Solutions to the Exercises in Stochastic Analysis

Lecturer: Xue-Mei Li

1 Problem Sheet 1

In these solution I avoid using conditional expectations. But do try to give alternative proofs once we learnt conditional expectations.

Exercise 1 For $x, y \in \mathbf{R}^d$ define $p_t(x,y) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|2}{2t}}$. Prove that $P_t(x, dy) = p_t(x,y)dy$ satisfies the Chapman-Kolmogorov equation: for Γ Borel subset of $\mathbf{R}^d$,

$$P_{t+s}(x,\Gamma) = \int_{\mathbf{R}^d} P_s(y,\Gamma)P_t(x,dy).$$

Solution: One one hand

$$P_{t+s}(x,\Gamma) = (2\pi(t+s))^{-\frac{d}{2}} \int_\Gamma e^{-\frac{|x-z|^2}{2(t+s)}} dz.$$

On the other hand,

$$\begin{aligned}&\int_{\mathbf{R}^d} P_s(y,\Gamma)P_t(x,dy)\\ &= (2\pi t)^{-\frac{d}{2}}(2\pi s)^{-\frac{d}{2}} \int_{\mathbf{R}^d}\int_\Gamma e^{-\frac{|y-z|^2}{2s}} e^{-\frac{|y-x|^2}{2t}} dz dy.\end{aligned}$$

We now complete the squares in y.

$$-\frac{|y-z|^2}{2s} - \frac{|y-x|^2}{2t} = -\frac{1}{2}\frac{t+s}{st}\left|y - \frac{tz+sx}{t+s}\right|^2 - \frac{1}{2}\frac{|x-z|^2}{t-s}.$$

next we change the variable $y - \frac{tz+sx}{t+s}$ to $\tilde{y}$, then

$$\begin{aligned}
&\int_{\mathbf{R}^d} P_s(y,\Gamma)P_t(x,dy) \\
&= (2\pi t)^{-\frac{d}{2}}(2\pi s)^{-\frac{d}{2}} \int_{\mathbf{R}^d}\int_\Gamma e^{-\frac{1}{2}\frac{t+s}{st}|\tilde{y}|^2} e^{-\frac{1}{2}\frac{|x-z|^2}{t-s}} dz d\tilde{y} \\
&= (2\pi t)^{-\frac{d}{2}}(2\pi s)^{-\frac{d}{2}} \int_{\mathbf{R}^d} e^{-\frac{1}{2}\frac{t+s}{st}|\tilde{y}|^2} d\tilde{y} \int_\Gamma e^{-\frac{1}{2}\frac{|x-z|^2}{t-s}} dz \\
&= (2\pi(t-s))^{-fd2} \int_\Gamma e^{-\frac{|x-z|^2}{2(t-s)}} dz.
\end{aligned}$$

□

Exercise 2 Let $(X_t, t \geq 0)$ be a Markov process with $X_0 = 0$ and transition function $p_t(x,y)dy$ where $p_t(x,y)$ is the heat kernel. Prove the following statements.

(1) For any $s < t$, $X_t - X_s \sim P_{t-s}(0, dy)$;

(2) Prove that (X_t) has independent increments.

(3) For every number $p > 0$ there exists a constant $c(p)$ such that

$$\mathbb{E}|X_t - X_s|^p = c(p)|t-s|^{\frac{p}{2}}.$$

(4) State Kolomogorov's continuity theorem and conclude that for almost surely all ω, $X_t(\omega)$ is locally Hölder continuous with exponent α for any number $\alpha < 1/2$.

(5) Prove that this is a Brownian motion on $\mathbf{R}^d$.

Solution: Let f be a bounded measurable function.

(1) Since $(x_s, x_t) \sim P_s(0,dx)P_{t-s}(x,dy)$,

$$\begin{aligned}
\mathbb{E}f(x_t - x_s) &= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(y-x)p_s(0,x)p_{t-s}(x,y)dxdy \\
&= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(z)p_s(0,x)p_{t-s}(0,z)dxdz \\
&= \int_{\mathbf{R}^d} f(z)p_{t-s}(0,z)dz \int_{\mathbf{R}^d} p_s(0,x)dx = \int_{\mathbf{R}^d} f(z)p_{t-s}(0,z)dz.
\end{aligned}$$

Hence $x_t - x_s \sim P_{t-s}(0,dz)$.

(2) Let us fix $0 = t_0 < t_1 < t_2 < \ldots < t_n$ and Borel sets $A_i \in \mathcal{B}(\mathbf{R})$, $i = 1, \ldots, n$. Let $f_i(x) = \mathbf{1}_{x \in A_i}$ where A_i are Borel measurable set. Then we obtain

$$\begin{aligned}
&\mathbb{P}(X_{t_1} \in A_1, \ldots, X_{t_n} - X_{t_{n-1}} \in A_n) \\
&= \int_{\mathbf{R}} \cdots \int_{\mathbf{R}} f_1(x_1) f_2(x_2 - x_2) \ldots f_n(x_n - x_{n-1}) \\
&\qquad \times p_{t_1}(0, x_1) p_{t_2 - t_1}(x_1, x_2) \ldots p_{t_n - t_{n-1}}(x_{n-1}, x_n)\, dx_n \ldots dx_1 \ ,
\end{aligned}$$

where in the last line we have used the identity (ii). Introducing new variables: $y_1 = x_1$, $y_2 = x_2 - x_1$, ..., $y_n = x_{n-1} - x_n$, we obtain

$$\begin{aligned}
\mathbb{P}(X_{t_1} \in A_1, \ldots, X_{t_n} - X_{t_{n-1}} \in A_n) &= \int_{\mathbf{R}} \cdots \int_{\mathbf{R}} f_1(y_1) f_2(y_2) \ldots f_n(y_n) \\
&\qquad \times p_{t_1}(0, y_1) p_{t_2 - t_1}(0, y_2) \ldots p_{t_n - t_{n-1}}(0, y_n)\, dy_n \ldots dy_1 \\
= \prod_{i=1}^{n} \int_{\mathbf{R}} f_i(y_i) p_{t_i - t_{i_1}}(0, y_i)\, dy_i \ .
\end{aligned}$$

This means that $\{X_{t_i - t_{i-1}}, i = 1, \ldots, n\}$ are independent random variables.

(3)

$$\begin{aligned}
\mathbb{E}|x_t - x_s|^p =& \frac{1}{(2\pi(t-s))^{\frac{d}{2}}} \int_{\mathbf{R}^d} |z|^p e^{-\frac{|z|^2}{2(t-s)}} dz \\
=& \frac{1}{(2\pi(t-s))^{\frac{d}{2}}} \int_{\mathbf{R}^d} (t-s)^{\frac{p}{2}} |y|^p e^{-\frac{|y|^2}{2}} |t-s|^{\frac{d}{2}} dy \\
=& (t-s)^{\frac{p}{2}} \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbf{R}^d} |y|^p e^{-\frac{|y|^2}{2}} dy.
\end{aligned}$$

The integral is finite. Let

$$c(p) = \int_{\mathbf{R}^d} |y|^p p_1(0, y) dy,$$

which is the pth moment of a $N(0, I_{d \times d})$ distributed variable.

(4) and (5) are straight forward application of Kolmogorov's theorem. □

Exercise 3 If (B_t) is a Brownian motion prove that (B_t) is a Markov process with transition function $p_t(x, y) dy$.

Solution: Let us denote for simplicity $f_i(x) = \mathbf{1}_{x \in A_i}$. Furthermore, we define the random variables $Y_i = X_{t_i} - X_{t_{i-1}}$, for $i = 1, \ldots, n$, where we have postulated $t_0 = 0$. From the properties of the Brownian motion we obtain that the random variables $\{Y_i\}$ are independent and moreover $Y_i \sim \mathcal{N}(0, t_i - t_{i-1})$. Thus, we have

$$
\begin{aligned}
&\mathbb{P}[X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n] = \mathbb{E} \prod_{i=1}^{n} f_i(X_{t_i}) \\
&\quad = \mathbb{E}[f_1(Y_1) f_2(Y_2 + Y_1) \ldots f_n(Y_n + \ldots + Y_1)] \\
&\quad = \int_{\mathbf{R}} \cdots \int_{\mathbf{R}} f_1(y_1) f_2(y_2 + y_1) \ldots f_n(y_n + \ldots + y_1) \\
&\qquad\qquad \times p_{t_1}(0, y_1) p_{t_2 - t_1}(0, y_2) \ldots p_{t_n - t_{n-1}}(0, y_n) \, dy_n \ldots dy_1 \; .
\end{aligned}
$$

Now we introduce new variables: $x_1 = y_1$, $x_2 = y_2 + y_1$, …, $x_n = y_n + \ldots + y_1$, and obtain that the last integral equals

$$
\begin{aligned}
&\int_{\mathbf{R}} \cdots \int_{\mathbf{R}} f_1(x_1) f_2(x_2) \ldots f_n(x_n) \\
&\qquad \times p_{t_1}(0, x_1) p_{t_2 - t_1}(0, x_2 - x_1) \ldots p_{t_n - t_{n-1}}(0, x_n - x_{n-1}) \, dx_n \ldots dx_1 \; .
\end{aligned}
$$

Noticing that $p_t(0, y - x) = p_t(x, y)$ and recalling the definition of the functions f_i, so the finite dimensional distribution agrees with that of the Markov process determined by the heat kernels. The two processes must agree.

□

Exercise 4 Let $(X_t, t \geq 0)$ be a continuous real-valued stochastic process with $X_0 = 0$ and let $p_t(x, y)$ be the heat kernel on $\mathbf{R}$. Prove that the following statements are equivalent:

(i) $(X_t, t \geq 0)$ is a one dimensional Brownian motion.

(ii) For any number $n \in \mathbf{N}$, any sets $A_i \in \mathcal{B}(\mathbf{R})$, $i = 1, \ldots, n$, and any $0 < t_1 < t_2 < \ldots < t_n$,

$$
\begin{aligned}
&\mathbb{P}[X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n] \\
&\quad = \int_{A_1} \cdots \int_{A_k} p_{t_1}(0, y_1) p_{t_2 - t_1}(y_1, y_2) \ldots p_{t_n - t_{n-1}}(y_{n-1}, y_n) \, dy_n \ldots dy_1 \; .
\end{aligned}
$$

Solution: This follows from the previous exercises. □

Exercise 5 A zero mean Gaussian process B_t^H is a fractional Brownian motion of Hurst parameter H, $H \in (0,1)$, if its covariance is

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) .$$

Then $\mathbb{E}|B_t^H - B_s^H|^p = C|t-s|^{pH}$. If $H = 1/2$ this is Brownian motion (Otherwise this process is not even a semi-martingale). Show that (B_t^H) has a continuous modification whose sample paths are Hölder continuous of order $\alpha < H$.

Solution: Since $\mathbb{E}|B_t^H - B_s^H|^p = C|t-s|^{pH} = C|t-s|^{1+(pH-1)}$, we can apply the Kolmogorov continuity criterion to obtain that B_t^H has a modification whose sample paths are Hölder continuous of order $\alpha < (pH-1)/p$. This means that for any $\alpha < H$ we can take p large enough to have $\alpha < (pH-1)/p$. This finishes the proof. $\square$

Exercise 6 Let (B_t) be a Brownian motion on $\mathbf{R}^d$. Let T be a positive number. For $t \in [0,T]$ set $Y_t = B_t - \frac{t}{T}B_T$. Compute the probability distribution of Y_t.

Solution:

$$\begin{aligned}
&\mathbb{E}f(B_t - \frac{t}{T}B_T) \\
&= \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f\left(x - \frac{t}{T}y\right)(2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}} (2\pi(T-t))^{-\frac{d}{2}} e^{-\frac{|y-x|^2}{2(T-t)}} dxdy.
\end{aligned}$$

We observe that

$$|x|^2 = \left|x - \frac{t}{T}y\right|^2 + \frac{2t}{T}\langle x, y\rangle - \frac{t^2}{T^2}|y|^2$$

and that

$$|x-y|^2 = \left|x - \frac{t}{T}y\right|^2 - 2\frac{(T-t)}{T}\langle x, y\rangle + \frac{2t(T-t)}{T^2}|y|^2 + \frac{(T-t)^2}{T^2}|y|^2.$$

Thus,

$$\frac{|x|^2}{t} + \frac{|y-x|^2}{(T-t)} = \frac{1}{t}\left|x - \frac{t}{T}y\right|^2 + \frac{1}{T-t}\left|x - \frac{t}{T}y\right|^2 + \frac{1}{T}|y|^2.$$

Finally

$$
\begin{aligned}
&\mathbb{E}f(B_t - \frac{t}{T}B_T) \\
&= (2\pi t)^{-\frac{d}{2}}(2\pi(T-t))^{-\frac{d}{2}} \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f\left(x - \frac{t}{T}y\right) e^{-\frac{1}{2t}\left|x-\frac{t}{T}y\right|^2} e^{-\frac{1}{2(T-t)}\left|x-\frac{t}{T}y\right|^2} e^{-\frac{|y|^2}{2T}} dxdy \\
&= (2\pi t)^{-\frac{d}{2}}(2\pi(T-t))^{-\frac{d}{2}} \int_{\mathbf{R}^d} f(z) e^{-\frac{1}{2t}|z|^2} e^{-\frac{1}{2(T-t)}|z|^2} dz \int_{\mathbf{R}^d} e^{-\frac{|y|^2}{2T}} dy \\
&= \int_{\mathbf{R}^d} f(z) p_t(0,z) p_{T-t}(z,0) dz \cdot (2\pi T)^{\frac{d}{2}}.
\end{aligned}
$$

Finally we see

$$B_t - \frac{t}{T}B_T \sim \frac{p_t(0,z)p_{T-t}(z,0)}{p_T(0,0)} dz.$$

$\square$

2 Brownian motion, conditional expectation, and uniform integrability

Exercise 7 Let (B_t) be a standard Brownian motion. Prove that

(a) (i) for any $t > 0$, $\mathbb{E}[B_t] = 0$ and $\mathbb{E}[B_t^2] = t$;
  (ii) for any $s, t \geq 0$, $\mathbb{E}[B_s B_t] = s \wedge t$, where $s \wedge t = \min(s, t)$.

