A bayesian solution to the Behrens-Fisher problem

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Summary

- multiparameter bayesian models; case of a normal with unknown mean and variance,
- comparison of two normal means: the Behrens-Fisher problem,
- a solution using the Gibbs sampler,
- "Modern" Behrens-Fisher problems.

Multiparameter models

Generally many parameters are involved in a statistical model.

Some of them are not of interest: **nuisance parameters.**

2 parameters case: consider $\boldsymbol{\theta} = (\theta_1, \theta_2)$, $\pi(\boldsymbol{\theta})$ is the associated **joint prior** distribution. We compute the **joint posterior** using the bayes rule:

$$\pi(\boldsymbol{\theta}|x) = \pi(\theta_1, \theta_2|x) \propto \pi(\theta_1, \theta_2) p(x|\theta_1, \theta_2).$$

As a consequence, if one wants to make inference on θ_1, we integrate out θ_2

$$\pi(\theta_1|x) = \int_{\Theta_2} \pi(\theta_1, \theta_2|x) d\theta_2$$

Terminology:

- **posterior marginal** of θ_1: $\pi(\theta_1|x)$.
- **posterior conditional** of θ_1 given θ_2: $\pi(\theta_1|\theta_2, x)$.

A remark on invariant prior specification

Location

If the parameter of interest is a location parameter θ, i.e., $x|\theta \sim p(x-\theta)$. A proper non informative prior has to be invariant w.r.t translations, i.e.,

$$\pi(\theta - \theta_0) = \pi(\theta), \ \forall \theta_0.$$

Hence $\pi(\theta) = constant$.

remark:

- if the parameter space is unbounded, this prior is not a p.d.f, this is an **improper prior**. This is fine as long as the posterior is proper.
- sufficient condition for obtaining a proper posterior is that the prior predictive distribution is finite for any x.

A remark on invariant prior specification

Scale

If the parameter of interest is a scale parameter, i.e., $x|\theta \sim \frac{1}{\theta} p(\frac{x}{\theta})$. The prior has to be scale invariant.

$$\pi(\theta) = \frac{1}{c}\pi\left(\frac{\theta}{c}\right),\ \forall c > 0.$$

Hence, we choose $\pi(\theta) \propto \frac{1}{\theta}$.

Normal model with unknown mean and variance

Reminder: important distributions.

- the Gamma distribution with parameters (λ, α)

$$x \sim \Gamma(\lambda, \alpha)$$
$$p(x) = \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \; x > 0, \lambda, \alpha > 0.$$

- the Inverse Gamma distribution with parameters (λ, α)

Let $y = 1/x$ where $x \sim \Gamma(\lambda, \alpha)$, then

$$y \sim \Gamma^{-1}(\lambda, \alpha)$$
$$p(y) = \frac{\lambda^{\alpha} y^{-(\alpha+1)} e^{-\lambda/y}}{\Gamma(\alpha)}, \; y > 0, \lambda, \alpha > 0.$$

We observe $(x_1, \ldots, x_n)|\mu, \sigma^2 \sim \mathcal{N}(\mu, \sigma^2), \; i.i.d.$ Consider the following prior distribution:

$$\pi(\mu, \sigma^2) = \pi(\mu)\pi(\sigma^2) \propto \frac{1}{\sigma^2}.$$

Normal model with unknown mean and variance

The joint posterior is easily obtained

$$\pi(\mu, \sigma^2 | \bar{x}) \propto \sigma^{-n-2} \exp\left\{-\frac{1}{2\sigma^2}[(n-1)s^2 + n(\bar{x}-\mu)^2]\right\}$$

We can write

$$\pi(\mu, \sigma^2 | \bar{x}) = \pi(\mu | \sigma^2, \bar{x}) \pi(\sigma^2 | \bar{x})$$

The conditional posterior of μ given σ is

$$\pi(\mu | \sigma^2, \bar{x}) \propto \exp\left\{-\frac{n}{2\sigma^2}(\bar{x}-\mu)^2\right\}$$
$$\mu | \sigma^2, \bar{x} \sim \mathcal{N}(\bar{x}, \sigma^2/n).$$

Normal model with unknown mean and variance

Marginal posterior of σ^2

$$
\begin{aligned}
\pi(\sigma^2|\bar{x}) &= \int_{\mathbb{R}} \pi(\mu, \sigma^2|\bar{x}) d\mu \\
&\propto \sigma^{-n-2} \exp\left\{-\frac{1}{2\sigma^2}(n-1)s^2\right\} \sqrt{2\pi\sigma^2/n} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{-\frac{n}{2\sigma^2}(\bar{x}-\mu)^2\right\} d\mu \\
&\propto (\sigma^2)^{-[(n-1)/2+1]} \exp\left\{-\frac{1}{2\sigma^2}(n-1)s^2\right\}.
\end{aligned}
$$

I.e., $\sigma^2|\bar{x} \sim \Gamma^{-1}\left((n-1)s^2/2, (n-1)/2\right)$.

Normal model with unknown mean and variance

Marginal posterior of μ

$$\pi(\mu|\bar{x}) = \int \pi(\mu, \sigma^2|\bar{x}) d\sigma^2$$
$$= \int \pi(\mu|\sigma^2, \bar{x})\pi(\sigma^2|\bar{x}) d\sigma^2$$
$$\propto \left(1 + \frac{n(\mu - \bar{x})^2}{(n-1)s^2}\right)^{-n/2}.$$

which is a generalized student (mixture of normals for different values of inverse-gamma distributed variances).

$$\mu|\bar{x} \sim t\left(\bar{x}, \frac{s}{\sqrt{n}}, n-1\right).$$

Comparing means of two samples (1)

