Université Denis Diderot Paris 7

Probability and Processus

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Exercises sheet 5 : Martingales

Exercise 1 Let $(X_n, n \geq 1)$ be independent, and such that $\mathbb{E}[X_i] := m_i, \mathrm{Var}(X_i) := \sigma_i^2, i \geq 1$. As usual, we set $S_n = \sum_{i=1}^n X_i, \mathcal{F}_n = \sigma(X_1, ..., X_n)$.

1. Find sequences $(b_n), (c_n)$ of real numbers such that $S_n^2 + b_n S_n + c_n$ is a $(\mathcal{F}_n)$-martingale.
2. Assume moreover that $\lambda \in \mathbb{R}$ is such that $\exp(\lambda X_i) \in \mathbb{L}^1$ for any $i \geq 1$, and set $G_i(\lambda) := \mathbb{E}\left[\exp(\lambda X_i)\right], i \geq 1$. Find a sequence $(a_n^\lambda)_{n\geq 0}$ such that $\left(\exp(\lambda S_n - a_n^\lambda)\right)_{n\geq 0}$ is a $(\mathcal{F}_n)$-martingale.

Exercise 2 Let $(\mathcal{F}_n)$ be a filtration and (M_n) a UI $(\mathcal{F}_n)$-martingale. Show that $(M_n, n \geq 0)$ converges a.s. and in L^1 towards a limiting M_∞.
Show that for any $n \in \mathbb{N}$, $M_n = E[M_\infty \mid \mathcal{F}_n]$

Exercise 3 Find an example of a martingale $(M_n, n \geq 0)$ such that almost surely $M_n \to M_\infty$ for some integrable r.v. M_∞, and such that $(\mathbb{E}[M_n])_{n\geq 0}$ does not converge to $\mathbb{E}[M_\infty]$.

Exercise 4

1. Set $X_0 = 0$ and for $k \geq 0$,

$$\mathbb{P}(X_{k+1} = 1 \mid X_k = 0) = \mathbb{P}(X_{k+1} = -1 \mid X_k = 0) = \frac{1}{2k}, \qquad \mathbb{P}(X_{k+1} = 0 \mid X_k = 0) = 1 - \frac{1}{2k}$$
$$\mathbb{P}(X_{k+1} = kX_k \mid X_k \neq 0) = \frac{1}{k}, \qquad \mathbb{P}(X_{k+1} = 0 \mid X_k \neq 0) = 1 - \frac{1}{k}$$

   Show that $(X_n, n \geq 0)$ is a martingale. Does it converge a.s.? in probability? in $\mathbb{L}^1$?

2. Find a martingale $(X_n, n \geq 0)$ such that $X_n \to -\infty$ a.s.
   Hint : You may look for $X_n = \xi_1 + ... + \xi_n$ with $(\xi_i)_{i\geq 1}$ independent, centered (but not identically distributed).

Exercise 5 Let $a > 0$ be fixed, $(\xi_i, i \geq 1)$ be i.i.d., $\mathbb{R}^d$-valued r.v., with each $\xi_i \sim \text{Unif}(B(0,a))$. Set $S_n = x + \sum_{i=1}^n \xi_i$.

1. Let f be a superharmonic function. Show that $(f(S_n), n \geq 0)$ defines a supermartingale.
2. Show that if $d \leq 2$, any nonnegative superharmonic function is in fact constant. Does this result remain true when $d > 2$?

Exercise 6 Let $(V_i, i \geq 1)$ be nonnegative i.i.d.r.r.v, such that $\mathbb{E}[V_i] = 1, \mathbb{P}(V_i = 1) < 1$. We set $X_0 = 1$, $X_n = \prod_{i=1}^n V_i$, and $\mathcal{F}_n = \sigma(V_i, i \leq n)$.

1. Show that (X_n) is a $(\mathcal{F}_n)$-martingale.
2. Does (X_n) converge? In what sense?

Exercise 7
Consider the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_n), P)$ where $\Omega = \mathbb{N}^*, \mathcal{F} = \mathcal{P}(\mathbb{N}^*)$, $P(\{n\}) = \frac{1}{n} - \frac{1}{n+1}$, $\mathcal{F}_n = \sigma\{\{1\}, ..., \{n\}, [n+1, +\infty[\}$. Define a sequence X of r.r.v. such that $X_n = (n+1)\mathbf{1}_{[n+1,+\infty[}$ for any $n \in \mathbb{N}$.