(b) (scaling invariance) For any $a > 0$, $\frac{1}{\sqrt{a}} B_{at}$ is a Brownian motion;

(c) (Translation Invariance) For any $t_0 \geq 0$, $B_{t_0+t} - B_{t_0}$ is a standard Brownian motion;

(d) If $X_t = B_t - tB_1, 0 \leq t \leq 1$, then $\mathbb{E}(X_s X_t) = s(1-t)$ for $s \leq t$. Compute the probability distribution of X_t.

Hint: break X_t down as the sum of two independent Gaussian random variables, then compute its characteristic function).

Solution: (a) (i) Since the distribution of B_t is $\mathcal{N}(0, t)$, we have $\mathbb{E}[B_t] = 0$ and $\mathbb{E}[B_t^2] = t$.
(a) (ii) We fix any $t \geq s \geq 0$. Then we have

$$\mathbb{E}[B_s B_t] = \mathbb{E}[(B_t - B_s)B_s] + \mathbb{E}[B_s^2] \ .$$

Since the Brownian motion has independent increments, the random variables $B_t - B_s$ and B_s are independent and we have

$$\mathbb{E}[(B_t - B_s)B_s] = \mathbb{E}[B_t - B_s]\mathbb{E}[B_s] = 0 \ .$$

Furthermore, from (i) we know that $\mathbb{E}[B_s^2] = s$. Hence, we conclude

$$\mathbb{E}[B_s B_t] = s \ ,$$

which is the required identity.

(b) Let us denote $W_t = \frac{1}{\sqrt{a}} B_{at}$. Then W has continuous sample paths and independent increments, which follows from the same properties of B. Furthermore, for any $t > s \geq 0$, we have

$$W_t - W_s = \frac{1}{\sqrt{a}} \left(B_{at} - B_{as}\right) \sim \mathcal{N}(0, \frac{a(t-s)}{a}) = \mathcal{N}(0, t-s) \ ,$$

which finishes the proof.

(c) If we denote $W_t = B_{t_0+t} - B_{t_0}$, then W has continuous sample paths and independent increments, which follows from the same properties of B. Moreover, for any $t > s \geq 0$, we have

$$W_t - W_s = B_{t_0+t} - B_{t_0+s} \sim \mathcal{N}(0, (t_0 + t) - (t_0 + s)) = \mathcal{N}(0, t - s) ,$$

which finishes the proof.

(d) For $t = 0$ or $t = 1$ we have $X_t = 0$. For $1 > t \geq s > 0$ we have

$$\begin{aligned} &\mathbb{E}[X_t X_s] = \mathbb{E}[(B_t - tB_1)(B_s - sB_1)] \\ =&\mathbb{E}[B_t B_s] - s\mathbb{E}[B_t B_1] - t\mathbb{E}[B_1 B_s] + st\mathbb{E}[B_1 B_1] \\ =&s - st - st + st = s(1 - t) . \end{aligned}$$

Let us take $t \in (0, 1)$. Then we have $X_t = (1 - t)B_t - t(B_1 - B_t)$. Since the random variables B_t and $B_1 - B_t$ are independent, the distribution of X_t is $\mathcal{N}(0, (1 - t)^2 t + t^2(1 - t)) = \mathcal{N}(0, t(1 - t))$. □

Exercise 8 Let $X \in L^1(\Omega, \mathcal{F}, P)$. Prove that the family of random variable $\{\mathbb{E}\{X|\mathcal{G}\} : \mathcal{G} \subset \mathcal{F}\}$ is L^1 bounded, i.e. $\sup_{\mathcal{G}\subset\mathcal{F}} \mathbb{E}\left(|\mathbb{E}\{X|\mathcal{G}\}|\right) < \infty$.

Solution: Let us take any $\mathcal{G} \subset \mathcal{F}$. Then using the Jensen's inequality we have

$$\mathbb{E}\left(|\mathbb{E}\{X|\mathcal{G}\}|\right) \leq \mathbb{E}\left(\mathbb{E}\{|X||\mathcal{G}\}\right) = \mathbb{E}|X| < \infty ,$$

which proves the claim. □

Exercise 9 Let $X \in L^1(\Omega; \mathbf{R})$. Prove that the family of functions

$$\{\mathbb{E}\{X|\mathcal{G}\} : \ \mathcal{G} \text{ is a sub } \sigma\text{-algebra of } \mathcal{F}\}$$

is uniformly integrable.

Solution: We note that, if a set of measurable sets A_C satisfies $\lim_{C\to\infty} \mathbb{P}[A_C] = 0$, then $\lim_{C\to\infty} \mathbb{E}[\mathbf{1}_{A_C}|X|] = 0$ (since $X \in L^1$, dominated convergence).

Let us take any $\mathcal{G} \subset \mathcal{F}$ and consider the family of events

$$A(C, \mathcal{G}) = \{\omega : |\mathbb{E}[X|\mathcal{G}](\omega)| \geq C\} .$$

Applying the Markov's and Jensen's inequalities we obtain

$$\mathbb{P}(A(C, \mathcal{G})) \leq C^{-1}\mathbb{E}\left(|\mathbb{E}[X|\mathcal{G}]|\right) \leq C^{-1}\mathbb{E}[\mathbb{E}[|X||\mathcal{G}]] = C^{-1}\mathbb{E}|X| \to 0 ,$$

as $C \to \infty$ (since $\mathbb{E}|X| < \infty$).

For any $\varepsilon > 0$, there exist a $\delta > 0$ such that if $\mathbb{P}[A] < \delta$, then $\mathbb{E}(\mathbf{1}_A|X|) < \varepsilon$. For this δ, take $C > \frac{\mathbb{E}|X|}{\delta}$, then $\mathbb{P}[A(C,\mathcal{G})] < \delta$ for any $\mathcal{G} \subset \mathcal{F}$, which implies

$$\sup_{\mathcal{G}\subset\mathcal{F}} \mathbb{E}[\mathbf{1}_{A(C,\mathcal{G})}|X|] < \varepsilon.$$

Finally, we conclude

$$\sup_{\mathcal{G}\subset\mathcal{F}} \mathbb{E}[\mathbf{1}_{A(C,\mathcal{G})}|\mathbb{E}[X|\mathcal{G}]|] \leq \sup_{\mathcal{G}\subset\mathcal{F}} \mathbb{E}[\mathbf{1}_{A(C,\mathcal{G})}|X|] < \varepsilon \ ,$$

which proves the claim. □

Exercise 10 Let $(\mathcal{G}_t, t \geq 0)$, $(\mathcal{F}_t, t \geq 0)$ be filtrations with the property that $\mathcal{G}_t \subset \mathcal{F}_t$ for each $t \geq 0$. Suppose that (X_t) is adapted to $(\mathcal{G}_t)$. If (X_t) is an $(\mathcal{F}_t)$-martingale prove that (X_t) is an $(\mathcal{G}_t)$-martingale.

Solution: The fact that (X_t) is $(\mathcal{G}_t)$-adapted, follows from the inclusion $\mathcal{G}_t \subset \mathcal{F}_t$ for each $t \geq 0$. Furthermore, for any $t > s \geq 0$, using the tower property of the conditional expectation, we obtain

$$\mathbb{E}[X_t|\mathcal{G}_s] = \mathbb{E}[\mathbb{E}[X_t|\mathcal{F}_s]|\mathcal{G}_s] = \mathbb{E}[X_s|\mathcal{G}_s] = X_s \ .$$

This shows that (X_t) is an $(\mathcal{G}_t)$-martingale. □

Exercise 11 (Elementary processes) Let $0 = t_0 < \cdots < t_n < t_{n+1}$, H_i be bounded $\mathcal{F}_{t_i}$-measurable functions, and

$$H_t(\omega) = H_0(\omega)\mathbf{1}_{\{0\}}(t) + \sum_{i=1}^{n} H_i(\omega)\mathbf{1}_{(t_i,t_{i+1}]} \ . \tag{1}$$

Prove that $H : \mathbf{R}_+ \times \Omega \to \mathbf{R}$ is Borel measurable. Define the stochastic integral

$$I_t \equiv \int_0^t H_s dM_s \equiv \sum_{i=1}^{n} H_i(M_{t_{i+1}\wedge t} - M_{t_i\wedge t})$$

and prove that

$$\mathbb{E}\left[\int_0^t H_s dB_s\right] = 0 \ , \qquad \mathbb{E}\left[\int_0^t H_s dB_s\right]^2 = \mathbb{E}\left[\int_0^t (H_s)^2 ds\right] \ . \tag{2}$$

Solution: First, we will prove that the function (1) is Borel-measurable. To this end, we take any Borel set $A \in \mathcal{B}(\mathbf{R})$ and we have to show that the set $\{(t,\omega) : H(t,\omega) \in A\}$ is measurable in the product space $(\mathbf{R}_+, \mathcal{B}(\mathbf{R}_+)) \times (\Omega, \mathcal{F})$. We can rewrite this set in the following way:

$$\begin{aligned}\{(t,\omega) : H(t,\omega) \in A\} = &(\{0\} \times \{\omega : H_0(\omega) \in A\}) \\ &\cup_{i=1}^n ((t_i, t_{i+1}] \times \{\omega : H_i(\omega) \in A\}) \ .\end{aligned}$$

Since the sets $\{0\}$ and $(t_i, t_{i+1}]$, $i = 1, \dots, n$, belong to $\mathcal{B}(\mathbf{R}_+)$, and $\{\omega : H_i(\omega) \in A\} \in \mathcal{F}_{t_i} \subset \mathcal{F}$, the claim now follows from the fact that the product of two measurable sets is measurable in the product space.

Next, we will show the identities (2). Let us denote $I_t = \int_0^t H_s dB_s$. Then we have

$$\mathbb{E}[I_t] = \sum_{i=1}^n \mathbb{E}\left[H_i(B_{t_{i+1}\wedge t} - B_{t_i \wedge t})\right] = \sum_{i=1}^n \mathbb{E}\left[H_i\right] \mathbb{E}\left[B_{t_{i+1}\wedge t} - B_{t_i \wedge t}\right] = 0 \ ,$$

where in the second equality we have used the independence of $B_{t_{i+1}\wedge t} - B_{t_i \wedge t}$ from $\mathcal{F}_{t_i}$, which follows from the properties of the Brownian motion and the fact that $B_{t_{i+1}\wedge t} - B_{t_i \wedge t} = 0$ if $t_i \geq t$. Furthermore, in the last equality we have used $\mathbb{E}\left[B_{t_{i+1}\wedge t} - B_{t_i \wedge t}\right] = 0$.

For the variance of the stochastic integral we have

$$\begin{aligned}\mathbb{E}[I_t^2] = &\sum_{i=1}^n \mathbb{E}\left[H_i^2(B_{t_{i+1}\wedge t} - B_{t_i \wedge t})^2\right] \\ &+ \sum_{i \neq j} \mathbb{E}\left[H_i H_j (B_{t_{i+1}\wedge t} - B_{t_i \wedge t})(B_{t_{j+1}\wedge t} - B_{t_j \wedge t})\right] \ .\end{aligned}$$

In the same way as before, using the independence of the increments of the Brownian motion, we obtain that the second sum is 0. Thus, we have

$$\begin{aligned}\mathbb{E}[I_t^2] &= \sum_{i=1}^n \mathbb{E}\left[H_i^2(B_{t_{i+1}\wedge t} - B_{t_i \wedge t})^2\right] \\ &= \sum_{i=1}^n \mathbb{E}\left[H_i^2\right] \mathbb{E}\left[(B_{t_{i+1}\wedge t} - B_{t_i \wedge t})^2\right] \\ &= \sum_{i=1}^n \mathbb{E}\left[H_i^2\right] (t_{i+1} \wedge t - t_i \wedge t) = \mathbb{E}\left[\int_0^t (H_s)^2 ds\right] \ ,\end{aligned}$$

where in the second line we have used the fact that H_i^2 and $(B_{t_{i+1}\wedge t} - B_{t_i \wedge t})^2$ are independent. $\square$

3 Martingales and Conditional Expectations

A given non-specific filtration $\{\mathcal{F}_t\}$ is used unless otherwise stated.

Exercise 12 Let μ be a probability measure. If $\{X_n\}$ is u.i. and $X_n \to X$ in measure, prove that X_n is L^1 bounded and $X \in L^1$.

