Brownian Motion
Exercises

Exercise 9.1. Let Z be a standard normal random variable. For all $t \geq 0$, let $X_t = \sqrt{t}Z$. The stochastic process $X = \{X_t : t \geq 0\}$ has continuous paths and $\forall t \geq 0$, $X_t \sim N(0, t)$. Is X a Brownian motion? Justify. (ref. Baxter and Rennie, p. 49)

Exercise 9.2. Let W and $\widetilde{W}$ be two independent Brownian motion and ρ is a constant contained in the unit interval. For all $t \geq 0$, let $X_t = \rho W_t + \sqrt{1-\rho^2}\widetilde{W}_t$. The stochastic process $X = \{X_t : t \geq 0\}$ has continuous paths and $\forall t \geq 0$, $X_t \sim N(0, t)$. Is X a Brownian motion? Justify. (ref Baxter and Rennie, p. 49)

Exercise 9.3. Let W be a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, P)$. Let $X_t = \exp\left[\sigma W_t - \frac{\sigma^2}{2}t\right]$. Show that $X = \{X_t : t \geq 0\}$ is a martingale.

Exercise 9.4. Let W be a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, P)$. . Show that $\{W_t^2 - t : t \geq 0\}$ is a martingale.

Exercise 9.5. Let W be a Brownian motion. Show that

$$\operatorname{Cov}[W_t, W_s] = \min(s, t).$$

Exercise 9.6. Let W be a Brownian motion. Show that

$$\begin{array}{cc}
(i) & \text{For all } s > 0,\ \{W_{t+s} - W_s : t \geq 0\} \\
(ii) & \{-W_t : t \geq 0\} \\
(iii) & \left\{cW_{\frac{t}{c^2}} : t \geq 0\right\} \\
(iv) & \left\{V_0 = 0 \text{ and } V_t = tW_{\frac{1}{t}} \text{if } t > 0\ : t \geq 0\right\}
\end{array}$$

are Brownian motions.

Exercise 9.7. Let $\mathbf{B}$ be a four-dimensional Brownian motion with

$$\operatorname{Corr}[\mathbf{B}_t] = \begin{pmatrix} 1 & 0.5 & 0.8 & 0.1 \\ 0.5 & 1 & 0.3 & 0.4 \\ 0.8 & 0.3 & 1 & 0.1 \\ 0.1 & 0.4 & 0.1 & 1 \end{pmatrix}.$$

Find the matrix $\mathbf{A}$ such that $\mathbf{B} = \mathbf{AW}$ and $\mathbf{W}$ is a four-dimensional Brownian motion with independent components.

Solutions

1 Exercise 9.1

No since $0 \leq s \leq t < \infty$,

$$\begin{aligned}
\text{Var}\left[X_t - X_s\right] &= \text{Var}\left[\sqrt{t}Z - \sqrt{s}Z\right] \\
&= \left(\sqrt{t} - \sqrt{s}\right)^2 Var\left[Z\right] \\
&= t - 2\sqrt{t}\sqrt{s} + s \\
&\neq t - s.
\end{aligned}$$

2 Exercise 9.2

Yes. Suffices to verify that (i) the time increments are independent and (ii) for all $0 \leq s \leq t < \infty$, $X_t - X_s \sim N(0, t-s)$.
(ii)

$$X_t - X_s = \underbrace{\rho\underbrace{(W_t - W_s)}_{N(0,t-s)}}_{N(0,\rho^2(t-s))} + \underbrace{\sqrt{1-\rho^2}\underbrace{\left(\widetilde{W}_t - \widetilde{W}_s\right)}_{N(0,t-s)}}_{N(0,(1-\rho^2)(t-s))}$$

Since the two terms in the right hand side are two independent Gaussian random variables with an expectation of zero, their sum is also a zero-expectation Gaussian random variable. Finally

$$\text{Var}\left[X_t - X_s\right] = \rho^2\left(t-s\right) + \left(1-\rho^2\right)\left(t-s\right) = t - s$$

which completes the first part.