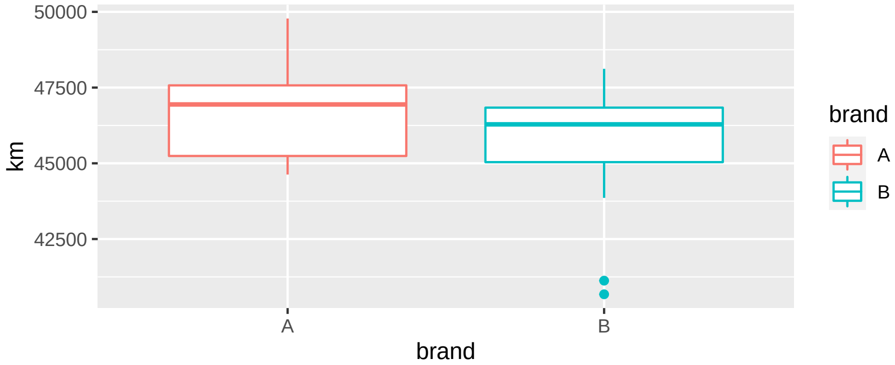
(Example inspired from Simar, L. (2002)). A firm wants to compare the quality of two differents brand of tires w.r.t their lifespan (number of kilometers to drive before tires are too damaged). This lifespan will vary from on tire to the other because of fluctuations in the production process (considering condition of experiments are controlled). We believe that a normal distribution can modelled these fluctuations.

```
head(dat)
```

```
##          km brand
## 1 45072.57     A
## 2 46789.13     A
## 3 44629.33     A
## 4 49780.34     A
## 5 47098.21     A
## 6 44661.46     A
```

Comparing means of two samples (2)

```
## Warning: package 'ggplot2' was built under R version 3.6.3
```



Comparing means of two samples (3)

Modelling hypothesis: two **independent** samples of sizes n_1, n_2.

$$x_{1i}|\mu_1, \sigma_1^2 \overset{i.i.d}{\sim} \mathcal{N}(\mu_1, \sigma_1^2),\ 1 \leq i \leq n_1.$$
$$x_{2j}|\mu_2, \sigma_2^2 \overset{i.i.d}{\sim} \mathcal{N}(\mu_2, \sigma_2^2),\ 1 \leq j \leq n_2.$$

Comparing means of two samples (4)

Question: compare the mean of two normal populations based on two independent random samples of resistance measures on tires produced by the two companies.

$$\delta = \mu_1 - \mu_2.$$

Consider 3 situations:

- known variances,
- unknown but equal variances,
- unknown and unequal variances (this problem is known as the **Berhens-Fisher problem**).

Comparing means of two samples: with known variances (1)

We decide to represent the prior information on the two means μ_1, μ_2 by two independent normal distributions.

$$\mu_k \sim \mathcal{N}(m_{k0}, \eta_{k0}^{-1}),\ k = \{1, 2\},\ \mu_1 \perp \mu_2,$$

where η_{k0} represents the **precision** $(\eta_{k0} = 1/\sigma_{k0}^2)$, $k = 1, 2$. We know that $\bar{x}_1$ and $\bar{x}_2$ are **sufficient** statistics. Then,

$$\mu_k | \bar{x}_k \sim \mathcal{N}(m_k^*, v_k^*).$$

where

$$m_k^* = \frac{m_{k0}\eta_{k0} + \bar{x}_k \eta_{n_k}}{\eta_{k0} + \eta_{n_k}},$$
$$v_k^* = (\eta_{k0} + \eta_{n_k})^{-1}.$$

with $\eta_{n_k}^{-1} = \frac{\sigma_k^2}{n_k}$.

Comparing means of two samples: with known variances (2)

Given the independence between the two samples, using properties of normals random variables, the posterior distribution of the parameter of interest δ is:

$$\delta|\bar{x}_1, \bar{x}_2 \sim \mathcal{N}(m_1^* - m_2^*, v_1^* + v_2^*).$$

Remark

- a) if $\eta_{k0} \to 0, \; k = 1, 2.$ Non informative prior, this posterior becomes

$$\delta|\bar{x}_1, \bar{x}_2 \sim \mathcal{N}\left(\bar{x}_1 - \bar{x}_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right).$$

- b) we have analogous result to the classical frequentist paradigm.

Here we are interested by posterior probability of $\delta > 0$ to decide if tires from the one company are better than the ones of the other.

Comparing means of two samples: with unknown but equal variances (1)

Posterior law of δ is still analytically tractable. Suppose $\sigma_1^2 = \sigma_2^2 = \sigma^2$

The parameters of the model are (μ_1, μ_2, σ^2) and $(\bar{x}_1, \bar{x}_2, s^2)$ are sufficient statistics for (μ_1, μ_2, σ^2), s^2 is the standard unbiased estimator of σ^2 (pooled sample variance estimator).

$$s^2 = v^{-1}(v_1 s_1^2 + v_2 s_2^2).$$

with

$$v_k = n_k - 1, k = 1, 2,$$
$$s_k^2 = v_k^{-1} \sum_{i=1}^{n_k} (x_{ki} - \bar{x}_k)^2, \; k = 1, 2,$$
$$v = v_1 + v_2 = n_1 + n_2 - 2.$$

Comparing means of two samples: with unknown but equal variances (2)

For simplicity set a **non informative prior**: we take independent non informative prior on (μ_1, μ_2, σ^2).

$$\pi(\mu_1, \mu_2, \sigma^2) \propto \frac{1}{\sigma^2}.$$

The likelihood function is obtained taking into account the sampling independence

$$\begin{aligned} p(\bar{x}_1, \bar{x}_2, s^2 | \mu_1, \mu_2, \sigma^2) &= p(\bar{x}_1 | s^2, \mu_1, \mu_2, \sigma^2) p(\bar{x}_2 | s^2, \mu_1, \mu_2, \sigma^2) p(s^2 | \mu_1, \mu_2, \sigma^2) \\ &= p(\bar{x}_1 | \mu_1, \sigma^2) p(\bar{x}_2 | \mu_2, \sigma^2) p(s^2 | \sigma^2). \end{aligned}$$

where

$$\bar{x}_k | \mu_k, \sigma^2 \sim \mathcal{N}\left(\mu_k, \frac{\sigma^2}{n_k}\right), \; k = 1, 2.$$

$$s^2 | \sigma^2 \sim \Gamma\left(\frac{v}{2\sigma^2}, \frac{v}{2}\right).$$

Comparing means of two samples: with unknown but equal variances (3)

