Lusternik
and
Sobolev

Elements of Functional Analysis

ELEMENTS OF FUNCTIONAL ANALYSIS

(Authorised third English translation from second extensively enlarged and rewritten Russian edition)

L. A. LUSTERNIK
and
V. J. SOBOLEV

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PREFACE TO THE FIRST RUSSIAN EDITION

DURING THE past few decades, the methods of **functional analysis**, especially the researches of Soviet mathematicians, have found extensive applications in various branches of pure and applied mathematics. The basic concepts of functional analysis now form an integral part of the mathematical courses at universities. An article by L. A. Lusternik appeared as the first comprehensive exposition of functional analysis in the first volume of the journal, *Uspekhi Matem. Nauk*. This was followed by a cycle of articles published in the *UMN*. It was, therefore, appropriate that no claim was made to completeness, not even in the sections dealing with functional analysis. Based on these articles V. J. Sobolev developed a lecture series on functional analysis, which he delivered several times at the State University of Voronezh.

These lectures *inter alia* covered a number of topics which were not dealt with in the article referred to above, for example, the principle of contraction mappings, the general form of linear functionals, the spaces with basis, the theory of completely continuous operators, the spectral representation of bounded self-adjoint operators, etc. The present text is the outgrowth of these lectures. The last section of the article, which dealt with the calculus of variations from the functional-analytic viewpoint, has been incorporated in the last chapter of the book in an expanded form.

The first chapter deals with the metric spaces. However, the approach is from a different angle, for example, as in the works of ALEXANDROV, *An Introduction to the General Theory of Sets and Functions* and HAUSDORFF, *Theory of Sets*. In a sense, this chapter is an extension of these works.

The second and third chapters are devoted to normed linear spaces and linear operators.

The motivation of the fourth chapter is the exposition of the theory of completely continuous operators in spaces with basis. An account has, however, been given of all spaces important for functional analysis.

The fifth chapter is focused on the spectral theory of self-adjoint operators. This topic is of key importance in modern analysis, and much has been written on it. A few works on this topic have been cited in the bibliography given at the end of this text. In the interest of curricular economy, the treatment here is confined to the exposition of fundamental notions and the discussion of bounded self-adjoint operators, presenting as much material as considered adequate for concentration on the methods of spectral theory.

The last chapter deals with operations for abstract functions. Differentiation and its applications have been discussed at length.

The authors have taken pains to illustrate the theory by application to different branches of mathematics—Theory of Differential and Integral Equations, Approximation Methods in Analysis, Theory of Almost Periodic Functions, Calculus of Variations etc.

At several places, examples have become somewhat lengthy. These have to be, however, included in order to make explicit the relations between functional analysis and other branches of mathematics.

INTERNATIONAL MONOGRAPHS ON ADVANCED MATHEMATICS & PHYSICS

ELEMENTS OF FUNCTIONAL ANALYSIS

PREFACE TO THE SECOND RUSSIAN EDITION

THE FIRST edition of the present text had appeared over a decade before. During this period, the **functional analysis** not only made a tremendous and diversified advance but also developed such intensively penetrating ideas and methods that it has to be recognized as a separate mathematical discipline. The functional analysis came to be used more extensively in mechanics and engineering, not to speak of quasi-mathematical subjects such as physics, which has been one of the earliest employers of the apparatus of functional-analytic concepts and methods for furthering its theoretical investigations. The importance of functional analysis and its place in the system of mathematical disciplines, therefore, do not call for any elaborate discussion.

The development of functional analysis and its ever growing spell on a wide range of mathematicians, physicists and researchers in mechanics has led to the appearance of a series of excellent texts and monographs devoted to a general treatment of this subject. Of them, it suffices to mention the works of KANTOROVICH and AKILOV [15], KOLMOGOROV and FOMIN [17], SMIRNOV [33], VULIKH [38], AKHIEZER and GLAZMAN [2], RIESZ and B. SZ.-NAGY [32], DUNFORD and SCHWARTZ [9], and so on. However, no apology is needed for offering the second edition of the present text, as it is in no way a repetition from the existing literature. In contradistinction to other titles, it retains its basic character of giving a transparent and expository treatment of the fundamentals, thus bringing the subject within the easy access of students and teachers.

This second edition is a considerable enlargement of the first edition due to the inclusion of substantial new material. A number of corrections and revisions have been incorporated and a few of those problems abridged which either fall out of the general plan of the book or lack the illustrative character. The most significant of the augumented part in the second edition relates to SOBOLEV's spaces and inclusion theorems for these spaces, RIESZ-SCHAUDER theory of linear operators, equations with completely continuous operators in arbitrary BANACH spaces, SCHAUDER's fixed point principle, the spectral theory of non-bounded linear operators in HILBERT spaces etc. At the same time, in line with the first edition, the second edition does not handle some of the esoteric topics, such as normed rings, representation theory, semi-ordered spaces and the generalized functions and their applications. Acting on the well-known K. PRUTKOV's principle on the fallacy to envelop the unfathomable, the authors make no claim of all-pervasiveness and refer the reader interested in such topics to other available monographs.

In the preparation of the second edition, the authors have drawn on various sources, important ones of which being the works of KANTOROVICH and

AKHIEZER, SMIRNOV, RIESZ and SZ.-NAGY. The exposition of the spectral theory of linear operators in HILBERT spaces is based on the plan and ideas of PLESNER [28, 29], who happened to be a strong protagonist of the spectral theory of operators and also functional analysis, in general, from the very inception of this extensively developed mathematical discipline.

The manuscript of this text was read by A. I. PEROV, D. A. RAIKOV and YA. B. RUTITSKII and many valuable remarks were offered by them. A number of their suggestions have fruitfully gone into the improvement of the text. These are gratefully acknowledged.

CHAPTER 0

INTRODUCTION

THE GENERALIZATION OF FUNDAMENTAL CONCEPTS OF ANALYSIS, GEOMETRY AND ALGEBRA

AT THE turn of the century a new discipline, the **functional analysis**, emerged in analysis.

The fundamental concepts and methods of functional analysis have progressively evolved from some of the most classical directions of analysis, e.g., the calculus of variations, the theory of differential equations, the theory of representation and approximation of functions, the numerical methods of analysis and especially the theory of integral equations.

The fascination of functional analysis lies in carrying over the concepts and methods of elementary analysis as well as of related branches of algebra and geometry to the objects of a more general and complex nature with the bonus of widening the ramifications of geometric and algebraic methods. This generalization permits the extension of a unified treatment to the problems previously tackled in isolation in different branches of analysis. Moreover, it enables us to establish the interplay and the connections between various quite different mathematical theories, and to derive thereby new mathematical results. This assertion is justified by the fact that certain existence theorems for differential and integral and other equations were of late established by the methods of functional analysis or by the functional analytic development of approximation methods of analysis.

These generalizations became possible because, in the process of development, the fundamental concepts of analysis revealed themselves to be very general. These concepts and methods often found analogies in algebra and geometry. Thus, for example, the solution of various problems of algebra and analysis is found by successive approximations, and the definition of a functional and the conditions for the existence of an extremum in the calculus of variations are completely analogous to the definitions of a function (of one or more variables), of a maximum or minimum (an extremum) of a function, and the conditions for the existence of such an extremum in the differential calculus.

The analogies between ordinary linear differential equations and linear difference equations on the one hand, and the systems of linear algebraic equations on the other hand, are well known. These analogies are more explicit in the theory of integral equations that has chronologically developed later.

The century also saw a generalization of geometry, parallel to that of analysis. The motivation stremmed from the development of non-Euclidean geometry traced to N. I. LOBACHEVSKII and the capability of n-dimensional spaces to impart a geometrical meaning to the functions of several variables. One bonus was the manifestation of fresh analogies between analysis and geometry, opening new possibilities of geometrized analysis and motivating further extension of geometric concepts. By way of examples, a collection of solutions of n-order ordinary homogeneous linear differential equation is isomorphic to an n-dimensional vector space, and a collection of solutions of a linear homogeneous partial differential equation is geometrically analogous to the infinite-dimensional extension of an n-dimensional vector space. Another example of a deeper analogy between analysis and geometry is the development of functions with respect to the elements of an orthogonal system, in many ways consistent with the orthogonal systems of vectors in the Euclidean space, both cases being distinctive only in the terminology. The decomposition of a vector into its components corresponds to the development of a function into a Fourier series, the theorem of PYTHAGORAS to the PARSEVAL-STEKLOV theorem, and so on. The geometrical representation of an infinite orthogonal system of functions again involves an infinite-dimensional extension of the Euclidean space.

The development of analysis and geometry not only enlarged the scope of analogies between the concepts of various branches of analysis as well as between those of analysis and geometry, but also made it explicit that these analogies are the consequences of the inter-relationship of the concepts basic to these theories. The notions of a function, limit process, neighbourhood and metric, serve as the examples of those concepts, which are drawn upon in these theories in various explicit or implicit forms.

As already remarked it is not only the generalization but also the geometrization of the fundamental concepts and methods of classical analysis that is characteristic of functional analysis. Functions of various classes are regarded as points or vectors of **function spaces.** As stated in the foregoing, such considerations motivated further extension of the geometric concepts, like those of an infinte-dimensional Euclidean space, a vector space and other spaces. This ultimately led to the emergence of the general concepts of a metric, a normed linear, and a topological space, covering the geometric entities as well as different function spaces.

The introduction of abstract spaces permitted us to interpret a host of problems of analysis in geometric terminology. Such a geometric representation of analytic theories is widely resorted to not only in mathematical literature but in wide-ranging fields such as physics and mechanics, where many a conjectures was suggested by analogies with the n-dimensional geometry. The proofs of many theorems were realized via the geometric methods, and thus analysis acquired a new medium of geometric language. The generalization of algebraic concepts took place simultaneously with that of the geometric concepts.

On the one hand, the algebraic operations on numbers were carried over to objects of more general nature, like matrices, operators, etc., that yielded and deepened the notions of group, ring, field, linear manifolds, etc. In connection with the applications of algebraic concepts to analysis, the algebraic mappings with a limit process came to be considered. Thus, there evolved a special discipline, **topological algebra** (A. HAAR, A. KOLMOGOROV, L.S. PONTRYAGIN, and others), where various algebraic structures, like topological groups, rings, fields, etc., in which limits are defined, are treated from a unified point of view.

On the other hand, algebraic operations involving a limit process are central to analysis. The elementary chapters of algebra have the same role in ordinary classic analysis as played by the corresponding generalizations of algebraic concepts in functional analysis.

Thus, the theory of linear operators, dealt with at length in this book, corresponds to linear algebra. The approximation of nonlinear objects by linear ones, a basic method of analysis, is also carried over to functional analysis (see, Ch. 7). The passage from polynomials on rings (matrix rings, operators, etc.) to arbitrary functions of these arguments corresponds to the passage from polynomials whose arguments are numbers to arbitrary functions. This forms the base for such important disciplines as calculus of matrices, calculus of operators and the spectral theory of linear operators (see, Ch. 7).

The functional analysis, by now matured into a huge independent mathematical discipline, continues to assimilate, sharpen and generalize the methods of other still younger mathematical disciplines. It suffices to indicate in this direction the intensive recent developments in the theories of linear topological spaces and representation of groups as well as some other contemporary achievements in functional analysis.

CHAPTER 1

METRIC SPACES

1.1 FUNCTION SPACES. ORDER RELATIONS

THE NOTION of a function is one of the basic ideas in mathematical analysis. The definition of function given in analysis says: Let X and Y be two sets of real numbers; if to every number $x \in X$ there is assigned uniquely a number $y \in Y$ according to some rule, then it is said that a *single-valued function* $y = f(x)$ is defined on X and its **range** is contained in Y. The set X is called the **domain** of the function.

It is easy to see that for the notion of a function it is not necessary that X and Y be sets of real numbers. If we take X and Y to be sets of elements of distinct character, we arrive at a very general concept of a function, examples of which are found in various branches of mathematical analysis.

Examples. 1. Let $y = f(x_1, x_2, \ldots, x_n)$ be a real-valued function of n real variables. Then X is a set of all (ordered) n-tuple of real numbers and Y is a set of real numbers.

2. Let $\mathbf{y} = f(x)$ be a vector function, assigning an n-dimensional vector $\mathbf{y}$ to every real number x. Here X is a set of real numbers, and Y is a set of n-dimensional vectors.

3. In the calculus of variations, we consider the functionals

$$I(\gamma) = \int_a^b F(x, y, y')\, dx,$$

where γ is a curve defined by $y = f(x)$ with $f(x)$ belonging to the class C_1 of those functions which have a continuous derivative and passing through the two given points $A(a, y_a)$ and $B(b, y_b)$. In this case, X is a set of such curves, and Y is a set of real numbers.

4. In the theory of integral equations, we consider expressions of the form

$$y(t) = \int_a^b K(t, s)\, x(s) ds,$$

assuming that the kernel $K(t, s)$ is defined and continuous on the square $a \leqslant t, s \leqslant b$. Then we can regard the above equation as a rule that associates a function $x(t)$ continuous on $[a, b]$ with other functions continuous on the same interval. Here X and Y are sets of continuous functions.

We shall now introduce a general definition of functional relationship.

Given two arbitrary sets X and Y and let there be a rule, which assigns to each element $x \in X$ a unique well-defined element $y \in Y$. Then we say that the given **operator** $y = f(x)$ [written also as $y = fx$] is defined on X such that its range lies in Y[1]. We can also say that this defines a **mapping** of X

---

1. We agree to say that this condition is satisfied *on a set* or *in a set* according as it holds for all elements of this set or perhaps not for all elements of the set, respectively.

onto Y. In particular, if the values of an operator are real numbers, the operator is called a **functional.**

The element $y \in Y$ corresponding to $x \in X$ under the mapping $y = f(x)$ is called the **image** of x and x the **inverse image** of the element $y \in Y$.

If the mapping $y = f(x)$ carries X into Y then, obviously, every element $y \in Y$ has atleast one inverse image x. In that case, if every $y \in Y$ has only one inverse image $x \in X$, the mapping of X onto Y defined by $y = f(x)$ is said to be **one-one.**

It is almost impossible to adduce properties of operators by definitions of such general nature, hence the motivation for introduction of certain additional premises.

Apart from the concept of a function, the notions of a limit and continuity also are basic in analysis. A set in which the limit of a sequence can be defined by various processes is called a **space.**

Spaces whose elements are functions or number sequences are said to be **function spaces.** The study of certain classes of operators defined in function spaces also forms the basic content matter of functional analysis.

We shall elaborate certain other notions employed in functional analysis.

Given a set X of objects $a, b, c, \ldots$ of certain nature. Let us introduce between certain pairs (a, b) of elements of X a relation $a \prec b$, with the properties:

(i) If $a \prec b$ and $b \prec c$, then $a \prec c$ (*transitivity*);

(ii) $a \prec a$;

(iii) If $a \prec b$ and $b \prec a$, then $a = b$.

Then X is said to be **partially ordered** (or **semi-ordered**) by $\prec$ and a and b satisfying the relation $a \prec b$ or $b \prec a$ are said to be **congruent.**

A set X is said to be **totally** (or **linearly**) ordered if for each pair of its elements (a, b), either $a \prec b$ or $b \prec a$ holds.

A subset Y of a partially ordered set is said to be **bounded above** if there is an element b such that $y \prec b$ for all $y \in Y$. The element b is called an **upper bound** for Y. The smallest of all upper bounds is called a **least upper bound** or **supremum** for a set.

The terms **bounded below,** a **greatest lower bound** or **infimum** for a set are defined analogously.

Finally, an element $z_0 \in X$ is said to be **maximal** if there exists in X no element $x \neq z_0$ satisfying the relation $z_0 \prec x$.

There holds the highly important next lemma.

1.11. Zorn's lemma. *If in a nonempty partially ordered set X each linearly ordered subset Y has an upper bound, then X has a maximal element z_0.*

A set is said to be **linearly (totally) ordered,** if any of its subsets has a minimal element, i.e. an element that precedes all the elements of the subset.

1.12. Zermelo's theorem. *Every set can be well ordered by introducing* *certain order relations.*

The proof of ZERMELO's theorem rests upon the so-called *Zermelo's axiom of arbitrary choice* (AC), which says : *if any system of non-empty pairwise-disjoint sets is given, then there is a new set having exactly one common element with each of the sets of the system.*

ZORN's lemma, ZERMELO's AC and well-ordering theorem are equivalent to each other. For greater details, see [5] and [25].

Example. Let M be a certain non-empty set and let $T = \{t\}$ be the collection of its subsets t. Assume that $t_1 \prec t_2$, if $t_1 \subset t_2$. It is evident that the order relation introduced in this manner satisfies the properties (i) - (iii) above. It is also plain that when M contains more than two elements, under such ordering the set M is not ordered (not to speak of linearly ordered).

If S is any subset of T, then it has an upper bound and its least upper bound is the set

$$S = \bigcup_{t \in S} t.$$

T contains a maximal element : this is the set M itself considered as a subset and ZORN's lemma is trivial in this case. ZERMELO's theorem also asserts that this can be well-ordered by introducing in this some order relation, but how this is to be accomplished does not follow from the theorem, because of the non-constructive character of its proof.

1.2. METRIC SPACES

IN MATHEMATICAL analysis we encounter limit processes introduced for one and the same sequence in the context of different problems. Most important of these, however, is the idea of a limit for sequences of real numbers which can be directly extended to sequences of complex numbers and n-dimensional vectors. For a sequence of functions we have a series of convergence concepts, like simple (point-wise), uniform, in the mean, etc. All these have in common the property that convergence of a sequence of elements x_n (which can be numbers, vectors or functions) to an element x implies an unrestricted *approaching* of x_n to x, that is, an unrestricted *decreasing of the distance* between these elements for increasing n. Depending upon the defination given of the distance between the elements x_n and x, we obtain different definitions of a limit. However, an expedient representation of certain sets of elements yields a general definition of the distance between elements, covering also the particular cases in question. This facilitates to introduce the concept of a limiting process by means of this distance defined in the set and turn this set into a space.

The concept of a metric space owes its motivation to CANTOR's general theory of sets developed in 1880's, origin to FRECHET in 1906 and further cultivation to HAUSDORFF in 1914.

1.21. Metric space. A set X is called a **metric space** if with every pair (x, y) of this set is associated a non-negative real number $\rho_X(x, y)$, with the properties :

(i) $\quad \rho_X(x, y) = 0, \quad$ iff $\quad x = y \quad$ (*identity*);

(ii) $$\rho_X(x, y) = \rho_X(y, x) \qquad (\textit{symmetry});$$

(iii) $$\rho_X(x, y) + \rho_X(y, z) \geqslant \rho_X(x, z) \qquad (\textit{triangle inequality}).$$

The number $\rho_X(x, y)$ is callled the **distance** between the elements x and y and is referred to as the **(usual) metric** or the **(usual) distance function** in X. The conditions (i) thro' (iii) are called the **metric axioms.** These axioms obviously express the fundamental properties of the distance between the points of three-dimensional Euclidean space.

In the sequel, if the necessity arises of explicitly representing any metric space X, then the $\rho_X(x, y)$ will be simply written as $\rho(x, y)$.

The elements of a metric space are also called **points.**

Finally, it may be remarked that any set Y lying in a metric space X and having the same distance between the elements as in X is itself a metric space and called a **subspace** of X.

1.22. Limit of a sequence. An element x of a metric space X is called the **limit** of a sequence of elements $x_1, x_2, \ldots, x_n, \ldots$ of X, if $\rho(x_n, y) \to 0$ as $n \to \infty$. In this case, we write $x_n \to x$ or $\lim\limits_n x_n = x$.

In relation to the convergent sequences of points of a metric space, it is possible to state some general theorems that follow immediately.

THEOREM 1. *If a sequence $\{x_n\}$ of points of a metric space X converges to a point $x \in X$, then every subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ also converges to the same point.*

The proof is trivial.

THEOREM 2. *A sequence $\{x_n\}$ of points of a metric space can converge to at most one limit.*

PROOF. Let $x_n \to x$ *and* $x_n \to y$. Then, $\rho(x, y) \leqslant \rho(x_n, x) + \rho(x_n, y) < \varepsilon$; whatever be $\varepsilon > 0$ and for sufficiently large n. Since x and y are fixed and ε is an arbitrary positive number, the inequality is possible only if, $\rho(x, y) = 0$, that is, $x = y$.

THEOREM 3. *If a sequence $\{x_n\}$ of points of X converges to a point $x \in X$, then the set of numbers $\rho(x_n, \theta)$ is bounded for every fixed point θ of the space X.*

In fact, by the triangle axiom, for any n we have :

$$\rho(x_n, \theta) \leqslant \rho(x_n, x) + \rho(x, \theta) \leqslant L + \rho(x, \theta) = K,$$

because $\{\rho(x_n, y)\}$, as a convergent number sequence, is bounded and, consequently, the numbers $\rho(x_n, x)$ are not greater than a constant L.

A set of all points x of the space X, satisfying the inequality $\rho(x, a) < r$ [corr. $\rho(x, a) \leqslant r$], is called an **open** [corr. **closed**] **sphere with centre** a **and radius** r, and denoted by $S(a, r)$ [corr. $\overline{S}(a, r)$]. Further every sphere with **centre** in x is called a **neighbourhood** of the point x.

It is easy to see that x is the limit of a sequence $\{x_n\}$ iff every neighbourhood of the point x contains, from a certain index onwards, all points of the sequence $\{x_n\}$. A set lying inside of a sphere is called **bounded**.

It happens sometimes that in some spaces the limit of a sequence of elements is directly defined. If a metric can be introduced in this space such that a sequence of limits defined by it coincides with the initial limit, then the given space is called **metrizable.**

1.23. Closure. It is now in order to introduce in the general metric spaces, various important ideas encountered in the theory of point sets on a line. Given a set $M \subset X$, a point $a \in X$ is called an **accumulation (limit) point** of this set if each neighbourhood of x contains at least one point of the set $M \setminus a$, i.e. if $S(a, r) \cap (M \setminus a) \neq \emptyset$ for every r. The set of all limit points of M is called the **closure** of M, and denoted by $\overline{M}$.

It is not difficult to establish that closed point sets of a metric space have the same basic properties as the closed numerical point sets, namely:

(i) $$\overline{M \cup N} = \overline{M} \cup \overline{N};$$

(ii) $$M \subset \overline{M};$$

(iii) $$\overline{\overline{M}} = \overline{M} = M.$$

(iv) The closure of an empty set is empty.

A set M is said to be **closed** if $\overline{M} = M$. A set M is called **open** if its complement $X \setminus M$ is closed. A set M is called **dense** in a set G, if $G \subseteq \overline{M}$. In particular, M is called **everywhere dense** in a space X or simply **densely defined** if $\overline{M} = X$. Finally, M is said to be **nowhere dense** in X, if every sphere of this space contains a certain sphere free from points of the set M. For a detailed discussion of the properties of closed and open sets in metric spaces, see [3].

1.24. Continuous functions. Given two metric spaces X and Y and a function $y = f(x)$ defined on a set $M \in X$ with range in Y. The function $f(x)$ is said to be **continuous at a point** $x_0 \in M$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\rho_Y[f(x), f(x_0)] < \varepsilon$ for every $x \in M$ satisfying $\rho_X(x, x_0) < \delta$.

From the definition of the continuity of $f(x)$, it follows: If $x_n \to x_0$ $(x_n, x_0 \in M)$, then $f(x_n) \to f(x_0)$. The converse also holds : If

$$f(x_n) \to f(x_0)$$

for any sequence $\{x_n\} \subseteq M$ convergent to $x_0 \in M$, then the function $f(x)$ is continuous at the point x_0. The proof of this assertion is just the same as for real functions of real variables.

1.25. Homeomorphism. Let X and Y be given metric spaces and let there be a one-one mapping of X onto Y. If this mapping is continuous, then X and Y are said to be **homeomorphic.**

1.3 EXAMPLES OF METRIC SPACES

1.31. The real. line R. Let $X = R$, where R is the set of all real numbers (the real line). If $x \in R$, $y \in R$, then we put $\rho(x, y) = |x - y|$, and the metric axioms hold. Convergence in X is the customary convergence of the numerical sequences.

1.32. The Euclidean n-dimensional space E_n. Let X be an n-dimensional arithmetic space, that is, the set of all n-tuple of real numbers. If $x = \{\xi_1, \xi_2, \ldots, \xi_n\}$ and $y = \{\eta_1, \eta_2, \ldots, \eta_n\}$, then we set

$$\rho(x, y) = \sqrt{\sum_{i=1}^{n} (\xi_i, \eta_i)^2}.$$

The metric axioms are obviously satisfied. Let $x_k = \{\xi_1^{(k)}, \xi_2^{(k)}, \ldots, \xi_n^{(k)}\}$, $k = 1, 2, \ldots$, and let $\rho(x, x_k) \to 0$, that is, let

$$\sqrt{\sum_{i=1}^{n} (\xi_i^{(k)} - \xi_i)^2} \to 0 \quad \text{as } k \to \infty.$$

This is equivalent to the condition $\xi_i^{(k)} \to \xi_i$, $i = 1, 2, \ldots, n$ as $k \to \infty$.

Thus, convergence in this space is coordinate-wise.

A space X with this metric is called the **n-dimensional Euclidean space.** We denote it by E_n.

1.33. The space C [0, 1] of continuous functions with the Chebyshev metric. Let X be a set of all continuous functious defined on the interval [0, 1][1].

Introduce a metric by setting $\rho(x, y) = \max_t |x(t) - y(t)|$. It is plain that the metric axioms are satisfied. Since $\rho(x, y) \geqslant 0$ and $\rho(x, y) = 0$ only if $x(t) \equiv y(t)$, hence $\rho(x, y) = \rho(y, x)$ is also evident. It remains to verify the triangle inequality. We have

$$\begin{aligned} |x(t) - z(t)| &= |[x(t) - y(t)] + [y(t) - z(t)]| \\ &\leqslant |x(t) - y(t)| + |y(t) - z(t)| \\ &\leqslant \max_t |x(t) - y(t)| + \max_t |y(t) - z(t)| \\ &= \rho(x, y) + \rho(y, z) \qquad \text{for every } t \in [0, 1]. \end{aligned}$$

Therefore, $$\rho(x, z) = \max_t |x(t) - z(t)| \leqslant \rho(x, y) + \rho(y, z).$$

The set of all continuous functions defined on the interval [0, 1] with the above metric is called the **space of continuous functions** and denoted by C [0, 1]. This is also called the **space of continuous functions with the CHEBYSHEV metric,** because the distance between the functions coincides with the CHEBYSHEV deviation.

We consider convergence in the space C [0, 1]. Given a sequence $\{x_n(t)\}$ of elements of C [0, 1] such that this sequence converges to $x(t)$ (that is, $\rho(x_n, x) \to 0$ as $n \to \infty$). This means, $\max_t |x_n(t) - x(t)| \to 0$ as $n \to \infty$, that is, there is a natural number $n_0 = n_0(\varepsilon)$ for every $\varepsilon > 0$, such that $\max_t |x_n(t) - x(t)| < \varepsilon$ for all $n \geqslant n_0(\varepsilon)$. Hence $|x_n(t) - x(t)| < \varepsilon$ for all $n \geqslant n_0(\varepsilon)$ and for all $t \in [0, 1]$. This, however, implies that the sequence $\{x_n(t)\}$ converges **uniformily** to the function $x(t)$. The converse also holds : If a sequence $\{x_n(t)\}$ converges uniformly to $x(t)$, then $\rho(x_n, x) \to 0$. Thus, the convergence in the space C[0, 1] is a **uniform convergence** in the interval [0, 1].

1.34. The space m of bounded number sequences. Let X be the set of bounded number sequences

$$x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$$

1. If the interval of the independent variable t is $[a, b]$, it can be transformed into the interval [0, 1] by introducing a new independent variable $\tau = (t-a)/(b-a)$.

implying that for every x there is a constant K_X such that $|\xi_i| \leqslant K_X$ for all i. Let $x = \{\xi_i\}$ and $y = \{\eta_i\}$ belong to X. Introduce the metric by

$$\rho(x, y) = \sup_i |\xi_i - \eta_i|.$$

Evidently, the verification of only triangle inequality is required. We have

$$\xi_i - \zeta_i| \leqslant |\xi_i - \eta_i| + |\eta_i + \zeta_i|$$
$$\leqslant \sup_i |\xi_i - \eta_i| + \sup_i |\eta_i - \zeta_i| = \rho(x, y) + \rho(y, z).$$

Consequently, also $\sup_i |\xi_i - \zeta_i| = \rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$.

The space obtained in this manner is the **space m of bounded number sequences.**

Let x_n and x be the elements of m, $x_n = \{\xi_i^{(n)}\}$, $x = \{\xi_i\}$ and $\rho(x_n, x) \to 0$ as $n \to \infty$. This means that there is an index $n_0 = n_0(\varepsilon)$ for every $\varepsilon > 0$, such that

$$\rho(x_n, x) = \sup_i |\xi_i^{(n)} - \xi_i| < \varepsilon \text{ for } n \geqslant n_0(\varepsilon).$$

Hence, for $n \geqslant n_0(\varepsilon)$ and for every i,

$$\xi_i^{(n)} - \xi_i| < \varepsilon.$$

As is easily seen, the converse also holds : If $|\xi_i^{(n)} - \xi_i| < \varepsilon$ for $n \geqslant n_0(\varepsilon)$ and for all i, then $\rho(x_n, x) \to 0$ as $n \to \infty$. Thus, convergence in the space m is **uniform (index) coordinate-wise convergence.**

1.35. The space c of convergent number sequences. Let X be the set of convergent number sequences

$$x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\},$$

where $\lim_i \xi_i = \xi$. Let $x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$, $y = \{\eta_1, \eta_2, \ldots, \eta_n, \ldots\}$. Set

$$\rho(x, y) = \sup_i |\xi_i - \eta_i|.$$

We shall designate the space so obtained as space c.

It is evident that the space c of convergent number sequences is a subspace of the space m of bounded number sequences. Hence, this implies that the metric axioms are satisfied in c and convergence in c is *uniform (index) coordinate-wise convergence.*

1.36. The space $M[0, 1]$ of bounded real functions. Consider a set of all bounded functions $x(t)$ of a real variable t, defined on the segment $[0, 1]$. Introduce the metric by setting

$$\rho(x, y) = \sup_t |x(t) - y(t)|.$$

It can be easily verified that all metric axioms are satisfied. The set of all real bounded functions with such a metric is designated as the space $M[0. 1]$. It is easy to see that the convergence in the space $M[0, 1]$ is **uniform convergence on the segment $[0, 1]$.** It is also clear that $C[0, 1] \subset M[0, 1]$.

1.37. The space $\tilde{M}[0, 1]$ of bounded measurable functions. Before considering this space, let us introduce a concept.

Let $\alpha(t)$ be a function measurable on $[0, 1]$. Denote by $\mathcal{E}$ the class of all sets E of measure zero lying in $[0, 1]$, and consider on $\mathcal{E}$ the function:

$$\sup_{[0,1]\setminus E} \alpha(t) = \mu(E).$$

We shall show that if this function is finite for every $E \in \mathcal{E}$, then it takes minimum values on some set E_α. Let

$$\mu_0 = \inf_{E \in \mathcal{E}} \mu_0(E).$$

By the definition of the greatest lower bound, it is possible to find a sequence of sets $\{E_n\} \subset \mathcal{E}$, such that

$$\mu_0 \leqslant \sup_{[0,1]\setminus E_n} \alpha(t) < \mu_0 + \frac{1}{n}.$$

Let
$$E_\alpha = \bigcup_{n=1}^{\infty} E_n, \quad \text{then } mE_\alpha = 0, \text{ and}$$

$$\mu_0 \leqslant \sup_{[0,1]\setminus E_\alpha} \alpha(t) \leqslant \sup_{[0,1]\setminus E_n} \alpha(t) < \mu_0 + \frac{1}{n}.$$

Since this inequality is valid for every n, $\mu_0 = \mu_0(E\alpha)$. The number μ_0 is said to be the *essential maximum* of the function $\alpha(t)$ on [0, 1] and assumes the notation

$$\operatorname*{vrai\,max}_{[0,1]} \alpha(t) = \min_{E \in \mathcal{E}} \left\{ \sup_{[0,1]\setminus E} \alpha(t) \right\}.$$

Let X be a set of all measurable functions $x(t)$, $y(t)$, $z(t), \ldots,$ on [0, 1] whose essential maxima are finite. We regard both the functions $x(t)$ and $y(t)$ of X to be identical, if they are equal almost everywhere (a.e.).

For $x(t)$, $y(t) \in X$, set

$$\rho(x, y) = \operatorname*{vrai\,max}_{[0,1]} |x(t) - y(t)|.$$

We shall verify that the metric axioms are satisfied.

(i) Since
$$\sup_{[0,1]\setminus E} |x(t) - y(t)| \geqslant 0,$$

it follows that $\rho(x, y) \geqslant 0$; moreover, it is obvious that $\rho(x, y) = 0$, if $x(t) = y(t)$ a.e. Conversely, let $\rho(x, y) = 0$. Then, for a certain set E_{xy} of measure zero, we get

$$\sup_{[0,1]\setminus E_{xy}} |x(t) - y(t)| = 0,$$

that is, $x(t) = y(t)$ outside of E_{xy} and, consequently, $x(t)$ and $y(t)$ are equal a.e.

(ii) $\rho(x, y) = \rho(y, x)$ is trivial.

(iii) Let $x(t)$, $y(t)$ and $z(t)$ be functions of X, and let E_{xz}, E_{yz} be sets of measure zero such that

$$\rho(x, z) = \sup_{[0,1]\setminus E_{xz}} |x(t) - z(t)|, \quad \rho(y, z) = \sup_{[0,1]\setminus E_{yz}} |y(t) - z(t)|.$$

Setting $E_{xy} = E_{xz} \cup E_{yz}$, we get

$$\begin{aligned}
\sup_{[0,1]\setminus E_{xy}} |x(t) - y(t)| &\leqslant \sup_{[0,1]\setminus E_{xy}} |x(t) - z(t)| + \sup_{[0,1]\setminus E_{xy}} |z(t) - y(t)| \\
&\leqslant \sup_{[0,1]\setminus E_{xz}} |x(t) - z(t)| + \sup_{[0,1]\setminus E_{yz}} |z(t) - y(t)| \\
&= \rho(x, z) + \rho(z, y).
\end{aligned}$$

What is more, $\rho(x, y) = \operatorname*{vrai\,max}_{[0,1]} |x(t) - y(t)| \leqslant \rho(x, z) + \rho(z, y)$ and the triangle inequality is proved.

The space obtained is designated as the space $\tilde{M}[0, 1]$.

We now take up the aspect of convergence in this space. Let $x_n(t)$, $x(t) \in M[0, 1]$ and let $\rho(x_n, x) \to 0$ as $n \to \infty$. This signifies, for a given $\varepsilon_k > 0$, that

$$\rho(x_n, x) = \min_{E} \left\{ \sup_{[0,1]\setminus E} |x_n(t) - x(t)| \right\} < \varepsilon_k$$

for $n \geqslant n_0(\varepsilon_k)$. Then, we find a set E_k of measure zero, such that

$$\sup_{[0,1]\setminus E_k} |x_n(t) - x(t)| < \varepsilon_k \quad \text{where } n \geqslant n_0(\varepsilon_k).$$

Hence $\quad |x_n(t) - x(t)| < \varepsilon_k$ when $n \geqslant n_0(\varepsilon_k)$ for any $t \in [0,1]\setminus E_k$.

We now make use of the sequence

$$\{\varepsilon_m\}, \quad \varepsilon_m \to 0 \text{ as } m \to \infty,$$

and the corresponding set E_m. Let $\varepsilon > 0$ be arbitrary. We obtain

$$|x_n(t) - x(t)| < \varepsilon_k < \varepsilon$$

for $n \geqslant n_0(\varepsilon_k)$ and all $t \in [0,1]\setminus \bigcup_{m=1}^{\infty} E_m$. Thus, $x_n(t) \to x(t)$ a.e. on $[0,1]$, that is, $x_n(t)$ converges uniformly a.e. on the indicated set of regular measure.

Conversely, let $\{x_n(t)\}$ be uniformly convergent a.e. to $x(t)$. Consequently, for every $\varepsilon > 0$, we can find an index $n_0(\varepsilon)$ and a set E_ε of measure zero, such that $|x_n(t) - x(t)| < \varepsilon$ for $n \geqslant n_0(\varepsilon)$ and any $t \in [0,1]\setminus E_\varepsilon$. But, then, also

$$\sup_{t \in [0,1]\setminus E_\varepsilon} |x_n(t) - x(t)| \leqslant \varepsilon$$

for $n \geqslant n_0(\varepsilon)$. Hence, it immediately follows that

$$\min_E \{ \sup_{t\in[0,1]\setminus E} |x_n(t) - x(t)| \} \leqslant \varepsilon$$

for $n \geqslant n_0(\varepsilon)$, that is, $\rho(x_n, x) \to 0$ as $n \to \infty$.

Consequently, the convergence in the space $\tilde{M}[0,1]$ is *uniform convergence* a.e.

1.38. The space s of all sequences of numbers. We now take up an example of a metric space. Let X be the set of all sequences of real numbers. In this set, we introduce the concept of a limit by saying that $x_n = \{\xi_i^{(n)}\}$ converges to $x = \{\xi_i\}$, if $\xi_i^{(n)} \to \xi_i$ for all $i = 1, 2, 3, \ldots$ (in general, non-uniformly with respect to i). Thus. we obtain a certain non-metric space, which we designate as the space s.

We shall now show that the space s is metrizable. Let $x = \{\xi_i\}$, $x \in s$ and $y = \{\eta_i\}$, $y \in s$. We put

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}.$$

The identity and symmetry axioms are obvious. The triangle inequality follows from the inequality

$$\frac{|a+b|}{1+|a+b|} \leqslant \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|},$$

which is proved as follows: Let a and b have the same sign. It can be assumed that $a > 0$ and $b > 0$, then

$$\frac{|a+b|}{1+|a+b|} = \frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b}$$
$$< \frac{a}{1+a} + \frac{b}{1+b} = \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}.$$

Now, let a and b have different signs and assume that $|a| \geqslant |b|$. Then $|a+b| \leqslant |a|$.

Consider the function $f(x) = x/(1+x)$. Then, $f'(x) = 1/(1+x)^2 > 0$; thus $f(x)$ is a monotone increasing function. Hence

$$\frac{|a+b|}{1+|a+b|} \leqslant \frac{|a|}{1+|a|} \leqslant \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}.$$

Reverting to the triangle inequality, we obtain

$$\rho(x, z) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \zeta_i|}{1 + |\xi_i - \zeta_i|} = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i + \eta_i - \zeta_i|}{1 + |\xi_i - \eta_i + \eta_i - \zeta_i|}$$

$$\leqslant \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|} + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\eta_i - \zeta_i|}{1 + |\eta_i - \zeta_i|}$$

$$= \rho(x, y) + \rho(y, z). \qquad \blacksquare$$

To prove that convergence is coordinate-wise in the sense of the metric supplied (it is, in general, non-uniform with respect to the index), let $x_n = \{\xi_i^{(n)}\}$, $x = \{\xi_i\}$ and $x_n \to x$, getting

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|} < \varepsilon$$

for $n \geqslant n_0(\varepsilon)$. But, then, for every fixed i, we get all the more

$$\frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|} < \varepsilon$$

for $n \geqslant n_0(\varepsilon)$, and since ε is arbitrary and i fixed, $|\xi_i^{(n)} - \xi_i| \to 0$ as $n \to \infty$. Conversely, let $|\xi_i^{(n)} - \xi_i| \to 0$ as $n \to \infty$ and for every i. We select an arbitrary $\varepsilon > 0$ and define an m, such that

$$\sum_{i=m+1}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{2}. \qquad \text{Then,}$$

$$\rho(x_n, x) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|}$$

$$= \sum_{i=1}^{m} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|} + \sum_{i=m+1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|}$$

$$< \sum_{i=1}^{m} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|} + \frac{\varepsilon}{2}.$$

Since the number of terms in the first sum is finite and fixed, we can choose an $n_0(\varepsilon)$ such that

$$\sum_{i=1}^{m} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i|}{1 + |\xi_i^{(n)} - \xi_i|} < \frac{\varepsilon}{2}$$

for $n \geqslant n_0(\varepsilon)$. However, then, for $n \geqslant n_0(\varepsilon)$, we get $\rho(x_n, x) < \varepsilon$. $\blacksquare$

This proof implies that convergence in the sense of the introduced metric coincides with the convergence defined earlier in the space s, and, consequently, introduction of this metric leads to the metrization of the space s.

1.39. The space S [0, 1] of convergence in measure. Let X be the collection of all measurable functions $x(t)$ defined on the interval $[0, 1]$. We shall regard the two functions to be identical if they coincide a.e.

Introduce the metric by the equality

$$\rho(x, y) = \int_0^1 \frac{|x(t) - y(t)|}{1 + |x(t) - y(t)} \, dt.$$

As in the preceding example, the fulfilment of metric axioms is assured.

The space so obtained is known as the **space** $S[0, 1]$. It can be shown that the convergence in $S[0, 1]$ is the **convergence in measure** (or the **asymptotic convergence**). The convergence in measure is defined in [25].

1.3(10). The space $L_2[0, 1]$ of p-integrable functions. Let X be the set of all functions $x(t)$ which belong to $L_p[0, 1]$[1]. We shall regard two functions to be identical if they differ only on a set of measure zero.

If $x(t) \in L_p[0, 1]$ and $y(t) \in L_p[0, 1]$, then we put

$$\rho(x, y) = \left(\int_0^1 |x(t) - y(t)|^p \, dt \right)^{1/p}.$$

It is easy to verify that the identity and symmetry axioms are satisfied. The validity of the triangle inequality follows from the MINKOWSKI[2] inequality for integrals. The space obtained is denoted by $L_p[0, 1]$.

The space $L_2[0, 1]$ is called the **Hilbert function space.**

Let $x_n(t) \in L_p[0, 1]$, $n = 1, 2, \ldots$, and let $\{x_n(t)\}$ converge to $x(t)$, $x(t) \in L_p[0, 1]$, that is, let

$$\int_0^1 |x_n(t) - x(t)|^p \, dt \to 0$$

as $n \to \infty$. Then, it is said to be the **mean convergence of p-th order** of the sequence $\{x_n(t)\}$ of functions to the function $x(t)$. For $p = 2$, we speak simply of **mean convergence.**

1.3(11). The space l_p ($p \geqslant 1$) of real number sequences. Let X be the set of sequences $x = (\xi_1, \xi_2, \ldots, \xi_n, \ldots)$ of real numbers belonging to l_p.[2] If $x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$ and $y = \{\eta_1, \eta_2, \ldots, \eta_n, \ldots\} \in l_p$, then, we define the metric as

$$\rho(x, y) = \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i|^p \right)^{1/p}.$$

It is trivial to verify that symmetiy and identity axioms are defined. The triangle inequality axiom follows from the MINKOWSKI inequality for sums.

The space obtained is denoted by l_p. The space l_2 is called the **Hilbert coordinate space.**

We can show[3] that the convergence of a sequence $\{x_n\}$, $x_n = \{\xi_i^{(n)}\}$ to an element $x = \{\xi_i\}$ in the space l_p implies that

(i) $\xi_i^{(n)} \to \xi_i$ as $n \to \infty$ for all i; and

1. See also Appendix I.
2. See, Appendix I.
3. Cf. the compactness criterion in the space l_p, dealt with in Chap. 5.25.

(ii) for every $\varepsilon > 0$ there exists an index $N_0(\varepsilon)$, such that

$$\sum_{i=N+1}^{\infty} |\xi_i^{(n)}|^p < \varepsilon^p \quad \text{for } N \geqslant N_0(\varepsilon) \text{ and for all } n.$$

1.3(12). **The space $l_p^{(n)}$.** Let X be an n-dimensional arithmetic space, that is, the set of all possible n-tuple of real numbers, and let $x = \{\xi_1, \xi_2, \ldots, \xi_n\}$ and $y = \{\eta_1, \eta_2, \ldots, \eta_n\}$. Put

$$\rho(x, y) = \left(\sum_{i=1}^{n} |\xi_i - \eta_i|^p \right)^{1/p}.$$

We designate the space so obtained by $l_p^{(n)}$. In particular, $l_2^{(n)}$ is the n-dimensional Euclidean space.

We can assume[1] that $l_p^{(n)} \subset l_p$, if we regard every element $\{\xi_1, \xi_2, \ldots, \xi_n\} \in l_p^{(n)}$ as an element $\{\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots\} \in l_p$. It is immediate from above that metric axioms are satisfied for $l_p^{(n)}$. Convergence in the space $l_p^{(n)}$ *is coordinate-wise convergence.*

1.3(13). **The complex spaces.** Together with the real spaces $C\,[0, 1]$, $L_p[0, 1]$, c, l_p it is possible to consider the complex spaces $C[0, 1]$, $L_p[0, 1]$, c, l_p, corresponding to the real spaces. The elements of complex space $C[0, 1]$ are complex-valued continuous functions of a real variable and those of $L_p[0, 1]$ are complex-valued functions, the p-th powers of moduli of which are summable. The elements of complex space c (corr. l_p) are sequences of complex numbers, which converge (corr. whose series of p-th powers of moduli converge).

All definitions given in the foregoing for real spaces can be directly extended to the corresponding complex spaces.

1.3(14). **Non-metrizable spaces.** Finally, we shall give an example of a set where the idea of convergence of a sequence can be introduced without an appeal to the concept of metric for defining this convergence.

Consider the set $F[0, 1]$ of all real functions, defined on the interval $[0, 1]$. A sequence $\{x_n(t)\} \subset F[0, 1]$ is regarded convergent to $x(t) \in F[0, 1]$, if for any fixed t, we have

$$x_n(t) \to x(t).$$

Thus, the convergence of a sequence of functions in the set $F[0, 1]$, is a *point-wise convergence.* This convergence is non-metrizable. In fact, assume that it is possible to introduce a metric in $F\,[0, 1]$, such that the convergence defined by this metric is point-wise convergence of a sequence of functions. Let M be a set of all continuous functions in the metric space $F[0, 1]$. On the one hand, by properties of closure in the metric space, $\overline{\overline{M}} = \overline{M}$. On the other, $\overline{\overline{M}} \neq \overline{M}$, because $\overline{M}$ is a set of continuous functions and their limits in the sense of uniform convergence, that is, it is a first category[2] set of functions, whereas $\overline{\overline{M}}$ is a set of functions of first class and their limits, in other words, a second category[2] set of functions.

1.4. COMPLETE SPACES. SOME EXAMPLES

1.41. Definitions. A sequence $\{x_n\}$ of elements in a metric space X is called a **Cauchy sequence** (or **fundamental sequence**), if there is an index $n_0(\varepsilon)$

1. Isometric spaces are discussed in Chap. 1.5.
2. For Baire's category, see [25] and also Chap. 1.6.

for each $\varepsilon > 0$, such that $\rho(x_n, x_m) < \varepsilon$ whenever $n, m \geqslant n_0(\varepsilon)$.

THEOREM. *If a sequence $\{x_n\}$ converges to a limit x_0, then it is a Cauchy (fundamental) sequence.*

PROOF. In fact, let $x_0 = \lim_n x_n$. Then there is an index $n_0(\varepsilon)$ for any $\varepsilon > 0$, such that $\rho(x_n, x_0) < \varepsilon/2$ for $n \geqslant n_0(\varepsilon)$. Hence

$$\rho(x_n, x_m) \leqslant \rho(x_n, x_0) + \rho(x_m, x_0) < \varepsilon \text{ for } n \geqslant n_0(\varepsilon) \text{ and } m \geqslant n_0(\varepsilon). \quad \blacksquare$$

The converse of this theorem is not true for an arbitrary metric space, since there exist metric spaces which contain a CAUCHY sequence but have no element which will be its limit.

Examples. 1. Let X be the set of rational numbers, in which metric is defined by

$$\rho(r_1, r_2) = |r_1 - r_2|.$$

Then, X is a metric space.

Consider the sequence $\quad r_1 = \frac{1}{2}, \quad r_2 = \frac{1}{4}, \ldots, r_n = \frac{1}{2^n}, \ldots$

This is a CAUCHY sequence and converges to the limit $r_0 = 0$. Now, take the sequence

$$r_n = \left(1 + \frac{1}{n}\right)^n.$$

This is a CAUCHY sequence but has no limit in X, because

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

is not a rational number.

2. Let X be the space of the polynomials $P(t)$, $0 \leqslant t \leqslant 1$ with the CHEBYSHEV metric, that is

$$\rho(P, Q) = \max_t |P(t) - Q(t)|$$

for $P(t)$ and $Q(t) \in X$. Let $\{P_n(t)\}$ be a sequence of polynomials converging uniformly to a continuous function which is not a polynomial. The sequence $\{P_n(t)\}$ is obviously a CAUCHY sequence but it possesses no limit in X.

A metric space X is said to be **complete** if every CAUCHY sequence in X converges to some limit point in X.

Let us remark that a *closed set of a complete space is itself a complete space.*

We shall now show certain metric spaces to be complete.

1.42. The space E_n. The completeness of E_n, the n-dimensional Euclidean space, follows from the CAUCHY test for the existence of a limit for a sequence of points in this space.

1.43. The space $C\,[0, 1]$. Given a sequence $\{x_n(t)\}$, where $x_n(t) \in C\,[0, 1]$, $n = 1, 2, \ldots$, and let $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$. This signifies that the sequence $\{x_n(t)\}$ satisfies CAUCHY's condition of uniform convergence on $[0, 1]$. Let $x_0(t)$ be the limit of the sequence $\{x_n(t)\}$. Being the limit of a uniformly convergent sequence of continuous functions, the function $x_0(t)$ is also continuous on $[0, 1]$. Thus, $x_0(t) \in C[0, 1]$ and $\rho(x_n, x_0) \to 0$. Consequently, the space $C[0, 1]$ is complete.

1.44 The space m. Let $\{x_n\}$ be a CAUCHY sequence of elements of m, and let $x_n = \{\xi_i^{(n)}\}$. Since $x_n \in m$, $|\xi_i^{(n)}| \leqslant K_n$ for $i = 1, 2, \ldots$ Further, since $\{x_n\}$ is a CAUCHY sequence, for every $\varepsilon > 0$, there is an index $n_0(\varepsilon)$ such that $\rho(x_n, x_k) < \varepsilon$ for $n, k \geqslant n_0(\varepsilon)$; in other words,

$$\sup_i |\xi_i^{(n)} - \xi_i^{(k)}| < \varepsilon \quad \text{for } n, k \geqslant n_0(\varepsilon).$$

Therefore,
$$|\xi_i^{(n)} - \xi_i^{(k)}| < \varepsilon \tag{1}$$

for $n, k \geqslant n_0(\varepsilon)$ and for every i.

Let i be fixed. Then, by (1), the number sequence $\{\xi_i^{(1)}, \xi_i^{(2)}, \ldots, \xi_i^{(n)}, \ldots\}$ satisfies the CAUCHY test for existence of a limit and, therefore, converges to some number ξ_i. Thus, we obtain a number sequence $\{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$.

Consider Ineq. (1) and let $k \to \infty$, getting in the limit,

$$|\xi_i^{(n)} - \xi_i| \leqslant \varepsilon \quad \text{for n} \geqslant n_0(\varepsilon) \text{ and for all } i. \tag{2}$$

Hence,
$$|\xi_i| \leqslant |\xi_i^{(n_0)} - \xi_i| + |\xi_i^{(n_0)}| \leqslant \varepsilon + K_{n_0},$$

for all i. This, however, implies that $\{\xi_i\}$ is a bounded numerical sequence, that is, $x_0 = \{\xi_i\} \in m$. We find from (2) that $\sup_i |\xi_i^{(n)} - \xi_i| \leqslant \varepsilon$ for $n \geqslant n_0(\varepsilon)$, that is, $\rho(x_n, x) \leqslant \varepsilon$ for $n \geqslant n_0(\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, $x_n \to x$ as $x \to \infty$. The completness of the space m is, therefore, proved.

1.45. The space c. We shall show that the space c considered as a subset of the space m, is closed in m. By the remarks on p. 10 this implies the space c to be complete.

Let $\{x_n\}$, with $x_n = \{\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_i^{(n)}, \ldots\}$, be a sequence of elements in c and let $x_n \to x_0$, where $x_3 = \{\xi_1^{(0)}, \xi_2^{(0)}, \ldots, \xi_i^{(0)}, \ldots\}$. We shall show that $\{\xi_i^{(0)}\}$ is a convergent sequence. In fact,

$$|\xi_i^{(0)} - \xi_j^{(0)}| \leqslant |\xi_i^{(0)} - \xi_i^{(n)}| + |\xi_i^{(n)} - \xi_j^{(n)}| + |\xi_j^{(n)} - \xi_j^{(0)}|$$
$$\leqslant 2\rho(x_n, x_0) + |\xi_i^{(n)} - \xi_j^{(n)}|.$$

Given an arbitrary number $\varepsilon > 0$. Let us first choose n so large that $\rho(x_n, x_0) < \varepsilon/4$, and then, fix this n. Since $\{\xi_i^{(n)}\}$ is a convergent sequence, there is an index n_0, such that for $i, j \geqslant n_0$, we obtain

$$| \xi_i^{(n)} - \xi_j^{(n)} | < \frac{\varepsilon}{2}. \quad \text{But, then,} \quad | \xi_i^{(0)} - \xi_j^{(0)} | < \varepsilon$$

for $i, j \geqslant n_0$, that is $\{\xi_i^{(0)}\}$ is a convergent sequence. Thus, $x_0 \in c$ which was also required to be proved.

1.46. The space $\tilde{M}[0, 1]$. Let $\{x_n\}$ be a CAUCHY sequence of elements in $\tilde{M}[0, 1]$. Then,

$$\inf_{E, \text{mes } E = 0} \{ \sup_{t \in [0, 1] \setminus E} | x_n(t) - x_m(t) | \} < \varepsilon$$

for any $\varepsilon > 0$ and $n, m \geqslant n_0(\varepsilon)$. Hence, there is a set E_{n_0} of measures zero such that

$$\sup_{t \in [0, 1] \setminus E_{n_0}} x_n(t) - x_m(t) | < \varepsilon, \quad \text{for } m, n \geqslant n_0(\varepsilon),$$

that is, $| x_n(t) - x_m | < \varepsilon$ for $n, m \geqslant n_0 (\varepsilon)$ a.e. on $[0, 1]$. Thus, we easily conclude (see p. 12) that the sequence $\{x_n(t)\}$ of bounded measurable functions is uniformly a CAUCHY sequence a.e. on $[0, 1]$. Therefore, there exists a bounded measurable function $x_0(t)$ which is the limit of this sequence a.e. on $[0, 1]$. Thence, it is plain that $\rho(x_n, x_0) \to 0$, and the space $\tilde{M}$ $[0, 1]$ is complete.

1.47. The spaces $L_p[0, 1]$ and l_p. The proof of the completeness of spaces $L_2[0, 1]$ and l_2 is customarily given in texts dealing with the theory of functions of a real variable (see, [25]). By analogous method, the spaces L_p $[0,1]$ and l_p can be proved to be complete. These proofs will not, however, be derived here and the assertion on the completeness of $L_p[0, 1]$ and l_p will be realized as a corollary to a general theorem of functional analysis (see Chap. 5.3).

1.5 THE COMPLETION OF METRIC SPACES

IT IS well known that the property of completeness of the field of real numbers plays an important role in mathematical analysis. The property of completeness of metric spaces occupies a similar crucial place in functional analysis. We shall, therefore, now examine a process according to which every non-complete metric space can be extended to a complete space, analogous to the extension of a set of rational numbers to a set of all real numbers. As a starting point, we introduce a concept which will be useful also in the sequel.

Given two metric spaces X and Y. Let the distance between the elements x_1 and x_2 in X be $\rho_X (x_1, x_2)$ and that between the elements y_1 and y_2 in Y be $\rho_Y (y_1, y_2)$.

If a one-one correspondence can be shown to exist between the elements of the spaces X and Y, such that the distance between two elements of one space is equal to the distance between the corresponding elements of the other space, then X and Y are said to be **isometric**.

It is easy to infer that the isometric spaces are *metrically identical*, that is, the two isometric spaces can be regarded identical from the view point of problems connected only with the distance between elements, e.g., those of covergence, completeness, and so on; in other words, distinctions between isometric spaces concern only the nature of their elements.

This can be said not only of the isometric spaces X and Y, but also of the isometric sets contained in these spaces, and in problems connected only with metric, the results obtained for a certain set in a metric space remain valid for all sets isometric to it.

THEOREM *Given a metric space X_0. Assume that this space is non-complete, that is in this space there is a Cauchy sequence but no element which will be its limit.*

Then there exists another complete metric space X, such that it has a subset X' everywhere dense in X and isometric to X_0.

The space X is called the **completion** of the space X_0.

PROOF. Consider all the possible sequences

$$\{x_n\}, \{y_n\}, \{z_n\}, \ldots,$$

formed of elements in X_0 and being CAUCHY sequences. We associate any two CAUCHY sequences $\{x_n\}$ and $\{x'_n\}$ to the same class and such that $\rho(x_n, x'_n) \to 0$ as $n \to \infty$, We consider this class $\tilde{x}$ as an element of a new space X. Let $\tilde{x}$ and $\tilde{y}$ be two elements in X. Then, we choose any sequences $\{x_n\}$ and $\{y_n\}$ from the classes $\tilde{x}$ and $\tilde{y}$.

It is required to show that $\lim\limits_{n} \rho(x_n, y_n)$ exists. In fact,

$$\rho(x_n, y_n) \leqslant \rho(x_n, x_m) + \rho(x_m, y_m) + \rho(y_m, y_n).$$

Hence, $$\rho(x_n, y_n) - \rho(x_m, y_m) \leqslant \rho(x_n, x_m) + \rho(y_n, y_m). \tag{1}$$

If the indices n and m are interchanged, then

$$\rho(x_m, y_m) - \rho(x_n, y_n) \leqslant \rho(x_n, x_m) + \rho(y_n, y_m). \tag{2}$$

It follows from (1) and (2) that

$$|\rho(x_m, y_m) - \rho(x_n, y_n)| \leqslant \rho(x_n, x_m) + \rho(y_n, y_m).$$

The right side of this inequality tends to zero as $m, n \to \infty$, implying that the numerical sequence $\{\rho(x_n, y_n)\}$ satisfies the CAUCHY condition and, consequently, the limit $\lim\limits_{n} \rho(x_n, y_n)$ exists.

Now, we define a distance in X by

$$\rho(\tilde{x}, \tilde{y}) = \lim_{n} \rho(x_n, y_n)$$

and show this distance to be independent of the choice of the sequences $\{x_n\}$

and $\{y_n\}$ from the respective classes.

Let $\{x'_n\}$ and $\{y'_n\}$ be two other sequences also of the classes $\tilde{x}$ and $\tilde{y}$. Then,

$$\rho(x_n, y_n) \leqslant \rho(x_n, x'_n) + \rho(x'_n, y'_n) + \rho(y'_n, y_n).$$

Hence,
$$\lim_n \rho(x_n, y_n) \leqslant \lim_n \rho(x'_n, y'_n).$$

Similarly, we obtain, which is also the converse:

$$\lim_n \rho(x'_n, y'_n) \leqslant \lim_n \rho(x_n, y_n).$$

Consequently,
$$\lim_n \rho(x_n, y_n) = \lim_n \rho(x'_n, y'_n).$$

Let us veryfy that the metric axioms are satisfied.

(i) Since $\rho(x_n, y_n) \geqslant 0$, it also follows that

$$\rho(\tilde{x}, \tilde{y}) = \lim_n \rho(x_n, y_n) \geqslant 0.$$

Further, the equality $\rho(\tilde{x}, \tilde{y}) = \lim_n \rho(x_n, y_n) = 0$ implies by definition, that the sequences $\{x_n\}$ and $\{y_n\}$ belong to the same class.

Since $\{x_n\}$ is any sequence of the class $\tilde{x}$ and $\{y_n\}$ of $\tilde{y}$, we have $\tilde{x} = \tilde{y}$.

(ii) $\rho(\tilde{x}, \tilde{y}) = \rho(\tilde{y}, \tilde{x})$ is trivial.

(iii) If $\{x_n\} \in \tilde{x}$, $\{y_n\} \in \tilde{y}$, $\{z_n\} \in \tilde{z}$, then, evidently,

$$\rho(\tilde{x}, \tilde{z}) = \lim_n \rho(x_n, z_n) \leqslant \lim_n \rho(x_n, y_n) + \lim_n \rho(y_n, z_n)$$
$$= \rho(\tilde{x}, \tilde{y}) + \rho(\tilde{y}, z).$$

We now show that X is a complete space. For this, choose an arbitrary Cauchy sequence $\{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n, \ldots\}$ of elements in X, that is, such that $\rho(\tilde{x}_n, \tilde{x}_m) \to 0$ as $n, m \to \infty$. From every class $\tilde{x}_n$, extract an arbitrary sequence $\{x_1^{(n)}, x_2^{(n)}, \ldots, x_k^{(n)}, \ldots\}$. Since this is a Cauchy sequence, k_n can be chosen such that

$$\rho(x_p^{(n)}, x_{k_n}^{(n)}) \leqslant \frac{1}{n} \quad \text{for } p > k_n.$$

Now, consider the sequence $\{x_{k_1}^{(1)}, x_{k_2}^{(2)}, \ldots, x_{k_n}^{(n)}, \ldots\}$ and show this to be a Cauchy sequence. We have

$$\rho(x_{k_n}^{(n)}, x_{k_m}^{(m)}) \leqslant \rho(x_{k_n}^{(n)}, x_p^{(n)}) + \rho(x_p^{(n)}, x_p^{(m)}) + \rho(x_p^{(m)}, x_{k_m}^{(m)}). \quad (3)$$

Given $\varepsilon > 0$ arbitrary. Since $\rho(\tilde{x}_n, \tilde{x}_m) \to 0$ as $n, m \to 0$, there is an index

n_0, such that

$$\rho(\tilde{x}_n, \tilde{x}_m) = \lim_p \rho(x_p^{(n)}, x_p^{(m)}) < \frac{\varepsilon}{2}$$

for $n, m \geqslant n_0$. Then,

$$\rho(x_p^{(n)}, x_p^{(m)}) < \frac{\varepsilon}{2} \quad \text{for } n, m \geqslant n_0 \text{ and sufficiently large } p. \tag{4}$$

For this, n_0 can be taken such that $1/n_0 < \varepsilon/4$. With n and m fixed and satisfying the condition $n, m \geqslant n_0$, p is taken so large that $p > k_m$ and $p > k_n$. Then, by the choice of k_n and k_m

$$\rho(x_p^{(n)}, x_{k_n}^{(n)}) < \frac{1}{n} < \frac{\varepsilon}{4}, \quad \rho(x_p^{(m)}, x_{k_m}^{(m)}) < \frac{1}{m} < \frac{\varepsilon}{4}.$$

From (3), (4) and (5), it follows that

$$\rho(x_{k_n}^{(n)}, x_{k_m}^{(m)}) < \varepsilon \quad \text{for } n, m \geqslant n_0, \tag{5}$$

that is, $\{x_{k_n}^{(n)}\}$ is a CAUCHY sequence.

Denoting the class containing the sequence $\{x_{k_n}^{(n)}\}$ by $\tilde{x}$, it is to be shown that $\tilde{x}_n \to \tilde{x}$. Obviously,

$$\rho(\tilde{x}_n, x) = \lim_p \rho(x_p^{(n)}, x_{k_p}^{(p)}) \leqslant \lim_p \rho(x_p^{(n)}, x_{k_n}^{(n)})$$

$$+ \lim_p \rho(x_{k_n}^{(n)}, x_{k_p}^{(p)}) < \frac{1}{n} + \lim_p \rho(x_{k_n}^{(n)}, x_{k_p}^{(p)}), \tag{6}$$

Since $\{x_{k_n}^{(n)}\}$ is a CAUCHY sequence, there exists n_0 for $\varepsilon > 0$ given, such that

$$\rho(x_{k_n}^{(n)}, {}_{k_p}^{(p)}) < \frac{\varepsilon}{2} \quad \text{for } n, p \geqslant n_0.$$

Hence
$$\lim_p \rho(x_{k_n}^{(n)}, x_{k_p}^{(p)}) \leqslant \frac{\varepsilon}{2} \tag{7}$$

for $n \geqslant n_0$. For this, it can be assumed without squeeze of generality, that $1/n_0 < \varepsilon/2$. From (6) and (7) it follows, for $n \geqslant n_0$, that $\rho(\tilde{x}_n, \tilde{x}) < \varepsilon$, that is, the sequence $\{\tilde{x}_n\}$ converges to the element $\tilde{x}$, proving that the space X is complete.

Now consider **stationary (repetitive)** sequences, that is, a sequence of the form $\{x, x, \ldots, x, \ldots\}$, which are evidently, CAUCHY sequences and, consequently, each one of them is contained in a certain class belonging to X. Obviously, there can be exactly one stationary sequence in one and the same class. Now, if $\{x, x, \ldots, x, \ldots\} \in \tilde{x}$, $\{y, y, \ldots, y, \ldots\} \in \tilde{y}$, then, evidently, $\rho(x, y) = \rho(\tilde{x}, \tilde{y})$.

We now show that X_0 is isometric to some subset X' of the space X, everywhere dense in X. Let X' contain all the classes $\tilde{x}$, among whose sequences appears the stationary (repetitive) sequence $\{x, x, \ldots, x, \ldots\}$. There is a one-one correspondence between the classes $\tilde{x} \in X'$ and the elements x of X_0 of which the stationary sequence in x is formed. Moreover, $\rho(\tilde{x}, \tilde{y}) = \rho(x, y)$ if $\{x\} \in \tilde{x}$ and $\{y\} \in \tilde{y}$. Therefore, this one-one correspondence between X_0 and X' is isometric.

It is easily verifiable that X' is dense in X, that is, for every number $\varepsilon > 0$ and for every element $\tilde{x} \in X$ there is an element $\tilde{x}_\varepsilon \in X'$ such that $\rho(\tilde{x}, \tilde{x}_\varepsilon) \leqslant \varepsilon$. In fact, let $\tilde{x}$ be any class containing the Cauchy sequence $\{x_1, x_2, \ldots, x_n, \ldots\}$. Choose an n such that $\rho(x_n, x_m) < \varepsilon$ for $m > n$. Construct the stationary sequence $\{x_n, x_n, \ldots, x_n, \ldots\}$ and denote by $\tilde{x}_\varepsilon$ the class containing this sequence. Obviously, $\tilde{x}_\varepsilon \in X'$. Further, $\rho(\tilde{x}, \tilde{x}_\varepsilon) = \lim_m \rho(x_m, x_n) \leqslant \varepsilon$ and this is what we sought to establish.

It remains to show that the completion of a space X_0 is uniquely defined to within isometry, that is, there exists, to within isometry, a complete space X containing an everywhere dense subset isometric to X_0. In fact, let Y be another complete space in which X_0 is everywhere dense. Then, every point $\tilde{y} \in Y$ is the limit of some sequence $\{x_1, x_2, \ldots, x_n, \ldots\} \subset X_0$. Since this is a Cauchy sequence, it defines some element $\tilde{x} \in X$. We associate this element $\tilde{x}$ with the element $\tilde{y}$. Conversely, let an element $\tilde{\xi} \in X$ be given and let $\{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$ be some fundamental sequence in the class $\tilde{\xi}$. Since this fundamental sequence lies in the complete space Y, it defines some element $\tilde{\eta} \in Y$. Associating this element with $\tilde{\xi}$, we obtain a correspondence between the elements of spaces X and Y, which is, obviously, one-one. Further, since

$$\rho(\tilde{x}, \tilde{\xi}) = \lim_n \rho(x_n, \xi_n) = \rho(\tilde{y}, \tilde{\eta})^1,$$

the correspondence between X and Y is isometric, giving the required proof.

Examples. 1. Consider the space l'_p consisting of all ordered systems $\{\xi_1, \xi_2, \ldots, \xi_{k_1}, 0, 0, 0, \ldots\}$, with k_1 any natural number and ξ_i arbitrarily real. If

$$x = \{\xi_1, \xi_2, \ldots, \xi_{k_1}, 0, \ldots\}, \qquad y = \{\eta_1, \eta_2, \ldots, \eta_{k_2}, 0, \ldots\}$$

and if $k_2 \geqslant k_1$, then we set

$$\rho(x, y) = \left(\sum_{i=1}^{k_1} | \xi_i - \eta_i |^p + \sum_{i=k_1+1}^{k_2} |\eta_i|^p \right)^{1/p}.$$

l'_p is a subspace of l_p and, besides, is non-complete, because for example, though the sequence

$$x_1 = \{1\}, \quad x_2 = \left\{1, \frac{1}{2}\right\}, \ldots, x_n = \left\{1, \frac{1}{2}, \ldots, \frac{1}{2^n}\right\}, \ldots$$

1. It is easy to prove that if $x_n \to x$ and $y_n \to y$ in a metric space, then also

$$\rho(x_n, y_n) \to \rho(x, y).$$

is a CAUCHY sequence

$$\left[\text{for} \quad \rho(x_n, x_m) = \left(\sum_{i=m}^{n-1} \frac{1}{2^{ip}}\right)^{1/p} \to 0 \quad \text{as} \quad m, n \to \infty, m < n\right],$$

it has no limit in l'_p.

Let us denote the completion of l'_p by X. Since, on the other hand, it is evident that l'_p lies everywbere dense in the complete space l_p, X is isometric to l_p. Thus, the completion of l'_p is l_p (more precisely: the space l_p is isometric to l'_p).

2. Let $C_0\,[0, 1]$ be the space of all polynomials defined on the interval $[0, 1]$. Let the CHEBYSHEV metric

$$\rho\,(p, q) = \max_t |\,p(t) - q(t)\,|$$

be introduced in this space. Obviously, $C_0\,[0, 1]$ is not complete. Since $C_0[0, 1]$ is everywhere dense in the complete space $C\,[0, 1]$, $C\,[0, 1]$, is the completion of $C_0\,[0, 1]$. verifying an isometric correspondence between them.

3. Let $L'_p\,[0, 1]$ be the collection of all continuous functions defined on $[0, 1]$ with the metric

$$\rho(x, y) = \left(\int_0^1 |\,x(t) - y(t)\,|^p\,dt\right)^{1/p}.$$

The space $L'_p\,[0, 1]$ is not complete, because a sequence of continuous functions which converges in the mean (p-th power) to a discontinuous function, is a fundamental sequence in $L'_p\,[0, 1]$ but has no limit point in this space. We make $L'_p\,[0, 1]$ complete and get $L_p\,[0, 1]$ isometric to it.

1.6 THEOREMS ON COMPLETE SPACES

WE NOW seek to establish the upcoming theorems, analogues of CANTOR's lemma on contracting system of intervals.

THEOREM 1. *Let a nested sequence of closed spheres [i.e. each of which contains all that follow, e.g.,* $\bar{S}_1 \supset \bar{S}_2 \supset, \ldots, \bar{S}_n, \ldots$*] be given in a complete metric space X. If the radii of these spheres tend to zero, then these spheres have a unique common point.*

Let us consider the spheres

$$\bar{S}\,(a_1, \varepsilon_1), \bar{S}\,(a_2, \varepsilon_2), \ldots, \bar{S}_n\,(a_n, \varepsilon_n), \ldots$$

By hypothesis, $\quad \bar{S}_1 \supset \bar{S}_2 \supset \ldots \bar{S}_n \supset \ldots \quad [\bar{S}_n = \bar{S}\,(a_n, \varepsilon_n]$.

We consider the sequence of centres of these spheres

$$a_1, a_2, \ldots, a_n, \ldots$$

Since $\bar{S}_{n+p} \subset \bar{S}_n$, so $a_{n+p} \in \bar{S}(a_n, \varepsilon_n)$. Therefore, $\rho(a_{n+p}, a_n) \leqslant \varepsilon_n$. Consequently, $\rho(a_{n+p}, a_n) \to 0$ as $n \to \infty$, that is, the sequence of centres of these spheres is a CAUCHY sequence.

Since the space X is complete, this sequence converges to a certain limit point $a \in X$. Take any arbitrary sphere $\bar{S}_k$ (k fixed). Then, the points $a_k, a_{k+1}, \ldots, a_{k+n}, \ldots$, belong to this sphere. Since the sphere $\bar{S}_k$ is closed, the limit point a of the sequence $a_k, a_{k+1}, \ldots, a_{k+n}, \ldots$ also belongs to $\bar{S}_k$, and hence $a = \lim_n a_n$ belongs to all the spheres.

Now, assume that there is a point b common to all the spheres but different from a, such that $\rho(a, b) = \delta > 0$. Since the points a and $b \in \bar{S}_n$, $n = 1, 2, \ldots$, so we must have

$$\delta = \rho(a, b) \leqslant \rho(a, a_n) + \rho(a_n, b) \leqslant 2\varepsilon_n.$$

This is, however, not possible, since $\varepsilon_n \to 0$ as $n \to \infty$. ■

REMARK. It is possible to somewhat generalize the theorem proved. Let us denote the **diameter** of a bounded set F in a metric space by

$$d(F) = \sup_{x, y \in F} \rho(x, y).$$

THEOREM 1′. *Given a nested sequence of closed sets in a complete metric space X, whose diameters tend to zero. Then, these sets have a unique common point.*

The proof of this theorem is essentially the same as that for Theorem 1.

As is known, the properties of rational nnmbers established by CANTOR's lemma can be employed for defining the completeness or continuity of the sets of real numbers or rational numbers. Analogously, the theorem on the nested sphercs characterizes the completeness of a metric space.

THEOREM 2. *If in a metric space X any nested sequence of closed spheres, whose diameters tend to zero, has a nonempty intersection, then the space X is complete.*

Let $\{x_n\}$ be a given fundamental sequence. Choose n_k such that

$$\rho(x_{n_k+p}, x_{n_k}) < \frac{1}{2^k}$$

for any $p > 0$. Let $\bar{S}_k$ be a closed sphere with radius $1/2^{k-1}$ and centre x_{n_k}. We have $\bar{S}_{k+1} \subset \bar{S}_k$. In fact, if $x \in \bar{S}_{k+1}$, then

$$\rho(x, x_{n_k}) \leqslant \rho(x_n, x_{_{k+1}}) + \rho(x_{n_{k+1}}, x_{n_k}) \leqslant \frac{1}{2^k} + \frac{1}{2^k} = \frac{1}{2^{k-1}},$$

that is, $x \in \bar{S}_k$.

The radius of the sphere $\bar{S}_k$ tends to zero. Consequently, by hypothesis, there exists a point x_0 in the intersection of all the spheres $\bar{S}_k$. Show that x_0 is a limit point of the sequence $\{x_n\}$. The subsequence $\{x_{n_k}\}$ converges to x_0, because x_{n_k}. $x_0 \in \bar{S}_k$ and, consequently,

$$\rho(x_{n_k}, x_0) \leqslant \frac{1}{2^{k-1}} \to 0.$$

But, then, the whole of the sequence $\{x_n\}$ converges to x_0, because

$$\rho(x_n, x_0) \leqslant \rho(x_n, x_{n_k}) + \rho(x_{n_k}, x_0),$$

and both the summands can be made arbitrarily small, if n, n_k are chosen sufficiently large. The theorem is proved.

A set M is said to be of the **first category** or **meagre** if it can be written as a *countable* union of nowhere dense sets. Otherwise, this is said to be of the **second category.**

The above relation is mutually exclusive, that is, a set is either of the first category or of the second category. For example, the set of rational points of a straight line is of the first category, that of all irrational points is of the second category, as is borne out by the theorem that follows.

THEOREM 3. *A nonempty complete metric space is a set af the second category.*

PROOF. Assume the contrary. Let the complete space X be representable in the form $X = \bigcup\limits_{n=1}^{\infty} M_n$, where M_n, $n = 1, 2, \ldots$, are nowhere dense. Choose a sphere $\bar{S}(a, 1)$ of radius 1 with centre in an arbitrary point a. Since M_1 is nowhere dense, there exists inside of $\bar{S}(a, 1)$ a sphere $\bar{S}(a_1, r_1)$ of radius $r_1 < \frac{1}{2}$ which contains no point of the set M_1. Since the set M_2 is nowhere dense, there exists inside of $\bar{S}(a_1, r_1)$ a sphere $\bar{S}(a_2, r_2)$, of radius $r_2 < \frac{1}{2^2}$ which contains no points of the set M_2, and so on.

We thus obtain a sequence of closed spheres: $\bar{S}(a_1, r_1), \bar{S}(a_2, r_2), \ldots,$ $\bar{S}(a_n, r_n), \ldots$, each of which contains all that follow and whose radii approach zero. For this, $\bar{S}(a_n, r_n)$ contains no points of the sets $M_1, M_2, \ldots, M_n$. By Theorem 1, there exists a point $a_0 \in X$ common to all the spheres. On the other hand, this point a_0 belongs to none of the sets M_n, therefore,

$$a_0 \bar{\in} X = \bigcup_{n=1}^{\infty} M_n,$$

a contradiction proving the theorem.

1.7 THE CONTRACTION MAPPING PRINCIPLE

THE WELL-KNOWN successive approximation method, or the iterative process, finds wide applications in proofs of existence theorems for solutions of the algebraic, differential, integral and other equations.

Besides its other extensive applications, this method has a key role in obtaining approximate solution of equations. The method of successive approximation of various types of equations belongs to the general set-up of functional analysis and leads to the principle of **contraction mappings**, formulated by the Polish mathematician, S. BANACH in 1922.

THEOREM 1. *In a complete metric space X, given an operator A which takes the elements of the space X again into the elements of this space. Further, for all x and y in X, let*

$$\rho[A(x), A(y)] \leqslant \alpha\rho(x, y), \tag{1}$$

with $\alpha < 1$ and not depending on x and y. Then, there is a unique point x_0, such that $A(x_0) = x_0$.

The point x_0 is called a **fixed point** of A.

PROOF. Consider an arbitrary fixed element $x \in X$, and set

$$x_1 = A(x), x_2 = A(x_1), x_3 = A(x_2), \ldots, x_n = A(x_{n-1}), \ldots$$

It is to be shown that $\{x_n\}$ is a CAUCHY sequence. For this, note that

$$\rho(x_1, x_2) = \rho[A(x), A(x_1)] \leqslant \alpha\rho(x, x_1) = \alpha\rho[x, A(x)],$$
$$\rho(x_2, x_3) = \rho[A(x_1), A(x_2)] \leqslant \alpha\rho(x_1, x_2) \leqslant \alpha^2\rho[x, A(x)],$$
$$\cdots\cdots\cdots$$
$$\rho(x_n, x_{n+1}) \leqslant \alpha^n\rho\,[x, A(x)]$$
$$\cdots\cdots\cdots$$

Further,
$$\begin{aligned}\rho(x_n, x_{n+p}) &\leqslant \rho(x_n, x_{n+1}) + \rho\,(x_{n+1}, x_{n+2}) + \ldots + \rho(x_{n+p-1}, x_{n+p})\\ &\leqslant (\alpha^n + \alpha^{n+1} + \ldots + \alpha^{n+p-1})\,\rho[x, A(x)]\\ &= \frac{\alpha^n - \alpha^{n+p}}{1-\alpha}\,\rho[x, A(x)].\end{aligned} \tag{2}$$

Since, by hypothesis $\alpha < 1$, it follows that $\rho(x_n, x_{n+p}) < \frac{\alpha_n}{1-\alpha}\ \rho[x, A(x)]$. This, in turn, implies that $\rho(x_n, x_{n+p}) \to 0$ as $n \to \infty$, $p > 0$. Thus, $\{x_n\}$ is a CAUCHY sequence. Since the space X is complete, there is an element $x_0 \in X$, the limit of this sequence :

$$x_0 = \lim_n x_n.$$

We shall show that $A(x_0) = x_0$. In fact,

$$\begin{aligned}\rho[x_0, A(x_0)] &\leqslant \rho(x_0, x_n) + \rho[x_n, A(x_0)]\\ &= \rho(x_0, x_n) + \rho[A(x_{n-1}), A(x_0)] \leqslant \rho(x_0, x_n) + \alpha\rho(x_{n-1}, x_0).\end{aligned}$$

But
$$\rho(x_0, x_n) < \varepsilon/2 \quad \text{and} \quad \rho(x_0, x_{n-1}) < \varepsilon/2$$

for any given ε and sufficiently large n. Hence

$$\rho[x_0, A(x_0)] < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\rho[x_0, A(x_0)] = 0$, that is, $A(x_0) = x_0$, proving the theorem.

Assume that there exist two elements $x_0 \in X$ and $y_0 \in X$, verifying

$$A(x_0) = x_0 \quad \text{and} \quad A(y_0) = y_0.$$

Then,
$$\rho(x_0, y_0) = \rho[A(x_0), A(y_0)] \leqslant \alpha\rho(x_0, y_0).$$

If we assume $\rho(x_0, y_0) > 0$, then by the reasonings in the foregoing it follows that $1 \leqslant \alpha$; but this violates the hypothesis.

If we pass on to the limit in (2) as $p \to \infty$, we estimate the error in the *n-th* approximation :

$$\rho(x_n, x_0) \leqslant \frac{\alpha^n}{1-\alpha} \rho[x, A(x)]. \tag{3}$$

REMARK 1. We can construct an approximating sequence $\{x_n\}$ which converges to the fixed point x_0, starting from any element $x \in X$. The rate with which $\{x_n\}$ converges to its limit is described by the choice of x.

REMARK 2. Sometimes it is useful to consider a mapping A such that Ineq. (1) is satisfied not in the entire space but only in some closed neighbourhood $\overline{S}(\bar{x}, r)$ of any of its points $\bar{x}$. Then, the contraction mapping principle can be applied subject to the additional condition that the operator A carries this sphere into itself and hence the approximating sequence does not fall outside the neighbourhood considered. Let, for example, in addition to Ineq. (1), the inequality

$$\rho[\bar{x}, A(\bar{x})] \leqslant (1-\alpha)r$$

be satisfied. If $x \in \overline{S}(\bar{x}, r)$, then also $A(x) \in \overline{S}(\bar{x}, \bar{r})$, since

$$\rho(A(x), \bar{x}] \leqslant \rho[A(x), A(\bar{x})] + \rho(A(\bar{x}), \bar{x})$$

$$\leqslant \alpha\rho\,(x, \bar{x}) + [\rho(\bar{x}, A(\bar{x})\,] \leqslant \alpha r + (1-\alpha)r = r.$$

Hence it is possible to consider A as an operator acting on a complete metric space $\overline{S}(\bar{x}, r)$ (see p.16) and satisfying in this space the condition (i). But, then, by proof, A has in $\overline{S}(\bar{x}, r)$ a unique fixed point.

Some examples of the application of the contraction mapping principle are given in what follows.

1.71. Solution of systems of linear algebraic equations by the iteration method. We consider an n-dimensional arithmetic space. If $x = \{\xi_1, \xi_2, \ldots, \xi_n\}$ and $y = \{\eta_1, \eta_2, \ldots, \eta_n\}$, then we put $\rho(x, y) = \max\limits_i |\xi_i - \eta_i|$. It is routine to show that the metric space m_n so defined is complete. In this space, consider the operator $y = A(x)$, defined by means of the equation

$$\eta_i = \sum_{j=1}^{n} a_{ij}\,\xi_j + b_i, \qquad i = 1, 2, \ldots, n. \quad \text{Then,}$$

$$\rho(y_1, y_2) = \rho[A(x_1), A(x_2)] = \max_i |\eta_i^{(1)} - \eta_i^{(2)}|$$

$$= \max_i \left| \sum_{j=1}^{n} a_{ij} (\xi_j^{(1)} - \xi_j^{(2)}) \right| \leqslant \max_i \sum_{j=1}^{n} | a_{ij} | \, | \xi_j^{(1)} - \xi_j^{(2)} |$$

$$\leqslant \max_i \sum_{j=1}^{n} | a_{ij} | \cdot \max_j | \xi_j^{(1)} - \xi_j^{(2)} | = \max_i \sum_{j=1}^{n} | a_{ij} | \, \rho(x_1, x_2).$$

Now, if it is assumed that

$$\sum_{j=1}^{n} | a_{ij} | < 1 \tag{4}$$

for all i, then the contraction mapping principle becomes applicable and, consequently, the operator A has a unique fixed point. This leads to the next theorem.

THEOREM 2. *If* $\sum_{j=1}^{n} | a_{ij} | < 1$ *holds for a matrix* (a_{ij}) *for all* i, *then the system of equations*

$$\xi_i - \sum_{j=1}^{n} a_{ij} \xi_j = b_i, \qquad i = 1, 2, \ldots, n$$

has a unique solution $x_0 = (\xi_1^0, \xi_2^0, \ldots, \xi_n)$. *This solution can be obtained by an iteration procedure, starting from an arbitrary vector* $x = (\xi_1, \xi_2, \ldots, \xi_n)$.

