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Unfolding intensity function of a Poisson process in models with approximately specified folding operator

Zbigniew Szkutnik

Department of Applied Mathematics, University of Mining and Metallurgy, Al. Mickiewicza 30, 30-059 Cracow, Poland (e-mail: szkutnik@mat.agh.edu.pl)

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Abstract. Quasi-maximum likelihood histogram sieve estimators of the intensity function of an indirectly observed Poisson process are studied. The setup differs from the standard one in that the exact form of the folding operator may not be known. Instead, approximate knowledge on its discretized version is available. Conditions for strong L^2-consistency are given and admissible discretization rates are studied. In non-folding problems, the number of histogram bins may essentially increase at the usual maximal rate while folding reduces the allowed discretization rates. It is shown that, even in moderately ill-posed problems, the discretization effects may be critical for the strong L^2-convergence and that there is an essential need both for further regularization and for imposing stronger conditions on the estimated function. Not surprisingly, the most restrictive factor is the low approximation power of piecewise constant functions. A regularization method is proposed which suitably modifies the discrete approximation of the folding operator and ensures the strong consistency. Since no penalty term is being introduced, the EM algorithm can be used in its factorized, efficient form. Convergence rates are obtained in terms of the discrete problem.

Key words: Quasi-Maximum Likelihood, Poisson process, intensity function, unfolding, discretization

1 Introduction

Let $(E_0, \mathscr{B}_0, \mu)$ be a metric, measure space, $\mathscr{B}_0$ its Borel σ-algebra and μ a fixed measure. For a family of Poisson processes $\{N_t^0, t > 0\}$ on E_0 with mean measures α_t admitting the representation $d\alpha_t = tf\,d\mu$, we shall be dealing with the problem of estimating the intensity function $f : E_0 \to [0, \infty)$ of the process N_t^0.

It is a common experimental problem that one cannot observe N_t^0 directly. The data distortion mechanism is usually described by a transition function K from $(E_0, \mathscr{B}_0)$ to another measure space $(E_1, \mathscr{B}_1)$, in which the observations are actually taken. In effect, one observes a Poisson process N_t^1 on E_1 with the mean measure (Karr (1986), Reiss (1993))

$$v_t(dy) = \int_{E_0} \alpha_t(dx) K(x, dy).$$

We admit $K(x, E_1) < 1$, which means that some data points can be lost. Given observed N_t^1 on E_1, we want to estimate the unknown function f and to study asymptotics as the observation time t tends to infinity.

Problems of this kind are omnipresent in applications and have been receiving much attention for many years. Well known examples include the Wicksell problem of stereology, the image reconstruction in positron emission tomography (PET) and many others (Reiss (1993), Vardi and Lee (1993)). We give only a short review of some recent papers which contain many other references.

We begin with a minimax approach. Johnstone and Silverman (1990, 1991), using a minimax approach over suitable function classes and singular value decomposition (SVD) of the folding operator, obtained minimax estimators and convergence rates for the minimax risk in the L^2-norm. Both the function classes and the estimators are defined in terms of the singular functions of the folding operator. The second paper deals with the case of binned observations in E_1 and studies the effect of the discretization rate on the accuracy of estimation. The general conclusion is that if the number of bins increases sufficiently fast, then the discretization in E_1 does not degrade the accuracy of estimation. Koo and Chung (1998) use a similar SVD approach and establish bounds on the Kullback-Leibler and L^2 distances between the true f and the estimators under some smoothness assumptions on $\log f$. The true f is assumed to belong to a special exponential family of distributions. It is essential for SVD approaches that the folding operator be exactly known in order to construct a suitable basis of singular functions. The minimax approach is also well studied in Korostelev and Tsybakov (1993). Wavelet-based methodology for inverse problems is discussed, for example, by Donoho (1995). Finally, one should mention maximum entropy methods, which intrinsically involve a kind of regularization. A critical review of this approach is given in Donoho et al. (1992).

In many cases, a discrete version of the problem, as described below, is solved via a maximum likelihood (ML) approach combined with the EM algorithm which has independently been invented in diverse areas of applications (Vardi and Lee (1993)). Smoothed versions of EM have been developed as well (eg Eggermont an LaRiccia (1995)), which is important, as the unfolding problems are typically ill-posed (in the sense of Hadamard) and the solutions require a kind of regularization or smoothing. Smoothing via a penalized likelihood approach, seriously deteriorates, however, the numerical performance of the EM algorithm (Hudson et al. (1994)) because it looses, in that case, its efficient, factorized form. Note also, that the EM algorithm converges to a ML solution for a fixed discretization. In PET-like applications, the binnings in E_0 and E_1, as defined by the geometry of the camera, are usually considered fixed and fine enough for the approximation to be satisfactory. There is some evidence, however, that oscillations often observed in the high iterations of the EM algorithm, or 'speckling' of reconstructed PET images, might be related to discretization effects (Mair et al. (1996)). It will be seen in section 2 that those effects may also be critical, when the strong L^2-consistency is studied.

Recently, an EM-equivalent approach has been exploited by D'Agostini (1995) for unfolding problems in high energy physics (HEP), with a Bayesian justification and, apparently, independently from the immense EM-related statistical literature. In contrast to PET-like problems, however, the numbers of bins involved in HEP problems are usually small and the binnings become finer as the sample size increases. A more essential special feature of HEP problems is that the folding operator is not known exactly and only its discrete approximation can be obtained, usually via Monte Carlo simulations. This makes the standard SVD approach inapplicable.

Variance estimation, especially important in HEP-like applications has been receiving relatively less attention. An overview of existing results, in particular those related to the EM algorithm is given in Segal et al. (1994). Recent reference on the same topic is Maitra and O'Sullivan (1998). More practical approaches are given in Blobel (1984), D'Agostini (1995) and Szkutnik (1997) in the HEP context. Asymptotic normality of quasi-maximum likelihood (QML) estimators in a setup with fixed binnings has been proved in Szkutnik (1996a).

HEP-like unfolding problems is our main motivation in this paper and conditions on discretization rates which ensure strong consistency and take into account both the degree of ill-posedness and the approximation effects will be our main interest.

Our approach to the problem will be via the method of sieves and exponential inequalities for deviation probabilities. Although we use different techniques, related results can be found in Wong and Shen (1995).

Let us now describe the discretization of the problem in some detail and define the QML histogram sieve estimator for f. Let $E_0 = A_1 \cup \cdots \cup A_n$ and $E_1 = B_1 \cup \cdots \cup B_m$ be partitions of E_0 and E_1 into disjoint and measurable histogram bins.

Denote $g_i^0 = \nu_t(B_i)/t$, $i = 1, \ldots, m$ and $f_j = \int_{A_j} f(x)\mu(dx)/\mu(A_j)$, $j = 1, \ldots, n$. Thus, f_j is the mean value of f in the bin A_j. Then, with $g^0 = [g_1^0, \ldots, g_m^0]^T$ and $\theta^0 = [f_1, \ldots, f_n]^T$, we have $g^0 = C^0\theta^0$ with $C^0 = [c_{ij}^0]$, where

$$c_{ij}^0 = \frac{\int_{B_i}\int_{A_j} K(x, dy) f(x)\mu(dx)}{\int_{A_j} f(x)\mu(dx)}\mu(A_j) := d_{ij}^0\mu(A_j).$$

Observe that if no data points are being lost, then $d_{ij}^0 = \mathrm{P}(Y \in B_i \mid X \in A_j)$, where $X \in E_0$ is a (non-observable) random point associated with N_t^0 and $Y \in E_1$ is the corresponding (observable) random point associated with N_t^1. If $\sum_i d_{ij}^0 < 1$, then $1 - \sum_i d_{ij}^0$ is the probability of a data point X being lost, given $X \in A_j$. The matrix $D^0 = [d_{ij}^0]$, being a discrete version of the folding kernel, depends on the unknown function f and a kind of approximation $D = [d_{ij}]$ to D^0 must be constructed, even if $K(x, y)$ were exactly known.

In the discrete setup, we thus have the following Poisson regression model for the vector $\bar{n} = (n_1, \ldots, n_m)$ of observed bin counts

$$P_g^t(\bar{n}) = \prod_{i=1}^{m} (tg_i)^{n_i} (n_i!)^{-1} e^{-tg_i} \tag{1}$$

with $g = C\theta,\ C = D\,\mathrm{diag}(\mu(A_j)),\ \theta \in \Theta_n \subset \mathscr{R}^n$. Since C only approximates C^0, the model (1) is only approximately correct, and it can happen that the true g^0 is not of the form $C\theta$ with $\theta \in \Theta_n$, even if $\theta^0 \in \Theta_n$.

We shall work with a family of QML estimators. (Wong and Shen (1995) used a term η-ML estimator in a similar context. We use the term QML estimators, as in Pfaff (1982), even if Pfaff actually assumed a constant γ in the definition which follows.) Define a QML estimator $\hat{\theta}_t$ of θ^0 in our model (1) as any statistic $\hat{\theta}_t(\bar{n})$ with values in Θ_n such that

$$P_{\hat{g}_t}^t(\bar{n}) \geq \gamma(t) \sup_{g \in \mathscr{G}} P_g^t(\bar{n}), \tag{2}$$

where $\hat{g}_t = C\hat{\theta}_t$, $\mathscr{G} = \{g : g = C\theta, \theta \in \Theta_n\}$, $\gamma(t) \in (0,1]$ and $\gamma(t)$ may tend to zero as $t \to \infty$.

It is worth noting here that the concept of QML estimators is essentially more general than that of ML estimators and that those two types of estimators are not asymptotically equivalent. For instance, the asymptotic distributional properties of QML and ML estimators are different, as can be seen from the following simple example. Let N be a random variable, Poisson distributed with parameter $t\lambda$. Then the ML estimator $\tilde{\lambda}_t = N/t$ of λ is asymptotically normally distributed. Define another estimator $\hat{\lambda}_t = (1 + c_t)\tilde{\lambda}_t$, where $c_t = t^{1/2}U/N$ and U is a random variable. $\hat{\lambda}_t$ will be a QML estimator if $\log(1 + c_t) \geq c_t - N^{-1}\log\gamma^{-1}$. Using $\log(1 + x) \geq x - x^2/2$, one can rewrite this QML condition in a slightly stronger form as $c_t^2 \leq 2N^{-1}\log\gamma^{-1}$, or $U^2 \leq 2Nt^{-1}\log\gamma^{-1}$. N/t tends to λ almost surely. If $\lambda > 0$ and $\mathrm{P}(U^2 \leq \lambda\log\gamma^{-1}) = 1$, then $\hat{\lambda}_t$ is indeed a QML estimator. However, since $t^{1/2}(\hat{\lambda}_t - \lambda) = t^{1/2}(\tilde{\lambda}_t - \lambda) + U$, the estimator $\hat{\lambda}_t$ does not have to be asymptotically normal.

