of [11], an L-analogue of Lemma III.1.9 of [16], and Zorn's Lemma. And the proof of (2) can be obtained from the categorical equivalence of 6.2(5) above, together with the categorical equivalence of compact Hausdorff spaces with compact Hausdorff-separated spaces in the sense of [11] over complete Boolean algebras (Section 4 of reference [25] of [11]). There is a further corollary we can add which follows from 6.2(6) above:

(3) If $L \in |\mathbf{CBOOL}|$, then an L-space is H-compact, HR-regular, and sober $\Leftrightarrow$ it is compact and Hausdorff-separated in the sense of Subsections 6.3–6.4 of [11].

7 Fixed-basis Stone representation theorems for Boolean algebras

This section shows how Boolean algebras can be categorically represented by certain compact/H-compact, regular/HR-regular, L-sober topological spaces. Crucial use is made of the representation of distributive lattices by L-topological spaces given in Section 5, in which a prominent role is played by the notion of L-spectrum created in 5.17 above. The classical Stone representation theorem for Boolean algebras is generalized in poslat topology in the sense that it is reformulated and incorporated into a schemum or class of representation theorems, each theorem indexed by the base lattice L of membership values, the classical theorem being recovered by stipulating $L = \{\bot, \top\}$. This implies that compactness/H-compactness, regularity/HR-regularity, and L-sobriety comprise poslat counterparts to traditional compactness and Hausdorff, a point reinforcing Section 6.

From [16] comes the modern form of the classical theory, the material concerning fixed-basis spaces is modified from [35], and Stone's original theorem is in [48-50, 52]. Proofs for this section are found in [42].

7.1 Theorem. The following sets of conditions are equivalent for a space $(X, \mathfrak{T})$ in **TOP**:

(1) compact, Hausdorff, and totally disconnected;

(2) compact and totally separated;

(3) compact, T_0 , and zero-dimensional;

(4) coherent and Hausdorff.

7.2 Definition (Classical Stone spaces). A space $(X, \mathfrak{T})$ in **TOP** is a **Stone space** iff it satisfies one 7.1(1-4). The full subcategory of **TOP** of all Stone spaces is denoted **STN-TOP**.

7.3 Proposition. $(X, \mathfrak{T})$ in **TOP** is a Stone space iff it is coherent and regular.

7.4 Remark. Proposition 7.3, taken from [35], indicates how to generalize the concept of Stone spaces to the fixed-basis setting. We define the notion of L-Stone spaces below using the localic approach; but it is necessary at times to use the Hutton/Hutton-Reilly approach, which is equivalent for appropriate base lattices (see 3.7, 4.5, 6.8) above.

7.5 Definition (L-**Stone spaces**).

(1) Let $L \in |\mathbf{SFRM}|$. A space (X, τ) in L-**TOP** is an L-Stone space iff it is L-coherent and regular. The full subcategory of L-**TOP** of all L-Stone spaces is denoted **STN**-L-**TOP**.

(2) Let $L \in |\mathbf{DMRG}|$. A space (X, τ) in L-**TOP** is an HR-L-Stone space iff it is HR-L-coherent H-compact, L-sober, and is a coherent lattice and HR-regular. The full subcategory of L-**TOP** of all HR-L-Stone spaces is denoted **HR-STN**-L-**TOP**.

For $L \in |\mathbf{CBOOL}|$, it follows from 3.7 and 4.5 above that L-Stone spaces and **STN**-L-**TOP** are equivalent to the HR-L-Stone spaces and **HR-STN**-L-**TOP**, respectively. *Sequens*, the reader will need to recall the notions of $K(A)$ for a complete lattice A (4.1(1) above), closed L-subset (4.1(4) above), L-closure (3.5(3) above), and L-spectrum (5.17 above).

7.6 Lemma. Let $L \in |\mathbf{DMRG}|$, $(X, \tau) \in |L\text{-}\mathbf{TOP}|$, and $\tau \cap \tau'$ denote the open and closed (or clopen) L-subsets of (X, τ). Then:

(1) (X, τ) is H-compact $\Rightarrow$ $K(\tau) \supset \tau \cap \tau'$, and the converse holds if L is a complete Boolean algebra.

(2) (X, τ) is HR-regular $\Rightarrow$ $K(\tau) \subset \tau \cap \tau'$, and the converse holds if τ is coherent.

7.7 Lemma. Let $L \in |\mathbf{CBOOL}|$ and $(X, \tau) \in |L\text{-}\mathbf{TOP}|$, and consider the following statements:

(i) $K(\tau)$ is a Boolean subalgebra of L^X (and so is closed under the complementation of L^X derived pointwise from L);

(ii) $K(\tau) = \tau \cap \tau'$;

(iii) (X, τ) is H-compact and HR-regular;

(iv) (X, τ) is compact and regular.

Then the following conclusions hold:

(1) (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii) $\Leftrightarrow$ (iv);

(2) If τ is coherent, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv);

(3) If any of (i–iv) hold, then (X, τ) is H-compact.

7.8 Lemma (Role of L-Spectra). Let $L \in |\mathbf{CBOOL}|$ and $A \in |\mathbf{DLAT}|$. Then the following are equivalent:

(1) L-Spec(A) is an L-Stone space;

(2) $A \in |\mathbf{BOOL}|$.

7.9 Lemma. Let $L \in |\mathbf{CBOOL}|$. Then **STN-L-TOP** is a full subcategory of **COH-L-TOP**.

7.10 Theorem (Schemum of Stone representation theorems for Boolean algebras). Let $L \in |\mathbf{CBOOL}|$. The following hold:

(1) **STN-L-TOP** $\approx$ $\mathbf{BOOL}^{op}$;

(2) **STN-TOP** $\approx$ $\mathbf{BOOL}^{op}$ (Corollary II.4.4 of [16]);

(3) **STN-L-TOP** $\approx$ **STN-TOP**.

7.11 Remark (Justification of fuzzy sets to traditional mathematics). To traditional mathematics we can say that fuzzy sets, i.e. lattice-valued subsets, give a much richer representation of Boolean algebras than traditional topology can alone—for each complete Boolean algebra L, there is a category of representing L-topological spaces (7.10(1)). Furthermore, fuzzy sets explain the classical Stone representation theorem for Boolean algebras: the membership lattice implicit in Stone's theorem is $\{\perp, \top\}$, and Stone's theorem hinges on this lattice being a complete Boolean algebra; in this way fuzzy sets as lattics-valued sets pinpoint the properties of $\{\perp, \top\}$ crucial to, yet hidden in, Stone's classical theorem.

7.12 Remark (Justification of compactness, regularity, L-sobriety to fuzzy sets). To fixed-basis topology we can say that compact/H-compact, regular/HR-regular, L-sober spaces are a proper generalization of the classical compact Hausdorff spaces (7.10(3)) indeed, each Boolean algebra generates such a space (through L-spectrum) which is additionally L-coherent. This reinforces the point made in Section 6.

8 Compactification reflectors for entire fixed-basis categories of topology

As indicated in 1.1 and 1.2 above, the categorical/localic way to generate the Stone-Čech compactification reflector for **TOP** is to combine the compactification reflector for **LOC** with the classical Ω and PT functors; see the details in [16, 42]. Using the $L\Omega$ and LPT functors for the fixed-basis case, we generalize or reformulate the major steps of this approach to obtain a schemum of Stone-Čech compactification reflectors indexed by $L \in |\mathbf{FRM}|$, which includes the classical reflector indexed by $L = \{\bot, \top\}$, such that each reflector is on the entirety of an L-**TOP**—and not merely on some proper subcategory—and also satisfies all the appropriate topological criteria as discussed in Section 1above.

The general fixed-basis case proved in [42] gives attention to many key issues: the Stone-Čech compactification of a locale is isomorphic to the frame of that locale's completely regular ideals; the general fixed-basis case lends itself to several interpretations w.r.t. the notions of closure and density when justifying the topological criteria, while no possible ambiguity exists for the crisp case; and generally many details of the general fixed-basis case are less trivial than their counterparts in the traditional case.

The plan of this section is as follows: state in 8.1 the Stone-Čech compactification for **LOC**; summarize all the main results under one theorem heading in 8.2; state some corollary results in 8.3; spend most of the rest of the section giving ideas/results needed for the statements of the main results to be well-formed (e.g. closure and density); and then state some consequences which include detailed analyses of the relationship of traditional comapctification to fixed-basis compactification. Through it all the roles of compactness of Section 4, the regularity and complete regularity of Section 3, and the L-sobriety of Section 2 are paramount; and for certain bases L, including $L \in |\mathbf{CBOOL}|$, various aspects of the Hutton/Hutton-Reilly compactness and separation axioms scheme, together with L-sobriety, are equivalent.

This section relies heavily on the notion of a poslat subspace topology on a crisp subset given in [54]—given a space (X, τ) and $A \subset X$, then the corresponding subspace is $(A, \tau(A))$ or $(A, \tau_{|A})$, where $\tau(A) \equiv \tau_{|A} \equiv \{u|_A : u \in \tau\}$ is the subspace topology on A from τ. As shown in [43], this notion identifies the subobjects of each category of the form L-**TOP** for $L \in |\mathbf{SFRM}|$. A closely related point: this section uses the notion of embedding studied in [43].

Finally, a full development with details of proof for this section is given in [42], with these exceptions: the error of Proposition 8.7(3) in [42] is corrected below by means of a reworked Definition 8.6 and Proposition 8.7; new material is added in Discussions 8.14–8.15; and the proof of naturality of the compactification reflector of 8.2V (which was left to the reader in [42]) is provided in Section 9 below—this detail of proof is given in the variable-basis case, where the additional complexity of change of basis merits its inclusion in the exposition.

8.1 Lemma. The following are in [16].

(1) The category **FRM** is algebraic. Hence each categorical regular epimorphism in **FRM** is an epimorphism in **SET** (i.e. a surjection).

(2) The inclusion functor $\hookrightarrow$: **LOC** $\leftarrow$ **K-REG-LOC** has a left adjoint β : **LOC** $\rightarrow$ **K-REG-LOC**, i.e. $\beta \dashv \hookrightarrow$. The unit $\eta : A \rightarrow \beta A$ of this adjunction is a regular monomorphism in **LOC** iff A is completely regular.

(3) For locales, compactness + regularity $\Rightarrow$ c-reg. Thus for $L \in |\mathbf{FRM}|$, (X, τ) is compact and regular $\Rightarrow (X, \tau)$ is c-reg. (Cf. 6.5(2) above).

8.2 Theorem (Compactification reflectors for each L-TOP). Let $L \in |\mathbf{FRM}|$, let $(X, \tau) \in |L\text{-}\mathbf{TOP}|$, let β, η be as given in 8.1, and let $H_\tau : X \rightarrow Lpt(\beta\tau)$ by $H_\tau(x) = \Psi_L(x) \circ \eta$. Then the following hold:

I. **Existence of compact objects and associated morphisms.**

(1) $\beta_s(X, \tau) \equiv LPT(\beta(L\Omega(X, \tau))) \in |\mathbf{K\text{-}REG\text{-}SOB\text{-}}L\text{-}\mathbf{TOP}|$.

(2) $H_\tau : (X, \tau) \rightarrow LPT(\beta(L\Omega(X, \tau)))$ is a morphism in L-**TOP**.

II. **Conditions for openness.** If η is a regular monomorphism and we identify

$$H_\tau^{\rightarrow}(X, \tau) \equiv (H_\tau^{\rightarrow}(X), \Phi_L^{\rightarrow}(\beta\tau)(H_\tau^{\rightarrow}(X)))$$

then H_τ is an L-open mapping w.r.t. $H_\tau^{\rightarrow}(X, \tau)$, where $H_\tau^{\rightarrow}(X, \tau)$ is a subspace of $\beta_s(X, \tau)$ formed by putting on $H_\tau^{\rightarrow}(X)$ the subspace topology $\Phi_L^{\rightarrow}(\beta\tau)(H_\tau^{\rightarrow}(X))$ induced by restricting to $H_\tau^{\rightarrow}(X)$ the L-subsets in $\Phi_L^{\rightarrow}(\beta\tau)$ as maps.

III. **Conditions for embeddings.** H_τ is a homeomorphism onto

$$(H_\tau^{\rightarrow}(X), \Phi_L^{\rightarrow}(\beta\tau)(H_\tau^{\rightarrow}(X)))$$

iff (X, τ) is c-reg and L-T_0.

IV **Existence of topological compactifications.** If

$$(X, \tau) \in |\mathbf{CREG}\text{-}T_0\text{-}L\text{-}\mathbf{TOP}|$$

then (X, τ) has a topological compactification in the sense of 1.1(1) above and 8.9 below, namely $(\beta_s(X, \tau), H_\tau)$.

V. **Existence of (unique) categorical compactifications.** The inclusion functor

$$\hookrightarrow : L\text{-}\mathbf{TOP} \leftarrow \mathbf{K\text{-}REG\text{-}SOB\text{-}}L\text{-}\mathbf{TOP}$$

has a left adjoint, the L-**compactification functor**, namely

$$\beta_s \equiv LPT \circ \beta \circ L\Omega$$

such that:

(1) H_τ is the unit of $\beta_s \dashv \hookrightarrow$ and maps its domain onto a dense subspace of its codomain;

(2) H_τ is an embedding iff its domain is c-reg and L-T_0.

8.3 Corollary. The following hold:

(1) If $L \in |\mathbf{FRM}|$, then **K-REG-SOB-L-TOP** is a reflective [regular mono-reflective] subcateory of **L-TOP** [**CREG-T_0-L-TOP**].

(2) (**Classical Stone-Čech compactification**). The inclusion functor

$$\hookrightarrow : \mathbf{TOP} \to \mathbf{K\text{-}HAUS\text{-}TOP}$$

has a left adjoint, the 2-**compactification functor** $PT \circ \beta \circ \Omega$, such that the unit

$$H_{\mathfrak{T}} : (X, \mathfrak{T}) \to \beta_s(X, \mathfrak{T})$$

maps $(X, \mathfrak{T})$ onto a dense subspace of $\beta_s\,(X, \mathfrak{T})$, and is an embedding iff $(X, \mathfrak{T})$ is completely regular and Hausdorff.

We now give ancillary results and ideas which support Theorem 8.2 and are important in their own right. Of particular significance are Definition 8.6, Proposition 8.7, Definition 8.8, and Definition 8.9—these are needed to make the statement of 8.2.IV well-defined (for until [42] no definition of density of an L-subset in an L-topological space had been given); and in fact, the statement of 8.2.IV must be understood in light of Definition 8.9. This section concludes with remarks and discussions giving detailed analyses of the relationship between traditional compactifications and L-compactifications.

8.4 Lemma. Let $L \in |\mathbf{SFRM}|$. If an object of **L-TOP** is regular [c-reg], then each crisp subset equipped with the L-subspace topology in the sense of [54] is also regular [c-reg].

8.5 Discussion. We have nowhere stated which of the several notions of closure in fuzzy topology we are using for the poslat analogue of the density criterion of the topological definition of compactification (1.1(1) above). The L-closure of a L-subset has been defined in 3.5(3) above; but for *crisp subsets* of a poslat topological space there are several notions of closure and density, and it is with the density of crisp subsets of a poslat topological space that we are concerned in 8.2.IV. For this reason we collect in Definition 8.6 all the possible notions of closure of crisp subsets in a poslat space, show in what sense these are equivalent in 8.7, and then define density in 8.8. It is shown in [42] that the statement of 8.2.IV is valid with the definition of density given in 8.8.

8.6 Definition (Notions of closure and closedness of crisp subsets). Let $L \in |\mathbf{SFRM}|$, $(X, \tau) \in |L\text{-}\mathbf{TOP}|$, $A \subset X$ be a crisp subset of X, and $\alpha \in L$.

(1) The **α-closure** [**α^*-closure**] of A [32, 33, 17–19, 47], denoted $Cl_\alpha(A)$ [$Cl_{a^*}(A)$], is the crisp subset

$$\{x \in X : (u \in \tau,\, u(x) > \alpha\ [\geq \alpha]) \Rightarrow u \wedge \chi_A \neq \perp\}$$

(2) The L-closure $\overline{\chi_A}$ of A is taken as in 3.5(3) above, where $L \in |\mathbf{DMRG}|$.

(3) A is **open** [**closed**] if χ_A [χ_{X-A}] $\in \tau$.

(4) A is **$\perp$-closed** iff $A = Cl_\perp(A)$; and A is **$\perp$-open** iff $X - A$ is $\perp$-closed. The latter is equivalent to saying that $\exists\, u \in \tau,\ A = [u > \perp]$.

8.7 Proposition. Let $L \in |\mathbf{SFRM}|$, $(X, \tau) \in |L\text{-}\mathbf{TOP}|$, and $A \subset X$.

(1) The following are equivalent:

(i) $\forall\, \alpha < \top,\ Cl_\alpha(A) = X$;
(ii) $\forall\, \alpha > \perp,\ Cl_{\alpha^*}(A) = X$;
(iii) $Cl_\perp(A) = X$ (where $Cl_\perp(A)$ is the α-closure of A for $\alpha = \perp$);
(iv) $\bigvee \{u \in \tau : u \leq \chi_{X-A}\} = \perp$.

If $L \in |\mathbf{DMRG}|$, then (i-iv) are equivalent to

1. (v) $\overline{\chi_A} = X$.

(2) $X - Cl_\perp(A)$ [$Cl_\perp(A)$] is an $\perp$-open [$\perp$-closed] (crisp) subset of (X, τ) in the sense of 8.6(4).

(3) Let $\alpha < \top$ [$\alpha > \perp$] and $B = Cl_\alpha(A)$ [$Cl_{\alpha^*}(A)$] in (X, τ). Then A is an α-dense [α^*-dense] subset of $(B, \tau(B))$.

Proof. Each statement is proved in [42] (or its references) save (2). To show $X - Cl_\perp(A)$ is $\perp$-open, we note

$$\forall\, x \notin Cl_\perp(A),\ \exists\, u_x \in \tau\, .\!\ni.\ u_x(x) > \perp,\ u_x \wedge \chi_A = \perp$$

Further we claim:

$$\forall\, x \notin Cl_\perp(A),\ u_x \wedge \chi_{Cl_\perp(A)} = \perp$$

For if not (for some x), $\exists\, y \in Cl_\perp(A)\ .\!\ni.\, u_x(y) > \perp$, which forces $u_x \wedge \chi_A \neq \perp$, a contradiction. The claim implies that $Cl_\perp(Cl_\perp(A)) = Cl_\perp(A)$, i.e. $Cl_\perp(A)$ is $\perp$-closed, which in turn implies $X - Cl_\perp(A)$ is $\perp$-open. We also observe that if we put

$$w = \bigvee_{x \notin Cl_\perp(A)} u_x$$

then $X - Cl_\perp(A) = [w > \perp]$. In any case, (2) is proved. $\square$

8.8 Definition (Density for poslat topology). Let $L \in |\mathbf{SFRM}|$, $(X,\tau) \in |L\text{-}\mathbf{TOP}|$, and $A \subset X$. Then A is **dense** in (X,τ) if it satisfies any of the conditions of 8.7(1)(i–iv). If $L \in |\mathbf{DMRG}|$, then A is **dense** in (X,τ) if it satisfies any of the conditions of 8.7(1)(i–v).

We can now give the definition of topological compactifications for fixed-basis topology in 8.9 below.

8.9 Definition (Topological definition of compactification for fixed-basis topology). Let $(X,\tau) \in |L\text{-}\mathbf{TOP}|$. Then $((X^*,\tau^*),h)$ is a **compactification** of (X,τ) iff $(X^*,\tau^*) \in |L\text{-}\mathbf{TOP}|$ is compact and:

(1) $h : (X,\tau) \to (X^*,\tau^*)$ is an L-homeomorphic embedding [43], i.e. given the subspace topology $\tau^*(h^{\to}(X))$ on $h^{\to}(X)$ in the sense of [43, 54],

$$h : (X,\tau) \to (h^{\to}(X), \tau^*(h^{\to}(X)))$$

is injective and continuous and $h^{-1} : (X,\tau) \to (h^{\to}(X), \tau^*(h^{\to}(X)))$ is continuous.

(2) $h^{\to}(X)$ is dense in (X^*,τ^*) in the sense of 8.8.

The proof of 8.2.IV allows a strengthening of the statement of 8.2.I(2) as follows:

8.10 Proposition. If $L \in |\mathbf{FRM}|$ and $(X,\tau) \in |L\text{-}\mathbf{TOP}|$, then H is a morphism in L-**TOP** from (X,τ) onto a dense subspace of (X,τ), where density is taken in the sense of 8.8..

8.11 Remark (Justification of fuzzy sets to traditional mathematics). To traditional mathematics we can say that fuzzy sets, i.e. lattice-valued subsets, give a much richer compactification theory than traditional topology can alone—each traditional topological space has an entire class of unique compactifications indexed by the objects of **FRM**: given a traditional space, it can be viewed as a fixed-basis space with base lattice $\{\bot,\top\}$, hence it is an object in L-**TOP** for each frame L, and in L-**TOP** it has a unique compactification via β_s; this can be made more precise using functors—see 2.15 and 6.3 above and 8.14 below. Furthermore, fuzzy sets explains the classical Stone-Čech theorem: the membership lattice implicit in the classical theorem is $\{\bot,\top\}$, and the theorem hinges on this lattice being a frame; and fuzzy sets as lattice-valued sets thus pinpoint the properties of $\{\bot,\top\}$ crucial to, yet hidden in, the classical theorem.

8.12 Remark (Justification of compactness, regularity, complete regularity, and L-sobriety to fuzzy sets). To fuzzy sets we simply repeat the point made several times before: compact, regular, L-sober [c-reg, L-sober] spaces are a proper generalization of traditional compact Hausdorff [Tihonov $\equiv$ completely regular Hausdorff, respectively] spaces. More specifically w.r.t. compactness, the compactness of [3] is the only compactness axiom in the extant literature boasting a categorical reflection on *all* of L-**TOP** for *each* L in **FRM**. Again, the reader should review our remarks in Section 1 above and the criteria for a "good" fuzzification of compactness given in [39]. We note, using Sections 3–4, that if $L \in |\mathbf{CBOOL}|$, then throughout this section H-compactness [HR-regularity, HR-cr] replaces compactness [regularity, c-reg], and one can then rightly speak of H-compactifications. Finally, we should emphasize that the roles of L-spatiality and L-sobriety are essential in at least two ways: first, these concepts set up the basic adjunction $L\Omega \dashv LPT$ which transfers the localic compactification to L-topology; and second, the technical role of L-spatiality and L-sobriety in assuring that Φ_L and Ψ_L are bijections can be constantly seen in the proofs of [42].

8.13 Remark. As analyzed in detail in [39], compactness has an excellent Tihonov theorem, namely the Goguen-Tihonov theorem [9], a true generalization of the classical theorem. For those interested in Tihonov theorems in which the cardinality of the indexing set is unrestricted *regardless of the base lattice* L, we recall the following result from [35]: if $\{(X_\gamma, \tau_\gamma)\}_\gamma$ is any collection of compact, L-sober spaces such that $\times_l \tau_\gamma$ (the localic product of the τ_γ's) is L-spatial, then the Goguen-Wong product space $(\times X_\gamma, \times \tau_\gamma)$ is compact and L-sober, yielding a cardinality-free Tihonov theorem for **K-SOB-L-TOP** for any L in **SFRM**. We also note that from the Chapter IV.2 notes of [16] comes the claim that the Tihonov theorem for **K-HAUS-TOP** is a consequence of the Stone-Čech compactification, so we pose the question: for L in **FRM**, does **K-REG-SOB-L-TOP** have a cardinality-free Tihonov theorem courtesy of $\beta_s \dashv \hookrightarrow$?

8.14 Discussion (Classical vis-a-vis L-compactifications at the object-level). We now resolve the question posed in Remark 6.8(2) regarding the object-level relationship between compact, regular, sober, L-spaces and compact Hausdorff spaces (recall that the categorical relationship is given in [42] and Section 6 above). The resolution of this question stems from the following: extensive discussions with U. Höhle, the L-2-soberifications and 2-L-soberifications of Discussions 2.15 and 6.3, the concept of density, and notions of subspaces. The discussion below will give an alternate description of the "soberifications" in Discussion 6.3 of "compact spaces" as well as further clarify the relationship between the objects of "L-2-compactifications" and the objects of (classical) 2-compactifications of ordinary topological spaces mentioned in Remark 8.11 above as part of the justification to traditional mathematics of fuzzy sets.

To begin this discussion, let $L \in |\mathbf{DSFRM}|$ and let (X, τ) be a compact, regular, sober L-space, and let

$$A = \{x \in X : \forall u \in \tau,\ u(x) \in \{\bot, \top\}\}$$

i.e. $A = \bigcap_{u \in \tau} u^{\leftarrow}\{\bot, \top\}$. The suggestion to study this subset of X is an insight of U. Höhle that is important to the sequel. Using the subspace notation of 5.2.2 of [43] and the $S_\bot$ functor of 6.2.1(3) of [43]—the latter is the S_α functor with $\alpha = \bot$—we make explicit two spaces based on A both of which we regard as "subspaces". The **internal subspace** is

$$(A, \tau_{|A}) \in |L\textbf{-TOP}|$$

and the **external subspace** is

$$(A, S_\bot(\tau_{|A})) \in |\textbf{TOP}|$$

Note that the internal subspace is a true subspace of (X, τ) as defined in the fourth paragraph at the beginning of this section. We now examine the relationship between these two "subspaces" as well as the relationship between the internal subspace and (X, τ), beginning with a proposition listing some needed facts.

8.14.1 Proposition. Let $L \in |\textbf{DSFRM}|$. The following hold:

(1) $2pt(\tau)$ is bijective with A.

(2) $2pt(\tau_{|A})$ is bijective with A.

(3) $\tau_{|A}$ is 2-spatial.

(4) τ is 2-spatial.

(5) $\Phi_2^{\rightarrow}(\tau) \approx \Phi_2^{\rightarrow}(\tau_{|A})$.

(6) The following bijections hold::

$$\tau_{|A} \approx \Phi_2^{\rightarrow}(\tau_{|A}) \approx \Phi_2^{\rightarrow}(\tau) \approx \tau \approx \Phi_L^{\rightarrow}(\tau),$$

$$\tau_{|A} \approx S_\bot(\tau_{|A}), \quad S_\bot(\Phi_2^{\rightarrow}(\tau_{|A})) \approx \Phi_2^{\rightarrow}(\tau_{|A}) \approx \Phi^{\rightarrow}(S_\bot(\tau_{|A}))$$

where $\Phi : \mathfrak{T} \to \wp(Pt(\mathfrak{T}))$ denotes the ordinary comparison map from [16] defined by $\Phi(U) = \{p \in Pt(\mathfrak{T}) : p(U) = \top\}$.

(7) $(A, \tau_{|A})$ is a compact, regular, and L-sober space; and $(A, S_\bot(\tau_{|A}))$ is a compact, regular, sober space, and hence compact Hausdorff.

(8) $(A, \tau_{|A})$ is a dense subspace of (X, τ).

Proof. *Ad(1).* $\forall x \in A$, $\Psi_L(x)$ is a 2-point on τ. By L-sobriety of (X, τ), X, and hence A injects nto $Lpt(\tau)$, and hence into $2pt(\tau)$. But given $p \in 2pt(\tau) \subset Lpt(\tau)$, L-sobriety guarantees $\exists\, x \in X$, $\Psi_L(x) = p$. But since $\Psi_L(x)$ is a 2-point on τ, then $x \in A$.

Ad (2). It suffices to show $2pt(\tau_{|A}) \approx 2pt(\tau)$. Given $p \in 2pt(\tau)$, put $p : \tau_{|A} \to 2$ by $p(u_{|A}) = p(u)$. Since restriction preserves $\bigvee$ and $\wedge$, it follows $p \in 2pt(\tau_{|A})$. Injectivity follows using (1), and surjectivity is immediate from the inverse correspondence—$p : \tau \to 2$ by $p(u) = p(u_{|A})$.

Ad (3). Let $u_{|A} \neq v_{|A}$. Then $\exists\, x \in A$, $u(x) \neq v(x)$. This implies $[u = \top] \neq [v = \top]$. This implies using (2) that

$$\begin{aligned}
\Phi_2(u_{|A}) &= \{p \in 2pt(\tau_{|A}) : p(u) = \top\} \\
&= \{x \in A : u_{|A}(x) = \top\} \\
&= \{x \in A : u(x) = \top\} \\
&= [u = \top] \\
&\neq [v = \top] \\
&= \Phi_2(v_{|A})
\end{aligned}$$

So $\Phi_2 : \tau_{|A} \to 2$ is injective.

Ad (4). This follows from Lemma 6.1 above.

Ad (5). This follows from the steps of the proof of (3).

Ad (6). Most of these follow from the preceding statements. The last bijection follows from the steps of (3), noting $|u = \top| = [u > \bot]$.

Ad (7). This follows from Sections 2–4 and the above claims; cf. the proof of Proposition 6.3.5 above.

Ad (8). Using $\tau_{|A} \approx \tau$ from (6), we establish density using 8.7(1)(iii). Let $x \in X$ and suppose $x \notin Cl_\bot(A)$. Then

$$\exists\, u \in \tau \,.\!\ni .\, u(x) > \bot \text{ and } u \wedge \chi_A = \bot$$

This means that $u \neq \bot$ and yet $u_{|A} = \bot_{|A}$. But this violates the bijection $\tau_{|A} \approx \tau$. □

We now link the internal and external subspaces of (X, τ) with the L-2 and 2-L soberification functors given in Discussion 2.15 above (cf. 6.3).

8.14.2 Theorem (External and internal subspaces and soberifications). Let $L \in |\mathbf{DSFRM}|$.

(1) The following hold:

 (a) $(A, \tau_{|A})$ is L-homeomorphic to $(2pt(\tau), \Phi_2^{\rightarrow}(\tau))$, which is the G_χ image of the 2-L-soberification of (X, τ) (where G_χ is given and studied in [43]).

(b) $\left(A, S_\perp\left(\tau_{|A}\right)\right)$ is the 2-L-soberification of (X, τ), i.e. the image of (X, τ) under $PT \circ L\Omega$.

(2) (X, τ) is L-homeomorphic to each of the following:

(a) the L-soberification of $\left(A, \tau_{|A}\right)$, i.e. the image of $\left(A, \tau_{|A}\right)$ under $LPT \circ L\Omega$.

(b) the L-2-soberification of $\left(A, S_\perp\left(\tau_{|A}\right)\right)$, i.e. the image of $\left(A, S_\perp\left(\tau_{|A}\right)\right)$ under $LPT \circ \Omega$.

Proof. The proofs follow from the preceding proposition, the universality of appropriate adjunctions, and Discussion 2.15. As an example, to prove (2)(b), we note from Proposition 8.14.1(6) that $\tau_{|A} \approx \tau$. From the universality of $L\Omega \dashv LPT$, it follows that $\left(Lpt\left(\tau_{|A}\right), \Phi_L^{\rightarrow}\left(\tau_{|A}\right)\right)$ is L-homemorphic to $\left(Lpt\left(\tau\right), \Phi_L^{\rightarrow}\left(\tau\right)\right)$, which in turn is L-homeomorphic to (X, τ) by sobriety. □

The following definition and theorems make explicit the relationships between traditional and fixed-basis compactifications and the role of fixed-basis soberifications. These results lean heavily on Discussions 2.15 and 6.3, as well as 8.14.1, and 8.14.2.

8.14.3 Definition (L-2-Compactification and 2-L-compactification). Let $L \in |\mathbf{FRM}|$, $(X, \tau) \in |L\text{-}\mathbf{TOP}|$, $(X, \mathfrak{T}) \in |\mathbf{TOP}|$, and β be the compactification functor for **LOC**. Then

$$PT \circ \beta \circ L\Omega : L\text{-}\mathbf{TOP} \to \mathbf{K\text{-}HAUS\text{-}TOP}$$

is the **2-L-compactification functor,** where object-wise, we have the following sequence of correspondences:

$$(X, \tau) \mapsto \tau \mapsto \beta\tau \mapsto PT\left(\beta\tau\right)$$

And

$$LPT \circ \beta \circ \Omega : \mathbf{TOP} \to \mathbf{K\text{-}REG\text{-}SOB\text{-}}L\mathbf{\text{-}TOP}$$

is the **L-2-compactification functor**, where object-wise, we have the following sequence of correspondences:

$$(X, \mathfrak{T}) \mapsto \mathfrak{T} \mapsto \beta\mathfrak{T} \mapsto LPT\left(\beta\mathfrak{T}\right)$$

Recall that the traditional Stone-Čech compactification functor is $PT \circ \beta \circ \Omega$ (8.3(2) above), and that the L-compactification functor is $LPT \circ \beta \circ L\Omega$ (8.2V above). The following theorem makes 8.11 more precise w.r.t. poslat compactifications of ordinary topological spaces.

8.14.4A Theorem (Comparison of compactifications of ordinary spaces). Let $(X, \mathfrak{T}) \in |\mathbf{TOP}|$ and $L \in |\mathbf{FRM}|$. Then the following hold:

(1) The traditional Stone-Čech compactification $PT(\beta\Upsilon)$ is the external subspace of the L-2-compactification $LPT(\beta\Upsilon)$.

(2) The L-space $2PT(\beta\Upsilon)$ is the internal subspace of the L-2-compactification $LPT(\beta\Upsilon)$.

(3) The point-sets $Pt(\beta\Upsilon)$ and $2pt(\beta\Upsilon)$ of the two subspaces are identical, and their respective topologies $\Phi^{\to}(\beta\Upsilon)$ and $\Phi_2^{\to}(\beta\Upsilon)$ are isomorphic.

(4) The internal subspace $2PT(\beta\Upsilon)$ is dense in the L-2-compactification $LPT(\beta\Upsilon)$.

(5) The L-2-compactification $LPT(\beta\Upsilon)$ is L-homeomorphic to the L-2-soberification of the traditional Stone-Čech compactification $PT(\beta\Upsilon)$.

(6) The traditional Stone-Čech compactification $PT(\beta\Upsilon)$ is homeomorphic to the 2-L-soberification of the L-2-compactification $LPT(\beta\Upsilon)$.

Proof. *Ad(2).* Following the terminology preceding 8.14.1 above, the internal subspace of $LPT(\beta\Upsilon)$ is based on the subset A given by

$$A = \{p \in Lpt(\beta\Upsilon) : \forall\, b \in \beta\Upsilon,\ \Phi_L(b)(p) \in \{\bot, \top\}\}$$

Noting $\Phi_L(b)(p) = p(b)$, then it follows

$$A = \{p \in Lpt(\beta\Upsilon) : \forall\, b \in \beta\Upsilon,\ p(b) \in \{\bot, \top\}\}$$

which means A is precisely $2pt(\beta\Upsilon)$. Now the internal L-subspace topology on A inherited from $LPT(\beta\Upsilon)$ is precisely $\Phi_L^{\to}(\beta\Upsilon)|_A$, which is precisely $\Phi_2^{\to}(\beta\Upsilon)$.

Ad(1). Using the proof of (2), we now prove (1). Note that the point-set $Pt(\beta\Upsilon)$ of the external subspace is precisely $2pt(\beta\Upsilon) = A$. And the external ordinary topology $S_\bot(\Phi_L^{\to}(\beta\Upsilon)|_A)$ is precisely $S_\bot(\Phi_2^{\to}(\beta\Upsilon)) = \Phi^{\to}(\beta\Upsilon)$, where $\Phi : \beta\Upsilon \to \wp(Pt(\beta\Upsilon))$ is the traditional comparison map from [16] defined by $\Phi(U) = \{p \in Pt(\Upsilon) : p(U) = \top\}$—we let the reader verify this last equality.

Ad (3). From (1), we have $S_\bot(\Phi_2^{\to}(\beta\Upsilon)) = \Phi^{\to}(\beta\Upsilon)$. Now the functor $S_\bot$ acts isomorphically from 2-topologies to ordinary topologies (via $u \mapsto [u > \bot]$).

Ad(4). This is an instantiation of 8.14.1(8).

Ad (5). This is an instantiation of 8.14.2(2)(b).

Ad (6). This is an instantiation of 8.14.2(1)(b), given that the compact, regular L-topology $\Phi_L^{\to}(\beta\Upsilon)$ is 2-spatial. □

8.14.4B Theorem (Comparison of compactifications of L-spaces). Let $L \in |\mathbf{FRM}|$ and $(X, \tau) \in |L\text{-}\mathbf{TOP}|$. Then the following hold:

(1) The 2-L-compactification $PT(\beta\tau)$ is the external subspace of the L-compactification $LPT(\beta\tau)$.

(2) The L-space $2PT(\beta\tau)$ is the internal subspace of the L-compactification $LPT(\beta\tau)$.

(3) The point-sets $Pt(\beta\tau)$ and $2pt(\beta\tau)$ of the two subspaces are identical, and their respective topologies $\Phi^{\rightarrow}(\beta\tau)$ and $\Phi_2^{\rightarrow}(\beta\tau)$ are isomorphic.

(4) The internal subspace $2PT(\beta\tau)$ is dense in the L-compactification $LPT(\beta\tau)$.

(5) The L-compactification $LPT(\beta\tau)$ is L-homeomorphic to the L-2-soberification of the 2-L-compactification $PT(\beta\tau)$.

(6) The 2-L-compactification $PT(\beta\tau)$ is homeomorphic to the 2-L-soberification of the L-compactification $LPT(\beta\tau)$.

Proof. The proof, similar and dual to the proof of 8.14.4A, is left to the reader. □

8.14.5 Application (Fuzzy unit intervals). Using Application 2.15.8, it follows from 8.14.4A(5) that if L is a complete Boolean algebra, then $\mathbb{I}(L)$ is the L-2-compactification of $[0,1]$.

8.14.6 Summary. The above comments and results of Discussion 8.14 are a rigorous and detailed development of U. Höhle's suggestion to the author that the object-level difference between L-compactifications and traditional Stone-Čech compactifications should be "essentially" the addition of more points of density, i.e. the traditional compactification adds a set of 2-points to compactify the given ordinary space, while an L-compactification adds a generally larger set of L-points to compactify a given L-space. Furthermore, if we start with an ordinary topological space, then fuzzy sets contribute a rich schemum of compactifications, more precisely, an L-2-compactification for every frame L which "essentially" differs from the classical compactification by adding more points of density. This clarifies in detail the comments in Remark 8.11 concerning the compactifications of ordinary spaces contributed by fuzzy sets.

Discussion 8.15 (Compactifications and extensions of mappings). Some of the ideas of 2.15, 6.3, and 8.14 may be used to further clarify the relationship between traditional compactifications and fixed-basis compactifications w.r.t. morphisms. We make special use of the internal and external subspaces $(A, \tau_{|A})$ and $(A, S_{\perp}(\tau_{|A}))$ of a compact, regular, sober L-space (X,τ). First, we define an equivalence relation on the morphisms of a category $\mathbf{C}$.

8.15.1 Definition (Morphism classes of a category). We say $f, g \in \mathbf{C}$ are **equivalent**, or $f \equiv g$, iff $\exists\, h, k$ isomorphisms in $\mathbf{C}$, $f = h \circ g \circ k$. It is easy to see that $\equiv$ is an equivalence relation. We let $[f]$ denote the equivalence class containing f of morphisms in $\mathbf{C}$ equivalent to f, and call this a **morphism class** of $\mathbf{C}$.

8.15.2 Proposition. If categories $\mathbf{C}$ and $\mathbf{D}$ are equivalent, then their morphism classes are in a bijection.

Proof. If f of $\mathbf{C}$ goes over to g of $\mathbf{D}$, and comes back to f', then f and f' will be in the same class—f and f' determine a naturality square in which the other two sides are units which are $\mathbf{C}$-isomorphisms. This is half of the proof of bijection. The other half is similar and lives on the fact that the units on the $\mathbf{D}$-side are $\mathbf{D}$-isomorphisms. □

8.15.3 Set-Up. Let us consider the internal and external subspaces $(A, \tau_{|A})$ and $(A, S_\perp(\tau_{|A}))$ of a compact, regular, sober L-space (X, τ); and let $(Y, \mathfrak{T})$ be an ordinary topological space in **SOB-TOP** and (Z, σ) be its L-2-soberification in **SOB-L-TOP**. Now let families $\mathcal{F}$ and $\mathcal{F}'$ of morphism classes be defined as follows:

$$\mathcal{F} = \{[f] \mid f : (A, S_\perp(\tau_{|A})) \to (Y, \mathfrak{T}) \text{ is continuous}\}$$
$$\mathcal{F}' = \{[f'] \mid f' : (X, \tau) \to (Z, \sigma) \text{ is } L\text{-continuous}\}$$

8.15.4 Theorem (Extension of mappings). Let $L \in |\mathbf{DSFRM}|$. Then $\mathcal{F}$ and $\mathcal{F}'$ are bijective.

Proof. We first note that $(A, S_\perp(\tau_{|A}))$ is sober, that (X, τ) is a 2-topological sober L-space (Remark 6.4), and that (Z, σ) is a 2-topological sober L-space (2.15.7(1)). Next we note that $\mathcal{F}$ is a subclass of the morphism classes of **SOB-TOP**, that $\mathcal{F}'$ is a subclass of the morphism classes of **2-TOP-SOB-L-TOP**, and that the morphism classes of **SOB-TOP** are in a bijection with the morphism classes of **2-TOP-SOB-L-TOP** (apply 2.15.7(2) followed by 8.15.2). Finally, we observe that this bijection restricts to these subclasses $\mathcal{F}$ and $\mathcal{F}'$ by the results of Discussion 6.3 and Discussion 8.14. □

8.15.5 Remark (Possible applications of extensions of mappings). The following are restatements and consequences of Theorem 8.15.4.

(1) Given that the functor $S_\perp$ identifies $(A, \tau_{|A})$ with $(A, S_\perp(\tau_{|A}))$, and that the latter is dense in (X, τ) (8.14.1(8)), then 8.15.4 says that each continuous mapping $f : (A, S_\perp(\tau_{|A})) \to (Y, \mathfrak{T})$ has, up to L-homeomorphism, a unique extension to some $f' : (X, \tau) \to (Z, \sigma)$; and conversely, each such f' has, up to homeomorphism, a unique restriction to some f.

(2) The theorem implies that L-compactifications behave w.r.t extension of maps similarly to the way traditional compactifications behave w.r.t. extension of maps. This is another way of philosophically confirming Theorem 8.2.

(3) Note that in the theorem we are allowed to specify the ordinary sober space$(Y, \mathfrak{T})$. As suggested by U. Höhle, if we choose $(Y, \mathfrak{T})$ to be $\mathbb{R}$, then the theorem gives us an L-extension of the Stone-Weierstraß Theorem. Let us midrash this suggestion using Applications 2.15.4 and 8.14.5 as a guide, and letting L be a complete Boolean algebra. Then by Application 2.15.8, we have that (Z, σ) is L-homeomorphic to the fuzzy real line $\mathbb{R}(L)$. Thus, the following subclasses of morphism classes are bijective:

$$\mathcal{F} = \{[f] \mid f : (A, S_{\perp}(\tau_{|A})) \to \mathbb{R} \text{ is continuous}\}$$

$$\mathcal{F}' = \{[f'] \mid f' : (X, \tau) \to \mathbb{R}(L) \text{ is } L\text{-continuous}\}$$

This may be seen as additional justification for the fuzzy real line (see [18–22, 43] and the author's chapter on $\mathbb{R}(L)$).

9 Compactification reflectors for variable-basis categories of topology

It is the purpose of this section to extend the compactifications for fixed-basis topology given in the preceding section and [42] to variable-basis topology. More precisely, it is the purpose of this section to give a compactification reflector for certain subcategories of **LOC-TOP** defined in Section 1. This requires that several previous concepts be extended, which we label as steps:

S1. The lattice-theoretic category **LOC** must be extended to $\mathbf{LOC^{1.5}}$ (defined *sequens*), thus "internalizing" all lattice-theoretic bases for topology within a lattice-theoretic category.

S2. The compactification reflector for **LOC** must be extended to $\mathbf{LOC^{1.5}}$.

S3. The fixed-basis adjunctions $L\Omega \dashv LPT$ must be extended to the variable-basis adjunction between certain subcategories of **SLOC-TOP** and corresponding subcategories of $\mathbf{SLOC^{1.5}}$ (defined *sequens*) in order to appropriately use restrictions of the compactification reflector for $\mathbf{LOC^{1.5}}$.

S4. The fixed-basis compactification functors β_s must be extended to the variable-basis case.

We now define lattice-theoretic categories, defined along the lines of the functor categories of [39, 40], needed for variable-basis compactifications.

9.1 Definition. We let **1.5** be the category having two objects and no morphisms (other than the identity morphisms), i.e. **1.5** may be regarded as the preordered category $\{l, t\}$ with the trivial ordering. We note that **1.5** is isomorphic to its dual $\mathbf{1.5}^{op}$, and so we use them interchangably as convenient. The notation l, t is motivated by the "(L, τ)" of a topological space $(X, L, \tau) \in |\mathbf{C}\text{-}\mathbf{TOP}|$ for any subcategory **C** of **LOQML**.

9.2 Definition. The categories $\mathbf{SLOC^{1.5}}$ and $\mathbf{LOC^{1.5}}$ are functor categories having base **SLOC** and **LOC**, respectively, and exponent **1.5**; i.e. an object $\mathbf{A}$ is a functor $\mathbf{A} : \mathbf{1.5} \to \mathbf{SLOC}, \mathbf{LOC}$, respectively, a morphism $\zeta : \mathbf{A} \to \mathbf{B}$ is a natural transformation, the composition is that of natural transformations, and the identity transformation is the identity morphism of **SLOC, LOC**, respectively, in each component. It is also convenient to consider $\left[\mathbf{SLOC^{1.5}}\right]^{op}$, $\left[\mathbf{LOC^{1.5}}\right]^{op}$, which is the category $\mathbf{SFRM^{1.5}}$, $\mathbf{FRM^{1.5}}$, respectively. We also write $\mathbf{A}$ as (L, A) (so that $\mathbf{A}(l) = L$, $\mathbf{A}(t) = A$), and a morphism ζ as $(\zeta_l^{op}, \zeta_t^{op})$, thereby reserving ζ_l, ζ_t as notation for the concrete maps which are the components of the morphism ζ^{op} in the dual category.

9.3 Remark. The reader can check that $\mathbf{SLOC^{1.5}}$ $\left[\mathbf{LOC^{1.5}}\right] \cong \mathbf{SLOC} \times \mathbf{SLOC}$ $[\mathbf{LOC} \times \mathbf{LOC}]$, and so we use these categories interchangably as convenient.

9.4 Definition (Compact objects in $\mathbf{LOC^{1.5}}$). We say (L, A) in $\mathbf{SLOC^{1.5}}$ is **compact** iff A is compact as defined above in 4.1(2); and (L, A) is **(completely) regular** iff A is (completely) regular as defined in (3.10(1)) 3.5(1) above. We let $\mathbf{K\text{-}REG\text{-}SLOC^{1.5}}$ $\left[\mathbf{K\text{-}REG\text{-}LOC^{1.5}}\right]$ be the full subcategory of $\mathbf{SLOC^{1.5}}$ $\left[\mathbf{LOC^{1.5}}\right]$ of compact, regular objects.

9.5 Proposition (Compactification reflector for $\mathbf{K\text{-}REG\text{-}LOC^{1.5}}$). The inclusion functor $\hookrightarrow\ : \mathbf{LOC^{1.5}} \leftarrow \mathbf{K\text{-}REG\text{-}LOC^{1.5}}$ has a left adjoint $\beta^{1.5} : \mathbf{LOC^{1.5}} \to \mathbf{K\text{-}REG\text{-}LOC^{1.5}}$. The unit $\eta^{1.5} : (L, A) \to \beta^{1.5}(L, A)$ of this adjunction is a regular monomorphism in $\mathbf{LOC^{1.5}}$ iff (L, A) is completely regular.

Proof. We need only define $\beta^{1.5}$ on objects, providing we subsequently satisfy the Lifting Criterion of 1.3.0 of [43]—the action on morphisms will then be correctly stipulated by the Naturality Criterion to insure the adjunction; and the regular monomorphism claim is left to the reader. Let (L, A) be given, put $\beta^{1.5}(L, A) = (L, \beta A)$, where β is the functor given in 8.1(2) above, and put $\eta^{1.5} : (L, A) \to (L, \beta A)$ by

$$\eta^{1.5} = (id_L, \eta)$$

where η is given in 8.1(2) above. Now let (M, B) and $(f, g) : (L, A) \to (M, B)$ be given, where (M, B) is compact. Then $\left(\overline{f}, \overline{g}\right) : (L, \beta A) \to (M, B)$ and

$$(f, g) = \left(\overline{f}, \overline{g}\right) \circ \eta^{1.5}$$

where $\overline{f} = f$ and $\overline{g}$ is the unique map guaranteed by $\beta \dashv\ \hookrightarrow$ satisfying $g = \overline{g} \circ \eta$. The reader can check that $\left(\overline{f}, \overline{g}\right)$ is the unique morphism from $(L, \beta A)$ to (M, B) satisfying the preceding display. $\square$

Steps 1 and 2 are done. Now to apply the $\beta^{1.5} \dashv\ \hookrightarrow$ of 9.5 to variable-basis topology, we must make careful restriction of the morphisms allowed in variable-basis topology. To discuss and justify such restrictions, we need the following lemma, a variation of Lemma 1.3.2 of [43], whose proof is left to the reader.

9.6 Lemma (Properties of adjoints). Let $\psi : L \leftarrow M$ in **SFRM** be given and let ψ^* be the right adjoint guaranteed by the Adjoint Functor Theorem (AFT).

(1) $\psi \circ \psi^* = id_L \Leftrightarrow \psi$ is surjective.

(2) If ψ also admits a left adjoint $^*\psi$, then $\psi \circ {}^*\psi = id_L \Leftrightarrow \psi$ is surjective.

9.7 Definition (Subcollections of SLOC and SLOC-TOP and prefixes).

(1) $\mathbf{SLOC}^{\mathbf{1.5}}_{right}$ comprises the following data:

  (i) $|\mathbf{SLOC}^{\mathbf{1.5}}_{right}| = |\mathbf{SLOC}^{\mathbf{1.5}}|$;

  (ii) each morphism ζ in $\mathbf{SLOC}^{\mathbf{1.5}}_{right}$ is a morphism of $\mathbf{SLOC}^{\mathbf{1.5}}$ satisfying:

   i. $\zeta_l^* \in$ **SFRM**;

   ii. $\zeta_l \circ \zeta_l^* = id$ (where id refers to the identity on the codomain of ζ_l^*).

(2) $\mathbf{SLOC}^{\mathbf{1.5}}_{left}$ comprises the following data:

  (i) $|\mathbf{SLOC}^{\mathbf{1.5}}_{left}| = |\mathbf{SLOC}^{\mathbf{1.5}}|$;

  (ii) each morphism ζ in $\mathbf{SLOC}^{\mathbf{1.5}}_{left}$ is a morphism of $\mathbf{SLOC}^{\mathbf{1.5}}$ satisfying:

   i. $\exists\, {}^*\zeta_l \in$ **SFRM**;

   ii. $\zeta_l \circ {}^*\zeta_l = id$ (where id refers to the identity on the codomain of $^*\zeta_l$).

(3) $\mathbf{SLOC}_{right}$**-TOP** comprises all objects of **SLOC-TOP** together with those morphisms (f, ϕ) such that ϕ^{op} satisfies the conditions on ζ_l in (1)(ii) above.

(4) $\mathbf{SLOC}_{left}$**-TOP** comprises all objects of **SLOC-TOP** together with those morphisms (f, ϕ) such that ϕ^{op} satisfies the conditions on ζ_l in (2)(ii) above.

(5) Prefixes for these categories—we anticipate 9.8 below—select out further subcategories based on conditions required of the objects; for example,

$$\mathbf{K\text{-}REG\text{-}SLOC}_{right}\mathbf{\text{-}TOP}$$

denotes the full subcategory of $\mathbf{SLOC}_{right}$**-TOP** of all compact, regular spaces, where a space (X, L, τ) is regular [c-reg, compact, T_0, etc] iff (X, τ) is regular [c-reg, compact, L-T_0, etc, resp.] as an L-space in the sense of Sections 3,4 above. Prefixes regarding sobriety and spatiality await 9.14 below.

9.8 Proposition (Subcategories of SLOC and SLOC-TOP). $\mathbf{SLOC}^{1.5}_{right}$ and $\mathbf{SLOC}^{1.5}_{left}$ are subcategories of $\mathbf{SLOC}^{1.5}$ with the composition and identities of $\mathbf{SLOC}^{1.5}$; and $\mathbf{SLOC}_{right}$**-TOP** and $\mathbf{SLOC}_{left}$**-TOP** are subcategories of **SLOC-TOP** with the composition and identities of **SLOC-TOP**.

9.9 Examples (Examples of morphisms in $\mathbf{SLOC}^{1.5}_{right}$, $\mathbf{SLOC}^{1.5}_{left}$, $\mathbf{SLOC}_{right}$-TOP, $\mathbf{SLOC}_{left}$-TOP). The restrictions defining the categories in 9.7 allow for a rich variety of morphisms; and in particular, these categories possess many morphisms for which ζ_l or ϕ^{op} are not isomorphisms (and hence not identities). Thus these categories are appropriate frameworks for variable-basis topology, and compactification reflectors in these settings constitute significant extensions of those given in Section 8 above. The following examples are stated for ζ_l only—these obviously can be stated for ϕ^{op} as well. These examples rely in part on 9.6 above.

(1) **Examples of non-isomorphisms satisfying 9.7(1)(ii):**

(a) In 7.1.7.2 of [43], let $M = L_1$, $L = L_2$, and $\zeta_l : L \leftarrow M$ by $\zeta_l = {}^*\psi$. Then $\zeta_l^* = \psi$; both ζ_l, ζ_l^* preserve arbitrary $\bigvee$ and arbitrary $\bigwedge$, so that $\zeta_l, \zeta_l^* \in$ **SFRM;** and ζ_l is surjective, so that $\zeta_l \circ \zeta_l^* = id_L$. There are many other such linear examples.

(b) Let M be any semiframe with $\perp \in M^b$ [32, 33]—or equivalently $\perp$ is meet-irreducible in M, let $L = \{\perp, \top\}$, and put $\zeta_l : L \leftarrow M$ by

$$\zeta_l(\alpha) = \begin{cases} \top, & \alpha > \perp \\ \perp, & \alpha = \perp \end{cases}$$

Then $\zeta_l^* : L \to M$ by

$$\zeta_l^*(\beta) \begin{cases} \top, & \beta = \top \\ \perp, & \beta = \perp \end{cases}$$

Note ζ_l preserves arbitrary $\bigvee$ and finite $\wedge$, ζ_l^* preserves preserves arbitrary $\bigvee$ and arbitrary $\bigwedge$, so that $\zeta_l, \zeta_l^* \in$ **SFRM**; and ζ_l is surjective, so that $\zeta_l \circ \zeta_l^* = id_L$. There are many complete lattices satisfying the condition required of M.

(2) **Examples of non-isomorphisms satisfying 9.7(2)(ii):**

(a) In 7.1.7.2 of [43], let $M = L_1$, $L = L_2$, and $\zeta_l : L \leftarrow M$ by $\zeta_l = \psi^*$. Then ${}^*\zeta_l = \psi$; both $\zeta_l, {}^*\zeta_l$ preserve arbitrary $\bigvee$ and arbitrary $\bigwedge$, so that $\zeta_l, {}^*\zeta_l \in$ **SFRM;** and ζ_l is surjective, so that $\zeta_l \circ {}^*\zeta_l = id_L$. There are many other such linear examples.

(b) Let M be any complete lattice such that $\top$ is κ-separated for any cardinality κ. This condition on M means that

$$\forall A \subset M, \ |A| \leq \kappa, \ \bigvee A < \top$$

Now let $L = \{\bot, \top\}$ and put $\zeta_l : L \leftarrow M$ by

$$\zeta_l(\alpha) = \begin{cases} \top, & \alpha = \top \\ \bot, & \alpha < \top \end{cases}$$

Then $\zeta_l^* : L \to M$ as in (1)(b) above; both $\zeta_l, {}^*\zeta_l$ preserve arbitrary $\bigvee$ and arbitrary $\bigwedge$, so that $\zeta_l, {}^*\zeta_l \in$ **SFRM**; and ζ_l is surjective, so that $\zeta_l \circ {}^*\zeta_l = id_L$. There are many examples of M satisfying this separation condition, e.g. $[0, 1/2] \cup \{1\}$ in the real line.

(3) **Examples of non-isomorphisms satisfying both 9.7(1)(ii) and 9.7(2)(ii)**. Every surjection in **DMRG** and **CLAT** has both right and left adjoints by AFT, and the composition property of (ii) is satisfied (by 9.6); so all such morphisms satisfy both 9.7(1)(ii) and 9.7(2)(ii) (since **DMRG**, **CLAT** $\hookrightarrow$ **SLOC**). There are many such surjections which are not isomorphisms. As a corollary, these comments hold for all surjections in **CBOOL**. An illustrative example is the following. Let

$$L = \mathcal{P}\{a, b\} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

$$M = \mathcal{P}\{a, b, c\} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

and put $\zeta_l : L \leftarrow M$ as follows:

$$\emptyset, \{b\} \mapsto \emptyset, \quad \{a\}, \{a, b\} \mapsto \{a\}, \quad \{c\}, \{b, c\} \mapsto \{b\},$$
$$\{a, c\}, \{a, b, c\} \mapsto \{a, b\}$$

Trivially ζ_l is non-injective and surjective, and it is left to the reader to check that ζ_l preserves all joins and meets, i.e. that $\zeta_l : L \leftarrow M$ is a complete Boolean morphism.

We are now in a position to carry out Step 3, namely, describe the adjunction between spaces and ordered pairs of semiframes.