The conditions (4) are sufficient for the convergence of the iterative process for the system considered. If in an n-dimensional space we introduce another metric, we obtain different conditions of convergence. Let, for example,

$$\rho(x, y) = \sqrt{\sum_{i=1}^{n} (\xi_i - \eta_i)^2}.$$

For such a metric

$$\rho(y_1, y_2) = \rho[(A(x_1), A(x_2)] = \sqrt{\sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} a_{ij} (\xi_j^{(1)} - \xi_j^{(2)}) \right\}^2}$$

$$\leqslant \sqrt{\sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} a_{ij}^2 \sum_{j=1}^{n} (\xi_j^{(1)} - \xi_j^{(2)})^2 \right\}} = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2} \, \rho(x_1, x_2).$$

Hence the conditions for the convergence of the sequence of successlve approximations are given this time by

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 < 1.$$

1.72. Existence and uniqueness of the solution of an integral equation

THEOREM. *Let $K(t, s)$ be a real-valued function defined and measurable in the square $a \leqslant t, s \leqslant b$, and such that*

$$\int_a^b \int_a^b K^2(t, s)\, dt\, ds < \infty, \tag{5}$$

and let $f(t) \in L_2[a, b]$. Then, the integral equation

$$x(t) = f(t) + \lambda \int_a^b K(t, s)\, x(s)\, ds$$

has a unique solution $x(t) \in L_2[a, b]$ for every sufficiently small value of the parameter λ.

PROOF. Consider the operator

$$A(x) = f(t) + \lambda \int_a^b K(t, s)\, x(s)\, ds,$$

and show that this operator takes every function $x(t) \in L_2[a, b]$ into a function, which again belongs to this space. Since $f(t) \in L_2[a, b]$, it suffices to show that the operator

$$A_0(x) = \int_a^b K(t, s)\, x(s)\, ds$$

takes every function $x(t) \in L_2[a, b]$ into a function belonging to the same space.

From hypothesis (5) and FUBIN's theorem (see, for example, [25]), it follows that $K^2(t, s)$ is s-integrable on $[a, b]$ for almost all t in $[a, b]$, implying that for all t in $[a, b]$ there is the integral

$$\int_a^b K(t, s)\, x(s)\, ds = y(t).$$

Then, by the CBS inequality, we get

$$y^2(t) = \left(\int_a^b K(t, s)\, x(s)\, ds \right)^2 \leqslant \int_a^b K^2(t, s)\, ds \int_a^b x^2(s)\, ds.$$

Since the function

$$\int_a^b x^2(s)\, ds \text{ is a constant, and } \int_a^b K^2(t, s)\, ds$$

is, by hypothesis (5) and FUBIN's theorem, t-integrable on $[a, b]$, $y^2(t)$ is also t-integrable on $[a, b]$ and, moreover,

$$\int_a^b y^2(t)\,dt \leqslant \int_a^b \int_a^b K^2(t, s)\,ds \int_a^b x^2(s)\,ds.$$

Now, estimate $\rho[A(x), A(y)]$ by

$$\rho[A(x), A(y)] = \left\{\int_a^b \left(\lambda \int_a^b K(t, s)\,x(s)\,ds - \lambda \int_a^b K(t, s)\,y(s)\,ds\right)^2 dt\right\}^{\frac{1}{2}}$$

$$= |\lambda| \left\{\int_a^b \left(\int_a^b K(t, s)\,[x(s)-y(s)]\,ds\right)^2 dt\right\}^{\frac{1}{2}}$$

$$= |\lambda| \left(\int_a^b \int_a^b K^2(t, s)\,dt\,ds\right)^{\frac{1}{2}} \left(\int_a^b [x(s)-y(s)]^2\,ds\right)^{\frac{1}{2}}$$

$$= |\lambda| \left(\int_a^b \int_a^b K^2(t, s)\,dt\,ds\right)^{\frac{1}{2}} \rho(x, y).$$

If
$$|\lambda| < \frac{1}{\left(\int_a^b \int_a^b K^2(t, s)\,dt\,ds\right)^{\frac{1}{2}}}, \tag{6}$$

then the contraction mapping principle holds, proving the existence and uniqueness of the solution of the considered integral equation for those values of λ which satisfy Ineq. (6).

1.73. Application to partial differential equations. As the third example, we consider the *Cauchy problem* for quasilinear hyperbolic second-order differential equations in two unknown variables.

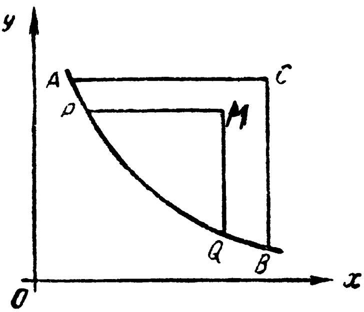

Fig. 1.

In the x, y-plane (Fig. 1), given a smooth curve AB such that any straight line parallel to x- or y-axis, intersects it at not more than one point. It is required to determine the function $u(x, y)$, which satisfies in the curvilinear triangle ABC, the equation

$$u_{xy} = f(x, y, u, u_x, u_y)$$

and is such that u, $u_x \equiv p$ and $u_y \equiv q$ take continuous values along the given AB. Without loss of generality, these values can be taken to be identically zero (see [6], Vol. II, Chap. V, Sec. 5). It is known that the solution of this CAUCHY problem reduces to the solution of the nonlinear integral equation

$$u(x, y) = \int\limits_{MP} \int\limits_{Q} f(\xi, \eta, u(\xi, \eta), u_x(\xi, \eta), u_y(\xi, \eta)) \, d\xi \, d\eta.$$

Consider the space X, whose element is the function $u(x, y)$ defined in the closed curvilinear triangle $\overline{ABC}$, continuous in this domain and having continuous first order partial derivatives. The distance is given by

$$\rho(u, v) = \max_{\overline{ABC}} | u(x, y) - v(x, y) | + \max_{\overline{ABC}} | u_x(x, y) - v_x(x, y) | + \max_{\overline{ABC}} | u_y(x, y) - v_y(x, y) |.$$

It is easy to verify that with the distance function so defined, X is a complete metric space, the convergence in which means that a sequence of functions and a sequence of their derivatives converge uniformly in $\overline{ABC}$ to a limit function and its derivatives.

Now, assume that in a space of independent variables x, y, u, p, q obeying the conditions that the point $M(x, y)$ does not extend beyond $\overline{ABC}$ and the variables u, p, q take the bounds $| u | \leqslant a$, $| p | \leqslant a$, $| q | \leqslant a$ with a some constant, the function $f(x, y, u, p, q)$ is continuous in the collection of variables and, what is more, in the variables u, p, q satisfies the LIPSCHITZ condition

$$| f(x, y, u, p, q) - f(x, y, \tilde{u}, \tilde{p}, \tilde{q}) | \leqslant L\{ | u - \tilde{u} | + | p - \tilde{p} | + | q - \tilde{q} |\},$$

L some constant.

This implies, in particular, that $f(x, y, u, p, q)$ is bounded in the considered domain.

Introduce in the space X, the operator

$$v(x, y) = U(u) = \int\limits_{MP} \int\limits_{Q} f(\xi, \eta, u, u_x, u_y) \, d\xi \, d\eta.$$ Remarking that

$$v_x(x, y) = \int\limits_{QM} f(x, \eta, u, u_x, u_y) \, d\eta,$$

$$v_y(x, y) = \int\limits_{PM} f(\xi, y, u, u_x, u_y) \, d\xi$$

yields $\quad |v(x, y)| \leqslant Kd^2, \quad |v_x(x, y)| \leqslant Kd, \quad |v_y(x, y)| \leqslant Kd,$

where $K = \sup |f(x, y, u, p, q)|$, with $M(x, y) \in \overline{ABC}$ and $|u| \leqslant a$, $|p| < a$, $|q| \leqslant a$, and d is the greatest distance between AC and BC. If the relations

$$Kd^2 \leqslant a/3 \qquad \text{and} \qquad Kd \leqslant a/3$$

are assumed to be satisfied, then the operator U takes the closed sphere $\overline{S}(\theta, a)$, where $\theta(x, y)$ is a function identically equal to zero, of the space X into itself. Moreover,

$$\frac{\max}{ABC} |v - \tilde{v}| \leqslant \frac{\max}{ABC} \int\limits_{MP} \int\limits_{Q} |f(\xi, \eta, u, u_x, u_y) - f(\xi, \eta, \tilde{u}, \tilde{u}_x, \tilde{u}_y)| \, d\xi \, d\eta$$

$$\leqslant \frac{\max}{ABC} \int\limits_{MP} \int\limits_{Q} L\{|u - \tilde{u}| + |u_x - \tilde{u}_x| + |u_y - \tilde{u}_y|\} d\xi \, d\eta$$

$$\leqslant Ld^2 \rho(u, \tilde{u}),$$

and, analogously,

$$\max |v_x - \tilde{v}_x| \leqslant L\, d\rho(u, \tilde{u}), \quad \max |v_y - \tilde{v}_y| \leqslant L\, d\rho(u, \tilde{u}).$$

Now, if it is assumed that $Ld^2 < \frac{1}{3}$ and $Ld < \frac{1}{3}$, in particular for sufficiently small d, then the operator U effects a contraction mapping of the sphere $\overline{S}(0, a)$ into itself, leading to next theorem.

THEOREM 3. *Given a smooth curve AB with the property that the straight lines parallel to the coordinate axes intersect it at more than one point, and the equation*

$$u_{xy} = f(x, y, u, u_x, u_y), \tag{7}$$

where the function on the right in the domain $M(x, y) \in \overline{ABC}$, $|u| \leqslant a$, $|u_x| \leqslant a$, $|u_y| \leqslant a$, is continuous in the collection of the first two variables and in the remaining three variables satisfies a Lipschitz condition uniformly in x and y.

Then, if the triangle $\overline{ABC}$ is sufficiently small, there exists in it a solution of Eq. (7), which together with the first derivative vanishes on AB.

By employing the contraction mapping principle we obtain also other results, some of which will be interpreted in the sequel. This principle is the simplest of a series of fixed point principles. The other principle due to SCHAUDER [26] will be discussed later when dealing with the compact sets in metric spaces.

1.8 SEPARABLE SPACE

A SPACE X is said to be **separable**, if it contains a countable everywhere dense set; in other words, if there is in X a sequence

$$\{x_1, x_2, \ldots, x_n, \ldots\} \tag{1}$$

such that for any $x \in X$ we find a subsequence $\{x_{n_1}, x_{n_2}, \ldots, x_{n_k} \ldots\}$ of (1), which converges to x. If X is a metric space, then separability can be defined as follows: There exists a sequence (1) in X such that we find an element x_{n_0} of (1) for every $x \in X$ and every $\varepsilon > 0$, verifying $\rho(x, x_{n_0}) < \varepsilon$.

1.81. The separability of the n-dimensional Euclidean space E_n. In fact, the set E_n^0 which consists of all points of the space E_n with rational coordinates is countable and everywhere dense in E_n.

1.82. The separability of the space $C[0, 1]$. In the spacc $C[0, 1]$, consider the set C_0 consisting of all polynomials with rational coefficients. C_0 is countable. It is easy to show that C_0 is everywhere dense in $C[0, 1]$. In fact, take any function $x(t) \in C[0, 1]$. By the WEIERSTRASS theorem, there is a polynomial $p(t)$ such that

$$\max_t |x(t) - p(t)| < \varepsilon/2,$$

$\varepsilon > 0$ a preassigned number. On the other hand, there exists, evidently, another polynomial $p_0(t)$ with rational coefficients, such that

$$\max_t |p(t) - p_0(t)| < \varepsilon/2.$$

Hence,
$$\rho(x, p_0) = \max_t |x(t) - p_0(t)| < \varepsilon. \qquad \blacksquare$$

1.83. The separability of the space l_p. Let E_0 be the set of all elements x of the form $\{r_1, r_2, \ldots, r_n, 0, 0, \ldots\}$, where r_i are any rational numbers and n is an arbitrary natural number. E_0 is countable. It is easy to show that E_0 is everywhere dense in l_p. In fact, take any element $x = \{\xi_i\} \in l_p$ and let an arbitrary $\varepsilon > 0$ be given. We find first a natural number n, such that

$$\sum_{k=n+1}^{\infty} |\xi_k|^p < \varepsilon^p/2.$$

Next, take an element $x_0 = \{r_1, r_2, \ldots, r_n, 0, 0, \ldots\}$ such that

$$\sum_{k=1}^{n} |\xi_k - r_k|^p < \varepsilon^p/2.$$

Then,
$$[\rho(x, x_0)]^p = \sum_{k=1}^{n} |\xi_k - r_k|^p + \sum_{k=n+1}^{\infty} |\xi_k|^p < \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon^p,$$

whence
$$\rho(x, x_0) < \varepsilon. \qquad \blacksquare$$

1.84. The separability of the space L_p [0, 1]. In fact, from the property of absolute discontinuity of the LEBESGUE integral (see [25]), it follows that any function $x(t)$ of the space $L_p[0, 1]$ is a limit in the mean of p-th power of a sequence of bounded measurable functions $x_n(t)$ defined by

$$x_n(t) = \begin{cases} x(t), & \text{if} \quad |x(t)| \leqslant n, \\ 0, & \text{if} \quad |x(t)| > n. \end{cases}$$

Further, it easily follows from a theorem of LUZIN (C-property) that every bounded measurable function is the limit in the mean of p-th power of a sequence of continuous functions. Therefore, a set of functions continuous on [0, 1] is everywhere dense in $L_p[0, 1]$. On the other hand, a countable set of polynomials with rational coefficients is everywhere dense in $C[0, 1]$ in the sense of the metric of this space, and all the more in the sense of the metric of $L_p[0, 1]$. But, then, the considered set of polynomials is everywhere dense in L_p [0, 1], proving the separability of $L_p[0, 1]$.

1.85. The separability of the space s. Let E_0 be a set of elements x of the form $(r_1, r_2, \ldots, r_n, 0, 0, \ldots)$, r_i arbitrary rational numbers and n any natural number. E_0 is countable. We shall show that a subsequence can be extracted from E_0 convergent to the arbiltrarily chosen element

$$x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\} \in s.$$

For every ξ_n we construct a sequence $\{r_n^{(k)}\}$, $k = 1, 2, \ldots$, of rational numbers, which converges to ξ_n as $k \to \infty$. Consider a sequence $\{x^{(k)}\}$ of the elements of E_0 of the form

$$x^{(k)} = \{r_1^{(k)}, r_2^{(k)}, \ldots, r_k^{(k)}, \ldots, 0, 0, \ldots\}.$$

As is obvious, $x^{(k)} \to x$ as $k \to \infty$. To prove this assertion, we must show that the n-th component of $x^{(k)}$ converges to the n-th component of x as $n \to \infty$. This is, however, evident, because

$$|\xi_n - r_n^{(k)}| < \varepsilon,$$

taking $k > n$ sufficiently large.

1.86. The inseparability of the space m. We consider the set of elements $x = \{\xi_i\} \in m$, ξ_i being equal to 0 or 1. The set of these elements has the power of the continuum. We choose two distinct elements $x = \{\xi_i\}$ and $y = \{\eta_i\}$ from this set. Then $\rho(x, y) = \sup_i |\xi_i - \eta_i| = 1$, and we have a continuum of elements, apart from each other by a distance of unity. This implies immediately that m is inseparable.

In fact, assume that in m there is a countable everywhere dense set E_0. We construct a sphere of radius $\varepsilon = 1/3$ around every element of E_0. Then, all the elements of the space m lie inside of these spheres. Since these spheres form a countable set, in at least one of these spheres there must exist two distinct elements x and y of the considered continual set. Let x_0 be the center of such a sphere. Then,

$$1 = \rho(x, y) \leqslant \rho(x, x_0) + \rho(x_0, y) \leqslant \frac{1}{3} + \frac{1}{3} = \frac{2}{3},$$

which is impossible. Consequently, m is inseparable. However, it can be proved that the *space c, which is a subspace of the space m, is separable.*

CHAPTER 2

NORMED LINEAR SPACE

2.1. LINEAR SPACES

WHILE CONSIDERING various concrete spaces, it is observed that the elements of these spaces (functions, number sequences, etc.) can be added together and multiplied by a number, yielding elements of the same space. Starting from such concrete cases, we shall arrive at a general definition of the linear spaces.

2.11. Definitions. Let E be a set of elements of certain nature, satisfying the axioms:

(i) E is an abelian group (written additively). This means that a definite *sum* $x + y$ of every two elements $x, y \in E$ is an element of the same set, where the operation of addition satisfies the following conditions:

(*a*) $x + y = y + x$ (*commutativity*);

(*b*) $x + (y+z) = (x+y) + z$ (*associativity*);

(*c*) There exists a uniquely defined element 0, such that $x + 0 = x$ for any x in E;

(*d*) For every element $x \in E$ there exists a uniquely defined element $(-x)$ of the same space, such that

$$x + (-x) = 0.$$

Instead of $x + (-y)$, it is customary to write $x - y$.

The neutral element 0 is said to be the **null element** or **zero** of group E and the element $-x$ is called the **inverse element** of x.

(ii) A *scalar multiplication* is defined, that is, there is defined a multiplication of elements $x, y, z, \ldots$ of E by real (complex) numbers $\lambda, \mu, \nu, \ldots$, where λx (called their **product**) is again an element of E and satisfies the conditions:

(*a*) $\lambda(\mu x) = (\lambda\mu) x$ (*associativity*);

(*b*) $\left.\begin{aligned} \lambda(x+y) &= \lambda x + \lambda y \\ (\lambda+\mu)x &= \lambda x + \mu x \end{aligned}\right\}$ (*distributivity*);

(*c*) $1 \cdot x = x.$

The set E satisfying the axioms (i) and (ii) is called a **linear** or **vector space.**† This is said to be a **real** or **complex** space according as the multiplication of elements in E is defined by real or complex numbers.†

†The term *space* on p. 5 has a different sense. However, in all linear spaces considered in the sequel, the notion of the limit point of a sequence shall be introduced.

Examples. 1. The collection E_n of n-dimensional real vectors forms a real linear space.

2. The collection of complex-valued solutions of an ordinary homogeneous linear differential equation of order n forms a complex linear space.

3. The collection of elements of the real (complex) spaces $C[0, 1]$, $L_p[0, 1]$ forms a real (complex) linear space.

4. The collection of elements of the real (complex) spaces m, c, l_p forms a real (complex) linear space. For this, the sum of elements $x = \{\xi_i\}$ and $y = \{\eta_i\}$ is denoted as usual by the element

$$x + y = \{\xi_1 + \eta_1, \xi_2 + \eta_2, \ldots, \xi_n + \eta_n, \ldots\},$$

and the product of the element x with the number λ by the element

$$\lambda x = \{\lambda\xi_1, \lambda\xi_2, \ldots, \lambda\xi_n, \ldots\}.$$

Some of the consequences from the axioms of a linear space are:

(i) $0 \cdot x = 0$.† In fact,

$$x = 1 \cdot x = (1+0 \cdot x) = 1 \cdot x + 0 \cdot x = x + 0 \cdot x.$$

Therefore, $$x + (-x) = x + 0 \cdot x + (-x),$$

or, $$0 = 0 + 0 \cdot x = 0 \cdot x.$$

(ii) $$(-1)x = -x,$$

because $$(-1)x + x = (-1+1)x = 0 \cdot x = 0.$$

(iii) $$\lambda \cdot 0 = 0,$$

because $$\begin{aligned}\lambda \cdot 0 = \lambda[x+(-x)] &= \lambda x + \lambda(-x) \\ &= \lambda x + (-\lambda)x = \lambda x - \lambda x = 0.\end{aligned}$$

(iv) If $\lambda x = \mu x$ and $x \neq 0$, then $\lambda = \mu$. In fact, if $\lambda x = \mu x$, then

$$\lambda x - \mu x = 0 \quad \text{or} \quad (\lambda - \mu)x = 0.$$

Therefore, if it is assumed that $\lambda \neq \mu$, then

$$x = \frac{1}{\lambda-\mu}(\lambda-\mu)x = \frac{1}{\lambda-\mu}\, 0 = 0,$$

a contradiction.

It may be remarked that if E is a linear space, then the commutativity of addition is a consequence of the rest of the axioms. In fact,

$$\begin{aligned}(x + y) - (y+x) &= (x+y) + (-1)(y+x) \\ &= (x+y)+(-1)\, y + (-1)x = x + [y+(-1)y] + (-1)x \\ &= x + 0 + (-1)x = x + (-1)x = 0.\end{aligned}$$

Finally, the two linear spaces E and E' are said to be **isomorphic** if between their elements there can be established a one-one correspondence, preserving the algebraic operations, that is, such that, if $x \leftrightarrow x'$ and $y \leftrightarrow y'$, then

$$x + y \leftrightarrow x' + y' \quad \text{and} \quad \lambda x \leftrightarrow \lambda x':$$

The linear spaces admit the notion of linear dependence and linear inde-

†By the letter 0 we denote the number zero as well as the null element of a linear space. However, it shall be clear from the text in what context it is being used.

pendence of elements. The elements $x_1, x_2, \ldots, x_n$ of a linear space are said to be **linearly independent**, if any relation of the form $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0$ implies that $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. Conversely, if $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0$ is possible with at least one of the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ not zero, then the elements $x_1, x_2, \ldots, x_n$ are said to be **linearly dependent**.

Let, in the latter case, for example, $\lambda_n \neq 0$. Then,

$$x_n = -\frac{\lambda_1}{\lambda_n} x_1 - \frac{\lambda_2}{\lambda_n} x_2 - \ldots - \frac{\lambda_{n-1}}{\lambda_n} x_n - 1$$

or, setting $-\lambda_i/\lambda_n = \alpha_i$,

$$x_n = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_{n-1} x_{n-1}.$$

In this case, we say that the element x_n is a **linear combination** of the elements $x_1, x_2, \ldots, x_{n-1}$.

2.12. Linear manifolds. A nonempty set L of the elements of a linear space E is called a **linear manifold** if together with the elements $x_1, x_2, \ldots x_n$ the set L also contains every linear combination $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$ of these elements.

It is noteworthy that every linear manifold contains the null element 0. In fact, since L is not empty, it contains, some element x. Because L is a linear manifold, it contains also the element $-x = (-1)x$ and, consequently, also the element $x + (-x) = 0$.

Consider the elements $x_1, x_2, \ldots, x_k$ of a linear space. The collection of all possible sums $\sum_{i=1}^{k} \alpha_i x_i$, obviously, forms a certain linear manifold L_0 in E. In fact, if the elements y_j take the form $y_j = \sum_{i=1}^{k} \alpha_i^{(j)} x_i$, then any linear combination of these elements by virtue of the equation

$$\lambda_1 y_1 + \lambda_2 y_2 + \ldots + \lambda_n y_n = \sum_{i=1}^{k} \beta_i x_i$$

has the same form. The linear manifold L_0 constructed in this manner is obviously the *smallest linear manifold* containing the elements $x_1, x_2, \ldots, x_k$ (smallest in the sense that every other linear manifold L containing the elements $x_1, x_2, \ldots, x_k$ contains L_0).

The definition of the smallest linear manifold containing given elements can be easily extended to the case of an infinite set, e.g., a countable set of elements. In fact, let $\{x_1, x_2, \ldots, x_n \ldots\}$ be a countable set of elements in E. The *smallest linear manifold* L_0 containing these elements is a set of all possible sums of the form $\sum_{i=1}^{k} \lambda_i x_i$, λ_i arbitrary and k an arbitrary natural number. The smallest linear manifold containing the given elements is also called the **linear manifold spanned by the given elements**, or the **linear span** of these elements.

If a linear manifold L of a space E is defined by a finite number of elements, then this is said to be *finite-dimensional.*

If L is defined by the elements $x_1, x_2, \ldots, x_n$ and these elements are linearly independent, then n is said to be the **dimension** of L. In this case, a collection of elements $x_1, x_2, \ldots, x_n$ is called a **basis** of L.† If, however, $x_1, x_2, \ldots, x_n$, are linearly dependent, then the **dimension** of L is the maximum number of linearly independent elements from the collection $x_1, x_2, \ldots, x_n$.

In other words, L is of n-dimension if it contains n-linearly independent elements, and any $n + 1$ elements of this linear manifold is linearly dependent.

If a space E (linear manifold L) for any number n has n linearly independent elements, then E (linear manifold L) is said to be *infinite-dimensional.* For example, it is plain that the space $C[0, 1]$ is infinite-dimensional.

2.13. Direct sum. Now, in order to introduce the representation of a linear space as a direct sum of two or more linear manifolds, let E be a linear space, and let $L_1, L_2, \ldots, L_n$ be linear manifolds belonging to E. If every element $x \in E$ has a unique representation of the form

$$x = x_1 + x_2 + \ldots + x_n, \qquad x_i \in L_i, \qquad i = 1, 2, \ldots, n, \tag{1}$$

E is said to be the **direct sum** of its linear manifolds $L_1, \ldots, L_n$, and the expression (1) is called a **decomposition of the element** x into the elements of the linear manifolds $L_1, \ldots, L_n$. In this case

$$E = \sum_{i=1}^{n} \oplus L_i.$$

It is routine to check that, if

$$E = \sum_{i=1}^{n} \oplus L_i, \quad \text{and} \quad L_i = \sum_{k=1}^{m_i} \oplus L_k^{(i)},$$

then,

$$E = \sum_{i=1}^{n} \sum_{k=1}^{m_i} \oplus L_k^{(i)}.$$

In fact, then, every element $x \in E$ is expressible in the form

$$x = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} (x_1^{(i)} + x_2^{(i)} + \ldots + x_{m_i}^{(i)})$$

$$x_i \in L_i, \qquad x_k^{(i)} \in L_k^{(i)},$$

and this representation is unique, because if

$$x = \sum_{i=1}^{n} \tilde{x}_i = \sum_{i=1}^{n} (\tilde{x}_1^{(i)} + \tilde{x}_2^{(i)} + \ldots + \tilde{x}_{m_i}^{(i)})$$

†The definition of a basis for some infinite-dimensional spaces will be given later.

be another such representation, then by virtue of the unique decomposition of $x \in E$ into the elements of the linear manifolds, $L_1, \ldots, L_n$, we get

$$x_i = x_1^{(i)} + x_2^{(i)} + \ldots + x_{m_i}^{(i)} = \tilde{x}_1^{(i)} + \tilde{x}_2^{(i)} + \ldots + \tilde{x}_{m_i}^{(i)} = \tilde{x}_i,$$

and the unique decomposition of the $x_i \in L_i$ into the elements of linear manifolds $L_1^{(i)}, L_2^{(i)}, \ldots, L_{m_i}^{(i)}$, yields

$$x_k^{(i)} = \tilde{x}_k^{(i)}, \qquad i = 1, 2, \ldots, n, \qquad k = 1, 2, \ldots, m_i.$$

It is not difficult to show that *if* $E = L_1 \oplus L_2$, *then* L_1 *and* L_2 *have in common only the null element* 0 *of the space.* In fact, if L_1 and L_2 could contain another common element u, then the element $x \in E$ with $x = y + z$, $y \in L_1$, $z \in L_2$, would have also the representation

$$x = (y-u) + (z+u), \quad y - u \in L_1, \quad z + u \in L_2,$$

which being different from the previous representation is impossible by hypothesis.

Conversely : *If every* $x \in E$ *can be represented in the form*

$$x = y + z, \qquad y \in L_1, \qquad z \in L_2, \tag{2}$$

and if $L_1 \cap L_2 = 0$, *then* $E = L_1 \oplus L_2$.

For proof of this assertion, it will suffice to establish the uniqueness of the representation (2). However, if

$$x = y + z = \tilde{y} + \tilde{z}; \quad y, \tilde{y} \in L_1, \quad z, \tilde{z} \in L_2,$$

then $\quad y - \tilde{y} = \tilde{z} - z, \quad y - \tilde{y} \in L_1; \quad \tilde{z} - z \in L_2.$

By hypothesis, it follows that

$$y - \tilde{y} = \tilde{z} - z = 0, \qquad \text{that is,} \quad y = \tilde{y}, \quad z = \tilde{z}. \qquad \blacksquare$$

In a number of cases, the notion of the direct sum of two or more spaces is found to be useful. Let $E_1, E_2, \ldots, E_n$ be linear spaces. Consider the set X of all ordered n-tuples $x = (x_1, x_2, \ldots, x_n)$ of elements of the given spaces, where $x_i \in E_i$, $i = 1, 2, \ldots, n$. If

$$x = (x_1, x_2, \ldots, x_n), \qquad y = (y_1, y_2, \ldots, y_n)$$

and a scalar λ are given, then we put

$$x + y = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n),$$
$$\lambda x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n).$$

It is easy to verify that the operations of addition and scalar multiplication defined in this manner satisfy all the axioms of a linear space, so that the set X of ordered n-tuples is a linear space.

If all the spaces E_i are metric spaces, then X can be metrized, e.g. by setting $\rho(x, y) = \max_i \rho(x_i, y_i)$, or

$$\rho(x, y) = \sqrt{\sum_{i=1}^{n} \rho^2(x_i, y_i)},$$

$\rho(x_i, y_i)$ being the distance between the points x_i and y_i of the space E_i.

The completeness of the spaces $E_1, E_2, \ldots, E_n$ implies the completeness of the space X. The proof is left to the reader.

Example. Let E_i be a real line for every i. Then $\sum_{i=1}^{n} \oplus E_i$, metrized by the latter method, is the n-dimensional Euclidean space.

2.14. Factor (quotient) spaces. Consider a linear space E and some linear manifold $L_0 \in E$. The space E as an additive group can be decomposed into cosets L of L_0 such that two elements x_1 and x_2 belong to one and the same L iff $x_1 - x_2$ belongs to L_0.

If x' is an arbitrary element of L, then every other element of L is expressible in the form $x = x' + x_0$, with $x_0 \in L_0$. Hence, it can be stated that L is constructed from the linear manifold L_0 through *displacement* by x'.

Let us construct a factor group E/L_0. Its elements are the set L, formed by displacement of the manifold L_0.

The operation of addition in E/L_0 is defined in the following manner. Let L_1 and L_2 be the elements of E/L_0; then the sum $L_1 + L_2$ is a coset formed of all possible sums $x_1 + x_2$, where $x_1 \in L_1$, $x_2 \in L_2$; $L_1 + L_2$ is, indeed, a coset, for if $x_1 + x_2$ and $x_1' + x_2'$ be two elements of this set, then

$$(x_1+x_2) - (x_1' + x_2') = (x_1 - x_1') + (x_2 - x_2') = x_0 + y_0 \in L_0,$$

because $x_0, y_0 \in L_0$ and L_0 is a linear manifold. Consequently, $L_1 + L_2 \subset L$, where L is some coset. If y is any element of this coset, then take an element of the form $x_1 + x_2$ appearing in L (plausible, because $L \supset L_1 + L_2$), to receive

$$y - (x_1 + x_2) = x_0 \in L_0, \qquad \text{whence} \quad y = x_1 + x_2 + x_0 = x_1 + \tilde{x}_2,$$

$x_1 \in L_1$, $\tilde{x}_2 \in L_2$. Hence $L \subset L_1 + L_2$. Consequenty, $L_1 + L_2 = L$.

Analogously, it can be shown that λL, a collection of elements of the form λx, $x \in L$ and $\lambda \neq 0$, is also a coset. Further, by definition, $0 \cdot L = L_0$ for any $L \in E/L_0$. It is routine to verify that E/L_0 satisfies all axioms of a linear space. For this, the role of the zero of the space E/L_0 is played by L_0. Finally, note that if $L \in E/L_0$ contains 0, the null element of the space E, then L coincides with L_0, because in this case any element $x \in L$ takes the form

$$x = 0 + x_0 = x_0 \in L.$$

The converse of this assertion also holds.

The space E/L_0 is called the **factor (quotient) space of** E **modulo** L_0.

Example. Consider in $C\,[0,1]$ a linear manifold C_0 of all continuous functions that vanish for $t = \frac{1}{2}$. The corresponding factor space is isomorphic to all real numbers.

In fact, let $x(t)$ and $y(t)$ belong to the same coset in C_0. This means that $x(\frac{1}{2}) - y(\frac{1}{2}) = 0$ or $x(\frac{1}{2}) = y(\frac{1}{2})$. Thus, a function is adjoined to the coset, having an identical value at the point $t = \frac{1}{2}$. Taking, by representation, $x(t) =$ const. in every coset, a one-one correspondence is obtained between the set of constants and the set of cosets. It is obvious that this correspondence is isomorphic.

It is easy to show that if the space $E = E_1 \oplus E_2$ then E/E_1 is isomorphic to E_2.

2.15. The relation between the real and the complex spaces. Besides the algebraic operations, the operation of taking the conjugate complex number

$$\overline{a+bi} = a - bi$$

is also basic to complex numbers. It is natural to consider those spaces complex on which the analogous operation, called **involution,** is defined.

In general, we define : **Involution** is a mapping, defined in a linear complex space E, which assigns all the elements $x, y, z, \ldots \in E$ to the elements $\bar{x}, \bar{y}, \bar{z}, \ldots \in E$, with the following properties :

(i) $\overline{x+y} = \bar{x} + \bar{y}$

(ii) $\overline{\lambda x} = \bar{\lambda} \cdot \bar{x}$ (λ complex)

(iii) $\overline{(\bar{x})} = \bar{\bar{x}} = x.$†

All the elements $x \in E$, admitting $\bar{x} = x$ are called **real.** The elements $x \in E$, for which $\bar{x} = -x$, are called **pure imaginary.** Thus, obiviously : If x is real, then ix is pure imaginary, and if y is pure imaginary, then $(1/i)y$ is real. Thus, the collection of pure imaginary elements y coincides with the collection of elements of the form ix, x real.

Each element $x \in E$ has exactly one representation in the form $x = u + iv$, u and v real. In fact, put

$$u = \frac{x+\bar{x}}{2}, \quad v = \frac{x-\bar{x}}{2i}.$$

Then $x = u + iv$; moreover,

$$\bar{u} = \frac{1}{2}\,\overline{(x+\bar{x})} = \frac{1}{2}\,(\bar{x}+\bar{\bar{x}}) = \frac{1}{2}(\bar{x}+x) = u,$$

and
$$\bar{v} = -\frac{1}{2i}\overline{(x-\bar{x})} = -\frac{1}{2i}\,(\bar{x}-\bar{\bar{x}}) = \frac{1}{2i}\,(x-\bar{x}) = v,$$

that is, u and v are real.

The representation of the element $x \in E$ in the form $x = u + iv$ is unique, that is, if

$$x = u + iv = t + is, \tag{3}$$

then $u = t$ and $v = s$. In fact, (3) implies that $u - t = i(s-v)$; u, v, t, s real. Furthermore,

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†. If the convergence of a sequence of elements is defined in E, then an additional requirement is introduced, viz.,

(iv) $x_n \to x$ implies that $\bar{x}_n \to \bar{x}$.

$$\overline{u - t} = \bar{u} - \bar{t} = u - t; \quad \overline{i\,(s - v)} = \bar{i}\,(\bar{s} - \bar{v}) = -\,i\,(s - v);$$

Hence, $u - t = i\,(s-v)$, that is, $i\,(s-v) = -\,i\,(s-v)$, and, therefore,

$$s - v = 0, \quad s = v. \quad \text{Thus,} \quad u - t = 0 \text{ and } u = t.$$

We have thus proved that the space E is the direct sum of two real linear spaces. Hence, many investigations on complex spaces reduce to the consideration of real spaces.

Note that an n-dimensional complex space is a $2n$-dimensional real space.

If we speak of linear spaces in what follows, we mean real linear spaces, unless specified otherwise.

2.2 NORMED LINEAR SPACES

2.21. Definitions. If a linear space is at the same time a metric space, it is called a **metric linear space.** The BANACH spaces (B-spaces) form an important class of metric linear spaces.

A set E is called a linear **normed** space, if:

(i) E is a linear system with real (complex) numbers as ring multipliers;

(ii) To every element x of the linear system E is assigned a unique real number, called the **norm** of this element and denoted by $\| x \|$, satisfying the following properties (*axioms of a normed linear space*) :

(a) $\| x \| \geqslant 0$ and $\| x \| = 0$, iff $x = 0$;

(b) $\| x + y \| \leqslant \| x \| + \| y \|$ (*triangle inequality*);

(c) $\| \lambda x \| = | \lambda | \, \| x \|$ (*homogeneity of the norm*).

In a normed linear space, a metric (distance) can be introduced by

$$\rho\,(x, y) = \| x-y \|.$$

It is trivial to verify that this distance satisfies all the metric axioms. After introducing the metric, we define the convergence of a sequence of elements $\{x_n\}$ to x, namely

$$x = \lim x_n \quad \text{or} \quad x_n \to x, \quad \text{if} \quad \| x_n - x \| \to 0 \quad \text{as } n \to \infty.$$

This convergence in a normed linear space is called **convergence in norm.**

A real (complex) **Banach space** or **B-space** is a real (complex) normed linear space which is *complete* in the sense of convergence in norm.

Examples. 1. The n-dimensional vector space E_n is a B-space.

In fact, defining the sum of elements and the product of elements by a number in the usual manner and the norm by

$$\| x \| = \left(\sum_{i=1}^{n} \xi_i^2 \right)^{1/2},$$

we find that E_n is a B-space; moreover, the metric in this space coincides with the metric introduced earlier in E_n.

2. $C[0, 1]$ is a BANACH space. We define the addition of functions and the multiplication of functions by a real number in the usual way. Further, we set

$$\| x \| = \max_t \, | x\,(t) |.$$

The metric obtained coincides with the one introduced earlier in $C[0, 1]$.

3. l_p is a B-space. In fact, defining the addition of elements and the multiplication of elements by a real number as indicated eaıliear (p. 36) and by setting

$$\| x \| = \left(\sum_{i=1}^{\infty} | \xi_i |^p \right)^{1/p},$$

we obtain a BANACH space, whose metric coincides with the previous metric.

4. L_p [0, 1] is a BANACH space. Hence, we set

$$\| x \| = \left(\int_0^1 | x(t) |^p \, dt \right)^{1/p}$$

for $x(t) \in L_p$ [0, 1]. The metric in the space obtained coincides with the earlier metric in L_p [0, 1].

5. m is a B-space. In fact, setting $\| x \| = \sup_i | \xi_i |$ for $x = \{\xi_i\}$, we again obtain a BANACH space with the metric already known.

6. $\tilde{M}$ [0, 1] is a BANACH space. For bounded functions $x(t)$ measurable on [0, 1], let us set

$$\| x \| = \text{vrai max} \, | (x(t) |.$$

7. Consider a space of functions $x(t)$, defined and continuous on [0, 1], and having there continuous derivatives to k-th order inclusive. The norm in this function space is defined by

$$\| x \| = \max \{\max_i | (x(t) |, \max_t | (x'(t) |, \ldots, \max_t | (x^{(k)}(t) | \}.$$

The space obtained, denoted by $C^k[0, 1]$, is a Banach space. This space finds extensive applications in the calculus of variations.

Note that the relations

$$\| (x_n + y_n) - (x + y) \| \leqslant \| x_n - x \| + \| y_n - y \|,$$
$$\| \lambda_n x_n - \lambda x \| \leqslant | \lambda_n | \, \| x_n - x \| + | \lambda_n - \lambda | \, \| x \|$$

imply that $\quad x_n + y_n \to x + y, \quad \lambda_n x_n \to \lambda x$

as $\quad x_n \to x, \; y_n \to y \;$ and $\; \lambda_n \to \lambda$

Furthermore,

$$\| x \| = \| y + (x - y) \| \leqslant \| y \| + \| x - y \|,$$

or, $\quad \| x \| - \| y \| \leqslant \| x - y \|.$

By interchanging x and y,

$$\| y \| - \| x \| \leqslant \| x - y \|$$

and, consequently,

$$| \, \| x \| - \| y \| \, | \leqslant \| x - y \|.$$

This implies that $\quad \| x_n \| \to \| x \|$, if $x_n \to x$,

and in particular, that $\{ \| x_n \| \}$ is a bounded numerical sequence.

Since normed linear spaces are metric spaces, all concepts introduced in metric spaces (e.g., a sphere, bounded set, separability, compactness, linear dependence of elements, linear manifold, etc.) have a meaning as well as the theorems proved earlier for metric spaces carry over to them.

All the results established earlier for complete metric spaces remain valid for BANACH spaces.

A set of elements of a linear space E having the form

$$y = tx, \qquad x \in E, \qquad x \neq 0, \qquad -\infty < t < \infty$$

is called a **real line** defined by the given element x, and a set of elements of the form

$$y = (1-t)x_1 + tx_2, \qquad x_1, x_2 \in E, \qquad 0 \leqslant t \leqslant 1$$

is called a **segment** connecting the points x_1 and x_2. A set K in a space E is called **convex**, if it contains the segment defined by any two of its points.

Let M be some point set in a linear space E. A set of elements of the form $x + a$, $x \in M$ and a a fixed element in E, is called the **desplacement** of the set M and denoted by $M + a$. It is easy to verify that if K is a convex set, then its displacement is also a convex set.

It is plain that in a normed linear space, an open (closed) sphere is a convex set. In fact, let $x_1, x_2 \in S(a, r)$, that is,

$$\| x_1 - a \| < r, \qquad \| x_2 - a \| < r.$$

Select any element of the form

$$y = (1-t)x_1 + tx_2, \qquad 0 < t < 1,$$

to receive

$$\begin{aligned} \| y-a \| &= \| (1-t)x_1 + tx_2 - a \| \\ &= \| (1-t)x_1 + tx_2 - (1-t)\, a - ta \| \\ &\leqslant \| (1-t)(x_1-a) \| + \| t(x_2-a) \| \\ &= (1-t) \| x_1 - a \| + t \| x_2 - a \| \\ &< (1-t)r + tr = r. \end{aligned}$$

Thus, $\| y - a \| < r$. Consequently, $y \in S(a, r)$.

Note two obvious properties of a sphere in the BANACH space: for any point $x \neq 0$ a sphere of radius $r > \| x \|$ with center in the origin contains this point, whereas a sphere of radius $r' < \| x \|$ with center in the origin does not contain this point.

Since the normed linear space E is a special case of a linear space, all the concepts introduced in a linear space (e.g. linear dependence and independence of elements, linear manifold, decomposition of E into direct sums, etc.) have a meaning for E.

Let L be a linear manifold of a normed linear space E. In addition, if L is a closed set, then L is called **a subspace**.

If L is a finite-dimensional linear manifold in a normed linear space, then as will be shown below, $\bar{L} = L$. This equality does not hold for infinite-dimensional linear manifolds.

Let, for example, $E = C\,[0, 1]$ and let L be a linear manifold spanned by the elements

$$x_0 = 1, x_1 = t, \ldots, x_n = t^n, \ldots$$

Then L is a set of all polynomials, and

$$\bar{L} = C[0, 1] \neq L.$$

Let E_1 and E_2 be two given normed spaces. In what follows, these spaces will be called **isomorphic**, if there is a one-one and mutually continuous isomorphic mapping of E_1 onto E_2. This leads to the next important theorem.

THEOREM. *All finite-dimensional normed linear spaces of a given dimension n are isomorphic to the n-dimensional Euclidean space E_n and consequently, are isomorphic to each other.*

PROOF. Let E be an n-dimensional normed linear space and let $x_1, x_2, \ldots, \ldots, x_n$ be the basis of this space; then, any element $x \in E$ is uniquely expressible in the form

$$x = \xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_n x_n.$$

Set the element $x \in E$ in correspondence to the element

$$\bar{x} = \{\xi_1, \xi_2, \ldots, \xi_n\} \in E_n.$$

It is evident that the correspondence established in this manner between the elements x and $\bar{x}$ is one-one. Moreover, this correspondence is an ismorphism of the linear space E onto E_n. Let us show that it is mutually continuous.

For any $x \in E$, we have

$$\| x \| = \| \sum_{i=1}^{n} \xi_i x_i \| \leqslant \sum_{i=1}^{n} | \xi_i | \, \| x_i \|$$
$$\leqslant \left(\sum_{i=1}^{n} \| x_i \|^2 \right)^{1/2} \left(\sum_{i=1}^{n} \xi_i^2 \right)^{1/2} = \beta \, \| \bar{x} \|. \tag{1}$$

In particular,

$$\| x-y \| \leqslant \beta \, \| \bar{x}-\bar{y} \|, \tag{2}$$

β not depending on x and y.

We now establish an equality of the opposite sign.

On the surface S of a unit sphere $\sum_{i=1}^{n} \xi_i^2 = 1$ of E_n, consider the function

$$f(\bar{x}) = f(\xi_1, \xi_2, \ldots, \xi_n) = \| x \| = \| \xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_n x_n \|.$$

Since all the ξ_i cannot vanish simultaneously on S, the linear independence of $x_1, x_2, \ldots, x_n$ implies that

$$f(\xi_1, \xi_2, \ldots, \xi_n) > 0.$$

The inequality

$$| f(\xi_1, \xi_2, \ldots, \xi_n) - f(\eta_1, \eta_2, \ldots, \eta_n) | = | \, \| x \| - \| y \| \, |$$
$$\leqslant \| x-y \| \leqslant \beta \| \bar{x}-\bar{y} \|$$

verifies $f(\xi_1, \xi_2, \ldots, \xi_n)$ to be a continuous function. By the WEIERSTRASS theorem, this function attains its minimum α on S. Plainly, $\alpha > 0$. Consequently, for $\bar{x} \in S$,

$$f(\bar{x}) = \| x \| \geqslant \alpha,$$

whence, for any $\bar{x} \in E_n$,

$$f(\bar{x}) = \| x \| = \| \bar{x} \| \left\| \sum_{i=1}^{n} \frac{\xi_i x_i}{\sqrt{\sum_{k=1}^{n} \xi_k^2}} \right\| \geqslant \alpha \| \bar{x} \|. \tag{3}$$

From (1) and (3) it follows that the mapping of E onto E_n is one-one.

The homeomorphism between E and E_n implies that in a finite-dimensional BANACH space the convergence in norm reduces to a coordinate-wise convergence and hence such a space is always complete.

For the subspaces of a normed linear space, the next important statement due to F. RIESZ holds.

LEMMA. *Let L be a subspace of normed linear space E, which does not coincide with E. Then there exists in E, for any given $\varepsilon > 0$, an element y with norm equal to* 1, *such that*

$$\| x-y \| > 1-\varepsilon \text{ for all } x \in L.$$

PROOF. In fact, let y_0 be any element in E not belonging to L, and let

$$d = \inf_{x \in L} \| y_0 - x \|.$$

Then $d > 0$ because otherwise y_0 would be the limit point of L and, consequently, enter in L, contrary to assumption. For any $\varepsilon > 0$ there is an element $x_0 \in L$, such that $d \leqslant \| y_0 - x_0 \| < d + d\varepsilon$. Set

$$y = \frac{y_0 - x_0}{\|y_0 - x_0\|}.$$

The element $y \bar{\in} L$ (since otherwise y_0 would enter in L) and $\| y \| = 1$. Take any element x in L. Let

$$\xi = x_0 + \| y_0 - x_0 \| \, x.$$

Then,
$$\| y-x \| = \left\| \frac{y_0-x_0}{\|y_0-x_0\|} - x \right\| = \frac{1}{\|y_0-x_0\|} \| y_0 - \xi \|$$
$$> \frac{1}{d+d\varepsilon} \| y_0 - \xi \| \geqslant \frac{d}{d+d\varepsilon} = 1 - \frac{\varepsilon}{1+\varepsilon} > 1 - \varepsilon,$$

giving the required proof.

Let E be a normed linear space, L_0 be its subspace and E/L_0 be the corresponding factor space. The factor space E/L_0 is endowed with the norm

$$\| L \| = \inf_{x \in L} \| x \|$$

for every $L \in E/L_0$.

Show that $\| L \|$ satisfies all the norm axioms.

(i) Evidently, $\| L \| \geqslant 0$. In order to demonstrate that $\| L \| = 0$ iff $L = L_0$, first remark that L is a closed set. In fact, let $\{x_n\}$ be a sequence of elements in L, convergent to $x \in E$. For any n and m, $x_n - x_m \in L_0$. Thereupon, $x_n - x_m \to x_n - x$, as $m \to \infty$.

Since L_0 is closed, it follows that $x_n - x \in L_0$. However, then x together with x_n is contained in L.

Now, let
$$\| L \| = \inf_{x \in L} \| x \| = 0.$$

Then there exists in L a sequence $\{x_n\} \to 0$, that is, $x_n \to 0$. As a consequence of L being closed, it must contain also 0; but then $L = L_0$. This implies that $\| L_0 \| = 0$ and, evidently, the first norm axiom is completely proved.

(ii) Let $\varepsilon > 0$. The definition of $\| L_1 \|$ and $\| L_2 \|$ implies the existence of elements $x_1 \in L_1$ and $x_2 \in L_2$, such that
$$\| x_1 \| \leqslant \| L_1 \| + \frac{\varepsilon}{2}, \quad \| x_2 \| \leqslant \| L_2 \| + \frac{\varepsilon}{2}.$$

Thereupon
$$\| x_1 + x_2 \| \leqslant \| x_1 \| + \| x_2 \| \leqslant \| L_1 \| + \| L_2 \| + \varepsilon.$$

Moreover,
$$\inf_{x \in L_1 + L_2} \| x \| \leqslant \inf_{x_1 \in L_1,\, x_2 \in L_2} \| x_1 + x_2 \| \leqslant \| L_1 \| + \| L_2 \| + \varepsilon,$$

or,
$$\| L_1 + L_2 \| \leqslant \| L_1 \| + \| L_2 \| + \varepsilon.$$

Take note of the arbitrariness of ε, to get the triangle inequality
$$\| L_1 + L_2 \| \leqslant \| L_1 \| + \| L_2 \|.$$

(iii) $\| \lambda L \| = | \lambda | \, \| L \|$. In fact, for $\lambda \neq 0$
$$\| \lambda L \| = \inf_{x \in L} \| \lambda x \| = | \lambda | \inf_{x \in L} \| x \| = | \lambda | \, \| L \|.$$

If, however, $\lambda = 0$, then for any L,
$$\| \lambda L \| = \| L_0 \| = 0 = | \lambda | \, \| L \|,$$

which completely proves the third norm axiom.

Finally, it remains to show that the convergence along a norm introduced in the space E/L_0 of a sequence of class $\{L_n\}$ to class L is equivalent to the condition that there exists a sequence of elements $\{x_n\}$, $x_n \in L_n$, such that $x_n \to x$, $x \in L$.

Let $\| L_n - L \| \to 0$, that is, $\| L_n - L \| = \varepsilon_n$, $\varepsilon_n \to 0$.

Then $L_n - L$ contains the element $y_n - x$ such that $y_n \in L_n$, $x \in L$ and $\| y_n - x \| < 2\varepsilon_n$. For this, any fixed element $x_0 \in L$ (not depending on n) can be taken as x.

In fact, if $\| y_n - x \| \leqslant 2\varepsilon_n$, where $y_n \in L_n$, $x \in L$, then

$$\| (y_n - x + x_0) - x_0 \| \leqslant 2\varepsilon_n,$$

and since $x_0 \in L$, $x \in L$, hence $x_0 - x \in L_0$ and $x_n = y_n - x + x_0 \in L_n$.

Thus for an element $x_0 \in L$, there exists a sequence $\{x_n\}$, $x_n \in L_n$, such that $x_n \to x_0$.

Conversely, let there exist a sequence $\{x_n\}$, $x_n \in L_n$, such that $x_n \to x$, $x \in L$. Since

$$\| L_n - L \| = \inf_{y_n \in L_n,\, y \in L} \| y_n - y \| \leqslant \| x_n - x \|,$$

hence $\| L_n - L \| \to 0$, and the statement is proved.

It is now trivial to show that *if E is a complete space, then E/L_0 is also complete.*

Let $\{L_n\}$ be a CAUCHY sequence in E/L_0. Select in every class L_n an element x_n, such that

$$\| x_n - x_m \| \leqslant 2 \| L_n - L_m \|,$$

to receive a CAUCHY sequence of elements $\{x_n\}$ in E. Since the space E is complete, there is an element $x \in E$, such that $x_n \to x$. However, then, $L_n \to L$, where L is the class containing the element x, proving that E/L_0 is a complete space.

Finally, note that if $E_1, E_2, \ldots, E_n$ are normed linear spaces and E is a direct sum of these spaces, then E is also normable.

For example, for $x = \{x_1, x_2, \ldots, x_n\}$, set

$$\| x \| = \| x_1 \| + \| x_2 \| + \ldots + \| x_n \|.$$

It is plain that if $E = E_1 \oplus E_2$, then the normed linear spaces E_1 and E/E_2 are isomorphic.

2.22. Series of elements of a Banach Space. Let $x_1, x_2, \ldots, x_n, \ldots$ be the elements of a BANACH space E. An expression of the form $\sum_{n=1}^{\infty} x_n$ is called a **series**, made up of the elements of the space E. Consider the n-th **partial sum** $s_n = x_1 + x_2 + \ldots + x_n$.

If the sequence of partial sums $\{s_n\}$ converges, then $\sum_{n=1}^{\infty}$ is said to be a **convergent** series.

E being a complete space, it suffices for the sequence $\{s_n\}$ to be convergent that this is a CAUCHY sequence. This, in turn, implies the following sufficiency condition for the covergence of a series : Let $\| x_n \| \leqslant a_n$ and let the numerical series $\sum_{n=1}^{\infty} a_n$ be convergent; then, the series $\sum_{n=1}^{\infty} x_n$ is also convergent. The proof, evidently, follows from the inequality

$$\| s_{n+p} - s_n \| = \| x_{n+1} + \ldots + x_{n+p} \| \leqslant a_{n+1} + \ldots + a_{n+p},$$

2.3 LINEAR TOPOLOGICAL SPACES

A normed linear space, some of whose properties are indicated in the foregoing, constitutes a special case of a metric linear space. A metric linear space, in turn, forms an important special case of a more generalized topological space. Of late, linear topological spaces have found increasing applications in various problems of functional analysis, theory of differential equations and certain other branches of mathematics. The object here is to describe only certain simple concepts related to linear topological spaces. For a more detailed presentation of the properties of these spaces, see [15].†

A set $X = \{x, y, z, \dots\}$ is called a **linear topological space**, if the following four axioms are satisfied :

(i) X is a *topological space*, that is a *topology* is defined in X by a system Y of subsets (called *open* sets), with the properties :

(*a*) The empty open set and the whole space belong to Y;

(*b*) Any union of open sets is an open set;

(*c*) Any finite intersection of open sets is an open set.

Any open set that contains a point $x \in X$ is called a **neighbourhood** (hereafter abbreviated nhood) of this point.

A point x of the set $M \subset X$ is said to be the **interior point** of this set, if it is contained in M together with some nhood $U(x)$. It is obvious that every point of an open set G is an interior point : in this case, for example, the set G itself can be treated as the nhood $U(x)$. The converse is also true : If every point of the set M is an interior point, then M is an *open set*. This follows from the equality

$$M = \bigcup_{x \in M} U(x), \qquad U(x) \subset M$$

and property (*b*) of the open set.

(ii) X is a separated topological space ; this means that for every pair of points $(x, y) \in X$, there is a nhood of the point x, not containing y.

The acquaintance with the idea of open sets paves the way for introducing the notion of the limit point of a set. A point $a \in X$ is called an **accumulation point** or **cluster point** or **limit point** of a set $M \subset X$, if every nhood of a contains at least one point of M other than a. The set M' of all limit points of M is called the **complement** (or **derived set**) of M.

The set $\bar{M} = M \cup M'$ is called the **closure** of the set M. A set M is said to be **closed** if it coincides with the closure. It can be shown that many of the properties of closure operations and closed sets are possessed in common by a topological space and a number field ; for example, the complement of an open set being a closed set, the validity of properties (i) thro' (iv) of closure established on p. 8, any finite union of closed sets being closed, and so on.

†BOURBAKI, N., *Espaces vectoriels topologiques*, Actualites Scientifiques et Industrielles 1189 and 1229, Hermann, Paris, 1953 and 1955.

The limit of a sequence of points $x_1, x_2, \ldots, x_n, \ldots$, can also be defined in a topological space. Namely, a point x is a **limit** of this sequence, if from a certain index onwards all points of the sequence $\{x_n\}$ are contained in every nhood of the point x. It is routine to show that the limit thus defined is unique.

(iii) X is a real linear space (regarded also a complex space, but this aspect will not be gone into here).

(iv) The operations of addition of elements and multiplication of elements by a real number are continuous in the topology of the space X. This signifies that:

(*a*) For every pair of elements $(x, y) \in X$ and every nhood $U(x+y)$ of the element $x + y$, there is a nhood $U(x)$ of the element x and a nhood $U(y)$ of the element y, such that

$$U(x) + U(y) \subset U(x+y)$$

(the symbol $A + B$, where A and B are sets of a linear space X, denotes a set of elements in X of the form $a + b$, $a \in A$, $b \in B$);

(*b*) For every real number λ, every element $x \in X$ and every nhood W of the element λx, there exist a number $\delta > 0$ and a nhood V of the element x, such that $\alpha V \subset W$ for every α, satisfying the inequality

$$|\alpha - \lambda| < \delta$$

(the symbol αV denotes a point set of the form αy, $y \in V$).

Let x_0 be a fixed element of a linear topological space and let G be an open set. Then, $x_0 + G$ is also an open set.

Taking any point $y \in x_0 + G$, we get $y = x_0 + x$, $x \in G$. Thereupon $y - x_0 \in G$. Since G is an open set, it is a nhood of $y - x_0$ and because of the continuity of addition there is a nhood $V(y)$ of the point y and a nhood $W(-x_0)$ of x_0, such that

$$V(y) + W(-x_0) \subset U(y-x_0) = G.$$

In particular, $\quad V(y) + (-x_0) \subset G, \quad$ that is, $V(y) \subset G + x_0$.

Thus, every point of the set $x_0 + G$ together with some nhood belongs to this set, that is, $x_0 + G$ is open.

Analogously, it can be proved that λG is an open set for every real number λ and every open set G.

It follows from what has been established that if $U(x)$ is a nhood of the point x of a linear topological space X, then $U(x) - x$ is a nhood of zero of the space X. Conversely, if $V(0)$ is a nhood of zero of the space X, then $V(0) + x$ is a nhood of a point x of the same space. Hence, in order to assign a collection of all the nhoods of all points of a linear space, that is, a collection of all its open sets defining the topology of this space, it is sufficient to give a collection of all nhoods of zero.

The set A of a linear space X is said to be **symmetric**, if $x \in A$ implies $-x \in A$. If U is a nhood of zero of a linear topological space X, then it is evident that $-U \cap U$ is also a nhood of zero and, moreover, symmetric.

Finally, if may be remarked that for defining the topology of a space it is not necessary to determine all the nhoods of zero. It is sufficient to provide only a **basic** system of nhoods of zero, such that for every nhood U of zero there is a basic system V of nhoods of zero completely contained in U. Generally, any two systems S and $\tilde{S}$ of nhoods in X, are called **equivalent** if for any nhood $U \in S$ there is a nhood $\tilde{U} \in \tilde{S}$, such that $\tilde{U} \subset U$; also, conversely, for every nhood $\tilde{V} \in \tilde{S}$ there is a nhood $V \in S$ such that $V \subset \tilde{V}$. It is obvious that two equivalent systems of nhood induce the same topology in X.

Examples. 1. Let X be a collection of real functions defined on the line $-\infty < t < +\infty$, infinitely differentiable there and vanishing outside of some finite segments.† The addition of functions and the multiplication of functions by a number are defined in the customary manner. As a nhood of zero, take note of the following set: for every $\varepsilon > 0$ and every n nhoods of zero $U(n, \varepsilon)$ there is a collection of functions $x(t) \in X$, such that $|x^{(k)}(t)| < \varepsilon$ for $k = 0, 1, 2, \ldots, n$. It is trivial to verify that this satisfies all axioms concerning a linear topological space.

2. A normed linear space is a linear topological space. Every open set (in the sense of metric, defined by a norm) containing the point zero is a nhood of zero.

The question arises as to the conditions under which a linear topological space is normable, that is, a norm can be introduced in it such that the collection of nhoods of zero of the normed linear space obtained coincides with the collection of nhoods of zero which defined the initial topology of the space.

An answer to this problem has been provided by an important theorem due to A. N. KOLMOGOROV.

A set A of a linear topological space is said to be **bounded**, if for every nhood $U(0)$ of zero there is a number $\lambda > 0$ such that the set λA lies completely inside of the considered nhood of zero. The boundedness of A is equivalent to the condition: For any sequence $\{x_n\} \subset A$ and any sequence $\{\lambda_n\}$ of real numbers the sequence of elements $\{\lambda_n x_n\}$ converges to zero. The proof of this proposition is, however, bypassed. This implies, in particular, that if A is bounded, then $-A$ is also bounded.

THEOREM (A. N. KOLMOGOROV). *In order that a linear topological space X be normable it is necesssary and sufficient that there exists in it a convex bounded nhood of zero.*

Let U be a nhood of zero in X, endowed with the indicated properties. For any $x \in X$, set

$$\|x\| = \inf_{\lambda > 0,\, x \in \lambda U} \lambda.$$

It is required to show that the norm so introduced has all the requisite properties.

In the first place, $\|0\| = 0$, because $0 \in \lambda U$ for every $\lambda > 0$. Let $x \neq 0$. Then, for some n_0, $x \,\bar{\in}\, \frac{1}{n_0} U$. In fact, if $x \in \frac{1}{n} U$ for any n, then

$$y_n = nx \in U \qquad \text{for} \qquad n = 1, 2, \ldots,$$

†For every function, this segment is the same.

and hence the sequence $\{y_n\}$ is bounded. Thereupon, $(1/n)\, y_n \to 0$. However, this is impossible, since $\frac{1}{n}\, y_n = x \neq 0$. Thus $x \;\overline{\in}\; \frac{1}{n_0}\, U$, hence

$$\| x \| \geqslant \frac{1}{n_0} > 0,$$

and the first property of norm is established.

Now, let $\| x \| = \alpha$, $\| y \| = \beta$; $x, y \neq 0$, then $\| x/\alpha \| = 1$ and, consequently,

$$\frac{x}{\alpha} \in (1+\varepsilon)\, U$$

for arbitrarily small $\varepsilon > 0$. Analogously,

$$\frac{y}{\beta} \in (1+\varepsilon)\, U.$$

By the convexity of U and hence also $(1+\varepsilon)\, U$, we get

$$\frac{\alpha}{\alpha+\beta}\,\frac{x}{\alpha} + \frac{\beta}{\alpha+\beta}\,\frac{y}{\beta} \in (1+\varepsilon)\, U,$$

or,

$$\frac{x+y}{\alpha+\beta} \in (1+\varepsilon)\, U.$$

Consequently, $x + y \in (\alpha+\beta)\,(1+\varepsilon)U$. Thereupon, $\| x+y \| \leqslant (\alpha+\beta)(1+\varepsilon)$, and since $\varepsilon > 0$ is arbitrary,

$$\| x+y \| \leqslant \alpha + \beta = \| x \| + \| y \|.$$

If, however, either x or y or both the elements vanish, then the equality

$$\| x+y \| = \| x \| + \| y \|$$

is evident. Consequently, the second property of norm is also proved.

The symmetry of $x \in \lambda U$ implies also the inverse symmetry of $-x \in \lambda U$. Hence $\| -x \| = \| x \|$.

Consider an element αx, $\alpha > 0$. Let $x \in \lambda U$. Then $\alpha x \in \alpha\lambda U$ and, conversely, $\alpha x \in \alpha\lambda U$ implies $x \in \lambda U$. Hence,

$$\| \alpha x \| = \inf_{\alpha x \in \mu U} \mu = \inf_{\alpha x \in \alpha\lambda U} \alpha\lambda = \alpha \inf_{x \in \lambda U} \lambda = \alpha\, \| x \|.$$

In the general case

$$\| \alpha x \| = \| \pm \,|\, \alpha \,|\, x \| = \| \,|\, \alpha \,|\, x \| = |\, \alpha \,|\, \| x \|$$

and the third property of norm is proved.

For completing the proof of theorem, it is sufficient to show that for any nhood $V(0)$ of zero in X there is a sphere $\| x \| < \rho$ completely contained in $V(0)$ and conversely : for every sphere $\| x \| < \rho$ there is a nhood $W(0)$ of zero completely contained in this sphere.

Take an arbitrary nhood $V(0)$ of zero. Since the nhood U of zero, by means of which a norm has been introduced, is a bounded set, there is a number $r > 0$, such that $rU \subset V(0)$. On the other hand, the unit sphere

$\| x \| < 1$, evidently, lies in the nhood U, implying that the sphere $\| x \| < r$ lies in rU and, by the same token, in the nhood $V(0)$ of zero.

Conversely, given a sphere $\| x \| < \rho$. By the definition of norm, it follows that a nhood $\rho' U$ of zero, ρ' being an arbitrary number smaller than ρ, is contained completely in this sphere.

The sufficiency condition of the theorem stands completely proved. The necessity condition is trivial and is, therefore, left to the reader.

2.4 ABSTRACT HILBERT SPACE

2.41. Introduction. In an n-dimensional real (complex) vector space E_n, besides the operations of addition of vectors and multiplication of vectors by a real (complex) number, we also introduce the **scalar** (or **inner**) product of two vectors. Namely, the **scalar product** of the vectors

$$x = \{\xi_1, \xi_2, \ldots, \xi_n) \qquad \text{and} \qquad y = \{\eta_1, \eta_2, \ldots, \eta_n)$$

in E_n is given by the number

$$(x, y) = \sum_{i=1}^{n} \xi_i \bar{\eta}_i.$$

The **norm** or **length** of a vector $x = \{\xi_1, \xi_2, \ldots, \xi_n\}$ is expressible as a scalar product of the form

$$\| x \| = \sqrt{\sum_{i=1}^{n} | \xi_i |^2} = \sqrt{(x, x)}.$$

In the analysis, the scalar product of functions has extensive applications. It is therefore, essential to consider a class of those linear spaces, where an inner product is defined. Such spaces are called **Hilbert spaces** and are defined by the following axioms.

2.42. Axioms of abstract Hilbert space. Let H be a set of certain elements $x, y, z, \ldots$. Assume that :

(i) H is a complex linear space ;

(ii) A complex number (x, y), called the **scalar product** or **inner product** of x and y, is associated with every pair of elements x and y, with the properties :

(*a*) $(x, y) = \overline{(y, x)}$ (in particular, (x, x) is real ; the bar denotes complex conjugation) ;

(*b*) $(x_1+x_2, y) = (x_1, y) + (x_2, y)$;

(*c*) $(\lambda x, y) = \lambda(x, y)$ for every complex number λ ;

(*d*) $(x, x) \geqslant 0$ for all $x \in H$; moreover, $(x, x) = 0$, iff $x = 0$. The number $\| x \| = \sqrt{(x, x)}$ is called the **norm** of x.†

†In what follows (p. 56) it shall be shown that it satisfies all requirements of the norm of a normed linear space.

(iii) H is complete in the sense of the metric $\rho(x, y) = \| x-y \|$.
If the axioms (i) — (iii) above are satisfied, the set H is called **an inner product space** (or **unitary space**). An n-dimensional unitary space is a complex Euclidean space.

(iv) There exist in H, n-linearly independent elements for every natural number n, that is, H is infinite-dimensional.

If H satisfies the axioms (i)—(iv), then it is called an **abstract separable Hilbert space**, hereafter a **Hilbert space** for brevity.

Examples. 1. The complex space l_2 becomes a HILBERT space, if

$$(x, y) = \sum_{j=1}^{\infty} \xi_j \bar{\eta}_j$$

for each pair of its elements $x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$ and $y = \{\eta_1, \eta_2, \ldots, \eta_n, \ldots\}$. The convergence of this series follows from the CBS† inequality for series.

2. The complex space $L_{2,\rho}\,[0, 1]$. This is a space of complex-valued functions, defined and measurable on $[0, 1]$ and also such that

$$\int_0^1 \rho(t) \mid x(t) \mid^2 dt < +\infty,$$

where $\rho(t)$ is real, $\rho(t) \geqslant 0$ a.e. on $[0, 1]$ and also $\rho(t) > 0$ on a set of measure zero. $L_{2,\rho}\,[0, 1]$ becomes a HILBERT space if, for $x, y \in L_{2,\rho}$ we set

$$(x, y) = \int_0^1 \rho(t)\, x(t)\, \bar{y}(t)\, dt.$$

The existence of this integral for every $x(t)$ and $y(t)$ in $L_{2,\rho}\,[0, 1]$ follows from the CBS† inequality for integrals. In particular, for $\rho(t) \equiv 1$ we obtain the complex space L_2 with the scalar product

$$(x, y) = \int_0^1 x(t)\, \bar{y}(t)\, dt.$$

A real HILBERT space is analogously defined. For this the scalar product of two elements must be real. The real spaces l_2, L_2, $L_{2,\rho}$, are real HILBERT spaces.

To consider briefly the simplest properties of a HILBERT space, it is convenient to start with the deduction of the relations

$$(x, y_1+y_2) = (x, y_1) + (x, y_2), \qquad (x, \lambda y) = \bar{\lambda}\,(x, y)$$

from axioms (i) through (iii). The second relation implies, in particular, that

$$\| \lambda x \| = | \lambda | \, \| x \|. \tag{1}$$

Next, to prove the CBS inequality for the scalar product, for any x, $y \in H$,

†Referred to variously as the SCHWARZ, or CAUCHY-BUNYAKOVSKII or CAUCHY-BUNYAKOVSKII-SCHWARZ (abridged to CBS) inequality.

$y \neq 0$ and any complex λ we have

$$(x+\lambda y,\ x+\lambda y) \geqslant 0,$$

or,
$$(x, x) + \bar{\lambda}\,(x, y) + \lambda(y, x) + |\lambda|^2\,(y, y) \geqslant 0.$$

Set
$$\lambda = -\frac{(x, y)}{(y, y)},$$

then
$$(x, x) - \frac{|(x, y)|^2}{(y, y)} \geqslant 0,$$

or
$$|(x, y)| \leqslant \|x\| \cdot \|y\|, \tag{2}$$

giving the required inequality. For $y = 0$, Ineq. (2) is trivial. Furthermore,

$$\begin{aligned}\|x+y\|^2 &= (x+y,\ x+y) = (x, x) + (x, y) + (y, x) + (y, y)\\ &\leqslant \|x\|^2 + 2\,\|x\|\,\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2,\end{aligned}$$

or,
$$\|x+y\| \leqslant \|x\| + \|y\|. \tag{3}$$

Axiom (ii) (d) and the relations (1) and (3) verify that the norm introduced by means of the scalar product satisfies all the norm axioms of a normed linear space and, therefore, the distance introduced by this norm satisfies all the axioms of a metric space.

As regards the extent of difference between HILBERT spaces and B-spaces, it can be remarked briefly that the norm in a HILBERT space is defined by an inner product. The completeness of HILBERT space, though an essential requirement, is superfluous because by adding new elements an inner product space can always be extended to be a complete HILBERT space.

The conventions set up above lead to the next proposition.

LEMMA 1. *The scalar product is a continuous function with respect to norm convergence.*

In fact, let $x_n \to x$ and $y_n \to y$, then the numbers $\|x_n\|$, $\|y_n\|$ are bounded; let M be their upper bound. Then,

$$\begin{aligned}|(x_n, y_n) - (x, y)| &= |(x_n, y_n) - (x_n, y) + (x_n, y) - (x, y)|\\ &\leqslant |(x_n, y_n) - (x_n, y)| + |(x_n, y) - (x, y)|\\ &= |(x_n, y_n - y)| + |(x_n - x, y)|\\ &\leqslant \|x_n\|\ \|y_n - y\| + \|x_n - x\|\ \|y\|\\ &\leqslant M\,\|y_n - y\| + \|y\|\,\|x_n - x\|.\end{aligned}$$

Now, since $\|y_n - y\| \to 0$ and $\|x_n - x\| \to 0$ as $n \to \infty$, it follows also that $|(x_n, y_n) - (x, y)| \to 0$ as $\to \infty$. ■

2.43. Orthogonality. Two elements x and $y \in H$ are called **orthogonal** ($x \perp y$), if their inner product vanishes (that is, if $(x, y) = 0$); x is said to be **orthogonal to a subspace** $L \subset H$ ($x \perp L$), if it is orthogonal to every $y \in L$.

LEEMA 1. *If $x \in H$ and L is some subspace of H, then x has a unique representation*

$$x = y + z, \tag{4}$$

with $y \in L$ and *$z \perp L$.*

Proof. Obviously, $y = x$ and $z = 0$ for $x \in L$. It is, therefore, possible to assume that $x \bar{\in} L$. Let $d = \inf_{y' \in L} \| x - y' \|^2$ and let $\{y_n\}$ be a sequence in L, such that $d_n = \| x - y_n \|^2 \to d$ as $n \to \infty$. Further, let h be any non-zero element of L. Then, $y_n + \varepsilon h \in L$ for any complex ε and, therefore, $\| x - (y_n + \varepsilon h) \|^2 \geqslant d$, that is,

$$\| x - y_n \|^2 - \bar{\varepsilon} (x - y_n, h) - \varepsilon (h, x - y_n) + | \varepsilon |^2 \| h \|^2 \geqslant d.$$

Set
$$\varepsilon = \frac{| x - y_n, h |}{\| h \|^2},$$

to receive
$$\| x - y_n \|^2 - \frac{| x - y_n, h |^2}{\| h \|^2} \geqslant d,$$

whence
$$| (x - y_n, h) |^2 \leqslant \| h \|^2 (d_n - d),$$

or,
$$|(x - y_n, h) | \leqslant \| h \| \sqrt{(d_n - d)}. \tag{5}$$

Ineq. (5) is, evidently, satisfied also for $h = 0$. This inequality implies for any $h \in L$, that
$$| y_n - y_m, h) | \leqslant | (y_n - x, h) | + |(x - y_m, h) |$$
$$\leqslant (\sqrt{d_n - d} + \sqrt{d_m - d}) \| h \|,$$

and setting, in particular $h = y_n - y_m$, it leads to
$$\| y_n - y_m \| \leqslant (\sqrt{d_n - d} + \sqrt{d_m - d}).$$

Therefore, $\{y_n\}$ is a Cauchy sequence and, hence, owing to the completeness of H converges to some element $y \in H$. Since L is closed, $y \in L$. Taking the limit in Ineq. (5), we get $(x - y, h) = 0$, and since h is an arbitrary element of L, $x - y \perp L$. Setting $x - y = z$, we obtain the required representation

$$x = y + z.$$

The uniqueness of this representation remains to be proved. Let $x = y + z$, $x = y' + z'$, where $y, y' \in L$ and $z, z' \perp L$. Then $y - y' = z' - z$ and

$$\| y - y' \|^2 = (z' - z, y - y') = 0, \tag{6}$$

because $y - y' \in L$ and $z' - z \perp L$. However, Eq. (6) signifies that $y = y'$ and, consequently, also $z = z'$. The lemma is thus completely proved.

The element y in (4) is called the **projection** of x on L. It is obvious that the collection M of all elements, orthogonal to L, is itself a subspace. That M forms a linear manifold is plain. It is also closed owing to the continuity of the scalar product. We can, therefore, speak of the element z of (4) as the projection of x on M. This subspace M is called the **orthogonal complement** of L and denoted by $H \dot{-} L$. Further, H is called the **orthogonal sum** of the subspaces L and M and written as $H = L \dot{+} M$. Obviously, the orthogonal sum is a special case of the direct sum. Thus the lemma gives the decomposition of an element into its projections onto two complementary orthogonal subspaces.

LEMMA 2. *In order that a linear manifold M is everywhere dense in H, it is necessary and sufficient that no element exists which is different from zero and orthogonal to all the elements of M.*

Necessity. In the first place it is plain that $x \perp M$ implies $x \perp \overline{M}$. However, by hypothesis $\overline{M} = H$ and, consequently, $x \perp H$, in particular, $x \perp x$, implying $x = 0$, proving the necessity,

Sufficiency. Assume that M is not everywhere dense in H. Then $\overline{M} \neq H$ and there is an $x \bar{\in} \overline{M}$ and $x \in H$. By Theorem 1, $x = y + z$, $y \in \overline{M}$, $z \perp \overline{M}$, and since $x \bar{\in} \overline{M}$, it follows that $z \neq 0$, a contradiction proving the necessity.

2.44. Orthonormal system. A system $e_1, e_2, \ldots, e_n, \ldots$ of elements of the space H is called **orthonormal** (or **orthonormalized**), if

$$(e_i, e_j) = \delta_{ij},$$

δ_{ij} the KRONECKER delta, unity for $i = j$ and zero for $i \neq j$. The system $\{e^{i2\pi nt}\}$, $n = 0, \pm 1, \pm 2, \ldots$, is an example of an orthonormal system in the complex space L_2 [0, 1].

An infinite system of elements of a linear space is called **linearly independent**, if every finite subset of this system is linearly independent.

By means of an inductive process, the **Schmidt orthogonalization process** described below, any system of linearly independent elements $h_1, h_2, \ldots, \ldots, h_n, \ldots$ can be converted into an orthonormal system.

Set $e_1 = h_1 / \| h_1 \|$. Let $g_2 = h_2 - c_{21} e_1$, and let the number c_{21} be so chosen that g_2 is orthogonal to e_1. Obviously, this leads to $c_{21} = (h_2, e_1)$. Further, put $e_2 = g_2 / \| g_2 \|$. Here $\| g_2 \| \neq 0$, because otherwise $g_2 = 0$ and, therefore, h_1 and h_2 would be linearly dependent, violating the assumption. Assuming that $e_1, e_2, \ldots, e_{k-1}$ are already constructed, take $g_k = h_k - \sum_{i=1}^{k-1} c_{ki} e_i$ and choose c_{ki}'s by putting $c_{ki} = (h_k, e_i)$ such that g_k is orthogonal to $e_1, e_2, \ldots, e_{k-1}$. Then, set $e_k = g_k / \| g_k \|$ and again $\| g_k \| \neq 0$, and so on.

Example. If the collection of powers $1, t, t^2, \ldots, t^n, \ldots$ is orthonormalized in the real function space $L_{2,\rho}$ $[a, b]$, square-summable with weight $\rho(t)$, we get the CHEBYSHEV system of polynomials

$$p_0 = \text{const}, p_1(t), p_2(t), \ldots, p_n(t), \ldots,$$

which are orthonormal with weight $\rho(t)$:

$$\int_a^b \rho(t) \; p_i(t) \, p_j(t) \, dt = \delta_{ij}.$$

We obtain to within the constant factors the LEGENDRE polynomials for $\rho(t) \equiv 1$, $a = -1$, $b = +1$; the HERMITE polynomials for $\rho(t) = e^{-t^2}$, $a = -\infty$, $b = +\infty$ and the LAGUERRE polynomials for $\rho(t) = e^{-t}$, $a = 0$, $b = \infty$.

Let L be the subspace spanned by the orthonormal system $e_1, e_2, \ldots,$

$e_n, \ldots$ and let $x \in L$. Consequently, there is a linear combination $\sum_{i=1}^{n} \alpha_i e_i$ for every $\varepsilon > 0$, such that $\| x - \sum_{i=1}^{n} \alpha_i e_i \| < \varepsilon$. However,

$$\| x - \sum_{i=1}^{n} \alpha_i e_i \|^2 = \| x \|^2 - \sum_{i=1}^{n} \overline{\alpha_i} (x, e_i) - \sum_{i=1}^{n} \alpha_i (e_i, x)$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} (e_i, e_j) = \| x \|^2 - \sum_{i=1}^{n} \overline{\alpha_i}\, c_i - \sum_{i=1}^{n} \alpha_i \overline{c_i} + \sum_{i=1}^{n} | \alpha_i |^2,$$

where $c_i = (x, e_i)$.

The numbers c_i are called **Fourier coefficients of the element** x with respect to the orthonormal system $\{e_i\}$. Further, the preceding equality implies

$$\left\| x - \sum_{i=1}^{n} \alpha_i e_i \right\|^2 = \| x \|^2 - \sum_{i=1}^{n} | c_i |^2 + \sum_{i=1}^{n} | \alpha_i - c_i |^2.$$

It thereby follows that the norm of the difference $x - \sum_{i=1}^{n} \alpha_i e_i$ takes its least value when the α_i are the Fourier coefficients of x with respect to the system $\{e_i\}$. In this case,

$$0 \leqslant \left\| x - \sum_{i=1}^{n} c_i e_i \right\|^2 = \| x \|^2 - \sum_{i=1}^{n} c_i^2 < \varepsilon, \tag{7}$$

and since ε can be chosen arbitrarily small, it follows that

$$x = \lim_{x \to \infty} \sum_{i=1}^{n} c_i e_i = \sum_{i=1}^{\infty} c_i e_i.$$

The convergence of the series $\sum_{i=1}^{\infty} | c_i |^2$ also follows from (7), and furthermore, $\sum_{i=1}^{\infty} | c_i |^2 = \| x \|^2$.

Now, let x be any element in H. Let z denote the projection of x on L. Then, $z = \sum_{i=1}^{\infty} c_i e_i$, where $c_i = (z, e_i) = (x, e_i)$ and $\sum_{i=1}^{\infty} | c_i |^2 = \| z \|^2$. Since $x = z + y$, $z \in L$, $y \perp L$, it follows that

$$\| x \|^2 = \| z \|^2 + \| y \|^2 \geqslant \| z \|^2.$$

Consequently, for any element x in H, the inequality

$$\sum_{i=1}^{\infty} | c_i |^2 \leqslant \| x \|^2 \tag{8}$$

holds, where $c_i = (x, e_i)$ $(i = 1, 2, \ldots)$. This relation is called the **Bessel inequality.**

2.45. Complete orthonormal system (V. A. STEKLOV). The concept of complete orthonormal syetems was inaugurated by V. A. STEKLOV while investigating the question of the representation of functions by the elements of an orthonormal system.

Given an orthonormal system $\{e_i\}$ in the space H. If there is no nonzero element $x \in H$ which is orthonormal to every element of $\{e_i\}$, then this system is called **complete.** In other words, a complete orthonormal system cannot be extended to a larger orthonormal system by adding new elements; that is, it is maximal (with respect to inclusion). An orthonormal system $\{e_i\}$ is said to be **closed**, if the subspace L spanned by this system, coincides with H. A Fourier series with respect to a closed system, constructed for any $x \in H$, converges to this element and for every $x \in H$, the **Parseval-Steklov**† **equality**

$$\sum_{i=1}^{\infty} c_i^2 = \| x \|^2 \tag{9}$$

holds.

Eq. (9) is a necessary and sufficient condition for the completeness of $\{e_i\}$ for any $x \in H$.

A closed (complete) orthonormal system is called also an **orthonormal basis** for a HILBERT space.

If an orthonormal system is complete, then it is closed. In fact, there exist in it no nonzero element which is orthogonal to a linear manifold L, spanned by this system. However, then, by Lemma 2, $\bar{L} = H$ and the system is complete.

Conversely, a closed orthonormal system $\{e_i\}$ is complete, since for such system, we have

$$\| x \|^2 = \sum_{i=1}^{\infty} c_i^2,$$

and if $x \perp e_i$, $i = 1, 2, \ldots$, that is, $c_i = 0$, $i = 1, 2, \ldots$, then $\| x \| = 0$, signifying that the system $\{e_i\}$ is complete.

The system of trigonometric functions

$$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \sin t, \frac{1}{\sqrt{\pi}} \cos 2t, \ldots$$

serves as an example of a complete orthonormal system in the real space $L_2[-\pi, \pi]$.

It is esay to show that a *complete orthonormal system exists in every*

†Called **Parseval's relation** or **equation** (also **the equation of closure**) according to common practice.

separable Hilbert space. Let $G = \{g_1, g_2, \ldots, g_n, \ldots\}$ be any countable everywhere complete set in the space H, all g_n ($n=1, 2, \ldots,$) being nonzero. Set

$$e_1 = g_1 / \| g_1 \|,$$

and let L_1 be a one-dimensional subspace spanned by the element e_1. Let g_{n_2} be the first element of G, not belonging to L_1 and h_2 a projection of g_{n_2} on $H \dot{-} L_1$. Set

$$e_2 = h_2 / \| h_2 \|.$$

Let L_2 be a subspace spanned by the elements e_1 and e_2, and g_{n_3} the first element of G not belonging to L_2. Let h_3 be a projection of g_{n_3} on $H \dot{-} L_2$. Set

$$e_3 = h_3 / \| h_3 \|$$

and continuing in this manner we obtain an orthogonal system $e_1, e_2, \ldots, e_n, \ldots$, and since every element g_n belongs to a certain L_m in virtue of the space containing these being a subspace, it follows that the subspace defined by the system $\{e_i\}$ coincides with the subspace defined by $\{g_i\}$, that is, with the space H. In addition, the system $\{e_i\}$ is countable, because were it to contain a finite number of p elements, then, as is known from linear algebra, there would not exist $p + 1$ linearly independent elements in H, contradicting the axiom (iv).