(i) Let $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 < \infty$. Since $X_{t_2} - X_{t_1}$ and $X_{t_4} - X_{t_3}$ have a Gaussian distribution,it suffices to show that the covariance is null:

$$\begin{aligned}
& \text{Cov}\left[X_{t_2} - X_{t_1}; X_{t_4} - X_{t_3}\right] \\
= \; & \text{Cov}\left[\rho\left(W_{t_2} - W_{t_1}\right) + \sqrt{1-\rho^2}\left(\widetilde{W}_{t_2} - \widetilde{W}_{t_1}\right); \rho\left(W_{t_4} - W_{t_3}\right) + \sqrt{1-\rho^2}\left(\widetilde{W}_{t_4} - \widetilde{W}_{t_3}\right)\right] \\
= \; & \rho^2\text{Cov}\left[W_{t_2} - W_{t_1}; W_{t_4} - W_{t_3}\right] + \rho\sqrt{1-\rho^2}\text{Cov}\left[W_{t_2} - W_{t_1}; \widetilde{W}_{t_4} - \widetilde{W}_{t_3}\right] \\
& + \rho\sqrt{1-\rho^2}\text{Cov}\left[\widetilde{W}_{t_2} - \widetilde{W}_{t_1}; W_{t_4} - W_{t_3}\right] + \left(1-\rho^2\right)\text{Cov}\left[\widetilde{W}_{t_2} - \widetilde{W}_{t_1}; \widetilde{W}_{t_4} - \widetilde{W}_{t_3}\right] \\
= \; & 0
\end{aligned}$$

since the Brownian increment independence implies that

$$\operatorname{Cov}\left[W_{t_2}-W_{t_1};W_{t_4}-W_{t_3}\right]=0$$

and

$$\operatorname{Cov}\left[\widetilde{W}_{t_2}-\widetilde{W}_{t_1};\widetilde{W}_{t_4}-\widetilde{W}_{t_3}\right]=0.$$

The independence between the two Brownian motions implies that

$$\operatorname{Cov}\left[W_{t_2}-W_{t_1};\widetilde{W}_{t_4}-\widetilde{W}_{t_3}\right]=0$$

and

$$\operatorname{Cov}\left[\widetilde{W}_{t_2}-\widetilde{W}_{t_1};W_{t_4}-W_{t_3}\right]=0.$$

3 Exercise 9.3

(i) Integrability

$$\begin{aligned}
\mathrm{E}\left[|X_t|\right] &= \mathrm{E}\left[\left|\exp\left[\sigma W_t-\frac{\sigma^2}{2}t\right]\right|\right]=\mathrm{E}\left[\exp\left[\sigma W_t-\frac{\sigma^2}{2}t\right]\right] \\
&= \int_{-\infty}^{\infty}\exp\left[\sigma w-\frac{\sigma^2}{2}t\right]\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}\exp\left[-\frac{w^2}{2t}\right]dw \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}\exp\left[-\frac{w^2-2t\sigma w+\sigma^2t^2}{2t}\right]dw \\
&= \int_{-\infty}^{\infty}\underbrace{\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}\exp\left[-\frac{(w-t\sigma)^2}{2t}\right]}_{N(t\sigma,t)\text{ density function}}dw=1<\infty
\end{aligned}$$

(ii) Since X_t is a continuous function of $\mathcal{F}_t-$measurable random variables, X_t is itself $\mathcal{F}_t-$measurable.

(iii) For all $0 \leq s \leq t \leq \infty$,

$$\begin{aligned}
\mathrm{E}\left[X_t \middle| \mathcal{F}_s\right] &= X_s \mathrm{E}\left[\frac{X_t}{X_s}\middle| \mathcal{F}_s\right] \text{ car } X_s > 0 \\
&= X_s \mathrm{E}\left[\frac{\exp\left[\sigma W_t - \frac{\sigma^2}{2} t\right]}{\exp\left[\sigma W_s - \frac{\sigma^2}{2} s\right]}\middle| \mathcal{F}_s\right] \\
&= X_s \mathrm{E}\left[\exp\left[\sigma\left(W_t - W_s\right) - \frac{\sigma^2}{2}\left(t-s\right)\right]\middle| \mathcal{F}_s\right] \\
&= X_s \mathrm{E}\left[\exp\left[\sigma\left(W_t - W_s\right) - \frac{\sigma^2}{2}\left(t-s\right)\right]\right] \text{ since } W_t - W_s \text{ is independent of } \mathcal{F}_s. \\
&= X_s \int_{-\infty}^{\infty} \exp\left[\sigma w - \frac{\sigma^2}{2}\left(t-s\right)\right] \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t-s}} \exp\left[-\frac{w^2}{2\left(t-s\right)}\right] dw \\
&= X_s \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t-s}} \exp\left[-\frac{w^2 - 2\left(t-s\right)\sigma w + \sigma^2\left(t-s\right)^2}{2\left(t-s\right)}\right] dw \\
&= X_s \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t-s}} \exp\left[-\frac{\left(w - \left(t-s\right)\sigma\right)^2}{2\left(t-s\right)}\right]}_{N\left(\left(t-s\right)\sigma, t-s\right) \text{ density function}} dw = X_s
\end{aligned}$$

4 Exercise 9.4

First, $W_t^2 - t$ is $\mathcal{F}_t-$measurable since it is a continuous function of W_t which is $\mathcal{F}_t-$measurable.

