Let $W = (X^2 + Y^2 + Z^2)^{1/2}$ (take the positive square root so that $W \ge 0$). Find the distribution function of W.

We take $\Omega = \mathbb{R}^3$, $\mathcal{F} = \mathcal{B}(\mathbb{R}^3)$, $X(x, y, z) = x$, $Y(x, y, z) = y$, $Z(x, y, z) = z$, and

$$P(B) = \iiint\limits_B f(x, y, z)\, dx\, dy\, dz, \qquad B \in \mathcal{B}(\mathbb{R}^3),$$

where

$$\begin{aligned} f(x, y, z) &= f_1(x) f_2(y) f_3(z) = g(x) g(y) g(z) \\ &= (2\pi)^{-3/2} \exp\left[-\tfrac{1}{2}(x^2 + y^2 + z^2)\right]. \end{aligned}$$

Thus

$$F(w) = P\{W \le w\} = P\{X^2 + Y^2 + Z^2 \le w^2\}$$

If $w \ge 0$,

$$F(w) = \iiint\limits_{x^2+y^2+z^2 \le w^2} (2\pi)^{-3/2} \exp\left[-\frac{1}{2}(x^2 + y^2 + z^2)\right] dx\, dy\, dz$$

or in spherical coordinates,

$$\begin{aligned} F(w) &= \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^{w} (2\pi)^{-3/2} \exp\left[-\frac{1}{2} r^2\right] r^2 \sin\phi\, dr \\ &= (2\pi)^{-3/2} (2\pi)(2) \int_0^w r^2 \exp\left[-\frac{1}{2} r^2\right] dr. \end{aligned}$$

Thus W is absolutely continuous, with density

$$f(w) = \begin{cases} \dfrac{2}{\sqrt{2\pi}} w^2 \exp\left[-\dfrac{1}{2} w^2\right], & w \ge 0, \\ 0, & w < 0. \end{cases}$$

4.9.3 Example. Let $X_1, \dots, X_n$ be independent random variables, each with density f and distribution function F; that is, $\Omega = \mathbb{R}^n$, $\mathcal{F} = \mathcal{B}(\mathbb{R}^n)$, $X_i(x_1, \dots, x_n) = x_i$, $1 \le i \le n$,

$$P(B) = \int_B f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n, \qquad B \in \mathcal{B}(\mathbb{R}^n).$$

Let T_k be the kth smallest of the X_i; for example, if $n = 4$, $X_1(\omega) = 2$, $X_2(\omega) = 1.4$, $X_3(\omega) = -7$, $X_4(\omega) = 8$, then $T_1(\omega) = \min_i X_i(\omega) = X_3(\omega)$

$= -7$, $T_2(\omega) = X_2(\omega) = 1.4$, $T_3(\omega) = X_1(\omega) = 2$, $T_4(\omega) = \max_i X_i(\omega) = X_4(\omega) = 8$. [Note that

$$P\{X_i = X_j\} = \iint\limits_{x_i = x_j} f(x_i) f(x_j)\, dx_i\, dx_j = 0 \qquad \text{for} \qquad i \neq j,$$

and therefore

$$P\{X_i = X_j \text{ for at least one } i \neq j\} \leq \sum_{i \neq j} P\{X_i = X_j\} = 0.$$

Thus ties occur with probability 0 and can be ignored.]

Find the individual distribution functions of the T_k, and the joint distribution function of $(T_1, \ldots, T_n)$.