Then we compute the posterior of (μ_1, μ_2, σ^2) using

$$\pi(\mu_1, \mu_2, \sigma^2 | \bar{x}_1, \bar{x}_2, s^2) \propto p(\bar{x}_1, \bar{x}_2, s^2 | \mu_1, \mu_2, \sigma^2) \pi(\mu_1, \mu_2, \sigma^2).$$

which is

$$\pi(\mu_1, \mu_2, \sigma^2 | \bar{x}_1, \bar{x}_2, s^2) = \frac{1}{\sqrt{2\pi\sigma^2/n_1}} \exp\left(-\frac{(\bar{x}_1 - \mu_1)^2}{2\sigma^2/n_1}\right) \frac{1}{\sqrt{2\pi\sigma^2/n_2}} \exp\left(-\frac{(\bar{x}_2 - \mu_2)^2}{2\sigma_2^2/n_2}\right)$$
$$\times \left(\frac{v}{2\sigma^2}\right)^{v/2} \frac{(s^2)^{(v/2-1)} \exp\left\{-\frac{v}{2\sigma^2} s^2\right\}}{\Gamma(v/2)} \frac{1}{\sigma^2}.$$

Comparing means of two samples: with unknown but equal variances (4)

We obtain

$$\pi(\mu_1, \mu_2, \sigma^2 | \bar{x}_1, \bar{x}_2, s^2) = \pi(\mu_1 | \bar{x}_1, \sigma^2)\pi(\mu_2 | \bar{x}_2, \sigma^2)\pi(\sigma^2 | s^2),$$

where

$$\mu_k | \sigma^2, \bar{x}_k \sim \mathcal{N}\left(\bar{x}_k, \frac{\sigma^2}{n_k}\right).$$

$$\sigma^2 | s^2 \sim (vs^2)\chi_v^{-2}.$$

Comparing means of two samples: with unknown but equal variances (5)

Remark that for the chosen model, s^2 is an exhaustive statistic for σ^2 and conditionally to σ^2, $\bar{x}_k$ is exhaustive statistic for μ_k.

$$\begin{cases} \mu_k|\sigma^2, \bar{x}_1, \bar{x}_2 \sim \mu_k|\sigma^2, \bar{x}_k \\ \sigma^2|s^2, \bar{x}_1, \bar{x}_2 \sim \sigma^2|s^2 \end{cases}$$

Comparing means of two samples: with unknown but equal variances (6)

The posterior law of σ^2 is proportional to a χ_v^{-2}, it is therefore an inverse gamma

$$\sigma^2|s^2 \sim \Gamma^{-1}\left(\frac{vs^2}{2}, \frac{v}{2}\right)$$
$$\pi(\sigma^2|s^2) = \frac{1}{\Gamma(v/2)}\left(\frac{vs^2}{2}\right)^{v/2}(\sigma^2)^{-[v/2+1]}\exp\left(-\frac{vs^2}{2\sigma^2}\right)$$

In particular we have

$$\mathbb{E}\left(\sigma^2|s^2\right) = s^2\frac{v}{v-2}$$

The conditional posterior distribution of δ given σ^2

$$\delta|\sigma^2, \bar{x}_1, \bar{x}_2 \sim \mathcal{N}\left(\bar{x}_1 - \bar{x}_2, \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right)$$

Comparing means of two samples: with unknown but equal variances (7)

We know compute the marginal posterior distribution for δ (we need to get rid off σ^2)

$$
\begin{aligned}
\pi(\delta|\bar{x}_1, \bar{x}_2, s^2) &= \int_0^\infty \pi(\delta, \sigma^2|\bar{x}_1, \bar{x}_2, \sigma^2) d\sigma^2 \\
&= \int_0^\infty \pi(\delta|\sigma^2, \bar{x}_1, \bar{x}_2, s^2)\pi(\sigma^2|\bar{x}_1, \bar{x}_2, s^2) d\sigma^2 \\
&= \int_0^\infty \pi(\delta|\sigma^2, \bar{x}_1, \bar{x}_2)\pi(\sigma^2|s^2) d\sigma^2.
\end{aligned}
$$

Comparing means of two samples: with unknown but equal variances (8)

We need now solve this integral. For that we need the properties of the Gamma function from which we get:

$$\pi(\delta|\bar{x}_1, \bar{x}_2, s^2) = \frac{\left(vs^2(1/n_1 + 1/n_2)\right)^{-1/2}}{B(1/2, v/2)} \left(1 + \frac{[\delta - (\bar{x}_1 - \bar{x}_2)]^2}{vs^2[1/n_1 + 1/n_2]}\right)^{-(v+1)/2}.$$

which is the density of a generalized student

$$\delta|\bar{x}_1, \bar{x}_2, s^2 \sim t(\bar{x}_1 - \bar{x}_2, s^2(1/n_1 + 1/n_2), v).$$

where $v = n_1 + n_2 - 2$. In particular we have

$$\mathbb{E}\left[\delta|\bar{x}_1, \bar{x}_2, s^2\right] = \bar{x}_1 - \bar{x}_2.$$
$$V\left[\delta|\bar{x}_1, \bar{x}_2, s^2\right] = s^2\left(1/n_1 + 1/n_2\right)\frac{v}{v-2}.$$

We therefore find analogous results to classical ones, but here we can compute the posterior probabilities that δ take any values in a set $A \subset \mathbb{R}$. $P\left[\delta \in A|\bar{x}_1, \bar{x}_2, s^2\right]$

Comparing means of two samples: with unknown but equal variances (9)

Numerical application: Compute the probability that $\delta > 0$.

```
library(LaplacesDemon, quietly = TRUE)
M =  10000
diff_mean = x1.bar - x2.bar
nu1  = n1-1
nu2 = n2-1
nu = nu1+ nu2 - 2
pooled_var = (s1^2*nu1 + s2^2*nu2)/nu
y  = rst(n = M, mu = diff_mean, sigma =
           sqrt(pooled_var*(1/n1+1/n2)), nu = nu)
```

```
## posterior probability of 0.9124  that the average lifetime
##     of tires of brand A is larger than that of brand B
```

Comparing means of two samples: with unknown variances (1)

In the frequentist framework, there are **no exact solution** for finite sample size, only asymptotic approximations are obtained (this is the Berhens-Fisher problem). In the bayesian framework we can get an easy solution. Here this is a four parameter model but the parameter of interest is still $\delta = \mu_1 - \mu_2$.

