Université Denis Diderot Paris 7

Probability and Processus

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Exercises sheet 4 : Martingales

Exercise 1 Let $X_i, i \geq 1$ be square-integrable, $\mathcal{F}_n := \sigma(X_1, ..., X_n)$, and assume that $S_n = \sum_{i=1}^n X_i$ defines a $(\mathcal{F}_n)$-martingale. Prove that $E[X_i X_j] = 0$ for any $i \neq j$.

Exercise 2 Let $X_i, i \geq 0$ be integrable random variables, and $\mathcal{F}_n := \sigma(X_0, ..., X_n)$. Assume that for $n \geq 1$,

$$E[X_{n+1} \mid \mathcal{F}_n] = aX_n + bX_{n-1},$$

where $a \in (0, 1)$ and $a + b = 1$. For what value(s) of α does $S_n := \alpha X_n + X_{n-1}$ defines a $(\mathcal{F}_n)$-martingale?

Exercise 3 Let $(X_n, n \geq 0)$ be a $(\mathcal{F}_n)$-supermartingale and $(H_n, n \geq 0)$ a $(\mathcal{F}_n)$-previsible, nonnegative and bounded process. What can be said about the process $H \bullet X$ defined by $(H \bullet X)_0 = 0$, and, for any $n \in \mathbb{N}^*$,

$$(H \bullet X)_n = \sum_{k=1}^{n} H_k (X_k - X_{k-1}).$$

How could one relax the boundedness assumption on H?

Exercise 4 Branching processes and martingales.
Let (Z_n) be a Galton-Watson process with branching law μ, such that $\sum_{i \geq 0} i\mu(i) =: m \in (0, \infty)$. We let $\mathcal{F}_n := \sigma(Z_0, ..., Z_n)$.

1. Show that $W_n := m^{-n} Z_n$ defines a $(\mathcal{F}_n)$-martingale
2. Let η denote the probability of extinction, $G(s) := \sum \mu(i)s^i$, and $V_n := \eta^{Z_n}$. Show that $G(\eta) = \eta$, then show that (V_n) is a $(\mathcal{F}_n)$-martingale.
3. In this question we consider a non-critical ($m \neq 1$) branching process with immigration. More precisely, if ν is the immigration law, such that $\sum i\nu(i) = \ell \in [0, \infty)$, we let $(\zeta_i, i \geq 1)$ be i.i.d with law ν and independent of $(\xi_{i,n}, i \geq 1, n \in \mathbb{N})$. The variable ζ_n represents the number of individuals immigrating in the population at generation n. In other words, for any $n \geq 1$, $Z_n = \zeta_n + \sum_{n=1}^{Z_{n-1}} \xi_{i,n-1}$. Show that $M_n := m^{-n} \left(Z_n - \ell \frac{1-m^n}{1-m} \right)$ defines a $(\mathcal{F}_n)$-martingale.

Exercise 5 De Mere's martingale
Consider a fair game of heads and tails (in which a winning bet of \$$k$ is rewarded \$$2k$). A player adopts the following strategy. He bets \$1 at the first hand. If this first bet is lost he then bets \$2 at the second hand. If he loses his n first bets, he bets \$$2^{n+1}$ at the $(n+1)$-th hand. Moreover, as soon as the player wins one bet, he stops playing (or equivalently, after he wins a bet, he bets \$0 on every subsequential hand).
Denote by Y_n the net profit of the player just after the n-th hand has been played.

1. Show that (Y_n) is a martingale in an adequate filtration.

2. Show that almost surely, the game ends in finite time. What is the expectation of the duration of the game? What is the net profit of the player at the moment when he stops playing?

3. *The glitch* : Compute the expectation of his maximal loss during the game. Comment.

4. What would change if the player decides to triple his bet after every hand that he has lost?
   What changes if the game is unfair? Why do you think casinos have (in effect) forbidden this strategy by limiting the maximum possible bet?

5. (*) For a fair game, consider the following strategy. Fix real numbers $x_1, ..., x_k$ and write them in a list. At the first hand bet $x_1 + x_k$. If the hand is won erase x_1, x_k from the list. Otherwise add the number $x_{k+1} = x_1 + x_k$ at the end of the list. Then repeat the mechanism with the new list. Show that almost surely the game stops and one wins $\sum_{i=1}^{k} x_i$ (hint : one may consider the martingale defined by the sum of the entries in the list at the n-th hand). Compute the expectation of the maximal loss before the game ends.