Coming back to our model (1) and the QML criterion (2), with $\hat{\theta}_t = [\hat{\theta}_t^{(1)}, \ldots, \hat{\theta}_t^{(n)}]$, the histogram sieve estimator for f is constructed as

$$\hat{f}_t(x) = \sum_{i=1}^{n} \hat{\theta}_t^{(i)} \mathbf{1}_{A_i}(x).$$

The consistency of this family of estimators has been studied in Szkutnik (1996b) in the case of fixed binnings. In this paper, conditions for strong L^2-consistency will be given with binnings becoming finer as t increases. In order to ensure measurability and separability of all involved random quantities, we shall always assume that $\hat{\theta}_t$ changes only when the partitions of E_0 or E_1 change or when a new random point in E_1 is observed, that the length of the time intervals between two consequtive changes of the partitions of E_0 or E_1 has a positive lower bound and that the time points t_k at which the partitions change are either nonrandom or they are measurable functions of the observable process N_t^1.

A special case of our QML estimators in the folding model is the ML estimator in the model with no folding – the case studied, for example, with different methods by Karr (1986). He considers independent replications of the process, which corresponds to a sequence $\{t_k\}$ of discrete values of t in our setup, say $t_k = k$. He assumes $f \in L^1(E_0, \mathscr{B}, \mu)$ and proves strong L^1-consistency of the ML sieve estimator when $\max_i \mu(A_i) \to 0$ and $\sum_{k=1}^{\infty} n^4/k^2 < \infty$, which requires at least $n = o(t^{1/4})$. This is clearly a too restrictive, technical condition. One can expect in this case that $n = o(t)$ should suffice, as it does in the histogram sieve density estimation based on a fixed number of i.i.d. observations. Indeed, our conditions assumed below allow n to increase essentially at this maximal rate.

In section 2, we formulate and discuss a general Theorem 1 which is subsequently applied to some well-known unfolding problems, discretized in a standard way. This shows an essential need for some further regularization which is discussed, along with smoothness restrictions on f, in section 3. The discussion results in Theorem 2, in which conditions for the strong L^2-consistency of additionally regularized QML estimators are given for the more practically relevant case of Poisson processes on $\mathscr{R}^d$. Finally, Theorem 3 gives the convergence rates for the setup from Theorem 2. Section 4 contains the proofs and some auxiliary results.

2 General and projection-type approximations

As t increases, we make the partitions $\{A_j\}$ and $\{B_i\}$ of E_0 and E_1 finer and construct a QML estimator for each given t using the histogram sieve, as described in section 1. For the sake of simplicity, we not always add the subscript t to symbols denoting the quantities which can change in time. So, for example, we still write m, n, g, C, θ^0, although all those quantities do change with t. Similarly, we write γ for $\gamma(t)$. In the sequel, for any square matrix M, we denote by $\lambda_{\min}(M)$ the minimal eigenvalue of M and $\|\cdot\|$ stands for the Euclidean norm or the operator norm, depending on the context. For t-dependent quantities, say a_t and b_t, we write $a_t \asymp b_t$, if a_t/b_t remains bounded and cut away from zero as t increases to infinity.

The following theorem provides some general, sufficient conditions for the strong L^2-consistency of our QML estimators. Note that, in Theorem 1, C is an arbitrary discrete approximation of the folding operator. The link between C and K is only indirectly set in the first part of A6 and the second part of A5. The assumptions are expressed in terms of C rather than K which reflects the fact that, as it will be seen later, it is extremely difficult to precisely relate the decay rate of the squared singular values of the linear operator defined by K to that of the eigenvalues of C^TC, the latter being crucial for the strong L^2-consistency. We thus keep that aspect outside Theorem 1. On the other hand, the theorem itself suggests a way of overcoming the difficulty.

Recall that, as pointed out in the introduction, even if K is known exactly, discretization necessarily leads to an approximation C to C^0. For known kernels, Theorem 1 may thus be used for studying discretization effects inherently associated with standard projection-type approximations. This helps to identify the crucial factors and suggests some modifications to C. Theorem 1 may then be used again to prove that strong L^2-consistency can be achieved with suitably modified, or regularized, projection-type approximation to K. Further, the same type of additional regularization step may also be used, even if only in a less formal way, in HEP-like problems, where K is not known exactly and a projection-type approximation becomes a starting point.

Theorem 1: *For a sequence of discretized models (1) with $\theta^0 \in \Theta_n$, let $\hat{\theta}_t$ be a QML estimator of θ^0, as defined in (2) and $g^0 = [g_1^0, \ldots, g_m^0]^T$ be the true vector of intensities for the given binning in a measurable space $(E_1, \mathscr{B}_1)$. Denote $g^* = C\theta^0$ and assume that:*

A1. *E_0 is a compact metric space, μ is a finite measure on the Borel σ-algebra $\mathscr{B}$ in E_0 and $f \in L^2(E_0, \mathscr{B}_0, \mu)$.*
A2. *$m \geq n$ and $\log \gamma(t)^{-1} = O(m \log mt)$, as $t \to \infty$.*
A3. *$\max_{1 \leq j \leq n} \operatorname{diam}(A_j) \to 0$, as $t \to \infty$.*
A4. *$\mu(A_j) \asymp n^{-1}$, $j = 1, \ldots, n$.*
A5. *$g_i^0 \asymp m^{-1}$ and $g_i \asymp m^{-1}$, $i = 1, \ldots, m$, for $g = [g_1, \ldots, g_m]^T = C\theta$, $\theta \in \Theta_n$.*
A6. *$\|g^* - g^0\| = o(\sqrt{m} n \lambda_{\min}(C^T C))$, $m = o(t)$, and $n^{-1} = O(t^\beta \lambda_{\min}(C^T C))$ for some $0 < \beta < 1$, as $t \to \infty$.*

Then, with probability one, $\|\hat{f}_t - f\|_{L^2} \to 0$ as $t \to \infty$.

Proof: see section 4.

In the special case of the ML estimator in the non-folding model we have $E_0 = E_1$ and $C = \operatorname{diag}(\mu(A_j))$, so that $\lambda_{\min}(C^T C) \asymp n^{-2}$ and $\|g^* - g^0\| = 0$. A6 then reduces to $n = m = O(t^\beta)$. Since β may be arbitrarily close to 1, this is essentially the same as the standard assumption $n = o(t)$, usually made for consistency of the histogram density estimator.

Let us also note that, in the non-folding model and with $m = t/\log t$, the second part of A2 becomes $\log \gamma(t)^{-1} = O(t)$. The quantity corresponding to $\log \gamma(t)^{-1}$ in the definition of η-ML estimator in Wong and Shen (1995) would be $t\eta_t$ which is required there to be $o(t)$. This shows that the concept of QML estimators is very similar to that of η-ML estimators.

Let us now discuss the meaning of the other assumptions. A1, A3 and A4 are standard in the present context. Especially, A1 and A3 ensure that the L^2-projections of $f \in L^2(E_0, \mathscr{B}_0, \mu)$ onto the spaces of piecewise constant functions converge to f in L^2 (cf Lemma 6 in section 4). The assumption $m \geq n$ ensures the identifiability of the discretized model, if C is a full-rank matrix, and A5 is essentially a postulate of a 'reasonable' binning in E_1 and a specific 'closeness' of D and D^0, as will be seen below.

It can easily be shown that, for the second part of A5 to hold true, it is sufficient, for example, that the following conditions are fulfilled along with A1 and A4

B1. There exist constants a and b such that $0 < a \leq f(x) \leq b$ for all $x \in E_0$.
B2. There exists a constant $\eta > 0$ such that $\sum_{i=1}^m d_{ij} \geq \eta$ for $j = 1, \ldots, n$.
B3. $\max_{1 \leq i \leq m} \sum_{j=1}^n d_{ij} / \min_{1 \leq i \leq m} \sum_{j=1}^n d_{ij} = O(1)$, as $t \to \infty$.

The parameter sets Θ_n may then be enclosed in $[a, b]^n$. The condition B2 means that, for every bin A_j, the fraction of lost data points has a positive lower bound. One may be surprised that the latter restriction is imposed on the approximating operator D only and that there is no corresponding assumption on the folding kernel K which can be sub-stochastic and can potentially harm the identifiability. In fact, a corresponding restriction on K is implicitly imposed. It propagates to K from D via the approximation part of

A6. The following remark sheds some light on the meaning of B3. If one replaces d_{ij} in B3 with d_{ij}^0, then such modified condition will be fulfilled because, due to A4 and B1, $\mathrm{P}(A_j) \asymp n^{-1}$, $j = 1, \ldots, n$. This can easily be shown using $g_i^0 \asymp m^{-1}$ and

$$\mathrm{P}(B_j) = \sum_{j=1}^{n} \mathrm{P}(B_i|A_j) \cdot \mathrm{P}(A_j) = \sum_{j=1}^{n} d_{ij}^0 \mathrm{P}(A_j).$$

Thus, B3 is a postulate of a specific 'closeness' of D and D_0.

Assumption A6 sets admissible rates for the numbers of bins n and m. These two rates need not to be identical. There are two main factors which influence the rates: the degree of ill-posedness of the folding operator, which is related to $\lambda_{\min}(C^T C)$, and the accuracy of the approximation, both of the true operator with its discrete approximation C and of the true f with its L^2-projections onto the finite dimensional spaces of piecewise constant functions. The first part of A6 is a feasibility condition. It will be seen below that it may not be fulfilled, even in moderately ill-posed problems, if no further restrictions on f are imposed and no further regularization is made. The second part of A6 sets discretrization rates in relation with t.