1. Show that $(X_n, n \geq 0)$ is a nonnegative $(\mathcal{F}_n)$-martingale. Check that $X_n \to 0$ a.s. Does X_n converge in L^1 ?
2. What is $\sup_{n\geq 0} X_n(k); k \in \mathbb{N}^*$? Is (X_n) uniformly integrable?

Exercise 8 Let $(U_n)_{n\geq 0}$ be i.i.d. Bernoulli with

$$P(U_n = 1) = p, \quad P(U_n = 0) = q = 1 - p, 0 < p < 1.$$

Let $T = \inf\{n \geq 0 : U_n = 1\}$, and for $n \geq 0$, $X_n = q^{-n}\mathbf{1}_{\{T>n\}}$

1. Show (X_n) is a martingale in a well-choosen filtration.
2. Show that (X_n) converges a.s. to 0.
3. Is the martingale (X_n) bounded in L^1 ? in L^2 ?
4. Does (X_n) converge in L^1 ?
5. Is the sequence $(Y_n = \sqrt{X_n})_{n\geq 0}$ UI?

Exercise 9 Let $\{X_n, n \in \mathbb{N}\}$ be i.i.d., $\sim$ Unif$[0, 2]$. For $n \geq 1$, let $Y_n = \prod_{i=1}^n X_i$.

1. Show that (Y_n) converges a.s. towards a r.r.v Y_∞.

2. Let $Q_n = \sqrt{Y_n}$. Show one can find a real $q > 1$ such that $q^n Q_n$ converges a.s. towards a r.r.v.

3. Conclude that $Y_\infty = 0$ a.s. Is the class $\{Y_n, n \in \mathbb{N}^*\}$ UI ?

Exercise 10 Let $(Y_n)_{n\geq 0}$ be nonnegative, independent r.r.v. defined on $(\Omega, \mathcal{F}, P)$, such that $\mathbb{E}[Y_n] = 1\ \forall n \in \mathbb{N}$. Set, for $n \geq 0$, $X_n = \prod_{i=0}^n Y_i$. We assume that (Y_n) is $(\mathcal{F}_n)$ adapted.

1. Show (X_n) (resp. $\sqrt{X_n}$), is a $(\mathcal{F}_n)$-martingale, (resp. supermartingale).

2. Show that the infinite product $\prod_{k\geq 0} E(\sqrt{Y_k})$ converges in $\mathbb{R}_+$. Denote by ℓ the limit.

3. Assume $\ell = 0$. Show then that $\sqrt{X_n} \to 0$ a.s. Is the martingale (X_n) UI in that case ?

4. Assume $\ell > 0$. Show then that $\sqrt{X_n}$ is Cauchy in L^2. Is (X_n) UI in that case?

Exercise 11 (*) Let μ and ν be probability measures on $(\Omega, \mathcal{F})$, and $\mathcal{F}_n$ a filtration generating $\mathcal{F}$ (i.e. such that $\sigma(\cup \mathcal{F}_n) = \mathcal{F}$).
Let μ_n, ν_n the restrictions of μ, ν to $\mathcal{F}_n$. We assume that for any n, $\mu_n << \nu_n$ so we can set $X_n = \frac{d\mu_n}{d\nu_n}$.

1. Show that (X_n) is a $(\mathcal{F}_n)$-martingale.

2. For $X = \limsup_{n\to\infty} X_n$, show that ν-a.s., $X < \infty$ and $X_n \to X$.
   One should be careful however, that $\mu(X = \infty)$ may still be positive!