Exercise 6 Pólya's urns and martingales
At time 0, an urn contains one white ball and one black ball. At each positive integer time, a ball from the urn is selected uniformly at random, and is then replaced in the urn along with a ball of same colour. Note that, after step n, there are $n+2$ balls in the urn. Among them, there are, say, $W_n + 1$ white balls, where W_n is the number of white balls which have been picked so far.

1. Show $P(W_n = k) = (n+1)^{-1}$ for any $k \in \{0, ..., n\}$.

2. Show that $M_n = (W_n + 1)/(n+2)$ (the proportion of white balls after step n), defines a martingale in a well-choosen filtration.

3. Show that M_n converges almost surely and express its limit.

4. Show that for any $\theta \in (0,1)$,

$$N_n^\theta := \frac{(n+1)!}{W_n!(n-W_n)!}\theta^{W_n}(1-\theta)^{n-W_n}$$

defines a martingale in a well-choosen filtration.

5. (*) Show that, starting instead with $w \in \mathbb{N}^*$ white balls and $b \in \mathbb{N}^*$ black balls initially in the urn, the limiting proportion of white balls has density

$$f_{r,w}(x) := \frac{(r+w-1)!}{(r-1)!(w-1)!}x^{w-1}(1-x)^{b-1}\mathbf{1}_{[0,1]}(x).$$

Hint : Letting as above W_n be the number of white balls which have been picked after step n, you may start by showing that

$$\begin{aligned}\mathbb{P}(W_n = m) \quad = \quad & \binom{n}{m}\frac{w \times (w+1) \times \ldots \times (w+(m-1))}{(w+b) \times (w+b+1) \times \ldots \times (w+b+(m-1))} \\ & \times \frac{b \times (b+1) \times \ldots \times (b+(n-m-1))}{(w+b+m) \times \ldots \times (w+b+(n-1))}.\end{aligned}$$

6. What is the relationship between $f_{w,r}$, $f_{w+1,r}$ and $f_{w,r+1}$?

Exercise 7 Wright-Fisher model and martingales
The Wright-Fisher model is a model of an evolving haploid population of constant size N. We consider the neutral case (neither type has a selective advantage). In other words, each individual in generation $n+1$ chooses its direct ancestor uniformly at random among the N individuals of generation n.
Suppose there exist only 2 types of individuals, for instance if we focus our study to the 2 alleles of one gene present in the population. We shall refer to them as original allele, and mutant allele.
Let X_n denote the proportion in generation n of individuals carrying the mutant allele. Enfin, on pose $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$.

1. Find the conditional law of NX_{n+1} knowing X_n.

2. Show (X_n) is a $(\mathcal{F}_n)$-martingale.

3. Let $Y_n = (1-N^{-1})^{-n}X_n(1-X_n)$. Show that (Y_n) is a $(\mathcal{F}_n)$-martingale.

Exercise 8 Stopping time : one example.

Let $(X_n)_{n\geq 0}$ be i.i.d uniform on $[0,1]$ random variables defined on $(\Omega, \mathcal{F}, P)$. For $n \geq 0$, let $\mathcal{F}_n = \sigma(X_k; k \leq n)$, and consider the random variable $T = \inf\{n \geq 1 : X_n > X_0\}$. Show T is a stopping time with respect to the filtration $\mathcal{F}_n$.

Exercise 9 Properties of stopping times.
Let $(\Omega, (\mathcal{F}_n), \mathcal{F}, P)$ be a filtered space, and T, S be $(\mathcal{F}_n)$-stopping times. We denote by $\mathcal{F}_T$ (resp. $\mathcal{F}_S$) the σ-field of events occuring prior to T:

$$\mathcal{F}_T = \{A \subset \Omega : \ \forall n \in \mathbb{N} \ \ A \cap \{T \leq n\} \in \mathcal{F}_n\} \, .$$

Verify that $\mathcal{F}_T$ is indeed a σ-field, then show that

(i) $S \wedge T$, $S \vee T$ are stopping times. What about $S + T$? $T - 2$? $(S + T)/2$?

(ii) If T is constant ($T = p$ for some $p \in \mathbb{N}$), then $\mathcal{F}_T = \mathcal{F}_p$,

(iii) If $S \leq T$, $\mathcal{F}_S \subset \mathcal{F}_T$,

(iv) $\mathcal{F}_{S\wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$,

(v) $\{S < T\} \in \mathcal{F}_S \cap \mathcal{F}_T$, $\{S = T\} \in \mathcal{F}_S \cap \mathcal{F}_T$.