Up to now, the assumptions have been formulated in terms of the discrete approximating operator only. Nothing has been explicitly assumed neither about the true folding operator nor about the way the approximating operator C was constructed. The assumption A6 can be put into more explicit form and our results set in relation with existing ones, if we are more specific about the true operator and the nature of the discrete approximation. Even if we are mostly interested in problems with only partially known folding operator and allow for a rather arbitrary approximating operator, it is instructive to see how Theorem 1 works in applications to some standard problems. Our plan is to use a standard discrete approximation of the folding operator and to apply Theorem 1 to problems well studied elsewhere, like deconvolution, the Wicksell problem and PET. This will help us to identify the critical points of our approach and to propose some remedies for the problems encountered.

Assume that the folding operator $\mathscr{K}$ is a bounded operator from $L^2(E_0, \mathscr{B}_0, \mu)$ into $L^2(E_1, \mathscr{B}_1, \mu_1)$, transforming the intensity function f of N_t^0 into an intensity function h of N_t^1 with respect to μ_1. Assume also that μ and μ_1 are diffuse measures. In many applications $\mathscr{K}$ can be written as

$$(\mathscr{K}f)(y) = \int_{E_0} k(x, y) f(x)\, d\mu(x) \tag{3}$$

with some kernel $k(x, y)$. Further, assume that the elements d_{ij}^0 of the true D^0, as defined in the introduction, are approximated with

$$d_{ij} = \frac{\int_{B_i} \int_{A_j} k(x, y) \mu(dx) \mu_1(dy)}{\mu(A_j)} \tag{4}$$

This will be called later a projection-type approximation and corresponds to a typical HEP-application, where one usually obtains d_{ij}'s via a Monte Carlo simulation, by generating data points from flat distributions in the bins A_j and counting the folded outcomes in the bins B_i. The quality of this approximation depends, of course, on the variability of f within the bins A_j. For f constant within the bins, the analytical approximation error is zero and only the Monte Carlo errors matter.

Denote by $\mathscr{P}_n$ the L^2-projection operator from $L^2(E_0, \mathscr{B}_0, \mu)$ onto the subspace of functions constant in the bins A_j and by $\mathscr{P}_m$ the L^2-projection operator from $L^2(E_1, \mathscr{B}_1, \mu_1)$ onto the subspace of functions constant in the bins B_i. Further, write $h_i^0 = g_i^0/\mu_1(B_i)$, $h_i^* = g_i^*/\mu_1(B_i)$,

$$h^0(y) = \sum_{i=1}^{m} h_i^0 \mathbf{1}_{B_i}(y), \quad h^*(y) = \sum_{i=1}^{m} h_i^* \mathbf{1}_{B_i}(y)$$

and assume that $\max_i \mu_1(B_i) \asymp m^{-1}$. Since the norms of the projection operators $\mathscr{P}_m$ are uniformly bounded by 1, we have

$$\begin{aligned} \|g^* - g^0\| &\leq O(m^{-1/2}) \|h^*(\cdot) - h^0(\cdot)\|_{L^2} \\ &= O(m^{-1/2}) \|\mathscr{P}_m \mathscr{K} \mathscr{P}_n f - \mathscr{P}_m \mathscr{K} f\|_{L^2} \\ &\leq O(m^{-1/2}) \|\mathscr{P}_n f - f\|_{L^2} \end{aligned}$$

so that the first part of A6 can be replaced with

$$\|\mathscr{P}_n f - f\|_{L^2} = o(mn\lambda_{\min}(C^T C))$$

The left-hand side of this formula is the error of the best L^2-approximation of f with a function constant in bins A_j and depends on the smoothness of f and on the mesh size. For example, if $E_0 \subset \mathscr{R}^d$ and μ is the Lebesgue measure, then the mesh size is of the order of $n^{-1/d}$. If the first derivatives of f are squared-integrable, then the approximation error is also of the order of $n^{-1/d}$ (cf Theorems 6.1 and 12.7 in Schumaker (1993)) and the first part of A6 further simplifies to

$$n^{-1/d} = o(mn\lambda_{\min}(C^T C)).$$

The decay rate of the eigenvalues of $C^T C$ is a measure of ill-posedness of the approximate, discretized unfolding problem: the faster the rate, the more difficult the unfolding problem. It can be related to the decay rate of singular values of the exact operator $\mathscr{K}$ by means of the theory of approximate solutions of operator eigenvalue problems.

Let us start with the following observation: if $\mathscr{K}^*\mathscr{K}$ is a compact operator, then the square roots of its eigenvalues provide singular values of $\mathscr{K}$. Our idea is, as above, to approximate $\mathscr{K}$ with $\mathscr{K}_{mn} = \mathscr{P}_m \mathscr{K} \mathscr{P}_n$, then to find its matrix representation C_+ on finite-dimensional subspaces $F_{(n)} \subset L^2(E_0, \mathscr{B}_0, \mu)$ and $F_{(m)} \subset L^2(E_1, \mathscr{B}_1, \mu_1)$ spanned by functions constant in bins and to approximate the eigenvalues of $\mathscr{K}^*\mathscr{K}$ with those of $\mathscr{K}_{mn}^*\mathscr{K}_{mn}$. We will represent the latter operator by the matrix $C_+^T C_+$, which is closely related to $C^T C$. The theory of the so-called Bubnov-Galerkin method can then be used to assess the quality of the approximation.

In order that, with some bases in $F_{(n)}$ and $F_{(m)}$, $C_+^T C_+$ be the matrix representation of $\mathscr{K}_{mn}^* \mathscr{K}_{mn}$, if C_+ is the matrix representation of $\mathscr{K}_{mn}$, the bases in $F_{(n)}$ and $F_{(m)}$ must be orthonormal. We thus take $u_j(x) = \mu^{-1/2}(A_j)\mathbf{1}_{A_j}(x)$, $j = 1, \ldots, n$ and $v_i(y) = \mu_1^{-1/2}(B_i)\mathbf{1}_{B_i}(y)$, $i = 1, \ldots, m$ as orthonormal bases in $F_{(n)}$ and $F_{(m)}$. With those bases, $\mathscr{K}_{mn} : F_{(n)} \to F_{(m)}$ is represented by the matrix

$$C_+ = \mathrm{diag}(\mu_1^{-1/2}(B_i))C\,\mathrm{diag}(\mu^{-1/2}(A_j)).$$

Assuming for simplicity that all $\mu_1(B_i)$ are equal and all $\mu(A_j)$ are equal, we conclude that the matrix representation $C_+^T C_+$ of $\mathscr{K}_{mn}^* \mathscr{K}_{mn}$ is proportional to mnC^TC and write the first part of A6 as

$$n^{-1/d} = o(\lambda_{\min}(C_+^T C_+)).$$

Let us write $T = \mathscr{K}^*\mathscr{K}$ and $T_n^{BG} = \mathscr{P}_n T$. The idea of the Bubnov-Galerkin method is to replace the eigenvalue equation $Tf = \lambda f$, $f \in L^2(E_0, \mathscr{B}_0, \mu)$, with a sequence of approximate equations $T_n^{BG} f = \lambda^{(n)} f$, $f \in F_{(n)}$ and to approximate the eigenvalues λ of T with $\lambda^{(n)}$. With this approach, it can be shown (Krasnoselskij et al. (1972), Ch. 18.4) that $0 \le \lambda_k - \lambda_k^{(n)} \le \|T - T_n^{BG}\|$, where λ_k and $\lambda_k^{(n)}$ denote, respectively, the k-th eigenvalue of T and T_n^{BG} and that $\|T - T_n^{BG}\| \to 0$ as $n \to \infty$. This means that the decay rate of $\lambda_n^{(n)}$ obtained via the Bubnov-Galerkin method is never slower than that of λ_n. The two rates would be identical if $\|T - T_n^{BG}\| = \|(I - \mathscr{P}_n)T\|$ were $o(\lambda_n)$.

The approximate operator used in our projection-type approach is $T_n = \mathscr{K}_{mn}^* \mathscr{K}_{mn} = \mathscr{P}_n \mathscr{K}^* \mathscr{P}_m \mathscr{K} \mathscr{P}_n$. As an operator on $F_{(n)}$ it can be written as $\mathscr{P}_n \mathscr{K}^* \mathscr{P}_m \mathscr{K}$. For the norm of $T_n^{BG} - T_n$ we have

$$\|T_n^{BG} - T_n\| = \|\mathscr{P}_n \mathscr{K}^*(I - \mathscr{P}_m)\mathscr{K}\| \le \|\mathscr{K}^*\| \cdot \|(I - \mathscr{P}_m)\mathscr{K}\|$$

which tends to zero if, for example, E_1 is compact and metric, μ_1 is finite and $\max_i \mathrm{diam}(B_i) \to 0$ (then $\|\mathscr{P}_m g - g\|_{L^2} \to 0$ for any $g \in L^2(E_1, \mathscr{B}_1, \mu_1)$) and if $\mathscr{K}$ is compact (cf Krasnoselskij et al. (1972), Lemma 15.4). Both T_n^{BG} and T_n are compact and selfadjoint. According to a theorem of Weil (cf Riesz and Nagy (1953)), the difference between our approximate eigenvalues obtained from $T_n f = \lambda^{(n)} f$ and those obtained from $T_n^{BG} f = \lambda^{(n)} f$ does not exceed the norm of the difference $T_n^{BG} - T_n$ and thus tends to zero as m tends to infinity.

All this provides some insight into the behaviour of $\lambda_k^{(n)}$ obtained from $T_n f = \lambda^{(n)} f$. If the discretization rate in E_1 is sufficiently higher than that in E_0, then $\|T_n^{BG} - T_n\|$ can approach zero faster than λ_n, in which case our $\lambda_n^{(n)}$ decays not slower than the exact λ_n, as for the Bubnov-Galerkin method. Otherwise, the regularizing impact of the discretization in E_1 may allow for a slower decay rate of $\lambda_n^{(n)}$. It is, however, very difficult to precisely relate the decay rates of λ_n and $\lambda_n^{(n)}$. The best one can hope for the decay rate of $\|(I - \mathscr{P}_n)T\|$ is $O(n^{-1/d})$, because of the weak approximation power of piecewise constant functions. As it will be seen shortly, for PET and for the Wicksell problem this is exactly the decay rate of λ_n, which means that there is no simple way to conclude about possibly equal decay rates of λ_n and $\lambda_n^{(n)}$.

In the three specific examples which follow, we shall temporarily assume that the decay rates of λ_n and $\lambda_n^{(n)}$ are the same. The examples are described in detail in Johnstone and Silverman (1991), including singular value decompositions.