If $\{e_i\}$ is a complete orthonormal system and x and y are elements in H with the respective Fourier coefficients c_i and d_i, $i = 1, 2, \ldots$, then it is easy to verify that

$$(x, y) = \sum_{i=1}^{\infty} c_i \bar{d}_i.$$

2.46. Isomorphism between separable Hilbert spaces. Consider a separable HILBERT space H and let $\{e_i\}$ be a complete orthonormal system in this space. If x is some element in H, then one can assign to this element a sequence of numbers, $\{c_1, c_2, \ldots, c_n, \ldots\}$, the Fourier coefficients of x with respect to $\{e_i\}$. As shown earlier, the series

$$\sum_{i=1}^{\infty} | c_i |^2$$

converges, and consequently, the sequence $\{c_1, c_2, \ldots, c_n, \ldots\}$ can be treated as some element $\tilde{x}$ of the complex space l_2. Thus, to every element $x \in H$ there is assigned some element $\tilde{x} \in l_2$; moreover, the assumption on the completeness of the system implies

$$\| x \|_H = \left(\sum_{i=1}^{\infty} | c_i |^2 \right)^{1/2} = \| \tilde{x} \|_{l_2}, \tag{10}$$

where the subscripts H and l_2 appear in the sense of any norm taking space.

Further, it is evident that if $x \in H$ corresponds to $\tilde{x} \in l_2$ and $y \in H$ to $\tilde{y} \in l_2$, then $x \pm y$ corresponds to $\tilde{x} \pm \tilde{y}$. Thereupon, (10) also implies

$$\| x - y \|_H = \| \tilde{x} - \tilde{y} \|_{l_2}. \tag{11}$$

Now, let $\tilde{z} = \{\zeta_i\}$ be an arbitrary element in l_2. Considering in H the elements $z_n = \sum\limits_{i=1}^{\infty} \zeta_i e_i$, $n = 1, 2, \ldots,$ we have

$$\| z_n - z_m \|^2 = \left\| \sum_{i=m+1}^{\infty} \zeta_i e_i \right\|^2 = \sum_{i=m+1}^{\infty} | \zeta_i |^2,$$

whence $\| z_n - z_m \| \to 0$ as $n, m \to \infty$. Thus, $\{z_n\}$ is a CAUCHY sequence in the sense of the metric of H and, in virtue of the completeness of H, converges to some element z of this space. Since

$$(z, e_i) = \lim_n (z_n, e_i) = \zeta_i,$$

it follows that the ζ_i are Fourier coefficients of z with respect to the chosen orthonormal system $\{e_i\}$. Thus, each $\tilde{z} \in l_2$ is assigned to some element $z \in H$. By the same token, a one-one correspondence is obtained between the elements of H and l_2.

The formula (11) shows that this correspondence between H and l_2 is an isometric correspondence. Moreover, since it is evident that if x corresponds to $\tilde{x}$, λx corresponds to $\lambda \tilde{x}$, and taking note of the fact that earlier assertions concerning the operation of addition are preserved under the considered correspondence, H and l_2 are found to be isomorphic. This leads to the next theorem.

THEOREM 2. *Every complex (real) separable Hilbert space is isomorphic and isometric to a complex (real) space l_2 and, consequently, all complex (real) separable spaces are isomorphic and isometric to each other.*

Thereupon, as a special case, it follows that

THEOREM 3. (RIESZ-FISHER). *Real spaces L_2 (0, 1) and l_2 are isomorphic and isometric.*

2.5 GENERALIZED DERIVATIVES* AND SPACES OF S. L. SOBOLEV

IN SEVERAL problems of mathematical physics, especially those of differential equations and the quantum field theory, it is expedient to introduce the so-called generalized solutions of linear partial differential equations. A collection of ordinary solutions of such equations when regarded a functional space with a certain metric is, generally speaking, a non-complete space. The process of completing this space leads to the generalized solution, being an element of the space completed.

*Also called distributional derivatives.

Thus, for example, the collection of solutions of the problem of free oscillations of an infinite string, expressible in the form

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2},$$

takes the form
$$u(x, t) = \varphi(x+at) + \psi(x-at), \tag{1}$$

where φ and ψ are twice differentiable functions. While completing the collection of such solutions, for example, in a uniformly convergent metric, we are led to a collection of generalized solutions also, having the form (1), φ and ψ this time being arbitrary continuous functions.

The problem of the construction of generalized solutions gave rise to the idea of **generalized** (or **distributional**) derivatives first introduced by SOBOLEV in 1936. In what follows, the basic ideas of the theory of generalized derivatives and spaces of S. L. SOBOLEV [34] are presented in the context of some simple cases.

Let G be a bounded convex domain on a plane. Consider a function $\varphi(x, y)$ defined and continuous together with derivatives of order up to and including l, in some domain containing G interior to it (in this case, it is said that the function $\varphi(x, y)$ is continuous together with its derivatives of order upto l in G). To introduce a norm in the set of such functions, put

$$\| \varphi \| = \left(\iint_G | \varphi(x, y) |^p \, dx\, dy + \sum_{l_1+l_2=l} \iint_G \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial x^{l_2}} \right|^p dx\, dy \right)^{1/p}.$$

It is easy to verify that all the norm axioms are satisfied and we get a normed linear space denoted by $\tilde{W}_p^{(l)}$ which is not complete. The completion of this space in the norm introduced, is defined by the space $W_p^{(l)}$ introduced by S. L. SOBOLEV.

Let f_0 be an element in $W_p^{(l)}$, not belonging to $W_p^{(l)}$, implying that there is a sequence of functions $\{\varphi_n(x, y)\} \subset \tilde{W}_p^{(l)}$, such that

$$\| \varphi_n - f_0 \|_{W_p^{(l)}} \to 0 \quad \text{as} \quad n \to \infty.$$

Thereupon,
$$\| \varphi_m - \varphi_n \|_{W_p^{(l)}} \to 0 \quad n, m \to \infty,$$

that is,
$$\iint_G | \varphi_m(x, y) - \varphi_n(x, y) |^p \, dx\, dy \to 0$$

and
$$\iint_G \left| \frac{\partial^l \varphi_m}{\partial x^{l_1} \partial y^{l_2}} - \frac{\partial^l \varphi_n}{\partial x^{l_1} \partial y^{l_2}} \right|^p dx\, dy \to 0,$$

$$l_1 + l_2 = l, \qquad n, m \to \infty.$$

Thus, the sequences $\{\varphi_n(x, y)\}$ and $\left\{\dfrac{\partial^l\varphi_n(x, y)}{\partial x^{l_1}\,\partial y^{l_2}}\right\}$ in $L_p(G)$ (see Appendix II) are mean CAUCHY sequences with exponent p. In virtue of the completeness of $L_p(G)$, there exist functions $\varphi_0(x, y)$ and $\varphi_0^{(l_1, l_2)}(x, y) \in L_p(G)$, limits of the indicated sequences. An element f_0 identified with $\varphi_0(x, y)$ and $\varphi_0^{(l_1, l_2)}(x, y)$ is said to be the l-th **order generalized derivative** of $\varphi_0(x, y)$ and is denoted in the conventional differential notation by $\partial^l\varphi_0(x, y)/\partial x^{l_1}\,\partial y^{l_2}$. Since by definition $\| f_0 \| = \lim\limits_n \| \varphi_n \|$, the norm of f_0 or all the same of $\varphi_0(x, y)$ can be set in the previous form

$$\| \varphi_0 \|_{W_p^{(l)}} = \left(\iint\limits_G | \varphi_0(x, y) |^p \, dx\, dy + \sum_{l_1+l_2=1} \iint\limits_G \left| \frac{\partial^l\varphi_0}{\partial x^{l_1}\,\partial y^{l_2}} \right|^p dx\, dy \right)^{1/p};$$

the generalized l-th order derivative of the function $\varphi_0(x, y)$ appears here under the summation sign.

Thus, if $\varphi_0^{(l_1, l_2)}(x, y) \in L_p(G)$ is a generalized l-th order derivative of $\varphi_0(x, y) \in L_p(G)$, then there exist in $\bar{G}$ a sequence of continuously differentiable up to l-th order of functions $\varphi_n(x, y)$ converging in the mean with exponent p to $\varphi_0(x, y)$ and such that the sequence $\{ \partial^l\varphi_n(x, y)/\partial x^{l_1}\,\partial y^{l_2} \}$ also converges in the mean with exponent p to $\varphi_0^{(l_1, l_2)}(x, y)$.

The definition of generalized derivative implies its uniqueness as an element of $L_p(G)$. If the function $\varphi_0(x, y) \in L_p(G)$ is continuously differentiable up to and including l-th order in the ordinary sense, then it is possible to take a sequence $\{\varphi_n(x, y)\}$ where $\varphi_n(x, y) \equiv \varphi_0(x, y)$ for all n and, consequently,

$$\frac{\partial^l\varphi_0}{\partial x^{l_1}\,\partial y^{l_2}} = \varphi_0^{(l_1, l_2)}(x, y) = \frac{\partial^l\varphi_n}{\partial x^{l_1}\,\partial y^{l_2}},$$

that is, in this case the generalized derivative is the classical derivative.

The generalized derivatives are frequently defined in a different manner. Again, let $\varphi(x, y)$ and $\psi(x, y)$ have in $\bar{G}$ derivatives continuous up to l-th order, $\psi(x, y)$ vanishing at some boundary of the plane G_ρ, which consists of the points of a domain lying at a distance not exceeding ρ from its boundary. Then, applying GREEN'S formula a certain number of times, we obtain

$$\iint\limits_G \frac{\partial^l\psi}{\partial x^{l_1}\,\partial y^{l_2}}\, \varphi(x, y)\, dx\, dy = (-1)^l \iint\limits_G \psi(x, y)\, \frac{\partial^l\varphi}{\partial x^{l_1}\,\partial y^{l_2}}\, dx\, dy.$$

Now, let $\varphi(x, y)$ be an arbitrary function of $L_p(G)$. If there is a function $\chi(x, y) \in L_p(G)$ such that for every function $\psi(x, y)$ having the properties indicated above, the equation

$$\iint\limits_G \frac{\partial^l\psi}{\partial x^{l_1}\,\partial y^{l_2}}\, \varphi(x, y)\, dx\, dy = (-1)^l \iint\limits_G \psi(x, y)\, \chi(x, y)\, dx\, dy$$

remains valid, then $\chi(x, y)$ is called the l-th **order generalized derivative** of the function $\varphi(x, y)$.

2.51. The equivalence of the two definitions of generalized derivative. In order to establish an equivalence between these two definitions, the upcoming auxiliary notions and assertions are prerequisite.

As usual, let r denote the distance between the points $P(x, y)$ and $Q(\xi, \eta)$. The function

$$\omega_h(x, y; \xi, \eta) = \begin{cases} c_h e^{\frac{r^2}{r^2-h^2}}, & r < h, \\ 0, & r \geqslant h, \end{cases}$$

as a continuous function of x and y, has continuous derivatives of all orders and vanishes outside of a circle K_h of radius h with center at the point $Q(\xi, \eta)$. In virtue of $\omega_h(x, y; \xi, \eta)$ being symmetric in the points P and Q, all the foregoing assertions remain true, if $\omega_h(x, y; \xi, \eta)$ is treated as a function of ξ and η in the circle K_h' with center at the point $P(x, y)$. In this connection, note that ω_h differentiable with respect to x can be regarded differentiable with respect to ξ with inverted sign, and the same holds also for y and η. Finally, select a distance c_h such that

$$\iint\limits_{K_h'} \omega_h(x, y; \xi, \eta)\, d\xi\, d\eta = 1.$$

Since
$$\iint\limits_{K_h'} \omega_h(x, y; \xi, \eta)\, d\xi\, d\eta = c_h \iint\limits_{K_h'} e^{\frac{r^2}{r^2-h^2}}\, d\xi\, d\eta$$

$$= c_h \int\limits_0^{2\pi} d\varphi \int\limits_0^h e^{\frac{r^2}{r^2-h^2}}\, r\, dr = 2\pi c_h \int\limits_0^h e^{\frac{r^2}{r^2-h^2}}\, r\, dr,$$

it follows that
$$c_h = \frac{1}{2\pi}\left(\int\limits_0^h e^{\frac{r^2}{r^2-h^2}}\, r\, dr\right)^{-1},$$

whence it is seen that the choice of c_h for a given h is independent of the location of the point $P(x, y)$ on the plane.

A function of two pairs of variables x, y and ξ, η with the properties defined is called the **averaging kernel**, an example of which is the function $\omega_h(x, y; \xi, \eta)$.

Let $\varphi(x, y)$ be an arbitrary function in $L_p(G)$. Extend this to the entire plane by setting $\varphi(x, y) = 0$ for $P(x, y) \bar{\in} G$. The function

$$\varphi_h(x, y) = \iint\limits_{K_h'} \omega_h(x, y; \xi, \eta)\, \varphi(\xi, \eta)\, d\xi\, d\eta$$

is called the **mean value** of $\varphi(x, y)$.

It is not difficult to verify that an integral defining the function $\varphi_h(x, y)$ is uniformly convergent in the entire plane. In fact, if R_δ is a circle with radius δ and center at every point of the plane, then, by setting $q = p/(p-1)$, we get

$$\left| \iint_{R_\delta} \omega_h(x, y\,;\, \xi, \eta)\, \varphi(\xi, \eta)\, d\xi\, d\eta \right|$$

$$\leqslant \left(\iint_{R_\delta} |\varphi(\xi, \eta)|^p\, d\xi\, d\eta \right)^{1/p} \left(\iint_{R_\delta} \omega_h(x, y\,;\, \xi, \eta)^q\, d\xi\, d\eta \right)^{1/q}$$

$$\leqslant \left(\iint_{G} |\varphi(\xi, \eta)|^p\, d\xi\, d\eta \right)^{1/p} \left(\iint_{R_\delta} \omega_h(x, y\,;\, \xi, \eta)^q\, d\xi\, d\eta \right)^{1/q}$$

$$= \| \varphi \|_{L_p} \left(\iint_{R_\delta} \omega_h(x, y\,;\, \xi, \eta)^q\, d\xi\, d\eta \right)^{1/q},$$

and the last integral, $\omega_h(x, y : \xi, \eta)$ being bounded, can be made arbitrarily small for sufficiently small δ. directly for all points $P(x, y)$ on the plane.

Analogously, it is possible to prove the uniform convergence of the integral

$$\iint_{K_h'} \frac{\partial^l \omega_h}{\partial x^{l_1}\, \partial y^{l_2}}\, \varphi(\xi, \eta)\, d\xi\, d\eta$$

for any l and $l_1 + l_2 = l$.

Thereupon, it follows that $\varphi_h(x, y)$ is an infinitely differentiable continuous function.

It may also be noted that if $\{\varphi_\alpha\,(x, y)\}$ belongs to a bounded set of $L_p(G)$, then. as shown in the previous estimate, the mean-value function

$$\varphi_\alpha\,(x, y)\,]_h = \iint_{K_h'} \omega_h(x, y;\, \xi, \eta)\, \varphi_\alpha\,(\xi, \eta)\, d\xi\, d\eta$$

is uniformly bounded and uniformly continuous.

The upcoming two lemmas relate to the mean-value functions.

LEMMA 1. *For any function $\varphi(x, y) \in L_p(G)$ and for any $h > 0$,*

$$\| \varphi_h \|_{L_p} \leqslant \| \varphi \|_{L_p}.$$

PROOF. Set $\varphi_h(x, y)$ in the form

$$\varphi_h\,(x, y) = \iint_{K_h'} \omega_h(x, y;\, \xi, \eta)^{1/p}\, \varphi(\xi, \eta)\, \omega_h\,(x, y;\, \xi, \eta)^{1/q}\, d\xi\, d\eta\,.$$

Apply HÖLDER's inequality to the integral, to receive $[q = p/(p-1)]$

$$|\varphi_h(x, y)| \leqslant \left(\iint_{K'_h} \omega_h(x, y; \xi, \eta) \, |\varphi(\xi, \eta)|^p \, d\xi \, d\eta \right)^{1/p}$$

$$\times \left(\iint_{K'_h} \omega_h(x, y; \xi, \eta) \, d\xi \, d\eta \right)^{1/q}$$

$$= \left(\iint_{K'_h} \omega_h(x, y; \xi, \eta) \, |\varphi(\xi, \eta)|^p \, d\xi \, d\eta \right)^{1/p},$$

since

$$\iint_{K'_h} \omega_h(x, y; \xi, \eta) \, d\xi \, d\eta$$

is unity. Raising this inequality to power p and integrating over G, we get

$$\iint_G |\varphi_h(x, y)|^p \, dx \, dy$$

$$\leqslant \iint_G \left\{ \iint_{K'_h} \omega_h(x, y; \xi, \eta) \, |\varphi(\xi, \eta)|^p \, d\xi \, d\eta \right\} dx \, dy.$$

Since the function $\omega_h(x, y; \xi, \eta)$ vanishes outside of K'_h and so does the function $\varphi(\xi, \eta)$ outside of G, the domain of integration of the interior integral can be regarded equal to G and then the order of integration can be changed by FUBINI's theorem. We have

$$\iint_G |\varphi_h(x, y)|^p \, dx \, dy$$

$$\leqslant \iint_G |\varphi(\xi, \eta)|^p \left\{ \iint_G \omega_h(x, y; \xi, \eta) \, dx \, dy \right\} d\xi \, d\eta$$

$$\leqslant \iint_G |\varphi(\xi, \eta)|^p \left\{ \iint_{K'_h} \omega_h(x, y; \xi, \eta) \, dx \, dy \right\} d\xi \, d\eta \, .$$

The interior integral again becomes equal to 1 and, consequently,

$$\iint_G |\varphi_h(x, y)|^p \, dx \, dy \leqslant \iint_G |\varphi(\xi, \eta)|^p \, d\xi \, d\eta,$$

whence the desired inequality is immediate.

REMARK. Let G^* be a subdomain of G. Then,

$$\iint_{G^*} |\varphi_h(x, y)|^p \, dx \, dy \leqslant \iint_{G^*} |\varphi(x, y)|^p \, dx \, dy + \alpha(h),$$

where $\alpha(h) \to 0$ as $h \to 0$, if the boundary of G^* is sufficiently smooth. In fact, as also under the proof of the lemma, we get

$$\iint_{G^*} \varphi_h(x, y) |^p \, dx \, dy$$

$$\leqslant \iint_{G^*} \left\{ \iint_{K_h'} \omega_h(x, y; \xi, \eta) \, |\varphi(\xi, \eta)|^p \, d\xi \, d\eta \right\} dx \, dy.$$

Let G_h^* be a collection of the points of domain $G \setminus G^*$ lying from the boundary of G^* at a distance not exceeding h. Then,

$$\iint_{G^*} |\varphi_h(x, y)|^p \, dx \, dy$$

$$\leqslant \iint_{G^*} \left\{ \iint_{G^* \cup G_h^*} \omega_h(x, y; \xi, \eta) \, |\varphi(\xi, \eta)|^p \, d\xi \, d\eta \right\} dx \, dy$$

$$\leqslant \iint_{G^* \cup G_h^*} |\varphi(\xi, \eta)|^p \left\{ \iint_{K_h} \omega_h(x, y; \xi, \eta) \, dx \, dy \right\} d\xi \, d\eta$$

$$= \iint_{G^* \cup G_h^*} |\varphi(\xi, \eta)|^p \, d\xi \, d\eta$$

$$= \iint_{G^*} |\varphi(\xi, \eta)|^p \, d\xi \, d\eta + \iint_{G_h^*} |\varphi(\xi, \eta)|^p \, d\xi \, d\eta.$$

Granting the boundary of G^* to be sufficiently smooth, it can be shown that mes $G^*h \to 0$ as $h \to 0$. Then, owing to the absolute continuity of the LEBESGUE integral, we have

$$\alpha(h) = \iint_{G_h^*} |\varphi(\xi, \eta)|^p \, d\xi \, d\eta \to 0 \quad \text{as } h \to 0 .$$

LEMMA 2. *For any function* $\varphi(x, y) \in L_p(G)$, $\| \varphi_h - \varphi \|_{L_p} \to 0$ as $h \to 0$.

PROOF. Again, let $\varphi(x, y)$ be continuous in G and, consequently, uniformly continuous in any of its closed subdomains. For any subdomain $G' \subset G$, we have

$$\iint_G |\varphi_h - \varphi|^p \, dx \, dy = \iint_{G'} |\varphi_h - \varphi|^p \, dx \, dy + \iint_{G \setminus G'} |\varphi_h - \varphi|^p \, dx \, dy.$$

By MINKOWSKI's inequality and taking note of the preceding remark, we get

$$\int\int_{G\setminus G'} |\varphi_h - \varphi|^p \,dx\,dy \leqslant \left\{\left(\int\int_{G\setminus G'} |\varphi_h|^p \,dx\,dy\right)^{1/p} + \left(\int\int_{G\setminus G'} |\varphi|^p \,dx\,dy\right)^{1/p}\right\}^p \leqslant 2^p \int\int_{G\setminus G'} |\varphi|^p \,dx\,dy + \gamma(h), \quad (2)$$

where $\gamma(h) \to 0$ as $h \to 0$.

Let $\varepsilon > 0$ be any given number. Choose again G', such that

$$2^p \int\int_{G\setminus G'} |\varphi(x, y)|^p \,dx\,dy < \frac{\varepsilon^p}{4}.$$

With G' fixed, take h_0 such that $\gamma(h) < \varepsilon^p/4$ for $h < h_0$. Then,

$$\int\int_{G\setminus G'} |\varphi(x, y)|^p \,dx\,dy < \frac{\varepsilon^p}{2}. \quad (3)$$

On the other hand, taking a third domain G'', $G' \subset G'' \subset G$, such that $\overline{G}' \subset G''$, $\overline{G}'' \subset G$ and assuming $h < h_0$ to be so small that $G' \cup G'_h$ does not occur as the limit of G'', we get

$$|\varphi_h(x, y) - \varphi(x, y)| = \left|\int\int_{K'_h} \omega_h(x, y; \xi, \eta)\, \varphi(\xi, \eta)\, d\xi\, d\eta - \int\int_{K'_h} \omega_h(x, y; \xi, \eta)\, \varphi(x, y)\, d\xi\, d\eta\right|$$

$$\leqslant \int\int_{K'_h} |\varphi(\xi, \eta) - \varphi(x, y)|\, \omega_h(x, y; \xi, \eta)\, d\xi\, d\eta$$

$$\leqslant \max_{K'_h} |\varphi(\xi, \eta) - \varphi(x, y)| < \frac{\varepsilon}{(2 \operatorname{mes} G)^{1/p}}$$

owing to the function $\varphi(x, y)$ being uniformly continuous in the domain $\overline{G}''$, if $h < h_0$ is sufficiently small. Thereupon,

$$\int\int_{G'} |\varphi_h(x, y) - \varphi(x, y)|^p \,dx\,dy < \frac{\varepsilon^p}{2}. \quad (4)$$

From (3) and (4) it follows that

$$\int\int_{G} |\varphi_h(x, y) - \varphi(x, y)| \,dx\,dy < \varepsilon^p.$$

However, since $\varepsilon > 0$ is arbitrary, the lemma is proved for the continuous function $\varphi(x, y)$.

Now, if $\varphi(x, y)$ is an arbitrary function in $L_p(G)$, a function ψ continuous in G, such that $\| \varphi-\psi \|_{L_p} < \varepsilon/3$, is again found. Thereupon,

$$\| \varphi-\varphi_h \|_{L_p} \leqslant \| \varphi-\psi \|_{L_p} + \| \psi-\psi_h \|_{L_p} + \| \psi_h-\varphi_h \|_{L_p}$$
$$\leqslant \| \psi-\psi_n \|_{L_p} + \frac{2\varepsilon}{3},$$

since by Lemma 1, we have also $\| \varphi_h-\psi_h \|_{L_p} < \varepsilon/3$.

Further, by what has already been proved, it is possible to take δ so small that $\| \psi-\psi_h \| < \varepsilon/3$ for $h < \delta$. Then, $\| \varphi-\varphi_h \|_{L_p} < \varepsilon$ for such h, and the lemma is completely proved.

We now proceed to show that both the defined generalized derivatives are equivalent. Let $\varphi_0^{(l_1, l_2)}(x, y)$ be a generalized derivative of $\varphi_0(x, y)$ in the sense of the first definition. Hence, there is a sequence $\{\varphi_n(x, y)\}$ of functions continuously differentiable to l-th order, such that $\| \varphi_n-\varphi \|_{L_p} \to 0$ and

$$\left\| \frac{\partial^l \varphi_n}{\partial x^{l_1} \partial y^{l_2}} - \varphi_0^{(l_1, l_2)}(x, y) \right\|_{L_p} \to 0 \quad \text{as} \quad n \to \infty.$$

Taking the limit in the equation

$$\iint\limits_G \varphi_n(x, y) \frac{\partial^l \psi}{\partial x^{l_1} \partial y^{l_2}} \, dx \, dy = (-1)^l \iint\limits_G \frac{\partial^l \varphi_n}{\partial x^{l_1} \partial y^{l_2}} \psi(x, y) \, dx \, dy,$$

$\psi(x, y)$ being any l-times continuously differentiable function vanishing on the boundary of G, we obtain

$$\iint\limits_G \varphi_0(x, y) \frac{\partial^l \psi}{\partial x^{l_1} \partial y^{l_2}} \, dx \, dy = (-1)^l \iint\limits_G \varphi_0^{(l_1, l_2)}(x, y) \psi(x, y) \, dx \, dy,\dagger$$

and $\varphi_0^{(l_1, l_2)}(x, y)$ is a generalized derivative of $\varphi_0(x, y)$ in the sense of the second definition.

Let $[\partial^l \varphi_0(x, y)]/\partial x^{l_1} \partial y^{l_2} = \chi(x, y)$ be a generalized derivative of $\varphi_0(x, y)$ in the sense of the second definition. Consider the mean value function $\varphi_{0,h}(x, y)$. We have

$$\frac{\partial^l \varphi_{0,h}(x, y)}{\partial x^{l_1} \partial y^{l_2}} = \iint\limits_{K'_h} \frac{\partial^l \omega_h(x, y; \xi, \eta)}{\partial x^{l_1} \partial y^{l_2}} \varphi_0(\xi, \eta) \, d\xi \, d\eta$$
$$= (-1)^l \iint\limits_{K'_h} \frac{\partial^l \omega_h(x, y; \xi, \eta)}{\partial \xi^{l_1} d\eta^{l_2}} \varphi_0(\xi, \eta) \, d\xi \, d\eta. \tag{5}$$

†From HÖLDER's inequality, it is immediate that

$$\iint\limits_G \alpha_n(x, y) \beta(x, y) \, dx \, dy \to \iint\limits_G \alpha_0(x, y) \beta(x, y) \, dx \, dy,$$

if $\| \alpha_n(x, y) - \alpha_0(x, y) \|_{L_p} \to 0$ and $\beta(x, y)$ is any bounded measurable function [or, if $\beta(x, y) \in L_q(G)$].

Let an arbitrary subdomain G' of G be fixed, such that $\bar{G}' \subset G$, and let h be so small that a circle of radius h with center at a point of G' remains interior to G. Then, $\omega_h(x, y; \xi, \eta)$ can be taken as a function $\psi(x, y)$ figuring in the generalized derivative of second definition, and Eq. (5) for the point $(x, y) \in G_h$ can be expressed in the form

$$\frac{\partial^l \varphi_{0,h}(x, y)}{\partial x^{l_1}\, dy^{l_2}} = \iint_{K'_h} \omega_h(x, y; \xi, \eta)\, \chi(\xi, \eta)\, d\xi\, d\eta. \tag{6}$$

By Lemma 2, Eq. (6) implies $\dfrac{\partial^l \varphi_{0,h}(x, y)}{\partial x^{l_1}\, \partial y^{l_2}} \to \chi(x, y)$ as $h \to 0$ for any G' lying strictly interior to G. The passage to domain G itself involves more complex reasonings [33]. Without squeeze of generality, it can be assumed that the origin lies inside of G. Denote by G_k a domain obtained from G by similar transformations in the origin $k / (k-1)$, $k = 2, 3, \ldots$

The formula for coordinate transformation is given by

$$x' = \frac{k}{k-1}\, x, \quad y' = \frac{k}{k-1}\, y,$$

and it is easy to see that to every function $f(x, y) \in L_p(G)$ there corresponds the function

$$f_k(x, y) = f\left(\frac{k-1}{k}\, x, \frac{k-1}{k}\, y\right) \in L_p(G_k),$$

and conversely. Let $\varphi(x, y) \in L_p(G)$ have an l-th order generalized derivative $\chi(x, y) \in L_p(G)$ in the sense of the second definition. Then, noting that

$$\frac{\partial^l \psi}{\partial x^{l_1}\, \partial y^{l_2}} = \left(\frac{k}{k-1}\right)^l \frac{\partial^l \psi_k(x', y')}{\partial x'^{l_1}\, \partial y'^{l_2}}$$

for any l-times continuously differentiable function $\psi(x, y)$, by the change of variables in the equation

$$\iint_G \varphi(x, y)\, \frac{\partial^l \psi(x, y)}{\partial x^{l_1}\, \partial y^{l_2}}\, dx\, dy = (-1)^l \iint_G \chi(x, y)\, \psi(x, y)\, dx\, dy,$$

we obtain

$$\left(\frac{k}{k-1}\right)^l \iint_{G_k} \varphi_k(x', y')\, \frac{\partial^l \psi_k(x', y')}{\partial x'^{l_1}\, \partial y'^{l_2}}\, dx'\, dy'$$
$$= (-1)^l \iint_{G_k} \chi_k(x', y')\, \psi_k(x', y')\, dx'\, dy',$$

implying that $\varphi_k(x, y)$ has the generalized derivative $\left(\dfrac{k-1}{k}\right)^l \chi_k(x, y)$ in the sense of the second definition.

Let us show that in G as $k \to \infty$ the function $\varphi_k(x, y)$ converges in mean to $\varphi(x, y)$, and the function $[\partial^l \varphi_k(x, y)]/\partial x^{l_1} \partial y^{l_2}$ to $\partial^l \varphi/\partial x^{l_1} \partial y^{l_2}$. In fact,

$$\int_G\int |\varphi(x, y) - \varphi_k(x, y)|^p \, dx\, dy$$

$$= \int_G\int \left|\varphi(x, y) - \varphi\left(x - \frac{x}{k}, y - \frac{y}{k}\right)\right|^p dx\, dy,$$

and the convergence to zero of the integral on the right side of equation is nothing else than the continuity in mean of the function $\varphi(x, y) \in L_p(G)$. Furthermore,

$$\left(\int_G\int \left|\frac{\partial^l \varphi(x, y)}{\partial x^{l_1} \partial y^{l_2}} - \frac{\partial^l \varphi_k(x, y)}{\partial x^{l_1} \partial y^{l_2}}\right|^p dx\, dy\right)^{1/p}$$

$$= \left(\int_G\int \left|\chi(x, y) - \left(\frac{k-1}{k}\right)^l \chi_k(x, y)\right|^p dx\, dy\right)^{1/p}$$

$$\leqslant \left[1 - \left(\frac{k-1}{k}\right)^l\right] \left(\int_G\int \left|\chi\left(x - \frac{x}{k}, y - \frac{y}{k}\right)\right|^p dx\, dy\right)^{1/p}$$

$$+ \left(\int_G\int \left|\chi(x, y) - \chi\left(x - \frac{x}{k}, y - \frac{y}{k}\right)\right|^p dx\, dy\right)^{1/p}.$$

The second member on the right again vanishes in virtue of the mean continuity of the function $\chi(x, y)$†. Concerning the first member, then, the factor $\left[1 - \left(\frac{k-1}{k}\right)^l\right] \to 0$ and the integrals appearing there are totally bounded, being norms of a sequence of functions $\{\chi_k(x, y)\}$, convergent in mean.

Since $G \subset G_k$ for every fixed k, it follows by what has been proved in the foregoing, that

$$\varphi_{k,h}(x, y) \to \varphi_k(x, y), \quad \frac{\partial^l \varphi_{k,h}(x, y)}{\partial x^{l_1} \partial y^{l_2}} \to \frac{\partial^l \varphi_k(x, y)}{\partial x^{l_1} \partial y^{l_2}}$$

in G as $h \to 0$. On the other hand, as just shown,

$$\varphi_k(x, y) \to \varphi(x, y), \quad \frac{\partial^l \varphi_k(x, y)}{\partial x^{l_1} \partial y^{l_2}} \to \chi(x, y)$$

in G as $k \to \infty$. Thereupon, it is easy to see that there is a sequence

†See Appendix I.

$\{\varphi_{k}, {}_{h_k}(x, y)\}$ of l-times continuously differentiable functions convergent in mean in the domain G to $\varphi(x, y)$, whose l-th order derivatives converge to $\chi(x, y)$, that is $\chi(x, y)$ is a generalized derivative in the sense of the first definition.

From the second definition of the generalized derivatives we can obtain the the following conclusions:

(*a*) *If* $\psi(x, y) = \dfrac{\partial^l \varphi(x, y)}{\partial x^{l_1}\, dy^{l_2}}$ *and* $\chi(x, y) = \dfrac{\partial^k \psi(x, y)}{\partial x^{k_1}\, \partial y^{k_2}}$,

then $\chi(x, y) = \dfrac{\partial^{l+k} \varphi(x, y)}{\partial x^{l_1+k_1}\, \partial y^{l_2+k_2}}$;

(*b*) *The generalized derivative does not depend on the order of differentiation*;

(*c*) *The operation of generalized differentiation is a distributive operation.*

It can also be proved that the *Liebniz' formula for differentiation of the product is valid for generalized derivatives.*

2.52. Sobolev's formula. The existence of generalized derivatives does not follow from the existence of derivatives a.e. in the ordinary sense. This is shown by the upcoming example due to SOBOLEV.

Given $\varphi(x)$ on [0, 1] and assume this to have a generalized derivative $\chi(x)$ there. Then, for any function $\psi(x)$, which is continuously differentiable and vanishes together with its derivatives on a finite segment, we have

$$\int_a^b \varphi(x)\, \psi'(x)\, dx = -\int_a^b \chi(x)\, \psi(x)\, dx.$$

Let $\omega(x) = \int_a^x \chi(\xi)\, d\xi$. Then, evidently,

$$-\int_a^b \chi(x)\, \psi(x)\, dx = \int_a^b \omega(x)\, \psi'(x)\, dx,$$

whence $$\int_a^b [\varphi(x) - \omega(x)]\, \psi'(x)\, dx = 0.$$

Since $\psi(x)$ is an arbitrary continuously differentiable function, vanishing on a finite segment, the last equation implies $\varphi(x) = \omega(x) + c$, and because $\omega(x)$ is an indefinite integral of a symmetric function, $\omega(x)$ is absolutely continuous. Now, in order to achieve the desired example, it is sufficient to take any function not absolutely continuous and having derivatives a.e.

It is easier to reduce examples of functions having generalized derivatives of higher orders rather than of lower orders.

Let $F(x, y) = f(x) + f(y)$, where $f(x)$ does not have a generalized derivative. Then, evidently, $F(x, y)$ has a generalized derivative of second order and not of first order. In fact, for any function $\psi(x, y)$ with the requisite properties, we have

$$\iint_G F(x, y) \frac{\partial^2 \psi}{\partial x\, \partial y}\, dx\, dy$$

$$= \iint_G f(x) \frac{\partial^2 \psi}{\partial x\, \partial y}\, dx\, dy + \iint_G f(y) \frac{\partial^2 \psi}{\partial x\, \partial y}\, dx\, dy.$$

However,

$$\iint_G f(x) \frac{\partial^2 \psi}{\partial x\, \partial y}\, dx\, dy$$

$$= \int_a^b f(x) \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial^2 \psi}{\partial x\, \partial y}\, dy\, dx = \int_a^b f(x) \left[\frac{\partial \psi}{\partial x}\right]_{\varphi_1(x)}^{\varphi_2(x)} dx = 0,$$

since $\psi'_x\,[x, \varphi_1(x)] = \psi'_x\,[x, \varphi_2(x)] = 0$, where $\varphi_1(x)$ and $\varphi_2(x)$ are boundary values of the ordinate domain G. Analogously,

$$\iint_G f(y) \frac{\partial^2 \psi}{\partial x\, \partial y}\, dx\, dy = 0,$$

and hence

$$\iint_G F(x, y) \frac{\partial^2 \psi}{\partial x\, \partial y}\, dx\, dy = 0 = \iint_G 0 \cdot \psi(x, y)\, dx\, dy,$$

that is, $\partial^2 F/\partial x \partial y$ exists and is identically zero.

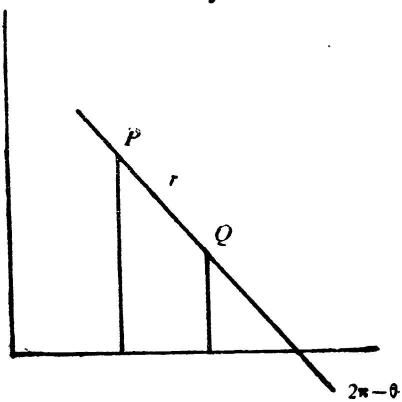

Fig. 2.

The following fact has a significantly greater depth: If a function $f(x, y) \in L_p(G)$ has all the l-th order generalized derivatives, then this has also the generalized derivatives of $(l-1)$-th order. To make this fact explicit, certain preliminary reasonings are presented.

Let $u(x, y)$ together with the derivatives of order up to and including l be continuous in a domain $\bar{G}$. We make an appeal to **Sobolev's integral formula** which expresses $u(x, y)$ in terms of an l-th order derivative of this function. Consider on a plane two points, $P(x, y)$ and $Q(\xi, \eta)$ (Fig. 2). Let r denote the distance between these points and θ the angle formed by the radius vector induced from the point P to the point Q with the positive x-axis. Evidently,

$$\xi = x + r \cos \theta, \qquad \eta = y + r \sin \theta.$$

Hence $\quad u(\xi, \eta) = u(x + r \cos \theta, \; y + r \sin \theta) = v(x, y, r, \theta),$

or, for brevity, $\quad u(Q) = v(P, r, \theta).$

Explicitly, $\quad v(P, 0, 0) = u(P).$

Choose any arbitrary interior point of the domain G as the origin of a system of Cartesian coordinates, and let K_R be a circle of some radius R with center at this point, completely contained within G. Introduce a function

$$\omega_R(Q) = \begin{cases} ce^{-\frac{R^2}{R^2-r^2}}, & \text{if } r < R, \\ 0, & \text{if } r \geqslant R. \end{cases} \qquad (r^2 = \xi^2 + \eta^2).$$

The constant c is so chosen, that

$$\iint_{K_R} \omega_R(Q)\, d\xi\, d\eta = 1.$$

Note that $\omega_R(Q)$ is an infinitely differentiable function.

Consider the integral

$$\iint_{K_R} u(Q)\, \omega_R(Q)\, dQ,$$

and transform it by means of some integration by parts. Let $P(x, y)$ be another arbitrary point of G. In the integral, replace $u(Q)$ by $v(P, r, \theta)$, $\omega_R(Q)$ by $\chi_R(P, r, \theta)$ and change over to polar coordinates with center at the point P and the polar axis directed along x. We obtain

$$\iint_{K_R} \omega_R(Q)\, u(Q)\, dQ = \iint_G \omega_R(Q)\, u(Q)\, dQ$$

$$= \iint_G \chi_R(P, r, \theta)\, v(P, r, \theta)\, d\xi\, d\eta = \int_0^{2\pi} d\theta \int_0^{\infty} v(P, r, \theta)\, \chi_R(P, r, \theta)\, r\, dr.$$

Note that our all integrals are proper. Write

$$-\int_r^{\infty} \rho \chi_R(P, \rho, \theta)\, d\rho$$

in terms of $\varkappa(r)$. The function $\varkappa(r)$ is an anti-derivative of $r\chi_R(P, r, \theta)$ and by integrating the interior integral by parts, we obtain

$$\iint_{K_R} \omega_R(Q)\, u(Q)\, dQ = \int_0^{2\pi} \left\{ v(P, r, \theta)\, \varkappa(r) \Big|_0^{\infty} - \int_0^{\infty} \frac{\partial v(P, r, \theta)}{\partial r} \varkappa(r)\, dr \right\} d\theta$$

$$= \int_0^{2\pi} \left\{ v(P, 0, \theta) \int_0^{\infty} \rho\chi_\rho(P, \rho, \theta)\, d\rho \right\} d\theta$$

$$+ \int_0^{2\pi} \left\{ \int_0^{\infty} \frac{\partial v(P, r, \theta)}{\partial r} \left[\int_r^{\infty} \rho\chi_R(P, \rho, \theta)\, d\rho \right] dr \right\} d\theta.$$

However, $v(P, 0, \theta) = u(P)$,

$$\int_0^{2\pi} \int_0^{\infty} \rho\chi_R(P, \rho, \theta)\, d\rho\, d\theta = \iint_G \omega_R(Q)\, dQ = 1,$$

yielding $u(P) = \iint_G u(Q)\, \omega_R(Q)\, dQ$

$$- \int_0^{2\pi} \int_0^{\infty} \frac{\partial v(P, r, \theta)}{\partial r} \frac{1}{r} \left[\int_r^{\infty} \rho\chi_R(P, \rho, \theta)\, d\rho \right] r\, dr\, d\theta,$$

or, $$u(P) = \iint_G u(Q)\, \omega_R(Q)\, dQ$$

$$- \iint_G \frac{\partial u(Q)}{\partial r} \frac{1}{r} \left[\int_r^{\infty} \rho\chi_R(P, \rho, \theta)\, d\rho \right] dQ.$$

For brevity, set

$$- \int_r^{\infty} \rho\chi_R(P, \rho, \theta)\, d\rho = C(P, Q),$$

to receive

$$u(P) = \iint_G u(Q)\, \omega_R(Q)\, dQ + \iint_G \frac{\partial u}{\partial r} \frac{1}{r} C(P, Q)\, dQ.$$

Evidently, the function $C(P, Q)$ is bounded for $P, Q \in G$ and, as is not difficult to see, this is continuous for $P \neq Q$ but has, as $P \to Q$, a distinct limit depending on the magnitude of the angle θ. Since

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cos(r, x) + \frac{\partial u}{\partial y} \cos(r, y),$$

the preceding formula assumes the form

$$u(P) = \iint_G u(Q)\, \omega_R(Q)\, dQ$$
$$+ \iint_G \left\{ A_{10}^{(1)}(P, Q) \frac{\partial u}{\partial x} + A_{01}^{(1)}(P, Q) \frac{\partial u}{\partial y} \right\} dQ. \tag{7}$$

Here $A_{i_1 i_2}^{(1)}(P, Q)$, $(i_1, i_2 = 0, 1, \ldots)$, takes the form

$$A_{i_1 i_2}^{(1)}(P, Q) = \frac{1}{r} B_{i_1 i_2}^{(1)}(P, Q),$$

$B_{i_1 i_2}^{(1)}(P, Q)$ being bounded functions.

Let us apply the formula derived in place of the function $u(P)$ to its partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$. Then,

$$\frac{\partial u}{\partial x} = \iint_G \frac{\partial u}{\partial x} \omega_R(Q)\, dQ$$
$$+ \iint_G \left\{ A_{10}^{(1)}(P, Q) \frac{\partial^2 u}{\partial x^2} + A_{01}^{(1)}(P, Q) \frac{\partial^2 u}{\partial x\, \partial y} \right\} dQ, \tag{8}$$

$$\frac{\partial u}{\partial y} = \iint_G \frac{\partial u}{\partial y} \omega_R(Q)\, dQ$$
$$+ \iint_G \left\{ A_{10}^{(1)}(P, Q) \frac{\partial^2 u}{\partial x\, \partial y} + A_{01}^{(1)}(P, Q) \frac{\partial^2 u}{\partial y^2} \right\} dQ. \tag{9}$$

Substituting (8) and (9) in Eq. (7), changing the order of integration and introducing the notations

$$\iint_G A_{l_1 l_2}^{(1)}(P, Q)\, dQ = C_{l_1 l_2}^{(1)}(P), \tag{10}$$

$$\iint_G A_{l_1 l_2}^{(1)}(P, S)\, A_{l_1' l_2'}^{(1)}(S, Q)\, dS = A_{l_1+l_1',\, l_2+l_2'}^{(2)}(P, Q), \tag{11}$$

we get
$$u(P) = \iint_G u(Q)\, \omega_R(Q)\, dQ$$
$$+ \sum_{l_1+l_2=1} C_{l_1 l_2}^{(1)}(P) \iint_G \frac{\partial u}{\partial x^{l_1}\, \partial y^{l_2}} \omega_R(Q)\, dQ$$
$$+ \sum_{l_1+l_2=2} \iint_G A_{l_1 l_2}^{(2)}(P, Q) \frac{\partial^2 u}{\partial x^{l_1}\, \partial y^{l_2}}\, dQ.$$

Here $\sum\limits_{l_1+l_2=k}$ denotes summation over all the values of indices l_1 and l_2 from 0 to k such that $l_1 + l_2 = k$.

Continuing in this manner, we arrive at the formula

$$u(P) = \iint_G u(Q)\,\omega_R(Q)\,dQ + \sum_{l_1+l_2=1} C^{(1)}_{l_1 l_2}(P) \iint_G \frac{\partial u}{\partial x^{l_1}\,\partial y^{l_2}}\,\omega_R(Q)\,dQ$$

$$+ \sum_{l_1+l_2=2} C^{(2)}_{l_1 l_2}(P) \iint_G \frac{\partial u}{\partial x^{l_1}\,\partial y^{l_2}}\,\omega_R(Q)\,dQ + \ldots$$

$$\ldots + \sum_{l_1+l_2=l-1} C^{(l-1)}_{l_1 l_2}(P) \iint_G \frac{\partial^{l-1} u}{\partial x^{l_1}\,\partial y^{l_2}}\,\omega_R(Q)\,dQ$$

$$+ \sum_{l_1+l_2=l} \iint_G A^{(l)}_{l_1 l_2}(P, Q) \frac{\partial^l u}{\partial x^{l_1}\,\partial y^{l_2}}\,dQ,$$

and the formulae for $C^{(k)}_{l_1 l_2}(P)$ and $A^{(k)}_{l_1 l_2}(P, Q)$, analogous to (10) and (11).

From (10) and (11) it is possible to verify that $A^{(2)}_{l_1 l_2}(P, Q)$ satisfies the inequality

$$|A^{(2)}_{l_1 l_2}(P, Q)| \leqslant \alpha \ln r + \beta,$$

where α and β are constants and $A^{(k)}_{l_1 l_2}(P, Q)$ is bounded for $k > 2$ and continuous for $P \neq Q$ (for example, see [22]).

The function $C^{(k)}_{l_1 l_2}(P)$ is continuous in the domain $\bar{G}$.

Since $\omega_R(Q)$ is a continuously infinitely differentiable function and vanishes on the boundary of G, hence applying GREEN's formula we can write

$$\iint_G \frac{\partial^l u}{\partial x^{l_1}\,\partial y^{l_2}}\,\omega_R(Q)\,dQ = \iint_G u(Q) \frac{\partial^l \omega_R}{\partial x^{l_1}\,\partial y^{l_2}}\,dQ,$$

finally getting

$$u(P) = \iint_G u(Q)\,\omega_R(Q)\,dQ + \sum_{l_1+l_2=1} C^{(1)}_{l_1 l_2}(P) \iint_G u(Q) \frac{\partial \omega_R}{\partial x^{l_1}\,\partial y^{l_2}}\,dQ$$

$$+ \sum_{l_1+l_2=2} C^{(2)}_{l_1 l_2}(P) \iint_G u(Q) \frac{\partial^2 \omega_R}{\partial x^{l_1}\,\partial y^{l_2}}\,dQ + \ldots$$

$$\ldots + \sum_{l_1+l_2=l-1} C^{(l-1)}_{l_1 l_2}(P) \iint_G u(P) \frac{\partial^{l-1} \omega_R}{\partial x^{l_1}\,\partial y^{l_2}}\,dQ$$

$$+ \sum_{l_1+l_2=l} \iint_G A^{(l)}_{l_1 l_2}(P, Q) \frac{\partial^l u}{\partial x^{l_1}\,\partial y^{l_2}}\,dQ.$$

SOBOLEV [34] obtained this formula in a somewhat different form and for a more general case.

We extend this formula to functions having generalized but not ordinary derivatives, via the upcoming Lemma 3.

LEMMA 3. *Let $A(P, Q)$ be a function of the form*

$$\text{(i)} \qquad A(P, Q) = \frac{B(P, Q)}{r},$$

or,

$$\text{(ii)} \qquad A(P, Q) = B(P, Q)\,(\alpha \ln r + \beta),$$

$B(P, Q)$ a bounded measurable function. Let

$$v(P) = \iint_G A(P, Q)\, u(Q)\, dQ.$$

Then, the mean convergence

$$\| u_n(P) - u(P) \|_{L_p} \to 0$$

implies the mean convergence

$$\| v_n(P) - v(P) \|_{L_p} \to 0.$$

PROOF. In fact, for the case (i) we have

$$| v_n(P) - v(P) | \leqslant M \iint_G | u_n(Q) - u(Q) |\, r^{-1}\, dQ$$

$$= M \iint_G | u_n(Q) - u(Q) |\, r^{-\frac{1}{p}} r^{-\frac{1}{q}}\, dQ$$

$$\leqslant M \left(\iint_G | u_n(Q) - u(Q) |^p\, r^{-1}\, dQ \right)^{1/p} \left(\iint_G r^{-1}\, dQ \right)^{1/q}.$$

Here $M = \sup\limits_G | B(P, Q) |$. If polar coordinates are introduced with poles at the point P, then

$$\left(\iint_G r^{-1}\, dQ \right)^{1/q} \leqslant (2\pi d)^{1/q},$$

d the diameter of G. Thus,

$$| v_n (P) - v (P) | \leqslant M\, (2\pi d)^{1/q} \left(\iint_G | u_n (Q) - u (Q) |^p\, r^{-1}\, dQ \right)^{1/p}.$$

Thence,

$$\iint_G |v_n (P) - v (P) |^p\, dP$$

$$\leqslant M\, (2\pi d)^{p/q} \iint_G \left\{ \iint_G | u_n (Q) - u (Q) |^p\, r^{-1}\, dQ \right\} dP$$

$$= M\, (2\pi d)^{p/q} \iint_G \left\{ | n_n (Q) - u (Q) |^p \cdot \iint_G r^{-1}\, dP \right\} dQ,$$

and again
$$\iint_G r^{-1}\, dP < 2\pi d,$$

if we introduce the polar coordinates, but we have right now poles at the point Q. Thus,

$$\iint_G |v_n(P) - v(P)|^p\, dP \leqslant M(2\pi d)^{\frac{p}{q}+1} \iint_G |u_n(Q) - u(Q)|^p\, dQ, \tag{12}$$

and the lemma is proved. Analogously, the proof for the function $A(P, Q)$ in the form (ii) is derived.

Now, let the function $\varphi(P)$ have all the l-th order generalized derivatives. It is sought to find a sequence of functions $\{\varphi_m(P)\}$, continuously differentiable l number of times, such that $\varphi_m(P)$ converges in the mean to $\varphi(P)$, and $\frac{\partial^l \varphi_m}{\partial x^{l_1} \partial y^{l_2}}$ to $\frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}}$ for all $l_1, l_2 = 0, 1, \ldots, l$, $l_1 + l_2 = l$. By SOBOLEV's formula,

$$\varphi_m(P) = \sum_{k=0}^{l-1} \sum_{l_1+l_2=k} C_{l_1 l_2}^{(k)}(P) \iint_G \varphi_m(Q) \frac{\partial^k \omega_R}{\partial x^{l_1} \partial y^{l_2}}\, dQ + \sum_{l_1+l_2=l} \iint_G A_{l_1 l_2}^{(l)}(P, Q) \frac{\partial^l \varphi_m}{\partial x^{l_1} \partial y^{l_2}}\, dQ\,.$$

By the lemma, it is possible to take the limit under the sign of the integrals appearing in this formula, so that

$$\varphi(x, y) = \sum_{k=0}^{l-1} \sum_{l_1+l_2=k} C_{l_1 l_2}^{(k)}(P) \iint_G \varphi(Q) \frac{\partial^k \omega_R}{\partial x^{l_1} \partial y^{l_2}}\, dQ + \sum_{l_1+l_2=l} \iint_G A_{l_1 l_2}^{(l)}(P, Q) \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}}\, dQ. \tag{13}$$

Thus, SOBOLEV's formula has been extended to the functions belonging to the space $W_p^{(l)}$.

2.53. Inclusion theorem. THEOREM (S. L. SOBOLEV). *The space $W_p^{(k)}$ for every $k < l$ is imbedded in the space $W_p^{(l)}$, that is, every function $\varphi(x, y)$, having all l-th order generalized derivatives, has also for $k < l$ all k-th order generalized derivatives, and furthermore,*

$$\|\varphi\|_{W_p^{(k)}} \leqslant C_k \|\varphi\|_{W_p^{(l)}}.$$

PROOF. Let $\varphi(x, y) \in W_p^{(l)}$ and let $\{\varphi_n(x, y)\}$ be a sequence of functions, continuously differentiable l number of times, such that

$$\varphi_n(x, y) \to \varphi(x, y), \qquad \frac{\partial^l \varphi_n}{\partial x^{l_1} \partial y^{l_2}} \to \varphi^{(l_1, l_2)}(x, y)$$

with respect to the metric space $L_p(G)$. Applying SOBOLEV's integral formula to $\partial^{l-1}\varphi_n/\partial x^{k_1} \partial y^{k_2}$, we get

$$\frac{\partial^{l-1}\varphi_n}{\partial x^{k_1} \partial y^{k_2}} = \iint_G \omega_R(Q) \frac{\partial^{l-1}\varphi_n}{\partial x^{k_1} \partial y^{k_2}} dQ + \sum_{l_1+l_2=1} \iint_G A_{l_1 l_2}^{(1)}(P, Q) \frac{\partial^l \varphi_n}{\partial x^{k_1+l_1} \partial y^{k_2+l_2}} dQ.$$

Transforming the first integral by GREEN's formula, we obtain

$$\frac{\partial^{l-1}\varphi_n}{\partial x^{k_1} \partial y^{k_2}} = (-1)^{l-1} \iint_G \varphi_n(Q) \frac{\partial^{l-1}\omega_R}{\partial x^{k_1} \partial y^{k_2}} dQ + \sum_{l_1+l_2=1} \iint_G A_{l_1 l_2}^{(1)}(P, Q) \frac{\partial^l \varphi_n}{\partial x^{k_1+l_1} \partial y^{k_2+l_2}}. \tag{14}$$

By Lemma 2, the right-hand side of Eq. (14) tends to some limit

$$\chi(P) = \iint_G \varphi(Q) \frac{\partial^{l-1}\omega_R}{\partial x^{k_1} \partial y^{k_2}} dQ + \sum_{l_1+l_2=1} \iint_G A_{l_1 l_2}^{(1)}(P, Q)\, \varphi^{(k_1+l_1,\, k_2+l_2)}(Q)\, dQ. \tag{15}$$

However, this means that $\varphi_n(x, y) \xrightarrow{L_p(G)} \varphi(x, y)$ and $\partial^{(l-1)} \varphi_n/\partial x^{l_1} \partial y^{l_2} \xrightarrow{L_p(G)} \chi(x, y)$, that is, $\chi(x, y)$ is an $(l-1)$-th order generalized derivative of $\varphi(x, y)$, proving the existence of an $(l-1)$-th order generalized derivative. This, in turn, implies the existence of generalized derivatives of $(l-2)$-th order and so on, so that the first part of the theorem is proved.

Making use of (15), expressing the $(l-1)$-th order generalized derivative in terms of the l-th generalized derivative and the function itself, leads to

$$\frac{\partial^{l-1}\varphi}{\partial x^{k_1} \partial y^{k_2}} = (-1)^{l-1} \iint_G \varphi(Q) \frac{\partial^{l-1}\omega_R}{\partial x^{k_1} \partial y^{k_2}} dQ + \sum_{l_1+l_2=1} \iint_G A_{l_1 l_2}^{(1)}(P, Q) \frac{\partial^l \varphi}{\partial x^{k_1+l_1} \partial y^{k_2+l_2}} dQ.$$

Taking advantage of the obvious inequality

$$(a_1 + a_2 + \ldots + a_\nu)^p \leqslant \nu^p(a_1^p + a_2^p + \ldots + a_\nu^p), \quad a_i > 0,$$

we obtain

$$\left|\frac{\partial^{l-1}\varphi}{\partial x^{k_1}\,\partial y^{k_2}}\right|^p \leqslant 3^p\left\{\left(\iint_G |\varphi(Q)|\left|\frac{\partial^{l-1}\omega_R}{\partial x^{k_1}\,\partial y^{k_2}}\right| dQ\right)^p + \sum_{l_1+l_2=1}\left(\iint_G |A_{l_1 l_2}^{(l)}(P,Q)|\left|\frac{\partial^l\varphi}{\partial x^{k_1+l_1}\partial y^{k_2+l_2}}\right. dQ\right)^p\right\}.$$

By HÖLDER's inequality,

$$\left(\iint_G |\varphi(Q)|\left|\frac{\partial^{l-1}\omega_R}{\partial x^{k_1}\,\partial y^{k_2}}\right| dQ\right)^p \leqslant a\iint_G |\varphi(Q)|^p\,dQ,$$

a some constant. Reiterating the reasonings of earlier lemmas, we obtain the estimate

$$\iint_G\left(\iint_G |A_{l_1 l_2}^{(1)}(P,Q)|\left|\frac{\partial^l\varphi}{\partial x^{k_1+l_1}\partial y^{k_2+l_2}}\right| dQ\right)^p dP \leqslant a_{l_1 l_2}^{(l)}\iint_G\left|\frac{\partial^l\varphi}{\partial x^{k_1+l_1}\partial y^{k_2+l_2}}\right|^p dQ,$$

$a_{l_1 l_2}^{(l)}$ also some constants. These inequalities yield

$$\iint_G\left|\frac{\partial^{l-1}\varphi}{\partial x^{k_1}\,\partial y^{k_2}}\right|^p dP$$

$$\leqslant b_{k_1k_2}^{(l-1)}\left\{\iint_G |\varphi(Q)|^p\,dQ + \sum_{l_1+l_2=l}\iint_G\left|\frac{\partial^l\varphi}{\partial x^{k_1+l_1}\partial y^{k_2+l_2}}\right|^p dQ\right\}$$

$$\leqslant b_{k_1k_2}^{(l-1)}\left\{\iint_G |\varphi(Q)|^p\,dQ + \sum_{l_1+l_2=1}\iint_G\left|\frac{\partial^l\varphi}{\partial x^{l_1}\,\partial y^{l_2}}\right|^p dQ\right\}.$$

Thereupon by summing this inequality over all $(l-1)$-th order derivatives, we get .

$$\iint_G |\varphi(Q)|^p\,dQ + \sum_{k_1+k_2=l-1}\iint_G\left|\frac{\partial^{l-1}\varphi}{\partial x^{k_1}\,\partial y^{k_2}}\right|^p dQ \leqslant A^p\left\{\iint_G |\varphi(Q)|^p\,dQ + \sum_{l_1+l_2=l}\iint_G\left|\frac{\partial^l\varphi}{\partial x^{l_1}\,\partial y^{l_2}}\right|^p dQ\right\}$$

for suitably chosen constants.

We thereby arrive at the desired inequality

$$\|\varphi\|_{W_p^{(l-1)}} \leqslant A\,\|\varphi\|_{W_p^{(l)}}\,.$$

The theorem is completely proved

For a more elaborate formulation of SOBOLEV's inclusion theorem, see [34].

CHAPTER 3

LINEAR OPERATORS

3.1 LINEAR OPERATORS

LINEAR OPERATORS defined in a linear space form an important and most extensively studied class of operators.

3.11. Definitions. Given two linear topological spaces E_x and E_y (real or complex) and an operator $y = A(x)$ defined on E_x with range in E_y. For convenience it is customary to write also $y = Ax$. The operator $y = Ax$ is called **linear**, if:

(i) It is *additive*, that is,

$$A(x_1+x_2) = Ax_1 + Ax_2, \tag{1}$$

for all x_1 and x_2 in E_x;

(ii) A is *homogeneous*, that is,

$$A(\lambda x) = \lambda A x,$$

for all $x \in E_x$ and every real (complex) λ whenever E_x is real (complex).

It is evident that the continuity of A in the case of a metric space means that there is a $\delta > 0$ for every $\varepsilon > 0$, such that the collection of images of elements in the sphere $S(x, \delta)$ lies in $S(Ax, \varepsilon)$.

In the sequel, the set of all linear continuous operators mapping E_x into E_y shall be denoted by $(E_x \to E_y)$.

Examples. 1. Consider a square matrix (a_{ik}) of order n, $(i, k = 1, \dots, n)$. The equations

$$\eta_i = \sum_{k=1}^{n} a_{ik}\,\xi_k, \qquad i = 1, \dots, n,$$

evidently, define a certain operator $y = Ax$ which transforms an element $x = (\xi_1, \xi_2, \dots, \xi_n)$ of the n-dimensional Euclidean space E_n into an element $y = \{\eta_1, \eta_2, \dots, \eta_n\}$ of the same space. A is a continuous linear operator.

In fact, the additivity of A follows from the equality

$$\sum_{k=1}^{n} a_{ik}(\xi_k^{(1)} + \xi_k^{(2)}) = \sum_{k=1}^{n} a_{ik}\,\xi_k^{(1)} + \sum_{k=1}^{n} a_{ik}\,\xi_k^{(2)}, \qquad i = 1, \dots, n,$$

which is equivalent to $A(x_1+x_2) = Ax_1 + Ax_2$, it being homogeneous is obvious and its continuity follows from

$$\sqrt{\sum_{i=1}^{n} (\eta_i^{(m)} - \eta_i)^2} \leqslant \sqrt{\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2}\ \sqrt{\sum_{k=1}^{n} (\xi_k^{(m)} - \xi_k)^2},$$

where $x_m = \{\xi_k^{(m)}\}, \qquad y_m = Ax_m = \{\eta_i^{(m)}\}, \qquad x_m \to x,$

which is evidently obtained by means of the CBS inequality for sums.

2. Set

$$y(t) = \int_0^1 K(t, s)\, x(s)\, ds,$$

where $K(t, s)$ is a function continuous on the square $0 \leqslant t, s \leqslant 1$. If $x(t) \in C[0, 1]$, then, evidently, also, $y(t) \in C[0, 1]$. Consequently, the operator $y = Ax$ maps the space $C[0, 1]$ into itself. As is easily seen, A is a linear continuous operator. In fact,

$$\text{(i)} \quad A(x_1+x_2) = \int_0^1 K(t, s)\, [x_1(s)+x_2(s)]\, ds$$

$$= \int_0^1 K(t, s)\, x_1(s)\, ds + \int_0^1 K(t, s)\, x_2(s)\, ds = Ax_1 + Ax_2,$$

and the additivity condition is satisfied.

(ii) The homogeneity of the operator A is obvious.

(iii) Let $\{x_n(t)\}$ coverge to $x(t)$ in the sense of convergence in $C[0, 1]$, that is, uniformly on $[0, 1]$. Since in the case of uniform convergence we can take the limit under the integral sign, it follows that

$$\lim_n \int_0^1 K(t, s)\, x_n(s)\, ds = \int_0^1 K(t, s)\, x(s)\, ds,$$

that is, $\lim_n Ax_n = Ax$, and the continuity of A is also proved.

3. Let E_x be the space of functions defined and continuous on a closed plane curve Γ with continuous curvature, endowed with a metric defined by $\rho(x_1, x_2) = \max_\Gamma |x_1(t) - x_2(t)|$. Further, let E_u be a space of functions of two variables, defined and continuous in a closed domain G bounded by Γ, with a metric defined by $\rho(u_1, u_2) = \max_G |u_1(\xi, \eta) - u_2(\xi, \eta)|$. To every function $x(t) \in E_x$ we assign that function $u(\xi, \eta) \in E_u$ which is a solution of the DIRICHLET problem for G with the boundary condition $x(t)$. As is known, this problem is solvable uniquely under the assumptions made here. This correspondence defines a certain operator $u = Ax$. On the basis of the well-known properties of harmonic functions, A is a linear continuous operator defined on E_x with range in E_u.

4. Let $E = C[0, 1]$. In this space, consider the operator $y = Ax$ defined by

$$y(t) = \int_0^t x(\tau)\, d\tau.$$

Obviously, A is a linear continuous operator defined on the entire E. Further, let another operator $y = Bx$ defined by

$$y(t) = \frac{d}{dt} x(t)$$

be considered in the same space. This operator is not defined for all $x(t) \in E$, and if Bx exists, y does not always belong to E. If, however, the domain of B is taken to be a linear manifold of functions having continuous derivatives lying everywhere dense in $C[0, 1]$), the range of B is also contained in $C[0, 1]$.

The operator B is, evidently, additive and homogeneous. However, this is not continuous in the domain of definition, because the derivative of a limit element of a uniformly convergent sequence of functions need not be equal to the limit of the derivatives of these functions, even if all these derivatives exist and are continuous.

3.12. Simplest properties of linear operators. Let A be a linear continuous operator. Put $x = \xi + \zeta$ and, consequently, $\zeta = x - \xi$. Then

$$Ax = A\xi + A\zeta = A\xi + A(x-\xi),$$

whence

$$A(x-\xi) = Ax - A\xi. \tag{2}$$

Set $x = \xi$ in (2). Then $A(0) = Ax - Ax = 0$. Set $x = 0$ in (2). Then $A(-\xi) = -A\xi$.

THEOREM 1. *An additive operator $y = Ax$, defined on a linear real space E_x with values in the linear real space E_y, continuous at a single point $x_0 \in E_x$, is continuous on the entire space E_x.*

PROOF. Let x be any point of E_x and let $x_n \to x$. Then, $x_n - x + x_0 \to x_0$, and since A is continuous at x_0, it follows that

$$\lim_n A(x_n - x + x_0) = Ax_0.$$

However,

$$A(x_n - x + x_0) = Ax_n - Ax + Ax_0,$$

owing to the additivity of A. Therefore,

$$\lim_n Ax_n - Ax + Ax_0 = Ax_0, \quad \text{whence} \quad \lim_n Ax_n = Ax. \qquad \blacksquare$$

THEOREM 2. *An additive and continuous operator Ax, defined in a real space E_x, is homogeneous.*

PROOF. Now, let $t = n$ be a positive integer. Then,

$$A(nx) = Ax + Ax + \ldots + Ax = nAx.$$

Let $t = m$ be a negative integer. Then,

$$A(mx) = -A(-mx) = -(-m)\,Ax = mAx.$$

Let $t = m/n$ be a rational number, then

$$A\left(\frac{m}{n}x\right) = mA\left(\frac{1}{n}x\right).$$

Set $(1/n)\,x = \xi$. Then, $x = n\xi$, and

$$Ax = A(n\xi) = nA\xi = nA\left(\frac{1}{n}x\right),$$

whence $A\left(\frac{1}{n}x\right) = \frac{1}{n}Ax$. Consequently,

$$A\left(\frac{m}{n}x\right) = mA\left(\frac{1}{n}x\right) = \frac{m}{n}Ax.$$

Finally, let t be any real number. It is necessary to consider only the case when t is irrational. There is a sequence $\{r_n\}$ of rational numbers, such that $r_n \to t$. Hence, $\lim_n r_n x = tx$, and since A is continuous, it follows that

$$A(tx) = A(\lim_n r_n x) = \lim_n A(r_n x) = \lim_n r_n Ax = tAx. \qquad \blacksquare$$

3.13. The space of operators. The algebraic operations can be introduced in the set of linear continuous operators, defined on the linear space E_x with values in the linear space E_y. Let A and B be such operators. We define the addition by

$$(A+B)\,x = Ax + Bx,$$

and the scalar multiplication by

$$(\lambda A)x = \lambda Ax.$$

Evidently, this set of linear operators is a linear space, because all the required axioms are satisfied with respect to these operations. In particular, the element O of this space is the **null (zero) operator**, such that $Ox = 0$ for all $x \in E$. The limit of a sequence is defined in a linear operator space by assuming, for example, that $A_n \to A$, if $A_n x \to Ax$ for every $x \in E_x$. This space will be considered in a somewhat greater detail in the upcoming discussions with additional restrictions on E_x and E_y.

3.14 The ring of linear continuous operators. Now, consider any linear space E and the set $(E \to E)$ of all posible linear continuous operators with domain and range in E. As shown above, these operators form a certain linear space. The product of linear operators A and B in $(E \to E)$ is defined by

$$(AB)x = A(Bx).$$

As is easily seen, this is again a linear continuous operator. The product of a finite number of linear operators is defined inductively. In particular,

$$A \cdot A = A^2,\ A^2 \cdot A = A^3, \ldots$$

It is obvious that $(AB)C = A(BC)$, $(A+B)C = AC + BC$ and also $C(A+B) = CA + CB$. Further, there exists an **identity operator** I, defined by $Ix = x$ for all x, and such that $AI = IA = A$ for every operator A.

Thus, the set $(E \to E)$ forms a *non-commutative* ring with identity, because, in general, $AB \neq BA$.

Example. Let $E = C[0, 1]$. Consider the operators

$$y(t) = \int_0^1 ts\, x(s)\, ds = Ax \qquad \text{and} \qquad y(t) = tx(t) = Bx.$$

Then,
$$ABx = \int_0^1 ts \cdot sx(s)\, ds = t \cdot \int_0^1 s^2 x(s)\, ds,$$
$$BAx = t \int_0^1 ts\, x(s)\, ds = t^2 \int_0^1 sx(s)\, ds.$$

Thus, $AB \neq BA$.

The concept of the **inverse operator** is of key importance. Thus, as in the definition of the inverse element in a ring, a linear continuous operator B is said to be a **left inverse** of A, if $BA = I$ and an operator C a **right inverse** of A, if $AC = I$. If A has a left inverse B and a right inverse C, these are necessarily equal, because $B = B(AC) = (BA)\, C = C$. In this case, it is said that the operator A has an inverse denoted by A^{-1}. Thus, if A^{-1} exists, then by definition $A^{-1} A = AA^{-1} = I$. An operator can have at most one inverse. We shall revert to the concept of an inverse operator somewhat later.

3.15. Functions of operators. The operator $A^n = \underbrace{AA \ldots A}_{n \text{ times}}$ represents the simplest example of an **operator-function.** It is a special case of a more general function, namely of the polynomial operator

$$p_n(A) = a_0 I + a_1 A + a_2 A^2 + \ldots + a_n A^n.$$

The defining of the operator function $f(A)$ is more involved than that of a polynomial operator and a different method has to be resorted to for this purpose. For example, let E be an n-dimensional Euclidean space and A an operator mapping E into itself and defined by a symmetric matrix $\mathcal{A}$. By means of a unitary transformation U, the matrix $\mathcal{A}$ gets reduced to the diagonal form

$$\Lambda = U\mathcal{A}U^{-1},$$

that is,
$$\Lambda = \begin{Vmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot \\ 0 & 0 & 0 & \ldots & \lambda_n \end{Vmatrix}.$$

Now, let $f(t)$ be an arbitrary function of a real variable t, defined on the segment $[m, M]$, with $m = \min_i \lambda_i$ and $M = \max_i \lambda_i$. Set

$$f(\Lambda) = \begin{Vmatrix} f(\lambda_1) & 0 & 0 & \ldots & 0 \\ 0 & f(\lambda_2) & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot\cdot\cdot & \cdot \\ 0 & 0 & 0 & \ldots & f(\lambda_n) \end{Vmatrix},$$

and
$$f(\mathcal{A}) = U^{-1} f(\Lambda)\, U.$$

Thus, a function of the matrix $f(\mathcal{A})$ is set in correspondence to every function $f(t)$ of a real variable t, defined on $[m, M]$. Obviously, the null matrix corresponds to the function $f(t) \equiv 0$, the unit (identity) matrix E to the function $f(t) \equiv 1$ and the matrix $\mathcal{A}^m$ to the function $f(t) = t^m$. Further, if $f(t) = f_1(t) + f_2(t)$, then $f(\mathcal{A}) = f_i(\mathcal{A}) + f_2(\mathcal{A})$, and if $\varphi(t) = f_1(t) \cdot f_2(t)$, then $\varphi(\mathcal{A}) = f_1(\mathcal{A}) \cdot f_2(\mathcal{A})$. These equalities imply that the relations

$$U(B+C)\, U^{-1} = UBU^{-1} + UCU^{-1},$$
$$U(BC)\, U^{-1} = (UBU^{-1})\,(UCU^{-1}),$$

and, consequently, $f(\Lambda) = f_1(\Lambda) + f_2(\Lambda)$, $\varphi(\Lambda) = f_1(\Lambda) f_2(\Lambda)$ hold for any two matrices B and C. We can also construct the theory of functions of matrices in another way by going over from polynomials of matrices to power series of matrices. In this way we can, however, define only analytic functions of a matrix. A. I. LAPPO-DANILEVSKI has carried out extensive research work in this direction. This theory of analytic functions of matrices has been applied to investigations in systems of differential equations [21]. The construction of matrix functions by reduction of the matrix to diagonal form has been carried out for infinite matrices and then also for arbitrary self-adjoint operators in the HILBERT space, in the spectral theory of these operators (see Chap. 5.8).

As another example, we consider the differential operators. Let $E = C[0, 1]$ and let $D = d/dt$ be the operator which associates with every continuous differentiable function $x(t)$ the derivative of this function: $Dx(t) = (d/dt)\, x(t)$. Then, the expression

$$p_n(D)\, x(t) = a_0\, x(t) + a_1 \frac{dx(t)}{dt} + \ldots + a_n \frac{d^n x(t)}{dt}$$

has a sense for n-times differentiable functions, where $p_n(s)$ is any polynomial of degree n in the argument s. For infinitely differentiable functions, the relation

$$\sum_{n=0}^{\infty} a_n D^n x(t) = \sum_{n=0}^{\infty} a_n \frac{d^n x(t)}{dt^n}$$

has a sense, if only the series on its right-hand side is convergent and its sum belongs to $C[0, 1]$. It would, in particular, hold if $x(t)$ is a polynomial of degree m, because then $D^n x(t) = 0$ for $n > m$ and the series becomes a finite sum.

Polynomials of differential operators find application in the theory of linear differential equations. The simplest example is the so-called symbolic method for solving a differential equation with constant coefficients. Other applications to linear differential equations (ordinary and partial) are encountered in the operational calculus [7].

Formal operations with series and functions of a differential operator, defined by means of series, were widely prevalent in the first half of the 19th century for the purpose of determining certain formulae in the theory of quadrature, integration, etc. Let us make this explicit by examples.

At first, note that the operator

$$I + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \ldots + \frac{h^nD^n}{n!} + \ldots = e^{hD},$$

if applied to $x(t)$, regarded analytic, gives the function $x(t+h)$. In fact,

$$\begin{aligned} e^{hD}\, x(t) &= x(t) + hx'(t) + \frac{h^2}{2!}\, x''(t) + \ldots + \frac{h^n}{n!}\, x^{(n)}(t) + \ldots \\ &= x(t+h). \end{aligned}$$

Hence, if Δ_h denotes the operator taking the difference with step h,

$$\Delta_h\, x(t) = x(t+h) - x(t),$$

then,

$$e^{hD}\, x(t) = x(t+h) = x(t) + \Delta_h\, x(t) = (I+\Delta_h)\, x(t),$$

or,

$$I + \Delta_h = e^{hD}.$$

Developing formally into a power series, it is found that

$$hD = \ln\,(I+\Delta_h) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Delta_h^n}{n}.$$

This is the Gregory formula which expresses the differential operator by difference operators.

Next, consider the operators

$$Jx(t) = \int_0^1 x(t+\tau)\, d\tau \qquad \text{and} \qquad Sx(t) = \sum_{i=1}^{n} c_i\, x(t+\tau_i),$$

where $\sum_{i=1}^{n} c_i = 1$. It is not difficult to verify that the operators J and S are expressible as functions of the operator D, viz.

$$J = \int_0^1 e^{\tau D}\, d\tau = \frac{e^D - I}{D}, \qquad S = \sum_{i=1}^{n} c_i\, e^{\tau_i D}.$$

Thereupon,

$$J - S = J(I - J^{-1} S) = J\left[I - \sum_{i=1}^{n} c_i \frac{De^{\tau_i D}}{e^D - 1} \right].$$

However, $ze^{\alpha z} / (e^z - 1)$ is a generating function for the Bernoulli polynomials:

$$\frac{ze^{\alpha z}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{z^k}{k!}\, B_k(\alpha).$$

Therefore, the preceding formula assumes the form

$$J - S = J\left[I - \sum_{i=1}^{n} c_i \left(\sum_{k=0}^{\infty} \frac{D^k}{k!} B_k(\tau_i)\right)\right]$$

$$= -J\left[\sum_{i=1}^{n} c_i \sum_{k=1}^{\infty} \frac{D^k}{k!} B_k(\tau_i)\right],$$

because
$$\sum_{i=1}^{n} c_i B_0(\tau_i) = \sum_{i=1}^{n} c_i = 1.$$

Since $J[D^k x(t)] = \int_0^1 x^{(k)}(t+\tau)\, d\tau = x^{(k-1)}(t+1) - x^{(k-1)}(t)$,

we arrive at the generalized EULER-MACLAURIN summation formula

$$\int_0^1 x(t+\tau)\, d\tau = \sum_{i=1}^{n} c_i\, x(t+\tau_i) - \sum_{k=1}^{\infty}\left[\sum_{i=1}^{n} c_i \frac{B_k(\tau_i)}{k!}\right] \times [x^{(k-1)}(t+1) - x^{(k-1)}(t)].$$

In particular, for $t = 0$

$$\int_0^1 x(\tau)\, d\tau = \sum_{i=1}^{n} c_i\, x(\tau_i) - \sum_{k=1}^{\infty}\left[\sum_{i=1}^{n} c_i \frac{B_k(\tau_i)}{k!}\right] \times [x^{(k-1)}(1) - x^{(k-1)}(0)].$$

This deduction of the formulae of GREGORY and EULER-MACLAURIN can be regarded legitimate, if $x(t)$ to which these formulae are applied, is a polynomial. Then the resulting infinite series turn into finite sums and all the formal transformations carried out by us are justified. For arbitrary and infinitely differentiable functions, the formulae of GREGORY and EULER-MACLAURIN needed in the sequel can be proved, for example, by estimating the remainder terms.

3.2 LINEAR OPERATORS IN NORMED LINEAR SPACES

LET E_x and E_y be normed linear spaces. Since a normed linear space is a particular case of a topological linear space, it follows that the earlier definition of a linear operator defined on E_x with range in E_y is preserved in the case of normed linear spaces, and Theorems 1 and 2 proved in the previous section also remain valid.

We only remark that since the convergence in E_x and E_y is a norm convergence, the continuity of an operator A means that

$$\| Ax_n - Ax \| \to 0 \quad \text{as} \quad \| x_n - x \| \to 0.$$

The operators considered in Exs. 1 and 2 of Chap. 3.1 are linear continuous operators, transforming the normed linear spaces E_n and $C[0, 1]$ into themselves, respectively. The operator of Ex. 3 is also a linear continuous operator, transforming a normed linear space of continuous functions defined along Γ, into a normed linear space of functions harmonic in G, if the norms in these spaces are defined by

$$\| x \| = \max_{\Gamma} | x(t) | \qquad \text{and} \qquad \| u \| = \max_{G} | u(\xi, \eta) |,$$

respectively.

As another example, consider the infinite matrix (a_{ik}), $i, k = 1, 2, \ldots$, such that

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} | a_{ik} |^q < \infty, \qquad q > 1.$$

Then, the system of equations

$$\eta_i = \sum_{k=1}^{\infty} a_{ik} \xi_k, \qquad i = 1, 2, \ldots,$$

by means of which $y = \{\eta_i\}$ is set in correspondence to every element $x = \{\xi_i\} \in l_p$, defines a linear continuous operator $y = Ax$, defined on l_p with range in l_q, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, that is, $A \in (l_p \to l_q)$.

We first show that, in fact, $y \in l_q$, if $x \in l_p$. Making use of HÖLDER's inequality for sums, we get

$$\sum_{i=1}^{n} | \eta_i |^q = \sum_{i=1}^{n} \left| \sum_{k=1}^{\infty} a_{ik} \xi_k \right|^q$$

$$\leqslant \sum_{i=1}^{n} \left\{ \left(\sum_{k=1}^{\infty} | \xi_k |^p \right)^{1/p} \left(\sum_{k=1}^{\infty} | a_{ik} |^q \right)^{1/q} \right\}^q$$

$$= \| x \|^q \sum_{i=1}^{n} \sum_{k=1}^{\infty} | a_{ik} |^q \leqslant \| x \|^q \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} | a_{ik} |^q.$$

Since this inequality holds for any n, it is possible to take the limit as $n \to \infty$. Thus,

$$\| y \|^q = \sum_{i=1}^{\infty} | \eta_i |^q \leqslant \| x \|^q \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} | a_{ik} |^q,$$

and, hence, $y \in l_q$.

Let us now show that A is a linear continuous operator. Let

$$x_1 = \{\xi_i^{(1)}\} \in l_p \qquad \text{and} \qquad x_2 = \{\xi_i^{(2)}\} \in l_p.$$

Then, the equality

$$\sum_{i=1}^{\infty} a_{ik} \, (\xi_k^{(1)}+\xi_k^{(2)}) = \sum_{i=1}^{\infty} a_{ik} \, \xi_k^{(1)} + \sum_{i=1}^{\infty} a_{ik} \, \xi_k^{(2)}$$

implies $A(x_1+x_2) = Ax_1 + Ax_2$, that is, the additivity of the operator A. The homogeneity of this operator is obvious. Now, let

$$x_n = \{\xi_i^{(n)}\}, \qquad \text{and} \qquad x = \{\xi_i\} \in l_p,$$

$$Ax_n = y_n = \{\eta_i^{(n)}\}, \quad Ax = y = \{\eta_i\} \in l_q.$$

Then, $\| y_n - y \| = \left(\sum_{i=1}^{\infty} | \eta_i^{(n)} - \eta_i |^q \right)^{1/q}$

$$\leqslant \left(\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} | a_{i_k} |^q \right)^{1/q} \cdot \left(\sum_{i=1}^{\infty} | \xi_k^{(n)} - \xi_k |^p \right)^{1/p}$$

$$= \left(\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} | a_{i_k} |^q \right)^{1/q} \| x_n - x \|.$$

Thereupon,

$$\| y_n - y \| \to 0$$

as $\| x_n - x \| \to 0$, which proves the continuity of the operator A.

An operator A is called **bonnded** (or **continuous**) if there is a constant M, such that $\| Ax \| \leqslant M \| x \|$ for all $x \in E_x$. Here the norm $\| Ax \|$ is taken in the sense of the metric of E_y, containing the range of A, but $\| x \|$ is taken in the sense of the metric of E_x.

According to this definition, a bounded operator maps a bounded set of elements $\{x\} \subset E_x$ into a bounded set of elements $\{Ax\} \subset E_y$.

THEOREM 1. *In order that an additive and homogeneous operator A be continuous, it is necessary and sufficient that it is bounded.*

Necessity. Let A be a continuous operator. Assume that this is not bounded. Then, there is a sequence $\{x_n\}$ of elements, such that

$$\| Ax_n \| > n \| x_n \|.$$

Construct the elements

$$\xi_n = \frac{x_n}{n \| x_n \|}; \quad \xi_n \to 0 \text{ because}$$

$$\| \xi_n \| = \frac{1}{n \| x_n \|} \| x_n \| = \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand,

$$\| A\xi_n \| = \frac{1}{n \| x_n \|} \| Ax_n \| > 1,$$

implying that $A\xi_n$ does not tend to $A0 = 0$. Thus, A cannot be continuous at the origin, contradicting the assumption.

Sufficiency. Let the additive operator A be bounded, that is, $\| Ax \| \leqslant M \| x \|$. Let $x_n \to x$, that is, $\| x_n - x \| \to 0$, then also

$$\| Ax_n - Ax \| = \| A(x_n - x) \| \leqslant M \| x_n - x \| \to 0, \quad \text{i.e.,} \quad Ax_n \to Ax;$$

consequently, A is continuous. ■

The lemma that follows is frequently found very useful.