Exercise 10 Gambler's ruin
Let $X_1, ..., X_n$ i.i.d variables such that $P(X_1 = 1) = p, P(X_1 = -1) = 1 - p =: q$. The random walk (S_n) on $\{0, ..., N\}$ *absorbed at the boundaries* $\{0, N\}$ is defined as follows. For $i \geq 1$, as long as S_{i-1} belongs to $\{1, ..., N - 1\}$ its i-jump is X_i. Values 0 and N are called absorbing for the walk, meaning that $S_n = 0, (\text{resp.} S_n = N) \Rightarrow S_{n+k} = 0 (\text{resp.} S_{n+k} = N) \ \forall k \in \mathbb{N}$.
This celebrated example was first studied by de Moivre, who was modeling a gambler who would bet \$1 at every hand until either he is ruined or he reaches a total fortune of N. De Moivre's martingale is defined by $Y_n = (q/p)^{S_n}$.

1. Verify (Y_n) is indeed a $\sigma(X_1, ..., X_n)$-martingale.
2. For $k \in \{0, ..., N\}$, compute $P(\text{walk is absorbed at } 0 \mid S_0 = k)$.

Exercise 11
Let $(X_n, n \geq 0)$ (taking values in $\{0, 1, 2, ..., 8\}$) be such that for any $n \in \mathbb{N}$, the conditional law of X_n knowing X_{n+1} is $\text{Bin}(8, \frac{X_n}{8})$.
We define $\mathcal{F}_n = \sigma(X_1, ..., X_n)$.

1. Show that (X_n) is a $(\mathcal{F}_n)$-martingale.
2. Let $T = \inf\{n \geq 0 : X_n = 0 \text{ or } X_n = 8\}$ (with the convention $\inf \emptyset = +\infty$). Show that T is a $(\mathcal{F}_n)$-stopping time.

3. Show that

$$\mathbb{P}(T = n + 1 \mid T > n) \geq \mathbb{P}(X_{n+1} = 8 \mid X_n \neq 0, X_n \neq 8) \geq \frac{1}{8^8}.$$

   Explain how we can deduce that

$$T \leq \inf\{n \geq 1 : \xi_n = 1\},$$

   where $(\xi_n)_{n\geq1}$ are i.i.d, $\sim$ Ber(8^{-8}).

4. Deduce that, regardless of the initial condition, $(X_n)_{n\geq0}$ converges almost surely as $n \to \infty$ to a limiting X_∞.

5. Do we have

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X_\infty]?$$

6. For $a \in \{0, 1, ..., 8\}$, deduce the distribution of X_∞ under $\mathbb{P}(\cdot \mid X_0 = a)$.

7. Show that $(Y_n := (7/8)^{-n} X_n(8 - X_n))_{n\geq0}$ also is a $(\mathcal{F}_n)$-martingale, and that $(Y_n)_{n\geq0}$ also converges almost surely as $n \to \infty$ to a limiting Y_∞. Do we have

$$\lim_{n\to\infty} \mathbb{E}[Y_n] = \mathbb{E}[Y_\infty]?$$

Exercise 12
Assume $(\xi_i, i \geq 1)$ are i.i.d.r.r.v's and let

$$S_n = \xi_1 + ... + \xi_n, \qquad \mathcal{F}_n = \sigma(\xi_1, ..., \xi_n).$$

Assume in addition that $\delta \in \overline{\mathbb{R}}_+^*$,

$$\forall \theta \in (-\delta, \delta) \qquad \phi(\theta) = \mathbb{E}[\exp(\theta \xi_1)] < \infty.$$

Finally for $\theta \in (-\delta, \delta)$, we let $\psi(\theta) = \log(\phi(\theta))$.

I. Let $\theta \in (-\delta, \delta)$ and $X_n^\theta := \exp(\theta S_n - n\psi(\theta))$. Show that $(X_n^\theta, n \geq 0)$ is a $(\mathcal{F}_n)$-martingale.

II. In this section only we assume $\xi_1 \sim \mathcal{N}(0, 1)$. For $\theta \in \mathbb{R}$, compute $\phi(\theta), \psi(\theta)$.

For $\theta \neq 0$, show that almost surely $X_n^\theta \to 0$ as $n \to \infty$. Do we have $\mathbb{E}[X_n^\theta] \to 0$?

III. In this section only we assume

$$\xi_1 \in \{-1, 1\}, \mathbb{P}(\xi_1 = 1) = p \in [1/2, 1],$$

so $(S_n, n \geq 0)$ is a simple random walk (symmetric when $p = 1/2$ and asymmetric when $p > 1/2$). We let $T_1 := \inf\{n : S_n = 1\}$.