We begin with a deconvolution example. If $E_0 \subset \mathscr{R}$, $\mathscr{K}$ is a convolution operator and the Fourier coefficients of the density $k(\,)$ are of the order of $(1+|i|)^{-s}$ with $s \geq 0$, then the squared singular values of $\mathscr{K}$ decay as $(1+|i|)^{-2s}$, $i = 0, \pm 1, \pm 2, \ldots$ and the assumption A6 takes then the form

$$n^{2s} = o(n) \quad m = o(t) \quad mn^{2s} = O(t^{\beta})$$

which can only be fulfilled if $s < 1/2$. For those s, if we take $n \asymp t^{\alpha_1}$ and $m \asymp t^{\alpha_2}$ with $\alpha_1 \leq \alpha_2 < 1$, the assumption A6 will be fulfilled if $\alpha_1 < \min\{\alpha_2, (1-\alpha_2)/(2s)\}$. Observe that for $s < 1/2$ the maximal rate for n is obtained by taking $\alpha_2 = 1/(2s+1)$ and $\alpha_1 < \alpha_2$. The larger s, the smoother the convolution kernel and more difficult the deconvolution problem, which is reflected by a slower rate of n. For s close to zero, both α_1 and α_2 can be close to one. As s approaches $1/2$, the discretization rates can approach $t^{1/2}$. Note that, due to the limited approximation power of piecewise constant functions, the rate n^{-1} for the approximation error $\|\mathscr{P}_n f - f\|_{L^2}$ cannot be improved, even if further smoothness assumptions are imposed on f. This means that the only way to make A6 feasible with $s \geq 1/2$ is to modify the discrete operator C and decrease the decay rate of its singular values.

For the Wicksell problem, we again have $d = 1$ and the squared singular values of $\mathscr{K}$ decay as $i^{-1}, i = 1, 2, \ldots$. The first part of A6 takes the form $n^{-1} = o(n^{-1})$ which, again, cannot be fulfilled, unless we modify C.

For the PET problem, $d = 2$ and the singular values are indexed by a double index. In the non-increasing sequence of squared singular values, each of the values $1/k$ appears exactly k times, $k = 1, 2, \ldots$. This means that the squared singular values of $\mathscr{K}$ decay as $i^{-1/2}$, $i = 1, 2, \ldots$ and the first part of A6 takes the form $n^{-1/2} = o(n^{-1/2})$, which cannot be satisfied neither, without modifications in C.

In all the examples, the difficulty comes from the weak approximation properties of piecewise constant functions and is thus intrinsic, as long as histogram sieves are used.

A known phenomenon observed in simulation studies of discretized PET problems is a severe speckling of reconstructed images in the high iterations of the EM algorithm. As suggested in Mair et al. (1996), this could possibly be explained by approximation effects related to the discretization, especially because speckling is *not* observed, if data points are generated according to the *discrete* model. Our analysis seems to confirm this conjecture from a different point of view. Discretization effects can be critical for the strong L^2-consistency, as seen in the examples. The numerical stability problems with the EM algorithm and the problems with the strong L^2-consistency may thus possibly be two different sides of the same discretization problem.

One possible method of avoiding the speckling effect in the high iterations of the EM algorithm is its early stopping, before the speckling begins (Vardi and Lee (1993)). In effect, EM does not really produce a ML estimate, but rather its approximation. Those approximate ML estimators nicely fit into our QML setup which does not require that the likelihood function be strictly maximized.

Even if Theorem 1 only provides some sufficient conditions for the strong L^2-consistency, which do not have to be necessary, the three specific examples analysed do suggest, that some further regularization is necessary, even in moderately ill-posed problems, if histogram sieves and the standard, projection-type matrix approximation of the folding operator are used. The discretization process itself can be looked at as a regularization procedure and discretization rates as regularization parameters. Since this kind of regularization is apparently not sufficient, we have to add an additional regularization step.

3 Regularized projection-type approximation

The form of the assumption A6 suggests, that we can try to modify the discrete operator C so that the decay rate of the eigenvalues of C^TC remains under control and, at the same time, the distance between g^* and g^0 does not increase too much so that both A6 and A5 are fulfilled. We shall also be forced to limit the admissible function class for f to those functions which can well be approximated by the initial eigenfunctions of $\mathscr{K}^*\mathscr{K}$ because, in that case, modifications of the smallest eigenvalues of C^TC will not change g^* too much, thus making A5 and A6 feasible. Note that the same idea makes successful the SVD approaches, in which the function classes are defined in terms of expansions with respect to singular functions of the folding operator with coefficients decreasing 'sufficiently fast' (see eg Johnstone and Silverman (1991) or Koo and Chung (1998)). The number of problems for which a mathematically tractable SVD is known is, however, limited.

In our approach, the exact operator $\mathscr{K}$ and/or its SVD may not be known which makes a fundamental difference, when compared to the standard SVD setup. Consequently, the singular functions of $\mathscr{K}$ will not be used in the estimation procedure which is based on the histogram sieves and the QML criterion. (They will, however, naturally emerge in the discussion of the consistency problem.) In addition to being applicable to HEP-like problems with only approximately known kernel, our methodology also provides an alternative to SVD-based methods in cases with a known kernel but difficult to obtain or mathematically intractable SVD.

An advantage of modifying the operator when compared to the maximum penalized likelihood methodology is that the EM algorithm can be used with the modified operator in its efficient, factorized form. This is not the case for the maximum penalized likelihood approach which seriously deteriorates the numerical performance of the EM algorithm (cf Hudson et al. (1994)).

We shall modify the discrete approximating operator by changing the singular values in its SVD. Let $\{\lambda_i\}$ be the descending sequence of eigenvalues of C^TC and $W = [w_1 \vdots \cdots \vdots w_n]$ the corresponding matrix of column eigenvectors. Assume that $\lambda_n > 0$. (It should explicitly be mentioned here that in the standard formulation of the PET problem one usually has $n > m$ which implies that $\lambda_n = 0$ so that our approach does not apply to such formulated PET problems.) With $V = CW\,\mathrm{diag}(\lambda_i^{-1/2})$, we decompose C as

$$C = V\,\mathrm{diag}(\lambda_i^{1/2})\,W^T$$

and define the regularized matrix operator as

$$C_r = V\,\mathrm{diag}[(\lambda_i + \delta_i)^{1/2}]W^T \tag{5}$$

with a set of non-negative regularization parameters δ_i, $i = 1, \ldots, n$ selected such that the sequence $\{\lambda_i + \delta_i\}$ is descending as well. With $g^* = C_r\theta^0$, we have

$$\|g^* - g^0\| \le \|C\theta^0 - g^0\| + \|(C_r - C)\theta^0\|.$$

If $E_0 \subset \mathscr{R}^d$ and the original matrix C has been constructed as a projection-type approximation, the first term can be bounded above, as in section 2, by $O(m^{-1/2}n^{-1/d})$, if the first derivatives of f are squared-integrable. For the second term, since $(\lambda_i + \delta_i)^{1/2} - \lambda_i^{1/2} \le \delta_i\lambda_i^{-1/2}/2$, we have

$$\|(C_r - C)\theta^0\| = \|\mathrm{diag}((\lambda_i + \delta_i)^{1/2} - \lambda_i^{1/2})W^T\theta^0\| \le \left(\sum_{i=1}^{n} \frac{\delta_i^2}{4\lambda_i} a_i^2\right)^{1/2}$$

where $a_i = w_i^T\theta^0$ is the i-th coordinate of θ^0 in the basis of normalized eigenvectors. It is thus sufficient for the first part of A6 that

$$n^{-1/d} + \left(m\sum_{i=1}^{n} \frac{\delta_i^2}{4\lambda_i} a_i^2\right)^{1/2} = o(mn(\lambda_n + \delta_n)) \tag{6}$$

or, assuming slightly more, that

$$n^{-1/d} = o(mn(\lambda_n + \delta_n)) \tag{7}$$

and

$$\left(m\sum_{i=1}^{n} \frac{\delta_i^2}{4\lambda_i} a_i^2\right)^{1/2} = o(m^{-1}) \tag{8}$$

Note that, if (8) holds true, then the second term in (6) tends to zero at least as fast as the first one. Further, if (8) is fulfilled with $a_i = w_i^T\theta$ and for all $\theta \in \Theta_n$, then the second part of A5 is satisfied with C_r, if it was satisfied for C, ie if $0 < A \le mC\theta \le B$ as $m \to \infty$, for some constants A and B and for all $\theta \in \Theta_n$. (The inequalities between a vector and a number are understood to hold for all vector's components.) To see this, observe that, because of (8), $m\|C_r\theta - C\theta\| = o(1)$ so that $A/2 \le mC_r\theta \le 2B$, say, for large m.

In what follows, we shall assume that the parameter sets Θ_n satisfy

$$\Theta_n \subset \left\{\theta \in \mathscr{R}^n : \sum_{i=1}^{n} i^{2a} \frac{(w_i^T\theta)^2}{n} < M\right\} \tag{9}$$

with positive constants M and a.

For an analysis of the inequality condition in (9), it will be useful to treat θ^0 and w_i's as piecewise constant functions in L^2 (ie the k-th vector component becomes the value of the corresponding step function in the bin A_k). Since we are only interested in rates, assume for simplicity that $\mu(E_0) = 1$ and $\mu(A_j) = n^{-1},\ j = 1, \ldots, n$. Then θ^0 approximates f and $n^{1/2}w_i$ approximates the i-th eigenelement of $\mathscr{K}^*\mathscr{K}$. The factor $n^{1/2}$ is needed in order to properly normalize w_i's as functions in L^2.

Let $f = \sum_i c_i\phi_i$ be the formal expansion of f with respect to singular functions ϕ_i of $\mathscr{K}$. Then $c_i = \langle f, \phi_i \rangle_{L^2} \approx \langle \theta^0, n^{1/2}w_i \rangle_{L^2} = n^{-1/2}a_i$. The inequality in (9) is a discrete version of a similar condition assumed eg by Johnstone and Silverman (1991) and Koo and Chung (1998), namely

$$\sum_i i^{2a}c_i^2 < M' \tag{10}$$

with positive constants M' and a, which can amount to the imposition of smoothness and integrability conditions on f. Whether c_i can be replaced with $n^{-1/2}a_i$ in (10), thus leading to (9), is a difficult question. Using Theorem 18.6 from Krasnoselskij et al. (1972), one can show for the Bubnov-Galerkin method that $|c_i - n^{-1/2}a_i| = O(n^{-1/d})$, if both f and ϕ_i posess squared-integrable first derivatives (we omit the details here). This error estimate is, however, not sufficient to infer about the decay rates of $n^{-1/2}a_i$ from those of c_i. In effect, we have to impose a rather vague condition on f: we restrict the class of functions f to those functions, for which $\theta^0 \in \Theta_n$ for sufficiently large n.