9.10 Discussion (Setting variable-basis functors up). We are going to construct two adjunctions—$\Omega_{right} \dashv PT_{right}$ and $\Omega_{left} \dashv PT_{left}$, but we will supply details of proof only for the first, the second being analogous and left to the reader. We put

$$\begin{aligned} \Omega_{right} &: \ \mathbf{SLOC}_{right}\text{-}\mathbf{TOP} \to \mathbf{SLOC}^{1.5}_{right}, \\ \Omega_{left} &: \ \mathbf{SLOC}_{left}\text{-}\mathbf{TOP} \to \mathbf{SLOC}^{1.5}_{left} \end{aligned}$$

by formally setting

$$\Omega_{right}(X, L, \tau), \Omega_{left}(X, L, \tau) = (L, \tau)$$

$$\Omega_{right}(f, \phi), \Omega_{left}(f, \phi) = \left(\phi^{op}, \left[(f, \phi)^{\leftarrow}|_{\text{codomain topology}}\right]^{op}\right)$$

where $(f, \phi)^{\leftarrow}$ is restricted to the topology of the codomain of (f, ϕ). We now put

$$\begin{aligned} PT_{right} &: \mathbf{SLOC}_{right}\text{-}\mathbf{TOP} \leftarrow \mathbf{SLOC}^{\mathbf{1.5}}_{right}, \\ PT_{left} &: \mathbf{SLOC}_{left}\text{-}\mathbf{TOP} \leftarrow \mathbf{SLOC}^{\mathbf{1.5}}_{left} \end{aligned}$$

by formally defining for objects

$$PT_{right}(L, A), PT_{left}(L, A) = \left(Lpt(A), L, \Phi_L^{\rightarrow}(A)\right)$$

where $Lpt(A), \Phi_L^{\rightarrow}(A)$ are as defined in Section 2 above. As for the action on morphisms, we set

$$PT_{right}(\zeta) = \left(PT_{right}(\zeta)_{pt}, \zeta_l^{op}\right), \quad PT_{left}(\zeta) = \left(PT_{left}(\zeta)_{pt}, \zeta_l^{op}\right)$$

where

$$PT_{right}(\zeta)_{pt} : Lpt(A) \to Mpt(B) \quad \text{by} \quad PT_{right}(\zeta)_{pt}(p) = \zeta_l^{*} \circ p \circ \zeta_t$$

$$PT_{left}(\zeta)_{pt} : Lpt(A) \to Mpt(B) \quad \text{by} \quad PT_{left}(\zeta)_{pt}(p) = {}^{*}\zeta_l \circ p \circ \zeta_t$$

9.11 Lemma (Variable-basis functors). Each of Ω_{right}, Ω_{left}, PT_{right}, PT_{left} is a functor.

Proof. We check only that PT_{right} makes continuous morphisms; all other details are left to the reader (who needs merely to note the relationship between right adjointness and composition). Let $\zeta : (L, A) \to (M, B)$. We wish to show

$$PT_{right}(\zeta) : \left(Lpt(A), L, \Phi_L^{\rightarrow}(A)\right) \to \left(Mpt(B), M, \Phi_M^{\rightarrow}(B)\right)$$

is continuous. Let $v \in \Phi_M^{\rightarrow}(B)$. Then

$$\exists\, b \in M,\ v = \Phi_M(b)$$

It must be shown that

$$\exists\, a \in L,\ PT_{right}(\zeta)^{\leftarrow}(v) = \Phi_L(a)$$

Given $b \in M$ above, choose

$$a = \zeta_t(b)$$

and let $p \in Lpt(A)$. Then:

$$\begin{aligned} PT_{right}(\zeta)^{\leftarrow}(\Phi_M(b))(p) &= \left[\zeta_l \circ \Phi_M(b) \circ PT_{right}(\zeta)_{pt}\right](p) \\ &= \zeta_l\left(\Phi_M(b)\left(PT_{right}(\zeta)_{pt}(p)\right)\right) \\ &= \zeta_l\left[\left(PT_{right}(\zeta)_{pt}(p)\right)(b)\right] \\ &= \zeta_l\left({}^*\zeta_l\left(p\left(\zeta_t(b)\right)\right)\right) \\ &= p\left(\zeta_t(b)\right) \\ &= p(a) \\ &= \Phi_L(a)(p) \end{aligned}$$

Thus $PT_{right}(\zeta)^{\leftarrow}(v) = \Phi_L(a)$, and this completes the proof. $\square$

9.12 Lemma (Variable-basis adjunctions).

$$\Omega_{right} \dashv PT_{right} \quad \text{and} \quad \Omega_{left} \dashv PT_{left}$$

Restated,

$$\mathbf{SLOC}_{right}\text{-}\mathbf{TOP} \dashv \mathbf{SLOC}^{\mathbf{1.5}}_{right} \quad \text{and} \quad \mathbf{SLOC}_{left}\text{-}\mathbf{TOP} \dashv \mathbf{SLOC}^{\mathbf{1.5}}_{left}$$

Proof. We prove only the "right" case; the "left" case is left to the reader. As outlined in 1.3.0 of [43], we must prove both the lifting criterion and the naturality criterion. For the lifting criterion, let $(X, L, \tau) \in |\mathbf{SLOC}_{right}\text{-}\mathbf{TOP}|$ and let

$$\eta : (X, L, \tau) \to \left(Lpt(\tau), L, \Phi_L^{\rightarrow}(\tau)\right)$$

by choosing

$$\eta = (\Psi_L, id_L)$$

where $\Psi_L : X \to Lpt(\tau)$ is taken from Section 2 above. It follows that (Ψ_L, id_L) satisfies the conditions required of morphisms in $\mathbf{SLOC}_{right}\text{-}\mathbf{TOP}$. We claim this η is the unit of the claimed adjunction. So let the following be given: $(M, B) \in \left|\mathbf{SLOC}^{\mathbf{1.5}}_{right}\right|$ and $(f, \phi) : (X, L, \tau) \to \left(Mpt(B), M, \Phi_M^{\rightarrow}(B)\right)$ in $\mathbf{SLOC}_{right}\text{-}\mathbf{TOP}$. We must show

$$\exists!\,\overline{(f,\phi)} \equiv \left(\overline{f}, \overline{\phi}\right) : (L, \tau) \to (M, B),\ (f, \phi) = PT_{right}\left(\overline{f}, \overline{\phi}\right) \circ \eta$$

where we will take each of $\overline{f}$ and $\overline{\phi}$ as indicating (non-concrete) maps in **SLOC**. It follows immediately that we must choose $\overline{f} = \phi$, in which case $\overline{f}^{op}$ satisfies the special restrictions of 9.7(1)(ii) above since ϕ^{op} satisfies the special restrictions of 9.7(3) above. Now to determine $\overline{\phi} : \tau \to B$ in **SLOC** from

$$f = PT_{right}\left(\overline{f}, \overline{\phi}\right)_{pt} \circ \Psi_L$$

let $b \in M$ and $x \in X$. Then we have the following:

$$\begin{aligned} f(x)(b) &= \left[PT_{right}\left(\overline{f},\overline{\phi}\right)_{pt}\left(\Psi_L(x)\right)\right](b) \\ &= \left[\left(\overline{f}^{op}\right)^{*} \circ \Psi_L(x) \circ \overline{\phi}^{op}\right](b) \\ &= \left(\phi^{op}\right)^{*}\left(\Psi_L(x)\left(\overline{\phi}^{op}(b)\right)\right) \\ &= \left(\phi^{op}\right)^{*}\left(\overline{\phi}^{op}(b)(x)\right) \end{aligned}$$

From this it follows:

$$\begin{aligned} \phi^{op}\left(f(x)(b)\right) &= \phi^{op}\left(\left(\phi^{op}\right)^{*}\left(\overline{\phi}^{op}(b)(x)\right)\right) \\ &= \overline{\phi}^{op}(b)(x) \end{aligned}$$

Rewritten we have

$$\overline{\phi}^{op}(b)(x) = \phi^{op}\left(\Phi_M(b)\left(f(x)\right)\right)$$

$$\overline{\phi}^{op}(b) = \phi^{op} \circ \Phi_M(b) \circ f \qquad 9.12.1$$

This implies that $\overline{\phi}^{op}(b) : X \to L$ and hence $\overline{\phi}^{op} : \tau \leftarrow B$ are uniquely determined in **SET**; $\overline{\phi}^{op}$ is a semiframe map since ϕ^{op}, $\Phi_M \in$ **SFRM**; and therefore $\overline{\phi} : \tau \to B$ is uniquely determined in **SLOC** as required. This completes the proof of the lifting criterion.

We now show that the action of Ω_{right} on morphisms is compatible with the "bar" operator $\overline{(f,\phi)}$ given above, thus satisfying the naturality criterion and completing the proof that $\Omega_{right} \dashv PT_{right}$. To this end, let

$$(f,\phi) : (X, L, \tau) \to (Y, M, \sigma)$$

in $\mathbf{SLOC}_{right}$-**TOP** be given. Now the "bar" operator of the lifting proof assigns to (f,ϕ) the following unique arrow in $\mathbf{SLOC}^{\mathbf{1.5}}_{right}$:

$$!\,\overline{(\Psi_M, id_M)\circ(f,\phi)} : (L,\tau) \to (M,\sigma) \qquad 9.12.2$$

But Ω_{right} assigns the arrow

$$\left(\phi^{op}, \left[(f,\phi)^{\leftarrow}|_{\sigma}\right]^{op}\right) : (L,\tau) \to (M,\sigma) \qquad 9.12.3$$

We must show these arrows are the same. Because of 9.12.1, the morphism of 9.12.2 may be rewritten as

$$\left(\phi^{op},\ \left[\phi^{op} \circ \Phi_M(\ ) \circ \Psi_M \circ f\right]^{op}\right) \qquad 9.12.4$$

So we must verify that 9.12.3 and 9.12.4 are in agreement, and it suffices to show the second components agree. To that end, let $x \in X$ and $v \in \sigma$. Then

$$\begin{aligned} \phi^{op}\left(\Phi_M(v)\left(\Psi_M(f(x))\right)\right) &= \phi^{op}\left(\Psi_M(f(x))\right)(v) \\ &= \phi^{op}\left(v(f(x))\right) \\ &= (f,\phi)^{\leftarrow}(v)(x) \end{aligned}$$

It follows that

$$[(f,\phi)^{\leftarrow}|_{\sigma}]^{op} = [\phi^{op} \circ \Phi_M(\ ) \circ \Psi_M \circ f]^{op}$$

This completes the proof of naturality and the proof of the lemma. □

9.13 Remark. The above proof of adjunction is even *more variable-basis* than the various arguments explicitly suggest. For example, the derivation of $\overline{\phi}$, in the proof that (Ψ_L, id_L) is the unit, implicitly uses M-valued sets and mappings that are quite unexpected. Starting with the displays just before 9.12.1, we have

$$\begin{aligned}
\overline{\phi}^{op}(b)(x) &= \phi^{op}(f(x)(b)) \\
&= \phi^{op}(\Phi_M(b)(f(x))) \\
&= \phi^{op}(\Phi_M(b) \circ f)(x) \\
&= \phi^{op}(\Psi_M(x)(\Phi_M(b) \circ f)) \\
&= \phi^{op}(\Psi_M(x)(f_M^{\leftarrow}(\Phi_M(b))))
\end{aligned}$$

where as expected, $\Phi_M : B \to M^{Mpt(B)}$, but where unexpectedly,

$$f_M^{\leftarrow} : M^X \leftarrow M^{Mpt(B)}, \quad \Psi_M : X \to Mpt\,[f_M^{\leftarrow}(\Phi_M^{\rightarrow}(B))]$$

9.14 Definition (Variable-basis sobriety and spatiality). We say a space (X, L, τ) is **sober** iff (X, τ) is L-sober in the sense of Section 2; and we say a functor (L, A) is **spatial** iff A is L-spatial in the sense of Section 2. In categorical notation, the prefix "**SOB**" ["**SPAT**"] indicates the full subcategory of the category in question of all its sober [spatial] objects.

9.15 Theorem (Variable-basis sobriety-spatiality representation theorems). Sobriety [spatiality] is equivalent to the unit η [counit ε] of both $\mathbf{SLOC}_{right}\text{-}\mathbf{TOP} \dashv \mathbf{SLOC}^{1.5}_{right}$ and $\mathbf{SLOC}_{left}\text{-}\mathbf{TOP} \dashv \mathbf{SLOC}^{1.5}_{left}$ being an isomorphism in the appropriate category. Furthermore, both $\mathbf{SLOC}_{right}\text{-}\mathbf{TOP} \dashv \mathbf{SLOC}^{1.5}_{right}$ and $\mathbf{SLOC}_{left}\text{-}\mathbf{TOP} \dashv \mathbf{SLOC}^{1.5}_{left}$ restrict to the following categorical equivalences:

$$\mathbf{SOB}\text{-}\mathbf{SLOC}_{right}\text{-}\mathbf{TOP} \cong \mathbf{SPAT}\text{-}\mathbf{SLOC}^{1.5}_{right}$$

$$\mathbf{SOB}\text{-}\mathbf{SLOC}_{left}\text{-}\mathbf{TOP} \cong \mathbf{SPAT}\text{-}\mathbf{SLOC}^{1.5}_{left}$$

Proof. The proof, based on Lemmas 9.11–9.12, is a modification of the proof of Theorem 6.12 of [40], and is left to the reader. □

We are now able to implement Step 4 above of our goal—variable-basis compactifications. In order to avail ourselves both of the compactification reflector of 9.5 and the adjunctions of 9.12, we must restrict ourselves to obtaining compactification reflectors for the categories $\mathbf{LOC}_{right}$-**TOP** and $\mathbf{LOC}_{left}$-**TOP**. With regard to **LOC-TOP**, this is no restriction whatsoever w.r.t. objects or spaces: every space of **LOC-TOP** has a compactification—in fact *two* compactifications (!); this follows from the fact that

$$|\mathbf{LOC}_{right}\text{-}\mathbf{TOP}| = |\mathbf{LOC}_{left}\text{-}\mathbf{TOP}| = |\mathbf{LOC}\text{-}\mathbf{TOP}|$$

Referring now to the definitions of topological and categorical compactifications given in 1.1 at the beginning of this chapter, some rather surprising conclusions follow from the results we are about to present:

(1) The compact space of the right compactification is the same as the compact space of the left compactification.

(2) The morphism of the right compactification is the same as the morphism of the left compactification.

(3) The right topological compactification is therefore the same as the left topological compactification.

(4) The right categorical compactification is *not* the same as the left categorical compactification. Basicly, this means that the morphisms of **LOC-TOP** which are factored uniquely through the unit of the right compactification reflector need not be the same as those which factor uniquely through the unit of the left compactification reflector—the former are the morphisms of $\mathbf{LOC}_{right}$-**TOP** and the latter are morphisms of $\mathbf{LOC}_{left}$-**TOP**, and these two classes of morphisms are different.

9.16 Discussion (Setting right and left compactification reflectors up). We use the functors set up previously in this section. Furthermore, as we shall see, most of the topological issues in variable-basis compactification have already been resolved in Section 8 and [42] in the fixed-basis case—issues such as embedding and density; i.e. most of what we subsequently do in this section is try to make the categorical issues clear. Now the basic idea, object-wise, is the following sequence of correspondences:

$$\begin{aligned} (X, L, \tau) &\mapsto (L, \tau) \\ &\mapsto (L, \beta\tau) \\ &\mapsto (Lpt\,(\beta\tau)\,, L, \Phi_L^{\rightarrow}\,(\beta\tau)) \end{aligned}$$

This leads us to propose the following two adjuctions as giving the compactification reflectors for variable-basis topology:

$$PT_{right} \circ \beta^{\mathbf{1.5}} \circ \Omega_{right} \dashv \hookrightarrow_{right}$$

$$PT_{left} \circ \beta^{\mathbf{1.5}} \circ \Omega_{left} \dashv \hookrightarrow_{left}$$

where the inclusion functors are respectively defined on

$$\textbf{K-REG-SOB-LOC}_{right}\textbf{-TOP}$$

$$\textbf{K-REG-SOB-LOC}_{left}\textbf{-TOP}$$

and the proposed left adjoints are respectively defined on $\textbf{LOC}_{right}$**-TOP** and $\textbf{LOC}_{left}$**-TOP**. We will abbreviate the respectively proposed left adjoints by $\beta^{\mathbf{1.5}}_{right}$ and $\beta^{\mathbf{1.5}}_{left}$. We now state the right and left variable-basis versions of Theorem 8.2.

9.17 Theorem (Right and left compactification reflectors for variable-basis topology). Let

$$(X, L, \tau) \in |\textbf{LOC-TOP}|$$

let H_τ be as given in 8.2, let $\beta^{\mathbf{1.5}}, \eta^{\mathbf{1.5}}$ be as given in 9.5, and let $\mathcal{H}_\tau : (X, L, \tau) \to (Lpt\,(\beta\tau)\,, L, \Phi_L^{\to}\,(\beta\tau))$ by $\mathcal{H}_\tau = (H_\tau, id_L)$. Then the following hold:

I. **Existence of right compact objects and associated morphisms.**

(1) $\beta^{\mathbf{1.5}}_{right}(X, L, \tau) \equiv PT_{right}(\beta^{\mathbf{1.5}}(\Omega_{right}(X, L, \tau))) \in$

$$|\textbf{K-REG-SOB-LOC}_{right}\textbf{-TOP}|$$

(2) $\mathcal{H}_\tau : (X, L, \tau) \to PT_{right}(\beta^{\mathbf{1.5}}(\Omega_{right}(X, L, \tau)))$ is a morphism in $\textbf{LOC}_{right}$**-TOP**.

II. **Conditions for openness.** If $\eta^{\mathbf{1.5}}$ is a regular monomorphism, then $\mathcal{H}_\tau$ is an open morphism in the sense of [43].

III. **Conditions for embeddings.** $\mathcal{H}_\tau$ is an embedding in the sense of [43] iff (X, L, τ) is c-reg and T_0.

IV **Existence of right topological compactifications.** If

$$(X, L, \tau) \in |\textbf{CREG-}T_0\textbf{-LOC}_{right}\textbf{-TOP}|$$

then (X, L, τ) has a (right) topological compactification in the sense of 1.1(1) above and 9.18 below, namely $(\beta^{\mathbf{1.5}}_{right}(X, L, \tau), \mathcal{H}_\tau)$.

V. **Existence of right categorical compactifications.** The inclusion functor

$$\hookrightarrow_{right} : \textbf{LOC}_{right}\textbf{-TOP} \leftarrow \textbf{K-REG-SOB-LOC}_{right}\textbf{-TOP}$$

has a left adjoint, namely

$$\beta^{\mathbf{1.5}}_{right} \equiv PT_{right} \circ \beta^{\mathbf{1.5}} \circ \Omega_{right}$$

such that:

(1) $\mathcal{H}_\tau$ is the unit of $\beta^{\mathbf{1.5}}_{right} \dashv \hookrightarrow_{right}$ and maps its domain onto a dense subspace of its codomain;

(2) $\mathcal{H}_\tau$ is an embedding in the sense of [43] iff its domain is c-reg and T_0.

VI. **Existence of left topological and categorical compactifications.** Each statement of I–V holds if "right" is everywhere replaced by "left" in both modifiers and subscripts.

9.18 Definition (Topological definition of variable-basis compactifications). Let $(X, L, \tau) \in |\mathbf{C}\text{-}\mathbf{TOP}|$ where $\mathbf{C}$ is a subcategory of $\mathbf{LOQML}$. Then

$$((X^*, L^*, \tau^*), (h, \phi))$$

is a **compactification** of (X, L, τ) iff (X^*, L^*, τ^*) is a compact object of **C-TOP** and the following hold:

(1) $(h, \phi) : (X, L, \tau) \to (X^*, L^*, \tau^*)$ is a homeomorphic embedding in the sense of [43]; and

(2) $h^{\to}(X)$ is dense in (X^*, L^*, τ^*) in the sense of 8.8 above.

Proof of 9.17 (I–IV). This is immediate from 8.2 and [42].

Proof of 9.17 (V). We first satisfy the Lifting Criterion of 1.3.0 [43]. Let

$$(X, L, \tau) \in |\mathbf{LOC}_{right}\text{-}\mathbf{TOP}|$$

$$(Y, M, \sigma) \in |\mathbf{K}\text{-}\mathbf{REG}\text{-}\mathbf{SOB}\text{-}\mathbf{LOC}_{right}\text{-}\mathbf{TOP}|$$

$$(f, \phi) : (X, L, \tau) \to (Y, M, \sigma) \ \in \ \mathbf{LOC}_{right}\text{-}\mathbf{TOP}$$

be given. It must be shown $\exists\,!\ \left(\overline{f}, \overline{\phi}\right) : \left(Lpt\left(\beta\tau\right), L, \Phi_L^{\to}\left(\beta\tau\right)\right) \to (Y, M, \sigma)$,

$$(f, \phi) = \left(\overline{f}, \overline{\phi}\right) \circ \mathcal{H}_\tau$$

Since $\mathcal{H}_\tau = (H_\tau, id_L)$, this forces $\overline{\phi} = \phi$, and it forces $\overline{f}$ to be the $\overline{f}$ of 8.2(V) and its proof in [42]. So the Lifting Criterion is just the fixed-basis case in disguise, and it guarantees that $\hookrightarrow_{right}$ has a left adjoint whose action on objects coincides with $\beta^{\mathbf{1.5}}_{right}$.

It remains to show that the action of this left adjoint on morphisms is that of $\beta^{\mathbf{1.5}}_{right}$, i.e. to satisfy the Naturality Criterion of 1.3.0 of [43]; and this is more subtle than the Lifting Criterion. To this end, we now give three needed lemmas, the first two of which are standard.

9.17.1 Lemma (Composition of adjunctions). Let

$$F : \mathcal{A} \to \mathcal{B},\ G : \mathcal{A} \leftarrow \mathcal{B},\ H : \mathcal{B} \to \mathcal{C},\ J : \mathcal{B} \leftarrow \mathcal{C}$$

be functors such that $F \dashv G$ and $H \dashv J$ with units η and δ, respectively. Then the following hold:

(1) $H \circ F \dashv G \circ J$; and

(2) $\forall f : A \to B$ in $\mathcal{A}$, then $GJHF(f)$ is a solution of

$$(\quad) \circ \left[G\left(\delta_{F(A)}\right) \circ \eta_A\right] = \left[G\left(\delta_{F(B)}\right) \circ \eta_B\right] \circ f$$

Proof. (1) is well-known; and (2) is proved by first building the naturality square for $H \dashv J$, applying G to this square, hooking that square to the naturality square for $F \dashv G$, and then taking the outer square. $\square$

9.17.2 Lemma (Condition for left adjoint). Let $F, \hat{F} : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{C} \leftarrow \mathcal{D}$ be functors satisfying the following conditions:

(1) $F \dashv G$ with unit η;

(2) F and $\hat{F}$ agree on objects; and

(3) $\forall f : A \to B$ in $\mathcal{C}$, $\hat{F}(f)$ is a solution of

$$G(\quad) \circ \eta_A = \eta_B \circ f$$

Then F and $\hat{F}$ agree on morphisms and $\hat{F} \dashv G$.

Proof. Applying (1), the lifting and naturality criteria of 1.3.0 of [43], and the "bar" operator of those criteria, we have

$$F(f) = \overline{\eta_B \circ f}$$

is the unique solution of

$$G(\quad) \circ \eta_A = \eta_B \circ f$$

Thus (3) applies to force $\hat{F}(f) = F(f)$, $\hat{F} = F$, and $\hat{F} \dashv G$. $\square$

9.17.3 Lemma (Characterization of unit $\mathcal{H}_\tau$).

$$\forall\ (X, L, \tau) \in |\mathbf{SLOC}_{right}\text{-}\mathbf{TOP}|,$$

$$PT_{right}\left(\eta^{\mathbf{1.5}}_{(L,\tau)}\right) \circ (\Psi_L, id_L) = \mathcal{H}_\tau \qquad 9.17.3.1$$

Proof. Abbreviating $\eta^{\mathbf{1.5}}_{(L,\tau)}$ by $\eta^{\mathbf{1.5}}$, we recall

$$\eta^{\mathbf{1.5}} = (id_L, \eta : \tau \to \beta\tau)$$

in which case

$$\left(\eta^{\mathbf{1.5}}\right)_l = id_L\,,\ \left(\eta^{\mathbf{1.5}}\right)_\tau = \eta^{op}$$

It follows

$$\begin{aligned} PT_{right}\left(\eta^{\mathbf{1.5}}\right) &= \left(PT_{right}\left(\eta^{\mathbf{1.5}}\right)_{pt}, \left(\eta^{\mathbf{1.5}}\right)_l\right) \\ &= \left(PT_{right}\left(\eta^{\mathbf{1.5}}\right)_{pt}, id_L\right) \end{aligned}$$

Now

$$\mathcal{H}_\tau = (H_\tau, id_L)$$

and so the second components on each side of 9.17.3.1 is id_L. It remains to check that the first components are the same, i.e. that

$$PT_{right}\left(\eta^{\mathbf{1.5}}\right)_{pt} \circ \Psi_L = H_\tau \qquad 9.17.3.2$$

Letting $p \in Lpt(\tau)$, we have

$$\begin{aligned} PT_{right}\left(\eta^{\mathbf{1.5}}\right)_{pt}(p) &= \left[\left(\eta^{\mathbf{1.5}}\right)_l\right]^* \circ p \circ \left(\eta^{\mathbf{1.5}}\right)_\tau \\ &= id_L \circ p \circ \eta^{op} \\ &= p \circ \eta^{op} \end{aligned}$$

Now let $x \in X$. Then

$$\begin{aligned} PT_{right}\left(\eta^{\mathbf{1.5}}\right)_{pt}(\Psi_L(x)) &= \Psi_L(x) \circ \eta^{op} \\ &= H_\tau(x) \end{aligned}$$

This completes the proof of 9.17.3.2 and hence of 9.17.3.1. □

Resumption of proof of 9.17(V). We now complete the proof of the Naturality Criterion. From the Lifting Criterion, we know that $\hookrightarrow$ has a left adjoint whose action on objects coincides with that of $PT_{right} \circ \beta^{\mathbf{1.5}} \circ \Omega_{right}$. Hence by 9.17.2, it suffices to show that

$$\forall\ (f, \phi) : (X, L, \tau) \to (Y, M, \sigma)\ \in \mathbf{LOC}_{right}\text{-}\mathbf{TOP}$$

$PT_{right}\left(\beta^{\mathbf{1.5}}\left(\Omega_{right}(f, \phi)\right)\right)$ is a solution of

$$(\quad) \circ \mathcal{H}_\tau = \mathcal{H}_\sigma \circ (f, \phi) \qquad 9.17a$$

But since $\Omega_{right} \dashv PT_{right}$ and $\beta^{\mathbf{1.5}} \dashv \hookrightarrow$, it follows from 9.17.1 that

$$(\quad) \circ \left[PT_{right}\left(\eta^{\mathbf{1.5}}_{(L,\tau)}\right) \circ (\Psi_L, id_L)\right] =$$

$$\left[PT_{right}\left(\eta^{1.5}_{(M,\sigma)}\right)\circ(\Psi_M, id_M)\right]\circ(f,\phi) \qquad 9.17b$$

Now it can be seen that 9.17a follows from 9.17b by 9.17.3. This completes the proof of naturality and of the adjunction.

Proof of 9.17(VI). The "left" case is completely analogous to the "right" case, including the Lifting Criterion and Lemma 9.17.3, and is left to the reader. □

9.19 Corollary. Theorem 8.2 is corollary to Theorem 9.17.

9.20 Remark (Additional compactification reflectors). For $\mathbf{C} \hookrightarrow \mathbf{LOC}$, the categories $\mathbf{C}_{right}$-**TOP** and $\mathbf{C}_{left}$-**TOP** have compactification reflectors by restricting the reflectors of 9.17. For certain **C**, these compactification reflectors can be generated by other than compactness and regularity; e.g. for $\mathbf{C} = \mathbf{CBOOL}$, the reflectors on $\mathbf{CBOOL}_{right}$-**TOP** and $\mathbf{CBOOL}_{left}$-**TOP** can be described using H-compactness [HR-regularity, HR-cr] *en lieu* of compactness [regularity, c-reg]. Cf. 8.12 and 9.9(3) above.

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CHAPTER 8

Uniform Spaces

W. Kotzé

Introduction

Uniform spaces are the carriers of notions such as uniform convergence, uniform continuity, precompactness, etc.. In the case of metric spaces, these notions were easily defined. However, for general topological spaces such distance- or size-related concepts cannot be defined unless we have somewhat more structure than the topology itself provides. So uniform spaces lie between pseudometric spaces and topological spaces, in the sense that a pseudometric induces a uniformity and a uniformity induces a topology.

The history of uniform spaces goes back to the late thirties. The approach of A. Weil in 1937 [20] is the so-called "entourage" or "surrounding" one, and many books on general topology contain this treatment. See for example: [2] and [21]. In 1940, J. W. Tukey [19] introduced another approach through uniform coverings. This approach is particularly favoured by J. R. Isbell [12].

In Sections 1 and 2 of this chapter we look at these two approaches as well as an equivalent one through families of functions which is less common in the literature. It is this latter approach which motivates the concept of L-fuzzy uniform spaces as was conceived by Bruce Hutton in [8, 10] - a point which is often missed.

Sections 1 and 2 contain no reference to the fuzzy case, but will hopefully be of use to the reader who is not a specialist in (crisp) uniformities. Furthermore, the juxtaposition with Sections 3, 4 and 5 is intended as an illustration of non-trivial fuzzifications of an important classical theory.

The fuzzification of the concept as established by R. Lowen [14, 15] is summarized as well, but discussed further in Chapter 9 on "Extensions of Uniform Space Notions" by M. H. Burton and J. Gutiérrez García. In fact that chapter could be read as a sequel to this one. Another fuzzification of the concept, not discussed here for sake of space, was introduced by U. Höhle [5, 6].

U. Höhle et al. (eds.), *Mathematics of Fuzzy Sets*

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1 Uniform spaces

A mapping f between pseudometric spaces (X, d) and (Y, ρ) is uniformly continuous if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(f(x_1), f(x_2)) < \varepsilon$ whenever $d(x_1, x_2) < \delta(\varepsilon)$. That is, images under f of points $\delta(\varepsilon)$-close in X are ε-close in Y.

Since orders of nearness are not defined for a general topological space, the notion of uniform continuity is meaningless here. The same, of course, can be said about size-dependent notions such as completeness, precompactness and boundedness.

The mathematical construct employed in studying this kind of properties is called a *uniform space.* Uniform spaces are between pseudometric spaces and topological spaces in the sense that any pseudometric space generates a topological space and any uniform space generates a topological space.

If X is a set we recall the following notations.

1. $\Delta = \Delta(X) \stackrel{\text{def}}{=} \{(x, x) : x \in X\}$.

2. If $U \subseteq X \times X$ then $U_s \stackrel{\text{def}}{=} \{(x, y) : (y, x) \in U\}$. We call U *symmetric* if $U = U_s$.

3. If $U, V \subseteq X \times X$ then

$$U \circ V \stackrel{\text{def}}{=} \{(x, y) \in X \times X : \exists z \in X \text{ s.t. } (x, z) \in V \text{ and } (z, y) \in U\}.$$

4. $U^n \stackrel{\text{def}}{=} \underbrace{U \circ U \circ \cdots \circ U}_{n \text{ factors}}$.

   If $\Delta \subseteq U$ then $U \subseteq U^2 \subseteq U^3 \subseteq \ldots \subseteq U^n$.

1.1 Diagonal uniformities

The reader is referred to [21] as a general reference. The motivation for the following definitions is:

In a metric space (X, d) the sets

$$U_\varepsilon = \{(x, y) : d(x, y) < \varepsilon\} \text{ with } \varepsilon > 0$$

have the properties:

1. $U_\varepsilon \cap U_\delta = U_{\min(\varepsilon,\delta)}$;

2. $\Delta \subseteq U_\varepsilon$ for each $\varepsilon > 0$;

3. $(U_\varepsilon)_s = U_\varepsilon$;

4. $U_\varepsilon \circ U_\delta \subseteq U_{(\varepsilon+\delta)}$, so given ε, $U_{\frac{\varepsilon}{2}} \circ U_{\frac{\varepsilon}{2}} \subseteq U_\varepsilon$.

Definitions 1.1.1 *A* diagonal uniformity *on a set X is a collection $\mathbb{U} \subseteq 2^{X\times X}$ satisfying the following properties:*

(DU1) $\mathbb{U}$ *is a filter on $X \times X$. In other words:*

> *(a)* $\emptyset \notin \mathbb{U}$,
> *(b)* $U_1, U_2 \in \mathbb{U} \Rightarrow U_1 \cap U_2 \in \mathbb{U}$,
> *(c)* $U \subseteq V, U \in \mathbb{U} \Rightarrow V \in \mathbb{U}$.

(DU2) If $U \in \mathbb{U}$ then $\Delta \subseteq U$.

(DU3) If $U \in \mathbb{U}$ then $U_s \in \mathbb{U}$.

(DU4) If $U \in \mathbb{U}$ then there exists $V \in \mathbb{U}$ such that $V \circ V \subseteq U$.

The elements of $\mathbb{U}$ are sometimes called entourages or surroundings.

If X is a set and $\mathbb{B} \subseteq 2^{X\times X}$, then $\mathbb{B}$ is called a uniform base *on X iff*

(DUB1) $\mathbb{B}$ *is a filter base. In other words:*

> *(a)* $\emptyset \notin \mathbb{B}$,
> *(b)* $B_1, B_2 \in \mathbb{B} \Rightarrow \exists B_3 \in \mathbb{B},\ B_3 \subseteq B_1 \cap B_2$.

(DUB2) If $B \in \mathbb{B}$ then $\Delta \subseteq B$.

(DUB3) If $B \in \mathbb{B}$ then there exists $D \in \mathbb{B}$ such that $D_s \subseteq B$.

(DUB4) If $B \in \mathbb{B}$ then there exists $D \in \mathbb{B}$ such that $D \circ D \subseteq B$.

Proposition 1.1.2 *1. If $\mathbb{B}$ is a uniform base then*

$$\langle \mathbb{B} \rangle \stackrel{\text{def}}{=} \{U \subseteq X \times X : B \subseteq U \text{ for some } B \in \mathbb{B}\}$$

is a uniformity.

2. If $\mathbb{U}$ is a uniformity and $\mathbb{B} \subseteq \mathbb{U}$ satisfies $\langle \mathbb{B} \rangle = \mathbb{U}$ then $\mathbb{B}$ is a uniform base.

Examples 1.1.3 *1. Let (X,d) be a pseudometric space and $\varepsilon > 0$. Let*

$$U_\varepsilon^d = \{(x,y) : d(x,y) < \varepsilon\};$$

$$\mathbb{B}_d = \{U^d_\varepsilon : \varepsilon > 0\};$$
$$\mathbb{U}_d = \langle \mathbb{B}_d \rangle.$$

Then $\mathbb{U}_d$ is a uniformity on X. Note that $\mathbb{U}_d = \mathbb{U}_{2d}$ and so different pseudometrics can generate the same uniformity.

2. *Let $\mathbb{B} = \{\Delta\}$ and $\mathbb{U} = \langle \mathbb{B} \rangle$. Then $\mathbb{B}$ is a uniform base and $\mathbb{U}$ is a uniformity called the* discrete uniformity *on X.*

3. *If $\mathbb{U} = \{X \times X\}$ then $\mathbb{U}$ is called the* trivial uniformity *on X.*

4. *For $r \in \mathbb{R}$, let $B_r = \Delta \cup \{(x, y) \in \mathbb{R} \times \mathbb{R} : x > r,\ y > r\}$ then $\mathbb{B} = \{B_r : r \in \mathbb{R}\}$ is a uniform base on $\mathbb{R}$.*

1.2 The uniform topology

Let $x \in X$, $A \subseteq X$ and $U \subseteq X \times X$ we define

$$U[x] \stackrel{\text{def}}{=} \{y \in X : (x, y) \in U\}$$

and

$$U[A] \stackrel{\text{def}}{=} \bigcup_{x \in A} U[x] = \{y \in X : \exists x \in A \text{ s.t. } (x, y) \in U\}.$$

Note that for U^d_ε as in Examples 1.1.3 above, $U^d_\varepsilon[x] = B(x, \varepsilon)$, the open ball with centre x and radius ε.

Theorem 1.2.1 *Let $(X, \mathbb{U})$ be a uniform space. Then the collection $\{U[x] : U \in \mathbb{U}\}$ is a neighbourhood base at x, or $\tau_{\mathbb{U}} = \{G \subseteq X : \forall x \in G,\ \exists U \in \mathbb{U} \text{ s.t. } U[x] \subseteq G\}$ is a topology on X.*

$\tau_{\mathbb{U}}$ is called the *uniform topology* generated by $\mathbb{U}$ and $(X, \tau_{\mathbb{U}})$ is called a *uniformizable* topological space.

Examples 1.2.2 1. *In the case of a pseudometric space (X, d) with the uniformity $\mathbb{U}_d$ as in Examples 1.1.3, number 1, above, the uniform topology generated is clearly the original pseudometric topology.*

2. *The discrete uniformity on X generates the discrete topology on X since*

$$\Delta[x] = \{y \in X : (x, y) \in \Delta\} = \{x\}.$$

3. *The trivial uniformity on X generates the indiscrete topology on X since $(X \times X)[x] = X$.*

4. *The topology generated by the uniformity on $\mathbb{R}$ as discussed in number 4 of Examples 1.1.3 is also the discrete topology since*

$$B_r[x] = \{x\} \textit{ for } x \leq r.$$

1.3 Uniformly continuous functions

Definition 1.3.1 *Let $(X, \mathbb{U})$ and $(Y, \mathbb{V})$ be uniform spaces and let $\varphi : X \to Y$ be a mapping. The mapping φ is said to be* uniformly continuous *iff for all $V \in \mathbb{V}$ there exists $U \in \mathbb{U}$ such that whenever $(x, y) \in U$, $(\varphi(x), \varphi(y)) \in V$. That is:*

$$\forall V \in \mathbb{V},\ \exists U \in \mathbb{U}:\ (\varphi \times \varphi)(U) \subseteq V.$$

Or

$$\forall V \in \mathbb{V},\ (\varphi \times \varphi)^{\leftarrow}(V) \in \mathbb{U}.$$

Theorem 1.3.2 *If $\varphi : (X, \mathbb{U}) \to (Y, \mathbb{V})$ is a uniformly continuous function then $\varphi : (X, \tau_{\mathbb{U}}) \to (Y, \tau_{\mathbb{V}})$ is continuous.*

Proposition 1.3.3 *If $f : (X, \mathbb{U}) \to (Y, \mathbb{V})$ and $g : (Y, \mathbb{V}) \to (Z, \mathbb{W})$ are two uniformly continuous mappings then the mapping $g \circ f : (X, \mathbb{U}) \to (Z, \mathbb{W})$ is also uniformly continuous.*

2 Equivalent approaches to uniformity

We'll now introduce equivalent approaches to uniform spaces.

2.1 Families of functions

First we'll prove the existence of a bijection between the following two families:

$$\mathcal{A} = \{U \subseteq X \times X : \Delta \subseteq U\}$$

and

$$\mathcal{B} = \{f : 2^X \to 2^X : \forall A \in 2^X,\ A \subseteq f(A) \text{ and } f \text{ preserves arbitrary unions.}\}$$

That is, $f \in \mathcal{B}$ iff

(i) $A \subseteq f(A)$ for each $A \in 2^X$;

(ii) $f\left(\bigcup_{i \in J} A_i\right) = \bigcup_{i \in J} f(A_i)$ for each family $\{A_i\}_{i \in J} \subseteq 2^X$.

We can start defining a map $\Upsilon : \mathcal{A} \longrightarrow \mathcal{B}$ in the following way:

$$\begin{array}{rcccccc} \Upsilon: & \mathcal{A} & \to & \mathcal{B} & & & \\ & U & \mapsto & \Upsilon(U): & 2^X & \to & 2^X \\ & & & & A & \mapsto & [\Upsilon(U)](A) = U[A] \end{array}$$

It is not difficult to see that $\Upsilon(U) \in \mathcal{B}$. In fact:
For each $A \in 2^X$, since $\Delta \subseteq U$ it is clear that $A \subseteq U[A] = [\Upsilon(U)](A)$.

On the other hand, for each $\{A_i\}_{i\in J} \subseteq 2^X$ we have

$$\begin{aligned}
[\Upsilon(U)]\left(\bigcup_{i\in J} A_i\right) &= U\left[\bigcup_{i\in J} A_i\right] = \bigcup_{x\in \bigcup_{i\in J} A_i} U[x] = \bigcup_{i\in J}\bigcup_{x\in A_i} U[x] \\
&= \bigcup_{i\in J} U\left[A_i\right] = \bigcup_{i\in J} [\Upsilon(U)]\left(A_i\right)
\end{aligned}$$

We now define a mapping $\Phi : \mathcal{B} \to \mathcal{A}$ in the following way:

$$\begin{array}{rcccl}
\Phi & : & \mathcal{B} & \to & \mathcal{A} \\
 & & f & \mapsto & \Phi(f) = \{(x,y) \in X \times X : y \in f(\{x\})\} = \bigcup_{x\in X} (\{x\} \times f(\{x\}))
\end{array}$$

Since $\{x\} \subseteq f(\{x\})$ for each $x \in X$ it is clear that $\Delta \subseteq \Phi(f)$ and so $\Phi(f) \in \mathcal{A}$.

Finally it is not difficult to check that $\Phi \circ \Upsilon = \mathrm{Id}_{\mathcal{A}}$ and $\Upsilon \circ \Phi = \mathrm{Id}_{\mathcal{B}}$ and therefore $\Phi = \Upsilon^{-1}$ is a bijection. In fact we have for each $U \in \mathcal{A}$:

$$\begin{aligned}
(\Phi \circ \Upsilon)(U) &= \Phi(\Upsilon(U)) = \{(x,y) \in X \times X : y \in [\Upsilon(U)](\{x\})\} \\
&= \{(x,y) \in X \times X : y \in U[x]\} = \{(x,y) \in X \times X : (x,y) \in U\} \\
&= U
\end{aligned}$$

and for each $f \in \mathcal{B}$ and $A \in 2^X$:

$$\begin{aligned}
((\Upsilon \circ \Phi)(f))(A) &= (\Upsilon(\Phi(f))(A) \\
&= \Phi(f)[A] = \{y \in X : \exists x \in A \text{ s.t. } (x,y) \in \Phi(f)\} \\
&= \{y \in X : \exists x \in A \text{ s.t. } y \in f(\{x\})\} \\
&= \left\{y \in X : y \in \bigcup_{x\in A} f(\{x\})\right\} \\
&= \{y \in X : y \in f(A)\} = f(A)
\end{aligned}$$

Therefore both Υ and Φ are bijective.

Now if we consider in $\mathcal{A}$ and $\mathcal{B}$ respectively the orders given by

$$U \leq V \iff U \subseteq V$$

and

$$f \leq g \iff f(A) \subseteq g(B) \quad \text{for all } A \in 2^X$$

then both Υ and Φ are order preserving.

If $U \subseteq V$ it is clear that $[\Upsilon(U)](A) = U[A] \subseteq V[A] = [\Upsilon(V)](A)$ for each $A \in 2^X$.

Conversely, if $[\Upsilon(U)](A) = U[A] \subseteq V[A] = [\Upsilon(V)](A)$ for each $A \in 2^X$ then

$$U = \bigcup_{x \in X} (\{x\} \times U[x]) \subseteq \bigcup_{x \in X} (\{x\} \times V[x]) = V.$$

On the other hand, if $f \leq g$ then

$$\Phi(f) = \{(x,y) \in X \times X : y \in f(\{x\})\} \subseteq \{(x,y) \in X \times X : y \in g(\{x\})\} = \Phi(g).$$

Conversely, if $\Phi(f) = \bigcup_{x \in X} (\{x\} \times f(\{x\})) \subseteq \Phi(g) = \bigcup_{x \in X} (\{x\} \times g(\{x\}))$ then $f(\{x\}) \subseteq g(\{x\})$ for each $x \in X$ and therefore $f \leq g$.

Taking into account that any diagonal uniformity $\mathbb{U}$ on X is just a subset of $\mathcal{A}$ satisfying axioms (DU1), (DU3) and (DU4), we can also think of any uniformity as a certain subset of $\mathcal{B}$ (the image of $\mathcal{U}$ under Υ). Consequently we have the following two theorems:

Theorem 2.1.1 *If $\mathbb{U} \subseteq \mathcal{A}$ is a diagonal uniformity then $\mathbb{B} = \Upsilon(\mathbb{U}) \subseteq \mathcal{B}$ satisfies the following properties:*

(B1) if $f_1, f_2 \in \mathbb{B}$ then there exists $f_3 \in \mathbb{B}$ such that $f_3 \leq f_1$ and $f_3 \leq f_2$;

(B2) if $f \in \mathbb{B}$, $g \in \mathcal{B}$ and $f \leq g$ then $g \in \mathbb{B}$;

(B3) if $f \in \mathbb{B}$ then $f^{\leftarrow} \in \mathbb{B}$, (where $f^{\leftarrow}(A) = \cap\{B \subseteq X : f(B^c) \subseteq A^c\}$ for each $A \in 2^X$);

(B4) if $f \in \mathbb{B}$ then there exists $g \in \mathbb{B}$ such that $g \circ g \leq f$.

PROOF.

(B1) Let $f_1 = \Upsilon(U_1), f_2 = \Upsilon(U_2) \in \mathbb{B}$ with $U_1, U_2 \in \mathbb{U}$. For each $A \subseteq X$ we have

$$\begin{aligned}[\Upsilon(U_1 \cap U_2)](A) &= (U_1 \cap U_2)[A] \subseteq U_1[A] \cap U_2[A] \\ &= [\Upsilon(U_1)](A) \cap [\Upsilon(U_2)](A) = f_1(A) \cap f_2(A)\end{aligned}$$

and so $f_3 = \Upsilon(U_1 \cap U_2)$ satisfies what we needed, since $U_1 \cap U_2 \in \mathbb{U}$.

(B2) Let $f = \Upsilon(U) \in \mathbb{B}$ with $U \in \mathbb{U}$ and $g = \Upsilon(V) \in \mathcal{B}$ such that $f \leq g$.
For each $x \in X$ we have $f(\{x\}) = U[x] \subseteq V[x] = g(\{x\})$ and therefore

$$U = \bigcup_{x \in X} (\{x\} \times U[x]) \subseteq \bigcup_{x \in X} (\{x\} \times V[x]) = V.$$

Since $\mathbb{U}$ is a filter it follows that $V \in \mathbb{U}$ and so $\Upsilon(V) = g \in \mathbb{B}$.

(B3) Let $f = \Upsilon(U) \in \mathbb{B}$ with $U \in \mathbb{U}$. It follows from (DU3) that $U_s \in \mathbb{U}$. Now for each $A \subseteq X$

$$\begin{aligned}[\Upsilon(U_s)](A) = U_s[A] = \bigcup_{x\in A} U_s[x] &= \bigcup_{x\in A} \{y \in X : (x,y) \in U_s\} \\ &= \bigcup_{x\in A} \{y \in X : (y,x) \in U\}\end{aligned}$$

Let $y \in \bigcup_{x\in A} \{y \in X : (y,x) \in U\}$ and $B \subseteq X$ such that $[\Upsilon(U)](B^c) = U[B^c] \subseteq A^c$. There exists $x \in A$ such that $(y,x) \in U$. Hence $x \notin U[B^c]$ and so $y \in B$. Therefore

$$\begin{aligned}[\Upsilon(U_s)](A) &= \bigcup_{x\in A} \{y \in X : (y,x) \in U\} \subseteq \cap\{B \subseteq X : [\Upsilon(U)](B^c) \subseteq A^c\} \\ &= [\Upsilon(U)]^{\leftarrow}(A).\end{aligned}$$

Conversely, let $y \in \cap\{B \subseteq X : [\Upsilon(U)](B^c) \subseteq A^c\}$. Since $y \notin \{y\}^c$, it follows that $[\Upsilon(U)](\{y\}) = U[y] \nsubseteq A^c$, that is $U[y] \cap A \neq \emptyset$.

Hence there exists $x \in A$ such that $(y,x) \in U$ and so $y \in U_s[A] = [\Upsilon(U_s)](A)$. Therefore

$$[\Upsilon(U)]^{\leftarrow}(A) = \cap\{B \subseteq X : [\Upsilon(U)](B^c) \subseteq A^c\} \subseteq [\Upsilon(U_s)](A)$$

and consequently

$$[\Upsilon(U_s)](A) = [\Upsilon(U)]^{\leftarrow}(A).$$

This shows that $f^{\leftarrow} = \Upsilon(U_s) \in \mathbb{B}$.

(B4) Let $f = \Upsilon(U) \in \mathbb{B}$ with $U \in \mathbb{U}$. It follows from (DU4) that there exists $V \in \mathbb{U}$ such that $V \circ V \subseteq U$. Now for each $A \subseteq X$

$$\begin{aligned}[\Upsilon(V) \circ \Upsilon(V)](A) &= V[V[A]] = \{y \in X : \exists z \in V[A] \text{ s.t. } (z,y) \in V\} \\ &= \{y \in X : \exists x \in A, z \in X \text{ s.t. } (x,z) \in V \text{ and } (z,y) \in V\} \\ &= \{y \in X : \exists x \in A \text{ s.t. } (x,y) \in V \circ V\} = (V \circ V)[A] \\ &\subseteq U[A] = [\Upsilon(U)](A).\end{aligned}$$

This shows that there exists $g = \Upsilon(V) \in \mathbb{B}$ such that $g \circ g \leq f$. □

Theorem 2.1.2 *If $\mathbb{B} \subseteq \mathcal{B}$ satisfies (B1), (B2), (B3) and (B4) then $\mathbb{U} = \Phi(\mathbb{B}) \subseteq \mathcal{A}$ is a diagonal uniformity.*

PROOF.

(DU1) It is clear that $\emptyset \notin \mathbb{U}$ and $X \times X \in \mathbb{U}$.

Let $U_1 = \Phi(f_1), U_2 = \Phi(f_2) \in \mathbb{U}$. There exists $f_3 \in \mathbb{B}$ such that $f_3 \leq f_1$ and $f_3 \leq f_2$. Therefore

$$U_3 = \Phi(f_3) \subseteq \Phi(f_1) \cap \Phi(f_2) = U_1 \cap U_2.$$

Let $U = \Phi(f) \in \mathbb{U}$ and $V \subseteq X \times X$ such that $U \subseteq V$. It is clear that $V \in \mathcal{A}$ and so $\Upsilon(V) \in \mathcal{B}$. On the other hand $f = \Upsilon(U) \subseteq \Upsilon(V)$, thus $\Upsilon(V) \in \mathbb{B}$. We therefore conclude that $V = \Phi(\Upsilon(V)) \in \Phi(\mathbb{B}) = \mathbb{U}$.

Consequently $\mathbb{U}$ is a filter.

(DU2)
This is just because $\mathbb{U} \subseteq \mathcal{A}$.

(DU3) Let $U = \Phi(f) \in \mathbb{U}$ with $f \in \mathbb{B}$. It follows from (U3) that $f^{\leftarrow} \in \mathbb{B}$. Now we have

$$\begin{aligned}\Phi(f^{\leftarrow}) &= \{(x,y) \in X \times X : y \in f^{\leftarrow}(\{x\})\} \\ &= \{(x,y) \in X \times X : y \in \cap\{B \subseteq X : f(B^c) \subseteq \{x\}^c\}\}\end{aligned}$$

Let $y \in \cap\{B \subseteq X : f(B^c) \subseteq \{x\}^c\}$. Since $y \notin \{y\}^c$, it follows that $f(\{y\}) \not\subseteq \{x\}^c$, that is $x \in f(\{y\})$.

Conversely, if $x \in f(\{y\})$, for each $B \subseteq X$ such that $f(B^c) \subseteq \{x\}^c$ we have $y \in B$, (if $y \in B^c$ then $x \in f(\{y\}) \subseteq f(B^c) \subseteq \{x\}^c$).

Therefore

$$\begin{aligned}\Phi(f^{\leftarrow}) &= \{(x,y) \in X \times X : y \in \cap\{B \subseteq X : f(B^c) \subseteq \{x\}^c\} \\ &= \{(x,y) \in X \times X : x \in f(\{y\})\} = [\Phi(f)]_s = U_s\end{aligned}$$

This shows that $U_s = \Phi(f^{\leftarrow}) \in \mathbb{U}$.

(DU4) Let $U = \Phi(f) \in \mathbb{U}$ with $f \in \mathbb{B}$. It follows from (U4) that there exists $g \in \mathbb{B}$ such that $g \circ g \leq f$. Now we have

$$\begin{aligned}\Phi(g) \circ \Phi(g) &= \{(x,y) \in X \times X : \exists z \in X \text{ s.t. } (x,z) \in \Phi(g) \text{ and } (z,y) \in \Phi(g)\} \\ &= \{(x,y) \in X \times X : \exists z \in X \text{ s.t. } z \in g(\{x\}) \text{ and } y \in g(\{z\})\} \\ &= \{(x,y) \in X \times X : y \in g(g(\{x\}))\} = \Phi(g \circ g) \\ &\subseteq \Phi(f) = U.\end{aligned}$$

This shows that there exists $V = \Phi(g) \in \mathbb{U}$ such that $V \circ V \leq U$. □

A consequence of these two theorems is that any uniformity on X can be described, without passing to $X \times X$, in the following way

Definition 2.1.3 *Let $\mathcal{B}$ denote the family of mappings $f : 2^X \to 2^X$ such that:*

(i) $A \subseteq f(A)$ *for each* $A \in 2^X$*;*

(ii) $f\left(\bigcup_{i\in J} A_i\right) = \bigcup_{i\in J} f(A_i)$ *for each family* $\{A_i\}_{i\in J} \subseteq 2^X$.

A subset $\mathbb{B} \subseteq \mathcal{B}$ *is called a* uniformity *on* X *iff it satisfies the following axioms:*

(B1) if $f_1, f_2 \in \mathbb{B}$ *then there exists* $f_3 \in \mathbb{B}$ *such that* $f_3 \leq f_1$ *and* $f_3 \leq f_2$*;*

(B2) if $f \in \mathbb{B}$, $f \leq g$ *and* $g \in \mathcal{B}$ *then* $g \in \mathbb{B}$*;*

(B3) if $f \in \mathbb{B}$ *then* $f^{\leftarrow} \in \mathbb{B}$, *(where* $f^{\leftarrow}(A) = \cap\{B \in 2^X : f(B^c) \leq A^c\}$ *for each* $A \in 2^X$*);*

(B4) if $f \in \mathbb{B}$ *then there exists* $g \in \mathbb{B}$ *such that* $g \circ g \leq f$*;*

The pair $(X, \mathbb{B})$ *is called a* uniform space.

2.2 Coverings

Another approach to uniformity is through covers, as first introduced by Tukey [19].

Definition 2.2.1 *If* $\mathcal{C}$ *is a covering of* X*, then another covering* $\mathcal{C}_1$ *of* X *is a* refinement *of* $\mathcal{C}$ *iff for every* $W \in \mathcal{C}_1$*, there exists* $V \in \mathcal{C}$ *such that* $W \subseteq V$*. We write*

$$\mathcal{C}_1 \prec \mathcal{C}$$

and also say $\mathcal{C}_1$ refines $\mathcal{C}$.

A cover $\mathcal{C}_1$ *is a* star refinement *of* $\mathcal{C}$ *(or* $\mathcal{C}_1$ star-refines $\mathcal{C}$*) iff for every* $W \in \mathcal{C}_1$ *there exists* $V \in \mathcal{C}$ *such that* $St(W, \mathcal{C}_1) \subseteq V$*, where*

$$St(W, \mathcal{C}_1) \stackrel{\text{def}}{=} \cup\{S \in \mathcal{C}_1 : S \cap W \neq \emptyset\}.$$

We write

$$\mathcal{C}_1 \prec^* \mathcal{C}.$$

Clearly

$$\mathcal{C}_1 \prec^* \mathcal{C} \Rightarrow \mathcal{C}_1 \prec \mathcal{C}.$$

Theorem 2.2.2 *If* $\mathcal{K}$ *is a family of coverings* $\mathcal{C}$ *of* X *such that for every* $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{K}$ *there exists* $\mathcal{C}_3 \in \mathcal{K}$ *such that* $\mathcal{C}_3 \prec^* \mathcal{C}_1$ *and* $\mathcal{C}_3 \prec^* \mathcal{C}_2$*, then* $\{D_{\mathcal{C}} : \mathcal{C} \in \mathcal{K}\}$ *is a base for a diagonal uniformity on* X*, where*

$$D_{\mathcal{C}} \stackrel{\text{def}}{=} \cup\{V \times V : V \in \mathcal{C}\}.$$

PROOF.

(DUB1) Given $D_{\mathcal{C}_1}$ and $D_{\mathcal{C}_2}$, choose $\mathcal{C}_3 \in \mathcal{K}$ which star-refines both $\mathcal{C}_1$ and $\mathcal{C}_2$. Then $D_{\mathcal{C}_3} \subseteq D_{\mathcal{C}_1} \cap D_{\mathcal{C}_2}$:

If $W \in \mathcal{C}_3$, then there exists $Q \in \mathcal{C}_1$ and $V \in \mathcal{C}_2$ such that $\mathrm{St}\,(W, \mathcal{C}_3) \subseteq Q$ and $\mathrm{St}\,(W, \mathcal{C}_3) \subseteq V$. So $W \times W \subseteq Q \times Q$ and $W \times W \subseteq V \times V$ and hence $W \times W \subseteq D_{\mathcal{C}_1} \cap D_{\mathcal{C}_2}$. Thus $D_{\mathcal{C}_3} \subseteq D_{\mathcal{C}_1} \cap D_{\mathcal{C}_2}$.

(DUB2) Clearly $\Delta \subseteq D_{\mathcal{C}}$ for each $\mathcal{C} \in \mathcal{K}$.

(DUB3) We have $(D_{\mathcal{C}})_s = D_{\mathcal{C}}$ for each $\mathcal{C} \in \mathcal{K}$.

(DUB4) Given $D_{\mathcal{C}}$, then choose $\mathcal{C}_1 \in \mathcal{K}$ which star-refines $\mathcal{C}$. Then $D_{\mathcal{C}_1} \circ D_{\mathcal{C}_1} \subseteq D_{\mathcal{C}}$:

We have

$$D_{\mathcal{C}_1} \circ D_{\mathcal{C}_1} = \{(x,y) : \exists z \in X \text{ s.t. } (x,z) \in D_{\mathcal{C}_1} \text{ and } (z,y) \in D_{\mathcal{C}_1}\}.$$

Thus $(x,z) \in Q \times Q$, $(z,y) \in V \times V$ for $Q, V \in \mathcal{C}_1$ and $z \in Q \cap V$. So $V \subseteq \mathrm{St}\,(Q, \mathcal{C}_1)$. Since $\mathcal{C}_1 \prec^* \mathcal{C}$, there exists $W \in \mathcal{C}$ s. t. $Q \cup V \subseteq W$. Thus $(x,y) \in W \times W$ and so $D_{\mathcal{C}_1} \circ D_{\mathcal{C}_1} \subseteq D_{\mathcal{C}}$.

□

We had above:

$$\mathcal{A} \underset{\Phi}{\overset{\Upsilon}{\rightleftharpoons}} \mathcal{B}$$

where Υ and Φ are bijections. We can now define a map $\Psi : \mathcal{K} \to \mathcal{B}$ with $\mathcal{K}$ as in Theorem 2.2.2 as follows:

For $\mathcal{C} \in \mathcal{K}$, let $\Psi(\mathcal{C}) : 2^X \to 2^X$ be defined by

$$[\Psi(\mathcal{C})]\,(A) \overset{\text{def}}{=} \mathrm{St}\,(A, \mathcal{C}) = \cup\{U \in \mathcal{C} : A \cap U \neq \emptyset\}.$$

This is well-defined because:

(i) $A \subseteq \mathrm{St}\,(A, \mathcal{C})$,

(ii)

$$\begin{aligned} [\Psi(\mathcal{C})]\,(\bigcup_i A_i) &= \mathrm{St}\,(\bigcup_i A_i, \mathcal{C}) \\ &= \cup\{U \in \mathcal{C} : U \cap \bigcup_i A_i \neq \emptyset\} \\ &= \bigcup_i \cup\{U \in \mathcal{C} : U \cap A_i \neq \emptyset\} \end{aligned}$$

$$= \bigcup_i [\Psi(\mathcal{C})]\,(A_i).$$

So we have

$$\begin{array}{ccc} \mathcal{A} & \overset{\Upsilon}{\underset{\Phi}{\rightleftharpoons}} & \mathcal{B} \\ & & \uparrow \quad \Psi \\ & & \mathcal{K} \end{array}$$

Can we complete this diagram?

Theorem 2.2.3 *If* $\mathbb{U}$ *is a diagonal uniformity then*

$$\Xi(U) = \mathcal{C}_U, \ U \in \mathcal{U}$$

where

$$\mathcal{C}_U = \{U[x] : x \in X\}$$

is a map from $\mathbb{U}$ *into* $\mathcal{K}$.

PROOF.

For $U \in \mathbb{U}$, $\mathcal{C}_U = \{U[x] : x \in X\}$ is a cover of X ($x \in U[x]$ since $\Delta \subseteq U$). Are they members of $\mathcal{K}$? In other words, does every pair of these have a common star-refinement?

Take $\mathcal{C}_{U_1}$ and $\mathcal{C}_{U_2}$ with $U_1, U_2 \in \mathbb{U}$. Then $U_1 \cap U_2 \in \mathbb{U}$. Choose $U_3 \in \mathbb{U}$ such that $U_3 \circ U_3 \subseteq U_1 \cap U_2$ and a symmetric $U_4 \in \mathbb{U}$ such that $U_4 \circ U_4 \subseteq U_3$. Then

$$U_4 \circ U_4 \circ U_4 \subseteq U_3 \circ U_3 \subseteq U_1 \cap U_2$$

and

$$\begin{aligned} \mathrm{St}\,(U_4[x], \mathcal{C}_{U_4}) &= \cup\{W \in \mathcal{C}_{U_4} : U_4[x] \cap W \neq \emptyset\} \\ &= \bigcup_y \{U_4[y] : U_4[x] \cap U_4[y] \neq \emptyset\}. \end{aligned}$$

So

$$\begin{aligned} z \in \mathrm{St}\,(U_4[x], \mathcal{C}_{U_4}) &\iff \exists y,\ (y,z) \in U_4 \text{ and } \exists w,\ (x,w) \in U_4 \text{ and } (y,w) \in U_4 \\ &\iff \exists y,\ (y,z) \in U_4 \text{ and } (x,y) \in U_4 \circ U_4 \\ &\iff (x,z) \in U_4 \circ U_4 \circ U_4 \\ &\Rightarrow (x,z) \in U_1 \cap U_2 \\ &\iff z \in U_1[x] \cap U_2[x]. \end{aligned}$$

Thus $\mathcal{C}_{U_4} \prec^* \mathcal{C}_{U_1}$ and $\mathcal{C}_{U_4} \prec^* \mathcal{C}_{U_2}$.

