Solution to selected problems.

Chapter 1. Preliminaries

1. $\forall A \in \mathcal{F}_S$, $\forall t \geq 0$, $A \cap \{T \leq t\} = (A \cap \{S \leq t\}) \cap \{T \leq t\}$, since $\{T \leq t\} \subset \{S \leq t\}$. Since $A \cap \{S \leq t\} \in \mathcal{F}_t$ and $\{T \leq t\} \in \mathcal{F}_t$, $A \cap \{T \leq t\} \in \mathcal{F}_t$. Thus $\mathcal{F}_S \subset \mathcal{F}_T$.

2. Let $\Omega = \mathbb{N}$ and $\mathcal{F} = \mathcal{P}(\mathbb{N})$ be the power set of the natural numbers. Let $\mathcal{F}_n = \sigma(\{2\}, \{3\}, \ldots, \{n+1\})$, $\forall n$. Then $(\mathcal{F}_n)_{n\geq 1}$ is a filtration. Let $S = 3 \cdot 1_3$ and $T = 4$. Then $S \leq T$ and

$$\{\omega : S(\omega) = n\} = \begin{cases} \{3\} & \text{if } n = 3 \\ \emptyset & \text{otherwise} \end{cases}$$

$$\{\omega : T(\omega) = n\} = \begin{cases} \Omega & \text{if } n = 4 \\ \emptyset & \text{otherwise} \end{cases}$$

Hence $\{S = n\} \in \mathcal{F}_n$, $\{T = n\} \in \mathcal{F}_n$, $\forall n$ and S, T are both stopping time. However $\{\omega : T - S = 1\} = \{\omega : 1_{\{3\}}(\omega) = 1\} = \{3\} \notin \mathcal{F}_1$. Therefore $T - S$ is not a stopping time.

3. Observe that $\{T_n \leq t\} \in \mathcal{F}_t$ and $\{T_n < t\} \in \mathcal{F}_t$ for all $n \in \mathbb{N}$, $t \in \mathbb{R}_+$, since T_n is stopping time and we assume usual hypothesis. Then
(1) $\sup_n T_n$ is a stopping time since $\forall t \geq 0$, $\{\sup_n T_n \leq t\} = \cap_n \{T_n \leq t\} \in \mathcal{F}_t$.
(2) $\inf_n T_n$ is a stopping time since $\{\inf_n T_n < t\} = \cup \{T_n < t\} \in \mathcal{F}_t$
(3) $\limsup_{n\to\infty}$ is a stopping time since $\limsup_{n\to\infty} = \inf_m \sup_{n\geq m} T_n$ (and (1), (2).)
(4) $\liminf_{n\to\infty}$ is a stopping time since $\liminf_{n\to\infty} = \sup_m \inf_{n\geq m} T_n$ (and (1), (2).).

4. T is clearly a stopping time by exercise 3, since $T = \liminf_n T_n$. $\mathcal{F}_T \subset \mathcal{F}_{T_n}$, $\forall n$ since $T \leq T_n$, and $\mathcal{F}_T \subset \cap_n \mathcal{F}_{T_n}$ by exercise 1. Pick a set $A \in \cap_n \mathcal{F}_{T_n}$.$\forall n \geq 1$, $A \in \mathcal{F}_{T_n}$ and $A \cap \{T_n \leq t\} \in \mathcal{F}_t$. Therefore $A \cap \{T \leq t\} = A \cap (\cap_n \{T_n \leq t\}) = \cap_n (A \cap \{T_n \leq t\}) \in \mathcal{F}_t$. This implies $\cap_n \mathcal{F}_{T_n} \subset \mathcal{F}_T$ and completes the proof.

5. (a) By completeness of L^p space, $X \in L^P$. By Jensen's inequality, $E|M_t|^p = E|E(X|\mathcal{F}_t)|^p \leq E\left[E(|X|^p|\mathcal{F}_t)\right] = E|X|^p < \infty$ for $p > 1$.

(b) By (a), $M_t \in L^p \subset L^1$. For $t \geq s \geq 0$, $E(M_t|\mathcal{F}_s) = E(E(X|\mathcal{F}_t)|\mathcal{F}_s) = E(X|\mathcal{F}_s) = M_s$ a.s. $\{M_t\}$ is a martingale. Next, we show that $\{M_t\}$ is continuous. By Jensen's inequality, for $p > 1$,

$$E|M_t^n - M_t|^p = E|E(M_\infty^n - X|\mathcal{F}_t)|^p \leq E|M_\infty^n - X|^p, \qquad \forall t \geq 0. \tag{1}$$

It follows that $\sup_t E|M_t^n - M_t|^p \leq E|M_\infty^n - X|^p \to 0$ as $n \to \infty$. Fix arbitrary $\varepsilon > 0$. By Chebychev's and Doob's inequality,

$$P\left(\sup_t |M_t^n - M_t| > \varepsilon\right) \leq \frac{1}{\varepsilon^p} E(\sup_t |M_t^n - M_t|^p) \leq \left(\frac{p}{p-1}\right)^p \frac{\sup_t E|M_t^n - M_t|^p}{\varepsilon^p} \to 0. \tag{2}$$

Therefore M^n converges to M uniformly in probability. There exists a subsequence $\{n_k\}$ such that M_{n_k} converges uniformly to M with probability 1. Then M is continuous since for almost all ω, it is a limit of uniformly convergent continuous paths.

6. Let $p(n)$ denote a probability mass function of Poisson distribution with parameter λt. Assume λt is integer as given.

$$\begin{aligned} E|N_t-\lambda t| =& E(N_t-\lambda t)+2E(N_t-\lambda t)^- = 2E(N_t-\lambda t)^- = 2\sum_{n=0}^{\lambda t}(\lambda t-n)p(n) \\ =& 2e^{-\lambda t}\sum_{n=0}^{\lambda t}(\lambda t-n)\frac{(\lambda t)^n}{n!} = 2\lambda t e^{-\lambda t}\left(\sum_{n=0}^{\lambda t}\frac{(\lambda t)^n}{n!}-\sum_{n=0}^{\lambda t-1}\frac{(\lambda t)^n}{n!}\right) \\ =& 2e^{-\lambda t}\frac{(\lambda t)^{\lambda t}}{(\lambda t-1)!} \end{aligned} \tag{3}$$

7. Since N has stationary increments, for $t\ge s\ge 0$,

$$E(N_t-N_s)^2 = EN_{t-s}^2 = Var(N_{t-s})+(EN_{t-s})^2 = \lambda(t-s)[1+\lambda(t-s)]. \tag{4}$$

As $t\downarrow s$ (or $s\uparrow t$), $E(N_t-N_s)^2\to 0$. N is continuous in L^2 and therefore in probability.

8. Suppose τ_α is a stopping time. A 3-dimensional Brownian motion is a strong Markov process we know that $P(\tau_\alpha<\infty)=1$. Let's define $W_t := B_{\tau_\alpha+t}-B_{\tau_\alpha}$. W is also a 3-dimensional Brownian motion and its argumented filtration $\mathcal{F}_t^W$ is independent of $\mathcal{F}_{\tau_\alpha+}=\mathcal{F}_{\tau_\alpha}$. Observe $\|W_0\|=\|B_{\tau_\alpha}\|=\alpha$. Let $S=\inf_t\{t>0:\|W_t\|\le\alpha\}$. Then S is a F_t^W stopping time and $\{S\le s\}\in\mathcal{F}_s^W$. So $\{S\le s\}$ has to be independent of any sets in $\mathcal{F}_{\tau_\alpha}$. However $\{\tau_\alpha\le t\}\cap\{S\le s\}=\emptyset$, which implies that $\{\tau_\alpha\le t\}$ and $\{S\le s\}$ are dependent. Sine $\{\tau_\alpha\le t\}\in\mathcal{F}_{\tau_\alpha}$, this contradicts the fact that $\mathcal{F}_{\tau_\alpha}$ and $\mathcal{F}_t^W$ are independent. Hence τ_α is not a stopping time.

9. (a) Since L^2-space is complete, it suffices to show that $S_t^n=\sum_{i=1}^n M_t^i$ is a Cauchy sequence w.r.t. n. For $m\ge n$, by independence of $\{M^i\}$,

$$E(S_t^n-S_t^m)^2 = E\left(\sum_{i=n+1}^m M_t^i\right)^2 = \sum_{i=n+1}^m E\left(M_t^i\right)^2 = t\sum_{i=n+1}^m\frac{1}{i^2}. \tag{5}$$

, Since $\sum_{i=1}^\infty 1/i^2<\infty$, as $n,m\to\infty$, $E(S_t^n-S_t^m)^2\to\infty$. $\{S_t^n\}_n$ is Cauchy and its limit M_t is well defined for all $t\ge 0$.

(b)First, recall Kolmogorov's convergence criterion: *Suppose $\{X_i\}_{i\ge1}$ is a sequence of independent random variables. If $\sum_i Var(X_i)<\infty$, then $\sum_i(X_i-EX_i)$ converges a.s.*
For all i and t, $\triangle M_t^i=(1/i)\triangle N_t^i$ and $\triangle M_t^i>0$. By Fubini's theorem and monotone convergence theorem,

$$\sum_{s\le t}\triangle M_s = \sum_{s\le t}\sum_{i=1}^\infty\triangle M_s^i = \sum_{i=1}^\infty\sum_{s\le t}\triangle M_s^i = \sum_{i=1}^\infty\sum_{s\le t}\frac{\triangle N_s^i}{i} = \sum_{i=1}^\infty\frac{N_t^i}{i}. \tag{6}$$

Let $X_i=(N_t^i-t)/i$. Then $\{X_i\}_i$ is a sequence of independent random variables such that $EX_i=0$, $Var(X_i)=1/i^2$ and hence $\sum_{i=1}^\infty Var(X_i)<\infty$. Therefore Kolmogorov's criterion implies that $\sum_{i=1}^\infty X_i$ converges almost surely. On the other hand, $\sum_{i=1}^\infty t/i=\infty$. Therefore, $\sum_{s\le t}\triangle M_s=\sum_{i=1}^\infty N_t^i/i=\infty$.

10. (a) Let $N_t = \sum_i \frac{1}{i}(N_t^i - t)$ and $L_t = \sum_i \frac{1}{i}(L_t^i - t)$. As we show in exercise 9(a), N, M are well defined in L^2 sense. Then by linearity of L^2 space M is also well defined in L^2 sense since

$$M_t = \sum_i \frac{1}{i}\left[(N_t^i - t) - (L_t^i - t)\right] = \sum_i \frac{1}{i}(N_t^i - t) - \sum_i \frac{1}{i}(L_t^i - t) = N_t - L_t. \tag{7}$$

Both terms in right size are martingales change only by jumps as shown in exercise 9(b). Hence M_t is a martingale which changes only by jumps.

(b) First show that given two independent Poisson processes N and L, $\sum_{s>0} \triangle N_s \triangle L_s = 0$ a.s., i.e. N and L almost surely don't jump simultaneously. Let $\{T_n\}_{n\geq 1}$ be a sequence of jump times of a process L. Then $\sum_{s>0} \triangle N_s \triangle L_s = \sum_n \triangle N_{T_n}$. We want to show that $\sum_n \triangle N_{T_n} = 0$ a.s. Since $\triangle N_{T_n} \geq 0$, it is enough to show that $E\triangle N_{T_n} = 0$ for $\forall n \geq 1$.
Fix $n \geq 1$ and let μ_{T_n} be a induced probability measure on $\mathbb{R}_+$ of T_n. By conditioning,

$$E(\triangle N_{T_n}) = E\left[E\left(\triangle N_{T_n}|T_n\right)\right] = \int_0^\infty E\left(\triangle N_{T_n}|T_n = t\right)\mu_{T_n}(dt) = \int_0^\infty E\left(\triangle N_t\right)\mu_{T_n}(dt), \tag{8}$$

where last equality is by independence of N and T_n. It follows that $E\triangle N_{T_n} = E\triangle N_t$. Since $\triangle N_t \in L^1$ and $P(\triangle N_t = 0) = 1$ by problem 25, $E\triangle N_t = 0$, hence $E\triangle N_{T_n} = 0$.

Next we show that the previous claim holds even when there are countably many Poisson processes. assume that there exist countably many independent Poisson processes $\{N^i\}_{i\geq 1}$. Let $A \subset \Omega$ be a set on which more than two processes of $\{N^i\}_{i\geq 1}$ jump simultaneously. Let Ω_{ij} denotes a set on which N^i and N^j don't jump simultaneously. Then $P(\Omega_{ij}) = 1$ for $i \neq j$ by previous assertion. Since $A \subset \cup_{i>j}\Omega_{ij}^c$, $P(A) \leq \sum_{i>j} P(\Omega_{ij}^c) = 0$. Therefore jumps don't happen simultaneously almost surely.