Solution: By the u.i., there exists a number C such that $\int_{|X_n|\ge C} |X_n| d\mu \le 1$.

$$\int |X_n| d\mu \le \int_{|X_n|\ge C} |X_n| d\mu + \int_{|X_n|\le C} |X_n| d\mu \le 1 + C.$$

Take an almost surely convergence sub-sequence if necessary, we may and will assume that $X_n \to X$. Then

$$\mathbb{E}[|X|] = \mathbb{E}[\liminf_{n\to\infty} |X_n|] \le \liminf_{n\to\infty} \mathbb{E}|X_n| \le \sup_n \mathbb{E}|X_n|,$$

which is finite. □

Exercise 13 If X is an L^1 function, prove that $X_t := \mathbb{E}\{X|\mathcal{F}_t\}$ is an $\mathcal{F}_t$-martingale.

Solution: By the definition of conditional expectations, each $X_t \in L^1$. By the definition $X_t \in \mathcal{F}_t$ for each t. By the tower property if $s < t$, $\mathbb{E}(X_t|\mathcal{F}_s) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}(X|\mathcal{F}_s) = X_s$. □

Exercise 14 (Discrete Martingales) If (M_n) is a martingale, $n \in \mathbf{N}$, its quadratic variation is the unique process (discrete time Doob-Meyer decomposition theorem) $\langle M\rangle_n$ such that $\langle M\rangle_0 = 0$ and $M_n^2 - \langle M\rangle_n$ is a martingale. Let (X_i) be a family of i.i.d.'s with $\mathbb{E}X_i = 0$ and $\mathbb{E}(X_i^2) = 1$. Then $S_n = \sum_{i=1}^n X_i$ is a martingale w.r.t. its natural filtration. This is the random walk.

Prove that S_n is a martingale and that its quadratic variation is $\langle S\rangle_n = n$.

Solution: Let $\mathcal{F}_n$ denotes the natural filtration of $\{X_n\}$. Then

$$\mathbb{E}\{S_n|\mathcal{F}_{n-1}\} = \mathbb{E}\{S_{n-1}|\mathcal{F}_{n-1}\} + \mathbb{E}\{X_n|\mathcal{F}_{n-1}\} = S_{n-1} + 0.$$

We used the fact that X_n is independent of $\mathcal{F}_{n-1}$. Similarly,

$$\begin{aligned}
&\mathbb{E}\{(S_n)^2 - n|\mathcal{F}_{n-1}\} \\
&= \mathbb{E}\{(S_{n-1})^2|\mathcal{F}_{n-1}\} + 2\mathbb{E}\{S_{n-1}X_n|\mathcal{F}_{n-1}\} + \mathbb{E}\{(X_n)^2|\mathcal{F}_{n-1}\} - n \\
&= (S_{n-1})^2 + 2S_{n-1}\mathbb{E}\{X_n|\mathcal{F}_{n-1}\} + \mathbb{E}[(X_n)^2] - n \\
&= (S_{n-1})^2 - (n-1).
\end{aligned}$$

So $(S_n)^2 - n$ is an $\mathcal{F}_n$ martingale. □

Exercise 15 Let $\phi : \mathbf{R}^d \to \mathbf{R}$ be a convex function. Show that

(a) If (X_t) is a sub-martingale and ϕ is increasing s.t. $\phi(X_t) \in L^1$, then $(\phi(X_t))$ is a sub-martingale.

(b) If (X_t) is a martingale and $\phi(X_t) \in L^1$, then $\phi(X_t)$ is a sub-martingale.

(c) If (X_t) is an L^p integrable martingale, for a number $p \geq 1$, prove that $(|X_t|^p)$ is a sub-martingale.

(d) If (X_t) is a real valued martingale, prove that $(X_t \vee 0)$ is a sub-martingale.

Solution: Since ϕ is convex, it is continuous and hence it is measurable. This means that in what follows the process $\phi(X_t)$ is adapted.

(a) For $t > s$, using the Jensen's inequality we obtain

$$\mathbb{E}[\phi(X_t)|\mathcal{F}_s] \geq \phi(\mathbb{E}[X_t|\mathcal{F}_s]) \geq \phi(X_s) ,$$

where in the last inequality we have used $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ and the fact that ϕ is increasing.

(b) The claim can be shown in the same way as (a), but now

$$\phi(\mathbb{E}[X_t|\mathcal{F}_s]) = \phi(X_s) .$$

(c), (d) The claims follow from (b) and the fact that the maps $x \mapsto |x|^p$ and $x \mapsto x \vee 0$ are convex. $\square$

Exercise 16 (Limit of Martingales) Let $\{(M_n(t), t \in [0,1]), n \in \mathbf{N}\}$ be a family of martingales. Suppose that for each t, $\lim_{n\to\infty} M_n(t) = M_t$ almost surely and $\{M_n(t), n \in \mathbf{N}\}$ is uniformly integrable for each t. Prove that $M(t)$ is a martingale.

Solution: For any $t \in [0,1]$ and any $A \in \mathcal{F}$ fixed, the family $\{M_n(t)\mathbf{1}_A, n \in \mathbf{N}\}$ is uniformly integrable. Indeed,

$$\lim_{C\to\infty} \sup_n \int_{\{|M_n(t)|\mathbf{1}_A \geq C\}} |M_n(t)|\mathbf{1}_A \, d\mathbb{P} \leq \lim_{C\to\infty} \sup_n \int_{\{|M_n(t)| \geq C\}} |M_n(t)| \, d\mathbb{P} = 0 ,$$

where the inequality follows from $|M_n(t)|\mathbf{1}_A \leq |M_n(t)|$ and the last equality follows from the fact that $\{M_n(t), n \in \mathbf{N}\}$ are uniformly integrable.

Since, $\lim_{n\to\infty} M_n(t)\mathbf{1}_A = M_t\mathbf{1}_A$ almost surely, the uniform integrability implies convergence in L^1. Take $A = \Omega$, we see that M_t is integrable. For any $0 \leq s < t \leq 1$ and any $A \in \mathcal{F}_s$ we have

$$\mathbb{E}[M(t)\mathbf{1}_A] = \lim_{n\to\infty} \mathbb{E}[M_n(t)\mathbf{1}_A] = \lim_{n\to\infty} \mathbb{E}[M_n(s)\mathbf{1}_A] = \mathbb{E}[M(s)\mathbf{1}_A] ,$$

where the second equality holds, because M_n is a martingale. This implies that

$$\mathbb{E}[M(t)|\mathcal{F}_s] = M(s) \ ,$$

which finishes the proof. $\square$

Exercise 17 Let $a > 0$ be a real number. For $0 = t_1 < \cdots < t_n < t_{n+1} < \ldots$ and $i = 1, \ldots, n, \ldots$ let H_i be bounded $\mathcal{F}_{t_i}$ measurable functions and, let H_0 be a bounded $\mathcal{F}_0$-measurable function. Define

$$H_t(\omega) = H_0(\omega)\mathbf{1}_{\{0\}}(t) + \sum_{i=1}^{\infty} H_i(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(t).$$

Let $(M_t, t \leq a)$ a martingale with $M_0 = 0$, and (H_t) an elementary process. Define the elementary integral

$$I_t \equiv \int_0^t H_s dM_s \equiv \sum_{i=1}^{\infty} H_i(M_{t_{i+1}\wedge t} - M_{t_i \wedge t}).$$

Prove that $(I_t, 0 \leq t \leq a)$ is an $\mathcal{F}_t$- martingale.

Solution: Note: The sum is always a finite sum. Please refer also to Exercise 11.

It is clear that $M_{t_{i+1}\wedge t} - M_{t_i\wedge t} \in \mathcal{F}_t$. There exists an n s.t. $t < t_{n+1}$. Then

$$M_{t_{i+1}\wedge t} - M_{t_i\wedge t} = 0, \quad \forall i \geq n+1.$$

$$\int_0^t H_s dM_s \equiv \sum_{i=1}^{n} H_i(M_{t_{i+1}\wedge t} - M_{t_i\wedge t}),$$

and for $1 \leq i \leq n$, $H_i \in \mathcal{F}_{t_i} \subset \mathcal{F}_t$. In particular, $I_t \in \mathcal{F}_t$ for each t and the process (I_t) is $(\mathcal{F}_t)$-adapted.

The finite number of bounded random variables $\{|H_i|, i = 1, \ldots, n\}$ is bounded by a common constant C. Furthermore, since each s, M_s is integrable,

$$\mathbb{E}\left|\int_0^t H_s dM_s\right| \leq \sum_{i=1}^{n} \mathbb{E}\left(|H_i||M_{t_{i+1}\wedge t} - M_{t_i\wedge t}|]\right) \leq C\sum_{i=1}^{n}\left(\mathbb{E}|M_{t_{i+1}\wedge t}| + \mathbb{E}|M_{t_i\wedge t}|\right) < \infty,$$

proving that for each t, I_t is integrable.

Finally, we prove the martingale property. First note that if $(M_s, s \geq 0)$ is a martingale then

$$E\{M_t|\mathcal{F}_s\} = M_{s\wedge t}, \quad \forall s, t \geq 0 \tag{3}$$

Take $t \geq s \geq 0$ and assume that $s \in [t_k, t_{k+1})$, for some $k \in \{0, \dots, n\}$.

Explanation: We only need to consider two cases: (1) $i \leq k$ in which case $H_i \in \mathcal{F}_s$ in which case we can take H_i out of the conditional expectation and (2) $i \geq k+1$ in which case $s < t_i$ and we may use tower property, to condition in addition w.r.t $\mathcal{F}_{t_i}$ and take H_i out.

Then we have

$$\begin{aligned}\mathbb{E}[I_t|\mathcal{F}_s] &= \sum_{i=1}^{n} \mathbb{E}[H_i(M_{t_{i+1}\wedge t} - M_{t_i \wedge t})|\mathcal{F}_s] \\ &= \sum_{i=1}^{k} \mathbb{E}[H_i(M_{t_{i+1}\wedge t} - M_{t_i \wedge t})|\mathcal{F}_s] + \sum_{i=k+1}^{n} \mathbb{E}[H_i(M_{t_{i+1}\wedge t} - M_{t_i \wedge t})|\mathcal{F}_s] \ .\end{aligned}$$

For $i \leq k$, we have $t_i \leq s$, and do the random variable $H_i \in \mathcal{F}_s$ and

$$\begin{aligned}&\sum_{i=1}^{k} \mathbb{E}[H_i(M_{t_{i+1}} - M_{t_i})|\mathcal{F}_s] = \sum_{i=1}^{k} H_i \mathbb{E}[(M_{t_{i+1}} - M_{t_i})|\mathcal{F}_s] \\ &= \sum_{i=1}^{k} H_i(M_{t_{i+1}\wedge s} - M_{t_i \wedge s}) = \sum_{i=1}^{\infty} H_i(M_{t_{i+1}\wedge s} - M_{t_i \wedge s}) = I_s.\end{aligned}$$

If $i \geq k+1$ then $s \leq t_i$, we may use the tower property of the conditional expectation:

$$\begin{aligned}&\sum_{i=k+1}^{n} \mathbb{E}[H_i(M_{t_{i+1}\wedge t} - M_{t_i} \wedge t)|\mathcal{F}_s] = \sum_{i=k+1}^{n} \mathbb{E}\left[\mathbb{E}\left[H_i(M_{t_{i+1}\wedge t} - M_{t_i \wedge t})|\mathcal{F}_{t_i}\right]|\mathcal{F}_s\right] \\ &= \sum_{i=k+1}^{n} \mathbb{E}[H_i \mathbb{E}[(M_{t_{i+1}\wedge t} - M_{t_i \wedge t})|\mathcal{F}_{t_i}]|\mathcal{F}_s] \\ &= \sum_{i=k+1}^{n} \mathbb{E}\left[H_i \overbrace{\left(M_{t_{i+1}\wedge t \wedge t_i} - M_{t_i \wedge t \wedge t_i}\right)}^{0} \Big| \mathcal{F}_s\right] = 0.\end{aligned}$$

Combing all these equalities we conclude

$$\mathbb{E}[I_t|\mathcal{F}_s] = I_s \ ,$$

and (I_t) is a martingale. $\square$

4 Stopping times, progressively measurability

The filtration $\mathcal{F}_t$ satisfies the usual conditions.

Exercise 18 Let S, T be stopping times. Prove that

(1) $S \wedge T$, $S \vee T$, aS where $a > 1$, are stopping times.

(2) If T is stopping time, then there exists a sequence of stopping times T_n such that T_n takes only a finite number of values and T_n decreases to T.

Solution: (1) For any $t \geq 0$ we have

$$\begin{aligned} \{S \wedge T \leq t\} &= \{S \leq t\} \cup \{T \leq t\} \in \mathcal{F}_t \ , \\ \{S \vee T \leq t\} &= \{S \leq t\} \cap \{T \leq t\} \in \mathcal{F}_t \ , \\ \{aS \leq t\} &= \{S \leq t/a\} \in \mathcal{F}_{t/a} \subset \mathcal{F}_t \ , \end{aligned}$$

which proves the claim.
(2) For any $n \in \mathbf{N}$ we define the stopping time

$$T_n = \begin{cases} i2^{-n}, & \text{if } (i-1)2^{-n} \leq T < i2^{-n} \text{ for some } i < n2^{-n} \ , \\ +\infty, & \text{if } T \geq n \ . \end{cases}$$

These stopping times satisfy the required conditions. □

Exercise 19 Prove that T is a stopping time iff $\{T < t\} \in \mathcal{F}_t$, for any $t > 0$.

