Sum of Two Independent Normal Random Variables

Example 7.5 It is an interesting and important fact that the convolution of two normal densities with means μ_1 and μ_2 and variances σ_1 and σ_2 is again a normal density, with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. We will show this in the special case that both random variables are standard normal. The general case can be done in the same way, but the calculation is messier. Another way to show the general result is given in Example 10.17.

Suppose X and Y are two independent random variables, each with the standard *normal* density (see Example 5.8). We have

$$f_X(x) = f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \ ,$$

and so

$$\begin{aligned} f_Z(z) &= f_X * f_Y(z) \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-(z-y)^2/2} e^{-y^2/2}\, dy \\ &= \frac{1}{2\pi} e^{-z^2/4} \int_{-\infty}^{+\infty} e^{-(y-z/2)^2}\, dy \\ &= \frac{1}{2\pi} e^{-z^2/4} \sqrt{\pi} \left[\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(y-z/2)^2}\, dy \ \right] . \end{aligned}$$

The expression in the brackets equals 1, since it is the integral of the normal density function with $\mu = 0$ and $\sigma = \sqrt{2}$. So, we have

$$f_Z(z) = \frac{1}{\sqrt{4\pi}} e^{-z^2/4} \ .$$

□

Sum of Two Independent Cauchy Random Variables

Example 7.6 Choose two numbers at random from the interval $(-\infty, +\infty)$ with the Cauchy density with parameter $a = 1$ (see Example 5.10). Then

$$f_X(x) = f_Y(x) = \frac{1}{\pi(1 + x^2)} \ ,$$

and $Z = X + Y$ has density

$$f_Z(z) = \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \frac{1}{1 + (z-y)^2} \frac{1}{1 + y^2}\, dy \ .$$

This integral requires some effort, and we give here only the result (see Section 10.3, or Dwass[3]):

$$f_Z(z) = \frac{2}{\pi(4+z^2)}\ .$$

Now, suppose that we ask for the density function of the *average*

$$A = (1/2)(X+Y)$$

of X and Y. Then $A = (1/2)Z$. Exercise 5.2.19 shows that if U and V are two continuous random variables with density functions $f_U(x)$ and $f_V(x)$, respectively, and if $V = aU$, then

$$f_V(x) = \left(\frac{1}{a}\right) f_U\left(\frac{x}{a}\right)\ .$$

Thus, we have

$$f_A(z) = 2f_Z(2z) = \frac{1}{\pi(1+z^2)}\ .$$

Hence, the density function for the average of two random variables, each having a Cauchy density, is again a random variable with a Cauchy density; this remarkable property is a peculiarity of the Cauchy density. One consequence of this is if the error in a certain measurement process had a Cauchy density and you averaged a number of measurements, the average could not be expected to be any more accurate than any one of your individual measurements! □

Rayleigh Density

Example 7.7 Suppose X and Y are two independent standard normal random variables. Now suppose we locate a point P in the xy-plane with coordinates (X,Y) and ask: What is the density of the square of the distance of P from the origin? (We have already simulated this problem in Example 5.9.) Here, with the preceding notation, we have

$$f_X(x) = f_Y(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\ .$$

Moreover, if X^2 denotes the square of X, then (see Theorem 5.1 and the discussion following)

$$\begin{aligned} f_{X^2}(r) &= \begin{cases} \frac{1}{2\sqrt{r}}(f_X(\sqrt{r}) + f_X(-\sqrt{r})) & \text{if } r > 0, \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} \frac{1}{\sqrt{2\pi r}}(e^{-r/2}) & \text{if } r > 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

[3] M. Dwass, "On the Convolution of Cauchy Distributions," *American Mathematical Monthly,* vol. 92, no. 1, (1985), pp. 55–57; see also R. Nelson, letters to the Editor, ibid., p. 679.