Martingales

Exercises

Exercise 4.1. Show that a sum of martingales is a martingale.

Exercise 4.2.
a) Is any Markovian process is a martingale? If yes, prove it. Otherwise, construct a counter-example.
b) Is any martingale is Markovian? If yes, prove it. Otherwise, construct a counter-example.

Exercise 4.3. Let $M = \{M_t : t \in \{0, 1, 2, ...\}\}$ be a square-integrable martingale[1] existing on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F}$ is the filtration filtration $\{\mathcal{F}_t : t \in \{0, 1, 2, ...\}\}$. The predictable process $\eta = \{\eta_t : t \in \{0, 1, 2, ...\}\}$ is constructed on the same space, that is, for all $t \in \{1, 2, ...\}$, η_t is $\mathcal{F}_{t-1}-$measurable, η_0 being $\mathcal{F}_0-$measurable. Moreover, we assume that for all $t \in \{0, 1, 2, ...\}$, the random variable η_t is square-integrable,that is, $\mathrm{E}\left[|\eta_t^2|\right] < \infty$. Show that the process $N = \{N_t : t \in \{0, 1, 2, ...\}\}$ defined as

$$N_t = N_0 + \sum_{k=1}^{t} \eta_k \left(M_k - M_{k-1}\right)$$

is a martingale provided that N_0 is $\mathcal{F}_0-$measurable.

Exercise 4.4. Let X be a square-integrable random variable $\left(\mathrm{E}^{\mathbb{P}}\left[|X|\right] < \infty\right)$ constructed on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Prove that the stochastic process $\{M_t : t \in \{0, 1, 2, ...\}\}$ defined as

$$M_t = \mathrm{E}^{\mathbb{P}}\left[X \,|\mathcal{F}_t\right], \ \ t \geq 0$$

is a martingale.

Exercise 4.5. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which two filtrations are constructed, $\{\mathcal{F}_t : t \geq 0\}$ and $\{\mathcal{G}_t : t \geq 0\}$, satisfying

$$\mathcal{F}_t \subseteq \mathcal{G}_t$$

a) Let $M = \{M_t : t \geq 0\}$ be a $\{\mathcal{F}_t : t \geq 0\}-$martingale and let $N = \{N_t : t \geq 0\}$ be a $\{\mathcal{G}_t : t \geq 0\}-$martingale. Is M a $\{\mathcal{G}_t : t \geq 0\}-$martingale ? Is N a $\{\mathcal{F}_t : t \geq 0\}-$martingale ? Justify.
b) Let τ be a $\{\mathcal{F}_t : t \geq 0\}-$stopping time and θ be a $\{\mathcal{G}_t : t \geq 0\}-$stopping time. Is θ a $\{\mathcal{F}_t : t \geq 0\}-$stopping time ? Is τ a $\{\mathcal{G}_t : t \geq 0\}-$stopping time ? Justify.

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[1] $\forall t \geq 0,\ E\left[M_t^2\right] < \infty$

Exercise 4.6. Let $\varepsilon_1, \varepsilon_2, ...$ be a sequence of independent random variables with zero mean and variance $\mathrm{Var}[\varepsilon_i] = \sigma_i^2$. Let

$$S_n = \sum_{i=1}^{n} \varepsilon_i \text{ and } T_n^2 = \sum_{i=1}^{n} \sigma_i^2.$$

Prove that $\{S_n^2 - T_n^2\}_{n\in\mathbb{N}}$ is a martingale.

Exercise 4.7. Let $\{X_t : t \geq 0\}$ be a $\{\mathcal{G}_t : t \geq 0\}$ $-$martingale and $\mathcal{F}_t = \sigma(X_s,\, s \leq t)$. Prove that $\{X_t : t \geq 0\}$ is also a $\{\mathcal{F}_t : t \geq 0\}$ $-$martingale.

Solutions

1 Exercise 4.1

Let $X = \{X_t : t \in \{0, 1, ...\}\}$ and $Y = \{Y_t : t \in \{0, 1, ...\}\}$, two martingales on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Since $\forall t \in \{0, 1, ...\}$,

$$\mathrm{E}^{\mathbb{P}}[|X_t + Y_t|] \leq \underbrace{\mathrm{E}^{\mathbb{P}}[|X_t|]}_{\substack{<\infty \\ \text{since } X \text{ is a martingale}}} + \underbrace{\mathrm{E}^{\mathbb{P}}[|Y_t|]}_{\substack{<\infty \\ \text{since } Y \text{ is a martingale}}} < \infty,$$

the $(M1)$ condition is verified.

Since two $\mathcal{F}_t-$measurable random variables is $\mathcal{F}_t-$measurable, then $\forall t \in \{0, 1, ...\}$, X_t+Y_t is $\mathcal{F}_t-$measurable. Therefore, the stochastic process $X + Y$ is adapted to the filtration $\mathbb{F}$.