Going back to the main proof, by (a) and the fact that N and L don't jump simultaneously, $\forall t > 0$,

$$\sum_{s\leq t} |\triangle M_s| = \sum_{s\leq t} |\triangle N_s| + \sum_{s\leq t} |\triangle L_s| = \infty \qquad a.s. \tag{9}$$

11. Continuity: We use notations adopted in Example 2 in section 4 (P33). Assume $E|U_1| < \infty$. By independence of U_i, elementary inequality, Markov inequality, and the property of Poisson process, we observe

$$\begin{aligned}
&\lim_{s\to t} P(|Z_t - Z_s| > \epsilon) = \lim_{s\to t} \sum_k P(|Z_t - Z_s| > \epsilon\,|N_t - N_s = k)P(N_t - N_s = k) \\
\leq &\lim_{s\to t} \sum_k P(\sum_{i=1}^k |U_i| > \epsilon)P(N_t - N_s = k) \leq \lim_{s\to t} \sum_k \left[kP(|U_1| > \frac{\epsilon}{k})\right] P(N_t - N_s = k) \\
\leq &\lim_{s\to t} \frac{E|U_1|}{\epsilon} \sum_k k^2 P(N_t - N_s = k) = \lim_{s\to t} \frac{E|U_1|}{\epsilon}\{\lambda(t-s)\} = 0
\end{aligned} \tag{10}$$

Independent Increment: Let F be a distribution function of U. By using independence of $\{U_k\}_k$ and strong Markov property of N, for arbitrary $t, s : t \geq s \geq 0$,

$$\begin{aligned} &E\left(e^{iu(Z_t-Z_s)+ivZ_s}\right) = E\left(e^{iu\sum_{k=N_s+1}^{N_t}U_k+iv\sum_{k=1}^{N_s}U_k}\right) \\ =&E\left(E\left(e^{iu\sum_{k=N_s+1}^{N_t}U_k+iv\sum_{k=1}^{N_s}U_k}|\mathcal{F}_s\right)\right) \\ =&E\left(e^{iv\sum_{k=1}^{N_s}U_k}E\left(e^{iu\sum_{k=N_s+1}^{N_t}U_k}|\mathcal{F}_s\right)\right) = E\left(e^{iv\sum_{k=1}^{N_s}U_k}E\left(e^{iu\sum_{k=N_s+1}^{N_t}U_k}\right)\right) \\ =&E\left(e^{iv\sum_{k=1}^{N_s}U_k}\right)E\left(e^{iu\sum_{k=N_s+1}^{N_t}U_k}\right) = E\left(e^{ivZ_s}\right)E\left(e^{iu(Z_t-Z_s)}\right). \end{aligned} \tag{11}$$

This shows that Z has independent increments.

Stationary increment: Since $\{U_k\}_k$ are i.i.d and independent of N_t,

$$\begin{aligned} E\left(e^{iuZ_t}\right) =&E\left(E\left(e^{iuZ_t}|N_t\right)\right) = E\left[\left(\int e^{iux}F(dx)\right)^{N(t)}\right] = \sum_{n\geq 0}\frac{(\lambda t)^n e^{-\lambda t}}{n!}\left(\int e^{iux}F(dx)\right)^n \\ =&\exp\left\{-t\lambda\left(\int[1-e^{iux}]F(dx)\right)\right\}. \end{aligned} \tag{12}$$

By (11) and (12), set $v = u$,

$$\begin{aligned} E\left(e^{iu(Z_t-Z_s)}\right) =&E\left(e^{iuZ_t}\right)/E\left(e^{iuZ_s}\right) = \exp\left\{-(t-s)\lambda\left(\int[1-e^{iux}]F(dx)\right)\right\} \\ =&E\left(e^{iu(Z_{t-s})}\right) \end{aligned} \tag{13}$$

Hence Z has stationary increments.

12. By exercise 12, a compound Poisson process is a Lévy process and has independent stationary increments.

$$\begin{aligned} E|Z_t-\lambda tEU_1| \leq&E(E(|Z_t||N_t)) + \lambda tE|U_1| \leq \sum_{n=1}^{\infty}E\left(\sum_{i=1}^{n}|U_i||N_t=n\right)P(N_t=n)+\lambda tE|U_1| \\ =&E|U_1|EN_t+\lambda tE|U_1| = 2\lambda tE|U_1| < \infty, \end{aligned} \tag{14}$$

For $t \geq s$, $E(Z_t|\mathcal{F}_s) = E(Z_t - Z_s + Z_s|\mathcal{F}_s) = Z_s + E(Z_{t-s})$. Since

$$EZ_t = E(E(Z_t|N_t)) = \sum_{n=1}^{\infty}E\left(\sum_{i=1}^{n}U_i|N_t=n\right)P(N_t=n) = \lambda tEU_1 \tag{15}$$

$E(Z_t - EU_1\lambda t|\mathcal{F}_s) = Z_s - EU_1\lambda s$ a.s. and $\{Z_t - EU_1\lambda t\}_{t\geq 0}$ is a martingale.

13. By Lévy decomposition theorem and hypothesis, Z_t can be decomposed as

$$Z_t = \int_{\mathbb{R}} xN_t(\cdot, dx) + t(\alpha - \int_{|x|\geq 1}x\nu(dx)) = Z'_t + \beta t \tag{16}$$

where $Z'_t = \int_{\mathbb{R}} xN_t(\cdot, dx)$, $\beta = \alpha - \int_{|x|\geq 1} x\nu(dx)$. By theorem 43, $E(e^{iuZ'_t}) = \int_{\mathbb{R}}(1-e^{iux})\nu(dx)$. Z'_t is a compound Poisson process (See problem 11). Arrival rate (intensity) is λ since $E(\int_{\mathbb{R}} N_t(\cdot, dx)) = t\int_{\mathbb{R}} \nu(dx) = \lambda t$. Since Z_t is a martingale, $EZ'_t = -\beta t$. Since $EZ'_t = E(\int_{\mathbb{R}} xN_t(\cdot, dx)) = t\int_{\mathbb{R}} x\nu(dx)$, $\beta = -\int_{\mathbb{R}} x\nu(dx)$. It follows that $Z_t = Z'_t - \lambda t\int_{\mathbb{R}} x\frac{\nu}{\lambda}(dx)$ is a compensated compound Poisson process. Then problem 12 shows that $EU_1 = \int_{\mathbb{R}} x\frac{\nu}{\lambda}(dx)$. It follows that the distribution of jumps $\mu = (1/\lambda)\nu$.

14. Suppose $EN_t < \infty$. At first we show that $Z_t \in L^1$.

$$\begin{aligned} E|Z_t| \leq E\left(\sum_{i=1}^{N_t} |U_i|\right) &= \sum_{n=0}^{\infty} E\left(\sum_{i=1}^{n} |U_i|\right) P(N_t = n) = \sum_{n=0}^{\infty}\sum_{i=1}^{n} E\left(|U_i|\right) P(N_t = n) \\ &\leq \sup_i E|U_i| \sum_{n=0}^{\infty} nP(N_t = n) = EN_t \sup_i E|U_i| < \infty. \end{aligned} \tag{17}$$

Then

$$\begin{aligned} E(Z_t|\mathcal{F}_s) &= E(Z_s + \sum_{i=1}^{\infty} U_i 1_{\{s<T_i\leq t\}}|\mathcal{F}_s) = Z_s + E(\sum_{i=1}^{\infty} U_i 1_{\{s<T_i\leq t\}}|\mathcal{F}_s) \\ &= Z_s + \sum_{i=1}^{\infty} E(U_i E(\{1_{s<T_i\leq t\}}|\mathcal{F}_s \vee \sigma(U_i : i \geq 1))|\mathcal{F}_s) = Z_s + \sum_{i=1}^{\infty} E(U_i E(1_{\{s<T_i\leq t\}}|\mathcal{F}_s)|\mathcal{F}_s) \\ &= Z_s + \sum_{i=1}^{\infty} E(U_i|\mathcal{F}_s)E(1_{\{T_i\leq t\}}|\mathcal{F}_s)1_{\{T_i>s\}} = Z_s, \qquad a.s. \end{aligned} \tag{18}$$

since $E(U_i|\mathcal{F}_s)1_{\{T_i>s\}} = 0$ a.s. Note that if we drop the assumption $EN_t < \infty$, Z_t is not a martingale in general. (Though it is a local martingale.)

15. By Lévy decomposition theorem (theorem 42),

$$Z_t = \int_{|x|<1} x(N_t(\cdot, dx) - t\nu(dx)) + \alpha t + \sum_{0<s\leq t} \triangle X_s 1_{\{|\triangle X_s|\geq 1\}}. \tag{19}$$

The first term in right side is a martingale (theorem 41). Then $\alpha = -\int_{|x|>1} x\nu(dx)$ since Z_t is a martingale. Note that α is well defined by a condition $\sum \beta_k^2\alpha_k < \infty$. Hence $Z_t = \int_{\mathbb{R}} x(N_t(\cdot, dx) - t\nu(dx))$. Let $A = \mathbb{R}\backslash\{\beta_k\}_k$. Then $EN_1^A = \int_A \nu(dx) = 0$ by theorem 38. Therefore mass of $N(\cdot, dx)$ is concentrated on countable set $\{\beta_k\}_k$. Fix $k \geq 1$ and let N_t^k denote $N_t(\cdot, \beta_k)$. Then N_t^k is a Poisson process and its rate is α_k since by theorem 38, $EN_1^k = \int_{\{\beta_k\}} \nu(dx) = \alpha_k$. Therefore,

$$Z_t = \beta_k(N_t^k - \alpha_k t) \tag{20}$$

To verify that $Z_t \in L^2$, we observe that $E\left(\sum_{k=m}^{n} \beta_k(N_k^t - \alpha_k t)\right)^2 = t\sum_{k=m}^{n} \beta_k^2\alpha_k \to 0$ as $n, m \to \infty$since $\sum_{k=1}^{\infty} \beta_k^2 < \infty$. Therefore, Z_t is a Cauchy limit in L^2. Since L^2-space is complete, $Z_t \in L^2$.

16. Let $\mathcal{F}_t$ be natural filtration of B_t satisfying usual hypothesis. By stationary increments property of Brownian motion and symmetry,

$$W_t = B_{1-t} - B_1 \stackrel{d}{=} B_1 - B_{1-t} \stackrel{d}{=} B_{1-(1-t)} = B_t \tag{21}$$

This shows W_t is Gaussian. W_t has stationary increments because $W_t - W_s = B_{1-s} - B_{1-t} \stackrel{d}{=} B_{t-s}$. Let $\mathcal{G}_t$ be a natural filtration of W_t. Then $\mathcal{G}_t = \sigma(-(B_1 - B_{1-s}) : 0 \leq s \leq t)$. By independent increments property of B_t, $\mathcal{G}_t$ is independent of $\mathcal{F}_{1-t}$. For $s > t$, $W_s - W_t = -(B_{1-t} - B_{1-s}) \in \mathcal{F}_{1-t}$ and hence independent of $\mathcal{G}_t$.

17. a) Fix $\varepsilon > 0$ and ω such that $X.(\omega)$ has a sample path which is right continuous, with left limits. Suppose there exists infinitely many jumps larger than ε at time $\{s_n\}_{n\geq 1} \in [0,t]$. (If there are uncountably many such jumps, we can arbitrarily choose countably many of them.) Since $[0,t]$ is compact, there exists a subsequence $\{s_{n_k}\}_{k\geq 1}$ converging to a cluster point $s_* \in [0,1]$. Clearly we can take further subsequence converging to $s^* \in [0,1]$ monotonically either from above or from below. To simplify notations, Suppose $\exists \{s_n\}_{n\geq 1} \uparrow s^*$. (The other case $\{s_n\}_{n\geq 1} \downarrow s^*$ is similar.) By left continuity of X_t, there exists $\delta > 0$ such that $s \in (s^* - \delta, s^*)$ implies $|X_s - X_{s^*-}| < \varepsilon/3$ and $|X_{s-} - X_{s^*-}| < \varepsilon/3$. However for $s_n \in (s^* - \delta, s^*)$,

$$|X_{s_n} - X_{s^*-}| = |X_{s_n-} - X_{s^*-} + \triangle X_{s_n}| \geq |\triangle X_{s_n}| - |X_{s_n-} - X_{s^*-}| > \frac{2\varepsilon}{3} \tag{22}$$

This is a contradiction and the claim is shown.

b) By a), for each n there is a finitely many jumps of size larger than $1/n$. But $J = \{s \in [0,t] : |\triangle X_s| > 0\} = \cup_{n=1}^{\infty} \{s \in [0,t] : |\triangle X_s| > 1/n\}$. We see that cardinality of J is countable.

18. By corollary to theorem 36 and theorem 37, we can immediately see that J^ε and $Z - J^\varepsilon$ are Lévy processes. By Lévy -Khintchine formula (theorem 43), we can see that $\psi_{J^\varepsilon}\psi_{Z-J^\varepsilon} = \psi_Z$. Thus J^ε and $Z - J^\varepsilon$ are independent. (For an alternative rigorous solution without Lévy -Khintchine formula, see a proof of theorem 36 .)

19. Let $T_n = \inf\{t > 0 : |X_t| > n\}$. Then T_n is a stopping time. Let $S_n = T_n 1_{\{X_0 \leq n\}}$. We have $\{S_n \leq t\} = \{T_n \leq t, X_0 \leq n\} \cup \{X_0 > n\} = \{T_n \leq t\} \cup \{X_0 > n\} \in \mathcal{F}_t$ and S_n is a stopping time. Since X is continuous, $S_n \to \infty$ and $X^{S_n} 1_{\{S_n > 0\}} \leq n$, $\forall n \geq 1$. Therefore X is locally bounded.

20. Let H be an arbitrary unbounded random variable. (e.g. Normal) and let T be a positive random variable independent of H such that $P(T \geq t) > 0$, $\forall t > 0$ and $P(T < \infty) = 1$. (e.g. Exponential). Define a process $Z_t = H 1_{\{T \geq t\}}$. Z_t is a càdlàg process and adapted to its natural filtration with $Z_0 = 0$. Suppose there exists a sequence of stopping times $T_n \uparrow \infty$ such that Z^{T_n} is bounded by some $K_n \in \mathbb{R}$. Observe that

$$Z_t^{T_n} = \begin{cases} H & T_n \geq T > t \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

Since Z^{T_n} is bounded by K_n, $P(Z^{T_n} > K_n) = P(H > K_n)P(T_n \geq T > t) = 0$. It follows that $P(T_n \geq T > t) = 0$, $\forall n$ and hence $P(T_n \leq T) = 1$. Moreover $P(\cap_n \{T_n \leq T\}) = P(T = \infty) = 1$. This is a contradiction.