For those functions, we can replace (8) with

$$m^3 n \max_{1 \le i \le n} \frac{\delta_i^2}{\lambda_i i^{2a}} = o(1) \tag{11}$$

In the remaining part of this section, we shall demonstrate the feasibility of our regularization approach. For a specific, natural definition of δ_i's, conditions for the strong L^2-consistency will be given in terms of the discretization rates, the decay rate of the singular values of the discretized operator, the degree of regularization and the size of the function class for f.

Recall from section 2 that $mn\lambda_i$ are equal to the squared singular values of $\mathscr{K}_{mn}$ and assume that they decay as i^{-b}. This transforms (11) to

$$m^2 n \max_{1 \le i \le n} \frac{\delta_i}{i^{a-b/2}} = o(1) \tag{12}$$

Let us select a positive parameter α and define

$$\delta_i = \begin{cases} 0 & \text{if } mn\lambda_i \ge n^{-(1/d-\alpha)} \\ m^{-1}n^{-(1+1/d-\alpha)} - \lambda_i & \text{otherwise} \end{cases} \tag{13}$$

In words, we keep untouched those eigenvalues of C^TC, for which $mn\lambda_i$ is greater or equal $n^{-(1/d-\alpha)}$ and assign the value $n^{-(1/d-\alpha)}$ to the remaining ones, so that (7) holds true.

For non-zero δ_i, we have $mn\delta_i = n^{\alpha-1/d} - mn\lambda_i$ and, as far as the rates are concerned, $mn\lambda_i$ can be replaced with i^{-b}. The condition (12) takes then the form

$$m \max_{n^{(1/d-\alpha)/b} \le i \le n} \frac{n^{\alpha-1/d} - i^{-b}}{i^{a-b/2}} = o(1) \tag{14}$$

The range for i in the last formula follows from the definition of δ_i's which are equal zero for $i < n^{(1/d-\alpha)/b}$ (strictly speaking, for $i = o(n^{(1/d-\alpha)/b})$). Further,

$$\max_{n^{(1/d-\alpha)/b} \le i \le n} \frac{n^{\alpha-1/d} - i^{-b}}{i^{a-b/2}} < \frac{1}{n^{(1/d-\alpha)(a/b+1/2)}},$$

and (14) can be replaced with

$$\frac{m}{n^{(1/d-\alpha)(a/b+1/2)}} = o(1).$$

This, and consequently the first part of A6, will be satisfied if, for example, $m \asymp n$, $a > b(d - 1/2)$ and $\alpha < 1/d - 2b/(2a+b)$.

The second part of A6 takes the form $mn^{1/d-\alpha} = O(t^{\beta})$, $\beta < 1$, which will be satisfied, for example, if $m \asymp n$ and $n \asymp t^{\beta/(1+1/d-\alpha)}$.

We have thus proved the following theorem. (Recall that $E_0 = A_1 \cup \cdots \cup A_n$, $E_1 = B_1 \cup \cdots \cup B_m$ and θ^0 is a vector of mean values of f in the bins A_i.)

Theorem 2: *Let $\mathscr{K} : L^2(E_0, \mathscr{B}_0, \mu) \to L^2(E_1, \mathscr{B}_1, \mu_1)$ be a bounded and compact linear operator of the form (3) with $E_0 \subset \mathscr{R}^d$, $E_1 \subset \mathscr{R}^k$, Borel σ-algebras $\mathscr{B}_0$, $\mathscr{B}_1$ and Lebesgue measures μ and μ_1. Further, let $\mathscr{K}_{mn}$ be the projection-type approximation to $\mathscr{K}$ based on (4) and let the squared singular values of $\mathscr{K}_{mn}$ decay as i^{-b}. Denote by $\hat{f}_t$ any QML estimator of f based on a regularized version of $\mathscr{K}_{mn}$, as defined by (5) and (13) with a positive α and with parameter sets satisfying (9) with some positive constants a and M. Assume that conditions A2, A3, A4 and A5 from Theorem 1 hold true and that:*

C1. *E_0 is compact.*
C2. *f and its first derivatives are squared-integrable.*
C3. $\max_i \mu_1(B_i) \asymp m^{-1}$.
C4. *$a > b(d - 1/2)$ and $\alpha < 1/d - 2b/(2a+b)$.*
C5. *$m \asymp n \asymp t^{\beta/(1+1/d-\alpha)}$, with some $0 < \beta < 1$.*

Then, with probability one, $\|\hat{f}_t - f\|_{L^2} \to 0$ as $t \to \infty$ for those f, for which $\theta^0 \in \Theta_n$ for sufficiently large n.

If $d = 1$, as for deconvolution or for the Wicksell problem, then C4 takes the form $a > b/2$, $0 < \alpha < (2a-b)/(2a+b)$ and C5 becomes $m \asymp n \asymp t^{\beta/(2-\alpha)}$. Observe that in this case $0 < \alpha < 1$ and, correspondingly, the maximal allowed discretization rate is between $t^{1/2}$ and t, depending on the degree of regularization. The larger the decay rate $b/2$ of the singular values of the discretized operator, the smaller the function class for f, because $a > b/2$. For

a close to $b/2$, which corresponds to nearly maximal admissible function class for f, the regularization parameter α will be close to zero and the maximal allowed discretization rate close to $t^{1/2}$. On the other extreme, the larger the value of a or the smaller the function class for f, the stronger regularization and faster discretization rates are allowed.

If $d = 2$, as for PET, then we have $a > 3b/2$, $0 < \alpha < (2a - 3b)/(4a + 2b)$ and $m \asymp n \asymp t^{\beta/(3/2-\alpha)}$ so that $0 < \alpha < 1/2$ and the maximal allowed discretization rate is between $t^{2/3}$ and t, depending on the degree of regularization.

For fixed values of a and b, the choice of α and β can be optimized with respect to convergence rates of the risk. Define the risk of the estimator $\hat{f}_t$ by

$$R(\hat{f}_t, f) = \mathrm{E}\|\hat{f}_t - f\|_{L^2}.$$

Theorem 3: *Under the conditions of Theorem 2 and for all f such that $\theta^0 \in \Theta_n$ for sufficiently large n,*

$$R(\hat{f}_t, f) = O(t^{-r})$$

as $t \to \infty$, with

$$r = \frac{1}{1 + 1/d - \alpha} \min\left\{\frac{\beta}{d}, \frac{1}{2}\left(1 + \frac{1}{d} - \alpha\right) - \frac{\beta}{2}\left(\frac{1}{d} - \alpha\right)\right\}.$$

For $d = 1$, r can be made arbitrarily close to $(2a + b)/(4a + 4b)$, when $m \asymp n \asymp t^{(2a+b)/(4a+4b)}$ and when α approaches $1 - 2b/(2a + b)$.

For $d > 1$, r can be made arbitrarily close to $(2a + b)/[d(2a + 3b)]$, when $m \asymp n \asymp t^s$ with s approaching $(2a + b)/(2a + 3b)$ and when α approaches $1/d - 2b/(2a + b)$.

Proof: see section 4.

It can be shown that allowing for different discretization rates in E_0 and E_1 does not improve the convergence rates over the case $m \asymp n$. We omit this elementary, although somewhat tedious, analysis here.

Strong inequalities in the upper bounds for α are needed in the proof to Theorem 1, in order that D^* be negligible in (23). If α is on the boundary, Theorem 1 does not guarantee the convergence of the estimator to f anymore, although one may expect that there is no immediate breakdown, but rather a gradual deterioration of the convergence, so that the limit values of r should, in fact, be attainable.

The rates obtained in Theorem 3 are qualitatively similar to those obtained by Johnstone and Silverman (1991). Take, for example, the Wicksell problem. If we assume $b = 1$, the true decay rate of the squared singular values of the exact operator, then we have $r = (a + 1/2)/(2a + 2)$, while Johnstone and Silverman have obtained $a/(2a + 2)$. One should, however, remember that the meaning of a in the last two expressions is not exactly the same. In our case, a defines the size of the parameter sets Θ_n, while in Johnstone and Silverman (1991) a is directly related to the function class for f. One may conjecture that, in order to have comparable function classes, we would have to take a larger value of a, although it is difficult to quantify (cf formulas (9) and (10) and the discussion thereafter).

The choice of α, a and M in practical applications is a nontrivial problem. Some guidance can be gained from simulation experiments and this is probably the best one can expect if the folding kernel is only approximately known. An analysis of problems with a known folding kernel can also provide some guidance but, if the kernel and the SVD are known, then this knowledge can probably better be used in SVD-based approaches, like those in Johnstone and Silverman (1991) and Koo and Chung (1998). A Monte Carlo evaluation of our regularization strategy combined with the EM algorithm is a subject of an ongoing project. Our experience with the Wicksell problem shows that the choice of α does not substantially affect the results and that even $\alpha = 0$ works fine in practical applications. The choice of a and M for Θ_n is, however, a critical issue.

It seems extremely difficult, if possible at all, to analyse the convergence rates in the setup with only approximately known folding kernel and the QML histogram sieve estimators in precise terms of smoothness classes for f. Difficult problems like under which assumptions on f and $\mathscr{K}$ (10) implies (9) are intrinsic to this setup.

4 Auxiliary results and proofs

In the sequel, we write $\mathscr{R}_+^m$ for $(0, \infty)^m$ and $\mathscr{P}(\lambda)$ for the Poisson distribution with parameter λ. For $x, y \in \mathscr{R}^m$, we write $\rho(x, y) := \max_{1 \le i \le m} |x_i - y_i|$ and denote, respectively, by $B_\rho(x, r)$ and $\bar{B}_\rho(x, r)$ the open and closed ball in $(\mathscr{R}^m, \rho)$, centered at x and with radius r. Notice the obvious fact: $B_\rho(x, R)$ can be covered with N balls $\bar{B}_\rho(\xi_i, r)$ where $N < (1 + R/r)^m < (2R/r)^m$.

Our first lemma is a specialized version of the fluctuation inequality for separable random functions from Pfaff (1982).