Condition $(M3)$: $\forall s, t \in \{0, 1, ...\}$ such that $s < t$

$$\mathrm{E}^{\mathbb{P}}[X_t + Y_t | \mathcal{F}_s] = \mathrm{E}^{\mathbb{P}}[X_t | \mathcal{F}_s] + \mathrm{E}^{\mathbb{P}}[Y_t | \mathcal{F}_s] = X_s + Y_s.$$

The proof can be generalized to the sum of more than 2 martingales using induction.

2 Exercise 4.2

a) It is not all Markovian processes that are martingales.

Counter-example. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$,consider the random walk X define as

$$X_0 = 0 \text{ and } \forall t \in \{0, 1, 2, ...\}, X_t = \sum_{n=1}^{t} \xi_n$$

where the sequence of independent and identically distributed random variables $\{\xi_t : t \in \{1, 2, ...\}\}$ has a positive expectation $\mathrm{E}^{\mathbb{P}}[\xi_t] = \mu > 0$. Since X is a random walk, it is Markovian. But X cannot be a martingale since its expectation varies over time. Indeed,

$$\mathrm{E}^{\mathbb{P}}[X_t] = \mathrm{E}^{\mathbb{P}}\left[\sum_{n=1}^{t} \xi_n\right] = \sum_{n=1}^{t} \mathrm{E}^{\mathbb{P}}[\xi_n] = \sum_{n=1}^{t} \mu = t\mu. \blacksquare$$

b) It is not all martingales that are Markovian. **Counter-example** :

ω	$X_1(\omega)$	$X_2(\omega)$	$X_3(\omega)$	$\mathbb{P}$
ω_1	-1	-2	-4	$\frac{1}{8}$
ω_2	-1	-2	0	$\frac{1}{8}$
ω_3	-1	0	0	$\frac{1}{8}$
ω_4	-1	0	0	$\frac{1}{8}$
ω_5	1	0	2	$\frac{1}{8}$
ω_6	1	0	-2	$\frac{1}{8}$
ω_7	1	2	2	$\frac{1}{8}$
ω_8	1	2	2	$\frac{1}{8}$

Indeed,

$$\begin{aligned}
\sigma\{X_1\} &= \sigma\{\{\omega_1,\omega_2,\omega_3,\omega_4\},\{\omega_5,\omega_6,\omega_7,\omega_8\}\} \\
\sigma\{X_1,X_2\} &= \sigma\{\{\omega_1,\omega_2\},\{\omega_3,\omega_4\},\{\omega_5,\omega_6\},\{\omega_7,\omega_8\}\} \\
\sigma\{X_1,X_2,X_3\} &= \sigma\{\{\omega_1\},\{\omega_2\},\{\omega_3,\omega_4\},\{\omega_5\},\{\omega_6\},\{\omega_7,\omega_8\}\}
\end{aligned}$$

The process $X = \{X_t : t \in \{1,2,3\}\}$ is not Markovian on the space $(\Omega,\mathcal{F},\mathbb{P})$ where $\mathcal{F}$ is the $\sigma-$algebra generated by all elements of Ω. Indeed,

$$\begin{aligned}
\mathbb{P}(X_3 = 2 \,|X_2 = 0) &= \frac{\mathbb{P}(X_3 = 2 \text{ and } X_2 = 0)}{\mathbb{P}(X_2 = 0)} \\
&= \frac{\mathbb{P}\{\omega_5\}}{\mathbb{P}\{\omega_3,\omega_4,\omega_5,\omega_6\}} = \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}
\end{aligned}$$

but

$$\begin{aligned}
\mathbb{P}(X_3 = 2 \,|X_2 = 0 \text{ and } X_1 = -1) &= \frac{\mathbb{P}(X_3 = 2,\ X_2 = 0 \text{ and } X_1 = -1)}{\mathbb{P}(X_2 = 0 \text{ and } X_1 = -1)} \\
&= \frac{\mathbb{P}(\varnothing)}{\mathbb{P}\{\omega_3,\omega_4\}} = \frac{0}{\frac{1}{4}} = 0.
\end{aligned}$$

However, the process $X = \{X_t : t \in \{1,2,3\}\}$ is a martingale on the space $(\Omega,\mathcal{F},\mathbb{P})$ if the filtration is generated by the process itself. Indeed, the condition $(M1)$ is trivially satisfied,