21. a) let $a = (1-t)^{-1}(\int_t^1 Y(s)ds)$ and Let $M_t = Y(\omega)1_{(0,t)}(\omega) + a1_{[t,1)}(\omega)$. For arbitrary $B \in \mathcal{B}([0,1])$, $\{\omega : M_t \in B\} = ((0,t) \cap \{Y \in B\}) \cup (\{a \in B\} \cap (t,1))$. $(0,t) \cap \{Y \in B\} \subset (0,t)$ and hence in $\mathcal{F}_t$. $\{a \in B\} \cap (t,1)$ is either $(t,1)$ or $\emptyset$ depending on B and in either case in $\mathcal{F}_t$. Therefore M_t is adapted.
Pick $A \in \mathcal{F}_t$. Suppose $A \subset (0,t)$. Then clearly $E(M_t : A) = E(Y : A)$. Suppose $A \supset (t,1)$. Then

$$\begin{aligned} E(M_t : A) &= E(Y : A \cap (0,t)) + E\left(\frac{1}{1-t}\int_t^1 Y(s)ds : (t,1)\right) \\ &= E(Y : A \cap (0,t)) + \int_t^1 Y(s)ds = E(Y : A \cap (0,t)) + E(Y : (t,1)) = E(Y : A). \end{aligned} \tag{24}$$

Therefore for $\forall A \in \mathcal{F}_t$, $E(M_t : A) = E(Y : A)$ and hence $M_t = E(Y|\mathcal{F}_t)$ a.s.

b) By simple calculation $EY^2 = 1/(1-2\alpha) < \infty$ and $Y \in L^2 \subset L^1$. It is also straightforward to compute $M_t = (1-t)^{-1}\int_t^1 Y(s)ds = (1-\alpha)^{-1}Y(t)$ for $\omega \in (t,1)$.

c) From b), $M_t(\omega) = Y(\omega)1_{(0,t)}(\omega) + 1/(1-\alpha)^{-1}Y(t)1_{(t,1)}(\omega)$. Fix $\omega \in (0,1)$. Since $0 < \alpha < 1/2$, $Y/(1-\alpha) > Y$ and Y is a increasing on $(0,1)$. Therefore,

$$\sup_{0<t<1} M_t = \left(\sup_{0<t<\omega} \frac{Y(t)}{1-\alpha}\right) \vee \left(\sup_{\omega \le t<1} Y(\omega)\right) = \frac{Y(\omega)}{1-\alpha} \vee Y(\omega) = \frac{Y(\omega)}{1-\alpha} \tag{25}$$

For each $\omega \in (0,1)$, $M_t(\omega) = Y(\omega)$ for all $t \ge \omega$ and especially $M_\infty(\omega) = Y(\omega)$. Therefore

$$\|\sup_t M\|_{L^2} = \frac{1}{1-\alpha}\|Y\|_{L^2} = \frac{1}{1-\alpha}\|M_\infty\|_{L^2} \tag{26}$$

22. a) By simple computation, $\frac{1}{1-t}\int_t^1 s^{-1/2}ds = 2/(1+\sqrt{t})$ and claim clearly holds.

b) Since T is a stopping time, $\{T > \varepsilon\} \in \mathcal{F}_\varepsilon$ and hence $\{T > \varepsilon\} \subset (0,\varepsilon]$ or $\{T > \varepsilon\} \supset (\varepsilon,1)$ for any $\varepsilon > 0$. Assume $\{T > \varepsilon\} \subset (0,\varepsilon]$ for all $\varepsilon > 0$. Then $T(\omega) \le \varepsilon$ on $(\varepsilon,1)$ for all $\varepsilon > 0$ and it follows that $T \equiv 0$. This is contradiction. Therefore there exists ε_0 such that $\{T > \varepsilon_0\} \supset (\varepsilon_0,1)$. Fix $\varepsilon \in (0,\varepsilon_0)$. Then $\{T > \varepsilon\} \supset \{T > \varepsilon_0\} \supset (\varepsilon_0,1)$. On the other hand, since T is a stopping time, $\{T > \varepsilon\} \subset (0,\varepsilon]$ or $\{T > \varepsilon\} \supset (\varepsilon,1)$. Combining these observations, we know that $\{T > \varepsilon\} \supset (\varepsilon,1)$ for all $\varepsilon \in (0,\varepsilon_0)$. $\forall \varepsilon \in (0,\varepsilon_0)$, there exists $\delta > 0$ such that $\varepsilon - \delta > 0$ and $\{T > \varepsilon - \delta\} \supset (\varepsilon - \delta, 1)$. Especially $T(\varepsilon) > \varepsilon - \delta$. Taking a limit of $\delta \downarrow 0$, we observe $T(\varepsilon) \ge \varepsilon$ on $(0,\varepsilon_0)$.

c) Using the notation in b), $T(\omega) \ge \omega$ on $(0,\varepsilon_0)$ and hence $M_T(\omega) = \omega^{-1/2}$ on $(0,\varepsilon_0)$. Therefore, $EM_T^2 \ge E(1/\omega : (0,\varepsilon_0)) = \int_0^{\varepsilon_0} \omega^{-1}d\omega = \infty$. Since this is true for all stopping times not identically equal to 0, M cannot be locally a L^2 martingale.

d) By a), $|M_t(\omega)| \le \omega^{-1/2} \vee 2$ and M has bounded path for each ω. If $M_T 1_{\{T>0\}}$ were a bounded random variable, then M_T would be bounded as well since $M_0 = 2$. However, from c) $M_T \notin L^2$ unless $T \equiv 0$ a.s. and hence $M_T 1_{\{T>0\}}$ is unbounded.

23. Let M be a positive local martingale and $\{T_n\}$ be its fundamental sequence. Then for $t \ge s \ge 0$, $E(M_t^{T_n}1_{\{T_n>0\}}|\mathcal{F}_s) = M_s^{T_n}1_{\{T_n>0\}}$. By applying Fatou's lemma, $E(M_t|\mathcal{F}_s) \le M_s$ a.s. Therefore a positive local martingale is a supermartingale. By Doob's supermartingale convergence theorem, positive supermartingale converges almost surely to $X_\infty \in L^1$ and closable. Then by Doob's optional stopping theorem $E(M_T|\mathcal{F}_S) \leq M_S$ a.s. for all stopping times $S \leq T < \infty$. If equality holds in the last inequality for all $S \leq T$, clearly M is a martingale since deterministic times $0 \leq s \leq t$ are also stopping time. Therefore any positive *honest* local martingale makes a desired example. For a concrete example, see example at the beginning of section 5, chapter 1. (p. 37)

24. a) To simplify a notation, set $\mathcal{F}^1 = \sigma(Z^{T-}, T) \vee \mathcal{N}$. At first, we show $\mathcal{G}_{T-} \supset \mathcal{F}^1$. Since $\mathcal{N} \subset \mathcal{G}_0$, $\mathcal{N} \subset \mathcal{G}_{T-}$. Clearly $T \in \mathcal{G}_{T-}$ since $\{T \leq t\} = (\Omega \cap \{t < T\})^c \in \mathcal{G}_{T-}$.

$\forall t > 0$, $\{Z_t^{T-} \in B\} = (\{Z_t \in B\} \cap \{t < T\}) \cup (\{Z_{T-} \in B\} \cap \{t \geq T\} \cap \{T < \infty\})$. $\{Z_t \in B\} \cap \{t < T\} \in \mathcal{G}_{T-}$ by definition of $\mathcal{G}_{T-}$. Define a mapping $f : \{T < \infty\} \to \{T < \infty\} \times \mathbb{R}_+$ by $f(\omega) = (\omega, T(\omega))$. Define a σ-algebra $\mathcal{P}^1$ consists of subsets of $\Omega \times \mathbb{R}_+$ that makes all left continuous processes with right limits (càglàd processes) measurable. Especially $\{Z_{t-}\}_{t\geq 0}$ is $\mathcal{P}$-measurable i.e, $\{(\omega, t) : Z_t(\omega) \in B\} \in \mathcal{P}$. Without proof, we cite a fact about $\mathcal{P}$. Namely, $f^{-1}(\mathcal{P}) = \mathcal{G}_{T-} \cap \{T < \infty\}$. Since $Z_{T(\omega)-}(\omega) = Z_- \circ f(\omega)$, $\{Z_{T-} \in B\} \cap \{T < \infty\} = f^{-1}(\{(\omega, t) : Z_t(\omega) \in B\}) \cap \{T < \infty\} \in \mathcal{G}_{T-} \cap \{T < \infty\}$. Note that $\{T < \infty\} = \cap_n \{n < T\} \in \mathcal{G}_{T-}$. Then $Z^{T-} \in \mathcal{G}_{T-}$ and $\mathcal{G}_{T-} \supset \mathcal{F}^1$.

Next We show $\mathcal{G}_{T-} \subset \mathcal{F}^1$. $\mathcal{G}_0 \subset \mathcal{F}^1$, since $Z_0^{T-} = Z_0$. Fix $t > 0$. We want to show that for all $A \in \mathcal{G}_t$, $A \cap \{t < T\} \in \mathcal{F}^1$. Let $\Lambda = \{A : A \cap \{t < T\} \in \mathcal{F}^1\}$. Let

$$\Pi = \{\cap_{i=1}^n \{Z_{s_i} \leq x_i\} : n \in \mathbb{N}_+, 0 \leq s_i \leq t, x_i \in \mathbb{R}\} \cup \mathcal{N} \tag{27}$$

Then Π is a π-system and $\sigma(\Pi) = \mathcal{G}_t$. Observe $\mathcal{N} \subset \mathcal{F}^1$ and

$$(\cap_{i=1}^n \{Z_{s_i} \leq x_i\}) \cap \{t < T\} = (\cap_{i=1}^n \{Z_{s_i}^{T-} \leq x_i\}) \cap \{t < T\} \in \mathcal{F}^1, \tag{28}$$

$\Pi \subset \Lambda$. By Dynkin's theorem ($\pi - \lambda$ theorem), $\mathcal{G}_t = \sigma(\Pi) \subset \Lambda$ hence the claim is shown.

b) To simplify the notation, let $\mathcal{H} \triangleq \sigma(T, Z^T) \wedge \mathcal{N}$. Then clearly $H \subset \mathcal{G}_T$ since $\mathcal{N} \subset \mathcal{G}_T$, $T \in \mathcal{G}_T$, and $Z_t^T \in \mathcal{G}_T$ for all t. It suffices to show that $\mathcal{G}_T \subset \mathcal{H}$. Let

$$\mathcal{L} = \Big\{ A \in \mathcal{G}_\infty : E[1_A|\mathcal{G}_T] = E[1_A|\mathcal{H}] \Big\} \tag{29}$$

Observe that $\mathcal{L}$ is a λ-system (Dynkin's system) and contains all the null set since so does $\mathcal{G}_\infty$. Let

$$\mathcal{C} = \left\{ \bigcap_{j=1}^n \{Z_{t_j} \in B_j\} : n \in \mathbb{N}, t_j \in [0, \infty), B_j \in \mathcal{B}(\mathbb{R}) \right\}. \tag{30}$$

Then $\mathcal{C}$ is a π-system such that $\sigma(\mathcal{C}) \vee \mathcal{N} = \mathcal{G}_\infty$. Therefore by Dynkin's theorem, provided that $\mathcal{C} \subset \mathcal{L}$, $\sigma(\mathcal{C}) \subset \mathcal{L}$ and thus $\mathcal{G} \subset \mathcal{L}$. For arbitrary $A \in \mathcal{G}_T \subset \mathcal{G}_\infty$, $1_A = E[1_A|\mathcal{G}_T] = E[1_A|\mathcal{H}] \in \mathcal{H}$ and hence $A \in \mathcal{H}$.

It remains to show that $\mathcal{C} \subset \mathcal{L}$. Fix $n \in \mathbb{N}$. Since $\mathcal{H} \subset \mathcal{G}_T$, it suffices to show that

$$E\Bigg[\prod_{j=1}^n 1_{\{Z_{t_j} \in B_j\}} \Bigg| \mathcal{G}_T\Bigg] \in \mathcal{H}. \tag{31}$$

[1] $\mathcal{P}$ is called a predictable σ-algebra. Its definition and properties will be discussed in chapter 3.