1. Compute $\phi(\theta)$ for $\theta \in \mathbb{R}$. Verify that $\psi(\theta) \geq 0$ for any $\theta > 0$.
2. Show that T_1 is a $(\mathcal{F}_n)$-stopping time, and recall why it is amost surely finite.
3. For $\theta > 0$ show that
$$\mathbb{E}[\phi(\theta)^{-T_1}] = \exp(-\theta).$$
4. Set $s = \phi(\theta)^{-1}$, then $x := \exp(-\theta)$ to obtain
$$s(1-p)x^2 - x + sp = 0.$$
Solve and deduce that if $p < 1$,
$$\mathbb{E}[s^{T_1}] = \frac{1 - \sqrt{1 - 4p(1-p)s^2}}{2(1-p)s}.$$
What is $\mathbb{E}[s^{T_1}]$ when $p = 1$?
5. Deduce from the preceding question that $\mathbb{E}[T_1] < \infty$ if and only if $p > 1/2$.

Exercise 13

Let $(X_i, i \geq 1)$ be i.i.d. with

$$\mathbb{P}(X_1 = 1) = p \in (0,1), \qquad \mathbb{P}(X_1 = -1) = 1 - p =: q.$$

Set for $n \geq 0$, $\mathcal{F}_n = \sigma(X_1, ..., X_n)$, and introduce $S_0 = 0$, $S_n := \sum_{i=1}^n X_i$, $n \geq 1$.
For k and ℓ positive integers, let $T_{k,\ell} := \inf\{n \geq 1 : S_n = k \text{ ou } S_n = -\ell\}$.

1. Show that $T_{k,\ell}$ is a $(\mathcal{F}_n)$-stopping time.
2. Show that $T_{k,\ell} < \infty$ a.s. Explain why we may assume without loss of generality that $p \leq 1/2$.
3. Let $\phi(\lambda) = \log(pe^{\lambda} + qe^{-\lambda})$, and fix $u > 0$. Show that
$$\phi(\lambda) = u \Leftrightarrow p(e^{\lambda})^2 - e^u e^{\lambda} + q = 0.$$
Deduce that the equation $\phi(\lambda) = u$ exactly has two solutions
$$\log\left(\frac{1}{2p}(e^u - \sqrt{\Delta})\right) =: \lambda_1^{(u)} \;<\; \lambda_2^{(u)} := \log\left(\frac{1}{2p}(e^u + \sqrt{\Delta})\right),$$
with $\Delta = e^{2u} - 4pq > 0$.
4. Let λ be such that $\phi(\lambda) > 0$. Show that $\left(Y_n^{(\lambda)} = \exp\left(\lambda S_n - n\phi(\lambda)\right)\right)_{n\geq 0}$ is a $(\mathcal{F}_n)$-martingale.

5. Deduce that for any $a \in \mathbb{R}$, and $\lambda_1^{(u)} < \lambda_2^{(u)}$ as above, $(Z_n^{(u,a)} := aY_n^{(\lambda_1^{(u)})} + (1-a)Y_n^{(\lambda_2^{(u)})})$ also is a $(\mathcal{F}_n)$-martingale.

6. Let us *choose* a such that

$$ae^{\lambda_1^{(u)}k} + (1-a)e^{\lambda_2^{(u)}k} = ae^{-\lambda_1^{(u)}\ell} + (1-a)e^{-\lambda_2^{(u)}\ell},$$

i.e.

$$a = \frac{e^{\lambda_2^{(u)}k} - e^{-\lambda_2^{(u)}\ell}}{e^{\lambda_2^{(u)}k} - e^{-\lambda_2^{(u)}\ell} + e^{-\lambda_1^{(u)}\ell} - e^{\lambda_1^{(u)}k}}.$$

What is $Z_{T_{k,\ell}}^{(u,a)}$? Conclude that

$$\mathbb{E}[\exp(-uT_{k,\ell})] = \frac{1}{ae^{\lambda_1^{(u)}k} + (1-a)e^{\lambda_2^{(u)}k}}.$$

What did we compute?

Exercise 14 Doob's inequalities For a process $(X_n)_{n\geq 0}$ we define

$$\overline{X}_n := \max_{0\leq k\leq n} X_k^+.$$

1. Assume $(X_n)_{n\geq 0}$ is a submartingale. Establish that

$$P(\overline{X}_n \geq \lambda) \leq \frac{1}{\lambda}\mathbb{E}[X_n \mathbf{1}_{\{\overline{X}_n \geq \lambda\}}] \leq \frac{1}{\lambda}\mathbb{E}[X_n^+].$$

Hint : For the first inequality above, introduce the bounded stopping time $N := \inf\{k : X_k \geq \lambda\} \wedge n$, and show that $\mathbb{E}[X_0] \leq \mathbb{E}[X_N] \leq \mathbb{E}[X_n]$.