3. Introduce $\rho = \frac{\mu+\nu}{2}$ (so that $\mu << \rho, \nu << \rho$), and similarly $\rho_n = \frac{\mu_n+\nu_n}{2}$. Set $Y_n = \frac{d\mu_n}{d\rho_n}, Z_n = \frac{d\nu_n}{d\rho_n}$.

   a. Show that $X_n = \frac{dY_n}{dZ_n}$.

   b. Show that ρ-a.s.,
   $$Y_n \underset{n\to\infty}{\longrightarrow} Y, \qquad Z_n \underset{n\to\infty}{\longrightarrow} Z$$

   c. Establish that
   $$Y = \frac{d\mu}{d\rho}, \qquad Z = \frac{d\nu}{d\rho}.$$

   Hint : It suffices to check that for any $n \in \mathbb{N}$, for any $A \in \mathcal{F}_n$, $\mu(A) = \int_A Y d\rho, \nu(A) = \int_A Z d\rho$.

d. Set $W = \frac{1}{Z}\mathbf{1}_{\{Z>0\}}$. Show that ν-a.s., $YW = X$.

e. Show that for any $A \in \mathcal{F}$,

$$\int_A \mathbf{1}_{\{Z=0\}} Y d\rho = \int_A \mathbf{1}_{\{X=\infty\}} d\mu.$$

f. Using $1 = ZW + \mathbf{1}_{\{Z=0\}}$, deduce from the preceding questions that for any $A \in \mathcal{F}$,

$$\mu(A) = \int_A X d\nu + \mu(A \cap \{X = \infty\}).$$

Note : We have established that $\mu = \mu_r + \mu_s$, where $\mu_r(A) := \int_A X d\nu$ is absolutely continuous with respect to μ, while $\mu_s(A) = \mu(A \cap \{X = \infty\})$ is singular with respect to μ.

4. Assume in this question that μ and ν are, on $\mathbb{R}^{\mathbb{N}}$, product measures (i.e., measures which make the coordinates $(\xi_n(\omega) := \omega_n, n \in \mathbb{N})$ independent). For $x \in \mathbb{R}$, let $F_n(x) = \mu(\xi_n \leq x)$, $G_n(x) = \nu(\xi_n \leq x)$. We assume $F_n << G_n$ and let $q_n := \frac{dF_n}{dG_n}$. Finally, we assume that $\mathcal{F}_n = \sigma(\xi_k, k \leq n)$ and let μ_n (resp. ν_n) be the restriction of μ (resp. of ν) to $\mathcal{F}_n$.

   Establish Kakutani's dichotomy theorem :
   $\mu << \nu$ or $\mu \perp \nu$ according to wether

$$\prod_{m\geq 1} \int \sqrt{q_m} dG_m > 0 \quad \text{or} \quad = 0.$$

Exercise 12 Let $(X_n)_{n\geq 1}$ a sequence of i.i.d.r.r.v, $\sim \mathcal{N}(0,1)$, and $(\alpha_n)_{n\geq 1}$ a sequence of real numbers. Set

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \qquad \mathcal{F}_n = \sigma(X_1, ..., X_n), \quad M_n = \exp\left(\sum_{k=1}^n \alpha_k X_k - \frac{1}{2}\sum_{k=1}^n \alpha_k^2\right), n \geq 1.$$

1. Show that (M_n) is a $(\mathcal{F}_n)$-martingale and that (M_n) converges a.s.
2. Assume in addition that $\sum_{k\geq 1} \alpha_k^2 = +\infty$. Show then that $\lim_{n\to\infty} M_n = 0$, a.s. Is (M_n) UI?

Exercise 13 The following is a model for a reinforced random walk on the set $\{-1, 0, 1\}$. We let $W : \mathbb{N} \to \mathbb{R}_+^*$ be the reinforcement function and we define $V : \mathbb{N} \to \mathbb{R}_+^*$ by

$$V(0) = 0, \qquad V(n) = \sum_{i=0}^{n-1} \frac{1}{W(i)}, n \geq 1.$$

The walk is $(X_n, n \geq 0)$ and the filtration is $\mathcal{F}_n := \sigma(X_i, i \leq n)$. We also define $Z_n^+ = \sum_{i=0}^n \mathbf{1}_{X_i=1}$ (resp. $Z_n^- = \sum_{i=0}^n \mathbf{1}_{X_i=-1}$), the total occupation time at 1 (resp. -1) of the walk X up to time n. We are yet to define the transitions of $(X_n, n \geq 0)$: for all $n \in \mathbb{N}$, $X_{2n} = 0$ and

$$P(X_{2n+1} = 1 \mid \mathcal{F}_{2n}) = \frac{W(Z_{2n}^+)}{W(Z_{2n}^+) + W(Z_{2n}^-)}, \quad P(X_{2n+1} = -1 \mid \mathcal{F}_{2n}) = \frac{W(Z_{2n}^-)}{W(Z_{2n}^+) + W(Z_{2n}^-)}.$$