Now

$$P\{T_k \leq x\} = \sum_{i=1}^{n} P\{T_k \leq x, T_k = X_i\} \qquad \text{by 4.5.2} \tag{1}$$

and, for example,

$$\begin{aligned} P\{T_k \leq x, T_k = X_1\} = P\{&X_1 \leq x, \text{ exactly } k-1 \text{ of the random} \\ &\text{variables } X_2, \ldots, X_n \text{ are less than } X_1 \\ &\text{and the remaining } n-k \text{ random variables} \\ &\text{are greater than } X_1\}. \end{aligned} \tag{2}$$

But, using Fubini's theorem,

$$\begin{aligned} &P\{X_1 \leq x, X_2 < X_1, \ldots, X_k < X_1, X_{k+1} > X_1, \ldots, X_n > X_1\} \\ &\quad = \int_{x_1=-\infty}^{x} \int_{x_2=-\infty}^{x_1} \cdots \int_{x_k=-\infty}^{x_1} \int_{x_{k+1}=x_1}^{\infty} \\ &\qquad \cdots \int_{x_n=x_1}^{\infty} f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n \\ &\quad = \int_{-\infty}^{x} f(x_1)(F(x_1))^{k-1}(1 - F(x_1))^{n-k}\, dx_1. \end{aligned} \tag{3}$$

Now by symmetry, (2) is the sum of $\binom{n-1}{k-1}$ terms, each of which has the same value as (3), since we may select the $k-1$ random variables to be less than X_1 in $\binom{n-1}{k-1}$ ways. Also, each term in the summation (1) has the same value as (2). Thus

$$P\{T_k \le x\} = \int_{-\infty}^{x} n\binom{n-1}{k-1} f(x_1)(F(x_1))^{k-1}(1 - F(x_1))^{n-k}\, dx_1$$

so that T_k is absolutely continuous, with density

$$f_{T_k}(x) = n\binom{n-1}{k-1} f(x)(F(x))^{k-1}(1 - F(x))^{n-k}.$$

We now find the joint distribution function of $T_1, \ldots, T_n$. Let $b_1 < b_2 < \cdots < b_n$. Then

$$\begin{aligned}
&P\{T_1 \le b_1, \ldots, T_n \le b_n\} \\
&\quad = n!P\{T_1 \le b_1, \ldots, T_n \le b_n, X_1 < X_2 < \cdots < X_n\} \qquad \text{by symmetry} \\
&\quad = n!P\{X_1 \le b_1, X_1 < X_2 \le b_2, X_2 < X_3 \le b_3, \ldots, X_{n-1} < X_n \le b_n\} \\
&\quad = n! \int_{-\infty}^{b_1} f(x_1)\, dx_1 \int_{x_1}^{b_2} f(x_2)\, dx_2 \cdots \int_{x_{n-1}}^{b_n} f(x_n)\, dx_n \\
&\quad = \int_{-\infty}^{b_1} \cdots \int_{-\infty}^{b_n} g(x_1, \ldots, x_n)\, dx_1, \cdots dx_n,
\end{aligned}$$

where

$$g(x_1, \ldots, x_n) = \begin{cases} n!f(x_1)\cdots f(x_n), & x_1 < x_2 < \cdots < x_n, \\ 0 & \text{elsewhere.} \end{cases}$$

Thus $(T_1, \ldots, T_n)$ is absolutely continuous with density g. (Note that f_{T_k} can be found from g (see 4.8.4), but the calculation is not any simpler than the direct method we have used above.)

4.9.4 Example. Let X be an absolutely continuous random variable with density f, assumed to be piecewise-continuous. Let D be a Borel subset of $\mathbb{R}$ such that D includes the range of X, and let g be a Borel measurable function from D to $\mathbb{R}$.

If $Y = g \circ X$, we wish to find the distribution of Y. [*Distribution* is a generic term; to say that we know the distribution of Y means that we know how to calculate $P\{Y \in B\}$ for all Borel sets B. Thus the distribution may be specified by giving the induced probability measure P_Y or the distribution function F_Y. If Y is absolutely continuous, its density is adequate, and if Y is discrete, the probability function suffices. If $Y: (\Omega, \mathscr{F}) \to (\Omega', \mathscr{F}')$ is an arbitrary random object, the distribution of Y means the probability measure P_Y, defined by $P_Y(B) = P\{Y \in B\}$, $B \in \mathscr{F}'$.]

Assume that D is an open interval I, and g is either strictly increasing or strictly decreasing, with inverse h. Assume also that g has a continuous