Note The image of $\mathbb{U}$ under Ξ in $\mathcal{K}$ has the additional property that, if $\mathcal{C} \in \Xi(\mathbb{U})$ and $\mathcal{C} \prec \mathcal{C}'$, where $\mathcal{C}'$ is another covering of X, then there exists a subcover of $\mathcal{C}'$ which is a member of $\Xi(\mathbb{U})$:

Consider $\Xi(U) = \mathcal{C}_U$ and $\mathcal{C}_U \prec \mathcal{C}'$. So for every $x \in X$ there exists $W_x \in \mathcal{C}'$ such that $U[x] \subseteq W_x$. Then

$$U = \bigcup_{x \in X} (\{x\} \times U[x]) \subseteq \bigcup_{x \in X} (\{x\} \times W_x)$$

and, since $\mathbb{U}$ is a diagonal uniformity, $\bigcup_{x \in X} (\{x\} \times W_x) \in \mathbb{U}$. Furthermore,

$$\begin{aligned} \bigcup_{x \in X} (\{x\} \times W_x)[z] &= \{y : (z,y) \in \bigcup_{x \in X} (\{x\} \times W_x)\} \\ &= \{y : \exists x \in X,\ (z,y) \in \{x\} \times W_x\} \\ &= \{y : \exists x,\ z = x,\ y \in W_x\} \\ &= W_z. \end{aligned}$$

So $\{W_x : x \in X\}$, which is a subcover of $\mathcal{C}'$, is an element of $\Xi(\mathbb{U})$.

Likewise, if $\mathcal{C} \in \Xi(\mathbb{U})$ and $\mathcal{C} \prec^* \mathcal{C}'$, then a subcover of $\mathcal{C}'$ is a member of $\Xi(\mathbb{U})$.

□

Theorem 2.2.4 *If $\mathcal{K}$ has the additional property that:*

$$\mathcal{C} \in \mathcal{K} \text{ and } \mathcal{C} \prec^* \mathcal{C}' \Rightarrow \mathcal{C}' \in \mathcal{K}$$

then $\Psi(\mathcal{K}) \in \mathcal{B}$ satisfies (B1), ... (B4) of Theorem 2.1.1.

PROOF.

(B1) Let $f_1 = \Psi(\mathcal{C}_1)$, $f_2 = \Psi(\mathcal{C}_2)$ with $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{K}$. Then there exists $\mathcal{C}_3 \in \mathcal{K}$ such that $\mathcal{C}_3 \prec^* \mathcal{C}_1$ and $\mathcal{C}_3 \prec^* \mathcal{C}_2$. Put $f_3 = \Psi(\mathcal{C}_3)$. Then for every $A \in 2^X$

$$[\Psi(\mathcal{C}_3)](A) = \mathrm{St}\,(A, \mathcal{C}_3) \subseteq \mathrm{St}\,(A, \mathcal{C}_1) = [\Psi(\mathcal{C}_1)](A).$$

$$\begin{aligned} [x \in \mathrm{St}\,(A, \mathcal{C}_3) &\iff \exists U \in \mathcal{C}_3 \text{ s.t. } x \in U \text{ and } U \cap A \neq \emptyset \\ &\Rightarrow \exists V \in \mathcal{C}_1 \text{ s.t. } \mathrm{St}\,(U, \mathcal{C}_3) \subseteq V,\ x \in U,\ U \cap A \neq \emptyset \\ &\Rightarrow \exists V \in \mathcal{C}_1 \text{ s.t. } x \in V,\ V \cap A \neq \emptyset \\ &\iff x \in \mathrm{St}\,(A, \mathcal{C}_1)] \end{aligned}$$

Likewise $[\Psi(\mathcal{C}_3)](A) \subseteq [\Psi(\mathcal{C}_2)](A)$ for all $A \in 2^X$. Hence $f_3 \leq f_1$ and $f_3 \leq f_2$.

(B2) Let $f = \Psi(\mathcal{C}), \mathcal{C} \in \mathcal{K}$ and $g \in \mathcal{B}$ with $f \leq g$. So

$$[\Psi(\mathcal{C})](A) = \mathrm{St}\,(A, \mathcal{C}) \subseteq g(A) \text{ for all } A \in 2^X.$$

So $\{g(\mathcal{C}) : U \in \mathcal{C}\}$ is a cover of X of which $\mathcal{C}$ is a star-refinement. By the additional property of $\mathcal{K}$, we have that $\mathcal{G} = \{g(U) : U \in \mathcal{C}\} \in \mathcal{K}$. Furthermore $\Psi(\mathcal{G}) = g$:

$$\begin{aligned} [\Psi(\mathcal{G})](A) &= \mathrm{St}\,(A, \mathcal{G}) \\ &= \bigcup_{x \in A} \mathrm{St}\,(x, \mathcal{G}) \subseteq \bigcup_{x \in A} g(\{x\}) = g(A). \end{aligned}$$

But also

$$\begin{aligned} \bigcup_{x \in A} \mathrm{St}\,(x, \mathcal{G}) &= \bigcup_{x \in A} \cup\{g(U) : U \in \mathcal{C},\ \{x\} \cap g(U) \neq \emptyset\} \\ &\supseteq \bigcup_{x \in A} g(\{x\}) \\ &= g(\bigcup_{x \in A} \{x\}) = g(A). \end{aligned}$$

(B3) If $f = \Psi(\mathcal{C}), \mathcal{C} \in \mathcal{K}$, consider

$$\begin{aligned} f^{\leftarrow}(A) &= \cap\{B \subseteq X : f(B^c) \subseteq A^c\} \\ &= \cap\{B \subseteq X : [\Psi(\mathcal{C})](B^c) \subseteq A^c\} \\ &= \cap\{B \subseteq X : \mathrm{St}\,(B^c, \mathcal{C}) \subseteq A^c\}. \end{aligned}$$

Now $\mathrm{St}\,(\mathrm{St}\,(A, \mathcal{C})^c, \mathcal{C}) \subseteq A^c$. So $f^{\leftarrow}(A) \subseteq \mathrm{St}\,(A, \mathcal{C})$.

Conversely, if $x \in \mathrm{St}\,(A, \mathcal{C})$ then there exists $U \in \mathcal{C}$ such that $A \cap U \neq \emptyset$ and $x \in U$. Let $B \subseteq X$ such that $\mathrm{St}\,(B^c, \mathcal{C}) \subseteq A^c$. Then $x \notin B^c$ or $x \in B$. Thus

$$f^{\leftarrow}(A) = \mathrm{St}\,(A, \mathcal{C}) = [\Psi(U)](A) = f(A) \text{ for all } A \in 2^X \text{ or } f^{\leftarrow} = f = \Psi(\mathcal{C}).$$

(B4) Let $f = \Psi(\mathcal{C}), \mathcal{C} \in \mathcal{K}$ and let U^* be a star-refinement of $\mathcal{C}$. Then if $g = \Psi(\mathcal{C}^*)$,

$$\begin{aligned} g \circ g(A) &= [\Psi(\mathcal{C}^*) \circ \Psi(\mathcal{C}^*)](A) \\ &= \mathrm{St}\,(\mathrm{St}\,(A, \mathcal{C}^*), \mathcal{C}^*) \\ &\subseteq \mathrm{St}\,(A, \mathcal{C}) = [\Psi(\mathcal{C})](A) = f(A). \end{aligned}$$

So $g \circ g \leq f$.

□

Note

In the diagram

$$\begin{tikzcd} \mathbb{U} \arrow[dr, "\Xi"'] & \mathbb{B} \arrow[l, "\Phi"'] \\ & \mathcal{K} \arrow[u, "\Psi"'] \end{tikzcd}$$

$\Xi \circ \Phi \circ \Psi : \mathcal{K} \to \mathcal{K}$ is:

$$\Xi \circ \Phi \circ \Psi(\mathcal{C}) = \Xi \circ \Phi(f)$$

where

$$\Phi(f) = \{(x,y) : y \in f(\{x\})\} = \{(x,y) : y \in \mathrm{St}\,(\{x\}, \mathcal{C})\} = U$$

and

$$\begin{aligned} \Xi(\Phi(f)) &= \{U[x] : x \in X\} \\ &= \{\{y : (x,y) \in U\} : x \in X\} \\ &= \{\mathrm{St}\,(\{x\}, \mathcal{C}) : x \in X\}. \end{aligned}$$

We can now define a uniformity on X also as follows:

Definition 2.2.5 *A* uniformity *on X is a family $\mathbb{K}$ of coverings of X satisfying:*

(K1) every two members of $\mathbb{K}$ has a star-refinement in $\mathbb{K}$,

(K2) if $\mathcal{C} \in \mathbb{K}$ and $\mathcal{C} \prec^* \mathcal{C}'$, then $\mathcal{C}' \in \mathbb{K}$.

The pair $(X, \mathbb{K})$ is called a uniform space.

2.3 Uniformly continuous functions

Definition 2.3.1 *Let $(X, \mathbb{B}_1)$ and $(Y, \mathbb{B}_2)$ be two uniform spaces in the sense of Definition 2.1.3 and let $\varphi : X \to Y$ be a mapping. Then φ is said to be* uniformly continuous *iff for every $g \in \mathbb{B}_2$ there exists $f \in \mathbb{B}_1$ such that $\varphi \circ f \le g \circ \varphi$. In other words*

$$\forall A \in 2^X, \ \varphi \circ f(A) \subseteq g \circ \varphi(A).$$

Definition 2.3.2 *Let $(X, \mathbb{K}_X)$ and $(Y, \mathbb{K}_Y)$ be two uniform spaces in the sense of Definition 2.2.5 and let $\varphi : X \to Y$ be a surjective mapping. Then φ is said to be* uniformly continuous *iff for every $\mathcal{C}_Y \in \mathbb{K}_Y$, there exists $\mathcal{C}_X \in \mathbb{K}_X$ such that $\varphi(\mathcal{C}_X) \prec^* \mathcal{C}_Y$, where $\varphi(\mathcal{C}_X) = \{\varphi(W) : W \in \mathcal{C}_X\}$.*

We can prove:

Theorem 2.3.3 *A map $\varphi : (X, \mathbb{B}_1) \to (Y, \mathbb{B}_2)$ is uniformly continuous in the sense of Definition 2.3.1 iff $\varphi : (X, \Phi(\mathbb{B}_1)) \to (Y, \Phi(\mathbb{B}_2))$ is uniformly continuous in the sense of Definition 1.3.1.*

PROOF.

($\Rightarrow$): Let $V \in \Phi(g),\ g \in \mathbb{B}_2$. Hence

$$V = \{(y_1, y_2) \in Y \times Y : y_2 \in g(\{y_1\})\}.$$

Now by Definition 2.3.1, $\exists f \in \mathbb{B}_1$ such that $\varphi \circ f(A) \subseteq g(\varphi(A))$ for all $A \in 2^X$. Put

$$U = \{(x_1, x_2) : x_2 \in f(\{x_1\})\}.$$

Then

$$\begin{aligned}(x_1, x_2) \in U &\Rightarrow \varphi(x_2) \in \varphi \circ f(\{x_1\}) \in g(\varphi(x_1)) \\ &\Rightarrow (\varphi(x_1), \varphi(x_2)) \in V.\end{aligned}$$

($\Leftarrow$): Let $g \in \mathbb{B}_2$ and consider

$$\Phi(g) = \{(y_1, y_2) \in Y \times Y : y_2 \in g(\{y_1\})\}.$$

By Definition 1.3.1, $\exists U \in \Phi(\mathbb{B}_1)$ such that $\varphi \times \varphi(U) \subseteq \Phi(g)$. Put $f = \gamma(U)$. So $\forall A \in 2^X$,

$$f(A) = [\gamma(U)](A) = U[A] = \{x_2 \in X : \exists x_1 \in A \text{ s.t. } (x_1, x_2) \in U\}.$$

Then $\varphi \circ f(A) = \{\varphi(x_2) : \exists x_1 \in A \text{ s.t. } (x_1, x_2) \in U\}$. But if $(x_1, x_2) \in U$ then $(\varphi(x_1), \varphi(x_2)) \in \Phi(g)$. So $\varphi(x_2) \in g(\varphi(x_1))$ and thus $\varphi \circ f(A) \subseteq g \circ \varphi(A)$ for $A \in 2^X$.

□

Theorem 2.3.4 *A map $\varphi : (X, \mathbb{K}_X) \to (Y, \mathbb{K}_Y)$ is uniformly continuous in the sense of Definition 2.3.2 iff $\varphi : (X, \Phi \circ \Psi(\mathbb{K}_X)) \to (Y, \Phi \circ \Psi(\mathbb{K}_Y))$ is uniformly continuous in the sense of Definition 1.3.1.*

PROOF.

($\Rightarrow$): Let $V \in \Phi \circ \Psi(\mathbb{K}_Y)$. So $V = \Phi \circ \Psi(\mathcal{C}_Y) = \{(y_1, y_2) : y_2 \in \mathrm{St}(\{y_1\}, \mathcal{C}_Y)\}$ for $\mathcal{C}_Y \in \mathbb{K}_Y$. We now have, by Definition 2.3.2, that $\exists \mathcal{C}_X \in \mathbb{K}_X$ such that $\varphi(\mathcal{C}_X) \prec^* \mathcal{C}_Y$, i. e. $\forall \varphi(Q), Q \in \mathcal{C}_X, \exists W \in \mathcal{C}_Y$ such that $\mathrm{St}(\varphi(Q), \varphi(\mathcal{C}_X)) \subseteq W$. Put $U = \{(x_1, x_2) : x_2 \in \mathrm{St}(\{x_1\}, \mathcal{C}_X)\}$. Then $U \in \Phi \circ \Psi(\mathbb{K}_X)$. If $x_1 \in Q, \mathrm{St}(\{\varphi(x_1)\}, \varphi(\mathcal{C}_X)) \subseteq \mathrm{St}(\{\varphi(x_1)\}, \mathcal{C}_Y)$. So

$$\begin{aligned}(x_1, x_2) \in U &\Rightarrow x_2 \in \mathrm{St}(\{x_1\}, \mathcal{C}_X) \\ &\Rightarrow \varphi(x_2) \in \mathrm{St}(\{\varphi(x_1)\}, \varphi(\mathcal{C}_X)) \\ &\Rightarrow \varphi(x_2) \in \mathrm{St}(\{\varphi(x_1)\}, \mathcal{C}_Y) \\ &\Rightarrow (\varphi(x_1), \varphi(x_2)) \in V.\end{aligned}$$

($\Leftarrow$): Let $\mathcal{C}_Y \in \mathbb{K}_Y$ with $\mathcal{C}'_Y \prec^* \mathcal{C}_Y$. Put

$$V = \Phi \circ \Psi(\mathcal{C}'_Y) = \{(y_1, y_2) : y_2 \in \mathrm{St}(\{y_1\}, \mathcal{C}'_Y)\}.$$

By Definition 1.3.1, $\exists U \in \Phi \circ \Psi(\mathcal{C}'_X)$ for some $\mathcal{C}'_X \in \mathbb{K}_X$ such that

$$\begin{aligned}(x_1, x_2) \in U \quad &\Rightarrow \quad (\varphi(x_1), \varphi(x_2)) \in V \\ &\Rightarrow \quad \varphi(x_2) \in \mathrm{St}\,(\{\varphi(x_1)\}, \mathcal{C}'_Y).\end{aligned}$$

Now $\Xi(U) = \{U[x], x \in X\} \in \mathbb{K}_X$ and let $\mathcal{C}_X \in \mathbb{K}_X$ be such that $\mathcal{C}_X \prec^* \Xi(U)$. Then consider $\varphi(\mathcal{C}_X) = \{\varphi(W) : W \in \mathcal{C}_X\}$. Now if $W \in \mathcal{C}_X$, $\mathrm{St}\,(W, \mathcal{C}_X) \subseteq S$ for some $S \in \Xi(U)$ where $S = U[x]$ for some $x \in X$. So

$$\begin{aligned}\mathrm{St}\,(\varphi(W), \varphi(\mathcal{C}_X)) \quad &\subseteq \quad \varphi(U[x]) \\ &\subseteq \quad \mathrm{St}\,(\{\varphi(x)\}, \mathcal{C}'_Y) \\ &\subseteq \quad T \text{ for some } T \in \mathcal{C}_Y, \quad \text{since } \mathcal{C}'_Y \prec^* \mathcal{C}_Y.\end{aligned}$$

Thus $\varphi(\mathcal{C}_X) \prec^* \mathcal{C}_Y$.

□

3 Hutton L-fuzzy uniformities

3.1 Basic notions

Developing the idea contained in Definition 2.1.3, Hutton arrived at the following concept in [8, 10]:

Definition 3.1.1 *Let L be a complete lattice and X be a set. Let $\mathcal{B}$ denote the family of mappings $f : L^X \to L^X$ such that:*

(i) $\mu \leq f(\mu)$ for each $\mu \in L^X$;

(ii) $f\left(\bigvee_{i \in J} \mu_i\right) = \bigvee_{i \in J} f(\mu_i)$ for each family $\{\mu_i\}_{i \in J} \subseteq L^X$.

A non empty subset $\mathcal{U} \subseteq \mathcal{B}$ is called a Hutton L-fuzzy quasi-uniformity *on X iff it satisfies the following axioms:*

(HU1) if $f_1, f_2 \in \mathcal{U}$ then there exists $f_3 \in \mathcal{U}$ such that $f_3 \leq f_1$ and $f_3 \leq f_2$;

(HU2) if $f \in \mathcal{U}$, $f \leq g$ and $g \in \mathcal{B}$ then $g \in \mathcal{U}$;

(HU4) if $f \in \mathcal{U}$ then there exists $g \in \mathcal{U}$ such that $g \circ g \leq f$;

If the complete lattice L has an order-reversing involution ^{c} (i. e. L is a complete de Morgan algebra) then:

An L-fuzzy quasi-uniformity is called a Hutton L-fuzzy uniformity *if*

(HU3) if $f \in \mathcal{U}$ then $f^{-1} \in \mathcal{U}$, (where $f^{-1}(\mu) = \bigwedge\{\nu \in L^X : f(\nu^c) \leq \mu^c\}$ for each $\mu \in L^X$).

The pair $(X, \mathcal{U})$ is called a Hutton L-fuzzy (quasi)-uniform space.

Note

1. *The reverse order of* (HU 4) *and* (HU 3) *above is deliberate to be in compliance with Definition 2.1.3.*

2. *In [8] Hutton specified L to be a completely distributive lattice. However, no distributivity of any kind is needed in the definition given above, which can be easily shown to be equivalent to Hutton's definition using Definition 1 and Lemmas 1–3 of [8] under the assumption of L being completely distributive. In particular, complete distributivity guarantees that in the presence of (HU2), Hutton's intersection axiom is equivalent to (HU1).*

Definition 3.1.2 *A mapping* $\varphi : (X,\mathcal{U}_X) \to (Y,\mathcal{U}_Y)$ *is called* (quasi-)uniformly continuous *if for every* $g \in \mathcal{U}_Y$ *there exists* $f \in \mathcal{U}_X$ *such that* $f(\mu) \leq g(\varphi(\mu))$ *for each* $\mu \in L^X$.

Proposition 3.1.3 *Let* $(X,\mathcal{U})$ *be a Hutton L-fuzzy quasi-uniform space. Then the mapping* $Int: L^X \to L^X$ *defined by* $Int(\mu) = \bigvee\{\nu \in L^X : \exists f \in \mathcal{U} \text{ s.t. } f(\nu) \leq \mu\}$ *for each* $\mu \in L^X$ *is an L-interior operator (cf. Subsection 6.1 in [7]).*

The corresponding L-topology $\tau_{\mathcal{U}} = \{\mu \in L^X : Int(\mu) = \mu\}$ *is called the topology* generated by $\mathcal{U}$.

Proposition 3.1.4 *If a mapping* $\varphi : (X,\mathcal{U}_X) \to (Y,\mathcal{U}_Y)$ *is (quasi-)uniformly continuous, then the mapping* $\varphi : (X,\tau_{\mathcal{U}_X}) \to (Y,\tau_{\mathcal{U}_Y})$ *is continuous.*

3.2 Further results in the case of complete distributivity of the underlying lattice

In [3] Erceg showed that a Hutton L-fuzzy quasi-uniform space is metrizable (in his sense) iff the quasi-uniformity has a countable base.

In [13] Katsaras showed (for $L = [0,1]$), that the $[0,1]$-topology generated by a uniformizable crisp topology is Hutton uniformizable.

In [8] Hutton also shows that an L-topological space is uniformizable iff it is completely regular. (He also characterised complete regularity of a L-topological space by maps into the fuzzy unit interval $I(L)$- also a concept due to him.)

Artico and Moresco [1] gave generalisations to L-topological spaces of the well-known theorems:

1. a completely regular space is normal iff the family of finite open covers is a basis for a compatible uniformity;

2. a completely regular compact space admits exactly one compatible uniformity.

In [8, 11, 16] it is shown that the higher order separation axioms are described by H-uniformities. In [16] it was shown that for a completely distributive lattice L with order-reversing involution, the canonical topology on the fuzzy real line $\mathbb{R}(L)$ is H-uniformizable. Furthermore, this uniformity induces a pseudometric which induces the canonical topology. The result is [18] that $\mathbb{R}(L)$ and $I(L)$ are metrizable in the Erceg sense.

Fuzzy addition is uniformly continuous with respect to the H-uniformity which underlies the canonical topology on $\mathbb{R}(L)$ and $I(L)$; usual addition is uniformly continuous with respect to the uniformity which underlies the canonical L-topology on the reals. Furthermore, if $L_1 \simeq L_2$ then $\mathbb{R}(L_1)$ is isomorphic to $\mathbb{R}(L_2)$ in some sense [17]. These results are solely dependent on the Hutton approach.

In [17] Rodabaugh also points out that a Hutton L-uniformity produces a uniform covering in terms of "shadings". This is in the spirit of our discussions of equivalent approaches to uniformity in Section 2 above.

4 Lowen fuzzy uniform spaces

4.1 I-Fuzzy uniform spaces

An alternative approach to the concept of a fuzzy uniformity was developed by Lowen in [14] in the case $L = I = [0, 1]$. We give a brief outline here. See also Chapter 9 in this volume.

We recall here some definitions which will be needed in the following.

If $\sigma, \psi \in I^{X\times X}$ we define,

$$\sigma_s(x, y) \stackrel{\text{def}}{=} \sigma(y, x),$$

$$(\sigma \circ \psi)(x, y) \stackrel{\text{def}}{=} \sup_{z\in X} \psi(x, z) \wedge \sigma(z, y).$$

If $U, V \subseteq X$ and $\sigma = 1_U, \psi = 1_V$ then

$$\begin{aligned}(1_U)_s(x, y) = 1 \quad &\Longleftrightarrow 1_U(y, x) = 1\\ &\Longleftrightarrow (y, x) \in U\\ &\Longleftrightarrow (x, y) \in U_s\\ &\Longleftrightarrow (1_{U_s})(x, y) = 1.\end{aligned}$$

Therefore $(1_U)_s = 1_{U_s}$.

$$\begin{aligned}(1_V \circ 1_U)(x, y) = 1 \quad &\Longleftrightarrow \sup_{z\in X} 1_U(x, z) \wedge 1_V(z, y) = 1\\ &\Longleftrightarrow \exists z \in X :\ (x, z) \in U \text{ and } (z, y) \in V\\ &\Longleftrightarrow (x, y) \in V \circ U\\ &\Longleftrightarrow 1_{V\circ U}(x, y) = 1.\end{aligned}$$

Therefore $1_V \circ 1_U = 1_{V\circ U}$.

Consequently the above definitions are natural generalisations of the standard notions.

Definitions 4.1.1 *A subset $\mathcal{U} \subseteq I^{X\times X}$ is called a* Lowen fuzzy uniformity *on X iff it satisfies the following properties*

(LU1) $\mathcal{U}$ *is a saturated prefilter;*

(LU2) $\forall\ \sigma \in \mathcal{U},\ \forall\ x \in X,\ \sigma(x,x) = 1$ *(hence $c(\mathcal{U}) = 1$);*

(LU3) $\forall\ \sigma \in \mathcal{U},\ \sigma_s \in \mathcal{U}$;

(LU4) $\forall\ \sigma \in \mathcal{U},\ \forall\ \varepsilon \in I_0,\ \exists \psi \in \mathcal{U} : \psi \circ \psi \leq \sigma + \varepsilon.$

The pair $(X,\mathcal{U})$ is called a Lowen fuzzy uniform space.

A subset $\mathcal{B} \subseteq I^{X\times X}$ is called a Lowen fuzzy uniform base *on X iff it satisfies the following properties*

(LUB1) $\mathcal{B}$ *is a prefilter base;*

(LUB2) $\forall\ \sigma \in \mathcal{B},\ \forall\ x \in X,\ \sigma(x,x) = 1$;

(LUB3) $\forall\ \sigma \in \mathcal{B},\ \forall\ \varepsilon > 0,\ \exists \psi \in \mathcal{B} : \psi \leq \sigma_s + \varepsilon$;

(LUB4) $\forall\ \sigma \in \mathcal{B},\ \forall\ \varepsilon > 0,\ \exists \psi \in \mathcal{B} : \psi \circ \psi \leq \sigma + \varepsilon.$

Definition 4.1.2 *If $\mathcal{U}$ is a Lowen fuzzy uniformity on X, then we say that $\mathcal{B} \subseteq \mathcal{U}$ is a* base for $\mathcal{U}$ *iff $\mathcal{B}$ is a prefilter base and $\widetilde{\mathcal{B}} = \widehat{\langle\mathcal{B}\rangle} = \mathcal{U}$. Where for any prefilter $\mathcal{F}$, $\widehat{\mathcal{F}}$ denotes the* saturation *of $\mathcal{F}$, that is,*

$$\widehat{\mathcal{F}} = \left\{ \sup_{\varepsilon\in I_0} (\mu_\varepsilon - \varepsilon) : \{\mu_\varepsilon\}_{\varepsilon\in I_0} \subseteq \mathcal{F} \right\}.$$

Proposition 4.1.3 *1. If $\mathcal{B}$ is a Lowen fuzzy uniform base then $\widetilde{\mathcal{B}}$ is a Lowen fuzzy uniformity.*

2. If $\mathcal{B}$ is a base for a Lowen fuzzy uniformity $\mathcal{U}$, then $\mathcal{B}$ is a Lowen fuzzy uniform base.

Proposition 4.1.4 *If $\mathcal{U}$ is a Lowen fuzzy uniformity on X then*

$$\mathcal{B} \stackrel{\text{def}}{=} \{\sigma \in \mathcal{U} : \sigma = \sigma_s\}$$

is a Lowen fuzzy uniform base for $\mathcal{U}$.

4.2 Fuzzy neighbourhood spaces

In [15] Lowen introduced a special kind of I-topological spaces — so-called *fuzzy neighborhood spaces**. Here we only assemble some facts regarding this type of neighbourhood spaces.

*See also Section 5 in Chapter 5 in this Volume

Definitions 4.2.1 *A collection* $\{\mathcal{N}_x\}_{x\in X}$, *where* $\mathcal{N}_x \subseteq I^X$ *for each* $x \in X$, *is called a* fuzzy neighbourhood system *iff the following conditions are fulfilled for each* $x \in X$:

(FN1) $\mathcal{N}_x$ *is a saturated prefilter;*

(FN2) $\mu(x) = 1$ *for all* $\mu \in \mathcal{N}_x$ *(hence* $c(\mathcal{N}_x) = 1$*);*

(FN3) for each $\mu \in \mathcal{N}_x$ *and each* $\varepsilon \in I_0$ *there exists* $\{\nu_z\}_{z\in X}$ *such that* $\nu_x \in \mathcal{N}_z$ *for all* $z \in X$ *and*

$$\sup_{z\in X} \nu_x(z) \wedge \nu_z(y) \leq \mu(y) + \varepsilon \quad \text{for all } y \in X.$$

$\mathcal{N}_x$ *is called a* fuzzy neighbourhood prefilter *on* X *and the elements of* $\mathcal{N}_x$ *are called* fuzzy neighbourhoods of x.

A collection $\{\beta_x\}_{x\in X}$, *where* $\beta_x \subseteq I^X$ *for each* $x \in X$, *is called a* fuzzy neighbourhood base *iff the following conditions are fulfilled for each* $x \in X$:

(FNB1) β_x *is a prefilter base;*

(FNB2) $\mu(x) = 1$ *for all* $\mu \in \beta_x$;

(FNB3) for each $\mu \in \beta_x$ *and each* $\varepsilon \in I_0$ *there exists* $\{\nu_z\}_{z\in X}$ *such that* $\nu_x \in \beta_z$ *for all* $z \in X$ *and*

$$\sup_{z\in X} \nu_x(z) \wedge \nu_z(y) \leq \mu(y) + \varepsilon \quad \text{for all } y \in X.$$

β_x *is called a* fuzzy neighbourhood base in x *and elements of* β_x *are called* basic fuzzy neighbourhoods of x.

Definition 4.2.2 *If* $\mathcal{N} = \{\mathcal{N}_x\}_{x\in X}$ *is a fuzzy neighbourhood system then we say that* $\beta = \{\beta_x\}_{x\in X}$ *is a* base for $\mathcal{N}$ *iff for each* $x \in X$, β_x *is a prefilter base and* $\widetilde{\beta_x} = \mathcal{N}_x$.

Proposition 4.2.3 *1. If* $\{\beta_x\}_{x\in X}$ *is a fuzzy neighbourhood base, then* $\left\{\widetilde{\beta_x}\right\}_{x\in X}$ *is a fuzzy neighbourhood system.*

2. If $\{\beta_x\}_{x\in X}$ *is a base for the fuzzy neighbourhood system* $\{\mathcal{N}_x\}_{x\in X}$ *then* $\{\beta_x\}_{x\in X}$ *is a fuzzy neighbourhood base.*

Theorem 4.2.4 *If* $\mathcal{N} = \{\mathcal{N}_x\}_{x\in X}$ *is a fuzzy neighbourhood system on* X *then the mapping*

$$\begin{array}{rcccccl} ^{-}: & I^X & \longrightarrow & I^X & & & \\ & \mu & \longmapsto & \overline{\mu}: & X & \longrightarrow & I \\ & & & & x & \longmapsto & \overline{\mu}(x) = \inf_{\nu\in\mathcal{N}_x} \sup \mu \wedge \nu \end{array}$$

is a fuzzy closure operator.

Proposition 4.2.5 *If $\beta = \{\beta_x\}_{x\in X}$ is a base for the fuzzy neighbourhood system $\mathcal{N} = \{\mathcal{N}_x\}_{x\in X}$ then for each $\mu \in I^X$ and each $x \in X$ we have*

$$\overline{\mu}(x) = \inf_{\nu\in\widetilde{\beta_x}} \mu \wedge \nu = \inf_{\nu\in\langle\beta_x\rangle} \mu \wedge \nu = \inf_{\nu\in\beta_x} \mu \wedge \nu$$

If $\mathcal{N} = \{\mathcal{N}_x\}_{x\in X}$ is a fuzzy neighbourhood system then the above fuzzy closure operator generates a I-topology[†‡] and is denoted by $\tau_{\mathcal{N}}$.

The I-topological space which is generated by some fuzzy neighbourhood system, will be called a *fuzzy neighbourhood space.*

4.3 Fuzzy uniform topology

As in the case of usual uniformities, any Lowen fuzzy uniform space generates a I-topological space. In particular we'll see that this I-topological space is a fuzzy neighbourhood space.

We first recall some definitions which will be needed in the following.

For $\sigma \in I^{X\times X}$, $\mu \in I^X$ and $x \in X$, we define $\sigma\langle x\rangle \in I^X$ by

$$\sigma\langle x\rangle(y) \stackrel{\text{def}}{=} \sigma(y,x)$$

and $\sigma\langle\mu\rangle \in I^X$ by

$$\sigma\langle\mu\rangle(x) \stackrel{\text{def}}{=} \sup \mu \wedge \sigma\langle x\rangle = \sup_{y\in X} \mu(y) \wedge \sigma(y,x).$$

If $A \subseteq X$ and $U \subseteq X$ then

$$\begin{aligned} 1_U\langle 1_A\rangle(x) = 1 \quad &\Longleftrightarrow \sup_{y\in X}\ 1_A(y) \wedge 1_U(y,x) = 1 \\ &\Longleftrightarrow \exists y \in A : (y,x) \in U \\ &\Longleftrightarrow x \in U[A] \\ &\Longleftrightarrow 1_{U[A]}(x) = 1. \end{aligned}$$

Therefore $1_U\langle 1_A\rangle = 1_{U[A]}$.

Thus the definition of $\sigma\langle\mu\rangle$ is a natural generalisation of the standard notion.

Let $\sigma \in I^{X\times X}$ and $\beta \in I_1$. Then

$$\sigma^\beta \stackrel{\text{def}}{=} \{(x,y) : \sigma(x,y) > \beta\}.$$

In the following lemma we collect some basic facts concerning these operations.

Lemma 4.3.1 *Let $\sigma, \psi \in I^{X\times X}$, $\nu, \mu \in I^X$, $\varepsilon \in I$, $\beta \in I_1$, $x \in X$ and $n \in \mathbb{N}$. Then*

[†]Concerning the change of the original terminology proposed by C.L. Chang the reader is referred to the introduction of Chapter 3.

[‡]See also Theorem 3.8 in Chapter 5 in this Volume

(1) $\nu \leq \sigma\langle\nu\rangle$;

(2) $(\sigma + \varepsilon)\langle\nu\rangle \leq \sigma\langle\nu\rangle + \varepsilon$;

(3) $\sigma\langle\mu \vee \nu\rangle = \sigma\langle\mu\rangle \vee \sigma\langle\nu\rangle$;

(4) $\sigma\langle\psi\langle\nu\rangle\rangle = (\sigma \circ \psi)\langle\nu\rangle$;

(5) $\sup\ \sigma\langle\nu\rangle \wedge \mu = \sup\ \mu \wedge \sigma_s\langle\mu\rangle$;

(6) $(\sigma^\beta)_s = (\sigma_s)^\beta$;

(7) $(\sigma\langle\nu\rangle)^\beta = \sigma^\beta\left(\nu^\beta\right)$;

(8) $\sigma\langle x\rangle^\beta = \sigma_s^\beta(x)$;

(9) $\left(\sigma^\beta\right)^n = (\sigma^n)^\beta$.

Theorem 4.3.2 *Let* $(X,\mathcal{U})$ *be a fuzzy uniform space. Then the mapping* $^- :$ $I^X \to I^X$ *defined by*

$$\overline{\mu} = \inf_{\sigma \in \mathcal{U}} \sigma\langle\mu\rangle$$

is a fuzzy closure operator.

Proposition 4.3.3 *If* $\mathcal{B}$ *is a base for the fuzzy uniformity* $\mathcal{U}$ *then for all* $\mu \in I^X$ *we have*

$$\overline{\mu} = \inf_{\sigma \in \widehat{\mathcal{B}}} \sigma\langle\mu\rangle = \inf_{\sigma \in \langle\mathcal{B}\rangle} \sigma\langle\mu\rangle = \inf_{\sigma \in \mathcal{B}} \sigma\langle\mu\rangle$$

The above fuzzy closure operator determines an I-topology on X, $\tau_{\mathcal{U}}$, associated with $\mathcal{U}$ which is called the *fuzzy uniform topology.*

Now, we'll see that any fuzzy uniform topological space is a fuzzy neighbourhood space.

Theorem 4.3.4 *Let* $(X,\mathcal{U})$ *be a fuzzy uniform space. If we define for each* $x \in X$,

$$\mathcal{U}_x \stackrel{\text{def}}{=} \{\sigma\langle x\rangle : \sigma \in \mathcal{U}\} \subseteq I^X,$$

then $\{\mathcal{U}_x\}_{x \in X}$ *is a fuzzy neighbourhood system.*

Thus $\{\mathcal{U}_x\}_{x \in X}$ is a fuzzy neighbourhood system and therefore we can define the generated fuzzy closure operator.

It is a routine to check that these two fuzzy closure operators are the same because

$$\overline{\mu}(x) = \inf_{\sigma \in \mathcal{U}} \sigma\langle\mu\rangle(x) = \inf_{\sigma \in \mathcal{U}} \sup\ \mu \wedge \sigma\langle x\rangle = \inf_{\nu \in \mathcal{U}_x} \sup\ \mu \wedge \nu$$

Consequently, any fuzzy uniform topological space is a fuzzy neighbourhood space.

4.4 Uniformly continuous functions

We can extend the notion of uniform continuity in uniform spaces to the fuzzy setting in a natural way as follows:

Definition 4.4.1 *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be fuzzy uniform spaces and $f : X \to Y$ a mapping. f is said to be uniformly continuous iff*

$$\forall\, \psi \in \mathcal{V},\ \exists \sigma \in \mathcal{U},\ : (f \times f)[\sigma] \leq \psi$$

or

$$\forall\, \psi \in \mathcal{V},\ (f \times f)^{-1}[\psi] \in \mathcal{U}.$$

Proposition 4.4.2 *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be fuzzy uniform spaces and $\mathcal{B}$ and $\mathcal{C}$ be bases for $\mathcal{U}$ and $\mathcal{V}$ respectively. If $f : X \to Y$ is a mapping, then f is uniformly continuous if and only if*

$$\forall\, \psi \in \mathcal{C},\ \forall\, \varepsilon \in I_0,\ \exists \sigma \in \mathcal{B} \text{ such that } \sigma - \varepsilon \leq (f \times f)^{-1}[\psi].$$

Corollary 4.4.3 *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be fuzzy uniform spaces and $f : X \to Y$ a mapping. Then f is uniformly continuous if and only if*

$$\forall\, \psi \in \mathcal{V},\ \forall\, \varepsilon \in I_0,\ \exists \sigma \in \mathcal{U} \text{ such that } \sigma - \varepsilon \leq (f \times f)^{-1}[\psi].$$

Theorem 4.4.4 *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be fuzzy uniform spaces and $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ be a uniformly continuous function. Then $f : (X,\tau_{\mathcal{U}}) \to (Y,\tau_{\mathcal{V}})$ is continuous.*

Proposition 4.4.5 *If $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ and $g : (Y,\mathcal{V}) \to (Z,\mathcal{W})$ are two uniformly continuous functions, then $(g \circ f) : (X,\mathcal{U}) \to (Z,\mathcal{W})$ is uniformly continuous.*

5 L-Uniform spaces

Our intension in this section is to present an approach to L-uniformities which covers U. Höhle's as well as J. Gutiérrez García's approach to L-valued uniformities. This approach is based an L-filters developed in Subsection 6.2 in [7]. For simplicity let us assume that the underlying lattice L is given by a complete Heyting algebra. In particular, the semigroup operations $*$ and $\otimes$ coincide and are determined by $\wedge$ – i.e. $* = \otimes = \wedge$ (cf. notation in Theorem 6.2.11 in [7]).

Referring to J. Gutiérrez García (cf. [4]) we start with the following

Definition 5.1 *Let X be a set. A map $\nu : L^{X\times X} \longmapsto L$ is called an* L–uniformity *on X iff it satisfies the following properties:*

(GU1) $\quad \nu$ *is an* L*-filter on* $X \times X$.

(GU2) $\quad \nu(f) \;\leq\; \bigwedge_{x\in X} f(x,x) \qquad \forall x \in X.$

(GU3) $\quad \nu(f) \;\leq\; \nu(f^{-1})$ *where* $f^{-1}(x,y) \;=\; f(y,x)$.

(GU4) $\quad \nu(f) \;\leq\; \bigvee\{\nu(g) \mid g\circ g \leq f\}$
where $g\circ g(x,y) \;=\; \vee\{g(x,z)\wedge g(z,y) \mid z\in X\}$.

We call (X,ν) *a* L-uniform space.

Definition 5.2 *Let* (X,ν_1) *and* (Y,ν_2) *be* L*-uniform spaces and* $\varphi : X \longmapsto Y$ *be a mapping. Then* φ *is said to be* L*-uniformly continuous if*

$$\nu_2(f) \;\leq\; \nu_1(\varphi\times\varphi)^{\leftarrow}(f)) \quad .$$

Obviously L-uniform spaces and L-uniformly continuous maps form a category L-**UNIF**.

Definition 5.3 (Crisp systems of L–valued entourages)
A 1*-filter* $\mathbf{V}$ *on* $X\times X$ *(cf. Definition 6.2.3 in [7]) is said to be a* crisp system of L-valued entourages *iff* $\mathbf{V}$ *satisfies the following conditions*

(V1) $\quad d(x,x) \;=\; 1 \qquad \forall x\in X \,,\, \forall d\in\mathbf{V}$.

(V2) $\quad d\in\mathbf{V} \;\Rightarrow\; d^{-1}\in\mathbf{V}$.

(V3) $\quad \forall d_1\in\mathbf{V}\;\exists d_2\in\mathbf{V}$ *s.t.* $d_2\circ d_2 \;\leq\; d_1$.

We call $(X,\mathbf{V})$ *an* L*-uniform space in the sense of Höhle or* LC-uniform space *(where* C *stands for crisp systems). If* $(X,\mathbf{V}_1)$ *and* $(Y,\mathbf{V}_2)$ *be* LC*-uniform spaces, then a mapping* $\varphi : X \longmapsto Y$ *is said to be* LC*-uniformly continuous iff* $(\varphi\times\varphi)^{\leftarrow}(d) \in \mathbf{V}_1 \;\; \forall d\in\mathbf{V}_2$.

Once again LC-uniform spaces and LC-uniformly continuous functions form a category LC-**UNIF**.

Proposition 5.4 *Every crisp system* $\mathbf{V}$ *of* L*-valued entourages on* X *determines an* L*-uniformity* $\nu_{\mathbf{V}}$ *by*

$$\nu_{\mathbf{V}}(f) \;=\; \bigvee_{d\in\mathbf{V}}\Big(\bigwedge_{(x,y)\in X\times X} d(x,y)\to f(x,y)\Big) \quad .$$

where $\to$ *denotes the "implication" in the underlying Heyting algebra* L.

Proof. Referring to Remark 6.2.3 in [7] it is a matter of routine to check that (V1) and (V3) imply (GU2) and (GU3). In order to verify (V3) $\Rightarrow$ (GU4) we proceed as follows: Because of $(V3)$ the the following relation holds:

$$\nu_{\mathbf{V}}(f) \;\leq\; \bigvee_{d\in\mathbf{V}}\Big(\bigwedge_{(x,y)\in X\times X} (d\circ d)(x,y)\to f(x,y)\Big) \quad .$$

Further we observe:

$$\bigwedge_{(x,y)\in X\times X} (d\circ d)(x,y) \to f(x,y) = \bigwedge_{(x,z)\in X\times X} d(x,z) \to \Big(\bigwedge_{y\in X} d(z,y) \to f(x,y)\Big).$$

Now we define a map $f_d : X \times X \longmapsto L$ by

$$f_d(x,z) = \bigwedge_{y\in X} d(z,y) \to f(x,y)$$

and observe for all $d \in \mathbf{V}$:

$$f_d \circ d \leq f$$

$$\bigwedge_{(x,y)\in X\times X} (d\circ d)(x,y) \to f(x,y) \leq \nu_{\mathbf{V}}(f_d) \wedge \nu_{\mathbf{V}}(d) \quad .$$

Hence (GU4) follows.

Referring to Proposition 5.4 it is not difficult to see that L**C-UNIF** is isomorphic to a full subcategory of L-**UNIF**.

Theorem 5.5 *Every L-uniformity ν on X induces an L-topology τ_ν on X by*

$$\mu_p(h) = \bigvee_{f\in L^{X\times X}} \nu(f) \wedge \Big(\bigwedge_{x\in X} f(p,x) \to h(x)\Big) \quad \forall\, h \in L^X$$

where $(\mu_p)_{p\in X}$ is the L-neighbourhood system corresponding to τ_ν (cf. Subsection 6.1 in [7]).

Proof. The L-neighbourhood axioms (U0)–(U3) follow immediately from (GU1) and (GU2). In order to verify the neighbourhood axiom (U4) (cf. Subsection 6.1 in [7]) we proceed as follows:

$$\begin{aligned}
\mu_p(\mu_(h)) &= \bigvee_{g\in L^{X\times X}} \nu(g) \wedge \Big(\bigwedge_{z\in X} g(p,z) \to \mu_z(h)\Big) \\
&\geq \bigvee_{g,f\in L^{X\times X}} (\nu(g)\wedge\nu(f)) \wedge \Big(\bigwedge_{z\in X} g(p,z) \to \Big(\bigwedge_{x\in X} f(z,x) \to h(x)\Big)\Big) \\
&= \bigvee_{g,f\in L^{X\times X}} (\nu(g)\wedge\nu(f)) \wedge \Big(\bigwedge_{x\in X} (g\circ f)(x) \to h(x)\Big).
\end{aligned}$$

Now we invoke (GU4) and obtain: $\mu_p(h) \leq \mu_p(\mu\_(h))$; hence (U4) is verified.

It is not difficult to show that the natural L-topology τ_ν associated with an L-uniformity ν is *stratified* (cf. Theorem 5.2.7 in [7]). Hence the previous theorem gives rise to a functor from the category L-**UNIF** of L-uniform spaces to the category **S**L-**TOP** of stratified L-topological spaces — a category which can be considered as the L–valued version of R. Lowen's category **fts** of fuzzy topological spaces.

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CHAPTER 9

Extensions Of Uniform Space Notions

M. H. Burton and J. Gutiérrez García*

Introduction

A large part of mathematics is based on the notion of a set and on binary logic. Statements are either true or false and an element either belongs to a set or not. In order to accommodate the idea of a sliding transition between the two states: true and false, and to generalise the concept of a subset of a given set, Zadeh introduced the notion of a fuzzy subset in a now-famous paper: [52]. For the record, let us recall that if X is a set and A is a subset of X then the *characteristic function*, denoted 1_A, is defined by

$$1_A(x) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Thus $1_A \in 2^X$. In [52], an element $\mu \in I^X$, where I denotes the closed unit interval, was called a *fuzzy set* in X, with $\mu(x)$ being interpreted as the degree to which x belongs to the fuzzy set μ. Since the elements $\mu \in I^X$ are generalisations of subsets of X, it is more accurate to refer to them as *fuzzy subsets* of X and we shall adopt this terminology here.

It was soon shown that analogues of the basic notions of set theory, such as: intersection, union and complement, could be defined in such a way that the resulting theory of fuzzy subsets resembled the structure of ordinary powersets. Furthermore, in each theorem, if the fuzzy subsets are just characteristic functions, in which case we could say that the fuzzy subset is *set-generated*, then the theorem reduces to a standard theorem. In this sense the theory of sets was extended to the theory of fuzzy sets.

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Since then, mathematicians have been attempting to extend useful mathematical theories to the fuzzy setting, replacing subsets by fuzzy subsets and standard notions by analogous fuzzy notions. The theories of: fuzzy topology, uniformities, metrics, norms, to name a few, are all under construction. Of course there are many ways to extend a standard notion and the value of the resulting theory will be judged by the strength of its links with other theories. However, one basic requirement of an extension is:

- In the extended theory, if the fuzzy objects referred to by a theorem are generated by standard objects then the theorem reduces to a standard theorem.

Our purpose here is to outline how some aspects of the theory of uniform spaces are being extended to the fuzzy setting. Since a uniform space gives rise to a topological space and topological spaces provide the appropriate setting for the abstract study of continuity and convergence, any extension of the theory of uniform spaces must be linked to a suitable extension of the theory of topological spaces.

In [34], R. Lowen extended the theory of convergence in topological spaces to the realm of stratified I-topological spaces. This was accomplished with the aid of filters on I^X which he called *prefilters* on X. In [35], the basic theory of *fuzzy neighbourhood spaces* is developed. This was a successful attempt to extend the notion of a topology, defined in terms of neighbourhoods, where families of filters on I^X are the analogues of neighbourhood systems. A similar concept appeared under the name probabilistic topology in [22, 24] and a comparison of both structures can be found in [26].

Fuzzy uniformities have two roots tracing back to R. Lowen [32, 36] and to B. Hutton [28]. Lowen defined I-fuzzy uniformities as a fuzzification of the *entourage* approach to uniformities, while Hutton followed a variation of the *covering* approach to uniformities and based his concept of L-fuzzy uniformities on a completely distributive lattice L provided with an order reversing involution. A third fuzzification was introduced by U. Höhle in [23, 24].

We intend to show that the theory of uniform spaces extends to the wider context of I-fuzzy uniform spaces and we base our investigations on fuzzy neighbourhood spaces and on Lowen's convergence theory. It must be emphasised that we are therefore dealing with the special case in which the underlying lattice is the real unit interval. In particular, our purpose is to extend the basic uniform space notions of Cauchy filter, boundedness, precompactness and completeness to the I-fuzzy uniform space setting. Among other things we prove the important theorem that a I-fuzzy subset μ is compact if and only if μ is precompact and complete.

1 Preliminaries

We present the basic theorems of fuzzy set theory.

If $\mathcal{S} \subseteq I^X$ we define the fuzzy analogues of intersection and union:

$$\Big(\bigwedge_{\mu\in\mathcal{S}} \mu\Big)(x) \stackrel{\text{def}}{=} \inf_{\mu\in\mathcal{S}} \mu(x),$$
$$\Big(\bigvee_{\mu\in\mathcal{S}} \mu\Big)(x) \stackrel{\text{def}}{=} \sup_{\mu\in\mathcal{S}} \mu(x).$$

We also write $\inf_{\mu\in\mathcal{S}} \mu$ for $\bigwedge_{\mu\in\mathcal{S}} \mu$ and $\sup_{\mu\in\mathcal{S}} \mu$ for $\bigvee_{\mu\in\mathcal{S}} \mu$.

For $\mu, \nu \in I^X, x \in X$ we define the fuzzy analogues of containment, complement, empty set and universal set:

$$\begin{aligned}
\mu \le \nu &\iff \forall x \in X,\ \mu(x) \le \nu(x),\\
\mu'(x) &\stackrel{\text{def}}{=} 1 - \mu(x),\\
0(x) &\stackrel{\text{def}}{=} 0,\\
1(x) &\stackrel{\text{def}}{=} 1.
\end{aligned}$$

It can be shown that the basic theorems of set theory, such as the associative, distributive and de Morgan laws, also apply to fuzzy subsets. A notable omission is the notion of the powerset and the reader is referred to [25] for a discussion of this and other topics relating to the foundations of fuzzy sets.

Let X and Y be sets, let $f : X \to Y$ be a function, $\mu \in I^X, \nu \in I^Y$. We define the *image*, denoted $f[\mu]$, of μ and the *preimage*, denoted $f^{-1}[\nu]$, of ν by

$$\begin{aligned}
f[\mu](y) &\stackrel{\text{def}}{=} \sup_{x\in f^{-1}[\{y\}]} \mu(x)\\
f^{-1}[\nu] &\stackrel{\text{def}}{=} \nu \circ f.
\end{aligned}$$

That the behaviour of these newly–defined notions mimics the standard notions is confirmed by the following theorem.

Theorem 1.1 *Let X, Y, Z be sets, $f : X \to Y$, $g : Y \to Z$, $\mu \in I^X$, $\nu \in I^Y$, $\lambda \in I^Z$, $\mathcal{F} \subseteq I^X$, $\mathcal{G} \subseteq I^Y$. Then*

1. $(g \circ f)[\mu] = g[f[\mu]]$, $(g \circ f)^{-1}[\lambda] = f^{-1}[g^{-1}[\lambda]]$;
2. $f^{-1}\left[\sup_{\nu\in\mathcal{G}} \nu\right] = \sup_{\nu\in\mathcal{G}} f^{-1}[\nu]$, $f^{-1}\left[\inf_{\nu\in\mathcal{G}} \nu\right] = \inf_{\nu\in\mathcal{G}} f^{-1}[\nu]$;
3. $f\left[\sup_{\mu\in\mathcal{F}} \mu\right] = \sup_{\mu\in\mathcal{F}} f[\mu]$, $f\left[\inf_{\mu\in\mathcal{F}} \mu\right] \le \inf_{\mu\in\mathcal{F}} f[\mu]$;
4. $f^{-1}[\nu'] = (f^{-1}[\nu])'$, $f[\mu'] \ge (f[\mu])'$;
5. $\nu_1 \le \nu_2 \Longrightarrow f^{-1}[\nu_1] \le f^{-1}[\nu_2]$, $\mu_1 \le \mu_2 \Longrightarrow f[\mu_1] \le f[\mu_2]$;
6. $f[f^{-1}[\nu]] \le \nu$ *with equality if f is surjective;*

7. *$f^{-1}[f[\mu]] \geq \mu$ with equality if f is injective;*

8. *$f[f^{-1}[\nu] \wedge \mu] = \nu \wedge f[\mu]$.*

The proofs are routine and some can be found in [45].

If $(X_j : j \in J)$ is a family of sets and $(\mu_j : j \in J)$ is a family of fuzzy subsets with $\mu_j \in I^{X_j}$ for each $j \in J$, then we define the *product* of the fuzzy subsets by

$$\Big(\prod_{j\in J} \mu_j\Big)((x_j : j \in J)) \stackrel{\text{def}}{=} \bigwedge_{j\in J} \mu_j(x_j).$$

If $\mu \in I^X$ and $\alpha \in I$ then we define

$$\mu^\alpha \stackrel{\text{def}}{=} \{x \in X : \mu(x) > \alpha\}, \quad \mu_\alpha \stackrel{\text{def}}{=} \{x \in X : \mu(x) \geq \alpha\}$$

and call these the *strong α-cut* and the *weak α-cut* of μ respectively. It is often useful to ascertain which properties of a fuzzy subset are inherited by its α-cuts and, conversely, which properties of the members of the family, $(\mu^\alpha : \alpha \in I)$ (or the family $(\mu_\alpha : \alpha \in I)$), confer an analagous property on μ. A theorem of the form

$$\mu \text{ has property } P \iff \forall \alpha \in I,\ \mu^\alpha \text{ has property } P'$$

is called an *α-level theorem* and such theorems are enormously useful as facilitators of proofs of theorems in the extended theory. We record some basic properties of α-cuts.

Theorem 1.2 *Let $\mu, \nu \in I^X, \mathcal{S} \subseteq I^X$. Then*

1. $\mu = \nu \iff \forall \alpha \in I,\ \mu^\alpha = \nu^\alpha$;

2. $\mu = \sup_{\alpha\in(0,1)} \alpha \wedge 1_{\mu^\alpha}$;

3. $(\mu \wedge \nu)^\alpha = \mu^\alpha \cap \nu^\alpha,\ (\mu \vee \nu)^\alpha = \mu^\alpha \cup \nu^\alpha$;

4. $(\mu \wedge \nu)_\alpha = \mu_\alpha \cap \nu_\alpha,\ (\mu \vee \nu)_\alpha = \mu_\alpha \cup \nu_\alpha$;

5. $\bigcup_{\mu\in\mathcal{S}} \mu_\alpha \subseteq \Big(\bigvee_{\mu\in\mathcal{S}} \mu\Big)_\alpha,\ \Big(\bigwedge_{\mu\in\mathcal{S}} \mu\Big)_\alpha = \bigcap_{\mu\in\mathcal{S}} \mu_\alpha$;

6. $\bigcup_{\mu\in\mathcal{S}} \mu^\alpha = \Big(\bigvee_{\mu\in\mathcal{S}} \mu\Big)^\alpha,\ \Big(\bigwedge_{\mu\in\mathcal{S}} \mu\Big)^\alpha \subseteq \bigcap_{\mu\in\mathcal{S}} \mu^\alpha$;

7. $(\mu')_\alpha = \big(\mu^{(1-\alpha)}\big)',\ (\mu')_\alpha = \big(\mu_{(1-\alpha)}\big)'$;

8. $\mu_\alpha = \bigcap_{\beta<\alpha} \mu^\beta,\ \mu^\alpha = \bigcup_{\beta<\alpha} \mu_\beta$.

2 I-Fuzzy uniform spaces

The necessary facts regarding filters on I^X can be found in [9, 26]. Also $\top$-filters have been defined in the chapter "Axiomatics foundations of fixed-basis fuzzy topology" (cf. Remark 6.2.3 in [27]) but, for convenience, we record the basics here.

If $\mathbb{F} \subseteq 2^X$ and $\mathcal{F} \subseteq I^X$ we define

$$\langle \mathbb{F} \rangle \overset{\text{def}}{=} \{G \in 2^X : \exists F \in \mathbb{F},\ F \subseteq G\},\ \langle \mathcal{F} \rangle \overset{\text{def}}{=} \{\mu \in I^X : \exists \nu \in \mathcal{F},\ \nu \leq \mu\}.$$

If $\mathcal{F}$ is a filter base on I^X we define the *characteristic*, $c(\mathcal{F})$, of $\mathcal{F}$ by

$$c(\mathcal{F}) \overset{\text{def}}{=} \inf_{\nu \in \mathcal{F}} \sup \nu.$$

If $\mathcal{F}$ and $\mathcal{G}$ are filters on I^X we say that

$$\mathcal{F} \sim \mathcal{G} \iff \forall \mu \in \mathcal{F},\ \forall \nu \in \mathcal{G},\ \mu \wedge \nu \neq 0.$$

If $\mathcal{F} \sim \mathcal{G}$ we say that $\mathcal{F}$ and $\mathcal{G}$ are *compatible*, in which case we can define

$$(\mathcal{F}, \mathcal{G}) \overset{\text{def}}{=} \langle \{\mu \wedge \nu : \mu \in \mathcal{F},\ \nu \in \mathcal{G}\} \rangle.$$

It is easy to check that $\mathcal{F} \vee \mathcal{G}$ is the smallest filter on I^X which is finer than both $\mathcal{F}$ and $\mathcal{G}$ – i.e. $\mathcal{F} \vee \mathcal{G}$ is the smallest upper bound of $\{\mathcal{F}, \mathcal{G}\}$. This observation does not imply that the set of all filters on I^X is a lattice; in particular the supremum of two different ultrafilters does not exist.

We define

$$c(\mathcal{F}, \mathcal{G}) \overset{\text{def}}{=} \begin{cases} c(\mathcal{F} \vee \mathcal{G}), & \text{if } \mathcal{F} \sim \mathcal{G} \\ 0, & \text{otherwise.} \end{cases}$$

If $\mu \neq 0$ then we write $(\mathcal{F}, \mu)$ instead of the more cumbersome: $\mathcal{F} \vee \langle \{\nu\} \rangle$.

If $\mathcal{F}$ is a filter base on I^X and $c(\mathcal{F}) > 0$, we define

$$\widehat{\mathcal{F}} \overset{\text{def}}{=} \left\{ \sup_{\varepsilon \in I_0} (\mu_\varepsilon - \varepsilon) : (\mu_\varepsilon)_{\varepsilon \in I_0} \in \mathcal{F}^{I_0} \right\}$$

and, as in [42], we call $\widehat{\mathcal{F}}$ the *saturation* of $\mathcal{F}$. We say that $\mathcal{F}$ is *saturated* if $\widehat{\mathcal{F}} = \mathcal{F}$. The saturation condition is also well known under the name κ-*condition* which appears in the $(\mathcal{F}1)$-axiom of *1-filters* (cf. Definition 1.5 in [24] and Definition 2.8 in [26]). It is not difficult to see that in the particular case $L = I$, $* = T_m$, where $T_m(x, y) = \max\{x + y - 1, 0\}$,

saturated filters with characteristic 1 are exactly 1-filters.

If $f : X \to Y$, $\mathcal{F}$ is a filter on I^X, $\mathcal{G}$ is a filter on I^Y then we define

$$f[\mathcal{F}] \overset{\text{def}}{=} \{f[\nu] : \nu \in \mathcal{F}\},\ f^{-1}[\mathcal{G}] \overset{\text{def}}{=} \{f^{-1}[\nu] : \nu \in \mathcal{G}\}$$

and it is routine to check that $f[\mathcal{F}]$ is a filter base and so is $f^{-1}[\mathcal{G}]$ if f is surjective. The reader is referred to [34, 35, 36] for the basic notation and theory of filters on I^X (where the term: "saturation" was not yet employed), as well as to [9, 21, 15, 16] for more details.

If $\mathcal{F}$ is a filter on I^X with $c(\mathcal{F}) > 0$ and $0 < \alpha \leq c(\mathcal{F})$ then we define

$$\mathcal{F}^\alpha \stackrel{\text{def}}{=} \{\nu^\beta : \nu \in \mathcal{F},\ \beta < \alpha\}.$$

If $0 \leq \alpha < c$ then we define

$$\mathcal{F}_\alpha \stackrel{\text{def}}{=} \langle\{\nu^\alpha : \nu \in \mathcal{F}\}\rangle$$

and the reader can check that these are both filters on X.

If $\mathbb{F}$ is a filter on X and $\alpha > 0$ we define

$$\mathbb{F}_\alpha \stackrel{\text{def}}{=} \langle\{\alpha 1_F : F \in \mathcal{F}\}\rangle,\ \mathbb{F}^\alpha \stackrel{\text{def}}{=} \{\nu \in I^X : \forall \beta < \alpha,\ \nu^\beta \in \mathbb{F}\}.$$

and a quick check reveals that these are both filters on I^X. Further properties of these filters are recorded in [9, 21].