LEMMA. *Let a given linear (not necessarily bounded) operator A map a Banach space E_x into a Banach space E_y. Denote by E_n the set of those $x \in E_x$, for which $\| Ax \| \leqslant n \| x \|$. Then $E_x = \bigcup_{n=1}^{\infty} E_n$, and at least one of the sets E_n is everywhere dense.*

PROOF. In the first place, each of the sets E_n is not empty, because, for instance, $0 \in E_n$ for every n; furthermore, it is evident that every $x \in E_x$, $x \neq 0$, occurs in one of the sets E_n; for this it is sufficient to take n as the least integer, greater than $\| Ax \| / \| x \|$. Hence, it is possible to write

$$E_x = \bigcup_{n=1}^{\infty} E_n.$$

Since a complete space E_x cannot be a countable sum of nowhere dense sets (Theorem 3, p. 25), at least one of the sets E_{n_0} is everywhere dense. Consequently, there is a sphere $S(x_0, r)$, containing $S(x_0, r) \cap E_{n_0}$, everywhere dense.

Consider a sphere $\bar{S}(x_1, r_1)$ lying completely inside of $S(x_0, r)$ and such that $x_1 \in E_{n_0}$. Take any element x with norm $\| x \| = r_1$. The element $x_1 + x \in \bar{S}(x_1, r_1)$, because $\| (x_1 + x) - x_1 \| = r_1$.

Since $\bar{S}(x_1, r_1) \subset \bar{E}_n$, there is a sequence $\{z_k\}$ of elements in $S(x_1, r_1) \cap E_{n_0}$, such that $z_k \to x_1 + x$ as $k \to \infty$ (this sequence can be stationary, if $x_1 + x \in E_{n_0}$). Consequently, $x_k = z_k - x_1 \to x$. For this, it is possible to assume that $r_1/2 \leqslant \| x_k \|$, because $x_k \to x$ and $\| x \| = r_1$; besides,

$$\| x_k \| \leqslant r_1.$$

Since z_k and $x_1 \in E_{n_0}$,

$$\| Ax_k \| = \| Az_k - Ax_1 \| \leqslant \| Az_k \| + \| Ax_1 \| \leqslant n_0 (\| z_k \| + \| x_1 \|).$$

Furthermore,

$$\| z_k \| = \| x_k + x_1 \| \leqslant \| x_k \| + \| x_1 \| \leqslant r_1 + \| x_1 \|.$$

Hence

$$\| Ax_k \| \leqslant n_0 (r_1 + 2 \|x_1\|) \leqslant \frac{2n_0(r_1 + 2 \| x_1 \|)}{r_1} \| x_k \|.$$

Denote by n the least integer, greater than $[2n_0(r_1 + 2 \| x_1 \|)] / r_1$. Then, $\| Ax_k \| \leqslant n \| x_k \|$, implying that all $x_k \in E_n$.

Thus, any element x with norm equal to r_1, can approximate the elements in E_n. Now, let x be any element in E_x. Considering $\xi = r_1 (x / \| x_1 \|)$, we have $\| \xi \| = r_1$.

By the results deduced so far, there is a sequence $\{\xi_k\} \subset E_n$, convergent to ξ. Then,

$$x_k = \xi_k \frac{\| x \|}{r_1} \to x,$$

$$\| Ax_k \| = \frac{\| x \|}{r_1} \| A\xi_k \| \leqslant \frac{\| x \|}{r_1} n \| \xi_k \| = n \| x_k \|.$$

Thus, $x_k \in E_n$. Consequently, E_n is everywhere dense in E, and the lemma is proved.

If in an arbitrary BANACH space a linear operator is not necessarily continuous, then every linear operator is continuous in a finite-dimensional space.

In fact, let $e_1, \ldots, e_n$ be a basis in E and, consequently, every element x of this space have the form $x = \sum\limits_{i=1}^{n} \xi_i e_i$. In virtue of the homeomorphism of every n-dimensional BANACH space onto n-dimensional Euclidean spaces, if

$$x_k = \sum_{i=1}^{n} \xi_i^{(k)} e_i \to x = \sum_{i=1}^{n} \xi_i e_i,$$

then $\xi_i^{(k)} \to \xi_i$, $i = 1, \ldots, n$. But, then,

$$Ax_k = \sum_{i=1}^{n} \xi_i^{(k)} Ae_i \to \sum_{i=1}^{n} \xi_i Ae_i = Ax,$$

as was sought to demonstrate.

3.21. The norm of an operator. Let A be a bounded linear operator. The smallest number M, satisfying the inequality $\| Ax \| \leqslant M \| x \|$ (such a constant exists owing to A being bounded) is called the **norm** of A and denoted by $\| A \|$.

Thus, by definition, A has the following two properties:

(i) $\| Ax \| \leqslant \| A \| \, \| x \|$ for all $x \in E_x$. (1)

(ii) For every $\varepsilon > 0$, there is an element x_ε such that

$$\| Ax_\varepsilon \| > (\| A \| - \varepsilon) \| x_\varepsilon \|.$$

Show that
$$\| A \| = \sup_{\| x \| \leqslant 1} \| Ax \|, \tag{2}$$

or, what is the same,
$$\| A \| = \sup_{x \neq 0} \frac{\| Ax \|}{\| x \|}. \tag{2'}$$

In fact, if $\| x \| \leqslant 1$, then $\| Ax \| \leqslant \| A \| \cdot \| x \| \leqslant \| A \|$ and, therefore,

$$\sup_{\| x \| \leqslant 1} \| Ax \| \leqslant \| A \|. \tag{3}$$

On the other hand, there is an element x_ε for every $\varepsilon > 0$, such that

$$\| Ax_\varepsilon \| > (\| A \| - \varepsilon) \| x_\varepsilon \|.$$

Put $\xi_\varepsilon = x_\varepsilon / \| x_\varepsilon \|$, then

$$\| A\xi_\varepsilon \| = \frac{1}{\| x_\varepsilon \|} \| Ax_\varepsilon \| > \frac{1}{\| x_\varepsilon \|} (\| A \| - \varepsilon) \| x_\varepsilon \| = \| A \| - \varepsilon.$$

Since $\| \xi_\varepsilon \| = 1$, it follows that

$$\sup_{\| x \| \leqslant 1} \| Ax \| \geqslant \| A\xi_\varepsilon \| \geqslant \| A \| - \varepsilon.$$

Consequently,
$$\sup_{\| x \| \leqslant 1} \| Ax \| \geqslant \| A \| . \tag{4}$$

From (3) and (4) follows (2).

We determine the norm of an integral operator with the continuous kernel

$$y(t) = \int_0^1 K(t, s)\, x(s)\, ds,$$

regarding this as an operator mapping $C[0, 1]$ into $C[0, 1]$. Put

$$Ax = \int_0^1 K(t, s)\, x(s)\, ds,$$

then,
$$\begin{aligned} \| Ax \| &= \max_t \left| \int_0^1 K(t, s)\, x(s)\, ds \right| \\ &\leqslant \max_t \int_0^1 | K(t, s) |\, ds \cdot \max_s | x(s) | \\ &= \max_t \int_0^1 | K(t, s) |\, ds \, \| x \| . \end{aligned}$$

Consequently,
$$\| A \| \leqslant \max_t \int_0^1 | K(t, s) |\, ds. \tag{5}$$

Since $\int_0^1 | K(t, s) |\, ds$ is a continuous function, it attains the maximum at some point t_0 of the interval $[0, 1]$. Put

$$z_0(s) = \operatorname{sign} K(t_0, s) .$$

Let $x_n(s)$ be a continuous function, such that $|x_n(s)| \leqslant 1$ and $x_n(s) = z_0(s)$ everywhere, except on a set E_n of measure less than $1/2Mn$, where $M = \max_{t, s} | K(t, s) |$. Then, $|x_n(s) - z_0(s)| \leqslant 2$, everywhere on E_n.

We have

$$\left| \int_0^1 K(t, s)\, z_0(s)\, ds - \int_0^1 K(t, s)\, x_n(s)\, ds \right|$$

$$\leqslant \int_0^1 | k(t, s) | \cdot | x_n(s) - z_0(s) | \, ds = \int_{E_n} | K(t, s) | \cdot | x_n(s) - z_0(s) | \, ds$$

$$\leqslant 2 \max_{t, s} | K(t, s) | \cdot \frac{1}{2M_n} = \frac{1}{n}.$$

This inequality holds for every $t \in [0, 1]$. Conosequently,

$$\int_0^1 K(t, s)\, z_0(s)\, ds < \int_0^1 K(t, s)\, x_n(s)\, ds + \frac{1}{n} \leqslant \| A \| \cdot \| x_n \| + \frac{1}{n}$$

for all $t \in [0, 1]$. Putting in this inequality $t = t_0$,

$$\int_0^1 | K(t_0, s) | \, ds \leqslant \| A \| \, \| x_n \| + \frac{1}{n}.$$

Since $\| x_n \| \leqslant 1$, the preceding inequality in the limit as $n \to \infty$, yields $\int_0^1 | K(t, s) | \, ds \leqslant \| A \|$, that is

$$\max_t \int_0^1 | K(t, s) | \, ds \leqslant \| A \| . \tag{6}$$

From (5) and (6) it follows that $\| A \| = \max_t \int_0^1 | K(t, s) | \, ds$.

Let a linear manifold L be given in a normed linear space E_x. This linear manifold itself can be regarded as a linear space, not necessarily complete. Assume that an additive operator A is defined on L with values in some normed linear space E_y (that is, it is an operator acting from L into E_y). The operator A is called **bounded** on L, if there is a constant M such that $\| Ax \| \leqslant M \cdot \| x \|$ for all $x \in L$. The smallest constant M is called the **norm** of A on L and denoted† by $\| A \|_L$.

THEOREM 2. *A bounded linear operator A_0, defined on a linear manifold L, which is everywhere dense in a normed linear space E_x, with values in a complete normed linear space E_y, can be extended to the entire space without increase of norm.*

PROOF. In other words, A can be defined on E_x, such that $Ax = A_0 x$ for $x \in L$, and $\| A \|_{E_x} = \| A_0 \|_L$.

†In agreement with this, the norm of the operator on the entire space is sometimes denoted by $\| A \|_{E_x}$.

Let x be an element in E_x, not belonging to L. Since L is everywhere dense in E_x, there is a sequence $\{x_n\} \subset L$, such that $\| x_n - x \| \to 0$ as $n \to \infty$ and, hence, $\| x_n - x_m \| \to 0$ as $n, m \to \infty$. However, then

$$\| A_0 x_n - A_0 x_m \| = \| A_0(x_n - x_m) \| \leqslant \| A_0 \|_L \| x_n - x_m \| \to 0$$

as $n, m \to \infty$, that is, $\{A_0 x_n\}$ is a CAUCHY sequence and, consequently, by the completeness of E_y, converges to some limit denoted as Ax. Let $\{\xi_n\} \subset L$ be another sequence convergent to x. Evidently, $\| x_n - \xi_n \| \to 0$, whence $\| A_0 x_n - A_0 \xi_n \| \to 0$. Consequently, $A_0 \xi_n \to Ax$, implying that A is defined uniquely by the elements of E_x. If $x \in L$, select $x_n = x$ for all n, and then

$$Ax = \lim_n A_0 x_n = A_0 x.$$

The operator A is additive, since

$$A(x_1+x_2) = \lim_n A_0 (x_n^{(1)} + x_n^{(2)}) = \lim_n A_0 x_n^{(1)} + \lim_n A_0 x_n^{(2)}$$
$$= Ax_1 + Ax_2,$$

and is bounded, since the inequality

$$\| A_0 x_n \| \leqslant \| A_0 \|_L \| x_n \|$$

on taking the limit, yields

$$\| Ax \| \leqslant \| A_0 \|_L \| x \| .$$

From this inequality, however, it follows that

$$\| A \|_{E_x} \leqslant \| A_0 \|_L .$$

Since under extension of the operator, the norm, evidently, cannot decrease, it follows that

$$\| A \|_{E_x} = \| A_0 \|_L ,$$

and the theorem is completely proved.

The indicated process of the extension of an operator is called its **extension (or completion) by continuity**, exhibiting that the extension by continuity of a bounded linear operator leads to a unique linear operator and both the operators are norm identical, that is, there is nothing about their respective norms which will serve to distinguish them.

3.3 LINEAR FUNCTIONALS

3.31. Basic concepts. If the range of an operator consists of real numbers, then the operator is called a **functional**. A functional $f(x)$ defined on a linear topological space is said to be **linear,** if

(i) $f(x_1+x_2) = f(x_1) + f(x_2)$; and

(ii) $f(x_n) \to f(x)$ as $x_n \to x$ in the sense of convergence in a linear space E.

Since the set R of real numbers is a BANACH space, all previous definitions and theorems derived for continuous linear operators are preserved for linear functionals.

As for the linear operators, the following theorems hold for linear functionals.

THEOREM 1′. *If an additive functional $f(x)$, defined on a linear space E, is continuous at a single point of this space, then it is also continuous and hence linear throughout E.*

THEOREM 2′. *Every linear functional is homogeneous.*

THEOREM 3′. *In order that an additive functional defined on a normed linear space E be linear, it is necessary and sufficient that it is bounded*:

$$|f(x)| \leqslant M \|x\|.$$

The smallest of the constants M in the last inequality is called the **norm** of the functional $f(x)$ and denoted by $\|f\|$. Thus,

$$|f(x)| \leqslant \|f\| \cdot \|x\|.$$

Finally, $\|f\| = \sup\limits_{\|x\| \leqslant 1} |f(x)|$, or what is the same,

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|}.$$

Examples. 1. Let $E = L_p\,[0, 1]$. Then, $f(x) = \int\limits_0^1 x(t)\,dt$ is a linear function, In fact, in order that $f(x)$ has a meaning for every $x \in L_p\,[0, 1]$, it follows from HÖLDER's inequality that

$$\left|\int_0^1 x(t)\,dt\right| \leqslant \left(\int_0^1 |x(t)|^p\,dt\right)^{1/p} \left(\int_0^1 dt\right)^{1/q} = \|x\|.$$

The last inequality implies the boundedness of $f(x)$, the additivity of $f(x)$ is obvious.

2. Let E be E_k, that is, a k-dimensional Euclidean space. For the element $x = \{\xi_i\}$ this space admits $f(x) = c_1\xi_1 + c_2\xi_2 + , \ldots , + c_k\xi_k$, where $c_1, c_2, \ldots, c_k$ are some constants. The additivity of the functional $f(x)$ is again obvious Since $x_n \to x$ signifies that $\xi_i^{(n)} \to \xi_i$ for all $i = 1, 2, \ldots, k$, it follows that

$$\lim_n f(x_n) = \lim_n \sum_{i=1}^k c_i \xi_i^{(n)} = \sum_{i=1}^k c_i \xi_i = f(x),$$

and the continuity of $f(x)$ is proved.

The norm of a linear functional admits geometrical interpretation. Since in a k-dimensional Euclidean space, the equation of plane

$$c_1\xi_1 + c_2\xi_2 + \ldots + c_k\xi_k = c$$

can be expressed in the form $f(x) = c$, hence, by geometric analogy, in an arbitrary linear space E, the collection of points of this space, satisfying the equation, $f(x) = c$, where f is a linear functional on E, is called a **hyperplane.**

The hyperplanes $f(x) = c_0$ and $f(x) = c_1$ are naturally said to be **parallel.**

The hyperplane $f(x) = c$ divides the space into two half-spaces; the collection of points x where $f(x) \leqslant c$ and the collection of points x where $f(x) \geqslant c$. Conventionally, the former of these half-spaces is said to lie to the left and the latter to the right of the hyperplane $f(x) = c$.

The hyperplane $f(x) = \| f \|$ has the property that all of the unit sphere $\| x \| \leqslant 1$ lies completely to the left of this hyperplane (because $f(x) \leqslant \| f \|$ holds for the points of the sphere $\| x \| \leqslant 1$). On the other hand, none of the parallel hyperplanes $f(x) = \| f \| - \varepsilon$ has this property.

In analogy with the theory of convex bodies of a k-dimensional Euclidean space, the hyperplane $f(x) = \| f \|$ is called a **support of the sphere** $\| x \| \leqslant 1$.

3.4 THE SPACE OF BOUNDED LINEAR OPERATORS

It is already remarked that all possible linear bounded operators, defined on exactly one linear space E_x with range in a linear space E_y, form a linear space $(E_x \to E_y)$.

If, in addition, it is assumed that E_x and E_y are normed spaces, then the norm can also be introduced in the space $(E_x \to E_y)$.

In fact, for every linear bounded operator A mapping E_x into E_y, a norm is defined by the method indicated in Chap. 3.21. It is not difficult to show that this norm satisfies the norm axioms. Indeed,

(i) $\| A \| = \sup\limits_{\| x \| \leqslant 1} \| Ax \| \geqslant 0$ is trivial. If $\| A \| = 0$, (that is, if $\sup\limits_{\| x \| \leqslant 1} \| Ax \| = 0$), then $\| Ax \| = 0$ for all x, such that $\| x \| \leqslant 1$.

However, then, because of the homogeneity of A, $Ax = 0$ for all x and therefore, $A = 0$.

(ii) $$\| \lambda A \| = \sup_{\| x \| \leqslant 1} \| \lambda \cdot Ax \| = | \lambda | \sup_{\| x \| \leqslant 1} \| Ax \| = | \lambda | \, \| A \|.$$

(iii) $$\| A + B \| = \sup_{\| x \| \leqslant 1} \| Ax + Bx \| \leqslant \sup_{\| x \| \leqslant 1} \| Ax \| + \sup_{\| x \| \leqslant 1} \| Bx \| = \| A \| + \| B \|.$$

Thus, a space of linear bounded operators is a normed linear space.

In a particular case, when $E_y = R$ is a set of real numbers, that is, when we consider a space of linear functionals defined on E_x, this space of linear functionals is called the **conjugate** (or **dual**) of E_x and denoted by E_x^*.

THEOREM 1. *If E_y is complete, the space of bounded linear operators is also complete and is, consequently, a Banach space.*

PROOF. Given a CAUCHY sequence $\{A_n\}$ of linear operators with respect to the norm in a space of linear operators, that is, such that $\| A_n - A_m \| \to 0$ as $n, m \to \infty$. Then,

$$\| A_n x - A_m x \| \leqslant \| A_n - A_m \| \, \| x \| \to 0$$

for every x and as $n, m \to \infty$.

Therefore, the sequence $\{A_n x\}$ of elements of E_y is a CAUCHY sequence for every fixed x. Now since E_y is complete, $\{A_n x\}$ has some limit y.

Thus, $y \in E_y$ is associated with every $x \in E_x$ and we obtain some operator A, defined by the equation $Ax = y$, which is also additive and bounded.

(i) The additivity of A is immediate from

$$A(x_1 + x_2) = \lim_n A_n(x_1 + x_2) = \lim_n A_n x_1 + \lim_n A_n x_2 = Ax_1 + Ax_2.$$

(ii) By hypothesis, $\| A_n - A_m \| \to 0$ as $n, m \to \infty$. Hence

$$| \| A_n \| - \| A_m \| | \to 0$$

as $n, m \to \infty$, that is, $\{\| A_n \|\}$ is a CAUCHY number sequence and is, consequently, bounded. Thus, there is a constant K such that $\| A_n \| \leqslant K$ for all n. Consequently, $\| A_n x \| \leqslant K \| x \|$ for all n and, hence,

$$\| Ax \| = \lim_n \| A_n x \| \leqslant K \| x \|,$$

proving the boundedness of A. Since A is, in addition, additive and homogeneous, A is a bounded linear operator.

Now, we shall prove that A is the limit of the sequence $\{A_n\}$ in the sense of norm convergence in a space of linear operators. There is an index n_0 for very $\varepsilon > 0$, such that

$$\| A_{n+p} x - A_n x \| < \varepsilon \tag{1}$$

for $n \geqslant n_0$, $p > 0$ and all x with the norm $\| x \| \leqslant 1$. Taking the limit in (1) as $p \to \infty$, we get

$$\| Ax - A_n x \| \leqslant \varepsilon$$

for $n \geqslant n_0$ and all x with $\| x \| \leqslant 1$. Hence, for $n \geqslant n_0$,

$$\| A_n - A \| = \sup_{\| x \| \leqslant 1} \| (A_n - A) x \| \leqslant \varepsilon.$$

Consequently,
$$A = \lim_n A_n,$$
in the sense of norm convergence in a space of bounded linear operators, and the completeness of this space is proved.

COROLLARY. *The space E^* which is conjugate to a normed linear space E, is a Banach space.*

3.41. Uniform and pointwise convergence of operators. The convergence of a sequence of linear bounded operators in the sense of convergence in the norm in a space of linear operators is called **uniform convergence.** This terminology is justified in that if $A_n \to A$ in the sense of norm convergence, then $A_n x \to Ax$ uniformly throughout the sphere $\| x \| \leqslant r$. In fact, for $\varepsilon > 0$ given, select n_0 such that $\| A_n - A \| < \varepsilon / r$ for $n \geqslant n_0$. Then,

$$\| A_n x - Ax \| \leqslant \| A_n - A \| \; \| x \| < \frac{\varepsilon}{r} r = \varepsilon$$

for all $x \in \bar{S}(\theta, r)$, which is also the required proof. Conversely, if $A_n x \to Ax$ uniformly on some sphere $\| x \| \leqslant r$, then also $A_n x \to Ax$ uniformly in the unit sphere, and, thereupon, it immediately follows that

$$\| A_n - A \| \to 0.$$

In the space $(E_x \to E_y)$ of bounded linear operators, we introduce still another convergence of a sequence of operators, namely : A sequence of linear bounded operators $\{A_n\}$ is said to **converge pointwise** to a linear operator A (or convergent to itself), if for every fixed x the sequence $\{A_n x\}$ converges to Ax (or to itself). Obviously, the uniform convergence of the sequence $\{A_n\}$ implies the pointwise convergence of this sequence. The converse is not true, as shown by the example below.

Let E be a HILBERT space H with an orthonormal basis $\{e_1, e_2, \ldots, e_n, \ldots\}$. Let A_n be an operator projected on the subspace H_n, spanned by the elements $e_1, e_2, \ldots, e_n$. Then,

$$A_n x = \sum_{i=1}^{n} (x, e_i)\, e_i \to \sum_{i=1}^{\infty} (x, e_i)\, e_i = x$$

for every $x \in H$ and, consequently, $A_n \to I$ in the sense of pointwise convergence.

On the other hand, for $\varepsilon_0 < 1$, any n and $p > 0$, we have

$$\| A_n e_{n+1} - A_{n+p} e_{n+1} \| = \| e_{n+1} \| = 1 > \varepsilon_0,$$

and, consequently, the uniform convergence of the sequence $\{A_n\}$ in the unit sphere $\| x \| \leqslant 1$ of the space H, does not hold.

THEOREM 2. *If the spaces E_x and E_y are complete, then the space of bounded linear operators is also complete in the sense of pointwise convergence.*

PROOF. Since $\{A_n x\}$ is a CAUCHY sequence for every x, there exists a limit $y = \lim\limits_{n} A_n x$ for every x. Thus, we obtain an operator $Ax = y$, with domain E_x and range in E_y. There is no contention about A being a linear operator. The proof of the boundedness of A follows from the next theorem.

THEOREM 3. (BANACH-STEINHAUS). *If a sequence of bounded linear operators is a Cauchy sequence at every point x of a Banach space E_x, then the sequence $\{\| A_n \|\}$ of norms of these operators is bounded.*

PROOF. Assume the contrary. Then the set $\{\| A_n x \|\}$ is not bounded on any closed sphere $\| x - x_0 \| \leqslant \varepsilon$. In fact, if

$$\| A_n x \| \leqslant c$$

for all n and all x in some sphere $\overline{S}(x_0, \varepsilon)$, then the element

$$x = \frac{\varepsilon}{\| \xi \|} \xi + x_0$$

would belong to this sphere for every $\xi \in E_x$, and, consequently,

$$\| A_n x \| \leqslant c, \qquad n = 1, 2, \ldots,$$

or, $$\frac{\varepsilon}{\|\xi\|}\|A_n\xi\| - \|A_n x_0\| \leqslant \left\| \frac{\varepsilon}{\|\xi\|} A_n \xi + A_n x_0 \right\| \leqslant c.$$

Hence, $$\|A_n \xi\| \leqslant \frac{c + \|A_n x_0\|}{\varepsilon} \|\xi\|.$$

Since the norm sequence $\{\|A_n x_0\|\}$ is bounded owing to $\{A_n x_0\}$ being a convergent sequence, it follows that

$$\|A_n \xi\| \leqslant c_1 \|\xi\|, \qquad n = 1, 2, \ldots,$$

and, consequently, $\|A_n\| \leqslant c_1$, $n = 1, 2, \ldots$ But this contradicts the hypothesis.

Now, let $\bar{S}_0(x_0, \varepsilon_0)$ be any closed sphere in E_x. The sequence $\{\|A_n x\|\}$ is not bounded on it and, hence, there is an index n_1 and an element $x_1 \in \bar{S}_0$, such that

$$\|A_{n_1} x_1\| > 1.$$

By the continuity of the operator A_{n_1}, this inequality holds in some closed sphere $\bar{S}_1(x_1, \varepsilon_1) \subset S_0$. The sequence $\{\|A_n x\|\}$ is again not bounded on $\bar{S}_1$ and, therefore, there is an index n_2, $n_2 > n_1$, and an element $x_2 \in \bar{S}_1$, such that

$$\|A_{n_2} x_2\| > 2.$$

Since A_{n_2} is continuous, this inequality must hold in some closed sphere $\bar{S}_2(x_2, \varepsilon_2) \subset \bar{S}_1$, and so on. If we continue this process and let $\varepsilon_n \to 0$ as $n \to \infty$, then there is a point $\bar{x}$ belonging to all the spheres $\bar{S}_n(x_n, \varepsilon_n)$. At this point,

$$\|A_{n_k} \bar{x}\| \geqslant k,$$

which contradicts the hypothesis that $\{A_n x\}$ converges for every $x \in E_x$. Hence, the theorem is proved.

Now we revert to the operator

$$Ax = \lim_n A_n x.$$

The inequality $\quad \|A_n x\| \leqslant M \|x\|, \qquad n = 1, 2, \ldots,$

that follows from the BANACH-STEINHAUS theorem, implies that

$$\|Ax\| \leqslant M \|x\|$$

on taking the limit as $n \to \infty$, that is, the operator A is bounded.

REMARK. In the formulation of the BANACH-STEINHAUS theorem, in place of the convergence to itself of the sequence of operators $\{A_n\}$ at every point $x \in E_x$, it is possible to prescribe the requirement of the boundedness of this sequence at every point of the space. For this the proof of the theorem needs no modification.

Thus, every sequence of linear bounded operators, converging pointwise to itself, has a limit, which also is a bounded linear operator, that is, the space of linear operators is complete in the sense of pointwise convergence.

The theorem that follows is often found useful.

THEOREM 4. *In order that an operator sequence* $\{A_n\}$ *be pointwise convergent to an operator* A_0, *it is necessary and sufficient that*

(i) *The sequence* $\{\| A_n \|\}$ *is bounded; and*

(ii) $A_n x \to A_0 x$ *for every* x *in some set* X, *the linear combination of whose elements lies everywhere dense in* E_x.

PROOF. The necessity of the hypothesis (i) stands proved in the foregoing BANACH-STEINHAUS theorem, that of (ii) is obvious. It remains to prove only the sufficiency of these conditions.

Let $M = \sup\limits_{n=0,1,\ldots} \| A_n \|$, and let $L(X)$ be a linear hull of the set X. Owing to the linearity of the operators A_n and A_0 and the hypothesis (ii), $A_n x \to A_0 x$ for every $x \in L(X)$.

Now take an element ξ of the space E_x, not belonging to $L(X)$. For $\varepsilon > 0$ given, there is an element $x \in L(X)$, such that $\| \xi - x \| < \varepsilon/4M$. We have

$$\begin{aligned}\| A_n\xi - A_0\xi \| &\leqslant \| A_n\xi - A_n x \| + \| A_n x - A_0 x \| + \| A_0 x - A_0\xi \| \\ &\leqslant \| A_n x - A_0 x \| + (\|A_n\| + \|A_0\|) \| x - \xi \| \\ &< \| A_n x - A_0 x \| + \frac{\varepsilon}{2}.\end{aligned}$$

Since $A_n x \to A_0 x$, there is an index n_0, such that $\| A_n x - A_0 x \| < \varepsilon/2$ for $n \geqslant n_0$. Hence, for $n \geqslant n_0$, we get $\| A_n\xi - A_0\xi \| < \varepsilon$, and the theorem is proved.

3.42. Application to interpolation theory. The BANACH-STEINHAUS theorem proved above offers possibilities of numerous applications. The next theorem serves as an example.

THEOREM 5. *Given some points on the segment* $[0, 1]$, *forming the infinite triangular matrix*

$$T = \left| \begin{matrix} t_1^{(1)} & 0 & 0 & \ldots \\ t_1^{(2)} & t_2^{(2)} & 0 & \ldots \\ t_1^{(3)} & t_2^{(3)} & t_3^{(3)} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{matrix} \right| . \tag{2}$$

For a given function $x(t)$, *defined on* $[0, 1]$, *we can construct the Lagrange interpolation polynomials* $L_n x$, *whose partition points are the points of the n-th row of* (2):

$$L_n x = \sum_{k=1}^{n} x\,(t_k^{(n)})\, l_k^{(n)}\,(t),$$

where $$l_k^{(n)}(t) = \frac{\omega_n(t)}{\omega_n'(t)\,(t-t_k^{(n)})}, \qquad \omega_n(t) = \prod_{k=1}^{n} (t-t_k^{(n)}).$$

Whatever be the matrix (2), *there is a continuous function* $x(t)$, *such that the* $L_n x$ *do not uniformly converge to* $x(t)$ *as* $n \to \infty$.

PROOF. Regard the $L_n x$ as operators mapping the function $x(t) \in C[0, 1]$ into the elements of the same space, and put

$$\lambda_n = \max_t \lambda_n(t), \qquad \text{where} \quad \lambda_n(t) = \sum_{k=1}^{n} |\, l_k^{(n)}(t)\,|.$$

Then, it is routine to prove [24] that

$$\|\, L_n \,\| = \lambda_n.$$

On the other hand, the BERNSTEIN inequality

$$\lambda_n > \frac{\ln n}{8\sqrt{\pi}}$$

holds. Consequently,

$$\|\, L_n \,\| \to \infty$$

as $n \to \infty$. This proves the theorem in hand, because if $L_n x \to x$ for all $x \in C[0, 1]$, then the norm $\|\, L_n \,\|$ must be bounded.

3.5 INVERSE OPERATORS

3.51. Inverse operators and algebraic equations. The concept of inverse operators and its importance are briefly sketched in Sec. 3.14. Presently, it shall be shown that the existence and uniqueness problem for the solution of the functional equation

$$Ax = y \tag{1}$$

is related to the notion of an inverse operator. Here y is the known element of the linear space E, and x belonging to the same space is unknown. Since systems of linear algebraic equations, linear differential and integral equations etc. can be expressed by (1), it is plain that the determination of an operator, inverse of a given operator, is of considerable importance.

Thus, consider Eq. (1) and assume A to have an inverse operator A^{-1}. Set $x = A^{-1}y$. Substitute this into (1) to receive the identity

$$AA^{-1}y = y,$$

that is, $y = y$. Consequently, $x = A^{-1}y$ is a solution of (1).

Let x_1 be another solution of (1), that is, let $Ax_1 = y$. Applying the operator A^{-1} to both sides of this equality we obtain $x_1 = A^{-1}y = x$. Consequently, the solution $x = A^{-1}y$ is unique.

If A has a right inverse C, it is easy to verify that $x = Cy$ is a solution of

Eq. (1); the uniqueness problem, however, remains open. Assume that A has a left inverse B. Thereupon, if Eq. (1) has a solution x, that is, $Ax = y$, then applying from left the operator B to this equality, we get $x = By$, that is, the solution is unique. However, the existence problem for the solution remains open.

An analysis of the preceding reasonings exhibits that the inverse operator (whether right or left) is to be applied not to every $x \in E$ but only to an element of the form Ax, that is, to the images of the elements in the space E. The collection of these images is some linear manifold (a subspace of E). By generalizing this situation, we arrive at the following more general definition of inverse operators.

Given two linear spaces E_x and E_y and an operator A which maps E_x onto E_y. If there is an operator A^{-1} with domain E_y and range in E_x, such that

$$A^{-1}Ax = x \tag{2}$$

for every $x \in E_x$, and

$$AA^{-1}y = y \tag{2'}$$

for every $y \in E_y$, then the operators A and A^{-1} are said to be **mutually inverse** (or **one-one**). This definition implies, in particular, that $(A^{-1})^{-1} = A$. If A^{-1} satisfies only either of the conditions (2) and (2′), it is called a **left** (or **right**) inverse of the operator A.

It is easy to show that *the inverse of a linear operator, if existing, is also linear.* In fact, let

$$x = A^{-1}(y_1+y_2) - A^{-1}y_1 - A^{-1}y_2.$$

By the additivity of A, we get

$$Ax = AA^{-1}(y_1+y_2) - AA^{-1}y_1 - AA^{-1}y_2 = (y_1+y_2) - y_1 - y_2 = 0.$$

Thereupon, $x = A^{-1}Ax = A^{-1}0 = 0$, that is, $A^{-1}(y_1+y_2) = A^{-1}y_1 + A^{-1}y_2$, proving A^{-1} to be additive. Analogously, the homogeneity of A^{-1} is established. However, the continuity of an operator A in some topology does not imply, generally speaking, the continuity of its inverse in the same or a different topology, that is, an operator inverse to a bounded linear operator may not be a bounded linear operator.

3.52. Theorems on inverse operator. We now take up a few theorems, giving the sufficiency conditions for the existence of inverse linear bounded operators.

As a preliminary, some remarks will be in order. Let a linear bounded operator A map E_x onto E_y one-one. Then, there is an inverse operator A^{-1}, which is linear. In fact, for every $y \in E_y$ there exists exactly one pre-image $x \in E_x$. Setting in correspondence to every element $y \in E_y$ its pre-image $x \in E_x$, we obtain the operator A^{-1}, which in the sense of its definition satisfies the condition (2), implying in turn the linearity of A^{-1}.

THEOREM 1. (BANACH). *Let a linear operator A map a normed linear*

(Banach) space E_x onto a normed linear (Banach) space E_y, satisfying for every $x \in E_x$ the condition

$$\| Ax \| \geqslant m \| x \|, \qquad m > 0, \tag{3}$$

m some constant. Then, there is an inverse bounded linear operator A^{-1}.

PROOF. The condition (3) implies that A maps E_x onto E_y in a one-one fashion: if $Ax_1 = y$ and $Ax_2 = y$, then $A(x_1 - x_2) = 0$ and by (3),

$$m \| x_1 - x_2 \| \leqslant \| A(x_1 - x_2) \| = 0,$$

whence $x_1 = x_2$. Hence, as shown above, there is a linear operator A^{-1}. This operator is bounded, as is immediate from (3):

$$\| A^{-1}y \| \leqslant \frac{1}{m} \| AA^{-1}y \| = \frac{1}{m} \| y \|$$

for every $y \in E_y$. The theorem is proved.

Now, consider two bounded linear operators A and B, which map a normed linear space E into itself. Then, the product AB has a meaning. It is to be shown that

$$\| AB \| \leqslant \| A \| \| B \|. \tag{4}$$

For any $x \in E$,

$$\| ABx \| \leqslant \| A \| \| Bx \| \leqslant \| A \| \| B \| \| x \|,$$

whence also follows the above assertion.

Now, let $A_n, A, B_n, B \in (E \to E)$ and $A_n \to A$, $B_n \to B$ in the sense of uniform convergence. Then, $A_nB_n \to AB$. In fact,

$$\begin{aligned} \| A_nB_n - AB \| &\leqslant \| A_nB_n - A_nB \| + \| A_nB - AB \| \\ &\leqslant \| A_n \| \| B_n - B \| + \| B \| \| A_n - A \|. \end{aligned}$$

$\{\| A_n \|\}$ is a convergent numerical sequence and is, therefore, bounded; also $\| A_n - A \| \to 0$ and $\| B_n - B \| \to 0$. Consequently,

$$\| A_nB_n - AB \| \to 0. \qquad \blacksquare$$

THEOREM 2. *Let a bounded linear operator A map E into E and let $\| A \| \leqslant q < 1$. Then, the operator $I + A$ has an inverse, which is a bounded linear operator.*

PROOF. In the space of operators with domain E and range as well in E, consider the series

$$I - A + A^2 - A^3 + \ldots + (-1)^n A^n + \ldots \tag{5}$$

Since $\| A^2 \| \leqslant \| A \|^2$ and, analogously, $\| A^n \| \leqslant \| A \|^n$, it follows for the partial sums S_n of the series (5), that

$$\begin{aligned} \| S_{n+p} - S_n \| &= \| (-1)^{n+1} A^{n+1} + (-1)^{n+2} A^{n+2} + \ldots + (-1)^{n+p} A^{n+p} \| \\ &\leqslant \| A \|^{n+1} + \| A \|^{n+2} \| + \ldots + \| A \|^{n+p} \\ &\leqslant q^{n+1} + q^{n+2} \ldots + q^{n+p} \to 0 \end{aligned}$$

for $n \to \infty$, $p > 0$. Hence, the sequence of the partial sums of the series (5) is a CAUCHY sequence and, therefore, owing to the completeness of the space of operators, converges also to some limit, that is, the series (5) is convergent.

Let S be the sum of the series (5). Then,

$$\begin{aligned} S(I+A) &= \lim_n S_n(I+A) \\ &= \lim_n (I+A+A^2+\ldots+A^n-A-A^2-\ldots-A^{(n+1)}) \\ &= \lim_n (I-A^{n+1}) = I, \end{aligned}$$

that is, $S = (I+A)^{-1}$. It is plain that S is a linear operator. Besides, it is bounded, since

$$\| S \| \leqslant \sum_{n=0}^{\infty} \| A \|^n \leqslant \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}.$$

Thus, $(I+A)^{-1}$ is a bounded linear operator and the theorem is proved.

THEOREM 3. *Suppose that the operator $A \in (E_x \to E_y)$ has an inverse A^{-1} and there is an operator ΔA such that,*

$$\| \Delta A \| < \| A^{-1} \| - 1.$$

Then, the operator $B = A + \Delta A$ has an inverse B^{-1}, furthermore

$$\| B^{-1} - A^{-1} \| \leqslant \frac{\|\Delta A\|}{1 - \|A^{-1}\| \, \|\Delta A\|} \| A^{-1} \|^2. \tag{6}$$

PROOF. In fact, $A + \Delta A = A(I+A^{-1}\,\Delta A)$. Since $\| A^{-1}\,\Delta A \| < 1$, the operator $I + A^{-1}\,\Delta A$, has the inverse

$$(I+A^{-1}\,\Delta A)^{-1} = \sum_{n=0}^{\infty} (-A^{-1}\,\Delta A)^n.$$

Thus, evidently, $(I+A^{-1}\,\Delta A)^{-1} A^{-1}$ is an operator, inverse to $A(I+A^{-1}\,\Delta A) = A + \Delta A$. Moreover,

$$\begin{aligned} \| (A+\Delta A)^{-1} - A^{-1} \| &\leqslant \| A^{-1} \| \, \| (I+A^{-1}\,\Delta A)^{-1} - I \| \\ &\leqslant \sum_{n=1}^{\infty} \| A^{-1}\,\Delta A \|^n \, \| A^{-1} \| \\ &= \frac{\|A^{-1}\,\Delta A\|}{1-\|A^{-1}\,\Delta A\|} \| A^{-1} \| \leqslant \frac{\|\Delta A\|}{1-\|A^{-1}\|\,\|\Delta A\|} \| A^{-1} \|^2. \quad \blacksquare \end{aligned}$$

Example. Consider the integral operator

$$Ax = x(t) - \int_0^1 K(t, s)\, x(s)\, ds \tag{7}$$

with a continuous kernel $K(t, s)$, which maps the space $C[0, 1]$ into itself. Let $K_0(t, s)$ be a degenerate kernel close to $K(t, s)$ and A_0 an integral operator coriesponding to the kernel $K_0(t, s)$:

$$A_0 x = x(t) - \int_0^1 K_0(t, s)\, x(s)\, ds. \tag{8}$$

Consider the equations
$$Ax = y, \tag{7'}$$
and
$$A_0 x = y. \tag{8'}$$
Set
$$\omega = \max_{t,s} | K(t, s) - K_0(t, s) | .$$

If $\Delta A = A - A_0$, then it is plain that $\| \Delta A \| \leqslant \omega$. As is known†, the solution of Eq. (8) with a degenerate kernel reduces to the solution of a linear algebraic system. Assume that the system has a solution and write this in the form

$$x_0(t) = Ry,$$

R an operator defined by the matrix (r_{ij}), inverse of the matrix of linear algebraic system under reference. Let r be the norm of R. Now, if $\omega < 1/r$, then by the theorems proved, the integral Eq. (7) with a continuous kernel has a solution; if $x(t)$ be this solution, then

$$\| x(t) - x_0(t) \| \leqslant \frac{\omega}{1 - \omega r} r^2 .$$

Conversely, if it is known that Eq. (7) is solvable, then the theorems already established can be used for proving the existence of solutions of approximating equations with a degenerate kernel and for estimating the error in approximate solutions.

Finally, we arrive at the next theorem.

THEOREM 4 (BANACH). *If a bounded linear operator A maps the whole of the Banach space E_x onto the whole of the Banach space E_y in a one-one fashion, then there exists a bounded linear operator A^{-1}, inverse to the operator A, which maps E_y onto E_x.*

PROOF. It is necessary to prove only that A^{-1} is bounded.

By the lemma of Chap. 3.2, E_y is representable in the form

$$E_y = \bigcup_{k=1}^{\infty} Y_k,$$

where Y_k is a collection of elements $y \in E_y$, verifying

$$\| A^{-1} y \| \leqslant k \| y \|,$$

and at least one of the sets Y_k is everywhere dense in E_y. Let this set be Y_n.

Take any element $y \in E_y$. Let $\| y \| = l$; then, there exists $y_1 \in Y_n$ such that

$$\| y - y_1 \| \leqslant l/2, \qquad \| y_1 \| \leqslant l.$$

This can be accomplished, since $\bar{S}(0, l) \cap Y_n$ is everywhere dense in $\bar{S}(0, l)$ and $y \in \bar{S}(0, l)$. Further, there exists an element $y_2 \in Y_n$ such that

$$\| (y - y_1) - y_2 \| \leqslant l/2^2, \qquad \| y_2 \| \leqslant l/2.$$

† See [22], for example.

Continuing this process, the elements $y_k \in Y_n$ are constructed, such that

$$\| y - (y_1 + y_2 + \ldots + y_k) \| \leqslant l/2^k, \qquad \| y_k \| \leqslant l/2^{k-1}.$$

Thus, $y = \lim_k \sum_{i=1}^{k} y_i$. Set $x_k = A^{-1}y_k$, then $\| x_k \| \leqslant n \| y_k \| \leqslant \frac{nl}{2^{k-1}}$. Since

$$\| s_{k+p} - s_k \| = \left\| \sum_{i=k+1}^{k+p} x_i \right\| < \frac{nl}{2^{k-1}},$$

and E_x is a complete space, the sequence $\{s_k\}$, $s_k = \sum_{i=1}^{k} x_i$, converges as $k \to \infty$ to some limit $x \in E_x$. Consequently,

$$x = \lim_k \sum_{i=1}^{k} x_i = \sum_{i=1}^{\infty} x_i .$$

Furthermore,

$$Ax = A\left(\lim_k \sum_{i=1}^{k} x_i \right) = \lim_k \sum_{i=1}^{k} Ax_i = \lim_k \sum_{i=1}^{k} y_i = y.$$

Hence,

$$\| A^{-1} y \| = \| x \| = \lim_k \left\| \sum_{i=1}^{\infty} x_i \right\| \leqslant \lim_k \sum_{i=1}^{k} \| x_i \|$$

$$\leqslant \sum_{i=1}^{\infty} \frac{nl}{2^{i-1}} = 2nl = 2n \| y \| .$$

Since y is any element of E_y, the boundedness of A^{-1} is proved.

We have so far indicated the cases when an operator inverse to a bounded linear operator, though linear, is defined not on the entire space E_y, but only on some linear manifold and is not bounded there. Exactly in the same way, an operator inverse to an unbounded linear operator, defined on some linear manifold everywhere dense in E_x, can be a bounded linear operator defined on the whole of E_y. A detailed treatment of these cases in arbitrary BANACH spaces is beyond the scope of this text. We shall, therefore, confine ourselves to two simple examples to make explicit the assertions made.

Examples. 1. Let $E = C[0, 1]$, and let

$$Ax = \int_0^t x(\tau)\, d\tau.$$

Then A is a bounded linear operator, but

$$A^{-1}y = \frac{d}{dt}\, y(t)$$

is an unbounded operator defined on the linear manifold of continuously differentiable functions, such that $y(0) = 0$.

2. Let $E = C[0, 1]$ and let $Ax = (d/dt)\{[p(t)]\,(dx/dt)\} + q(t)\,x$ be an unbounded STURM-LIOUVILLE operator defined on the linear manifold of twice continuously differentiable functions, such that $x(0) = x(1) = 0$. The inverse operator

$$A^{-1}y = \int_0^1 G(t, \tau)\, y(\tau)\, d\tau,$$

$G(t, \tau)$ GREEN's function, is a bounded linear operator defined on the entire $C[0, 1]$.

3.53. Operators depending on a parameter. Equations of the form

$$Ax - \lambda x = y \quad \text{or} \quad (A - \lambda I)x = y \tag{9}$$

are frequently encountered in various branches of mathematics. Here A is a linear operator and λ some parameter. Together with Eq. (9), consider the equation

$$Ax - \lambda x = 0 \quad \text{or} \quad (A - \lambda I)\, x = 0, \tag{10}$$

called the **homogeneous equation** corresponding to Eq. (9). This equation has always a zero solution, $x = 0$, called the **trivial solution.**

Assume that the operator $A - \lambda I$ has an inverse operator $(A - \lambda I)^{-1} = R_\lambda$ for some λ. The operator R_λ which depends on the parameter λ, is called the **resolvent** of (9). Then, for this λ, Eq. (9) has for every y the unique solution

$$x = R_\lambda y.$$

The corresponding homogeneous Eq. (10) has in this case only the trivial solution $x = 0$.

Those λ for which Eq. (9) has a unique solution for every y and the operator R_λ is bounded, are called **regular values** of Eq. (9) or of the operator A. If Eq. (10) has also a non-trivial solution for a given λ, then λ is called an **eigenvalue** or **characteristic number** of (9) or of A. The non-trivial solution is called an **eigenvector** of Eq. (9) or of operator A for the eigenvalue λ. If λ is an eigenvalue of the operator A and if Eq. (9) has a solution for some y, then this solution is not unique, because if x_0 is a solution of Eq. (9), $Ax_0 - \lambda x_0 = y$ and e is an eigenvector of A, corresponding to the eigenvalue λ, $Ae - \lambda e = 0$, then $A(x_0 + e) - \lambda(x_0 + e) = y$ and $x_0 + e$ is also a solution of Eq. (9).

All the non-regular values of λ are called the **points of the spectrum** of the operator A and the collection of non-regular values of λ is called the **spectrum** of A. In particular, all the eigenvalues belong to the spectrum. However, in more general cases, the collection of eigenvalues does not exhaust the spectrum of an operator.

From Theorems 2 and 3 comes the next proposition.

If the inequality $(1 / |\lambda|)\, \|A\| = q < 1$ *holds for* λ, *then* $A - \lambda I$ *has an inverse operator; moreover,*

$$R_\lambda = -\frac{1}{\lambda}\left(I + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \dots \right).$$

If λ is a regular value, then $\lambda + \Delta\lambda$ for $|\Delta\lambda| < \|(A-\lambda I)^{-1}\|^{-1}$ is also a regular value. This implies that the collection of regular values is an open set and, hence, the spectrum of an operator is a closed set.

Example. Consider in the space $C[0, 1]$ the integral equation

$$x(t) = y(t) + \lambda \int_0^1 K(t, s)\, x(s)\, ds, \tag{11}$$

$K(t, s)$ being continuous in the square $0 \leqslant t, s \leqslant 1$. Put $1/\lambda = \mu$, and rewrite Eq. (11) in the form, in which we considered above the functional equation; we obtain

$$\int_0^1 K(t, s)\, x(s)\, ds - \mu x(t) = -\mu y(t);$$

or, $Ax - \mu x = -\mu y$, using there the notation $Ax = \int_0^1 K(t, s)\, x(s)\, ds$. Furthermore,

$$R_\mu = R_{1/\lambda} = -\frac{1}{\mu}\left(I + \frac{A}{\mu} + \frac{A^2}{\mu^2} + \dots \right) = -\lambda(I+\lambda A+\lambda^2 A^2+\dots).$$

Note that $A^p z = \int_0^1 K_p(t, s)\, z(s)\, ds$, where $K_p(t, s)$ is the p-th *iterate* of the kernel $K(t, s)$. Consequently,

$$R_{\frac{1}{\lambda}}\, z = -\lambda z(t) - \lambda^2 \int_0^1 K(t, s)\, z(s)\, ds - \lambda^3 \int_0^1 K_2(t, s)\, z(s)\, ds - \dots$$

Hence, the solution of Eq. (11) is

$$\begin{aligned} x(t) &= R_{\frac{1}{\lambda}}\left(-\frac{1}{\lambda} y\right) \\ &= y(t) + \lambda \int_0^1 K(t, s)\, y(s)\, ds + \lambda^2 \int_0^1 K_2(t, s)\, y(s)\, ds + \dots \end{aligned}$$

Thus, we get the same solution

$$x(t) = y(t) + \lambda \int_0^1 R(t, s, \lambda)\, y(s)\, ds,$$

as in the theory of integral equations; here $R(t, s, \lambda)$, the *resolvent* of the kernel $K(t, s)$, is defined by

$$R(t, s, \lambda) = K(t, s) + \lambda K_2(t, s) + \lambda^2 K_3(t, s) + \dots$$

The equations for the resolvent $R(t, s, \lambda)$, appearing in the theory of integral equations, are the conditions that $R_{1/\lambda}$ is the right or left inverse operator of $\lambda A - I$.

3.6 BANACH SPACES WITH BASIS

3.61. Definition. Let E be an infinite-dimensional space of type B. A sequence of elements $e_1, e_2, \dots, e_n, \dots$ in E is called a **basis** for this space, if every element $x \in E$ has a unique representation of the form

$$x = \sum_{i=1}^{\infty} \xi_i e_i,$$

ξ_i real. The uniqueness of the representation is, evidently, equivalent to the condition that

$$\sum_{i=1}^{\infty} \xi_i e_i = 0$$

iff $\xi_i = 0$ for all i.

Examples. 1. Let $E = l_p$. Then, the collection of elements $e_1 = \{1, 0, 0, 0, \ldots$, $e_2 = \{0, 1, 0, 0, \ldots\}, \ldots$ forms a basis in l_p, since for every $x \in l_p$ there holds the unique representation

$$x = \sum_{i=1}^{\infty} \xi_i e_i,$$

if $x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$. In fact,

$$\sum_{i=1}^{n} \xi_i e_i = \{\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots\},$$

and hence

$$\left\| x - \sum_{i=1}^{n} \xi_i e_i \right\| = \| \{0, 0, \ldots, 0, \xi_{n+1}, \xi_{n+2}, \ldots\} \| = \left(\sum_{i=n+1}^{\infty} | \xi_i |^p \right)^{1/p} \to 0$$

as a remainder convergent series. Consequently,

$$x = \lim_n \sum_{i=1}^{n} \xi_i e_i = \sum_{i=1}^{\infty} \xi_i e_i.$$

Furthermore, if $x = \sum_{i=1}^{\infty} \xi_i e_i = \sum_{i=1}^{\infty} \xi_i' e_i$,

that is, $$0 = \sum_{i=1}^{\infty} (\xi_i - \xi_i')\, e_i = \{\xi_1 - \xi_1', \xi_2 - \xi_2', \ldots\},$$

then $\xi_i = \xi_i'$, $i = 1, 2, \ldots$, as was required to prove.

2. Let $E = C[0, 1]$. Consider in $C[0, 1]$ the sequence of elements

$$t, \quad 1 - t, \quad u_{00}(t), \quad u_{10}(t), \quad u_{11}(t), \quad u_{20}(t), \quad u_{21}(t), \quad u_{22}(t), \ldots, \tag{1}$$

where $u_{kl}(t)$, $k = 1, 2, \ldots$, $0 \leqslant l < 2^k$ are defined in the following way: $u_{kl}(t) = 0$, if t is located outside the interval $(l/2^k, (l+1)/2^k)$, but inside of this interval $u_{kl}(t)$ has a graph in the form of a triangular isosceles with height equal to 1 (the graph of the function $u_{22}(t)$ is given in Fig. 3).

Take a function $x(t) \in C[0, 1]$ representable in the form of the series

$$x(t) = a_0 t + a_1(1-t) + \sum_{k=0}^{\infty} \sum_{l=0}^{2^k-1} a_{kl}\, u_{kl}(t), \tag{2}$$

where $a_0 = x(1)$, $a_1 = x(0)$ and the coefficients a_{k_l} admit a unique geometric construction depicted in Fig. 4.

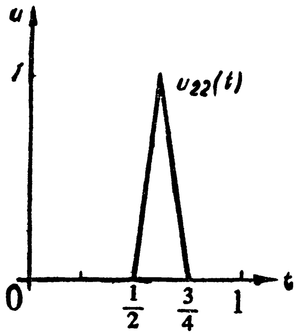

Fig. 3.

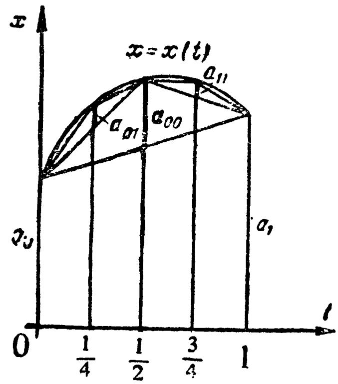

Fig. 4.

The graph of the partial sums of the series (2)

$$a_0 t + a_1(1-t) + \sum_{k=0}^{s-1} \sum_{l=0}^{2^k-1} a_{kl}\, u_{kl}(t)$$

is, evidently, an open polygon with $2^s + 1$ vertices lying on the curve $x = x(t)$ at the points with equidistant abscissae. The collection of functions (1) forms a basis in $C[0, 1]$.

If a space E has a basis, then it is, evidently, a separable space. A countable everywhere dense set in a space with basis is a set of linear combinations of the form $\sum_{i=1}^{n} r_i e_i$ with rational coefficients r_i. It is natural to assume that every separable B-space has a basis. However, though a basis is constructed for all known concrete separable BANACH spaces, the existence of basis in an arbitrary separable B space is not proved.

Thus, let $E = E_x$ be a B-spacc with a basis $e_1, e_2, \ldots, e_n, \ldots$ Consider a linear space E_y, whose elements are all possible numbers of the sequence $y = \{\eta_1, \eta_2, \ldots, \eta_n, \ldots\}$, such that the series $\sum_{i=1}^{\infty} \eta_i e_i$ is convergent.

Introduce a norm in E_y by setting

$$\| y \| = \sup_n \left\| \sum_{i=1}^{n} \eta_i e_i \right\|.$$

It is to be shown that E_y is a B-space. In fact, it is an easy exercise to verify that the norm axioms are satisfied. Now, given a CAUCHY sequence

$$\{ y_k \} \subset E_y, \qquad y_k = \{\eta_i^{(k)}\}_{i=1,2,\dots}$$

Then, for a given $\varepsilon > 0$,

$$\| y_m - y_k \| = \sup_n \left\| \sum_{i=1}^{n} (\eta_i^{(m)} - \eta_i^{(k)}) e_i \right\| < \varepsilon \qquad \text{for } m, k \geqslant m_0(\varepsilon),$$

and, consequently,

$$\left\| \sum_{i=1}^{n} (\eta_i^{(m)} - \eta_i^{(k)}) e_i \right\| < \varepsilon \tag{3}$$

for $m, k \geqslant m_0(\varepsilon)$ and every n. Thereupon,

$$\| (\eta_n^{(m)} - \eta_n^{(k)}) e_n \| = \left\| \sum_{i=1}^{n} (\eta_i^{(m)} - \eta_i^{(k)}) e_i - \sum_{i=1}^{n-1} (\eta_i^{(m)} - \eta_i^{(k)}) e_i \right\|$$

$$\leqslant \left\| \sum_{i=1}^{n} (\eta_i^{(m)} - \eta_i^{(k)}) e_i \right\| + \left\| \sum_{i=1}^{n-1} (\eta_i^{(m)} - \eta_i^{(k)}) e_i \right\| < 2\varepsilon,$$

and, hence, $| \eta_n^{(m)} - \eta_n^{(k)} | < 2\varepsilon / \| e_n \|$, for $m, k \geqslant m_0(\varepsilon)$ and every n. Consequently, the number sequence $\{ \eta_n^{(m)} \}_{m=1,2,\dots}$ converges to some limit $\eta_n^{(0)}$, and this holds for every n.

Taking limit in Ineq. (3) as $k \to \infty$, yields

$$\left\| \sum_{i=1}^{n} (\eta_i^{(m)} - \eta_i^{(0)}) e_i \right\| \leqslant \varepsilon \tag{4}$$

for $m \geqslant m_0(\varepsilon)$ and every n. Put

$$s_n^{(m)} = \sum_{i=1}^{n} \eta_i^{(m)} e_i, \qquad s_n^{(0)} = \sum_{i=1}^{n} \eta_i^{(0)} e_i.$$

Taking note of (4), we obtain

$$\left\| s_{n+p}^{(0)} - s_n^{(0)} \right\| \leqslant \left\| s_{n+p}^{(m)} - s_n^{(m)} \right\| + 2\varepsilon$$

for $m \geqslant m_0(\varepsilon)$, every n and $p > 0$.

Now, given $\delta > 0$, an arbitrary number. Take again ε and, by the same token, also $m_0(\varepsilon)$, such that $2\varepsilon < \delta/2$; then, let $m \geqslant m_0(\varepsilon)$ be held fixed and select n_0, such that $\left\| s_{n+p}^{(m)} - s_n^{(m)} \right\| < \delta/2$ for $n \geqslant n_0$ and every $p > 0$ (this is plausible, the series $\sum_{i=1}^{\infty} \eta_i^{(m)} e_i$ being convergent). Then,

$$\left\| s_{n+p}^{(0)} - s_n^{(0)} \right\| < \delta$$

for $n \geqslant n_0$ and every $p > 0$, that is, the series $\sum_{i=1}^{\infty} \eta_i^{(0)} e_i$ converges and, consequently,

$$y_0 = \{\eta_1^{(0)}, \eta_2^{(0)}, \ldots, \eta_n^{(0)}, \ldots\} \in E_y.$$

Besides, since Ineq. (4) implies that

$$\sup_n \left\| \sum_{i=1}^{n} (\eta_i^{(m)} - \eta_i^{(0)}) e_i \right\| \leqslant \varepsilon \qquad \text{for } m \geqslant m_0,$$

that is, $\| y_m - y_0 \| \leqslant \varepsilon$ for $m \geqslant m_0$, it follows that E_y is complete.

Evidently, a unique element $y_x = \{ \xi_1, \xi_2, \ldots, \xi_n, \ldots\} \in E_y$ corresponds to every $x = \sum_{i=1}^{\infty} \xi_i e_i \in E_x$. Conversely, a unique element $x_y \in E_x$, $x_y = \sum_{i=1}^{\infty} \eta_i e_i$ corresponds to every element $y = \{ \eta_i \} \in E_y$.

Thus, it is possible to regard that a definite operator $x = Ay$ maps one-one the space E_y onto E_x. It is plain that A is a linear operator. Besides, A is bounded. In fact,

$$\| Ay \| = \| x \| = \left\| \sum_{i=1}^{\infty} \eta_i e_i \right\| \leqslant \sup_n \left\| \sum_{i=1}^{n} \eta_i e_i \right\| = \| y \|.$$

Consequently, we have a linear operator A which maps one-one the space E_y onto E_x. By Banach's theorem there is an inverse operator $y = A^{-1} x$ which is also a bounded linear operator.

Let $x = \sum_{i=1}^{\infty} \xi_i e_i$ be any element in E_x. Determine the functional f_k, setting $f_k(x) = \xi_k$. Evidently, the functional f_k is additive. Furthermore,

$$\begin{aligned} | f_k(x) | &= | \xi_k | = \frac{|\xi_k| \, \|e_k\|}{\|e_k\|} = \frac{\left\| \sum_{i=1}^{k} \xi_i e_i - \sum_{i=1}^{k-1} \xi_i e_i \right\|}{\|e_k\|} \\ &\leqslant 2 \sup_n \left\| \sum_{i=1}^{n} \xi_i e_i \right\| \frac{1}{\|e_k\|} = \frac{2 \, \|y\|}{\| e_k \|} \\ &= \frac{2 \| A^{-1} x \|}{\| e_k \|} \leqslant \frac{2 \| A^{-1} \|}{\|e_k\|} \, \| x \|, \end{aligned}$$

which implies the boundedness and, consequently, the linearitly of f_k, as well as that

$$\| f_k \| \leqslant 2 \, \frac{\|A^{-1}\|}{\|e_k\|}.$$

Constructing for every k the functional f_k, we obtain an infinite sequence ot linear functionals $f_1, f_2, \ldots, f_n, \ldots, \subset E^*$; moreover, every element $x \in E$ is expressible in the form

$$x = \sum_{i=1}^{\infty} f_i(x)\, e_i.$$

Set, in particular, $x = e_j$. Then,

$$\xi_i = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases}$$

that is,

$$f_i(e_j) = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{5}$$

Thus, we obtain two sequences: a sequence of elements $\{e_i\}$ and a sequence of functionals $\{f_i\}$, which satisfy Eq. (5). These two sequences are called **biorthogonal.**

Now, take any linear functional $f \in E^*$. Since

$$x = \sum_{i=1}^{\infty} f_i(x)\, e_i = \lim_n \sum_{i=1}^{\infty} f_i(x)\, e_i,$$

it follows that

$$f(x) = \lim_n \sum_{i=1}^{n} f[f_i(x)\, e_i] = \lim_n \sum_{i=1}^{n} f_i(x) f(e_i)$$

$$= \sum_{i=1}^{\infty} f_i(x) f(e_i).$$

Write $f(e_i) = c_i$. Then, for every linear functional $f \in E^*$, there holds the representation

$$f(x) = \sum_{i=1}^{\infty} c_i f_i(x),$$

or,

$$f = \sum_{i=1}^{\infty} c_i f_i. \tag{6}$$

The representation (6) is, evidently, unique. The series (6) converges for for every $x \in E$.

Again, let x be any element in E. Then,

$$x = \sum_{j=1}^{\infty} \xi_j e_j = \sum_{j=1}^{n} \xi_j e_j + \sum_{j=n+1}^{\infty} \xi_j e_j,$$

and the following two uniquely defined elements

$$y_n = \sum_{j=1}^{n} \xi_j e_j \qquad \text{and} \qquad z_n = \sum_{j=n+1}^{\infty} \xi_j e_j$$

can be associated to every element $x \in E$. These equalities yield the two operators

$$y_n = S_n x \qquad \text{and} \qquad z_n = R_n x,$$

with domain E and range in E.

Evidently, S_n and R_n are bounded linear operators for every fixed n. In fact, the linearity of these is obvious and the boundedness is implied by the inequalities

$$\| S_n x \| \leqslant \sup_m \left\| \sum_{j=1}^{m} \xi_j e_j \right\| = \| A^{-1} x \| \leqslant \| A^{-1} \| \| x \|,$$

and analogously, $$\| R_n x \| \leqslant 2 \| A^{-1} \| \| x \|.$$

CHAPTER 4

LINEAR FUNCTIONALS

THIS CHAPTER aims at elaborating some simple properties of functionals defined in a normed linear space.

The starting point is some of the theorems proved earlier, which remain true both for operators and functionals. These are recapitulated below with suitable adaptations where called for in the case of functionals.

THEOREM (BANACH-STEINHAUS). *If a sequence of linear functionals, defined on a Banach space E, is bounded at every point $x \in E$, then the sequence of norms $\{\| f_n \|\}$ of these functionals is also bounded.*

THEOREM 1. *If a sequence of linear functionals $\{f_n(x)\}$ is a Cauchy sequence at every point of a Banach space E, then there is a linear functional $f(x)$, such that $f_n(x) \to f(x)$ for every $x \in E$.*

THEOREM 2. *In order that a sequence $\{f_n\}$ of linear functionals be convergent to a functional f_0 at every point x of a Banach space E, it is necessary and sufficient that*

(i) *the norm sequence $\{\| f_n \|\}$ is bounded; and*

(ii) *$f_n(x) \to f_0(x)$ for every x from some set $M \subset E$, a linear combination of whose elements lies everywhere dense in E.*

THEOREM 3. *A linear functional f_0, defined on a linear manifold L, which is everywhere dense in a normed linear space E and is bounded there, can be extended to the entire space without increasing the norm and also uniquely.*

4.1 HAHN-BANACH EXTENSION THEOREM AND ITS COROLLARIES

THE NEXT theorem admits the possibility that a linear functional defined initially on a linear manifold L of a normed linear space E, not necessarily everywhere dense in E, can be extended to the entire space with preservation of the norm.

THEOREM 4 (BANACH-HAHN). *Every linear* bounded *functional $f(x)$ defined on a linear manifold L of a normed linear space E, can be extended to the entire space with preservaton of the norm, that is, we can construct a linear functional $F(x)$, defined on E and such that*

$$\text{(i)} \quad F(x) = f(x) \text{ for } x \in L; \qquad \text{(ii)} \quad \| F \|_E = \| f \|_L.$$

PROOF. Take an element $x_0 \overline{\in} L$, and consider a set $(L; x_0) = L_1$ of elements of the form $x + tx_0$, $x \in L$, and t any real number.

Evidently, the set L_1 is a linear manifold. It is to be shown that each of its elements has a unique representation in the form $x + tx_0$. Assume that

$u \in L_1$ has two representations: $u = x_1 + t_1x_0$ and $u = x_2 + t_2x_0$, and also that $t_1 \neq t_2$ (otherwise, $x_1 + t_1x_0 = x_2 + t_1x_0$ would imply $x_1 = x_2$ and the representation would be unique). We have

$$x_1 - x_2 = (t_2 - t_1)\, x_0 \quad \text{and} \quad x_0 = \frac{x_1 - x_2}{t_2 - t_1}.$$

This is, however, impossible, since $x_0 \bar{\in} L$, but x_1 and $x_2 \in L$. Thus, $t_1 = t_2$ and, hence, $x_1 = x_2$, proving the representation to be unique.

Now take any two elements x' and $x'' \in L$. We have

$$\begin{aligned} f(x') - f(x'') = f(x' - x'') &\leqslant \|f\|\, \|x' - x''\| \\ &\leqslant \|f\| \{ \|x' + x_0\| + \|x'' + x_0\| \}, \end{aligned}$$

whence $f(x') - \|f\|\, \|x' + x_0\| \leqslant f(x'') + \|f\|\, \|x'' + x_0\|$.

Since x' and x'' are arbitrary in L, independent of each other, it follows that

$$\sup_{x \in L} \{ f(x) - \|f\|\, \|x + x_0\| \} \leqslant \inf_{x \in L} \{ f(x) + \|f\|\, \|x + x_0\| \}.$$

Consequently, there is a real number c, satisfying the inequality

$$\sup_{x \in L} \{ f(x) - \|f\|\, \|x + x_0\| \} \leqslant c \leqslant \inf_{x \in L} \{ f(x) + \|f\|\, \|x + x_0\| \}. \tag{1}$$

Now, take any element $u \in L_i$. By what has been proved, this has the form $u = x + tx_0$, $x \in L$ and t a uniquely defined real number. Introduce a new functional $\varphi(u)$, defined for all elements $u = x + tx_0$ by the equality

$$\varphi(u) = f(x) - tc,$$

c some fixed real number which satisfies (1). Evidently, f and φ coincide on L. It is also obvious that $\varphi(u)$ is additive. To show that $\varphi(u)$ is bounded and has the same norm as $f(x)$, consider the two cases:

(i) For $t > 0$. From $(x/t) \in L$ and (1) it follows that

$$\begin{aligned} |\varphi(u)| = t \left| f\left(\frac{x}{t}\right) - c \right| &\leqslant t \left\{ \|f\| \left\| \frac{x}{t} + x_0 \right\| \right\} \\ &= \|f\|\, \|x + tx_0\| = \|f\| \cdot \|u\|, \end{aligned}$$

thus,

$$|\varphi(u)| \leqslant \|f\| \cdot \|u\|. \tag{2}$$

(ii) For $t < 0$. Then, (1) yields

$$\begin{aligned} f\left(\frac{x}{t}\right) - c &\geqslant -\|f\| \cdot \left\| \frac{x}{t} + x_0 \right\| \\ &= -\frac{1}{|t|} \|f\| \cdot \|x + tx_0\| = \frac{1}{t} \|f\| \cdot \|u\|, \end{aligned}$$

whence

$$\varphi(u) = t \left\{ f\left(\frac{x}{t}\right) - c \right\} \leqslant t \cdot \frac{1}{t} \|f\|\, \|u\| = \|f\|\, \|u\|,$$

that is, we again arrive at (2).

Thus, Ineq. (2) remains valid for all $u \in (L; x_0) = L_1$. Substitute $-u$ for u in (2), to receive $-\varphi(u) \leqslant \|f\| \, \|u\|$. Thereupon, and from (2), $|\varphi(u)| \leqslant \|f\| \cdot \|u\|$, whence $\|\varphi\| \leqslant \|f\|$. However, since the functional φ is an extension of f from L to L_1,

$$\|\varphi\| \geqslant \|f\|. \qquad \text{Consequently,} \qquad \|\varphi\| = \|f\|.$$

(note that we have determined the norm of the functional φ with respect to that linear manifold on which φ is defined). Thus the functional $f(x)$ is extended to $L_1 = (L; x_0)$ with preservation of the norm.

REMARK. If the space E is separable, then the proof of the HAHN-BANACH theorem can be completed in the following way. Let N be a countable everywhere dense set in E. Select those elements of this set, which do not fall in L, and arrange them in the sequence

$$x_0, x_1, x_2, \ldots, x_n, \ldots$$

Extending the functional $f(x)$ successively to the manifolds $(L; x_0) = L_1$, $(L_1; x_1) = L_2, \ldots$ so on and so forth, we ultimately construct a certain linear functional φ_ω defined on the linear manifold L_ω, which is everywhere dense in E and is equal to the union of all L_n; moreover, $\|\varphi_\omega\| = \|f\|$. Then, extending the functional φ_ω by continuity to the entire E (Theorem 3), we arrive at the desired functional F. This completes the proof of the BANACH-HAHN theorem in the general case.

Consider all possible extensions of the functional f with preservation of the norm. As already shown, such extensions exist. In a set Φ of these extensions, introduce a partial ordering by assuming that $f' \prec f''$, if a linear manifold L', on which f' is defined, is a partial linear manifold of L'' on which f'' is defined and, if $f'(x) = f''(x)$ for $x \in L'$. Evidently, the relation $f' \prec f''$ has all the properties of ordering.

Now, let $\{f_\alpha\}$ be an arbitrary ordered subset of the set Φ. This subset has an upper bound, which is the functional f_*, defined on a linear manifold $L_* = \bigcup_\alpha L_\alpha$, where L_α is the domain of f_α, and also

$$f_*(x) = f_{\alpha_0}(x),$$

if $x \in L_*$ is an element of L_{α_0}. Evidently, f_* is a linear functional and $\|f_*\| = \|f\|$, that is, $f_* \in \Phi$. Thus, it is seen that all the hypotheses of ZORN's lemma are satisfied and Φ has a maximal element F. This functional is defined on the entire E, because in the opposite case it could be extendable, violating the maximality of F in Φ.

The theorem is completely proved.

REMARK. Since the number c, satisfying (1), may be preassigned and there may not be only a single maximal element in Φ, the extension of a linear functional by the BANACH-HAHN theorem is, generally, not unique.

G.A. SUKHOMLINOV† has generalized the theorem on the extension of functionals to spaces of complex numbers and quaternions.

COROLLARY 1. *Let E be a normed linear space and $x_0 \neq 0$ be any fixed element in E. Then, there is a linear* bounded *functional $f(x)$, defined on the entire space E, and such that*

$$\text{(i) } \| f \| = 1 \quad \textit{and} \quad \text{(ii) } f(x_0) = \| x_0 \|.$$

PROOF. Consider a set of elements $\{tx_0\} = L$, where t runs through all possible real numbers. The set L is a subspace of E, spanned by x_0. A functional $\varphi(x)$, defined on L, has the following form: if $x = tx_0$, then

$$\varphi(x_0) = t \, \| x_0 \|. \tag{3}$$

Evidently, (i) $\varphi(x_0) = \| x_0 \|$, and (ii) $| \varphi(x) | = | t | \, \| x_0 \| = \| x \|$, whence $\| \varphi \| = 1$.

Now, if the functional $\varphi(x)$ is extended to the entire space with preservation of the norm, we get a functional $f(x)$ having the requisite properties.

COROLLARY 2. *Given a linear manifold L and an element $x_0 \in \overline{L}$ in a normed linear space E. Let $d > 0$ be the distance from x_0 to L ($d = \inf\limits_{x \in L} \| x_0 - x \|$). Then, there is a functional $f(x)$ defined everywhere on E, and such that*

$$\text{(i) } f(x) = 0 \text{ for } x \in L; \quad \text{(ii) } f(x_0) = 1\,; \quad \text{and} \quad \text{(iii) } \| f \| = 1/d.$$

PROOF. Consider a set $(L; x_0)$. Each of its elements is uniquely representable in the form $u = x + tx_0$ where $x \in L$ and t real. Construct the functional $\varphi(u)$ by the rule : If $u = x + tx_0$, then $\varphi(u) = t$. Evidently, $\varphi(x) = 0$, if $x \in L$ and $\varphi(x_0) = 1$. To determine $\| \varphi \|$, we have

$$| \varphi(u) | = | t | = \frac{| t | \, \| u \|}{\| u \|} = \frac{| t | \, \| u \|}{\| x + tx_0 \|}$$
$$= \frac{\| u \|}{\left\| \frac{x}{t} + x_0 \right\|} = \frac{\| u \|}{\left\| x_0 - \left(- \frac{x}{t} \right) \right\|} \leqslant \frac{\| u \|}{d},$$

whence
$$\| \varphi \| \leqslant 1/d. \tag{4}$$

Furthermore, there is a sequence $\{x_n\} \subset L$, such that

$$\lim_n \| x_n - x_0 \| = d.$$

Then, we have $| \varphi(x_n - x_0) | \leqslant \| \varphi \| \, \| x_n - x_0 \|$. Since $| \varphi(x_n - x_0) | = |\varphi(x_n) - \varphi(x_0) | = 1$, it also follows that $1 \leqslant \| \varphi \| \, \| x_n - x_0 \|$. Hence, by taking the limit, we get $1 \leqslant \| \varphi \| \, d$, or

$$\| \varphi \| \geqslant 1/d. \tag{5}$$

Thereupon (4) and (5) yield $\| \varphi \| = 1/d$. Thus, by extending $\varphi(x)$ to the entire space with preservation of the norm, we obtain a functional $f(x)$ with the requisite properties.

† *Mat. Sbornik* 3 (45), 1938.

Corollary 1 proves the existence of a functional in an arbitrary normed linear space, which does not vanish identically. On the other hand, it implies that $x = 0$ if for some element x of a normed linear space E, the equality $f(x) = 0$ is satisfied for every linear functional in the conjugate space E^*.

This corollary can be given a geometric interpretation as follows.

Through every point x_0 on the surface of a sphere $\| x \| \leqslant r$, that is, such that $\| x_0 \| = r$, a supporting plane for this sphere can be drawn.

This theorem is a generalization of the theorem proved by MINKOWSKI for the n-dimensional spaces.

In fact, the equation of the suporting plane for such a sphere must have the form $f(x) = r \| f \|$. However, there is a functional f_0 for x_0 with $\| f_0 \| = 1$, such that

$$f_0(x_0) = \| x_0 \| = r. \tag{6}$$

The plane $f_0(x) = r$ is a supporting plane and in virtue of (6) passes through x_0.

Corollary 2 is of interest for it makes explicit the permissibility of approximating a given element x_0 by means of linear combinations of other given elements $\{x_1, x_2, \ldots, x_n, \ldots\} \subset E$. In fact, it yields the next theorem.

THEOREM. *In order that x_0 be the limit of some sequence of linear combinations of the form $\sum_{i=1}^{n} c_i x_i$, it is necessary and sufficient that $f(x_0) = 0$ for all linear functionals f which vanish for $x_1, x_2, \ldots$*

PROOF. In fact, let $f(x_i) = 0$, $i = 1, 2, \ldots$, imply $f(x_0) = 0$. Then, x_0 cannot have a distance $d > 0$ from the linear manifold L, spanned by the elements $\{x_i\}$, because otherwise, by Corollary 2, there would exist a functional f_0 such that $f_0(x_i) = 0$, $i = 1, 2, \ldots$, whereas $f_0(x_0) = 1$. However, if $d = 0$, then this means that x_0 is either a limit point of a linear manifold L, or $x_0 \in L$ and, consequently, x_0 can be approximated by the elements of the form $\sum_{i=1}^{n} c_i x_i$.

Conversely, let x_0 be the limit of a sequence of elements in L and let $(x_i)f = 0$ for some functional f. Then, setting

$$x_0 = \lim_n \xi_n, \qquad \xi_n = \sum_{i=1}^{k_n} c_i^{(n)} x_i,$$

we get
$$f(\xi_n) = \sum_{i=1}^{k_n} c_i^{(n)} f(x_i) = 0,$$

and, consequently,
$$f(x_0) = f(\lim_n \xi_n) = \lim_n f(\xi_n) = 0.$$

4.2 THE GENERAL FORM OF LINEAR FUNCTIONALS IN CERTAIN FUNCTIONAL SPACES

FOR SEVERAL concrete functional spaces, it is possible to indicate the general form of linear functionals defined on these speces. The general form of the functionals finds fruitful applications in various studies of functional spaces.

4.21. Linear functionals on the n-dimensional space E_n. Let f be a linear functional defined on E_n. For $x = \sum_{i=1}^{n} \xi_i e_i \in E_n$, where $\{e_1, e_2, \ldots, e_n\}$ is a basis of E_n, we have

$$f(x) = f\left(\sum_{i=1}^{n} \xi_i e_i\right) = \sum_{i=1}^{n} \xi_i f(e_i) = \sum_{i=1}^{n} \xi_i f_i.$$

Conversely, $f(x)$ expressed in the form

$$f(x) = \sum_{i=1}^{n} \xi_i f_i, \tag{1}$$

where the f_i are arbitrary, is, evidently, a linear functional on E_n. Thus, the expression (1) yields a general form of the linear functionals defined on an n-dimensional space. Since f_i can be regarded as the components of an n-dimensional vector f, the space E_n^*, the dual of E_n, is also an n-dimensional space with a metric, generally speaking, different from the metric of E_n.

For example, let $\| x \| = \max_i | \xi_i |$; then

$$|f(x)| = \left|\sum_{i=1}^{n} \xi_i f_i\right| \leqslant \sum_{i=1}^{n} |\xi_i| \, |f_i| \leqslant \left(\sum_{i=1}^{n} |f_i|\right) \| x \|,$$

whence
$$\| f \| \leqslant \sum_{i=1}^{n} | f_i |. \tag{2}$$

On the other hand, if we select an element $x_0 = \sum_{i=1}^{n} \operatorname{sgn} f_i \cdot e_i \in E_n$, then $\| x_0 \| = 1$ and

$$f(x_0) = \sum_{i=1}^{n} \operatorname{sgn} f_i \cdot f_i = \sum_{i=1}^{n} | f_i | = \left(\sum_{i=1}^{n} | f_i |\right) \| x_0 \|,$$

whence
$$\| f \| \geqslant \sum_{i=1}^{n} | f_i |. \tag{3}$$

From (2) and (3) it follows that $\| f \| = \sum_{i=1}^{n} | f_i |$.

If an Euclidean metric is introduced in E_n, then it is easy to verify that the metric in E_n^* is also Euclidean.

In analogy with the terminology used in tensor algebra, the elements of E_n are called **contavariants** and those of E_n^* **covariants**. A linear functional $f(x)$ is representable in the form of the scalar product

$$f(x) = (x, f), \quad \text{where } x \in E_n, f \in E_n^* .$$

4.22. The general form of Linear functionals on s. Let $f(x)$ be a linear functional defined on s (see p. 12). Put $e_n = \{\xi_i^{(n)}\}$, where $\xi_n^{(n)} = 1$ and $\xi_i^{(n)} = 0$ for $i \neq n$. Further, let $f(e_n) = a_n$. Since the convergence in s is coordinatewise, the equality

$$x = \lim_n \sum_{k=1}^{n} \xi_k e_k = \sum_{k=1}^{\infty} \xi_k e_k$$

holds for the element $x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$. Thereupon, because of the continuity of $f(x)$,

$$f(x) = \sum_{k=1}^{\infty} \xi_k f(e_k) = \sum_{k=1}^{\infty} a_k \xi_k.$$

Since this series must converge for every number sequence $\{\xi_k\}$, the a_k must be equal to zero from a certain index on and, consequently,

$$f(x) = \sum_{k=1}^{n} a_k \xi_k .$$

Conversely, since such expression for every real a_k and every natural number n, is a linear functional on s, it follows that every linear functional defined on s has the general form

$$f(x) = \sum_{k=1}^{n} a_k \xi_k.$$

The numbers n and a_k, $k = 1, 2, \ldots, n$, are uniquely defined by the functional f.

4.23. The general form of linear functionals on $C[0, 1]$. **The Riesz Theorem.** Let $f(x)$ be a linear functional defined on $C[0, 1]$. Since every continuous function, defined on $[0, 1]$, is bounded and since

$$\sup_{0 \leqslant t \leqslant 1} x(t) = \max_{0 \leqslant t \leqslant 1} x(t)$$

for every continuous function $x(t)$, the space $C[0, 1]$ can be regarded as a subspace of $M[0, 1]$, where

$$\| x \| = \rho(x, \theta) = \sup_{0 \leqslant t \leqslant 1} | x(t) |.$$

Extend the functional $f(x)$, defined on $C[0, 1]$, to the whole of $M[0, 1]$ with preservation of the norm; denote this extension by $F(x)$. Consider the function

$$u_t(\xi) = \begin{cases} 1 & \text{for} \quad 0 \leqslant \xi < t, \\ 0 & \text{for} \quad t \leqslant \xi \leqslant 1. \end{cases}$$

Obiviously, $u_t(\xi) \in M[0, 1]$. Let $F[u_t(\xi)] = g(t)$. To show that the function $g(t)$ is of bounded variation, partition the interval $[0, 1]$ into

$$t_0 = 0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = 1.$$

Construct the sum $\sum_{i=1}^{n} | g(t_i) - g(t_{i-1}) |$, and put $\varepsilon_i = \operatorname{sgn} [g(t_i) - g(t_{i-1})]$.

Then

$$\sum_{i=1}^{n} | g(t_i) - g(t_{i-1}) | = \sum_{i=1}^{n} \varepsilon_i [g(t_i) - g(t_{i-1})]$$

$$= \sum_{i=1}^{n} \varepsilon_i [F(u_{t_i}) - F(u_{t_{i-1}})]$$

$$= F\left[\sum_{i=1}^{n} \varepsilon_i (u_{t_i} - u_{t_{i-1}}) \right].$$

Thereupon, $\sum_{i=1}^{n} | g(t_i) - g(t_{i-1}) | \leqslant \| F \| \left\| \sum_{i=1}^{n} \varepsilon_i (u_{t_i} - u_{t_{i-1}}) \right\| \leqslant \| f \|$,

since $\quad \| F \| = \| f \| \quad$ and $\quad \left\| \sum_{i=1}^{n} \varepsilon_i (u_{t_i} - u_{t_{i-1}}) \right\| = 1.$

Thus,

$$\sum_{i=1}^{n} | g(t_i) - g(t_{i-1}) | \leqslant \| f \|,$$

and, consequently, $g(t)$ is of bounded variation. Take any continuous function $x(t)$, defined on $[0, 1]$, and construct the function

$$z_n(t) = \sum_{k=1}^{n} x\left(\frac{k}{n}\right)\left[u_{\frac{k}{n}}(t) - u_{\frac{k-1}{n}}(t) \right],$$

$z_n(t)$ a step function. We get

$$F(z_n) = \sum_{k=1}^{n} x\left(\frac{k}{n}\right)\left[g\left(\frac{k}{n}\right) - g\left(\frac{k-1}{n}\right) \right].$$

Hence

$$\lim_n F(z_n) = \lim_n \sum_{k=1}^{n} x\left(\frac{k}{n}\right)\left[g\left(\frac{k}{n}\right) - g\left(\frac{k-1}{n}\right)\right] = \int_0^1 x(t)\,dg(t).$$

On the other hand, as $n \to \infty$ the sequence $\{z_n(t)\}$ converges uniformly to $x(t)$, that is, $\| z_n - x \| \to 0$, and since $F(x)$ is continuous, it follows that $F(z_n) \to F(x)$. Hence

$$F(x) = \int_0^1 x(t)\,dg(t).$$

By assumption $x(t)$ is continuous, hence $F(x) = f(x)$. Therefore,

$$f(x) = \int_0^1 x(t)\,dg(t) \tag{4}$$

Furthermore, the function $g(t)$ can be replaced by a function $\bar{g}(t)$, which coincides with it at the point of continuity and is semi-continuous from the left: $\bar{g}(t-0) = \bar{g}(t)$.