Solution: If T is a stopping time, then $\{T < t\} = \cup_{n \geq 1} \{T \leq t - \frac{1}{n}\} \in \mathcal{F}_t$, because $\{T \leq t - \frac{1}{n}\} \in \mathcal{F}_{t-1/n} \subset \mathcal{F}_t$.

Conversely, if $\{T < t\} \in \mathcal{F}_t$, for any $t > 0$, then

$$\{T \leq t\} = \cap_{n \geq 1} \{T < t + \frac{1}{n}\} \in \mathcal{F}_{t+} = \mathcal{F}_t \ ,$$

because the filtration is right-continuous. □

Exercise 20 Let $(M_t, t \in I)$ be an $(\mathcal{F}_t)$-martingale. Let τ be a bounded stopping time that takes countably many values. Let Y be a bounded $\mathcal{F}_\tau$ measurable random variable. Let $N_t = Y(M_t - M_{t \wedge \tau})$. Prove that (N_t) is a martingale.

Solution: Since Y is bounded, N_t is an integrable process. Furthermore, it is adapted, since, for any $t \geq 0$ and any Borel set B, we have

$$\{N_t \in B\} = (\{Y(M_t - M_{t\wedge\tau}) \in B\} \cap \{\tau \leq t\}) \cup (\{0 \in B\} \cap \{\tau > t\}) \in \mathcal{F}_t \ .$$

Let us now take any $0 \leq s < t$. Then we have

$$\begin{aligned}\mathbb{E}[N_t|\mathcal{F}_s] &= \mathbb{E}[Y(M_t - M_{t\wedge\tau})\mathbf{1}_{\tau>s}|\mathcal{F}_s] + \mathbb{E}[Y(M_t - M_{t\wedge\tau})\mathbf{1}_{\tau\leq s}|\mathcal{F}_s] \\ &= I_1 + I_2 \ .\end{aligned}$$

For the first term we have, by the optional stopping theorem,

$$I_1 = \mathbb{E}[\mathbb{E}[Y(M_t - M_{t\wedge\tau})\mathbf{1}_{\tau>s}|\mathcal{F}_\tau]|\mathcal{F}_s] = \mathbb{E}[Y \overbrace{\mathbb{E}[M_t - M_{t\wedge\tau}|\mathcal{F}_\tau]}^{0} \mathbf{1}_{\tau>s}|\mathcal{F}_s] = 0.$$

For the second term we can get, again by the optional stopping theorem,

$$\begin{aligned}I_2 &= \mathbb{E}[Y(M_t - M_\tau)\mathbf{1}_{\tau\leq s}|\mathcal{F}_s] = Y\mathbf{1}_{\tau\leq s}\mathbb{E}[M_t - M_\tau|\mathcal{F}_s] \\ &= \mathbf{1}_{\tau\leq s}Y(M_s - M_{\tau\wedge s}) = N_s \ .\end{aligned}$$

This finishes the proof. □

Exercise 21 Show that for $s < t$ and $A \in \mathcal{F}_s$, $\tau = s\mathbf{1}_A + t\mathbf{1}_{A^c}$ is a stopping time.

Solution: Indeed, for any $r \geq 0$ we have

$$\{\tau \leq r\} = \begin{cases} \emptyset, & \text{if } r < s \\ A, & \text{if } s \leq r < t \\ \Omega, & \text{if } r \geq t \end{cases} \in \mathcal{F}_r \ .$$

□

Exercise 22 Let $0 = t_1 < \cdots < t_{n+1} < \ldots$ with $\lim_{n\to\infty} t_n = \infty$. For each $i = 0, 1, \ldots$, let H_i be a real valued $\mathcal{F}_{t_i}$ measurable random variable and H_0 an $\mathcal{F}_0$-measurable random variable. For $t > 0$, we define

$$X_t(\omega) = H_0(\omega)\mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{\infty} H_i(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(t).$$

Prove that $(X_t, t \geq 0)$ is progressively measurable.

Solution: To this end, we take any Borel set $A \in \mathcal{B}(\mathbf{R})$ and we have to show that for any $t \geq 0$, the set $\{(s,\omega) : H(s,\omega) \in A\}$ is measurable in the product space $([0,t], \mathcal{B}([0,t])) \times (\Omega, \mathcal{F}_t)$. We can rewrite this set in the following way:

$$\begin{aligned}\{(s,\omega) \in [0,t] \times \Omega : H(s,\omega) \in A\} = &(\{0\} \times \{\omega : H_0(\omega) \in A\}) \\ &\cup_{t_i \leq t} \left((t_i \wedge t, t_{i+1} \wedge t] \times \{\omega : H_i(\omega) \in A\}\right) .\end{aligned}$$

Since the sets $\{0\}$ and $(t_i, t_{i+1}]$, $i = 1, \dots, n$, belong to $\mathcal{B}([0,t])$, and $\{\omega : H_i(\omega) \in A\} \in \mathcal{F}_{t_i} \subset \mathcal{F}$, the claim now follows from the fact that the product of two measurable sets is measurable in the product space. □

Exercise 23 Let $s < t \leq u < v$ and let (H_s) and (K_s) be two elementary processes. We define: $\int_s^t H_r dB_r = \int_0^t H_r dB_r - \int_0^s H_r dB_r$. Prove that

$$\mathbb{E}\left[\int_s^t H_r dB_r \int_u^v K_r dB_r\right] = 0.$$

Solution: Recall that the stochastic process $(\int_0^t H_r dB_r, t \geq 0)$ is measurable w.r.t. $\mathcal{F}_t$ and is a martingale. We use the tower property to obtain the following:

$$\begin{aligned}\mathbb{E}\left[\int_s^t H_r dB_r \int_u^v K_r dB_r\right] &= \mathbb{E}\left[\mathbb{E}\left[\int_s^t H_r dB_r \int_u^v K_r dB_r \Big| \mathcal{F}_u\right]\right] \\ &= \mathbb{E}\left[\int_s^t H_r dB_r \mathbb{E}\left[\int_u^v K_r dB_r \Big| \mathcal{F}_u\right]\right] = 0 ,\end{aligned}$$

because the stochastic integral is a martingale, and hence the inner expectation vanishes. □

Exercise 24 Let $f : \mathbf{R}_+ \to \mathbf{R}$ be a differentiable function with $f' \in L^1([0,1])$. Let $\Delta_n : 0 = t_1^{(n)} < t_2^{(n)} < \cdots < t_{N_n}^{(n)} = t$ be a sequence of partitions of $[0,t]$ with $\lim_{n\to\infty} |\Delta_n| \to 0$. Prove that

$$\lim_{n\to\infty} \sum_{j=1}^{N_n} \left(f(t_j^{(n)}) - f(t_{j-1}^{(n)})\right)^2 = 0 .$$

Hint: f is uniformly continuous on $[0,t]$ and $f(t) - f(s) = \int_s^t f'(r)\, dr$.

Solution: We have a simple estimate

$$\sum_{j=1}^{N_n} \left(f(t_j^{(n)}) - f(t_{j-1}^{(n)})\right)^2 \leq \max_j \left|f(t_j^{(n)}) - f(t_{j-1}^{(n)})\right| \sum_{j=1}^{N_n} \left|f(t_j^{(n)}) - f(t_{j-1}^{(n)})\right| .$$

Firstly, we have $\max_j \left| f(t_j^{(n)}) - f(t_{j-1}^{(n)}) \right| \to 0$, as $n \to \infty$, because of the uniform continuity of f on $[0,t]$. Secondly, we can estimate

$$\begin{aligned}\sum_{j=1}^{N_n} \left| f(t_j^{(n)}) - f(t_{j-1}^{(n)}) \right| = \sum_{j=1}^{N_n} \left| \int_{t_{j-1}^{(n)}}^{t_j^{(n)}} f'(r)\, dr \right| &\le \sum_{j=1}^{N_n} \int_{t_{j-1}^{(n)}}^{t_j^{(n)}} \left| f'(r) \right|\, dr \\ &= \int_0^t \left| f'(r) \right|\, dr < \infty\ ,\end{aligned}$$

which follows from the properties of f. Thus, from these facts the claim follows. □

5 Martingales and Optional Stopping Theorem

Let $(\mathcal{F}_t)$ be a filtration satisfying the usual assumptions, unless otherwise stated..

Exercise 25 Use the super-martingale convergence theorem to prove the following statement. Let (X_n) be a sub-martingale sequence and $\sup_n \mathbb{E}(X_n^+) < \infty$. Then $\lim_{n\to\infty} X_n$ exists almost surely.

Solution: The process $Y_n = -X_n$ is a super-martingale. Furthermore, $\sup_n \mathbb{E}(Y_n^-) = \sup_n \mathbb{E}(X_n^+) < \infty$. Thus, from the super-martingale convergence theorem, the limit $\lim_{n\to\infty} Y_n = -\lim_{n\to\infty} X_n$ exists almost surely. □

Exercise 26 Let $(M_t, 0 \le t \le t_0)$ be a continuous local martingale with $M_0 = 0$. Prove that $(M_t, 0 \le t \le t_0)$ is a martingale if $\sup_{t\le t_0} M_t \in L^1$.

Solution: Since $M_0 = 0$, there is a sequence of stopping times $T_n \uparrow \infty$ almost surely, such that $(M_t^{T_n})$ is a martingale. Hence, for any $t > s$, we have

$$\mathbb{E}[M_t^{T_n}|\mathcal{F}_s] = M_s^{T_n} \to M_s \ ,$$

almost surely, as $n \to \infty$. Observe that $M_t^{T_n} \le \sup_{t\le t_0} M_t$, the condition that $\sup_{t\le t_0} M_t \in L^1$ allows us to apply the dominated convergence theorem to obtain

$$\mathbb{E}[M_t^{T_n}|\mathcal{F}_s] \to \mathbb{E}[M_t|\mathcal{F}_s] \ ,$$

almost surely, as $n \to \infty$. This shows that $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$, what means that (M_t) is a martingale. □

Exercise 27 Let $\{B_t\}_{t\ge 0}$ be one dimensional Brownian Motion starting at x ($B_0 = x$), and let $a < x < b$. In this question we are going to find the probability of the Brownian Motion hitting b before a using the Optional Stopping Theorem (OST). Set

$$T_a = \inf\{t : B_t = a\}, \qquad T_b = \inf\{t : B_t = b\}, \quad \text{and} \quad T = T_a \wedge T_b.$$

(a) Give an easy arguments why T_a, T_b and T are all stopping times with respect to the natural filtration of the Brownian Motion.

(b) One would like to compute $\mathbb{E}[B_T]$ using OST, but $(B_t, t \ge 0)$ is not a uniformly integrable martingale and apply OST would require for T to be a bounded stopping time. Instead we are using the limiting argument.

(b1) Let $n \in \mathbb{N}$ use OST to prove that $\mathbb{E}[B_{T\wedge n}] = x$.

(b2) Conclude that $\mathbb{E}[B_T] = x$.

(c) Compute $\mathbb{P}(T_a > T_b)$.

Solution: (a) T_a is a stopping time since it is the hitting time by (B_t) of the closed set $\{a\}$ and the same can be said for T_b. That $T = T_a \wedge T_b$ is the stopping time by Exercise 18.
$(b1)$ Note that $0 \leq T \wedge n$ and both 0 and $T \wedge n$ are bounded stopping times. Hence by the OST $\mathbb{E}[B_{T\wedge n}|B_0] = B_0 = x$ almost surely. Taking the expectation on the both parts and using the tower law we get the result.
$(b2)$ It is clear that pointwise $B_{T\wedge n} \to B_T$ as $n \to \infty$. Moreover $B_{T\wedge n}$ is bounded between a and b hence we can use DCT to conclude:

$$\mathbb{E}[B_T] = \lim_{n\to\infty} \mathbb{E}[B_{T\wedge n}] = x$$

(c) Denote $\mathbb{P}(T_a > T_b)$ by p, then:

$$\begin{aligned} x = \mathbb{E}[B_T] &= \mathbb{E}[B_{T_b}\mathbb{1}_{\{T_a>T_b\}} + B_{T_a}\mathbb{1}_{\{T_b>T_a\}}] = \mathbb{E}[b\mathbb{1}_{\{T_a>T_b\}} + a\mathbb{1}_{\{T_b>T_a\}}] \\ &= bp + a(1-p) \end{aligned}$$

From here we deduce that $\mathbb{P}(T_a > T_b) = \frac{x-a}{b-a}$. □

Exercise 28 If $(M_t, t \in I)$ is a martingale and S and T are two stopping times with T bounded, prove that

$$M_{S\wedge T} = \mathbb{E}\{M_T|\mathcal{F}_S\}.$$

Solution: We can write

$$\mathbb{E}\{M_T|\mathcal{F}_S\} = \mathbb{E}\{\mathbf{1}_{T<S}M_T|\mathcal{F}_S\} + \mathbb{E}\{\mathbf{1}_{T\geq S}M_T|\mathcal{F}_S\} \ .$$

Note, that $\mathbf{1}_{T<S}M_T$ is $\mathcal{F}_S$-measurable. Thus, for the first term we have

$$\mathbb{E}\{\mathbf{1}_{T<S}M_T|\mathcal{F}_S\} = \mathbf{1}_{T<S}M_T = \mathbf{1}_{T<S}M_{S\wedge T} \ .$$

One can see that $\{T \geq S\} \in \mathcal{F}_{S\wedge T}$. Indeed, for any $t \geq 0$ one has

$$\{T \geq S\} \cap \{T \wedge S \leq t\} = \{S \leq T \wedge t\} = \{S \leq T \leq t\} \cup (\{S \leq t\} \cap \{T > t\}) \in \mathcal{F}_t \ ,$$

because every set belongs to $\mathcal{F}_t$. Thus, we conclude

$$\mathbb{E}\{\mathbf{1}_{T\geq S}M_T|\mathcal{F}_S\} = \mathbb{E}\{\mathbf{1}_{T\geq S}M_T|\mathcal{F}_{S\wedge T}\} = \mathbf{1}_{T\geq S}\mathbb{E}\{M_T|\mathcal{F}_{S\wedge T}\} = \mathbf{1}_{T\geq S}M_{S\wedge T} \ ,$$

where we have used the Doob's optional stopping theorem. Combining all these equalities together, we obtain the claim. □

Exercise 29 Prove the following: (1) A positive right continuous local martingale $(M_t, t \geq 0)$ with $M_0 = 1$ is a super-martingale. (2) A positive right continuous local martingale $(M_t, t \geq 0)$ is a martingale if $\mathbb{E}|M_0| < \infty$ and $\mathbb{E}M_t = \mathbb{E}M_0$ for all $t > 0$.