Lemma 1: *Assume that $(\xi_T)_{T \in \mathscr{R}^m}$ is a separable random function on $(\Omega, \mathscr{C}, \mathrm{P})$ satisfying*

$$\mathrm{E_P}|\xi_{T_1}(\omega) - \xi_{T_2}(\omega)|^{2m} \le L\rho(T_1, T_2)^{2m}$$

for every $T_1, T_2 \in \bar{B}_\rho(T_0, \varepsilon)$, where $L < \infty$ is a real constant. Then,

$$\mathrm{P}\left\{\omega \in \Omega : \sup_{T_1, T_2 \in \bar{B}_\rho(T_0, \varepsilon)} |\xi_{T_1}(\omega) - \xi_{T_2}(\omega)| > \delta\right\} \le L 2^{-4m} (\varepsilon/\delta)^{2m}.$$

Proof: Only minor modifications are needed in the proof given on pages 1000–1001 in Pfaff (1982). Observe that, in our specialized setup, the cardinality of the set U_{n+1} used in Pfaff (1982) can be given exactly as $(2^{n+1}/2^{n(\varepsilon)})^m$. Elementary analysis shows then that the constant K in Pfaff (1982) can be selected as $2^{3m/2}(1 - 2^{-1/4})^{2m}/(1 - 2^{-m/2})$ which can further be bounded above by 2^{-4m}. Since the random function $(\xi_T)_{T \in \mathscr{R}^m}$ is uniformly P-continuous, closed balls can be used instead of open ones.

The following lemma gives an exponential bound for large deviation probabilities for a normalized Poisson random variable, which is further used in establishing an upper bound for the absolute moments of that random variable. The inequality in Lemma 2 resembles Bernstein-type inequalities discussed, for example, by Bennett (1962). Note, however, that in order to use the Bernstein's approach, one needs an upper bound for the absolute moments of the random variables involved (cf formula (7) in Bennett (1962)). We reverse the approach by first directly proving the inequality in Lemma 2. An upper bound for the absolute moments then follows easily in Lemma 3.

Lemma 2: *Let $X \sim \mathscr{P}(t\lambda)$. Then, for any $\tau > \lambda$,*

$$\mathrm{P}\left(\left|\frac{X}{t} - \lambda\right| \geq \tau\right) \leq \exp\left[-\frac{\tau}{3}t\right].$$

Proof: Notice that $\mathrm{P}(|X/t - \lambda| \geq \tau) = \mathrm{P}(X \geq t\lambda + t\tau)$ and define $Y = X - t\lambda - t\tau$. Obviously, $\mathrm{E}(Y) < 0$ and $\mathrm{P}(Y > 0) > 0$. The infimum r of the moment generating function of Y can easily be found to be $r = \exp[-t\lambda(-\tau/\lambda + (1 + \tau/\lambda)\log(1 + \tau/\lambda))]$. Since $\tau/\lambda > 1$ and $-x + (1 + x) \cdot \log(1 + x) \geq x/3$ for $x > 1$, we get $r < \exp[-\tau t/3]$. A standard large deviation argument for Y (cf eg Billingsley (1979), Ch. 1.9) completes the proof.

Lemma 3: *Let $X \sim \mathscr{P}(t\lambda)$ with $\lambda < 1$. Then, for any positive integer k and for all $t > 3k$,*

$$\max_{1 \leq i \leq k} \mathrm{E}\left|\frac{X}{t} - \lambda\right|^i < 2.$$

Proof: Using Theorem 3.2.1 from Chung (1974) and our Lemma 2, we obtain

$$\mathrm{E}\left|\frac{X}{t} - \lambda\right|^i \leq 1 + \sum_{n=1}^{\infty} \mathrm{P}\left(\left|\frac{X}{t} - \lambda\right|^i \geq n\right) \leq 1 + \sum_{n=1}^{\infty} \exp\left[-\frac{n^{1/k}}{3}t\right]$$
$$< 1 + \int_0^{\infty} \exp\left[-\frac{t}{3}u^{1/k}\right] \mathrm{d}u = 1 + k!\left(\frac{3}{t}\right)^k \leq 2,$$

which completes the proof.

For the model (1), let us define

$$X_w^t(\bar{n}) := \log P_w^t(\bar{n}) - \mathrm{E}[\log P_w^t(\bar{n})], \quad w \in \mathscr{R}^m. \tag{15}$$

Our next lemma is a fluctuation inequality for $X_w^t(\bar{n})$. It is a modified version of Lemma 1 from Szkutnik (1996b).

Lemma 4: *Assume that there exist constants c_1 and c_2 such that, in model (1), $0 < c_1 \leq g_i \leq c_2 < 1$ for $i = 1, 2, \ldots, m$. Let $w_0 \in \mathscr{R}_+^m$ and $r > 0$. Then, for*

$t > 6m$ and for all $\delta > 0$,

$$\mathrm{P}\left\{\bar{n} : \sup_{w,u \in \bar{B}_\rho(w_0,r)} |X_w^t(\bar{n}) - X_u^t(\bar{n})| > \delta\right\} \leq \left(\frac{rtm}{2c_1\delta}\right)^{2m} \frac{1}{2^m}.$$

Proof: Let us define $J := \{(j_1, \ldots, j_m) : \sum_\ell j_\ell = 2m, j_\ell \geq 0\}$. Using the multinomial formula, the mutual independence of n_j and Lemma 3, we obtain

$$\begin{aligned}
\mathrm{E}|X_w^t(\bar{n}) - X_u^t(\bar{n})|^{2m} &= \mathrm{E}\left|\sum_{i=1}^m (n_i - tg_i)\log\frac{w_i}{u_i}\right|^{2m} \\
&\leq t^{2m} \sum_{(j_1,\ldots,j_m)\in J} c(j_1,\ldots,j_m) \prod_{i=1}^m \left|\log\frac{w_i}{u_i}\right|^{j_i} \mathrm{E}\left|\frac{n_i}{t} - g_i\right|^{j_i} \\
&\leq t^{2m} 2^m \left(\sum_{i=1}^m \left|\frac{w_i - u_i}{\xi_i}\right|\right)^{2m} \leq c_1^{-2m} t^{2m} 2^m m^{2m} \rho(w,u)^{2m},
\end{aligned}$$

where $c(j_1, \ldots j_m)$ are the coefficients from the multinomial formula and ξ_i are between w_i and u_i. Application of Lemma 1 completes the proof.

The next lemma will be used in the stochastic part of the consistency proof.

Lemma 5: *Let $\xi(t,\omega), t \geq 0$ be a real valued stochastic process with right-continuous trajectories. Assume that for every $\varepsilon > 0$ there exist constants $d > 0$ and $T > 0$ such that, for all $t > T$, $\mathrm{P}(|\xi(t,\omega)| > \varepsilon) < \exp(-dt^\alpha)$ for some $\alpha > 0$. Then, with P-probability one, $\lim_{t\to\infty} \xi(t,\omega) = 0$.*

Proof: Denote by Γ the set of rational valued series $\{t_k\}$ for which $t_k > k^{1/\alpha}$ and let $A := \{\omega : \lim_{t\to\infty} \xi(t,\omega) = 0\}$. Making use of the right-continuity assumption, we may write

$$A = \bigcap_{\{t_k\}\in\Gamma} \{\omega : \lim \xi(t_k,\omega) = 0\}.$$

For any $\{t_k\} \in \Gamma$, by means of the Borel-Cantelli lemma, $\mathrm{P}\{\omega : \lim \xi(t_k,\omega) = 0\} = 1$, because

$$\sum_{k=1}^\infty \mathrm{P}(|\xi(t_k,\omega)| > \varepsilon) < C + \sum_{k=k_0}^\infty \exp(-dt_k^\alpha) < C + \sum_{k=k_0}^\infty \exp(-dk) < \infty,$$

with k_0 such that $t_{k_0} > T$. Hence, $\mathrm{P}(A) = 1$ since A is an intersection of countably many measure-one sets.

Our last lemma will cover the nonstochastic part of the consistency proof.

Lemma 6: *Consider a sequence of partitions of a compact metric space E, ie for the m-th partition the sets $A_i^{(m)}$ are disjoint and $E = A_1^{(m)} \cup \cdots \cup A_{n_m}^{(m)}$. Further,* *consider a real-valued function* $f \in L^2(E, \mathscr{B}, \mu)$, *where* $\mathscr{B}$ *is the Borel* σ*-algebra and the measure* μ *is finite. For every* m, *define*

$$f^{(m)}(x) = \sum_{i=1}^{n_m} \frac{\int_{A_i^{(m)}} f \, d\mu}{\mu(A_i^{(m)})} \mathbf{1}_{A_i^{(m)}}(x)$$

and assume that $d_m := \max_{1 \le i \le n_m} \operatorname{diam}(A_i^{(m)}) \to 0$, *as* $m \to \infty$. *Then,* $f^{(m)} \to f$ *in* $L^2(E, \mathscr{B}, \mu)$ *as* $m \to \infty$.

Proof: Let us fix $\varepsilon > 0$ and assume, without loss of generality, that $\mu(E) = 1$ and, for all $A_i^{(m)}$, $\mu(A_i^{(m)}) > 0$. The family $\mathscr{C}$ of continuous, squared-integrable and $\mathscr{B}$-measurable functions is dense in $L^2(E, \mathscr{B}, \mu)$. Select $g \in \mathscr{C}$ such that $\|f - g\|_{L^2} \le \varepsilon/3$ and define

$$g^{(m)}(x) = \sum_{i=1}^{n_m} \frac{\int_{A_i^{(m)}} g \, d\mu}{\mu(A_i^{(m)})} \mathbf{1}_{A_i^{(m)}}(x).$$

Let $\mathscr{A}_m$ denote the finite σ-algebra generated by the m-th partition. Clearly, $f^{(m)}$ is a version of $E(f|\mathscr{A}_m)$ and $g^{(m)}$ is a version of $E(g|\mathscr{A}_m)$. In other words, $g^{(m)}$ is the L^2-projection of g onto the m-dimensional subspace of squared-integrable, $\mathscr{A}_m$-measurable functions. We have $\|f - f^{(m)}\|_{L^2} \le \|f - g\|_{L^2} + \|g - g^{(m)}\|_{L^2} + \|f^{(m)} - g^{(m)}\|_{L^2}$. Since $f^{(m)} - g^{(m)} = E(f - g|\mathscr{A}_m)$, we obtain (cf Barra (1971), p. 44) $\|f^{(m)} - g^{(m)}\|_{L^2} \le \|f - g\|_{L^2}$ and, consequently $\|f - f^{(m)}\|_{L^2} \le 2\varepsilon/3 + \|g - g^{(m)}\|_{L^2}$. Hence, it will be sufficient to show that, for any $g \in \mathscr{C}$, $\|g - g^{(m)}\|_{L^2} \le \varepsilon/3$ for large m.