Thus, we are led to the next theorem, the well-known RIESZ representation theorem.

THEOREM (F. RIESZ). *Every linear functional, defined on* $C[0, 1]$, *can be represented by the Stieltjes integral* (4), *where* $g(t)$ *is a function of bounded variation, defined by the functional* $f(x)$.

Conversely, as can be seen easily, *the functional*

$$\varphi(x) = \int_0^1 x(t)\,dh(t),$$

$h(t)$ *any function of bounded variation, is a linear functional on* $C[0, 1]$.

Obviously, $\varphi(x)$ is additive. Since it is possible to take the limit for uniformly convergent sequences of functions under the STIELTJES integral, it follows that $\varphi(x)$ is also continuous. Thus (4) is the general form of linear functionals on $C[0, 1]$, in the sense that if $g(t)$ runs through all the functions of bounded variation, then this formula expresses all the linear functionals on $C[0, 1]$.

Now, determine the norm of $f(x)$. We have

$$\sum_{i=1}^{n} | g(t_i) - g(t_{i-1}) | \leqslant \| f \|,$$

whence

$$\operatorname*{var}_0^1 \{g\} \leqslant \| f \|. \tag{5}$$

On the other hand, (4) implies

$$|f(x)| = \left| \int_0^1 x(t)\, dg(t) \right| \leqslant \max_{0 \leqslant t \leqslant 1} |x(t)| \operatorname*{var}_0^1 \{g\} = \operatorname*{var}_0^1 \{g\} \, \| x \|.$$

Thereupon,
$$\| f \| = \operatorname*{var}_0^1 \{g\}. \tag{6}$$

Thus (5) and (6) imply
$$\| f \| = \operatorname*{var}_0^1 \{g\}. \tag{7}$$

If two functions of bounded variation are regarded identical in case they differ by at most an additive constant at all points of continuity, then it is easy to prove that the correspondence between the linear functionals on $C[0, 1]$ and the functions of bounded variation on $[0, 1]$, established by (4), becomes one-one.

If the function $g(t)$ is replaced by $\bar{g}(t)$ in (4), Ineq. (6) remains valid, but (5) is merely strengthened. Thus Eq. (7) is preserved.

A. A. Markov† extended the Riesz theorem by finding the general form of linear functionals on the space $C(K)$ of all functions continuous on a certain compact set K.

4.24. The general form of linear functionals on l_p. Let $f(x)$ be a linear functional, defined on l_p. Since the elements $e_k = \{\xi_i^{(k)}\}$, where $\xi_k^{(k)} = 1$ and $\xi_i^{(k)} = 0$ for $i \neq k$, form a basis of l_p, every element $x \in l_p$ can be expressed in the form

$$x = \sum_{k=1}^{\infty} \xi_k e_k.$$

The linearity of $f(x)$ implies

$$f(x) = \sum_{k=1}^{\infty} \xi_k f(e_k).$$

Set $f(e_k) = c_k$. Then, the numbers c_k are uniquely defined by f, so that the preceding equality becomes

$$f(x) = \sum_{k=1}^{\infty} c_k \xi_k. \tag{8}$$

This makes explicit the characteristics of c_k. Put $x_n = \{\xi_k^{(n)}\}$, where

$$\xi_k^{(n)} = \begin{cases} |c_k|^{q-1} \operatorname{sgn} c_k, & \text{if} \quad k \leqslant n, \\ 0 \quad , & \text{if} \quad k > n. \end{cases}$$

†*Mat. Sbornik* **4** (46), 1938.

Here q is chosen such that the equality $[(1/p) + (1/q)] = 1$ holds. Then,

$$f(x_n) = \sum_{k=1}^{n} |c_k|^q.$$

On the other hand,

$$f(x_n) \leqslant \|f\| \cdot \|x_n\| = \|f\| \left(\sum_{k=1}^{n} |c_k|^{(q-1)p} \right)^{1/p} = \|f\| \left(\sum_{k=1}^{n} |c_k|^q \right)^{1/p}.$$

Thus,
$$\sum_{k=1}^{n} |c_k|^q \leqslant \|f\| \left(\sum_{k=1}^{n} |c_k|^q \right)^{1/p},$$

whence,
$$\left(\sum_{k=1}^{n} |c_k|^q \right)^{1/q} \leqslant \|f\|.$$

This inequality holds for every n. Consequently,

$$\left(\sum_{k=1}^{\infty} |c_k|^q \right)^{1/q} \leqslant \|f\|. \tag{9}$$

Thus, $\{c_k\} \in l_q$. Conversely, take an arbitrary sequence $\{d_k\} \in l_q$. Then,

$$\varphi(x) = \sum_{k=1}^{\infty} d_k \xi_k$$

is a linear functional on l_p. Trivially, $\varphi(x)$ is additive and the boundedness of $\varphi(x)$ follows from HÖLDER's inequality. Thus: *The formula* (8) *represents the general form of linear functionals on the space* l_p.

Let us now calculate the norm of the functional f. By HÖLDER's inequality, (8) directly leads to

$$|f(x)| = \left| \sum_{k=1}^{\infty} c_k \xi_k \right| \leqslant \left(\sum_{k=1}^{\infty} |c_k|^q \right)^{1/q} \left(\sum_{k=1}^{\infty} |\xi_k|^p \right)^{1/p}$$

$$= \left(\sum_{k=1}^{\infty} |c_k|^q \right)^{1/q} \|x\|.$$

Consequently,
$$\|f\| \leqslant \left(\sum_{k=1}^{\infty} |c_k|^q \right)^{1/q}. \tag{10}$$

Equating (9) and (10), we finally get

$$\|f\| = \left(\sum_{k=1}^{\infty} |c_k|^q \right)^{1/q}.$$

CORROLARY. *Consider the space l_2. Every linear functional defined on l_2 can be written in the general form*

$$f(x) = \sum_{k=1}^{\infty} c_k \xi_k,$$

where
$$\sum_{k=1}^{\infty} c_k^2 < +\infty \quad \text{and} \quad \| f \| = \left(\sum_{k=1}^{\infty} | c_k |^2 \right)^{1/2}.$$

In the functional analysis, besides the spaces l_p, we also consider the space l, whose elements are all possible numerical sequences $x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$, such that $\sum_{k=1}^{\infty} | \xi_k | < \infty$ and also $\| x \| = \sum_{k=1}^{\infty} | \xi_k |$. It is possible to show that *every linear functional in the space l has the form*

$$f(x) = \sum_{k=1}^{\infty} c_k \xi_k,$$

$\{c_k\}$ a bounded sequence of real numbers. The norm of the functional f is defined by

$$\| f \| = \sup_k | c_k |.$$

4.25. The general form of linear functionals in the space L_p [0, 1]. Consider an arbitrary linear functional $f(x)$ defined on L_p [0, 1] ($p > 1$). Put

$$u_t(\xi) = \begin{cases} 1 & \text{for } 0 \leqslant \xi < t, \\ 0 & \text{for } t \leqslant \xi \leqslant 1, \end{cases}$$

and let $g(t) = f[u_t(\xi)]$. Show that $g(t)$ is an absolutely continuous function. For this purpose let $\delta_i = (\tau_i, t_i)$, $i = 1, 2, \ldots, n$, be an arbitrary system of non-overlapping intervals in [0, 1]. Choose the numbers ε_i defined as above (p. 125). Then,

$$\begin{aligned}
\sum_{i=1}^{n} | g(t_i) - g(\tau_i) | &= \sum_{i=1}^{n} \varepsilon_i [g(t_i) - g(\tau_i)] \\
&= f\left\{ \sum_{i=1}^{n} \varepsilon_i [u_{t_i}(\xi) - u_{\tau_i}(\xi)] \right\} \\
&\leqslant \| f \| \left\| \sum_{i=1}^{n} \varepsilon_i [u_{t_i}(\xi) - u_{\tau_i}(\xi)] \right\| \\
&= \| f \| \left(\int_0^1 \left| \sum_{i=1}^{n} \varepsilon_i [u_{t_i}(\xi) - u_{\tau_i}(\xi)] \right|^p d\xi \right)^{1/p} \\
&\leqslant \| f \| \left(\sum_{i=1}^{n} \int_{\delta_i} d\xi \right)^{1/p} = \| f \| \left(\sum_{i=1}^{n} \operatorname{mes} \delta_i \right)^{1/p},
\end{aligned}$$

whence it follows that $g(t)$ is absolutely continuous, implying that $g(t)$ has an a.e. L-integrable derivative and is equal to the LEBESGUE integral of this derivative. Put $g'(t) = \alpha(t)$, so that

$$g(t) - g(0) = \int_0^t \alpha(\tau)\, d\tau.$$

However, then $$g(0) = f[u_0(\xi)] = 0,$$

since $u_0(\xi) \equiv 0$ is the null element of $L_p[0, 1]$. Consequently,

$$g(t) = \int_0^t \alpha(\tau)\, d\tau.$$

We find, in view of the functions $u_t(\tau)$ defined above, that

$$f[u_t(\tau)] = g(t) = \int_0^t \alpha(\tau)\, d\tau = \int_0^1 u_t(\tau)\, \alpha(\tau)\, d\tau,$$

and since f is a linear functional, it follows that

$$f(z_n) = \int_0^1 z_n(\tau)\, \alpha(\tau)\, d\tau,$$

by putting $$z_n(\tau) = \sum_{k=1}^{n} c_k \left[u_{\frac{k}{n}}(\tau) - u_{\frac{k-1}{n}}(\tau)\right].$$

Let $x(t)$ be an arbitrary, bounded and measurable function, then there is a sequence $\{z_m(t)\}$ of step functions, such that $z_m(t) \to x(t)$ a.e. as $m \to \infty$, where $\{z_m(t)\}$ can be assumed to be uniformly bounded.

By the LEBESGUE theorem on the integration of a bounded sequence, we get

$$\begin{aligned}\lim_m f(z_m) &= \lim_m \int_0^1 z_m(t)\, \alpha(t)\, dt = \int_0^1 \lim_m z_m(t)\, \alpha(t)\, dt \\ &= \int_0^1 x(t)\, \alpha(t)\, dt.\end{aligned}$$

Since, on the other hand, $z_n(t) \to x(t)$ a.e. and $z_n(t)$ is uniformly bounded, it follows that

$$\| z_m - x \| = \left(\int_0^1 | z_m(t) - x(t) |^p\, dt \right)^{1/p} \to 0$$

as $m \to \infty$. Therefore, $f(z_m) \to f(x)$ and, consequently,

$$f(x) = \int_0^1 x(t)\, \alpha(t)\, dt.$$

Consider now the functions $x_n(t)$, defined by the equality

$$x_n(t) = \begin{cases} |\alpha(t)|^{q-1} \operatorname{sgn} \alpha(t) & \text{if} \quad |\alpha(t)| \leqslant n, \\ 0 & \text{if} \quad |\alpha(t)| > n, \end{cases}$$

where q is the number conjugate to p, that is, $(1/p) + (1/q) = 1$. The function $x_n(t)$ is bounded and measurable. Consequently,

$$f(x_n) = \int_0^1 x_n(t)\,\alpha(t)\,dt,$$

and
$$|f(x_n)| \leqslant \|f\| \cdot \|x_n\| = \|f\| \left(\int_0^1 |x_n(t)|^p\,dt\right)^{1/p}.$$

On the other hand,

$$|f(x_n)| = f(x_n) = \int_0^1 x_n(t)\,\alpha(t)\,dt$$
$$= \int_0^1 |x_n(t)|\,|\alpha(t)|\,dt \geqslant \int_0^1 |x_n(t)|\,|x_n(t)|^{1/(q-1)}\,dt$$
$$= \int_0^1 |x_n(t)|^{q/(q-1)}\,dt = \int_0^1 |x_n(t)|^p\,dt.$$

Consequently,

$$\int_0^1 |x_n(t|^p\,dt \leqslant \|f\|\ \|x_n\| = \|f\| \left(\int_0^1 |x_n(t)|^p\,dt\right)^{1/p},$$

whence
$$\left(\int_0^1 |x_n(t)|^p\,dt\right)^{1/q} \leqslant \|f\|.$$

However, obviously, $|x_n(t)| \to |\alpha(t)|^{q-1}$ a.e. on $[0, 1]$ as $n \to \infty$, since $\alpha(t)$ is L-integrable and, consequently, becomes infinite only for a point set of measure zero. Proceeding to the limit as $n \to \infty$, we get

$$\left(\int_0^1 |\alpha(t)|^{(q-1)p}\,dt\right)^{1/p} \leqslant \|f\|,$$

or,
$$\left(\int_0^1 |\alpha(t)|^q\,dt\right)^{1/q} \leqslant \|f\|, \tag{11}$$

whence
$$\alpha(t) \in L_q\,[0, 1].$$

Now, let $x(t)$ be any function in $L_p\,[0, 1]$. Then, there exists $\int_0^1 x(t)\,\alpha(t)\,dt$.

Furthermore, a sequence $\{x_m(t)\}$ of bounded functions can be found, such that

$$\int_0^1 |x(t) - x_m(t)|^p \, dt \to 0$$

as $m \to \infty$. By HÖLDER's inequality,

$$\int_0^1 x_m(t)\, \alpha(t)\, dt \to \int_0^1 x(t)\, \alpha(t)\, dt$$

as $m \to \infty$. Since the $x_m(t)$ are bounded measurable functions,

$$\int_0^1 x_m(t)\, \alpha(t)\, dt = f(x_m).$$

Consequently,
$$f(x_m) \to \int_0^1 x(t)\, \alpha(t)\, dt$$

as $m \to \infty$. On the other hand, $f(x_m) \to f(x)$. But, then,

$$f(x) = \int_0^1 x(t)\, \alpha(t)\, dt. \tag{12}$$

Thus: *Every functional defined on* $L_p[0, 1]$ *can be represented in the form* (12). Conversely, if $\beta(t)$ is an arbitrary function belonging to $L_p[0, 1]$, then

$$\varphi(x) = \int_0^1 x(t)\, \beta(t)\, dt$$

is a linear functional defined on $L_p[0, 1]$. In fact, the additivity of $\varphi(x)$ is obvious and its boundedness is implied by HÖLDER's inequality.

Thus, the formula (12) represents the general form of a linear functional defined on $L_p[0, 1]$ for an arbitrary fixed function $\alpha(t) \in L_q[0, 1]$.

The norm of such a linear functional can be easily found. The formula (12) implies

$$\begin{aligned} |f(x)| &= \left| \int_0^1 x(t)\, \alpha(t)\, dt \right| \\ &\leqslant \left(\int_0^1 |x(t)|^p \, dt \right)^{1/p} \cdot \left(\int_0^1 |\alpha(t)|^q \, dt \right)^{1/q} \\ &= \left(\int_0^1 |\alpha(t)|^q \, dt \right)^{1/q} \|x\|. \end{aligned}$$

Consequently,
$$\|f\| \leqslant \left(\int_0^1 |\alpha(t)|^q \, dt \right)^{1/q}. \tag{13}$$

Comparing (13) and (11), the conclusion is that $\| f \| = (\int_0^1 | \alpha(t) |^q \, dt)^{1/q}$. It is customary to regard $L[0, 1]$ as the space of L-integrable functions, where

$$\| x \| = \int_0^1 | x(t) | \, dt.$$

The general form of the linear functionals, defined on $L[0, 1]$, then becomes $f(x) = \int_0^1 x(t) \, \alpha(t) \, dt$, where $\alpha(t)$ is bounded a.e. For the norm, we get

$$\| f \| = \operatorname*{vrai\,max}_{[0,1]} | \alpha(t) |.$$

4.26. The general form of linear functionals on Hilbert spaces. In a HILBERT space H, consider a linear functional $f(x)$. Since H is a linear complex space, it is natural to grant that $f(x)$ can take complex values. Moreover, a complex functional is said to be **linear** if this is *additive*, *homogeneous* and *continuous* (note that in case of complex functionals these three conditions are independent).

Let H be a HILBERT space and $f(x)$ an arbitrary linear functional defined on H. Denote by L the set of zeros of this functional, that is, the collection of the elements $x \in H$, such that $f(x) = 0$. It is plain that L is a subspace. In fact, the additivity and homogeneity of the functional $f(x)$ imply L to be a linear manifold and the continuity of $f(x)$ implies L to be closed.

For an arbitrary element $x \in H$ and $\overline{\in}\ L$ let x_0 denote the projection of x on the subspace $H \dot{-} L$. Then, $f(x_0) = \alpha$ and also, evidently, $\alpha \neq 0$ We have $f(x_1) = 1$, putting $x_1 = x_0/\alpha$. If now $x \in H$ is arbitrary and $f(x) = \beta$, then

$$f(x) - \beta f(x_1) = 0, \quad \text{or} \quad f(x - \beta x_1) = 0,$$

whence $x - \beta x_1 = z$, where $z \in L$, or $x = \beta x_1 + z$. This equality shows that H is the orthogonal sum of the subspace L and one-dimensional subspace spanned by the element x_1.

Since $x_1 \perp z$, it follows that $(x, x_1) = \beta \| x_1 \|^2$, or since $\beta = f(x)$,

$$f(x) = \left(x, \frac{x_1}{\| x_1 \|^2} \right).$$

If $x_1 / \| x_1 \|^2$ is denoted by u, then,

$$f(x) = (x, u), \tag{14}$$

that is, we get the representation of an arbitrary linear functional $f(x)$ as an inner product of the element x with a fixed element u. The element u is defined uniquely by f because if $f(x) = (x, v)$, then $(x, u - v) = 0$ for every $x \in H$, implying that $u = v$. Further, Eq. (14) yields

$$| f(x) | = | (x, u) | \leqslant \| x \| \cdot \| u \|, \quad \text{whence } \| f \| \leqslant \| u \|.$$

Since, on the other hand, $f(u) = (u, u) = \| u \|^2$, it follows that $\| f \|$ cannot be smaller than $\| u \|$, hence $\| f \| = \| u \|$.

We have thus proved: *Every linear functional $f(x)$ in the Hilbert space H can be represented uniquely in the form*

$$f(x) = (x, u),$$

where the element u is defined uniquely by the functional f. Moreover,

$$\| f \| = \| u \|. \tag{15}$$

Conversely, *for any $u \in H$, the relation* (14) *defines a linear functional $f(x)$ with norm* (15).

Thus: *the formula* (14) *constitutes the general form of linear functionals on a Hilbert space.*

4.3 CONJUGATE SPACES AND ADJOINT OPERATORS

4.31. Examples of conjugate spaces. As shown in the foregoing, a collection of all linear functionals $f(x)$ defined on a normed linear space E forms a BANACH space E^*, which is called the **conjugate** or the **dual** of the space E. Making use of the general form of linear functionals, it is possible in certain cases to determine the space E^* to within isomorphism.

(i) Let $E = C[0,1]$. Consider a set of functions $g(t)$ of bounded variation defined on $[0, 1]$ and vanishing at the point $t = 0$. Assume that $g(\tau) = g(\tau - 0)$ at the jump point τ. Obviously, this set is a linear space, if the sum of two functions, and the product of a fuction and a real number are defined in the customary way. Introduce the norm of such a function by putting $\| g \| = \underset{0}{\overset{1}{\mathrm{var}}}\, \{g\}$. It is easy to notice that all the norm axioms are satisfied.

The normed linear space obtained in this manner is the space V of *functions of bounded variation.*

Consider, on the other hand, the space $E^* = C^*[0, 1]$ of all linear functionals defined on $C[0, 1]$. As shown above, a certain function $g(t)$, $g(0) = 0$, of bounded variation corresponds uniquely to every linear functional $f \in C^*[0, 1]$. Conversely, a functional $f \in C^*[0, 1]$ is associated with every function $g(t)$, $g(0) = 0$, of bounded variation. Hence, a one-one correspondence exists between the set of all linear functionals of $C^*[0, 1]$ and the set of all elements of the space of functions of bounded variation. Since it is evident that the sum of functions $g_1 + g_2$ corresponds to the sum of functionals $f_1 + f_2$ and the function $\lambda g(t)$ to the functional λf in case the functionals f_1, f_2 correspond to the functions g_1, g_2, it follows that this association between $C^*[0, 1]$ and the space of functions of bounded variation is an isomorphism. Furthermore, since

$$\| f \| = \underset{0}{\overset{1}{\mathrm{var}}}\, \{g\} = \| g \|,$$

this correspondence is isometric too.

From the standpoint of functional analysis, these two spaces are non-distinct; hence, it is frequently said that a space, which is the dual of a space of continuous functions, is a space of functions of bounded variation.

(ii) Let $E = L_p\,[0, 1]$. In addition, consider a space $L_q\,[0, 1]$ where $q = 1/(p - 1)$. Since a function $\alpha(t) \in L_q\,[0, 1]$ corresponds uniquely to every functional $f \in L_p^*\,[0, 1]$ and conversely, a one-one correspondence is established betwecn $L_p^*\,[0, 1]$ and $L_q\,[0, 1]$. It is verified, as in the foregoing, that this correspondenc is isomorphic and isometric, that is, $L_p^*\,[0, 1] = L_q\,[0, 1]$, to within isomorphism and isometry. In particular, for $p = 2$, $L_2^*\,[0, 1] = L_2\,[0, 1]$. Therefore, $L_2\,[0, 1]$ is called a **self-conjugate space**.

(iii) It is easy to see that $l_p^* = l_q$ and, in particular, $l_2^* = l_2$.

It was remarked in Chap. 3.4 (p.100) that the dual of a normed linear, not necessarily complete, space is a BANACH space, that is a normed linear complete space. Since $L_p\,[0, 1]$ is the dual of $L_q\,[0, 1]$, $[(1/p) + (1/q)] = 1$, and l_p of l_q, in consequence of (ii) and (iii) we get fresh evidence of $L_p\,[0, 1]$ and l_p being the complete spaces.

(iv) A linear functional in a HILBERT space is spanned by elements of the same space. A HILBERT space is, therefore, self-conjugate.

Similarly, the n-dimensional Euclidean space is also self-conjugate.

4.32 Reflexive spaces. Let E be a normed linear space and let E^* be the dual of the space E. Since E^* is also a normed linear space, it is possible to construct $E^{**} = (E^*)^*$, and so on, that is, **second, third**, conjugate spaces of E. Consider, in detail, E^{**}. This is the space of linear functionals F, defined on E,* whose elements are linear functionals defined on E. Consider a linear functional $f(x)$ defined on E. Here the functional f is fixed and x is a variable element in E. According to another approach to $f(x)$, $x \in E$ is regarded fixed and f varying in E. For example, let

$$f(x) = \int_0^1 x(t)\, dg(t).$$

Then, we have case (i) for fixed $g(t)$ and varying $x(t)$, and case (ii) if $x(t)$ is fixed and $g(t)$ varies.

For x fixed and f varying, some real number corresponds to every $f \in E^*$. Consequently, $f(x)$ can be treated as a functional F_x, defined on E^*, for fixed x and variable f. Hence, it is possible to write $f(x) = F_x((f)$. It is plain that F_x is a linear functional, and consequently, also $F_x \in E^*$. In fact,

$$F_x(f_1+f_2) = (f_1+f_2)\,(x) = f_1(x) + f_2(x) = F_x(f_1) + F_x(f_2),$$

and $|\,F_x(f)\,| = |\,f(x)\,| \leqslant \|\,x\,\| \cdot \|\,f\,\|$. Thereupon, it follows, in particular, that

$$\|\,F_x\,\| \leqslant \|\,x\,\|. \tag{1}$$

Furthermore, by Corollary 1 to the HAHN-BANACH theorem, for every x there is a linear functional f_0 with norm equal to 1, such that $f_0(x) = \|\,x\,\|$ and for such a functional, $|\,F_x(f_0)\,| = |\,f_0(x)\,| = \|\,x\,\|$, or equivalently, $|\,F_x(f_0)\,| =$

$\| f_0 \| \cdot \| x \|$. Hence

$$\| F_x \| \geqslant \| x \| . \tag{2}$$

Comparing (1) and (2), the conclusion is

$$\| F_x \| = \| x \| . \tag{3}$$

Also, evidently,

$$F_{x_1 + x_2}(f) = F_{x_1}(f) + F_{x_2}(f) \text{ and } F_{\lambda x}(f) = \lambda F_x(f).$$

Hence, there is set in *natural correspondence* to every $x \in E$ a well-defined functional $F_x \in E^{**}$ and this correspondence between the space E and thc set $\{F_x\} \subset E^{**}$ is isomorphic and isometric (the bi-uniqueness between E and $\{F_x\}$ stems from (3)], that is, $E \subset E^{**}$. If $E = E^{**}$ under such correspondence, then the space E is called **regular or reflexive.**

Examples. 1. The n-dimensional Euclidean space is reflexive. In fact, if E is an n-dimensional Euclidean space, then E^* is also an n-dimensional Euclidean space, so is E^{**} too. However, if an n-dimensional space is part of another n-dimensional space, then both of them would coincide. Therefore, $E \subseteq E^{**}$ implies $E = E^{**}$.

2. The space $L_p\,[0, 1]$ $(p > 1)$ is reflexive. In fact,

$$L_p^{**}\,[0, 1] = (L_p^*\,[0, 1])^* = (L_q\,[0, 1])^* = L_p\,[0, 1].$$

3. The space l_p $(p > 1)$ is reflexive by the reasonings of Example 2.

4. The space $C[0, 1]$ is not reflexive. For proving this by contradiction, we asssume that $C\,[0, 1]$ is reflexive. Then any linear functional $F(f)$, defined on the space V of functions of bounded variation, must have the form $F_x(f) = f(x)$ for a suitably chosen element $x \in C\,[0, 1]$. Recalling the general form of linear functionals $f[x)$ defined on $C\,[0, 1]$, it is found that every linear functional $F(f)$ assumes the form

$$F_x(f) = f(x) = \int_0^1 x(t)\, df(t), \tag{4}$$

$f(t)$ a function of bounded variation associated with the functional $f(x)$ in $C^*\,[0, 1]$. The functional

$$F_{x_0}(f) = f(t_0 + 0) - f(t_0 - 0)$$

assigns to every function $f(t)$ of bounded variation its jump at the point t_0. Obviously, $Fx_0(f)$ is additive. Furthermore,

$$| F_{x_0}(f) | = | f(t_0+0) - f(t_0-0) | \leqslant \operatorname*{var}_0^1 \{ f \} = \| f \|.$$

Consequently, $Fx_0(f)$ is bounded and has a norm not greater than 1. Moreover, it is evident that $F_{x_0}(f) \not\equiv 0$. In fact, it suffices to consider $Fx_0(f_1)$ with

$$f_1(t) = \begin{cases} 0 & \text{for} \quad 0 \leqslant t \leqslant t_0, \\ 1 & \text{for} \quad t_0 \leqslant t \leqslant 1. \end{cases}$$

Because of (4), there must exist a continuous function $x_0(t)$, such that

$$F_{x_0}(f) = \int_0^1 x_0(t)\, df(t). \tag{5}$$

Then, $\quad F_{x_0}(f_0) = 0 \quad$ when $\quad f_0(t) = \int_0^t x_0(\tau)\, d\tau,$

since $f_0(t)$ is continuous on $[0, 1)$. But, on the other hand, $Fx_0(f) \not\equiv 0$ implies $x_0(t) \not\equiv 0$, and that

$$Fx_0(f_0) = \int_0^1 x_0(t)\, df_0(t) = \int_0^1 x_0^2(t)\, dt > 0,$$

a contradiction. This contradiction stems from the assumption that every linear functional $F \in C^{**}[0, 1]$ has the form F_x, that is, the space $C[0, 1]$ is reflexive. Consequently, the space $C[0, 1]$ is not reflexive.

A. I. PLESSNER has proved; *Under the natural correspondence, either* $E = E^{**}$ *or all the spaces in the sequence* $E, E^*, E^{**}, E^{***}, \ldots$ *are different.* For proof, see [9].

4.33. The Adjoint operators. Consider a bounded linear operator $y = Ax$, which maps a normed linear space E_x into a normed linear space E_y. Further, let $\varphi(y)$ be a linear functional defined on E_y. Then, $\varphi(y)$ is defined for $y = Ax$, x an element in E_x, and for $y = Ax$, we have

$$\varphi(y) = \varphi(Ax) = f(x),$$

$f(x)$ a functional defined on E_x. Obviously, $f(x)$ is linear. Hence, the functional $f \in E_x^*$ corresponds to every $\varphi \in E_y^*$.

The collection of all correspondences so obtained forms a certain operator with domain E_y^* and range contained in E_x^*. This operator is denoted by A^* and called the **adjoint** of A. The equality $\varphi(y) = f(x)$ is expresed in the form

$$f = A^*\varphi.$$

Examples. 1. Let A be an operator in $(E \rightarrow E)$, where E is an n-dimensional space. Then, A is defined by a matrix (a_{ij}) of order n and the equality $y = Ax$, where $x = \{\xi_1, \xi_2, \ldots, \xi_n\}$ and $y = \{\eta_1, \eta_2, \ldots, \eta_n\}$ is expressed in the form

$$\eta_i = \sum_{j=1}^{n} a_{ij}\, \xi_j.$$

Consider a linear functional $f \in E^*$:

$$f = (f_1, f_2, \ldots, f_n), \qquad f(x) = \sum_{i=1}^{n} f_i\, \xi_i.$$

Hence,

$$f(Ax) = \sum_{i=1}^{n} f_i\, \eta_i = \sum_{i=1}^{n} f_i \sum_{j=1}^{n} a_{ij}\, \xi_j = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, f_i\, \xi_j$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij}\, f_i \right) \xi_j = \sum_{j=1}^{n} g_j\, \xi_j, \quad \text{where} \quad g_j = \sum_{i=1}^{n} a_{ij}\, f_i.$$

The vector $g = (g_1, g_2, \ldots, g_n)$ is an element of E^* and is obtained from the vector $f = (f_1, f_2, \ldots, f_n)$ of the same space by the linear transformation $g = A^*f$, where A^* is the transposed matrix of A. Consequently, the transposed matrix corresponds to the adjoint operator in the n-dimensional space.

2. Consider in $L_2[0, 1]$, the operator

$$Ax = y(t) = \int_0^1 K(t, s)\, x(s) ds,$$

$K(t, s)$ a continuous kernel. An arbitrary linear functional $f(y) \in L_2[0, 1]$ has the form

$$f(y) = (y, f) = \int_0^1 y(t) f(t)\, dt, \qquad f(t) \in L_2[0, 1].$$

Therefore,
$$f(Ax) = \int_0^1 f(t) \left\{ \int_0^1 K(t, s)\, x(s)\, ds \right\} dt$$
$$= \int_0^1 x(s) \left\{ \int_0^1 K(t, s) f(t)\, dt \right\} ds = \int_0^1 x(s)\, g(s)\, ds,$$

where
$$g(t) = \int_0^1 K(s, t) f(s)\, ds.$$

Thus, in the given case, the passage to the adjoint operator signifies the interchange in the kernel [the kernel $K(s, t)$ is called the **transpose** of the kernel $K(t, s)$].

THEOREM 1. *The operator A^*, adjoint to a bounded linear operator A, which maps the normed linear space E_x into the normed linear space E_y, is also a bounded linear operator, and* $\| A^* \| = \| A \|$.

PROOF. To start with, it is evident that A^* is additive. Furthermore,

$$| (A^*\varphi)(x) | = | f(x) | = | \varphi(Ax) | \leqslant \| \varphi \| \, \| Ax \| \leqslant \| \varphi \| \, \| A \| \cdot \| \, \| x \|,$$

whence $\| A^*\varphi \| \leqslant \| A \| \, \| \varphi \|$. Consequently, A^* is a bounded operator, and

$$\| A^* | \leqslant \| A \|. \tag{6}$$

Let x_0 be an arbitrary element in E_x. Then, by Corollary 1 to the HAHN-BANACH theorem, there is a functional $\varphi_0 \in E_y^*$ with the norm $\| \varphi_0 \| = 1$, such that $\varphi_0(Ax_0) = \| Ax_0 \|$. Thereupon,

$$\| Ax_0 \| = \varphi_0 (Ax_0) = f_0(x_0) \leqslant \| f_0 \| \, \| x_0 \|$$
$$= \| A^*\varphi_0 \| \cdot \| x_0 \| \leqslant \| A^* \| \cdot \| \varphi_0 \| x_0 \| = \| A^* \| \cdot \| \, \| x_0 \|.$$

Consequently,
$$\| A \| \leqslant \| A^* \|. \tag{7}$$

From (6) and (7) it follows that $\| A \| = \| A^* \|$, and the theorem is proved.

The notion of adjoint operator can be extended also to an unbounded linear operator, defined on a linear manifold L_x, everywhere dense in the normed linear space E_x and with its range in the space E_y. Suppose that A is such an operator and that $\varphi \in E_y^*$. Consider

$$\varphi (Ax) = f_0 (x), \quad x \in L_x.$$

Then $f_0 (x)$ is, evidently, an additive and homogeneous functional, defined on L_x. For an arbitrary functional φ in E_y^*, the functional f_0 is not generally bounded. Hewever, if for some $\varphi \in E_y^*$ the functional f_0 is bounded, then it can be extended by continuity up to the linear functional f defined on the entire E_x.

Thus, it is found that on a certain manifold $L_y^* \subset E_y^*$ defining the operator A^*, the linear functional $f \in E_x^*$ is set in correspondence to the linear functional $\varphi \in L_y^*$. This operator A^* is also called the **adjoint** of the unbounded linear operator A. It is easy to verify that L_y^* is a linear manifold and that A^* is a linear operator on this manifold, generally not bounded there.

Example. In the space $L_q(G)$, G a bounded measurable domain on a plane, consider a differential operator

$$A = \frac{\partial^l}{\partial x^{l_1}\, \partial y^{l_2}}, \quad l_1 + l_2 = l,$$

defined on a linear manifold $L_0 \subset L_q(G)$ of l-times continuously differentiable functions which vanish at some boundary of a region of the domain G. The manifold L_0 is everywhere dense in $L_q(G)$ and the operator A is distributive there but not bounded. The range of the operator is regarded to belong to the same space $L_q(G)$.

Suppose that for some function $v(x, y) \in L_p(G)$ $[(1/p) + (1/q) = 1]$ the equality

$$\iint_G \frac{\partial^l u(x, y)}{\partial x^{l_1}\, \partial y^{l_2}} v(x, y)\, dx\, dy = \iint_G Auv\, dx\, dy = \iint_G uw\, dx\, dy$$

holds for every function $u(x, y) \in L_0$, where $w(x, y) \in L_p$. The functional

$$f(u) = \iint_G u(x, y)\, w(x, y)\, dx\, dy$$

as a functional defined on $L_0 \subset L_q(G)$ is, evidently, distributive and also bounded, since

$$|f(u)| = \left| \iint_G uw\, dx\, dy \right| \leqslant \|u\|_{L_q} \|w\|_{L_p},$$

and can be extended to the entire $L_q(G)$. By the same token we obtain A^*, $A^*v = w$, the adjoint of A, defined on some set of functions $v(x, y) \in L_p(G)$, with range in the same space. Recalling the second definition of the generalized derivatives, it observed that A^*v differs from the generalized derivative $\partial^l v/\partial x^{l_1} \partial y^{l_2}$ only by the factor $(-1)^l$, Thus, the operator of generalized differentiation can be treated also as the adjoint of a differential operator defined on a set of l-times continuously differentiable functions, which vanish at the boundary of a region of the domain G.

4.34. The Matrix form of operators in spaces with basis. In a BANACH space E with basis, given a bounded linear operator A, which maps E into itself.

Take $x \in E$. Then,

$$x = \lim_n x_n, \text{ where } x_n = \sum_{i=1}^{n} \xi_i e_i.$$

Consequently,
$$y = Ax = A(\lim_n x_n) = \lim_n \sum_{i=1}^{n} \xi_i A e_i.$$

Since Ae_i again forms an element of E, it can be expanded in basis elements

$$Ae_i = \sum_{k=1}^{\infty} a_{ki}\, e_k,$$

to receive
$$y = Ax = \lim_n \sum_{i=1}^{n} \xi_i \left(\sum_{k=1}^{\infty} a_{ki} e_k \right). \tag{8}$$

However, $y \in E$ and, consequently, this also can be expanded in basis elements

$$y = \sum_{k=1}^{\infty} \eta_k \, e_k. \tag{9}$$

Now, let $\{ f_i \}$ be a sequence of functionals biorthogonal to the sequence $\{e_i)$. Then (8) and (9) imply

$$\eta_m = f_m(y) = f_m \left\{ \lim_n \sum_{i=1}^{\infty} \xi_i \left(\sum_{k=1}^{\infty} a_{ki} e_k \right) \right\}$$

$$= \lim_n f_m \left\{ \sum_{i=1}^{n} \xi_i \left(\sum_{k=1}^{\infty} a_{ki} e_k \right) \right\}$$

$$= \lim_n \sum_{i=1}^{n} \xi_i \sum_{k=1}^{\infty} a_{ki} f_m(e_k)$$

$$= \lim_n \sum_{i=1}^{n} a_{mi} \xi_i = \sum_{i=1}^{\infty} a_{mi} \xi_i. \tag{10}$$

Eq. (10) shows that the operator A is uniquely defined by the infinite matrix (a_{mi}) (by means of this matrix, the components of the element $y = Ax$ are uniquely defined with respect to the components of the element x). It is intended to exhibit next that the notion of the transpose of a matrix can be extended to the notion of an adjoint operator.

Now consider an adjoint operator A^*, mapping E^* into itself.

Let $f = A^*\varphi$, that is, $\varphi(Ax) = f(x)$ for every $x \in E$. Furthermore, let $\varphi = \sum_{i=1}^{\infty} c_i f_i$ and $f = \sum_{i=1}^{\infty} d_i f_i$, to receive

$$\varphi(Ax) = \varphi \left\{ \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ki} \xi_i \right) e_k \right\}$$

$$= \lim_n \varphi \left\{ \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ki} \xi_i \right) e_k \right\} = \lim_n \sum_{k=1}^{n} \left(\sum_{i=1}^{n} a_{ki} \xi_i \right) \varphi(e_k)$$

$$= \lim_n \sum_{k=1}^{n} \left(\sum_{i=1}^{\infty} a_{ki} \xi_i \right) c_k = \lim_n \sum_{i=1}^{\infty} \left(\sum_{k=1}^{n} a_{ki} c_k \right) \xi_i.$$

On the other hand,

$$\varphi(Ax) = f(x) = \sum_{i=1}^{\infty} d_i f_i(x) = \sum_{i=1}^{\infty} d_i \xi_i.$$

Consequently,
$$\sum_{i=1}^{\infty} d_i\xi_i = \lim_{n} \sum_{i=1}^{\infty} \left(\sum_{k=1}^{n} a_{ki}c_k \right) \xi_i. \tag{11}$$

Let $x = e_m$, that is, $\xi_m = 1$, $\xi_i = 0$, for $i \neq m$. Then, (11) yields

$$d_m = \lim_{n} \sum_{k=1}^{n} a_{km}c_k = \sum_{k=1}^{\infty} a_{km}c_k.$$

The last equality shows that the matrix corresponding to the adjoint operator is the transpose of the matrix corresponding to the original operator. Such representation of operators and their adjoints holds, for instance, in the space l_2.

It is easy to find from the matrix representation of operators that :

(i) $(A+B)^* = A^* + B^*$; (ii) $(AB)^* = B^*A^*$;

(iii) $(A^{-1})^* = (A^*)^{-1}$, if A^{-1} exists.

Besides, these formulae easily establish that the space has an *unconditional basis.*

4.35. The inner product, orthogonal elements, biorthogonal systems. Let $x \in E$ and let f be a linear functional on E, that is, $f \in E^*$. The expression

$$f(x) = (x, f) = (f, x) \tag{12}$$

is a **bilinear (sesquilinear)** functional of the two variables x and f, that is, it is linear in x as well as in f. This bilinear functional, when E is a HILBERT space, means also that $E^* = E$, and (12) represents the inner product of x and f (see the formula (20) of Chap. 4.2). In the general case also when $E^* \neq E$ the expression (12) is called the **scalar** (or **inner**) **product** of $x \in E$ and $f \in E^*$.

If $(x, f) = (f, x) = 0$, then $x \in E$ and $f \in E^*$ are said to be **orthogonal.**

THEOREM 2. *Let λ_0 be an eigenvalue of the linear operator $A \in (E \to E)$ and x_0 the corresponding eigenvector. Furthermore, let μ_0 be an eigenvalue of the adjoint operator A^*, with the eigenvector f_0.*

Then, the eigenvectors x_0 and f_0 are orthogonal, if $\lambda_0 \neq \mu_0$.

This theorem is a generalization of the theorem on the orthogonality of eigenfunctions of adjoint integral equations.

The relation between the operators A and A^* can be expressed by their inner product in the form

$$(Ax, f) = (x, A^*f),$$

which holds for all $x \in E$, $f \in E^*$. Then, by hypothesis,

$$Ax_0 = \lambda_0 x_0, \qquad A^* f_0 = \mu_0 f_0.$$

Thereupon and from the foregoing equalities, it follows that

$$\lambda_0(x_0, f_0) = \mu_0(x_0, f_0), \qquad \text{or} \qquad (\lambda_0 - \mu_0)(x_0, f_0) = 0.$$

However, by hypothesis, $\lambda_0 \neq \mu_0$. Consequently, $(x_0, f_0) = 0$. As indicated earlier, the sequences $\{x_n\}$, $x_n \in E$ and $\{f_n\}$, $f_n \in E^*$ are called **biorthogonal**, if

$$(x_i, f_j) = \delta_{ij}. \tag{13}$$

By the same token, x_i and f_j are orthogonal for $i \neq j$.

In Chap. 3.6 we have considered the examples of biorthogonal sequences. The basis elements $e_1, e_2, \ldots, e_n, \ldots$ and the functionals $f_1, f_2, \ldots, f_n, \ldots$ are defined by the equation : $f_k(x) = \xi_k$ for $x = \sum_{k=1}^{\infty} \xi_k e_k$. In a self-conjugate space, say a HILBERT space, both the biorthogonal sequences are contained in one and the same space. If $f_n = x_n$, then their biorthogonality reduces to the usual orthogonality.

Let $\{x_n\}$ and $\{f_n\}$ be biorthogonal and let x be represented by the series

$$x = \sum_{i=1}^{\infty} \xi_i x_i. \tag{14}$$

Then, $$(x, f_k) = \lim_n \left(\sum_{i=1}^{n} \xi_i x_i, f_k \right) = \lim_n \sum_{i=1}^{n} \xi_i (x_i, f_k).$$

Thus, by (13), $\sum_{i=1}^{n} \xi_i (x_i, f_k) = \xi_k$, for $n \geqslant k$, since all the terms except $\xi_k(x_k, f_k) = \xi_k$ vanish in this sum. Hence, $(x, f_k) = \xi_k$, and (14) assumes the form

$$x = \sum_{i=1}^{\infty} (x, f_i)\, x_i. \tag{15}$$

Analogously, if $$f = \sum_{i=1}^{\infty} d_n f_n, \tag{16}$$

then, $$d_n = (x_n, f).$$

The series (15) and (16) are called the **Fourier series** of the respective biorthogonal sequences.

P. L. CHEBYEHEV and A. A. MARKOV were the first to investigate the non-trivial examples of biorthogonal sequences of functions in connection with the interpolation problems.

Let us show that for every linearly independent system of elements

$$(x_1, x_2, \ldots, x_n) \subset E$$

there is a system of linear functionals $\{f_1, f_2, \ldots, f_n\} \subset E^*$, biorthogonal to the given system.

Let $L_1 = L(x_2, x_3, \ldots, x_n)$ be a linear manifold, spanned by the elements $x_2, x_3, \ldots, x_n$. Since x_1 lies apart from L_1 by a distance $d > 0$ (because of the linear independence of $x_1, x_2, \ldots, x_n$ and L_1 being closed), there is a linear functional $f_1(x)$, such that $f_1(x) = 0$ on L_1, in particular on the elements $x_2, x_3, \ldots, x_n$, and $f_1(x_1) = 1$.

By iterating this process for the manifold $L_2 = L(x_1, x_3, \ldots, x_n)$ and the elements $x_2, \ldots$ the desired system of functionals is realized.

Conversely, given a system $\{f_1, f_2, \ldots, f_n\} \subset E^*$ of linearly independent linear functionals, such that

$$\lambda_1 f_1(x) + \lambda_2 f_2(x) + \ldots + \lambda_n f_n(x) = 0$$

for any $x \in E$ implies $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. Then, there is a system of elements $\{x_1, x_2, \ldots, x_n\} \subset E$, biorthogonal to the given system of functionals.

To start with, let $n = 1$. Since $f_1(x) \neq 0$, there is an element x_0, such that $f_1(x_0) = \alpha \neq 0$. Then the element $x_1 = x_0/\alpha$ has the requisite properties.

Assuming the assertion to have been proved for $n - 1$ linearly independent functionals, let it be proved for n functionals. Suppose that the elements $\{x_2, x_3, \ldots, x_n\}$ form a system, biorthogonal to the functionals $f_2, f_3, \ldots, f_n$. Denote by M_1 a linear manifold, defined by the system of equations

$$f_2(x) = 0, \quad f_3(x) = 0, \ldots, f_n(x) = 0.$$

For every $x \in E$, the element

$$u = x - \sum_{i=2}^{n} c_i x_i, \quad \text{with} \quad c_i = f_i(x)$$

belongs to this manifold. In M_1, there is an element x_0, such that $f_1(x_0) = \alpha \neq 0$. Contrarily, $f_1(u)$ would vanish for all u:

$$f_1(x) - \sum_{i=2}^{n} c_i f_1(x_i) = 0, \quad \text{or} \quad f_1(x) = \sum_{i=2}^{n} f_1(x_i)\, f_i(x)$$

for every $x \in E$, implying f_1 to be a linear combination of the functionals $f_2, f_3, \ldots, f_n$, which, by hypothesis, is impossible.

Thus, there exists an element x_0, such that

$$f_1(x_0) = \alpha \neq 0, \quad f_2(x_0) = f_3(x_0) = \ldots = f_n(x_0) = 0.$$

Putting $x_1 = x_0/\alpha$ we obtain the first element of the biorthogonal system.

Continuing the process for the manifold

$$M_2 = \{x : f_1(x) = 0, f_3(x) = 0, \ldots, f_n(x) = 0\}$$

and the functional f_2, we obtain the element x_2, and so on.

4.36. The spaces conjugate to complex linear spaces. All notions introduced in this section can be extended also to a complex linear space E. The collection of all *linear complex functionals* on E is again denoted as the space E^*, conjugate of E.

The number $f(x)$, as before, is the inner product (x, f) with $x \in E$ and $f \in E^*$. In order that the properties of this inner product remain preserved for a complex HILBERT space, it is necessary that (x, f) be a linear functional in x and adjointly-linear in f: $(x, \lambda f) = \bar{\lambda}(x, f)$, defining also the multiplication of f by a complex number λ in E^*: λf is a linear functional of φ on E, such that

$$\varphi(x) = \bar{\lambda} f(x).$$

The notion of the adjoint operator A^* of an operator A in $(E \to E)$ also carries over to a complex space: A^* is an operator in $(E^* \to E^*)$, such that

$$(Ax, f) = (x, A^* f)$$

for every $x \in E$ and $f \in E^*$.

All properties of the adjoint operators can be directly extended to the complex case with a slight modification: The theorem of the orthogonality of the eigenvectors x_0 and f_0 of the operators A and A^* holds, respectively, for $Ax_0 = \lambda_0 x_0$, $A^* f_0 = \mu_0 f_0$, if $\lambda_0 \neq \bar{\mu}_0$.

4.4 WEAK CONVERGENCE OF SEQUENCES OF FUNCTIONALS AND ELEMENTS

4.41. Weak convergence. Let E be a normed linear space. A sequence $\{f_n\}$ of linear functionals in E^* is said to be **weakly convergent** to a linear functional $f_0 \in E^*$ if $f_n(x) \to f_0(x)$ for every $x \in E$. Thus, for linear functionals the notion of weak convergence is equivalent to that of pointwise convergence for operators.

In the terminology of weak convergence, Theorems 1 and 2 given at the start of the present chapter can be restated, as

THEOREM 1. *A sequence $\{f_n\}$ of linear functionals, weakly convergent to itself, converges weakly to some linear functional f_0.*

THEOREM 2. *In order that a sequence $\{f_n\}$ of linear functionals be weakly convergent to a linear functional f_0, it is necessary and sufficient that*

(i) *the sequence $\{\|f_n\|\}$ is bounded: and*

(ii) *$f_n(x) \to f_0(x)$ for every x of some manifold M, the linear combinations of whose elements are everywhere dense in E.*

It also warrants to note that Theorem 1 implies the weak completeness of the space E^*, the conjugate of a BANACH space E.

4.42. Application to the theory of quadrature formulae [14]. Consider in the space $C[0, 1]$ the functional

$$f(x) = \int_0^1 x(t)\, d\sigma(t),$$

$\sigma(t)$ some non-decreasing function, and the sequence of functionals

$$f_n(x) = \sum_{k=1}^{k_n} c_k^{(n)} x\,(t_k^{(n)}), \qquad n = 1, 2, 3, \ldots,$$

$c_k^{(n)}$ so chosen that $f(x)$ and $f_n(x)$ coincide for all polynomials of degree less than or equal to n:

$$f(x) = f_n(x), \qquad \text{if } x(t) = \sum_{p=0}^{n} a_p t^p.$$

The f_n so constructed are employed for the approximation of the functional f. The relation $f(x) \approx f_n(x)$ which becomes an equality for all polynomials of degree less than or equal to n is called a **quadrature formula.**

Consider a sequence of quadrature formulae

$$f(x) \approx f_n(x), \qquad n = 1, 2, 3, \ldots$$

The problem that now arises is whether the sequence of $f_n(x)$ converges to the value $f(x)$ as $n \to \infty$ for any $x(t) \in C\,[0, 1]$. In other words: Does the sequence $\{f_n\}$ of functionals converge weakly to the functional f?

THEOREM 3. *In order that the convergence of a sequence of quadrature formulae holds, that is, in order that*

$$\lim_n \sum_{k=1}^{k_n} c_k^{(n)} x\,(t_k^{(n)}) = \int_0^1 x(t)\, d\sigma(t)$$

for every continuous function $x(t)$, it is necessary and sufficient that

$$\sum_{k=1}^{k_n} |\, c_k^{(n)} \,| \leqslant K = \text{const}, \quad \textit{for every } n.$$

PROOF. By definition of the functionals f_n,

$$f_m(x) = f(x), \quad m \geqslant n,$$

for every polynomial $x(t)$ of degree n. Furthermore, evidently,

$$\| f_n \| = \sum_{k=1}^{k_n} |\, c_k^{(n)} \,| \leqslant K.$$

Thus, the sequence of functionals $\{f_n\}$ converges to the functional f on a set of all polynomials, everywhere dense in the space $C\,[0, 1]$, and the norms of

f_n are bounded. However, then, the proof of the theorem is immediate from Theorem 2 of the present section.

THEOREM 4. (V. A. STEKLOV). *If all the coefficients* $c_k^{(n)}$ *of quadrature formulae are positive, then the sequence of quadrature formulae* $f(x) \approx f_n(x)$, $n = 1, 2, \ldots$, *is convergent for every continuous function* $x(t)$.

In fact, $f_n(x_0) = f(x_0)$ for any n and $x_0(t) \equiv 1$. Hence,

$$\sum_{k=1}^{k_n} |c_k^{(n)}| = \sum_{k=1}^{k_n} c_k^{(n)} = \int_0^1 d\sigma = \sigma(1) - \sigma(0),$$

and thereby the hypotheses of the preceding theorem are satisfied.

4.43. Weak convergence of sequences of clements of a space. Elementary theorems. We now introduce the notion of weak convergence for sequences of elements of normed linear spaces.

Let E be a normed linear space, $\{x_n\}$ a sequence of elements in E and $x_0 \in E$. If $f(x_n) \to f(x_0)$ as $n \to \infty$ for every functional $f \in E^*$, then it is said that the sequence $\{x_n\}$ is **weakly convergent to the element** x_0, in symbols

$$x_n \xrightarrow{w} x_0.$$

We also say that x_0 *is the weak limit of a sequence of elements* $\{x_n\}$.

A sequence cannot converge weakly to two different limits, that is the weak limit of a sequence is unique.

Assume that $x_n \xrightarrow{w} x_0$ and $x_n \xrightarrow{w} \xi_0$, that is, $f(x_n) \to f(x_0)$ and $f(x_n) \to f(\xi_0)$ for every linear functional $f \in E^*$. Consequently, $f(x_0) = f(\xi_0)$, or $f(x_0 - \xi_0) = 0$, implying that $x_0 = \xi_0$.

It is easy to see that *every subsequence* $\{x_{n_k}\}$ *also converges weakly to* x_0, *if* $x_n \xrightarrow{w} x_0$. In contrast to weak convergence of elements (functions), the convergence of a sequence of elements (functions) with respect to the norm of the given space is called **strong convergence.**

It is evident that thc strong convergence of a sequence $\{x_n\}$ to an element x_0 always implies the weak convergence of this sequence to the same element. The converse statement is, however, not true, generally speaking. A sequence can converge weakly to some element without converging strongly to the same element. For example, consider the sequence of elements $\{\sin n\pi t\}$ in $L_2[0, 1]$. Put $x_n(t) = \sin n\pi t$, to receive

$$f(x_n) = \int_0^1 \sin n\pi t \, \alpha(t) \, dt,$$

where $\alpha(t)$ is a square-integrable function, uniquely defined with respect to the functional f. Obviously, $f(x_n)$ is the n-th FOURIER coefficient of $\alpha(t)$ relative to $\{\sin n\pi t\}$. Consequently, $f(x_n) \to 0$ as $n \to \infty$, so that $x_n \xrightarrow{w} 0$ as $n \to \infty$, On the other hand, it is plain that $\{x_n\}$ cannot converge strongly. In fact,

$$\| x_n - x_m \|^2 = \int_0^1 [\sin n\pi t - \sin m\pi t]^2 \, dt = 1.$$

However, the upcoming theorem demonstrates the indistinguishability of strong and weak convergence in finite-dimensional spaces.

THEOREM 5. *In a finite-dimensional space, the notions of weak and strong convergence are equivalent.*

PROOF. It will suffice to prove that in a finite-dimensional space, weak convergence of a sequence to some element implies its strong convergence to the same element. Let E be a finite-dimensional space and $\{x_n\}$ a given sequence, such that $x_n \xrightarrow{w} x_0$. Since E is finite-dimensional, there is a finite system of linearly independent elements $e_1, e_2, \ldots, e_k$, such that every $x \in E$ can be represented in the form $x = \xi_1 e_1 + \xi_2 e_2 + \ldots + \xi_k e_k$, ξ_i real. Let

$$x_n = \xi_1^{(n)} e_1 + \xi_2^{(n)} e_2 + \ldots + \xi_k^{(n)} e_k,$$

and
$$x_0 = \xi_1^{(0)} e_1 + \xi_2^{(0)} e_2 + \ldots + \xi_k^{(0)} e_k.$$

Now, consider the functionals $f_i \in E^*$, such that $f_i(e_i) = 1$, $f_i(e_j) = 0$ for $j \neq i$, to receive

$$f_i(x_n) = \xi_i^{(n)} \quad \text{and} \quad f_i(x_0) = \xi_i^{(0)}, \quad i = 1, 2, \ldots, k.$$

But since $f(x_n) \to f(x_0)$ for every linear functional f, then also $f_i(x_n) \to f_i(x_0)$, that is
$$\xi_i^{(n)} \to \xi_i^{(0)}, \qquad i = 1, 2, \ldots, k.$$

However, in a finite-dimensional space weak convergence implies convergence in the norm. Consequently, $x_n \to x_0$ strongly. ∎

There also exist some infinite-dimensional spaces in which strong and weak convergence of elements are equivalent. The example of such an infinite-dimensional space is the space l of sequences $\{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}$, such that the series $\sum_{n=1}^{\infty} | \xi_n |$ converges.

The following interesting result is due to M. I. KATZ: *If the space E is separable, then it is possible to introduce in this an equivalent norm, such that the weak convergence $x_n \xrightarrow{w} x_0$ and $\| x_n \| \to \| x_0 \|$ in the new norm imply the strong convergence of the sequence $\{x_n\}$ to x_0.*

THEOREM 6. *If the sequence $\{x_n\}$ converges weakly to x_0, there is a sequence of linear combinations $\left\{ \sum_{k=1}^{k_n} c_k^{(n)} x_k \right\}$, converging strongly to x_0.*

In other words, x_0 belongs to a closed linear manifold L, spanned by the elements $x_1, x_2, \ldots, x_n, \ldots$

Assume the contrary, viz. that x_0 does not belong to L. Then, by Corollary 2 to the BANACH-HAHN theorem (p. 121), there is a linear functional $f \in E^*$, such that $f(x_0) = 1$ and $f(x_n) = 0$ for $n = 1, 2, \ldots$. But this means that $f(x_n)$ does not converge to $f(x_0)$, contradicting the hypothesis that $x_n \xrightarrow{w} x_n$.

THEOREM 7. *Let A be a bounded linear operator with domain E_x and range in E_y, both normed linear spaces. If the sequence $\{x_n\} \subset E_x$ converges weakly to $x_0 \in E_x$, then the sequence $\{Ax_n\} \subset E_y$ converges weakly to $Ax_0 \in E_y$.*

Select any functional $\varphi \in E_y^*$. Then, $\varphi(Ax_n) = f(x_n)$, $f \in E_x^*$. Analogously, $\varphi(Ax_0) = f(x_0)$. Since $x_n \xrightarrow{w} x_0$, $f(x_n) \to f(x_0)$, that is, $\varphi(Ax_n) \to \varphi(Ax_0)$. Since φ is an arbitrary functional in E_y^*, it follows that $Ax_n \xrightarrow{w} Ax_0$. Thus: *every bounded linear operator is not only strongly but also weakly continuous.* ■

THEOREM 8. *If a sequence $\{x_n\}$ converges weakly to x_0, then the norms of the elements of this sequence are bounded.*

We regard the x_n $(n = 1, 2, \ldots)$ as the elements of the space E^{**}. Then the weak convergence of $\{x_n\}$ to x_0 means that the sequence of functionals $\{x_n\} \subset E^{**}$ converges to $x_0 \in E^{**}$ for all $f \in E^*$. But then by the BANACH-STEINHAUS theorem the norm sequence $\{\| x_n \|\}$ is bounded, which is also the required proof.

REMARK. *If x_0 is the weak limit of the sequence $\{x_n\}$, then*

$$\| x_0 \| \leqslant \varliminf_n \| x_n \|;$$

moreover, the existence of this finite lower limit follows from the preceding theorem.

In fact, assume that $\| x_0 \| > \varliminf_n \| x_n \|$. Then, there is a number c such that $\| x_0 \| > c > \varliminf_n \| x_n \|$. Consequently, there is a sequence $\{x_{n_i}\}$, such that $\| x_0 \| > c > \| x_{n_i} \|$. Construct a linear functional f_0, such that $\| f_0 \| = 1$ and $f_0(x_0) = \| x_0 \| > c$. Then, $f_0(x_{n_i}) \leqslant \| f_0 \| \cdot \| x_{n_i} \| = \| x_{n_i} \| < c$ for all i. Consequently, $f_0(x_n)$ does not converge to $f_0(x_0)$, contradicting the hypothesis that $x_n \xrightarrow{w} x_0$.

The cases admitting the strict inequality $\| x_0 \| < \varliminf_n \| x_n \|$ are plausible, as evidenced from the following example.

In the space $L_2[0, 1]$, consider the functions $x_n(t) = \sqrt{2} \sin n\pi t$, to receive $\| x_n \| = 1$, such that also $\lim_n \| x_n \| = 1$. On the other hand, for every linear functional f,

$$f(x_n) = \sqrt{2} \int_0^1 \alpha(t) \sin n\pi t \, dt = \sqrt{2}\, e_n,$$

c_n the FOURIER coefficients of $\alpha(t) \in L_2\,[0, 1]$. Thus, $f(x_n) \to 0$ as $n \to \infty$ for every linear f, that is, $x_n \xrightarrow{w} 0$. Consequently, $x_0 = 0$ and

$$\| x_0 \| = 0 < 1 = \lim_n \| x_n \|.$$

THEOREM 9. *In order that the sequence $\{x_n\}$ converges weakly to x_0, it is necessary and sufficient that*

(i) *the sequence $\{\| x_n \|\}$ be bounded; and*
(ii) *$f(x_n) \to f(x_0)$ for every f of a certain set Γ of linear functionals, linear combinations of whose elements are everywhere dense in E^*.*

This theorem is a particular case of Theorem 2 of the present section. For verifying this, it is necessary to remark that weak convergence of a sequence $\{x_n\} \subset E$ to $x_0 \in E$ is, evidently, equivalent to weak convergence of the same sequence to x_0 if the $\{x_n\}$ and x_0 are regarded as linear functionals defined on E^*.

4.44. Weak convergenee in concrete spanes.

(i) Weak convergence in l_q.

THEOREM 10. *In order that a sequence $\{x_n\}$, $x_n = \{\xi_i^{(n)}\}$, $\xi_i^{(n)} \in l_p$, converges weakly to $x_0 = \{\xi_i^{(0)}\}$, $\xi_i^{(0)} \in l_p$, it is necessary and sufficient that*

(i) *the sequence $\{\| x_n \|\}$ be bounded ; and*

(ii) *$\xi_i^{(n)} \to \xi_i^{(0)}$ as $n \to \infty$ and for all i (in general, however, nonuniformly).*

To prove this, note that the linear combinations of the $f_i = \{0, 0, \ldots \ldots, 0, 1, 0, \ldots\}$, $i = 1, 2, \ldots$, are everywhere dense in $l_q = l_p^*$. Hence by the general criterion in Theorem 9, in order that $x_n \xrightarrow{w} x_0$ it is necessary and sufficient that hypothesis (i) is satisfied and that $f_i(x_n) = \xi_i^{(n)} \to f_i(x_0) = \xi_i^{(0)}$ for every i.

Thus, weak convergence in l_p is equivalent to coordinate-wise convergence together with the boundedness of norms.

(ii) Weak converqence in L_p.

THEOREM 11. *In order that a sequence $\{x_n(t)\} \subset L_p[0, 1]$ converges weakly to $x_0(t) \in L_p[0, 1]$, it is necessary and sufficient that*

(i) *the sequence $\{\| x_n \|\}$ is bounded, and*

(ii) $$\int_0^t x_n(\tau)\, d\tau \to \int_0^t x_0(\tau)\, d\tau \quad \textit{for any } t \in [0, 1].$$

Hypotheses (i) of the theorem in hand and Theorem 9 are equivalent. Therefore, hypothesis (ii) remains to be examined. For this purpose, put

$$\alpha_\tau(t) = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant \tau, \\ 0 & \text{for } \tau < t \leqslant 1. \end{cases}$$

Then, the linear combinations of the functions $\alpha_\tau(t)$, that is, the sums

$$\sum_{i=1}^{n} c_i[\alpha_{\tau_i}(t) - \alpha_{\tau_{i-1}}(t)],$$

where $0 = \tau_0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n = 1$, are everywhere dense in $L_q[0, 1] = L_p^*[0, 1]$. Consequently, in order that $x_n(t) \xrightarrow{w} x_0(t)$, it is necessary and sufficient that hypothesis (i) is satisfied, and that

$$\int_0^1 x_n(t)\, \alpha_\tau(t)\, dt \to \int_0^1 x_0(t)\, \alpha_\tau(t)\, dt, \quad \text{or} \quad \int_0^\tau x_n(t)\, dt \to \int_0^\tau x_0(t)\, dt,$$

as $n \to \infty$ and for every $\tau \in [0, 1]$. ∎

(iii) *Weak convergence in Hilbert spaces.* Since in a HILBERT space H every linear functional $f(x)$ represents an inner product, hence $x_n \xrightarrow{w} x_0$ in this space, implying that $(x_n, y) \to (x, y)$ for every $y \in H$.

As observed earlier, if $x_n \to x_0$, $y_n \to y_0$ then $(x_n, y_n) \to (x_0, y_0)$, that is, the inner product is continuous over the collection of both the arguments with respect to strong convergence. If, however, $x_n \xrightarrow{w} x_0$, $y_n \xrightarrow{w} y_0$ then, generally, (x_n, y_n) does not converge to (x_0, y_0). Thus, for example, if $x_n = y_n = e_n$, $\{e_n\}$ an arbitrary orthonormal sequence, then $e_n \xrightarrow{w} 0$, but

$$(e_n, e_n) = \| e_n \|^2 = 1 \text{ not tending to } 0 = (0, 0).$$

However, if $x_n \to x_0$, $y_n \xrightarrow{w} y_0$, then $(x_n, y_n) \to (x_0, y_0)$. In fact, in this case, the norm $\| y_n \|$ is totally bounded.

Let $M = \sup_n \| y_n \|$. Then

$$\begin{aligned} |(x_n, y_n) - (x_0, y_0)| &\leqslant |(x_n - x_0, y_n)| + |(x_0, y_n - y_0)| \\ &\leqslant M \| x_n - x_0 \| + |(x_0, y_n - y_0)|, \end{aligned}$$

and both the members on the right tend to zero. Finally, notice that if $x_n \xrightarrow{w} x_0$ and $\| x_n \| \to \| x_0 \|$, then $x_n \to x_0$, since

$$\begin{aligned} \| x_n - x_0 \|^2 &= (x_n - x_0, x_n - x_0) \\ &= [(x_n, x_n) - (x_0 - x_0)] + [(x_0, x_0) - (x_0, x_n)] + [(x_0, x_0) - (x_n, x_0)] \end{aligned}$$

and all the members on the right again tend to zero.

CHAPTER 5

COMPACT SETS IN METRIC AND NORMED SPACES

MORE THAN a century back the Czech mathematician, BERNHARD BOLZANO observed that every bounded infinite set of points on the real line has at least one limit point. He drew attention to the importance of this assertion for a rigorous development of mathematical analysis. The idea of choosing a convergent sequence from some set which consists of not points but functions or curves, has since found application in the existence proof for solutions of ordinary differential equations, in the calculus of variations, etc. This has also led to the general definition of compactness of a set lying in some space.

5.1 DEFINITIONS. GENERAL THEOREMS

A SET K lying in a metric space X is called **compact** if every sequence of elements of this set contains a convergent subsequence. If the limits of the sequences referred belong to K, then the set K is called **sequentially compact.** However, if these limits belong to the space X and possibly not to the set K, then K is called **compact in** X or **compact with respect to** X. It is, obviously, necessary and sufficient for K to be sequentially compact that K should be closed and compact in X.

If, in particular, every infinite subset of the space X contains a sequence which converges to an element of X, then the space X is said to be **compact.** A compact metric space is also called **compact.** It is plain that every compact metric space is complete.

Examples. 1. Let $X = [0, 1]$. Obviously, by the BOLZANO theorem, X is a compact space.

2. $X = E_1$, the one-dimensional Euclidean space (the real line), is not compact. In fact, its subset $M = \{1, 2, 3, \ldots, n, \ldots\}$ contains no convergent sequence. By the BOLZANO theorem, however, every bounded set in this space is compact.

3. Similarly, we find that E_n, the n-dimensional Euclidean space, is not compact. But every bounded set of elements of this space is compact.

4. $X = C[0, 1]$ is not compact. Moreover, there exists in $C[0, 1]$ a bounded non-compact set (p. 153).

5. The space $X = l_2$ is not compact. What is more, there exists in this space a bounded non-compact set, e.g. the closed unit sphere $\bar{S}(0, 1) = S$.

In fact, consider the sequence of points in $S: e_1 = \{1, 0, 0, \ldots\}$, $e_2 = \{0, 1, 0, \ldots\}, \ldots$ Then $\| e_i - e_j \| = \sqrt{2}$ for $i \neq j$. Hence, the sequence $\{ e_i \}$ and every subsequence of it is divergent, proving also that S is not compact. However, a non-trivial example of a set compact in the space l_2 is furnished by the so called *Hibert cube* U which is defined as the collection U of points $x = \{ \xi_1, \xi_2, \ldots, \xi_n, \ldots \}$, whose coordinates satisfy the condition $0 \leqslant \xi_n \leqslant 1/n$. The compactness of the set U stems from the general criterion for compactness formulated in the sequel (p. 167).

For a compact set it is possible to prove an analogue of the theorem on closed spheres of a metric space, without assuming X to be complete. The upcoming theorem provides a case in point.

THEOREM (CANTOR'S). *Given a nested sequence $K_1 \supset K_2 \supset \ldots \supset K_n \supset \ldots$ of nonempty closed compact sets in a metric space X. Then, the intersection $K = \bigcap_{i=1}^{\infty} K_i$ is nonempty.*

PROOF. In fact, select a point x_i in each set K_i to form the sequence $\{ x_i \} \subset K_i$. Since K_i is compact, a convergent subsequence $\{x_{i_k}\}$ can be extracted from $\{x_i\}$. Let $x_0 = \lim_k x_{i_k}$. Since for any fixed n, from the index $i_k > n$ onward, all terms of this sequence belong to K_n and K_n is closed, it follows that $x_0 \in K_n$. However, then, $x_0 \in \bigcap_{i=1}^{\infty} K_i$, and we are through.

5.11. The existence theorem for an extremum. Proofs of the theorems on continuous functions defined on a closed interval are based on compactness. A number of these theorems can be extended to continuous functionals defined on compact sets of an arbitrary metric space. The upcoming theorem, which is a generalization of the well-known WEIERSTRASS theorem, serves as an example in this direction.

THEOREM 1. *Let K be a sequentially compact set of the space X and $f(x)$ a continuous functional defined on this set. Then*

(i) *the functional $f(x)$ is bounded on K; and*
(ii) *the functional $f(x)$ assumes its least upper (supremum) and greatest lower (infimum) bounds on K.*

PROOF. (i). It is required to show that the functional $f(x)$ is bounded above (boundedness below can be demonstrated analogously). Assume the contrary. Then, there is a sequence $\{x_n\}$ of points of K, such that $f(x_n) > n$. Since the set K is sequentially compact, $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$, which converges to a point $x_0 \in K$. However, then, $f(x_{n_k}) > n_k$ and, consequently, $f(x_{n_k}) \to \infty$ as $k \to \infty$. On the other hand,

$$f(x_{n_k}) \to f(x_0) \qquad \text{as } k \to \infty,$$

since the functional is continuous everywhere on K and, in particular, at the point x_0, a contradiction proving that $f(x)$ is bounded.

(ii) Let $\beta = \sup_{x \in K} f(x)$, implying that $f(x) \leqslant \beta$ for all $x \in K$, and that there is a point $x_\varepsilon \in K$ for all $\varepsilon > 0$, such that $f(x_\varepsilon) > \beta - \varepsilon$. Hence there exists $\{x_n\}$, such that

$$\beta - (1/n) < f(x_n) \leqslant \beta. \tag{1}$$

Since K is sequentially compact, $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$, convergent to the point $x_0 \in K$. Then

$$\beta - (1/n_k) < f(x_{n_k}) \leqslant \beta, \text{ and, therefore, } \lim_{k \to \infty} f(x_{n_k}) = \beta.$$

On the other hand, $\lim_{k\to\infty} f(x_{n_k}) = f(x_0)$, since $f(x)$ is continuous at all points of the set K and, in particular, at the point x_0. Hence, $f(x_0) = \beta$, yielding the desired proof.

Analogously, it can be proved that if $\alpha = \inf_{x\in K} f(x)$, then there is a point $\xi_0 \in K$, such that $f(\xi_0) = \alpha$.

REMARK. *If a continuous functional $f(x)$ is defined on some set M, sequentially noncompact, then* $\sup_{x\in K} f(x)$ *and* $\inf_{x\in K} f(x)$ *cannot be attained.*

For example, consider in $C[0, 1]$ the set M of all functions $x(t)$, such that $x(0) = 0$, $x(1) = 1$ and $\max_t |x(t)| \leqslant 1$. The continuous functional

$$f(x) = \int_0^1 x^2(t)\, dt,$$

though continuous on M, does not attain its greatest lower bound on M.

In fact, if $x(t) = t^n$, then $f(x) = 1/(2n+1)$. Thus, $\inf_M f(x) = 0$. But, evidently, $f(x) > 0$ for every continuous curve $x = x(t)$, which joins the points (0, 0) and (1, 1). (This implies, in particular, that the set of curves considered is not compact, even if it is a bounded and closed set in $C[0, 1]$).

Thus, the compactness of the set on which a continuous functional is defined, is an essential hypothesis for Theorem 1. The assumption of the existence of a greatest lower bound or a least upper bound of a functional on a noncompact set can lead to an erroneous inference as the above example shows.

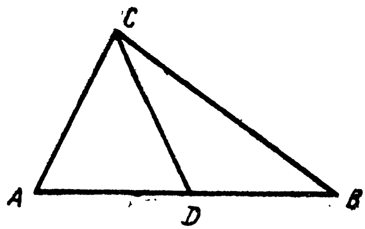

Fig. 5.

By way of another example of this type we give a *pseudo-proof* of the fifth postulate of EUCLID. As is well known, this postulate is equivalent to the assumption that the sum of the angles of a triangle is π. We can rigorously prove that this sum cannot be greater than π. Here we shall show that the sum of the angles of some triangle is π. Let $\varkappa$ be the supremum of the sum of the angles of the triangle, $\varkappa \leqslant \pi$, and let there exist a triangle ABC (see Fig. 5) on which the sum of angles attains its maximum value $\varkappa$. If an arbitrary interior point D of the side AB is joined with the vertex C by a segment CD, then CD divides the triangle into two triangles, ADC and DCB, and the sum of the angles of either triangle is not greater than $\varkappa$. On the other hand, the sum of the angles of both the triangles is equal to $\varkappa + \pi$. Hence, $\varkappa + \pi \leqslant 2\varkappa$. However, since $\varkappa$ is not greater than π, it follows that $\varkappa = \pi$. Thus, there is a triangle whose angle-sum is π, and the fifth postulate of EUCLID is proved.

Here the mistake lies in the hypothesis on the existence of a triangle, the sum of whose angles attains the supremum (which, as already remarked, is equivalent to the fifth postulate of EUCLID). In the LOBACHEVSKI geometry the difference between π and the angle-sum of a triangle is proportional to the area of the triangle, and if this difference converges to zero, the triangle shrinks to a point.

Theorem 1 can be generalized to semi-continuous functionals. A functional $f(x)$ said to be **lower (upper) semi-continuous**, if the condition $x_n \to x$ implies that $f(x) \leqslant \underline{\lim}_{n} f(x_n)$ [corres. $f(x) \geqslant \overline{\lim}_{n} f(x_n)$]. For such functionals we have:

THEOREM 2. *A functional $f(x)$, lower (upper) semi-continuous and defined on a set which is sequentially compact, is bounded below (above) this set and takes its greatest lower (least upper) limit on the set.*

This theorem finds extensive application in the calculus of variations, since the semi-continuous functionals considered there constitute the most important class of functionals.

5.12. Criterion for compactness of sets in metric spaces. For prescribing a general criterion for compactness of a set lying in a metric space, we introduce the definition which says: A set N in a metric space X is called an **ε-net for the set** M of the same space, if there is a point $x_\varepsilon \in N$ for every point $x \in M$, such that $\rho(x, x_\varepsilon) < \varepsilon$. (In particular, M can coincide with the entire space X.)

THEOREM 3. (HAUSDORFF). *For a set K in a metric space X to be compact, it is necessary, and in the case of completeness of X, sufficient that there is a finite ε-net* for *the set K for every $\varepsilon > 0$.*

Necessity. Assume that K is compact and let x_1 be any point of K. If $\rho(x, x_1) < \varepsilon$ for all $x \in K$, then a finite ε-net is already constructed. If, however, this is not the case, then there is a point $x_2 \in K$, such that $\rho(x_1, x_2) \geqslant \varepsilon$. If either $\rho(x, x_1) < \varepsilon$ or $\rho(x, x_2) < \varepsilon$ for every point $x \in K$, then a finite ε-net has been found. If this does not hold, then there is a point x_3, such that

$$\rho(x_1, x_3) \geqslant \varepsilon, \qquad \rho(x_2, x_3) \geqslant \varepsilon.$$

Continuing in this manner, a set of points $x_1, x_2, \ldots, x_n$ is obtained, such that $\rho(x_i, x_j) \geqslant \varepsilon$, for $i \neq j$. There arise two possibilities. Either, this process terminates at the k-th step, that is, there holds one of the inequalities

$$\rho(x, x_i) < \varepsilon, \qquad i = 1, 2, \ldots, k$$

for every $x \in K$, and the $x_1, x_2, \ldots, x_k$ form a finite ε net† for K; or the process continues indefinitely. The second possibility can be ruled out, for otherwise an infinite sequence of points $x_1, x_2, \ldots, x_n, \ldots$ would be obtained such that $\rho(x_i, x_j) \geqslant \varepsilon$ for $i \neq j$, and neither this sequence nor any one of its subsequences would converge, violating the hypothesis that the set K is compact.

Sufficiency. Assume that the space X is complete and that there is a finite ε-net for K for every $\varepsilon > 0$. Select a number sequence $\{\varepsilon_n\}$, $\lim_n \varepsilon_n = 0$, and for every ε_n construct a finite ε_n-net,

$$\{x_1^{(n)}, x_2^{(n)}, \ldots, x_{k_n}^{(n)}\}$$

for the set K. Choose also any infinite subset $T \subset K$ and describe a closed sphere of radius ε_1 around each of the points $\{x_1^{(1)}, x_2^{(1)}, \ldots, x_{k_1}^{(1)}\}$ in the ε-net. Then, each of the points in T falls into one of these spheres. Since the number of spheres is finite, at least one of these spheres contains an infinite set of point T. Denote such a subset of T by T_1. Again, we describe a closed sphere of radius ε_2 around each of the points $\{x_1^{(2)}, x_2^{(2)}, \ldots, x_{k_2}^{(2)}\}$ in the ε-net. A reiteration of the preceding arguments yields another infinite set $T_2 \subset T_1$ lying completely in a sphere of radius ε_2. Continuing this process, a nested sequence $T_1 \supset T_2 \supset \ldots \supset T_n \supset \ldots$ of infinite subsets is obtained where the subset T_n is contained in a closed sphere of radius ε_n, and, consequently, the distance between any two points of T_n does not exceed $2\varepsilon_n$.

Now, choose a point $\xi_1 \in T_1$, a point $\xi_2 \in T_2$ other than ξ_1, a point $\xi_3 \in T_3$ other than ξ_1 and ξ_2 and so on, to obtain a sequence of points

$$T_\omega = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\}.$$

This is a Cauchy sequence. In fact, $\xi_n \in T_n$ and $\xi_{n+p} \in T_{n+p} \subset T_n$ for every natural number p. Consequently, $\rho(\xi_{n+p}, \xi_n) < 2\varepsilon_n \to 0$ as $n \to \infty$, $p > 0$. Since by hypothesis the space X is complete, the sequence T_ω converges to some point $\xi \in X$, proving that the set K is compact.

Corollary 1. *For a set K of a complete metric space X to be compact, it is necessary and sufficient that there is a compact ε-net for the set K for every $\varepsilon > 0$.*

Let N be a compact $\varepsilon/2$-net for K. Applying the preceding theorem to N, there is a finite $\varepsilon/2$-net N_0 for N. Then, N_0 is a finite ε-net for K. In fact, there is a point $\xi \in N$ for every point $x \in K$, such that $\rho(x, \xi) < \varepsilon/2$. This again implies that there is a point $x_\varepsilon \in N_0$ for every point $\xi \in N$, such that $\rho(\xi, x_\varepsilon) < \varepsilon/2$. Therefore, there is a point x_ε for every point $x \in K$ in N_0, such that

$$\rho(x, x_\varepsilon) \leqslant \rho(x, \xi) + \rho(\xi, x_\varepsilon) < (\varepsilon/2) + (\varepsilon/2) = \varepsilon,$$

that is, N_0 is a finite ε-net for K.

†It is relevant to note that this ε-net consists of the points of the set K.

Since the space X is complete, we conclude by the preceding theorem that K is compact.

COROLLARY 2. *A compact space X is separable.*

Choose a sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$, and construct for every ε_n a finite ε_n-net

$$N_n = \{x_i^{(n)}\}_{i=1, 2, \dots, k_n}.$$

Let $N = \bigcup_{n=1}^{\infty} N_n$. Evidently, N is a countable set everywhere dense in X.

COROLLARY 3. *A compact set K of a metric space is bounded.*

Let $N_1 = \{x_1, \dots, x_n\}$ be a 1-net for K and a a fixed element of the space X. Further, let $d = \max_i \rho(a, x_i)$. Then, evidently, $\rho(x, a) \leqslant 1 + d$ for every point $x \in K$, proving the corollary.

Let us deduce two more criteria for a set to be sequentially compact, which are also admissible in the definition of this concept.

A system $\{G_\alpha\}$ of open sets of a space X is called a **cover (or covering) of the set** $M \subset X$, if every point $x \in M$ belongs to at least one set G_α of this system.

THEOREM 4. *In order that a closed set F of a metric space X be sequentially compact, it is necessary and sufficient that every covering of F by open sets of X contains a covering consisting of a finite number of these open sets.*

Necessity. Let a system $\{G_\alpha\}$ of open sets be a covering of a sequentially compact set F, such that it is not possible to extract from it a finite covering. Select a sequence $\{\varepsilon_n\}$ that converges to zero. Let $x_1^{(1)}, x_2^{(1)}, \dots, x_{k_1}^{(1)}$ be an ε_1-net for the set F. Then, $F = \bigcup_{i=1}^{k_1} F_i$, with $F_i = \bar{S}(x_i^{(1)}, \varepsilon_1) \cap F$. It is plain that F_i is a sequentially compact set of diameter not exceeding $2\varepsilon_1$. If F cannot be covered by any finite subsystem in $\{G_\alpha\}$, then the same is true for at least one of the sets of F_i, which may be denoted as F_{i_1}.

Continuing this process, we extract $F_{i_1 i_2}$ of diameter not exceeding $2\varepsilon_2$, from the sequentially compact F_{i_1}, and here also it is not possible to extract from $\{G_\alpha\}$ any finite covering of $F_{i_1 i_2}$, and so on and so forth. Thus, a nested sequence of closed compact sets is obtained: $F_{i_1} \supset F_{i_1 i_2} \supset \dots \supset F_{i_1 i_2 \dots i_n} \supset \dots$, whose diameters tend to zero.

Let x_0 be a point belonging to all these sets. Since the system $\{G_\alpha\}$ is a covering of F and $x_0 \in F$, there is a set G_{α_0} which contains this point. G_{α_0} being an open set, there is a neighbourhood $S(x_0, \varepsilon)$ of the point x_0, contained completely in G_{α_0}. Now, select n so large that the diameter of $F_{i_1 i_2 \dots i_n}$ is less than ε. Then, evidently, $F_{i_1 i_2 \dots i_n} \subset S(x_0, \varepsilon)$, a contradiction, because, on the one hand, by construction, it is impossible for $\{G_\alpha\}$ to contain any finite covering of $F_{i_1 i_2 \dots i_n}$, whereas on the other hand, this set is covered by G_{α_0}.

Consequently, every system that covers a set F has a finite cover. The necessity is proved.

Sufficiency. Assume that every covering of F by open sets contains a finite covering. Let M be a subset of F, having no limit point. Then, for every point $x \in F$ there is a neighbourhood $S(x, \varepsilon_x)$ which does not contain, except perhaps the point x itself, any of the points of M. This neighbourhood forms a covering of M. Extract from this the finite coverings $S(x_1, \varepsilon_1)$, $S(x_2, \varepsilon_2)$, . . . , $S(x_n, \varepsilon_n)$. Since the entire set M is located in these neighbourhoods and each neighbourhood cannot contain more than one point of M, it follows that M must be finite. Consequently, every infinite subset $M \subset F$ must have a limit point, that is, F is compact.

A system of sets is called **centralized** (or is said to **have the finite intersection property**), if any finite subsystem of this system has nonempty intersection.