For this, let $t_0 = 0$ and $t_{n+1} = \infty$ and write

$$E\left[\prod_{j=1}^{n} 1_{\{Z_{t_j}\in B_j\}}\middle|\mathcal{G}_T\right] = \sum_{k=1}^{n+1} 1_{\{T\in[t_{k-1},t_k)\}} \prod_{j<k} 1_{\{Z_{t_j\wedge T}\in B_j\}} E\left[\prod_{j\geq k} 1_{\{Z_{t_j\vee T}\in B_j\}}\middle|\mathcal{G}_T\right].$$

Let $\xi \triangleq \prod_{j\geq k} 1_{\{Z_{t_j\vee T-T}\in B_j\}} \in \mathcal{G}_\infty$. Then by the strong Markov property of Z,

$$E\left[\prod_{j\geq k} 1_{\{Z_{t_j\vee T}\in B_j\}}\middle|\mathcal{G}_T\right] = E\left[\xi\circ\theta_T\middle|\mathcal{G}_T\right] = E_{Z_T}[\xi]. \tag{32}$$

This verifies (31) and completes the proof. (This solution is by Jason Swanson).

c) Since $\mathcal{G}_{T-}\subset\mathcal{G}_T$ and $Z_T 1_{\{T<\infty\}}\in\mathcal{G}_T$, $\sigma\{\mathcal{G}_{T-}, Z_T\}\subset\mathcal{G}_T$ if $T<\infty$ a.s. To show the converse, observe that $Z_t^T = Z_t^{T-} + \triangle Z_T 1_{\{t\geq T\}}$. Since Z_t^{T-}, Z_{T-}, $1_{\{t\geq T\}}\in\mathcal{G}_{T-}$ for all $t\geq 0$, $Z_t^T\in\sigma(\mathcal{G}_{T-}, Z_T)$ for all $t\geq 0$. Therefore $\mathcal{G}_T = \sigma(\mathcal{G}_{T-}, Z_T)$.

d) Since $\mathcal{N}\subset\mathcal{G}_{T-}$, $Z_T = Z_{T-}$ a.s. for $T<\infty$ implies $\mathcal{G}_T = \sigma(\mathcal{G}_{T-}, Z_T) = \sigma(\mathcal{G}_{T-}) = \mathcal{G}_{T-}$.

25. Let Z be a Lévy process. By definition Lévy process is continuous in probability, i.e. $\forall t$, $\lim_n P(|Z_t - Z_{T-1/n}| > \varepsilon) = 0$. $\forall\varepsilon>0$, $\forall t>0$, $\{|\triangle Z_t|>\varepsilon\} = \cup_n\cap_{n\geq m}\{|Z_t - Z_{T-1/n}|>\varepsilon\}$. Therefore,

$$P(|\triangle Z_t|>\varepsilon) \leq \liminf_{n\to\infty} P(|Z_t - Z_{T-1/n}|>\varepsilon) = 0 \tag{33}$$

Since this is true for all $\varepsilon>0$, $P(|\triangle Z_t|>0)=0$, $\forall t$.

26. To apply results of exercise 24 and 25, we first show following almost trivial lemma.
Lemma Let T be a stopping time and $t\in\mathbb{R}_+$. If $T\equiv t$, then $\mathcal{G}_T = \mathcal{G}_t$ and $\mathcal{G}_{T-} = \mathcal{G}_{t-}$.

Proof. $\mathcal{G}_{T-} = \{A\cap\{T>s\} : A\in\mathcal{G}_s\} = \{A\cap\{t>s\} : A\in\mathcal{G}_s\} = \{A : A\in\mathcal{G}_s,\ s<t\} = \mathcal{G}_{t-}$. Fix $A\in\mathcal{G}_t$. $A\cap\{t\leq s\} = A\in\mathcal{G}_t\subset\mathcal{G}_s$ $(t\leq s)$, or $\emptyset\in\mathcal{G}_s$ $(t>s)$ and hence $\mathcal{G}_t\subset\mathcal{G}_T$. Fix $A\in\mathcal{G}_T$, $A\cap\{t\leq s\}\in\mathcal{G}_s$, $\forall s>0$. Especially, $A\cap\{t\leq t\} = A\in\mathcal{G}_t$ and $\mathcal{G}_T\subset\mathcal{G}_t$. Therefore $\mathcal{G}_T = \mathcal{G}_t$. □

Fix $t>0$. By exercise 25, $Z_t = Z_{t-}$ a.s.. By exercise 24 (d), $\mathcal{G}_t = \mathcal{G}_{t-}$ since t is a bounded stopping time.

27. Let $A\in\mathcal{F}_t$. Then $A\cap\{t<S\} = (A\cap\{t<S\})\cap\{t<T\}\in\mathcal{F}_{T-}$ since $A\cap\{t<S\}\in\mathcal{F}_t$. Then $\mathcal{F}_{S-}\subset\mathcal{F}_{T-}$.

Since $T_n\leq T$, $\mathcal{F}_{T_n-}\subset\mathcal{F}_{T-}$ for all n as shown above. Therefore, $\vee_n\mathcal{F}_{T_n-}\subset\mathcal{F}_{T-}$. Let $A\in\mathcal{F}_t$. $A\cap\{t<T\} = \cup_n(A\cap\{t<T_n\})\in\vee_n\mathcal{F}_{T_n-}$ and $\mathcal{F}_{T-}\subset\vee_n\mathcal{F}_{T_n-}$. Therefore, $\mathcal{F}_{T-} = \vee_n\mathcal{F}_{T_n-}$.

28. Observe that the equation in theorem 38 depends only on the existence of a sequence of simple functions approximation $f1_\Lambda\geq 0$ and a convergence of both sides in $E\{\sum_j a_j N_j^{\Lambda_j}\} = t\sum_j a_j\nu(\Lambda_j)$. For this, $f1_\Lambda\in L^1$ is enough. (Note that we need $f1_\Lambda\in L^2$ to show the second equation.)

29. Let M_t be a Lévy process and local martingale. M_t has a representation of the form

$$M_t = B_t + \int_{\{|x|\le 1\}} x\,[N_t(\cdot,dx) - t\nu(dx)] + \alpha t + \int_{\{|x|>1\}} xN_t(\cdot,dx) \tag{34}$$

First two terms are martingales. Therefore WLOG, we can assume that

$$M_t = \alpha t + \int_{\{|x|>1\}} xN_t(\cdot,dx) \tag{35}$$

M_t has only finitely many jumps on each interval $[0,t]$ by exercise 17 (a). Let $\{T_n\}_{n\ge1}$ be a sequence of jump times of M_t. Then $P(T_n<\infty)=1$, $\forall n$ and $T_n\uparrow\infty$. We can express M_t by a sum of compound Poisson process and a drift term (See Example in P.33):

$$M_t = \sum_{i=1}^{\infty} U_i 1_{\{t\ge T_i\}} - \alpha t. \tag{36}$$

Since M_t is local martingale by hypothesis, M is a martingale if and only if $M_t\in L^1$ and $E[M_t]=0$, $\forall t$. There exists a fundamental sequence $\{S_n\}$ such that M^{S_n} is a martingale, $\forall n$. WLOG, we can assume that M^{S_n} is uniformly integrable. If $U_1\in L^1$, $M_t\in L^1$ for every α and M_t becomes martingale with $\alpha_* = EN_tEU_1/t$. Furthermore, for any other α, M_t can't be local martingale since for any stopping time T, $M_t^T = L_t^T + (\alpha_*-\alpha)(t\wedge T)$ where L_t is a martingale given by $\alpha=\alpha_*$ and M_t^T can't be a martingale if $\alpha\ne\alpha_*$. It follows that if M_t is only a local martingale, $U_1\notin L^1$.

By exercise 24,

$$\mathcal{F}_{T_1-} = \sigma(M^{T_1-},T_1)\vee\mathcal{N} = \sigma(T_1)\vee\mathcal{N} \tag{37}$$

since $M_t^{T_1-} = M_t1_{\{t<T_1\}} + M_{T_1-}1_{\{t\ge T_1\}} = -\alpha(t\wedge T_1)$ and $M_\infty^{T_1-} = -\alpha T_1$. Then

$$\{S_n<T_1\} = \cup_{r\in\mathbb{Q}_+}\{S_n\le r\}\cap\{r<T_1\} \in \mathcal{F}_{T_1-} = \sigma(T_1)\vee\mathcal{N} \tag{38}$$

Therefore,

$$\begin{aligned} E|M_{S_n\wedge T_1}| =& E|U_1 1_{\{S_n\ge T_1\}} - \alpha(S_n\wedge T_1)| \ge E|U_1 1_{\{S_n\ge T_1\}}| - \alpha E|S_n\wedge T_1| \\ =& E[E(|U_1|1_{\{S_n\ge T_1\}}|\sigma(T_1)\vee\mathcal{N})] - \alpha E|S_n\wedge T_1| \\ =& E[1_{\{S_n\ge T_1\}}E(|U_1||\sigma(T_1)\vee\mathcal{N})] - \alpha E|S_n\wedge T_1| \\ \ge& E|U_1|P(S_n\ge T_1) - \alpha ET_1 = \infty \end{aligned} \tag{39}$$

This is a contradiction. Therefore, M is a martingale.

30. Let $T_z = \inf\{s>0 : Z_t\ge z\}$. Then T_z is a stopping time and $P(T_z<\infty)=1$. Let's define a coordinate map $\omega(t) = Z_t(\omega)$. Let $R = \inf\{s<t\ :\ Z_s\ge z\}$. We let

$$Y_s(\omega) = \begin{cases} 1 & s\le t\ ,\ \omega(t-s) < z-y \\ 0 & \text{otherwise} \end{cases}\quad ,\ Y_s'(\omega) = \begin{cases} 1 & s\le t\ ,\ \omega(t-s) > z+y \\ 0 & \text{otherwise} \end{cases} \tag{40}$$

So that

$$Y_R \circ \theta_R(\omega) = \begin{cases} 1 & R \le t\ ,\ Z_t < z - y \\ 0 & \text{otherwise} \end{cases} \quad , \ Y'_R \circ \theta_R(\omega) = \begin{cases} 1 & R \le t\ ,\ Z_t > z + y \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

Strong Markov property implies that on $\{R < \infty\} = \{T_z \le t\} = \{S_t \ge z\}$,

$$E_0(Y_R \circ \theta_R | \mathcal{F}_R) = E_{Z_R} Y_R, \qquad E_0(Y'_R \circ \theta_R | \mathcal{F}_R) = E_{Z_R} Y'_R \tag{42}$$

Since $Z_R \ge z$ and Z is symmetric,

$$E_a Y_s = E_a 1_{\{Z_{t-s} < z-y\}} < E_a 1_{\{Z_{t-s} > z+y\}} = E_a Y'_s, \qquad \forall a \ge z,\ s < t. \tag{43}$$

By taking expectation,

$$\begin{aligned} &P_0(T_z \le t, Z_t < z - y) = E(E_0(Y_R \circ \theta_R | \mathcal{F}_R) : R < \infty) = E(E_{Z_R} Y_R : R < \infty) \\ \le &E(E_{Z_R} Y'_R : R < \infty) = E(E_0(Y'_R \circ \theta_R | \mathcal{F}_R) : R < \infty) = P_0(T_z \le t, Z_t > z + y) \\ = &P(Z_t > z + y) \end{aligned} \tag{44}$$

31. We define T_z, R, $\omega(t)$ as in exercise 30. We let

$$Y_s(\omega) = \begin{cases} 1 & s \le t\ ,\ \omega(t-s) \ge z \\ 0 & \text{otherwise} \end{cases} \tag{45}$$

Since $Z_R \ge z$ and Z is symmetric,

$$E_a Y_s = E_a 1_{\{Z_{t-s} \ge z\}} \ge \frac{1}{2} \qquad \forall a \ge z,\ s < t. \tag{46}$$

By the same reasoning as in exercise 30, taking expectation yields

$$\begin{aligned} &P_0(Z_t \ge z) = P_0(T_z \le t, Z_t \ge z) = E(E_0(Y_R \circ \theta_R | \mathcal{F}_R) : R < \infty) = E(E_{Z_R} Y_R : R < \infty) \\ \ge &E(\frac{1}{2} : R < \infty) = \frac{1}{2} P(R < \infty) = \frac{1}{2} P(S_t \ge z) \end{aligned} \tag{47}$$

Chapter 2. Semimartingales and Stochastic Integrals

1. Let $x_0 \in \mathbb{R}$ be a discontinuous point of f. Wlog, we can assume that f is a right continuous function with left limit and $\triangle f(x_0) = d > 0$. Since $\inf_t\{B_t = x_0\} < \infty$ and due to Strong Markov property of B_t, we can assume $x_0 = 0$. Almost every Brownian path does not have point of decrease (or increase) and it is a continuous process. So $B.$ visit $x_0 = 0$ and changes its sign infinitely many times on $[0, \epsilon]$ for any $\epsilon > 0$. Therefore, $Y_t = f(B_t)$ has infinitely many jumps on any compact interval almost surely. Therefore,

$$\sum_{s \le t} (\triangle Y_s)^2 = \infty \qquad a.s.$$

2. By dominated convergence theorem,

$$\lim_{n\to\infty} E_Q[|X_n - X| \wedge 1] = \lim_{n\to\infty} E_P\left[|X_n - X| \wedge 1 \cdot \frac{dQ}{dP}\right] = 0$$

$|X_n - X| \wedge 1 \to 0$ in $L^1(Q)$ implies that $X_n \to X$ in Q-probability.

3. B_t is a continuous local martingale by construction and by independence between X and Y,

$$[B, B]_t = \alpha^2 [X, X]_t + (1 - \alpha^2)[Y, Y]_t = t$$

So by Lévy theorem, B is a standard 1-dimensional Brownian motion.

$$[X, B]_t = \alpha t, \qquad [Y, B]_t = \sqrt{1 - \alpha^2} t$$

4. Assume that $f(0) = 0$. Let $M_t = B_{f(t)}$ and $\mathcal{G}_t = \mathcal{F}_{f(t)}$ where $B.$ is a one-dimensional standard Brownian motion and $\mathcal{F}_t$ is a corresponding filtration. Then

$$E[M_t|\mathcal{G}_s] = E\left[B_{f(t)}|\mathcal{F}_{f(s)}\right] = B_{f(s)} = M_s$$

Therefore M_t is a martingale w.r.t. $\mathcal{G}_t$. Since f is continuous and B_t is a continuous process, M_t is clearly continuous process. Finally,

$$[M, M]_t = [B, B]_{f(t)} = f(t)$$

If $f(0) > 0$, then we only need to add a constant process A_t to $B_{f(t)}$ such that $2A_t M_0 + A_t^2 = -B_{f(0)}^2$ for each ω to get a desired result.