We keep the same approach with independent and non informative prior over all the parameters

$$\pi(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2) \propto \frac{1}{\sigma_1^2}\frac{1}{\sigma_2^2}.$$

For the likelihood function

$$p(\bar{x}_1, \bar{x}_2, s_1^2, s_2^2 | \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) \propto p(\bar{x}_1 | \mu_1, \sigma_1^2) p(\bar{x}_2 | \mu_2, \sigma_2^2) p(s_1^2 | \sigma_1^2) p(s_2^2 | \sigma_2^2),$$

where

$$\bar{x}_k | \mu_k, \sigma_k^2 \sim \mathcal{N}(\mu_k, \sigma_k / n_k),\ \ k = 1, 2,$$
$$s_k^2 | \sigma_k^2 \sim \frac{\sigma_k^2}{v_k} \chi_{v_k}^2 \sim \Gamma(\frac{v_k}{2\sigma_k^2}, \frac{v_k}{2}), k = 1, 2.$$

Comparing means of two samples: with unknown variances (2)

Similarly one can show that the posterior factorizes as follows

$$\pi(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2 | \bar{x}_1, \bar{x}_2, s_1^2, s_2^2) = \pi(\mu_1 | \sigma_1^2, \bar{x}_1)\pi(\mu_2 | \sigma_2^2, \bar{x}_2)\pi(\sigma_1^2 | s_1^2)\pi(\sigma_2^2 | s_2^2),$$

where

$$\mu_k | \bar{x}_k, \sigma_k^2 \sim \mathcal{N}(\bar{x}_k, \sigma_k / n_k).$$
$$\sigma_k^2 | s_k^2 \sim \Gamma^{-1}\left(\frac{v_k}{2s_k^2}, \frac{v_k}{2}\right).$$

Comparing means of two samples: with unknown variances (3)

The joint posterior marginal distribution of (μ_1, μ_2).

$$\pi(\mu_1, \mu_2|\bar{x}_1, \bar{x}_2, s_1^2, s_2^2) = \pi(\mu_1|\bar{x}_1, s_1^2)\pi(\mu_2|\bar{x}_2, s_2^2).$$

where marginal posterior of μ_k are obtained as follows:

$$\begin{aligned}
\pi(\mu_k|\bar{x}_k, s_k^2) &= \int_0^\infty \pi(\mu_k|\bar{x}_k, s_k^2)\pi(\sigma_k^2|s_k^2)d\sigma_k^2 \\
&= \int_0^\infty f_N(\mu_k|\bar{x}_k, \sigma_k^2)f_{i\gamma}\left(\sigma_k^2|\frac{v_k s_k^2}{2}, \frac{v_k}{2}\right)d\sigma_k^2.
\end{aligned}$$

By definition of the generalized student law we recognized that

$$\mu_k|\bar{x}_k, s_k^2 \sim t\left(\bar{x}_k, \frac{s_k^2}{n_k}, v_k\right).$$

Comparing means of two samples: with unknown variances (4)

The marginal a posteriori of $\delta = \mu_1 - \mu_2$.

This law is obtained by simple transformation, let us do the following change of variables $(\mu_1, \mu_2) \to (\delta, \mu_2)$. Hence

$$\pi(\delta, \mu_2 | \bar{x}_1, \bar{x}_2, s_1^2, s_2^2) = \pi_{\mu_1}(\delta + \mu_2 | \bar{x}_1, s_1^2) \pi_{\mu_2}(\mu_2 | \bar{x}_2, s_2^2).$$

Then we need to integrate over μ_2:

$$\begin{aligned}
\pi(\delta | \mu_1, \mu_2 \bar{x}_1, \bar{x}_2, s_1^2, s_2^2) &= \int_{-\infty}^{\infty} \pi(\delta, \mu_2 | \bar{x}_1, \bar{x}_2, s_1^2, s_2^2) d\mu_2 \\
&= \int_{-\infty}^{\infty} \frac{\left[v_1 s_1^2 / n_1\right]^{-1/2}}{B(1/2, v_1/2)} \left(1 + \frac{n_1 \left(\delta + \mu_2 - \bar{x}_1\right)^2}{v_1 s_1}\right)^{-(v_1+1)/2} \\
&\times \frac{\left[v_2 s_2^2 / n_2\right]^{-1/2}}{B(1/2, v_2/2)} \left(1 + \frac{n_2 \left(\mu_2 - \bar{x}_2\right)^2}{v_2 s_2}\right)^{-(v_1+1)/2} d\mu_2.
\end{aligned}$$

Marginal a posteriori of the ratio of variances

We can compare the variances using the following ratio:

$$\gamma = \frac{\sigma_2^2}{\sigma_1^2}.$$

We need to get the posterior of this new parameter. From the conditional distribution and from properties of

$$\frac{\sigma_k^2}{(v_k s_k^2)} | s_k^2 \sim \chi_{v_k}^{-2}, \; k = 1, 2.$$

$$\frac{(v_k s_k^2)}{\sigma_k^2} | s_k^2 \sim \chi_{v_k}^2.$$

Marginal a posteriori of the ratio of variances

Given the independence of the two samples and between the prior information on σ_1^2 and σ_2^2, these χ^2 are independent. Then we usually have

$$\gamma \frac{s_1^2}{s_2^2} | s_1^2 \sim F_{\eta_1, \eta_2}.$$

$$\mathbb{E}(\gamma | s_1^2, s_2^2) = \frac{s_1^2}{s_2^2} \frac{\eta_2}{\eta_2 - 2}.$$

$$\mathbb{V}(\gamma | s_1^2, s_2^2) = \left(\frac{s_1^2}{s_2^2} \right)^2 \frac{2\eta_2^2(\eta_1 + \eta_2 - 2)}{\eta_1(\eta_2 - 2)^2(\eta_2 - 4)}.$$

Posterior probabilities of γ

$$P\left(\frac{s_1^2}{s_2^2} F_{\frac{\alpha}{2}, \eta_1, \eta_2} \leq \gamma \leq \frac{s_1^2}{s_2^2} F_{1-\frac{\alpha}{2}, \eta_1, \eta_2} | s_1^2, s_2^2 \right) = 1 - \alpha.$$

Gibbs sampler in a nutshell

- Most often the posterior are P-variate distributions not analytically tractable.

We would like to

- obtain posterior marginal distributions,
- compute their properties such as their means or a tail-areas.

If we could generate a sample of size M from the joint posterior

$$\left\{ \left(\theta_1^{(m)}, \ldots, \theta_P^{(m)} \right); 1 \leq m \leq M \right\},$$

then the $\left\{ \theta_1^{(m)}; 1 \leq m \leq M \right\}$ is a sample from the marginal posterior $\pi(\theta_1|x)$.