$(M2)$ is also satisfied from the construction of the filtration. Need to verify the last condition $(M3^*)$. But

$$\mathrm{E}^{\mathbb{P}}\left[X_3\left|\sigma\left\{X_1, X_2\right\}\right.\right] = X_2$$

since

$$\begin{aligned}
\forall\omega &\in \{\omega_1,\omega_2\}, \, \mathrm{E}^{\mathbb{P}}\left[X_3\left|\sigma\left\{X_1, X_2\right\}\right.\right](\omega) = \frac{-4\times\frac{1}{8}+0\times\frac{1}{8}}{\frac{1}{4}} = -2 = X_2(\omega) \\
\forall\omega &\in \{\omega_3,\omega_4\}, \, \mathrm{E}^{\mathbb{P}}\left[X_3\left|\sigma\left\{X_1, X_2\right\}\right.\right](\omega) = \frac{0\times\frac{1}{8}+0\times\frac{1}{8}}{\frac{1}{4}} = -0 = X_2(\omega) \\
\forall\omega &\in \{\omega_5,\omega_6\}, \, \mathrm{E}^{\mathbb{P}}\left[X_3\left|\sigma\left\{X_1, X_2\right\}\right.\right](\omega) = \frac{2\times\frac{1}{8}-2\times\frac{1}{8}}{\frac{1}{4}} = 0 = X_2(\omega) \\
\forall\omega &\in \{\omega_7,\omega_8\}, \, \mathrm{E}^{\mathbb{P}}\left[X_3\left|\sigma\left\{X_1, X_2\right\}\right.\right](\omega) = \frac{2\times\frac{1}{8}+2\times\frac{1}{8}}{\frac{1}{4}} = 2 = X_2(\omega)
\end{aligned}$$

and

$$\mathrm{E}^{\mathbb{P}}\left[X_2\left|\sigma\left\{X_1\right\}\right.\right] = X_1$$

since

$$\begin{aligned}
\forall\omega &\in \{\omega_1,\omega_2,\omega_3,\omega_4\} \\
\mathrm{E}^{\mathbb{P}}\left[X_2\left|\sigma\left\{X_1\right\}\right.\right](\omega) &= \frac{-2\times\frac{1}{8}-2\times\frac{1}{8}+0\times\frac{1}{8}+0\times\frac{1}{8}}{\frac{1}{2}} = -1 = X_1(\omega) \\
\forall\omega &\in \{\omega_5,\omega_6,\omega_7,\omega_8\} \\
\mathrm{E}^{\mathbb{P}}\left[X_2\left|\sigma\left\{X_1\right\}\right.\right](\omega) &= \frac{0\times\frac{1}{8}+0\times\frac{1}{8}+2\times\frac{1}{8}+2\times\frac{1}{8}}{\frac{1}{2}} = 1 = X_1(\omega). \blacksquare
\end{aligned}$$

3 Exercise 4.3

Verification of $(M1)$:

$$
\begin{aligned}
\mathrm{E}\left[|N_t|\right] &= \mathrm{E}\left[\left|N_0+\sum_{k=1}^{t}\eta_k\left(M_k-M_{k-1}\right)\right|\right] \\
&\le \mathrm{E}\left[|N_0|\right]+\sum_{k=1}^{t}\mathrm{E}\left[|\eta_k|\left|M_k-M_{k-1}\right|\right] \\
&\le \mathrm{E}\left[|N_0|\right]+\sum_{k=1}^{t}\left(\mathrm{E}\left[|\eta_k|^2\right]\right)^{1/2}\left(\mathrm{E}\left[\left|M_k-M_{k-1}\right|^2\right]\right)^{1/2} \\
&\quad \text{from Cauchy-Schwarz inequality} \\
&= \mathrm{E}\left[|N_0|\right]+\sum_{k=1}^{t}\left(\mathrm{E}\left[|\eta_k|^2\right]\right)^{1/2}\left(\mathrm{E}\left[M_k^2-2M_kM_{k-1}+M_{k-1}^2\right]\right)^{1/2} \\
&= \mathrm{E}\left[|N_0|\right]+\sum_{k=1}^{t}\left(\mathrm{E}\left[|\eta_k|^2\right]\right)^{1/2}\left(\mathrm{E}\left[M_k^2\right]-2\mathrm{E}\left[M_kM_{k-1}\right]+\mathrm{E}\left[M_{k-1}^2\right]\right) \\
&= \mathrm{E}\left[|N_0|\right]+\sum_{k=1}^{t}\left(\mathrm{E}\left[|\eta_k|^2\right]\right)^{1/2}\left(\mathrm{E}\left[M_k^2\right]-2\mathrm{E}\left[M_kM_{k-1}\right]+\mathrm{E}\left[M_{k-1}^2\right]\right) \\
&\quad \begin{aligned} \text{since } \mathrm{E}\left[M_kM_{k-1}\right] &= \mathrm{E}\left[\mathrm{E}\left[M_kM_{k-1}\left|\mathcal{F}_{k-1}\right.\right]\right] \\ &= \mathrm{E}\left[M_{k-1}\mathrm{E}\left[M_k\left|\mathcal{F}_{k-1}\right.\right]\right] \\ &= \mathrm{E}\left[M_{k-1}^2\right] \end{aligned} \\
&= \underbrace{\mathrm{E}\left[|N_0|\right]}_{<\infty}+\sum_{k=1}^{t}\underbrace{\left(\mathrm{E}\left[|\eta_k|^2\right]\right)^{1/2}}_{<\infty}\underbrace{\left(\mathrm{E}\left[M_k^2\right]-\mathrm{E}\left[M_{k-1}^2\right]\right)^{1/2}}_{<\infty} \\
&< \infty
\end{aligned}
$$

