MATHEMATICS

Proof of the Box-Muller method

Asked 4 years, 8 months ago Active 1 month ago Viewed 15k times

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This is Exercise 2.2.2 from *Achim Klenke: »Probability Theory — A Comprehensive Course«*.

> *Exercise (Box–Muller method):* Let U and V be independent random variables that are uniformly distributed on $[0, 1]$. Define
>
> $$X := \sqrt{-2\log(U)}\,\cos(2\pi V) \quad \text{and} \quad Y := \sqrt{-2\log(U)}\,\sin(2\pi V)\,.$$
>
> Show that X and Y are independent and $\mathcal{N}_{0,1}$-distributed.

Solution: Define random variable $R := \sqrt{-2\log(U)}$, then

$$\begin{aligned}\mathbf{P}\left[R \le r\right] &= \mathbf{P}\left[-2\log(U) \le r^2\right] = \\ &= \mathbf{P}\left[\log(U) \ge -\frac{r^2}{2}\right] = \\ &= 1 - \mathbf{P}\left[U < \exp\left(-\frac{r^2}{2}\right)\right].\end{aligned}$$

U is uniformly defined on $[0, 1]$, so the distribution of R is

$$\mathbf{P}[R \le r] = 1 - \int_0^{\exp(-r^2/2)} dt = 1 - \exp\left(-\frac{r^2}{2}\right).$$

For the density of R we get: $f_R(t) = \exp\left(-\frac{t^2}{2}\right) \cdot t$ with $t > 0$.

We also define the random variable $\Phi := 2\pi V$. Since V is uniformly distributed on $[0, 1]$, $f_\Phi(t) = \frac{1}{2\pi}$

with $0 < t \leq 2\pi$.

Since U, V are independent, R, Φ must also be independent and

$$f_{R,\Phi}(t_1, t_2) = f_R(t_1) f_\Phi(t_2) = \frac{1}{2\pi} \exp\left(-\frac{t_1^2}{2}\right) \cdot t_1 \,.$$

With

$$\begin{aligned} g\colon (0, \infty) \times (0, 2\pi] &\to \mathbb{R}^2 \\ (r, \phi) &\mapsto \big(r\cos(\phi), r\sin(\phi)\big) \end{aligned}$$

we see that

$$(X, Y) = g(R, \Phi)\,,$$

so we want to find the image measure

$$\mathbf{P}_{X,Y} = \mathbf{P}_{R,\Phi} \circ g^{-1}\,.$$

We use the transformation formula for densities:

$$f_{X,Y}(\tau_1, \tau_2) = \frac{f_{R,\Phi}(g^{-1}(\tau_1, \tau_2))}{|\det(g'(g^{-1}(\tau_1, \tau_2)))|}$$

g is just the transformation for polar coordinates. With

$$t_1 = \sqrt{\tau_1^2 + \tau_2^2} = |\det(g'(g^{-1}(\tau_1, \tau_2)))|$$

we finally get

$$f_{X,Y}(\tau_1, \tau_2) = \frac{1}{2\pi} \exp\left(-\frac{\tau_1^2 + \tau_2^2}{2}\right) = \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\tau_1^2}{2}\right)}_{=f_X(\tau_1)} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\tau_2^2}{2}\right)}_{=f_Y(\tau_2)}\,,$$

that is: X, Y are $\mathcal{N}_{0,1}$-distributed and independent. $\square$

Could you please check my proof? I'm sorry that it's so long — it seems right to me, but I'm self-studying and really need to catch any eventual mistakes... Thank you!

probability-theory proof-verification

asked Jan 19 '15 at 7:39

Ystar

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Just today someone posted this question again. I think I answered it here several years ago. – Michael Hardy Jun 3 '15 at 20:28

I also asked this question yesterday. I solved it using the method I was attempting in my question. There was a little bit of algebra involved, but it actually wasn't too bad. I also found this link which discuss the problem in 2.4.3.mathematik.uni-ulm.de/numerik/teaching/ss09/NumFin/Script/… – user75514 Jun 4 '15 at 13:54

Looks about right to me. There's a bit of algebra involved in the transformation formula for densities (like the comment above mentions) that is skimmed over but the end result looks correct. – brian.keng Nov 23 '15 at 0:02

3 How do we know that the joint probability density factors out into probability densities of X and Y? Shouldn't we