If $\mathcal{F}$ is a filter on I^X we say that (see Definition 3.17 on page 73 in [19]):

$$\mathcal{F} \text{ is } \textit{prime} \iff \forall \mu, \nu \in \mathcal{F},\ (\mu \vee \nu \in \mathcal{F} \Rightarrow \nu \in \mathcal{F} \text{ or } \nu \in \mathcal{F}).$$

We quote two useful results from [34].

Theorem 2.1 *Let $\mathcal{F}$ be a filter on I^X and let*

$$\mathcal{P}(\mathcal{F}) \stackrel{\text{def}}{=} \{\mathcal{G} \in I^X : \mathcal{G} \textit{ is a prime filter and } \mathcal{F} \subseteq \mathcal{G}\}.$$

Then $\mathcal{P}(\mathcal{F})$ has minimal elements.

If $\mathbb{F}$ is a filter on X we define

$$\mathbb{P}(\mathbb{F}) \stackrel{\text{def}}{=} \{\mathbb{K} : \mathbb{K} \text{ is an ultrafilter}, \mathbb{F} \subseteq \mathbb{K}\}.$$

Theorem 2.2 *Let $\mathcal{F}$ be a filter on I^X and let*

$$\mathcal{P}_m(\mathcal{F}) \stackrel{\text{def}}{=} \{\mathcal{G} \in \mathcal{P}(\mathcal{F}) : \mathcal{G} \textit{ is minimal }\}.$$

Then

$$\mathcal{P}_m(\mathcal{F}) = \{\mathcal{F} \vee \mathbb{F}_1 : \mathbb{F} \in \mathbb{P}(\mathcal{F}_0)\}.$$

This last theorem, which characterises the minimal prime filters finer than a given filter, has found a number of applications.

Corollary 2.3 *There is a bijection between $\mathcal{P}_m(\mathcal{F})$ and $\mathbb{P}(\mathcal{F}_0)$. In particular, every saturated prime filter on I^X with characteristic 1 can be identified with an ultrafilter on X and vice versa.*

For a filter $\mathcal{F}$ on I^X we define the *lower characteristic*, $\bar{c}(\mathcal{F})$, of $\mathcal{F}$ by

$$\bar{c}(\mathcal{F}) \stackrel{\text{def}}{=} \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} c(\mathcal{G}).$$

Further properties of the lower characteristic can be found in [9].

The convergence theory established in [34, 35] paved the way for the extension of the notion of a uniform space to the fuzzy setting. The idea is to replace subsets $D \subseteq X \times X$ with fuzzy subsets $\sigma \in I^{X \times X}$. If $\sigma, \psi \in I^{X \times X}$, we define $\sigma \circ \psi$ by:

$$(\sigma \circ \psi)(x, y) \stackrel{\text{def}}{=} \sup_{z \in X} \psi(x, z) \wedge \sigma(z, y)$$

and it is easy to check that this reduces to the standard definition in the set-generated case.

Definition 2.4 *In [36] R. Lowen defines a* fuzzy uniform space *to be a pair* $(X, \mathcal{D})$ *where:*

(FUS1) $\mathcal{D}$ is a saturated filter on $I^{X \times X}$;

(FUS2) $\forall \sigma \in \mathcal{D},\ \forall x \in X,\ \sigma(x, x) = 1$;

(FUS3) $\forall \sigma \in \mathcal{D},\ \sigma_{\mathrm{S}} \in \mathcal{D}$, where $\sigma_{\mathrm{S}}(x, y) = \sigma(y, x)\ \forall\, x, y \in X$;

(FUS4) $\forall \sigma \in \mathcal{D},\ \forall \varepsilon > 0,\ \exists \psi \in \mathcal{D},\ \psi \circ \psi \leq \sigma + \varepsilon$.

To conform with the latest terminology we call $(X, \mathcal{D})$ an *I-fuzzy uniform space*. The basic theory of I-fuzzy uniform spaces is developed by Lowen in [36, 38, 39] and we give an outline here.

If $(X, \mathcal{D})$ is an I-fuzzy uniform space, $\sigma \in \mathcal{D}$ and $x \in X$ we define

$$\begin{aligned} \sigma\langle x\rangle &\stackrel{\text{def}}{=} \sigma(y, x) \text{ and} \\ \mathcal{D}_x &\stackrel{\text{def}}{=} \{\sigma\langle x\rangle : \sigma \in \mathcal{D}\}. \end{aligned}$$

Then $\mathcal{D}_x$ is a fuzzy neighbourhood prefilter (an I-fuzzy neighbourhood filter) at x and $(\mathcal{D}_x : x \in X)$ is a fuzzy neighbourhood system (I-fuzzy neighbourhood system) in the sense of [35]. It therefore follows, from the theory developed in [35], that, for $\mu \in I^X$

$$\begin{aligned} \bar{\mu} &= \inf_{\sigma \in \mathcal{D}_x} \sup_{y \in X} \mu(y) \wedge \sigma(y, x) \\ &= \inf_{\sigma \in \mathcal{D}} \sup \mu \wedge \sigma\langle x\rangle \end{aligned}$$

is the closure of μ with respect to the *I-fuzzy uniform topology*, $\tau_{\mathcal{D}}$, generated by $\mathcal{D}$. In particular, $\tau_{\mathcal{D}}$ is a stratified, transitive I–topology on X (Proposition 5.1 in [26]).

Alternatively, we can define

$$\sigma\langle\mu\rangle \stackrel{\text{def}}{=} \sup \mu \wedge \sigma\langle x\rangle = \sup_{y \in X} \mu(y) \wedge \sigma(y, x).$$

and

$$\sigma\langle\mathcal{F}\rangle \stackrel{\text{def}}{=} \langle\{\sigma\langle\nu\rangle : \nu \in \mathcal{F}\}\rangle.$$

It then follows that

$$\bar{\mu} = \inf_{\sigma \in \mathcal{D}} \sigma\langle\mu\rangle$$

and so we have a closure operator in terms of [36].

Thus the theory of I-fuzzy uniform spaces is intimately related to the theory of I-fuzzy neighbourhood spaces and papers on one topic often contain useful results and insights on the other.

If $(X, \mathcal{D})$ is an I-fuzzy uniform space and $\mathcal{F}$ is a filter on I^X then we define the *$\mathcal{D}$-adherence* of $\mathcal{F}$, denoted $\operatorname{adh}_{\mathcal{D}} \mathcal{F}$, and the *$\mathcal{D}$-limit* of $\mathcal{F}$, denoted $\lim_{\mathcal{D}} \mathcal{F}$ by

$$\begin{aligned} \operatorname*{adh}_{\mathcal{D}} \mathcal{F} &\stackrel{\text{def}}{=} \inf_{\nu \in \mathcal{F}} \bar{\nu} \\ \lim_{\mathcal{D}} \mathcal{F} &\stackrel{\text{def}}{=} \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \operatorname*{adh}_{\mathcal{D}} \mathcal{G}. \end{aligned}$$

We collect together some basic facts in the following lemma.

Lemma 2.5 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space, $\mathcal{F}$ a filter on I^X and $x \in X$. Then*

1. $\left(\operatorname*{adh}_{\mathcal{D}} \mathcal{F}\right)(x) = c(\mathcal{D}_x, \mathcal{F})$;
2. $\left(\lim_{\mathcal{D}} \mathcal{F}\right)(x) = \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} c(\mathcal{D}_x, \mathcal{G})$;
3. *If $\mathcal{F}$ is prime then* $\lim_{\mathcal{D}} \mathcal{F} = \operatorname*{adh}_{\mathcal{D}} \mathcal{F}$;
4. $\sup \operatorname*{adh}_{\mathcal{D}} \mathcal{F} \leq c(\mathcal{F})$;
5. $\sup \lim \mathcal{F} \leq \bar{c}(\mathcal{F})$;
6. *If $\mathcal{F} \subseteq \mathcal{G}$ then* $\operatorname*{adh}_{\mathcal{D}} \mathcal{G} \leq \operatorname*{adh}_{\mathcal{D}} \mathcal{F}$;
7. $\operatorname*{adh}_{\mathcal{D}} \mathcal{F} = \sup_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \operatorname*{adh}_{\mathcal{D}} \mathcal{G}$;
8. *If $\mathcal{F}$ is a filter base on I^X and $\sigma \in \mathcal{D}$ then*

$$\operatorname*{adh}_{\mathcal{D}} \langle\mathcal{F}\rangle = \operatorname*{adh}_{\mathcal{D}} \mathcal{F} = \operatorname*{adh}_{\mathcal{D}} \widehat{\mathcal{F}} = \operatorname*{adh}_{\mathcal{D}} \sigma\langle\mathcal{F}\rangle.$$

If $((X_j, \mathcal{D}_j) : j \in J)$ is a family of I-fuzzy uniform spaces, we define the *product I-fuzzy uniformity* , denoted $\mathcal{D} = \prod_{j \in J} \mathcal{D}_j$ to be that I-fuzzy uniformity whose sub-basic elements are of the form

$$(p_j \times p_j)^{-1}[\sigma_j] \text{ with } \sigma_j \in \mathcal{D}_j \text{ and } j \in J.$$

The papers [17, 36, 9] contain useful theorems regarding product spaces.

If $(X, \mathcal{D})$ is an I-fuzzy uniform space and $\mathcal{F}$ is a filter on I^X then we say that

$$\mathcal{F} \textit{ is } \mathcal{D}\textit{-convergent} \iff \bar{c}(\mathcal{F}) = \sup \lim_{\mathcal{D}} \mathcal{F}.$$

In view of Lemma 2.5(5), this is equivalent to

$$\mathcal{F} \text{ is } \mathcal{D}\text{-convergent} \iff \bar{c}(\mathcal{F}) \leq \sup \lim_{\mathcal{D}} \mathcal{F}.$$

If $\mu \in I^X$ we say that

$$\mathcal{F} \textit{ is } \mathcal{D}\textit{-convergent in } \mu \iff \bar{c}(\mathcal{F}) \leq \sup \mu \wedge \lim_{\mathcal{D}} \mathcal{F}.$$

The following proposition is from [35] and is an example of what we might call *an α-level theorem.*

Theorem 2.6 *If $(X, \mathcal{D})$ is an I-fuzzy uniform space, $\mathcal{F}$ is a filter on I^X and $\alpha \leq \bar{c}(\mathcal{F})$ then*

$$\lim_{\mathcal{D}} \mathcal{F} \geq \alpha \iff \mathcal{F}_0 \overset{\mathcal{D}^\alpha}{\to} x.$$

In the sequel we will see further examples of how these α-level theorems provide a link between fuzzy and crisp theories.

If $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ are I-fuzzy uniform spaces then they each are endowed with an I-fuzzy uniform topology, $\tau_{\mathcal{D}}$ and $\tau_{\mathcal{E}}$ respectively, which is obtained from the I-fuzzy neighbourhood structure or the associated closure operator as described above. The continuity of a function $f : (X, \mathcal{D}) \to (Y, \mathcal{E})$ is defined in the natural way. Namely:

$$f \text{ is } \textit{continuous} \iff \forall \nu \in \tau_{\mathcal{E}},\ f^{-1}[\nu] \in \tau_{\mathcal{D}}.$$

We say that

$$\begin{aligned} f \text{ is } \textit{uniformly continuous} &\iff \forall \psi \in \mathcal{E},\ (f \times f)^{-1}[\psi] \in \mathcal{D} \\ &\iff \forall \psi \in \mathcal{E},\ \exists \sigma \in \mathcal{D},\ (f \times f)[\sigma] \leq \psi. \end{aligned}$$

If we let US denote the category of uniform spaces with uniformly continuous maps and let FUS denote the category of I-fuzzy uniform spaces with uniformly continuous maps, then

$$\omega_u : US \to FUS, \quad (X, \mathbb{D}) \mapsto (X, \mathbb{D}^1)$$

is a functor if we let ω_u leave maps unchanged. Similarly,

$$\iota_u : FUS \to US, \quad (X, \mathcal{D}) \mapsto (X, \mathcal{D}^1)$$

is a functor if we let ι_u leave maps unchanged. The functor ω_u embeds the category US into the category FUS in the sense that ι_u is a right adjoint for ω_u and hence $\omega_u[US]$ is a coreflective subcategory of FUS (see [14](Theorem 5)). We call an I-fuzzy uniform space $(X, \mathcal{D})$ *uniformly generated* if there exists a uniformity $\mathbb{D}$ on X such that $(X, \mathcal{D}) = \omega_u[(X, \mathbb{D})]$. In other words, if $\mathcal{D} = \mathbb{D}^1$. Similarly, a filter $\mathcal{F}$ on I^X will be called *uniformly generated* if there exists a filter $\mathbb{F}$ on X such that $\mathcal{F} = \mathbb{F}^1$. We will call a statement about fuzzy subsets and filters on I^X an ω_u*-extension* of a uniform space notion if it reduces to a standard notion if the fuzzy subsets and filters referred to are set–generated and uniformly generated.

We intend to show that there are ω_u-extensions of various uniform space notions to the I-fuzzy uniform space setting.

To start with, we define an I-fuzzy uniform space, $(X, \mathcal{D})$, to be *Hausdorff* iff $\inf\limits_{\sigma \in \mathcal{D}} \sigma = 1_\Delta$, where Δ denotes the diagonal in $X \times X$. This is easily seen to be an ω_u-extension of the standard notion. In [36] this is shown that:

Theorem 2.7 *An I-fuzzy uniform space $(X, \mathcal{D})$ is Hausdorff iff for every prime filter $\mathcal{F}$ on I^X, $(\mathrm{adh}\,\mathcal{F})^0$ is empty or a singleton.*

It is interesting to note that the separation condition quoted in the previous theorem implies T. Kubiak's Hausdorff separation axiom I–T_2 (cf. [30], see also Proposition 4.12 in [31]†). Further, it is an easy exercise to prove the following α-level theorem:

Theorem 2.8 *An I-fuzzy uniform space $(X, \mathcal{D})$ is Hausdorff iff $\forall \alpha \in I_0$, $(X, \mathcal{D}^\alpha)$ is Hausdorff.*

In [8] a method of constructing an I-fuzzy uniformity from a family of uniformities is described:

Theorem 2.9 *Let $(\mathcal{D}(\alpha) : \alpha \in (0,1))$ be a family of uniformities on a set X satisfying:*

1. $0 < \beta \leq \alpha \Rightarrow \mathcal{D}(\beta) \subseteq \mathcal{D}(\alpha)$;
2. $\forall \alpha \in (0,1]$, $\mathcal{D}(\alpha) = \bigcup\limits_{\beta < \alpha} \mathcal{D}(\beta)$.

Let

$$\mathcal{D} \stackrel{\text{def}}{=} \{\sigma \in I^{X \times X} : \forall \alpha \in (0,1), \forall \beta < \alpha,\ \sigma^\beta \in \mathcal{D}(\alpha)\}.$$

Then $\mathcal{D}$ is the unique I-fuzzy uniformity on X such that $\forall \alpha \in (0,1]$, $\mathcal{D}^\alpha = \mathcal{D}(\alpha)$.

This theorem was used many times in the development of the theory to follow.

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†Other variations of the Hausdorff separation axiom for L–topological spaces can be found in Subsection 6.3 in [27].

3 Cauchy filters

The notion of completeness plays a central role in the theory of uniform spaces (see [48]) and this, in turn is described in terms of Cauchy filters or nets. The notions of completeness, precompactness and compactness are linked by the celebrated theorem

$$\text{compactness} = \text{completeness} + \text{precompactness}$$

and in [9, 10, 11] these notions are extended to the context of I-fuzzy uniform spaces in such a way that the above theorem still holds. We outline the development of this theory in the subsequent sections.

If $(X, \mathbb{D})$ is a uniform space, a filter $\mathbb{F}$ on X is called $\mathbb{D}$-*Cauchy* if it contains elements which are arbitrarily $\mathbb{D}$-small. More precisely:

Definition 3.1 $\mathbb{F}$ *is* $\mathbb{D}$-*Cauchy* $\iff \forall U \in \mathbb{D},\ \exists F \in \mathbb{F},\ F \times F \subseteq U.$

In [9] the generalisation of this notion is based on the following theorem.

Theorem 3.2 *Let $(X, \mathbb{D})$ be a uniform space and let $\mathbb{F}$ be a filter on X. Then*

$$\mathbb{F} \textit{ is } \mathbb{D}\textit{-Cauchy} \iff \forall U \in \mathbb{D},\ \bigcap_{\mathbb{K} \in \mathbb{P}(\mathbb{F})} \bigcap_{K \in \mathbb{K}} U(K) \neq \emptyset.$$

The proof of this theorem and all other results in this section can be found in [9]. For an ultrafilter the condition simplifies.

Corollary 3.3 *If $(X, \mathbb{D})$ is a uniform space and $\mathbb{F}$ is an ultrafilter on X then*

$$\mathbb{F} \textit{ is } \mathbb{D}\textit{-Cauchy} \iff \forall U \in \mathbb{D},\ \bigcap_{F \in \mathbb{F}} U(F) \neq \emptyset.$$

Definition 3.4 *If $(X, \mathcal{D})$ is an I-fuzzy uniform space and $\mathcal{F}$ is a filter on I^X we say that*

$$\mathcal{F} \textit{ is } \mathcal{D}\text{-Cauchy} \iff \bar{c}(\mathcal{F}) \leq \inf_{\sigma \in \mathcal{D}} \sup \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \inf_{\nu \in \mathcal{G}} \sigma\langle \nu \rangle.$$

We note that if $\mathcal{F}$ is prime then $\mathcal{P}_m(\mathcal{F}) = \{\mathcal{F}\}$ and $\bar{c}(\mathcal{F}) = c(\mathcal{F})$. Therefore

$$\mathcal{F} \text{ is } \mathcal{D}\text{-Cauchy} \iff c(\mathcal{F}) \leq \inf_{\sigma \in \mathcal{D}} \sup \inf_{\nu \in \mathcal{F}} \sigma\langle \nu \rangle.$$

The following technical lemma simplifies some of the calculations.

Lemma 3.5 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space and let $\mathcal{F}$ be a filter on I^X. Then*

1. $\inf_{\sigma \in \mathcal{D}} \sup \inf_{\nu \in \mathcal{F}} \sigma\langle \nu \rangle = \inf_{\sigma \in \mathcal{D}} \sup \inf_{\nu \in \mathcal{F}} (\sigma \circ \sigma)\langle \nu \rangle;$

2. $\inf_{\sigma \in \mathcal{D}} \sup \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \inf_{\nu \in \mathcal{F}} \sigma\langle \nu \rangle = \inf_{\sigma \in \mathcal{D}} \sup \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \inf_{\nu \in \mathcal{G}} (\sigma \circ \sigma)\langle \nu \rangle;$

3. $\inf_{\sigma\in\mathcal{D}} \sup \inf_{\nu\in\mathcal{F}} \sigma\langle\nu\rangle = \inf_{\sigma\in\mathcal{D}} \sup \operatorname{adh}_{\mathcal{D}} \sigma\langle\mathcal{F}\rangle$;

4. $\inf_{\sigma\in\mathcal{D}} \sup \inf_{\mathcal{G}\in\mathcal{P}_m(\mathcal{F})} \inf_{\nu\in\mathcal{G}} \sigma\langle\nu\rangle = \inf_{\sigma\in\mathcal{D}} \sup \inf_{\mathcal{G}\in\mathcal{P}_m(\mathcal{F})} \operatorname{adh}_{\mathcal{D}} \sigma\langle\mathcal{G}\rangle$

In view of the last lemma, if $(X,\mathcal{D})$ is an I-fuzzy uniform space and $\mathcal{F}$ is a filter on I^X, then

$$\mathcal{F} \text{ is } \mathcal{D}\text{-Cauchy} \iff \bar{c}(\mathcal{F}) \leq \inf_{\sigma\in\mathcal{D}} \sup \inf_{\mathcal{G}\in\mathcal{P}_m(\mathcal{F})} \operatorname{adh}_{\mathcal{D}} \sigma\langle\mathcal{G}\rangle.$$

If $\mathcal{F}$ is prime then

$$\mathcal{F} \text{ is } \mathcal{D}\text{-Cauchy} \iff c(\mathcal{F}) \leq \inf_{\sigma\in\mathcal{D}} \sup \operatorname{adh}_{\mathcal{D}} \sigma\langle\mathcal{F}\rangle.$$

We have an α-level theorem for Cauchy filters.

Theorem 3.6 *Let $(X,\mathcal{D})$ be an I-fuzzy uniform space and let $\mathcal{F}$ be a filter on I^X with $\bar{c}(\mathcal{F}) = \bar{c} > 0$ and $c(\mathcal{F}) = c$. Then*

$$\begin{aligned} \mathcal{F} \textit{ is } \mathcal{D}\textit{-Cauchy} &\iff \mathcal{F}_0 \textit{ is } \mathcal{D}^{\bar{c}}\textit{-Cauchy} \\ &\iff \forall\alpha < \bar{c},\ \mathcal{F}_0 \textit{ is } \mathcal{D}^{\alpha}\textit{-Cauchy.} \end{aligned}$$

Furthermore, if $\mathcal{F}$ is prime then

$$\begin{aligned} \mathcal{F} \textit{ is } \mathcal{D}\textit{-Cauchy} &\iff \mathcal{F}_0 \textit{ is } \mathcal{D}^{c}\textit{-Cauchy} \\ &\iff \forall\alpha < c,\ \mathcal{F}_0 \textit{ is } \mathcal{D}^{\alpha}\textit{-Cauchy.} \end{aligned}$$

In view of the hypothesis of the previous theorem, it is important to note that the condition: $0 < \bar{c}(\mathcal{F})$ is non-trivial. For example, in the case of an atomless complete Boolean algebra $\mathbb{B}$, every filter $\mathcal{F}$ on $\mathbb{B}^X$ fulfils the condition $\bar{c}(\mathcal{F}) = 0$.

Corollary 3.7 *Let $(X,\mathbb{D})$ be a uniform space and let $\mathbb{F}$ be a filter on X. Then*

$$\mathbb{F} \textit{ is } \mathbb{D}\textit{-Cauchy} \iff \mathbb{F}^1 \textit{ is } \mathbb{D}^1\textit{-Cauchy.}$$

The last result confirms that we have a w_u-extension of the standard notion.

Corollary 3.8 *Let $(X,\mathcal{D})$ be an I-fuzzy uniform space, $\mathbb{F}$ a filter on X and $\alpha > 0$. Then*

$$\mathbb{F} \textit{ is } \mathcal{D}^{\alpha}\textit{-Cauchy} \iff \mathbb{F}_{\alpha} \textit{ is } \mathcal{D}\textit{-Cauchy.}$$

Definition 3.9 *If $(X,\mathcal{D})$ is an I-fuzzy uniform space and $\mathcal{F}$ is a filter on I^X then we say that*

$$\mathcal{F} \textit{ is } \text{strong } \mathcal{D}\text{-Cauchy} \iff \forall\sigma\in\mathcal{D},\ \forall\varepsilon > 0,\ \exists\nu\in\mathcal{F},\ \nu\times\nu \leq \sigma + \varepsilon.$$

This is the most "natural" generalisation of the notion of a Cauchy filter. However, it is not the same as the notion that we have just defined. The terminology is justified by the following theorem.

Theorem 3.10 *If $(X, \mathcal{D})$ is an I-fuzzy uniform space and $\mathcal{F}$ is a filter on I^X then*

$$\mathcal{F} \text{ is strong } \mathcal{D}\text{-Cauchy } \Rightarrow \mathcal{F} \text{ is } \mathcal{D}\text{-Cauchy.}$$

Moreover, if $\mathcal{F}$ is prime then

$$\mathcal{F} \text{ is strong } \mathcal{D}\text{-Cauchy } \iff \mathcal{F} \text{ is } \mathcal{D}\text{-Cauchy.}$$

So the two notions coincide for prime filters. Another example of this coincidence is provided by the following theorem.

Theorem 3.11 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space, $\mathbb{F}$ a filter on X and $\alpha > 0$. Then*

1. *$\mathbb{F}$ is $\mathcal{D}^\alpha$-Cauchy $\iff$ $\mathbb{F}_\alpha$ is strong $\mathcal{D}$-Cauchy.*

2. *$\mathbb{F}_\alpha$ is $\mathcal{D}$-Cauchy $\iff$ $\mathbb{F}_\alpha$ is strong $\mathcal{D}$-Cauchy.*

Strong Cauchy filters are stable under refinements:

Lemma 3.12 *If $\mathcal{F}$ is a strong Cauchy filter and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G}$ is also strong Cauchy.*

The notion of a strong Cauchy filter on I^X is also a w_u-extension of the notion of a Cauchy filter on X:

Theorem 3.13 *Let $(X, \mathbb{D})$ be a uniform space and $\mathbb{F}$ a filter on X. Then*

$$\mathbb{F} \text{ is } \mathbb{D}\text{-Cauchy } \iff \mathbb{F}^1 \text{ is strong } \mathbb{D}^1\text{-Cauchy.}$$

We establish a necessary condition for a filter to be strong Cauchy.

Theorem 3.14 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space, $\mathcal{F}$ a filter on I^X, $c(\mathcal{F}) = c$ and $\bar{c}(\mathcal{F}) = \bar{c} > 0$. Then*

1. *$\mathcal{F}$ is strong $\mathcal{D}$-Cauchy $\Rightarrow$ $\mathcal{F}^c$ is $\mathcal{D}^c$-Cauchy;*

2. *$\mathcal{F}$ is strong $\mathcal{D}$-Cauchy $\Rightarrow$ $\mathcal{F}^{\bar{c}}$ is $\mathcal{D}^{\bar{c}}$-Cauchy.*

Theorem 3.14 can be used to prove the following result.

Lemma 3.15 *There exists an I-fuzzy uniform space $(X, \mathcal{D})$ and a filter $\mathcal{F}$ on I^X which is*

1. *$\mathcal{D}$-Cauchy but not strong $\mathcal{D}$-Cauchy;*

2. *$\mathcal{D}$-Cauchy and there is a filter $\mathcal{G} \supseteq \mathcal{F}$, which is not $\mathcal{D}$-Cauchy.*

Lemma 3.15 shows that the two notions are not the same. However, in the sequel we shall see that completeness and precompactness can be described in terms of *prime* filters and therefore, since these two different notions coincide for prime filters, they both generate the same theory.

We show, in the following theorems, that the basic theory of Cauchy filters on X carries over to this setting. Convergent filters are Cauchy and the same is true here.

Theorem 3.16 *Let $(X,\mathcal{D})$ be an I-fuzzy uniform space, $\mathcal{F}$ a filter on I^X and $\mu \in I^X$. Then*

$$\mathcal{F} \text{ is } \mathcal{D}\text{-convergent in } \mu \;\Rightarrow \mathcal{F} \text{ is } \mathcal{D}\text{-Cauchy.}$$

The following theorems are fuzzy analogues of standard theory.

Theorem 3.17 *Let $\mathcal{D}$ and $\mathcal{E}$ be I-fuzzy uniformities on a set X with $\mathcal{D} \subseteq \mathcal{E}$. Let $\mathcal{F}$ be filter on I^X. Then*

$$\mathcal{F} \text{ is } \mathcal{E}\text{-Cauchy } \Rightarrow \mathcal{F} \text{ is } \mathcal{D}\text{-Cauchy.}$$

Theorem 3.18 *Let $(X,\mathcal{D})$ be an I-fuzzy uniform space, $\mathcal{F}$ and $\mathcal{G}$ prime filters on I^X with $\mathcal{F} \subseteq \mathcal{G}$. Then*

$$\mathcal{F} \text{ is } \mathcal{D}\text{-Cauchy } \Rightarrow \mathcal{G} \text{ is } \mathcal{D}\text{-Cauchy.}$$

Theorem 3.19 *Let $(X,\mathcal{D})$ and $(Y,\mathcal{E})$ be I-fuzzy uniform spaces, $f : X \to Y$ a uniformly continuous function and $\mathcal{F}$ a prime $\mathcal{D}$-Cauchy filter on I^X. Then $f[\mathcal{F}]$ is a prime $\mathcal{E}$-Cauchy filter base on I^Y.*

Theorem 3.20 *Let $((X_j,\mathcal{D}_j) : j \in J)$ be a family of I-fuzzy uniform spaces. Let $X \overset{\text{def}}{=} \prod_{j\in J} X_j$, $\mathcal{D} \overset{\text{def}}{=} \prod_{j\in J} \mathcal{D}_j$ and let $\mathcal{F}$ be a prime filter on I^X. Then*

$$\mathcal{F} \text{ is } \mathcal{D}\text{-Cauchy } \iff \forall j \in J,\ p_j[\mathcal{F}] \text{ is } \mathcal{D}_j\text{-Cauchy.}$$

We therefore have an extension of the notion of a Cauchy filter which generates a theory which is analogous to standard theory. With this in place, we can assemble the theory of precompact, complete and bounded fuzzy subsets.

4 Precompactness

We follow [17] and define compactness for fuzzy subsets as follows.

Definition 4.1 *If $(X,\mathcal{D})$ is an I-fuzzy uniform space and $\mu \in I^X$ we say:*

$$\mu \text{ is } \mathcal{D}\text{-compact} \iff \text{for each filter } \mathcal{F} \text{ on } I^X, c(\mathcal{F},\mu) \le \sup(\mu \wedge \operatorname{adh}\mathcal{F}).$$

Equivalently

$$\mu \text{ is } \mathcal{D}\text{-compact} \iff \begin{cases} \text{for each filter } \mathcal{F} \text{ on } I^X \text{ with } \mu \in \mathcal{F}, \\ c(\mathcal{F}) \le \sup(\mu \wedge \operatorname{adh} \mathcal{F}). \end{cases}$$

The next result, which is from [17], characterises compactness in terms of prime filters.

Theorem 4.2 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space and let $\mu \in I^X$. Then*

$$\mu \textit{ is } \mathcal{D}\textit{-compact} \iff \begin{cases} \textit{for each prime filter } \mathcal{F} \textit{ on } X, \\ c(\mathcal{F}, \mu) \le \sup(\mu \wedge \operatorname{adh} \mathcal{F}). \end{cases}$$

Corollary 4.3 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space and let $\mu \in I^X$. Then*

$$\mu \textit{ is } \mathcal{D}\textit{-compact} \iff \begin{cases} \textit{for each prime filter } \mathcal{F} \textit{ with } \mu \in \mathcal{F}, \\ c(\mathcal{F}) \le \sup(\mu \wedge \operatorname{adh} \mathcal{F}). \end{cases}$$

Definition 4.4 *If $(X, \mathbb{D})$ is a uniform space and $A \subseteq X$ we let $\wp_f(X)$ denote the collection of finite subsets of X and we say that*

$$A \textit{ is } \mathbb{D}\textit{-precompact} \iff \forall U \in \mathbb{D},\ \exists F \in \wp_f(X),\ A \subseteq U(F) \overset{\text{def}}{=} \bigcup_{x \in F} U(x).$$

Equivalently:

Theorem 4.5 *A is $\mathbb{D}$-precompact iff every ultrafilter $\mathbb{F}$ with $A \in \mathbb{F}$ is $\mathbb{D}$-Cauchy.*

We use Theorem 4.5 as the basis for the definition of a precompact fuzzy subset.

Definition 4.6 *If $(X, \mathcal{D})$ is an I-fuzzy uniform space and $\mu \in I^X$ we say that*

*μ is $\mathcal{D}$-*precompact *iff every prime filter $\mathcal{F}$ on I^X with $\mu \in \mathcal{F}$ is $\mathcal{D}$-Cauchy.*

We first check that the definition forms a w_u-extension of the crisp notion.

Theorem 4.7 *Let $(X, \mathbb{D})$ be a uniform space with $A \subseteq X$. Then*

$$A \textit{ is } \mathbb{D}\textit{-precompact} \iff 1_A \textit{ is } \mathbb{D}^1\textit{-precompact.}$$

The proof of this theorem, and all results in this section, can be found in [11].

The next theorem characterises precompactness in terms of prime filters and furthermore, in terms of certain types of prime filters.

Theorem 4.8 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space with $\mu \in I^X$. Then the following are equivalent.*

1. *μ is $\mathcal{D}$-precompact;*

2. *for every prime filter* $\mathcal{F}$ *on* I^X, $c(\mathcal{F},\mu) \leq \inf_{\sigma\in\mathcal{D}} \sup \operatorname{adh}\sigma\langle\mathcal{F}\rangle$;

3. *for every prime filter* $\mathcal{F}$ *on* I^X, $c(\mathcal{F},\mu) \leq \inf_{\sigma\in\mathcal{D}} \sup \inf_{\nu\in\mathcal{F}} \sigma\langle\nu\rangle$;

4. *for every prime filter* $\mathcal{F}$ *on* I^X *with* $\mu \in \mathcal{F}$, $c(\mathcal{F}) \leq \inf_{\sigma\in\mathcal{D}} \sup \inf_{\nu\in\mathcal{F}} \sigma\langle\nu\rangle$;

5. *for every prime filter* $\mathcal{F}$ *on* I^X *with* $c(\mathcal{F}) = 1$, $c(\mathcal{F},\mu) \leq \inf_{\sigma\in\mathcal{D}} \sup \operatorname{adh}\sigma\langle\mathcal{F}\rangle$;

6. *for every prime, 1-filter* $\mathcal{F}$ *on* I^X, $c(\mathcal{F},\mu) \leq \inf_{\sigma\in\mathcal{D}} \sup \operatorname{adh}\sigma\langle\mathcal{F}\rangle$;

7. *every prime, saturated filter* $\mathcal{F}$ *with* $\mu \in \mathcal{F}$ *is* $\mathcal{D}$*-Cauchy.*

Next we find an α-level theorem for precompactness.

Theorem 4.9 *Let* $(X,\mathcal{D})$ *be an* I*-fuzzy uniform space and let* $\mu \in I^X$. *Then*

1. μ *is* $\mathcal{D}$*-precompact* $\iff \forall\alpha \in I_0,\ \mu_\alpha$ *is* $\mathcal{D}^\alpha$*-precompact;*

2. μ *is* $\mathcal{D}$*-precompact* $\iff \forall\alpha \in I_0,\ \mu^\alpha$ *is* $\mathcal{D}^\alpha$*-precompact;*

The next theorem characterises precompactness of fuzzy subsets in a way which is reminiscent of Definition 4.4.

Theorem 4.10 *Let* $(X,\mathcal{D})$ *be an* I*-fuzzy uniform space with* $\mu \in I^X$. *Then*

$$\mu \text{ is } \mathcal{D}\text{-precompact} \iff \forall\sigma \in \mathcal{D},\ \forall\varepsilon > 0,\ \exists F \in \wp_f(X),\ \mu \leq \sup_{y\in F} \sigma\langle y\rangle + \varepsilon.$$

The basic theory of precompact sets lifts to the fuzzy setting, as evidenced by the following results. Each of these is the fuzzy analogue of a result from the theory of precompact sets.

Theorem 4.11 *Let* $(X,\mathcal{D})$ *be an* I*-fuzzy uniform space,* $n \in \mathbb{N}, [n] \stackrel{\text{def}}{=} \{1,2,\ldots,n\}$ *and* $\mu,\nu,\nu_i \in I^X$. *Then*

1. $\nu \leq \mu$ *and* μ *is* $\mathcal{D}$*-precompact* $\Rightarrow \nu$ *is* $\mathcal{D}$*-precompact;*

2. $\forall i \in [n],\ \nu_i$ *is* $\mathcal{D}$*-precompact* $\Rightarrow \sup_{i\in[n]} \nu_i$ *is* $\mathcal{D}$*-precompact;*

3. μ *is* $\mathcal{D}$*-compact* $\Rightarrow \mu$ *is* $\mathcal{D}$*-precompact.*

Theorem 4.12 *Let* $(X,\mathcal{D})$ *be an* I*-fuzzy uniform space with* $\mu \in I^X$. *Then*

$$\mu \text{ is } \mathcal{D}\text{-precompact} \Rightarrow \bar{\mu} \text{ is } \mathcal{D}\text{-precompact.}$$

Theorem 4.13 *Let* $(X,\mathcal{D})$ *and* $(Y,\mathcal{E})$ *be* I*-fuzzy uniform spaces,* $\mu \in I^X$ *and let* $f : X \to Y$ *be uniformly continuous. Then*

$$\mu \text{ is } \mathcal{D}\text{-precompact} \Rightarrow f[\mu] \text{ is } \mathcal{E}\text{-precompact.}$$

Theorem 4.14 *Let $((X_j, \mathcal{D}_j) : j \in J)$ be a family of I-fuzzy uniform spaces with $\mu_j \in I^{X_j}$ for each $j \in J$. Let $X \stackrel{\text{def}}{=} \prod_{j \in J} X_j$ and $\mathcal{D} \stackrel{\text{def}}{=} \prod_{j \in J} \mathcal{D}_j$. Then*

1. $\forall j \in J,\ \mu_j$ *is* $\mathcal{D}_j$*-precompact* $\Rightarrow \prod_{j \in J} \mu_j$ *is* $\mathcal{D}$*-precompact;*

2. $\mu \in I^X$ *is* $\mathcal{D}$*-precompact* $\Rightarrow \forall j \in J,\ p_j[\mu]$ *is* $\mathcal{D}_j$*-precompact.*

5 Boundedness

We recall the definition of boundedness of subsets of a uniform space.

Definition 5.1 *If $(X, \mathbb{D})$ is a uniform space, $U \in \mathbb{D}$, $K \in \wp_f(X)$ and $B \subseteq X$, we define*

$$U^n(K) \stackrel{\text{def}}{=} \bigcup_{x \in K} U^n(x)$$

and we say that

$$B \text{ is } \mathbb{D}\text{-bounded} \iff \forall U \in \mathbb{D},\ \exists K \in \wp_f(X),\ \exists n \in \mathbb{N},\ B \subseteq U^n(K).$$

It follows immediately that a precompact set is bounded. We show that we can obtain a characterisation of boundedness in terms of filters.

Definition 5.2 *If $\mathbb{F}$ is an ultrafilter on a set X we say that*

$$\mathbb{F} \text{ is weak } \mathbb{D}\text{-Cauchy} \iff \forall U \in \mathbb{D},\ \exists n \in \mathbb{N},\ \exists F \in \mathbb{F},\ F \times F \subseteq U^n.$$

We show that weak Cauchy filters are characterised by a condition which is a weakening of the condition in Theorem 3.2 which characterises Cauchy filters.

Theorem 5.3 *Let $(X, \mathbb{D})$ be a uniform space and $\mathbb{F}$ an ultrafilter on X. Then*

$$\mathbb{F} \text{ is weak } \mathbb{D}\text{-Cauchy} \iff \forall U \in \mathbb{D},\ \exists n \in \mathbb{N},\ \bigcap_{F \in \mathbb{F}} U^n(F) \neq \emptyset.$$

The proof of this theorem, and all other results in this section, can be found in [12]. We show next that bounded sets, like precompact sets (Theorem 4.5), can be characterised in terms of ultrafilters.

Theorem 5.4 *Let $(X, \mathbb{D})$ be a uniform space with $B \subseteq X$. Then*

$$B \text{ is } \mathbb{D}\text{-bounded} \iff \text{every ultrafilter } \mathbb{F} \text{ with } B \in \mathbb{F} \text{ is weak } \mathbb{D}\text{-Cauchy.}$$

We can use Theorem 5.4 to validate elementary properties of bounded sets.

Theorem 5.5 *Let $(X, \mathbb{D})$ be a uniform space, $A, B, B_i \subseteq X, n \in \mathbb{N}$. Then*

1. $A \subseteq B$, B *is* $\mathbb{D}$*-bounded* $\Rightarrow A$ *is* $\mathbb{D}$*-bounded;*

2. $\forall i \in [n]$, B_i *is* $\mathbb{D}$*-bounded* $\Rightarrow \bigcup_{i\in[n]} B_i$ *is* $\mathbb{D}$*-bounded;*

3. B *is* $\mathbb{D}$*-precompact* $\Rightarrow B$ *is* $\mathbb{D}$*-bounded;*

4. B *is* $\mathbb{D}$*-bounded* $\Rightarrow \bar{B}$ *is* $\mathbb{D}$*-bounded.*

We now extend these notions to the fuzzy setting.

Definition 5.6 *If* $(X, \mathcal{D})$ *is an* I*-fuzzy uniform space and* $\mathcal{F}$ *is a prime filter on* I^X *we say that*

$$\mathcal{F} \text{ is weak } \mathcal{D}\text{-Cauchy} \iff c(\mathcal{F}) \leq \inf_{\sigma\in\mathcal{D}} \sup \sup_{n\in\mathbb{N}} \inf_{\nu\in\mathcal{F}} \sigma^n\langle\nu\rangle.$$

It follows immediately that a $\mathcal{D}$-Cauchy prime filter on I^X is weak $\mathcal{D}$-Cauchy. We first establish an α-level theorem for weak $\mathcal{D}$-Cauchy prime filters on I^X.

Theorem 5.7 *Let* $(X, \mathcal{D})$ *be an* I*-fuzzy uniform space and let* $\mathcal{F}$ *be a prime filter on* I^X *with* $c(\mathcal{F}) = c$*. Then*

$$\mathcal{F} \text{ is weak } \mathcal{D}\text{-Cauchy} \iff \mathcal{F}_0 \text{ is weak } \mathcal{D}^c\text{-Cauchy}.$$

The following characterisation is the analogue of Definition 5.2.

Theorem 5.8 *Let* $(X, \mathcal{D})$ *be an* I*-fuzzy uniform space and let* $\mathcal{F}$ *be a prime filter on* I^X*. Then*

$$\mathcal{F} \text{ is weak } \mathcal{D}\text{-Cauchy} \iff \forall\sigma \in \mathcal{D},\ \exists\varepsilon > 0,\ \exists\nu \in \mathbb{N},\ \nu \times \nu \leq \sigma^n + \varepsilon.$$

We check that the definition is a w_u-extension of the crisp notion.

Theorem 5.9 *Let* $(X, \mathbb{D})$ *be a uniform space and let* $\mathbb{F}$ *be an ultrafilter on* X*. Then*

$$\mathbb{F} \text{ is weak } \mathbb{D}\text{-Cauchy} \iff \mathbb{F}^1 \text{ is weak } \mathbb{D}^1\text{-Cauchy}.$$

We note some elementary properties of weak-Cauchy filters.

Theorem 5.10 *Let* $(X, \mathcal{D})$ *and* $(Y, \mathcal{E})$ *be* I*-fuzzy uniform spaces. Further let* $f : (X, \mathcal{D}) \to (Y, \mathcal{E})$ *be uniformly continuous and let* $\mathcal{F}$ *be a prime weak* $\mathcal{D}$*-Cauchy filter on* I^X*. Then* $\langle f[\mathcal{F}]\rangle$ *is a prime weak* $\mathcal{E}$*-Cauchy filter on* I^Y*.*

Theorem 5.11 *Let* $((X_j, \mathcal{D}_j) : j \in J)$ *be a family of* I*-fuzzy uniform spaces,* $X \stackrel{\text{def}}{=} \prod_{j\in J} X_j$, $\mathcal{D} \stackrel{\text{def}}{=} \prod_{j\in J} \mathcal{D}_j$ *and* $\mathcal{F}$ *a prime filter on* I^X*. Then*

$$\mathcal{F} \text{ is weak } \mathcal{D}\text{-Cauchy} \iff \forall j \in J,\ p_j[\mathcal{F}] \text{ is weak } \mathcal{D}_j\text{-Cauchy}.$$

We now extend the notion of boundedness to the fuzzy setting.

Definition 5.12 *If $(X, \mathcal{D})$ is an I-fuzzy uniform space we define $\mu \in I^X$ to be* $\mathcal{D}$-bounded $\iff \forall \varepsilon > 0,\ \forall \sigma \in \mathcal{D},\ \exists F \in \wp_f(X),\ \exists n \in \mathbb{N},\ \mu \leq \sup_{x \in F} \sigma^n\langle x \rangle + \varepsilon.$

This definition is a w_u-extension of the notion of bounded sets:

Theorem 5.13 *Let $(X, \mathbb{D})$ be a uniform space and let $B \subseteq X$. Then*

$$B \text{ is } \mathbb{D}\text{-bounded} \iff 1_B \text{ is } \mathbb{D}^1\text{-bounded.}$$

Next we establish the basic theory of bounded fuzzy subsets.

Theorem 5.14 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space, $m \in \mathbb{N}$ and let $\mu, \nu, \nu_i \in I^X$. Then*

1. *$\nu \leq \mu$ and μ is $\mathcal{D}$-bounded $\Rightarrow \nu$ is $\mathcal{D}$-bounded;*
2. *$\forall i \in [m]$, ν_i is $\mathcal{D}$-bounded $\Rightarrow \sup_{i \in [m]} \nu_i$ is $\mathcal{D}$-bounded;*
3. *μ is $\mathcal{D}$-precompact $\Rightarrow \mu$ is $\mathcal{D}$-bounded.*

Theorem 5.15 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space and let $\mu \in I^X$. Then*

$$\mu \text{ is } \mathcal{D}\text{-bounded} \Rightarrow \bar{\mu} \text{ is } \mathcal{D}\text{-bounded.}$$

Theorem 5.16 *Let $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ be I-fuzzy uniform spaces and let $f : (X, \mathcal{D}) \to (Y, \mathcal{E})$ be uniformly continuous. Then*

$$\mu \text{ is } \mathcal{D}\text{-bounded} \Rightarrow f[\mu] \text{ is } \mathcal{E}\text{-bounded.}$$

The next theorem is the fuzzy analogue of Theorem 5.4

Theorem 5.17 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space and let $\mu \in I^X$. Then*

$$\mu \text{ is } \mathcal{D}\text{-bounded} \iff \text{every prime filter } \mathcal{F} \text{ with } \mu \in \mathcal{F} \text{ is weak } \mathcal{D}\text{-Cauchy.}$$

We have an α-level theorem for bounded fuzzy subsets:

Theorem 5.18 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space and let $\mu \in I^X$. Then*

$$\mu \text{ is } \mathcal{D}\text{-bounded} \iff \forall \alpha \in I_0,\ \mu_\alpha \text{ is } \mathcal{D}^\alpha\text{-bounded.}$$

Boundedness of fuzzy subsets can be characterised in terms of prime filters, as the following theorem shows.

Theorem 5.19 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space with $\mu \in I^X$. Then the following are equivalent.*

1. *μ is $\mathcal{D}$-bounded;*
2. *for every prime filter $\mathcal{F}$ on I^X the filter $(\mathcal{F}, \mu)$ is weak $\mathcal{D}$-Cauchy;*

3. *for every prime filter* $\mathcal{F}$, $c(\mathcal{F},\mu) \le \inf_{\sigma\in\mathcal{D}} \sup \sup_{n\in\mathbb{N}} \inf_{\nu\in\mathcal{F}} \sigma^n\langle\nu\rangle$;

4. *for every prime filter* $\mathcal{F}$ *on* I^X *with* $c(\mathcal{F}) = 1$, $c(\mathcal{F},\mu) \le \inf_{\sigma\in\mathcal{D}} \sup \sup_{n\in\mathbb{N}} \inf_{\nu\in\mathcal{F}} \sigma^n\langle\nu\rangle$;

5. *for every prime, 1-filter on* I^X, $c(\mathcal{F},\mu) \le \inf_{\sigma\in\mathcal{D}} \sup \sup_{n\in\mathbb{N}} \inf_{\nu\in\mathcal{F}} \sigma^n\langle\nu\rangle$;

6. *every prime, saturated filter* $\mathcal{F}$ *on* I^X *with* $\mu \in \mathcal{F}$ *is weak* $\mathcal{D}$*-Cauchy.*

Finally, we record the behaviour of products of bounded fuzzy subsets.

Theorem 5.20 *Let* $((X_j, \mathcal{D}_j) : j \in J)$ *be a family of* I*-fuzzy uniform spaces with* $\mu_j \in I^{X_j}$ *for each* $j \in J$. *Let* $X \stackrel{\text{def}}{=} \prod_{j\in J} X_j$ *and let* $\mathcal{D} \stackrel{\text{def}}{=} \prod_{j\in J} \mathcal{D}_j$. *Then*

1. $\forall j \in J$, μ_j *is* $\mathcal{D}_j$*-bounded* $\Rightarrow \prod_{j\in J} \mu_j$ *is* $\mathcal{D}$*-bounded;*

2. $\mu \in I^X$ *is* $\mathbb{D}$*-bounded* $\iff \forall j \in J$, $p_j[\mu]$ *is* $\mathcal{D}_j$*-bounded.*

6 Completeness.

We recall the definition of completeness.

Definition 6.1 *If* $(X, \mathbb{D})$ *is a uniform space and* $A \subseteq X$ *then* A *is* $\mathbb{D}$-complete *iff every* $\mathbb{D}$*-Cauchy filter* $\mathbb{F}$ *on* X *is convergent in* A.

With this in mind, we define completeness of fuzzy subsets of an I-fuzzy uniform space as follows.

Definition 6.2 *If* $(X, \mathcal{D})$ *is an* I*-fuzzy uniform space and* $\mu \in I^X$ *then we define* μ *to be* $\mathcal{D}$-complete *iff every* $\mathcal{D}$*-Cauchy filter* $\mathcal{F}$ *on* I^X *is convergent in* μ.

We first note that completeness can be characterised in terms of prime filters.

Theorem 6.3 *Let* $(X, \mathcal{D})$ *be an* I*-fuzzy uniform space with* $\mu \in I^X$. *Then* μ *is* $\mathcal{D}$*-complete iff every prime* $\mathcal{D}$*-Cauchy filter* $\mathcal{G}$ *on* I^X *with* $\mu \in \mathcal{G}$ *is convergent in* μ.

The proof of this theorem, and all others in this section, can be found in [10]. We next establish an α-level theorem.

Theorem 6.4 *Let* $(X, \mathcal{D})$ *be an* I*-fuzzy uniform space with* $\mu \in I^X$. *Let* $J \stackrel{\text{def}}{=} (0, \sup \mu]$. *Then the following are equivalent.*

1. *μ is $\mathcal{D}$ complete;*

2. $\forall \alpha \in J,\ \forall \beta < \alpha,\ \forall \mathbb{F},\ \mathbb{F}$ *is a* $\mathcal{D}^\alpha$ – *Cauchy filter on* $X,\ \mu_\alpha \in \mathbb{F} \Rightarrow$ $\exists x \in \mu^\beta,\ \mathbb{F} \overset{\mathcal{D}^\beta}{\to} x;$

3. $\forall \alpha \in J,\ \forall \beta < \alpha,\ \forall \mathbb{F},\ \mathbb{F}$ *is a* $\mathcal{D}^\alpha$ – *Cauchy filter on* $X,\ \mu_\alpha \in \mathbb{F} \Rightarrow$ $\exists x \in \mu_\beta,\ \mathbb{F} \overset{\mathcal{D}^\beta}{\to} x;$

4. $\forall \alpha \in J,\ \forall \beta < \alpha,\ \forall \mathbb{F},\ \mathbb{F}$ *is a* $\mathcal{D}^\alpha$ – *Cauchy ultrafilter on* $X,\ \mu_\alpha \in \mathbb{F} \Rightarrow$ $\exists x \in \mu^\beta,\ \mathbb{F} \overset{\mathcal{D}^\beta}{\to} x;$

5. $\forall \alpha \in J,\ \forall \beta < \alpha,\ \forall \mathbb{F},\ \mathbb{F}$ *is a* $\mathcal{D}^\alpha$ – *Cauchy ultrafilter on* $X,\ \mu_\alpha \in \mathbb{F} \Rightarrow$ $\exists x \in \mu_\beta,\ \mathbb{F} \overset{\mathcal{D}^\beta}{\to} x.$

In the presence of a Hausdorff space, the α-level theorem simplifies considerably.

Theorem 6.5 *If $(X, \mathcal{D})$ is a Hausdorff I-fuzzy uniform space and $\mu \in I^X$ then*

$$\mu \text{ is } \mathcal{D}\text{-complete} \iff \forall \alpha \in (0, \sup \mu],\ \mu_\alpha \text{ is } \mathcal{D}^\alpha\text{-complete.}$$

The next theorem shows that the Hausdorff condition in Theorem 6.5 is necessary.

Theorem 6.6 *There is an I-fuzzy uniform space $(X, \mathcal{D})$ and a $\mathcal{D}$-complete $\mu \in I^X$ such that for some $\alpha \in (0, \sup \mu]$, μ_α is not $\mathcal{D}^\alpha$-complete.*

We note some further characterisations of completeness:

Theorem 6.7 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space with $\mu \in I^X$. Then the following are equivalent.*

1. *μ is $\mathcal{D}$-complete;*

2. *for every prime filter $\mathcal{F}$ on I^X, $(\mathcal{F}, \mu)$ is $\mathcal{D}$-Cauchy $\Rightarrow$ $(\mathcal{F}, \mu)$ is convergent in μ;*

3. *for every prime filter $\mathcal{F}$ on I^X, $(\mathcal{F}, \mu)$ is $\mathcal{D}$-Cauchy $\Rightarrow$ $c(\mathcal{F}, \mu) \leq \sup \mu \wedge \operatorname{adh} \mathcal{F}$;*

4. *every prime $\mathcal{D}$-Cauchy saturated filter $\mathcal{F}$ on I^X with $\mu \in I^X$ is $\mathcal{D}$-convergent in μ.*

We can characterise completeness in terms of strong-Cauchy filters:

Theorem 6.8 *Let$(X, \mathcal{D})$ be an I-fuzzy uniform space and $\mu \in I^X$. Then the following are equivalent.*

1. *μ is $\mathcal{D}$-complete;*

2. *for every strong $\mathcal{D}$-Cauchy filter $\mathcal{F}$ on I^X with $\mu \in \mathcal{F}$,* $\bar{c}(\mathcal{F}) \leq \sup(\mu \wedge \lim \mathcal{F})$;

3. *for every strong $\mathcal{D}$-Cauchy filter $\mathcal{F}$ on I^X with $\mu \in \mathcal{F}$,* $c(\mathcal{F}) \leq \sup(\mu \wedge \operatorname{adh} \mathcal{F})$.

We next check that the definition is a w_u-extension of the crisp notion.

Theorem 6.9 *Let $(X, \mathbb{D})$ be a uniform space with $A \subseteq X$. Then*

$$A \text{ is } \mathbb{D}\text{-complete} \iff 1_A \text{ is } \mathbb{D}^1\text{-complete.}$$

We now develop the theory of complete fuzzy subsets of an I-fuzzy uniform space. The celebrated theorem mentioned at the beginning has a fuzzy analogue, as recorded in the next theorem.

Theorem 6.10 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space with $\mu \in I^X$. Then*

$$\mu \text{ is } \mathcal{D}\text{-compact} \iff \mu \text{ is } \mathcal{D}\text{-complete and } \mathcal{D}\text{-precompact.}$$

Closed subsets of complete sets are complete. We have the same situation here.

Theorem 6.11 *Let $(X, \mathbb{D})$ be an I-fuzzy uniform space with $\mu, \nu \in I^X$. Then*

$$\mu \text{ is } \mathcal{D}\text{-complete and } \nu \text{ is } \mathcal{D}\text{-closed} \Rightarrow \mu \wedge \nu \text{ is } \mathcal{D}\text{-complete.}$$

A finite union of complete sets is complete. This result also carries over.

Theorem 6.12 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space, K a finite set and $\mu_i \in I^X$ for each $i \in K$. Then*

$$\forall i \in K,\ \mu_i \text{ is } \mathcal{D}\text{-complete} \Rightarrow \sup_{i \in K} \mu_i \text{ is } \mathcal{D}\text{-complete.}$$

Complete subsets of Hausdorff spaces are closed. This theorem, too, has its counterpart.

Theorem 6.13 *If $(X, \mathcal{D})$ is a Hausdorff I-fuzzy uniform space and $\mu \in I^X$ is $\mathcal{D}$-complete then μ is $\mathcal{D}$-closed.*

Corollary 6.14 *Let $(X, \mathcal{D})$ be a Hausdorff I-fuzzy uniform space with $\mu, \nu \in I^X$. Then*

1. *if μ, ν are $\mathcal{D}$-complete then $\mu \wedge \nu$ is $\mathcal{D}$-complete;*

2. if μ is $\mathcal{D}$-complete and $\nu \leq \mu$ then

$$\nu \text{ is } \mathcal{D}\text{-complete} \iff \nu \text{ is } \mathcal{D}\text{-closed.}$$

Complete subsets of uniform spaces are stable with respect to the formation of closures. A similar situation pertains to I-fuzzy uniform spaces.

Theorem 6.15 *Let $(X, \mathcal{D})$ be an I-fuzzy uniform space with $\mu \in I^X$. Then*

$$\mu \text{ is } \mathcal{D}\text{-complete } \Rightarrow \bar{\mu} \text{ is } \mathcal{D}\text{-complete.}$$

Finally, a product of complete sets is complete. This result also extends to the fuzzy setting.

Theorem 6.16 *Let $((X_j, \mathcal{D}(j)) : j \in J)$ be a family of I-fuzzy uniform spaces and let $\mu(j) \in I^X$ for each $j \in J$. Let $X \overset{\text{def}}{=} \prod_{j \in J} X_j$ and let $\mathcal{D} \overset{\text{def}}{=} \prod_{j \in J} \mathcal{D}(j)$. Then*

$$\forall j \in J,\ \mu(j) \text{ is } \mathcal{D}(j)\text{-complete } \Rightarrow \prod_{j \in J} \mu(j) \text{ is } \mathcal{D}\text{-complete.}$$

We have seen that the notions of: Hausdorff, Cauchy, precompactness, boundedness and completeness can be extended from the setting of uniform spaces to that of I-fuzzy uniform spaces in such a way that the basic theorems have fuzzy analogues. The theory of I-fuzzy uniform spaces is still developing and, for further enhancements to the theory, the reader is referred to the work of: T. M. G. Ahsanullah, G. Artico, M. H. Burton, O. A. El-Tantaway, J. Gutiérrez García, K. A. Hashem, U. Höhle, A. Kandil, A. S. Mashour, R. Moresco, M. A. de Prada Vicente, R. Lowen, E. Soetens, A. Sŏstak, P. Wuyts, which can be found in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 20, 21, 24, 29, 37, 41, 42, 44, 49, 50, 51].

Recently, in [14], M. H. Burton, M. A. De Prada Vicente and J. Gutiérrez García defined the notion of a *generalised uniform space* and the category, GUS, of generalised uniform spaces is, rather surprisingly, shown to be isomorphic to the category, FUS, of I-fuzzy uniform spaces. The way in which the categories: US, of uniform spaces, FUS and GUS embed into the newly-defined category, SUS, of *super uniform spaces* [20] is detailed in [21]. The process of extending the theory of uniform spaces to the category SUS is under way.

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CHAPTER 10

Fuzzy Real Lines And Dual Real Lines As Poslat Topological, Uniform, And Metric Ordered Semirings With Unity

S. E. RODABAUGH

Introduction

Nontrivial examples of objects and morphisms are fundamentally important to establishing the credibility of a new category or discipline such as lattice-dependent or fuzzy topology; and often the justifications of the importance of certain objects and the importance of certain morphisms are intertwined. In [33], we established classes of variable-basis morphisms between different fuzzy real lines and between different dual real lines, but left untouched the issue of the canonicity of these objects. In this chapter, we attempt to demonstrate the canonicity of these spaces stemming from the interplay between arithmetic operations and underlying topological structures. We shall summarize the definitions of fuzzy addition and fuzzy multiplication on the fuzzy real lines and indicate their joint-continuity—along with that of the addition and multiplication on the usual real line—with respect to the underlying poslat topologies, as well as the quasi-uniform and uniform continuity (in the case of fuzzy addition and addition) with respect to the underlying quasi-uniform, uniform, and metric structures. These results not only help establish fuzzy topology w.r.t. objects, but enrich our understanding of traditional arithmetic operations.

We now expand on the foregoing paragraph by touching on various historical and philosophical considerations.

It is a truism that the usual real line $\mathbb{R}$ and unit interval $\mathbb{I}$ have been seminal in the development of traditional axiomatic mathematics, and these spaces have been rightly recognized as canonical or universal examples. To cite some familiar facts justifying such status for these spaces, we have from general topology that

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both spaces are absolute (and absolute neighborhood) retracts and extendors for normal spaces, and $\mathbb{I}$ is an absolute (and absolute neighborhood) retract for Tihonov spaces (though not an extendor for such spaces—this means normality is strictly stronger than Tihonov); classical algebra tells us that $\mathbb{R}$ is, up to isomorphism, the only order-complete field; and category theory indicates that $\mathbb{I}$ is a universal object in the sense that **K-HAUS-TOP** is the compact hull of $\mathbb{I}$. And there is much additional confirmation from classical analysis and applied mathematics.