5. Since B is a continuous process with $\triangle B_0 = 0$, M is also continuous. M is local martingale since B is a locally square integrable local martingale.

$$[M, M]_t = \int_0^t H_s^2 ds = \int_0^t 1 ds = t$$

So by Lévy 's characterization theorem, M is also a Brownian motion.

6. Pick arbitrary $t_0 > 1$. Let $X^n_t = 1_{(t_0 - \frac{1}{n}, \infty)}(t)$ for $n \geq 1$, $X_t = 1_{[t_0, \infty)}$, $Y_t = 1_{[t_0,\infty)}$. Then X^n, X, Y are finite variation processes and Semimartingales. $\lim_n X^n_t = X_t$ almost surely. But

$$\lim_n [X^n, Y]_{t_0} = 0 \neq 1 = [X, Y]_{t_0}$$

7. Observe that

$$[X^n, Z] = [H^n, Y \cdot Z] = H^n \cdot [Y, Z], \qquad [X, Z] = [H, Y \cdot Z] = H \cdot [Y, Z]$$

and $[Y, Z]$ is a semimartingale. Then by the continuity of stochastic integral, $H^n \to H$ in ucp implies, $H^n \cdot [Y, Z] \to H \cdot [Y, Z]$ and hence $[X^n, Z] \to [XZ]$ in ucp.

8. By applying Ito's formula,

$$\begin{aligned}
[f_n(X) - f(X), Y]_t =& [f_n(X_0) - f(X_0), Y]_t + \int_0^t (f_n'(X_s) - f'(X_s)) d[X, Y]_s \\
&+ \frac{1}{2} \int_0^t (f"_n(X_s) - f"_n(X_s)) d[[X, X], Y]_s \\
=& (f_n(X_0) - f(X_0)) Y_0 + \int_0^t (f_n'(X_s) - f'(X_s)) d[X, Y]_s
\end{aligned}$$

Note that $[[X, X], Y] \equiv 0$ since X and Y are continuous semimartingales and especially $[X, X]$ is a finite variation process. As $n \to \infty$, $(f_n(X_0) - f(X_0))Y_0 \to 0$ for arbitrary $\omega \in \Omega$. Since $[X, Y]$ has a path of finite variation on compacts, we can treat $f_n(X) \cdot [X, Y]$, $f(X) \cdot [X, Y]$ as Lebesgue-Stieltjes integral computed path by path. So fix $\omega \in \Omega$. Then as $n \to \infty$,

$$\sup_{0 \leq s \leq t} |f_n(B_s) - f(B_s)| = \sup_{\inf_{s \leq t} B_s \leq x \leq \sup_{s \leq t} B_s} |f_n(x) - f(x)| \to 0$$

since $f_n' \to f$ uniformly on compacts. Therefore

$$\left| \int_0^t (f_n'(X_s) - f'(X_s)) d[X, Y]_s \right| \leq \sup_{0 \leq s \leq t} |f_n(B_s) - f(B_s)| \int_0^t d|[X, Y]|_s \longrightarrow 0$$

9. Let σ_n be a sequence of random partition tending to identity. By theorem 22,

$$\begin{aligned}
[M, A] &= M_0 A_0 + \lim_{n \to \infty} \sum_i \left(M^{T^n_{i+1}} - M^{T^n_i} \right) \left(A^{T^n_{i+1}} - A^{T^n_i} \right) \\
&\leq 0 + \lim_{n \to \infty} \sup_i \left| M^{T^n_{i+1}} - M^{T^n_i} \right| \sum_i \left| A^{T^n_{i+1}} - A^{T^n_i} \right| = 0
\end{aligned}$$

since M is a continuous process and $\sum_i \left| A^{T^n_{i+1}} - A^{T^n_i} \right| < \infty$ by hypothesis. Similarly

$$\begin{aligned}
[A, A] &= M_0^2 + \lim_{n \to \infty} \sum_i \left(A^{T^n_{i+1}} - A^{T^n_i} \right) \left(A^{T^n_{i+1}} - A^{T^n_i} \right) \\
&\leq 0 + \lim_{n \to \infty} \sup_i \left| A^{T^n_{i+1}} - A^{T^n_i} \right| \sum_i \left| A^{T^n_{i+1}} - A^{T^n_i} \right| = 0
\end{aligned}$$

Therefore,

$$[X, X] = [M, M] + 2[M, A] + [A, A] = [M, M]$$

10. X^2 is P-semimartingale and hence Q-semimartingale by Theorem 2. By Corollary of theorem 15, $(X_- \cdot X)^Q$ is indistinguishable form $(X_- \cdot X)^P$. Then by definition of quadric variation,

$$[X,X]^P = X^2 - (X_- \cdot X)^P = X^2 - (X_- \cdot X)^Q = [X,X]^Q$$

up to evanescent set.

11. **a)** $\Lambda = [-2,1]$ is a closed set and B has a continuous path, by Theorem 4 in chapter 1, $T(\omega) = \inf t : B_t \notin \Lambda$ is a stopping time.

b) M_t is a uniformly integrable martingale since $M_t = E[B_T|\mathcal{F}_t]$. Clearly M is continuous. Clearly N is a continuous martingale as well. By Theorem 23, $[M,M]_t = [B,B]_t^T = t \wedge T$ and $[N,N]_t = [-B,-B]_t^T = t \wedge T$. Thus $[M,M] = [N,N]$. However $P(M_t > 1) = 0 \neq P(N_t > 1)$. M and N does not have the same law.

12. Fix $t \in \mathbb{R}_+$ and $\omega \in \Omega$. WLOG we can assume that $X(\omega)$ has a càdlàg path and $\sum_{s \leq t} |\triangle X_s(\omega)| < \infty$. Then on $[0,t]$, continuous part of X is bounded by continuity and jump part of X is bounded by hypothesis. So $\{X_s\}_{s \leq t}$ is bounded. Let $K \subset \mathbb{R}$ be $K = [\inf_{s \leq t} X_s(\omega), \sup_{s \leq t} X_s(\omega)]$. Then f, f', f" is bounded on K. Since $\sum_{s \leq t} \{f(X_s) - f(X_{s-}) - f'(X_{s-})\triangle X_s\}$ is an absolute convergent series, (see proof of Ito's formula (Theorem 32), it suffices to show that $\sum_{s \leq t} \{f'(X_{s-})\triangle X_s\} < \infty$. By hypothesis,

$$\sum_{s \leq t} \left| f'(X_{s-})\triangle X_s \right| \leq \sup_{x \in K} |f'(x)| \sum_{s \leq t} |\triangle X_s| < \infty$$

Since this is true for all $t \in \mathbb{R}_+$ and almost all $\omega \in \Omega$, the claim is shown.

13. By definition, stochastic integral is a continuous linear mapping J_X: $\mathbf{S}_{ucp} \to \mathbb{D}_{ucp}$. (Section 4). By continuity, $H^n \to H$ under ucp topology implies $H^n \cdot X \to H \cdot X$ under ucp topology.

14. Let $\hat{A} = 1 + A$ to simplify notations. Fix $\omega \in \Omega$ and let $\hat{A}^{-1} \cdot X = Z$. Then $Z_\infty(\omega) < \infty$ by hypothesis and $\hat{A} \cdot Z = X$. Applying integration by parts to X and then device both sides by $\hat{A}$ yields

$$\frac{X_t}{\hat{A}} = Z_t - \frac{1}{\hat{A}} \int_0^t Z_s d\hat{A}_s$$

Since $Z_\infty < \infty$ exists, for any $\epsilon > 0$, there exists τ such that $|Z_t - Z_\infty| < \varepsilon$ for all $t \geq \tau$. Then for $t > \tau$,

$$\frac{1}{\hat{A}_t} \int_0^t Z_s d\hat{A}_s = \frac{1}{\hat{A}_t} \int_0^\tau Z_s d\hat{A}_s + \frac{1}{\hat{A}_t} \int_\tau^t (Z_s - Z_\infty) d\hat{A}_s + Z_\infty \frac{\hat{A}_t - \hat{A}_\tau}{\hat{A}_t}$$

Let's evaluate right hand side. As $t \to \infty$, the first term goes to 0 while the last term converges to Z_∞ by hypothesis. By construction of τ, the second term is smaller than $\varepsilon(A_t - A_\tau)/A_t$, which converges to ε. Since this is true for all $\varepsilon > 0$,

$$\lim_{t \to \infty} \frac{1}{\hat{A}} \int_0^t Z_s d\hat{A}_s = Z_\infty$$

Thus $\lim_{t \to \infty} X_t / \hat{A}_t = 0$.

15. If $M_0 = 0$ then by Theorem 42, there exists a Brownian motion B such that $M_t = B_{[M,M]_t}$. By the law of iterated logarithm,

$$\lim_{t\to\infty} \frac{M_t}{[M,M]_t} = \lim_{t\to\infty} \frac{B_{[M,M]_t}}{[M,M]_t} = \lim_{\tau\to\infty} \frac{B_\tau}{\tau} = 0$$

If $M_0 \neq 0$, Let $X_t = M_t - M_0$. Then $[X,X]_t = [M,M]_t - M_0^2$. So $[X,X]_t \to \infty$ and

$$\frac{M_t}{[M,M]_t} = \frac{X_t + M_0}{[X,X]_t + M_0^2} \to 0$$

17. Since integral is taken over $[t-1/n, t]$ and Y is adapted, X_t^n is also an adapted process. For $t > s \geq 1/n$ such that $|t-s| \leq 1/n$,

$$|X_t^n - X_s^n| = n\left|\int_s^t Y_\tau d\tau - \int_{s-1/n}^{t-1/n} Y_\tau d\tau\right| \leq 2n(t-s) \sup_{s-\frac{1}{n}\leq\tau\leq t} |Y_\tau|$$

Since X is constant on $[0,t]$ and $[1,\infty)$, let π be a partition of $[t,1]$ and $M = \sup_{s-\frac{1}{n}\leq\tau\leq t}|Y_\tau| < \infty$. Then for each n, total variation of X^n is finite since

$$\sup_\pi |X_{t_{i+1}}^n - X_{t_i}^n| \leq 2nM\sum_\pi (t_{i+1} - t_i) = 2nM(1-t)$$

Therefore X_n is a finite variation process and in particular a semimartingale. By the fundamental theorem of calculus and continuity of Y, $Y_s = \lim_{n\to\infty} n\int_{s-1/n}^s Y_s du = \lim_{n\to\infty} X_t^n$ for each $t > 0$. At $t = 0$, $\lim_n X_n = 0 = Y_0$. So $X_n \to Y$ for all $t \geq 0$. However Y need not be a semimartingale since $Y_t = (B_t)^{1/3}$ where B_t is a standard Brownian motion satisfies all requirements but not a semimartingale.

18. B_t has a continuous path almost surely. Fix ω such that $B_t(\omega)$ is continuous. Then $\lim_{n\to\infty} X_t^n(\omega) = B_t(\omega)$ by Exercise 17. By Theorem 37, the solution of $dZ_t^n = Z_{s-}^n dX_s^n$, $Z_0 = 1$ is

$$Z_t^n = \exp\left(X_t^n - \frac{1}{2}[X^n, X^n]_t\right) = \exp(X_t^n),$$

since X^n is a finite variation process. On the other hand, $Z_t = \exp(B_t - t/2)$ and

$$\lim_{n\to\infty} Z_t^n = \exp(B_t) \neq Z_t$$

19. A^n is a clearly càdlàg and adapted. By periodicity and symmetry of triangular function,

$$\int_0^{\frac{\pi}{2}} |dA_s^n| = \frac{1}{n}\int_0^{\frac{\pi}{2}} |d\sin(ns)| = \frac{1}{n}\cdot n\int_0^{\frac{\pi}{2n}} d(\sin(ns)) = \int_0^{\frac{\pi}{2}} d(\sin(s)) = 1$$

Since total variation is finite, A_t^n is a semimartingale. On the other hand,

$$\lim_{n\to\infty} \sup_{0\leq t\leq\frac{\pi}{2}} |A_t^n| = \lim_{n\to\infty} \frac{1}{n} = 0$$

20. Applying Ito's formula to $u \in C^2(\mathbb{R}^3 - \{0\})$ and $B_t \in \mathbb{R}^3\backslash\{0\}$, $\forall t$, a.s.