Solution: (1) Since $\mathbb{E}|M_0| < \infty$, there is a sequence of increasing stopping times T_n, such that the process $M_t^{T_n}$ is a martingale. In particular, $M_t^{T_n}$ is integrable. For $s < t$, we use Fatou's lemma to obtain

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[\liminf_{n\to\infty} M_{t\wedge T_n}|\mathcal{F}_s] \leq \liminf_{n\to\infty} \mathbb{E}[M_{t\wedge T_n}|\mathcal{F}_s] = \liminf_{n\to\infty} M_{s\wedge T_n} = M_s \ .$$

(2) Let $S \leq T$ be two stopping times, bonded by a constant K. Then, from (1), we have

$$\mathbb{E}M_0 \geq \mathbb{E}M_S \geq \mathbb{E}M_T \geq \mathbb{E}M_K = \mathbb{E}M_0 \ ,$$

which implies $\mathbb{E}M_T = \mathbb{E}M_S$. We know that $\mathbb{E}M_T = \mathbb{E}M_S$ for any two bounded stopping times $t \leq T$ implies that (M_t) is a martingale, completing the proof. $\square$

Exercise 30 Let $(M_t, t \geq 0)$ and $(N_t, t \geq 0)$ be continuous local martingales with $M_0 = 0$ and $N_0 = 0$.

(1) Let (A_t) and (A'_t) be two continuous stochastic processes of finite variation with initial values 0 and such that $(M_tN_t - A_t)$ and $(M_tN_t - A'_t)$ are local martingales. Prove that (A_t) and (A'_t) are indistinguishable.

(2) Prove that $\langle M, N\rangle_t$ is symmetric in (M_t) and (N_t) and is bilinear.

(3) Prove that $\langle M, N\rangle_t = \frac{1}{4}\Big(\langle M+N, M+N\rangle_t - \langle M-N, M-N\rangle_t\Big)$.

(4) Prove that $\langle M - M_0, N - N_0\rangle_t = \langle M, N\rangle_t$.

(5) Let T be a stopping time, prove that

$$\langle M^T, N^T\rangle = \langle M, N\rangle^T = \langle M, N^T\rangle.$$

Solution: (1) We use the following theorem. If $(M_t, 0 \leq t \leq T)$ is a continuous local martingale with $M_0 = 0$. Suppose that $(M_t, 0 \leq t \leq T)$ has finite total variation. Then $M_t = M_0$, any $0 \leq t \leq T$.

The process $A'_t - A_t = (M_tN_t - A_t) - (M_tN_t - A'_t)$ is a continuous local martingale, and at the same time a process of finite total variation. Then $A'_t - A_t = A'_0 - A_0 = 0$.

(2) The symmetry equality $\langle M,N\rangle = \langle N,M\rangle$ comes from that of the product: $MN = NM$. Next, we will prove

$$\langle M_1 + M_2, N\rangle = \langle M_1, N\rangle + \langle M_2, N\rangle \ .$$

The process

$$M_1N + M_2N - \langle M_1, N\rangle - \langle M_2, N\rangle$$

is a local martingale. Thus, the claim follows from the uniqueness of the bracket process. Similarly for $k \in \mathbf{R}$, $kM - k\langle M, \rangle$ is a local martingale and $\langle kM\rangle = k\langle M\rangle$.

(3) is a consequence of the bi-linearity, proved earlier.

(4) Since $(M_tN_0, t \geq 0)$ is a local martingale, the bracket process $\langle M, N_0\rangle$ vanishes. By the same reason we have $\langle M_0, N\rangle = \langle M_0, N_0\rangle = 0$. Hence, the claim follows from the bilinearity.

(5) By the definition $M^TN^T - \langle M^T, N^T\rangle$ and $(MN)^T - \langle M,N\rangle^T$ are local martingales. This implies $\langle M^T, N^T\rangle = \langle M,N\rangle^T$, from the uniqueness of the bracket process.

Furthermore, both $M^TN^T - \langle M^T, N^T\rangle$ and $(M - M^T)N^T$ are local martingales, hence their sum $MN^T - \langle M^T, N^T\rangle$ is a local martingale as well. This implies $\langle M, N^T\rangle = \langle M^T, N^T\rangle$, from the uniqueness of the bracket process. □

Exercise 31 Let $(M_t, t \in [0,1])$ and $(N_t, t \in [0,1])$ be bounded continuous martingales with $M_0 = N_0 = 0$. If (M_t) and (N_t) are furthermore independent prove that their quadratic variation process $(\langle M,N\rangle_t)$ vanishes.

Solution: Let us take a partition $0 = t_0 < t_1 < \ldots < t_n = 1$. Then we have

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{i=1}^n (M_{t_i} - M_{t_{i-1}})(N_{t_i} - N_{t_{i-1}})\Big]^2 \\
&= \sum_{i,j=1}^n \mathbb{E}\Big[(M_{t_i} - M_{t_{i-1}})(M_{t_j} - M_{t_{j-1}})\Big]\mathbb{E}\Big[(N_{t_i} - N_{t_{i-1}})(N_{t_j} - N_{t_{j-1}})\Big] \\
&= \sum_{i=1}^n \mathbb{E}\Big[M_{t_i} - M_{t_{i-1}}\Big]^2 \mathbb{E}\Big[N_{t_i} - N_{t_{i-1}}\Big]^2 = \sum_{i=1}^n \mathbb{E}\Big[M_{t_i}^2 - M_{t_{i-1}}^2\Big]\mathbb{E}\Big[N_{t_i}^2 - N_{t_{i-1}}^2\Big] \\
&\leq \sum_{i=1}^n \mathbb{E}\Big[M_{t_i}^2 - M_{t_{i-1}}^2\Big] \sup_j \mathbb{E}\Big[N_{t_j}^2 - N_{t_{j-1}}^2\Big] \leq \sum_{i=1}^n \mathbb{E}\Big[M_{t_i}^2 - M_{t_{i-1}}^2\Big]\mathbb{E}\sup_j \Big[N_{t_j}^2 - N_{t_{j-1}}^2\Big].
\end{aligned}$$

Because $N_{t_j}^2$ is uniformly continuous, $\sup_j |N_{t_j}^2 - N_{t_{j-1}}^2| \to 0$ almost surely and N_t^2 is bounded, $\mathbb{E}\sup_j \Big[N_{t_j}^2 - N_{t_{j-1}}^2\Big] \to 0$. Since the bracket process is the limit of $\sum_{i=1}^{n}(M_{t_i} - M_{t_{i-1}})(N_{t_i} - N_{t_{i-1}})$ (in probability), this implies $\langle M, N\rangle_t = 0$. $\square$

Exercise 32 Prove that (1) for almost surely all ω, the Brownian paths $t \mapsto B_t(\omega)$ has infinite total variation on any intervals $[a,b]$. (2) For almost surely all ω, $B_t(\omega)$ cannot have Hölder continuous path of order $\alpha > \frac{1}{2}$.

Solution: (1) We know that t is the quadratic variation of (B_t). We can choose a sequence of partitions $\{t_j^{(n)}\}_{j=1,\dots,M_n}$ such that for almost surely all ω,

$$\sum_{j=1}^{n}\left(B_{t_j^{(n)}}(\omega) - B_{t_{j-1}^{(n)}}(\omega)\right)^2 \to (b-a)\ , \tag{4}$$

as $n \to \infty$. Also for every ω,

$$\begin{aligned}\sum_{j=1}^{M_n}(B_{t_j^{(n)}}(\omega) - B_{t_{j-1}^{(n)}}(\omega))^2 &\le \max_j |B_{t_j^{(n)}}(\omega) - B_{t_{j-1}^{(n)}}(\omega)| \sum_{j=1}^{M_n}|B_{t_j^{(n)}}(\omega) - B_{t_{j-1}^{(n)}}(\omega)| \\ &\le \max_j |B_{t_j^{(n)}}(\omega) - B_{t_{j-1}^{(n)}}(\omega)||B(\omega)|_{\mathrm{TV}_{[a,b]}}.\end{aligned}$$

If $|B(\omega)|_{\mathrm{TV}_{[a,b]}}$ is finite then, since $t \mapsto B_t(\omega)$ is uniformly continuous on $[a,b]$,

$$\sum_{j=1}^{M_n}(B_{t_j^{(n)}}(\omega) - B_{t_{j-1}^{(n)}}(\omega))^2 \le \max_j |B_{t_j^{(n)}}(\omega) - B_{t_{j-1}^{(n)}}(\omega)|\mathrm{TV}_{[a,b]}(B(\omega)) \to 0\ .$$

But for almost surely all ω, this limit is $b-a$, and hence $\mathrm{TV}_{[a,b]}(B(\omega)) = \infty$ for almost surely all ω.
(2) Suppose that for some $\alpha > \frac{1}{2}$ and some number C, both may depend on ω,

$$|B_t(\omega) - B_s(\omega)| \le C(\omega)|t-s|^{\alpha}.$$

We take the partitions $\{t_j^{(n)}\}_{j=1,\dots,M_n}$, as in (1). For any ω we have

$$\begin{aligned}\sum_{j=1}^{M_n}(B_{t_j^{(n)}}(\omega) - B_{t_{j-1}^{(n)}}(\omega))^2 &\le C^2\sum_{j=1}^{M_n}|t_j^{(n)} - t_{j-1}^{(n)}|^{2\alpha} \le C^2|\Delta_n|^{2\alpha-1}\sum_{j=1}^{M_n}|t_j^{(n)} - t_{j-1}^{(n)}| \\ &= C^2(b-a)|\Delta_n|^{2\alpha-1} \to 0\ ,\end{aligned}$$

what contradicts with (4), unless $b-a=0$. Hence for almost surely all ω, the Brownian path is not Hölder continuous of order $\alpha > 1/2$ in any interval $[a,b]$. $\square$

6 Local Martingales and Stochastic Integration

If $M \in H^2$, $L^2(M)$ denotes the L^2 space of progressively measurable processes with the following L^2 norm:

$$|f|_{L^2(M)} := \mathbb{E}\left[\int_0^t (f_s)^2 d\langle M, M\rangle_s)\right] < \infty.$$

If (M_t) is a continuous local martingale we denote by $L^2_{\text{loc}}(M)$ the space of progressively measurable processes with

$$\int_0^t (K_s)^2 d\langle M, M\rangle_s < \infty, \qquad \forall t.$$

Exercise 33 Let (B_t) be a standard Brownian Motion and let $f_s, g_s \in L^2_{\text{loc}}(B)$. Compute the indicated bracket processes, the final expression should not involve stochastic integration.

1. $\langle \int_0^t f_s dB_s, \int_0^t g_s dB_s\rangle$.
2. $\langle \int_0^\cdot B_s dB_s, \int_0^\cdot B_s^3 dB_s\rangle_t$.

Solution: (1a) By the definition,

$$\langle \int_0^\cdot f_s dB_s, \int_0^\cdot g_s dB_s\rangle_t = \int_0^t f_s g_s d\langle B\rangle_s = \int_0^t f_s g_s ds\ .$$

(1b) By Itô isometry:

$$\langle \int_0^\cdot B_s dB_s, \int_0^\cdot B_s^3 dB_s\rangle_t = \int_0^t B_s^4 d\langle B\rangle_s = \int_0^t B_s^4 ds\ .$$

(2) We have the following identities:

$$\begin{aligned} I_t &= \int_0^t (2B_s+1) d\left(\int_0^s B_r d(B_r + r)\right) = \int_0^t (2B_s+1) B_s d(B_s+s) \\ &= 2\int_0^t B_s^2 dB_s + \int_0^t B_s dB_s + 2\int_0^t B_s^2 ds + \int_0^t B_s ds\ . \end{aligned}$$

□

Exercise 34 We say that a stochastic process belongs to $C^{\alpha-}$ if its sample paths are of locally Hölder continuous of order γ for every $\gamma < \alpha$. Let $(H_t, t \leq 1)$ be an adapted continuous and L^p bounded stochastic process where $p > 2$, with $H \in L^2([0,1] \times \Omega)$. Let (B_t) be a one dimensional Brownian motion and set $M_t = \int_0^t H_s dB_s$.