THEOREM 5. *In order that a closed set F of a metric space X be compact, it is necessary and sufficient that every centralized system of closed subsets of F has nonempty intersection.*

Necessity. Let F be sequentially compact and let $\{F_\alpha\}$ be a centralized system of closed subsets of F with empty intersection. Select $G_\alpha = CF_\alpha$†. Then G_α forms a covering and $\bigcup_\alpha G_\alpha = C \bigcap_\alpha F_\alpha = X$. Hence the system $\{G_\alpha\}$ is a cover of F and it is possible to extract from this system a finite subsystem of covers of F: $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$. Since $\bigcup_{i=1}^{n} G_{\alpha_i} \supset F$, it follows that

$$\bigcap_{i=1}^{n} F_{\alpha_i} = C \bigcup_{i=1}^{n} G_{\alpha_i} \subset CF. \tag{2}$$

However, on the other hand, $F_{\alpha_i} \subset F$ and, hence,

$$\bigcap_{i=1}^{n} F_{\alpha_i} \subset F. \tag{3}$$

From (2) and (3) it follows that $\bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset$, contradicting the assumption that $\{F_\alpha\}$ is a centralized system. Thus, the necessity is proved.

Sufficiency. Suppose that any centralized system of closed subsets of F has an empty intersection. Consider F to be covered by any $\{G_\alpha\}$. Introduce $F_\alpha = F \setminus G_\alpha = F \cap CG_\alpha$. The set F_α is closed, and $\bigcap_\alpha F_\alpha = F \setminus \bigcup_\alpha G_\alpha = \emptyset$. Hence $\{F_\alpha\}$ is not a centralized system and, consequently, there is a subsystem $F_{\alpha_1}, F_{\alpha_2}, \ldots, F_{\alpha_n}$ with empty intersection. Then, for the corres-

†CA means a complement of the set A,

ponding sets $G\alpha_1, G\alpha_2, \ldots, G\alpha_n$ we have

$$\bigcup_{i=1}^{n} G\alpha_i \supset F \setminus \bigcap_{i=1}^{n} F\alpha_i = F.$$

Thus, it is shown that from an arbitrary $\{G_\alpha\}$ that covers the set F, it is possible to extract a finite subsystem of covers.

THEOREM 6. *Every sequentially compact set of a metric space is a continuous image of the Cantor perfect set.*

In a metric space X, let K be a sequentially compact set. Consider a sequence $\{\varepsilon_n\}$ convergent to zero, and for every $n = 1, 2, 3, \ldots$ construct a finite ε_n-net $\{x_i^{(n)}\}$, $i = 1, 2, \ldots, m_n$ for K. As a supplementry assumption, if needed, it can be further regarded that $m_n = 2^{k_n}$ everywhere. Consider a sphere $S_i^{(1)}$ of radius ε_1 with centre at the point $x_i^{(1)}$. The set K is contained completely in this sphere and, what is more, in the closed sphere $\bar{S}_i^{(1)}$. Let $K_{i_1} = K \cap S_{i_1}^{(1)}$, $i_1 = 1, \ldots, m_1$. The set K can then be represented in the form of the sum of m_1 closed sets of diameter not exceeding $2\varepsilon_1$. Being a closed component of a sequentially compact set, every K_{i_1} is again a sequentially compact set. Continuing so, every K_{i_1} becomes representable in the form of the sum of m_2 closed sets $K_{i_1 i_2}$, $i_2 = 1, 2, \ldots, m_2$, of diameter not exceeding $2\varepsilon_2$, and so on. All these sets can be regarded to be non-empty.

Let us now revert to the CANTOR perfect set P_0. This set lies completely on a segment of k_1-th rank, $\Delta_{j_1, j_2, \ldots, j_{k_1}}$, $j_1 = 0, 1$ and also on the segments of (k_1+k_2)-th rank, $\Delta_{j_1, j_2, \ldots, j_{k_1+k_2}}$, and so on. Renumber the segments of k_1-th rank from left to right and denote them by $\tilde{\Delta}_{i_1}$, $i_1 = 1, 2, \ldots, m_1 = 2^{k_1}$. On every k_1-th rank segment $\tilde{\Delta}_{i_1}$, there lie $2^{k_2} = m_2$ segments of (k_1+k_2)-th rank. Again, they may be renumbered from left to right and denoted by $\tilde{\Delta}_{i_1 i_2}$, $i_2 = 1, 2, \ldots, m_2$, and so on. We obtain a one-one correspondence between the closed sets $F_{i_1 i_2, \ldots, i_s}$ of X and segments $\tilde{\Delta}_{i_1 i_2 \ldots i_s}$ of the interval $[0, 1]$.

Select an arbitrary point $t \in P_0$. This uniquely defines the system of segments $\tilde{\Delta}_{i_1}, \tilde{\Delta}_{i_1 i_2}, \tilde{\Delta}_{i_1 i_2 i_3}, \ldots,$ containing this point and contracting to it. Consider the corresponding system of closed sets $K_{i_1}, K_{i_1 i_2}, K_{i_1 i_2 i_3}, \ldots$ (with the same indices as those of the segments). Since every set that follows is contained in the preceding one and the diameters of sets tend to zero, there is a unique point $x \in K$ belonging to all those sets to which the point $t \in P_0$ corresponds.

It is to be shown that every point $x \in K$ is an image of some point $t \in P_0$. In fact, $x \in K_{i_1}$ for some value of i_1 (this index is, generally, not unique, since the sets K_{i_1} may intersect), analogously $x \in K_{i_1 i_2}$, and so on. The sets $K_{i_1}, K_{i_1 i_2}$ correspond to the segments $\tilde{\Delta}_{i_1}, \tilde{\Delta}_{i_1 i_2}, \ldots$. The point t belonging to all these segments, has for its image the point x under consideration. Thus, $x = \varphi(t)$ defines uniquely a mapping of the CANTOR perfect set P_0 into the sequentially compact set K. It remains to exhibit that this mapping is continuous.

Let $x_0 = \varphi(t_0)$ and let $S(x_0, \varepsilon)$ be a neighbourhood of the point x_0. Select those sets $K_{i_1 i_2 \ldots i_n}$ of the system which shrink to the point x_0 with diameter less than ε. Then $K_{i_1 i_2 \ldots i_n} \subset S(x_0, \varepsilon)$. Denote by δ the distance between t_0 and the nearest ends of the segments $\tilde{\Delta}_{i_1 i_2 \ldots i_n}$, which correspond to the sets $K_{i_1 i_2 \ldots i_n}$. If $|t - t_0| < \delta$, then $t \in \tilde{\Delta}_{i_1 i_2 \ldots i_n}$; consequently, $x = \varphi(t) \in K_{i_1 i_2 \ldots i_n} \subset S(x_0, \varepsilon)$ and, hence, $\rho(x, x_0) < \varepsilon$. The theorem is completely proved.

We now proceed to consider a mapping f of the compact set X onto a metric space Y.

THEOREM 7. *Every continuous image of a compact set is compact.*

In fact, let $\{y_n\}$ be an arbitrary sequence of $f(X) \subset Y$. For every y_n take one of its pre-images x_n. Since $\{x_n\} \subset X$ and X is compact, we can extract from $\{x_n\}$ a subsequence $\{x_{n_k}\}$, convergent to $x_0 \in X$. Since $f(x)$ is continuous, it follows that $f(x_{n_k}) = y_{n_k} \to y_0 = f(x_0) \in f(X)$. Thus, every sequence in $f(X)$ contains a convergent subsequence, and also the limit of such a subsequence belongs to $f(X)$. Consequently, $f(X)$ is compact.

This theorem, together with Theorem 6, completes the compactness criterion. Namely, *a metric space K is compact iff it is a continuous image of the Cantor perfect sets.*

5.2 CRITERIA FOR COMPACTNESS IN SOME FUNCTIONAL SPACES

5.21. The space $C[0, 1]$. The functions of a set M are said to be **uniformly bounded** if there is a constant c such that $|x(t)| \leqslant c$ with all $x(t) \in M$ and for every $t \in [0, 1]$. They are called **equicontinuous** if for every $\varepsilon > 0$ there is a $\delta > 0$, depending only on ε, such that the inequality $|t_1 - t_2| < \delta_2$ is satisfied for every $t_1, t_2 \in [0, 1]$ and the relation $|x(t_1) - x(t_2)| \leqslant \varepsilon$ holds for every function $x(t)$ of the considered set.

THEOREM 1. (ARZELÀ). *For a set $K \subset C[0, 1]$ to be compact, it is necessary and sufficient that the functions $x(t) \in K$ be uniformly bounded and equicontinuous.*

Necessity. Let K be compact. The uniform boundedness of the functions $x(t) \in K$ follows from Corollary 3 to Theorem 3 of the preceding section. It remains to prove the equicontinuity of these functions. Construct for a given $\varepsilon > 0$, a finite $\varepsilon/3$-net $\{x_1(t), x_2(t), \ldots, x_k(t)\}$ for K. Since the functions $x_i(t)$ are continuous on $[0, 1]$, they are also uniformly continuous on this interval.

For every function $x_i(t)$ select a number δ_i, such that $|x_i(t_1) - x_i(t_2)| < \varepsilon/3$ for $|t_1 - t_2| < \delta_i$, whatever be $t_1, t_2 \in [0, 1]$. Let δ be the smallest of the numbers δ_i, $i = 1, 2, \ldots, k$. Now, if $|t_1 - t_2| < \delta$, then

$$|x(t_1) - x(t_2)| \leqslant \max_{0\leqslant t\leqslant 1} |x(t) - x_i(t)| + |x_i(t_1) - x_i(t_2)|$$
$$+ \max_{0\leqslant t\leqslant 1} |x_i(t) - x(t)| < 2\rho(x, x_i) + \frac{\varepsilon}{3}$$

holds for every function $x(t) \in K$. If $x_i(t)$ is so chosen that $\rho(x, x_i) < \varepsilon/3$, then, $|x(t_1) - x(t_2)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary and this inequality is independent of the location of points t_1 and t_2 on $[0, 1]$ and of the choice of the function $x(t) \in K$, the equicontinuity of all the functions belonging to K is proved.

Sufficiency. In view of the hypotheses of the theorem, a $\delta > 0$ can be chosen for every $\varepsilon > 0$, such that $|x(t_1) - x(t_2)| < \varepsilon$ with $|t_1 - t_2| < \delta$ for every $t_1, t_2 \in [0, 1]$ and every function $x(t) \in K$. Select a natural number n such that $1/n < \delta$ and divide $[0, 1]$ into n equal parts:

$$\left[\frac{k}{n}, \frac{k+1}{n}\right], \qquad k = 0, 1, 2, \ldots, n-1,$$

then,
$$|x(t_1 - x(t_2)| < \varepsilon$$

for every function $x(t) \in K$ and any $t_1, t_2 \in [0, 1]$, such that $|t_1 - t_2| < (1/n)$, in particular, for t_1 and t_2 belonging to one and the same subinterval $[k/n, (k+1)/n]$.

Assign a continuous function $x_n(t)$ to every $x(t)$, such that: (i) $x_n(k/n) = x(k/n)$ for $k = 0, 1, \ldots, n-1$; and (ii) the functions $x_n(t)$ are linear on the interval $[k/n, (k+1)/n]$.

Thus, $x_n(t)$ is a polygonal curve with n sides inscribed in the curve $x(t)$. Let

$$x\left(\frac{k}{n}\right) \leqslant x\left(\frac{k+1}{n}\right).$$

Then, owing to the linearity of $x_n(t)$ on $[k/n, (k+1)/n]$, $x(k/n) \leqslant x_n(t) \leqslant x[(k+1)/n]$, whence $-\varepsilon < x(t) - x[(k+1)/n] \leqslant x(t) - x_n(t) \leqslant x(t) - x(k/n) < \varepsilon$. However, if $x[(k+1)/n] \geqslant x(k/n)$, then $-\varepsilon < x(t) - x(k/n) \leqslant x(t) - x_n(t) \leqslant x(t) - x[(k+1)/n] < \varepsilon$. Consequently,

$$|x(t) - x_n(t)| < \varepsilon$$

for all $t \in [0, 1]$, that is, $\rho(x, x_n) < \varepsilon$. Thus, the set N of functions $x_n(t)$ forms an ε-net for K. Further, owing to the uniform boundedness of the set K, we have

$$|x_n(t)| \leqslant |x(t)| + |x(t) - x_n(t)| < c + \varepsilon = c_1,$$

that is N is uniformly bounded.

Associate to every function $x_n(t) \in N$ the points of an $(n+1)$-dimensional space $\tilde{X}$, having as coordinates the ordinates of the vertices of a polygon, the graph of $x_n(t)$. This correspondence, as is easy to see, is one-one and also mutually continuous, so that if the sequence of functions $\{x_n^{(k)}(t)\}$ converges to $x_n^{(0)}(t)$ in the sense of the metric space $C[0, 1]$, the sequence of points $\{\tilde{x}^{(k)}\}$ converges to the points $\tilde{x}^{(0)}$ in the sense of the metric space E_{n+1}. However, the set $\tilde{N} = \{\tilde{x}\}$ is bounded and is, consequently, compact in E_{n+1}. Hence, the set $N = \{x_n(t)\}$ is compact in $C[0, 1]$.

Thus, for every $\varepsilon > 0$, a compact ε-net for K can be formed. However, then, owing to the completeness of $C[0, 1]$ and Corollary 1 to Theorem 3 of Sec. 1, K is compact.

The theorem proved admits extension to the case of *mapping a compact set into a compact set.*

Given two metric spaces X and Y and let F be a set of mappings f carrying X into Y. The mapping $f \in F$ is said to be **bounded**, if for every $x \in X$, $\rho(f(x), \theta) \leqslant c_f$, with θ some fixed element in Y and c_f a constant which depends, generally speaking, on f. The mapping $f \in F$ is said to be **uniformly continuous**, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\rho[f(x_1), f(x_2)] < \varepsilon$ holds for all points x_1 and x_2 in X satisfying $\rho(x_1, x_2) < \delta$.

Let $M(X, Y)$ be a set of all bounded mappings of X into Y. Transform $M(X, Y)$ into a metric space, setting $\rho(f, \varphi) = \sup_{x \in X} [f(x), \varphi(x)]$. It is plain that all metric axioms are satisfied. The convergence in $M(X, Y)$ represents on X the uniform convergence of the sequence of mappings $\{f_n(x)\} \subset M(X, Y)$ to the mapping $f(x) \in M(X, Y)$.

If Y is a complete space, $M(X, Y)$ is also complete.

In fact, if $\rho(f_n, f_m) \to 0$ as n and $m \to \infty$, then for every $\varepsilon > 0$ there is an index $n_0(\varepsilon)$, such that

$$\rho[f_n(x), f_m(x)] < \varepsilon \tag{1}$$

for $n, m \geqslant n_0(\varepsilon)$ and straightaway for all $x \in X$. We hold $x \in X$ fixed. By the completeness of Y, $\{f_n(x)\}$ is a CAUCHY sequence convergent to some $y \in Y$. Set $f(x) = y = \lim_n f_n(x)$ to receive some mapping of X into Y.

The passage to the limit in (1) as $m \to \infty$, yields $\rho[f_n(x), f(x)] \leqslant \varepsilon$ for $n \geqslant n_0(\varepsilon)$ and straightaway for all $x \in X$, implying that $f \in M(X, Y)$ and $f_n(x) \to f(x)$ uniformly on X.

Denote by $C(X, Y)$ a set of all uniformly continuous mappings of $M(X, Y)$. It is trivial to verify that the limit of a uniformly convergent sequence of uniformly continuous mappings is also a uniformly continuous mapping, implying that the set $C(X, Y)$ is closed in $M(X, Y)$.

Finally, one more definition may be introduced. The mapping f appearing in some family $Q \subset C(X, Y)$ is called equicontinuous **~~uniformly~~ continuous**, if for every $\varepsilon > 0$ there is $\delta > 0$ depending only on ε, such that $\rho[f(x_1), f(x_2)] < \varepsilon$ for $\rho(x_1, x_2) < \delta$, straightaway for all $f \in Q$ and independent of the choice of points x_1 and $x_2 \in X$.

THEOREM 2. *In order that a uniformly convergent sequence can be extracted from a family Q of continuous mappings of a compact set X into a compact set Y, it is necessary and sufficient that mappings of Q be uniformly convergent.*

It is proposed to prove only the sufficiency part of the formulated theorem.

We first note that Y being compact is a bounded set and, consequently, all mappings of the family Q are uniformly bounded.

Thus, $Q \subset C(X, Y)$. Since $C(X, Y)$ is closed in $M(X, Y)$, for the compactness of Q in $C(X, Y)$, it is sufficient to prove its compactness in $M(X, Y)$.

For any $\varepsilon > 0$, select $\delta > 0$ such that

$$\rho[f(x_1), f(x_2)] < \varepsilon/2 \tag{2}$$

for $\rho(x_1, x_2) < \delta$ and simultaneously for all $f \in Q$, which is possible because of the equicontinuity of the mappings. Thereupon, take a finite $\delta/2$-net $x_1 x_2, \ldots, x_n$ in the set X. Introduce the sets

$$X_i = S(x_i, \delta/2) \setminus \bigcup_{j \neq i} S(x_j, \delta/2).$$

These sets are non-intersecting, in the sum give all the X and the diameter of each of X_i does not exceed δ. Further, let $y_1, y_2, \ldots, y_n$ be an $\varepsilon/2$-net for compact Y. Consider all possible functions $g(x) \in M(X, Y)$, assuming on the sets X_i the constant values y_j. These functions form a finite ε-net for the set Q. In fact, take any mapping $f \in Q$, to receive for arbitrary $x \in X$ and every $g(x)$,

$$\rho[f(x), g(x)] \leqslant \rho[f(x), f(x_i)] + \rho[f(x_i), g(x_i)] + \rho[g(x_i), g(x)],$$

where x_i is so chosen that $x \in X_i$. Hence, by (2) and since x and x_i belong to one and the same X_i, $\rho[f(x), f(x_i)] < \varepsilon/2$, $\rho[g(x), g(x_i)] = 0$, whence

$$\rho[f(x), g(x)] < (\varepsilon/2) + \rho[f(x_i), g(x_i)].$$

Now, select $g(x)$ such that $g(x_i) = y_j$ satisfies the inequality $\rho(f(x_i), y_j) < \varepsilon/2$. Then $\rho[f(x), g(x)] < \varepsilon$ for every $x \in X$ and, consequently,

$$\rho(f, g) = \sup_x \rho[f(x), g(x)] \leqslant \varepsilon.$$

The set Q, as a subset of the complete metric space $M(X, Y)$, having a finite ε-net, is compact. Thus, the theorem is proved.

5.22. Space L_p [0, 1]. Let $x(t) \in L_p[0, 1]$. Extend the function $x(t)$ beyond the interval $[0, 1]$ and put $x(t) = 0$ if t lies outside this interval. Then for every segment $[a, b]$ of the numerical axis t, the integrals

$$\int_a^b |x(t)|\, dt \quad \text{and} \quad \int_a^b |x(t)|^p\, dt$$

have a meaning. With this setting, we formulate the next theorem.

Compactness in L_p [0, 1] (M. RIESZ' THEOREM). *For a family of functions $K = \{x(t)\} \subset L_p[0, 1]$ to be compact, it is necessary and sufficient that these functions are uniformly bounded in the norm and uniformly continuous in mean,* *namely that*

(i) $$\int_0^1 |x(t)|^p \, dt \leqslant c^p; \quad \text{and}$$

(ii) $$\int_0^1 |x(t+h) - x(t)|^p \, dt < \varepsilon^p \quad \text{for } 0 < h < \delta(\varepsilon) \text{ simultaneously}$$

for all the functions of the family.

Necessity. The necessity of hypothesis (i) is trivial. Therefore, it remains to show that hypothesis (ii) is satisfied. Since K is a compact set, for any $\varepsilon > 0$ there exists for this set a finite $\varepsilon/3$-net $x_1(t), x_2(t), \ldots, x_n(t)$. Since every function in L_p [0, 1] is continuous in mean, for every i there exists δ_i, such that

$$\int_0^1 |x_i(t+h) - x_i(t)|^p \, dt < \left(\frac{\varepsilon}{3}\right)^p$$

for $0 < h < \delta_i$. Let $\delta = \min\limits_i \delta_i$. Then

$$\int_0^1 |x_i(t+h) - x_i(t)|^p \, dt < \left(\frac{\varepsilon}{3}\right)^p$$

for $0 < h < \delta$ and for all $i = 1, 2, \ldots, n$.

Take an arbitrary function $x(t) \in K$. There is a function $x_i(t)$ such tha·

$$\int_0^1 |x(t) - x_i(t)|^p \, dt < \left(\frac{\varepsilon}{3}\right)^p.$$

For $0 < h < \delta$, we get

$$\left(\int_0^1 |x(t+h) - x(t)|^p \, dt\right)^{1/p}$$

$$\leqslant \left(\int_0^1 |x(t+h) - x_i(t+h)|^p \, dt\right)^{1/p}$$

$$+ \left(\int_0^1 |x_i(t+h) - x_i(t)|^p \, dt\right)^{1/p} + \left(\int_0^1 |x_i(t) - x(t)|^p \, dt\right)^{1/p}$$

$$< \left(\int_0^1 |x(t+h) - x_i(t+h)|^p\right)^{1/p} + \frac{2\varepsilon}{3}.$$

However, $\int_0^1 |x(t+h) - x_i(t+h)|^p \, dt$

$$= \int_h^1 |x(s) - x_i(s)|^p \, ds \leqslant \int_0^1 |x(s) - x_i(s)|^p \, ds < \left(\frac{\varepsilon}{3}\right)^p$$

taking advantage of the fact that $x(t)$ and $x_i(t)$ vanish outside $[0, 1]$. The last two inequalities imply

$$\left(\int_0^1 |x(t+h) - x(t)|^p \, dt\right)^{1/p} < \varepsilon \qquad \text{for } 0 < h < \delta,$$

and since $x(t)$ is any function in K, the necessity of hypothesis (ii) is proved.

Sufficiency. Consider the mean-valued STEKLOV function

$$x_h(t) = \frac{1}{2h} \int_{t-h}^{t+h} x(\tau) \, d\tau,$$

to receive

$$|x_h(t)| = \frac{1}{2h} \left| \int_{t-h}^{t+h} x(\tau) \, d\tau \right|$$

$$\leqslant \frac{1}{2h} \left(\int_{t-h}^{t+h} d\tau \right)^{1/p} \left(\int_{t-h}^{t+h} |x(\tau)|^p \, d\tau \right)^{1/p}$$

$$\leqslant \left(\frac{1}{2h}\right)^{1/p} \left(\int_0^1 |x(\tau)|^p \, d\tau \right)^{1/p}, \tag{3}$$

and

$$|x_h(t+u) - x_h(t)| = \frac{1}{2h} \left| \int_{t+u-h}^{t+u+h} x(\tau) \, d\tau - \int_{t-h}^{t+h} x(\tau) \, d\tau \right|$$

$$= \frac{1}{2h} \left| \int_{t-h}^{t+h} x(\tau+u) \, d\tau - \int_{t-h}^{t+h} x(\tau) \, d\tau \right|$$

$$\leqslant \frac{1}{2h} \int_{t-h}^{t+h} |x(\tau+u) - x(\tau)| \, d\tau$$

$$\leqslant \left(\frac{1}{2h}\right)^{1/p} \left(\int_{t-h}^{t+h} |x(\tau+u) - x(\tau)|^p \, d\tau \right)^{1/p}$$

$$\leqslant \left(\frac{1}{2h}\right)^{1/p} \left(\int_0^1 |x(\tau+u) - x(\tau)|^p \, d\tau \right)^{1/p}. \tag{4}$$

Hypotheses (i) and (ii) together with Ineqs. (3) and (4) imply that for h fixed the functions of the family $\{x_h(t)\}$ for $x(t) \in K$ are uniformly bounded and equicontinuous. Consequently, $\{x_h(t)\}$ is compact in the sense of uniform convergence, and, what is more, in the sense of p-th order mean convergence.

On the other hand,

$$|x(t) - x_h(t)| \leqslant \frac{1}{2h} \int_{t-h}^{t+h} |x(t) - x(\tau)| \, d\tau$$

$$= \frac{1}{2h} \int_{-h}^{h} |x(t) - x(t+\tau)| \, d\tau$$

$$\leqslant \left(\frac{1}{2h}\right)^{1/p} \left(\int_{-h}^{h} |x(t) - x(t+\tau)|^p \, d\tau \right)^{1/p}.$$

Thereupon,

$$\int_0^1 |x(t) - x_h(t)|^p \, dt \leqslant \frac{1}{2h} \int_0^1 \left\{ \int_{-h}^{h} |x(t) - x(t+\tau)|^p \, d\tau \right\} dt$$

$$= \frac{1}{2h} \int_{-h}^{h} \left\{ \int_0^1 |x(t) - x(t+\tau)|^p \, dt \right\} d\tau$$

$$< \frac{1}{2h} \varepsilon^p \int_{-h}^{h} d\tau = \varepsilon^p,$$

since [by hypothesis (ii)] $\int_0^1 |x(t+\tau) - x(t)|^p \, dt < \varepsilon^p$, if $|\tau| < \delta$. Thus $\{x_h(t)\}$ forms an ε-net for K and since this ε-net is compact, the set K itself is also compact in consequence of the HAUSDORFF theorem.

In what follows, another compactness criterion in the space $L_p\,[0, 1]$ is formulated without proof.

THEOREM (A. N. KOLMOGOROV). *A set $K \subset L_p\,[0, 1]$ is compact, iff*

(i) *the norm of the function $x(t) \in K$ is totally bounded;*

(ii) *for every $\varepsilon > 0$ there exists $\delta > 0$, such that $\| x - x_h \| < \varepsilon$ for $h < \delta$ and for all $x(t) \in K$.*

It is said that a family of functions $M = \{x(t)\}$ has *uniformly an absolutely continuous norm*, if for every $\varepsilon > 0$ there can be found $\delta > 0$, such that

$$\| x(t)\, \chi_E(t) \| < \varepsilon$$

whenever mes $E < \delta$ (here $\chi_E(t)$ is the characteristic function of the set E).

THEOREM (M. A. KRASNOSELSKII [18]). *Let a family $K \subset L_p\,[0, 1]$ have uniformly an absolutely continuous norm and be compact in the sense of measure convergence. Then this family is compact in the sense of mean convergence.*

5.23. Space Q. For illustration, proof of an existence theorem in the calculus of variations. Consider the collection of curves $\{q\}$, defined by the equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad (0 \leqslant t \leqslant 1) \tag{5}$$

$x(t)$, $y(t)$ and $z(t)$ being continuous functions of the parameter t. If q and p are two given curves expressible in the form (5), then we let the curve-points associated with the same value of the parameter correspond to each other. Let d be the maximum of the distance between the corresponding points of the two curves. The number d depends on the choice of the parametric representation of the two curves. Let us regard the distance $\rho(q, p)$ as the greatest lower bound of the numbers d over all possible parametric representations.

It is easy to verify that the distance between the two curves, defined in this manner, satisfies all the metric axioms. Denote the space so obtained by the space Q. This space plays an important role in the calculus of variations. It can be shown that the space Q is complete.

THEOREM (HILBERT). *The collection $K \subset Q$, consisting of rectifiable curves lying in a finite part of the space and having tatally bounded lengths, is compact.*

PROOF. Let the lengths of the curves $q \in K$ be not greater than l. Partition every $q \in K$ into n arcs of equal length, join the partitioning points by segments to receive a polygon q_n. At least one of the n arcs of the curve q and correspondingly at least one segment of the polygon q_n does not exceed l/n. The distance between the points of such an arc and the points of the corresponding segment is less than or equal to $2l/n$. Introduce a parametric representation of q and q_n, such that the vertices of q_n in both the representations correspond to numbers of the form k/n, $k = 0, 1, \ldots, n$, and we obtain a sub-arc of the curve q as well as, correspondingly, the associated segment of the polygon q_n, when t runs through the interval $(k/n, (k+1)/n$. The distance between two points of q and q_n, corresponding to the same value of the parameter, is not greater than $2l/n$. Consequently, also

$$\rho(q, q_n) \leqslant 2l/n.$$

Thus, the collection K_n of the polygons q_n forms a $2l/n$-net for K. However, every polygon is determined by the $3(n+1)$ coordinates of its $n+1$ vertices. By hypothesis, these coordinates are totally bounded. Therefore, K_n is compact. But, then, by Corollary 1 to Theorem 3 of Sec. 1, K is also compact.

This theorem is used for proving the existence of geodesic lines.

5.24. Conditions of compactness in spaces with basis. THEOREM 3. *For compactness of a set K in a Banach space E with basis, it is necessary and sufficient that K is bounded and that for every $\varepsilon > 0$ there exists an index n_0 such that $\| R_n x \| < \varepsilon$ for $n \geqslant n_0$ and every x in K*†.

Necessity. The boundedness of K is implied by Corollary (3) to Theorem 3 of Sec. 1. Therefore, the second hypothesis only remains to be satisfied.

Select some number $\eta > 0$ and form a finite η-net for K: $\{x_1, \ldots, x_k\}$. For

†For operators S and R see p. 117, and p. 115 for operator A^{-1}.

every $x \in K$ there exist x_i, belonging to η-net, such that $\| x-x_i \| < \eta$. We get

$$\begin{aligned}\| R_n x \| = \| x-S_n x \| &\leqslant \|x-x_i \| + \| x_i - S_n x \| \\ &\leqslant \| x-x_i \| + \| S_n x_i - S_n x \| + \| R_n x_i \| \\ &\leqslant (1+\|A^{-1}\|) \| x-x_i \| + \| R_n x_i \| \\ &< (1+\|A^{-1}\|) \eta + \| R_n x_i \|.\end{aligned}$$

For every fixed x, $R_n x \to 0$ as $n \to \infty$. Hence, there exists n_0 such that $\| R_n x_i \| < \eta$ $(n \geqslant n_0)$ for $i = 1, 2, \ldots, k$. Thus,

$$\| R_n x \| < (2+\|A^{-1}\|) \eta$$

for $n \geqslant n_0$. It is now sufficient to take $\eta = \varepsilon/(2+\| A^{-1}\|)$ for obtaining the required inequality, because n_0 is independent of whatever x is taken in K.

Sufficiency. Let us show that subject to the fulfilment of hypotheses of the theorem, there is for every $\varepsilon > 0$ a finite ε-net for K. For this, let ε be defined by choosing n_0 such that $\| R_{n_0} x \| < \varepsilon/2$ for all $x \in K$. Then consider the set K_{n_0} consisting of the elements of the form $S_{n_0} x$, with $x \in K$. K_{n_0} can be treated as a set in an n_0-dimensional space $E_{n_0} \subset E$, defined by the elements $e_1, e_2, \ldots, e_{n_0}$. In addition, since the inequality

$$\| S_{n_0} x \| \leqslant \| A^{-1} \| \| x \|$$

and the assumption on the boundedness of K imply that K_{n_0} is bounded, it is compact too and, therefore, there exists in E_{n_0} a finite $\varepsilon/2$-net for K_{n_0}. However, this net is, evidently, an ε-net for K, which is also the required proof.

5.25. Space l_p. THEOREM. *For the compactness of a set $K \subset l_p$, it is necessary and sufficient that K be bounded and that there exists an index n_0, depending only on ε, for every $\varepsilon > 0$, such that $\sum_{i=n+1}^{\infty} | \xi_i |^p < \varepsilon^p$ for $n \geqslant n_0$ and for all $x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\} \in K$.*

The proof is immediate from the preceding theorem by noticing that in l_p,

$$\left(\sum_{i=n+1}^{\infty} | \xi_i |^p \right)^{1/p} = \| R_n x \|.$$

Example. Consider in l_2 a set of elements $x = \{\xi_i\}$, such that $0 \leqslant \xi_n \leqslant 1/n$, equivalently a *fundamental parallelopiped* of coordinates in a HILBERT space. By the preceding criterion, this parallelopiped is compact. It has been proved by P. S. URYSOHN that every separable metric space is homeomorphic to some subset of the fundamental parallelopiped of space l_2 [36].

5.3 FINITE DIMENSION AND COMPACTNESS

AS IS known, every bounded set is compact in the n-dimensional Euclidean space. It shall be established that the compactness of bounded sets is a characteristic property of finite-dimensional normed liner spaces.

THEOREM 1. *For a subspace L of a normed linear space E to be of finite dimension, it is necessary and sufficient that every bounded set of elements in L is compact.*

Necessity. Let L be n-dimensional. Then, L is homeomorphic to the n-dimensional Euclidean space E_n. A bounded set $M \subset L$ goes over into a bounded set $N \subset E_n$ one-one and mutually continuously, and since N in E_n is compact, M in L is also compact.

Sufficiency. Assume that every bounded set of elements in L is compact. Let x_1 be an arbitrary element of L, $\| x_1 \| = 1$ and denote by L_1 a subspace spanned by x_1. If $L = L_1$, the theorem is proved. Otherwise, by the lemma of Chap. 2.3, there is an $x_2 \in L$, such that $\| x_2 \| = 1$ and $\| x_2 - x_1 \| \geqslant \frac{1}{2}$. The subspace L_2 of L, spanned by x_1, x_2, is then either equal to L and the theorem is proved; or L_2 does not coincide with L. Then, by the lemma there is an x_3 such that

$$\| x_3 \| = 1, \quad \| x_3 - x_1 \| \geqslant \tfrac{1}{2}, \quad \| x_3 - x_2 \| \geqslant \tfrac{1}{2}.$$

Continuing this process, there arise two possibilities: Either there exists an n with $L_n = L$ and the theorem is proved, or we obtain an infinite sequence $\{x_n\}$, such that $\| x_n \| = 1$ and $\| x_n - x_m \| \geqslant \frac{1}{2}$, n and m arbitrary, $n \neq m$. The latter possibility, however, contradicts the hypothesis, because otherwise we get a bounded ($\| x_n \| = 1$) but non-compact ($\| x_n - x_m \| \geqslant \frac{1}{2}$ for $n \neq m$) set.

5.4 THE PROBLEM OF THE BEST APPROXIMATION

THE PROBLEM of the best approximation of functions by their linear combinations was investigated by P. L. CHEBYSHEV. Keeping to the accepted terminology, we can consider approximations in the spaces $C[0, 1]$, $L_{2,\rho}$ †, L, and so on.

Presently, it is intended to deal with the problem of the best approximation of arbitrary elements x in a normed space E by linear combinations of a given finite system of linearly independent elements $x_1, x_2, \ldots, x_n \in E$ [1].

LEMMA. *If* $\sum_{i=1}^{n} \lambda_i^2$ *increases infinitely, then*

$$\varphi(\lambda_1, \lambda_2, \ldots, \lambda_n) = \| x - \lambda_1 x_1 - \lambda_2 x_2 - \ldots - \lambda_n x_n \| \to \infty.$$

PROOF. We have

$$\varphi(\lambda_1, \lambda_2, \ldots, \lambda_n) \geqslant \| \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \| - \| x \|.$$

The continuous function

$$\psi(\lambda_1, \lambda_2, \ldots, \lambda_n) = \| \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \|$$

of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ assumes its minimum μ (which is greater

†Though $L_{2,\rho}$ has been defined as a complex space on p. 55, it is treated here as a real space, definable analogously.

than zero because of the linear independence of $x_1, x_2, \ldots, x_n$) on the sphere $\sum_{i=1}^{n} \lambda_i^2 = 1$ of the n-dimensional Euclidean space (which is a sequentially compact set). Given an arbitrary $k > 0$. If

$$\sqrt{\sum_{i=1}^{n} \lambda_i^2} > \frac{1}{\mu}(k + \| x \|),$$

then

$$\varphi(\lambda_1, \lambda_2, \ldots, \lambda_n) \geqslant \left\| \sum_{i=1}^{n} \lambda_i x_i \right\| - \| x \|$$

$$= \sqrt{\sum_{j=1}^{n} \lambda_j^2} \left\| \sum_{i=1}^{n} \frac{\lambda_i}{\sqrt{\sum_{j=1}^{n} \lambda_j^2}} x_i \right\| - \| x \|$$

$$\geqslant \sqrt{\sum_{j=1}^{n} \lambda_j^2} \cdot \mu - \| x \| > k,$$

which proves the lemma.

THEOREM. *There exist real numbers* $\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}$, *such that*

$$\varphi(\lambda_1, \lambda_2, \ldots, \lambda_n) = \| x - \lambda_1 x_1 - \lambda_2 x_2 - \ldots - \lambda_n x_n \|$$

assumes its minimum for $\lambda_1 = \lambda_1^{(0)}, \lambda_2 = \lambda_2^{(0)}, \ldots, \lambda_n = \lambda_n^{(0)}$†.

The statement of the theorem is trivial, if x depends linearly on $x_1, x_2, \ldots, x_n$. Assume that x does not lie in the subspace spanned by $x_1, x_2, \ldots, x_n$. In the first place, it is plain that $\varphi(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a continuous function of its arguments, as implied by the inequality

$$| \varphi(\lambda_1, \lambda_2, \ldots, \lambda_n) - \varphi(\mu_1, \mu_2, \ldots, \mu_n) |$$

$$= \left| \| x - \sum_{i=1}^{n} \lambda_i x_i \| - \| x - \sum_{i=1}^{n} \mu_i x_i \| \right|$$

$$\leqslant \left\| \sum_{i=1}^{n} (\lambda_i - \mu_i) x_i \right\| \leqslant \sum_{i=1}^{n} | \lambda_i - \mu_i | \| x_i \|$$

$$\leqslant \max_{1 \leqslant i \leqslant n} | \lambda_i - \mu_i | \sum_{i=1}^{n} \| x_i \|.$$

By the preceding lemma, $\varphi(\lambda_1, \lambda_2, \ldots, \lambda_n) \geqslant \| x \|$ outside of some sphere

†That is, in a finite-dimensional space spanned by the elements $x_1, x_2, \ldots, x_n$, there is an element arbitrarily close to x.

$\sum_{i=1}^{n} \lambda_i^2 \leqslant r^2$. Since this is a sequentially compact sphere, $\varphi(\lambda_1, \lambda_2, \ldots, \lambda_n)$, being a continuous function, assumes on it its minimum ν at some point $(\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)})$. But $\nu \leqslant \varphi(0, 0, \ldots, 0) = \| x \|$. Hence ν is the least value of the function $\varphi(\lambda_1, \lambda_2, \ldots, \lambda_n)$ on the entire space of the points $\lambda_1, \lambda_2, \ldots, \lambda_n$, which also proves the theorem.

The linear combination, $\lambda_1^{(0)} x_1 + \lambda_2^{(0)} x_2 + \ldots + \lambda_n^{(0)} x_n$, giving the best approximation of the element x is, in general, not unique. To obtain uniqueness it is necessary to augment the restrictions on the approximating expression $\sum_{i=1}^{n} \lambda_i x_i$. For this purpose, consider in $C[0, 1]$, systems of functions which satisfy the so-called CHEBYSHEV condition. However, there can also exist certain spaces in which best approximation is everywhere uniquely defined.

A space E is said to be **strictly normed** if the equality $\| x+y \| = \| x \| + \| y \|$ for $x \neq \theta$, $y \neq \theta$ is possible only when $y = \alpha x$, with $\alpha > 0$.

It is now easy to show that the best approximation is uniquely defined in a strictly normed space. In fact, if there exist two linear combinations $\sum_{i=1}^{n} \lambda_i x_i$ and $\sum_{i=1}^{n} \mu_i x_i$ such that

$$\left\| x - \sum_{i=1}^{n} \lambda_i x_i \right\| = \left\| x - \sum_{i=1}^{n} \mu_i x_i \right\| = d,$$

where

$$d = \min \left\| x - \sum_{i=1}^{n} \lambda_i x_i \right\| > 0, \quad \text{then}$$

$$\left\| x - \sum_{i=1}^{n} \frac{\lambda_i + \mu_i}{2} x_i \right\| \leqslant \frac{1}{2} \left\| x - \sum_{i=1}^{n} \lambda_i x_i \right\| + \frac{1}{2} \left\| x - \sum_{i=1}^{n} \mu_i x_i \right\|$$

$$= \frac{1}{2} d + \frac{1}{2} d = d,$$

and since

$$\left\| x - \sum_{i=1}^{n} \frac{\lambda_i + \mu_i}{2} x_i \right\| \geqslant d,$$

then

$$\left\| x - \sum_{i=1}^{n} \frac{\lambda_i + \mu_i}{2} x_i \right\| = d.$$

Consequently,

$$\left\| x - \sum_{i=1}^{n} \frac{\lambda_i + \mu_i}{2} x_i \right\| = \left\| \frac{1}{2} \left(x - \sum_{i=1}^{n} \lambda_i x_i \right) \right\| + \left\| \frac{1}{2} \left(x - \sum_{i=1}^{n} \mu_i x_i \right) \right\|.$$

Thereupon, the space being strictly normed,

$$x - \sum_{i=1}^{n} \mu_i x_i = \alpha \left\{ x - \sum_{i=1}^{n} \lambda_i x_i \right\}.$$

If it were that $\alpha \neq 1$, then x would be a linear combination of the elements $x_1, x_2, \ldots, x_n$, a contradiction. Thus $\alpha = 1$, but then $\sum_{i=1}^{n} (\lambda_i - \mu_i) x_i = 0$, whence, by the linear independence of the elements $x_1, x_2, \ldots, x_n$,

$$\lambda_i = \mu_i, \qquad i = 1, 2, \ldots, n,$$

as was required to prove.

The spaces $L_p[0, 1]$ and l_p for $p > 1$ can be cited as examples of a strictly normed space. The space $C[0, 1]$ is not strictly normed. For verifying this, it is sufficient to consider two non-negative linearly independent functions $x(t)$ and $y(t) \in C[0, 1]$, taking the maximum values at one and the same point of the interval $[0, 1]$. For such functions, it is evident that

$$\| x+y \| = \| x \| + \| y \|,$$

although $y \neq \alpha x$. It is left as an exercise to the reader to verify that $L[0, 1]$ and l also are not strictly normed spaces.

5.5 WEAK COMPACTNESS

THE UPCOMING theorem is highly important and finds extensive applications in functional analysis.

THEOREM 1. *If the space E is separable, then every sphere in the conjugate space E^* is weakly compact, that is, a subsequence, converging weakly to some linear functional f_0, can be extracted from every sequence $\{f_n\}$ of linear functionals with bounded norms.*

PROOF. It is shown in the foregoing that E^* is complete in the sense of weak convergence of operators. Since the notions of weak and strong convergence are equivalent for linear functionals, it implies that the space E^* of linear functionals is complete in the sense of strong convergence. Hence, it will suffice to show that a weakly convergent CAUCHY subsequence can be chosen from every sequence $\{f_n\}$ of linear functionals with bounded norms. Without squeeze of generality, it can be assumed that $\| f_n \| \leqslant 1$. Let $x_1, x_2, \ldots, x_n, \ldots$ be a countable everywhere dense set in E. Since

$$f_n(x_1) \mid \leqslant \| f_n \| \, \| x_1 \| \leqslant \| x_1 \|,$$

$\{f_n(x_1)\}$ is a bounded number sequence. Hence, a convergent subsequence

$$f_{n_1^{(1)}}(x_1),\ f_{n_2^{(1)}}(x_1),\ \ldots,\ f_{n_k^{(1)}}(x_1),\ \ldots$$

can be extracted from this. Consider the sequence of functionals $\left\{ f_{n_k^{(1)}} \right\}$.

Since $| f_{n_k^{(1)}}(x_2) | \leqslant \| x_2 \|$, $\left\{ f_{n_k^{(1)}}(x_2) \right\}$ is a bounded number sequence.

Hence, a convergent subsequence

$$f_{n_1^{(2)}}(x_2),\ f_{n_2^{(2)}}(x_2),\ \ldots,\ f_{n_k^{(2)}}(x_2),\ \ldots$$

can be extracted from this.

Continuing this process, we can construct the sequence $\left\{ f_{n_k^{(3)}} \right\}$ convergent to x_3, so on and so forth. For this it is necessary to note that every succeeding subsequence forms part of the preceding subsequence, and hence converges to every element to which the preceding subsequences converge.

Now, we construct the *diagonal subsequence*

$$f_{n_1^{(1)}},\ f_{n_2^{(2)}},\ \ldots,\ f_{n_k^{(k)}},\ \ldots$$

It is easy to see that this subsequence is convergent for every x_m belonging to the countable everywhere dense set considered. In fact, it is sufficient to note that $f_{n_m^{(m)}}, f_{n_{m+1}^{(m+1)}}, \ldots,$ is a part of the sequence $\left\{ f_{n_k^{(m)}} \right\}$, which is convergent for x_m by construction. Therefore, all sequences $\left\{ f_{n_k^{(k)}} \right\}$ also converge for x_m.

Since the norms of the functionals of the sequence $\left\{ f_{n_k^{(k)}} \right\}$ are totally bounded and this sequence converges to the set $\{x_1, x_2, \ldots, x_n, \ldots\}$, everywhere dense in E, hence by Theorem 2 of Chap. 4.4 the sequence $\left\{ f_{n_k^{(k)}} \right\}$ is weakly convergent. The theorem is proved.

COROLLARY. *Every sphere in the spaces l_p and $L_p\,[0, 1]$ is weakly compact.*

This conclusion is immediate from $l_p = l_q^*$, $L_p\,[0, 1] = L_q^*\,[0, 1]$ and the separability of l_q and $L_q\,[0, 1]$.

REMARK. It is easily verifiable that *every sphere in E^* is weakly closed. Consequently, if it is weakly compact, then it is weakly compact-in-itself.*

5.6. UNIVERSALITY OF THE SPACE $C\,[0,1]$

THE SOVIET mathematician P. S. URYSOHN proved in 1923 that there exist *universal separable metric spaces*, that is, *spaces, an isometric subspace of which can be assigned to every separable metric space.* Later, the Polish mathematician, S. BANACH and S. MAZUR proved $C\,[0, 1]$ to be one of the universal spaces.

The proof of the BANACH-MAZUR theorem rests on the property of the weak compactness of conjugate spaces.

THEOREM. 1. *Every separable Banach space E is isometrically isomorphic to a subspace of $C\,[0, 1]$.*

PROOF. Let S denote the sphere $\| f \| \leqslant 1$ in E^*, the convergence in S being understood as weak convergence of linear functionals. Then, by Theorem 1 of Sec. 5, S is a sequentially compact set. Let $a_1, a_2, \ldots, a_n, \ldots$ be a countable everywhere dense set on the unit sphere $\| x \| \leqslant 1$ of the space E. For any functional $f \in S$, set $f(a_k) = \xi_k$, $|\xi_k| \leqslant 1$, $k = 1, 2, \ldots$ If

$$f_n \xrightarrow{w} f_0 \; (f_n, f_0 \in S), \quad \text{then} \quad \xi_k^{(n)} = f_n(a_k) \to f_0(a_k) = \xi_k^{(0)}.$$

Thus, an element $y = \{\xi_k\}$ $[\xi_k = f(a_k)]$ of the space s is associated with every $f \in S$, and $f_n \xrightarrow{w} f_0$ implies $y_n \to y_0$ for corresponding elements in s.

Let N be a point set in s, corresponding to $f \in S$. Then, N is a continuous image of a sequentially compact set and is, therefore, itself sequentially eompact. It is easy to remark that the inverse mapping of N onto S is also unique and continuous. In fact, let $f(a_k) = \varphi(a_k)$, $k = 1, 2, \ldots$ For any $x \in E$, $\| x \| \leqslant 1$, choose a_{k_0} such that $\| x - a_{k_0} \| < \varepsilon$. Then

$$|f(x) - \varphi(x)| \leqslant |f(x - a_{k_0})| + |f(a_{k_0}) - \varphi(a_{k_0})| + |\varphi(x - a_{k_0})| < 2\varepsilon,$$

whence, ε being arbitrary, $f(x) = \varphi(x)$, that is, $f = \varphi$.

Furthermore, if $f_n \,|\, (a_k) \,| \to f_0 \; (a_k)$, $k = 1, 2, \ldots$, then the boundedness of the norms of the functionals $(\| f_n \|, \| f_0 \| \leqslant 1)$ implies $f_n \xrightarrow{w} f_0$, proving $S \leftrightarrow N$ is one-one and continuous.

By Theorem 6 of Sec. 1, N being a sequentially compact set of a metric space, is a continuous image of the CANTOR perfect set P_0. Thus, the functional $f_t \in S$ corresponds to every $t \in P_0$, the collection of all f_t coincides with S, and thus $f_{t_n} \xrightarrow{w} f_t$ as $t_n \longrightarrow t$.

Select an arbitrary $x \in E$. By definition of the weak convergence of functionals, we get $f_{t_n}(x) \to f_t(x)$ as $t_n \to t$. Hence, $f_t(x)$ is a continuous function of $t \in P_0$ for x fixed. Let this be denoted by $\varphi_x(t)$:

$$\varphi_x(t) = f_t(x). \tag{1}$$

Extend the function $\varphi_x(t)$ defined on P_0 linearly and continuously onto the intervals adjacent to P_0. Then, $\varphi_x(t)$ is a continuous function defined on the interval $[0, 1]$, that is, it belongs to $C[0, 1]$. By the definition of the norm in $C[0, 1]$,

$$\| \varphi_x \|_C = \max_{0 \leqslant t \leqslant 1} |\varphi_x(t)|.$$

However, because of the linearity of $\varphi_x(t)$ on the intervals adjacent to P_0, the maximum of $\varphi_x(t)$ on $[0, 1]$ coincides with the maximum of $\varphi_x(t)$ on P_0. Therefore,

$$\| \varphi_x \|_C = \max_{t \in P_0} |\varphi_x(t)|.$$

On the other hand by (1), $|\varphi_x(t)| = |f_t(x)| \leqslant \| f_t \| \cdot \| x \| \leqslant \| x \|_E$ for $t \in P_0$ and, consequently,

$$\max_{t \in P_0} |\varphi_x(t)| \leqslant \| x \|_E \,. \tag{2}$$

Further, we can construct a functional f_0 with norm 1 for x given, which verifies $f_0(x) = \| x \|_E$. Since $f_0 \in S$, there exists $t_0 \in P_0$, verifying $f_{t_0} = f_0$. Consequently, $f_{t_0}(x) = \| x \|_E$, that is, $\varphi_x(t_0) = \| x \|_E$ and, hence,

$$\max_{t \in P_0} | \varphi_x(t) | \geqslant \| x \|_E. \tag{3}$$

From (2) and (3) it follows that

$$\| \varphi_x \|_C = \max_{t \in P_0} | \varphi_x(t) | = \| x \|_E. \tag{4}$$

From the construction of the function $\varphi_x(t)$ it is seen that if $x_1 \in E$ corresponds to $\varphi_{x_1}(t)$ and $x_2 \in E$ to $\varphi_{x_2}(t)$, then $x_1 + x_2$ and λx correspond to the functions $\varphi_{x_1}(t) + \varphi_{x_2}(t)$ and $\lambda\varphi_x(t)$, respectively. Consequently, we have an isomorphic mapping of E onto a subspace of $C[0, 1]$. However, since the function $\varphi_{x_1}(t) - \varphi_{x_2}(t)$ is associated with the element $x_1 - x_2$ because of this isomorphism, (4) implies that

$$\| x_1 - x_2 \|_E = \| \varphi_{x_1} - \varphi_{x_2} \|_C,$$

that is, the mapping of E onto a subspace of $C[0, 1]$ is not only isomorphic but also isometric. Hence, the theorem is completely proved.

THEOREM 2 (FRÈCHET) . *Every separable metric space X is isometric to a subspace of some separable Banach spece.*

PROOF. Let $M = \{x_0, x_1, x_2, \ldots, x_n, \ldots\}$ be a countable everywhere dense set in X. Associate with every $x \in X$ a point $y = \{\eta_i\}$ of the space m; $\eta_i = \rho(x, x_i) - \rho(x_0, x_i)$, $i = 1, 2, 3, \ldots$ By the triangle inequality, $|\eta_i| = | \rho(x, x_i) - \rho(x_0, x_i) | \leqslant | \rho(x, x_0)$ and, consequently, $\{\eta_i\}$ is a bounded sequence, that is, y is indeed a point of the space m.

Under this mapping, let the elements $y = \{\eta_i\}$ and $y' = \{\eta'_i\}$ of m correspond to the elements $x \in X$ and $x' \in X$. Then,

$$\begin{aligned} \| y - y' \| &= \sup_i | \eta_i - \eta'_i \\ &= \sup_i | [\rho(x, x_i) - \rho(x_0, x_i)] - [\rho(x', x_i) - \rho(x_0, x_i)] | \\ &= \sup_i | \rho(x, x_i) - \rho(x', x_i) | \leqslant \rho(x, x'). \end{aligned} \tag{5}$$

Now let ε be an arbitrary positive number $< \rho(x, x')$. There exists a point x_n of a countable everywhere dense set M, such that $\rho(x, x_n) < \varepsilon/2$. Consequently,

$$\rho(x', x_n) \geqslant \rho(x', x) - \rho(x, x_n) > \rho(x', x) - (\varepsilon/2) > 0,$$

and, hence,

$$\begin{aligned} | \eta_n - \eta'_n | = | \rho(x_n, x) - \rho(x_n, x') &> \rho(x', x_n) - (\varepsilon/2) \\ &> \rho(x, x') - (\varepsilon/2) - (\varepsilon/2) = \rho(x, x') - \varepsilon. \end{aligned}$$

Thereupon,
$$\| y - y' \| > \rho(x, x') - \varepsilon, \tag{6}$$

Since $\varepsilon > 0$ is arbitrary, (6) implies that

$$\| y - y' \| \geqslant \rho(x, x'). \tag{7}$$

Comparing (5) and (7),

$$\| y - y' \| = \rho(x, x'). \tag{8}$$

Thus, the distance between the points x and x' in X is equal to the distance between the corresponding points y and y' in m. Consequently, X is isometric to some $L \subset M$. It is evident that this subspace of m is separable.

Let E be a subspace of m, spanned by the elements of the set L. Then, E is a separable BANACH space. X is isometric to a subspace of this space, and the theorem is proved.

THEOREM 3. (BANACH-MAZUR). *Every separable metric space is isometric to a subspace of* C [0, 1].

The proof of this theorem is immediate from Theorems 1 and 2.

Finally, let us give another property of the universality of C[0, 1] in a somewhat different sense. M. G. KREIN has constructed [20] cones in the BANACH spaces in connection with various questions relating to the problems of moments and the theory of linear integral equations.

In the space E, a closed convex set $K \subset E$ is called a **cone**, if: (i) $x \in K$ ($x \neq 0$) imply that $\lambda x \in K$ for $\lambda \geqslant 0$ and $\lambda x \bar{\in} K$ for $\lambda < 0$; and (ii) $x, y \in K$ imply that $x + y \in K$. This cone K is called **normal** if $\| x + y \| \geqslant \delta$ holds for any two elements $x, y \in K$ with $\| x \| = \| y \| = 1$, where δ is a fixed nonnegative number. For example, the collection of all non-negative functions in C [0, 1] forms a normal cone. We have the underlying conclusion.

THEOREM 4 (M. G. KREIN). *If K is a normal cone in a separable space E, then there is a linear one-one mapping of E into a subspace of* C [0, 1], *such that only the elements of K are transformed into the non-negative functions.*

If E is not separable, then an analogous theorem holds. In that case C [0, 1] is replaced by the space $C(Q)$ of the functions continuous on some bicompact set Q.

CHAPTER 6

COMPLETELY CONTINUOUS OPERATORS

6.1 COMPLETELY CONTINUOUS OPERATORS

6.11. Definition. A linear operator A, with domain on a normed linear space E_x and range in a normed linear space E_y (acting from E_x onto E_y) is called **completely continuous**, if it maps each bounded set (of E_x) onto a compact set (of E_y).

Evidently, every completely continuous operator A is bounded. Further, by Theorem 7 of Chapter 5.1 every bounded linear operator A maps a comdact set into another compact set. Complete continuity is, in general, stronger than simple continuity. Thus, for example, the identity operator I in an infinite-dimensional space E is not completely continuous, because it maps the unit sphere onto itself, that is, onto a set which is not compact. It is immediate from the definition that the sum and product of two completely continuous operators are also completely continuous: for the product to be completely continuous, even this is sufficient that either of the operators in question be completely continuous.

Example. Let $E_x = E_y = C[0, 1]$, and let

$$Ax = y(t) = \int_0^1 K(t, s)\, x(s)\, ds,$$

$K(t, s)$ a kernel continuous on the square $0 \leqslant t, s \leqslant 1$. It is to be shown that the operator A is completely continuous. Let $\{x(t)\}$ be a bounded set of functions on $C[0, 1]$, $\| x \| \leqslant r$. It is obvious that the functions

$$y(t) = \int_0^1 K(t, s)\, x(s)\, ds,$$

$x(t)$ a function of the set under consideration, are uniformly bounded. In fact, if $K = \max_{t, s} | K(t, s) |$, then $| y(t) | \leqslant Kr$. Furthermore, the functions $y(t)$ are uniformly continuous. In fact, given $\varepsilon > 0$. Because of the uniform continuity of the kernel $K(t, s)$, there is a $\delta > 0$ such that

$$| K(t_1, s) - K(t_2, s) | < \varepsilon / r$$

for $| t_1 - t_2 | < \delta$ and every $s \in [0, 1]$. However, then,

$$|y(t_1) - y(t_2)| \leqslant \int_0^1 | K(t_1, s) - K(t_2, s) |\, | x(s) |\, ds < \varepsilon,$$

whenever $|t_1-t_2| < \delta$ for all the functions $y(t)$ considered, implying directly also the uniform continuity of the functions $y(t)$.

By ARZELÀ's theorem, the set of functions $\{y(t)\}$ is compact in the sense of the metric of the space $C[0, 1]$, proving that the operator A is completely continuous.

LEMMA. *If a sequence $\{x_n\}$ is weakly convergent to x_0 and compact, then this is strongly convergent to x_0.*

Assume the contrary. Then there exist a number $\varepsilon_0 > 0$ and an infinitely increasing sequence of indices $n_1, n_2, \ldots, n_k, \ldots$ such that $\| x_{n_k}-x_0 \| \geqslant \varepsilon_0$. Since the sequence $\{x_{n_i}\}$ is compact, it contains a subsequence $\{x_{n_{ij}}\}$, strongly convergent to some element u_0. What is more, $x_{n_{ij}} \xrightarrow{w} u_0$. Since at the same time $x_{n_{ij}} \xrightarrow{w} x_0$, it follows that $u_0 = x_0$.

Thus, on the one hand, $\| x_{n_{ij}} - x_0 \| \geqslant \varepsilon_0$, whereas on the other, $\| x_{n_{ij}} - x_0 \| \to 0$, a contradiction proving the lemma.

THEOREM 1. *A completely continuous operator A maps a weakly convergent sequence into a strongly convergent sequence.*

Let the sequence $\{x_n\}$ converge weakly to x_0. Then the norms of the elements of this sequence are bounded and $\{x_n\}$, as a bounded sequence, is carried by the operator A into a compact sequence $\{y_n\}$, where $y_n = Ax_n$.

On the other hand, by Theorem 7 of Chapter 4.4, $y_n = Ax_n \xrightarrow{w} Ax_0 = y_0$. However, then, by the lemma $y_n \to y_0$ and the theorem is proved.

Let A be a completely continuous operator mapping an infinite-dimensional space E into itself and let B be an arbitrary linear operator acting in the same space. Then, AB and BA are completely continuous operators.

In fact, the operator B carries an arbitrary bounded set $M \subset E$ into a bounded set $B(M)$, and this set is carried by the operator A into a compact set $A[B(M)]$. Consequently, the operator AB carries every bounded set into a compact set and is, therefore, completely continuous.

Analogously, it can be shown that the operator BA is also completely continuous.

Since the unit operator I is not completely continuous, it follows, in particular, that a completely continuous operator A cannot have a bounded inverse operator A^{-1}.

Finally, it is evident that if the operators A and B are completely continuous, then $\alpha A + \beta B$ is also a completely continuous operator.

THEOREM 2. *If a sequence $\{A_n\}$ of completely continuous operators which maps E_x into a complete space E_y, converges strongly to the operator A, that is, if $\| A_n-A \| \to 0$, then A is also a completely continuous operator.*

PROOF. It is required to prove that A maps each bounded set in E_x into a compact set in E_y. Let M be a bounded set in E_x and r a constant, such that $\| x \| \leqslant r$ for every $x \in M$. For a given $\varepsilon > 0$ there is an index n_0 such that $\| A_{n_0} - A \| < \varepsilon/r$. Let $A(M) = K$ and $A_{n_0}(M) = N$. The set N is an ε-net for K. In fact, take for every $y \in K$ one of its pre-images $x \in M$ and put $y_0 = A_{n_0}x \in N$, to receive $\| y - y_0 \| = \| Ax - A_{n_0}x \| \leqslant \| A - A_{n_0} \| \| x \| < \varepsilon/r \cdot r = \varepsilon$. Since, on the other hand, in virtue of A_{n_0} being completely continuous and M being bounded the set N is compact, it follows that K for every $\varepsilon > 0$ has a compact ε-net and is, therefore, itself compact. Thus, the operator A maps an arbitrary bounded set into a compact set and is, consequently, completely continuous, proving the theorem.

Example. If $E_x = E_y = L_2\,[0,1]$, then the operator

$$Ax = y(t) = \int_0^1 K(t, s)\, x(s)\, ds, \quad \text{with} \int_0^1 \int_0^1 K^2(t, s)\, dt\, ds < +\infty,$$

is completely continuous.

Assume first that $K(t, s)$ is a continuous kernel. Let M be a bounded set of $L_2\,[0, 1]$ and let

$$\int_0^1 x^2(t)\, dt \leqslant r^2$$

for all $x(t) \in M$. Consider the set of functions

$$y(t) = \int_0^1 K(t, s)\, x(s)\, ds, \quad x(t) \in M.$$

It is to be shown that the functions $y(t)$ are uniformly bounded and continuous. This implies the compactness of the set $\{y(t)\}$ in the sense of uniform convergence and also in the sense of convergence in the mean square.

We have

$$| y(t) | = \left| \int_0^1 K(t, s)\, x(s)\, ds \right|$$

$$\leqslant \left(\int_0^1 K^2(t, s)\, ds \right)^{1/2} \left(\int_0^1 x^2(s)\, ds \right)^{1/2} \leqslant Kr,$$

where $K = \max\limits_{t,\, s} | K(t, s) |$. Consequently, the functions $y(t)$ are uniformly bounded. Furthermore,

$$| y(t_1) - y(t_2) | \leqslant \left(\int_0^1 [K(t_1, s) - K(t_2, s)]^2\, ds \right)^{1/2} \left(\int_0^1 x^2(s)\, ds \right)^{1/2} < \varepsilon$$

for $| t_1 - t_2 | < \delta$, where δ is chosen such that

$$| K(t_1, s) - K(t_2, s) | < \varepsilon/r$$

for $|t_1 - t_2| < \delta$. The estimate $| y(t_1) - y(t_2) | < \varepsilon$ depends neither on the position of t_1, t_2 on $[0, 1]$ nor on the choice of $y(t) \in M$; consequently, the functions $y(t)$ are uniformly continuous.

Thus, in the case of a continuous kernel, the operator A is completely continuous, Now, assume $K(t, s)$ to be an arbitrary square-integrable kernel. Select a sequence of continuous kernels $\{K_n(t, s)\}$ which converges in the mean to $K(t, s)$, that is a sequence such that

$$\int_0^1 \int_0^1 \{K(t, s) - K_n(t, s)\}^2 \, dt \, ds \to 0, \quad \text{as } n \to \infty.$$

Put
$$A_n x = \int_0^1 K_n(t, s)\, x(s)\, ds.$$

Then,
$$\| Ax - A_n x \| = \left\{ \int_0^1 \left[\int_0^1 K(t, s)\, x(s)\, ds - \int_0^1 K_n(t, s)\, x(s)\, ds \right]^2 dt \right\}^{1/2}$$

$$= \left\{ \int_0^1 \left[\int_0^1 (K(t, s) - K_n(t, s))\, x(s)\, ds \right]^2 dt \right\}^{1/2}$$

$$\leqslant \left\{ \int_0^1 \left[\int_0^1 [K(t, s) - K_n(t, s)]^2\, ds \int_0^1 x^2(s)\, ds \right] dt \right\}^{1/2}$$

$$= \left\{ \int_0^1 \int_0^1 [K(t, s) - K_n(t, s)]^2\, dt\, ds \right\}^{1/2} \| x \|.$$

Hence,
$$\| A - A_n \| \leqslant \left\{ \int_0^1 \int_0^1 [K(t, s) - K_n(t, s)]^2\, dt\, ds \right\}^{1/2},$$

implying that $\| A - A_n \| \to 0$ as $n \to \infty$. Since all the A_n are cempletely continuous, A is also completely continuous by Theorem 2.

REMARK. *The limit of a weakly convergent sequence $\{A_n\}$ of completely continuous operators also cannot be completely continuous.* In fact, in an infinite-dimensional BANACH space E with basis $\{e_i\}$ consider the operators S_n, defined by the equality

$$S_n x = \sum_{i=1}^{n} \xi_i e_i \quad \text{for} \quad x = \sum_{i=1}^{\infty} \xi_i e_i .$$

The operator S_n maps E into a finite-dimensional space E_n and is, therefore, completely continuous. As $n \to \infty$ the sequence of operators S_n weakly converges to the identity operator I, which is not completely continuous.

THEOREM 3. *The range of a completely continuous operator A is separable.*

PROOF. In fact, let K_n be an image of the sphere $\| x \| \leqslant n$. Since A is completely continuous, K_n is a compact and therefore, also a separable set (see p. 155). Let T_n be a countable, everywhere dense set in K_n. Since

$K = \bigcup_{n=1}^{\infty} K_n$ is the range of A, $T = \bigcup_{n=1}^{\infty} T_n$ is a countable, everywhere dense set in K. ■

THEOREM 4. *If A is a completely continuous operator which maps E_x into E_y, then the adjoint operator A^*, mapping E_y^* into E_x^*, is completely continuous.*

PROOF. It will suffice to prove that the image $A^*(S_y^*)$ of the unit sphere S_y^* in the space E_y^* is compact.

Consider an image $A(\overline{S}_x)$ of a closed unit sphere in E_x. Since A is completely continuous, $A(\overline{S}_x)$ is a compact set. Next, consider on this set linear functionals belonging to S_y^*. If $f \in S_y^*$, $y \in A(\overline{S}_x)$, then

$$|f(y)| \leqslant \|f\| \|y\| = \|f\| \; \|Ax\| \leqslant \|f\| \; \|A\| \; \|x\| \leqslant \|A\|,$$

since $\|f\| \leqslant 1$, $\|x\| \leqslant 1$. Consequently, the functionals in S_y^* on the set $A(\overline{S}_x)$ are uniformly bounded. Further, for $y_1, y_2 \in A(\overline{S}_x)$, $f \in S_y^*$,

$$|f(y_1) - f(y_2)| = |f(y_1 - y_2)| \leqslant \|f\| \|y_1 - y_2\| \leqslant \|y_1 - y_2\|,$$

and, consequently, the functionals in S_y^* on $A(\overline{S}_x)$ are uniformly continuous. By the generalization of ARZELÁ's theorem (Chapter 5, p. 16) the set S_y^* is compact in the sense of uniform convergence to $A(\overline{S}_x)$.

Now, consider an arbitrary sequence $\{A^* f_n\} \subset A^*(S_y^*)$. Since the set S_y^* is compact, it is possible to extact from the sequence $\{f_n\}$ a subsequence $\{f_{n_i}\}$, uniformly convergent to $A(\overline{S}_x)$:

$$\sup_{x \in S_x} |f_{n_i}(Ax) - f_{n_j}(Ax)| \to 0, \quad \text{as } n_i, n_j \to \infty.$$

However,
$$\sup_{x \in \overline{S}_x} |f_{n_i}(Ax) - f_{n_j}(Ax)| = \sup_{x \in \overline{S}_x} |A^*(f_{n_i} - f_{n_j})x| = \|A^* f_{n_i} - A^* f_{n_j}\|.$$

Hence, the sequence $\{A^* f_{n_i}\}$ in the space E_x^* is convergent with respect to norm, proving that $A^*(S_y^*)$ is compact.

6.12. Approximation of completely continuous operators in a Banach space with a basis of finite-dimensional operators. Consider a completely continuous operator A, mapping a BANACH space E with basis into itself. Let S be a unit sphere of this space and K a collection of elements of the form $y = Ax$, where $x \in S$. Since A is completely continuous, K is a compact set. Then, by Theorem 3 of Chap. 5.2, there is for every number $\varepsilon > 0$ an index $n = n(\varepsilon)$ such that $\|R_n y\| < \varepsilon$ for all $y \in K$. Holding this n fixed, we get

$$Ax = y = S_n y + R_n y = S_n(Ax) + R_n(Ax) = A_1 x + A_2 x,$$

where A_1 and A_2 are, evidently, linear operators. Moreover, putting

$$y = \sum_{i=1}^{\infty} \eta_i e_i, \text{ we receive } A_1 x = S_n y = \sum_{i=1}^{n} \eta_i e_i,$$

whence it is seen that A_1 is a finite-dimensional operator, in the sense that the element A_1x belongs for every x to a finite-dimensional subspace defined by the basis elements $e_1, e_2, \ldots, e_n$. Furthermore,

$$\sup_{x \in S} \| A_2 x \| = \sup_{y \in K} \| R_n y \| < \varepsilon, \quad \text{implying } \| A_2 \| < \varepsilon.$$

Thus, a completely continuous operator A is decomposed into a sum of two operators, of which one is finite-dimensional and the norm of the other does not exceed a preassigned number which can be chosen arbitrarily small. Hence, it is sometimes said that a completely continuous operator in a space with basis is **almost finite-dimensional.**

6.2 LINEAR OPERATOR EQUATIONS WITH COMPLETELY CONTINUOUS OPERATORS

In this section, linear operator equations with completely continuous operators will be considered. It has been shown by F. Riesz that such equations admit the applications of basic consequences from the Fredholm theory of linear integral equations.

6.21. Two Lemmas. Let A be a completely continuous operator which maps a Banach space E into itself. Consider the equation

$$Ax - x = y, \tag{1}$$

or,

$$Tx = y, \tag{1'}$$

where $T = A - I$. Together with Eq. (1), consider

$$A^*f - f = g \tag{2}$$

or,

$$T^*f = g, \tag{2'}$$

where A^* is the adjoint operator of A and acts into the space E^*. As is already exhibited, A^* is also a completely continuous operator.

Lemma 1. *Let N be a subspace of null operator T, that is, a collection of elements x such that $Tx = 0$. Then, N is a finite-dimensional subspace of E.*

Let M be an arbitrary bounded set in N. For every $x \in N$, $Ax = x$, that is, the operator A leaves the elements of the subspace N invariant and, in particular, carries the set M into itself. The subspace N of E is then said to be **invariant** with respect to A.

On the other hand, A as a completely continuous operator carries M into a compact set. Consequently, every bounded set $M \subset N$ is compact, implying by Theorem 4 of Chapter 5.2, that N is a finite-dimensional subspace.

Remark. The elements of the subspace N are eigenvectors of the operator A, corresponding to the eigenvalue $\lambda_0 = 1$. The statement of the theorem and its proof are not affected under substitution of 1 by any other eigenvalue λ different from zero. Thus, it is shown : *The completely continuous operator A can have only a finite number of linearly independent eigenvectors, which correspond to one and the same eigenvalue.*

LEMMA 2. *Let $L = T(E)$, that is, let L be a collection of elements $y \in E$, representable in the form $y = Ax - x$. Then, L is a subspace.*

That L is a linear manifold, is obvious. It is necessary only to prove that L is closed.

Show first that there is a constant α depending only on A, such that whenever the equation

$$Tx = y \tag{1'}$$

is solvable, at least one of its solutions satisfies the inequality

$$\| x \| \leqslant \alpha \| y \|. \tag{3}$$

Let x_0 be one of the solutions of Eq. (1'). Then, every other solution of (1') takes the form: $x = x_0 + z$, where z is a solution of the homogeneous equation

$$T(x) = 0. \tag{1*}$$

Consider $\varphi(z) = \| x_0+z \|$, a bounded below, continuous functional. Let $\alpha = \inf \varphi(z)$ and let $\{z_n\} \subset N$ be the **minimising sequence**, that is,

$$\varphi(z_n) = \| x_0+z_n \| \to d. \tag{4}$$

The sequence $\{ \| x_0+z_n \| \}$ because of having a limit, is bounded. However, the sequence $\{ \| z_n \| \}$ is also then bounded, since

$$\| z_n \| = \| (z_n+x_0) - x_0 \| \leqslant \| z_n + x_0 \| + \| x_0 \|.$$

Thus, $\{z_n\}$ is a bounded sequence in a finite-dimensional space and, consequently, a convergent subsequence can be extracted from it. Disregarding, if necessary, superfluous terms in $\{z_n\}$, it can be assumed, without squeeze of generality, that $z_n \to z_0$. Then,

$$\varphi(z_n) \to \varphi(z_0). \tag{5}$$

From (4) and (5) it follows that $\varphi(z_0) = \| x_0 + z_0 \| = d$. Consequently, because of its solvability, Eq. (1') has always the solution $\tilde{x} = x_0 + z_0$ with the minimal norm.

In order to show that Ineq. (3) holds for $\tilde{x}$, consider the ratio $\| \tilde{x} \| / \| y \|$ and assume that this is not bounded. Then, there exist sequences y_n and $\tilde{x}_n$ such that

$$\| \tilde{x}_n \| / \| y_n \| \to \infty.$$

Since λy_n, evidently, corresponds to the minimal solution $\lambda \tilde{x}_n$, we can assume without loss of generality, that $\| \tilde{x}_n \| = 1$; then $\| y_n \| \to 0$. Since the sequence $\{\tilde{x}_n\}$ is bounded and A is a completely continuous operator, the sequence $\{A\tilde{x}_n\}$ is compact and, consequently, contains a convergent subsequence. Again without restricting generality, it can be assumed that

$$A\tilde{x}_n \to \tilde{x}_0. \tag{6}$$

However, thereupon, since $\tilde{x}_n = A\tilde{x}_n - y_n$, hence, $\tilde{x}_n \to \tilde{x}_0$ and, consequently,

$$A\tilde{x}_n \to A\tilde{x}_0. \tag{7}$$

From (6) and (7) it follows that $A\tilde{x}_0 = \tilde{x}_0$, that is, $\tilde{x}_0 \in N$. However, then, because of the minimality of the norm of the solution $\tilde{x}_n$, it says $\| \tilde{x}_n - \tilde{x}_0 \| \geqslant \| \tilde{x}_n \| = 1$, contradicting the convergence of $\{\tilde{x}_n\}$ to $\tilde{x}_0$. Thus, $\| \tilde{x} \| / \| y \|$ is bounded, and if $\alpha = \sup (\| \tilde{x} \| / \| y \|)$, then the inequality in hand is proved.

Now, given a sequence $\{y_n\} \subset L$, convergent to y_0. Passing on, if necessary, to a subsequence it can be assumed that $\| y_n - y_0 \| < 1/2^{n+1}$, whence $\| y_{n+1} - y_n \| < 1/2^n$. Let x_0 be a minimal solution of the equation $T_x = y_1$ and x_n, $n = 1, 2, \ldots$, a minimal solution of the equation $Tx = y_{n+1} - y_n$. Then, $\| x_n \| \leqslant \alpha \| y_{n+1} - y_n \| < \alpha/2^n$. This estimate implies that the series $\sum_{n=1}^{\infty} x_n$ converges, and if $\tilde{x}$ is the sum of this series, then

$$T\tilde{x} = T\left(\lim_n \sum_{k=0}^{n} x_k \right) = \lim_n \sum_{k=0}^{n} Tx_k$$

$$= \lim_n \left[y_1 + \sum_{k=1}^{n} (y_{k+1} - y_k) \right] = \lim_n y_{n+1} = y_0,$$

exhibiting that $y_0 \in L$. ■

THEOREM 1. *In order that for $y \in E$ given Eq.* (1′) *be solvable, it is necessary and sufficient that $f(y) = 0$ for every linear functional f, such that*

$$A^*f - f = 0. \tag{2*}$$

Necessity. Suppose that the equation $Ax - x = y$ is solvable, that is, y is expressible in the form $y = Ax_0 - x_0$ for some $x_0 \in E$. Select an arbitrary linear functional f, such that $A^*f - f = 0$. Then,

$$f(y) = f(Ax_0 - x_0) = f(Ax_0) - f(x_0) = A^*f(x_0) - f(x_0)$$
$$= (A^*f - f)\, x_0 = 0,$$

proving the necessity.

Sufficiency. It is required to show that $y \in L = T(E)$ satisfies hypotheses of the theorem. Assume the contrary, that is, $y \bar{\in} L$. Since L is closed, y lies at a distance $d > 0$ from L and by corollaries to the BANACH-HAHN theorem there is a linear functional f_0, such that $f_0(y) = 1$ and $f_0(z) = 0$ for every $z \in L$. The latter equality implies $f_0(Ax - x) = (A^* f_0 - f_0)\, x = 0$ for all $x \in E$, that is, $A^*f_0 - f_0 = 0$, a contradiction, because on the one hand by construction $f_0(y) = 1$, whereas on the other hand, by hypothesis $f_0(y) = 0$. Consequently, $y \in L$, proving the sufficiency. ■

REMARK. An equation $Tx = y$ with the property that it has a solution if $f(y) = 0$ for every f, satisfying the equality $T^*f = 0$, is said to be **normally solvable.** In the preceding theorem, it is proved in essence that: *For the equation $Tx = y$ to be normally solvable, it is sufficient that $L = T(E)$ be closed.* It can be proved that this condition is also necessary (see [11]).

COROLLARY. *If a conjugate homogeneous equation $A^*f - f = 0$ has only a trivial solution $f = 0$, then the equation*

$$Ax - x = y$$

has a solution for any right side.

THEOREM 2. *In order that Eq.* (2) *be solvable for $g \in E^*$ given, it is necessary and sufficient that $g(x) = 0$ for every element $x \in E$, such that*

$$Ax - x = 0. \tag{1**}$$

The necessity, evidently, follows from the equality $g(x) = (A^*f - f)\,x = f(Ax-x) = 0$. Therefore, it remains to prove the sufficiency.

Define the functional $f_0(y)$ on the subspace L by means of the equality $f_0(y) = g(x)$, x being one of the pre-images of the element y under the mapping T (that is, $Ax - x = y$). The functional f_0 satisfying hypotheses of the theorem is uniquely defined, since if u be another pre-image of the same element y, $Ax - x = Au - u$, then $A(x - u) - (x-u) = 0$, whence $g(x-u) = 0$, that is, $g(x) = g(u)$.

The additivity and homogeneity of the functional f are easily verifiable and its boundedness is proved in what follows. As established in the proof of Lemma 2, the inequality $\| x \| \leqslant \alpha \| y \|$ is satisfied for at least one of the pre-images x of the element y. However, then,

$$| f_0(y) | = | g(x) | \leqslant \| g \| \| x \| \leqslant \| g \| \alpha \| y \|,$$

and the boundedness of f_0 is proved. Extend f_0 by the BANACH-HAHN theorem on the entire space E, to receive a linear functional f, such that

$$f(Ax-x) = f(y) = f_0(y) = g(x),$$

or,

$$(A^* f-f)\, x = g(x),$$

that is, a solution of Eq. (2) ■

COROLLARY. *If the equation $Ax - x = 0$ has only a null solution $x = 0$, then the equation $A^* f - f = g$ is solvable only in the right side g.*

We have so far investigated the relationship between the given and conjugate equations. It is now intended to show that homogeneous and inhomogeneous equations having solutions in the identical space are also closely related.

THEOREM 3. *In order that the equation*

$$Ax - x = y \tag{1}$$

be solvable for every y, where A is a completely continuous operator mapping a Banach space E into itself, it is necessary and sufficient that the corresponding homogeneous equation

$$Ax - x = 0 \tag{1**}$$

has only a trivial solution $x = 0$. In this case, the solution of Eq. (1) *is uniquely defined, and the operator $T = A - I$ has a bounded inverse.*

Necessity. Denote by N_k the subspace of the null operator T^k. It is plain that $T^k x = 0$ implies $T^{k+1} x = 0$, that is, $N_k \subset N_{k+1}$.

Let the equation $Ax - x = y$ be solvable for every y and assume that the homogeneous equation $Ax - x = 0$ has a non-trivial solution x_1. Let x_2 be a solution of the equation $Ax - x = x_1$ and, in general, let x_{k+1} be a solution of the equation

$$Ax - x = x_k, \qquad k = 1, 2, 3, \ldots$$

We have

$$Tx_k = x_{k-1}, \quad T^2 x_k = x_{k-2}, \ldots, \quad T^{k-1} x_k = x_1 \neq 0,$$

whenever $T^k x_k = Tx_1 = 0$. Hence, $x_k \in N_k$, and $x_k \bar{\in} N_{k-1}$, that is, each subspace N_{k-1} is a proper part of the succeeding subspace N_k. Then, by Lemma 2 of Chap. 2.4 there is in the subspace N_k an element y_k with norm 1, such that $\| y_k - x \| \geqslant \frac{1}{2}$ for every $x \in N_{k-1}$. Consider the sequence $\{Ay_k\}$, which is compact since $\| y_k \| = 1$ and A is a completely continuous operator. On the other hand, let y_p and y_q be two such elements and $p > q$. Since

$$T^{p-1}(y_q + Ty_p - Ty_q) = T^{p-1} y_q + T^p y_p - T^p y_q = 0,$$

then $y_q + Ty_p - Ty_q \in N_{p-1}$, and hence

$$\| Ay_p - Ay_q \| = \| y_p - (y_q + Ty_p - Ty_q) \| \geqslant \tfrac{1}{2},$$

a contradiction arising from the assumption that Eq. (1) has in the presence of (1*) a nontrivial solution. The necessity is proved.

Sufficiency. Suppose that Eq. (1*) has only a trivial solution. By Corrollary to Theorem 2, the equation

$$A^* f - f = g \tag{2}$$

is then solvable for any right side. Since A^* is also a completely continuous operator and E^* a BANACH space, the necessity part of the theorem just proved becomes applicable to Eq. (2) and the equation

$$A^* f - f = 0 \tag{2*}$$

has only a trivial solution. However, then, Eq. (1) by Corollary to Theorem 1 has a solution for every y, and the sufficiency is proved.

Since by hypotheses of the theorem, Eq. (1) has a unique solution, there is an operator

$$T^{-1} = (A - I)^{-1},$$

inverse to the operator $A - I$. Because of uniqueness property, the unique solution is at the same time minimal, and hence

$$\| (A-I)^{-1} y \| \leqslant \alpha \| y \|. \qquad \blacksquare$$

THEOREM 4. *The equations*

$$Ax - x = 0, \tag{1*}$$

and

$$A^* f - f = 0, \tag{2*}$$

with completely continuous operators A and A^*, mapping a Banach space E (corr. E^*) into itself, have the same number of linearly independent solutions.

PROOF. Let $x_1, x_2, \ldots, x_n$ be a basis of the subspace N of solutions of Eq. (1*) and $f_1, f_2, \ldots, f_m$ a basis of the subspace of solutions of Eq. (2*).

Construct a system of functionals $\varphi_1, \varphi_2, \ldots, \varphi_n$ biorthogonal to $x_1, x_2, \ldots, x_n$, that is, such that

$$\varphi_i(x_j) = \delta_{ij}, \qquad i, j = 1, 2, \ldots, n,$$

and a system of elements $z_1, z_2, \ldots, z_m$, biorthogonal to $f_1, f_2, \ldots, f_m$. Assuming $n < m$, consider the operator

$$Ux = Ax + \sum_{i=1}^{n} \varphi_i(x)\, z_i.$$

Being a sum of completely continuous and finite-dimensional operators, this operator is completely continuous. In order to show that the equation $Ux - x = 0$ has only a trivial solution, assume x_0 to be a solution of this equation. Then,

$$f_k(Ux_0 - x_0) = 0, \qquad k = 1, 2, \ldots, m,$$

or,

$$f_k\left(Ax_0 - x_0 + \sum_{i=1}^{n} \varphi_i(x_0)\, z_i\right) = 0,$$

whence

$$(A^* f_k - f_k)\, x_0 + \sum_{i=1}^{n} \varphi_i(x_0)\, f_k(z_i) = 0;$$

and, consequently, taking account of the biorthogonality of $\{f_i\}$ and $\{z_i\}$,

$$\varphi_k(x_0) = 0, \qquad k = 1, 2, \ldots, n \quad (n < m).$$

Thus, $Ux_0 = Ax_0$ and, consequently, x_0 satisfies $Ax_0 - x_0 = 0$. Since $x_0 \in N$ and $\{x_i\}$ is a basis in N,

$$x_0 = \sum_{i=1}^{n} \xi_i\, x_i.$$

However, $\xi_i = \varphi_i(x_0) = 0$. Hence $x_0 = 0$, as we sought to exhibit.