¿From the earliest days of fuzzy sets it was recognized that the mathematics of fuzzy sets needed counterparts to, or generalizations of, $\mathbb{R}$ and $\mathbb{I}$. Several approaches to generalization emerged:

(1) Generalize these spaces by considering poslat (or L-) topologies on a set of "fuzzy numbers" [1–2, 4, 7, 9, 11–13, 35];

(2) Generalize these spaces by considering traditional topologies on a set of "fuzzy numbers" [5, 19–20];

(3) Generalize these spaces by considering poslat (or L-) topologies on the usual real numbers [30–31].

In regard to (1), a controversy presented itself: should fuzzy numbers be "probability density" functions on $\mathbb{R}$, with the attendant restrictions of continuity and convexity, requirements preventing the usual real numbers from being "crisp" fuzzy numbers and tying such numbers explicitly to $\mathbb{I}$ as the only allowed codomain of these numbers; or should fuzzy numbers be generalized "probability distributions" on $\mathbb{R}$, allowing any traditional number to be viewed as a crisp number or "step function" and allowing any complete quasi-monoidal lattice L as the codomain of such numbers? A further related fundamental issue: with the first viewpoint of (1), a standard poslat topology was never defined; while with the second viewpoint of (1), a standard topology (with various enrichments) was standard *ab initio* (with L a deMorgan quasi-monoidal lattice). When all these issues are considered, it is not surprising that the second viewpoint—that fuzzy numbers are "generalized probability distributions"—has completely dominated in the mathematics and mathematical foundations of fuzzy sets and fuzzy logic. The most common implementation of (1) is the L-fuzzy unit interval $\mathbb{I}(L)$ of [9] and the related L-fuzzy real line $\mathbb{R}(L)$ [4]. We point that [19–20] gives an important confirmation of the second point of view by showing the deep connections between $\mathbb{I}(L)$ and the well-known Helly spaces of traditional topology.

While the generalizations *à la* (1) are "second-order", i.e. the both the numbers are (essentially) fuzzy subsets and the topology comprises fuzzy subsets, the generalizations *à la* (2) and (3) are "first-order". To be more precise, [5] views fuzzy numbers along the lines of (1), but with $L = \mathbb{I}$, so that fuzzy numbers comprise a subset of the Tihonov cube $\mathbb{I}^{\mathbb{R}}$, in which case the fuzzy numbers can be endowed with the traditional subspace topology inherited from the Tihonov cube; so this yields the generalization *à la* (2). A closely related notion [19–20] involves the Helly spaces of traditional topology—fuzzy numbers along the lines of $\mathbb{I}(L)$ are viewed, modulo a quotient operation, as a traditional subspace of a classical product space $L^{\mathbb{I}}$, which is deeply related (via the Kubiak-Lowen functors [20]) to $\mathbb{I}(L)$.

Now the generalization *à la* (3) uses the operators which generate the poslat topology of $\mathbb{R}(L)$ to generate a "dual" poslat topology, or co-topology, on $\mathbb{R}$ and $\mathbb{I}$; such a topology is available for each deMorgan quasi-monoidal lattice L. It is shown in [30–31] that $\mathbb{R}(L)$ and $\mathbb{R}$ are essentially dual spaces.

The currently known extent to which $\mathbb{R}(L)$ successfully generalizes $\mathbb{R}$ stands thusly: $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are absolute extendors of perfectly normal L-topological spaces [16–18, 22] for L a meet-continuous deMorgan algebra, are T_0 for any deMorgan algebra and L-sober for L a complete Boolean algebra [32, 35], are L-T_2 for L a deMorgan algebra [21], are HR-regular through HR-perfectly normal for L a deMorgan frame [21, 22], and are metrizable for L a Hutton algebra [14, 27, 35]; and arithemetic operations have been developed for $\mathbb{R}(L)$ which are compatible with its canonical topology and uniform structure (the latter in the addition case) for L a complete deMorgan chain [26–29]. And $\mathbb{R}$ with each of its dual topologies possesses a full range of separation axioms as well as compatible arithmetic operations (namely the usual operations) [30–31] under various lattice-theoretic conditions on the base L.

It is the purpose of this chapter to give a plenary summary of the properties of the spaces $\mathbb{R}(L)$ and $\mathbb{R}$ with respect to arithmetic operations, including the interplay between arithmetic operations and underlying topological structures. Thus we shall summarize the definitions of fuzzy addition and fuzzy multiplication on the fuzzy real lines and indicate their joint-continuity—along with that of the usual addition and multiplication—on the usual real line with respect to the underlying standard poslat topologies and co-topologies, as well as the quasi-uniform and uniform continuity (in the case of the fuzzy addition and addition) with respect to the underlying quasi-uniform, uniform, and metric structures. These results enrich our understanding of the topological properties of the traditional arithmetic operations on $\mathbb{R}$.

It should be mentioned that new, alternative models $\mathbb{R}^*(L)$ and $\mathbb{I}^*(L)$ of the fuzzy real lines and unit intervals, in keeping with approach (1) above, have just been constructed which are fundamentally related to $\mathbb{R}(L)$ and $\mathbb{I}(L)$.

(1) The Klein fuzzification machinery of [11–13] constructs fuzzy real lines and unit intervals for all Hutton algebras L in which $\perp$ is meet-irreducible. These models are L-homeomorphic to $\mathbb{R}(L)$ and $\mathbb{I}(L)$ when L is a complete deMorgan chain. This machinery also constructs a fuzzy complex plane $(\mathbb{R} \times \mathbb{R})(L)$.

(2) The L-2-soberification functors of [35] construct fuzzy real lines $\mathbb{R}^*(L)$ and unit intervals $\mathbb{I}^*(L)$ for all semiframes L, along with jointly-continuous fuzzy additions and multiplications, which are L-homeomorphic to $\mathbb{R}(L)$ and $\mathbb{I}(L)$ when L is a complete Boolean algebra. This machinery also constructs a fuzzy complex plane $(\mathbb{R} \times \mathbb{R})^*(L)$.

The investigation of these new models is a future area of development.

Uniform continuity in the sense of Hutton may be found in [10, 14, 29]. Throughout this chapter, we follow the notations of [8, 33, 34] regarding various types of lattices and monoids, powerset operators, and L-continuity. We point out that the definitions of $\mathbb{R}(L)$, $\mathbb{I}(L)$, and their respective L-topologies can be made within the general context of **DQML**, the category of deMorgan quasi-monoidal lattices [33], but that most of the development has been in the lattice-theoretic setting (with $\otimes = \wedge$). It remains an open question as to how much of this development can be done for other choices of $\otimes$.

1 Preliminary notions of $\mathbb{R}(L)$ and $\mathbb{R}$ as poslat spaces

This section reviews basic definitions, notation, and tools fundamental to the algebraic, topological, and uniform structure of the fuzzy real lines. All other needed definitions (e.g. of various categories of lattices) can be found in [33].

The L-**fuzzy real line** $\mathbb{R}(L)$, for $L \in |\mathbf{DQML}|$ equipped with order-reversing involution $'$, is formally defined as the set of all equivalence classes $[\lambda]$ of antitone maps $\lambda : \mathbb{R} \to \mathbb{L}$ satisfying the boundary conditions

$$\bigvee_{t \in \mathbb{R}} \lambda(t) = \top, \qquad \bigwedge_{t \in \mathbb{R}} \lambda(t) = \bot$$

where the equivalence identifies two such maps λ, μ iff $\forall t \in \mathbb{R}$, $\lambda(t+) = \mu(t+)$. Where it causes no confusion, we often write λ for $[\lambda]$. The canonical overlying L-topology $\tau(L)$ is generated from the subbase

$$\{\mathfrak{L}_t, \mathfrak{R}_t : t \in \mathbb{R}\}$$

where

$$\mathfrak{L}_t : \mathbb{R}(L) \to L \quad \text{by} \quad \mathfrak{L}_t(\lambda) = (\lambda(t-))'$$

$$\mathfrak{R}_t : \mathbb{R}(L) \to L \quad \text{by} \quad \mathfrak{R}_t(\lambda) = \lambda(t+)$$

The L-**unit interval** $\mathbb{I}(L)$ is that subset of $\mathbb{R}(L)$ in which each $[\lambda]$ satisfies $\lambda(0-) = \top$ and $\lambda(1+) = \bot$, and which inherits the subspace L-topology from $\tau(L)$. See [33] for a formal treatment of subbase and subspace.

The canonical L-quasi-uniform and L-uniform structures on $\mathbb{R}(L)$ are given by the Hutton, or uniform covering approach, to fuzzy uniformities, as detailed in [10, 14, 27, 29]. We now summarize these structures on $\mathbb{R}(L)$. For L a completely distributive deMorgan algebra, i.e. a Hutton algebra, create

$$\mathcal{B} \equiv \left\{B_\varepsilon : L^{\mathbb{R}(L)} \to L^{\mathbb{R}(L)} \mid \varepsilon \in \mathbb{R},\ \varepsilon > 0\right\}$$

$$\mathcal{C} \equiv \left\{C_\varepsilon : L^{\mathbb{R}(L)} \to L^{\mathbb{R}(L)} \mid \varepsilon \in \mathbb{R},\ \varepsilon > 0\right\}$$

by

$$B_\varepsilon(a) = \mathfrak{R}_{t-\varepsilon}, \quad t = \bigvee \{s \in \mathbb{R} : a \leq (\mathfrak{L}_s)'\}$$

$$C_\varepsilon(a) = \mathfrak{L}_{t+\varepsilon}, \quad t = \bigwedge \{s \in \mathbb{R} : a \leq (\mathfrak{R}_s)'\}$$

Then $\mathcal{B}$ [$\mathcal{C}$] is a basis for the "right-handed" ["left-handed"] quasi-uniformity $\mathcal{Q}_{\mathfrak{R}}(L)$ [$\mathcal{Q}_{\mathfrak{L}}(L)$] generating the "right-handed" ["left-handed"] L-topology $\tau_{\mathfrak{R}}(L)$ [$\tau_{\mathfrak{L}}(L)$] on $\mathbb{R}(L)$, i.e. the L-topology

$$\{\underline{\perp}, \underline{\top}\} \cup \{\mathfrak{R}_t : t \in \mathbb{R}\} \quad [\{\underline{\perp}, \underline{\top}\} \cup \{\mathfrak{L}_t : t \in \mathbb{R}\}]$$

Using the Δ-operator of Definition 2.4 of [28] based on Raney's Theorem,

$$\{B_\varepsilon \Delta C_\varepsilon : \varepsilon \in \mathbb{R}, \varepsilon > 0\}$$

is the basis of the canonical uniformity $\mathcal{U}(L)$ generating the canonical L-topology $\tau(L)$ defined above. For convenience' sake we denote this basis $\mathcal{B} \Delta \mathcal{C}$.

These structures generate the canonical Erceg fuzzy pseudo-metric [3] on (the L-hyperspace of) $\mathbb{R}(L)$, which because of the L-T_0 axiom enjoyed by all fuzzy real lines [32], should be regarded as the canonical metric on $\mathbb{R}(L)$. Since $\mathcal{B} \Delta \mathcal{C}$ is a symmetric basis of $\mathcal{U}(L)$ having the property that

$$(B_\varepsilon \Delta C_\varepsilon) \circ (B_\delta \Delta C_\delta) \leq B_{\varepsilon+\delta} \Delta C_{\varepsilon+\delta}$$

the Erceg metric $d_L : L^{\mathbb{R}(L)} \times L^{\mathbb{R}(L)} \to \mathbb{R}$ on $\mathbb{R}(L)$ is defined by

$$d_L(a, b) = \bigwedge \{\varepsilon : b \leq (B_\varepsilon \Delta C_\varepsilon)(a)\}$$

If we choose characteristics of singleton subsets of crisp numbers

$$a = \chi_{\{[\lambda_r]\}}, \; b = \chi_{\{[\lambda_s]\}}$$

then

$$d_L(a, b) = |r - s|$$

So each such d_L extends the Euclidean metric on $\mathbb{R}$ as carried over to $\mathbb{R}\{\perp, \top\}$. The metric topology $\tau(d_L)$ turns to be $\tau(L)$.

The **L-dual real line** is $\mathbb{R}$ furnished with the canonical L-topology *co*-$\tau(L)$ generated from the subbase

$$\{\mathfrak{L}_\lambda, \mathfrak{R}_\lambda : [\lambda] \in \mathbb{R}(L)\}$$

where

$$\mathfrak{L}_\lambda : \mathbb{R} \to L \quad \text{by} \quad \mathfrak{L}_\lambda(t) = (\lambda(t-))'$$

$$\mathfrak{R}_\lambda : \mathbb{R} \to L \quad \text{by} \quad \mathfrak{R}_\lambda(t) = \lambda(t+)$$

As for the associated canonical L-quasi-uniform, L-uniform, and L-metric structures on $\mathbb{R}$, we need two digressions (and their notation): the embeddings of $\mathbb{R}$ into the fuzzy real lines; and the extended fuzzy real lines.

The embeddings of $\mathbb{R}$ into each of $(\mathbb{R}(L), L, \tau(L))$ and $(\mathbb{R}, L, co\text{-}\tau(L))$ for $L = \{\bot, \top\}$ are given in [33], the former sending $r \in \mathbb{R}$ to $[\lambda_r]$, where

$$\lambda_r(t) = \begin{cases} \top, & t < r \\ \bot, & t > r \end{cases}$$

If in the definition of $\mathbb{R}(L)$ we require only that representatives λ of $[\lambda]$ are antitone maps, i.e. do not require the boundary conditions, and define the (subbasis for) the canonical L-topology formally as for $\mathbb{R}(L)$, then we have a space, denoted $\mathbb{E}(L)$ or $(\mathbb{E}(L), L, \tau(L))$, that is L-homeomorphic to $\mathbb{I}(L)$. For this reason we call $\mathbb{E}(L)$ the **L-extended (fuzzy) real line**. The same ordering put on $\mathbb{R}(L)$ in the next section extends to $\mathbb{E}(L)$ and makes it a complete lattice, whereas $\mathbb{R}(L)$ is conditionally complete. Further, we can use $\mathbb{E}(L)$ to create the dual topology $co\text{-}\tau(L)$ defined above on $\mathbb{R}$: if we define a dual L-topology on $\mathbb{R}$ by using

$$\{\mathfrak{L}_\lambda, \mathfrak{R}_\lambda : [\lambda] \in \mathbb{E}(L)\}$$

as a subbasis, the resulting topology is the same as $co\text{-}\tau(L)$ defined above [30].

While the extended lines $\mathbb{E}(L)$ are of interest in their own right [30], we shall regard them only as a tool to describe the L-quasi-uniform and L-uniform structures on $\mathbb{R}$. In order to do this, we first define a specially restricted form of fuzzy addition on $\mathbb{E}(L)$.

The fuzzy addition on $\mathbb{R}(L)$ (and implicitly on $\mathbb{E}(L)$) as discussed in Section 4 below requires L a chain. However, translation of fuzzy numbers by crisp numbers in $\mathbb{E}(L)$ (and $\mathbb{R}(L)$) can be defined for any deMorgan quasi-monoidal lattice. Let $[\lambda] \in \mathbb{E}(L)$ and $[\lambda_r]$, for $r \in \mathbb{R}$, be given (recall previous paragraph), and put

$$(\lambda \oplus \lambda_r)(t) = \lambda(t - r), \quad [\lambda] \oplus [\lambda_r] = [\lambda \oplus \lambda_r]$$

This definition is a restriction of the fuzzy addition of Section 4 (as extended to $\mathbb{E}(L)$).

We can now define the L-quasi-uniform and L-uniform structures on $\mathbb{R}$. It will be seen that these are dual to their counterparts on $\mathbb{R}(L)$. For $\varepsilon > 0$, recall λ_ε is a crisp number as in the paragraphs above. For L a completely distributive deMorgan algebra, i.e. a Hutton algebra, create

$$co\text{-}\mathcal{B} \equiv \left\{B_{\lambda_\varepsilon} : L^{\mathbb{R}} \to L^{\mathbb{R}} \mid \varepsilon \in \mathbb{R},\ \varepsilon > 0\right\}$$

$$co\text{-}\mathcal{C} \equiv \left\{C_{\lambda_\varepsilon} : L^{\mathbb{R}} \to L^{\mathbb{R}} \mid \varepsilon \in \mathbb{R},\ \varepsilon > 0\right\}$$

by

$$B_{\lambda_\varepsilon}(a) = \mathfrak{R}_{\lambda \oplus \lambda_\varepsilon}, \quad \text{where } [\lambda] = \bigwedge \left\{[\mu] \in \mathbb{E}(L) : a \leq (\mathfrak{L}_\mu)'\right\}$$

$$C_{\lambda_\varepsilon}(a) = \mathfrak{L}_{\lambda \oplus \lambda_{(-\varepsilon)}}, \quad \text{where } [\lambda] = \bigvee \left\{[\mu] \in \mathbb{E}(L) : a \leq (\mathfrak{R}_\mu)'\right\}$$

Then $co\text{-}\mathcal{B}$ $[co\text{-}\mathcal{C}]$ is a basis for the canonical dual "right-handed" ["left-handed"] quasi-uniformity $co\text{-}\mathcal{Q}_{\mathfrak{R}}(L)$ $[co\text{-}\mathcal{Q}_{\mathfrak{L}}(L)]$ generating the canonical dual "right-handed" ["left-handed"] L-topology $co\text{-}\tau_{\mathfrak{R}}(L)$ $[co\text{-}\tau_{\mathfrak{L}}(L)]$ on $\mathbb{R}(L)$, i.e. the L-topology

$$\{\underline{\bot}, \underline{\top}\} \cup \{\mathfrak{R}_\lambda : [\lambda] \in \mathbb{R}(L)\} \quad [\{\underline{\bot}, \underline{\top}\} \cup \{\mathfrak{L}_\lambda : [\lambda] \in \mathbb{R}(L)\}]$$

Again, appeal to the the Δ-operator cited above shows

$$\{B_{\lambda_\varepsilon} \Delta C_{\lambda_\varepsilon} : \varepsilon \in \mathbb{R},\ \varepsilon > 0\}$$

is the basis of the canonical dual uniformity $co\text{-}\mathcal{U}(L)$ generating the canonical dual L-topology $co\text{-}\tau(L)$ defined above. For convenience' sake we denote this basis

$$(co\text{-}\mathcal{B})\ \Delta\ (co\text{-}\mathcal{C})$$

These structures generate the canonical Erceg fuzzy pseudo-metric [3] on (the L-hyperspace of) $\mathbb{R}$, which because of the L-T_0 axiom enjoyed by $\mathbb{R}$ [30]—a consequence of $\mathbb{R}$ satisfying the traditional T_0 axiom—should be regarded as the canonical L-metric on $\mathbb{R}$. Since $(co\text{-}\mathcal{B})\ \Delta\ (co\text{-}\mathcal{C})$ is a symmetric basis of $co\text{-}\mathcal{U}(L)$ having the property that

$$(B_{\lambda_\varepsilon} \Delta C_{\lambda_\varepsilon}) \circ (B_{\lambda_\delta} \Delta C_{\lambda_\delta}) \leq B_{\lambda_\varepsilon \oplus \lambda_\delta} \Delta C_{\lambda_\varepsilon \oplus \lambda_\delta}$$

the Erceg metric $d_L : L^{\mathbb{R}} \times L^{\mathbb{R}} \to \mathbb{R}$ on $\mathbb{R}$ is defined by

$$d_L(a,b) = \bigwedge \{\varepsilon : b \leq (B_{\lambda_\varepsilon} \Delta C_{\lambda_\varepsilon})(a)\}$$

If we choose characteristics of singleton subsets of real numbers r and s, i.e. choose

$$a = \chi_{\{r\}},\ b = \chi_{\{s\}}$$

then

$$d_L(a,b) = |r - s|$$

So each such d_L extends the Euclidean metric on $\mathbb{R}$; and the metric topology $\tau(d_L)$ turns to be $co\text{-}\tau(L)$.

2 Basic tools for fuzzy arithmetic operations

This section summarizes well-known tools necessary to study many algebraic and topological properties of $\mathbb{R}(L)$ [7, 11–13, 20, 26, 28, [95] in 32], including the hypergraph functor. While these tools are used to study separation, compactness, and connectedness in $\mathbb{R}(L)$, we shall be concerned only with their application to the construction and continuity of arithmetic operations. It is convenient to have the following subsets of L:

$$L^c = \{\alpha \in L : \alpha \text{ is comparable to each element of } L\}$$

$$L^b = \{\alpha \in L : \alpha < \beta \wedge \gamma \Rightarrow \alpha < \beta \text{ or } \alpha < \gamma\}$$

$$L^a = L^b \cap L^c$$

We note that the condition $\perp \in L^b$ holds iff $\perp$ is meet-irreducible.

2.1 Lemma (Representatives of fuzzy numbers). Each $[\lambda]$ contains a left-continuous representative and a right-continuous representative, denoted λ_- and λ_+, respectively. The following holds: $[\lambda] = [\mu] \Leftrightarrow \lambda_+ = \mu_+ \Leftrightarrow \lambda_- = \mu_-$.

2.2 Lemma (Partial order and order completeness). Putting $[\lambda] \le [\mu] \Leftrightarrow \lambda_+ \le \mu_+$, it follows that $[\lambda] \le [\mu] \Leftrightarrow \lambda_+ \le \mu_+ \Leftrightarrow \lambda_- \le \mu_-$. The following hold:

(1) this relation is a partial order on $\mathbb{R}(L)$;

(2) $(\bigvee_i \lambda_i)^+ = \bigvee_i \lambda_i^+$ and $(\bigwedge_i \lambda_i)^- = \bigwedge_i \lambda_i^-$;and

(3) $(\mathbb{R}(L), \le)$ is conditionally complete.

The next results summarize the fundamental link between a fuzzy number and those rays and intervals of the traditional real line which characterize that fuzzy number. This link is very powerful if L is a complete deMorgan chain.

2.3 Lemma (Fuzzy numbers and intervals). Let $\alpha, \beta \in L^a$ and $[\lambda], [\mu] \in \mathbb{R}(L)$.

(1) $\exists\, a(\lambda, \alpha) \in [-\infty, +\infty]$ such that $\forall$ left continuous member of $[\lambda]$,

$$\lambda(t) < \alpha' \Leftrightarrow t > a(\lambda, \alpha)$$

(2) $\exists\, b(\lambda, \alpha) \in [-\infty, +\infty]$ such that $\forall$ right continuous member of $[\lambda]$,

$$\lambda(t) > \alpha \Leftrightarrow t < b(\lambda, \alpha)$$

(3) The following are equivalent:

(i) $t_0 = a(\lambda, \alpha)\ [b(\lambda, \alpha)]$;

(ii) for each representative λ of $[\lambda]$,

$$\begin{array}{lcll} t > t_0\ [< t_0] & \Rightarrow & \lambda(t) < \alpha'\ [> \alpha] & \text{and} \\ t < t_0\ [> t_0] & \Rightarrow & \lambda(t) \ge \alpha'\ [\le \alpha] & \end{array}$$

(iii) for some representative λ of $[\lambda]$, the conclusion of (ii) holds.

(4) For L a complete deMorgan chain, the following are equivalent:

(i) $[\lambda] = [\le]\ [\mu]$;

(ii) $\forall \alpha \in L,\ a(\lambda, \alpha) = [\le]\ a(\mu, \alpha)$;

(iii) $\forall \alpha \in L,\ b(\lambda, \alpha) = [\le]\ b(\mu, \alpha)$.

(5) $\alpha \le \beta \Rightarrow \begin{cases} b(\lambda, \beta) \le b(\lambda, \alpha) \wedge a(\lambda, \beta'), \\ b(\lambda, \beta') \vee a(\lambda, \alpha) \le a(\lambda, \alpha). \end{cases}$

(6) $\alpha < \beta \Rightarrow a(\lambda, \beta') \le b(\lambda, \alpha)$.

(7) $\alpha \notin \{\bot, \top\} \Rightarrow a(\lambda, \bot) \le b(\lambda, \alpha) \le a(\lambda, \alpha') \le b(\lambda, \bot)$.

(8) $(b(\lambda,\alpha), a(\lambda,\alpha)) \subset \lambda^{\leftarrow}\{\alpha\} \subset [b(\lambda,\alpha), a(\lambda,\alpha)]$.

(9) For L a complete deMorgan chain, the following hold:

(i)

$$(a(\lambda,\perp), b(\lambda,\perp)) \subset \bigcup\{[b(\lambda,\alpha), a(\lambda,\alpha')] : \alpha \in L - \{\perp,\top\}\}$$

$$\subset [a(\lambda,\perp), b(\lambda,\perp)]$$

(ii) $b(\lambda,\alpha) = \bigvee\{a(\lambda,\beta') : \beta \in L,\ \alpha < \beta\}$;

(iii) $a(\lambda,\alpha') = \bigwedge\{b(\lambda,\beta) : \beta \in L,\ \beta < \alpha\}$;

(iv) $\begin{array}{lcl} \beta \searrow \alpha^{+} & \Rightarrow & a(\lambda,\beta') \nearrow (\lambda,\alpha)^{-} \quad \text{and} \\ \beta \nearrow \alpha^{-} & \Rightarrow & b(\lambda,\beta) \searrow a(\lambda,\alpha')^{+} \quad . \end{array}$

We now give results which select and restate precisely those properties most germane to the characterization and generation of fuzzy numbers.

2.4 Lemma (Operators generated by fuzzy numbers). Let L be a complete deMorgan chain and $[\lambda] \in \mathbb{R}(L)$. Then there are operators $a(\lambda,\ )$, $b(\lambda,\ )$: $L \to [-\infty,+\infty]$ satisfying the following properties:

(1) $a(\lambda,\ )$ is isotone, $b(\lambda,\ )$ is antitone;

(2) $a(\lambda,\top) = +\infty$, $b(\lambda,\top) = -\infty$;

(3) $\alpha \notin \{\perp,\top\} \Rightarrow a(\lambda,\alpha), b(\lambda,\alpha) \in (-\infty,+\infty)$;

(4) $\alpha \notin \{\perp,\top\} \Rightarrow a(\lambda,\perp) \le b(\lambda,\alpha) \le a(\lambda,\alpha') \le b(\lambda,\perp)$;

(5) $b(\lambda,\alpha) = \bigvee\{a(\lambda,\beta') : \alpha < \beta\}$, $a(\lambda,\alpha') = \bigwedge\{b(\lambda,\beta) : \beta < \alpha\}$.

2.5 Lemma (Operators generated by crisp numbers). Let $\perp \in L^{b}$. Then $[\lambda] \in \mathbb{R}\{\perp,\top\} \Leftrightarrow \exists r \in \mathbb{R},\ a(\lambda,\perp) = r = b(\lambda,\perp)$, in which case $\lambda = \lambda_r$ as classes.

2.6 Lemma (Operators generating fuzzy numbers). Let L be a complete deMorgan chain and suppose there are operators $a, b : L \to [-\infty,+\infty]$ satisfying the following properties:

(1) a is isotone, b is antitone;

(2) $a(\top) = +\infty$, $b(\top) = -\infty$;

(3) $\alpha \notin \{\perp,\top\} \Rightarrow a(\alpha), b(\alpha) \in (-\infty,+\infty)$;

(4) $\alpha \notin \{\perp,\top\} \Rightarrow a(\perp) \le b(\alpha) \le a(\alpha') \le b(\perp)$;

(5) $b(\alpha) = \bigvee \{a(\beta') : \alpha < \beta\}, a(\alpha') = \bigwedge \{b(\beta) : \beta < \alpha\}$.

If we define $\rho : [-\infty, +\infty] \to L$ by

$$\rho(t) = \begin{cases} \top, & t \le a(\bot) \\ \bigvee \{\alpha : t \in [b(\alpha), a(\alpha')]\}, & a(\bot) < t \le b(\bot) \\ \bot, & b(\bot) < t \end{cases}$$

and define $\lambda : \mathbb{R} \to L$ by

$$\lambda = \rho|_{\mathbb{R}}$$

then the following hold:

(I) ρ and λ are well-defined left-continuous maps;

(II) $[\lambda] \in \mathbb{R}(L)$;

(III) the **fundamental identities** hold:

$$\forall \alpha \in L,\ a(\lambda, \alpha) = a(\alpha) \text{ and } b(\lambda, \alpha) = b(\alpha)$$

2.7 Remark. If we replace "$\bigvee$" in the definition of ρ in Lemma 2.6 by "$\bigwedge$", then both ρ and λ become right-continuous maps equivalent to those of the Lemma, i.e. generating the same fuzzy number as a class.

2.8 Discussion (Hypergraph functor). Let L be a complete chain, and let L-**TOP** be the fixed-basis category of L-topological spaces and L-continuous maps between (in the sense of Chang-Goguen—see [8, 33] for definitions and notation). Given $(X, \tau) \in |L\text{-}\mathbf{TOP}|$, and $f \in L$-**TOP**, we define the following maps and functor, all denoted S:

$$S(X) = X \times (L - \{\top\})$$

$$S : L^X \to \mathcal{P}(S(X)) \quad \text{by} \quad S(u) = \{(x, \alpha) : u(x) > \alpha\}$$

$$S(\tau) = \{S(u) : u \in \tau\}, \quad S(X, \tau) = (S(X), S(\tau))$$

$$S(f) = f \times id_{L-\{\top\}}, \quad \text{i.e.} \quad S(f)(x, \alpha) = (f(x), \alpha)$$

Then $S : L\text{-}\mathbf{TOP} \to \mathbf{TOP}$ by these definitions is a functor with the distinctive property of **morphism-invertibility**, which we now describe. Let $(X, \tau), (Y, \sigma) \in |L\text{-}\mathbf{TOP}|$ and $f \in \mathbf{SET}(X, Y)$. Then morphism-invertibility means that

$$f : (X, \tau) \to (Y, \sigma) \text{ in } L\text{-}\mathbf{TOP} \quad \Leftrightarrow \quad S(f) : S(X, \tau) \to S(Y, \sigma) \text{ in } \mathbf{TOP}$$

Thus the hypergraph functor reduces certain issues of L-continuity to issues of traditional continuity; and it also shows the richness of L-continuity, for the associated continuity is not with respect to the traditional topologies $\{U : \chi_U \in \tau\}$,

$\{V : \chi_V \in \sigma\}$ respectively hidden inside τ, σ, but rather with respect to the much richer traditional topologies $S(\tau), S(\sigma)$.

The hypergraph functor as presented above appears in [23, 25, 32, 36] under the restriction of L being a complete chain. Attempts to weaken this restriction appear in [31], but much more significant generalizations are developed in [15]; and yet other versions are given in [8].

The above lemmas are the workhorses of this chapter, which may be regarded as an outline of their applications to certain algebraic, topological, and uniform questions concerning both $\mathbb{R}(L)$ and its dual $\mathbb{R}$.

3 $\mathbb{R}(L)$ as L-topological and L-uniform additive monoid

Throughout this section L is a complete deMorgan chain. The primary references are [26, 28].

3.1 Discussion (Interval arithematic construction of fuzzy addition). Let $[\lambda], [\mu] \in \mathbb{R}(L)$ be given. The strategy is to use these fuzzy numbers to define operators which then generate the fuzzy number which is their sum. Define $a, b : L \to [-\infty, +\infty]$ by

$$a(\alpha) = a(\lambda, \alpha) + a(\mu, \alpha), \quad b(\alpha) = b(\lambda, \alpha) + b(\mu, \alpha)$$

Then it can be shown from the operators $a(\lambda, \ ), a(\mu, \ ), b(\lambda, \ ), b(\mu, \ )$ having the properties of Lemmas 2.3–2.4 that the operators a, b satisfy the hypotheses of Lemma 2.6. So from Lemma 2.6 we obtain a well-defined map $\rho : [-\infty, +\infty] \to L$ which we then restrict to $\mathbb{R}$ to obtain a map we denote $\lambda \oplus \mu$. In accordance with Lemma 2.6, $\lambda \oplus \mu : \mathbb{R} \to L$ is a left-continuous map such that $[\lambda \oplus \mu] \in \mathbb{R}(L)$ and the **fundamental identities** obtain:

$$\begin{cases} a(\lambda \oplus \mu, \alpha) = a(\lambda, \alpha) + a(\mu, \alpha) \\ b(\lambda \oplus \mu, \alpha) = b(\lambda, \alpha) + b(\mu, \alpha) \end{cases} \tag{3.1}$$

We define **fuzzy addition** by

$$[\lambda] \oplus [\mu] = [\lambda \oplus \mu]$$

3.2 Example. Let $L = \mathbb{I} \equiv [0, 1]$ with $\alpha' = 1 - \alpha$,

$$\lambda(t) = \begin{cases} 1, & t \leq 1 \\ 2 - t, & 1 < t < 2 \\ 0, & t \geq 2 \end{cases}$$

$$\mu(t) = \begin{cases} 1, & t \le 2 \\ 5 - 2t, & 2 < t < 5/2 \\ 0, & t \ge 5/2 \end{cases}$$

Then $[\lambda] \oplus [\mu] = [\lambda \oplus \mu]$, where

$$(\lambda \oplus \mu)(t) = \begin{cases} 1, & t \le 3 \\ 3 - 2t/3, & 2 < t < 9/2 \\ 0, & t \ge 9/2 \end{cases}$$

The fundamental identities (3.1) above easily prove many important properties of $\oplus$.

3.3 Definition (Consistency). We say that $* : \mathbb{R}(L) \times \mathbb{R}(L) \to \mathbb{R}(L)$ is **consistent** iff $\forall [\lambda], [\mu], [\rho], [\sigma] \in \mathbb{R}(L)$ and $\forall \alpha \in L$, it is the case that

$$a(\lambda,\alpha) = a(\rho,\alpha) \text{ and } a(\mu,\alpha) = a(\sigma,\alpha) \Rightarrow a(\lambda * \mu,\alpha) = a(\rho * \sigma,\alpha)$$

and

$$b(\lambda,\alpha) = b(\rho,\alpha) \text{ and } b(\mu,\alpha) = b(\sigma,\alpha) \Rightarrow b(\lambda * \mu,\alpha) = b(\rho * \sigma,\alpha)$$

3.4 Theorem (Uniqueness of $\oplus$). $\oplus$ is the unique, consistent extension to $\mathbb{R}(L)$ of the usual $+$ on $\mathbb{R}$ as transferred to $\mathbb{R}\{\bot, \top\} \hookrightarrow \mathbb{R}(L)$.

3.5 Theorem (Additive structure of $(\mathbb{R}(L), \oplus)$). $(\mathbb{R}(L), \oplus)$ is an abelian, cancellation monoid with identity $[\lambda_0]$. A fuzzy number $[\lambda]$ has an additive inverse iff it is crisp, i.e. iff $\exists r \in \mathbb{R}$, $[\lambda] = [\lambda_r] \in \mathbb{R}\{\bot, \top\}$.

The order-theoretic properties of the usual addition, the fundamental identities (3.1), and the morphism-invertibility of the hypergraph functor mix nicely together to give the shortest known proof [26] of the subbasic-continuity of $\oplus$ on $\mathbb{R}(L) \times \mathbb{R}(L)$ equipped with the Goguen-Wong [6, 37] product topology $\tau(L) \times \tau(L)$, which is the categorical L-product topology guaranteed by L-**TOP** being topological [8, 33]. That is,

$$S(\oplus)^{\leftarrow}(S(\mathfrak{L}_t)) \in S(\tau(L) \times \tau(L)), \quad S(\oplus)^{\leftarrow}(S(\mathfrak{R}_s)) \in S(\tau(L) \times \tau(L))$$

¿From [33], this suffices to say that $\oplus$ is jointly L-continuous. Thus we have:

3.6 Theorem (Topological properties of $\oplus$). $\oplus : \mathbb{R}(L) \times \mathbb{R}(L) \to \mathbb{R}(L)$ is jointly L-continuous w.r.t. the canonical L-topology $\tau(L)$ on $\mathbb{R}(L)$ and the L-product topology $\tau(L) \times \tau(L)$ on $\mathbb{R}(L) \times \mathbb{R}(L)$.

¿From [24] comes an alternative description of $\oplus$ on $\mathbb{R}(L)$, namely the convolution approach.

3.7 Discussion (Convolution approach to $\oplus$). Let $[\lambda], [\mu] \in \mathbb{R}(L)$ with $\lambda, \mu : \mathbb{R} \to L$ left-continuous maps. Define

$$(\lambda \oplus \mu)(t) = \bigvee_{r+s=t} (\lambda(r) \wedge \mu(s)) \tag{3.7}$$

It can be shown that $\lambda \oplus \mu : \mathbb{R} \to L$ is left-continuous, that $[\lambda \oplus \mu] \in \mathbb{R}(L)$, and that this $\oplus$ coincides with that described above. We then define

$$[\lambda] \oplus [\mu] = [\lambda \oplus \mu]$$

Some algebraic properties are obtained directly from the convolution (3.7). The fundamental identities (3.1) of the interval arithmetic approach can be derived from the convolution (3.7), from which the other properties immediately follow. As for the joint (subbasic-)continuity of $\oplus$, this can obviously be derived from the fundamental identities as discussed above. But there is another proof of continuity using the convolution directly—it can be shown that the pre-images of the subbasic sets are as follows:

$$\oplus^{\leftarrow}(\mathfrak{L}_t) = \bigvee_{r+s=t} ((\pi_1)_L^{\leftarrow}(\mathfrak{L}_r) \wedge (\pi_2)_L^{\leftarrow}(\mathfrak{L}_s)) \tag{3.7a}$$

$$\oplus^{\leftarrow}(\mathfrak{R}_t) = \bigvee_{r+s=t} ((\pi_1)_L^{\leftarrow}(\mathfrak{R}_r) \wedge (\pi_2)_L^{\leftarrow}(\mathfrak{R}_s)) \tag{3.7b}$$

¿From 3.7(a,b) follows the subbasic-continuity of $\oplus$, and hence its continuity as indicated above. Even though this proof seems formally more direct than the hypergraph functor proof [26] outlined above Theorem 3.6, it is much longer in actual detail, showing the power of the categorical and interval arithmetic approach. The main utility, then, of the convolution approach is not in the continuity of $\oplus$, but in the way it combines synergistically with the interval arithmetic approach (fundamental identities) in [29] to build the uniform continuity of $\oplus$.

3.8 Discussion (Setting uniform continuity up). The canonical uniformity $\mathcal{U}(L)$ on $\mathbb{R}(L)$ generates via the reverse L-powerset operators

$$(\pi_1)_L^{\leftarrow}, (\pi_2)_L^{\leftarrow} : L^{\mathbb{R}(L)\times\mathbb{R}(L)} \leftarrow L^{\mathbb{R}(L)}$$

of the projections

$$\pi_1, \pi_2 : \mathbb{R}(L) \times \mathbb{R}(L) \to \mathbb{R}(L)$$

the canonical L-product quasi-uniformity $\mathcal{Q}_{\mathcal{U}(L)\times\mathcal{U}(L)}$ on $\mathbb{R}(L) \times \mathbb{R}(L)$ [29], which both induces the canonical L-product topology $\tau(L)\times\tau(L)$, as well as the canonical L-product uniformity $\mathcal{U}(L) \times \mathcal{U}(L)$, on $\mathbb{R}(L) \times \mathbb{R}(L)$. Since L-quasi-uniform continuity implies both L-continuity and L-uniform continuity—when the topology and uniformity are both generated from the quasi-uniformity, the

L-quasi-uniform continuity of $\oplus$ would then guarantee both its L-continuity and its L-uniform continuity. This quasi-uniform continuity of $\oplus$ requires using *both* approaches to fuzzy addition—that of the fundamental identities of (3.1), and that of the convolution of (3.7), together with the order properties of $+$, the identities of (3.7a,b), the fuzzy points of $\mathbb{R}(L) \times \mathbb{R}(L)$, and the properties of the Δ operator. Thus we obtain:

3.9 Theorem (Quasi-uniform and uniform properties of $\oplus$**).**
$\oplus : \mathbb{R}(L) \times \mathbb{R}(L) \to \mathbb{R}(L)$ is jointly L-quasi-uniformly continuous w.r.t. the canonical L-uniformity $\mathcal{U}(L)$ on $\mathbb{R}(L)$ and the canonical L-product quasi-uniformity $\mathcal{Q}_{\mathcal{U}(L)\times\mathcal{U}(L)}$ on $\mathbb{R}(L) \times \mathbb{R}(L)$. Hence, the following hold:

(1) $\oplus : \mathbb{R}(L) \times \mathbb{R}(L) \to \mathbb{R}(L)$ is jointly L-uniformly continuous w.r.t. the canonical L-uniformity $\mathcal{U}(L)$ on $\mathbb{R}(L)$ and the canonical L-product uniformity $\mathcal{U}(L) \times \mathcal{U}(L)$ on $\mathbb{R}(L) \times \mathbb{R}(L)$.

(2) $\oplus : \mathbb{R}(L) \times \mathbb{R}(L) \to \mathbb{R}(L)$ is jointly L-continuous w.r.t. the canonical L-topology $\tau(L)$ on $\mathbb{R}(L)$ and the L-product topology $\tau(L) \times \tau(L)$ on $\mathbb{R}(L) \times \mathbb{R}(L)$.

3.10 Remark. We have now discussed three distinct proofs of the L-continuity of $\oplus$: that stemming from the fundamental identities and the hypergraph functor; that stemming from the convolution; and that stemming from the quasi-uniform continuity of $\oplus$ (which in turn stems from both the fundamental identities and the convolution).

3.11 Summary (Additive structure on $\mathbb{R}(L)$**).**

$$(\mathbb{R}(L), \leq, \tau(L), \mathcal{U}(L), d_L, \oplus)$$

is an L-topological, uniform, metric, abelian, cancellation, conditionally order-complete monoid with identity.

3.12 Discussion (Further results on $\oplus$**).** The reader may wish to pursue other topics developed in the literature. For example, a fairly complete analysis of fuzzy translations is known [26]. Despite the lack of additive inverses for all fuzzy numbers, all translations are bijections; and all fuzzy translations are L-continuous (and L-uniformly continuous). There are classes of translations for which the inverse mapping is L-continuous iff the translation is by a crisp number. However, for these translations, the inverse mapping always enjoys a weakened L-continuity strictly between traditional continuity and full L-continuity. Characterizations of this weakened L-continuity are known from the fundamental identities and the hypergraph functor.

4 $\mathbb{R}(L)$ as L-topological complete fuzzy hyperfield

As in the preceding section, L is a complete deMorgan chain. The description of the multiplicative aspects of $\mathbb{R}(L)$ is much less tractable than the description of the additive side. Accordingly, the plan of attack for this section is as follows. First, we define the decomposition of L-numbers into their "negative" and "positive" parts. Second, we give a motivation of the definition of fuzzy multiplication from the requirement of distributivity over fuzzy addition. Third, we present the formal definition of fuzzy multiplication, together with the associated fundamental identities. And fourth, we catalog a few properties of fuzzy multiplication, including its joint L-continuity.

The eventual goal of this section is the result that $\mathbb{R}(L)$ is an L-topological, complete fuzzy hyperfield. A full description of the multiplicative aspects of $\mathbb{R}(L)$, let alone the full axiomatization of the additive and multiplicative aspects together defining **[complete] fuzzy hyperfield**, is quite beyond the scope of this section—the nine multi-part postulates defining this structure are given in [28], the primary reference for this section. Essentially, a [complete] fuzzy hyperfield is a set with an "addition" and "multiplication", where that set decomposes into two disjoint subsets, the subset of **crisp** elements and the subset of **fuzzy** elements; the crisp elements form a [complete] ordered field; the entire set forms a conditionally complete, ordered commutative semi-ring with unity whose identities are crisp elements; and a variety of properties hold, including several relationships between the crisp and fuzzy elements. Furthermore, an **L-topological [complete] fuzzy hyperfield** is a [complete] fuzzy hyperfield in which the addition and multiplication are jointly L-continuous with respect to some L-topology on the fuzzy hyperfield and the Goguen-Wong categorical L-product topology. In the case of $\mathbb{R}(L)$, the crisp elements precisely comprise the usual real line $\mathbb{R}$ embedded as $\mathbb{R}\{\perp, \top\} \hookrightarrow \mathbb{R}(L)$; the addition and multiplication are $\oplus$ and $\odot$, where $\oplus$ is that of Section 3 and $\odot$ is that defined *sequens*; and the L-topology is the canonical $\tau(L)$ defined in Section 1.

We may speak of the fuzzy hyperfield as an extension of the field of its crisp elements. If a fuzzy hyperfield is complete, then the field of its crisp elements is field-isomorphic and order-isomorphic to $\mathbb{R}$. So each complete fuzzy hyperfield is an extension of $\mathbb{R}$. But it is shown in [28] that there are uncountably many complete fuzzy hyperfield extensions of $\mathbb{R}$, at least one for each complete deMorgan chain L having at least three elements, and there are uncountably many L-topological complete fuzzy hyperfield extensions of $\mathbb{R}$, at least two for each complete deMorgan chain having at least three elements. This leads us to state:

4.0 Open Question. If the topological aspects are ignored and the lattice L is fixed, is there a unique complete fuzzy hyperfield extension of $\mathbb{R}$?

4.1 Definition (Decomposition of L-numbers). Let $[\lambda] \in \mathbb{R}(L)$ and put

$\lambda^+, \lambda^- : \mathbb{R} \to L$ by

$$\lambda^+(t) = \begin{cases} \top, & t \le 0 \\ \lambda(t), & t > 0 \end{cases} \qquad \lambda^-(t) = \begin{cases} \lambda(t), & t \le 0 \\ \bot, & t > 0 \end{cases}$$

Note that $[\lambda^+], [\lambda^-] \in \mathbb{R}(L)$ and depend only on the class $[\lambda]$.

4.2 Lemma (Addition of decomposants).

$$\forall\, [\lambda] \in \mathbb{R}(L),\ [\lambda] = [\lambda^-] \oplus [\lambda^+]$$

4.3 Discussion (Motivation of definition of multiplication). Let $[\lambda], [\mu] \in \mathbb{R}(L)$, and assume that fuzzy multiplication, written either with $\odot$ or with juxtaposition, can be defined so that it is distributive over $\oplus$. Then we would have

$$\begin{aligned} \lambda\mu &= \left(\lambda^+ \oplus \lambda^-\right)\left(\mu^+ \oplus \mu^-\right) \\ &= \lambda^+\mu^+ \oplus \lambda^+\mu^- \oplus \lambda^-\mu^+ \oplus \lambda^-\mu^- \end{aligned}$$

The goal is to define $\odot$, and then redescribe it by these four partial products, denoted respectively P_1, P_2, P_3, P_4, in combination with $\oplus$.

4.4 Discussion (Construction of fuzzy multiplication). Let $[\lambda]$, $[\mu] \in \mathbb{R}(L)$ be given. The strategy is to use these fuzzy numbers to define operators which then generate the fuzzy number which is their product. Define $a, b : L \to [-\infty, +\infty]$ by

$$\begin{aligned} a(\alpha) &= a\left(\lambda^+, \alpha\right) a\left(\mu^+, \alpha\right) + b\left(\lambda^+, \alpha\right) a\left(\mu^-, \alpha\right) \\ &\quad + a\left(\lambda^-, \alpha\right) b\left(\mu^+, \alpha\right) + b\left(\lambda^-, \alpha\right) b\left(\mu^-, \alpha\right) \end{aligned}$$

$$\begin{aligned} b(\alpha) &= b\left(\lambda^+, \alpha\right) b\left(\mu^+, \alpha\right) + a\left(\lambda^+, \alpha\right) b\left(\mu^-, \alpha\right) \\ &\quad + b\left(\lambda^-, \alpha\right) a\left(\mu^+, \alpha\right) + a\left(\lambda^-, \alpha\right) a\left(\mu^-, \alpha\right) \end{aligned}$$

Then it can be shown from the eight operators

$$\begin{gathered} a\left(\lambda^+, \ \right), a\left(\lambda^-, \ \right), a\left(\mu^+, \ \right), a\left(\mu^-, \ \right) \\ b\left(\lambda^+, \ \right), b\left(\lambda^-, \ \right), b\left(\mu^+, \ \right), b\left(\mu^-, \ \right) \end{gathered}$$

having the properties of Lemmas 2.3–2.4 that the operators a, b satisfy the hypotheses of Lemma 2.6. So from Lemma 2.6 we obtain a well-defined map $\rho : [-\infty, +\infty] \to L$ which we then restrict to $\mathbb{R}$ to obtain a map we denote $\lambda \odot \mu$. In accordance with Lemma 2.6, $\lambda \odot \mu : \mathbb{R} \to L$ is a left-continuous map such that $[\lambda \odot \mu] \in \mathbb{R}(L)$ and the **fundamental identities** obtain:

$$\begin{aligned} a(\lambda \odot \mu, \alpha) &= a\left(\lambda^+, \alpha\right) a\left(\mu^+, \alpha\right) + b\left(\lambda^+, \alpha\right) a\left(\mu^-, \alpha\right) \\ &\quad + a\left(\lambda^-, \alpha\right) b\left(\mu^+, \alpha\right) + b\left(\lambda^-, \alpha\right) b\left(\mu^-, \alpha\right) \qquad (4.4a) \end{aligned}$$

$$\begin{aligned} b(\lambda \odot \mu, \alpha) &= b(\lambda^+, \alpha)\, b(\mu^+, \alpha) + a(\lambda^+, \alpha)\, b(\mu^-, \alpha) \\ &\quad + b(\lambda^-, \alpha)\, a(\mu^+, \alpha) + a(\lambda^-, \alpha)\, a(\mu^-, \alpha) \end{aligned} \tag{4.4b}$$

We define **fuzzy multiplication** by

$$[\lambda] \odot [\mu] = [\lambda \odot \mu]$$

4.5 Example. Let $L = \mathbb{I} \equiv [0,1]$ with $\alpha' = 1 - \alpha$, adopt the convention that 2λ means $\lambda_2\lambda$ (where λ_2 is crisp 2) and λ^2 means $\lambda\lambda$, and let λ be the "fuzzy zero" given by

$$\lambda(t) = \begin{cases} 1, & t < -1 \\ \frac{1}{2}(1-t), & -1 < t < 1 \\ 0, & t > 1 \end{cases}$$

where the definition at each endpoints can be either way because of the equivalence relation. Then λ^2 is the "fuzzy zero" given by

$$\lambda^2(t) = \begin{cases} 1, & t < -2 \\ \frac{1}{4}\left(2 + \sqrt{-2t}\right), & -2 < t < 0 \\ \frac{1}{4}\left(2 - \sqrt{2t}\right), & 0 < t < 2 \\ 0, & t > 2 \end{cases}$$

and 2λ is the "fuzzy zero" given by

$$\lambda(t) = \begin{cases} 1, & t < -2 \\ \frac{1}{4}(2-t), & -2 < t < 2 \\ 0, & t > 2 \end{cases}$$

As with $\oplus$, the fundamental identities (4.1) of $\odot$ are instrumental in proving many important properties of $\odot$.

4.6 Theorem (Some multiplicative structures of $(\mathbb{R}(L), \odot)$). $(\mathbb{R}(L), \odot)$ is an abelian monoid with identity $[\lambda_1]$ satisfying the **fuzzy cancellation law**: if

$$[\lambda] \notin \mathbb{R}_0(L) \equiv \{[\sigma] \in \mathbb{R}(L) : \sigma(0+) = \top \text{ or } \sigma(0-) = \bot\}$$

and $[\mu], [\rho] \in \mathbb{R}(L)$, then

$$\lambda\mu = \lambda\rho \Rightarrow \mu = \rho$$

(The classes in $\mathbb{R}_0(L)$ are called **fuzzy zeroes.**)

4.7 Theorem (Algebraic structure of $(\mathbb{R}(L), \leq, \oplus, \odot)$). $(\mathbb{R}(L), \leq, \oplus, \odot)$ is a complete fuzzy hyperfield on the crisp real line $(\mathbb{R}\{\bot, \top\}, \leq, +, \cdot)$ for each complete deMorgan chain L with $|L| \geq 3$. In particular, $\odot$ is distributive over $\oplus$.

4.8 Definition (Piece-wise consistent operations). We say that $* : \mathbb{R}(L) \times \mathbb{R}(L) \to \mathbb{R}(L)$ is **piece-wise consistent (PC)** iff $\forall\, [\lambda], [\mu], [\rho], [\sigma] \in \mathbb{R}(L)$ and $\forall\, \alpha \in L$, all the following implications hold:

$$a(\lambda^+, \alpha) = a(\rho^+, \alpha) \text{ and } a(\mu^+, \alpha) = a(\sigma^+, \alpha) \;\Rightarrow$$
$$a(\lambda^+ * \mu^+, \alpha) = a(\rho^+ * \sigma^+, \alpha)$$
$$b(\lambda^+, \alpha) = b(\rho^+, \alpha) \text{ and } b(\mu^+, \alpha) = b(\sigma^+, \alpha) \;\Rightarrow$$
$$b(\lambda^+ * \mu^+, \alpha) = b(\rho^+ * \sigma^+, \alpha)$$
$$b(\lambda^+, \alpha) = b(\rho^+, \alpha) \text{ and } a(\mu^-, \alpha) = a(\sigma^-, \alpha) \;\Rightarrow$$
$$a(\lambda^+ * \mu^-, \alpha) = a(\rho^+ * \sigma^-, \alpha)$$
$$a(\lambda^+, \alpha) = a(\rho^+, \alpha) \text{ and } b(\mu^-, \alpha) = b(\sigma^-, \alpha) \;\Rightarrow$$
$$b(\lambda^+ * \mu^-, \alpha) = b(\rho^+ * \sigma^-, \alpha)$$
$$a(\lambda^-, \alpha) = a(\rho^-, \alpha) \text{ and } b(\mu^+, \alpha) = b(\sigma^+, \alpha) \;\Rightarrow$$
$$a(\lambda^- * \mu^+, \alpha) = a(\rho^- * \sigma^+, \alpha)$$
$$b(\lambda^-, \alpha) = b(\rho^-, \alpha) \text{ and } a(\mu^+, \alpha) = a(\sigma^+, \alpha) \;\Rightarrow$$
$$b(\lambda^- * \mu^+, \alpha) = b(\rho^- * \sigma^+, \alpha)$$
$$b(\lambda^-, \alpha) = b(\rho^-, \alpha) \text{ and } b(\mu^-, \alpha) = b(\sigma^-, \alpha) \;\Rightarrow$$
$$a(\lambda^- * \mu^-, \alpha) = a(\rho^- * \sigma^-, \alpha)$$
$$a(\lambda^-, \alpha) = a(\rho^-, \alpha) \text{ and } a(\mu^-, \alpha) = a(\sigma^-, \alpha) \;\Rightarrow$$
$$b(\lambda^- * \mu^-, \alpha) = b(\rho^- * \sigma^-, \alpha)$$

4.9 Examples (Piece-wise consistent operations on $\mathbb{R}(L)$). There are uncountably many natural PC binary operations on $\mathbb{R}(L)$. For example:

(1) any constant map from $\mathbb{R}(L) \times \mathbb{R}(L)$ to $\mathbb{R}(L)$ is PC; and

(2) each of the projections $\pi_1, \pi_2 : \mathbb{R}(L) \times \mathbb{R}(L) \to \mathbb{R}(L)$ is PC.

4.10 Theorem (Uniqueness of $\odot$). $\odot$ is the unique PC extension to $\mathbb{R}(L)$ of the usual $\cdot$ on $\mathbb{R}$, as transferred to $\mathbb{R}\{\perp, \top\} \hookrightarrow \mathbb{R}(L)$, which distributes over $\oplus$.

To properly address the joint L-continuity of $\odot$, we formally define the partial products maps

$$P_1, P_2, P_3, P_4 : \mathbb{R}(L) \times \mathbb{R}(L) \to \mathbb{R}(L)$$

as follows:

$$P_1(\lambda,\mu) = \lambda^+\mu^+, \quad P_2(\lambda,\mu) = \lambda^+\mu^-, \quad P_3(\lambda,\mu) = \lambda^-\mu^+, \quad P_4(\lambda,\mu) = \lambda^-\mu^-$$

Analogous to the fuzzy addition case, the order-theoretic properties of the usual multiplication, the fundamental identities (4.4), and the morphism-invertibility of the hypergraph functor mix together to give manageable proofs that each P_i is jointly L-subbasic continuous. So each of these maps is jointly L-continuous.

4.11 Lemma (Topological properties of partial products). Each of

$$P_1, P_2, P_3, P_4 : \mathbb{R}(L) \times \mathbb{R}(L) \to \mathbb{R}(L)$$

is jointly L-continuous w.r.t. the canonical L-topology $\tau(L)$ on $\mathbb{R}(L)$ and the L-product topology $\tau(L) \times \tau(L)$ on $\mathbb{R}(L) \times \mathbb{R}(L)$.

Since we have that composition of L-continuous maps is L-continuous, that $\oplus$ is jointly L-continuous, and that the cross of two L-continuous maps is jointly L-continuous [6, 37], the previous lemma and the identity

$$\odot = \oplus \circ ((\oplus \circ (P_1 \times P_2)) \times (\oplus \circ (P_3 \times P_4)))$$

guarantee the following theorem.

4.12 Theorem (Topological properties of $\odot$). $\odot : \mathbb{R}(L) \times \mathbb{R}(L) \to \mathbb{R}(L)$ is jointly L-continuous w.r.t. the canonical L-topology $\tau(L)$ on $\mathbb{R}(L)$ and the L-product topology $\tau(L) \times \tau(L)$ on $\mathbb{R}(L) \times \mathbb{R}(L)$.

4.13 Summary (Multiplicative structure on $\mathbb{R}(L)$). $(\mathbb{R}(L), \le, \tau(L), \odot)$ is an L-topological, abelian, conditionally order-complete monoid with identity and fuzzy cancellation; and $(\mathbb{R}(L), \le, \tau(L), \oplus, \odot)$ is an L-topological fuzzy hyperfield, which includes being a commutative semiring with unity. (See remarks at the beginning of this section.)

4.14 Discussion (Further results on $\odot$). The reader may wish to pursue other topics developed in the literature. These include the nine postulates of a fuzzy hyperfield and a fairly complete analysis of fuzzy contractions and expansions [28]. The topological behavior of the latter is analogous to that of fuzzy translations, including the inverse of certain contractions and expansion being L-continuous and having inverses which enjoy a weakened L-continuity strictly between traditional continuity and full L-continuity. Characterizations of this weakened L-continuity rest on the fundamental identities (4.4) and the hypergraph functor.

5 $\mathbb{R}$ as L-uniform additive group and L-topological field

Since the algebraic and order-theoretic aspects of $\mathbb{R}$ are well-known, the main point is the L-topological behavior of the arithmetic operations, namely the joint L-continuity of the arithmetic operations w.r.t. the canonical co-topologies. Such results enrich our topological understanding of these arithmetic operations. The primary references are [30–31].

5.1 Theorem (Topological properties of +). $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is jointly L-continuous w.r.t. the canonical L-topology $co\text{-}\tau(L)$ on $\mathbb{R}$ and the L-product topology $co\text{-}\tau(L) \times co\text{-}\tau(L)$ on $\mathbb{R} \times \mathbb{R}$.

5.1.1 Outline of the hypergraph functor proof. The following lemma is key to both the hypergraph functor proof and the convolution proof given ***sequens***: *if* $\alpha \in L$, $[\lambda] \in \mathbb{R}(L)$, *and* $t_0, s_0 \in \mathbb{R}$ *with* $t_0 + s_0 > a(\lambda, \alpha)$ $[< b(\lambda, \alpha)]$, *then* $\exists\, [\mu], [\nu] \in \mathbb{R}(L)$ *satisfying:*

(1) $[\lambda] = [\mu] \oplus [\nu]$;

(2) $t_0 > a(\mu, \alpha)$, $s_0 > a(\nu, \alpha)$ $[t_0 < b(\mu, \alpha)$, $s_0 < b(\nu, \alpha)]$.

Now it can be shown that $\forall \mathfrak{L}_\lambda, \mathfrak{R}_\lambda$,we have

$$S(+)^{\leftarrow}(S(\mathfrak{L}_\lambda)), S(+)^{\leftarrow}(S(\mathfrak{R}_\lambda)) \in S(co\text{-}\tau(L) \times co\text{-}\tau(L))$$

which establishes via morphism-invertibility the subbasic continuity of +, hence its continuity by [33]. The crucial step relates $S(\mathfrak{L}_\lambda)$ to

$$S((\pi_1)_L^{\leftarrow}(\mathfrak{L}_\mu) \wedge (\pi_2)_L^{\leftarrow}(\mathfrak{L}_\nu))$$

and $S(\mathfrak{R}_\lambda)$ to

$$S((\pi_1)_L^{\leftarrow}(\mathfrak{R}_\mu) \wedge (\pi_2)_L^{\leftarrow}(\mathfrak{R}_\nu))$$

where $[\lambda] = [\mu] \oplus [\nu]$, using the order-theoretic properties of $\oplus$, the above lemma, and the fundamental identities of $\oplus$. This gives the shortest proof to the L-continuity of +, but is closely elated to the following convolution proof.