(a) Prove that (M_t) belongs to $C^{\frac{1}{2}-\frac{1}{p}-}$.

(b) If (H_t) is a bounded process, i.e. $\sup_{(t,\omega)} |H_t(\omega)| < \infty$, prove that (M_t) belongs to $C^{\frac{1}{2}-}$.

Proof: First note that the bracket process of $\int_s^t H_r dB_r$ is $\int_s^t (H_r)^2 dr$.

(1) Since H is L^p bounded, there exists C s.t. $\sup_{s\in[0,1]} \mathbb{E}(|H_s|^p) < C$. By Burkholder-Davis-Gundy inequality,

$$\begin{aligned}\mathbb{E}|M_t - M_s|^p &\le c_p \mathbb{E}\left(\int_s^t (H_r)^2 dr\right)^{\frac{p}{2}} \le c_p (t-s)^{\frac{p}{2}} \frac{1}{t-s} \mathbb{E}\int_s^t (H_r)^p dr \\ &\le C c_p (t-s)^{\frac{p}{2}}.\end{aligned}$$

By Kolmogorov's continuity criterion, (M_t) has a modification which is Hölder continuous for any $\gamma < \frac{1}{p}(\frac{p}{2}-1) = \frac{1}{2} - \frac{1}{p}$.

(2) Similarly for any $p \ge 0$,

$$\mathbb{E}|M_t - M_s|^p \le c_p \mathbb{E}\left(\int_s^t (H_r)^2 dr\right)^{\frac{p}{2}} \le c_p (t-s)^{\frac{p}{2}} \sup_{r\in[0,1],\omega\in\Omega} (H_r(\omega))^p,$$

so that (M_t) has a modification which is Hölder continuous for any $\gamma < \frac{1}{2} - \frac{1}{p}$ for any $p > 0$ and concluding the second assertion, again by Kolmogorov's continuity theorem. □

Exercise 35 Let $T > 0$. Let f be a left continuous and adapted process such that $\mathbb{E}\int_0^T (f_s)^2 ds < \infty$. Prove that $(\int_0^t f_s dB_s, 0 \le t \le T)$ is a martingale.

Solution: We know that $(\int_0^t f_s dB_s, 0 \le t \le T)$ is a local martingale. Furthermore,

$$\mathbb{E}(\int_0^t f_s dB_s,)^2 \le c_2 \mathbb{E}\int_0^t (f_s)^2 ds \le c_2 \mathbb{E}\int_0^T (f_s)^2 ds < \infty.$$

by Burkholder-Davis-Gundy inequality. It is therefore an L^2 bounded process, and so a martingale.

□

Exercise 36 Let $M \in H^2$ and $H \in L^2(M)$. Let τ be a stopping time. Prove that

$$\int_0^{t\wedge\tau} H_s dM_s = \int_0^t \mathbf{1}_{s\le\tau} H_s dM_s = \int_0^t H_s dM_s^\tau.$$

Solution: For any $N \in H^2$, we use Exercise 30 to obtain the following identities

$$\int_0^t H_s d\langle M^\tau, N\rangle_s = \int_0^t H_s d\langle M, N\rangle_{\tau\wedge s}$$
$$= \int_0^t \mathbf{1}_{s\le\tau} H_s d\langle M, N\rangle_s = \int_0^{\tau\wedge t} H_s d\langle M, N\rangle_s \ .$$

The claim now follows from the fact, that the stochastic integral $I_t = \int_0^t H_s dM_s^\tau$ is defined uniquely by the identities

$$\langle I, N\rangle_t = \int_0^t H_s d\langle M^\tau, N\rangle_s \ , \quad t \ge 0 \ ,$$

for any $N \in H^2$. □

Exercise 37 Let $M \in H^2$ and $K \in \mathcal{E}$. Prove that the elementary integral $I_t := \int_0^t K_s dM_s$ satisfies

$$\langle I, N\rangle_t = \int_0^t K_s \ d\langle M, N\rangle_s, \qquad \forall t \ge 0 \ ,$$

for any $N \in H^2$.

Solution: See lecture notes. □

Exercise 38 Let $B_t = (B_t^1, \ldots, B_t^n)$ be an n-dimensional Brownian motion. Prove that

$$|B_t|^2 = 2\sum_{i=1}^n \int_0^t B_s^i dB_s^i + nt.$$

Solution: Either use $|B_t|^2 = \sum_{i=1}^n |B_t^i|^2$ apply the product to each component (B_t^i), then summing up or use the multi-dimensional version of the Itô's formula.

Let us consider the function $f : \mathbf{R}^n \to \mathbf{R}$, $f : x \mapsto |x|^2$. Then its derivatives are given by

$$\partial_i f(x) = 2x_i \ , \quad \partial_{ij}^2 f(x) = 2\delta_{i,j} \ .$$

Applying the Itô formula, we obtain

$$|B_t|^2 = f(B_t) = f(B_0) + \sum_{i=1}^n \int_0^t \partial_i f(B_s) dB_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \partial_{ij}^2 f(B_s) d < B^i, B^j >_s$$
$$= 0 + 2\sum_{i=1}^n \int_0^t B_s^i dB_s^i + \sum_{i=1}^n \int_0^t ds = 2\sum_{i=1}^n \int_0^t B_s^i dB_s^i + nt \ ,$$

which is the required identity. □

Exercise 39 Give an expression for $\int_0^t s\ dB_s$ that does not involve stochastic integration.

Solution: Applying the classical integration by parts formula, we obtain

$$\int_0^t s\ dB_s = tB_t - \int_0^t B_s\ ds.$$

□

Exercise 40 Write $\int_0^t (2B_s+1)d\left(\int_0^s B_r d(B_r+r)\right)$ as a function of (B_t), not involving stochastic integrals.

Solution: Firstly,

$$\int_0^t (2B_s+1)d\left(\int_0^s B_r d(B_r+r)\right) = \int_0^t (2B_s+1)B_s d(B_s+s)$$
$$= \int_0^t 2B_s^2 dB_s + \int_0^t B_s dB_s + \int_0^t (2B_s^2+B_s)ds.$$

From the Itô formula for the function x^3 we obtain

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds\ .$$

And the Itô formula for the function x^2 gives

$$\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t)\ .$$

Substituting these stochastic integrals into the formula obtained in (2a), we get

$$I_t = \frac{2}{3}B_t^3 + \frac{1}{2}(B_t^2 - t) + 2\int_0^t B_s^2 ds - \int_0^t B_s ds\ .$$

□

Exercise 41 If (H_t) and (K_t) are continuous martingales, prove that

$$\langle HK, B\rangle_t = \int_0^t H_r d\langle K, B\rangle_r + \int_0^t K_r d\langle H, B\rangle_r.$$

Solution: The product formula gives

$$H_t K_t = \int_0^t H_r dK_r + \int_0^t K_r dH_r + H_0 K_0 + \langle H, K \rangle_t \ .$$

Hence, by Itô isometry, we obtain

$$\begin{aligned} \langle HK, B \rangle_t &=< \int_0^\cdot H_r dK_r + \int_0^\cdot K_r dH_r, B \rangle_t \\ &= \int_0^t H_r d\langle K, B \rangle_r + \int_0^t K_r d\langle H, B \rangle_r \ , \end{aligned}$$

what is the required identity. □

7 Itô's Formula

Exercise 42 If (N_t) is a continuous local martingale with $N_0 = 0$, show that $(e^{N_t - \frac{1}{2}\langle N,N\rangle_t})$ is a local martingale, and $\mathbb{E}\Big(e^{N_t - \frac{1}{2}\langle N,N\rangle_t}\Big) \leq 1$.

Solution: Let us denote $X_t = e^{N_t - \frac{1}{2}\langle N,N\rangle_t}$. The process $(N_t - \frac{1}{2}\langle N,N\rangle_t)$ is a semi-martingale, and we can apply the Itô formula to the function e^x:

$$X_t = 1 + \int_0^t X_s dN_s \ .$$

Since the stochastic integral is a local martingale, we conclude that (X_t) is a local martingale.

Moreover, let T_n be the sequence of increasing stopping times such that $X_t^{T_n}$ is a uniformly integrable martingale. Then, applying the Fatou lemma, we get

$$\mathbb{E}[X_t] = \mathbb{E}[\lim_{n\to\infty} X_t^{T_n}] \leq \liminf_{n\to\infty} \mathbb{E}[X_t^{T_n}] = \mathbb{E}[X_0] = 1 \ ,$$

which is the required bound. □

Exercise 43 Show that a positive continuous local martingale (N_t) with $N_0 = 1$ can be written in the form of $N_t = \exp\left(M_t - \frac{1}{2}\langle M,M\rangle_t\right)$ where (M_t) is a continuous local martingale.

Solution: Let us define $M_t = \int_0^t N_s^{-1} dN_s$. Applying the Itô formula to the function $\log(x)$, we obtain

$$log(N_t) = log(N_0) + \int_0^t N_s^{-1} dN_s - \frac{1}{2}\int_0^t N_s^{-2} d\langle N\rangle_s = M_t - \frac{1}{2}\langle M\rangle_t \ ,$$

which implies the claim. □

Exercise 44 Let (B_t) be a one dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and let $g : \mathbf{R} \to \mathbf{R}$ be a bounded Borel measurable function. Find a solution to

$$x_t = x_0 + \int_0^t x_s g(B_s) dB_s \ .$$

Solution: We define $M_t = \int_0^t g(B_s) dB_s$. Then $dX_t = X_t dM_t$ whose solution is the exponential martingale:

$$e^{M_t - \frac{1}{2}\langle M\rangle_t}.$$

Let $f(x) = x$ and $y_t = M_t - \frac{1}{2}\langle M\rangle_t$. We see that

$$e^{y_t} = 1 + \int_0^t e^{y_s} dy_s + \frac{1}{2}\int_0^t e^{y_s} d\langle M\rangle_s = 1 + \int_0^t e^{y_s} dM_s,$$

proving the claim. $\square$

Exercise 45 1. Let $f, g : \mathbf{R} \to \mathbf{R}$ be C^2 functions, prove that

$$\langle f(B), g(B)\rangle_t = \int_0^t f'(B_s)g'(B_s)ds.$$

2. Compute $\langle \exp(M_t - \frac{1}{2}\langle M\rangle_t), \exp(N_t - \frac{1}{2}\langle N\rangle_t)\rangle$, where (M_t) and (N_t) are continuous local martingales.

Solution: (1) By Itô formula we have

$$\begin{aligned} f(B_t) &= f(0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds \ , \\ g(B_t) &= g(0) + \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)ds \ . \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \langle f(B), g(B)\rangle_t &= \langle \int_0^\cdot f'(B_s)dB_s, \int_0^\cdot g'(B_s)dB_s\rangle_t \\ &= \int_0^t f'(B_s)g'(B_s)ds \ , \end{aligned}$$

what is the required identity.
(2) Let us denote $X_t = \exp(M_t - \frac{1}{2}\langle M\rangle_t)$ and $Y_t = \exp(N_t - \frac{1}{2}\langle N\rangle_t)$. Then, applying the Itô formula to the functions e^x, we obtain

$$X_t = \exp(M_0) + \int_0^t X_s dM_s \ , \quad Y_t = \exp(N_0) + \int_0^t Y_s dN_s \ .$$

Hence, we conclude

$$\langle X, Y\rangle_t = \langle \int_0^\cdot X_s dM_s, \int_0^\cdot Y_s dN_s\rangle_t = \int_0^t X_s Y_s d\langle M, N\rangle_s \ .$$

$\square$

Exercise 46 Let $B_t = (B_t^1, \dots, B_t^n)$ be an n-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, with $n > 1$. We know that for any $t > 0$, $\{r_s \neq 0 \text{ for all } s \leq t\}$ has probability one. Prove that the process $r_t = |B_t|$ is a solution of the equation

$$r_t = r_0 + B_t + \int_0^t \frac{n-1}{2r_s} ds \ , \quad r_0 = 0.$$

Solution: (1) Let us take a function $f(x) = |x|$, for $x \in \mathbf{R}^n$. Its derivatives are given by

$$\frac{\partial f}{\partial x_i} f(x) = \frac{x_i}{|x|} \ , \quad \frac{\partial^2 f}{\partial x_i^2}(x) = \frac{|x|^2 - x_i^2}{|x|^3} \ ,$$

for $x \neq 0$. Note that f is C^2 on $\mathbf{R}^2 \setminus \{0\}$. Let τ be the first time that $X_t := x_0 + B_t$ hits 0. Then $\tau = \infty$ almost surely. Applying the Itô formula to f and X_t^τ, the latter equals X_s almost surely. We obtain

$$\begin{aligned} r_t &= r_0 + \sum_{i=1}^n \int_0^t \frac{X_s^i}{|X_s|} dB_s^i + \frac{1}{2} \sum_{i=1}^n \int_0^t \frac{|X_s|^2 - (X_s^i)^2}{|X_s|^3} ds \\ &= r_0 + \sum_{i=1}^n \int_0^t \frac{X_s^i}{|X_s|} dB_s^i + \frac{1}{2} \int_0^t \frac{n|X_s|^2 - |X_s|^2}{|X_s|^3} ds. \end{aligned}$$

To finish the proof, we have to show that $\tilde{B}_t := \sum_{i=1}^n \int_0^t \frac{X_s^i}{|X_s|} dB_s^i$ is a Brownian motion. Firstly it is a continuous martingale, starting at 0. It has the quadratic variation

$$\langle \tilde{B}, \tilde{B} \rangle_t = \sum_{i=1}^n \int_0^t \frac{(B_s^i)^2}{|B_s|^2} ds = t.$$

Hence, the claim follows from the Lévy characterization theorem. □

Exercise 47 Let $f : \mathbf{R} \to \mathbf{R}$ be C^2. Suppose that there is a constant $C > 0$ s.t. $f \geq C$. Let

$$h(x) = \int_0^x \frac{1}{f(y)} dy.$$

Suppose that $\lim_{x \to +\infty} h(x) = \infty$. Denote by $h^{-1} : [0, \infty) \to [0, \infty)$ its inverse.. Prove that $h^{-1}\Big(h(x_0) + B_t\Big)$ satisfies:

$$x_t = x_0 + \int_0^t f(x_s) dB_s + \frac{1}{2} \int_0^t f(x_s) f'(x_s) ds.$$

Solution: Let us denote $g(x) = h^{-1}\Big(h(x_0) + x\Big)$. Then its derivatives are given by

$$g'(x) = f(g(x)) \ , \quad g''(x) = f'(g(x))f(g(x)) \ .$$

The claim now follows by applying the Itô formula to $g(B_t)$. □

Exercise 48 Let (B_t) be a Brownian motion and τ its hitting time of the set $[2, \infty)$. Is the stochastic process $(\frac{1}{B_t-2}, t < \tau)$ a solution to a stochastic differential equation? (The time τ is the natural life time of the process $\frac{1}{B_t-2}$.)