Second,

$$\mathrm{E}\left[\left|W_t^2 - t\right|\right] \leq \mathrm{E}\left[W_t^2\right] + t = t + t = 2t < \infty.$$

Third, $\forall 0 \leq s \leq t$,

$$\begin{aligned}
\mathrm{E}\left[W_t^2 - t \,|\mathcal{F}_s\right] &= \mathrm{E}\left[\left(W_t - W_s + W_s\right)^2 - t \,|\mathcal{F}_s\right] \\
&= \mathrm{E}\left[\left(W_t - W_s\right)^2 + 2W_s\left(W_t - W_s\right) + W_s^2 - t \,|\mathcal{F}_s\right] \\
&= \underbrace{\mathrm{E}\left[\left(W_t - W_s\right)^2 |\mathcal{F}_s\right]}_{\substack{=\mathrm{E}\left[\left(W_t - W_s\right)^2\right] \\ =t-s}} + 2W_s \underbrace{\mathrm{E}\left[W_t - W_s \,|\mathcal{F}_s\right]}_{\substack{=\mathrm{E}\left[W_t - W_s\right] \\ =0}} + W_s^2 - t \\
&= W_s^2 - s. \blacksquare
\end{aligned}$$

5 Exercise 9.5

Without lost of generality, let $0 < s < t$.

$$\begin{aligned}
\text{Cov}[W_t, W_s] &= \text{Cov}[W_t - W_s + W_s, W_s] \\
&= \text{Cov}[W_t - W_s, W_s] + \text{Cov}[W_s, W_s] \\
&= \text{Cov}[W_t - W_s, W_s - W_0] + \text{Var}[W_s] \\
&= 0 + s \text{ because the increments of } W \text{ are independent} \\
&= \min(s,t) \text{ since } s < t. \blacksquare
\end{aligned}$$

6 Exercise 9.6

Let

$$Z_t = W_{t+s} - W_s.$$

$(MB1)$ $Z_0 = W_s - W_s = 0.$
$(MB2)$ Since $Z_{t_k} - Z_{t_{k-1}} = (W_{t_k+s} - W_s) - \left(W_{t_{k-1}+s} - W_s\right) = W_{t_k+s} - W_{t_{k-1}+s}$ and for $\forall 0 \leq t_0 < t_1 < ... < t_k$, the random variables $W_{t_1+s} - W_{t_0+s}$, $W_{t_2+s} - W_{t_1+s}$, ..., $W_{t_k+s} - W_{t_{k-1}+s}$ are independent, then for $\forall 0 \leq t_0 < t_1 < ... < t_k$, the random variables $Z_{t_1} - Z_{t_0}$, $Z_{t_2} - Z_{t_1}$, ..., $Z_{t_k} - Z_{t_{k-1}}$ are independent.
$(MB3)$ $\forall u, t \geq 0$ such that $u < t$, $Z_t - Z_u = (W_{t+s} - W_s) - (W_{u+s} - W_s) = W_{t+s} - W_{u+s}$ has a zero-expectation Gaussian distribution 0 with variance $(t+s) - (u+s) = t - u$.
$(MB4)$ $\forall \omega \in \Omega$, the path $t \to Z_t(\omega) = W_{t+s}(\omega) - W_s(\omega)$ is continuous since $t \to W_t(\omega)$ is continuous.