Verification de $(M2)$:

$$
N_t = N_0+\sum_{k=1}^{t}\underbrace{\underbrace{\eta_k}_{\mathcal{F}_{k-1}-\text{mesurable}}\underbrace{\left(M_k-M_{k-1}\right)}_{\mathcal{F}_k-\text{mesurable}}}_{\mathcal{F}_k-\text{mesurable donc }\mathcal{F}_t-\text{mesurable}} \text{ est } \mathcal{F}_t-\text{mesurable.}
$$

Verification de $(M3)$: firstly, note that for all $0 \leq s < t < \infty$,

$$N_t = N_s + \sum_{k=s+1}^{t} \eta_k \left(M_k - M_{k-1}\right).$$

Indeed,

$$\begin{aligned}
N_t &= N_0 + \sum_{k=1}^{t} \eta_k \left(M_k - M_{k-1}\right) \\
&= N_0 + \sum_{k=1}^{s} \eta_k \left(M_k - M_{k-1}\right) + \sum_{k=s+1}^{t} \eta_k \left(M_k - M_{k-1}\right) \\
&= N_s + \sum_{k=s+1}^{t} \eta_k \left(M_k - M_{k-1}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{E}\left[N_t \middle| \mathcal{F}_s \right] &= \mathrm{E}\left[N_s + \sum_{k=s+1}^{t} \eta_k \left(M_k - M_{k-1}\right) \middle| \mathcal{F}_s \right] \\
&= N_s + \sum_{k=s+1}^{t} \mathrm{E}\left[\eta_k \left(M_k - M_{k-1}\right) \middle| \mathcal{F}_s \right] \\
&= N_s + \sum_{k=s+1}^{t} \mathrm{E}\left[\mathrm{E}\left[\eta_k \left(M_k - M_{k-1}\right) \middle| \mathcal{F}_{k-1} \right] \middle| \mathcal{F}_s \right] \text{ from } (\mathrm{E}C3) \\
&= N_s + \sum_{k=s+1}^{t} \mathrm{E}\left[\eta_k \underbrace{\mathrm{E}\left[\left(M_k - M_{k-1}\right) \middle| \mathcal{F}_{k-1} \right]}_{=0} \middle| \mathcal{F}_s \right] \\
&= N_s.
\end{aligned}$$

4 Exercise 4.4

Verification of $(M1)$:

$$\begin{aligned}
\mathrm{E}^{\mathbb{P}}\left[|M_t|\right] &= \mathrm{E}^{\mathbb{P}}\left[\left|\mathrm{E}^{\mathbb{P}}\left[X\,|\mathcal{F}_t\right]\right|\right] \\
&\leq \mathrm{E}^{\mathbb{P}}\left[\left|\mathrm{E}^{\mathbb{P}}\left[|X|\,|\mathcal{F}_t\right]\right|\right] \text{ since } X \leq |X| \\
&= \mathrm{E}^{\mathbb{P}}\left[\mathrm{E}^{\mathbb{P}}\left[|X|\,|\mathcal{F}_t\right]\right] \text{ since } \mathrm{E}^{\mathbb{P}}\left[|X|\,|\mathcal{F}_t\right] \geq 0 \\
&= \mathrm{E}^{\mathbb{P}}\left[|X|\right] \text{ since } (EC3) \\
&< \infty \text{ by hypothesis}
\end{aligned}$$

Verification of $(M2)$: $M_t = \mathrm{E}^{\mathbb{P}}\left[X\,|\mathcal{F}_t\right]$ is $\mathcal{F}_t-$measurable because constant on the atoms $\mathcal{F}_t$.

Verification of $(M3)$: Let $0 \leq s \leq t$.

$$\begin{aligned}
\mathrm{E}^{\mathbb{P}}\left[M_t\,|\mathcal{F}_s\right] &= \mathrm{E}^{\mathbb{P}}\left[\mathrm{E}^{\mathbb{P}}\left[X\,|\mathcal{F}_t\right]|\mathcal{F}_s\right] \text{ from the defn. of } M_t \\
&= \mathrm{E}^{\mathbb{P}}\left[X\,|\mathcal{F}_s\right] \text{ from } (EC3) \text{ since } \mathcal{F}_s \subset \mathcal{F}_t \\
&= M_s \text{ from the definition of } M_s. \blacksquare
\end{aligned}$$