Solution: Note that $\tau = \inf\{t : B_t \in [2, \infty)\}$. Let τ_n be a sequence of stopping time increasing to τ. Let $f(x) = (x-2)^{-1}$. Where f is differentiable, $f'(x) = -(x-2)^{-2} = -f^2$ and $f''(x) = 2(x-2)^{-3} = 2f^3$. Applying the Itô formula to f and to the stopped process $X_t := (B_t - 2)^{-1}$ we obtain

$$X_t^{\tau_n} \equiv f(B_t^{\tau_n}) = -\frac{1}{2} - \int_0^t (X_s^{\tau_n})^2 dB_s + \int_0^t (X_s^{\tau_n})^3 ds \ ,$$

i.e.

$$X_t^{\tau_n} = -\frac{1}{2} - \int_0^{t\wedge\tau_n} (X_s)^2 dB_s + \int_0^{t\wedge\tau_n} (X_s)^3 ds \ ,$$

We see that on the set $\{t < \tau_n\}$,

$$X_t = -\frac{1}{2} - \int_0^t X_s^2 dB_s + \int_0^t X_s^3 ds \ .$$

which is the required equation. Since $\{t < \tau\} = \cup_n \{t < \tau_n\}$, the equation holds on $\{t < \tau\}$. □

Exercise 49 Let (X_t) be a continuous local martingale begins with 0. Prove that $\langle X^2, X\rangle_t = 2\int_0^t X_s d\langle X, X\rangle_s$.

Solution: Note that $X_t = \int_0^t dX_s$ and

$$X_t^2 = 2\int_0^t X_s X_s + \langle X, X\rangle_t.$$

Consequently,

$$\langle X^2, X\rangle_t = \langle \int_0^\cdot dX_s, 2\int_0^\cdot X_s X_s\rangle_t = 2\int_0^t X_s d\langle X, X\rangle_s.$$

□

8 Lévy's Characterisation Theorem, SDEs

Exercise 50 Let $\sigma_k = (\sigma_k^1, \dots, \sigma_k^d), b = (b^1, \dots, b^d)$ be C^2 functions from $\mathbf{R}^d$ to $\mathbf{R}^d$. Let

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{d} \left(\sum_{k=1}^{m} \sigma_k^i \sigma_k^j \right) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{l=1}^{d} b^l \frac{\partial}{\partial x_l}.$$

Suppose that for $C > 0$,

$$|\sigma(x)|^2 \le c(1 + |x|^2), \quad \langle b(x), x \rangle \le c(1 + |x|^2).$$

(a) Let $f(x) = |x|^2 + 1$. Prove that $\mathcal{L}f \le af$ for some a.

(b) If $x_t = (x_t^1, \dots, x_t^d)$ is a stochastic process with values in $\mathbf{R}^d$, satisfying the following relations:

$$x_t^i = x_0 + \sum_{k=1}^{m} \int_0^t \sigma_k^i(x_s) dB_s^k + \int_0^t b^i(x_s) ds,$$

and $f : \mathbf{R}^d \to \mathbf{R}$, prove that $f(x_t) - f(x_0) - \int_0^t \mathcal{L}f(x_s)ds$ is a local martingale and give the semi-martingale decomposition for $f(x_t)$.

Solution: (a) The partial derivatives of f are given by

$$\frac{\partial f}{\partial x_i}(x) = 2x_i \ , \quad \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = 2\delta_{i,j} \ .$$

Hence, we obtain the following bound

$$\begin{aligned} \mathcal{L}f(x) = \sum_{i=1}^{d} \sum_{k=1}^{m} (\sigma_k^i)^2 + 2 \sum_{i=1}^{d} b^i x_i &= |\sigma(x)|^2 + 2 < b(x), x > \\ &\le cf(x) + 2cf(x) = 3cf(x) \ , \end{aligned}$$

which is the required bond.

(b) Firstly, $\langle B^i, B^j \rangle_s = t$ if $i = j$ and vanishes otherwise. Hence,

$$\langle x^i, x^j \rangle_t = \left\langle \sum_{k=1}^{m} \int_0^\cdot \sigma_k^i(x_s) dB_s^k, \ \sum_{l=1}^{m} \int_0^\cdot \sigma_l^j(x_s) dB_s^l \right\rangle_t = \sum_{k=1}^{m} \int_0^t \sigma_k^i(x_s) \sigma_k^j(x_s) ds.$$

$$
\begin{aligned}
f(x_t) =& f(x_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(x_s) dx_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(x_s) d\langle x^i, x^j\rangle_s \\
=& f(x_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(x_s) \sum_{k=1}^m \sigma_k^i(x_s)\; dB_s^k + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(x_s) b^i(x_s) ds \\
&+ \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(x_s) \sum_{k=1}^m \sigma_k^i(x_s)\sigma_k^j(x_s) ds \\
=& f(x_0) + \sum_{k=1}^m \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}(x_s)\sigma_k^i(x_s)\; dB_s^k \\
&+ \int_0^t \left(\frac{1}{2}\sum_{i,j=1}^d \left(\sum_{k=1}^m \sigma_k^i(x_s)\sigma_k^j(x_s)\right) \frac{\partial^2 f}{\partial x_i \partial x_j}(x_s) + \sum_{i=1}^d \frac{\partial f}{\partial x_i}(x_s) b^i(x_s) \right) ds
\end{aligned}
$$

Thus $f(x_t) - f(x_0) - \int_0^t \mathcal{L}f(x_s)ds = \sum_{k=1}^m \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}(x_s)\sigma_k^i(x_s)\; dB_s^k$ is a local martingale and the semi-martingale decomposition is as given in the identity earlier. □

Exercise 51 Let T be a bounded stopping time and (B_t) a one dimensional Brownian motion. Prove that $(B_{T+s} - B_T, s \geq 0)$ is a Brownian motion.

Solution: Let us denote $W_s = B_{T+s} - B_T$. It is obvious, that $W_0 = 0$ and W_t has continuous sample paths. Moreover, by the OST we have, for any $s < t$,

$$\mathbb{E}[W_t|\mathcal{F}_{T+s}] = \mathbb{E}[B_{T+t}|\mathcal{F}_{T+s}] - B_T = B_{T+s} - B_T = W_s\ .$$

Let $\mathcal{G}_t = \mathcal{F}_{T+t}$. Then (W_t) is a $(\mathcal{G}_t)$-martingale. Next, we take $s < t$ and derive

$$
\begin{aligned}
\mathbb{E}[W_t^2|\mathcal{F}_{T+s}] &= \mathbb{E}[B_{T+t}^2 + B_T^2 - 2B_T B_{T+t}|\mathcal{F}_{T+s}] \\
&= \mathbb{E}[B_{T+t}^2 - (T+t)|\mathcal{F}_{T+s}] + (T-t) + B_T^2 - 2B_T B_{T+s} \\
&= (B_{T+s}^2 - (T+s)) + (T-t) + B_T^2 - 2B_T B_{T+s} \\
&= W_s^2 + t - s\ ,
\end{aligned}
$$

where we have used the fact that $(B_{T+t})^2 - (T+t)$ is a martingale. This implies that $W_t^2 - t$ is a $(\mathcal{G}_t)$-martingale, and hence $\langle W, W\rangle_t = t$. Using the Lévy characterization theorem, we conclude that W_t is a $(\mathcal{G}_t)$-Brownian motion. □

Exercise 52 Let $\{B_t, W_t^1, W_t^2\}$ be independent one dimensional Brownian motions.

1. Let (x_s) be an adapted continuous stochastic process. Prove that (W_t) defined below is a Brownian motion,
$$W_t = \int_0^t \cos(x_s) dW_s^1 + \int_0^t \sin(x_s) dW_s^2.$$

2. Let $sgn(x) = 1$ if $x > 0$ and $sgn(x) = -1$ if $x \leq 0$. Prove that (W_t) is a Brownian motion if it satisfies the following relation
$$W_t = \int_0^t sgn(W_s) dB_s.$$

3. Prove that the process (X_t, Y_t) is a Brownian motion if they satisfy the following relations,
$$\begin{aligned} X_t &= \int_0^t \mathbf{1}_{X_s > Y_s} dW_s^1 + \int_0^t \mathbf{1}_{X_s \leq Y_s} dW_s^2 \\ Y_t &= \int_0^t \mathbf{1}_{X_s \leq Y_s} dW_s^1 + \int_0^t \mathbf{1}_{X_s > Y_s} dW_s^2. \end{aligned}$$

Solution: (1) The process (W_t) is a continuous martingale with the quadratic variation
$$< W >_t = \int_0^t \left((\cos(x_s))^2 + (\sin(x_s))^2 \right) ds = t \; .$$
Hence, by Lévy characterization theorem, we conclude that (W_t) is a Brownian motion.
(2) Can be shown in the same way.
(3) The processes (X_t) and (Y_t) are martingales, and their quadratic variations are
$$\begin{aligned} < X >_t &= \int_0^t \left(\mathbf{1}^2_{X_s > Y_s} + \mathbf{1}^2_{X_s \leq Y_s} \right) ds = t \; , \\ < Y >_t &= \int_0^t \left(\mathbf{1}^2_{X_s \leq Y_s} + \mathbf{1}^2_{X_s > Y_s} \right) ds = t \; . \end{aligned}$$
The bracket process is equal to
$$< X, Y >_t = \int_0^t \mathbf{1}_{X_s > Y_s} \mathbf{1}_{X_s \leq Y_s} ds + \int_0^t \mathbf{1}_{X_s \leq Y_s} \mathbf{1}_{X_s > Y_s} ds = 0 \; ,$$
where we have used the fact that $\mathbf{1}_{X_s > Y_s} \mathbf{1}_{X_s \leq Y_s} = 0$. The claim now follows from the Lévy characterization theorem. $\square$

Exercise 53 Let $t \in [0, 1)$. Define $x_0 = 0$ and

$$x_t = (1-t)\int_0^t \frac{1}{1-s} dB_s.$$

(1) Prove that

$$\mathbb{E}\left[\int_0^t \frac{dB_s}{1-s}\right]^2 = \frac{t}{1-t}.$$

(2) Prove that

$$\left\langle \int_0^\cdot \frac{dB_s}{1-s} \right\rangle_t = \frac{t}{1-t}.$$

Define $A(t) = \frac{t}{1-t}$. Then $A^{-1}(r) = \frac{r}{r+1}$.

(3) Define

$$W_r = \int_0^{A^{-1}(r)} \frac{dB_s}{1-s}, 0 \le r < \infty.$$

Let $\mathcal{G}_r = \mathcal{F}_{A^{-1}(r)}$. Prove that (W_r) is an $(\mathcal{G}_r)$-martingale.

(4) For a standard one dimensional Brownian motion B_t, $\lim_{r\to 0} rB_{1/r} = 0$. Use this to prove that $\lim_{t\to 1} x_t = 0$.

(5) Prove that x_t solves

$$x_t = B_t - \int_0^t \frac{x_s}{1-s} ds.$$

(6) Prove that $\int_0^t \frac{x_s}{1-s} ds$ is of finite total variation on $[0, 1]$.

Hint: it is sufficient to prove that

$$\int_0^1 \frac{|x_s|}{1-s} ds < \infty$$

almost surely.