5.1.2 Outline of the convolution proof. Using the lemma in 5.1.1, one can show that

$$+^{\leftarrow}(\mathfrak{L}_\lambda) = \bigvee_{\mu \oplus \nu = \lambda} ((\pi_1)_L^{\leftarrow}(\mathfrak{L}_\mu) \wedge (\pi_2)_L^{\leftarrow}(\mathfrak{L}_\nu)) \tag{5.1.2a}$$

$$+^{\leftarrow}(\mathfrak{R}_\lambda) = \bigvee_{\mu \oplus \nu = \lambda} ((\pi_1)_L^{\leftarrow}(\mathfrak{R}_\mu) \wedge (\pi_2)_L^{\leftarrow}(\mathfrak{R}_\nu)) \tag{5.1.2b}$$

¿From 5.1.2(a,b) comes the subbasic continuity of $+$ and hence its joint L-continuity.

5.1.3 Outline of quasi-uniform and uniform continuity proof. The canonical (dual) uniformity $co\text{-}\mathcal{U}(L)$ on $\mathbb{R}$ generates, via the reverse L-powerset operators

$$(\pi_1)_L^{\leftarrow}, (\pi_2)_L^{\leftarrow} : L^{\mathbb{R}\times\mathbb{R}} \leftarrow L^{\mathbb{R}}$$

of the projections

$$\pi_1, \pi_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

the canonical L-product quasi-uniformity $\mathcal{Q}_{co\text{-}\mathcal{U}(L) \times co\text{-}\mathcal{U}(L)}$ on $\mathbb{R}\times\mathbb{R}$ [30], which both induces the canonical L-product topology $co\text{-}\tau(L) \times co\text{-}\tau(L)$, as well as the canonical L-product uniformity $co\text{-}\mathcal{U}(L) \times co\text{-}\mathcal{U}(L)$, on $\mathbb{R} \times \mathbb{R}$. Since L-quasi-uniform continuity implies both L-continuity and L-uniform continuity—when the topology and uniformity are both generated from the quasi-uniformity, the L-quasi-uniform continuity of $+$ would then guarantee both its L-continuity and its L-uniform continuity. This quasi-uniform continuity of $+$ requires using the fundamental identities (3.1) of fuzzy addition, the order properties of $+$, the identities 5.1.2(a,b), the fuzzy points of $\mathbb{R} \times \mathbb{R}$, and the properties of the Δ operator.

5.1.4 Outline of the stratification functor proof. The L-**characteristic** functor $G_\chi^L : \mathbf{TOP} \to L\text{-}\mathbf{TOP}$, the fixed-basis restriction of G_χ of [33], is defined as follows: given $(X, \mathfrak{T}) \in |\mathbf{TOP}|$ and $f \in \mathbf{TOP}$,

$$G_\chi^L(\mathfrak{T}) = \{\chi_U : U \in \mathfrak{T}\}, \quad G_\chi^L(X, \mathfrak{T}) = \left(X, G_\chi^L(\mathfrak{T})\right), \quad G_\chi^L(f) = f$$

And the L-**stratification** functor $G_k^L : L\text{-}\mathbf{TOP} \to L\text{-}\mathbf{TOP}_k$, the fixed-basis restriction of the functor G_χ of [33], is defined as follows: given $(X, \tau) \in |L\text{-}\mathbf{TOP}|$ and $f \in L\text{-}\mathbf{TOP}$,

$$G_k^L(\tau) = \tau \vee \{\underline{\alpha} : \alpha \in L\}, \quad G_k^L(X, \tau) = \left(X, G_k^L(\tau)\right), \quad G_k^L(f) = f$$

It can be shown that each of these functors has morphism-invertibility, i.e.

$$f : (X, \mathfrak{T}) \to (Y, \mathfrak{S}) \text{ in } \mathbf{TOP} \Leftrightarrow G_\chi^L(f) : G_\chi^L(X, \tau) \to G_\chi^L(Y, \sigma) \text{ in } L\text{-}\mathbf{TOP}$$

$$f : (X, \tau) \to (Y, \sigma) \text{ in } L\text{-}\mathbf{TOP} \Leftrightarrow G_k^L(f) : G_k^L(X, \tau) \to G_k^L(Y, \sigma) \text{ in } L\text{-}\mathbf{TOP}_k$$

Letting $G_\omega^L : \mathbf{TOP} \to L\text{-}\mathbf{TOP}_k$ [31, 33] by

$$G_\omega^L = G_k^L \circ G_\chi^L$$

it follows that G_ω^L is also morphism-invertible. Trivially G_χ^L preserves products; and from Theorem 5.2.1 of [33] we have G_k is a right-adjoint and so preserves products, and hence the restriction G_k^L preserves products. Thus G_ω^L preserves products. Now the following results from [30–31] apply these properties of G_ω^L to the question of the joint L-continuity of $+$ on $\mathbb{R} \times \mathbb{R}$ equipped with $co\text{-}\tau(L) \times co\text{-}\tau(L)$:

(1) for each complete deMorgan algebra L, $co\text{-}\tau(L) \subset G_\omega^L(\mathfrak{T})$, where $\mathfrak{T}$ is the traditional topology on $\mathbb{R}$;

(2) for each complete deMorgan chain L, $co\text{-}\tau(L) \supset G_\omega^L(\mathfrak{T})$, i.e. $co\text{-}\tau(L) = G_\omega^L(\mathfrak{T})$.

Since $+$ is continuous on $\mathbb{R} \times \mathbb{R}$ equipped with $\mathfrak{T} \times \mathfrak{T}$, the joint L-continuity of $+$ on $\mathbb{R} \times \mathbb{R}$ equipped with $co\text{-}\tau(L) \times co\text{-}\tau(L)$ now follows.

Sequens, let $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denote the traditional multiplication on $\mathbb{R}$; and for $i = 1, 2, 3, 4$, put

$$M_i(t,s) = \begin{cases} ts, & (t,s) \in i^{th} \text{ quadrant} \\ 0, & \text{otherwise} \end{cases}$$

Then $M = \sum_i M_i$. Because of the properties of the L-product topology and the L-continuity of $+$ from Theorem 5.1, M is jointly L-continuous iff each M_i is joint L-continuous.

We now state the joint L-continuity of M as our next theorem, following which we briefly discuss the three different proofs of the joint L-continuity of M.

5.2 Theorem (Topological properties of M). $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is jointly L-continuous w.r.t. the canonical L-topology $co\text{-}\tau(L)$ on $\mathbb{R}$ and the L-product topology $co\text{-}\tau(L) \times co\text{-}\tau(L)$ on $\mathbb{R} \times \mathbb{R}$.

5.2.1 Outline of the hypergraph functor proof. An analogue of the lemma in 5.1.1 relating $+$ to $\oplus$ must be proved for each M_i relative to P_i. For example, the "$a(\lambda, \alpha)$" lemma for M_1 reads as follows: *if* $\alpha \in L$, $[\lambda] \in \mathbb{R}(L)$, *and* $t_0, s_0 \in \mathbb{R}$ *with* $t_0, s_0 > 0$ *and* $t_0 s_0 > a(\lambda, \alpha)$, *and if* $[\rho] \in \mathbb{R}(L)$ is *defined by*

$$\rho(t) = \lambda^+ \left(t - \left(\frac{1}{2}\right) \left(t_0 s_0 - a\left(\lambda^+, \alpha\right)\right)\right)$$

then $\exists\ [\mu], [\nu] \in \mathbb{R}(L)$ *satisfying:*

(1) $[\rho] = [\mu] \odot [\nu]$;

(2) $t_0 > a(\mu, \alpha)$, $s_0 > a(\nu, \alpha)$.

Then this lemma and the "$b(\lambda, \alpha)$" lemma for M_1, together with the order-theoretic properties of P_1, the fundamental identities of P_1, and the hypergraph functor are used to prove the joint L-continuity of M_1. Similar proofs can be given for the other M_i's. And so the joint L-continuity of M on $\mathbb{R}$ follows.

5.2.2 Outline of the convolution proof. Using the lemmas from 5.1.2, a proof analogous to the convolution proof of $+$ can be made for M.

5.2.3 Outline of the stratification functor proof. The set-up of the proof outlined in 5.2.4 immediately proves that M is jointly L-continuous on $\mathbb{R}$; this is because M is jointly continuous on $\mathbb{R}$ with its traditional topology.

We therefore formally state:

5.3 Summary. $(\mathbb{R}, \leq, co\text{-}\tau(L), co\text{-}\mathcal{U}(L), d_L, +)$ is an L-topological, uniform, metric additive group, and $(\mathbb{R}, \leq, co\text{-}\tau(L), +, M)$ is a complete, ordered, metric, L-topological field.

See [31] for open questions concerning other possible families of canonical dual-topologies on $\mathbb{R}$ stemming from alternate L-topologies on $\mathbb{R}(L)$.

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CHAPTER 11

Fundamentals of a Generalized Measure Theory

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Introduction

In this chapter, we try to present a coherent survey on some recent attempts in building a theory of generalized measures. Our main goal is to emphasize a minimal set of axioms both for the measures and their domains, and still to be able to prove significant results. Therefore we start with fairly general structures and enrich them with additional properties only if necessary.

Concerning the algebraic basis, our starting point is a bounded lattice, and the uncertainty measure essentially preserves this structure, i.e., only the usual boundary conditions and the isotonicity are required. Each measure of this type can be represented in terms of a probability measure on the set of all pseudo-realizations. Although this representation is not unique in general, it allows us to introduce a well-defined integral of lattice-valued real numbers with respect to an uncertainty measure as the Lebesgue integral of its quasi-inverse with respect to a representing probability. For this purpose, the richer structure of a De Morgan algebra turns out to be useful. This integral generalizes the Choquet integral and, subsequently, the classical Lebesgue integral. Moreover, it is possible to prove a monotone convergence theorem (Section 1).

If the uncertainty measure is specified to be a plausibility, then the representation becomes unique on a smaller set, namely, the set of realizations, thus extending Shafer's classical result (Section 2).

Next we consider valuations as a further specification of plausibilities. In this context, the integral is additive, where the representing probability is concentrated on the set of coherent realizations, i.e., lattice homomorphisms. For this result, some distributivity of the lattice L is required. This parallels the situation in the classical Stone represention theorem were also coherent realizations are used (Section 3).

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In the framework of MV-algebras, a natural concept of additivity of uncertainty measures emerges, leading to special valuations and, therefore, preserving all the preceding results (Section 4).

Finally, measures on tribes of fuzzy subsets of a set X are presented. Many important tribes (e.g., those with respect to Frank t-norms with the notable exception of the minimum) turn out to be MV-algebras. However, these tribes are equipped with an additional semigroup operation which is respected by these measures. In this special setting it is possible to represent a measure as a Lebesgue integral with respect to its restriction to the collection of crisp sets in the tribe, the integrand being some Markov kernel (Section 5).

For the lattice theoretical background we usually refer to [4], concerning probability and the theory of σ-additive measures to [1, 15]. Our exposition does not contain the proofs of the results, but we try to give at least one precise reference in each case.

1 Uncertainty measures and integration

For a measure theory, as developed in this section, a minimal requirement for the domain of measures is a *bounded lattice* $(L, \leq)$, i.e., a lattice $(L, \leq)$ with universal bounds $\mathbf{0}$ and $\mathbf{1}$. If we consider measures with non-negative values only, the isotonicity (together with suitable boundary conditions) seems to be a minimal requirement. Since most results can be proven for finite measures only, we even can restict ourselves, without loss of generality, to normalized measures.

Definition 1.1 Let $(L, \leq)$ be a bounded lattice.

(i) A function $m : L \longrightarrow [0, 1]$ is called an *uncertainty measure* on L if it satisfies the following conditions:

(M1) $\quad m(\mathbf{0}) = 0, \quad m(\mathbf{1}) = 1,$ (Boundary condition)

(M2) $\quad \alpha \leq \beta \implies m(\alpha) \leq m(\beta).$ (Isotonicity)

(ii) If $(L, \leq)$ is a σ-complete lattice, then an uncertainty measure m on L is called *σ-smooth* if for each non-decreasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ in L we have

(M3) $\quad \sup_{n\in\mathbb{N}} m(\alpha_n) = m(\bigvee_{n\in\mathbb{N}} \alpha_n).$ (σ-smoothness)

A special case of a σ-smooth uncertainty measure was studied in [39]. It is remarkable that even in this general setting each uncertainty measure (where no additivity is required) can be represented by suitable probability (i.e., additive) measures. In order to prepare this, we first introduce the concept of pseudo-realizations [22, Definition 4.1(a)], which can be seen as suitable generalizations of the events in a random experiment.

Definition 1.2 Let $(L, \leq)$ be a bounded lattice. A function $\omega : L \longrightarrow \{0,1\}$ is called a *pseudo-realization* if it fulfills

$$\text{(R1)} \qquad \omega(\mathbf{0}) = 0, \qquad \omega(\mathbf{1}) = 1, \qquad \text{(Boundary condition)}$$

$$\text{(R2)} \qquad \alpha \leq \beta \implies \omega(\alpha) \leq \omega(\beta). \qquad \text{(Isotonicity)}$$

The set of all pseudo-realizations will be denoted $\mathbf{PR}(L)$.

The set $\mathbf{PR}(L)$ forms a compact subspace of the product space $\{0,1\}^L$, where $\{0,1\}$ is equipped with the discrete topology. It therefore makes sense to consider regular Borel probability measures [1] on $\mathbf{PR}(L)$ (also called *random sets* in L [31]).

The following general representation theorem can be found in [22, Proposition 4.1] (observe that the stronger hypothesis of L being a De Morgan algebra is not used in the proof of this proposition).

Theorem 1.3 *For each uncertainty measure m on a bounded lattice L there exists a regular Borel probability measure P on the space $\mathbf{PR}(L)$ of all pseudo-realizations such that for all $\alpha \in L$*

$$P(\{\omega \in \mathbf{PR}(L) \mid \omega(\alpha) = 1\}) = m(\alpha). \tag{1}$$

If L is σ-complete and m σ-smooth, then for each non-decreasing sequence $(\alpha_n)_{n \in \mathbb{N}}$ in L

$$P(\{\omega \in \mathbf{PR}(L) \mid \textit{for all } n \in \mathbb{N} \quad \omega(\alpha_n) = 0 \quad \textit{and} \quad \omega(\bigvee_{n \in \mathbb{N}} \alpha_n) = 1\}) = 0. \tag{2}$$

In general, there is no unique regular Borel probability measure P representing an uncertainty measure m on L (for a unique representation in a more special setting see Section 2).

Example 1.4 Consider the bounded lattice $L = \{\mathbf{0}, \alpha, \beta, \mathbf{1}\}$ with $\alpha, \beta \notin \{\mathbf{0}, \mathbf{1}\}$, $\alpha \vee \beta = \mathbf{1}$ and $\alpha \wedge \beta = \mathbf{0}$, and define the uncertainty measure m on L by $m(\alpha) = a$ and $m(\beta) = b$ with $a, b \in \,]0,1[$. We have exactly four pseudo-realizations, namely,

$$\omega_1 = \chi_{\{\alpha,\beta,\mathbf{1}\}}, \qquad \omega_2 = \chi_{\{\alpha,\mathbf{1}\}},$$
$$\omega_3 = \chi_{\{\beta,\mathbf{1}\}}, \qquad \omega_4 = \chi_{\{\mathbf{1}\}},$$

where χ_M denotes the characteristic function of the set M. Then there are infinitely many probability measures P on the set of pseudo-realizations satisfying (1), specified by

$$P(\{\omega_1\}) = p, \qquad P(\{\omega_2\}) = a - p,$$
$$P(\{\omega_3\}) = b - p, \qquad P(\{\omega_4\}) = 1 - (a + b) + p,$$

with $p \in [\max(a + b - 1, 0), \min(a, b)]$.

Since for a reasonable concept of an integral some complementation is useful, we start here with the following structure.

Definition 1.5

(i) A triple $(L, \leq, ^\perp)$ is called a *De Morgan algebra* if $(L, \leq)$ is a lattice with universal bounds $\mathbf{0}$ and $\mathbf{1}$, and $^\perp : L \longrightarrow L$ is an order reversing involution. We shall call $\alpha^\perp$ the *complement* of α.

(ii) Two elements α, β of a De Morgan algebra are said to be *orthogonal* if $\alpha \leq \beta^\perp$.

Special and important examples of De Morgan algebras are *quantum De Morgan algebras* [22, Definition 2.1], *weakly orthomodular lattices* [4], *Girard algebras* (called integral, commutative Girard monoids in [20]) and *MV-algebras* [10] (see Section 4), *Boolean algebras*, and *tribes of fuzzy sets* [9, Definition 2.5] (see Section 5).

The structure of the bounded lattice $(L, \leq)$ in Example 1.4 can be enriched to that of a De Morgan algebra in several ways. One possibility is to put $\beta = \alpha^\perp$, in which case we obtain a Boolean algebra (which can be related to a simple random experiment, e.g., tossing a coin). Another possibility is $\alpha^\perp = \alpha$ and $\beta^\perp = \beta$. Note, however, that the pseudo-realizations are always independent of the choice of the complement.

When trying to build an integration theory based on uncertainty measures on L, the so-called L-valued real numbers [17] turn out to be appropriate integrands. Observe that $[0,1]$-valued real numbers were expected to be "numbers of the future" in [35, page 123]. An important mathematical tool when dealing with them is the concept of a quasi-inverse [37, 38].

Definition 1.6 Let $(L, \leq, ^\perp)$ be a σ-complete De Morgan algebra.

(i) A map $F : \mathbb{R} \longrightarrow L$ is called an *L-valued real number* if F satisfies the following conditions:

$$\bigvee_{n \in \mathbb{N}} F(-n) = \mathbf{1}, \qquad \bigwedge_{n \in \mathbb{N}} F(n) = \mathbf{0}.$$

$$\text{for all } r \in \mathbb{R}: \quad \bigvee_{s \in \mathbb{Q} \cap]r, \infty[} F(s) = F(r). \qquad \text{(Right continuity)}$$

An L-valued real number F is said to be *non-negative* if $\bigwedge_{n \in \mathbb{N}} F(-\frac{1}{n}) = \mathbf{1}$.

The set of L-valued real numbers will be denoted $\mathcal{D}(\mathbb{R}, L)$, and the set of non-negative L-valued real numbers $\mathcal{D}^+(\mathbb{R}, L)$.

(ii) The *quasi-inverse* $F^{(q)} : \mathbf{PR}(L) \longrightarrow [-\infty, \infty]$ of an L-valued real number F is defined by

$$F^{(q)}(\omega) = \inf\{r \in \mathbb{Q} \mid \omega(F(r)) = 0\}.$$

Remark 1.7

(i) A canonical embedding $j_{\mathbb{R}} : \mathbb{R} \longrightarrow \mathcal{D}(\mathbb{R}, L)$ is given by $j_{\mathbb{R}}(x) = H_x$, where $H_x : \mathbb{R} \longrightarrow L$ is defined by

$$H_x(r) = \begin{cases} \mathbf{1} & \text{if } r \in \,]-\infty, x[\,, \\ \mathbf{0} & \text{if } r \in [x, \infty[\,, \end{cases}$$

and for each $\omega \in \mathbf{PR}(L)$ we have $(H_x)^{(q)}(\omega) = x$.

(ii) There is a canonical embedding $j_L : L \longrightarrow \mathcal{D}(\mathbb{R}, L)$ defined by $j_L(\alpha) = F_\alpha$, where $F_\alpha : \mathbb{R} \longrightarrow L$ is given by

$$F_\alpha(r) = \begin{cases} \mathbf{1} & \text{if } r \in \,]-\infty, 0[\,, \\ \alpha & \text{if } r \in [0, 1[\,, \\ \mathbf{0} & \text{if } r \in [1, \infty[\,, \end{cases}$$

and $(F_\alpha)^{(q)} = \chi_{\{\omega \in \mathbf{PR}(L) | \omega(\alpha) = 1\}}$.

(iii) Right continuous, non-increasing probability distribution functions and $[0, 1]$-valued real numbers are the same mathematical objects.

(iv) Let X be a non-empty set and $\mathcal{A}$ be a σ-algebra of subsets of X. Then $\mathcal{A}$-valued real numbers can be identified with $\mathcal{A}$-measurable functions from X to $\mathbb{R}$ in the following sense. If $\varphi : X \longrightarrow \mathbb{R}$ is an $\mathcal{A}$-measurable function, then $F : \mathbb{R} \longrightarrow \mathcal{A}$ defined by

$$F(r) = \{x \in X \mid \varphi(x) > r\} \tag{3}$$

is an $\mathcal{A}$-valued real number. Vice versa, if $F : \mathbb{R} \longrightarrow \mathcal{A}$ is an $\mathcal{A}$-valued real number, then the mapping $\varphi : X \longrightarrow \mathbb{R}$ defined by

$$\varphi(x) = \inf\{r \in \mathbb{R} \mid x \notin F(r)\} \tag{4}$$

is an $\mathcal{A}$-measurable function, and for all $r \in \mathbb{R}$ equation (3) holds. Moreover, we have

$$\varphi = F^{(q)} \circ j_X, \tag{5}$$

where the embedding $j_X : X \longrightarrow \mathbf{PR}(\mathcal{A})$ is given by $(j_X(x))(A) = \chi_A(x)$ for each $A \in \mathcal{A}$.

The representation of uncertainty measures in terms of regular Borel probability measures on the space of pseudo-realizations and the quasi-inverse, which is a monotone and, therefore, measurable function on this space, suggests the following concept of an integral (note that this integral always exists and is well-defined [22, Lemma 5.2]).

Definition 1.8 Let $(L, \leq, ^{\perp})$ be a σ-complete De Morgan algebra. If m is an uncertainty measure on L and F a non-negative L-valued real number, then the *integral* of F over L with respect to m is defined by

$$\int\limits_{L} F\, dm = \int\limits_{\mathbf{PR}(L)} F^{(q)}\, dP,$$

where P is some regular Borel probability measure on $\mathbf{PR}(L)$ which represents m in the sense of Theorem 1.3, and $F^{(q)}$ is the quasi-inverse of F.

Remark 1.9 An immediate consequence of Remark 1.7(ii) and Theorem 1.3 is that for each $\alpha \in L$ we get

$$\int\limits_{L} F_{\alpha}\, dm = m(\alpha),$$

i.e., the integral is an extension of the measure m. Moreover, the integral is also isotone with respect to the integrand (the order on $\mathcal{D}(\mathbb{R}, L)$ is just the pointwise usual order).

In the case if m is a σ-smooth uncertainty measure, the rather abstract integral on the right hand side of Definition 1.8 can be written in a more familiar way [22, Proposition 5.1].

Theorem 1.10 *If m is a σ-smooth uncertainty measure on a σ-complete De Morgan algebra L, the integral in Definition 1.8 can be written as*

$$\int\limits_{L} F\, dm = \int\limits_{0}^{\infty} m(F(t))\, d\lambda(t),$$

where λ is the Lebesgue measure on $\mathbb{R}$.

Remark 1.11

(i) If L equals a σ-algebra $\mathcal{A}$ of subsets of a non-empty set X as discussed in Remark 1.7(iv) then we have

$$\int\limits_{\mathcal{A}} F\, dm = \int\limits_{0}^{\infty} m(\{x \in X \mid \varphi(x) > t\})\, d\lambda(t),$$

where F and φ are related via (3). This means that the integral reduces to the Choquet integral [11] of the measurable function $\varphi : X \longrightarrow [0, \infty[$ with respect to m.

(ii) If, in addition, m is a probability measure on $\mathcal{A}$, then we obtain just the Lebesgue integral of φ with respect to m.

Observe that the least upper bound of two L-valued real numbers equals just the pointwise join thereof. This allows us to formulate a monotone convergence theorem (for a proof see [22, Proposition 5.2]).

Theorem 1.12 *Let m be a σ-smooth uncertainty measure on a De Morgan algebra L, and let $(F_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathcal{D}^+(\mathbb{R}, L)$ such that also $\bigvee_{n\in\mathbb{N}} F_n \in \mathcal{D}^+(\mathbb{R}, L)$. Then*

$$\sup_{n\in\mathbb{N}} \int_L F_n \, dm = \int_L \Big(\bigvee_{n\in\mathbb{N}} F_n\Big) \, dm.$$

In general, a lattice structure on $\mathcal{D}(\mathbb{R}, L)$ can be introduced if L is a De Morgan algebra which satisfies the additional condition that the meet $\wedge$ is distributive over countable joins:

$$\begin{aligned}(F \wedge G)(r) &= \bigvee \{F(s) \wedge G(s) \mid s \in \mathbb{Q} \cap]r, \infty[\},\\ (F \vee G)(r) &= F(r) \vee G(r).\end{aligned}$$

The sublattice $\mathcal{D}^+(\mathbb{R}, L)$ has a universal lower bound, namely H_0, but no upper bound. This drawback could be overcome considering $[0, \infty]$ rather than $[0, \infty[$, in which case also the formulation of the preceding theorem would be slightly simpler.

2 Plausibility measures

Definition 2.1 Let $(L, \leq)$ be a bounded lattice. An uncertainty measure m on L is called a *plausibility measure* if for each non-empty finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of L we have

$$\text{(PL)} \qquad m\Big(\bigwedge_{i=1}^{n} \alpha_i\Big) \leq \sum_{i=1}^{n} (-1)^{i-1} \sum_{1 \leq j_1 < \cdots < j_i \leq n} m\Big(\bigvee_{k=1}^{i} \alpha_{j_k}\Big).$$

Remark 2.2

(i) Each *possibility measure* [13], i.e., a function $m : L \longrightarrow [0, 1]$ satisfying (M1) such that for all $\alpha, \beta \in L$

$$m(\alpha \vee \beta) = \max(m(\alpha), m(\beta)),$$

is a plausibility measure. The converse is not true in general (see Example 2.5 below).

(ii) If L equals the class of fuzzy subsets $[0,1]^X$ of a non-empty set X, fix $\nu_0 \in [0,1]^X$ with $\sup\{\nu_0(x) \mid x \in X\} = 1$. Then $m_{\nu_0} : [0,1]^X \longrightarrow [0,1]$ given by

$$m_{\nu_0}(\mu) = \sup\{\min(\mu(x), \nu_0(x)) \mid x \in X\}$$

is a σ-smooth possibility measure (ν_0 is then called the possibility distribution of m_{ν_0} [45]).

Definition 2.3 Let $(L, \leq)$ be a bounded lattice. A function $\omega : L \longrightarrow \{0,1\}$ is called a *realization* if it fulfills (R1) and for all $\alpha, \beta \in L$

(R3) $\qquad \omega(\alpha \vee \beta) = \max(\omega(\alpha), \omega(\beta)).$

The set of all realizations will be denoted $\mathbf{R}(L)$.

Obviously, each realization is a pseudo-realization. The representation of a plausibility measure by a probability on the set of pseudo-realizations according to Theorem 1.3 is still not unique. However, uniqueness can be achieved if we restrict ourselves to realizations. For a proof see [22, Theorem 4.1] and, in a more restrictive setting, [19, 21] (compare also [18, 36]).

Theorem 2.4 *For each plausibility measure m on a bounded lattice L there exists a unique regular Borel probability measure P on the space $\mathbf{PR}(L)$ of all pseudo-realizations such that for all $\alpha \in L$ the representation property (1) holds and the support of P is contained in the subspace of all realizations, i.e.,*

$$P(\mathbf{R}(L)) = 1.$$

Example 2.5 Consider again the bounded lattice L and the uncertainty measure m of Example 1.4. Obviously, m is a plausibility measure if and only if $a + b \geq 1$ (but never a possibility measure). Since $\mathbf{R}(L) = \{\omega_1, \omega_2, \omega_3\}$, the unique probability measure P representing m with $P(\mathbf{R}(L)) = 1$ is obtained if we put $p = a + b - 1$, i.e.,

$$P(\{\omega_1\}) = a + b - 1, \qquad P(\{\omega_2\}) = 1 - b, \qquad P(\{\omega_3\}) = 1 - a.$$

If the De Morgan algebra satisfies some kind of distributivity then the integral somehow reflects the structure of the underlying plausibility measure [22, Proposition 5.3].

Theorem 2.6 *Let m be a σ-smooth plausibility measure on a De Morgan algebra L which satisfies the additional condition that the meet $\wedge$ is distributive over countable joins. Then for each non-empty finite subset $\{F_1, F_2, \ldots, F_n\}$ of $\mathcal{D}^+(\mathbb{R}, L)$ we have*

$$\int_L \Big(\bigwedge_{i=1}^{n} F_i\Big)\, dm \;\leq\; \sum_{i=1}^{n} (-1)^{i-1} \sum_{1 \leq j_1 < \cdots < j_i \leq n} \int_L \Big(\bigvee_{k=1}^{i} F_{j_k}\Big)\, dm.$$

3 Valuations

Definition 3.1 Let $(L, \leq)$ be a bounded lattice. An uncertainty measure m on L is called a *valuation* if for all $\alpha, \beta \in L$ we have

$$\text{(VAL)} \qquad m(\alpha \wedge \beta) + m(\alpha \vee \beta) = m(\alpha) + m(\beta).$$

Remark 3.2

(i) If the bounded lattice L is a chain, then trivially each uncertainty measure on L is a valuation.

(ii) Each valuation m on a distributive bounded lattice L is also plausibility measure [22, Proposition 3.5].

For valuations, we can obtain a unique representation by a probability measure with an even smaller support with a richer structure (a lattice homomorphism rather than a semi-lattice homomorphism), in analogy to the classical situation [22, Corollary 4.1].

Definition 3.3 Let $(L, \leq)$ be a bounded lattice. A realization $\omega : L \longrightarrow \{0, 1\}$ is called *coherent* if for all $\alpha, \beta \in L$

$$\text{(R4)} \qquad \omega(\alpha \wedge \beta) = \min(\omega(\alpha), \omega(\beta)).$$

The set of all coherent realizations will be denoted $\mathbf{CR}(L)$.

Theorem 3.4 *Let m be a plausibility measure on a bounded lattice L and P the unique regular Borel probability measure on the space $\mathbf{PR}(L)$ such that for all $\alpha \in L$ the representation property (1) holds. Then m is a valuation if and only if the support of P is contained in the subspace of all coherent realizations, i.e.,*

$$P(\mathbf{CR}(L)) = 1.$$

Example 3.5 Reconsider the bounded lattice L and the uncertainty measure m of Examples 1.4 and 2.5. Of course, m is a valuation if and only if $a + b = 1$. Since $\mathbf{CR}(L) = \{\omega_2, \omega_3\}$, the unique probability measure P representing m with $P(\mathbf{CR}(L)) = 1$ is obtained if we put $p = 0$, i.e.,

$$P(\{\omega_2\}) = a, \qquad P(\{\omega_3\}) = 1 - a.$$

There is a canonical extension of the addition of non-negative real numbers to $\mathcal{D}^+(\mathbb{R}, L)$ (see, e.g., [16]).

Definition 3.6 Let $(L, \leq, {}^\perp)$ be a σ-complete De Morgan algebra which satisfies the additional condition that the meet $\wedge$ is distributive over countable joins. Then the *sum* $F \oplus G$ of $F, G \in \mathcal{D}^+(\mathbb{R}, L)$ is defined by

$$(F \oplus G)(r) = \bigvee \{F(r_1) \wedge G(r_2) \mid r_1, r_2 \in \mathbb{Q}, r_1 + r_2 > r\}.$$

Remark 3.7

(i) It is not difficult to see that $(\mathcal{D}^+(\mathbb{R}, L), \oplus)$ is a commutative semigroup with unit element H_0.

(ii) For all $F, G \in \mathcal{D}^+(\mathbb{R}, L)$ and for all $\omega \in \mathbf{CR}(L)$ we have (for a proof see [22, Lemma 5.4])

$$(F \oplus G)^{(q)}(\omega) = F^{(q)}(\omega) + G^{(q)}(\omega).$$

Again, the same type of distributivity allows us to obtain the additivity of the integral with respect to valuations [22, Proposition 5.4].

Theorem 3.8 *Let $(L, \leq, {}^\perp)$ be a σ-complete De Morgan algebra which satisfies the additional condition that the meet $\wedge$ is distributive over countable joins, and let m be a valuation on L. Then for all $F, G \in \mathcal{D}^+(\mathbb{R}, L)$ we have*

$$\int_L (F \oplus G) dm = \int_L F\, dm + \int_L G\, dm.$$

4 Additive measures on MV-algebras

Definition 4.1

(i) A quadruple $(L, \leq, \sqcap, \rightarrow)$ is called a *bounded, integral, residuated, commutative ℓ-monoid* [4] if the following conditions are satisfied:

(a) $(L, \leq)$ is a bounded lattice with universal bounds $\mathbf{0}$ and $\mathbf{1}$;

(b) $(L, \sqcap)$ is a commutative monoid with unit element $\mathbf{1}$;

(c) For the binary operation $\rightarrow$ and for all $\alpha, \beta, \gamma \in L$ we have

$$\beta \sqcap \gamma \leq \alpha \iff \gamma \leq \beta \rightarrow \alpha. \qquad \text{(Residuation)}$$

(ii) A bounded, integral, residuated, commutative ℓ-monoid $(L, \leq, \sqcap, \to)$ is called a *Girard algebra* if for all $\alpha \in L$

$$(\alpha \to \mathbf{0}) \to \mathbf{0} = \alpha. \qquad \text{(Involution)}$$

(iii) A Girard algebra $(L, \leq, \sqcap, \to)$ is called an *MV-algebra* if for all $\alpha, \beta \in L$

$$\beta \sqcap (\beta \to \alpha) = \alpha \wedge \beta. \qquad \text{(Divisibility)}$$

In this chapter, we shall not deal with Girard algebras which are not MV-algebras. Girard algebras play, however, a crucial role in the context of conditioning operators (see Chapter 12).

Remark 4.2

(i) The operation $\to$ in a bounded, integral, residuated, commutative ℓ-monoid is usually called the *residual implication.*

(ii) In a Girard algebra it is possible to define the *residual complement* $^\perp$ and the *dual operation* $\sqcup$ associated with $\sqcap$ by, respectively,

$$\alpha^\perp = \alpha \to \mathbf{0},$$
$$\alpha \sqcup \beta = (\alpha^\perp \sqcap \beta^\perp)^\perp.$$

(iii) Each Girard algebra and, subsequently, each MV-algebra is a De Morgan algebra, and orthogonality is equivalent to *disjointness* with respect to $\sqcap$, i.e.,

$$\alpha \leq \beta^\perp \iff \alpha \sqcap \beta = \mathbf{0}.$$

(iv) The definition of MV-algebras given above is equivalent to the original definition in [10], where $+$ (addition), $\cdot$ (multiplication) and $^-$ (complementation) were used rather than $\sqcup$, $\sqcap$ and $^\perp$, respectively.

(v) An MV-algebra is a Boolean algebra if and only if $\sqcap = \wedge$.

Example 4.3

(i) Take the unit interval $[0, 1]$, the operation $\sqcap = T_{\mathbf{L}}$, i.e., the Łukasiewicz t-norm defined by

$$T_{\mathbf{L}}(a, b) = \max(a + b - 1, 0), \tag{6}$$

and the Łukasiewicz implication given by

$$a \to b = \min(1 - a + b, 1),$$

then $([0,1], \leq, \sqcap, \rightarrow)$ is an MV-algebra. Observe that here the residual complement $^{\perp}$ and the dual operation $\sqcup$ are given by, respectively,

$$a^{\perp} = 1 - a,$$
$$a \sqcup b = \min(a + b, 1).$$

(ii) The class $[0,1]^X$ of all fuzzy subsets of a non-empty set X, equipped with the pointwise extensions of the operations in (i), is also an MV-algebra. Moreover, each semi-simple MV-algebra is isomorphic to a sub-MV-algebra of $[0,1]^X$ for some set X (see [2, 3, 10]). A complete characterization of MV-algebras, using nonstandard real-valued functions, was given in [12].

In the following we list some of the more important properties of MV-algebras. In particular, property (v) below establishes a generalization of the classical disjoint decomposition of elements in L. This will be crucial in the context of additive uncertainty measures.

Remark 4.4 In an *MV*-algebra $(L, \leq, \sqcap, \rightarrow)$ the following properties hold [10, 20, 41]:

(i) $(L, \leq)$ is a distributive lattice,

(ii) $\alpha \sqcap \alpha^{\perp} = \mathbf{0}$ and $\alpha \sqcup \alpha^{\perp} = \mathbf{1}$,

(iii) $\alpha \rightarrow \beta = \alpha^{\perp} \sqcup \beta$,

(iv) $(\beta \rightarrow \alpha) \rightarrow \alpha = \alpha \vee \beta$,

(v) $\beta = (\alpha \wedge \beta) \sqcup (\alpha^{\perp} \sqcap \beta)$.

Definition 4.5 Let $(L, \leq, \sqcap, \rightarrow)$ be an MV-algebra. An uncertainty measure m on L is said to be *additive* if

(ADD) $\quad m(\alpha \sqcup \beta) = m(\alpha) + m(\beta)$ whenever $\alpha \sqcap \beta = \mathbf{0}$.

It is interesting to note that for functions $m : L \longrightarrow [0,1]$, where L is an MV-algebra, the additivity (ADD) implies the isotonicity (M2). The following result states that additive uncertainty measures on MV-algebras are special valuations [22, Proposition 3.4]. Moreover, on MV-algebras (ADD) is equivalent to a property, which in general is strictly stronger ([41, Proposition 2.1]). Also, additive uncertainty measures on MV-algebras are naturally related to measures on AF C^*-algebras [34].

Theorem 4.6 *Let m be an uncertainty measure on an MV-algebra L. Then we have:*

(i) If m is additive then m is a valuation.

(ii) m is additive if and only if for all $\alpha, \beta \in L$

$$m(\alpha \sqcap \beta) + m(\alpha \sqcup \beta) = m(\alpha) + m(\beta).$$

5 Measures on tribes of fuzzy sets

In this section, we shall consider uncertainty measures on certain classes of *fuzzy subsets* [43] of a non-empty set X, which are described by their so-called membership functions $\mu : X \longrightarrow [0,1]$ (of course, in this context each crisp subset A of X will be identified with its characteristic function χ_A).

Operations on fuzzy sets are usually based on *t-norms* [35], i.e., binary operations T on $[0,1]$ such that $([0,1], \leq, T)$ is a commutative, ordered semigroup with unit element 1. The most important examples of continuous t-norms are the minimum, the product, and the Łukasiewicz t-norm T_{L} given in (6). For each $s \in]0,1[\cup]1,\infty[$ the Frank t-norm T_s [14] is given by

$$T_s(a,b) = \log_s \left(1 + \frac{(s^a - 1)(s^b - 1)}{s - 1}\right).$$

The limit cases for s going to 0, 1 and ∞ are minimum, product and T_{L}, respectively, and we shall simply write for them T_0, T_1 and T_∞ in this context.

For each left continuous t-norm T, the quadruple $([0,1], \leq, T, \to)$ is a bounded, integral, residuated, commutative ℓ-monoid, where the residual implication $\to$ can be computed by

$$a \to b = \bigvee \{c \in [0,1] \mid T(a,c) \leq b\}.$$

Among the Frank t-norms, only T_{L} leads to an MV-algebra in which case we have $a^\perp = 1 - a$ (see Example 4.3(i)).

However, for an arbitrary t-norm T the dual t-conorm S is defined by

$$S(a,b) = 1 - T(1-a, 1-b).$$

Within the class of continuous t-norms, the pairs of Frank t-norms and their dual t-conorms (together with the ordinal sums thereof [35]) are the only solutions of the functional equation [14]

$$T(a,b) + S(a,b) = a + b,$$

a fact which will be crucial in the following. For this reason, we shall restrict ourselves to Frank t-norms only.

When working with fuzzy sets, the order as well as t-norms and t-conorms are extended pointwise.

Definition 5.1 Let $\sqcap = T_s$ be a Frank t-norm for some $s \in [0, \infty]$, and let $\mathcal{T} \subseteq [0,1]^X$ be a class of fuzzy subsets of the non-empty set X containing the empty set and such that for all $\mu, \nu \in \mathcal{T}$ we have

$$\mu \sqcap \nu \in \mathcal{T},$$

and define the complement of $\mu \in \mathcal{T}$ by

$$\mu^{\perp}(x) = 1 - \mu(x).$$

If $(\mathcal{T}, \leq, ^{\perp})$ is a σ-complete De Morgan algebra, then the quadruple $(\mathcal{T}, \leq, \sqcap, ^{\perp})$ will be called a *T_s-tribe* on X.

In the light of [9, Proposition 2.7], T_s-tribes in the sense given above are exactly T_s-tribes in the sense of [9, Definition 2.5], see also [23, 25]. Note, however, that tribes were introduced there with respect to arbitrary t-norms. The following considerations therefore remain valid for all (measurable) t-norms.

Remark 5.2

(i) Trivially, each σ-algebra of subsets of X is a T_s-tribe for each $s \in [0, \infty]$. In particular, for each T_s-tribe $\mathcal{T}$, the set $\mathcal{T}^{\vee}$ of all crisp subsets of X contained in $\mathcal{T}$ is a σ-algebra, and hence a T_s-tribe.

(ii) Conversely, given a σ-algebra $\mathcal{A}$ of crisp subsets of X, the family $\mathcal{A}^{\wedge}$ of all $\mathcal{A}$-measurable fuzzy subsets of X is a T_s-tribe for each $s \in [0, \infty]$.

(iii) A T_s-tribe $\mathcal{T}$ is said to be *generated* if there exists a σ-algebra $\mathcal{A}$ such that $\mathcal{T} = \mathcal{A}^{\wedge}$.

In the properties listed below the special structure of the Frank t-norms and t-conorms is essential.

Remark 5.3

(i) By definition, T_0-tribes are exactly σ-complete De Morgan algebras.

(ii) If $s \in \,]0, \infty[$, then each T_s-tribe is a $T_{\mathbf{L}}$-tribe, i.e., an MV-algebra [9, Proposition 2.7(i)].

(iii) If $s \in \,]0, \infty]$, a T_s-tribe $\mathcal{T}$ on X is generated if and only if it contains all constant fuzzy subsets of X [9, Proposition 3.3].

(iv) For $s \in \,]0, \infty]$ and for each T_s-tribe $\mathcal{T}$ we also have $\mathcal{T} \subseteq (\mathcal{T}^{\vee})^{\wedge}$, i.e., each element in $\mathcal{T}$ is $\mathcal{T}^{\vee}$-measurable [9, Theorem 3.2]. Additional results concerning T_s-tribes can be found in [28, 32, 33].

Definition 5.4 Let $(\mathcal{T}, \leq, \sqcap, ^{\perp})$ be a T_s-tribe for some $s \in [0, \infty]$. A σ-smooth uncertainty measure $\mathbf{m}$ on $\mathcal{T}$ is called a *T_s-measure* on $\mathcal{T}$ if for all $\mu, \nu \in \mathcal{T}$ we have

$$\text{(TM)} \qquad \mathbf{m}(\mu \sqcap \nu) + \mathbf{m}(\mu \sqcup \nu) = \mathbf{m}(\mu) + \mathbf{m}(\nu).$$

Remark 5.5

(i) Each T_s-measure is a σ-smooth valuation, i.e., a T_0-measure [26].

(ii) Each $T_{\mathbf{L}}$-measure on a T_s-tribe $\mathcal{T}$ with $s \in \,]0, \infty]$ is a σ-smooth additive uncertainty measure on (the MV-algebra) $\mathcal{T}$.

(iii) Each T_s-measure satisfies an additivity property analogous to (ADD) (introduced in the context of MV-algebras only) which, however, is significantly weaker than (TM) with the obvious exception if $\sqcap = T_{\mathbf{L}}$.

We have essentially three types of T_s-measures $\mathbf{m}$ on tribes $\mathcal{T}$ of fuzzy sets: in descending order of generality they are valuations (for arbitrary $s \in [0, \infty]$), T_s-measures with $s \in \,]0, \infty]$, and additive uncertainty measures (in the case $s = \infty$). Accordingly, there are three representation theorems where $\mathbf{m}(\mu)$ can be written as a (Lebesgue) integral with respect to the restriction of $\mathbf{m}$ to the crisp sets in $\mathcal{T}$, and where the integrand depends on μ in a way which corresponds to the respective level of generality.

The most general representation theorem uses the powerful concept of a $\mathcal{T}^{\vee}$-Markov kernel [1]. In our situation, a $\mathcal{T}^{\vee}$-Markov kernel is a two-place function $K : X \times (\mathcal{B} \cap [0, 1[) \longrightarrow [0, 1]$ which is $\mathcal{T}^{\vee}$-measurable in its first component, and a probability measure on $\mathcal{B} \cap [0, 1[$ in its second component. A complete proof can be found in [9, Theorem 5.8], see also [24].

Theorem 5.6 *Let $\mathcal{T}$ be a generated (T_0-)tribe on X. If $\mathbf{m} : \mathcal{T} \longrightarrow [0, 1]$ is a σ-smooth valuation on $\mathcal{T}$, then there exists a unique probability m on $\mathcal{T}^{\vee}$, namely, the restriction of $\mathbf{m}$ to $\mathcal{T}^{\vee}$, and an m-almost everywhere uniquely determined $\mathcal{T}^{\vee}$-Markov kernel K such that for all $\mu \in \mathcal{T}$*

$$\mathbf{m}(\mu) = \int_X K(x, [0, \mu(x)[\,)\, dm(x).$$

Recall that each T_s-measure on a generated tribe is also a σ-smooth valuation. Therefore, the Markov kernel representation given in Theorem 5.6 applies to all T_s-measures on generated tribes.

The other extreme are additive uncertainty measures on a $T_{\mathbf{L}}$-tribe (i.e, on an MV-algebra) $\mathcal{T}$ which have the simplest possible integral representation (which is just the construction suggested in [44]) even in the case of a non-generated $T_{\mathbf{L}}$-tribe $\mathcal{T}$. A full proof can be found in [9, Theorem 6.2], compare also [5, 6, 7, 8, 26, 27].

Theorem 5.7 *Let $\mathcal{T}$ be a $T_{\mathbf{L}}$-tribe on X. If $\mathbf{m} : \mathcal{T} \longrightarrow [0,1]$ is a σ-smooth additive uncertainty measure on $\mathcal{T}$, then there exists a unique probability m on $\mathcal{T}^{\vee}$, namely, the restriction of $\mathbf{m}$ to $\mathcal{T}^{\vee}$, such that for all $\mu \in \mathcal{T}$*

$$\mathbf{m}(\mu) = \int_X \mu \, dm.$$

The intermediate case, i.e., $s \in]0, \infty[$, leads to a representation of T_s-measures on generated tribes where the corresponding Markov kernel has a rather special form (which is independent of the index s). Again, a proof appeared in [9, Theorem 7.1], an earlier version in [26].

Theorem 5.8 *Let T_s be a Frank t-norm with $s \in]0, \infty[$ and $\mathcal{T}$ be a generated (T_s-)tribe on X. If $\mathbf{m} : \mathcal{T} \longrightarrow [0,1]$ is a T_s-measure on $\mathcal{T}$, then there exists a unique probability m on $\mathcal{T}^{\vee}$, namely, the restriction of $\mathbf{m}$ to $\mathcal{T}^{\vee}$, and an m-almost everywhere uniquely determined $\mathcal{T}^{\vee}$-measurable function $f : X \longrightarrow [0,1]$ such that for all $\mu \in \mathcal{T}$*

$$\mathbf{m}(\mu) = \int_{\{\mu > 0\}} [f + (1 - f) \cdot \mu] \, dm.$$

Concluding remarks

Following the main lines sketched in the introduction, we could of course not cover all possible aspects of generalized measures.

In particular, the range of all measures was the unit interval equipped with the usual addition. Several generalizations, e.g., decomposable measures where the addition is replaced by an Archimedean t-conorm, have not been mentioned here [29, 30, 40].

The concept of the additivity of an uncertainty measure is not as clear when we leave the safe ground of MV-algebras. This is of particular interest in the context of conditioning operators, where Girard algebras appear in a very natural way. First steps into this directions were done in [22, 42] and in Chapter 12.

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CHAPTER 12

On Conditioning Operators*

U. Höhle and S. Weber

Introduction

The construction of *conditional events* (so-called measure-free conditioning) has a long history and is one of the fundamental problems in non-deterministic system theory (cf. [6]). In particular, the iteration of measure-free conditioning is still an open problem. The present paper tries to make a contribution to this question. In particular, we give an axiomatic introduction of *conditioning operators* which act as binary operations on the universe of events. The corresponding axiom system of this type of operators focus special attention on the intuitive understanding that the event 'α given β' is somewhere in "between" 'α and β' and 'β implies α'. A detailed motivation of these axioms can be found in Section 2 and Proposition 6.4.

A further problem of measure-free conditioning is to express the uncertainty of conditional events. Referring to *Lewis' Triviality Result* (cf. [4]), it is interesting to see that standard methods from probability theory (e.g. conditional probabilities) can not be applied to this type of question. In our paper we overcome this problem by applying a concept of generalized probability theory which is not based on Boolean algebras, but on MV-algebras (see also previous Chapter 11 in this volume). In this context, we are able to show that Lewis' Triviality Result does not hold; i.e. there exist (additive) uncertainty measures which are compatible with certain conditioning operators.

The paper is organized as follows: After presenting the lattice-theoretic prerequisites (Section 1) we define conditional events $(\alpha \parallel \beta)$ as ordered intervals in the case of MV-algebras. In Section 3, we construct the canonical Girard extension of any Girard algebra (cf. Main Theorem 3.1). Subsequently, in Section 4 we show that there exists an intimate relationship between conditioning operators and mean value functions which are compatible with the residual

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complement in the underlying Girard algebra. In Section 5, the existence of conditioning operators on the canonical Girard algebra extension is established (cf. Theorem 5.3) which includes all interval based, conditional events (cf. Remark 5.6). Finally, Section 6 deals with the relationship between uncertainty measures and conditioning operators in which mean value measure extensions play a significant role (cf. Theorem 6.8).

1 Lattices of events

Let $(L, \leq)$ be a bounded lattice — i.e. $(L, \leq)$ is a lattice provided with universal bounds $0, 1$. With regard to possible applications to probability theory we interpret elements of L as *events*. In this context the universal upper (lower, resp.) bound represents the *sure* (*impossible*, resp.) *event*. In order to deal with various non-classical logical systems, we enrich the structure of bounded lattices as follows:

Definition 1.1 (a) A triple $(L, \leq, \sqcap)$ is called a bounded, integral, residuated, commutative ℓ–monoid iff the following conditions are satisfied:

(i) $(L, \leq)$ is a bounded lattice.

(ii) $(L, \sqcap)$ is a commutative monoid.

(iii) There exists a further binary operation $\rightarrow : L \times L \longmapsto L$ provided with the following property:
$$\beta \sqcap \gamma \leq \alpha \iff \gamma \leq \beta \rightarrow \alpha \quad . \qquad \text{(Residuation)}$$

(iv) The upper universal bound 1 acts as unity w.r.t. $\sqcap$. (Integrality)

□

It is easy to see that the binary operation appearing in (iii) is uniquely determined by the residuation property; hence $\rightarrow$ is also called the residual implication. Further, in any bounded, integral, residuated, commutative ℓ- monoid the universal lower bound 0 acts as zero, and the semigroup operation $\sqcap$ is distributive over finite meets.

Definition 1.2 (Girard algebra) A bounded, integral, residuated, commutative ℓ–monoid $\mathcal{G} = (L, \leq, \sqcap)$ is called a Girard algebra iff the following axiom is satisfied:

$$(\alpha \rightarrow 0) \rightarrow 0 = \alpha \quad . \qquad \text{(Involution)}$$

□

In any Girard algebra the residual complement and the dual operation associated with $\sqcap$ can be defined by

$$\alpha^{\perp} = \alpha \to 0 \quad , \quad \alpha \sqcup \beta = (\alpha^{\perp} \sqcap \beta^{\perp})^{\perp} \quad .$$

In case of disjoint (or orthogonal) events,

$$\alpha \perp \beta \iff \alpha \sqcap \beta = 0 \quad ,$$

the disjoint union will be denoted by $\alpha \dot{\sqcup} \beta$.

Definition 1.3 (MV-algebras)
(a) A Girard algebra $\mathcal{G}$ is called an MV-algebra iff the following axiom is satisfied:

$$\beta \sqcap (\beta \to \alpha) = \alpha \wedge \beta \quad . \qquad \text{(Divisibility)}$$

(b) An MV-algebra is called a Boolean algebra iff $\sqcap = \wedge$.

(c) An MV-algebra $\mathcal{M} = (L, \leq, \sqcap)$ has square roots iff there exists a unary operation $r : L \longmapsto L$ satisfying the following conditions

(S1) $\quad r(\alpha) \sqcap r(\alpha) = \alpha \quad ,$

(S2) $\quad \beta \sqcap \beta \leq \alpha \Longrightarrow \beta \leq r(\alpha) \quad .$ □

It is easy to see that the unary operation r is uniquely determined by the axioms (S1) and (S2). Therefore r is also called the square root operation on L. Obviously Boolean algebras have always square roots, and the square root operation coincides with the identity map in this context.

An MV-algebra $\mathcal{M}$ is called strict iff $\mathcal{M}$ has square roots and the square root operation r satisfies the following additional condition:

(S3) $\quad (r(0))^{\perp} = r(0) \quad .$ (Strictness)

In this context it is interesting to see that any MV-algebra with square roots is a either Boolean algebra or a strict MV-algebra algebra or a product of a Boolean algebra and a strict MV-algebra (cf. Theorem 2.21 in [1]).

In the following propositions we collect fundamental properties of Girard and MV-algebras. Their proofs can be found in Section 2 in [1] or in [7] or, in the context of Conditioning, in [8, 9].

Proposition 1.4 *In a Girard algebra $(L, \leq, \sqcap)$ the following properties hold:*

(a) $\quad \alpha \sqcap (\beta \vee \gamma) = (\alpha \sqcap \beta) \vee (\alpha \sqcap \gamma) \quad .$

(b) $\beta \to \alpha = \beta \to (\beta \sqcap (\beta \to \alpha))$.

(c) $\beta \to \alpha = \beta^{\perp} \sqcup \alpha = \beta^{\perp} \dot{\sqcup} (\alpha \wedge \beta)$.

(d) *The lower universal bound* 0 *acts as unity w.r.t.* $\sqcup$.

(e) $\alpha \dot{\sqcup} \alpha^{\perp} = 1$.

(f) $(\alpha \vee \beta)^{\perp} = \alpha^{\perp} \wedge \beta^{\perp}$, $(\alpha \wedge \beta)^{\perp} = \alpha^{\perp} \vee \beta^{\perp}$.

Proposition 1.5 *In an MV-algebra* $(L, \leq, \sqcap)$ *the following properties hold:*

(a) $(L, \leq)$ *is a distributive lattice.*

(b) $\alpha \sqcap (\beta \wedge \gamma) = (\alpha \sqcap \beta) \wedge (\alpha \sqcap \gamma)$.

(c) $(\beta \to \alpha) \to \alpha = \alpha \vee \beta$. *(MV-Property)*

(d) $\beta = (\alpha \wedge \beta) \dot{\sqcup} (\alpha^{\perp} \sqcap \beta)$. *(Disjoint Decomposition)*

(e) $\alpha \to (\alpha \sqcap \beta) = \alpha^{\perp} \vee \beta$.

Proposition 1.6 *In a strict MV-algebra* $(L, \leq, \sqcap)$ *the following properties hold:*

(a) $\alpha \leq r(\alpha)$, $\alpha \leq \beta \implies r(\alpha) \leq r(\beta)$.

(b) $r(\beta \to \alpha) = r(\beta) \to r(\alpha)$.

(c) $(r(\alpha^{\perp}))^{\perp} = r(\alpha) \sqcap r(0)$, $r(\alpha) \sqcap r(\alpha^{\perp}) = r(0)$.

(d) $(r(\alpha) \sqcap r(\beta))^{\perp} = r(\alpha^{\perp}) \sqcap r(\beta^{\perp})$.

(e) $r(\alpha) = \alpha \dot{\sqcup} (r(\alpha))^{\perp}$.

2 Interval based conditional events in the case of MV- algebras

Let $\mathcal{M} = (L, \leq, \sqcap)$ be an MV-algebra, and let α and β be events of L. The problem of defining the *conditional event* 'α *given* β' deals with the intuitive understanding that 'α given β' is neither the conjunction of α and β nor the implication 'β implies α', but 'α given β' is somewhere located between the conjunction and the implication of α and β. This understanding motivates the following definition: For any event $\alpha, \beta \in L$ the **interval based conditional event** $(\alpha \parallel \beta)$ of α **given** β is defined as the order interval:

$$(\alpha \parallel \beta) \;=\; [\alpha \wedge \beta \,,\, \beta \rightarrow \alpha] \quad .$$

The set of all such conditional events will be denoted by $\tilde{L}$. This construction was introduced in [3] for Boolean algebras and further developed in [6], and for MV-algebras in [8]. Because of Proposition 1.5(c) we have the following

Lemma 2.1 (Interval representation)

Order intervals of MV-algebras $(L, \leq, \sqcap)$ are in a one- to-one correspondence to conditional events in $\tilde{L}$ via:

$$[\alpha \,,\, \gamma] \;=\; (\alpha \parallel \gamma \rightarrow \alpha) \quad .$$

$\square$

Further , it is easy to see that the set $\tilde{L}$ of conditional events extends the set L of (unconditional) events in the following sense:

(I1) $\quad (\alpha \parallel 1) \;=\; \{\alpha\} \quad ,$

(I2) $\quad (\alpha \parallel \beta) = (\gamma \parallel \delta) \quad \Longleftrightarrow \quad \alpha \wedge \beta = \gamma \wedge \delta \quad , \quad \beta = \delta \quad .$

A special case of (I2) is the subsequent relation:

(I2′) $\quad (\alpha \wedge \beta \parallel \beta) \;=\; (\alpha \parallel \beta) \quad .$

In order to extend the partial ordering from the set L of all (unconditional) events to the set $\tilde{L}$ of all (interval based) conditional events, it follows immediately from (I1) that we cannot use the set-inclusion. We make the following choice:

$$(\alpha \parallel \beta) \preccurlyeq (\gamma \parallel \delta) \quad \Longleftrightarrow \quad \alpha \wedge \beta \leq \gamma \wedge \delta \quad , \quad \beta \rightarrow \alpha \leq \delta \rightarrow \gamma \quad ,$$

The reflexivity and transitivity of $\preccurlyeq$ is evident. The anti-symmetry follows from (I2) and Proposition 1.5(c). Moreover we observe:

$$\alpha \quad \leq \quad \beta \qquad \Longleftrightarrow \qquad (\alpha \parallel 1) \quad \preccurlyeq \quad (\beta \parallel 1) \quad ;$$

hence $\preccurlyeq$ fulfils the desired properties. Some order-theoretic aspects of $(\tilde{L}, \preccurlyeq)$ are collected in:

Lemma 2.2 (Lattice extension)

(a) The following monotonicity properties hold:

(I3) $\quad \alpha_1 \leq \alpha_2 \quad \Longrightarrow \quad (\alpha_1 \parallel \beta) \preccurlyeq (\alpha_2 \parallel \beta) \quad ,$

(I4) $\quad \beta_1 \leq \beta_2 \quad , \quad \alpha \wedge \beta_2 \leq \alpha \wedge \beta_1 \quad \Longrightarrow \quad (\alpha \parallel \beta_2) \preccurlyeq (\alpha \parallel \beta_1) \quad .$

(b) Furthermore, $(\tilde{L}, \preccurlyeq)$ *is a bounded lattice. In particular, the lattice operations and universal bounds are given by*

$$(\alpha \,\|\, \beta) \wedge (\gamma \,\|\, \delta) \;=\; [\,(\alpha \wedge \beta) \wedge (\gamma \wedge \delta)\,,\,(\beta \to \alpha) \wedge (\delta \to \gamma)\,] \quad ,$$

$$(\alpha \,\|\, \beta) \vee (\gamma \,\|\, \delta) \;=\; [\,(\alpha \wedge \beta) \vee (\gamma \wedge \delta)\,,\,(\beta \to \alpha) \vee (\delta \to \gamma)\,] \quad ,$$

$$\tilde{0} = (0 \,\|\, 1) = \{0\} \quad , \quad \tilde{1} = (1 \,\|\, 1) = \{1\} \quad .$$

(c) $(L, \leq)$ *is a sublattice of* $(\tilde{L}, \preccurlyeq)$.

PROOF. The assertions follow immediately from the definitions.