Using the **Monte Carlo Principle** we can compute quantities of interest since

$$\mathbb{E}\left[g(\theta_1)\right] = \int g(\theta_1)\pi(\theta_1|x)d\theta_1 \approx \frac{1}{M}\sum_{m=1}^{M} g\left(\theta_1^{(m)}\right).$$

Gibbs sampler in a nutshell

Our aim is to draw random samples from the posterior $\pi(\boldsymbol{\theta}|x)$ where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_P)' \in \mathbb{R}^P$.

- It could be difficult to obtain independent sample from the posterior but easier to find a way to generate a Markov chain which stationary distribution is our target posterior.
- *If we are in the specific situation* where we can draw samples from all the **full conditional posterior** distributions; i.e., $\pi(\theta_p|\theta_1, \ldots, \theta_{p-1}, \ldots, \theta_{p+1}, \ldots, \theta_P, x)$ **then** we can use the Gibbs sampler.

Markov chain

Markov Chain on the space Θ (state space) is a stochastic process satisfying the markov property

$$p(\theta^{(m+1)}|\theta^{(1)},\ldots,\theta^{(m)}) = p(\theta^{(m+1)}|\theta^{(m)})$$

The MC will explore the parameter space. The rule governing how to jump from one state to another is described with a transition kernel

Transition kernel

Consider a discrete state space of 3 states, i.e., θ can take 3 values. The corresponding transition matrix P is

$$\begin{pmatrix} p\left(\theta_A^{(m+1)}|\theta_A^{(m)}\right) & p\left(\theta_B^{(m+1)}|\theta_A^{(m)}\right) & p\left(\theta_C^{(m+1)}|\theta_A^{(m)}\right) \\ p\left(\theta_A^{(m+1)}|\theta_B^{(m)}\right) & p\left(\theta_B^{(m+1)}|\theta_B^{(m)}\right) & p\left(\theta_C^{(m+1)}|\theta_B^{(m)}\right) \\ p\left(\theta_A^{(m+1)}|\theta_C^{(m)}\right) & p\left(\theta_B^{(m+1)}|\theta_C^{(m)}\right) & p\left(\theta_C^{(m+1)}|\theta_C^{(m)}\right) \end{pmatrix}.$$

The rows sum to one and define a conditional probability mass function (conditional on the current state).

The columns are the marginal probabilities of being in a certain state in the next period.

This is naturally extended to continuous state spaces.

Stationary distribution

Let us denote as $\Pi^{(0)}$ the starting distribution (pmf)

at iteration m: $\Pi^{(m)}$ the distribution from which $\theta^{(m)}$ is drawn is

$$\Pi^{(m)} = \Pi^{(0)} \times P^m$$

We define the stationary distribution π to be some distribution such that $\pi = \pi P$.

our aim in bayesian statistics generate a Markov chain whose stationary distribution is our posterior $\pi(\theta|x)$. From the random draws from the posterior we can use Monte Carlo principles to compute quantities of interest.

Difficulty: when has the chain converge? has it converged to the posterior dist. ?

Monte carlo principles for Markov chains

Beware: our draws are **not independent**, SLNN have been used to justify Monte Carlo Integration.

But we have an analog to SLLN for markov chains: the **Ergodic Theorem.**

Let $\left\{\theta^{(m)},\ 1 \leq m \leq M\right\}$ be M values from an aperiodic, irreductible, positive reccurent markov chain and $\mathbb{E}(g[\theta]) < \infty$, then

$$\frac{1}{M}\sum_{m=1}^{M} g(\theta^{(m)}) \to \int_{\Theta} g(\theta)\pi(\theta|x)d\theta,\ M \to \infty,$$

where π is the stationary distribution.

Gibbs sampler in a nutshell

The algorithm is:

- *step 0:* initialize $\boldsymbol{\theta}^{(0)} = (\theta_1^{(0)}, \ldots, \theta_P^{(0)})$, set $t = 1$,
- *step 1:* for $p \in \{1, \ldots, P\}$ sample $\theta_p^{(m)}$ from $\pi(\theta_p | \theta_1^{(m)}, \ldots \theta_{p-1}^{(m)} \ldots \theta_{p+1}^{(m-1)}, \ldots, \theta_P^{(m-1)}, y)$
- *step 2:* set $m = m + 1$ and go back to step 1. Iterate until you obtain enough samples from the stationary distribution.

Reminder: Multivariate normal distribution

Let $X \sim \mathcal{N}_p(\mu, \Sigma)$, its pdf is given by:

$$f(x) = (2\pi)^{-p/2}|\Sigma|^{-\frac{1}{2}} \exp\left\{\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\}, \; f : \mathbb{R}^p \to \mathbb{R}.$$

Mahalanobis transformation

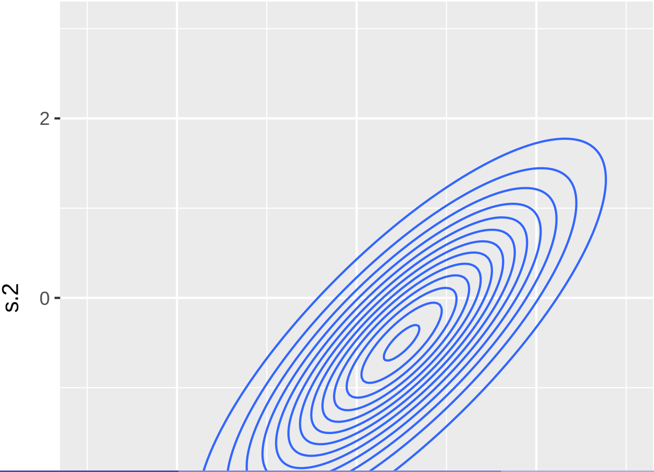
$$\begin{aligned} Y &= \Sigma^{-\frac{1}{2}}(X-\mu) \\ Y &\sim \mathcal{N}_p(0, I). \end{aligned}$$

Meaning that $Y_j \in \mathbb{R}$ the elements of Y are independent $\mathcal{N}(0,1)$. This implies that $f_Y(y) = \prod_{j=1}^p f_{Y_j}(y)$.

Geometry of the Multivariate Normal

The density of the $\mathcal{N}_p(\mu, \Sigma)$ forms ellipsoids of the form

$$(x-\mu)^t\Sigma^{-1}(x-\mu) = d^2.$$



Gibbs sampling for multivariate normal

Remark: on using properties of the multivariate normal for inverse probability inference.