$$u(B_t) = u(x) + \int_0^t \nabla u(B_s) \cdot dB_s + \frac{1}{2}\int_0^t \triangle u(B_s)ds = \frac{1}{\|x\|} + \sum_{i=1}^{3}\int_0^t \partial_i u(B_s)dB_s^i$$

and $M_t = u(B_t)$ is a local martingale. This solves **(a)**. Fix $1 \leq \alpha \leq 3$. Observe that $E(u(B_0)^\alpha) < \infty$. Let p, q be a positive number such that $1/p + 1/q = 1$. Then

$$\begin{aligned}
E^x(u(B_t)^\alpha) &= \int_{\mathbb{R}^3} \frac{1}{\|y\|^\alpha}\frac{1}{(2\pi t)^{3/2}}e^{-\frac{\|x-y\|^2}{2t}}dy \\
&= \int_{\{B(0;1)\cap B^c(x;\delta)\}} \frac{1}{\|y\|^\alpha}\frac{1}{(2\pi t)^{3/2}}e^{-\frac{\|x-y\|^2}{2t}}dy + \int_{\mathbb{R}^3\backslash\{B(0;1)\cap B^c(x;\delta)\}} \frac{1}{\|y\|^\alpha}\frac{1}{(2\pi t)^{3/2}}e^{-\frac{\|x-y\|^2}{2t}}dy \\
&\leq \sup_{y\in\{B(0;1)\cap B^c(x;\delta)\}} \frac{1}{(2\pi t)^{\frac{3}{2}}}e^{-\frac{\|x-y\|^2}{2t}} \cdot \int_{B(0;1)} \frac{1}{\|y\|^\alpha}dy \\
&\quad + \left(\int_{\mathbb{R}^3\backslash\{B(0;1)\cap B^c(x;\delta)\}} \frac{1}{\|y\|^{\alpha p}}dy\right)^{1/p}\left(\int_{\mathbb{R}^3\backslash\{B(0;1)\cap B^c(x;\delta)\}} \left(\frac{1}{(2\pi t)^{3/2}}e^{-\frac{\|x-y\|^2}{2t}}\right)^q dy\right)^{1/q}
\end{aligned}$$

Pick $p > 3/\alpha > 1$. Then the first term goes to 0 as $t \to \infty$. In particular it is finite for all $t \geq 0$. The first factor in the second term is finite while the second factor goes to 0 as $t \to \infty$ since

$$\int \frac{1}{(2\pi t)^{3/2}}\frac{1}{(2\pi t)^{3(q-1)/2}}e^{-\frac{\|x-y\|^2}{2t/q}}dy = \frac{1}{(2\pi t)^{3(q-1)/2}}\int \frac{1}{q^{3/2}}\frac{1}{(2\pi t/q)^{3/2}}e^{-\frac{\|x-y\|^2}{2t/q}}dy = \frac{1}{(2\pi t)^{3(q-1)/2}}\frac{1}{q^{3/2}}$$

and the second factor is finite for all $t \geq 0$. **(b)** is shown with $\alpha = 1$. **(c)** is shown with $\alpha = 2$. It also shows that

$$\lim_{t\to\infty} E^x(u((B_t)^2) = 0$$

which in turn shows **(d)**.

21. Applying integration by parts to $(A^\infty - A_.)(C^\infty - C_.)$,

$$(A_\infty - A_.)(C_\infty - C_.) = A_\infty C_\infty - \int_0^. (A_\infty - A_{s-})dC_s - \int_0^. (C_\infty - C_{s-})dA_s + [A, C].$$

Since A, C are processes with finite variation,

$$[A, C]_t = \sum_{0<s<t} \triangle A_s \triangle C_s$$

$$\int_0^t (A_\infty - A_{s-})dC_s = \int_0^t (A_\infty - A_s)dC_s + \sum_{0<s<t} \triangle A_s \triangle C_s$$

Substituting these equations,

$$(A_\infty - A_t)(C_\infty - C_t) = A_\infty C_\infty - \int_0^t (A^\infty - A_s)dC_s - \int_0^t (C^\infty - C_{s-})dA_s$$

Letting $t \to 0$ and we obtain **(a)**. **(b)** is immediate from this result.

22. Claim that for all integer $k \geq 1$ and prove by induction.

$$(A_\infty - A_s)^k = k! \int_s^\infty dA_{s_1} \int_{s_1}^\infty dA_{s_2} \dots \int_{s_{p-1}}^\infty dA_{s_p} \tag{48}$$

For $k = 1$, equation (48) clearly holds. Assume that it holds for $k = n$. Then

$$(k+1)! \int_s^\infty dA_{s_1} \int_{s_1}^\infty dA_{s_2} \dots \int_{s_{p-1}}^\infty dA_{s_{k+1}} = (k+1) \int_s^\infty (A_\infty - A_{s_1}) dA_{s_1} = (A_\infty - A_s)^{k+1}, \tag{49}$$

since A is non-decreasing continuous process, $(A_\infty - A) \cdot A$ is indistinguishable from Lebesgue-Stieltjes integral calculated on path by path. Thus equation (48) holds for $n = k + 1$ and hence for all integer $n \geq 1$. Then setting $s = 0$ yields a desired result since $A_0 = 0$.

24. **a)**

$$\begin{aligned}
\int_0^t \frac{1}{1-s} d\beta_s &= \int_0^t \frac{1}{1-s} dB_s - \int_0^t \frac{1}{1-s} \frac{B_1 - B_s}{1-s} ds \\
&= \int_0^t \frac{1}{1-s} dB_s - B_1 \int_0^t \frac{1}{(1-s)^2} ds + \int_0^t \frac{B_s}{(1-s)^2} ds \\
&= \int_0^t \frac{1}{1-s} dB_s - B_1 \int_0^t \frac{1}{(1-s)^2} ds + \int_0^t B_s d\left(\frac{1}{1-s}\right) \\
&= \frac{B_t}{1-t} - \left[\frac{1}{1-s}, B\right]_t - B_1 \left(\frac{1}{1-t} - \frac{1}{1}\right) \\
&= \frac{B_t - B_1}{1-t} + B_1
\end{aligned}$$

Arranging terms and we have desired result.

b) Using integration by arts, since $[1-s, \int_0^s (1-u)^{-1} d\beta_u]_t = 0$,

$$\begin{aligned}
X_t = (1-t) \int_0^t \frac{1}{1-s} d\beta_s &= \int_0^t (1-s) \frac{1}{1-s} d\beta_s + \int_0^t \int_0^s \frac{1}{1-u} d\beta_u (-1) ds \\
=& \beta_t + \int_0^t \left(-\frac{X_s}{1-s}\right) ds,
\end{aligned}$$

as required. The last equality is by result of (a).

27. By Ito's formula and given SDE,

$$d(e^{\alpha t} X_t) = \alpha e^{\alpha t} X_t dt + e^{\alpha t} dX_t = \alpha e^{\alpha t} X_t dt + e^{\alpha t}(-\alpha X_t dt + \sigma dB_t) = \sigma e^{\alpha t} dX_t$$

Then integrating both sides and arranging terms yields

$$X_t = e^{-\alpha t} \left(X_0 + \sigma \int_0^t e^{\alpha s} dB_s \right)$$

28. By the law of iterated logarithm, $\limsup_{t\to\infty} \frac{B_t}{t} = 0$ a.s. In particular, for almost all ω there exists $t_0(\omega)$ such that $t > t_0(\omega)$ implies $B_t(\omega)/t < 1/2 - \epsilon$ for any $\epsilon \in (0, 1/2)$. Then

$$\lim_{t\to\infty} \mathcal{E}(B_t) = \lim_{t\to\infty} \exp\left\{ t\left(\frac{B_t}{t} - \frac{1}{2}\right)\right\} \le \lim_{t\to\infty} e^{-\epsilon t} = 0, \qquad \text{a.s.}$$

29. $\mathcal{E}(X)^{-1} = \mathcal{E}(-X + [X,X])$ by Corollary of Theorem 38. This implies that $\mathcal{E}(X)^{-1}$ is the solution to a stochastic differential equation,

$$\mathcal{E}(X)_t^{-1} = 1 + \int_0^t \mathcal{E}(X)_{s-}^{-1} d(-X_s + [X,X]_s),$$

which is the desired result. Note that continuity assumed in the corollary is not necessary if we assume $\triangle X_s \neq -1$ instead so that $\mathcal{E}(X)^{-1}$ is well defined.

30. a) By Ito's formula and continuity of M, $M_t = 1 + \int_0^t M_s dB_s$. B_t is a locally square integrable local martingale and $M \in \mathbb{L}$. So by Theorem 20, M_t is also a locally square integrable local martingale.

b)

$$[M,M]_t = \int_0^t M_s^2 ds = \int_0^t e^{2B_s - s} ds$$

and for all $t \ge 0$,

$$E([M,M]_t) = \int_0^t E[e^{2B_s}]e^{-s} ds = \int_0^t e^s ds < \infty$$

Then by Corollary 3 of Theorem 27, M is a martingale.

c) Ee^{B_t} is calculated above using density function. Alternatively, using the result of b),

$$Ee^{B_t} = E(M_t e^{\frac{t}{2}}) = e^{\frac{t}{2}} EM_0 = e^{\frac{t}{2}}$$

31. Pick $A \in \mathcal{F}$ such that $P(A) = 0$ and fix $t \ge 0$. Then $A \cap \{R_t \le s\} \in \mathcal{F}_s$ since $\mathcal{F}_s$ contains all P-null sets. Then $A \in \mathcal{G}_t = \mathcal{F}_{R_t}$. If $t_n \downarrow t$, then by right continuity of R, $R_{t_n} \downarrow R_t$. Then

$$\mathcal{G}_t = \mathcal{F}_{R_t} = \cap_n \mathcal{F}_{R_{t_n}} = \cap_n \mathcal{G}_{t_n} = \cap_{s \ge t} \mathcal{G}_s$$

by the right continuity of $\{\mathcal{F}_t\}_t$ and Exercise 4, chapter 1. Thus $\{\mathcal{G}_t\}_t$ satisfies the usual hypothesis.

32. If M has a càdlàg path and R_t is right continuous, $\bar{M}_t$ has a càdlàg path. $\bar{M}_t = M_{R_t} \in \mathcal{F}_{R_t} = \mathcal{G}_t$. So $\bar{M}_t$ is adapted to $\{\mathcal{G}_t\}$. For all $0 \le s \le t$, $R_s \le R_t < \infty$. Since M is uniformly integrable martingale, by optional sampling theorem,

$$\bar{M}_s = M_{R_s} = E(M_{R_t} | \mathcal{F}_{R_s}) = E(\bar{M}_t | \mathcal{G}_s), \qquad \text{a.s.}$$

So $\bar{M}$ is $\mathcal{G}$-martingale.

Chapter 3. Semimartingales and Decomposable Processes

1. Let $\{T_i\}_{i=1}^n$ be a set of predictable stopping time. For each i, T_i has an announcing sequence $\{T_{i,j}\}_{j=1}^{\infty}$. Let $S_k := \max_i T_{i,k}$ and $R_k := \min_i T_{i,k}$. Then S_k, R_k are stopping time. $\{S_k\}$ and $\{R_k\}$ make announcing sequences of maximum and minimum of $\{T_i\}_i$.

2. Let $T_n = S + (1 - 1/n)$ for each n. Then T_n is a stopping time and $T_n \uparrow T$ as $n \uparrow \infty$. Therefore T is a predictable stopping time.

3. Let S, T be two predictable stopping time. Then $S \wedge T$, $S \vee T$ are predictable stopping time as shown in exercise 1. In addition, $\Lambda = \{S \vee T = S \wedge T\}$. Therefore without loss of generality, we can assume that $S \leq T$. Let $\{T_n\}$ be an announcing sequence of T. Let $R_n = T_n + n1_{\{T_n \geq S\}}$. Then R_n is a stopping time since $\{R_n \leq t\} = \{T_n \leq t\} \cap (\{t - T_n \leq n\} \cup \{T_n < S\}) \in \mathcal{F}_t$. R_n is increasing and $\lim R_n = T_\Lambda = S_\Lambda$.

4. For each $X \in \mathbb{L}$, define a new process X_n by $X^n = \sum_{k \in \mathbb{N}} X_{k/2^n} 1_{[k/2^n,(k+1)/2^n)}$. Since each summand is an optional set (See exercise 6) X^n is an optional process. As a mapping on the product space, X is a pints limit of X^n. Therefore X is optional. Then by the definition $\mathcal{P} \subset \mathcal{O}$.

5. Suffice to show that all càdlàg processes are progressively measurable. (Then we can apply monotone class argument.) For a càdlàg process X on $[0,t]$, define a new process X^n by putting $X^n_u = X_{k/2^n}$ for $u \in [(k-1)t/2^n, kt/2^n)$, $k = \{1, \ldots, 2^n\}$. Then on $\Omega \times [0,t]$,

$$\{X^n \in B\} = \cup_{k \in \mathbb{N}_+} \left[\{\omega : X_{k/2^n}(\omega) \in B\} \times \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) \right] \in \mathcal{F}_t \otimes \mathcal{B}([0,t]) \tag{50}$$

since X is adapted and X^n is progressively measurable process. Since X is càdlàg , $\{X^n\}$ converges pints to X, which therefore is also $\mathcal{F}_t \otimes \mathcal{B}([0,t])$ measurable.

6. **(1)** $(S,T]$**:** Since $1_{(S,T]} \in \mathbb{L}$, $(S.T] = \{(s,\omega) : 1_{(S,T]} = 1\} \in \mathcal{P}$ by definition.
(2) $[S,T)$**:** Since $1_{[S,T)} \in \mathbb{D}$, $[S.T) = \{(s,\omega) : 1_{[S,T)} = 1\} \in \mathcal{O}$ by definition.
(3) (S,T)**:** Since $[S,T] = \cup_n [S.T + 1/n)$, $[S,T]$ is an optional set. Then $(S,T) = [0,T) \cap [0,S]^c$ and (S,T) is optional as well.
(4) $[S,T)$ **when** S,T **is predictable :** Let $\{S_n\}$ and $\{T_n\}$ be announcing sequences of S and T. Then $[S,T) = \cap_m \cup_n (S_m, T_n]$. Since $(S_m, T_n]$ is predictable by (1) for all m, n, $[S,T)$ is predictable.