Let

$$Y_t = -W_t.$$

$(MB1)$ $Y_0 = -W_0 = 0.$
$(MB2)$ Since $Y_{t_k} - Y_{t_{k-1}} = W_{t_{k-1}} - W_{t_k}$ and $\forall 0 \leq t_0 < t_1 < ... < t_k$, the random variables $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$, ..., $W_{t_k} - W_{t_{k-1}}$ are independent, then $\forall 0 \leq t_0 < t_1 < ... < t_k$, the random variables $W_{t_0} - W_{t_1}$, $W_{t_1} - W_{t_2}$, ..., $W_{t_{k-1}} - W_{t_k}$ are independent, which implies that $Y_{t_1} - Y_{t_0}$, $Y_{t_2} - Y_{t_1}$, ..., $Y_{t_k} - Y_{t_{k-1}}$ are independent.
$(MB3)$ $\forall s, t \geq 0$ such that $s < t$, $Y_t - Y_s = W_s - W_t$ has a Gaussian distribution with expectation 0 and variance $t - s$.
$(MB4)$ $\forall \omega \in \Omega$, the path $t \to Y_t(\omega) = -W_t(\omega)$ is continuous since $t \to W_t(\omega)$ is continuous.

Let

$$X_t = cW_{\frac{t}{c^2}}.$$

$(MB1)$ $\quad X_0 = cW_0 = 0.$

$(MB2)$ $\quad$ Since $X_{t_k} - X_{t_{k-1}} = cW_{\frac{t_k}{c^2}} - cW_{\frac{t_{k-1}}{c^2}} -$ and $\forall 0 \leq t_0 < t_1 < ... < t_k$, the random variables $W_{\frac{t_1}{c^2}} - W_{\frac{t_0}{c^2}}$, $W_{\frac{t_2}{c^2}} - W_{\frac{t_1}{c^2}}$, ..., $W_{\frac{t_k}{c^2}} - W_{\frac{t_{k-1}}{c^2}}$ are independent, then $\forall 0 \leq t_0 < t_1 < ... < t_k$, the random variables $cW_{\frac{t_1}{c^2}} - cW_{\frac{t_0}{c^2}}$, $cW_{\frac{t_2}{c^2}} - cW_{\frac{t_1}{c^2}}$, ..., $cW_{\frac{t_k}{c^2}} - cW_{\frac{t_{k-1}}{c^2}}$ are independent, which implies that $X_{t_1} - X_{t_0}$, $X_{t_2} - X_{t_1}$, ..., $X_{t_k} - X_{t_{k-1}}$ are independent.

$(MB3)$ $\quad$ Since cW is Gaussian if W is, $\forall s, t \geq 0$ such that $s < t$, $X_t - X_s = c\left(W_{\frac{t}{c^2}} - W_{\frac{s}{c^2}}\right)$ is Gaussian with $\mathrm{E}\left[X_t - X_s\right] = c\mathrm{E}\left[W_{\frac{t}{c^2}} - W_{\frac{s}{c^2}}\right] = 0$ and variance $\mathrm{Var}\left[X_t - X_s\right] = c^2\mathrm{Var}\left[W_{\frac{t}{c^2}} - W_{\frac{s}{c^2}}\right] = c^2\left(\frac{t}{c^2} - \frac{s}{c^2}\right) = t - s$.

$(MB4)$ $\quad \forall \omega \in \Omega$, the path $t \to X_t(\omega) = cW_{\frac{t}{c^2}}(\omega)$ is continuous since $t \to W_t(\omega)$ is.

Let

$$V_t = \begin{cases} 0 & \text{if } t = 0 \\ tW_{\frac{1}{t}} & \text{if } t > 0. \end{cases}$$

$(MB1)$ $\quad V_0 = 0$ from definition of V.

$(MB3)$ $\quad \forall s, t \geq 0$ such that $s < t$,

$$\begin{aligned} V_t - V_u &= tW_{\frac{1}{t}} - sW_{\frac{1}{s}} \\ &= -s\left(W_{\frac{1}{s}} - W_{\frac{1}{t}}\right) + (t-s)\, W_{\frac{1}{t}} \end{aligned}$$

is a linear combination of two independent Gaussian random variable. Therefore, $V_t - V_u$ is Gaussian with

$$\mathrm{E}\left[V_t - V_s\right] = \mathrm{E}\left[tW_{\frac{1}{t}} - sW_{\frac{1}{s}}\right] = t\mathrm{E}\left[W_{\frac{1}{t}}\right] - s\mathrm{E}\left[W_{\frac{1}{s}}\right] = 0$$