Use the gibbs sampler to generate a sample of size 1000 from the joint distribution of (θ_1, θ_2) given by:

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

Draw a sample of size 1000 from this joint distribution.

Gibbs sampler for bivariate normal (solution)

From the properties of multivariate normal, we get:

$$\theta_1|\theta_2 \sim \mathcal{N}(\theta_2\rho, 1-\rho^2)$$
$$\theta_2|\theta_1 \sim \mathcal{N}(\theta_1\rho, 1-\rho^2)$$

Gibbs sampler for bivariate normal (solution)

```
burn_in = 500
M = 10000 + burn_in
rho = 0.8
theta1= theta2 = rep(0, length = M)
theta1[1] = theta2[1] =10 # initial values

for (i in 2:M)
{
  theta1[i] = rnorm(1,mean = rho*theta2[i-1], sd = sqrt(1-rho^2))
  theta2[i] = rnorm(1,mean = rho*theta1[i], sd = sqrt(1-rho^2))
}
theta1 = theta1[-c(1:burn_in)] # burn-in
theta2 = theta2[-c(1:burn_in)] # burn-in
post = cbind(theta1,theta2)
colnames(post) = c("theta1","theta2")
```

Checking convergence

we expect convergence to a stationary distribution which is also our posterior

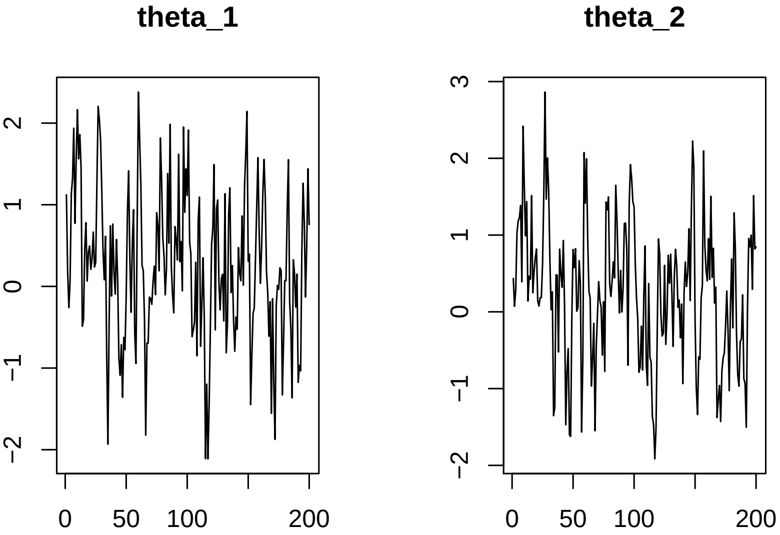
How to check?

- **visual inspection:** how well chains are mixing
- **autocorrelation:** high autocorrelation = slow mixing
- **Rubin, Gelman, multiple chains diagnotic**

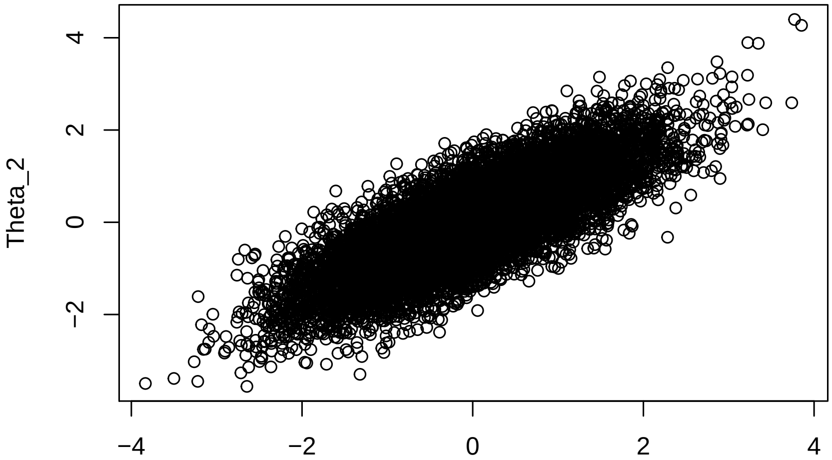
how to improve?

- **burnin:** discard first M generated values (till convergence of the chain to its stationary distribution)
- **thining:** keep every m-th observations in our chains to eliminate autocorrelation

Checking convergence



sample from the joint posterior



Gibbs sampler for Behrens-Fisher problem

Let us denote $D = (\bar{x}_1, \bar{x}_1, s_1^2, s_2^2)$.

The gibbs sampler can be written as:

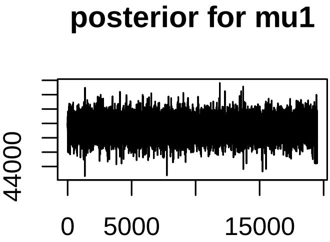
- step 0: initial values for $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$, set $m = 1$
- step 1:
  - $\mu_1^{(m)}$ draw from $\mu_1|\mu_2^{(m-1)}, \sigma_1^{2,(m-1)}, \sigma_2^{2,(m-1)}, D \sim \mathcal{N}(\bar{x}_1, \sigma_1^{2,(m-1)}/n_1)$
  - $\mu_2^{(m)}$ draw from $\mu_2|\mu_1^{(m)}, \sigma_1^{2,(m-1)}, \sigma_2^{2,(m-1)}, D \sim \mathcal{N}(\bar{x}_2, \sigma_2^{2,(m-1)}/n_2)$
  - $\sigma_1^{(m)}$ draw from $\sigma_1^2|\mu_1^{(m)}, \mu_2^{(m)}, \sigma_2^{2,(m-1)}, D \sim \Gamma^{-1}\left(\frac{(n_1-1)s_1^2 + n_1\left(\bar{x}_1 - \mu_1^{(m)}\right)^2}{2s_1^2}, \frac{n_1}{2}\right)$
  - $\sigma_2^{(m)}$ draw from $\sigma_2^2|\mu_1^{(m)}, \mu_2^{(m)}, \sigma_1^{2,(m)}, D \sim \Gamma^{-1}\left(\frac{(n_2-1)s_2^2 + n_2\left(\bar{x}_2 - \mu_2^{(m)}\right)^2}{2s_2^2}, \frac{n_2}{2}\right)$
- step 2: set $m \leftarrow m + 1$, iterate until $m = M$.