In other words, the walk is on $\{-1, 0, 1\}$ and starts at 0, all the jumps have size 1, and the walk takes value 1 at time $2n+1$ with a probability proportional to $W(Z_n^+)$.

1. Show that $(M_n = V(Z_n^+) - V(Z_n^-), n \geq 0)$ is a $(\mathcal{F}_n)$-martingale.

2. Let $\tau = \inf\{n \geq 0 : X_n = 1\}$ the time of the first visit at 1. Find a necessary and sufficient condition for having $\tau < \infty$ a.s. In such a case compute $E[V(Z_\tau^-)]$.

3. Suppose now that $\sum_{k \geq 0} W(k)^{-1} < \infty$. Check then that (M_n) converges a.s. towards a limiting M_∞. Compute $E[M_\infty]$. Show that $P(Z_\infty^+ \neq Z_\infty^-) > 0$.

Exercise 14 Let $X \sim \mathcal{N}(0, \sigma^2)$, with $\sigma^2 \in (0,1)$, and for $k \in \mathbb{N}$, let $\eta_k \sim \mathcal{N}(0, \varepsilon_k^2)$ with $\varepsilon_k > 0$. We assume that the variables $\{X, \eta_0, \eta_1, ...\}$ are independent. Define $Y_k = X + \eta_k, k \in \mathbb{N}$, $\mathcal{F}_n = \sigma(Y_0, ..., Y_n), n \in \mathbb{N}, \mathcal{F}_\infty = \sigma(Y_n; n \geq 0)$. Finally let $X_n := E(X \mid \mathcal{F}_n) = E(X \mid Y_0, ..., Y_n)$.

1. Show that (X_n) is a martingale and that (X_n) converges a.s. and in L^1 towards a r.r.v. X_∞. What is the relationship between X and X_∞ ?

2. Show (X_n) is bounded in L^2.
   Show that the three following properties are equivalent

$$a) X_n \xrightarrow[n\to\infty]{\mathbb{L}^2} X, \quad b) X_n \xrightarrow[n\to\infty]{\mathbb{L}^1} X, \quad c) X \text{ is } \mathcal{F}_\infty\text{-measurable}.$$

3. Compute $E(Y_iY_j), E(Y_i^2)$ and $E(XY_i)$ for $i, j \geq 0, i \neq j$. Show that for any $n \geq 0$, $i = 0, ..., n$ we have $E(Z_nY_i) = 0$, where

$$Z_n := X - \frac{\sigma^2}{1 + \sigma^2 \sum_{k=0}^n \varepsilon_k^{-2}} \sum_{j=0}^n \varepsilon_j^{-2} Y_j.$$

4. Show that for any $n \geq 0$ the variable Z_n is independent of $\sigma(Y_0, ..., Y_n)$, deduce that $X_n = X - Z_n$.

5. Compute $E((X - X_n)^2)$ and show that

$$X_n \xrightarrow[n\to\infty]{\mathbb{L}^2} X \quad \Leftrightarrow \quad \lim_{n\to\infty} \sum_{i=0}^n \varepsilon_i^{-2} = +\infty.$$

6. Discuss the case $\varepsilon_i = \varepsilon > 0$ for any $i \geq 0$.

Exercise 15
Suppose in a game between a gambler and a croupier the total capital in play is 1. After the nth hand the proportion of the capital held by the gambler is denoted $X_n \in [0,1]$, thus that held by the croupier is $1 - X_n$. We assume $X_0 = p \in (0,1)$. We assume that the rules of the game are such that after n hands, the probability for the gambler to win $(n+1)$th hand is X_n; if he does, he gains half of the capital the croupier held after the nth hand, while if he loses he gives half of his capital. Let $\mathcal{F}_n = \sigma(X_1, ..., X_n)$