Since the equation $Ux - x = 0$ has only a trivial solution, the equation $Ux - x = y$ is solvable for every right side and, in particular, for $y = z_{n+1}$. Let x' be a solution of this equation. Then, as above, on the one hand,

$$f_{n+1}(z_{n+1}) = f_{n+1}\left(Ax' - x' + \sum_{i=1}^{n} \varphi_i(x')\, z_i\right)$$

$$= (A^* f_{n+1} - f_{n+1})\, x' + \sum_{i=1}^{n} \varphi_i(x')\, f_{n+1}(z_i) = 0,$$

whereas, on the other hand, by construction $f_{n+1}(z_{n+1}) = 1$. The contradiction obtained proves the inequality $n < m$ to be impossible.

Assume, conversely, that is, $m < n$. Consider in the space E^*, the operator

$$U^*f = A^*f + \sum_{i=1}^{m} f(z_i)\,\varphi_i.$$

It is trivial to verify (taking note of $m < n$), that this operator is adjoint to the operator U, replacing n by m.

It is to be shown that the equation $U^*f - f = 0$ has only a trivial solution even for U^*. For all $k = 1, 2, \ldots, n$,

$$(U^*f - f)\,x_k = (A^*f - f)\,x_k + \sum_{i=1}^{m} f(z_i)\,\varphi_i(x_k)$$

$$f = (Ax_k - x_k) + f(z_k) = f(z_k). \tag{8}$$

Thus, if f_0 is a solution of the equation $U^*f - f = 0$, then from (8) it follows that $f_0(z_k) = 0$, $k = 1, 2, \ldots, m$. Consequently, $U^*f_0 = A^*f_0$ and f_0 is a solution of the equation $A^*f - f = 0$. However, then,

$$f_0 = \sum_{i=1}^{m} \alpha_i f_i = \sum_{i=1}^{m} f_0(z_i)\,f_i = 0.$$

Since U^* is a completely continuous operator, by Theorem 3 the equation $U^*f - f = g$ has a solution for any g, particularly for, $g = \varphi_{m+1}$. Thereupon, on the one hand, if f' is a solution of this equation, then

$$\begin{aligned}\varphi_{m+1}(x_{m+1}) &= (U^*f' - f')\,x_{m+1}\\ &= (A^*f' - f')\,x_{m+1} + \sum_{i=1}^{m} f'(z_i)\,\varphi_i(x_{m+1})\\ &= f'(Ax_{m+1} - x_{m+1}) = 0,\end{aligned}$$

whereas on the other, $\varphi_{m+1}(x_{m+1}) = 0$.

The contradiction obtained proves the inequality $m < n$ to be impossible. Thus, $m = n$. ■

The conelusions of Theorems 1—4 can be combined in a single proposition that follows, representing an extension of the well-known FREDHOLM theorem on linear integral equations to any equation with completely continuous operators,

Given the equations

$$Ax - x = y, \tag{1}$$

and

$$A^*f - f = g, \tag{2}$$

A a completely continuous linear operator acting in a Banach space E and A^* an adjoint operator acting in the conjugate space E^*. Then, either Eqs. (1)

and (2) *have a solution for any right side and in this case the homogeneous equations*

$$Ax - x = 0 \tag{1*}$$

$$A^* f - f = 0 \tag{2*}$$

have only a trivial solution, or the homogeneous equations have the same finite number of linearly independent solutions $x_1, x_2, \ldots, x_n; f_1, f_2, \ldots, f_n$; *and in that case for Eq.* (1) [*corr. Eq.* (2)] *to have a solution, it is necessary and sufficient that*

$$f_i(y) = 0, \quad (corr.\ g(x_i) = 0), \qquad i = 1, 2, \ldots, n;$$

The general solution of Eq. (1), then, takes the form

$$x = x_0 + \sum_{i=1}^{n} \alpha_i x_i,$$

x_0 any solution of Eq. (1), and $\alpha_1, \alpha_2, \ldots, \alpha_n$ arbitrary constants. Correspondingly, the general solution of Eq. (2) has the form

$$f = f_0 + \sum_{i=1}^{n} \lambda_i f_i,$$

f_0 any solution of Eq. (2) and $\lambda_1, \lambda_2, \ldots, \lambda_n$ arbitrary constants.

The result realized by a deeper study of Eq. (1) is stated next. Denote by L_k the range of the operator

$$T^k = (A-I)^k = A^k - C_k^1 A^{k-1} + \ldots + (-1)^k I = \pm (A_k - I),$$

where A_k is again a completely continuous operator.

By Lemma 2 every L_k is a subspace. If $y \in L_k$, then, $y = T^k x = T^{k-1}(Tx) = T^{k-1} z$; that is, $y \in L_{k-1}$ and, consequently, L_k forms a decreasing sequence.

Recalling that N_k denotes a subspace of null operator T^k, we arrive at:

THEOREM 5. *Among the subspaces,* L_k *represents only a finite number and precisely so does* N_k.

As a preliminary, show that if $L_m = L_{m+1}$, then $L_m = L_k$ for all $k > m$. Select an arbitrary element $y \in L_{m+1}$, to receive $y = T^{m+1} x = T(T^m x)$. However, since $L_m = L_{m+1}$, there is an element x', such that $T^m x = T^{m+1} x'$. Hence, $y = T(T^m x) = T(T^{m+1} x') = T^{m+2} x'$, that is, $y \in L_{m+2}$. Thereupon, $L_{m+2} = L_{m+1}$; analogously, $L_{m+3} = L_{m+2}, \ldots$ Irrespective of the assertions of the proposition, now assume that $L_n \neq L_{n+1}$ for every n. Since $L_{n+1} \subset L_n$, by the lemma of Chap. 2.2 there is in L_n an element x_n with norm 1, such that $\| x_n - y \| \geqslant \frac{1}{2}$ for all $y \in L_{n+1}$.

Consider the sequence $\{x_n\}$. This sequence belongs to the unit sphere of the space E and, hence, the sequence $\{Ax_n\}$ must be compact. However, on the other hand,

$$\begin{aligned} Ax_n - Ax_{n+p} &= x_n - Tx_n - x_{n+p} + Tx_{n+p} \\ &= x_n - (Tx_n + x_{n+p} - Tx_{n+p}). \end{aligned}$$

The element $y = Tx_n + x_{n+p} - Tx_{n+p} \in L_{n+1}$, since $Tx_n = T(T^n x) = T^{n+1}x \in L_{n+1}$, $x_{n+p} \in L_{n+p} \subset L_{n+1}$ and $Tx_{n+p} \in L_{n+p+1} \subset L_{n+1}$. However, then, by the construction of the sequence $\{x_n\}$,

$$\| Ax_n - Ax_{n+p} \| = \| x_n - y \| \geqslant \tfrac{1}{2},$$

and the sequence $\{Ax_n\}$ is non-compact. The contradiction obtained proves the first part of the theorem. The second part can be proved analogously which incidentally has been done in the proof of Theorem 3.

THEOREM 6. *For every completely continuous operator A, there exists a decomposition of the space E into a direct sum of the subspaces U and V*

$$E = U \oplus V; \tag{9}$$

furthermore: (i) *the subspace V is finite-dimensional*; (ii) *the operator $A - I$ maps U onto itself and V into itself one-one*; (iii) *A admits representation in the form of a sum of two operators A_u and A_v*:

$$A = A_u + A_v, \tag{10}$$

A_u and A_v being completely continuous linear operators, mapping E, respectively into U and V; $A_u - I$ is an inverse operator and

$$A_u A_v = A_v A_u = 0.$$

Let ν be the smallest of n natural numbers, such that $L_n = L_{n+1}$. Let $U = L_\nu$, $V = N_\nu$. As shown earlier U and V are subspaces. Since $T^\nu = \pm (A_\nu - I)$, A_ν being completely continuous, hence by Lemma 1 V, the subspace N_ν of the null operator $A_\nu - I$, is finite-dimensional.

Let $x \in U$ and $y = Tx$. Since $x \in L_\nu$, there is an element $x' \in E$ such that $x = T^\nu x'$. Then, $y = Tx = T^{\nu+1} x' \in L_{\nu+1} = L_\nu = U$ and, consequently, forms an element of U belonging to the same subspace.

Now let y be an arbitrary element in U: $y \in U = L_\nu = L_{\nu+1}$. There is an element $x' \in E$, such that $y = T^{\nu+1} x' = T(T^\nu x') = Tx$, where $x = T^\nu x' \in L_\nu = U$. Consequently, every element $y \in U$ is an image of some element $x \in U$ and the operator T maps U onto U.

Thereupon, Theorem 3 implies that the mapping $Tx = y$, $x \in U$, is one-one and there is a bounded inverse operator of T acting only on the elements of the subspace U.

Let $x \in V = N$. This signifies that $T^\nu x = 0$, or $T^{\nu-1}(Tx) = 0$. However, then, $Tx \in N_{\nu-1} \subset N_\nu$ and, consequently, T maps V into itself.

Now it is easy to prove Eq. (9). Let T_u be treated as the operator T acting in U. As indicated above, this operator has an inverse. For an arbitrary $x \in E$, put $u = T_u^{-\nu} T^\nu x$ and $v = x - u$. Evidently, $u \in U$. Furthermore, since

$$T^\nu v = T^\nu x - T^\nu u = 0, \quad \text{hence } v \in V.$$

Now, if $x = \tilde{u} + \tilde{v}$ be another representation of $x \in E$, where $\tilde{u} \in U$ and $\tilde{v} \in V$, then

$$T^\nu x = T^\nu \tilde{u} + T^\nu \tilde{v} = T^\nu \tilde{u}, \tag{11}$$

so that $T^{\nu}\tilde{v} = 0$. Since $\tilde{u} \in U$, (11) implies that

$$\tilde{u} = T_u^{-\nu} T^{\nu}\tilde{u} = T_u^{-\nu} T^{\nu}x,$$

and the uniqueness of the representation is proved.

Note that since T and T_u^{-1} are linear operators,

$$\| u \| = \| T_u^{-\nu}T^{\nu}x \| \leqslant c_1 \| x \|,$$

and, consequently, also

$$\| v \| \leqslant c_2 \| x \|.$$

Introduce the operators A_u and A_v, putting $A_u x = Au$ and $A_v x = Av$ for an arbitrary $x \in E$. In particular, $A_u v = A_v u = 0$. Since $A_u x = Tu + u$ and $A_v x = Tv + v$, the relations $T(U) = U$ and $T(V) \subset V$ proved earlier imply that $Au \in U$ and $Av \in V$. Hence, A_u and A_v map E, respectively, into U and V. It is plain that

(*a*) A_u and A_v are bounded linear operators;
(*b*) $A = A_u + A_v$;
(*c*) $A_u(A_v x) = A_v(A_u x) = 0$;
(*d*) A_v is completely continuous, since it maps E into a finite-dimensional subspace V; however, then, so is A_u.

Consider, finally, the equation

$$A_u x - x = y = u + v. \tag{12}$$

Suppose that x_0' is a solution of the equation $Ax - x = u$, which exists because of the invertibility of $T = A - I$ on U. Consider the element $x_0 = x_0' - v$, to receive

$$A_u x_0 - x_0 = A_u(x_0' - v) - x_0' + v = A_u x_0' - x_0' + v = u + v = y$$

and, consequently, Eq. (12) has always a solution. However, then, by Theorem 3 the operator $A_u - I$ has a bounded inverse. ∎

Before concluding this section, consider an equation containing a parameter. Since the equation

$$Ax - \lambda x = y, \qquad \lambda \neq 0 \tag{1_λ}$$

can be expressed in the form $(1/\lambda)\, Ax - x = (1/\lambda)\, y$ and $(1/\lambda)\, A$ is completely continuous together with A, the theorem proved for Eq. (1) remains valid for Eq. (1_λ).

Theorem 3 implies that for a given $\lambda \neq 0$, either the equation $Ax - \lambda x = y$ is solvable for any right side, or the homogeneous equation $Ax - \lambda x = 0$ has a nontrivial solution. Hence, every value of the parameter $\lambda \neq 0$ is either regular or is an eigenvalue and the operator A has no other non-zero point spectrum except the eigenvalues†.

†The dimension of the null subspace of the operator $A - \lambda I$ is called the **multiplicity** of the eigenvalue λ. From Lemma 1 it follows that every non-zero eigenvalue of a completely continuous operator is of finite multiplicity.

THEOREM 7. *If A is a completely continuous operator, then its spectrum consists of finite or countable point sets. All eigenvalues are located on the interval $[-\| A \|, \| A \|]$ and in the case of a countable spectrum these have only one limit point $\lambda = 0$.*

Consider the operator $T_\lambda = A - \lambda I$. Transforming this operator into the form $T_\lambda = -\lambda\left(I - \frac{1}{\lambda} A\right)$, it is seen that in virtue of the consequences of Chapter 3.5 when $(1/|\lambda|)\| A \| < 1$, the operator $I - (1/\lambda) A$ and, consequently, also T_λ has an inverse operator, that is, the spectrum of the operator A indeed lies on $[-\| A \|, \| A \|]$. Let $0 < \alpha < \| A \|$. For a conclusive proof, it will suffice to exhibit that there can exist only a finite number of eigenvalues λ, such that $| \lambda | \geqslant \alpha$. Assume the contrary. Then it is possible to select a sequence $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ of distinct eigenvalues, and also $\lambda_i | \geqslant \alpha$. Let $x_1, x_2, \ldots, x_n, \ldots$ be a sequence of eigenvectors corresponding to these eigenvalues

$$Ax_n = \lambda_n x_n.$$

It is required to show that the elements $x_1, x_2, \ldots, x_k$ for every k are linearily independent. For $k = 1$, this is trivial. Suppose that $x_1, x_2, \ldots, x_k$ are linearly independent.

If it is assumed that

$$x_{k+1} = \sum_{i=1}^{k} c_i x_i, \tag{13}$$

then with the operator A acting on Eq. (13), we get

$$\lambda_{k+1} x_{k+1} = \sum_{i=1}^{k} \lambda_i c_i x_i. \tag{14}$$

From (13) and (14) it follows (since $\lambda_{k+1} \neq 0$) that

$$\sum_{i=1}^{k} \left(1 - \frac{\lambda_i}{\lambda_{k+1}}\right) c_i x_i = 0.$$

However, this is impossible in virtue of the inequality

$$1 - \frac{\lambda_i}{\lambda_{k+1}} \neq 0$$

and $x_1, x_2, \ldots, x_k$ being linearly independent.

Let L_k be a subspace spanned by the elements $x_1, x_2, \ldots, x_k$. L_k is a proper subspace of the space L_{k+1}. Hence, there is an element $y_{k+1} \in L_{k+1}$, $\| y_{k+1} \| = 1$ such that $\| y_{k+1} - x \| \geqslant \frac{1}{2}$, for every $x \in L_k$. Estimate $\| Ay_m - Ay_n \|$ assuming, say $m > n$. We have

$$Ay_m - Ay_n = \lambda_m y_m + T_{\lambda_m} y_m - \lambda_n y_n - T_{\lambda_n} y_n = \lambda_m y_m - \tilde{x},$$

where $$\tilde{x} = \lambda_n y_n + T_{\lambda_n} y_n - T_{\lambda_m} y_m.$$

Now, note that

$$T_{\lambda_m} y_m = Ay_m - \lambda_m y_m = A\left(\sum_{i=1}^{m} c_i x_i\right) - \lambda_m \sum_{i=1}^{m} c_i x_i$$
$$= \sum_{i=1}^{m} c_i \lambda_i x_i - \sum_{i=1}^{m} c_i \lambda_m x_i = \sum_{i=1}^{m-1} (\lambda_i - \lambda_m) c_i x_i .$$

Hence, $T_{\lambda_m} y_m \in L_{m-1}$. Since $y_n \in L_n \subset L_m \subset L_{m-1}, T_{\lambda_n} y_n \in L_{n-1} \subset L_{m-1}$, it follows that $\tilde{x} \in L_{m-1}$. Put $\tilde{x} = \lambda_m \tilde{y}, \tilde{y} \in L_{m-1}$, to receive

$$\| Ay_m - Ay_n \| = \| \lambda_m y_m - \lambda_m \tilde{y} \| = | \lambda_m | \, \| y_m - \tilde{y} \| \geqslant \alpha \tfrac{1}{2},$$

and, consequently, neither $\{Ay_n\}$ nor any of its subsequences is convergent. On the other hand, since $\{y_n\}$ is a bounded set, $\{Ay_n\}$ is compact and, consequently, contains a convergent subsequence, a contradiction. ■

Theorem 7 characterizes the so-called discreteness of the spectra of completely continuous operators.

6.3 SCHAUDER PRINCIPLE AND ITS APPLICATIONS

LET M be a set of the BANACH space E and A an operator, generally nonlinear, defined on M and mapping M into itself.

The operator A is called **compact** on the set M if it carries every bounded subset of this set into a compact set. If, in addition, A is continuous on M, then it is said to be **completely continuous** on M (in case A is linear, then both the definitions coincide). Without restricting the generality, the set M can be regarded as bounded in what follows.

For completely continuous operators, which map a convex body of a BANACH space into itself, J. SCHAUDER has generalized the celebrated BROUWER's fixed-point theorem which states that every continuous mapping of a closed unit sphere of an n-dimensional Euclidean space into itself has at least one fixed point. Its extension by SCHAUDER to infinite n-dimensional normed linear spaces finds numerous applications in the existence proofs for solutions of differential equations.

6.31. Three Lemmas. The proof of SCHAUDER's theorem is sought to be prefaced by the three upcoming auxiliary lemmas.

Let M be a set of elements in a BANACH space and let $\{A_n\}$ be a sequence of operators, generally nonlinear, defined on M. This sequence is said to converge uniformly on M to the operator A_0, if for any number $\varepsilon > 0$ there is an index n_0 depending only on ε, such that

$$\| A_n x - A_0 x \| < \varepsilon$$

for $n \geqslant n_0$ and for every $x \in M$.

LEMMA 1. *If a sequence of operators $\{A_n\}$, completely continuous on M, converges uniformly on this set to an operator A_0, then A_0 is also completely continuous on M.*

In the first place, show that A_0 is continuous on M. Let $\{x_n\} \subset M$ converge to $x_0 \in M$. We have

$$\| A_9 x_m - A_0 x_0 \| \leqslant \| A_0 x_m - A_n x_m \| + \| A_n x_m - A_n x_0 \| + \| A_n x_0 - A_0 x_0 \|.$$

Since the sequence of operators $\{A_n\}$ converges uniformly on M to A_0, there exists, for $\varepsilon > 0$ given, an index n_0 such that

$$\| A_n x_m - A_0 x_m \| < \varepsilon/3, \qquad \| A_n x_0 - A_0 x_0 \| < \varepsilon/3$$

for $n \geqslant n_0$, such n being fixed. Since A_n is continuous, there is an index m_0 such that $\| A_n x_m - A_n x_0 \| < \varepsilon/3$ for $m \geqslant m_0$. However, then, $\| A_0 x_m - A_0 x_0 \| < \varepsilon$ for $m \geqslant m_0$; in other words, the operator A_0 is continuous.

For the compactness of A_0, the set $A_0(M)$ has to be compact.

For $\varepsilon > 0$ given, select n_0 such that $\| A_n x - A_0 x \| < \varepsilon$ for all $x \in M$. This is plausible owing to the uniform convergence of $\{A_n\}$ to A_0. Let $N = A_{n_0}(M)$. The set N is compact and forms an ε-net for $A_0(M)$ (see, proof of Theorem 2 of Sec. 1). Thereupon, it follows that $A_0(M)$ is compact.

Thus, A_0 is a continuous and compact operator. ■

LEMMA 2 (J. SCHAUDER). *Every operator A, completely continuous on the set M, is a uniform limit on this set of the sequence $\{A_k\}$ of continuous finite-dimensional operators (maps M into a finite-dimensional subspace E).*

Since A is a completely continuous operator, $A(M)$ is a compact set. Select a sequence of positive numbers $\{\varepsilon_k\}$, convergent to zero, and for every k construct an ε_k-net

$$N_k = \left\{ y_1^{(k)}, y_2^{(k)}, \ldots, y_{m_k}^{(k)} \right\}$$

for $A(M)$, consisting of the points of $A(M)$. Define on $A(M)$ the operator P_k, putting for $y \in A(M)$

$$P_k(y) = \frac{\sum\limits_{i=1}^{m_k} \mu_i^{(k)}(y)\, y_i^{(k)}}{\sum\limits_{i=1}^{m_k} \mu_i^{(k)}(y)}, \tag{1}$$

where
$$\mu_i^{(k)}(y) = \begin{cases} \varepsilon_k - \| y - y_i^{(k)} \|, & \text{if} \quad \| y - y_i^{(k)} \| < \varepsilon_k, \\ 0, & \text{if} \quad \| y - y_i^{(k)} \| \geqslant \varepsilon_k. \end{cases}$$

Eq. (1) has a meaning for every $y \in A(M)$, since all the $\mu_i^{(k)}(y) \geqslant 0$ and $u_i^{(k)}(y) > 0$ at least for a single i.

The operator $P_k(y)$ is continuous on $A(M)$. This is implied by the fact that all the $\mu_i^{(k)}(y)$ are continuous functions and, consequently, $\sum_{i=1}^{m_k} \mu_i^{(k)}(y)$ is also a continuous function of y. In addition, $\sum_{=1}^{m_k} \mu_i^{(k)}(y) > 0$ for every $y \in A(M)$, implying the continuity of the ratio $\mu_i^{(k)}(y) / \sum_{j=1}^{m_k} \mu_j^{(k)}(y)$, and consequently, also of the expression $\sum_{i=1}^{m_k} \mu_i^{(k)}(y)\, y_i^{(k)} / \sum_{i=1}^{m_k} \mu_i^{(k)}(y)$, that is, of the operator $P_k(y)$. Furthermore,

$$\| y - P_k(y) \| = \left\| y - \frac{\sum_{i=1}^{m_k} \mu_i^{(k)}(y)\, y_i^{(k)}}{\sum_{i=1}^{m_k} \mu_i^{(k)}(y)} \right\|$$

$$= \left\| \frac{\sum_{i=1}^{m_k} \mu_i^{(k)}(y)\,(y - y_i^{(k)})}{\sum_{i=1}^{m_k} \mu_i^{(k)}(y)} \right\| \leqslant \frac{\sum_{i=1}^{m_k} \mu_i^{(k)}(y)\, \| y - y_i^{(k)} \|}{\sum_{i=1}^{k} \mu_i^{(k)}(y)}$$

$$< \varepsilon_k \frac{\sum_{i=1}^{m_k} \mu_i^{(k)}(y)}{\sum_{i=1}^{m_k} \mu_i^{(k)}(y)} = \varepsilon_k,$$

since if $\| y - y_i^{(k)} \| \geqslant \varepsilon_k$ for some i, then the corresponding coefficients $\mu_i^{(k)}(y)$ vanish.

Now, put $A_k x = P_k(Ax)$ for $x \in M$, to receive a sequence of finite-dimensional operators $\{A_k\}$, such that

$$\| Ax - A_k x \| = \| Ax - P_k(Ax) \| < \varepsilon_k$$

for every $x \in M$. ∎

REMARK. *Since the elements $y_i^{(k)}$ belong to the set $A(M)$, the range of operators constructed in Lemma 2, belongs to a convex hull of $A(M)$.*

LEMMA 3. *Suppose that a sequence $\{A_n\}$ of operators, completely continuous on M, uniformly converges on M to an operator A_0. Further, let $K_n = A_n(M)$, $n = 0, 1, 2, \ldots$, then the set $K = \bigcup_{n=0}^{\infty} K_n$ is compact.*

By Lemma 1, A_0 is completely continuous. Since the sequence $\{A_n\}$ uniformly converges to A_0, there exists $u \in K_0$ such that $\| y - u \| < \varepsilon$ for any $\varepsilon > 0$ and every $y \in K_n$ with $n \geqslant n_0(\varepsilon)$. This is plausible because if y is an arbitrary element in K_n and x one of the pre-images of y under the map A_n, then it suffices to take u as $u = A_0(x)$.

Construct the set $N = \bigcup_{n=0}^{n_0} K_n$. It is obviously compact. It is required to show that this set is an ε-net for K.

Let $y \in K$. If $y \in \bigcup_{n=0}^{n_0} K_n$, there is nothing to prove. If, however, $y \in K_n$ for $n > n_0$, then as proved above, there exists $u \in K_0$ such that $\| y - u \| < \varepsilon$. Consequently, N is a compact ε-net for K, implying that K is compact.

6.32. The Schauder fixed point principle. THEOREM 1. *If A is a completely continuous operator, mapping a closed bounded convex set S of a Banach space E into itself, then there is a point x of S fixed under A, such that $A(x) = x$.*

Take a sequence $\{\varepsilon_n\}$ of positive numbers, convergent to zero and construct by Lemma 2 a sequence of continuous finite-dimensional operators A_n, uniformly convergent on S to the operator A.

By Remark appended below Lemma 2 and S being convex, $A_n x \in S$ for every $x \in S$. Let E_n be a finite-dimensional subspace, containing the set $A_n(S)$. Consider an operator A_n on the set $S_n = S \cap E_n$ of the subspace E_n. It is plain that S_n is also a closed convex set.

Since $A_n(S) \subset S$ and $A_n(S) \subset E_n$, it follows that $A_n(S) \subset S_n$, and what is more, $A_n(S_n) \subset S_n$.

Thus, A_n on the finite-dimensional space E_n maps a closed convex set S_n of this space into itself, and hence, by the BOLYA-BROUWER theorem (see, Appendix III), there exists a fixed point under this mapping, that is a point $x_n \in S_n$ such that $A_n x_n = x_n$. However, since $S_n \subset S$, x_n is a fixed point of the operator A_n also under the mapping by this operator of the set S. Since every $x_n \in A_n(S)$, the sequence $\{x_n\}$ belongs to the set $\tilde{S} = \bigcup_{n=1}^{\infty} A_n(s) \subset S$. By Lemma 3, $\tilde{S}$ is a compact set, so that one can extract from $\{x_n\}$ a convergent subsequence $\{x_{n_i}\}$ and the limit x_0 of this subsequence belongs to S, S being closed.

It is to be shown that x_0 is a fixed point of the operator A. We have

$$\| Ax_0 - x_0 \| \leqslant \| Ax_0 - Ax_{n_i} \| + \| Ax_{n_i} - A_{n_i} x_{n_i} \| + \| A_{n_i} x_{n_i} - x_0 \|$$

$$= \| Ax_0 - Ax_{n_i} \| + \| Ax_{n_i} - A_{n_i} x_{n_i} \| + \| x_{n_i} - x_0 \|.$$

For $\varepsilon > 0$ given, first select n' so large that for $n_i \geqslant n'$

$$\| x_{n_i} - x_0 \| \leqslant \varepsilon/3 \quad \text{and} \quad \| Ax_0 - Ax_{n_i} \| < \varepsilon/3.$$

Next, choose n'' so large that for $n_i \geqslant n''$, $\| Ax - A_{n_i} x \| < \varepsilon/3$ uniformly on

S and, in particular, for all x_{n_i}. Then, for $n_i \geqslant n_0 = \max(n', n'')$,

$$\| Ax_0 - x_0 \| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this is plausible only if $Ax_0 = x_0$, that is, x_0 is a fixed point of A. ■

As an example of the SCHAUDER principle we take up next the well-known PEANO's theorem on the existence of solutions of ordinary differential equations.

THEOREM 2. *Let the function $f(t, x)$ be continuous in the collection of variables in the domain $|t - t_0| \leqslant a$, $|x - x_0| \leqslant b$, and let β be the maximum of $|f(t, x)|$ in this domain. If $h = \min[a, (b/\beta)]$, then the equation*

$$\frac{dx}{dt} = f(t, x), \tag{2}$$

satisfying the condition $$x(t_0) = x_0, \tag{3}$$
has at least one solution on the interval $[t_0 - h, t_0 + h]$.

PROOF. Eq. (2) together with the initial condition (3) is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f[\tau, x(\tau)]\, d\tau. \tag{4}$$

Consider an operator A defined by

$$Ax = x_0 + \int_{t_0}^{t} f[\tau, x(\tau)]\, d\tau$$

on the sphere $\| x - x_0 \| \leqslant b$ of the space $C[t_0 - h, t_0 + h]$. It is required to show that A is completely continuous on this sphere.

To start with, if the sequence $\{x_n(t)\}$ belonging to the sphere $\| x - x_0 \| \leqslant b$ is uniformly convergent to the function $x(t)$, evidently, belonging to the same sphere, then by the continuity of the function $f(t, x)$,

$$f[t, x_n(t)] \to f[t, x(t)]$$

uniformly on $[t_0 - h, t_0 + h]$. Thereupon, because of the possibility of the integral taking the limit under uniform convergence, $Ax_n \to Ax$, that is, the operator A is continuous on the sphere $\| x - x_0 \| \leqslant b$.

Further, for every element $x(t)$ of the sphere $\| x - x_0 \| \leqslant b$, we have

$$|Ax(t)| \leqslant |x_0| + \left| \int_{t_0}^{t} f[\tau, x(\tau)]\, d\tau \right| \leqslant |x_0| + \beta\, |h|. \tag{5}$$

If t_1 and t_2 be any pair of points of the segment $[t_0 - h, t_0 + h]$, then

$$|Ax(t_1) - Ax(t_2)| \leqslant \left| \int_{t_1}^{t_2} f[\tau, x(\tau)]\, d\tau \right| \leqslant \beta\, |t_2 - t_1|. \tag{6}$$

Ineqs. (5) and (6) imply by ARZELÁ's theorem that the operator A carries the sphere $\| x - x_0 \| \leqslant b$ into a compact set.

Finally, it is to be shown that this sphere is mapped into itself by A. In fact,

$$| Ax(t) - x_0 | = \left| \int_{t_0}^{t} f[\tau, x(\tau)]\, d\tau \right| \leqslant \beta h \leqslant \beta (b/\beta) = b.$$

Thus, the operator A satisfies all hypotheses of the SCHAUDER theorem. Hence, there exists a fixed point of this operator, that is, the function $x(t)$ such that

$$x(t) \equiv x_0 + \int_{t_0}^{t} f[\tau, x(\tau)]\, d\tau.$$

This equality is equivalent to the two equalities

$$\frac{dx(t)}{dt} = f[t, x(t)], \qquad x(t_0) = x_0. \qquad \blacksquare$$

In proving PEANO'S theorem, we made an appeal to the SCHAUDER principle to establish the existence of the solution of Eq. (4). The fascination of the SCHAUDER fixed point principle consists in providing existence proofs for complex nonlinear integral and integro-differential equations under rather general conditions. The limitation of this principle is that it leads to pure existence results and does not provide either with uniqueness results or a successive approximation scheme furnished by contraction mapping principle for obtaining approximations to the solution.

6.4 COMPLETELY CONTINUOUS INCLUSION OPERATORS OF SOBOLEV

IT IS shown in the foregoing (pp. 80-1) that if the function $\varphi(x, y)$ belongs to the class $W_p^{(l)}$, then by implication it belongs also to the class $W_p^{(k)}$ for $k < l$.

Introduce an operator A, defined for all the functions $\varphi(x, y) \in W_p^{(l)}$ and transforming $\varphi(x, y)$ into itself, $\varphi(x, y)$ regarded beforehand as an element of the space $W_p^{(k)}$. It is plain that for distinct k, this operator is essentially distinct. The operator A is called an **inclusion operator.** The linearity of A is obvious and by the inequality on p. 80, A is bounded. The motivation of the theorem that follows is to show that A is also a completely continuous operator.

THEOREM (V. I. KONDRASHOV). *Let $\mathcal{M}$ be a bounded set in the space $W_p^{(l)}$. If $p > 2$, then the set $A(\mathcal{M})$ is compact in the sense of uniform convergence; if $p \leqslant 2$, then $A(\mathcal{M})$ is compact in the sense of the metric of the space $L_p(G)$.*

By SOBOLEV's formula,

$$A\varphi = \varphi(x, y) = \sum_{k=0}^{l-1} \sum_{k_1+k_2=k} C_{k_1k_2}^{(k)}(P) \int\int_G \varphi(Q) \frac{\partial^k \omega_k}{\partial x^{k_1} \partial y^{k_2}} dQ$$

$$+ \sum_{l_1+l_2=l} \int\int_G A_{l_1l_2}^{(l)}(P, Q) \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} dQ. \quad (1)$$

By the continuity of $\partial^k \omega_k / \partial x^{k_1} \partial y^{k_2}$, the first group of terms of formula (1) repesents an integral operator with a continuous kernel and is, consequently, a completely continuous operator in the metric of $C(G)$ as well as $L_p(G)$. Thus, it remains to investigate only the latter summand of (1). The kernel $A_{l_1l_2}^{(l)}(P, Q)$ of the integral operators appearing there takes the form

$$A(P, Q) = B(P, Q) / r, \quad \text{or}, \quad A(P, Q) = B(P, Q)(\alpha \ln r + \beta)$$

where $B(P, Q)$ is a bounded function : $|B(P, Q)| \leqslant C$. It is to be shown that an integral operator with this kernel is completely continuous.

Consider the case $p > 2$ and restrict to the former expression for $A(P, Q)$, getting

$$\left| \int\int_G A_{l_1l_2}^{(l)}(P, Q) \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} dQ \right| \leqslant C \int\int_G \frac{\left| \dfrac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right|}{r} dQ$$

$$\leqslant C \left(\int\int_G \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right|^p dQ \right)^{1/p} \left(\int\int_G r^{-q} dQ \right)^{1/q}$$

$$\leqslant C \| \varphi \|_{W_p^{(l)}} \left(\int_0^{2\pi} \int_0^{\rho} r^{-q+1} dr\, d\theta \right)^{1/q} \leqslant CK(2\pi)^{1/q} (B)^{1/q}. \quad (2)$$

Here the constant K is the bounded norm of the function $\varphi(x, y)$ in the space $W_p^{(l)}$ and B is the value of the integral $\int_0^{\rho} r^{-q+1} dr$, which converges if $q < 2$, that is, if $p > 2$.

Further, for brevity we introduce the notation

$$\psi(P) = \int\int_G A_{l_1l_2}^{(l)}(P, Q) \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} dQ, \text{ getting}$$

$$|\psi(P + \Delta P) - \psi(P)|$$

$$\leqslant C \left\{ \int\int_{G-G_\delta} \left| \frac{1}{r_{P+\Delta P, Q}} - \frac{1}{r_{P,Q}} \right| \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right| dQ \right.$$

$$\left. + \int\int_{G_\delta} \frac{1}{r_{P+\Delta P, Q}} \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right| dQ + \int\int_{G_\delta} \frac{1}{r_{P,Q}} \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right| dQ \right\}, \quad (3)$$

where G_δ is a part of the domain G, consisting of the points which are apart from the points P by a distance less than 2δ. In addition, the distance between the points P and $P + \Delta P$ is assumed to be not greater than δ.

By the assertions above, $1/r_{P,Q}$ under the sign of first integral occurring in the braces is a continuous function and, therefore, the first summand is arbitrarily small for sufficiently small ΔP. Concerning the second summand, introduce for it the polar coordinates with center $P + \Delta P$, to receive the estimate

$$\iint\limits_{G_\delta} \frac{1}{r_{P+\Delta P,Q}} \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right| dQ$$
$$\leqslant \left(\iint\limits_{G_\delta} \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right|^p dQ \right)^{1/p} \left(\iint\limits_{G_\delta} \left| \frac{1}{r_{P+\Delta P,Q}} \right|^q dQ \right)^{1/q}$$
$$\leqslant K (2\pi)^{1/q} \left(\int_0^{3\delta} \frac{1}{r^q} \, r \, dr \right)^{1/q} \leqslant K (2\pi)^{1/q} \frac{1}{2-q} (3\delta)^{2-q},$$

whose right-hand side can be made arbitrarily small for sufficiently small δ, if only $q < 2$, that is, $p > 2$. Analogously, the third integral is evaluated and the equicontinuity of the function $\psi(P)$ is established, proving the first part of the theorem.

Now, we pass on to the case $p \leqslant 2$, to receive as above,

$$|\psi(P+\Delta P) - \psi(P)|$$
$$\leqslant C \left\{ \iint\limits_{G - G_\delta} \left| \frac{1}{r_{P+\Delta P,Q}} - \frac{1}{r_{P,Q}} \right| \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right| dQ \right.$$
$$\left. + \iint\limits_{G_\delta} \frac{1}{r_{P+\Delta P,Q}} \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right| dQ + \iint\limits_{G_\delta} \frac{1}{r_{P,Q}} \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right| dQ \right\}. \tag{3}$$

Again, by the continuity of $(1 / r_{P,Q})$ the first integral can be made arbitrarily small with respect to the norm of the space L_p for sufficiently small ΔP. For the second summand, we obtain, on similar lines as before,

$$\left\| \iint\limits_{G_\delta} \frac{1}{r_{P+\Delta P,Q}} \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right| dQ \right\|_{L_p}^p$$
$$= \iint\limits_G \left[\iint\limits_{G_\delta} \frac{1}{r_{P+\Delta P,Q}} \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right| dQ \right]^p dP$$
$$\leqslant \iint\limits_G \left\{ \iint\limits_{G_\delta} \frac{1}{r_{P+\Delta P,Q}} \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right|^p dQ \right\}$$
$$\times \left\{ \iint\limits_{G_\delta} \frac{1}{r_{P+\Delta P,Q}} dQ \right\}^{p/q} dP$$

$$\leqslant (2\pi)^{p/q} (3\delta)^{p/q} \iint_G \left\{ \iint_G \frac{1}{r_{P+\Delta P,Q}} \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right|^p dQ \right\} dP$$

$$= (6\pi\delta)^{p/q} \iint_G \left\{ \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right|^p \iint_G \frac{1}{r_{P+\Delta P,Q}} dP \right\} dQ$$

$$\leqslant (6\pi\delta)^{p/q} 2\pi D K^p,$$

D the diameter of the domain G. The inequalities obtained imply that

$$\left\| \iint_{G_\delta} \frac{1}{r_{P+\Delta P,Q}} \left| \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2}} \right| dQ \right\|$$

can be made arbitrarily small, if δ is sufficiently small.

The norm of the third summand of (3) is evaluated in exactly the same way. Thus,

$$\| \psi(P+\Delta P) - \psi(P) \|_{L_p} \to 0 \quad \text{as} \quad \Delta P \to 0.$$

Similar calculations reveal the function $\psi(P)$ to be uniformly bounded in the mean. However, then, by RIESZ' theorem, the function $\psi(P)$ forms a compact family.

For establishing the complete continuity of the inclusion operator, it will now suffice to apply the theorem just proved to SOBOLEV's formula, expressing the k-th generalized derivative in terms of the l-th generalized derivative for $k < l$.

The treatment has been confined here to functions of two independent variables. This has been extended to functions of larger number of independent variables and also of more complex nature by SOBOLEV in [34].

The application of the inclusion theorem to problems in nuclear mathematical physics is now intended to be illustrated through an example.

Let G be a domain on a plane of the considered form. It is required to show that there is a λ, such that the equation $\Delta\varphi + \lambda\varphi = 0$ inside of G has a non-trivial solution, satisfying the condition $\varphi |_\Gamma = 0$, Γ being the boundary of G (DIRICHLET eigenfunction problem).

Replace the second condition $\varphi |_\Gamma = 0$ by a weaker condition requiring that $\varphi \in \dot{W}_2^{(1)}$ ($\dot{W}_2^{(1)}$, being a subspace of $W_2^{(1)}$, consists of the functions which constitute the limit of a sequence of functions in the sense of the metric of this space), vanishes in some neighbourhood (natural for every function) of the boundary of G.

Consider in $\dot{W}_2^{(1)} G$ the functional

$$J(\varphi) = \iint_G \left\{ \left(\frac{\partial\varphi}{dx}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 \right\} dx\, dy.$$

This functional is bounded (below) and, consequently, on the function

$\varphi(x, y) \in \dot{W}_2^{(1)}$, such that

$$\iint_G \varphi^2(x, y)\, dx\, dy = 1,$$

it has the greatest lower bound λ_0. Evidently, $\lambda_0 > 0$.

It is intended to show that the greatest lower bound of the functional J is attained on some function $\varphi_0(x, y) \in \dot{W}_2^{(1)}$. Let $\{\varphi_n(x, y)\} \subset \dot{W}_2^{(1)}$ be the minimizing sequence, that is, such that

$$J(\varphi_n) = \lambda_n \to \lambda_0, \quad \| \varphi_n \|_{L_2} = 1. \tag{4}$$

Since
$$\| \varphi_n \|^2_{W_2^{(1)}} = \| \varphi_n \|^2_{L_2} + J(\varphi_n) \tag{5}$$

and $\{J(\varphi_n)\}$ as a sequence convergent to the limit is bounded, $\{\varphi_n\}$ is a bounded sequence of the space $W_2^{(1)}$. By the complete continuity of the inclusion operator, $\{\varphi_n\}$ is compact in the space L_2. Disregarding, if necessary, some terms of the minimizing sequence, $\{\varphi_n\}$ can be assumed to converge in L_2. Hence, for any $\varepsilon > 0$ there is an index n_0 such that

$$\| \varphi_n - \varphi_m \|_{L_2} < \varepsilon \qquad \text{for } n \geqslant n_0.$$

Furthermore,

$$\left\| \frac{\varphi_n + \varphi_m}{2} \right\|^2_{L_2} + \left\| \frac{\varphi_n - \varphi_m}{2} \right\|^2_{L_2} = \frac{1}{2} \| \varphi_n \|^2_{L_2} + \frac{1}{2} \| \varphi_m \|^2_{L_2} = 1$$

Thereupon

$$\left\| \frac{\varphi_n + \varphi_m}{2} \right\|^2_{L_2} = 1 - \left\| \frac{\varphi_n - \varphi_m}{2} \right\|^2_{L_2} > 1 - \frac{\varepsilon^2}{4}, \quad n \geqslant n_0.$$

Since
$$\lambda_0 = \inf_{\varphi \in W_2^{(1)},\, \| \varphi \|_{L_2} = 1} J(\varphi),$$

then, by the quadratic homogeneity of $J(\varphi)$, we have

$$\inf_{\varphi \in W_2^{(1)}} \frac{J(\varphi)}{\| \varphi \|^2_{L_2}} = \inf_{\varphi \in W_2^{(1)}\, \| \varphi \|_{L_2} = 1} J(\varphi) = \lambda_0.$$

Hence,
$$J(\varphi) \geqslant \lambda_0 \| \varphi \|^2_{L_2},$$

and, in particular,

$$J\left(\frac{\varphi_n + \varphi_m}{2}\right) \geqslant \lambda_0 \left\| \frac{\varphi_n + \varphi_m}{2} \right\|^2_{L_2} > \lambda_0 \left(1 - \frac{\varepsilon^2}{4}\right).$$

Select n and m, so large that

$$J(\varphi_n) < \lambda_0 + \varepsilon, \qquad J(\varphi_m) < \lambda_0 + \varepsilon.$$

Then, $$J\left(\frac{\varphi_n-\varphi_m}{2}\right)=\frac{1}{2}J(\varphi_n)+\frac{1}{2}J(\varphi_m)-J\left(\frac{\varphi_n+\varphi_m}{2}\right)$$
$$<\lambda_0+\varepsilon-\lambda_0\left(1-\frac{\varepsilon^2}{4}\right)=\varepsilon\left(1-\lambda_0\frac{\varepsilon}{4}\right),$$

that is, $J\left(\frac{\varphi_n-\varphi_m}{2}\right)\to 0$, as $n, m\to\infty$. Thereupon (5) implies not only that $\|\varphi_n-\varphi_m\|_{L_2}\to 0$ but also that $\|\varphi_n-\varphi_m\|_{W_2^{(1)}}\to 0$ as $n, m\to\infty$. By completeness of $W_2^{(1)}$, there exists $\varphi_0(x, y)\in W_2^{(1)}$, forming the limit in $W_2^{(1)}$ of the sequence $\{\varphi_n\}$.

It is evident that $\|\varphi_0\|_{L_2}=1$, $\varphi_0(x, y)\in \dot{W}_2^{(1)}$. The inequality
$$|J(\varphi)^{1/2}-J(\psi)^{1/2}|\leqslant[J(\varphi-\psi)]^{1/2},$$
valid for any $\varphi, \psi\in W_2^{(1)}$, implies
$$|J(\varphi_n)^{1/2}-J(\varphi_0)^{1/2}|\leqslant[J(\varphi_n-\varphi_0)]^{1/2}\leqslant\|\varphi_n-\varphi_0\|_{W_2^{(1)}},$$
that is, $J(\varphi_n)\to J(\varphi_0)$ as $n\to\infty$. Since, on the other hand, $J(\varphi_n)\to\lambda_0$, hence $J(\varphi_0)=\lambda_0$, proving the existence in $W_2^{(1)}$ of a function realizing the minimum $J(\varphi)$, provided that $\|\varphi\|_{L_2}=1$.

It is to be shown now that the limit of the function $\varphi_0(x, y)$ satisfies the LAPLACE equation.

Let $\zeta(x, y)$ be an arbitrary function of $W_2^{(1)}$, which vanishes at some neighbourhood of the boundary of G. For any real t,
$$\frac{J(\varphi_0+t\zeta)}{\|\varphi_0+t\zeta\|_{L_2}^2}\geqslant\lambda_0,$$
whence $$J(\varphi_0)+2t\iint_G\left\{\frac{\partial\varphi_0}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial\varphi_0}{\partial y}\frac{\partial\zeta}{\partial y}\right\}d\xi\,d\eta+t^2J(\zeta)$$
$$\geqslant\lambda_0\left[\|\varphi_0\|_{L_2}^2+2t\iint_G\varphi_0(\xi,\eta)\,\zeta(\xi,\eta)\,d\xi\,d\eta+t^2\|\zeta\|_{L_2}^2\right].$$

Taking note of $J(\varphi_0)=\lambda_0$ and $\|\varphi_0\|_{L_2}=1$, we get
$$2t\left\{\iint_G\left[\frac{\partial\varphi_0}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial\varphi_0}{\partial y}\frac{\partial\zeta}{\partial y}\right]d\xi\,d\eta-\lambda_0\iint_G\varphi_0(\xi,\eta)\,\zeta(\xi,\eta)\,d\xi\,d\eta\right\}$$
$$+t^2\{J(\zeta)-\lambda_0\|\zeta\|_{L_2}^2\}\geqslant 0.$$

Thereupon, by usual reasonings,
$$\iint_G\left[\frac{\partial\varphi_0}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial\varphi_0}{\partial y}\frac{\partial\zeta}{\partial y}\right]d\xi\,d\eta-\lambda_0\iint_G\varphi_0(\xi,\eta)\,\zeta(\xi,\eta)\,d\xi\,d\eta=0.\qquad(6)$$

Let $\psi(t)$ be a function satisfying the conditions:

(i) $\psi(t) = 1$ for $0 \leqslant t \leqslant \frac{1}{2}$;

(ii) $\psi(t) = 0$ for $t \geqslant 1$;

(iii) $\psi(t)$ is monotone decreasing on the interval $[\frac{1}{2}, 1]$;

(iv) $\psi(t)$ has a continuous derivative of any order on $[0, \infty)$.

It is evident that such a function exists.

Further, let $Y_0(\sqrt{\lambda_0}\, r) = X(r)$ be the BESSEL function of the first-kind of order zero. As is known [39],

$$\Delta X(r) + \lambda_0 X(r) = 0,$$

and $X(r)$ has a single logarithmic singularity for $r = 0$. Set

$$\zeta(x, y) = \left[\psi\left(\frac{r}{h_1}\right) - \psi\left(\frac{r}{h_2}\right)\right] X(r).$$

For $r \leqslant \rho_1 = \frac{1}{2} \min(h_1, h_2)$, we have

$$\psi\left(\frac{r}{h_1}\right) - \psi\left(\frac{r}{h_2}\right) = 1 - 1 = 0, \quad \text{and} \quad \psi\left(\frac{r}{h_1}\right) = \psi\left(\frac{r}{h_2}\right) = 0$$

for $r \geqslant \rho_2 = \max(h_1, h_2)$. Consequently, $\zeta(x, y)$ vanishes inside the circle $r = \rho_1$ and outside the circle $r = \rho_2$. Hence, if ρ_2 is smaller than the distance between $P(x, y)$ and the boundary of G, then $\zeta(x, y)$ is a function continuously differentiable any number of times, which vanishes at some neighbourhood of the boundary of G.

Substitute this function into (6) and make use of the definition (ii) of the generalized derivative, to receive

$$\iint_G \varphi_0(\Delta\zeta + \lambda_0\zeta)\, d\xi\, d\eta = 0. \tag{7}$$

Let $$\Omega_h(r) = \frac{1}{C(h)} \left\{\Delta\left[\psi\left(\frac{r}{h}\right) X(r)\right] + \lambda_0 \psi\left(\frac{r}{h}\right) X(r)\right\},$$

where $$C(h) = \iint_G \left\{\Delta\left[\psi\left(\frac{r}{h}\right) X(r)\right] + \lambda_0 \psi\left(\frac{r}{h}\right) X(r)\right\} d\xi d\eta.$$

It can be shown that $C(h)$, as $h \to 0$, tends to a finite limit C_0. The properties of $\psi(t)$ imply that:

(*a*) $\Omega_h(r) \equiv 0$ for $r \geqslant h$ and for $r \leqslant h/2$ (since in the latter case $\psi(r/h) \equiv 1$ and

$$\Delta\left[\psi\left(\frac{r}{h}\right) X(r)\right] + \lambda_0 \psi\left(\frac{r}{h}\right) X(r) = \Delta X(r) + \lambda_0 X(r) = 0);$$

(*b*) $\Omega_h(r)$ has continuous derivatives of all order.

Treat $\Omega_h(r)$ as an averaging kernel. The formula (7) can be written in the form

$$C(h_1) \iint_G \varphi_0(\xi, \eta)\, \Omega_h(r)\, d\xi\, d\eta = C(h_2) \iint_G \varphi_0(\xi, \eta)\, \Omega_h(r)\, d\xi\, d\eta,$$

and, then, this induces an equality between the mean-valued functions

$$C(h_1)\,(\varphi_0)_{h_1} = C(h_2)\,(\varphi_0)_{h_2}, \quad \text{or} \quad (\varphi_0)_{h_2} = [C(h_1)/C(h_2)]\,(\varphi_0)_{h_1},$$

signifying that two distinct mean-valued functions differ only by a numerical factor. However, then, $\varphi_0(x, y)$, the limit of the mean-valued functions, also differs from them only by a numerical factor

$$\varphi_0(x, y) = (C(h)/C_0)\,[\varphi_0(x, y)]_h.$$

Since a mean-valued function has continuous derivatives of all order, hence, so has $\varphi_0(x, y)$.

Eq. (6) can now be expressed in the form

$$\iint_G (\Delta\varphi_0 + \lambda\varphi_0)\, \zeta(\xi, \eta)\, d\xi\, d\eta = 0.$$

Since $\zeta(\xi, \eta)$ is an arbitrary finitary, that is, infinitely differentiable, function in G, becoming zero in a neighbourhood of the boundary of G, it follows from the fundamental lemmas of calculus of variations, that

$$\Delta\varphi_0 + \lambda_0\varphi_0 = 0$$

inside G. The function φ_0 belongs to the class $\dot{W}_2^{(1)}$, defined above. It can be shown that for the case of two independent variables this implies that

$$\| \varphi_0 \|_\Gamma = 0.$$

CHAPTER 7

ELEMENTS OF SPECTRAL THEORY OF SELF-ADJOINT OPERATORS IN HILBERT SPACE

7.1. SELF-ADJOINT OPERATORS

7.11. Adjoint operators in Hilbert spaces. In the study of linear operators, defined in HILBERT spaces, it is possible, thanks to these being self-conjugate and inner product spaces, to isolate a class of operators having the singular property of symmetry or self-adjointness and investigate these operators in a greater depth than arbitrary linear operators in BANACH spaces. These operators play a significant role in analysis and theoretical physics and the literature devoted to them is extensive.

7.12. Adjoint operators. Let H be a HILBERT space and A a bounded linear operator, defined on H, with range in the same space. Consider a linear functional

$$f_y(x) = (Ax, y). \tag{1}$$

As a linear functional in the HILBERT space, $f_y(x)$ has also the form $f_y(x) = (x, y^*)$, where y^* is some element $\in H$, uniquely defined by f_y. It is evident that f_y varies with a change of y and so does y^* too, and thereby we get the operator $y^* = A^*y$, defined on H with range in the same space. This operator A^* is associated with A by

$$(Ax, y) = (x, A^*y) \tag{2}$$

and is called the **adjoint operator** of A. A^* is uniquely defined by (2). In fact, if for all x and y, $(Ax, y) = (x, A^*y) = (x, A_1^* y)$ holds, then this implies that $A^*y = A_1^* y$ for all y. However, this also signifies that

$$A^* = A_1^* .$$

It is easy to see that the definition of adjoint operators derived here formally coincides with the definition given in Chap. 4 for the case of BANACH spaces, but the BANACH spaces were assumed there to be real whereas HILBERT spaces are complex. However, it is easy to verify that theorems on adjoint operators proved in Chap. 4 remain valid also for complex spaces. In particular, A^* is a bounded operator, and

$$\| A^* \| = \| A \| . \tag{3}$$

It is intended to find the operator adjoint to A^*, denoting this by A^{**}. By (2),

$$(A^*x, y) = \overline{(y, A^*x)} = \overline{(Ay, x)} = (x, Ay)$$

for every $x, y \in H$. Hence, $A^{**} = A$; similarly, $A^{***} = A^*$, and so on, It is plain that: $(A+B)^* = A^* + B^*$; $(\lambda A)^* = \bar{\lambda} A^*$; $(AB)^* = B^*A^*$, *if A is invertible, then so is A^* and* $(A^*)^{-1} = (A^{-1})^*$

7.13. Self-adjoint operators. A bounded linear operator A coinciding with its adjoint, $A = A^*$, is called **self-adjoint** (or **Hermitian**).

Examples. 1. In an n-dimensional unitary space which can be regarded as a finite-dimensional analogue of a HILBERT space, the linear operators can be identified with the matrices (a_{ik}) with complex numbers as elements. The operator adjoint to (a_{ik}) is $(\bar{a}_{kt})$. A self-adjoint operator is a Hermitian matrix, that is, a matrix satisfying $a_{ik} = \bar{a}_{ki}$.

If (a_{ik}) is real, then self-adjointness implies symmetry.

Thus, self-adjoint operators in any inner product space are also called symmetric if the space is real and Hermitian if it is complex.

2. For a FREDHOLM operator in $L_2\,[0, 1]$ with the kernel $K(t, s)$ the adjoint operator is the FREEDHOLM operator with the kernel $\overline{K(s, t)}$. Self-adjointness is identical with

$$K(t, s) = \overline{K(s, t)}.$$

We have symmetry in the case of a real kernel.

3. Consider in $L_2\,[0, 1]$ an operator A, associating the function $Ax = tx(t) \in L_2\,[0, 1]$ with every function $x(t) \in L_2\,[0, 1]$. It is easy to verify that this operator is self-adjoint.

The 'modifier' bounded will be hereafter dropped. From the foregoing discussion, it follows that if A is a self-adjoint operator and λ a real number, then λA is also a self-adjoint operator; if A and B are self-adjoint operators, then $A + B$ is self-adjoint and AB is self-adjoint iff A and B commute. Finally, if $A_n \to A$ in the sense of norm convergence in the operator space or in the sense of weak convergence and all A_n are self-adjoint operators, then A is also self-adjoint.

If we consider (Ax, y), A a self-adjoint operator, as a functional of both x and y, then it is easy to verify that this functional, written as $A(x, y)$, satisfies the conditions

$$A(\alpha x_1+\beta x_2, y) = \alpha A(x_1, y) + \beta A(x_2, y),$$
$$A(x, y) = \overline{A(y, x)}.$$

Such a functional is said to be a **bilinear Hermitian form.** This form is bounded in the sense, that

$$|A(x, y)| \leqslant C_A \| x \| \cdot \| y \|,$$

C_A some constant (in the case considered, $C_A = \| A \|$).

Thus, *every self-adjoint operator A generates some bounded bilinear Hermitian form*

$$A(x, y) = (Ax, y) = (x, Ay).$$

Conversely, *if a bounded bilinear Hermitian form $A(x, y)$ is given, then it generates some self-adjoint operator A, satisfying the equality*

$$A(x, y) = (Ax, y).$$

In fact, keeping y fixed in $A(x, y)$, we obtain a linear functional of x. Consequently,

$$A(x, y) = (x, y^*),$$

y^* a uniquely defined element. Thus, we get an operator A, defined by

$$Ay = y^*, \qquad \text{and such that } (x, Ay) = A(x, y).$$

The linearity of A is obvious and its boundedness is easily verifiable. In fact,

$$|(x, Ay)| = |A(x, y)| \leqslant C_A \| x \| \cdot \| y \|.$$

Putting $x = Ay$ and contracting to $\| Ay \|$, we find

$$\| Ay \| \leqslant C_A \| y \|.$$

It is to be shown that A is self-adjoint. For every x and $y \in H$, we have $(x, Ay) = \overline{A(y, x)} = \overline{(y, Ax)} = (Ax, y)$, implying that $A = A^*$ and $A(x, y) = (Ax, y)$.

7.14. Quadratic forms. Put $y = x$ in a bilinear Hermitian form $A(x, y)$, to receive a quadratic form $A(x, x)$, which assumes real values for all x, and satisfies

$$A(\alpha x+\beta y, \alpha x+\beta y) = \alpha\bar{\alpha}A(x, x) + \bar{\alpha}\beta[A(x, y) + \bar{\alpha}\beta A(y, x) + \beta\bar{\beta}A(y, y)].$$

$A(x, x)$ is then called a **quadratic Hermitian form,** which corresponds to the Hermitian form $A(x, y)$. To every bilinear Hermitian form $A(x, y)$, a quadratic Hermitian form $A(x, x)$ can be associated. The converse is also true: the bilinear Hermitian form $A(x, y)$ is uniquely defined by a quadratic Hermitian form $A(x, x)$. This bilinear form is defined by

$$A(x, y) = \tfrac{1}{4} \{[A(x_1, x_1) - A(x_2, x_2)] + i[A(x_3, x_3) - A(x_4, x_4)]\},$$

where $\quad x_1 = x + y, \; x_2 = x - y, \;$ and $\; x_3 = x + iy, \; x_4 = x - iy.$

It is easy to show that the quadratic form $A(x, x)$ is bounded, that is,

$$|A(x, x)| \leqslant C_A \| x \|^2,$$

iff the corresponding bilinear Hermitian form is bounded.

Let $m = \inf\limits_{\| x \| = 1} (Ax, x)$ and $M = \sup\limits_{\| x \| = 1} |(Ax, x)$. The numbers m and M are called the **greatest lower** and the **least upper bounds** of the self-adjoint operator A. Show that

$$\| A \| = \max(| m |, | M |) = \sup_{\| x \| = 1} |(Ax, x)|.$$

In fact, let $\| x \| = 1$. Then,

$$|(Ax, x)| \leqslant \| Ax \| \cdot \| x \| \leqslant \| A \| \cdot \| x \|^2 = \| A \|,$$

and, consequently,

$$C_A = \sup_{\| x \| = 1} |(Ax, x)| \leqslant \| A \|. \tag{4}$$

On the other hand, for every $y \in H$, we have

$$(Ay, y) \leqslant C_A \; \| y \|^2.$$

Hence, if z is any element in H, different from zero, then putting

$$\lambda = \left(\frac{\| Az \|}{\| z \|} \right)^{1/2} \quad \text{and} \quad u = \frac{1}{\lambda} Az,$$

we get

$$\begin{aligned} \| Az \|^2 &= (A(\lambda z), u) = \tfrac{1}{4} \{(A(\lambda z+u), \lambda z+u) - (A(\lambda z-u), \lambda z-u)\} \\ &\leqslant \tfrac{1}{4} C_A \{\| \lambda z+u \|^2 + \| \lambda z-u \|^2\} = \tfrac{1}{2} C_A \{\| \lambda z \|^2 + \| u \|^2\} \\ &= \tfrac{1}{2} C_A \{\lambda^2 \| z \|^2 + (1/\lambda^2) \| Az \|^2 \} = C_A \| z \| \cdot \| Az \|, \end{aligned}$$

whence

$$\| Az \| \leqslant C_A \| z \|,$$

and, consequently,

$$\| A \| \leqslant C_A = \sup_{\| x \| = 1} | (Ax, x) |. \tag{5}$$

The desired equality is immediate from (4) and (5).

In particular, it follows from what has been stated that if the operators A and B be self-adjoint for all $x \in H$, satisfying

$$(Ax, x) = (Bx, x),$$

then $A = B$.

7.2. UNITARY OPERATORS. PROJECTION OPERATORS

THIS SECTION discusses two special classes of operators in a HILBERT space.

7.21. Unitary Operators. A linear operator U is called **unitary** if it maps a HILBERT space H onto all of H with preservation of the norm, that is, if

$$\| Ux \| = \| x \|. \tag{1}$$

It is plain that this mapping is one-one, since if $Ux_1 = Ux_2$, that is, if $U(x_1-x_2) = 0$, then $\| x_1-x_2 \| = \| U(x_1-x_2 \| = 0$ and $x_1 = x_2$. Hence, the operator U^{-1}, the inverse of the unitary operator, exists and this is also, evidently, unitary.

Furthermore, Eq. (1) yields $(Ux, Ux) = \| Ux \|^2 = \| x \|^2 = (x, x)$, whence $(U^*Ux, x) = (x, x) = (Ex, x)$; E denotes here as well throughout this chapter the identity operator. Since the quadratic forms of the operators U^*U and E are equal, these operators coincide†

$$U^*U = E. \tag{2}$$

Multiply (2) by U from left and U^{-1} from right, to receive

$$UU^* = E. \tag{3}$$

Thereupon, $U^* = U^{-1}$. From (2) it also follows that $(Ux, Uy) = (x, y)$. Conversely, from the conditions (2) and (3) it follows that U is a unitary operator, since these imply that $U^{-1} = U^*$ exists and, consequently, H is

†Note that for every linear operator A, the operator A^*A is self-adjoint.

mapped one-one onto H and that

$$\| Ux \|^2 = (Ux, Ux) = (U^*Ux, x) = (x, x) = \| x \|^2,$$

that is, U preserves the norm of the element.

By way of an example of a unitary operator in the coordinate HILBERT space l_2, we can take an infinite **unitary matrix** (u_{ij}), that is, such that its matrix elements satisfy the relations

$$\sum_{k=1}^{\infty} u_{k_i}\bar{u}_{k_j} = \delta_{ij}, \qquad \sum_{l=1}^{\infty} u_{il}\bar{u}_{jl} = \delta_{ij}. \tag{4}$$

Given a linear operator A acting in a HILBERT space, and a unitary operator U. The operator

$$B = UAU^{-1} = UAU^* \tag{5}$$

is called an operator **unitarily equivalent** to A. It is obvious from (5) that the operator, unitarily equivalent to a self-adjoint operator, is itself self-adjoint.

It is easy to verify that the norms of unitarily equivalent operators are equal.

7.22. Projection operators. Let L be a (closed) subspace of H. Every element $x \in H$ is uniquely representable in the form

$$x = y + z,$$

where $y \in L$, $z \perp L$. Put $Px = y$, to receive an operator defined on the whole of H with range in L. This operator is called the **operator of orthogonal projection upon** L or, simply, a **projection operator** (or **projector**) and denoted by P_L.

THEOREM 1 (*Projection Theorem*). *P is a self-adjoint operator with its norm equal to one and satisfies $P^2 = P$.*

In the first place, P is a linear operator. In fact, if $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$, where $y_1, y_2 \in L$ and $z_1, z_2 \perp L$, then

$$\alpha x_1 + \beta x_2 = (\alpha y_1+\beta y_2) + (\alpha z_1+\beta z_2),$$

where $$\alpha y_1 + \beta y_2 \in L, \qquad \alpha z_1 + \beta z_2 \perp L,$$

whence $$P(\alpha x_1+\beta x_2) = \alpha y_1 + \beta y_2 = \alpha Px_1 + \beta Px_2.$$

Further, owing to the orthogonality of y and z,

$$\| x \|^2 = \| y+z \|^2 = (y+z, y+z) = \| y \|^2 + \| z \|^2.$$

Consequently, $\| y \| \leqslant \| x \|$, that is, $\| Px \| \leqslant \| x \|$ for every x. Thereupon, $\| P \| \leqslant 1$. Since for $x \in L$, we have $Px = x$ and, consequently, $\| Px \| = \| x \|$, it follows that $\| P \| = 1$.

It remains to exhibit that P is a self-adjoint operator. Let x_1 and x_2 be any two elements in H and let y_1 and y_2 be their projections on L. We have

$$(Px_1, x_2) = (y_1, x_2) = (y_1, y_2).$$

Analogously, $(x_1, Px_2) = (x_1, y_2) = (y_1, y_2)$.

Consequently, $(Px_1, x_2) = (x_1, Px_2)$.

Finally, $Px \in L$ for every $x \in H$. Hence

$$P^2x = P(Px) = Px$$

for every $x \in H$, that is, $P^2 = P$.

The upcoming theorem shows that the converse is also true.

THEOREM 2. *Every self-adjoint operator P satisfying $P^2 = P$ is an orthogonal projection on some space L.*

Consider the set L of elements of the form $y = Px$, where x runs throughout H. Because of the additivity and homogeneity of P, L is a linear manifold. It is easy to show that L is closed. In fact, let $y_n \to y_0, y_n \in L$. Since $y_n \in L$, then $y_n = Px_n$ for some $x_n \in H$. Hence $Py_n = P^2x_n = Px_n = y_n$.

Owing to the continuity of P, $y_n \to y_0$ implies $Py_n \to Py_0$. However, since $Py_n = y_n$, hence $y_n \to Py_0$. Consequently, $y_0 = Py_0$, that is, $y_0 \in L$. Finally, $x - Px \perp Px$, since P is self-adjoint and the condition $P^2 = P$ implies that

$$(x - Px, Px) = (Px - P^2x, x) = 0.$$

Now it follows from the definition of L that P is the projection onto this subspace, as was required to prove. Note also that L consists of exactly those points $x \in H$, which satisfy $Px = x$.

From what has been proved it follows that together with P, $I - P$ is also a projection operator.

We shall now mention certain properties of projection operators. Two projection operators P_1 and P_2, are *orthogonal*, if $P_1 P_2 = 0$.† This is equivalent to $P_2 P_1 = 0$, since if $P_1P_2 = 0$, then $(P_1 P_2)^* = P_2P_1 = 0$ and vice versa.

THEOREM 3. *For the projections P_1 and P_2 to be orthogonal, it is necessary and sufficient that the corresponding subspaces L_1 and L_2 are orthogonal.*

PROOF. In fact, if $P_1P_2 = 0$, then

$$(x_1, x_2) = (P_1x_1, P_2x_2) = (P_2P_1x_1, x_2) = (0, x_2) = 0$$

for $x_1 \in L_1$, $x_2 \in L_2$. Conversely, if $L_1 \perp L_2$, then $P_2x \in L_2$ for every $x \in H$ and, consequently, $P_1P_2x = 0$, that is, $P_1P_2 = 0$.

LEMMA. 1. *In order that the sum of two projections P_{L_1} and P_{L_2} be a projection operator, it is necessary and sufficient that these operators are orthogonal. In this case $P_{L_1} + P_{L_2} = P_{L_1 \dot{+} L_2}$.*

Necessity. Let $P = P_{L_1} + P_{L_2}$ be a projection operator. Then,

$$(P_{L_1} + P_{L_2})^2 = P_{L_1} + P_{L_2}, \text{ whence } P_{L_1}P_{L_2} + P_{L_3}P_{L_1} = 0.$$

†Now and onwards 0 stands not only for the null number but also for the null operator,

Multiply on the left by P_{L_1}, to receive

$$P_{L_1}P_{L_2} + P_{L_1}P_{L_2}P_{L_1} = 0;$$

and multiplying now on the right by P_{L_1}, we have,

$$P_{L_1}P_{L_2}P_{L_1} = 0.$$

However, then, $P_{L_2}P_{L_1} = 0$.

Sufficiency. Let $P_{L_1}P_{L_2} = P_{L_2}P_{L_1} = 0$. Then

$$(P_{L_1} + P_{L_2})^2 = P_{L_1} + P_{L_2}.$$

Consequently, $P_{L_1} + P_{L_2}$ is a projection operator.

Since $P_{L_1}P_{L_2} = 0$ by hypothesis, the subspaces L_1 and L_2 are orthogonal. If $x \in H$, then

$$Px = P_{L_1}x + P_{L_2}x = x_1 + x_2 \text{ with } x_1 + x_2 \in L_1 \dotplus L_2. \tag{6}$$

Further, if $x = x_1 + x_2$ is an element in $L_1 \dotplus L_2$, then taking note of the equalities $P_{L_1}x_2 = 0$, $P_{L_2}x_1 = 0$, we get

$$\begin{aligned} x = x_1 + x_2 = P_{L_1}x_1 + P_{L_2}x_2 &= P_{L_1}(x_1 + x_2) + P_{L_2}(x_1 + x_2) \\ &= (P_{L_1} + P_{L_2})\, x. \end{aligned} \tag{7}$$

From (6) and (7) it follows that P is a operator of projection on $L_1 \dotplus L_2$, completely proving the lemma.

LEMMA. 2. *In order that the product of two projections P_{L_1} and P_{L_2} be a projection operator, it is necessary and sufficient that the operators P_{L_1} and P_{L_2} commute, that is, $P_{L_1}P_{L_2} = P_{L_2}P_{L_1}$. In this case, $P_{L_1}P_{L_2} = P_{L_1 \cap L_2}$.*

Neccessity. Since $P = P_{L_1}P_{L_2}$ is self-adjoint, hence

$$P_{L_1}P_{L_2} = (P_{L_1}P_{L_2})^* = P^*_{L_1}P^*_{L_2} = P_{L_2}P_{L_1},$$

and the commutativity is proved.

Sufficiency. If $P_{L_1}P_{L_2} = P_{L_2}P_{L_1}$, then $P = P_{L_1}P_{L_2}$ is self-adjoint. Furthermore,

$$(P_{L_1}P_{L_2})^2 = P_{L_1}P_{L_2}P_{L_1}P_{L_2} = P^2_{L_1}P^2_{L_2} = P_{L_1}P_{L_2},$$

and, consequently, P is a projection operator.

Let $x \in H$ be arbitrary. Then,

$$Px = P_{L_1}P_{L_2}x = P_{L_2}P_{L_1}x$$

belongs also to L_1 and L_2, that is, to $L_1 \cap L_2$. Now, let $y \in L_1 \cap L_2$. Then,

$$Py = P_{L_1}(P_{L_2}y) = P_{L_1}y = y.$$

All this signifies that P is a projection on $L_1 \cap L_2$, proving the lemma.

The projection operator P_2 is said to be a *part* of the projection P_1, if

$P_1P_2 = P_2$. If extended to the adjoint operator, it can be verified that this definition is equivalent to $P_2P_1 = P_2$, which directly shows that P_{L_2} is a part of P_{L_1}, iff L_2 is a subspace of L_1.

THEOREM 4. *For a projection operator P_{L_2} to be a part of the projection operator P_{L_1}, it is necessary and sufficient that the inequality $\| P_{L_2}x \| \leqslant \| P_{L_1}x \|$ is satisfied for all $x \in H$.*

In fact, $P_{L_2}P_{L_1}x = P_{L_2}x$ implies that

$$\| P_{L_2}x \| \leqslant \| P_{L_2} \| \cdot \| P_{L_1}x \| \leqslant \| P_{L_1}x \|.$$

Conversely, if this condition is satisfied, then, for every $x \in L_2$, we have

$$\| P_{L_1}x \| \geqslant \| P_{L_2}x \| = \| x \|,$$

and since $\| P_{L_1}x \| \leqslant \| x \|$ is also true, $\| P_{L_1}x \| = \| x \|$. Thereupon, $\| P_{H+L_1}x \|^2 = \| x \|^2 - \| P_{L_1}x \|^2 = 0$, and, consequently, $x \in L_1$. Therefore, $P_{L_2}x \in L_1$ for every $x \in H$, implying that $P_{L_1}P_{L_2}x = PL_2x$, that is, $P_{L_1}P_{L_2} = P_{L_2}$, as was required to prove.

LEMMA 3. *The difference $P_1 - P_2$ of two projections is a projection operator, iff P_2 is a part of P_1. In this case $L_{P_1-P_2}$ is the orthogonal complement of L_{P_2}* in L_{P_1}.

Necessity. If $P_1 - P_2$ is a projection operator, then

$$E - (P_1 - P_2) = (E - P_1) + P_2$$

is also a projection operator. However, then, by Lemma 1,

$$(E - P_1) P_2 = 0, \quad \text{that is, } P_1 P_2 = P_2 .$$

Sufficiency. Let P_2 be a part of P_1. Then, $E - P_1$ is orthogonal to P_2; by Lemma 1, $(E - P_1) + P_2$ is a projection operator and, consequently, so also is $P_1 - P_2$. Finally, the condition $P_1P_2 = P_2$ implies that $P_1 - P_2$ and P_2 are orthogonal. Howrver, then, because of Lemma 1,

$$L_{P_1} = L_{P_1-P_2} \dot{+} L_{P_2}. \qquad \blacksquare$$

7.3. POSITIVE OPERATORS. SQUARE ROOTS OF POSITIVE OPERATORS

7.31. Positive operators. A self-adjoint operator A is said to be **positive**, $A > 0$, if it is different from zero and its lower bound is non-negative, that is, if $(Ax, x) \geqslant 0$ for every $x \in H$ and $(Ax, x) > 0$ even for a single $x \in H$. If $A - B > 0$, then the self-adjoint operator A is said to be *greater* than the self-adjoint operator B, $A > B$; also that B is *smaller* than A. It is easy to verify that the inequality relation introduced in the set of self-adjoint operators has the following properties†:

(i) $A \geqslant B$ and $C \geqslant D$ imply $A + C \geqslant B + D$;

†The inequality $A \geqslant B$ signifies that either $A > B$ or $A = B$.

(ii) $A \geqslant 0$ and $\alpha \geqslant 0$ imply $\alpha A \geqslant 0$;

(iii) $A \geqslant B$ and $B \geqslant C$ imply $A \geqslant C$;

(iv) If $A \geqslant 0$ and A^{-1} exists, then $A^{-1} > 0$.

Further, it is obvious that AA^* and A^*A are positive operators for every linear operator A different from zero. In particular, $A^2 > 0$ for any self-adjoint operator A, $A \neq 0$. The latter implies that an operator of projection on a subspace of positive dimension serves as an example of positive operators.

THEOREM 1. *If two positive self-adjoint operators A and B commute, then their product is also a positive operator.*

PROOF. Put

$$A_1 = \frac{A}{\| A \|}, \quad A_2 = A_1 - A_1^2, \ldots, A_{n+1} = A_n - A_n^2, \ldots$$

and show that
$$0 \leqslant A_n \leqslant E \tag{1}$$
for every n. This is trivial for $n = 1$. Suppose that (1) is true for $n = k$. Then,

$$(A_k^2 (E-A_k) x, x) = [(E-A_k) A_k x, A_k x] \geqslant 0,$$

that is, $A_k^2 (E-A_k) \geqslant 0$. Analogously, $A_k(E-A_k)^2 \geqslant 0$. Hence,

$$A_{k+1} = A_k^2 (E-A_k) + A_k(E-A_k)^2 \geqslant 0,$$

and
$$E - A_{k+1} = (E-A_k) + A_k^2 \geqslant 0.$$

Consequently, (1) holds for $n = k + 1$.

Furthermore,

$$A_1 = A_1^2 + A_2 = A_1^2 + A_2^2 + A_3 = \ldots = A_1^2 + A_2^2 + \ldots + A_n^2 + A_{n+1},$$

whence
$$\sum_{k=1}^{n} A_k^2 = A_1 - A_{n+1} \leqslant A_1,$$

(since $A_{n+1} \geqslant 0$), that is, $\sum\limits_{k=1}^{n} (A_k x, A_k x) \leqslant (A_1 x, x)$. Consequently, the series $\sum\limits_{k=1}^{\infty} \| A_k x \|^2$ converges and $\| A_k x \| \to 0$ as $k \to \infty$. Hence,

$$\left(\sum_{k=1}^{n} A_k^2 \right) x = A_1 x - A_{n+1} x \to A_1 x.$$

Since B, evidently, commutes with all the A_k, it follows that

$$(ABx, x) = \| A \| (BA_1x, x) = \| A \| \lim_n \sum_{k=1}^{n} (BA_k^2 x, x)$$

$$= \| A \| \lim_n \sum_{k=1}^{n} (BA_kx, A_kx) \geqslant 0.$$

The theorem is proved and leads to:

THEOREM 1′. *If $\{A_n\}$ is a monotone increasing sequence of mutually commuting self-adjoint operators, bounded above by a self-adjoint operator B commuting with all the A_n:*

$$A_1 \leqslant A_2 \leqslant \ldots \leqslant A_n \leqslant \ldots \leqslant B,$$

then the sequence $\{A_n\}$ converges to a self-adjoint operator A and $A \leqslant B$. (An analogous statement holds for monotone decreasing sequences).

In fact, consider the self-adjoint operators $C_n = B - A_n$. They are positive, commute and form a monotone decreasing sequence. Consequently, for $m < n$, the operators $(C_m - C_n)\, C_m$ and $C_n(C_m - C_n)$ are also positive, whence

$$(C_m^2\, x, x) \geqslant (C_mC_nx, x) \geqslant (C_n^2\, x, x) \geqslant 0.$$

This implies that the monotone decreasing non-negative numerical sequence $\{(C_n^2x, x)\}$ has a limit. By the inequalities just obtained, the sequence $\{(C_mC_nx, x)\}$ also tends to the same limit as $n, m \to \infty$. Therefore,

$$\| C_mx - C_nx \|^2 = [(C_m - C_n)^2x, x] = (C_m^2\, x, x) - 2(C_mC_nx, x) + (C_n^2\, x, x) \to 0,$$

as $n, m \to \infty$. Thus, the sequence $\{C_nx\}$ and thereby also $\{A_nx\}$ converge to some limit Ax for arbitrary x, that is, $Ax = \lim_n A_nx$. Obviously, A is a self-adjoint operator, satisfying $A \leqslant B$. ■

7.32. Square roots of positive operators. The self-adjoint operator B is called a **square root** of the positive operator A, if $B^2 = A$. A theorem on the existence and uniqueness of square roots now follows.

THEOREM 2. *There exists a unique positive square root B of every positive self-adjoint operator A; it commutes with every operator commuting with A.*

PROOF. Without loss of generality, it can be assumed that $A \leqslant E$. Put $B_0 = 0$ and

$$B_{n+1} = B_n + \frac{1}{2}\,(A - B_n^2), \qquad n = 0, 1, \ldots \tag{2}$$

All the operators B_n are obviously self-adjoint, positive and commute with every operator commuting with A; in particular, the B_n commute with each other (that is, $B_nB_m = B_mB_n$). It is easy to verify that

$$E - B_{n+1} = \frac{1}{2}(E - B_n)^2 + \frac{1}{2}(E - A), \tag{3}$$

and
$$B_{n+1} - B_n = \frac{1}{2}[(E - B_{n-1}) + (E - B_n)](B_n - B_{n-1}). \tag{4}$$

From (3) it follows that $B_n \leqslant E$ for all n. Then, also $B_n \leqslant B_{n+1}$, which is evident for $n = 0$ in view of the inequality

$$B_1 = \tfrac{1}{2}A > 0 = B_0.$$

Further, Eq. (4) exhibits that $B_{n+1} - B_n \geqslant 0$, if $B_n - B_{n-1} \geqslant 0$. Consequently, $B_n \leqslant B_{n+1}$ for all n. Thus, $\{B_n\}$ is a bounded monotone increasing sequence of self-adjoint positive operators. This sequence converges in limit to some self-adjoint positive operator B. Taking limit, Eq. (2) yields

$$B = B + \tfrac{1}{2}(A - B^2), \quad \text{that is, } B^2 = A.$$

Finally, B commutes with any operator that commutes with A, implied by the fact that the B_n possess this property. Thus, the operator B has all the requisite properties, and the existence of a positive square root of A is proved.

Let B_1 be another positive square root of A, which commutes with A. Then, $B_1B = BB_1$. Hence, if x is any element in H and $y = (B - B_1)x$, then

$$\begin{aligned}(By, y) + (B_1y, y) &= [(B + B_1)y, y] = [(B + B_1)(B - B_1)x, y)] \\ &= [(B^2 - B_1^2)x, y] = 0.\end{aligned}$$

Since B and B_1 are positive, $(By, y) = (B_1y, y) = 0$. However, since the roots are positive, we have $B = C^2$, where C is a self-adjoint operator. Since

$$\| Cy \|^2 = (C^2y, y) = (By, y) = 0, \qquad \text{hence } Cy = 0.$$

Consequently, $By = C(Cy) = 0$ and analogously $B_1y = 0$. However, then, $\| B_1x - Bx \|^2 = [(B - B_1)^2x, x] = [(B - B_1)y, x] = 0$, that is, $Bx = B_1x$ for every $x \in H$ and the uniqueness of the square root is proved.

Example. In the space $L_2[0, 1]$, the operator A defined by $Ax(t) = tx(t)$ has the positive square root B, where $Bx(t) = +\sqrt{t}\,x(t)$.

7.4 SPECTRUM OF SELF-ADJOINT OPERATORS

CONSIDER A family of operators $A_\lambda = A - \lambda E$, A self-adjoint and λ a complex number.

Theorem 2 of Chap. 3.5 implies that: If $\| (1/\lambda)A \| < 1$ (that is, if $|\lambda| > \| A \|$), then λ is a **regular** value of A and, consequently, the entire spectrum of A lies inside and on the boundary of the circle $|\lambda| \leqslant \| A \|$. This is true for arbitrary linear operators acting into a BANACH space. For a self-adjoint operator defined on a HILBERT space, the plane comprising the spectrum of the operator is indicated more precisely in what follows.

If A is a self-adjoint operator, then all of its eigenvalues are real, since $Ax = \lambda x$ implies $(Ax, x) = \lambda(x, x)$, where both the scalar products (Ax, x) and (x, x) are real. Furthermore, the condition $A = A^*$, the fact that the eigenvalues are real and Theorem 2 of Chap. 4.3 imply that all eigenvectors corresponding to distinct eigenvalues of a self-adjoint operator are orthogonal.

THEOREM 1. *For the point λ to be a regular value of the self-adjoint operator A, it is necessary and sufficient that there is a positive constant c, such that*

$$\| A_\lambda x \| = \| Ax-\lambda x \| \geqslant c \| x \|, \tag{1}$$

for every $x \in H$.

Necessity. Suppose that there is a bounded operator $R_\lambda = A_\lambda^{-1}$ and that $\| R_\lambda \| = d$. For every $x \in H$, we have $\| x \| = \| R_\lambda A_\lambda x \| \leqslant d \| A_\lambda x \|$, whence $\| A_\lambda x \| \geqslant (1/d) \| x \|$, proving the necessity.

Sufficiency. Let $y = Ax - \lambda x$ and x run through H. Then y runs through some linear manifold L. By (1) there is a one-one correspondence between x and y, since if x_1 and x_2 go over to one and the same element y, then

$$A(x_1-x_2) - \lambda(x_1-x_2) = 0,$$

whence
$$\| x_1-x_2 \| \leqslant (1/c) \| A_\lambda (x_1-x_2) \| = 0.$$

Show that L is everywhere dense in H, In fact, if it were not so, then there would exist an element $x_0 \in H$, different from zero and such that $(x_0, y) = 0$ for every $y \in L$, implying $(x_0, Ax-\lambda x) = 0$, whence $(Ax_0-\bar{\lambda} x_0, x) = 0$, A being a self-adjoint operator, and since it would be true for every $x \in H$, then

$$Ax_0 - \bar{\lambda} x_0 = 0.$$

But this equality for x_0 different from zero is impossible, either for complex λ (because, then the eigenvalues of a self-adjoint operator would be complex), or for real λ (because, then $\bar{\lambda} = \lambda$ and $\| x_0 \| \leqslant (1/c) \| Ax_0-\lambda A x_0 \| = 0$).