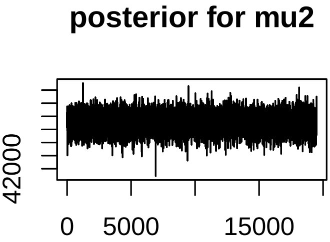
```
## ##
## ## Markov Chain Monte Carlo Package (MCMCpack)
## ## Copyright (C) 2003-2020 Andrew D. Martin, Kevin M. Quinn, and Jong He
## ##
```

Gibbs sampler for Behrens-Fisher problem

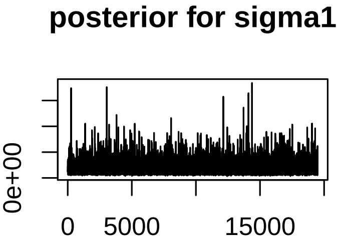
```
x1.bar = mean(x1); x2.bar = mean(x2); s1 =  sd(x1); s2 =  sd(x2)
M= 20000;
mu1 = mu2 = sigma1 = sigma2 = rep(0,M)

# starting values
mu1[1] = x1.bar; mu2[1] = x2.bar; sigma1[1] = s1^2; sigma2[1] = s2^2

# iteration loop
for (m in 2:M)
{
  mu1[m] = rnorm(1, x1.bar, sqrt(sigma1[m-1]/n1))
  mu2[m] = rnorm(1, x2.bar, sqrt(sigma2[m-1]/n2))

  scale_val = 0.5*((n1-1)*s1^2+n1*(x1.bar-mu1[m])^2)
  sigma1[m] = rinvgamma(1,shape = n1/2, scale = scale_val)

  scale_val = 0.5*((n2-1)*s2^2+n2*(x2.bar-mu2[m])^2)
  sigma2[m] = rinvgamma(1,shape = n2/2, scale = scale_val)
}
```

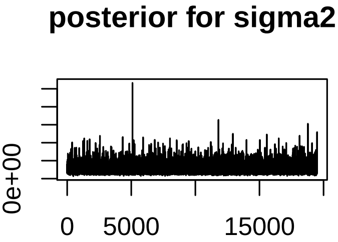
Marginal posterior for the means



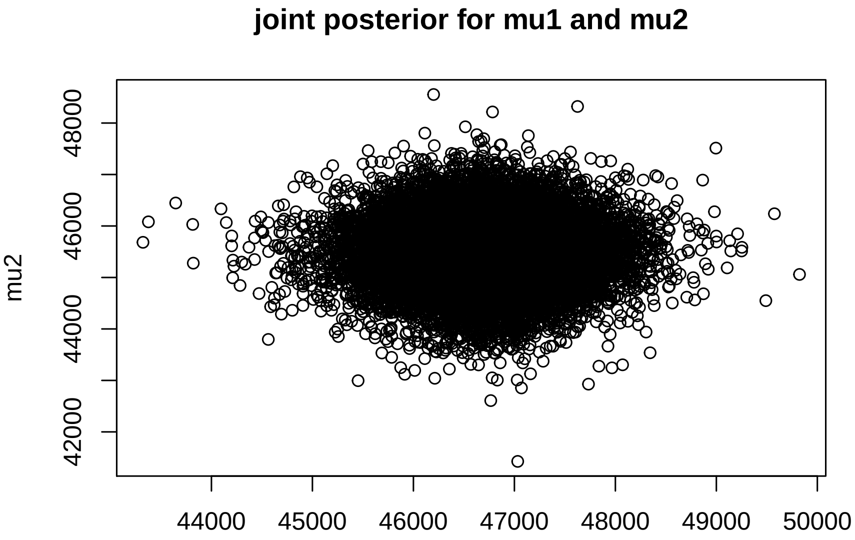


Marginal posterior for the variances





Joint posterior for the means



posterior inference for *delta*

```
delta= mu1-mu2
delta = delta[-c(1:burn_in)]
mean(delta); sd(delta)
```

```
## [1] 1193.113
```

```
## [1] 842.325
```

```
quantile(delta, c(0.025,0.05,0.5,0.95,0.975))
```

```
##      2.5%        5%       50%       95%     97.5%
## -467.4695 -189.3275 1188.9244 2581.7725 2859.6543
```

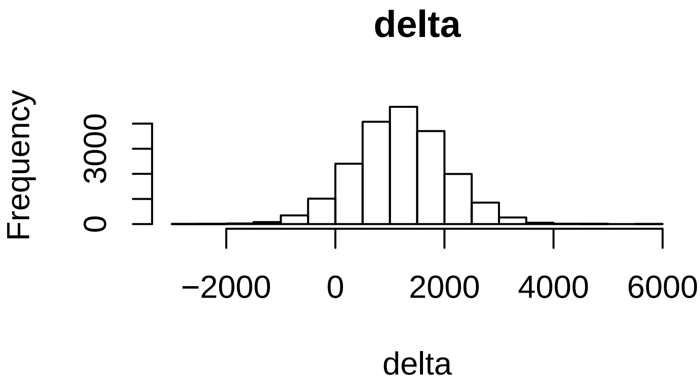
```
sum(delta>0)/length(delta)
```

```
## [1] 0.9248718
```

```
HPDinterval(as.mcmc(delta))
```

```
##          lower    upper
## var1 -454.5336 2870.633
## attr(,"Probability")
## [1] 0.95
```

posterior inference for delta



Bayesian inference using R-Jags

```r
library(coda)
library(rjags)
library(R2jags)

model_code = "
model {
  for (i in 1:n1)
  {
    x1[i] ~ dnorm(mu1,tau1)
  }
  for (j in 1:n2)
  {
    x2[j] ~ dnorm(mu2,tau2)
  }
  mu1 ~ dnorm(0,0.0001)
  mu2 ~ dnorm(0,0.0001)
  tau1 <- 1/s1
  tau2 <- 1/s2
  delta <- mu1-mu2
  s1 ~ dgamma(0.0001, 0.0001)
  s2 ~ dgamma(0.0001, 0.0001)
```

Overview of some recent "Behrens-Fisher" like problems

Actual researchs Behrens-Fisher for

- multivariate data:

$$x_{k1}, \ldots, x_{kn_k} \sim \mathcal{N}_p(\mu_k, \Sigma_k), \;\; k = 1, 2.$$

Hypothesis testing problem:

$$H_0 : \;\; \mu_1 = \mu_2, \;\; \text{vs } \mu_1 \neq \mu_2$$

Remark: if assume $\Sigma_1 = \Sigma_2$, then we have the Hotelling T^2-test.