1. Show (X_n) is a martingale.
2. Show that (X_n) converges a.s. and in $\mathbb{L}^2$ towards a limiting Z.
3. Show that $E(X_{n+1}^2) = E(3X_n^2 + X_n)/4$. Deduce that $E(Z^2) = E(Z) = p$. Deduce the law of Z.
4. For any $n \geq 0$, let $Y_n := 2X_{n+1} - X_n$. Find the conditional law of X_{n+1} knowing $\mathcal{F}_n$. Deduce
$$P(Y_n = 0 \mid \mathcal{F}_n) = 1 - X_n; \quad P(Y_n = 1 \mid \mathcal{F}_n) = X_n,$$
and express the law of Y_n.
5. Let $G_n := \{Y_n = 1\}, P_n := \{Y_n = 0\}$. Prove that $Y_n \to Z$ a.s. and deduce that
$$P\left(\liminf_{n\to\infty} G_n\right) = p, \quad P\left(\liminf_{n\to\infty} P_n\right) = 1 - p.$$
Are the variables $\{Y_n, n \geq 1\}$ independent ?
6. Interpret 4, 5, 6 in terms of gain/loss for the gambler.

Exercise 16 Let $Y := (Y_n)_{n\in\mathbb{N}^*}\}$ be independent and such that for $n \in \mathbb{N}^*$,
$$Y_n \sim \mathcal{N}\left(\sqrt{1-n^{-2}}, n^{-2}\right).$$
We set $M_0 := 1$ and for $n \in \mathbb{N}^*$, $M_n := M_{n-1}Y_n^2$. We finally let for any $n \in \mathbb{N}$, $\mathcal{F}_n := \sigma(Y_k, k \leq n)$.

1. Show $M := (M_n)_n$ is a $(\mathcal{F}_n)$-martingale. Does M converge almost surely ?
2. For $i \in \mathbb{N}^*$ set
$$b_i := \frac{1}{i}\sqrt{\frac{2}{\pi}}\exp\left(\frac{1-i^2}{2}\right) + \sqrt{1-\frac{1}{i^2}}\,\mathbb{P}\left(|Z| \leq \sqrt{i^2-1}\right), \qquad \text{where } Z \sim \mathcal{N}(0,1).$$

Setting $N_0 := 1$, and for $n \in \mathbb{N}^*$,

$$N_n := \frac{\sqrt{M_n}}{\prod_{i=1}^n b_i},$$

show that N is a $(\mathcal{F}_n)$-martingale.

3. Establish that $b_n = 1 - (2n^2)^{-1} + o(n^{-2})$ as $n \to \infty$ and deduce that $\sup_n \mathbb{E}\left[N_n^2\right] < \infty$.

4. Discuss convergence properties of N and establish that $N^2 = (N_n^2)_{n\in\mathbb{N}}$ is uniformly integrable.

5. Conclude that M converges in $\mathbb{L}^1$ (towards a limit denoted M_∞).

6. Show that $\mathbb{P}(M_\infty = 0) = 0$. (Hint : you may use Kolmogorov's 0–1 law.)

7. Define $\widetilde{Y} := (\widetilde{Y}_n)_{n\in\mathbb{N}^*}\}$ a sequence of independent random variables such that for $n \in \mathbb{N}^*$,

$$\widetilde{Y}_n \sim \mathcal{N}\left(\sqrt{1-n^{-1}}, n^{-1}\right),$$

and $\widetilde{\mathcal{F}}_n := \sigma(\widetilde{Y}_k, k \leq n)$.

Setting $\widetilde{M}_0 := 1$, and for $n \in \mathbb{N}^*$,

$$\widetilde{M}_n := \widetilde{M}_{n-1}\widetilde{Y}_n^2,$$

establish that $\widetilde{M} := \left(\widetilde{M}_n\right)_{n\in\mathbb{N}}$ is a $(\tilde{\mathcal{F}}_n)$-martingale. Show then that $\widetilde{M}$ converges almost surely to 0. Is $\widetilde{M}$ uniformly integrable ? (Hint : one could, similarly to the above, define an intermediate martingale $\widetilde{N}$.)