□

In the following considerations we are interested in extending also the monoidal structure of the given MV-algebra $\mathcal{M} = (L, \leq, \sqcap)$ to $(\tilde{L}, \preccurlyeq)$. Referring to the definition of the partial ordering $\preccurlyeq$ the immediate natural choice seems to be the following binary operation $\triangle$ on $\tilde{L}$ defined as follows:

$$(\alpha \,\|\, \beta) \,\triangle\, (\gamma \,\|\, \delta) \;=\; [\,(\alpha \wedge \beta) \sqcap (\gamma \wedge \delta)\,,\,(\beta \to \alpha) \sqcap (\delta \to \gamma)\,]\,.$$

It is not difficult to show that $(\tilde{L}, \preccurlyeq, \triangle)$ is a bounded, integral residuated, commutative ℓ–monoid. Because of

$$(\alpha \,\|\, \beta)^{\star} \;=\; (\alpha \,\|\, \beta) \to (0 \,\|\, 1) \;=\; (\alpha^{\perp} \sqcap \beta \,\|\, 1)$$

the residual complement is not involutory; hence $(\tilde{L}, \preccurlyeq, \triangle)$ fails to be a Girard algebra. On the other hand there exists an order reversing involution $' : L \longmapsto L$ on $\tilde{L}$ which is determined by

$$(\alpha \,\|\, \beta)' \;=\; [\,(\beta \to \alpha)^{\perp}\,,\,(\alpha \wedge \beta)^{\perp}\,] \;=\; (\alpha^{\perp} \sqcap \beta \,\|\, \beta) \quad . \tag{2.1}$$

Since $'$ is obviously an extension of the residual complement in L, we ask the following

Question: Does there exist a Girard algebra structure $\mathcal{G}$ on $(\tilde{L}, \preccurlyeq)$ such that the residual complement in $\mathcal{G}$ coincides with the order reversing involution defined in (2.1) and such that the given MV-algebra is a Girard subalgebra of $\mathcal{G}$?

As the following theorem shows we can give an affirmative answer to the previous question:

Theorem 2.3 (Girard algebra extension)

There exists a binary operation $\sqcap$ *on* $\tilde{L}$ *such that* $\tilde{\mathcal{M}} = (\tilde{L}, \preccurlyeq, \sqcap)$ *is a Girard algebra. In particular,* $\sqcap$ *and the corresponding* residual implication $\to$ *are given by*

$$(\alpha \,\|\, \beta) \sqcap (\gamma \,\|\, \delta) \;=$$
$$[\,(\alpha \wedge \beta) \sqcap (\gamma \wedge \delta)\,,\,((\alpha \wedge \beta) \sqcap (\delta \to \gamma)) \vee ((\beta \to \alpha) \sqcap (\gamma \wedge \delta))\,]\,,$$

$$(\alpha \,\|\, \beta) \to (\gamma \,\|\, \delta) \;=$$
$$[\,(\beta \vee \delta) \to \epsilon\,,\,(\beta \sqcap \delta) \to \epsilon\,]\,, \quad \textit{where} \quad \epsilon = (\beta \to \alpha) \to (\gamma \wedge \delta)\,.$$

Further the residual complements *are determined by:*

$$\text{(I5)} \quad (\alpha \,\|\, \beta)^{\perp} = [\,(\beta \to \alpha)^{\perp}, (\alpha \wedge \beta)^{\perp}\,] = (\alpha^{\perp} \sqcap \beta \,\|\, \beta) \quad ,$$

where, particularly:

$$(0 \,\|\, 0)^{\perp} = (0 \,\|\, 0) = L \quad .$$

Finally the dual operation $\sqcup$ *associated with* $\sqcap$ *has the following form:*

$$(\alpha \,\|\, \beta) \sqcup (\gamma \,\|\, \delta) = [\,((\alpha \wedge \beta) \sqcup (\delta \to \gamma)) \wedge ((\beta \to \alpha) \sqcup (\gamma \wedge \delta)), (\beta \to \alpha) \sqcup (\delta \to \gamma)\,] \,.$$

PROOF. The assertion follows immediately from Lemma 2.1 and the Main Theorem 3.1.

□

Corollary 2.4 *Let* $\mathcal{M} = (L, \leq, \sqcap)$ *be an MV-algebra, and* $\tilde{\mathcal{M}} = (\tilde{L}, \preccurlyeq, \sqcap)$ *be the Girard algebra extension from Theorem 2.3. Then:*

$\tilde{\mathcal{M}}$ *is an MV-algebra* $\iff$ $\mathcal{M}$ *is a Boolean algebra.*

PROOF. The assertion follows immediately from Lemma 2.1 and Corollary 3.2.

□

Remark 2.5 (Boolean case) Let $L = \mathbb{B}$ be a Booelan algebra. Referring to Remark 3.3 infra it is interesting to see that every interval based conditional event of $\tilde{\mathbb{B}}$ can be represented by a continuous 3-valued map. In this sense *conditional events* are *special lattice-valued maps.*

□

3 Canonical extension of Girard algebras

In this subsection we show that the construction in Theorem 2.3 can be extended to any Girard algebra. Since order intervals $[\alpha, \beta]$ can be identified with ordered pairs (α, β) with $\alpha \leq \beta$, we first fix the following notation and terminology: Let $\mathcal{G} = (L, \leq, \sqcap)$ be a Girard algebra. Then the set

$$L_1 = \{\, (\alpha, \beta) \in L \times L \,:\, \alpha \leq \beta \,\}$$

will be called the canonical extension of L where L is identified with its diagonal $L_{\Delta} = \{\, (\alpha, \alpha) \,:\, \alpha \in L \,\}$. Further we provide L_1 with the partial ordering $\leq$ which arises as restriction from the product ordering on $L \times L$ — i.e.

$$(\alpha, \beta) \leq (\gamma, \delta) \iff \alpha \leq \gamma \quad , \quad \beta \leq \delta \quad .$$

Obviously $(L_1, \leq)$ is a bounded lattice, and the lattice-theoretic operations and universal bounds are given by

$$\begin{aligned} (\alpha,\beta) \wedge (\gamma,\delta) &= (\alpha \wedge \gamma, \beta \wedge \delta) \ , \\ (\alpha,\beta) \vee (\gamma,\delta) &= (\alpha \vee \gamma, \beta \vee \delta) \ , \\ (0,0) = 0 \quad &, \quad (1,1) = 1 \ . \end{aligned}$$

If we identify L with its diagonal L_Δ, then $(L, \leq)$ is a sublattice of $(L_1, \leq)$.

Main Theorem 3.1 (Canonical Girard algebra extension)

Let $\mathcal{G} = (L, \leq, \sqcap)$ be a Girard algebra structure. Then there exists a unique *binary operation on L_1 (also denoted by $\sqcap$) satisfying the following conditions:*

(i) $\mathcal{G}_1 := (L_1, \leq, \sqcap)$ *is a Girard algebra,*

(ii) $(\alpha,\alpha) \sqcap (\beta,\beta) = (\alpha \sqcap \beta, \alpha \sqcap \beta)$,

(iii) $(0,\alpha)^\perp = (\alpha^\perp, 1)$.

In particular the monoidal structure of $\mathcal{G}_1$ is given by:

$$(\alpha,\beta) \sqcap (\gamma,\delta) = (\alpha \sqcap \gamma, (\alpha \sqcap \delta) \vee (\beta \sqcap \gamma)) \ . \tag{3.1}$$

$$(\alpha,\beta) \rightarrow (\gamma,\delta) = ((\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \delta), \alpha \rightarrow \delta) \ . \tag{3.2}$$

$$(\alpha,\beta)^\perp = (\beta^\perp, \alpha^\perp) \ . \tag{3.3}$$

$$(\alpha,\beta) \sqcup (\gamma,\delta) = ((\alpha \sqcup \delta) \wedge (\beta \sqcup \gamma), \beta \sqcup \delta) \ . \tag{3.4}$$

PROOF. (a) Let $\otimes$ be any binary operation on L_1 satisfying (i)—(iii). First we show:

$$(\alpha,\alpha) \rightarrow (0,1) = (\alpha^\perp, 1) \ . \tag{3.5}$$

Let us choose an element $(\gamma,\delta) \in L_1$ satisfying the following condition

$$(*) \qquad (\alpha,\alpha) \otimes (\gamma,\delta) \leq (0,1).$$

Because of (i) the binary operation $\otimes$ is distributive over finite joins; hence we obtain:

$$[(\alpha,\alpha) \otimes (\gamma,\gamma)] \vee [(\alpha,\alpha) \otimes (0,\delta)] \leq (0,1) \ .$$

Now we apply (ii) and the integrality (cf. (i)) of $\otimes$ and obtain that the relation $(*)$ is equivalent to $(\alpha \sqcap \gamma, \alpha \sqcap \gamma) \leq (0,1)$; — i.e. $\gamma \leq \alpha^\perp$. Hence the relation (3.5) follows from the residuation property of $\mathcal{G}_1$ (cf. (i)). (b) Because of Condition (iii) and Proposition 1.4(c) we conclude from (3.5)

$$(\alpha,\alpha) \otimes (0,1) = (0,\alpha) \ .$$

Now we invoke the associativity of $\otimes$ and obtain from the previous relation:

$$\begin{aligned}(\alpha,\alpha)\otimes(0,\beta) &= [(\alpha,\alpha)\otimes(\beta,\beta)]\otimes(0,1)\\ &= (\alpha\sqcap\beta\,,\,\alpha\sqcap\beta)\otimes(0,1) \quad = \quad (0\,,\,\alpha\sqcap\beta)\ .\end{aligned}$$

Now we apply again the distributivity of $\otimes$ over finite joins

$$\begin{aligned}&(\alpha_1,\beta_1)\otimes(\alpha_2,\beta_2) \quad = \\ &[(\alpha_1,\alpha_1)\otimes(\alpha_2,\alpha_2)] \vee [(\alpha_1,\alpha_1)\otimes(0,\beta_2)] \vee [(0,\beta_1)\otimes(\alpha_2,\alpha_2)] \vee \\ &\qquad\qquad\qquad \vee\ [(0,\beta_1)\otimes(0,\beta_2)] \quad = \\ &(\alpha_1\sqcap\alpha_2\,,\,(\alpha_1\sqcap\beta_2)\vee(\beta_1\sqcap\alpha_2)) \vee [(0,\beta_1)\otimes(0,\beta_2)] \qquad ;\end{aligned}$$

hence the relation (3.1) follows from the observation

$$(0,\beta_1)\otimes(0,\beta_2) \quad \leq \quad (0,1)\otimes(0,1) \quad = \quad (0,0)\ .$$

(c) Because of formula (3.1) it is a matter of routine to check that $(L_1,\sqcap)$ is a commutative monoid. In order to verify the residuation property we proceed as follows: Let us consider an element $(\gamma,\delta)\in L_1$ satisfying the following condition:

$$(**) \qquad (\alpha_1,\beta_1)\sqcap(\gamma,\delta) \quad \leq \quad (\alpha_2,\beta_2)\ .$$

Because of (3.1) and the componentwise defined partial ordering the relation $(**)$ is equivalent to

$$\alpha_1\sqcap\gamma \leq \alpha_2 \quad , \quad \alpha_1\sqcap\delta \leq \beta_2 \quad , \quad \beta_1\sqcap\gamma \leq \beta_2\ .$$

Now we invoke the residuation property of $\mathcal{G}$ and obtain:

$$\gamma \quad \leq \quad (\alpha_1\rightarrow\alpha_2)\wedge(\beta_1\rightarrow\beta_2) \quad , \quad \delta \quad \leq \quad \alpha_1\rightarrow\beta_2\ .$$

Because of $\alpha_2 \leq \beta_2$ it is not difficult to verify (3.2). Further (3.3) is a special case of (3.2) , and (3.4) follows immediately from (3.1) and (3.3).

□

Let $\mathcal{G}_1$ be the Girard algebra constructed in Theorem 3.1. If we identify L with its diagonal L_Δ, then it is easy to see that the given Girard algebra $\mathcal{G}$ is a Girard subalgebra of $\mathcal{G}_1$. Therefore $\mathcal{G}_1$ is called the **canonical Girard algebra extension** of $\mathcal{G}$.

Corollary 3.2 *Let $\mathcal{G}$ be a Girard algebra, and $\mathcal{G}_1$ be its canonical Girard algebra extension. Then the following assertions are equivalent:*

(i) $\mathcal{G}_1$ *is an MV-algebra.*

(ii) $\mathcal{G}$ *is a Boolean algebra.*

PROOF. (a)((i) $\Longrightarrow$ (ii)) Let α be an element of L. Because of $(\alpha \sqcap \alpha, 1) \leq (\alpha, 1)$ the divisibility property implies

$$(\alpha \sqcap \alpha, 1) \quad = \quad (\alpha, 1) \sqcap ((\alpha, 1) \to (\alpha \sqcap \alpha, 1)) \quad .$$

Now we apply (3.1) and (3.2) and obtain

$$(*) \qquad \begin{array}{rcl} (\alpha \sqcap \alpha, 1) & = & (\alpha, 1) \sqcap ((\alpha \to (\alpha \sqcap \alpha)), 1) \\ & = & (\alpha \sqcap \alpha, (\alpha \vee (\alpha \to (\alpha \sqcap \alpha)))) \quad . \end{array}$$

Since $\mathcal{G}$ is a subalgebra of $\mathcal{G}_1$, $\mathcal{G}$ is also an MValgebra. Therefore we can derive from Proposition 1.5(e) and $(*)$ the following relation

$$1 \quad = \alpha \vee (\alpha \to (\alpha \sqcap \alpha)) \quad = \quad \alpha \vee \alpha^{\perp} \quad ,$$

hence the distributivity of $\sqcap$ over finite joins implies the idempotency of α w.r.t. $\sqcap$; i.e. $\mathcal{G}$ is a Boolean algebra.
(b) ((ii) $\Longrightarrow$ (i)) Let us assume $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$. Since $\mathcal{G}$ is a Boolean algebra, it is not difficult to verify the following relations:

$$\begin{array}{rcl} \alpha_1 & = & \alpha_2 \wedge (\alpha_2 \to \alpha_1) \wedge (\beta_2 \to \beta_1) \\ \beta_1 & = & (\alpha_2^{\perp} \wedge \beta_1) \vee (\alpha_2 \wedge \beta_1) \;=\; [(\alpha_2 \to \alpha_1) \wedge \beta_1] \vee [(\alpha_2 \wedge \beta_1)] \;= \\ & = & [\beta_2 \wedge (\alpha_2 \to \alpha_1) \wedge (\beta_2 \to \beta_1)] \vee [\alpha_2 \wedge (\alpha_2 \to \beta_1)] \quad . \end{array}$$

Now we refer to (3.1) and (3.2) and obtain:

$$(\alpha_1, \beta_1) \quad = \quad (\alpha_2, \beta_2) \sqcap [(\alpha_2, \beta_2) \to (\alpha_1, \beta_1)] \quad ;$$

hence the divisibility property is verified.

□

Remark 3.3 (Representation by 3-valued maps) Let $\mathcal{G} = (\mathbb{B}, \leq, \wedge)$ be a Boolean algebra. Then Corollary 3.2 shows that the canonical Girard extension $\mathcal{G}_1$ of $\mathcal{G}$ is an MV- algebra. Moreover, it follows immediately from (3.1) that for every event $(\alpha, \beta) \in \mathbb{B}_1$ its square $(\alpha, \beta) \sqcap (\alpha, \beta)$ is *idempotent* w.r.t $\sqcap$. Hence Proposition 4.12 and Proposition 2.6(xii) in [1] (see also Lemma 6.2 in [1]) imply that $\mathcal{G}_1$ is a semisimple MValgebra — a result which was first established by H.T Nguyen , I.R. Goodman and E.A. Walker [6]. Further, let $\mathbf{3} = \{0, \frac{1}{2}, 1\}$ be the MV-algebra consisting of three elements (which is obviously the canonical Girard extension of the Boolean algebra $\mathbf{2}$ of two elements (cf. Remark 2.1(c) in [2])), and let (X, τ) be the Stone space corresponding to $\mathbb{B}$. Then it is not difficult to show that the canonical Girard $\mathcal{G}_1$ extension of $(\mathbb{B}, \leq, \wedge)$ is isomorphic (in the sense of MV-algebras) to the MV-algebra of all τ-continuous maps $f : X \longmapsto \mathbf{3}$ where $\mathbf{3}$ is provided with the discrete topology.

□

4 Conditioning and mean value generation

Definition 4.1 (Axioms for conditioning operators)

Let $\mathcal{G} = (L, \leq, \sqcap)$ be a Girard algebra. A binary operation $| : L \times L \longmapsto L$ is called a conditioning operator on $\mathcal{G}$ iff $|$ satisfies the following axioms:

(C1) $(\alpha \mid 1) = \alpha$,

(C2) $(\beta \sqcap (\beta \rightarrow \alpha) \mid \beta) = (\alpha \mid \beta)$,

(C3) $\alpha_1 \leq \alpha_2 \implies (\alpha_1 \mid \beta) \leq (\alpha_2 \mid \beta)$,

(C4) $\beta_1 \leq \beta_2 \, , \, \beta_2 \sqcap (\beta_2 \rightarrow \alpha) \leq \beta_1 \sqcap (\beta_1 \rightarrow \alpha) \implies (\alpha \mid \beta_2) \leq (\alpha \mid \beta_1)$,

(C5) $(\alpha \mid \beta)^\perp = (\alpha^\perp \sqcap \beta \mid \beta)$, particularly, $(0 \mid 0)^\perp = (0 \mid 0)$.

□

Remark 4.2

(a) Boolean algebras L *do not admit* conditioning operators, because L has no selfcomplemented element which could serve as $(0 \mid 0)$ in axiom (C5) . Moreover, MV-algebras without a selfcomplemented element do also not admit conditioning operators.

(b) We will show in Section 5 that in a wide class of Girard algebras the existence of conditioning operators is guaranteed.

□

For the sake of completeness we quote Lemma 2.1 in [2]:

Lemma 4.3 *Every conditioning operator fulfills the following property:*

$$\beta \sqcap (\beta \rightarrow \alpha) \quad \leq \quad (\alpha \mid \beta) \quad \leq \quad \beta \rightarrow \alpha \quad .$$

Definition 4.4 (Mean values in Girard algebras)

Let $\mathcal{G} = (L, \leq, \sqcap)$ be a Girard algebra. A a mean value function is an isotone, idempotent binary (not necessarily commutative) operation on L — i.e. a map $C : L \times L \longmapsto L$ satisfying the following axioms

(i) $C(\alpha, \alpha) = \alpha$, (Idempotency)

(ii) $C(\alpha_1, \beta_1) \leq C(\alpha_2, \beta_2)$ whenever $\alpha_1 \leq \alpha_2 \, , \, \beta_1 \leq \beta_2$. (Isotonocity)

A mean value function C is said to be compatible with the residual complement in L iff C satisfies the following additional condition

(iii) $\quad (C(\alpha,\beta))^{\perp} = C(\beta^{\perp},\alpha^{\perp})$.

$\square$

Theorem 4.5 *Let $\mathcal{G} = (L, \leq, \sqcap)$ be a Girard algebra and C be a mean value function on L which is compatible with the residual complement in L. Then C induces a conditioning operator $\mid$ on L by*

$$(\alpha \mid \beta) \quad = \quad C(\, \beta \sqcap (\beta \rightarrow \alpha)\, ,\, \beta \rightarrow \alpha\,)$$

PROOF. The axiom (C1)and (C2) follow from the idempotency of C, and Proposition 1.4(b). Further, it is easy to see that the axioms (C3) and (C4) are an immediate consequence of the isotonicity of C. In order to verify (C5) we first refer to Proposition 1.4(c) and observe:

$$(\beta \sqcap (\beta \rightarrow \alpha))^{\perp} \quad = \quad \beta \rightarrow (\beta \rightarrow \alpha)^{\perp} \quad = \quad \beta \rightarrow (\beta \sqcap \alpha^{\perp})\ .$$

Now we apply the residual complement compatibility of C and obtain:

$$(\alpha \mid \beta)^{\perp} \quad = \quad C(\beta \sqcap \alpha^{\perp}, \beta \rightarrow (\beta \sqcap \alpha^{\perp})) \quad = \quad (\alpha^{\perp} \sqcap \beta \mid \beta) \quad .$$

$\square$

The type of conditioning operator constructed in Theorem 4.5 is called a mean value based conditioning operator.

For the case of MV-algebras the construction was introduced in [9]. In this special situation we will now show that there exists a close relationship between a certain class of conditioning operators and mean value functions.

Theorem 4.6 (Mean values and measure-free conditioning)

Let $\mathcal{M} = (L, \leq, \sqcap)$ be an MV-algebra and $\mid$ be a conditioning operator. Then the follwoing assertions are equivalent:

(i) $\quad \mid$ *is a mean value based conditioning operator.*

(ii) $\quad \mid$ *satisfies the following condition*

$$(\mathrm{C3}') \qquad \alpha_1 \leq \alpha_2 \quad \Longrightarrow \quad (\alpha_1 \wedge \gamma \mid \gamma \rightarrow \alpha_1) \;\leq\; (\alpha_2 \wedge \gamma \mid \gamma \rightarrow \alpha_2) \quad .$$

PROOF. (a) ((i) $\Longrightarrow$ (ii)) Referring to the construction in Theorem 4.5 we obtain from Proposition 1.5(c):

$$(\alpha \wedge \gamma \mid \gamma \rightarrow \alpha) \quad = \quad C(\alpha \wedge \gamma, (\gamma \rightarrow \alpha) \rightarrow (\alpha \wedge \gamma)) \quad = \quad C(\alpha \wedge \gamma, \gamma) \quad .$$

If we fix $\gamma \in L$, then the isotonicity of C implies (C3$'$).
(b) ((ii) $\Longrightarrow$ (i)) The given conditioning operator determines a map $C : L \times L \longmapsto L$ by

$$C(\alpha, \beta) \quad = \quad (\alpha \wedge \beta \mid \beta \rightarrow \alpha) \qquad .$$

We show that C is a mean value function on L which is compatible with the residual complement. The idempotency of C follows immediately from (C1). Further, the condition (C3$'$) guarantees the isotonocity of C in its first variable. In order to verify the isotonocity in its second variable we proceed as follows: Let us fix a triple $(\alpha, \beta_1, \beta_2)$ with $\beta_1 \leq \beta_2$. The divisibility axiom of MValgebras implies

$$(\beta \rightarrow \alpha) \sqcap ((\beta \rightarrow \alpha) \rightarrow (\alpha \wedge \delta)) \quad = \quad (\beta \rightarrow \alpha) \wedge \alpha \wedge \delta \quad = \quad \alpha \wedge \delta \; .$$

Hence the triple $((\alpha \wedge \beta_1), \beta_1 \rightarrow \alpha, \beta_2 \rightarrow \alpha)$ satisfies the hypothesis of Axiom (C4). Therefore (C3) and (C4) imply

$$C(\alpha, \beta_1) \quad \leq \quad ((\alpha \wedge \beta_1) \mid \beta_2 \rightarrow \alpha) \quad \leq \quad C(\alpha, \beta_2) \quad ;$$

hence the isotonicity of C in its second variable is verified. Finally, we conclude from (C2), (C5) and the divisibility axiom:

$$\begin{aligned} (C(\alpha, \beta))^{\perp} \quad &= \quad ((\alpha^{\perp} \sqcap (\beta \rightarrow \alpha)) \mid (\beta \rightarrow \alpha)) \\ &= \quad ((\alpha^{\perp} \wedge \beta^{\perp}) \mid (\alpha^{\perp} \rightarrow \beta^{\perp})) \quad = \quad C(\beta^{\perp}, \alpha^{\perp}) \quad . \end{aligned}$$

Now it is sufficient to show that C induces $\mid$ in the sense of Theorem 4.5. Once again we apply (C2), the divisibility and MV-property of MV-algebras and obtain:

$$\begin{aligned} C(\beta \sqcap (\beta \rightarrow \alpha), \beta \rightarrow \alpha) \quad &= \quad C(\alpha \wedge \beta, \beta \rightarrow \alpha) \\ &= \quad (\alpha \wedge \beta \mid (\beta \rightarrow \alpha) \rightarrow (\alpha \wedge \beta)) \\ &= \quad (\alpha \wedge \beta \mid (\beta \vee (\alpha \wedge \beta))) \quad = \quad (\alpha \mid \beta) \quad . \end{aligned}$$

□

Example 4.7 (Mean value functions)

(a) Let $\mathcal{M} = (L, \leq, \sqcap)$ be an MV-algebra with a (unique) selfcomplemented element σ – i.e. $\sigma^{\perp} = \sigma$. Further let $(L, \leq)$ be a chain. Then the map $C : L \times L \longmapsto L$ defined by

$$C(\alpha, \beta) \quad = \quad \left\{ \begin{array}{ccc} \alpha \vee \beta & : & \alpha \vee \beta \leq \sigma \\ \alpha \wedge \beta & : & \sigma \leq \alpha \wedge \beta \\ \sigma & : & \text{elsewhere} \end{array} \right\}$$

is a mean value function on L which is compatible with the residual complement.

(b) Let $\mathcal{M} = (L, \leq, \sqcap)$ be a strict MV- algebra, and r be the square root operation on L. Then the map $C : L \times L \longmapsto L$ defined by

$$C(\alpha, \beta) \quad = r(\alpha) \sqcap r(\beta) \qquad \forall\, \alpha, \beta \in L$$

is a mean value function on L. Because of Proposition 1.6(d) C is compatible with the residual complement. In particular, C is commutative and bisymmetric operation. (cf. Section 5 in [9])

□

5 Measure-free conditioning on canonical extensions of Girard algebras

Even though not every Girard algebra admits an conditioning operator (cf. Remark 4.2(a)), we show in this section that there **exist** conditioning operators on the canonical Girard algebra extension of any Girard algebra. In this sense we give a non-dogmatic and far reaching solution to all of the problems arising in measure-free conditioning – e.g. the construction of the iteration of the conditioning process.

We start with a fundamental observation:

Lemma 5.1 (Mean value functions on the canonical extension)

Let $\mathcal{G} = (L, \leq, \sqcap)$ be a Girard algebra and $\mathcal{G}_1 = (L_1, \leq, \sqcap)$ the canonical Girard algebra extension of $\mathcal{G}$ (cf. Theorem 3.1). Every mean value function B on L (in the sense of Definition 4.4) induces a mean value function C on L_1 by:

$$C(\,(\alpha,\beta)\,,\,(\gamma,\delta)\,) \;=\; (\,B(\alpha\,,\,\beta\wedge\gamma)\,,\,(B(\delta^{\perp}\,,\,\beta^{\perp}\wedge\gamma^{\perp})\,)^{\perp}) \quad .$$

Moreover, C on L_1 is compatible with the residual complement in L_1 (w.r.t $\mathcal{G}_1$) and satisfies the further property:

(D) $\quad C(\,(\alpha,\alpha)\,,\,(\gamma,\gamma)\,) \;=\; (\alpha\,,\,\gamma)$ *whenever* $\alpha \leq \gamma$.

□

The proof of the previous lemma is straight forward and can be left to the reader. What is more important in this context, is the observation that every Girard algebra admits mean value functions.

Remark 5.2 (Existence of mean value functions) Let $\mathcal{G} = (L, \leq, \sqcap)$ be a Girard algebra. Then the following maps B_1 and B_2 defined by

$$B_1(\alpha,\beta) \;=\; \alpha \quad , \quad B_2(\alpha,\beta) \;=\; \beta$$

are mean value functions on L. Further the corresponding mean value functions C_i on L_1 ($i = 1, 2$) (in the sense of Lemma 5.1) are given by

$$C_1((\alpha,\beta),(\gamma,\delta)) = (\alpha,\delta) \quad ,$$

$$C_2((\alpha,\beta),(\gamma,\delta)) = (\beta\wedge\gamma,\beta\vee\gamma) \quad .$$

□

Theorem 5.3 (Existence of conditioning operators)
Let $\mathcal{G} = (L, \leq, \sqcap)$ be a Girard algebra. Then there exist conditioning operators on the canonical Girard algebra extension $\mathcal{G}_1 = (L_1, \leq, \sqcap)$ of $\mathcal{G}$ (cf. Theorem 3.1).

PROOF. See Theorem 4.5, Lemma 5.1 and Remark 5.2.

□

A more specific result is the following

Theorem 5.4 *Let $\mathcal{G}_1 = (L_1, \leq, \sqcap)$ be the canonical Girard algebra extension on $\mathcal{G}$. Then there exists a mean value function C on L_1 which is compatible with the residual complement in L_1 and satisfies the condition (D) in Lemma 5.1 Further the mean value based conditioning operator $|$ corresponding to C (cf. Theorem 4.5) satisfies the additional* diagonal property*:*

$$\text{(DP)} \qquad ((\alpha,\alpha)\,|\,(\beta,\beta)) = (\beta\sqcap(\beta\to\alpha), \beta\to\alpha) \quad .$$

PROOF. The assertion follows immediately from the constructions in Theorem 4.5, Lemma 5.1. and Remark 5.2

□

Proposition 5.5 *Let $\mathcal{G} = (L, \leq, \sqcap)$ be a Girard algebra and $\mathcal{G}_1 = (L_1, \leq, \sqcap)$ be the canonical Girard algebra extension (Theorem 3.1). Then any conditioning operator $|$ on L_1 satisfying the diagonal property (DP) listed in Theorem 5.4 leads to a mapping $\| : L \times L \longmapsto L_1$, given by:*

$$(\alpha \,\|\, \beta) = ((\alpha,\alpha)\,|\,(\beta,\beta)) = (\beta\sqcap(\beta\to\alpha), \beta\to\alpha) \quad ,$$

which satisfies the following properties:

(D1) $\quad (\alpha \,\|\, 1) = (\alpha,\alpha) \quad ,$

(D2) $\quad (\beta\sqcap(\beta\to\alpha) \,\|\, \beta) = (\alpha \,\|\, \beta) \quad ,$

(D3) $\quad \alpha_1 \leq \alpha_2 \implies (\alpha_1 \,\|\, \beta) \leq (\alpha_2 \,\|\, \beta) \quad ,$

(D4) $\quad \beta_1 \leq \beta_2\,,\ \beta_2\sqcap(\beta_2\to\alpha) \leq \beta_1\sqcap(\beta_1\to\alpha) \implies (\alpha \,\|\, \beta_2) \leq (\alpha \,\|\, \beta_1)\,,$

(D5) $\quad (\alpha \,\|\, \beta)^{\perp} = (\alpha^{\perp}\sqcap\beta \,\|\, \beta)$ *, particularly,* $(0 \,\|\, 0)^{\perp} = (0 \,\|\, 0) = (0,1)\,.$

PROOF. Since $\mathcal{G}$ is a Girard subalgebra of $\mathcal{G}_1$ the properties (D1) – (D5) follow immeadiately from (C1) – (C5).

□

Remark 5.6 The values $(\alpha \parallel \beta) \in L_1$ of the mapping $\parallel : L \times L \longmapsto L_1$ from Proposition 5.5 can be called conditional events. Furthermore, Proposition 5.5 extends the interval approach from Section 2, making the obvious identifications comparing the properties (D1)–(D5) with (I1), (I2′), (I3)–(I5) (cf. Section 2). Therefore we used the same symbol $\parallel$ abusing the notation.

□

6 Uncertainty measures and measure-free conditioning

Let $\mathcal{G} = (L, \leq, \sqcap)$ be a Girard algebra, and $^\perp$ and $\sqcup$ be the residual complement and the dual operation associated with $\sqcap$. A map $m : L \longmapsto [0,1]$ is called an uncertainty measure iff m satisfies the following conditions (cf. Section 3 in [2] and Chapter 11):

(M1) $\quad m(0) = 0 \quad , \quad m(1) = 1 \, .$

(M2) $\quad \alpha \leq \beta \implies m(\alpha) \leq m(\beta) \, .$

If m is an uncertainty measure, then the entity $m(\alpha)$ is usually interpreted as the *uncertainty that the event* α *occurs.* Here an uncertainty measure m will be called additive iff m satisfies the additonal axiom

(M3) $$\begin{aligned} &\alpha^\perp \sqcap (\alpha \sqcup \beta) = \beta \text{ and } \beta^\perp \sqcap (\alpha \sqcup \beta) = \alpha \\ &\implies m(\alpha \sqcup \beta) = m(\alpha) + m(\beta) \quad . \end{aligned}$$

An uncertainty measure m is said to be strictly positive iff $0 < m(\alpha)$ for all $\alpha \neq 0$.

Remark 6.1 (a) Every additive uncertainty measure m on a Girard algebra fulfils the following property

$$m(\alpha^\perp) = 1 - m(\alpha) \quad .$$

(b) In the case of MV-algebras, the divisibility property implies that an uncertainty measure m is additive iff m satisfies the more intuitive axiom

(M4) $\quad \alpha \sqcap \beta = 0 \implies m(\alpha \sqcup \beta) = m(\alpha) + m(\beta) \quad ,$

i.e. if m is a state in the sense of D. Mundici (cf. [5]), see also *Chapter 11* in this volume. But, in general, (M4) is a stronger property than (M3). $\square$

The (non-trivial) problem of additivity in Girard algebras will be discussed in a forthcoming paper.

In the present section we are rather interested in the problem to what extent conditioning operators are related to conditional probability measures. In the case of Boolean algebras there is a long history of attempts (cf. [6]). The main point seems to be *Lewis' Triviality Result* (cf. [4]), namely there does not exist any binary operation $\diamond$ on a Boolean algebra $\mathbb{B}$ with at least eight elements such that the following relation holds for all probability measures m on $\mathbb{B}$:

$$m(\alpha \diamond \beta) = \frac{m(\alpha \wedge \beta)}{m(\beta)} \qquad \text{whenever} \qquad m(\beta) > 0 .$$

This fits naturally with Remark 4.2(a). In the following considerations we investigate *Lewis Triviality Result* in the realm of MV-algebras provided with a selfcomplemented element. First we start with a definition:

Definition 6.2 Let $\mid$ be a conditioning operator on an MV-algebra $\mathcal{M} = (L, \leq, \sqcap)$, and m be a strictly positive uncertainty measure on $\mathcal{M}$. Then m is said to be a **conditional uncertainty measure** w.r.t. $\mid$ iff the following relation holds

$$\text{(CM)} \qquad m(\alpha \mid \beta) = \left\{ \begin{array}{ccc} \frac{m(\alpha\wedge\beta)}{m(\beta)} & : & \beta \neq 0 \\ \frac{1}{2} & : & \beta = 0 \end{array} \right\} .$$

$\square$

Example 6.3 On the real unit interval $[0,1]$ we consider a binary operation $\sqcap$ which is determined by Łukasiewicz' arithmetic conjunction, i.e.

$$\alpha \sqcap \beta = \max(\alpha + \beta - 1, 0) .$$

Then $\mathcal{M} = ([0,1], \leq, \sqcap)$ is an MV-algebra. Further, the binary operation $\mid$ defined by

$$(\alpha \mid \beta) = \left\{ \begin{array}{ccc} \frac{\alpha\wedge\beta}{\beta} & : & \beta \neq 0 \\ \frac{1}{2} & : & \beta = 0 \end{array} \right\}$$

is a conditioning operator on $\mathcal{M}$, based on the mean value function C, given by

$$C(\alpha, \beta) = \left\{ \begin{array}{ccc} \frac{\alpha}{\alpha+1-\beta} & : & (\alpha,\beta) \neq (0,1) \\ \frac{1}{2} & : & (\alpha,\beta) = (0,1) \end{array} \right\}$$

which is compatible with the residual complement $\alpha^{\perp} = 1 - \alpha$. Obviously, the identity map id of $[0,1]$ is a conditional uncertainty measure w.r.t. this conditioning operator which is also additive. $\square$

Proposition 6.4 (Motivation of the axioms (C1),(C2) and (C5))

Let $\mathcal{M} = (L, \leq, \sqcap)$ be an MV-algebra, $\mid$ be a binary operation on L satisfying the axioms (C3) and (C4), and $\mathcal{F}$ be a non empty family of strictly positive, additive uncertainty measures m on $\mathcal{M}$ satisfying (CM) s.t. $\mathcal{F}$ separates points in L, i.e.

$$\forall (\alpha, \beta) \in L \times L \text{ with } \alpha \neq \beta \ \exists m \in \mathcal{F} : \ m(\alpha) \neq m(\beta) \quad .$$

Then $\mid$ is a conditioning operator on $\mathcal{M}$, and all $m \in \mathcal{F}$ are (additive) conditional uncertainty measures w.r.t. $\mid$.

PROOF. Let us fix an uncertainty measure $m \in \mathcal{F}$. Then we obtain immediately from (M1) and (CM):

$$m(\alpha \mid 1) = m(\alpha) \quad , \quad m(\alpha \wedge \beta \mid \beta) = m(\alpha \mid \beta) \quad .$$

Since $\mathcal{F}$ separates points in L, the binary operation satisfies necessarily the axioms (C1) and (C2). In order to verify (C5) we proceed as follows: First, according to Remark 6.1(b), we apply the additivity axiom (M4) to the disjoint decomposition of β from Proposition 1.5(d) and obtain

$$m(\alpha^{\perp} \sqcap \beta) = m(\beta) - m(\alpha \wedge \beta) \quad .$$

Then we embark on (CM) and apply again the additivity of m:

$$m((\alpha \mid \beta)^{\perp}) = 1 - m(\alpha \mid \beta) = \frac{m(\beta) - m(\alpha \wedge \beta)}{m(\beta)} = m(\alpha^{\perp} \sqcap \beta \mid \beta) \, .$$

Since $\mathcal{F}$ separates points in L, the axiom (C5) follows. Now, because of (CM), it is trivial that in this context all $m \in \mathcal{F}$ are conditional uncertainty measures w.r.t. the conditioning operator $\mid$.

□

Addition. Because Example 6.3 it is important to note that in general the hypothesis of the previous proposition is non empty.

In the following conderations we investigate the relationship between conditioning operators and uncertainty measures. We start with the following important observation (cf. Proposition 3.6 in [2]):

Theorem 6.5 (Additive measure extension in the Boolean case)

Let $\mathcal{G} = (\mathbb{B}, \leq, \wedge)$ be a Boolean algebra, and m be a probability measure on $\mathcal{G}$. Further let $\mathcal{G}_1 = (\mathbb{B}_1, \leq, \sqcap)$ be the canonical MV- algebra extension of $\mathcal{G}$. Then m has a unique extension to an additive uncertainty measure $\tilde{m}$ on $\mathcal{G}_1$, *i.e. there exists a unique additive uncertainty measure* $\tilde{m}$ *on* $\mathcal{G}_1$ *such that the restriction of* $\tilde{m}$ *to* $\mathbb{B}$ *coincides with* m. *In particular,* $\tilde{m}$ *is given by*

$$\tilde{m}(\alpha,\beta) \quad = \quad \frac{1}{2}\cdot\Big(m(\alpha) + m(\beta)\Big) \quad .$$

□

Remark 6.6 Let $\mathcal{G} = (\mathbb{B}, \leq, \wedge)$ be a Boolean algebra with at least four elements. Further let $\mathcal{G}_1$ be the canonical MV-algebra extension of $\mathcal{G}$. Then there exist strictly positive probability measures m on $\mathcal{G}$ such that their (unique) additive extension $\tilde{m}$ to $\mathcal{G}_1$ is not a conditional uncertainty measure w.r.t. any conditioning operator $\mid$ on $\mathcal{G}_1$. For simplicity let us consider the following

Example: $\mathbb{B} = \{\varnothing, \{\omega\}, \{\bar{\omega}\}, \{\omega,\bar{\omega}\}\}$ and $m(\{\omega\}) = m(\{\bar{\omega}\}) = \frac{1}{2}$ where we assume $\omega \neq \bar{\omega}$. Further we consider the following pairs in $\mathcal{G}_1$:

$$\alpha = (\{\omega\},\{\omega\}) \quad , \quad \beta = (\{\omega\},\{\omega,\bar{\omega}\}) \quad , \quad 1 = (\{\omega,\bar{\omega}\}, \{\omega,\bar{\omega}\}) \quad .$$

Then the values of the additive extension $\tilde{m}$ of m are given by (cf. Theorem 6.5):

$$\tilde{m}(\alpha) = \tfrac{1}{2} \quad , \quad \tilde{m}(\beta) = \tfrac{3}{4} \quad , \quad \tilde{m}(1) = 1 \, .$$

Now, let $\mid$ denote any conditioning operator on $\mathcal{G}_1$. Then the axioms (C1) and (C4) imply $\alpha = (\alpha \mid 1) \leq (\alpha \mid \beta)$. We put $\gamma = \alpha^{\perp} \sqcap (\alpha \mid \beta)$ and assume that $\tilde{m}$ is a conditional uncertainty measure w.r.t. $\mid$. Now we invoke the additivity of $\tilde{m}$ and obtain:

$$\tilde{m}(\gamma) \quad = \quad \tilde{m}(\alpha \mid \beta) - \tilde{m}(\alpha) \quad = \quad \tfrac{1}{2}\cdot\tfrac{4}{3} - \tfrac{1}{2} \quad = \quad \tfrac{1}{6} \quad .$$

Since Theorem 6.5 implies that $\frac{1}{6}$ cannot be in the range of $\tilde{m}$, we conclude that $\tilde{m}$ is **not** a conditional uncertainty measure w.r.t. $\mid$.

□

It follows from the previous Remark 6.6 that already on canonical MV-algebra extensions the additivity of uncertainty measures and the compatiblity of uncertainty measures with conditioning operators in the sense of Definition 6.2 are mutually exclusive concepts. One way to overcome this obstacle is to remember that a non-trival (generalized) measure theory is also possible w.r.t. non-additive uncertainty measures (cf. Section 4 and Section 5 in [2] or *Chapter 11* in this volume). Therefore, if we drop the additivity axiom, we obtain the following partially positive result:

Theorem 6.7 (Conditional uncertainty measure extension)

Let $\mathcal{G} = (L, \leq, \sqcap)$ *be an* MV*-algebra and* $\mathcal{G}_1$ *be the canonical Girard algebra extension of* $\mathcal{G}$. *Further let* m *be a strictly positive, additive uncertainty* *measure on $\mathcal{G}$, and let $\mid$ be a conditioning operator on $\mathcal{G}_1$ which satisfies the diagonal property (DP) (cf. Theorem 5.4). Then there exists an uncertainty measure $\tilde{(m)}$ on $\mathcal{G}_1$ satisfying the following properties*

$$\text{(i)} \quad \tilde{m}(\alpha,\alpha) \;=\; m(\alpha) \quad , \text{i.e. } \tilde{m} \text{ is an extension of } m\,,$$

$$\text{(ii)} \quad \tilde{m}((\alpha,\alpha) \mid (\beta,\beta)) \;=\; \tfrac{\tilde{m}(\alpha\wedge\beta\,,\,\alpha\wedge\beta)}{\tilde{m}(\beta,\beta)} \qquad \text{whenever} \qquad \beta > 0\,.$$

In particular, $\tilde{m}$ can be given by

$$\tilde{m}(\alpha,\beta) \;=\; \frac{m(\alpha)}{1 - m(\beta) + m(\alpha)} \quad \text{whenever} \quad (\alpha,\beta) \neq (0,1)\,.$$

PROOF. (i) is obvious, and (ii) follows immediately from (DP) and the additivity of m.

□

The Theorem 6.5 and Theorem 6.7 are dealing implicitly with the extension of uncertainty measures from a given Boolean algebra to its canonical MV-algebra extension. It is interesting to see that these constructions are special cases of the following general theorem:

Theorem 6.8 (Mean value measure extensions)

Let $\mathcal{G} = (L, \leq, \sqcap)$ be a Girard algebra and $\mathcal{G}_1$ be the canonical Girard algebra extension of $\mathcal{G}$. Further let M be a mean value function on $[0,1]$. Each uncertainty measure m on $\mathcal{G}$ can then be extended to an uncertainty measure $\tilde{m}$ on $\mathcal{G}_1$, via:

$$\tilde{m}(\alpha,\beta) \;=\; M\,(m(\alpha)\,, m(\beta)\,) \quad .$$

Furthermore, if M is compatible with the usual complement in $[0,1]$ (i.e. $1-x$ is the complement of x), then we have

$$m(\alpha^{\perp}) \;=\; 1 - m(\alpha) \quad \Longrightarrow \quad \tilde{m}\,((\alpha,\beta)^{\perp}) \;=\; 1 - \tilde{m}\,(\alpha,\beta) \quad .$$

□

Finally, it it is interesting to observe that the arithmetic mean (cf. Theorem 6.5) and the mean value function which appears in Theorem 6.7 are members of a large family of mean value functions constructed in Example 4.2 in [9].

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CHAPTER 13

Applications Of Decomposable Measures

E. Pap

Introduction

We shall give a brief overview of some applications of special non–additive measures - so–called decomposable measures including the corresponding integration theory, which form the basis for *pseudo-analysis*. We present applications to optimization problems, nonlinear partial differential equations, optimal control.

We want to stress here some of the advantages of pseudo–analysis. Based on a unique theory we present a unified treatment of various (usually nonlinear) problems arising in such fields as system theory, optimization, difference equations, partial differential equations, control theory. The important fact is that this approach gives also solutions in the form which are not achieved by other theories (for example, Bellman difference equation, Hamilton–Jacobi equation with non–smooth Hamiltonians and discontinuous initial data). In some cases this method enables us to obtain exact solutions of mostly nonlinear equations in a form which is similar to the case of linear equations.

Some obtained principles (e.g. the principle of pseudo-linear superposition) allows us to transfer methods of linear equations to the theory of nonlinear equations.

Pseudo-analysis uses many mathematical tools from different fields as functional equations, variational calculus, measure theory, functional analysis, optimization theory, semiring theory, etc. and is still undergoing a rapid development in various directions (cf. [4, 14, 19, 23, 32, 35, 41]).

1 Non–additive measures and integrals

1.1 Null–additive measures

Let $\mathcal{D}$ be a family of subsets of a given set X such that $\emptyset \in \mathcal{D}$. A very general class of non–additive measures is determined by the following definition:

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A set function $m : \mathcal{D} \to [0,\infty]$ with $m(\emptyset) = 0$ is called *null–additive*, if and only if

$$m(A \cup B) = m(A) \quad \text{whenever} \quad A, B \in \mathcal{D}, A \cap B = \emptyset \quad \text{and} \quad m(B) = 0.$$

A detailed study of null - additive set functions can be found in [32, 41]. A set function $m : \mathcal{D} \to [0,\infty]$ is called *monotone* iff

$$m(A) \leq m(B) \quad \text{whenever} \quad A \subseteq B, \ A, B \in \mathcal{D}.$$

In applications monotone (positive) set functions are usually called *fuzzy measures* (see [14, 39, 41]). Recently there is obtained a close relation between some classes of non-additive measures and random sets, see [15].

Positive monotone set functions vanishing at the empty set were investigated by G.Choquet (1953-1954). He introduced an integral, which generalizes the *Lebesgue integral.* Let m be a monotone positive set function defined on a σ-algebra Σ.. The *Choquet integral* of a non-negative measurable function f on $A \in \Sigma$ is given by (see [8, 14, 32, 15])

$$(C)\int_A f\,dm = \int_0^\infty m(A \cap \{x|f(x) \geq r\})\,dr\ .$$

In 1974 Sugeno introduced another type of integral on a given subset $A \subset X$ defined as follows (cf. [39])

$$(S)\int_A f(x)dm = \sup_{r\in[0,+\infty]} \min[r, m(A \cap \{x|f(x) \geq r\})]\ .$$

In various applications both integrals are usually called *fuzzy integrals.*

1.2 Pseudo operations

1.2.1 Real setting

Now we restrict our interest to a special subclass of null–additive set functions determined by *decomposable measures* (see [10, 14, 18, 17, 27, 28, 29, 32, 39, 40, 41, 42]). This measures are based on pseudo–addition and serve as an important tool in pseudo–analysis.

We shall start with the operations on the real interval $[0,1]$. A *t–norm* (short for triangular norm) is a function $T : [0,1]^2 \to [0,1]$ such that

(T1) $T(x,y) = T(y,x)$. (Commutativity)

(T2) $T(x,T(y,z)) = T(T(x,y),z)$. (Associativity)

(T3) $T(x,y) \leq T(x,z)$ whenever $y \leq z$. (Monotonicity)

(T4) $T(x,1) = x$. (Boundary Condition)

Example 1.2.1 The following are the most important t-norms

$$\begin{aligned} T_{\mathbf{M}}(x,y) &= \min(x,y) \quad , \quad T_{\mathbf{L}}(x,y) = \max(0, x+y-1) \\ T_{\mathbf{P}}(x,y) &= xy \quad , \quad T_{\mathbf{W}}(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y)=1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

□

The dual concept of operation can be defined as follows: A *t–conorm* (short for triangular conorm) is a function $S : [0,1]^2 \to [0,1]$ such that

(S1) $S(x,y) = S(y,x)$. (Commutativity)

(S2) $S(x, S(y,z)) = S(S(x,y),z)$. (Associativity)

(S3) $S(x,y) \leq S(x,z)$ for $y \leq z$. (Monotonicity)

(S4) $S(x,0) = x$. (Boundary Condition)

We see that t-norms and t-conorms differs only by the boundary conditions.

Example 1.2.2 The following are the most important t-conorms :

$$\begin{aligned} S_{\mathbf{M}}(x,y) &= \max(x,y) \quad , \quad S_{\mathbf{L}}(x,y) = \min(1, x+y) \\ S_{\mathbf{P}}(x,y) &= x+y-xy \quad , \quad S_{\mathbf{W}}(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y)=0 \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

□

Many other important t-norms and t–conorms can be found in [12, 17]. The basic connectives in fuzzy logic and their corresponding operations in fuzzy set theory are based on triangular norms and triangular conorms (cf. [9, 14, 13]).

We extend now the considered interval for the previous operations. Let $[a,b]$ be a closed (in some cases semiclosed) subinterval of $[-\infty, +\infty]$. We consider here a total order $\leq$ on $[a,b]$ (although it can be taken in the general case a partial order, see subsection 1.2.2). The operation $\oplus$ (*pseudo-addition*) is a function $\oplus : [a,b] \times [a,b] \to [a,b]$ which is commutative, nondecreasing, associative and has a zero element, denoted by $\mathbf{0}$.
Let $[a,b]_+ = \{x : x \in [a,b], \ x \geq \mathbf{0}\}$. The operation $\odot$ (*pseudo-multiplication*) is a function $\odot : [a,b] \times [a,b] \to [a,b]$ which is commutative, positively nondecreasing, i.e. $x \leq y$ implies $x \odot z \leq y \odot z$, $z \in [a,b]_+$, associative and for which there exist a unit element $\mathbf{1} \in [a,b]$, i.e., for each $x \in [a,b]$

$$\mathbf{1} \odot x = x.$$

We suppose, further, $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e.

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a,b],\oplus,\odot)$ is called a *semiring* (see [7, 12, 20, 32]).

In this paper we will consider only special semirings with the following continuous operations:

Case I : Pseudo-addition $\oplus$ is idempotent and pseudo-multiplication is not idempotent.

$$(i) \quad x \oplus y = \min(x,y), \quad x \odot y = x + y,$$

on the interval $(-\infty,+\infty]$. We have $\mathbf{0} = +\infty$ and $\mathbf{1} = 0$.

$$(ii) \quad x \oplus y = \max(x,y), \quad x \odot y = x + y,$$

on the interval $[-\infty,+\infty)$. We have $\mathbf{0} = -\infty$ and $\mathbf{1} = 0$.

Case II : Both operations are not idempotent. Semirings with pseudo-operations defined by a monotone and continuous generator g (see [1, 25, 29, 32]). In this case we will consider only strict pseudo–additions $\oplus$ – i.e. $\oplus$ is continuous and strictly increasing in $(a,b) \times (a,b)$.

By Aczel's representation theorem (see [1], see also [21, 26]) for each strict pseudo - addition $\oplus$ there exists a monotone function g (generator for $\oplus$) , $g : [a,b] \to [-\infty,\infty]$ (or with values in $[0,\infty]$) such $g(\mathbf{0}) = 0$ and

$$u \oplus v = g^{-1}(g(u) + g(v)).$$

Using a generator g of a strict pseudo-addition $\oplus$ we can define a pseudo-multiplication $\odot$ by

$$u \odot v = g^{-1}(g(u)g(v)).$$

This is the only way to define a pseudo–multiplication $\odot$, which is distributive with respect to a given pseudo–addition $\oplus$ generated g.

Case III : Both operations are idempotent. Let $\oplus = \max$ and $\odot = \min$ on the interval $[-\infty,+\infty]$.

1.2.2 General setting

We can give a more general formulation of the preceding real situation.
Let P be a non-empty set equipped with a partial order $\leq$. The operation $\oplus$ is a function $\oplus : P \times P \to P$ which is commutative, nondecreasing, associative and has a zero element, denoted by $\mathbf{0}$. Let $P_+ = \{x : x \in P,\ x \geq \mathbf{0}\}$. The operation $\odot$ is a function $\odot : P \times P \to P$ which is commutative, positively nondecreasing, i.e., $x \leq y$ implies $x \odot z \leq y \odot z$, $z \in P_+$, associative and for which there exist a unit element $\mathbf{1} \in P$, i.e., for each $x \in P$

$$\mathbf{1} \odot x = x.$$

We suppose, further, $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is distributive with respect to $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $(P,\oplus,\odot)$ is called a *commutative semiring* ([23], [32]). We suppose that P is a complete lattice. The order convergence of nets on P induces the order topology . We consider on P a metric d for which we require a compatibility with the introduced topology in the following sense: the convergence of a net in the order topology is equivalent with the convergence with respect to the metric d; the monotonicity of the metric, i.e., $x \leq y$ implies $d(x,y) = \sup_{z\in[x,y]} \sup(d(x,z), d(z,y))$; and which satisfies at least one of the following conditions:

(a) $d(x \oplus y,\ x' \oplus y') \leq d(x,x') + d(y,y')$

(b) $d(x \oplus y,\ x' \oplus y') \leq \max\{d(x,x'),\ d(y,y')\}$.

Example 1.2.3 Let $(\mathbb{R}\cup\{-\infty\})^n$ be endowed with the partial order $\leq$ determined by

$$(x_1,\ldots,x_n) \ \leq\ (y_1,\ldots,y_n) \qquad \Longleftrightarrow \qquad x_i \leq y_i \quad \forall i = 1,\ldots,n\,.$$

Then $((\mathbb{R}\cup\{-\infty\})^n, \oplus, \odot)$ with $x \oplus y = \sup(x,y)$ and $x \odot y = (x_1 + y_1, \ldots, x_n + y_n)$ is a semiring. We can introduce the following metrics which are compatible with the order

$$d(x,y) = \sum_{i=1}^{n} (e^{\max(x_i,y_i)} - e^{\min(x_i,y_i)})$$

or

$$d(x,y) = \sum_{i=1}^{n} (\arctan(\max(x_i,y_i)) - \arctan(\min(x_i,y_i))).$$

□

Example 1.2.4 Let $(\mathbb{R}\cup\{+\infty\})^n$ be endowed with the partial order $\leq$ determined by

$$(x_1,\ldots,x_n) \ \leq\ (y_1,\ldots,y_n) \qquad \Longleftrightarrow \qquad x_i \leq y_i \quad \forall i = 1,\ldots,n\,.$$

Then $((\mathbb{R}\cup\{+\infty\})^n, \oplus, \odot)$ with $x \oplus y = \inf(x,y)$ and $x \odot y = (x_1 + y_1, \ldots, x_n + y_n)$ is a semiring. We can introduce the following metric which is compatible with the order

$$d(x,y) = \sum_{i=1}^{n} (e^{-\min(x_i,y_i)} - e^{-\max(x_i,y_i)}).$$

□

The results in next subsection can be formulated also in the general semiring settings, but we restrict ourself to the real case.

1.3 Pseudo–integral

Let X be a non-empty set. Let Σ be a $\sigma-$algebra of subsets of X.

A set function $m : \Sigma \to [a,b]_+$ (or semiclosed interval) is called $\oplus-$*decomposable measure* iff the following axioms hold:

(M1) $m(\emptyset) \;=\; \mathbf{0}$.

(M2) $m(A \cup B) \;=\; m(A) \oplus m(B)$ whenever $A \cap B \;=\; \emptyset$.

(M3) If $A_n \subseteq A_{n+1}$, then $m(\bigcup_{n=1}^{\infty} A_n) \;=\; \sup_n m(A_n))$ – i.e. m is continuous from below (cf. [9, 42]).

In the case when $\oplus$ is idempotent, it is possible that m is not defined on an empty set. A $\oplus-$decomposable measure m is $\sigma - \oplus-$decomposable, i.e.,

$$m(\bigcup_{i=1}^{\infty} A_i) = \bigoplus_{i=1}^{\infty} m(A_i)$$

hold for any sequence $\{A_i\}$ of pairwise disjoint sets from Σ.

Let m be a $\oplus-$decomposable measure. We suppose that $([a,b],\oplus)$ and $([a,b],\odot)$ are complete lattice ordered semigroups. Further, we suppose that $[a,b]$ is endowed with a metric d compatible with sup and inf, i.e., $\limsup x_n = x$ and $\liminf x_n = x$ imply $\lim_{n\to\infty} d(x_n, x) = 0$, and which satisfies at least one of the following conditions:

(a) $d(x \oplus y,\; x' \oplus y') \leq d(x,x') + d(y,y')$

(b) $d(x \oplus y,\; x' \oplus y') \leq \max\{d(x,x'),\; d(y,y')\}$.

Both conditions (a) and (b) imply that :

$$d(x_n, y_n) \to 0 \quad \text{implies} \quad d(x_n \oplus z,\; y_n \oplus z) \to 0.$$

Let f and h be two functions defined on the interval $[c,d]$ and with values in $[a,b]$. Then, we define for any $x \in [c,d]$ $(f \oplus h)(x) = f(x) \oplus h(x)$, $(f \odot h)(x) = f(x) \odot h(x)$ and for any $\lambda \in [a,b]$ $(\lambda \odot f)(x) = \lambda \odot f(x)$.

We define the characteristic function with values in a semiring

$$\chi_A(x) = \begin{cases} \mathbf{0}, & x \notin A \\ \mathbf{1}, & x \in A \end{cases}.$$

A mapping $e : X \to [a,b]$ is an *elementary (measurable) function* if it has the following representation

$$e = \bigoplus_{i=1}^{\infty} a_i \odot \chi_{A_i} \quad \text{for} \quad a_i \in [a,b]$$

and $A_i \in \Sigma$ disjoint if $\oplus$ is not idempotent.

Let ε be a positive real number, and $B \subset [a,b]$. A subset $\{l_i^\varepsilon\}$ of B is a $\varepsilon-$*net* in B if for each $x \in B$ there exists l_i^ε such that $d(l_i^\varepsilon, x) \leq \varepsilon$ If we have $l_i^\varepsilon \leq x$, then we shall call $\{l_i^\varepsilon\}$ a lower $\varepsilon-$net. If $l_i^\varepsilon \leq l_{i+1}^\varepsilon$ holds, then $\{l_i^\varepsilon\}$ is monotone.

The pseudo-integral of a bounded measurable function $f : X \to [a,b]$, for which, if $\oplus$ is not idempotent for each $\varepsilon > 0$ there exists a monotone $\varepsilon-$net in $f(X)$, is defined by

$$\int_X^{\oplus} f \odot dm = \lim_{n\to\infty} \int_X^{\oplus} \varphi_n(x) \odot dm,$$

where $\{\varphi_n\}$ is the sequence of elementary functions constructed in [28, 32].

Remark 1.3.1 If we replace decomposable measures by general fuzzy measures, then in a similar way it is possible to construct a general integral [14, 17, 24, 27, 41] which covers also the Choquet and Sugeno integral.

2 Pseudo–convolution and pseudo–Laplace transform

2.1 Pseudo-convolution

We have introduced in the papers [36, 38] (see also [3, 4, 23]) the notion of the pseudo-convolution of functions.

Let G be subset of $\mathbb{R}$ such that $(G, +)$ is a group with unit element 0 and $G_+ = \{x : x \in G, x \geq 0\}$. We shall consider functions whose domain will be G.

Definition 2.1.1 *The* pseudo-convolution of the first type *of two functions $f : G \to [a,b]$ and $h : G \to [a,b]$ with respect to a $\sigma - \oplus$–decomposable measure m and $x \in G_+$ is given in the following way*

$$f \star h(x) = \int_{G_+^x}^{\oplus} f(u) \odot dm_h(v),$$

where $G_+^x = \{(u,v) | u + v = x, v \in G_+, u \in G_+\}$, $m_h = m$ in the case of sup-*decomposable measure $m(A) = \sup_{x\in A} h(x)$ (and in the case of* inf*-decomposable measure $m(A) = \inf_{x\in A} h(x)$) and $dm_h = h \odot dm$ in the case of $\oplus$-decomposable measure m, where $\oplus$ has an additive generator g and $g \circ m$ is the Lebesgue measure (g-calculus [32]).*

We consider also the second type of pseudo-convolution *:*

$$f \star h(x) = \int_G^{\oplus} f(x-t) \odot dm_h(t).$$

for $x \in G$.