2. Assume $(M_n)_{n\geq 0}$ is a square-integrable martingale. Deduce from the preceding question that

$$\mathbb{P}(\max_{0\leq k\leq n} |M_k| \geq \lambda) \leq \frac{1}{\lambda^2}\mathbb{E}(M_n^2).$$

3. Assume $p > 1$, and (X_n) is a submartingale such that for any n, $X_n \in \mathbb{L}^p$. Establish that

$$\mathbb{E}[\overline{X}_n^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[(X_n^+)^p].$$

Hint : You may first work with $\overline{X}_n \wedge M$, and use that

$$\mathbb{E}[(\overline{X}_n \wedge M)^p] = \int_0^\infty px^{p-1}\mathbb{P}(\overline{X}_n \wedge M \geq x)dx$$

along with question 1.

4. Assume $p > 1$, and (M_n) is a submartingale such that for any n, $M_n \in \mathbb{L}^p$. Establish that
$$\mathbb{E}\left[\left(\max_{0\le k\le n} |M_k|\right)^p\right] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}[|M_n|^p].$$

Exercise 15 Let Y be a nonnegative $(\mathcal{F}_n)$-supermartingale. Show that
$$P(\max_{0\le k\le n} Y_k \ge x) \le \frac{1}{x} E[Y_0].$$

Exercise 16
Let $(\Omega, \mathcal{F}, (\mathcal{F}_n), P)$ a filtered space, and assume the sequence if r.r.v.'s $(X_n)_{n\ge 0}$ is $(\mathcal{F}_n)$-adapted. Let $\mathcal{T}_b$ denote the set of bounded $(\mathcal{F}_n)$-stopping times. Show that for any $\lambda > 0$,
$$P\left(\sup_{n\ge 0} |X_n| > \lambda\right) \le \frac{1}{\lambda} \sup_{T\in\mathcal{T}_b} E(|X_T|).$$

Exercise 17
Let $(\Omega, \mathcal{F}, (\mathcal{F}_n), (X_n), P)$, with (X_n) adapted. Show (X_n) is a $\mathcal{F}_n$-martingale, resp. supermartingale, resp. submartingale *iff* $E(X_T)$ is constant on $\mathcal{T}_b$, resp. nonincreasing on $\mathcal{T}_b$, resp. nondecreasing on $\mathcal{T}_b$.

Exercise 18 Suppose in this exercise that $(X_n, n \ge 1)$ are independent and satisfy $\sum_{n\ge 1} n^{-2}\mathrm{Var}(X_n) < \infty$, and set $S_n = \sum_{i=1}^n (X_i - E[X_i])/i$. Use Doob-Kolmogorov's inequality (ex.14.2) to establish that a.s.,
$$\sum_{i=1}^n \frac{X_i - E[X_i]}{i} \underset{n\to\infty}{\longrightarrow} Y,$$
for some real valued Y. Deduce that $n^{-1}\sum_{i=1}^n (X_i - E[X_i])$ converges almost surely to 0 as $n \to \infty$.

Exercise 19 Let $p \ge 2$, $(\Omega, \mathcal{F}, (\mathcal{F}_n), P)$ be a filtered space and (M_n) a $(\mathcal{F}_n)$-martingale. Assume there exists $A \ge 0$ such that for any $n \in \mathbb{N}$, $E[M_n^p] \le A$.

1. Show that $(E[M_n^2])$ is nondecreasing, and deduce that, as $m \to \infty$,
$$\sup_{n\ge 0} \mathbb{E}[(M_{m+n} - M_m)^2] \to 0.$$

2. For $\varepsilon > 0$ set $\mathcal{E}_m(\varepsilon) := \{|M_{m+n} - M_m| \geq \varepsilon \text{ for an } n \geq 1\}$.

   Use Doob-Kolmogorov's inequality (ex.14.2) for the martingale $(M_{m+n} - M_m)_{n\geq 0}$, to establish that $P(\mathcal{E}_m(\varepsilon)) \to 0$ as $m \to \infty$.

3. Deduce that $\lim_{\varepsilon\to 0} P\left(\bigcap_{m\in\mathbb{N}} \mathcal{E}_m(\varepsilon)\right) = 0$, and then that

   $$P\Big(\forall\varepsilon > 0,\ \exists m \in \mathbb{N} : |M_{m+i} - M_{m+j}| < \varepsilon)\ \forall i, j \in \mathbb{N}\Big) = 1,$$

   that is, (M_n) converges a.s.

4. Conclude that (M_n) also converges in $\mathbb{L}^p$.

5. (*) Explain how, when $p > 1$, Doob's martingale convergence theorem allows us to reach the same conclusions.