- high-dimensional: *Long Feng, Changliang Zou, Zhaojun Wang, Lixing Zhu. (2015) TWO-SAMPLE BEHRENS-FISHER PROBLEM FOR HIGH-DIMENSIONAL DATA. Statistica Sinica 25, 1297-1312.*
- non-euclidean data (compositional, manifold valued. . .)
- functional data anlaysis

Behrens-Fisher for functional data

In Functional Data Analysis our **observations are curves**.

Consider two samples, modelled as realization of a Gaussian Process $GP(\mu, \gamma)$, where $\mu(t)$ is the mean function, $\gamma(s, t)$ the autocovariance function, $t \in \mathcal{T}$ where $\mathcal{T}$ a compact interval.

$$y_{l1}(t), \ldots, y_{ln_l}(t)$$

$$H_0 : \mu_1(t) = \mu_2(t) \text{ vs } H_1 : \mu_1(t) \neq \mu_2(t)$$

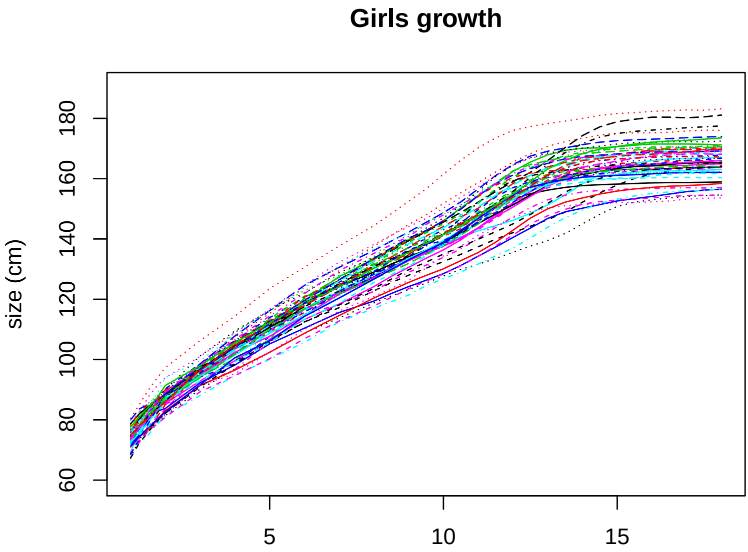
Behrens-Fisher for functional data (data example)

Data collected in the Berkeley Growth Study (Tuddenham and Snyder 1954).

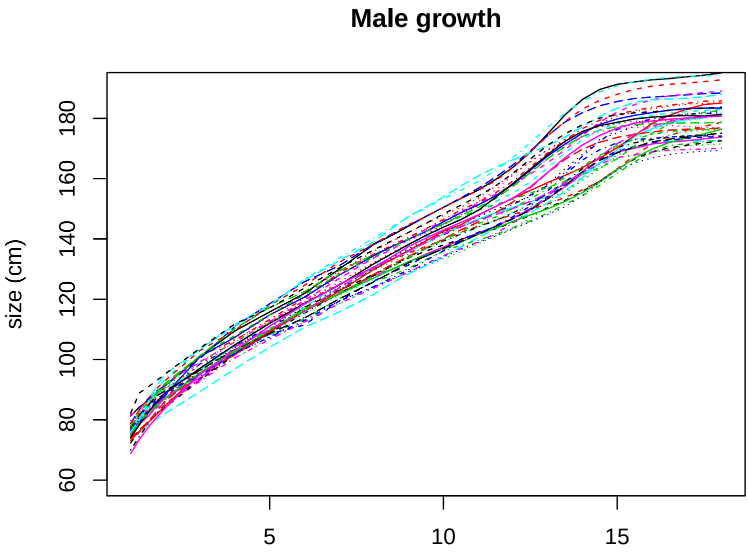
- Heights of 54 girls and 39 boys recorded at 31 ages from Year 1 to Year 18.
- Ages not equally spaced.
- Heights of a girl or a boy form a growth curve over ages.

```
##
## Attaching package: 'fda'

## The following object is masked from 'package:graphics':
##
##     matplot
```

Behrens-Fisher for functional data (data example)

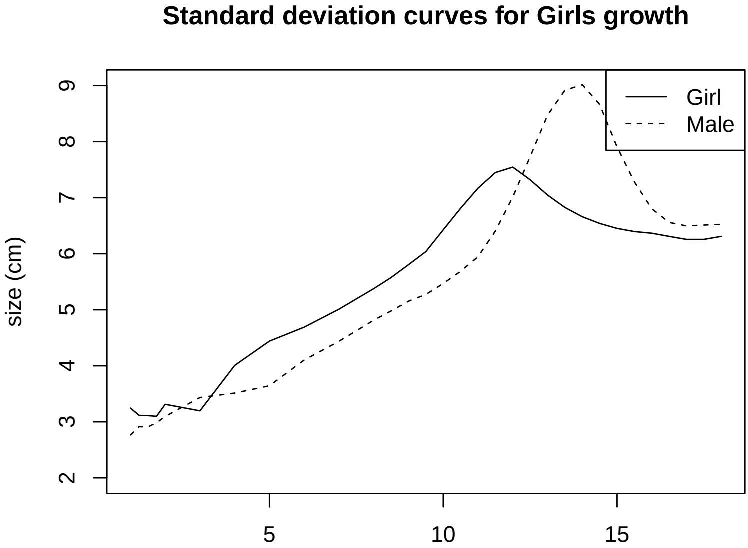


Behrens-Fisher for functional data (data example)



Behrens-Fisher for functional data (data example)

Behrens-Fisher for functional data (data example)



Behrens-Fisher for functional data (data example)

Questions:

- Did girls and boys grow in a same pace?
- Did girls and boys grow in a same pace over some growth period?
    - Baby period (Year 1-Year 4)?
    - Post-baby period (Year 4-Year 13)?
    - Teenage period (Year 13-Year 18)?
- Plots suggests that the covariance functions of girls and boys may not be the same.
- Motivating the two-sample Behrens-Fisher problem for functional data.