We shall denote $f \star h$ also by $\int^{\oplus} f(x-t) \odot h(t)dt$. The pseudo-convolution is a commutative and associative operation.

Example 2.1.2

(I) (i) For the semiring $((-\infty, \infty], \min, +)$ the integral is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \inf_{x \in \mathbb{R}} (f(x) + h(x)),$$

where the function h defines the inf-decomposable measure m. The domain of functions will be $\mathbb{R}$ (or some subset of $\mathbb{R}$) and the domain of the semiring can be any subinterval of $(-\infty, \infty]$ which contains 0 and ∞. The zero element for the $\oplus$ is ∞ and the unit element for the $\odot$ is 0. The pseudo–delta function is given by

$$\delta^{\min,+}(x) = \begin{cases} \mathbf{1} \ (=0) & \text{if } x = 0, \\ \mathbf{0} \ (=\infty) & \text{if } x \neq 0. \end{cases}$$

The pseudo-delta function is the unit element for the operation of pseudo-convolution of first type

$$f \star h(x) = \inf_{0 \le t \le x} (f(x-t) + h(t)).$$

(ii) For the semiring $([-\infty, \infty), \max, +)$ the pseudo-integral with respect to a sup-decomposable measure $m, m(A) = \sup_{x \in A} h(x)$, is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \sup_{x \in \mathbb{R}} (f(x) + h(x)),$$

and the pseudo-convolution of second type of the functions f and h will be

$$f \star h(x) = \sup_{x \in \mathbb{R}} (f(x-t) + h(t)).$$

(II) The pseudo-convolution of first type in the sense of the $g-$integral , i.e., when the pseudo-operations are represented by a generator g as $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot y = g^{-1}(g(x)g(y))$, is given in the following way

$$f \star h(x) = \int_{[0,x]}^{\oplus} f(t) \odot h(x-t)dt$$

$$= g^{-1}(\int_0^x g(g^{-1}(g(f(t)) \cdot g(h(x-t))))dt).$$

In the sequel, generators g will be monotone and continuous functions. For such functions $g^{-1}(g(s)) = g(g^{-1}(s)) = s$ holds. The pseudo-convolution of first type then becomes

$$(f \star h)(x) = g^{-1}\Big(\int_0^x g(f(t)) \cdot g(h(x-t))dt\Big).$$

Hence

$$(f \star h)(x) = g^{-1}(((g \circ f) * (g \circ h))(x)), \text{ i.e. } \; g \circ (f \star h) = (g \circ f) * (g \circ h),$$

where $*$ is the classical convolution.

(III) (a) For the semiring $([-\infty, \infty], \max, \min)$ the integral is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \sup_{x \in \mathbb{R}}(\min(f(x), h(x)),$$

where the function h defines the sup-decomposable measure m. The domain of functions will be $\mathbb{R}$ (or some subset of $\mathbb{R}$) and the domain of the semiring can be any subinterval of $[-\infty, \infty]$. The zero element for the $\oplus$ is $-\infty$ and the unit element for the $\odot$ is $+\infty$. The pseudo-delta function is given by

$$\delta^{\max,\min}(x) = \begin{cases} \mathbf{1} \ (= +\infty) & \text{if } x = 0, \\ \mathbf{0} \ (= -\infty) & \text{if } x \neq 0. \end{cases}$$

The pseudo-delta function is the unit element for the operation of convolution $f \star h(x) = \sup_{0 \le t \le x}(\min(f(x-t), h(t)))$.

(b) For the semiring $([-\infty, \infty], \min, \max)$ the integral is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot dm = \inf_{x \in \mathbb{R}}(\max(f(x), h(x))),$$

where the function h defines the inf-decomposable measure m. The domain of functions will be $\mathbb{R}$ (or some subset of $\mathbb{R}$) and the domain of the semiring can be any subinterval of $[-\infty, +\infty]$. The zero element for the $\oplus$ is $+\infty$ and the unit element for the $\odot$ is $-\infty$. The pseudo-delta function is given by

$$\delta^{\min,\max}(x) = \begin{cases} \mathbf{1} \ (= -\infty) & \text{if } x = 0, \\ \mathbf{0} \ (= +\infty) & \text{if } x \neq 0. \end{cases}$$

The pseudo-delta function is the unit element for the pseudo-convolution of first type $f \star h(x) = \inf_{0 \le t \le x}(\max(f(x-t), h(t)))$.

□

Remark 2.1.3 In many different mathematical theories as in optimization, probabilistic metric spaces, information, fuzzy numbers, system theory there are some basic operations with functions. We can consider all these operations as some kind of generalized pseudo-convolutions, which are based on some pseudo-operations (see [37]).

2.2 Pseudo-Laplace transform

Let P be a semiring either of I) type or II) or III) type from subsection 1.2.1 and $(G,+)$ a subgroup of $(\mathbb{R},+)$. We shall use the results from [36].

Definition 2.2.1 *The pseudo-character of the group $(G,+)$, $G \subset \mathbb{R}$ is a continuous map $\xi : G \to P$, of the group $(G,+)$ into semiring $(P, \oplus, \odot)$, with the property*

$$\xi(x+y) = \xi(x) \odot \xi(y), \quad x, y \in G.$$

Theorem 2.2.2 *Let ξ be a pseudo-character of the group $(G,+)$, $G \subset \mathbb{R}$. Then we have*

I) If $\oplus \in \{\max, \min\}$, then

a) for $\odot = +$, that $\xi(x,c) = c \cdot x$,

b) for $\odot = \cdot$, that $\xi(x,c) = c^x$ $(= e^{kx}$, where $k = \ln c)$.

II) If $\oplus$ is with an additive generator g such that $x \oplus y = g^{-1}(g(x) + g(y))$, then we have

$$\xi(x,k) = g^{-1}(e^{kx}).$$

III) a) If $\oplus = \max$ and $\odot = \min$, then $\xi(x,c) = c.$.

b) If $\oplus = \min$ and $\odot = \max$, then $\xi(x,c) = c$.

Proof. I) (i) In this case the equation $\xi(x+y) = \xi(x) \odot \xi(y)$ reduces on the Cauchy functional equation

$$\xi(x+y) = \xi(x) + \xi(y)$$

which has the solution (see [1]) $\xi(x,c) = cx$.

(ii) The general equation reduces on the following form

$$\xi(x+y) = \xi(x) \cdot \xi(y)$$

which has the solution (see [1]) $\xi(x,c) = c^x$.

II) The equation

$$\xi(x+y) = \xi(x) \odot \xi(y)$$

reduces by the representation of the operation $\odot$ on the following functional equation

$$\xi(x+y) = g^{-1}(g(\xi(x)) \cdot g(\xi(y))),$$

i.e.,

$$g(\xi(x+y)) = g(\xi(x)) \cdot g(\xi(y)).$$

Hence

$$(g \circ \xi)(x+y)) = (g \circ \xi)(x) \cdot (g \circ \xi)(y),$$

which has the solution (see [1]) $(g \circ \xi)(x) = a^x$, i.e., $\xi(x,a) = g^{-1}(e^{x \ln a})$.

III) (i) In this case the equation $\xi(x+y) = \xi(x) \odot \xi(y)$ reduces on the functional equation

$$\xi(x+y) = \min(\xi(x), \xi(y))$$

which has the solution $\xi(x,c) = c$.

(ii) The general equation reduces on the following form

$$\xi(x+y) = \max(\xi(x), \xi(y))$$

which has the solution $\xi(x,c) = c$.

□

Definition 2.2.3 *The semiring $B(G,P)$ consists in the cases I) and III) of the bounded (with respect to the order in P) functions, and the case II) of functions $f : G \to P$ with the property $g \circ f \in L_1(G)$ (the space $L_1(G)$ consists of Lebesgue integrable functions which satisfy the condition $\int_G |f(x)|dx < +\infty$.)*

Let f_1 and f_2 be from $B(G,P)$. We introduce the following operations

$$(f_1 \oplus f_2)(x) = f_1(x) \oplus f_2(x), \quad x \in G$$

$$(f_1 \star f_2)(x) = \int_{G\cap[0,x]}^{\oplus} f_1(x-t) \odot f_2(t)dt, \quad x \in G$$

(pseudo-convolution).

For $\lambda \in P$, and $f \in B(G,P)$ we define $(\lambda \odot f)(x) = \lambda \odot f(x), \quad x \in G.$

Definition 2.2.4 *Pseudo-Laplace transform $\mathcal{L}^{\oplus}(f)$ of a function $f \in B(G,P)$ is defined by*

$$(\mathcal{L}^{\oplus} f)(z) = \int_{G\cap[0,\infty)}^{\oplus} (\xi(x,z) \odot f(x))dm(x),$$

where ξ is the pseudo-character . For cases I) and III) we consider also pseudo-Laplace transform taking in the pseudo-integral the whole G instead of $G\cap[0,\infty)$.

Example 2.2.5 We give some examples, where the numeration correspond to the subsection 1.2.1.

I) (i)

$$(\mathcal{L}^{\oplus} f)(z) = \inf_{x\geq 0}(-xz + f(x)),$$

II) g-Laplace transform

$$(\mathcal{L}^{\oplus} f)(z) = g^{-1}(\int_0^{\infty} e^{-xz} g(f(x))dx).$$

□

We have proved in [36] the pseudo-exchange formula

$$\mathcal{L}^{\oplus}(f_1 \star f_2) = \mathcal{L}^{\oplus}(f_1) \odot \mathcal{L}^{\oplus}(f_2).$$

We shall prove here the exchange formula only for one case. For other cases see [36].

I) (ii) We have

$$\begin{aligned}
\mathcal{L}^{\oplus}(f_1 \star f_2)(z) &= \sup_{x\geq 0}(-zx + (f_1 \star f_2)(x)) \\
&= \sup_{x\geq 0}[-zx + \sup_{0\leq t\leq x}(f_1(x-t) + f_2(t))] \\
&= \sup_{x\geq 0}\sup_{0\leq t\leq x}[-zx + (f_1(x-t) + f_2(t))] \\
&= \sup_{t\geq 0}\sup_{x\geq t}[-z(x-t+t) + f_1(x-t) + f_2(t)] \\
&= \sup_{t\geq 0}\sup_{x\geq t}[-z(x-t) + f_1(x-t) - zt + f_2(t)] \\
&= \sup_{t\geq 0}[\sup_{x\geq t}[-z(x-t) + f_1(x-t)] + (-zt + f_2(t))] \\
&= \sup_{t\geq 0}[\sup_{x-t\geq 0}[-z(x-t) + f_1(x-t)] + (-zt + f_2(t))] \\
&= \sup_{t\geq 0}[\sup_{s\geq 0}[-zs + f_1(s)] + (-zt + f_2(t))] \\
&= \sup_{s\geq 0}(-zs + f_1(s)) + \sup_{t\geq 0}(-zt + f_2(t)) \\
&= \mathcal{L}^{\oplus}(f_1)(z) + \mathcal{L}^{\oplus}(f_2)(z) \\
&= \mathcal{L}^{\oplus}(f_1)(z) \odot \mathcal{L}^{\oplus}(f_2)(z)
\end{aligned}$$

We have used the equality $\sup_t(a+h(t)) = a+\sup_t h(t)$, and that the supremum is invariant with respect to translation, i.e., $\sup_x f(x) = \sup_{x-t} f(x-t)$.

□

In the paper [36] the following theorem for inverse pseudo-Laplace transform was proved.

Theorem 2.2.6 *If for $\mathcal{L}^{\oplus}(f) = F$, there exists $(\mathcal{L}^{\oplus})^{-1}(F)$, then it has the following form for the cases I, II:*

(i)

$$((\mathcal{L}^{\oplus})^{-1}(F))(x) = \inf_{z\geq 0}(xz + F(z)),$$

(ii)

$$((\mathcal{L}^{\oplus})^{-1}(F))(x) = \sup_{z\geq 0}(xz + F(z)).$$

We have for $\odot$ = product

(i)

$$((\mathcal{L}^{\oplus})^{-1}(F))(x) = \min_{z\geq 0}(e^{xz}F(z)),$$

(ii)

$$((\mathcal{L}^{\oplus})^{-1}(F))(x) = \sup_{z\geq 0}(e^{xz}F(z)),$$

II)

$$(\mathcal{L}^{\oplus})^{-1} = g^{-1} \circ \mathcal{L}^{-1} \circ g,$$

i.e.,

$$((\mathcal{L}^{\oplus})^{-1}(F))(x) = g^{-1}(\mathcal{L}^{-1}(g \circ F)(x)).$$

Remark 2.2.7 All considerations can be extended also to the case $(G, +) \subset (\mathbb{R}^n, +)$.

3 Optimization

In this section we give first an example of an application of pseudo–analysis to the theory of *optimization* ([5, 36]).

Example 3.1 We intend to find the maximum of the utility function

$$U(x_1, x_2, ..., x_N) = f_1(x_1) + f_2(x_2) + \cdots + f_N(x_N),$$

on the domain

$$R \;=\; \{(x_1, x_2, ..., x_N) : x_1 + x_2 + ... + x_N = x, \;\; x_i \geq 0, \; i = 1, ..., N\}\,.$$

Such problems often occur in mathematical economics and operation research.

Let

$$f(x) \;=\; \max_R[f_1(x_1) + f_2(x_2) + \cdots + f_N(x_N)]\,.$$

Applying the pseudo–Laplace transform (for the case $\oplus = \max$ and $\odot = +$)

$$F(z) \;=\; (\mathcal{L}^{\oplus} f)(z) \;=\; \max_{x\geq 0}(-xz + f(x))$$

we obtain by the pseudo–exchange formula (by induction)

$$\mathcal{L}^{\oplus}(f) \quad = \quad \sum_{i=1}^{N} \mathcal{L}^{\oplus}(f_i) \quad .$$

Applying the inverse of the pseudo–Laplace transform

$$((\mathcal{L}^{\oplus})^{-1}F)(x) \quad = \quad \min_{z\geq 0}(xz + F(z))$$

we obtain the solution

$$f(x) \quad = \quad \min_{z\geq 0}\left[xz+\sum_{i=1}^{N}(\mathcal{L}^{\oplus}f_i)(z)\right] \quad .$$

□

Further, there are some pseudo-integral representations of nonlinear operators which occurs in dynamical programming ([4, 12, 19, 23, 32, 38]).

Example 3.2 In the optimization problems often occurs the following operator which we shall call Bellman operator. Let X and Y be arbitrary non-empty subsets of $\mathbb{R}$. Let $k \in C(X \times Y)$.
Then the operator $B : C(Y) \to C(X)$ is defined by

$$(Bf)(x) = \max_{y\in Y}(k(x,y) + f(y)).$$

Using the pseudo-integral with respect to $\oplus = \max$ and $\odot = +$ we can rewrite the preceding operator in the form

$$(Bf)(x) = \int_Y^{\oplus} k(x,y)dm_f = \int_Y^{\oplus} k(x,y) \odot f(y)dy.$$

□

A set S is a *semimodule* over the semiring P if $(S,\oplus)$ is a commutative semigroup with the neutral element $\mathbf{0}$ and a multiplication $\odot : P \times S \to S$ which for all $\alpha,\beta \in P, x,y \in S$ satisfies the conditions: (i) $(\alpha \odot \beta) \odot x = \alpha \odot (\beta \odot x)$; (ii) $\alpha \odot (x \oplus y) = (\alpha \odot x) \oplus (\beta \odot y)$; (iii) $\mathbf{0} \odot x = \alpha \odot \mathbf{0} = \mathbf{0}$. If S is endowed with some topology we suppose that the involved operations are continuous.
The non-linear operator B is pseudo-linear in the following sense

Definition 3.3 *Let $B(X,P)$ denote the semimodule of all functions from X into the semiring $(P,\oplus,\odot)$ from cases (I)–(III) from Example 2.1.2, such that*

$$\int_X^{\oplus} f(x)dm \in P,$$

where m is a $\sigma-\oplus-$decomposable measure. A mapping $T : B(Y,P) \to B(X,P)$ is pseudo-linear if the following conditions are satisfied

$$T(\mathbf{0}) = \mathbf{0};\ T(f\oplus h) = T(f)\oplus T(h);\ T(\lambda\odot f) = \lambda\odot T(f)\ \ (\lambda \in P, f,h \in B(Y,P)).$$

The Bellman operator B can be extended on the whole $B(Y,P)$ by

$$(Bf)(x) = \sup_{y\in Y}(k(x,y) + f(y)).$$

As it is well-known from classical linear functional analysis the representation of a linear operator on function spaces as integral operator is crucial. In the same spirit now we represent pseudo-linear operators by pseudo-integral ([38]). We restrict ourselves to those cases in which the pseudo-addition $\oplus$ is idempotent, i.e., max or min (cases (I) and (III) in Example 2.1.2).

Theorem 3.4 *Let P be one of the semirings from cases (I) or (III) from Example 2.1.2, with the property that for every bounded subset $\{a_\alpha\}$ of P and $\lambda \in P$ we have*

$$\oplus_\alpha(\lambda \odot a_\alpha) = \lambda \odot \oplus_\alpha a_\alpha.$$

If $T : B(Y,P) \to B(X,P)$ is a pseudo-linear operator, then it satisfies the condition

$$T(\oplus_\alpha f_\alpha) = \oplus_\alpha T(f_\alpha)$$

if and only if there exists a unique function $k \in B(X \times Y, P)$ such that

$$(Tf)(x) = \int_Y^\oplus k(x,y) \odot f(y)dy.$$

4 Morphism between probability calculus and decision calculus

We have to stress also the very close relation of the probability and the dynamic programming which is achieved by pseudo–analysis (see [3, 4, 19]).
It is well-known that for the Gaussian law $\mathcal{N}_{m,\sigma}$, with mean m and standard deviation σ, the following convolution rule hold

$$\mathcal{N}_{m_1,\sigma_1} * \mathcal{N}_{m_2,\sigma_2} = \mathcal{N}_{m_1+m_2,\sqrt{\sigma_1^2+\sigma_2^2}},$$

where $*$ is the usual convolution. In the dynamic programming specially in the decision theory the quadratic form $\mathcal{Q}_{m,\sigma}$ defined for $m \in \mathbb{R}$ and $\sigma \in [0,\infty)$

$$\mathcal{Q}(x) = \frac{1}{2}(\frac{x-m}{\sigma})^2 \text{ for } \sigma \neq 0$$

and

$$\mathcal{Q}_{0,0}(x) = \begin{cases} 0 & \text{for } x = m \\ +\infty & \text{otherwise} \end{cases}$$

plays important role. In [4] it was given the analogous pseudo-convolution rule.

Theorem 4.1

$$\mathcal{Q}_{m_1,\sigma_1} \star \mathcal{Q}_{m_2,\sigma_2} = \mathcal{Q}_{m_1+m_2,\sqrt{\sigma_1^2+\sigma_2^2}},$$

with respect to the pseudo–convolution $\star$ based on the operations $\oplus = \min$ and $\odot = +$.

Remark 4.2 Hence there exists a morphism between the family of quadratic forms endowed with the pseudo-convolution based on $(\min, +)$ and the family of exponentials of quadratic forms endowed with the classical convolution. This morphism is a special case of the Cramer transform.

Definition 4.3 *The Cramer transform $\mathcal{C}$ is a function from $\mathcal{M}$, the set of positive measures, into C_x, the set of mappings from $\mathbb{R}$ into $\overline{\mathbb{R}}$ convex, lower semicontinuous and proper, defined by*

$$\mathcal{C} = (-\mathcal{L}^{\oplus}) \circ \log \circ \mathcal{L},$$

where $\oplus = \min$ and $\mathcal{L}$ is the classical Laplace transform.

Theorem 4.4 *If $\mu_1, \mu_2 \in \mathcal{M}$, then*

$$\mathcal{C}(\mu_1 * \mu_2) = \mathcal{C}(\mu_1) \star \mathcal{C}(\mu_2),$$

where $*$ is the classical convolution.

Remark 4.5 The morphism from Theorem 4.4 gives a tool adapted to optimization analogous to probability calculus, for more details see [2, 3, 4, 19]).

So, e.g.,it was introduced the cost measure $\mathcal{K}$ as a mapping from the set $\mathcal{U}$ of all open sets of a topological space U into $[0, \infty]$ endowed with the operations $(\min, +)$ such that

$$\mathcal{K}(U) = 0 \, , \quad \mathcal{K}(\emptyset) = +\infty \, , \quad \mathcal{K}(\bigcup_n A_n) = \inf_n \mathcal{K}(A_n)$$

for every sequence (A_n) from $\mathcal{U}$. Then as analogy of the conditional probability the conditional cost excess to take the best decision in A knowing that it have to be taken in B as

$$\mathcal{K}(A|B) = \mathcal{K}(A \cap B) - \mathcal{K}(B)$$

was introduced. A decision vector is a mapping from U into $\mathbb{R}^n$. □

5 Hamilton-Jacobi equation with non-smooth Hamiltonian

5.1 Two examples

In this section we show how the theory of pseudo–decomposable measures and their corresponding integrals (pseudo–analysis) together with the pseudo–linear superposition principle can be applied to some non–linear partial differential equations.

Example 5.1.1 An important nonlinear partial differential equation is the Burgers equation for a function $u = u(x,t)$

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{c}{2}\frac{\partial^2 u}{\partial x^2} = 0$$

for $x \in \mathbb{R}$ and $t > 0$, with the initial condition $u(x,0) = u_0(x)$, where c is the given positive constant. We intend to apply the g-calculus with the generator

$g(u) = e^{-u/c}$ ([29, 30, 32]) to this equation. The corresponding pseudo-addition and the distributive pseudo-multiplication are given by

$$u \oplus v = -c\ln(e^{-u/c} + e^{-v/c}) \quad , \quad u \odot v = u + v \ .$$

Then for solutions u_1 and u_2 the function $(\lambda_1 \odot u_1) \oplus (\lambda_2 \odot u_2)$ is also a solution of Burgers equation.
The solution of the given initial problem is

$$u(x,t) = \frac{c}{2}\ln(2\pi ct) \odot \int^{\oplus} \left[\frac{(x-s)^2}{2t}\right] \odot u_0(s)\,ds,$$

where the integral is given by

$$\int^{\oplus} f(x)\,dx = -c\ln\left(\int e^{-f(x)/c}\,dx\right).$$

The operator $L : u \to u_0$ is self-adjoint with to respect to the scalar product

$$(u,v)_{\oplus} = \int_{\mathbb{R}}^{\oplus} u(x) \odot v(x)\,dx.$$

□

Example 5.1.2 Taking $c \to 0$ in the Burgers equation we obtain the Hamilton–Jacobi equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = 0.$$

Then for solutions u_1 and u_2 the function $(\lambda_1 \odot u_1) \oplus (\lambda_2 \odot u_2)$, where

$$u \oplus v = \min(u,v)$$

and

$$u \odot v = u + v,$$

is also a solution of the preceding Hamilton–Jacobi equation. □

5.2 General case

We extend now the pseudo–superposition principle to a more general case.

Theorem 5.2.1 *If u_1 and u_2 are solutions of the Hamilton-Jacobi equation*

$$\frac{\partial u(x,t)}{\partial t} + H(\frac{\partial u}{\partial x}, x, t) = 0, \tag{1}$$

where $H \in C(\mathbb{R}^{n+2})$ and $\frac{\partial u}{\partial x}$ is the gradient of u, then $(\lambda_1 \odot u_1) \oplus (\lambda_2 \odot u_2)$ is also a solution of the Hamilton-Jacobi equation (1), with respect to the operations $\oplus = \min$ and $\odot = +$.

Proof. First, it is obvious that $u + \lambda$ is a solution of (1) if u is a solution of (1). Now, it is enough to prove that $u_1 \oplus u_2 = \min(u_1, u_2)$ is a solution of (1) if u_1 and u_2 are solutions of (1). We can represent $u_1 \oplus u_2$ in the following form

$$\min(u_1, u_2) = (u_1 - u_2)\mathcal{H}(u_2 - u_1) + u_2$$

where $\mathcal{H}$ is the Heviside function defined by

$$\mathcal{H}(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

Putting $u_1 \oplus u_2$ in the left side of (1) we obtain

$$\frac{\partial((u_1 - u_2)\mathcal{H}(u_2 - u_1))}{\partial t} + \frac{\partial u_2}{\partial t} + H(\frac{\partial(u_1 - u_2)\mathcal{H}(u_2 - u_1))}{\partial x} + \frac{\partial u_2}{\partial t}, x, t). \quad (2)$$

Using the equality

$$\frac{\partial((u_1 - u_2)\mathcal{H}(u_2 - u_1))}{\partial t} = \frac{\partial(u_1 - u_2)}{\partial t}\mathcal{H}(u_2 - u_1)$$

we transform (2)

$$(\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial t})(1 - \mathcal{H}(u_2 - u_1)) + \frac{\partial u_2}{\partial t} + H((\frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x})\mathcal{H}(u_2 - u_1) + \frac{\partial u_2}{\partial x}, x, t). \quad (3)$$

Now we use the following obvious equality for any arbitrary function F

$$F(v + \mathcal{H}(s)w) \quad = \quad F(v) + (F(v + w) - F(v))\mathcal{H}(s).$$

Taking in the preceding equality $F = H, v = \frac{\partial u_2}{\partial x}$ and $w = \frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x}$ and using this in (3) we obtain

$$\begin{aligned} &(\tfrac{\partial u_1}{\partial t} - \tfrac{\partial u_2}{\partial t})\mathcal{H}(u_2 - u_1) + \tfrac{\partial u_2}{\partial t} + (H(\tfrac{\partial u_1}{\partial x}, x, t) \\ &-H(\tfrac{\partial u_2}{\partial x}, x, t))\mathcal{H}(u_2 - u_1) + H(\tfrac{\partial u_2}{\partial x}, x, t). \end{aligned} \quad (4)$$

We can easily write (4) in the form

$$(\frac{\partial u_1}{\partial x} + H(\frac{\partial u_1}{\partial x}, x, t))(1 - \mathcal{H}(u_2 - u_1)) + (\frac{\partial u_2}{\partial t} + H(\frac{\partial u_2}{\partial t}, x, t))(1 - \mathcal{H}(u_2 - u_1)). \quad (5)$$

Since u_1 and u_2 are solutions of (1) we obtain that (5) reduces identically to zero, what means that $u_1 \oplus u_2$ is a solution of the equation (1).

□

More details can be found in [32, 36, 23].

We consider now the following Cauchy problem for the Hamilton–Jacobi (–Bellman) equation

$$\frac{\partial u}{\partial t} + H(\frac{\partial u}{\partial x}) = 0 \quad , \quad u(x, 0) = u_0(x) \quad (6)$$

where $x \in \mathbb{R}^n$, and the function $H : \mathbb{R}^n \to \mathbb{R}$ is convex (hence by boundedness of H it is also continuous). For control theory the important examples of the Hamiltonian H are non-smooth functions, e.g., max and $|.|$. The approach with pseudo–analysis avoids the use of the so called "viscosity solution" method, which does not give the exact solution of (6) (see [22]). We apply now the methods of pseudo-analysis.

For that purpose we define the family of operators $\{R_t\}_{t>0}$, for a function $u_0(x)$ bounded from below in the following way

$$u(t,x) = (R_t u_0)(x) = \inf_{z\in\mathbb{R}^n}(u_0(z) - t\mathcal{L}^{\oplus}(H)(\frac{x-z}{t})), \qquad (7)$$

where $\mathcal{L}^{\oplus}$ is considered on the whole $\mathbb{R}^n$. The operator R_t is pseudo-linear with respect to $\oplus = \min$ and $\odot = +$. We denote by L (or $L(q)$, $q \in \mathbb{R}^n$) the transformation $L(H)(q) = \sup_{p\in\mathbb{R}^n}(pq - H(p))$, which is called also Legendre transform of H. Note that

$$L(H)(q) = -\mathcal{L}^{\oplus}(H)(q),$$

where $\mathcal{L}^{\oplus}(H)(q) = \inf_{p\in\mathbb{R}^n}(-pq + H(p))$. We note that L is also convex for H convex, but it can be discontinuous.

Let $\mathcal{S}_0(\mathbb{R}^n)$ be the space of continuous functions $f : \mathbb{R}^n \to P$ (P is of type I (i), III) with the property that for each $\varepsilon > 0$ there exists a compact subset $K \subset \mathbb{R}^n$ such that $d(\mathbf{0}, \inf_{x\in\mathbb{R}^n\setminus K} f(x)) < \varepsilon$, with the metric $D(f,g) = \sup_x d(f(x), g(x))$. Let $C_0(\mathbb{R}^n)$ be the subspace of $\mathcal{S}_0(\mathbb{R}^n)$ of functions f with compact support $supp_{\mathbf{0}} = \overline{\{x : f(x) \neq \mathbf{0}\}}$.

First we suppose that u_0 is smooth and strongly convex. We use the notation $\langle x, y\rangle$ and $||x||$ for the scalar product and Euclidean norm in $\mathbb{R}^n$, respectively. For a function $F : \mathbb{R}^n \to [-\infty, +\infty]$ its *subgradient* at a point $u \in \mathbb{R}^n$ is a point $w \in \mathbb{R}^n$ such that $F(u)$ is finite and

$$\langle w, v-u\rangle + F(u) \leq F(v)$$

for all $v \in \mathbb{R}^n$, see [11]. Then we have by [19].

Lemma 5.2.2 *Let $u_0(x)$ be smooth and strongly convex and there exists $\delta > 0$ such that for all x the eigenvalues of the matrix $u_0''(x)$ of all second derivatives are not less than δ. Then*

1°) *For every $x \in \mathbb{R}^n$, $t > 0$, there exists a unique $\xi(t,x) \in \mathbb{R}^n$ such that $\frac{x-\xi(t,x)}{t}$ is a subgradient of the function H at the point $u_0'(\xi(t,x))$ and*

$$(R_t u_0)(x) = u_0(\xi(t,x)) + tL(H)(\frac{x-\xi(t,x)}{t}). \qquad (8)$$

2°) *The function $\xi(t,x)$ for $t > 0$ satisfies the Lipschitz condition on compact sets, and $\lim_{t\to 0}\xi(t,x) = x$.*

3°) *The Cauchy problem (6) has a unique C^1 solution given by (8), and*

$$\frac{\partial u}{\partial x}(t,x) = u_0'(\xi(t,x)).$$

Sketch of the proof.
1°) The uniqueness of a point $\xi(t,x)$ with the desired properties follows from the definition of the subgradient.
We shall prove now the existence of $\xi(t,x)$ which satisfies (8). For that purpose we shall prove that if $\xi(t,x)$ is the (unique) point at which the infimum in (7), by strong convexity of $u_0(\xi)$, is taken, then $\frac{x-\xi(t,x)}{t}$ is a subgradient of H at $u_0'(\xi)$. This point $\xi(t,x)$ satisfies the following variational inequality (see [11]):

$$\langle u_0'(\xi), \eta - \xi\rangle + tL(\frac{x-\eta}{t}) - tL(\frac{x-\xi}{t}) \geq 0 \tag{9}$$

for every η. Taking $f(\xi) = \frac{x-\xi}{t}$ and $h = \frac{x-\eta}{t}$, we have

$$\langle u_0'(\xi), f(\xi)\rangle - L(f(\xi)) \geq \langle u_0'(\xi), h\rangle - L(h)$$

for all h. Since H is the Legendre transform of L, we obtain that the left–hand side of the last inequality is equal to $H(u_0'(\xi))$, i.e.,

$$H(u_0'(\xi)) + L(f(\xi)) = (u_0'(\xi), f(\xi)). \tag{10}$$

But the last equation is the characteristic of the subgradient ([11]). Therefore $f(\xi) = \frac{x-\xi}{t}$ is a subgradient of the function H at $u_0'(\xi)$.

2°) By (9) we obtain at $\xi = \xi(t,x)$ and $\eta = \xi(\tau, y)$

$$\langle u_0'(\xi), \frac{x-\xi}{t} - \frac{y-\eta}{\tau}\rangle + L(\frac{y-\eta}{\tau}) - L(\frac{x-\xi}{t}) \geq 0,$$

$$\langle u_0'(\eta), \frac{y-\eta}{\tau} - \frac{x-\xi}{t}\rangle + L(\frac{x-\xi}{t}) - L(\frac{y-\eta}{\tau}) \geq 0.$$

Adding these two inequalities we obtain after some calculations

$$\langle u_0'(\xi) - u_0'(\eta), \tau x - ty\rangle \geq \langle u_0'(\xi) - u_0'(\eta), \tau\xi - t\eta\rangle, \tag{11}$$

i.e.

$$\langle u_0'(\xi) - u_0'(\eta), \tau(x-y) + (\tau - t)(y-\eta)\rangle \geq \tau\langle u_0'(\xi) - u_0'(\eta), \xi - \eta\rangle. \tag{12}$$

We prove :

(i) If (x,t), (y,t) belong to some compact subset C of the half–space $\{(t,x) : \mathbb{R}^n, t > 0\}$, then ξ, η also belong to some compact subset of $\mathbb{R}^n$.

First we remark that $\lim_{||q||\to+\infty} L(q) = +\infty$, since $L(q) \geq \max_{||p||=1}(pq - H(p))$. Therefore there exists $r > 0$ such that $L(q) > L(0)$ for $||q|| \geq r$. Let C be a compact subset of $\{(t,x) : t > 0, x \in \mathbb{R}^n\}$. Then there exists a compact subset K of $\mathbb{R}^n$, such that for $\xi \notin K$ and $(t,x) \in C$, we have $u_0(\xi) > u_0(x)$ and $||\frac{x-\xi}{t}|| \geq A$. Hence

$$u_0(\xi) + tL(\frac{x-\xi}{t}) > u_0(x) + tL(0).$$

Therefore for $(t,x) \in C$ the point $\xi(t,x)$ of infimum in (7) belongs to K, and so

$$u(t,x) = \min_{\xi\in K}(u_0(\xi) + tL(\frac{x-\xi}{t})).$$

Using (i) we have the inequalities

$$\langle u_0'(\xi) - u_0'(\eta), \xi - \eta\rangle \geq \delta||\xi - \eta||^2,$$

$$||u_0'(\xi) - u_0'(\eta)|| \leq 2||\xi - \eta|| \cdot \max_{\theta\in[\xi,\eta]} ||u_0'(\theta)||.$$

Hence by (11) we have for $\tau > t$

$$\delta||\xi - \eta||^2\tau \leq 2\lambda(\tau||x - y|| + (\tau - t)||y - \eta||)||\xi - \eta||,$$

where $\lambda = \lambda(C)$ is some constant.Therefore $\xi(t,x)$ satisfies the Lipschitz condition on compact sets.

$||\xi(t,x)||$ is bounded for an arbitrary but fixed x and $t \leq t_0$. We have $\lim_{t\to0}\xi(t,x) = x$.

3°) The uniqueness of a C^1 solution follows from the fact that $H(p)$ satisfies the Lipschitz condition
By (10) we can rewrite (7) in the following form

$$u(t,x) = u_0(\xi) + \langle u_0'(\xi), x - \xi\rangle - tH(u_0'(\xi)),$$

where $\xi = \xi(t,x)$. Hence by 2°) the function $u(t,x)$ is continuous and satisfies the initial condition.

Let $R > r$ (r is from 2°)) such that the ball $Q_R = \{q : ||q|| \leq R\}$ contains the set $\{q = \frac{x-\xi}{t} : (t,x) \in C, \xi \in K\}$. We construct a sequence of smooth strictly convex functions $L_n : \mathbb{R}^n \to \mathbb{R}$, such that it is monotone increasing and pointwise convergent in Q_R to L, and $L_n(q) > L_n(0)$ for every n for $||q|| \geq r$. Let H_n be the Legendre transform of L_n. H_n is also smooth and strictly convex and so the lemma trivially holds for the Cauchy problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + H_n(\frac{\partial u}{\partial x}) = 0, \\ &u(0,x) = u_0(x). \end{aligned} \tag{13}$$

We denote by $\xi_n(t,x)$ the corresponding function by 1°) to H_n. The function

$$u_n(t,x) = u_0(\xi_n(t,x)) + tL_n(\frac{x-\xi_n(t,x)}{t})$$

is the smooth solution of (13) with $\frac{\partial u_n}{\partial x}(t,x) = u_0'(\xi_n(t,x))$. We have $\xi_n(t,x) \in K$ for $(t,x) \in C$ and every n. Therefore

$$u_n(t,x) = \min_{\xi \in K}(u_0(\xi) + tL_n(\frac{x-\xi}{t})).$$

Hence the sequence $\{u_n(t,x)\}$, $(t,x) \in C$, of smooth functions is nondecreasing and bounded above by $u(t,x)$. Therefore it converges for $(t,x) \in C$, i.e., $\tilde{u}(t,x) = \lim_{n\to\infty} u_n(t,x) \leq u(t,x)$.

We have for $(t,x) \in C$

$$\lim_{n\to\infty} \xi_n(t,x) = \xi(t,x)$$

and $\tilde{u}(t,x) = u(t,x)$. Let $\xi_n = \xi_n(t,x)$ and $\xi = \xi(t,x)$.

By the monotonicity of the sequence $\{u_n\}$ and continuity of the members u_n, it follows the uniform convergence of the sequence $\{u_n\}$ to u on compact set C. Consequently the sequence $\{\xi_n\}$ converges uniformly, since

$$\begin{aligned}|u(t,x) - u_n(t,x)| &\geq (u_0(\xi) + L_n(\frac{x-\xi}{t})) - (u_0(\xi_n) + L_n(\frac{x-\xi_n}{t}))\\ &\geq \frac{\delta}{2}\|\xi - \xi_n\|^2.\end{aligned}$$

So we have proved that the sequence $\{u_n(t,x)\}$ of smooth functions converges uniformly to $u(t,x)$ on C, and their derivatives $\frac{\partial u_n}{\partial x}(t,x) = u_0'(\xi_n(t,x))$ and

$$\begin{aligned}\frac{\partial u_n}{\partial t} &= -H_n(u_0'(\xi_n(t,x))) = L_n(\frac{x-\xi_n}{t}) - \langle u_0'(\xi_n), \frac{x-\xi_n}{t}\rangle\\ &= \frac{1}{t}(u_n(t,x) - u_0(\xi_n) - \langle u_0'(\xi_n), x-\xi_n\rangle)\end{aligned}$$

converge uniformly to $u_0'(\xi(t,x))$ and $-H(u_0'(\xi(t,x)))$, respectively. Therefore $u(t,x)$ is the C^1 solution of the Cauchy problem (6).

□

The dual semimodul $(\mathcal{S}_0(\mathbb{R}^n))^*$ is the semimodul of continuous pseudo–linear P-valued functionals on $\mathcal{S}_0(\mathbb{R}^n)$ (with respect to pointwise operations). Analogously the dual semimodul $(C_0(\mathbb{R}^n))^*$ is the semimodul of continuous pseudo–linear P–valued functionals on $C_0(\mathbb{R}^n)$ (with respect to pointwise operations). We need the following representation theorem, see [19].

Theorem 5.2.3 *Let f be a function defined on $\mathbb{R}^n$ and with values in the semiring P of type I) (i) or III). Further let a functional $m_f : C_0(\mathbb{R}^n) \to P$ be given by*

$$m_f(h) = \int^{\oplus} f \odot dm_h = \inf_x (f(x) \odot h(x)).$$

Then

1) *The mapping $f \mapsto m_f$ is a pseudo-isomorphism of the semimodule of lower semicontinuous functions onto the semimodule $C_0^*(\mathbb{R}^n)$.*

2) *The space S_0^* is isometrically isomorphic with the space of bounded functions, i.e., for every $m_{f_1}, m_{f_2} \in S_0^*(\mathbb{R}^n)$ we have*

$$\sup_x d(f_1(x), f_2(x)) = \sup\{d(m_{f_1}(h), m_{f_2}(h)) : h \in C_0(\mathbb{R}^n), D(h, \mathbf{0}) \leq 1\}.$$

3) *The functionals m_{f_1} and m_{f_2} are equal if and only if $Clf_1 = Clf_2$, where*

$$Clf(x) = \sup\{\psi(x) : \psi \in C(\mathbb{R}^n), \psi \leq f\}.$$

The Cauchy problem

$$\frac{\partial u}{\partial t} + H(-\frac{\partial u}{\partial x}) = 0, \qquad (14)$$
$$u(0,x) = u_0(x),$$

is the adjoint problem of the Cauchy problem (6). Referring to Lemma 5.2.2 the classical resolving operator R_t^* of the Cauchy problem (14) on the smooth convex functions is given by

$$(R_t^* u_0)(x) = \inf_\xi (u_0(\xi) + tL(\frac{\xi - x}{t})).$$

We note that R_t^* is the adjoint of the resolving operator R_t with respect to bipseudo–linear functional

$$\int_{\mathbb{R}^n}^{\oplus} f \odot h dm.$$

Then in analogy to the theory of linear partial differential equation we can introduce the notion of a generalized weak solution (using Theorem 5.2.3).

Definition 5.2.4 *Let u_0 be a bounded from below function $u_0 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and m_{u_0} the corresponding functional (see [36]) from $S_0^*(\mathbb{R}^n)$. The generalized weak pseudo–solution of the Cauchy problem (6) is a continuous function from below $(R_t u_0)(x)$ which is defined uniquely by $m_{R_t u_0}(\varphi) = m_{u_0}(R_t^* \varphi)$ for all smooth convex functions φ.*

We can construct the solution for the case in which u_0 is a smooth strictly convex function by Lemma 5.2.2. Then it follows by Theorem 5.2.3 and Definition 5.2.4 (see [19], [36]).

Theorem 5.2.5 *For an arbitrary function $u_0(x)$ bounded from below the weak pseudo-solution of the Cauchy problem (6) is given by*

$$(R_t u_0)(x) = (R_t Clu_0)(x) = \inf_z (Clu_0(z) - t\mathcal{L}^{\oplus}(H)(\frac{x - z}{t})),$$

where

$$Clf(x) = \sup\{\psi(x) : \psi \in C(\mathbb{R}^n), \psi \leq f\}.$$

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CHAPTER 14

Fuzzy Random Variables Revisited

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Introduction

Fuzzy-valued functions, as an extension of set-valued functions, were considered in connection with a categorical theory of fuzzy sets, since the mid 1970's (see Negoita and Ralescu [16]. Indeed, even earlier, fuzzy-valued functions with some special properties were studied: fuzzy relations (Zadeh [28]).

However, fuzzy-valued functions defined on a *probability* space, came at a later date. Different concepts: fuzzy variables, linguistic variables, fuzzy random sets, fuzzy random variables, probabilistic sets, were introduced in the late 1970's.

Féron [5] defined a *fuzzy random set* as a random element with values L-fuzzy sets in a topological space. However, he only studied some algebraic properties, and he did not define any concept of expected value.

Nahmias [15] defined *fuzzy variables* as ordinary random variables on a space endowed with both a probability and a possibility measure. His is *not* a concept of fuzzy-valued function.

Hirota [6], defined *probabilistic sets*, but he did not discuss any concept of expected value, and he concentrated on properties which are very similar to those of fuzzy relations.

Kwakernaak [13] introduced the concept of *fuzzy random variable* whose values are fuzzy sets on the real line. His concept of measurability is such, however, that it cannot be extended beyond fuzzy numbers in $\mathbb{R}$. As a matter of fact, that concept of measurability is not the proper concept needed in this framework (see later). Also his extensive discussion is often obscuring in what is the simplest example of fuzzy random variable (i.e. a family of random intervals in $\mathbb{R}$).

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Since 1980 (see the early papers [20], [21]) we have defined and studied the concept of *fuzzy random variable* (FRV) whose values are fuzzy subsets of $\mathbb{R}^n$ (or, more generally, of a Banach space). We have investigated the proper notions of measurability, integrability, metric concepts. We have defined the *expected value* and studied limit properties (Puri and Ralescu [19]). Our concept is much more general, and it includes [13].

An important subject in this framework is that of *limit theorems* for fuzzy random variables (e.g. strong law of large numbers and central limit theorem). In [21] we have announced a strong law of large numbers (SLLN) for fuzzy random variables in $\mathbb{R}^n$. Independently, Kruse [11] proved an SLLN for fuzzy random variables whose values are *convex* fuzzy numbers (in $\mathbb{R}$). His methods, however, cannot be used in more general cases and his SLLN does *not* include the corresponding result for random sets, thus it does not include the classical SLLN.

The result announced in [21] was proved by Klement, Puri, and Ralescu [9], [10]. Note that the latter result is valid for fuzzy subsets of $\mathbb{R}^n$, *not necessarily convex*. It completely subsumes the result of Kruse [11] (see below) and it extends the SLLN for random sets of Artstein and Vitale [1].

Miyakoshi and Shimbo [14] gave a result similar to that of [11].

Let us recall that Klement, Puri and Ralescu [9], [10] were the first to prove a central limit theorem (CLT) for fuzzy random variables in $\mathbb{R}^n$.

A few years later, Boswell and Taylor [3] gave a CLT by using the concept of FRV of Kwakernaak [13] and a new concept of independence that they have proposed. We show below that their concept of independence is so strong that it collapses and so their result is nothing else but the classical CLT for random variables!

A law of large numbers for fuzzy variables (Nahmias, [15]) was given by Stein and Talati [23]. However, their result is just the well known Bernoulli law of large numbers.

Niculescu and Viertl [17] have proved an extension of the Bernoulli law of large numbers for a very particular (essentially translates of a single membership function) kind of fuzzy random variables in $\mathbb{R}$. We show that their result is a simple consequence of the Klement, Puri, Ralescu SLLN. That also gives us the extension of the Bernoulli law to more general fuzzy sets in $\mathbb{R}^n$.

Interesting extensions of the SLLN were obtained by Inoue [7]. An in-depth study of the expected value was done in [27] and various measurability concepts were investigated by Butnariu [4]. An ergodic theorem for fuzzy random variables in Banach space appears in Bán [2].

Recently, a number of papers have "rediscovered" some of the old results (a sample of these papers: [24], [29], [30]). Essentially, they do not contain anything new.

The remainder of this paper contains: Section 2, in which we very briefly describe the mathematical framework we need and we state the Klement, Puri, and Ralescu SLLN; Section 3, in which we critically discuss various other concepts and results as suggested above. Finally, in Section 4 we explore the Brunn-Minkowski and the Jensen inequalities for fuzzy random variables.

1 The law of large numbers

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $F(\mathbb{R}^n)$ be the space of all fuzzy sets $u : \mathbb{R}^n \to [0,1]$ whose levels $L_\alpha u = \{x|u(x) \geq \alpha\}$ are compact, and nonempty for $\alpha > 0$, and whose support is compact. Let d_∞ be the distance in $F(\mathbb{R}^n)$ (first introduced in Puri and Ralescu [18]):

$$d_\infty(u,v) = \sup_{\alpha>0} d_H(L_\alpha u, L_\alpha v) \tag{1}$$

where d_H is the Hausdorff distance. Let d_1 be the distance [9]:

$$d_1(u,v) = \int_0^1 d_H(L_\alpha u, L_\alpha v) d\alpha \tag{2}$$

Definition 1.1 ([21, 19]) Any measurable map $X : \Omega \to F(\mathbb{R}^n)$ is called a *fuzzy random variable* (FRV).

Definition 1.2 ([21, 19]) The *expected value* of the FRV X is the unique fuzzy set EX such that $L_\alpha(EX) = E(L_\alpha X)$ for $\alpha \in [0,1]$, where $E(L_\alpha X)$ is the Kudo [12] integral (see [1]).

Theorem 1.3 *(SLLN of [9], [10]). Let $\{X_k|k \geq 1\}$ be independent and identically distributed FRV's such that $E\|supp\, X_1\| < \infty$. Then*

$$d\left(\frac{X_1 + X_2 + \cdots + X_n}{n}, E(coX_1)\right) \to 0 \tag{3}$$

almost surely.

Here d stands for d_1 or d_∞ (in the latter case we have to assume that $\{X_k\}$ are separably valued).

2 Critical discussion of other results

(a) Kwakernaak [13] coined the term fuzzy random variable. He defined his concept for $F(\mathbb{R})$ only, i.e. $X : \Omega \to F(\mathbb{R})$. His measurability concept *cannot* be extended to $F(\mathbb{R}^n)$; actually that measurability concept is unrelated to any of the (equivalent) measurability concepts for spaces of sets (cf. [8]).

For example, consider the random set

$$X = [A, B] \cup [C, D]$$

where $A, B, C, D : \Omega \to \mathbb{R}$, and A, D are measurable, while $B = -C$ are non-measurable, and $A \leq B \leq 0 \leq C \leq D$. Then X is an FRV in the sense of Kwakernaak, but the distance $\operatorname{dist}(0, X) = C$ is *not* measurable.

Actually, in the sense of Definition 1.1, in the particular case $n = 1$ and convex values, an FRV X has levels $L_\alpha X = [U_\alpha, V_\alpha]$, where U_α, V_α are ordinary random variables.

(b) *The Kruse LLN* [11] is proved for FRV's which are *convex* and valued in a subset $F_1 \subseteq F(\mathbb{R})$. Also convergence is pointwise convergence of the membership functions. The first drawback is that space F_1 does not include the ordinary intervals - thus this SLLN does *not* extend the classical SLLN of probability theory.

At any rate, it is possible to show that in F_1 convergence in metric d_∞ implies pointwise convergence, thus the SLLN of [9] contains the result of [11].

Actually, pointwise convergence of levels implies pointwise convergence of the membership functions:

Proposition 2.1 *Let u_n, u be fuzzy subsets of $\mathbb{R}$, with compact and convex levels.*

If $L_\alpha u_n \to L_\alpha u$ for every $\alpha > 0$ (convergence in d_H) then $u_n \to u$ pointwise.

Proof: Let $x_0 \in \mathbb{R}$; we have three cases:

(1) $0 < u(x_0) < 1$. Let ε be such that $0 < \varepsilon < u(x_0)$. Then $x_0 \in L_{u(x_0)-\varepsilon}u$ and, since $L_{u(x_0)-\varepsilon}u_n \to L_{u(x_0)-\varepsilon}u$ we get $x_0 \in L_{u(x_0)-\varepsilon}u_n$ for n large enough. Thus $u_n(x_0) \geq u(x_0) - \varepsilon$. Similarly, since $x_0 \notin L_{u(x_0)+\varepsilon}u$, we get $x_0 \notin L_{u(x_0)+\varepsilon}u_n$ for n large, i.e. $u_n(x_0) < u(x_0) + \varepsilon$. In conclusion, $u(x_0) - \varepsilon \leq u_n(x_0) < u(x_0) + \varepsilon$ for n large enough, thus $u_n(x_0) \to u(x_0)$ in this case.

(2) $u(x_0) = 0$. For $\varepsilon > 0$, since $x_0 \notin L_\varepsilon u$, it follows that $x_0 \notin L_\varepsilon u_n$ for n large enough. Thus $u_n(x_0) \to u(x_0) = 0$.

(3) $u(x_0) = 1$. This case is dealt with similarly to case (2) and the proof ends.

Note: The above proof is correct *only* if u_n, u belong to F_1 (as defined in [11]).

Another problem is that we don't really know if class F_1 is closed with respect to operations of fuzzy arithmetic, and if $X \in F_1$ whether $EX \in F_1$ or not? The proper framework is to use a distance between fuzzy sets, such as (1) or (2).

(c) *The Boswell-Taylor CLT* [3]. These authors also use the Kwakernaak FRV concept, but they define their own notion of independence. Let us briefly show that this independence concept is so strong that it can *only* be satisfied if the FRV's are ordinary random variables. So their result collapses into the classical, well known CLT.

We don't even have to consider FRV's; random sets will suffice. Their concept is: X, Y are "independent" if every selector $f \in X$ and every selector $g \in Y$ are independent. They do this, to obtain $E(XY) = EX \cdot EY$, for "independent" X, Y. It is not difficult to show

Proposition 2.2 *(i) if $X, \{\xi\}$ are "independent", then $X = \{\eta\}$ (i.e. X is a single random variable).*

(ii) If X, Y are "independent", then $X = \{\eta\}$, $Y = \{\xi\}$.

Proof:

(i) Let us assume, without loss of generality, that $E\xi \geq 0$. Let $X = [U, V]$ where U, V are random variables. Then

$$\begin{aligned}
\{\xi\} \cdot [U,V] &= \begin{cases} [\xi U, \xi V] & \text{if} \quad \xi \geq 0 \\ [\xi V, \xi U] & \text{if} \quad \xi < 0 \end{cases} \\
&= [\xi U, \xi V] I_{(\xi \geq 0)} + [\xi V, \xi U] I_{(\xi < 0)} \\
&= [\xi U I_{(\xi \geq 0)} + \xi V I_{(\xi < 0)}, \xi V I_{(\xi \geq 0)} + \xi U I_{(\xi < 0)}]
\end{aligned}$$

where I denotes indicator functions.

Since $\xi, [U, V]$ "independent" implies ξ, U independent and ξ, V independent, we obtain

$$\begin{aligned}
E(\{\xi\} \cdot [U,V]) = \; & [EU \cdot E(\xi I_{(\xi \geq 0)}) \; + \; EV \cdot E(\xi I_{(\xi < 0)}) \; , \\
& EU \cdot E(\xi I_{(\xi < 0)}) \; + \; EV \cdot E(\xi I_{(\xi \geq 0)})]
\end{aligned}$$

Since $\{\xi\}$ "independent" of $[U, V]$ implies $\{\xi - E\xi\}$ "independent" of $[U, V]$, we can assume $E\xi = 0$. Then $E(\{\xi\} \cdot [U, V]) = E\xi \cdot E[U, V] = \{0\}$. The above calculation now implies

$$EU \cdot E(\xi I_{(\xi \geq 0)}) + EV(\xi I_{(\xi \geq 0)}) = 0 \quad (*)$$

(since $E\xi = E(\xi I_{(\xi \geq 0)}) + E(\xi I_{(\xi < 0)}) = 0$).

Assume $E(\xi I_{(\xi \geq 0)}) = \int_{\{\xi \geq 0\}} \xi dP \neq 0$ (otherwise, since $\xi \geq 0$ on the integration set, equality to zero would imply $\xi = 0$ almost everywhere on $\{\xi \geq 0\}$, i.e. that $\xi \leq 0$, contrary to our assumption). Thus $(*)$ gives $EU = EV$.

But $V - U \geq 0$ (since $[U, V]$ is an interval!), thus $V - U$ is nonnegative and has expectation $E(V - U) = 0$; it follows that $V = U = \eta$ almost everywhere, i.e. that $X = [U, V] = \{\eta\}$ (a single random variable).

(ii) Let $X = [A, B]$, $Y = [C, D]$ be "independent" and let $\xi \in [C, D]$ be a selector. Then ξ is "independent" of $[A, B]$. It follows from (i) that $[A, B] = \{\eta\}$ must be a single random variable. Now the hypothesis of (ii) becomes $\{\eta\}$ in dependent of $[C, D]$. Again, by using (i), it follows that $C = D$. In conclusion $A = B$ and $C = D$, i.e. both X and Y are single random variables.

(d) *The Niculescu-Viertl [17]* extension of the Bernoulli LLN can be deduced as a particular case of the Klement, Puri, Ralescu SLLN (Theorem 1). Their idea is to extend $\sum_{i=1}^{k} I_A(x_i) =$ "total number of x_i's which belong to A" (where $A \subset \mathbb{R}$, $x_1, \ldots, x_k \in \mathbb{R}$) to fuzzy sets, by using the extension principle. It can be shown that, at least for the class $\mathcal{F}_A$ of membership functions u such that $\sup_A u$

and $\sup_{\overline{A}} u$ are both attained, their concept can be expressed as $\sum_{i=1}^{k} I_A(u_i) =$ "number of u_i's which belong to A".

Then, a simple application of the Klement, Puri, Ralescu SLLN (Theorem 1) gives: If $\{X_i | i \geq 1\}$ are i.i.d. FRV's valued in $\mathcal{F}_A$, then

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} I_A(X_i) &\to E[I_A(X_1)] \\ &= \text{Probability } (X_1 \text{ is } A). \end{aligned}$$

If we take $\sum_{i=1}^{k} I_A(u_i)$ as the *definition* for "the number of u_i's which belong to A" (which agrees with the other definition if the u_i's are numbers or ordinary sets), then the restriction of values to $\mathcal{F}_A$ is unnecessary, and we obtain the general result.

Incidentally, the restriction $A \subset \mathbb{R}$ is also unnecessary and we can assume $A \subset \mathbb{R}^n$.

Also the restriction (of [17]) to continuous membership functions is unnecessary. Actually that restriction would imply that the classical Bernoulli LLN is *not* a particular case of the result in [17].

Actually an application of Theorem 1.3 gives us the more general result

$$\frac{1}{n}\sum_{i=1}^{n} v(X_i) \to E(v(X_1))$$

almost surely, where v is a fuzzy set. Here the sum is interpreted as "number of X_i's which are v" (e.g. "number of short people who are smart").

3 The Brunn-Minkowski and the Jensen inequalities

We first define the volume of a fuzzy set $u \in F(\mathbb{R}^n)$ as

$$V_n(u) = \left[\int_{[0,1]} \left(V_n\left(L_\alpha u\right)\right)^{1/n} d\alpha\right]^n \tag{4}$$

where V_n inside the integral is ordinary volume. Note that this concept of volume depends on the dimension n. We can now prove

Theorem 3.1 *(The Brunn-Minkowski inequality for fuzzy sets): Let $u, v \in F(\mathbb{R}^n)$. Then*

$$V_n[\lambda u + (1-\lambda)v] \geq \left[\lambda\left(V_n(u)\right)^{1/n} + (1-\lambda)\left(V_n(v)\right)^{1/n}\right]^n \tag{5}$$

where V_n denotes the volume of a fuzzy set.

Proof: We have successively

$$\begin{aligned} V_n[\lambda u + (1-\lambda)v] &= \left[\int_{[0,1]} (V_n(L_\alpha(\lambda u + (1-\lambda)v)))^{1/n}\, d\alpha\right]^n \\ &= \left[\int_{[0,1]} V_n\left(\lambda L_\alpha u + (1-\lambda)L_\alpha v)\right)^{1/n} d\alpha\right]^n \\ &\geq \left[\int_{[0,1]} \left[\lambda\left(V_n(L_\alpha u)\right)^{1/n} + (1-\lambda)\left(V_n(L_\alpha v\right)^{1/n}\right] d\alpha\right]^n \\ &= \left[\lambda\left(V_n(u)\right)^{1/n} + (1-\lambda)\left(V_n(v)\right)^{1/n}\right]^n . \end{aligned}$$

We have used the classical Brunn-Minkowski inequality as well as $L_\alpha(\lambda u + (1-\lambda)v) = \lambda L_\alpha u + (1-\lambda)L_\alpha v$.

It now follows that for a fuzzy random variable X we have

$$V_n(EX) \geq (E(V_n X)^{1/n})^n \tag{6}$$

which extends the result for random sets in [25].

Our next goal is to extend the Jensen inequality to FRV's. The main result is

Theorem 3.2 *If X is an FRV and φ is a convex function $\varphi : F(\mathbb{R}^n) \to \mathbb{R}$ (i.e. $\varphi(\lambda u + (1-\lambda)v) \leq \lambda\varphi(u) + (1-\lambda)\varphi(v)$) then we have*

$$\varphi(EX) \leq E\varphi(X) \tag{7}$$

Proof: Let us consider first that X is a simple FRV

$$X = \sum_{i=1}^{r} u_i I_{A_i} \tag{8}$$

where $u_i \in F(\mathbb{R}^n)$ and $(A_i)_i$ form a partition of Ω. Then (7) becomes

$$\varphi\left[\sum_{i=1}^{r} P(A_i)u_i\right] \leq \sum_{i=1}^{r} \varphi(u_i)P(A_i) \tag{9}$$

which is true since it follows by induction from the convexity of φ.

If X is now an integrable FRV, i.e. $E\|\mathrm{supp}\, X\| < \infty$ (which we assume in this Theorem), the result follows by using a Lebesgue dominated convergence result (see, for example, [21], [19], or [10]).

An important example of such a convex function φ is the Steiner point map; in that case φ is actually additive and positively homogeneous (i.e. "linear"). For this choice of φ (7) becomes an equality.

Our extension of the Jensen inequality seems to be unknown even if X is a random set.

Potential applications of these inequalities are in the field of statistics with inexact data and, more specifically, at the evaluation of the power function of statistical tests based on fuzzy data (see the recent papers [22] and [26]).

4 Conclusions

We showed that the proper concept of fuzzy random variable [19] (in $\mathbb{R}^n$ or in a Banach space) involves distances on spaces of fuzzy sets and measurability of random elements valued in a metric space.

The most general limit theorems for fuzzy random variables are those in [10]; all other results in this direction can be derived from them.

Finally, we extended the Brunn-Minkowski and the Jensen inequalities to fuzzy random variables.

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