Fix $x_0 \in E$. For every m, $x_0 \in A_{i_m}^{(m)}$, for some i_m. Denote by $B(x, r)$ the closed ball in E with centre x and radius r. By the mean value theorem

$$\begin{aligned}
\inf_{x \in B(x_0, d_m)} g(x)\mu(A_{i_m}^{(m)}) &\le \inf_{x \in A_{i_m}^{(m)}} g(x)\mu(A_{i_m}^{(m)}) \le \int_{A_{i_m}^{(m)}} g \, d\mu \\
&\le \sup_{x \in A_{i_m}^{(m)}} g(x)\mu(A_{i_m}^{(m)}) \\
&\le \sup_{x \in B(x_0, d_m)} g(x)\mu(A_{i_m}^{(m)}).
\end{aligned}$$

Hence, by continuity of g, $g^{(m)}(x_0) \to g(x_0)$ as $m \to \infty$. Since g is bounded, as continuous on compact E, so is $g^{(m)}$ and (Serfling (1980), Th. 1.3.7) dominated pointwise convergence $g^{(m)}(x) \to g(x)$ implies $g^{(m)} \to g$ in L^2. Hence, $\|g - g^{(m)}\|_{L^2} \le \varepsilon/3$ for large m which completes the proof.

A key tool in the proof of Theorem 1 will be the following Proposition 1.

Proposition 1: *For model (1), assume that there exist constants* c_1 *and* c_2 *such that* $0 < c_1 \le g_i \le c_2 < 1$ *for* $i = 1, \ldots, n$ *and let* $c_1 \le g^* \le c_2$. *Denote by* g^0

the true g and by $\hat{g}_t$ a QML estimator of g. Then, for all $h > 0$ and $t > 6m$

$$\mathrm{P}(t^{1/2}\|\hat{g}_t - g^*\| > h) \le F \exp\left[-\frac{h^2}{32c_2} + D^* t\right],$$

with

$$F = F(\gamma, c_1, c_2, m, t)$$
$$= (1 + 6\sqrt{c_2})2^m c_1^{-3m/2}[(2/\gamma)^{1/2} + (2m^2 c_1 t)^m] \exp[8m^2 c_2]$$

and

$$D^* = \frac{c_2}{c_1} \sum_{i=1}^{m} |g_i^* - g_i^0|.$$

Proof: The general idea of the proof is taken from Pfaff (1982). Let us fix $\tau = m^{1/2}(c_2 - c_1)$ and notice that for $h > t^{1/2}\tau$ we have $\mathrm{P}(t^{1/2}\|\hat{g}_t - g^*\| > h) = 0$. It is thus sufficient to consider $h \le t^{1/2}\tau$ only. For those h one can write

$$\mathrm{P}(t^{1/2}\|\hat{g}_t - g^*\| > h) \le \sum_{j=0}^{[t^{1/2}\tau - h]} \mathrm{P}(h + j \le t^{1/2}\|\hat{g}_t - g^*\| \le h + j + 1).$$

We shall first prove that for each t there exist positive C and c such that

$$P_0 := \mathrm{P}\left(\sup_{w \in G(\lambda, t)} P_w^t \ge \gamma P_{g^*}^t\right) \le C \exp[-c\lambda^2] \tag{16}$$

for $h \le \lambda \le t^{1/2}\tau$, where $G(\lambda, t) = \{\omega : \lambda \le t^{1/2}\|\omega - g^*\| \le \lambda + 1\}$. Let us take

$$\varepsilon = c_1^{3/2} t^{-1/2} \exp\left[-\frac{\lambda^2}{16mc_2} + \frac{D^*}{2m} t\right].$$

Since $G(\lambda, t) \subset \bar{B}_\rho(g^*, (\lambda + 1)t^{-1/2}) \subset \mathscr{R}^m$, the set $G(\lambda, t)$ can be covered with balls $\bar{B}_\rho(w_i, \varepsilon)$, $i = 1, \ldots, N$ with

$$N \le (\lambda + 1)^m t^{-m/2} 2^m \varepsilon^{-m}$$

Without loss of generality, we can assume that $B_i = \bar{B}_\rho(w_i, \varepsilon) \cap G(\lambda, t) \ne \emptyset$ and select $v^i \in B_i$ for $i = 1, \ldots, N$ such that

$$\sup_{w \in B_i} \mathrm{E}(\log P_w^t) < \mathrm{E}(\log P_{v^i}^t) + \frac{1}{2} \log 2 \tag{17}$$

The last requirement can be fulfilled because the supremum in (17) is finite.

Note further that the sets B_i cover $G(\lambda, t)$ and, in effect, with $X_w^t(\bar{n})$ defined by (15),

$$
\begin{aligned}
P_0 &\le \sum_{i=1}^{N} \mathrm{P}\left(\sup_{w\in B_i} P_w^t \ge \gamma P_{g^*}^t\right)\\
&\le \sum_{i=1}^{N}\left[\mathrm{P}\left(P_{v^i}^t \ge \frac{\gamma}{2}P_{g^*}^t\right) + \mathrm{P}\left(P_{v^i}^t < \frac{\gamma}{2}P_{g^*}^t \text{ and } \sup_{w\in B_i} P_w^t > 2P_{v^i}^t\right)\right]\\
&\le \sum_{i=1}^{N}\left[\mathrm{P}\left(P_{v^i}^t \ge \frac{\gamma}{2}P_{g^*}^t\right) + \mathrm{P}\left(\sup_{w\in B_i} X_w^t(\bar n) - X_{v^i}^t(\bar n) > \frac{1}{2}\log 2\right)\right],
\end{aligned}
$$

where (17) was used to obtain the last equality.

Further, we have

$$
P_0 \le \sum_{i=1}^{N}\left[\mathrm{P}\left(P_{v^i}^t \ge \frac{\gamma}{2}P_{g^*}^t\right) + \mathrm{P}\left(\sup_{w,v\in \bar B_\rho(w_i,\varepsilon)} |X_w^t(\bar n) - X_v^t(\bar n)| > \frac{1}{2}\log 2\right)\right] \tag{18}
$$

For any $g \in \mathscr{R}_+^m$, denote $g \odot g^* := (\sqrt{g_1 g_1^*}, \ldots, \sqrt{g_m g_m^*})$ and note that

$$
\begin{aligned}
(P_g^t P_{g^*}^t)^{1/2} &= \exp\left[-\frac{t}{2}\sum_{j=1}^{m}(\sqrt{g_j} - \sqrt{g_j^*})^2\right]\cdot P_{g\odot g^*}^t\\
&\le \exp\left[-\frac{t}{8c_2}\|g-g^*\|^2\right]\cdot P_{g\odot g^*}^t.
\end{aligned}
$$

Define $B = \{\bar n : P_{v^i}^t(\bar n) \ge (\gamma/2)P_{g^*}^t(\bar n)\}$. For the first probability in (18), we have

$$
\begin{aligned}
\mathrm{P}(B) &= \sum_{\bar n\in B}\frac{P_{g^0}^t}{P_{g^*}^t}(P_{g^*}^t)^{1/2}(P_{g^*}^t)^{1/2}\\
&\le \left(\frac{2}{\gamma}\right)^{1/2}\sum_{\bar n\in\mathcal{N}^m}(P_{v^i}^t P_{g^*}^t)^{1/2}\prod_{j=1}^{m}\left(\frac{g_j^0}{g_j^*}\right)^{n_j}\exp[-t(g_j^0 - g_j^*)]\\
&\le \left(\frac{2}{\gamma}\right)^{1/2}\exp\left[-\frac{t}{8c_2}\|v^i - g^*\|^2\right]\exp\left[-t\sum_{j=1}^{m}(g_j^0 - g_j^*)\right]\\
&\qquad\cdot \mathrm{E}_{P_{v^i\odot g^*}^t}\prod_{j=1}^{m}\left(\frac{g_j^0}{g_j^*}\right)^{n_j}\\
&\le \left(\frac{2}{\gamma}\right)^{1/2}\exp\left[-\frac{\lambda^2}{8c_2}\right]\exp\left[t\sum_{j=1}^{m}(g_j^* - g_j^0)\right]\\
&\qquad\cdot\prod_{j=1}^{m}\sum_{n=0}^{\infty}\left(\frac{g_j^0}{g_j^*}\right)^{n}\frac{(t\sqrt{v_j^i g_j^*})^n}{n!}\exp[-t\sqrt{v_j^i g_j^*}].
\end{aligned}
$$

After some further algebra, we finally obtain

$$\mathrm{P}(B) \leq \left(\frac{2}{\gamma}\right)^{1/2} \exp\left[tD^* - \frac{\lambda^2}{8c_2}\right].$$

For the second probability in (18) we apply Lemma 4 and get

$$\mathrm{P}\left(\sup_{w,v\in\bar{B}_\rho(w_i,\varepsilon)} |X_w^t(\bar{n}) - X_v^t(\bar{n})| > \frac{1}{2}\log 2\right) < \left(\frac{m\varepsilon t}{c_1}\right)^{2m} 2^m.$$

Combining everything,

$$\begin{aligned} P_0 &\leq (\lambda+1)^m t^{-m/2}\varepsilon^{-m}2^m\left[\left(\frac{2}{\gamma}\right)^{1/2}\exp\left(tD^* - \frac{\lambda^2}{8c_2}\right) + \left(\frac{m\varepsilon t}{c_1}\right)^{2m}2^m\right] \\ &= (\lambda+1)^m c_1^{-3m/2}2^m\left[\left(\frac{2}{\gamma}\right)^{1/2} + m^{2m}c_1^m t^m 2^m\right]\exp\left(\frac{tD^*}{2}\right)\exp\left(-\frac{\lambda^2}{16c_2}\right). \end{aligned}$$

Further, because

$$(\lambda+1)^m \exp\left(-\frac{\lambda^2}{16c_2}\right) \leq \exp(8m^2c_2)\exp\left(-\frac{\lambda^2}{32c_2}\right),$$

we obtain

$$P_0 \leq C(\gamma, c_1, c_2, m, t)\exp\left(\frac{tD^*}{2}\right)\exp\left(-\frac{\lambda^2}{32c_2}\right)$$

with

$$C(\gamma, c_1, c_2, m, t) = c_1^{-3m/2}2^m\left[\left(\frac{2}{\gamma}\right)^{1/2} + m^{2m}c_1^m t^m 2^m\right]\mathrm{e}^{8m^2c_2},$$

which proves (16).