and

$$\begin{aligned} \mathrm{Var}\left[V_t - V_s\right] &= \mathrm{Var}\left[-s\left(W_{\frac{1}{s}} - W_{\frac{1}{t}}\right) + (t-s)\, W_{\frac{1}{t}}\right] \\ &= \mathrm{Var}\left[-s\left(W_{\frac{1}{s}} - W_{\frac{1}{t}}\right)\right] + \mathrm{Var}\left[(t-s)\, W_{\frac{1}{t}}\right] \\ &\quad \text{since } W_{\frac{1}{s}} - W_{\frac{1}{t}} \text{ is independent of } W_{\frac{1}{t}} \\ &= s^2\mathrm{Var}\left[W_{\frac{1}{s}} - W_{\frac{1}{t}}\right] + (t-s)^2\, \mathrm{Var}\left[W_{\frac{1}{t}}\right] \\ &= s^2\left(\frac{1}{s} - \frac{1}{t}\right) + (t-s)^2\, \frac{1}{t} \\ &= s - \frac{s^2}{t} + t - 2s + \frac{s^2}{t} \\ &= t - s. \end{aligned}$$

If $s=0$, then $V_t = tW_{\frac{1}{t}}$ is Gaussian with

$$\mathrm{E}\left[V_t\right] = \mathrm{E}\left[tW_{\frac{1}{t}}\right] = t\mathrm{E}\left[W_{\frac{1}{t}}\right] = 0$$

and

$$\mathrm{Var}\left[V_t\right] = \mathrm{Var}\left[tW_{\frac{1}{t}}\right] = t^2\mathrm{Var}\left[W_{\frac{1}{t}}\right] = t^2\frac{1}{t} = t.$$

$(MB2)$ It suffices to show that $\forall 0 \leq t_1 < t_2 \leq t_3 < t_4$, the covariance between $V_{t_2} - V_{t_1}$ and $V_{t_4} - V_{t_3}$ is zero since these two random variables have a Gaussian distribution.
If $t_1 > 0$, then because $0 < \frac{1}{t_4} < \frac{1}{t_3} \leq \frac{1}{t_2} < \frac{1}{t_1}$,

$$\begin{aligned}
\mathrm{Cov}\left[V_{t_2} - V_{t_1}; V_{t_4} - V_{t_3}\right] &= \mathrm{Cov}\left[t_2W_{\frac{1}{t_2}} - t_1W_{\frac{1}{t_1}}; t_4W_{\frac{1}{t_4}} - t_3W_{\frac{1}{t_3}}\right] \\
&= t_2t_4\mathrm{Cov}\left[W_{\frac{1}{t_2}}; W_{\frac{1}{t_4}}\right] - t_2t_3\mathrm{Cov}\left[W_{\frac{1}{t_2}}; W_{\frac{1}{t_3}}\right] \\
&\quad -t_1t_4\mathrm{Cov}\left[W_{\frac{1}{t_1}}; W_{\frac{1}{t_4}}\right] + t_1t_3\mathrm{Cov}\left[W_{\frac{1}{t_1}}; W_{\frac{1}{t_3}}\right] \\
&\quad \text{since } \mathrm{Cov}\left(W_t, W_s\right) = \min(s,t). \\
&= t_2t_4\frac{1}{t_4} - t_2t_3\frac{1}{t_3} - t_1t_4\frac{1}{t_4} + t_1t_3\frac{1}{t_3} \\
&= t_2 - t_2 - t_1 + t_1 \\
&= 0
\end{aligned}$$

If $t_1 = 0$, then

$$\begin{aligned}
\mathrm{Cov}\left[V_{t_2} - V_{t_1}; V_{t_4} - V_{t_3}\right] &= \mathrm{Cov}\left[t_2W_{\frac{1}{t_2}}; t_4W_{\frac{1}{t_4}} - t_3W_{\frac{1}{t_3}}\right] \\
&= t_2t_4\mathrm{Cov}\left[W_{\frac{1}{t_2}}; W_{\frac{1}{t_4}}\right] - t_2t_3\mathrm{Cov}\left[W_{\frac{1}{t_2}}; W_{\frac{1}{t_3}}\right] \\
&= t_2t_4\frac{1}{t_4} - t_2t_3\frac{1}{t_3} \\
&= t_2 - t_2 \\
&= 0
\end{aligned}$$

$(MB4)$ $\forall\omega \in \Omega$, the path $t \to V_t(\omega) = tW_{\frac{1}{t}}(\omega)$ is continuous for all $t > 0$ since the functions $t \to W_t(\omega)$ and $t \to t$ are continuous and so is their product. Since $\lim_{t\to 0} tW_{\frac{1}{t}}(\omega) = 0$ almost-surely, the path $t \to V_t(\omega) = tW_{\frac{1}{t}}(\omega)$ is continuous for all t.