# Chapter 3 Exercises

**2. Z root t.**

- If $Z \sim N(0,1)$, then $\sqrt{t}Z \sim N(0,t)$.
- $\{X_t = \sqrt{t}Z\}_{t\geq 0}$ is *not* a Brownian motion, because for $0 \leq t_0 < t_1 \leq t_2 < t_3$, the increments $X_{t_1} - X_{t_0}$ and $X_{t_3} - X_{t_2}$ are both proportional to $Z$, and *not* independent.

**3. Correlated Brownian motion.** If $X_t = \rho W_t + \sqrt{1-\rho^2}\tilde{W}_t$, then:

- for each $s \geq 0$ and $t > s$,

$$X_{t+s} - X_s = \rho(W_{t+s} - W_s) + \sqrt{1-\rho^2}(\tilde{W}_{t+s} - \tilde{W}_s)$$

  so $X_{t+s} - X_s \sim N(0,t)$;
- for each $n \geq 1$ and times $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, the increments

$$X_{t_i} - X_{t_{i-1}} = \rho\left(W_{t_i} - W_{t_{i-1}}\right) + \sqrt{1-\rho^2}\left(\tilde{W}_{t_i} - \tilde{W}_{t_{i-1}}\right)$$

  are independent;
- $X_0 = 0$;
- $X_t$ is continuous in $t \geq 0$.

So $\{X_t\}_{t\geq 0}$ is $\mathbb{P}$-Brownian motion. Note that $X_t$ has correlation $\rho$ with $W_t$.

**4. New Brownian motions.**

- $\{-W_t\}_{t\geq 0}$ trivially satisfies the above properties, so is Brownian motion.
- $\{X_t = cW_{t/c^2}\}_{t\geq 0}$ trivially has independent Gaussian increments, satisfies $X_0 = 0$, and has continuous paths; lastly, $\mathbb{E}[X_t] = 0$ and $\mathrm{var}(X_t) = c^2 t/c^2 = t$, so $\{X_t\}_{t\geq 0}$ is Brownian motion.
- $\{X_t = \sqrt{t}W_1\}_{t\geq 0}$ is not Brownian motion; see Exercise 2 above.
- If $X_t = W_{2t} - W_t$, $t \geq 0$, then

$$X_{2t} - X_t = W_{4t} - W_2 t - (W2t - W_t) = W_{4t} - 2W_{2t} + W_t$$

and

$$\operatorname{cov}(X_{2t} - X_t, X_t - X_0) = -t \neq 0,$$

so $\{X_t\}_{t\geq 0}$ is not Brownian motion.

**5. Moment generating function.**

- $$\mathbb{E}\big[e^{\theta X}\big] = \int_{-\infty}^{\infty} e^{\theta x} \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

  Now

  $$\begin{aligned}(x-\mu)^2 - 2\sigma^2\theta x &= x^2 - 2(\mu + \sigma^2\theta)x + \mu^2 \\ &= (x - \mu - \sigma^2\theta)^2 + \mu^2 - (\mu + \sigma^2\theta)^2 \\ &= (x - \mu - \sigma^2\theta)^2 - 2\mu\sigma^2\theta - \sigma^4\theta^2\end{aligned}$$

  so

  $$\mathbb{E}\big[e^{\theta X}\big] = e^{-\frac{1}{2\sigma^2}(-2\mu\sigma^2\theta - \sigma^4\theta^2)} = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}.$$

- If $m(\theta) = \mathbb{E}\big[e^{\theta X}\big]$ then $\mathbb{E}[X^n] = m^{(n)}(0)$. Now

  $$\ln m(\theta) = \mu\theta + \frac{1}{2}\sigma^2\theta^2$$

  so

  $$\begin{aligned}m'(\theta) &= m(\theta)(\mu + \sigma^2\theta), \\ m''(\theta) &= m'(\theta)(\mu + \sigma^2\theta) + \sigma^2 m(\theta), \\ m^{(3)}(\theta) &= m''(\theta)(\mu + \sigma^2\theta) + 2\sigma^2 m'(\theta), \\ m^{(4)}(\theta) &= m^{(3)}(\theta)(\mu + \sigma^2\theta) + 3\sigma^2 m''(\theta).\end{aligned}$$

  So

  $$\begin{aligned}m'(0) &= \mu, \\ m''(0) &= \mu^2 + \sigma^2, \\ m^{(3)}(0) &= (\mu^2 + \sigma^2)\mu + 2\sigma^2\mu = \mu^3 + 3\mu\sigma^2, \\ m^{(4)}(0) &= (\mu^3 + 3\mu\sigma^2)\mu + 3\sigma^2(\mu^2 + \sigma^2) \\ &= \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 \\ &= \mathbb{E}\big[X^4\big].\end{aligned}$$

**10. Negative values.** $\{W_t\}_{t\geq 0}$ is standard Brownian motion, and $S_t = \mu t + \sigma W_t$. Then $S_T \sim N(\mu T, \sigma^2 T)$, so

$$\mathbb{P}[S_T < 0] = \mathbb{P}\left[\frac{S_T - \mu T}{\sqrt{\sigma^2 T}} < \frac{-\mu T}{\sqrt{\sigma^2 T}}\right] = \Phi\left(\frac{-\mu T}{\sqrt{\sigma^2 T}}\right) > 0.$$

**15. New martingales.**

Suppose that $0 \leq s < t$.

(a) If $\sigma^2 > 0$,

$$\begin{aligned}\mathbb{E}\left[e^{\sigma W_t}\middle|\,\mathcal{F}_s\right] &= \mathbb{E}\left[e^{\sigma W_s} e^{\sigma(W_t - W_s)}\middle|\,\mathcal{F}_s\right]\\ &= e^{\sigma W_s}\mathbb{E}\left[e^{\sigma(W_t - W_s)}\middle|\,\mathcal{F}_s\right]\\ &= e^{\sigma W_s} e^{\frac{1}{2}\sigma^2(t-s)}\\ &> e^{\sigma W_s}\end{aligned}$$

so $\left\{e^{\sigma W_t}\right\}$ is *not* a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ martingale.

(b) If $c \neq 1$, then

$$\mathbb{E}\left[cW_{t/c^2}\middle|\,\mathcal{F}_s\right] = cW_{s\wedge(t/c^2)} \neq cW_{s/c^2}$$

so $\left\{X_t = cW_{t/c^2}\right\}$ is *not* a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ martingale.
But note that $\{X_t\}$ is a Brownian motion, and in particular is a $\left(\mathbb{P}, \{\mathcal{F}_t^X\}_{t\geq 0}\right)$ martingale.

(c)

$$X_t = tW_t - \int_0^t W_u du$$

so

$$\begin{aligned}\mathbb{E}[X_t|\mathcal{F}_s] &= tW_s - \int_0^s W_u du - \int_s^t W_s du\\ &= sW_s - \int_0^s W_u du\\ &= X_s\end{aligned}$$

so $\{X_t\}$ *is* a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ martingale.

Note: we used

$$\mathbb{E}\left[\int_s^t W_u du \middle| \mathcal{F}_s\right] = \int_s^t \mathbb{E}[W_u | \mathcal{F}_s] \, du$$

which is ensured by Fubini's Theorem, because

$$\int_s^t \mathbb{E}[|W_u| | \mathcal{F}_s] \, du < \infty.$$