7. Pick a set $A \in \mathcal{F}_{S_n}$. Then $A = A \cap \{S_n < T\} \in \mathcal{F}_{T-}$ by theorem 7 and the definition of $\{S_n\}_n$. (Note: The proof of theorem 7 does not require theorem 6). Since this is true for all n, $\vee_n \mathcal{F}_{S_n} \subset \mathcal{F}_{T-}$. To show the converse let $\Pi = \{B \cap \{t < T\} : B \in \mathcal{F}_t\}$. Then Π is closed with respect to finite intersection. $B \cap \{t < T\} = \cup_n (B \cap \{t < S_n\})$. Since $(B \cap \{t < S_n\}) \cap \{S_n \leq t\} = \emptyset \in \mathcal{F}_t$,$B \cap \{t < S_n\} \in \mathcal{F}_{S_n}$. Therefore $B \cap \{t < T\} \in \vee_n \mathcal{F}_{S_n}$ and $\Pi \subset \vee_n \mathcal{F}_{S_n}$. Then by Dynkin's theorem, $\mathcal{F}_{T-} \subset \vee_n \mathcal{F}_{S_n}$.

8. Let S, T be stopping times such that $S \le T$. Then $\mathcal{F}_{S_n} \subset \mathcal{F}_T$. and $\vee_n \mathcal{F}_{S_n} \subset \mathcal{F}_T$. By the same argument as in exercise 7, $\mathcal{F}_{T-} \subset \vee_n \mathcal{F}_{S_n}$. Since $\mathcal{F}_{T-} = \mathcal{F}_T$ by hypothesis, we have desired result. (Note: $\{S_n\}$ is not necessarily an announcing sequence since $S_n = T$ is possible. Therefore $\vee \mathcal{F}_{S_n} \neq \mathcal{F}_{T-}$ in general.)

9. Let X be a Lévy process, $\mathcal{G}$ be its natural filtration and T be stopping time. Then by exercises 24(c) in chapter 1, $\mathcal{G}_T = \sigma(\mathcal{G}_{T-}, X_T)$. Since X jumps only at totally inaccessible time (a consequence of theorem 4), $X_T = X_{T-}$ for all predictable stopping time T. Therefore if T is a predictable time, $\mathcal{G}_T = \sigma(\mathcal{G}_{T-}, X_T) = \sigma(\mathcal{G}_{T-}, X_{T-}) = \mathcal{G}_{T-}$ since $X_{T-} \in \mathcal{G}_{T-}$. Therefore a completed natural filtration of a Lévy process is quasi left continuous.

10. As given in hint, $[M, A]$ is a local martingale. Let $\{T_n\}$ be its localizing sequence, that is $[M, A]^{T_n}_{\cdot}$ is a martingale for all n. Then $E([X, X]^{T_n}_t) = E([M, M]^{T_n}_t) + E([A, A]^{T_n}_t)$. Since quadratic variation is non-decreasing non-negative process, first letting $n \to \infty$ and then letting $t \to \infty$, we obtain desired result with monotone convergence theorem.

11. By the Kunita-Watanabe inequality for square bracket processes, $([X + Y, X + Y])^{1/2} \le ([X, X])^{1/2} + ([Y, Y])^{1/2}$, It follows that $[X + Y, X + Y] \le 2\,([X, X] + [Y, Y])$. This implies that $[X + Y, X + Y]$ is locally integrable and the sharp bracket process $\langle X + Y, X + Y\rangle$ exists. Since sharp bracket process is a predictable projection (compensator) of square bracket process, we obtain polarization identity of sharp bracket process from one of squared bracket process. Namely,

$$\langle X, Y\rangle = \frac{1}{2}(\langle X + Y . X + Y\rangle - \langle X, X\rangle - \langle Y, Y\rangle) \tag{51}$$

Then the rest of the proof is exactly the same as the one of theorem 25, chapter 2, except that we replace square bracket processes by sharp bracket processes.

12. Since a continuous finite process with finite variation always has a integrable variation, without loss of generality we can assume that the value of A changes only by jumps. Thus A can be represented as $A_t = \sum_{s \le t} \triangle A_s$. Assume that $C_\cdot = \int_0^\cdot |dA_s|$ is predictable. Then $T_n = \inf\{t > 0 : C_t \ge n\}$ is a predictable time since it is a debut of right closed predictable set. Let $\{T_{n,m}\}_m$ be an announcing sequence of T_n for each n and define S_n by $S_n = \sup_{1 \le k \le n} T_{k,n}$. Then S_n is a sequence of stopping time increasing to ∞, $S_n < T_n$ and hence $C_{S_n} \le n$. Thus $EC_{S_n} < n$ and C is locally integrable. To prove that C is predictable, we introduce two standard results.

lemma *Suppose that A is the union of graphs of a sequence of predictable times. Then there exists a sequence of predictable times $\{T_n\}$ such that $A \subset \cup_n [T_n]$ and $[T_n] \cap [T_m] = \emptyset$ for $n \neq m$.*

Let $\{S_n\}_n$ be a sequence of predictable stopping times such that $A \subset \cup_n [S_n]$. Put $T_1 = S_1$ and for $n \ge 2$, $B_n = \cap_{k=1}^{n-1} [S_k \neq S_n]$, $T_n = (S_n)_{B_n}$. Then $B_n \in \mathcal{F}_{S_n-}$, T_n is predictable, $[T_n] \cap [T_m] = \emptyset$ when $n \neq m$, and $A = \cup_{n \ge 1} [T_n]$. (Note: By the definition of the graph, $[T_n] \cap [T_m] = \emptyset$ even if $P(T_n = T_m = \infty) > 0$ as long as T_n and T_m are disjoint on $\Omega \times \mathbb{R}_+$) $\square$

lemma *Let X_t be a càdlàg adapted process and predictable. Then there exists a sequence of strictly positive predictable times $\{T_n\}$ such that $[\triangle X \neq 0] \subset \cup_n [T_n]$.*

Proof. Let $S_{n+1}^{1/k} = \inf\{t : t > T_n^{1/k}(\omega), |X_{S_n^{1/k}} - X_t| > 1/k$ or $|X_{S_n^{1/k}} - X_{t-}| > 1/k\}$. Then since X is predictable, we can show by induction that $\{S_n\}_{n\geq 1}$ is predictable stopping time. In addition $[\triangle X \neq 0] \subset \cup_{n,k\geq 1}[S_n^{1/k}]$. Then by previous lemma, there exists a sequence of predictable stopping times $\{T_n\}$ such that $\cup_{n,k\geq 1}[S_n^{1/k}] \subset \cup_n [T_n]$. Then $[\triangle X \neq 0] \subset \cup_n [T_n]$ □

Proof of the main claim: Combining two lemmas, we see that $\{\triangle A \neq 0\}$ is the union of a sequence of disjoint graphs of predictable stopping times. Since $A_t = \sum_{s\leq t} \triangle A_s$ is absolute convergent for each ω, it is invariant with respect to the change of the order of summation. Therefore $A_t = \sum_n \triangle A_{S_n} 1_{\{S_n \leq t\}}$ where S_n is a predictable time. $\triangle A_{S_n} 1_{\{S_n\leq t\}}$ is a predictable process since S_n is predictable and $A_{S_n} \in \mathcal{F}_{S_n -}$. This clearly implies that $|\triangle A_{S_n}| 1_{\{S_n \leq t\}}$ is predictable. Then C is predictable as well.

Note: As for the second lemma, following more general claim holds.

lemma *Let X_t be a càdlàg adapted process. Then X is predictable if and only if X satisfies the following conditions (1). There exists a sequence of strictly positive predictable times $\{T_n\}$ such that $[\triangle X \neq 0] \subset \cup_n [T_n]$. (2). For each predictable time T, $X_T 1_{\{T<\infty\}} \in \mathcal{F}_{T-}$.*

13. Let $\{T_i\}_i$ be a jump time of a counting process and define $S_i = T_i - T_{i-1}$. Then by corollary to theorem 23, a compensator A is given by

$$A_t = \sum_{i\geq 1} \left[\sum_{j=1}^{i} \phi_j(S_j) + \phi_{i+1}(t - T_i) \right] 1_{\{T_i \leq t < T_{i+1}\}}, \qquad \phi_i(s) = \int_0^s \frac{-1}{F_i(u-)} dF_i(u), \tag{52}$$

where $F_i(u) = P(S_i > u)$. For each ω, it is clear that If F_i has a density such that $dF_i(u) = f_i(u)$ then A_t is absolutely continuous. Conversely if $F_k(u)$ does not admit density then on $[T_{k-1}, T_k)$, A_t is not absolutely continuous.

14. $N_t - 1_{\{t\geq T\}} = 0$ is a trivial martingale. Since Doob-Meyer decomposition is unique, it suffices to show that $1_{\{t\geq T\}}$ is a predictable process, $1_{\{t\geq T\}}$ is a predictable process if and only if T is a predictable time. Let $S_n = T(1 - 1/n)$. Then $T \in \mathcal{F}_0$ implies $S_n \in \mathcal{F}_0$. In particular $\{S_n \leq t\} \in \mathcal{F}_0 \subset \mathcal{F}_t$ and S_n is a stopping time such that $S_n < T$ and $S_n \uparrow T$. This makes $\{S_n\}$ an announcing sequence of T. Thus T is predictable.

15. By the uniqueness of Doob-Meyer decomposition, it suffices to show that $N_t - \mu\lambda t$ is a martingale. since $\mu\lambda t$ is clearly a predictable process with finite variation. Let C_t be a Poisson process associated with N_t. Then by the independent and stationary increment property of compound

Poisson process,

$$
\begin{aligned}
E[N_t - N_s|\mathcal{F}_s] = E\left[\sum_{i=1}^{C_{t-s}} U_i\right] &= \sum_{k=1}^{\infty} E\left[\sum_{i=1}^{C_{t-s}} U_i | C_{t-s} = k\right] P(C_{t-s} = k) \\
=\mu \sum_{k=1}^{\infty} kP(C_{t-s} = k) = \mu\lambda(t-s)
\end{aligned}
\tag{53}
$$

Therefore $N_t - \mu\lambda t$ is a martingale and the claim is shown.

16. By direct computation,

$$
\lambda(t) = \lim_{h\to 0} \frac{1}{h} P(t \le T < t+h|T \ge t) = \lim_{h\to 0} \frac{1-e^{-\lambda h}}{h} = \lambda. \tag{54}
$$

A case that T is Weibull is similar and omitted.

17. Observe that $P(U > s) = P(T > 0, U > s) = \exp\{-\mu s\}$. $P(T \le t, U > s) = P(U > s) - P(T > t.U > s)$. Then

$$
\begin{aligned}
P(t \le T < t+h|T \ge t, U \ge t) &= \frac{P(t \le T < t+h, t \le U)}{P(t \le T, t \le U)} = \frac{\int_t^{t+h} e^{-\mu t}(\lambda + \theta t)e^{-(\lambda+\theta t)u} du}{\exp\{-\lambda t - \mu t - \theta t^2\}} \\
= \frac{-1[\exp\{-\lambda(t+h) - \theta t(t+h)\} - \exp\{-\lambda t - \theta t^2\}]}{\exp\{-\lambda t - \theta t^2\}} &= 1 - \exp\{-(\lambda+\theta t)h\}
\end{aligned}
\tag{55}
$$

and

$$
\lambda^{\#}(t) = \lim_{h\to 0} \frac{1 - \exp\{-(\lambda+\theta t)h\}}{h} = \lambda + \theta t \tag{56}
$$

18. There exists a disjoint sequence of predictable times $\{T_n\}$ such that $A = \{t > 0 : \triangle\langle M, M\rangle_t \neq 0\} \subset \cup_n [T_n]$. (See the discussion in the solution of exercise 12 for details.) In addition, all the jump times of $\langle M, M\rangle_\cdot$ is a predictable time. Let T be a predictable time such that $\langle M, M\rangle_T \neq 0$. Let $N = [M, M] - \langle M, M\rangle$. Then N is a martingale with finite variation since $\langle M, M\rangle$ is a compensator of $[M, M]$. Since T is predictable, $E[N_T|\mathcal{F}_{T-}] = N_{T-}$. On the other hand, since $\{\mathcal{F}_t\}$ is a quasi-left-continuous filtration, $N_{T-} = E[N_T|\mathcal{F}_{T-}] = E[N_T|\mathcal{F}_T] = N_T$. This implies that $\triangle\langle M, M\rangle_T = \triangle[M, M]_T = (\triangle M_T)^2$. Recall that M itself is a martingale. So $M_T = E[M_T|\mathcal{F}_T] = E[M_T|\mathcal{F}_{T-}] = M_{T-}$ and $\triangle M_T = 0$. Therefore $\triangle\langle M, M\rangle_T = 0$ and $\langle M, M\rangle$ is continuous.

19. By theorem 36, X is a special semimartingale. Then by theorem 34, it has a unique decomposition $X = M + A$ such that M is local martingale and A is a predictable finite variation process. Let $X = N + C$ be an arbitrary decomposition of X. Then $M - N = C - A$. This implies that A is a compensator of C. It suffices to show that a local martingale with finite variation is locally integrable. Set $Y = M - N$ and $Z = \int_0^t |dY_s|$. Let S_n be a fundamental sequence of Y and set $T_n = S_n \wedge n \wedge \inf\{t : Z_t > n\}$. Then $Y_{T_n} \in L^1$ (See the proof of theorem 38) and $Z_{T_n} \le n + |Y_{T_n}| \in L^1$. Thus $Y = M - N$ is has has a locally integrable variation. Then C is a sum of two process with locally integrable variation and the claim holds.