Exercise 17 Let $\{U_n\}_{n\in\mathbb{N}^*}$ be i.i.d., $\sim$ Unif$[0,1]$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and for $n \in \mathbb{N}^*$, $\mathcal{F}_n := \sigma(U_1, \ldots, U_n)$.
The sequence $X := (X_n)_{n\in\mathbb{N}}$ is defined inductively as follows :

$$X_0 = p \in (0,1), \qquad \text{and, for } n \geq 0, \quad X_{n+1} := \theta X_n + (1-\theta)\mathbf{1}_{[0,X_n]}(U_{n+1}),$$

with $\theta \in (0,1)$ being fixed.

1. Show that $0 < X_n < 1$, for any $n \in \mathbb{N}$.

2. Show that X is a $(\mathcal{F}_n)_{n\geq 0}$-martingale.

3. Establish that X converges almost surely and in $\mathbb{L}_p$ for any $p \geq 1$, towards a limiting random variable X_∞.

4. Show that for any $n \geq 0$,

$$\mathbb{E}\left[(X_{n+1} - X_n)^2\right] := (1-\theta)^2 \mathbb{E}\left[X_n(1 - X_n)\right].$$

5. Compute $\mathbb{E}\left[X_\infty(1 - X_\infty)\right]$. Deduce the law of X_∞.

Exercise 18 Let $(Y_n)_{n\in\mathbb{N}^*}$ be a sequence of random variables, and assume (Y_n) converges to a limiting Y.
Also, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the sequence of independent random variables $X := (X_n)_{n\in\mathbb{N}^*}$ is defined, and we assume that the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ (i.e. $S_0 = 0$ and $S_n := \sum_{j=1}^n X_j$) converges in distribution. Set $(\mathcal{F}_n)$ the natural filtration of X and $\phi_n(t) = \mathbb{E}(\exp(itS_n))$ for $t \in \mathbb{R}$.

1. Establish that $(\phi_{Y_n}(\cdot))_{n\geq 1}$ converges uniformly on every compact, *i.e.* show that for any $a > 0$,
$$\max_{t\in[-a,a]} |\phi_{Y_n}(t) - \phi_Y(t)| \underset{n\to\infty}{\longrightarrow} 0.$$
Establish moreover there exists $a > 0$ such that for any $n \geq 1$, $\min_{t\in[-a,a]} |\phi_{Y_n}(t)| \geq 1/2$.

2. Show that there exists $t_0 > 0$ such that if $t \in [-t_0, t_0]$ then $\left(\exp(itS_n)/\phi_n(t)\right)_{n\geq 0}$ is a $(\mathcal{F}_n)$-martingale (*i.e.* real and imaginary parts of $\left(\exp(itS_n)/\phi_n(t)\right)_n$ are both $(\mathcal{F}_n)$-martingales).

3. Show that we can choose $t_0 > 0$ such that for any $t \in [-t_0, t_0]$, $\lim_{n\to\infty} \exp(itS_n(\omega))$ exists $\mathbb{P}(d\omega)$-p.s..

4. (*) Set
$$C := \{(t, \omega) \in [-t_0, t_0] \times \Omega : \lim_{n\to\infty} \exp(itS_n(\omega)) \text{ exists } \}.$$
Explain why C is measurable (the σ-algebra of measurable subsets of $[-t_0, t_0] \times \Omega$ is the product of $\mathcal{B}([-t_0, t_0])$ with $\mathcal{F}$).

5. (*) Establish that $\int_{-t_0}^{t_0} \int_\Omega \mathbf{1}_C(t, \omega)\mathbb{P}(d\omega)dt = 2t_0$. Deduce $\lim_{n\to\infty} S_n(\omega)$ exists $\mathbb{P}(d\omega)$-a.s. (One will admit the following result : if $(c_n)_{n\in\mathbb{N}^*}$ is a sequence of reals such that $\lim_{n\to\infty} \exp(itc_n)$ exists for almost every $t \in [-t_0, t_0]$, then $\lim_{n\to\infty} c_n \in \mathbb{R}$ exists).