Solution: (1) By Itô isometry we obtain

$$\mathbb{E}\left[\int_0^t \frac{dB_s}{1-s}\right]^2 = \int_0^t \frac{ds}{(1-s)^2} = \frac{t}{1-t}\ .$$

(2) We have

$$\left\langle \int_0^\cdot \frac{dB_s}{1-s} \right\rangle_t = \int_0^t \frac{d<B>_s}{(1-s)^2} = \frac{t}{1-t}\ .$$

(3) It is obvious that (W_r) is integrable and adapted to $(\mathcal{G}_r)$. Moreover, for any $t > s$, we have

$$\mathbb{E}[W_t|\mathcal{G}_s] = \mathbb{E}\left[\int_0^{t/(t+1)} \frac{dB_r}{1-r}\Big|\mathcal{F}_{s/(s+1)}\right] = \int_0^{s/(s+1)} \frac{dB_r}{1-r} = W_s \ ,$$

because $s/(s+1) < t/(t+1)$. Thus the Itô integral (W_t) is a martingale with respect to $(\mathcal{F}_t)$. This shows that (W_r) is a martingale with respect to $(\mathcal{G}_r)$.
(4) By (2), we can calculate

$$\langle W \rangle_t = \frac{A^{-1}(t)}{1 - A^{-1}(t)} = t \ .$$

By Paul Lévy characterization theorem, we conclude that (W_t) is a Brownian motion. Then, $\lim_{r\to 0} rW_{1/r} = 0$. This implies

$$\lim_{t\to 1} X_t = \lim_{r\to 0} X_{r/(r+1)} = \lim_{r\to 0} rW_{1/r} = 0 \ .$$

(5) Since the function $\frac{1}{1-s}$ has finite total variation on $[0, c]$, for every $c < 1$, the integral $\int_0^t B_s d\frac{1}{1-s}$, for $t \leq c$, can be defined in the Stieltjes sense. Using integration by parts, we see that the process W_t is defined in the Stieltjes sense as well. We can use the classical analysis to derive the following equalities:

$$\begin{aligned}
\int_0^t \frac{x_s}{1-s} ds &= \int_0^t W_{A(s)} ds = sW_{A(s)}\Big|_0^t - \int_0^t s \, dW_{A(s)} \\
&= t\int_0^t \frac{dB_s}{1-s} - \int_0^t \frac{s}{1-s} dB_s = t\int_0^t \frac{dB_s}{1-s} - \int_0^t \frac{dB_s}{1-s} + \int_0^t dB_s \\
&= -x_t + B_t \ ,
\end{aligned}$$

which is the claimed equality, for any $t \leq c$. Passing $c \to 1$, we conclude that the equality holds for any $t \in [0, 1]$.
(6) Since, x_s is defined as an integral of a non-random process with respect to the Brownian motion, it has the normal distribution, whose parameters are easily seen to be 0 and $s(1-s)$ (the latter follows from (1)). Thus, we apply Fubini lemma to derive

$$\mathbb{E}\int_0^1 \frac{|x_s|}{1-s} ds = \int_0^1 \frac{\mathbb{E}|x_s|}{1-s} ds = C\int_0^1 \frac{\sqrt{s(1-s)}}{1-s} ds < \infty \ ,$$

for some constant $C > 0$. This implies that $\int_0^1 \frac{|x_s|}{1-s} ds < \infty$ almost surely. Defining the process $Y_t = \int_0^t \frac{x_s}{1-s} ds$ and taking any partition $0 = t_0 < t_1 < \ldots < t_n = 1$, we derive

$$\begin{aligned}\sum_{i=0}^{n-1} |Y_{t_{i+1}} - Y_{t_i}| = \sum_{i=0}^{n-1} \Big| \int_{t_i}^{t_{i+1}} \frac{x_s}{1-s} ds \Big| &\leq \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{|x_s|}{1-s} ds \\ &= \int_0^1 \frac{|x_s|}{1-s} ds < \infty ,\end{aligned}$$

from what the claim follows. $\square$

Problem Sheet 9: SDE and Grisanov

Exercise 54 Let us consider an SDE $dx_t = \sum_{k=1}^m \sigma_k(x_t)dB_t^k + \sigma_0(x_t)dt$ on $\mathbf{R}^d$ with infinitesimal generator $\mathcal{L}$. Let D be a bounded open subset of $\mathbf{R}^d$ with closure $\bar{D}$ and C^2 boundary ∂D. Let $x_0 \in D$. Suppose that there is a global solution to the SDE with initial value x_0 and denote by τ its first exit time from D. Suppose that $\tau < \infty$ almost surely. Let $g : D \to \mathbf{R}$ be a continuous function, and $u : \bar{D} \to \mathbf{R}$ be a solution to the Dirichlet problem $\mathcal{L}u = -g$ on D and $u = 0$ on the boundary.

Prove that $u(x_0) = \mathbb{E}\int_0^\tau g(x_s)ds$. [Hint: Use a sequence of increasing stopping times τ_n with limit τ].

Solution: Let D_n be a sequence of increasing open sets of $D_n \subset D_{n+1} \subset D$ and τ_n the first exit time of x_t from D_n. By Itô's formula,

$$\begin{aligned}
u(x_{t\wedge\tau_n}) &= u(x_0) + \int_0^{t\wedge\tau_n} \mathcal{L}u(x_s)ds + \sum_{k=1}^n \int_0^{t\wedge\tau_n} du(\sigma_k(x_s))dB_s^k \\
&= u(x_0) - \int_0^{t\wedge\tau_n} g(x_s)ds + \sum_{k=1}^n \int_0^{t\wedge\tau_n} du(\sigma_k(x_s))dB_s^k.
\end{aligned}$$

Since u, g are continuous on $\bar{D}_n$, the stochastic integral is a martingale. We have

$$\mathbb{E}[u(x_{t\wedge\tau_n})] = u(x_0) - \mathbb{E}\int_0^{t\wedge\tau_n} g(x_s)ds.$$

Since $\tau_n < \tau < \infty$, $\lim_{t\uparrow\infty}(t \wedge \tau_n) = t$ a.s. We take $t \to \infty$ followed by taking $n \to \infty$ and used dominated convergence theorem to take the limit inside the expectation, we see that

$$\lim_{n\to\infty}\lim_{t\to\infty} \mathbb{E}[u(x_{t\wedge\tau_n})] = \mathbb{E}u(x_\tau) = 0.$$

and

$$\lim_{n\to\infty}\lim_{t\to\infty} \mathbb{E}\int_0^{t\wedge\tau_n} g(x_s)ds = \mathbb{E}\int_0^\tau g(x_s)ds.$$

This completes the proof. □

Exercise 55 A sample continuous Markov process on $\mathbf{R}^2$ is a Markov process on the hyperbolic space (the upper half space model) if its infinitesimal generator is $\frac{1}{2}y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$. Prove that the solution to the following equation is a hyperbolic Brownian motion.

$$\begin{aligned}
dx_t &= y_t dB_t^1 \\
dy_t &= y_t dB_t^2.
\end{aligned}$$

[Hint: Show that if $y_0 > 0$ then y_t is positive and hence the SDE can be considered to be defined on the upper half plane. Compute its infinitesimal generator $\mathcal{L}$.]

Solution: Just observe that $y_t = y_0 e^{B_t^2-(1/2)t}$ so $y_t > 0$ if $y_0 > 0$. Compute the generator by Itô's formula. □

Exercise 56 Discuss the uniqueness and existence problem for the SDE

$$dx_t = \sin(x_t)dB_t^1 + \cos(x_t)dB_t^2.$$

Solution: The functions $\sin(x)$ and $\cos(x)$ are C^1 with bounded derivative, and hence Lipschitz continuous. For each initial value there is a unique global strong solution. □

Exercise 57 Let (x_t) and (y_t) be solutions to the following respective SDEs (in Stratnovitch form),

$$x_t = x_0 - \int_0^t y_s \circ dB_s, \qquad y_t = y_0 + \int_0^t x_s \circ dB_s.$$

Show that $x_t^2 + y_t^2$ is independent of t.

Solution: Let us rewrite the SDEs in the Itô form:

$$x_t = x_0 - \int_0^t y_s dB_s - \frac{1}{2} < y, B >_t \ , \quad y_t = y_0 + \int_0^t x_s dB_s + \frac{1}{2} < x, B >_t \ .$$

We can calculate the bracket processes in these expressions:

$$< y, B >_t = < \int_0^\cdot x_s dB_s, B >_t = \int_0^t x_s ds \ , \quad < x, B >_t = \int_0^t y_s ds \ .$$

Moreover, the quadratic variations of (x_t) and (y_t) are

$$< y >_t = < \int_0^\cdot x_s dB_s >_t = \int_0^t x_s^2 ds \ , \quad < x >_t = \int_0^t y_s^2 ds \ .$$

Applying now the Itô formula to the function $f(x, y) = x^2 + y^2$, we obtain

$$\begin{aligned} f(x_t, y_t) =& f(x_0, y_0) + 2\int_0^t x_s dx_s + 2\int_0^t y_s dy_s + < x >_t + < y >_t \\ =& f(x_0, y_0) - 2\int_0^t x_s y_s dB_s - \int_0^t x_s d < y, B >_s + 2\int_0^t x_s y_s dB_s \\ &+ \int_0^t y_s d < x, B >_s + < x >_t + < y >_t \\ =& f(x_0, y_0) - \int_0^t x_s^2 ds - \int_0^t y_s^2 ds + < x >_t + < y >_t = f(x_0, y_0) \ , \end{aligned}$$

which is independent of t. $\square$

Exercise 58 1. Let (h_t) be a deterministic real valued process with $\int_0^\infty (h_s)^2 ds < \infty$. Let (B_t) be an $\mathcal{F}_t$-Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Prove that

$$\exp\left(\int_0^t h_s dB_s - \frac{1}{2}\int_0^t (h_s)^2 ds\right)$$

is a martingale.

2. Let (h_t) be a bounded continuous and adapted real valued stochastic process. Prove that

$$\exp\left(\int_0^t h_s dB_s - \frac{1}{2}\int_0^t (h_s)^2 ds\right)$$

is a martingale.

3. Let Q, P be two equivalent martingale measures on $\mathcal{F}_t$ with

$$\frac{dQ}{dP} = \exp\left(\int_0^t h_s dB_s - \frac{1}{2}\int_0^t (h_s)^2 ds\right).$$

Define $\tilde{B}_t = B_t - \int_0^t h_s ds$ for (h_s) given in part (1). Prove that $(\tilde{B}_t)$ is an $(\mathcal{F}_t)$-Brownian motion with respect to Q.

Solution: Let

$$M_t := \exp\left(\int_0^t h_s dB_s - \frac{1}{2}\int_0^t (h_s)^2 ds\right)$$

By Itô's formula, M_t satisfies the equation $dx_t = h_t x_t dB_t$ with initial value 1 and so (M_t) is a local martingale. That it is a martingale follows from the following arguments.

First proof. We use Novikov's condition: If $\exp(\frac{1}{2}\langle N, N\rangle_t)$ is finite then $\exp\left(N_t - \frac{1}{2}\langle N, N\rangle_t\right)$ is a martingale. In our case

$$\mathbb{E}\exp\left(\frac{1}{2}\left\langle \int_0^\cdot h_s dB_s \right\rangle_t\right) = \mathbb{E}\exp\left(\frac{1}{2}\int_0^t (h_s)^2 ds\right) < \infty$$

in either case (1) or (2).

Second proof for part (1). Note that

$$\langle M, M\rangle_t = \int_0^t (h_s)^2 (M_s)^2 ds.$$

Since $M_t - 1$ is a local martingale starting from 0, we apply Burkholder-Davis-Gundy inequality,

$$\mathbb{E}(M_t)^2 \le 2\mathbb{E}(M_t - 1)^2 + 2 \le c\mathbb{E}\langle M, M\rangle_t + 2.$$

Thus

$$\mathbb{E}\langle M, M\rangle_t \le 2\int_0^t (h_s)^2 ds + c\int_0^t (h_s)^2 \mathbb{E}\langle M, M\rangle_s ds.$$

By a version of Grownall's inequality,

$$\mathbb{E}\langle M, M\rangle_t \le 2\int_0^t (h_s)^2 ds\ \exp\left(c\int_0^t (h_r)^2 dr\right).$$

This proves (1).

Second proof for part (2). From $M_t = 1 + \int_0^t h_s M_s dB_s$,

$$\mathbb{E}(M_t)^2 \le 2 + 2\int_0^t \mathbb{E}(h_s M_s)^2 ds.$$

If (h_s) is bounded, $\mathbb{E}(M_t)^2 \le 2e^{2|h|_\infty^2 t}$ and this proves that (M_t) is an L^2 bounded martingale for case (2).

To prove the question (3), let us define $N_t = \int_0^t h_s dB_s$. Then

$$\frac{dQ}{dP} = \exp\left(N_t - \frac{1}{2} < N >_t\right),$$

and $\tilde{B}_t = B_t - < B, N >_t$. First observe that the exponential martingale of (N_t) is a martingale, c.f. part (1) and Q is a probability measure, equivalent to P. By Girsanov theorem, $(\tilde{B}_t)$ is an $\mathcal{F}_t$ local -martingale with respect to Q. Since $< \tilde{B} >_t = < B >_t = t$, it follows from the Lévy characterisation theorem that $(\tilde{B}_t)$ is a Brownian motion with respect to Q. $\square$