Finally, we have

$$\begin{aligned} \mathrm{P}(t^{1/2}\|\hat{g}_t - g^*\| > h) &\leq C(\gamma, c_1, c_2, m, t)\exp\left[\frac{tD^*}{2}\right]\sum_{j=0}^{\infty}\exp\left[-\frac{(h+j)^2}{32c_2}\right] \\ &< C(\gamma, c_1, c_2, m, t)(1+6\sqrt{c_2})\exp\left[-\frac{h^2}{32c_2}\right]\exp\left[\frac{tD^*}{2}\right], \end{aligned}$$

because

$$\sum_{j=0}^{\infty} \exp\left[-\frac{(h+j)^2}{32c_2}\right] \le \exp\left[-\frac{h^2}{32c_2}\right]\left(1+\int_0^{\infty}\exp\left[-\frac{x^2}{32c_2}\right]dx\right)$$
$$= (1+2\sqrt{2\pi c_2})\exp\left[-\frac{h^2}{32c_2}\right].$$

This completes the proof of Proposition 1.

The following corollary will be useful in the proof of our main result.

Corollary: *For model (1), assume that A5 holds true and that* $c_2 < 1$. *Then, for any* $\varepsilon > 0$ *and for* $t > 6m$,

$$\mathrm{P}(\|\hat{\theta}_t - \theta^0\| > \varepsilon) \le F \exp\left[-\left(\frac{\varepsilon^2 \lambda_{\min}(C^T C)}{32c_2} - D^*\right)t\right]$$

with $c_1 \asymp m^{-1}$ *and* $c_2 \asymp m^{-1}$.

Proof: Observe first that, according to the assumption A5, $c_1 \le g_i \le c_2$ with $c_1 \asymp m^{-1}$ and $c_2 \asymp m^{-1}$ so that Proposition 1 applies. Since $\|C\hat{\theta}_t - C\theta^0\| \ge \lambda_{\min}^{1/2}(C^T C)\|\hat{\theta}_t - \theta^0\|$, we obtain

$$\mathrm{P}(t^{1/2}\lambda_{\min}^{1/2}(C^T C)\|\hat{\theta}_t - \theta^0\| > h) \le F\exp\left[-\frac{h^2}{32c_2} + D^* t\right].$$

Setting $\varepsilon = ht^{-1/2}\lambda_{\min}^{-1/2}(C^T C)$ proves the corollary.

Note that if conditions B1–B3 from section 2 are satisfied, one can take

$$c_1 = a \min_i \mu(A_i) \min_i \sum_{j=1}^{n} d_{ij}, \quad c_2 = b \max_i \mu(A_i) \max_i \sum_{j=1}^{n} d_{ij}$$

and have $c_1 \le g_i \le c_2$. If necessary, one can change a to a smaller, but still positive a' and b to a larger b' to also have the true g_i^0's between c_1 and c_2. Further, B3, A4 and A5 imply that $mc_2 = O(1)$ and B2, B3 and A4 lead to the conclusion that mc_1 is cut away from zero so that, indeed, $c_1 \asymp m^{-1}$ and $c_2 \asymp m^{-1}$.

We are now ready to prove our main result. Recall that, in Theorem 1, we make the partitions $\{A_j\}$ and $\{B_j\}$ finer as t increases.

Proof to Theorem 1: For a given partition $\{A_j\}$, define

$$f_t(x) = \sum_{i=1}^{n} \frac{\int_{A_i} f\, d\mu}{\mu(A_i)} \mathbf{1}_{A_i}(x). \tag{19}$$

Then, $\|\hat{f}_t - f\|_{L^2} \le \|\hat{f}_t - f_t\|_{L^2} + \|f_t - f\|_{L^2}$. The second, nonstochastic term tends to zero as $t \to \infty$ because of the assumptions A1, A2 and A4 and

Lemma 6. For the first one, we have

$$\|\hat{f}_t - f_t\|_{L^2}^2 \le \max_i \mu(A_i)\|\hat{\theta}_t - \theta^0\|^2.$$

Let us fix $\varepsilon > 0$ and denote, for brevity, $\mathrm{Max} = \max_i \mu(A_i)$. For fixed partitions $\{A_j\}$ and $\{B_i\}$ such that $c_2 < 1$, we obtain from the Corollary:

$$\begin{aligned}\mathrm{P}(\|\hat{f}_t - f_t\|_{L^2} > \varepsilon) &\le \mathrm{P}(\mathrm{Max}^{1/2}\|\hat{\theta}_t - \theta^0\| > \varepsilon)\\ &\le F\exp\left[-\left(\frac{\varepsilon^2\lambda_{\min}(C^TC)}{32c_2\,\mathrm{Max}} - D^*\right)t\right]\\ &\le \exp\left[-\left(\frac{\varepsilon^2\lambda_{\min}(C^TC)}{64c_2\,\mathrm{Max}} - D^*\right)t\right] \qquad (20)\end{aligned}$$

for all $t > t_0$, where

$$t_0 = \max\left\{6m, \frac{64c_2\,\mathrm{Max}}{\varepsilon^2\lambda_{\min}(C^TC)}\log F\right\}. \qquad (21)$$

According to Lemma 5, in order to prove Theorem 1, it is sufficient to show that for some positive d, s and T,

$$\mathrm{P}(\|\hat{f}_t - f_t\| > \varepsilon) < \exp(-dt^s) \qquad (22)$$

for all $t > T$. Since the partitions $\{A_j\}$ and $\{B_i\}$ change with t and we wish to use (20) for proving (22), we have to assure that $t_0 = o(t)$ as $t \to \infty$ with t_0 defined in (21).

Using $mc_2 = O(1)$ as $t \to \infty$, we see that, indeed, c_2 ultimately becomes smaller than 1 and obtain

$$\frac{64c_2\,\mathrm{Max}}{\varepsilon^2\lambda_{\min}(C^TC)} \le \frac{O(1)}{mn\lambda_{\min}(C^TC)}.$$

Further, some elementary algebra gives $\log F = O(m\log(mt))$.

In order to prove that the second expression in (21) is $o(t)$, it is thus sufficient to show that $n^{-1}\log(mt)\lambda_{\min}^{-1}(C^TC) = o(t)$. To this end, use A6 and write

$$\frac{1}{n\lambda_{\min}(C^TC)}\,\frac{\log mt}{t} = O(t^\beta)\,\frac{\log mt}{t} = O(1)\,\frac{\log mt}{t^{1-\beta}} = o(1).$$

Since $m = o(t)$ is assumed in A6, this completes the proof of $t_0 = o(t)$.

To prove (22), we shall show that

$$\frac{\varepsilon^2\lambda_{\min}(C^TC)}{64c_2\,\mathrm{Max}} - D^* > dt^{s-1} \qquad (23)$$

for some positive d and s and for large t. As above, we obtain, with a constant A

$$\frac{\varepsilon^2 \lambda_{\min}(C^T C)}{64 c_2 \,\mathrm{Max}} \geq Amn\lambda_{\min}(C^T C).$$

According to A6, $\lambda_{\min}(C^T C)$ approaches zero not faster than $n^{-1}t^{-\beta}$ which means that $mn\lambda_{\min}(C^T C)$ approaches zero not faster than $mt^{-\beta}$. Thus, it will suffice to show that $D^* = o(mn\lambda_{\min}(C^T C))$. Since $\sum_{i=1}^m |g_i^* - g_i^0| \leq m^{1/2}\|g^* - g^0\|$, we get the desired result from A6, taking into account that $c_2/c_1 = O(1)$. Application of Lemma 5 completes the proof of Theorem 1.

Proof to Theorem 3: With f_t defined by (19), we have

$$R(\hat{f}_t, f) \leq \|f_t - f\|_{L^2} + \mathrm{E}\|\hat{f}_t - f_t\|_{L^2}.$$

Under the assumptions of Theorem 2, we obtain $\|f_t - f\|_{L^2} = O(n^{-1/d})$. It follows from the proof to Theorem 1 that, for large t,

$$\mathrm{P}(\|\hat{f}_t - f_t\|_{L^2} > \varepsilon) \leq \exp[-A\varepsilon^2 mt^{1-\beta}]$$

with a positive constant A. Consequently,

$$\mathrm{E}\|\hat{f}_t - f_t\|_{L^2} = \int_0^\infty \mathrm{P}(\|\hat{f}_t - f_t\|_{L^2} > x)\,\mathrm{d}x \leq O(1)m^{-1/2}t^{-(1-\beta)/2}$$

and

$$R(\hat{f}_t, f) \leq O(1)(n^{-1/d} + m^{-1/2}t^{-(1-\beta)/2}).$$

An application of the assumption C5 proves the first part of the theorem.

Denote $x = 1/d - \alpha$. C4 implies $2b/(2a+b) < x < 1/d$. Then

$$r = \min\left\{\frac{\beta}{d(1+x)}, \frac{1}{2} - \frac{\beta x}{2(1+x)}\right\}.$$

For $d > 1$, $r = \beta/[d(1+x)]$ and we obtain the upper bound by setting $\beta = 1$ and $x = 2b/(2a+b)$ which, through C5, gives the discretization rate.

For $d = 1$, consider separately three cases for β. If $\beta \geq 2/3$, then $r = 1/2 - \beta x/[2(1+x)]$ and the upper bound $(a + 5b/6)/(2a+3b)$ is obtained with $\beta = 2/3$ and $x = 2b/(2a+b)$. If $\beta \leq 1/2$, then $r = \beta/(1+x)$ which has an upper bound $(a + b/2)/(2a+3b)$ attained at $\beta = 1/2$ and $x = 2b/(2a+b)$. If $1/2 < \beta < 2/3$, then the upper bound for r is $(a+b/2)/(2a+3b)$, attained at $\beta = (2a+3b)/(4a+4b)$ and $x = 2b/(2a+b)$, which is the best out of the three cases considered, because of C4. An application of C5 gives the discretization rate thus completing the proof.

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