20. Let $\{T_n\}$ be an increasing sequence of stopping times such that X^{T_n} is a special semimartingale as shown in the statement. Then by theorem 37, $X^*_{t\wedge T_n} = \sup_{s\leq t}|X^{T_n}_s|$ is locally integrable. Namely, there exists an increasing sequence of stopping times $\{R_n\}$ such that $(X^*_{t\wedge T_n})^{R_n}$ is integrable. Let $S_n = T_n \wedge R_n$. Then S_n is an increasing sequence of stopping times such that $(X^*_t)^{S_n}$ is integrable. Then X^*_t is locally integrable and by theorem 37, X is a special semimartingale.

21. Since $\mathbb{Q}\sim\mathbb{P}$, $d\mathbb{Q}/d\mathbb{P} > 0$ and $Z > 0$. Clearly $M\in L^1(\mathbb{Q})$ if and only if $MZ \in L^1(\mathbb{P})$. By generalized Bayes' formula.

$$E_{\mathbb{Q}}[M_t|\mathcal{F}_s] = \frac{E_{\mathbb{P}}[M_tZ_t|\mathcal{F}_s]}{E_{\mathbb{P}}[Z_t|\mathcal{F}_s]} = \frac{E_{\mathbb{P}}[M_tZ_t|\mathcal{F}_s]}{Z_s}, \qquad t\geq s \tag{57}$$

Thus $E_{\mathbb{P}}[M_tZ_t|\mathcal{F}_s] = M_sZ_s$ if and only if $E_{\mathbb{Q}}[M_t|\mathcal{F}_s] = M_s$.

22. By Rao's theorem. X has a unique decomposition $X = M + A$ where M is $(\mathcal{G},\mathbb{P})$-local martingale and A is a predictable process with path of locally integrable variation. $E[X_{t_{i+1}} - X_{t_i}|\mathcal{G}_{t_i}] = E[A_{t_{i+1}} - A_{t_i}|\mathcal{G}_{t_i}]$. So

$$\sup_\tau E\left[\sum_{i=0}^n |E[A_{t_i} - A_{t_i+1}|\mathcal{G}_{t_i}]|\right] < \infty \tag{58}$$

$Y_t = E[X_t|\mathcal{F}_t] = E[M_t|\mathcal{F}_t] + E[A_t|\mathcal{F}_t]$. Since $E[E[M_t|\mathcal{F}_t]|\mathcal{F}_s] = E[M_t|\mathcal{F}_s] = E[E[M_t|\mathcal{G}_s]|\mathcal{F}_s]] = E[M_s|\ fil_s]$, $E[M_t|\mathcal{F}_t]$ is a martingale. Therefore

$$\mathrm{Var}_\tau(Y) = E\left[\sum_{i=1}^n |E[E[A_{t_i}|\mathcal{F}_{t_i}] - E[A_{t_i+1}|\mathcal{F}_{t_i+1}]|\mathcal{F}_{t_i}]|\right] = \sum_{i=1}^n E\left[|E[A_{t_i} - A_{t_i+1}|\mathcal{F}_{t_i}]|\right] \tag{59}$$

For arbitrary σ-algebra $\mathcal{F},\mathcal{G}$ such that $\mathcal{F}\subset\mathcal{G}$ and $X\in L_1$,

$$E\left(|E[X|\mathcal{F}]|\right) = E\left(|E\left(E[X|\mathcal{G}]|\mathcal{F}\right)|\right) \leq E\left(E\left(|E[X|\mathcal{G}]||\mathcal{F}\right)\right) = E\left(|E[X|\mathcal{G}]|\right) \tag{60}$$

Thus for every τ and t_i, $E\left[|E[A_{t_i} - A_{t_i+1}|\mathcal{F}_{t_i}]|\right] \leq E\left[|E[A_{t_i} - A_{t_i+1}|\mathcal{G}_{t_i}]|\right]$. Therefore Var(X) $< \infty$ (w.r.t. $\{\mathcal{G}_t\}$)implies Var(Y) $< \infty$ (w.r.t $\{\mathcal{F}_t\}$) and Y is $(\{\mathcal{F}_t\},\mathbb{P})$-quasi-martingale.

23. We introduce a following standard result without proof.

Lemma. Let N be a local martingale. If $E([N,N]^{1/2}_\infty) < \infty$ or alternatively, $N\in\mathcal{H}^1$ then N is uniformly integrable martingale.

This is a direct consequence of Fefferman's inequality. (Theorem 52 in chapter 4. See chapter 4 section 4 for the definition of $\mathcal{H}^1$ and related topics.) We also take a liberty to assume Burkholder-Davis-Gundy inequality (theorem 48 in chapter 4) in the following discussion.

Once we accept this lemma, it suffices to show that a local martingale $[A,M]\in\mathcal{H}^1$. By Kunita-Watanabe inequality, $[A,M]^{1/2}_\infty \leq [A,A]^{1/4}_\infty[M,M]^{1/4}_\infty$. Then by Hölder inequality,

$$E\left([A,M]^{1/2}_\infty\right) \leq E\left([A,A]^{1/2}_\infty\right)^{\frac{1}{2}} E\left([M,M]^{1/2}_\infty\right). \tag{61}$$

By hypothesis $E\left([A,A]_\infty^{1/2}\right)^{\frac{1}{2}} < \infty$. By BDG inequality and the fact that M is a bounded martingale, $E([M,M]^{1/2}) \leq c_1 E[M_\infty^*] < \infty$ for some positive constant c_1. This complete the proof.

24. Since we assume that the usual hypothesis holds throughout this book (see page 3), let $\mathcal{F}_t^0 = \sigma\{T \wedge s : s \leq t\}$ and redefine $\mathcal{F}_t$ by $\mathcal{F}_t = \cap_\epsilon \mathcal{F}_{t+\epsilon}^0 \vee \mathcal{N}$. Since $\{T < t\} = \{T \wedge t < t\} \in \mathcal{F}_t$, T is $\mathcal{F}$-stopping time. Let $\mathbb{G} = \{\mathcal{G}_t\}$ be a smallest filtration such that T is a stopping time, that is a natural filtration of the process $X_t = 1_{\{T \leq t\}}$. Then $\mathbb{G} \subset \mathbb{F}$.

For the converse, observe that $\{T \wedge s \in B\} = (\{T \leq s\} \cap \{T \in B\}) \cup (\{T > s\} \cap \{s \in B\}) \in \mathcal{F}_s$ since $\{s \in B\}$ is $\emptyset$ or Ω, $\{T \in B\} \in \mathcal{F}_T$ and in particular $\{T \leq s\} \cap \{T \in B\} \in \mathcal{F}_s$ and $\{T > s\} \in \mathcal{F}_s$. Therefore for all t, $T \wedge s$, $(\forall s \leq t)$ is $\mathcal{G}_t$ measurable. Hence $\mathcal{F}_t^0 \subset \mathcal{G}_t$. This shows that $\mathbb{G} \subset \mathbb{F}$. (Note: we assume that $\mathbb{G}$ satisfies usual hypothesis as well.)

25. Recall that the $\{(\omega, t) : \triangle A_t(\omega) \neq 0\}$ is a subset of a union of disjoint predictable times and in particular we can assume that a jump time of predictable process is a predictable time. (See the discussion in the solution of exercise 12). For any predictable time T such that $E[\triangle Z_T | \mathcal{F}_{T-}] = 0$,

$$\triangle A_T = E[\triangle A_T | \mathcal{F}_{T-}] = E[\triangle M_T | \mathcal{F}_{T-}] = 0 \qquad a.s. \tag{62}$$

26. Without loss of generality, we can assume that $A_0 = 0$ since $E[\triangle A_0 | \mathcal{F}_{0-}] = E[\triangle A_0 | \mathcal{F}_0] = \triangle A_0 = 0$. For any finite valued stopping time S, $E[A_S] \leq E[A_\infty]$ since A is an increasing process. Observe that $A_\infty \in L_1$ because A is a process with integrable variation. Therefore A $\{A_S\}_S$ is uniformly integrable and A is in class (D). Applying Theorem 11 (Doob-Meyer decomposition) to $-A$, we see that $M = A - \tilde{A}$ is a uniformly integrable martingale. Then

$$0 = E[\triangle M_T | \mathcal{F}_{T-}] = E[\triangle A_T | \mathcal{F}_{T-}] - E[\triangle \tilde{A}_T | \mathcal{F}_{T-}] = 0 - E[\triangle \tilde{A}_T | \mathcal{F}_{T-}]. \qquad \text{a.s.} \tag{63}$$

Therefore $\tilde{A}$ is continuous at time T.

27. Assume A is continuous. Consider an arbitrary increasing sequence of stopping time $\{T_n\} \uparrow T$ where T is a finite stopping time. M is a uniformly integrable martingale by theorem 11 and the hypothesis that Z is a supermartingale of class D. Then $\infty > EZ_T = -EA_T$ and in particular $A_T \in L^1$. Since $A_T \geq A_{T_n}$ for each n. Therefore by Doob's optional sampling theorem, Lebesgue's dominated convergence theorem and continuity of A yields,

$$\lim_n E[Z_T - Z_{T_n}] = \lim_n E[M_T - N_{T_n}] - \lim_n E[A_T - A_{T_n}] = -E[\lim_n (A_T - A_{T_n})] = 0. \tag{64}$$

Therefore Z is regular.

Conversely suppose that Z is regular and assume that A is not continuous at time T. Since A is predictable, so is A_- and $\triangle A$. In particular, T is a predictable time. Then there exists an announcing sequence $\{T_n\} \uparrow T$. Since Z is regular,

$$0 = \lim_n E[Z_T - Z_{T_n}] = \lim E[M_T - M_{T_n}] - \lim E[A_T - A_{T_n}] = E[\triangle A_T]. \tag{65}$$

Since A is an increasing process and $\triangle A_T \geq 0$. Therefore $\triangle A_T = 0$ a.s. This is a contradiction. Thus A is continuous.

31. Let T be an arbitrary $\mathbb{F}^\mu$ stopping time and $\Lambda = \{\omega : X_T(\omega) \neq X_{T-}(\omega)\}$. Then by Meyer's theorem, $T = T_\Lambda \wedge T_{\Lambda^c}$ where T_Λ is totally inaccessible time and T_Λ^c is predictable time. By continuity of X, $\Lambda = \emptyset$ and $T_\Lambda = \infty$. Therefore $T = T_{\Lambda^c}$. It follows that all stopping times are predictable and there is no totally inaccessible stopping time.

32. By exercise 31, the standard Brownian space supports only predictable time since Brownian motion is clearly a strong Markov Feller process. Since $\mathcal{O} = \sigma([S,T[: S,T$are stopping time and $S \leq T)$ and $\mathcal{P} = \sigma([S,T[: S,T$are predictable times and $S \leq T)$, if all stopping times are predictable $\mathcal{O} = \mathcal{P}$.

35. $\mathcal{E}(M_t) \geq 0$ and $\mathcal{E}(M_t) = \exp\left[B_t^\tau - 1/2(t \wedge \tau)\right] \leq e$. So it is a bounded local martingale and hence martingale. If $\mathcal{E}(-M)$ is a uniformly integrable martingale, there exists $\mathcal{E}(-M_\infty)$ such that $E[\mathcal{E}(-M_\infty)] = 1$. By the law of iterated logarithm, $\exp(B_\tau - 1/2\tau)1_{\{\tau=\infty\}} = 0$ a.s. Then

$$E[\mathcal{E}(-M_\infty)] = E\left[\exp\left(-1 - \frac{\tau}{2}\right) 1_{\{\tau<\infty\}}\right] \leq e^{-1} < 1. \tag{66}$$

This implies that $\mathcal{E}(-M)$ is not a uniformly integrable martingale.

36. Clearly $\mathcal{F}_{T-} \subset \mathcal{F}_T$ and $\sigma\{\triangle M_T : M \text{ a martingale}\} \subset \mathcal{F}_T$. Therefore $\mathcal{F}_{T-} \vee \sigma\{\triangle M_T : M \text{ a martingale}\} \subset \mathcal{F}_T$. For the converse, recall Theorem 6 in chapter 1 and Theorem 5 in chapter 3.

$$\mathcal{F}_T = \sigma\{X_T; X \text{ all adapted càdlàg processes}\}$$
$$\mathcal{F}_{T-} = \sigma\{H_T; H \text{ predictable}\}$$

Pick an X_T. Assume first that X_T is bounded. Let $M_t = E[X_T|\mathcal{F}_{t\wedge T}]$. Note that X_T is bounded and in particular in L_1. So this process is well defined. Then M_t is a martingale such that $M_T = X_T$. Then $X_T = M_T = M_{T-} + \triangle M_T$ where M_{T-} is a left continuous process M_{t-} evaluated at T. $M_T \in \sigma\{\triangle M_T : M \text{ a martingale}\}$. Since $\{M_{t-}\}$ is a predictable process, $M_{T-} \in \mathcal{F}_{T-}$. Thus $X_T = M_T \in \mathcal{F}_{T-} \vee \sigma\{\triangle M_T : M \text{ a martingale}\}$. For unbounded X_T, set $X_T^n = X_T 1_{\{|X_T|<n\}}$. Then $X_T^n \to X_T$ a.s. while $X_T^n \in \mathcal{F}_{T-} \vee \sigma\{\triangle M_T : M \text{ a martingale}\}$ for each n. Then $X_T \in \mathcal{F}_{T-} \vee \sigma\{\triangle M_T : M \text{ a martingale}\}$ and $\mathcal{F}_T \subset \mathcal{F}_{T-} \vee \sigma\{\triangle M_T : M \text{ a martingale}\}$.

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