Finally, it is to be shown that L is closed. Let $\{y_n\} \subset L$, $y_n = A_\lambda x_n$ and $y_n \to y_0$. By (1),

$$\| x_n - x_m \| \leqslant (1/c) \| A_\lambda x_n - A_\lambda x_m \| = (1/c) \| y_n - y_m \| .$$

$\{y_n\}$ is a CAUCHY sequence and, hence, $\| y_n - y_m \| \to 0$ as $n, m \to +\infty$. However, then, $\| x_n - x_m \| \to 0$ as $n, m \to \infty$. Since H is a complete space, there exists a limit for $\{x_n\}$: $x = \lim_n x_n$. Moreover,

$$A_\lambda x = \lim_n A_\lambda x_n = \lim_n y_n = y, \text{ that is, } y \in L.$$

Thus, L is a closed linear manifold everywhere dense in H, that is, $L = H$. In addition, since the correspondence $y = A_\lambda x$ is one-one, there exists the inverse operator $x = A_\lambda^{-1} y = R_\lambda y$ defined on the entire H. Ineq. (1) yields

$$\| R_\lambda y \| = \| x \| \leqslant (1/c) \| A_\lambda x \| = (1/c) \| y \|$$

that is, R_λ is a bounded operator and

$$\| R_\lambda \| \leqslant 1/c.$$

COROLLARY. *The point λ belongs to the spectrum of a self-adjoint operator A, iff there exists a sequence $\{x_n\}$, such that*

$$\| Ax_n - \lambda x_n \| \leqslant c_n \| x_n \|, \quad c_n \to 0 \text{ as } n \to \infty. \tag{2}$$

Put $\| x_n \| = 1$ in (2), to receive

$$\| Ax_n - \lambda x_n \| \to 0, \qquad \| x_n \| = 1. \tag{3}$$

THEOREM 2. *The complex numbers $\lambda = \alpha + i\beta$ with $\beta \neq 0$ are regular values of a self-adjoint operator A.*

In fact, if $y = A_\lambda x = Ax - \lambda x$, then $(y, x) = (Ax, x) - \lambda (x, x)$; $(x, y) = \overline{(y, x)} = (Ax, x) - \bar{\lambda}(x, x)$. Hence $(x, y) - (y, x) = (\lambda - \bar{\lambda})(x, x) = 2i\beta \| x \|^2$, or, $2 | \beta | \| x \|^2 = | (x, y) - (y, x) | \leqslant | (x, y) | + | (y, x) | \leqslant 2 \| y \| \| x \|$ and, therefore,

$$\| y \| \geqslant | \beta | \| x \|, \quad \text{that is, } \| A_\lambda x \| \geqslant | \beta | \| x \|. \tag{4}$$

For completing the proof, it sufficcs to make use of Theorem 1.

THEOREM 3. *The spectrum of a self-adjoint operator A lies entirely on a segment $[m, M]$ of the real axis, where*

$$M = \sup_{\| x \| = 1} (Ax, x) \quad \text{and} \quad m = \inf_{\| x \| = 1} (Ax, x).$$

From Theorem 2 it follows that the spectrum can lie only on the real axis. Now, it is to be shown that the real λ lying outside of $[m, M]$ are regular.

For example, let $\lambda > M$, thus $\lambda = M + d$ with $d > 0$. We have

$$(A_\lambda x, x) = (Ax, x) - \lambda (x, x) \leqslant M(x, x) - \lambda (x, x) = - d \| x \|^2,$$

whence $$| (A_\lambda x, x) | \geqslant d \| x \|^2.$$

On the other hand, $| (A_\lambda x, x) | \leqslant \| A_\lambda x \| \cdot \| x \| \cdot$ Consequently,

$$\| A_\lambda x \| \geqslant d \| x \|,$$

evidencing the regularity of λ. The case $\lambda < m$ is tackled on similar lines.

THEOREM 4. *M and m belong to the point spectrum.*

We prove the theorem, say for the number M.

REMARK. If A is replaced by A_μ, then the spectrum is shifted by μ to the left and M and m change into $M - \mu$ and $m - \mu$. Thus, without loss of generality, it can be assumed that $0 \leqslant m \leqslant M$. Then, $M = \| A \|$ (see p. 207). It is to be exhibited that M is the point spectrum.

In fact, by definition of $M = \| A \|$, there is a sequence $\{x_n\}$ with $\| x_n \| = 1$, such that

$$(Ax_n, x_n) = M - \delta_n, \quad \delta_n \to 0 \text{ as } n \to \infty.$$

Further, $$\| Ax_n \| \leqslant \| A \| \| x_n \| = \| A \| = M.$$

Therefore, $\quad \| Ax_n - Mx_n \|^2 = (Ax_n - Mx_n, Ax_n - Mx_n)$

$$= \| Ax_n \|^2 - 2M(Ax_n, x_n) + M^2 \| x_n \|^2$$
$$\leqslant M^2 - 2M(M-\delta_n) + M^2 = 2M\delta_n,$$

or, $\quad \| Ax_n - Mx_n \| \leqslant \sqrt{2M\delta_n}.$

Consequently, $\quad \| Ax_n - Mx_n \| \to 0, \quad \| x_n \| = 1,$

which together with the Corollary to Theorem 1 shows that M belongs to the spectrum.

COROLLARY. *The spectrum of a self-adjoint operator is not empty.*†

Examples. 1. If A is the identity operator E, then its spectrum consists of the single eigenvalue 1 for which the corresponding eigenspace is $H_1 = H$. $R_\lambda = [1/(\lambda - 1)]\, E$ is a bounded operator for $\lambda \neq 1$.

2. The operator A in $(L_2[0, 1] \to L_2\,[0, 1])$ is defined by

$$Ax(t) = tx(t), \qquad 0 \leqslant t \leqslant 1.$$

Obviously, $m = 0$ and $M \leqslant 1$. Let us show that all the points of the segment $[0, 1]$ belong to the spectrum of A, implying that $M = 1$.

In fact, let $0 \leqslant \lambda \leqslant 1$ and $\varepsilon > 0$. Consider the interval $[\lambda, \lambda + \varepsilon]$ (or, $[\lambda - \varepsilon, \lambda]$) lying in $[0, t]$. Let

$$x_\varepsilon(t) = \begin{cases} 1/\sqrt{\varepsilon} & \text{for } t \in [\lambda, \lambda + \varepsilon], \\ 0 & \text{for } t \bar{\in} [\lambda, \lambda + \varepsilon]. \end{cases}$$

Since
$$\int_0^1 x_\varepsilon^2(t)\, dt = \int_\lambda^{\lambda+\varepsilon} \frac{1}{\varepsilon}\, dt = 1,$$

hence
$$x_\varepsilon(t) \in L_2\,[0, 1], \qquad \| x_\varepsilon \| = 1.$$

Furthermore,
$$A_\lambda x_\varepsilon(t) = (t-\lambda)\, x_\varepsilon(t),$$

whence
$$\| A_\lambda \; x_\varepsilon(t) \|^2 = \frac{1}{\varepsilon} \int_\lambda^{\lambda+\varepsilon} (t-\lambda)^2\, dt = \frac{\varepsilon^2}{3}.$$

We have $\| A_\lambda x_\varepsilon \| \to 0$ as $\varepsilon \to 0$. Consequently, $\lambda, 0 \leqslant \lambda \leqslant 1$, is the point spectrum.

At the same time, the operator A has no eigenvalue. In fact,

$$A_\lambda\, x(t) = (t-\lambda)\, x(t).$$

If $A_\lambda\, x(t) = 0$, then $(t-\lambda)\, x(t) \to 0$ almost everywhere on $[0, 1]$, and thus $x(t)$, is also equal to zero almost everywhere.

7.41. Invariant subspaces. A subspace L of H is called **invariant** under an operator A, if $x \in L$ implies $Ax \in L$. Let us illustrate this by concrete examples. Let λ be the eigenvalue of A and N_λ the collection of eigenvectors corresponding to this eigenvalue, to which the zero element is adjoined. N_λ is an

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†It has been demonstrated in the theory of normed rings [10] that the spectrum of an arbitrary bounded operator defined on a BANACH space is also not empty.

invariant subspace, since because of the equality $Ax = \lambda x$, $x \in N_\lambda$ implies that $Ax \in N_\lambda$.

If the subspace L is invariant under A, then we say that L **reduces** the operator A. Some properties of subspaces invariant under self-adjoint operators are given in what follows.

(i) *The invariance of L implies the invariance of its orthogonal complement* $M = H \dot{-} L$.

Let $x \in M$, implying $(x, y) = 0$ for every $y \in L$. However, Ay also belongs to L for $y \in L$ and, hence, $(x, Ay) = 0$. Thereupon, since A is self-adjoint, $(Ax, y) = 0$ for every $y \in L$. Consequently, $Ax \in M$, as was required to prove.

Denote by G_λ the range of the operator A_λ, that is, the collection of elements of the form $y = Ax - \lambda x$, λ an eigenvalue. It is easily verifiable that $H = \bar{G}_\lambda \dot{+} N_\lambda$. In fact, if $y \in G_\lambda$, $u \in N_\lambda$, then $(y, u) = (Ax - \lambda x, u) = (x, Au - \lambda u) = (x, 0) = 0$. Consequently, $G_\lambda \perp N_\lambda$. If $y \in \bar{G}_\lambda$ and $y \bar{\in} G_\lambda$, then $y = \lim\limits_n y_n$, where $y_n \in G_\lambda$. The equality $(y_n, u) = 0$ implies that $(y, u) = \lim\limits_n (y_n, u) = 0$. Consequently, $\bar{G}_\lambda \perp N_\lambda$.

Now, let $(y, u) = 0$ for every $y \in G_\lambda$. For any $x \in H$, $0 = (Ax - \lambda x, u) = (x, Au - \lambda u)$, whence $Au - \lambda u = 0$, that is, $u \in N_\lambda$. Consequently,

$$N_\lambda = H \dot{-} G_\lambda = H \dot{-} \bar{G}_\lambda. \qquad \blacksquare$$

The following proposition is immediate from property (i) just established: $\bar{G}_\lambda$ *is an invariant subspace under a self-adjoint operator A.*

Denote by N the orthogonal sum of all the subspaces N_λ or, what is the same, a closed linear span of all the eigenvectors of the operetor A. This is also an invariant subspace of the given operator. If H is separable, then it is possible to construct in every N_λ a completely finite or countable orthonormal system of eigenvectors. Since the eigenvectors of distinct N_λ are orthogonal, hence by uniting these systems, we obtain an orthogonal system of eigenvectors $\{x_n\}$, contained completely in the space N.

The operator A defines in the invariant subspace L an operator A_L in $(L \to L)$; namely, $A_L x = Ax$ for $x \in L$. It is easy to verify that A_L is also a self-adjoint operator.

(ii) *If the invariant subspaces L and M are orthogonal complements of each other, then the spectrum of A is the set-theoretic union of the spectra of operators* A_L *and* A_M.

Let λ belong to the point spectrum of A_L (or A_M). Then, there is a sequence of elements $\{x_n\} \subset L$ (or M) such that $\| x_n \| = 1$, $\| A_{L,\lambda} x_n \| \to 0$. However, $\| A_{L,\lambda} x_n \| = \| A_\lambda x_n \|$, hence λ belongs to the spectrum of A.

Now, let λ belong to the spectrum of neither A_L nor A_M. Then, there is a positive number c, such that

$$\| A_\lambda y \| = \| A_{L,\lambda} y \| \geqslant c \| y \|, \qquad \| A_\lambda z \| \geqslant c \| z \|,$$

for any $y \in L$ and $z \in M$. However, every $x \in H$ has the form $x = y + z$, with $y \in L$, $z \in M$ and $\| x \|^2 = \| y \|^2 + \| z \|^2$. Hence,

$$\| A_\lambda x \| = \| A_\lambda y + A_\lambda z \| = (\| A_\lambda y \|^2 + \| A_\lambda z \|^2)^{1/2}$$
$$\geqslant c\,(\| y \|^2 + \| z \|^2)^{1/2} = c \, \| x \|.$$

Thus, λ is not in the point spectrum of A.

7.42. Continuous spectra and point spectra. As already shown, the space H can be represented as the orthogonal sum of two spaees: N a closed linear hull of the set of all eigenvectors of a self-adjoint operator A and its orthogonal complement G. The subspace N is invariant under A, thus the spectrum of A is the set-theoretic union of the spectra of A_N and A_G. The spectrum of A_N is called the **discrete** or **point spectrum**† and that of A_G the **continuous spectrum** of A. If $N = H$, then the continuous spectrum is absent and A has a **pure point spectrum.** This happens in the case of completely continuous operators, as seen in Chap. 6. If the operator has no eigenvector, then the subspace N is empty, $H = G$ and the operator has a **purely continuous spectrum.** The operator A in Ex. 2 is of this type.

7.43. Operators with pure point spectrum. Let the self-adjoint operator A has a pure point spectrum. Then $N = H$ and, consequently, there exists in H a closed orthonormal system of eigenvectors $\{x_n\}$, such that

$$Ax_n = \lambda_n x_n, \tag{5}$$

where the λ_n are eigenvalues.††

Every x can be represented as the FOURIER series

$$x = \sum_{n=1}^{\infty} c_n x_n, \qquad c_n = (x, x_n). \tag{6}$$

Denote by P_n the projection operator, defined by

$$P_n x = (x, x_n)\, x_n = c_n x_n;$$

(P_n is an operator of projection on the real line tx_n, $-\infty < t < +\infty$).

The series (6) can also be written in the form

$$x = Ex = \sum_n P_n x,$$

or, in the operator form
$$E = \sum_n P_n. \tag{7}$$

Plainly,
$$P_n P_m = 0, \qquad m \neq n. \tag{8}$$

†The collection of all the eigenvalues of the operator A is frequently called the **point spectrum** of A. According to our definition, to the point spectrum of an operator is related also the limit points of the set of its eigenvalues.

††Assuming that H is separable.

By (5) and (6),
$$Ax = \sum_n \lambda_n c_n x_n = \sum_n \lambda_n P_n x, \tag{9}$$

(since $|\lambda| \leqslant \|A\|$, the sum $\sum\limits_n (\lambda_n c_n)^2$ together with $\sum\limits_n c_n^2$ is finite). Write (9) in the operator form
$$A = \sum_n \lambda_n P_n. \tag{10}$$

Then (9) and (6) imply
$$(Ax, x) = \sum_n \lambda_n c_n^2. \tag{11}$$

Thus, the quadratic form (Ax, x) is reduced to a sum of squares. The formula, (11) as implied by (9), can be written in the form
$$(Ax, x) = \sum_n \lambda_n (P_n x, x). \tag{12}$$

Suppose now that λ is not contained in the closed set $\{\lambda_n\}$ of eigenvalues. Then, there is a constant $d > 0$ such that $|\lambda - \lambda_n| > d$. We have
$$A_\lambda x = (A - \lambda E)\, x = \sum_n (\lambda_n - \lambda)\, P_n x.$$

Thereupon, by (8), it is easy to receive
$$R_\lambda x = A_\lambda^{-1} x = \sum_n \frac{1}{\lambda_n - \lambda} P_n x, \tag{13}$$

or, since $P_n x = c_n x_n$, $R_\lambda x = \sum\limits_n \frac{c_n}{\lambda_n - \lambda} x_n$. Since $\left|\frac{c_n}{\lambda_n - \lambda}\right| \leqslant \frac{|c_n|}{d}$,
$$\|R_\lambda\| \leqslant \frac{1}{d}\left(\sum_n c_n^2\right)^{1/2} = \frac{1}{d}\|x\|, \qquad \text{or} \quad \|R_\lambda\| \leqslant \frac{1}{d}.$$

Consequently, λ does not belong to the spectrum. Now, it is possible to write (13) in the form
$$R_\lambda = \sum_n \frac{1}{\lambda_n - \lambda} P_n. \tag{14}$$

These formulae are completely analogous to those for quadratic forms and for symmetric (Hermitian) matrices in the n-dimensional case, differing only in this that the finite sums are replaced here by the infinite series.

Hilbert investigated for the first time in [12] the general theory of self-adjoint operators and the corresponding forms (Ax, x), and regarded the latter as limits of quadratic forms with n variables as $n \to \infty$. As n tends to infinity, the finite sums can be transformed, following exactly the above deductions, into infinite sums as also into integral expressions to be dealt with in the sequel. This explains the emergence of point and continuous spectra. The pure point spectrum is particularly simple owing to its complete analogy with the finite-dimensional case. In this work, HILBERT investigated the important class of operators with a pure point spectrum, the class of completely continuous operators. Independently of the general conclusions of Chap. 6, the discrete character of the spectrum of completely continuous operators is sought to be established below.

THEOREM 5. *Every non-zero point of the spectrum of a self-adjoint completely continuous operator A is an eigenvalue of A.*

If $\lambda \neq 0$ is in the point spectrum of A, then there is a sequence of elements $\{x_n\} \subset H$ such that

$$\| x_n \| = 1 \quad \text{and} \quad \| Ax_n - \lambda x_n \| \to 0,$$

or, putting

$$Ax_n - \lambda x_n = y_n, \ \| y_n \| \to 0,$$
$$x_n = (1/\lambda)(Ax_n - y_n).$$

A maps the sequence $\{x_n\}$ into a compact sequence $\{Ax_n\}$. Therefore, there is a convergent subsequence $\{Ax_{n_k}\}$, and thus the subsequence

$$x_{n_k} = \frac{1}{\lambda}(Ax_{n_k} - y_{n_k}) \tag{15}$$

also converges. Let $x_{n_k} \to x$. Then $Ax_{n_k} \to Ax$; furthermore, $y_{n_k} \to 0$, hence (15) implies that

$$x = (1/\lambda) Ax \quad \text{or} \quad Ax = \lambda x.$$

Moreover,

$$\| x \| = \lim_n \| x_n \| = 1.$$

Consequently, x is an eigenvector and λ is an eigenvalue of A. ■

COROLLARY 1. *Every self-adjoint completely continuous operator has at least one eigenvalue other than* 0.

This statement is immediate from the theorem just proved and Corollary to Theorem 4.

COROLLARY 2. *Every nonempty space L invariant under a self-adjoint completely continuous operator A contains an eigenvector.*

The assertion implies that together with A, $A_L \in (L \to L)$ is also completely continuous. By virtue of Corollary 1, this operator has an eigenvalue λ. Hence, there exists in L an eigenvector of the operator A_L and thereby also of A.

COROLLARY 3. *A self-adjoint completely continuous operator has a pure point spectrum.*

In fact, an invariant subspace G, orthogonal to all eigenvectors, is empty. If G were not empty, it would have to contain, by Corollary 2, an eigenvector, which contradicts the definition of G.

THEOREM 6. *The set $\{\lambda_n\}$ of eigenvalues of a self-adjoint completely continuous operator A can have only one limit point $\lambda = 0$.*

Although this theorem is a particular case of Theorem 7 of Chap. 6.2, it can also be proved in an independent and simpler way.

In fact, if there were an infinite sequence of distinct eigenvalues $\{\lambda_n\}$, such that $|\lambda_n| \geqslant c > 0$, then owing to the orthogonality for the corresponding eigenvectors x_n, $\| x_n \| = 1$, we would get

$$\| Ax_n - Ax_m \|^2 = \| \lambda_n x_n - \lambda_m x_m \|^2 = \lambda_n^2 + \lambda_m^2 \geqslant 2c^2 \qquad \text{for } n \neq m.$$

But then the sequence $\{Ax_n\}$ would not be compact, contradicting the complete continuity of A.

7.5. SPECTRAL DECOMPOSITION OF A SELF-ADJOINT OPERATOR

7.51. Resolution of the identity. It is sought to generalize formulae (7), (10) and (14) of Sec. 4 for arbitrary self-adjoint operators.

LEMMA. *Let A and B be self-adjoint commuting operators and let $A^2 = B^2$. Denote by P the operetor of projection on a subspace of zero operator $A—B$. Then:*

(i) *Every linear bounded operator C commutating with $A—B$, commutes with P;*

(ii) *$Ax = 0$ implies that $Px = x$;*

(iii) *$A = (2P-E)B$.*

Let L be a subspace of zero operator $A-B$ and let P be an operator of projection on this subspace. Thereupon, if $y \in L$ and C commutes with $A-B$, then Cy also belongs to L, since $(A-B)\,Cy = C(A-B)\,y = 0$. Hence, $CPx \in L$ for every $x \in H$ and thus, $PCPx = CPx$, that is, $PCP = CP$. Similarly, $C^*P = PC^*P$, whence

$$PC = (C^*P)^* = (PC^*P)^* = PCP.$$

Consequently, $CP = PC$ and (i) is proved. In particular, $AP = PA$ and $BP = PB$.

Further, let $Ax = 0$. Then

$$\| Bx \|^2 = (Bx, Bx) = (B^2x, x) = (A^2x, x) = \| Ax \|^2 = 0,$$

that is, $Bx = 0$. Hence, $(A-B)\,x = 0$; consequently, $Px = x$ and (ii) is also proved.

Finally, $(A-B)\,(A+B) = A^2 - B^2 = 0$. Hence, for every x, $(A+B)\,x \in L$, and, consequently, $P(A+B)\,x = (A+B)\,x$, that is, $P(A+B) = A+B$. In addition, since $P(A-B) = (A-B)\,P = 0$,

$$P(A+B) - P(A-B) = A+B,$$

whence $A = (2P-E)B$. The lemma is completely proved.

THEOREM 1. *For every self-adjoint operator A, there exists a projection operator E_+, such that:*

(a) *any linear bounded operator C commuting with A, commutes with E_+;*

(*b*) $AE_+ \geqslant 0$, $A(E-E_+) \leqslant 0$;

(*c*) *if* $Ax = 0$, *then* $E_+x = x$.

Let E_+ be the projection of entire H on a subspace of zero operator $A-B$, where B is the positive square root of A^2. From the preceding lemma (*a*) and (*c*) are immediate; in particular, $AE_+ = E_+A$ and $BE_+ = E_+B$. However, in view of this lemma,

$$A = (2E_+-E)\,B.$$

Consequently, $AE_+ = BE_+ \geqslant 0$, $A(E-E_+) = -(E-E_+)\,B \leqslant 0$, since the product of two commuting positive operators is again a positive operator. Theorem 1 is completely proved.

REMARK. The equality $A = (2E_+-E)\,B$ implies $BE_+ = \frac{1}{2}(A+B)$, whence $AE_+ = \frac{1}{2}(A+B)$, $A(E-E_+) = \frac{1}{2}(A-B)$. The operator AE_+ is now denoted by A_+ and called the **positive part** of A. For $A(E-E_+)$ we write A_- and call it the **negative part** of A. Then,

$$A = A_+ + A_-.$$

Examples. 1. Let A be a symmetric matrix of order n with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, where $\lambda_1, \lambda_2, \ldots, \lambda_k < 0$ and $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n > 0$. From linear algebra, it is known that A is unitarily equivalent to the diagonal matrix

$$(\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_n) = \begin{Vmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \ldots & \lambda_n \end{Vmatrix},$$

that is,
$$A = U\,(\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_n)\;U^{-1};$$
then,
$$A_+ = U\,(0, 0, \ldots, 0, \lambda_{k+1}, \ldots, \lambda_n)\;U^{-1},$$
and
$$A_- = U\,(\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0)\;U^{-1}.$$

2. Let the operator A in $L_2[-1, +1]$ be defined by $Ax(t) = tx(t)$. Then

$$A_+x(t) = \frac{t+|t|}{2}\,x(t), \qquad A_-\,x(t) = \frac{t-|t|}{2}\,x(t).$$

THEOREM 2. *Every self-adjoint operator A generates a family of projection operators $\{E_\lambda\}$, depending on the real parameter λ, $-\infty < \lambda < +\infty$ and satisfying the conditions:*

(*a*) $AC = CA$ *implies* $E_\lambda C = CE_\lambda$ *for every* λ;

(*b*) $E_\lambda \leqslant E_\mu$, *or equivalently,* $E_\lambda E_\mu = E_\lambda$, *if* $\lambda < \mu$;

(*c*) E_λ *is continuous from the left with respect to* λ, *that is,* $E_{\lambda-0} = \lim\limits_{n\to\lambda-0} E_\mu = E_\lambda$;

(*d*) $E_\lambda = 0$ *for* $-\infty < \lambda \leqslant m$ *and* $E_\lambda = E$ *for* $M < \lambda < +\infty$, *where* m *and* M *are, respectively, the greatest lower and the least upper bounds of* A.

In this setting $\{E_\lambda\}$ is called a **resolution of the identity** generated by the operator A.

The proof of the theorem is sought to be prefaced by the examples that immediately follow.

1. Let A be a symmetric matrix of order n:

$$A = U(\lambda_1, \lambda_2, \ldots, \lambda_n)\, U^{-1},$$

where $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ and let e_i be its eigenvectors corresponding to the eigenvalues λ_i. Then E_λ for $\lambda_i < \lambda \leqslant \lambda_{i+1}$ is the operator of projection on the i-dimensional subspace spanned by the vectors $e_1, e_2, \ldots, e_i$. We have $E_\lambda = 0$ for $\lambda < \lambda_1$ and $E_\lambda = E$ for $\lambda > \lambda_1$.

2. Let A in $L_2[-1, 1]$ be defined by

$$Ax(t) = tx(t).$$

Then, $E_\lambda x(t) = \varphi_\lambda(t)\, x(t)$, with $\varphi_\lambda(t) = 0$ for $t > \lambda$ and $\varphi_\lambda(t) = 1$ for $t \leqslant \lambda$. Evidently, $E_\lambda = 0$ for $\lambda < -1$ and $E_\lambda = E$ for $\lambda > 1$.

Now, we proceed to prove Theorem 2.

PROOF. Let λ be a any real number and $A_\lambda = A - \lambda E$. Let E_λ denote the projection operator $E - E_+(\lambda)$. Then, $E_+(\lambda)$ is a projection operator constructed according to Theorem 1, for $A - \lambda E$.

Condition (*a*) is obviously satisfied. Thus, in particular, E_λ and E_μ commute for any λ and μ.

For the condition (*b*), examine the projection operator $P = E_\lambda\,(E - E_\mu)$, where $\lambda < \mu$. We have

$$E_\lambda P = E_\lambda^2\,(E - E_\mu) = E_\lambda\,(E - E_\mu) = P. \tag{1}$$

Analogously,
$$(E - E_\mu)\, P = P. \tag{2}$$

Furthermore, by definition of E_λ,

$$(A - \lambda E)\, E_\lambda \leqslant 0, \tag{3}$$

$$(A - \mu E)\,(E - E_\mu) \geqslant 0. \tag{4}$$

Put $Px = y$ for any $x \in H$, so that (1) and (2) imply $E_\lambda y = E_\lambda Px = Px = y$. Analogously, $(E - E_\mu)\, y = y$. By (3) and (4),

$$[(A - \lambda E)\, y, y] = [(A - \lambda E)\, E_\lambda y, y] \leqslant 0,$$

$$[(A - \mu E)\, y, y] = [(A - \mu E)\,(E - E_\mu)\, y, y] \geqslant 0.$$

Now, subtract the second equality from the first, to receive

$$[(\mu - \lambda)\, y, \lambda] \leqslant 0, \quad \text{or} \quad (\mu - \lambda)\, \| y \|^2 \leqslant 0.$$

Thereupon, taking note of the inequality $\lambda < \mu$, we finally get $y = Px = 0$, x any element in H. Consequently, $P = 0$, that is,

$$E_\lambda\,(E - E_\mu) = E_\lambda - E_\lambda E_\mu = 0,$$

which satisfies condition (*b*).

Consider the half-open interval $\Delta = [\lambda, \mu)$ of the real line. For the projection operator $E(\Delta) = E_\mu - E_\lambda$, we have $E_\mu E(\Delta) = E(\Delta)$, $(E - E_\lambda) E(\Delta) = E(\Delta)$. Therefore,

$$(A - \mu E) E(\Delta) = (A - \mu E) E_\mu E(\Delta) \leqslant 0,$$
$$(A - \lambda E) E(\Delta) = (A - \lambda E)(E - E_\lambda) E(\Delta) \geqslant 0,$$

and, consequently,
$$\lambda E(\Delta) \leqslant AE(\Delta) \leqslant \mu E(\Delta). \tag{5}$$

(We have made use of the fact that we can apply operator inequalities obeying the customary sign rules.)

Let us now take up condition (c). The expression $(E_\lambda x, x)$ is a non-decreasing function of λ for any $x \in H$. Therefore, $\lim\limits_{\lambda \to \mu - 0} (E_\lambda x, x)$ exists. This leads to the convergence of

$$\| E_\nu x - E_\lambda x \|^2 = [(E_\nu - E_\lambda) x, x] = (E_\nu x, x) - (E_\lambda x, x) \to 0,$$

for $\lambda < \nu < \mu$, $\lambda \to \mu$ and $\nu \to \mu$. Consequently, $\lim\limits_{\lambda \to \mu - 0} E_\lambda x = E_{\mu - 0} x$, for any $x \in H$. It is easy to verify that $E_{\mu-0}$ is a projection operator. In order to exhibit that

$$E_{\mu-0} = E_\mu,$$

let
$$E(\Delta_0) = E_\mu - E_{\mu-0}.$$

Then, $E(\Delta) = E_\mu - E_\lambda \to E(\Delta_0)$ as $\lambda \to \mu - 0$, in the sense of strong convergence of operators. Now, taking a limit in (5) which is obviously possible, we obtain

$$\mu E(\Delta_0) = AE(\Delta_0).$$

Now, let x be any element in H and $y = E(\Delta_0)x$. From the preceding equality, we have

$$(A - \mu E) y = 0,$$

implying, in view of the property (c) of Theorem 1, that $E_\mu y = 0$. Furthermore, $E_\mu E(\Delta) = E(\Delta)$ whence, proceeding to the limit,

$$E_\mu E(\Delta_0) = E(\Delta_0).$$

Consequently, $E(\Delta_0) x = E_\mu E(\Delta_0) x = E_\mu y = 0$. Since $x \in H$ is arbitrary, $E(\Delta_0) = 0$ and condition (c) is satisfied.†

The fulfilment of condition (d) is trivial to prove. Let $\lambda < m$ and $E_\lambda \neq 0$. Then, there is an element x, such that $E_\lambda x \neq 0$. Put $E_\lambda x = y$, to receive $E_\lambda y = y$, where it can be assumed that $\| y \| = 1$. Then,

$$\begin{aligned}(Ay, y) - \lambda &= (Ay, y) - \lambda (y, y) = [(A - \lambda E) y, y] \\ &= [(A - \lambda E) E_\lambda y, y] \leqslant 0, \quad \text{that is, } (Ay, y) \leqslant \lambda < m,\end{aligned}$$

†By the definition of E_λ, the zero operator $A - \lambda E$ belongs to the orthogonal complement of the subspace L_{E_λ}. If, however, E_λ is so defined that $A - \lambda E$ is contained in L_{E_λ}, which can be accomplished without violating the properties (a), (b) and (d), then E_λ is continuous from the right.

contradicting the definition of the number m. Consequently, $E_\lambda = 0$ for $\lambda < m$ and owing to the left-hand continuity, $E_m = 0$ too. Similarly, it can be shown that $E_\lambda = E$ for $\lambda > M$.

7.52. Spectral theorem for self-adjoint operators. THEOREM 3. *For every self-adjoint operator A, the equality*

$$A = \int_m^{M+\varepsilon} \lambda dE_\lambda \tag{6}$$

holds, where the Stieltjes integral on the right-hand side is understood to be the limit of the integral sums in the sense of strong convergence in the operator space and ε is any positive number.

PROOF. Let the half-open interval $[m, M+\varepsilon)$, $\varepsilon > 0$, be partitioned into the half-open intervals $\Delta_1, \Delta_2, \ldots, \Delta_n$, $\Delta_k = [\lambda_k, \mu_k)$. By (5), we get

$$\lambda_k E(\Delta_k) \leqslant AE(\Delta_k) \leqslant \mu_k E(\Delta_k)$$

for every Δ_k. Summing over all the $k = 1, 2, \ldots, n$ and noting that

$$\sum_{k=1}^n E(\Delta_k) = E,$$

we get
$$\sum_{k=1}^n \lambda_k E(\Delta_k) \leqslant A \leqslant \sum_{k=1}^n \mu_k E(\Delta_k).$$

Let $\nu_k \in [\lambda_k, \mu_k)$ be any number. Then

$$\sum_{k=1}^n (\lambda_k - \nu_k) E(\Delta_k) \leqslant A - \sum_{k=1}^n \nu_k E(\Delta_k) \leqslant \sum_{k=1}^n (\mu_k - \nu_k) E(\Delta_k).$$

Putting $\max_k (\mu_k - \lambda_k) = \delta$, these inequalities yield

$$-\delta E \leqslant A - \sum_{k=1}^n \nu_k E(\Delta_k) \leqslant \delta E,$$

that is,
$$-\delta(x, x) \leqslant \left(\left\| A - \sum_{k=1}^n \nu_k E(\Delta_k) \right\| x, x \right) \leqslant \delta(x, x).$$

Thereupon,
$$\left\| A - \sum_{k=1}^n \nu_k E(\Delta_k) \right\| \leqslant \delta,$$

that is,
$$A = \lim_{\delta \to 0} \sum_{k=1}^n \nu_k E(\Delta_k) = \int_m^{M+\varepsilon} \lambda \, dE_\lambda,$$

which is also the required proof. For completely continuous operators, this formula carries over to (10) of Chap. 7.4.

REMARK. Since the convergence of a sequence of operators $\{A_n\}$ to A in the sense of strong convergence in a space of operators implies weak convergence of $\{A_n\}$ to A, as well as the convergence of the quadratic forms $(A_n x, x)$ to (Ax, x), hence Theorem 3, for every $x \in H$, implies

$$\text{(i)} \qquad Ax = \lim \sum_{k=1}^{n} \nu_k E(\Delta_k)\, x = \int_m^{M+\varepsilon} \lambda dE_\lambda\, x,$$

$$\text{(ii)} \quad (Ax, x) = \lim \sum_{k=1}^{n} \nu_k (E(\Delta_k)\, x, x) = \int_m^{M+\varepsilon} \lambda d(E_\lambda\, x, x).$$

7.53. Operator Functions. Resolvent. Spectra. *7.53.1 Definition of $F(A)$.* We define

$$\int_m^{M+\varepsilon} F(\lambda)\, dE_\lambda,$$

where $F(\lambda)$ is an arbitrary complex step function defined on the closed interval $[m, M]$ and $\{E_\lambda\}$ is a resolution of the identity generated by the self-adjoint operator A. If λ_0 is a *jump point* of this function, then we agree that $F(\lambda_0) = F(\lambda_0+0)$. Extend $F(\lambda)$ to the half-open interval $[m, M+\varepsilon)$ by defining there $F(\lambda) = F(M)$. Let $F(\lambda_k) = \nu_k$ on $\Delta_k = [\lambda_k, \mu_k)$, $k = 1, 2, \ldots, n$, and

$$\bigcup_{k=1}^{n} \Delta_k = [m, M+\varepsilon).$$

By definition,
$$\int_m^{M+\varepsilon} F(\lambda)\, dE_\lambda = \sum_{k=1}^{n} \nu_k E(\Delta_k).$$

It is easy to see that we have also the equality

$$\int_m^{M+\varepsilon} F(\lambda)\, dE_\lambda = \sum_{k=1}^{p} \tilde{\nu}_k\, E(\tilde{\Delta}_k),$$

where $\tilde{\Delta}_k$ are any partial half-intervals, on which $F(\lambda)$ is fixed and whose sum yields $[m, M+\varepsilon)$. The operator $\int_m^{M+\varepsilon} F(\lambda)\, dE_\lambda$ is denoted by $F(A)$ and called a **function of the operator** A, corresponding to the function $F(\lambda)$ of the real variable λ. Thus, it is possible to associate the step functions of a real variable with the operator functions of A. This correspondence relation has the following properties:

(i) If $F(\lambda) = \alpha F_1(\lambda) + \beta F_2(\lambda)$,
then $F(A) = \alpha F_1(A) + \beta F_2(A)$; *(additivity)*

(ii) If $F(\lambda) = F_1(\lambda)\, F_2(\lambda)$,
then $F(A) = F_1(A)\, F_2(A)$; *(multiplicativity)*

(iii) $\overline{F}(A) = [F(A)]^*$, where bar on the function denotes transition to the complex conjugate function;

(iv) $\| F(A) \| \leqslant \max | F(\lambda) |$;

(v) $AB = BA$ implies $F(A)\,B = BF(A)$ for any linear bounded operator B.

To prove properties (i) and (ii), partition $[m, M+\varepsilon)$ into the sub-intervals Δ_k, on which both the functions $F_1(\lambda)$ and $F_2(\lambda)$ are constant. Then, for $F(\lambda) = \alpha F_1(\lambda) + \beta F_2(\lambda)$, we have

$$F(A) = \sum_{k=1}^{n} (\alpha c_k^{(1)} + \beta c_k^{(2)})\, E(\Delta_k)$$

$$= \alpha \sum_{k=1}^{n} c_k^{(1)}\, E(\Delta_k) + \beta \sum_{k=1}^{n} c_k^{(2)}\, E(\Delta_k) = \alpha F_1(A) + \beta F_2(A),$$

and for $F(\lambda) = F_1(\lambda)\, F_2(\lambda)$, because of the orthogonality of $E(\Delta_k)$ and $E(\Delta_l)$, $k \neq l$, we have

$$F(A) = \sum_{k=1}^{n} c_k^{(1)}\, c_k^{(2)}\, E(\Delta_k) = \left(\sum_{k=1}^{n} c_k^{(1)}\, E(\Delta_k) \right)\left(\sum_{l=1}^{n} c_l^2\, E(\Delta_l) \right)$$

$$= F_1(A)\, F_2(A).$$

Furthermore,

$$(F(A)\, x, y) = \left(\sum_{k=1}^{n} c_k\, E(\Delta_k)\, x, y \right) = \left(x, \sum_{k=1}^{n} \bar{c}_k E(\Delta_k)\, y \right)$$

$$= (x, \overline{F}(A)\, y), \quad \text{whence } [F(A)]^* = \overline{F}(A).$$

Finally, $|\,(F(A)\, x, x)| = \left| \left(\sum_{k=1}^{n} c_k\, E(\Delta_k)\, x, x \right) \right|$

$$\leqslant \sum_{k=1}^{n} |\, c_k \,|\, (E(\Delta_k)\, x, x) \leqslant \max |\, F(\lambda)\, |\, (x, x).$$

Hence, $$\| F(A) \| = \sup_{\| x \| = 1} |\, (F(A)\, x, x)\, | \leqslant \max |\, F(\lambda)\, |.$$

Property (v) is trivial.

From the definition of $F(A)$, it follows, in perticular, that $E(\Delta) = \chi_\Delta(A)$, where $\chi_\Delta(\lambda)$ is the characteristic function of the half-open interval Δ. Now, let $F(\lambda)$ be an arbitrary function continuous on $[m, M]$, and let it be extended to $[m, M+\varepsilon)$ by putting $F(\lambda) = F(M)$ for $\lambda \in (M, M+\varepsilon)$. There is a sequence of step functions $F_n(\lambda)$ uniformly convergent to $F(\lambda)$ on $[m, M+\varepsilon)$. Then, consider the associated operator functions $F_n(A)$, getting

$$\| F_n(A) - F_m(A) \| \leqslant \max |\, F_n(\lambda) - F_m(\lambda)\, | \to 0 \quad \text{as} \quad m, n \to \infty.$$

By virtue of the completeness of the operator space, there exists an operator $B = \lim_n F_n(A)$. The integral, set by definition, as $B = \int_m^{M+\varepsilon} F(\lambda)\, dE_\lambda$, denoted hereafter either by B or even $F(A)$ is called **a function of operator** A assigned to a continuous function $F(\lambda)$ of a real variable λ. The equivalence of $F(A)$ and B is given by a theorem on STIELTJES integrals. It is easy to verify in this that the definition of $F(A)$ is independent of the choice of the sequence $\{F_n(\lambda)\}$, convergent to $F(\lambda)$, and that the properties (i) thro' (v) are preserved also for continuous functions. In particular,

$$A^n = \int_m^{M+\varepsilon} \lambda^n \, dE_\lambda, \qquad n = 0, 1, 2, \ldots$$

7.53.2 Resolvent. The correspondence obtained between functions of a real variable and the operator functions admits extensive applications for making explicit a series of properties, especially the spectral properties of a self-adjoint operator. However, the discussion will be confined here to the following three theorems.

THEOREM 4. *In order that for a given* λ_0 *there exists the resolvent* $R_{\lambda_0} = (A - \lambda_0 E)^{-1}$, *it is necessary and sufficient that one of the following conditions is satisfied*:

(i) λ_0 *is not real*;
(ii) λ_0 *is real and lies outside the interval* $[m, M]$;
(iii) *If* $\lambda_0 \in [m, M]$, *then there exists a half-open interval* $[\alpha, \beta)$, $\alpha < \lambda_0 < \beta$, *inside of which* E_λ *is constant.*

In all these cases,

$$R_{\lambda_0} = \int_m^{M+\varepsilon} \frac{dE_\lambda}{\lambda - \lambda_0}.$$

PROOF. In cases (i) and (ii) the function $f(\lambda) = 1/(\lambda - \lambda_0)$ is continuous in $[m, M+\varepsilon)$ for sufficiently small ε. Hence

$$\int_m^{M+\varepsilon} \frac{dE_\lambda}{\lambda - \lambda_0} \int_m^{M+\varepsilon} (\lambda - \lambda_0)\, dE_\lambda = \int_m^{M+\varepsilon} dE_\lambda = E,$$

and, since
$$\int_m^{M+\varepsilon} (\lambda - \lambda_0)\, dE_\lambda = A - \lambda_0 E,$$

it follows that
$$\int_m^{M+\varepsilon} \frac{dE_\lambda}{\lambda - \lambda_0} = R_{\lambda_0}.$$

In the case (iii), partition $[m, M+\varepsilon)$ into three half-open intervals $[m, \alpha)$,

$[\alpha, \beta)$ and $[\beta, M+\varepsilon)$. Let $\varphi(\lambda) = 1/(\lambda-\lambda_0)$ on $[m, \alpha)$ and $[\beta, M+\varepsilon)$, and let $\varphi(\lambda)$ be linear on $[\alpha, \beta)$, also $\varphi(\alpha) = 1/(\alpha-\lambda_0)$, $\varphi(\beta) = 1/(\beta-\lambda_0)$.

Since E_λ is constant in the half-open interval $[\alpha, \beta)$,

$$\int_\alpha^\beta \psi(\lambda)\, dE_\lambda = 0$$

for every function $\psi(\lambda)$. Hence, we can write

$$\int_m^{M+\varepsilon} \varphi(\lambda)\, dE_\lambda = \int_m^{M+\varepsilon} \frac{dE_\lambda}{\lambda-\lambda_0}.$$

Consequently,

$$\int_m^{M+\varepsilon} \frac{dE_\lambda}{\lambda-\lambda_0} \int_m^{M+\varepsilon} (\lambda-\lambda_0)\, dE_\lambda = E.$$

This implies the existence of

$$R_{\lambda_0} = \int_m^{M+\varepsilon} \frac{dE_\lambda}{\lambda-\lambda_0}. \qquad \blacksquare$$

THEOREM 5. *If R_{λ_0} exists for a real λ_0, then λ_0 lies in the interior of an half-open interval $[\alpha, \beta)$, $\lambda_0 \neq \alpha$, in which E_λ is constant.*

PROOF. For an arbitrary $x \in H$, we construct the equality

$$(A-\lambda_0 E)\, x = \int_m^{M+\varepsilon} (\lambda-\lambda_0)\, dE_\lambda\, x,$$

and apply to its both sides the operator $R_{\lambda_0} E(\Delta)$, $\Delta = [\alpha, \beta)$ some half-open interval containing λ_0 interior to it. Then

$$E(\Delta)\, x = R_{\lambda_0}\left(\int_\alpha^\beta (\lambda-\lambda_0)\, dE_\lambda\, x \right).$$

Hence,
$$\| E(\Delta)\, x \| \leqslant \| R_{\lambda_0} \| \left\| \int_\alpha^\beta (\lambda-\lambda_0)\, dE_\lambda\, x \right\|.$$

However, it is easy to verify that

$$\left\| \int_\alpha^\beta (\lambda-\lambda_0)\, dE_\lambda\, x \right\| \leqslant c \, \| E(\Delta)\, x \|,$$

where $c = \max\,(\beta-\lambda_0, \lambda_0-\alpha)$. Consequently,

$$\| E(\Delta)\, x \| \leqslant c \, \| R_{\lambda_0} \| \, \| E(\Delta)\, x \|.$$

Now, select a half-open interval $[\alpha, \beta)$ so small that $c \| R_{\lambda_0} \| < \frac{1}{2}$, getting

$$\| E(\Delta) x \| \leqslant \tfrac{1}{2} \| E(\Delta) x \|.$$

However, this is possible only if $E(\Delta) x = 0$, and since $x \in H$ is arbitrary, $E(\Delta) = 0$. What is more, $E(\tilde{\Delta}) = 0$ for every half-open interval $\tilde{\Delta} \subset \Delta$, implying that E_λ is constant in $[\alpha, \beta)$. ■

COROLLARY. From Theorem 4 it is immediate that: *the set of regular real points of a self-adjoint operator A is open and, consequently, the spectrum of A is a closed set on the real line* (it is shown in Chap. 3 that the spectrum of arbitrary linear bounded operators in a real BANACH space is closed).

7.54. Eigenvalues of a self-adjoint operator. THEOREM 6. *In order that λ_0 be an eigenvalue of a self-adjoint operator A, it is necessary and sufficient that λ_0 is a jump point of E_λ.*

Necessity. For some $x_0 \neq 0$, let $Ax_0 - \lambda_0 x_0 = 0$. Then, $[(A-\lambda_0 E)^2 x_0, x_0] = 0$. Consequently,

$$\int_m^{M+\varepsilon} (\lambda-\lambda_0)^2 \, d\,(E_\lambda x_0, x_0) = 0.$$

Since the integrand is non-negative and the weight is monotone increasing, it also follows that

$$\int_\alpha^\beta (\lambda-\lambda_0)^2 \, d(E_\lambda x_0, x_0) = 0$$

for every half-open interval $[\alpha, \beta)$. In particular,

$$\int_{\lambda_0+\varepsilon}^{M+\varepsilon} (\lambda-\lambda_0)^2 \, d\,(E_\lambda x_0, x_0) = 0$$

for every $\varepsilon > 0$. Furthermore, since $(\lambda-\lambda_0)^2 \geqslant \varepsilon^2$ on the interval of integration,

$$\varepsilon^2 \int_{\lambda_0+\varepsilon}^{M+\varepsilon} d(E_\lambda (x_0, x_0) = \varepsilon^2 \, [(x_0, x_0) - (E_{\lambda_0+\varepsilon} x_0, x_0)] = 0.$$

Consequently, $(x_0, x_0) - (E_{\lambda_0+\varepsilon} x_0, x_0) = 0$, that is

$$E_{\lambda_0+\varepsilon} x_0 = x_0. \tag{7}$$

Similarly,

$$\int_m^{\lambda_0-\varepsilon} (\lambda-\lambda_0)^2 \, d\,(E_\lambda x_0, x_0) = 0,$$

whence taking into account that $E_m = 0$, we get

$$E_{\lambda_0-\varepsilon} x_0 = 0. \tag{8}$$

From (7) and (8) it follows that $(E_{\lambda_0+\varepsilon}-E_{\lambda_0-\varepsilon})\, x_0' = x_0$, and, since ε is arbitrary,

$$(E_{\lambda_0+0}- E_{\lambda_0})\, x_0 = x_0.$$

Consequently, λ_0 is, indeed, a jump point of E_λ, and the eigenvector x_0 belongs to the subspace, associated with the projection operator $E_{\lambda_0+0} - E_{\lambda_0}$.

Sufficiency. Let $E_{\lambda_0+0} \neq E_{\lambda_0}$ and let x_0 be any element of the subspace, associated with $E_{\lambda_0+0}-E_{\lambda_0}$. Then

$$(E_{\lambda_0+0}-E_{\lambda_0})\, x_0 = x_0,$$

that is, x_0 belongs to the orthogonal complement of the space $L_{E_{\lambda_0}}$ in $L_{E_{\lambda_0+0}}$. Therefore, $E_{\lambda_0+0}x_0 = x_0$, $E_{\lambda_0}x_0 = 0$. What is more, $E_\lambda\, x_0 = x_0$ for $\lambda > \lambda_0$ and, consequently, $E\,(\Delta)\, x_0 = x_0$ for $\Delta = [\lambda_0, \lambda_0+\varepsilon)$. However, then,

$$Ax_0 = AE\,(\Delta)\, x_0 = \int\limits_{\lambda_0}^{\lambda_0+\varepsilon} \lambda\, dE_\lambda\ x_0,$$

$$\lambda_0 x_0 = \lambda_0 E\,(\Delta)\, x_0 = \int\limits_{\lambda_0}^{\lambda_0+\varepsilon} \lambda_0\, dE_\lambda\ x_0,$$

and, consequently,

$$Ax_0 - \lambda_0 x_0 = \int\limits_{\lambda_0}^{\lambda_0+\varepsilon} (\lambda-\lambda_0)\, dE_\lambda\, x_0.$$

Thereupon, $\| Ax_0 - \lambda_0 x_0 \| \leqslant \varepsilon \| E\,(\Delta)\, x_0 \| \leqslant \varepsilon \| x_0 \|$, and since ε is arbitrary,

$$\| Ax_0 - \lambda_0 x_0 \| = 0.$$

Incidentally, it is found that the whole subspace associated with the operator $E_{\lambda_0+0} - E_{\lambda_0}$ consists of the eigenvectors of A corresponding to the eigenvalue λ_0.

7.6 NON-BOUNDED LINEAR OPERATORS. BASIC CONCEPTS AND DEFINITIONS

THE PRECEDING section dealt with linear bounded operators defined on the whole HILBERT space H. However, there is a highly important class of linear operators that do not obey this condition. An example is furnished by the differential operator $A = d/dt$, which is defined only on a set everywhere dense in $L_2\,[-\pi, \pi]$ of functions having square-integrable derivatives. The differential operator is not bounded on this set, since $\| Ax_n \| = n \| x_n \|$ for $x_n\,(t) = \sin nt$.

If a linear operator A is defined on an everywhere dense set of the space H and is bounded there, then A is uniformly continuous on this set and it can be extended in a unique manner by continuity to the entire space. The upcoming proposition and its converse hold for a certain class of operators.

THEOREM 1. *If a linear operator A is defined on the whole space H and if the* *equality* $(Ax, y) = (x, Ay)$ *is satisfied for all* $x, y \in H$, *then it is bounded and, consequently, continuous.*

Assume the contrary. Then there is a sequence $\{x_n\} \subset H$, such that

$$\| x_n \| = 1 \quad \text{and} \quad \| Ax_n \| \to \infty. \tag{1}$$

Consider the functional $f_n(x) = (Ax, x_n) = (x, Ax_n)$. This is additive and homogeneous and, in addition,

$$| f_n(x) | = | (Ax, x_n) | \leqslant \| Ax \| \| x_n \| = \|Ax\| = c_x.$$

By the BANACH-STEINHAUS theorem, the norm of this functional is totally bounded : $\| f_n \| \leqslant c$. However, $\| f_n \| = \| Ax_n \|$, implying $\| Ax_n \| \leqslant c$, which is impossible because of (1), a contradiction proving the theorem.

Now, consider an operator A defined on a linear manifold $D(A) \subset H$, everywhere dense in H, with values in the same space and linearity on $D(A)$:

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$$

for every $x, y \in D(A)$ and any numbers α and β.

The set $D(A)$ is called the **domain** of the operator. The set $R(A) = AD(A)$ is called the **range** of the operator. The operators A and B are regarded as **equal** or **coinciding** if $D(A) = D(B)$ and $Ax = Bx$ for every $x \in D(A)$. If, however, $D(A) \subset D(B)$ and $Ax = Bx$ for every $x \in D(A)$, then B is called an **extension** of A and A a **restriction** of B. This is symbolically denoted as $A \subset B$.

Let A and B be two linear operators with domains $D(A)$ and $D(B)$. If $L = D(A) \cap D(B)$, then both the operators have a meaning on elements of a linear manifold L. The operator

$$(A + B)\, x = Ax + Bx, \quad x \in L,$$

is the **sum** of the operators A and B. The manifold L always contains the null element and is, consequently, non-void but the sum of operators is non-trivial, only if L contains non-zero elements. This observation is, however, related to the next definition.

Now, suppose that there is a subset D in $D(A)$, such that $Ax \in D(B)$ for any $x \in D$. Then, the **product** of the operator B by the operator A on D is defined as

$$(BA)x = B(Ax).$$

The product AB is also defined analogously.

If A gives a one-one map of $D(A)$ onto $R(A)$, then there is an **inverse** operator A^{-1} (or, in short, inverse of A), with domain $R(A)$ and range $D(A)$. It is possible to regard that $R(A) = H$ and that A^{-1} is bounded, although A is a non-bounded linear operator. Conversely, it is possible that a bounded linear operator A may have an inverse that is not bounded. The examples of such operators, when considered as operators in the space $L_2[0, 1]$, are furnished on pp. 109-110.

7.61. Adjoint operators. Let A be a linear operator defined on a linear manifold $D(A)$ everywhere dense in H. If the scalar product (Ax, y) for a given fixed y and every $x \in D(A)$ is representable in the form

$$(Ax, y) = (x, y^*), \tag{2}$$

then it is said that y belongs to the domain $D(A^*)$ of the operator, *adjoint* of A, and the adjoint operator A^* itself is defined by

$$A^*y = y^*.$$

Since $D(A)$ is by assumption everywhere dense in H, the element y^* is defined uniquely by (2). Without much inconvenience, it can be verified that $D(A^*)$ is a linear manifold and A^* is a linear operator. Note that the domain of an adjoint operator need not always be void, this being dependent on whether it contains a null element.

Example. Let $H = L^2(G)$, where G is a bounded measurable domain on the x, y-plane. Consider an operator $A = \partial^l / \partial x^{l_1} \partial y^{l_2}$, defined on a linear manifold, everywhere dense in G, of functions $\psi(x, y)$, which are continuous together with partial derivatives to within order l and become zero in some neighbourhood of the boundary of G. Since $D(A)$ is everywhere dense in $L_2(G)$, there exists an adjoint operator A^*. Keeping in view the definition on p. 65. $D(A^*)$ is the collection of functions $\varphi(x, y)$ having l-th order generalized derivatives and A^* is a generalized differential operator $A^* \varphi = \partial^l \varphi / \partial x^{l_1} \partial y^{l_2}$.

A linear operator A defined on $D(A)$ is called **symmetric**, if for every $x, y \in D(A)$, the equality

$$(Ax, y) = (x, Ay)$$

is satisfied.

For the bounded operators, the notion of the symmetric operators is equivalent to the notion of self-adjoint operators. The position is somewhat different in the case of non-bounded operators, as will be seen presently.

As also for bounded operators, in order that A be symmetric, it is necessary and sufficient that (Ax, x) is real for every $x \in D(A)$. It is plain that for a symmetric operator, the inclusion $y \in D(A)$ implies $y \in D(A^*)$ and that for $y \in D(A)$,

$$A^*y = Ay.$$

Hence $A^* \supset A$, that is, A^* is an extension of A.

It is easily verifiable that if $A \subset B$, then $B^* \subset A^*$.

THEOREM 2. *If the operator A^{-1} exists and has, just as the operator A, an everywhere dense domain, then $(A^*)^{-1}$ exists and is equal to $(A^{-1})^*$.*

Let $y \in D\,[(A^{-1})^*]$. For every $x \in D(A)$,

$$(x, y) = (A^{-1}Ax, y) = (Ax, (A^{-1})^*y).$$

If this equality is read from the right to the left, then it is seen that $(A^{-1})^*y \in D(A^*)$ and that

$$A^*(A^{-1})^*y = y, \tag{3}$$

Similarly, if $x' \in D(A^{-1})$, $y' \in D(A^*)$, then

$$(x', y') = (AA^{-1} x', y') = (A^{-1}x', A^*y'),$$

whence, as above, it follows that $A^*y' \in D[(A^{-1})^*]$ and

$$(A^{-1})^* A^*y' = y'. \tag{4}$$

Eqs. (3) and (4) imply that $(A^*)^{-1}$ exists and equals $(A^{-1})^*$.

It can be shown that: $(\lambda A)^* = \bar{\lambda} A^*; (A+B)^* \supset A^* + B^*; (AB)^* \supset B^*A^*$.

Now, we revert to the problem of the commutativity of two operators. Let A be a linear operator with domain $D(A)$ and B be a bounded linear operator. It is said that B **permutes** (or **commutes**) with A, if $x \in D(A)$ implies $Bx \in D(A)$ and $ABx = BAx$. In a more generalized sense, a definition of two non-bounded commuting operators is given below.

Introduce an additional notion: Let A and B be two linear operators and let A commute with every bounded operator that commutes with B. In this case, A is called **self-commutative** with B.

7.62. Closed operators. Closure of operators. A non-bounded linear operator A does not have the property of continuity. Because of this, $x_n \to x_0$ does not imply, in general, that $\{Ax_n\}$ tends to any limit. However, many of the non-bounded linear operators have a rather weaker property which somewhat compensates for the absence of continuity.

A linear operator A with domain $D(A)$ is called **closed,** if $x_n \to x_0$, $\{x_n\} \subset D(A)$ and $Ax_n \to y_0$ imply $x_0 \in D(A)$ and $y_0 = Ax_0$.

The adjoint of an arbitrary linear operator can serve as an example of a closed operator; in particular, self-adjoint operators are closed.

In fact, let $y_n \in D(A^*)$ and let $y_n \to y_0$, $A^*y_n \to z_0$, to receive

$$(x, A^*y_n) = (Ax, y_n) \to (Ax, y_0)$$

for every $x \in D(A)$. On the other hand, $(x, A^*y_n) \to (x, z_0)$. Consequently, $(Ax, y_0) = (x, z_0)$ for every $x \in D(A)$. Thereupon, it follows that $y_0 \in D(A^*)$ and $A^*y_0 = z_0$.

The partial differential operator defined on p. 235 serves as an example of operators that are not closed.

It is said that the operator A admits **closure** if there exists a closed operator B, which is the extension of A (that is, $B \supset A$). Among the closed extensions that A allows, it is possible to distinguish the **smallest closed extension** which is contained in every other closed extension of A. This is termed as the closure of A and denoted by $\bar{A}$. The proof of the existence and uniqueness of a closure of a closable operator will not be attempted, instead the discussion that follows will remain confined to symmetric operators.

THEOREM 3. *For every symmetric operator A, the closure $\bar{A}$ can be constructed.*

PROOF. Denote by $D(\bar{A})$ a collection of elements $x \in H$, for which there exists a sequence $\{x_n\} \subset D(A)$, such that, $x_n \to x$, $Ax_n \to y$, y some element in H.

Evidently, $D(\bar{A})$ is a linear manifold and $D(A) \subseteq D(\bar{A})$. For $x \in D(\bar{A})$, put $\bar{A}x = y$. This definition is unique. Let $\{x'_n\} \subset D(A)$ be another sequence, such that $x'_n \to x$, $Ax'_n \to y'$. Then, for any $h \in D(A)$, making use of the symmetric property of the operator. we get

$$(h, y - y') = \lim_n (h, Ax_n - Ax'_n)$$
$$= \lim_n (Ah, x_n - x'_n) = (Ah, x - x) = 0.$$

Since $D(A)$ is everywhere dense in H, $y = y'$. The operator $\bar{A}$ is, evidently, linear and is an extension of A.

The operator $\bar{A}$ is symmetric, since for every $x, y \in D(\bar{A})$,

$$(x, \bar{A}y) = \lim_n (x_n, Ay_n) = \lim_n (Ax_n, y_n) = (\bar{A}x, y).$$

The operator $\bar{A}$ is closed. In fact, let $\{x_n\} \subseteq D(\bar{A})$, $x_n \to x$, $Ax_n \to y$. Since $x_n \in D(\bar{A})$, there is an element $x'_n \in D(A)$, such that

$$\| x_n - x'_n \| < 1/n, \qquad \| \bar{A}x_n - Ax'_n \| < 1/n.$$

But, then, $x'_n \to x$, $Ax'_n \to y$ and, consequently, $x \in D(\bar{A})$ and $\bar{A}x = y$ by the definition of $D(\bar{A})$ and $\bar{A}$.

Thus, the fact that every element $x \in D(A)$ must belong to the domain of every closed extension of A implies that $\bar{A}$ is the smallest closed extension of the symmetric operator A.

Thereupon, it also follows that the closure $\bar{A}$ is unique.

REMARK. Show that: *If $\bar{A}$ is a closure of a symmetric operator A, then* $(\bar{A})^* = A^*$.

Since $A \subseteq \bar{A}$, hence $(\bar{A})^* \subset A^*$. However, it is necessary to prove the converse also.

Let $y \in D(A^*)$ and x be any element of $D(A)$. We have

$$(\bar{A}x, y) = \lim_n (Ax_n, y) = \lim_n (x_n, A^*y) = (x, A^*y).$$

It is evidenced from this equality that $y \in D[(\bar{A})^*]$ and $(\bar{A})^* y = A^*y$, that is, $A^* \subseteq (\bar{A})^*$.

7.63. Graph of an operator. The idea of the graph of an operator is introduced here as we shall draw upon it later in our treatment of self-adjoint operators and the closure operation.

Consider two prototypes of a HILBERT space H, and let $\tilde{H}$ be the direct sum of these spaces, that is, a collection of pairs $\tilde{z} = \{x, y\}$, $x, y \in H$ with the customary definitions of linear operations. Further, define for $\tilde{z}_1, \tilde{z}_2 \in \tilde{H}$, the scalar product of these elements by

$$(\tilde{z}_1, \tilde{z}_2) = (x_1, x_2) + (y_1, y_2).$$

It is easy to verify that all properties of the scalar product hold. All the rest of the axioms of HILBERT space are also satisfied. Consequently, $\tilde{H}$ is also a HILBERT space.

Given a linear operator A in H, the set $\mathcal{G}r(A) \subset \tilde{H}$ of elements of the form $\{(x, Ax)\}$, $x \in D(A)$, is called the **graph of the operator** A.

It is easy to see that $\mathcal{G}r(A)$ is a linear manifold, uniquely defined by A. Conversely, if $\mathcal{G}r(A) = \mathcal{G}r(B)$ for A and B, then $A = B$. Finally, it is also verifiable that for A to be closed, it is necessary and sufficient that $\mathcal{G}r(A)$ is a closed subspace in $\tilde{H}$.

Consider in $\tilde{H}$, an operator $\tilde{U}$ defined by

$$\tilde{U}\{x, y\} = \{y, -x\}.$$

Plainly, $\tilde{U}^2 = -\tilde{E}$ and $\tilde{U}^* = -\tilde{U}$, whence $\tilde{U}^*\tilde{U} = \tilde{U}\tilde{U}^* = \tilde{E}$, that is, $\tilde{U}$ is a unitary operator.

LEMMA. *If A is an arbitrary linear operator defined on an everywhere dense linear manifold $D(A)$, then $\mathcal{G}r(A^*)$ is an orthogonal complement of the linear manifold $\tilde{U}$ $[\mathcal{G}r(A)]$.*

Let $\tilde{z} = \{x', y'\} \in \tilde{H} \dot{-} \overline{\tilde{U}[\mathcal{G}r(A)]}$, implying $(\{x', y'\}, \{Ax, -x\}) = 0$ for every $x \in D(A)$. Thereupon, $(x', Ax) = (y', x)$, and, consequently, $x' \in D(A^*)$ and $y' = A^*x'$, that is, $\{x', y'\} \in \mathcal{G}r(A^*)$.

Repeating the reasonings in the reverse order, it is found that $\{x', y'\} \in \mathcal{G}r(A^*)$ implies these elements to be orthogonal to every element in $\tilde{U}[\mathcal{G}r(A)]$. ■

THEOREM 4. *If A is a closed operator defined on a set $D(A)$, everywhere dense in H, then $D(A^*)$ is also everywhere dense and uniquely defines $(A^*)^* = A^{**}$. Moreover, $A^{**} = A$.*

Since A is closed, $\mathcal{G}r(A)$ is a closed linear manifold and, hence, $\tilde{U}[\mathcal{G}r(A)]$ is also closed. Thus,

$$\tilde{H} = \tilde{U}[\mathcal{G}r(A)] \dotplus \mathcal{G}r(A^*). \tag{5}$$

Applying the unitary operator $\tilde{U}$ to both sides of this equality and noting first that $\tilde{U}^2[\mathcal{G}r(A)] = -\tilde{E}\,\mathcal{G}r(A) = \mathcal{G}r(A)$ and next that the unitary operator carries orthogonal elements into orthogonal ones, we get

$$\tilde{U}(\tilde{H}) = \tilde{H} = \mathcal{G}r(A) \dotplus \tilde{U}[\mathcal{G}r(A^*)]. \tag{6}$$

First, it is to be shown that $D(A^*)$ is everywhere dense. If this were not so, then there would exist $y_0 \in H$, different from zero and orthogonal to $D(A^*)$. The element $\tilde{y}_0 = \{0, y_0\} \in \tilde{H}$ would then be orthogonal to $\tilde{U}[\mathcal{G}r(A^*)]$, since $(\{0, y_0\}, \tilde{U}\{y, A^*y\}) = (0, A^*y) - (y_0, y) = 0$ for every $\{y, A^*y\} \in \mathcal{G}r(A^*)$. Consequently, $\{0, y_0\} \in \mathcal{G}r(A)$, whence $y_0 = A0 = 0$. The contradiction obtained proves the assertion.

Since $D(A^*)$ is everywhere dense, A^{**} is uniquely defined. For proof of the equality $A^{**} = A$, it suffices to make use of the relation (6) and the lemma.

THEOREM 5. *An operator A^{**} exists iff an operator A defined on an everywhere dense set is closable. In this case $A^{**} = \bar{A}$.*

PROOF. If A admits the closure $\bar{A}$ then, by Theorem 4, $(\bar{A})^{**}$ exists and $(\bar{A})^{**} = \bar{A}$. However, $(\bar{A})^* = A^*$; consequently, $(\bar{A})^{**} = A^{**}$, whence $A^{**} = \bar{A}$, proving the first part of the theorem.

Suppose that A^{**} exists. Apply (5) to $\bar{A}^*$, getting

$$\tilde{H} = \tilde{U}[\mathcal{G}r(A^*)] \dot{+} \mathcal{G}r(A^{**}). \tag{7}$$

On the other hand, by applying the operator $\tilde{U}$ to both sides of the equality $\tilde{H} = \tilde{U}[\overline{\mathcal{G}r(A)}] \dot{+} \mathcal{G}r(A^*)$, we get

$$\tilde{H} = \tilde{U}[\mathcal{G}r(A^*)] \dot{+} \overline{\mathcal{G}r(A)}. \tag{8}$$

Comparing (7) and (8) it follows that $\mathcal{G}r(A) \subset \mathcal{G}r(A^{**})$, that is, A admits the closed extension A^{**}.

7.64. Invariant subspaces. Reducibility. The idea of invariable subspaces is applicable also to non-bounded operators.

A subspace L is said to be **invariant** with respect to the operator A, if:

(i) $x \in D(A)$ implies $Px \in D(A)$ ($P = P_L$);

(ii) $x \in D(A) \cap L$ implies $Ax \in L$ [that is, $PAPx = APx$ for all $x \in D(A)$].

From (i) and $\overline{D(A)} = H$, it follows that $D(A) \cap L$ is everywhere dense in L.

Let us show that if L is invariant also for a non-bounded symmetric operator, then $M = H \dot{-} L$ is invariant. In fact, let $x \in D(A)$ and $x = x_1 + x_2$, where $x_1 \in L$ and $x_2 \in M$. Since L is an invariant subspace, $x_1 \in D(A)$ and since $D(A)$ is a linear manifold, hence $x_2 = x - x_1 \in D(A)$.

Further, if $x \in D(A) \cap M$ and y is any element in $D(A) \cap L$, then $(Ax, y) = (x, Ay) = 0$, since $Ay \in L$ and $x \perp L$. Thus, the element Ax is orthogonal to $D(A) \cap L$ and since this manifold is everwhere dense in L, $Ax \perp L$, whence $Ax \in M$.

If L is invariant for A, then L is said to **reduce** A.

THEOREM 6. *A subspace L reduces a symmetric operator A iff the projection operator P on this subspace commutes with A.*

Suppose that L is an invariant subspace. Thereupon, if $x \in D(A)$, then $Px \in D(A)$, and $Px \in D(A) \cap L$. By condition (ii) of L being invariant,

$$PAPx = APx. \tag{9}$$

Since A is symmetric, $H \dot{-} L$ is also invariant for A. Hence, as in the foregoing, $(E-P)A(E-P)x = A(E-P)x$, or, on removing the parentheses and simplifying,

$$PAPx = PAx \tag{10}$$

From (9) and (10) it follows that $PAx = APx$, whence it is immediate that A commutes with the bounded operator P.

Conversely, let A and P be commuting. Then, as before, $x \in D(A)$ implies

$Px \in D(A)$. Further, $Ax = APx = PAx$ for $x \in D(A) \cap L$, that is, $Ax \in L$, proving that L is invariant.

7.7 SELF-ADJOINT OPERATORS AND EXTENSION OF SYMMETRIC OPERATORS. DEFICIENCY INDICES.

7.71. Self-Adjoint operators. A linear (not necessarily bounded) operator A is said to be **self-adjoint** if $A = A^*$. This definition implies that every self-adjoint operator is symmetric. The converse is not true, as shown below.

A number of assertions made for the spectrum of bounded self-adjoint operators extend also to non-bounded self-adjoint operators. Thus, all eigenvalues of a self-adjoint operator are real and eigenvectors corresponding to distinct eigenvalues are orthogonal; the point λ is a regular value of the operator, iff there is a number c such that

$$\| (A-\lambda E)\, x \| \geqslant c \, \| x \| \tag{1}$$

for every $x \in D(A)$. This is shown, for example, by proving the latter statement.

If λ is regular, then there is a bounded inverse operator $R_\lambda = (A-\lambda E)^{-1}$. Hence,

$$\| x \| = \| R_\lambda (A-\lambda E)\, x \| \leqslant \| R_\lambda \| \, \| (A-\lambda E)\, x \|,$$

and we obtain (1) where $c = 1 / \| R_\lambda \|$. Suppose that (1) is satisfied. Consider again a linear manifold L consisting of elements of the form $y = (A-\lambda E)\, x$, where x runs through $D(A)$. The correspondence between $D(A)$ and L in virtue of (1) is one-one. L is everywhere dense in H. If this were not so, then there would exist in H an element $x_0 \neq 0$, such that $(x_0, y) = 0$ for every $y \in L$, or

$$(x_0, Ax-\lambda x) = 0 \tag{2}$$

for every $x \in D(A)$. It would follow from (2) that $(Ax, x_0) = (x, \bar{\lambda} x_0)$, implying that $x_0 \in D(A^*) = D(A)$, and

$$A^* x_0 = Ax_0 = \bar{\lambda} x_0.$$

By the same reasonings as adopted in the case of bounded operators, this is impossible.

Finally, L is closed. Let $\{y_n\} \subset L$, $y_n \to y_0$. If $y_n = A_\lambda x_n$, then by (1), $\| x_n - x_m \| \leqslant (1/c) \| y_n - y_m \|$, whence $\| x_n - x_m \| \to 0$. The operator A being closed (self-adjoint operators are, in particular, closed), we get

$$x_0 = \lim_n x_n \in D(A), \qquad y_0 = A_\lambda x_0,$$

exhibiting that L is closed.† It is completely proved that λ is a regular point in exactly the same way as in the case of bounded operators.

† See, p. 241.

COROLLARY 1. *The point λ belongs to the spectrum of a self-adjoint operator, iff in $D(A)$ there exists a sequence $\{x_n\}$, such that $\| x_n \| = 1$, $\|Ax_n - \lambda x_n\| \to 0$ as $n \to \infty$.*

COROLLARY 2. *The set of regular points of a self-adjoint operator is open and, consequently, the spectrum is a closed set.*

COROLLARY 3. *Every non-real λ is a regular value of a self-adjoint operator and, consequently, the spectrum of this operator lies wholly on a real line.*

7.72. Extension of a symmetric operator. Deficiency indices. Given a symmetric operator A with domain $D(A)$, assumed, as usual, everywhere dense in H.

If A is not closed, then it is *a priori* closable. In what follows it shall, therefore, be assumed to be closed. Now, we describe a process permitting, in general, the construction of extensions of a symmetric operator A and, in particular, the extension of a symmetric operator to a self-adjoint operator.

Let B be a symmetric extension of the operator A. Then $A \subset B$ implies $B^* \subset A^*$ and since $B \subset B^*$, hence $B \subset A^*$. Thus, every symmetric extension of A is a part of the self-adjoint operator A^*:

$$D(B) \subset D(A^*), \qquad By = A^*y \text{ for } y \in D(B). \tag{3}$$

Since B is symmetric, (By, y) is real for every $y \in D(B)$. However, $(By, y) = (A^*y, y)$ and, hence $D(B) \subset \Gamma$, where Γ is a set† of the elements y in $D(A)$, for which the quadratic form (A^*y, y) takes real values.

Conversely, if L is a linear manifold, satisfying the condition $D(A) \subset L \subset \Gamma$, then B defined on L by $By = A^*y$ is a symmetric extension of A.

Let L_i be a linear manifold of elements of the form $y = (A+iE)\,x$, where i is an imaginary element and x runs through $D(A)$. Show that L_i is a subspace. Simple calculations yield $\| (A \pm iE)\, x \|^2 = \| Ax \|^2 + \| x \|^2$, whence $\| (A+iE)\, x \| \geqslant \| x \|$. Now, let $y_n = (A+iE)\, x_n$ and $y_n \to y_0$. Then $\| y_n - y_m \| \to 0$ and, consequently, also $\| x_n - x_m \| \to 0$ as $n, m \to 0$. By the completeness of H, $x_n \to x_0$.

Thus, $x_n \in D(A)$, $x_n \to x_0$, $Ax_n = y_n - ix_n \to y_0 - ix_0$. Since A is closed, $x_0 \in D(A)$ and $Ax_0 = y_0 - ix_0$. Thereupon, $y_0 \in L_i$, evidencing that L_i is closed.

The element z is orthogonal to L_i iff for every $x \in D(A)$, $(z, Ax + ix) = 0$ or, $(Ax, z) = (x, iz)$, that is, iff $z \in D(A^*)$ and $A^*z = iz$. Consequently, the orthogonal complement of L_i is N_i, a subspace of eigenvectors of A^* corresponding to the eigenvalues i,

$$H = L_i \dotplus N. \tag{4}$$

Similarly,
$$H = L_{-i} \dotplus N_{-i}. \tag{5}$$

† Note that Γ does not form a linear manifold.

LEMMA 1. *The domain $D(A^*)$ of an operator A^*, adjoint to a closed symmetric operator A, is the direct sum of a linear manifold $D(A)$ and a pair of the spaces N_i and N_{-i}:*

$$D(A^*) = D(A) \oplus N_i \oplus N_{-i}. \tag{6}$$

Let y be an arbitrary element in $D(A^*)$. Consider the element $A^*y - iy$. Then, by (5),

$$A^*y - iy = (Ax - ix) + \tilde{z}_0.$$

Taking note of $Ax = A^*x$ and $A^* \tilde{z}_0 = -i\tilde{z}_0$ the preceding relation yields

$$A^*\left(y - x - \frac{1}{2}\, i\tilde{z}_0\right) = i\,(y - x) + \tilde{z}_0 + A^*\left(-\frac{1}{2}\, i\tilde{z}_0\right)$$

$$= i\,(y - x) + \tilde{z}_0 - \frac{1}{2}\, i\,(-i)\,\tilde{z}_0 = i(y - x) + \frac{1}{2}\,\tilde{z}_0$$

$$= i\,(y - x) + \frac{1}{2}\, i\,(-i)\,\tilde{z}_0 = i\left(y - x - \frac{1}{2}\, i\tilde{z}_0\right).$$

Consequently, $y - x - \frac{1}{2}\, i\tilde{z}_0 = z \in N_i$. Thereupon, set $\frac{1}{2}i\tilde{z}_0 = \tilde{z}$, to receive

$$y = x \dotplus z + \tilde{z}, \tag{7}$$

which is the required representation of y. To show that this representation is unique, let

$$y = x_1 + z_1 + \tilde{z}_1 \tag{7$_1$}$$

be another representation of the same element y. Then,

$$(x - x_1) + (z - z_1) + (\tilde{z} - \tilde{z}_1) = 0. \tag{8}$$

Apply A^* to both sides of (8), getting

$$A\,(x - x_1) + i\,(z - z_1) - i\,(\tilde{z} - \tilde{z}_1) = 0. \tag{9}$$

Multiplying (8) by i and subtracting from (9), we get

$$[\,A\,(x - x_1) - i(x - x_1)\,] - 2i\,(\tilde{z} - \tilde{z}_1) = 0.$$

The members on the left-hand side of this equality are orthogonal, $A\,(x-x_1) - i\,(x - x_1) \perp 2i\,(\tilde{z} - \tilde{z}_1)$. Consequently, $2i\,(\tilde{z} - \tilde{z}_1) = 0$, or $\tilde{z} = \tilde{z}_1$. Similarly, $z = z_1$. Thereupon, also $x = x_1$. The lemma is completely proved.

It is possible to take any complex number λ in place of the imaginary element i. In that case, we obtain a different structure

$$D(A^*) = D(A) \oplus N_\lambda \oplus N_{\bar{\lambda}}.$$

The subspaces N_λ and $N_{\bar{\lambda}}$ for distinct λ are, generally speaking, distinct. However, it can be shown that if λ lies in the upper half-plane, then the dimension of N_λ coincides with the dimension of N_i and the dimension of $N_{\bar{\lambda}}$ with that of N_{-i}. The dimensions of N_i and N_{-i} are called the **deficiency indices** of the operator A and N_λ and $N_{\bar{\lambda}}$ are said to be **deficiency subspaces.**

LEMMA 2. *In order that $y \in D(A^*)$ belongs to the set* Γ (see p. 241), *it is necessary and sufficient that* $\| z \| = \| \tilde{z} \|$ *is satisfied in the representation* (7).

In fact, if $y = x + z + \tilde{z}$, then

$$(A^*y, y) = [A^*x + A^*(z + \tilde{z}), x + (z + \tilde{z})]$$
$$= (Ax, x) + (Ax, z + \tilde{z}) + (A^*(z + \tilde{z}), x)$$
$$+ (A^*(z + \tilde{z}), z + \tilde{z}).$$

Since (Ax, x) is real and

$$(Ax, z + \tilde{z}) + (A^*(z + \tilde{z}), x) = [x, A^*(z + \tilde{z})] + (A^*(z + \tilde{z}), x)$$

is also real as a sum of complex conjugate quantities, it follows that $\operatorname{Im}(A^*y, y) = \operatorname{Im}(A^*(z + \tilde{z}), z + \tilde{z})$. Furthermore,

$$(A^*(z + \tilde{z}), z + \tilde{z}) = (iz - i\tilde{z}, z + \tilde{z})$$
$$= i\| z \|^2 + i(z, \tilde{z}) - i(\tilde{z}, z) - i\| \tilde{z} \|^2.$$

Again $i(z, \tilde{z}) - i(\tilde{z}, z)$, as a sum of complex conjugate quantities, is real. Hence

$$\operatorname{Im}(A^*(z + \tilde{z}), z + \tilde{z}) = i(\| z \|^2 - \| \tilde{z} \|^2). \quad \blacksquare$$

THEOREM 1. *Every symmetric extension B of a closed symmetric operator A corresponding to two linear manifolds $T_i \subset N_i$ and $T_{-i} \subset N_{-i}$ and an operator U which isometrically maps T_i onto T_{-i} has the following properties*:

(*a*) *The domain $D(B)$ of B consists of all elements of the form*

$$y = x + z + Uz, \tag{10}$$

where $x \in D(A)$ and $z \in T_i$ are arbitrary elements;

(*b*) *The values of the operator B on the elements of the form* (10) *are given by*

$$By = Ax + iz - iUz. \tag{11}$$

Conversely, *given two linear manifolds $T_i \subset N_i$ and $T_{-i} \subset N_{-i}$ and an operator U mapping isometrically T_i onto T_{-i}, then the operator B defined on a set of elements of the form* (10) *by the formula* (11) *is a symmetric extension of the operator A.*

B is closed iff T_i and T_{i-1} are closed.

Let B be a symmetric extension of A and $y \in D(B)$. As already seen, $D(B) \subset D(A^*)$ and by (7) y assumes the form

$$y = x + z + \tilde{z}, \tag{12}$$

and since $y \in \Gamma$, Lemma 2 implies

$$\| z \| = \| \tilde{z} \|. \tag{13}$$

If y runs through $D(B)$, the element z runs through some linear manifold T_i and $\tilde{z}$ through a linear manifold T_{-i}. For this, it is possible to set in correspondence only a single element $\tilde{z} \in T_{-i}$ to $z \in T_i$. In fact, if

$$y_1 = x_1 + z_1 + \tilde{z}_1, \quad y_2 = x_2 + z_1 + \tilde{z}_2, \quad y_1, y_2 \in D(B),$$

then, $\quad y_1 - y_2 = x_1 - x_2 + 0 + (\tilde{z}_1 - \tilde{z}_2) \in D(B) \subset \Gamma,$

and hence, because of (13), $\| \tilde{z}_1 - \tilde{z}_2 \| = \| 0 \| = 0$, that is, $\tilde{z}_1 = \tilde{z}_2$.

With regard to z to which $\tilde{z}$ corresponds uniquely by (12), an isometrically isomorphic mapping of T_i onto T_{-i} is obtained. Denote by U this map, leading obviously to (10), but, then

$$By = A^*y = A^*(x + z + Uz) = Ax + iz - iUz,$$

and (11) is proved.

Conversely, let $T_i \subseteq N_i$ and $T_{-i} \subseteq N_{-i}$ be two linear manifolds and let U be an operator isometrically mapping T_i onto T_{-i}. The operator B consisting of elements (10) and defined by (11) is a symmetric extension of A, since the linear manifold $D(B)$ of elements (10) satisfies the condition $D(B) \subset \Gamma \cap D(A^*)$ and $By = A^*y$ is satisfied on $D(B)$.

In order to prove the ultimate assertion of the theorem, first note that for the operator B to be closed it is necessary and sufficient that the manifold L_i' of elements of the form $(B + iE)\, y$, $y \in D(B)$ be closed. The necessity stands established on p. 241. For the proof of sufficiency, assume that L_i' is closed but B is not closed. Closing B, associate a new limit element to L_i' and, consequently, L_i' is also not closed.

For every $y \in D(B)$,

$$(B + iE)y = (B + iE)\,(x + z + \tilde{z}) = (A + iE)x + 2iz$$

and, consequently,

$$L_i' = L_i \dotplus T_i, \tag{14}$$

where L_i is the collection of elements $(A + iE)x$, $x \in D(A)$. Since L_i is closed, L_i' turns out to be closed iff T_i is closed. ■

Assuming that the method indicated above permits extension of a symmetric operator A to a symmetric operator B, the question arises as to the specification of deficiency spaces and deficiency indices in this extension.

THEOREM 2. *Let B be a closed symmetric extension of a closed symmetric operator A with domain $D(B) = D(A) + T_i + U(T_i)$. Denote the deficiency indices of A by (m_1, m_2): $m_1 = \dim N_i$, $m_2 = \dim N_{-i}$, and those of B by (m_1', m_2') : $m_1' = \dim N_i'$, $m_2' = \dim N_{-i}'$, where N_i' and N_{-i}' are deficiency subspaces of B. Thereupon*

$$N_i = N_i' \dotplus T_i, \quad N_{-i} = N_{-i}' \dotplus T_{-i},$$

and, consequently, if $\dim T_i = \dim T_{-i} = l$, *then*

$$m_1 = m_1' + l, \qquad m_2 = m_2' + l.$$

PROOF. In fact, by (4), $H = L' + N_i'$. Making use of (14), $L' = L_i \dotplus T_i$. Then, $H = L_i \dotplus N_i' + T_i$.

On the other hand, $H = L_i \dotplus N_i$, whence

$$N_i = N_i' \dotplus T_i.$$

Similarly, it can be proved that $N_{-i} = N_{-i}' + T_{-i}$.

The relationship between the deficiency indices is immediate from these equalities.

It is now intended to deal with the so-called **maximal symmetric extensions** of symmetric operators, that is, such symmetric extensions as do not admit further extension with retention of symmetry.

Let A have deficiency indices (0, 0). This implies that the deficiency subspaces N_i and N_{-i} consist of only null elements and, consequently, $D(A^*) = D(A)$. However, this signifies that A is a self-adjoint operator and does not admit a symmetric extension; in other words, *every self-adjoint operator is maximally symmetric.*

Let the deficiency indices (m_1, m_2) of the operator A be finite. Assume first that $m_1 = m_2 = m \neq 0$. Select in N_i and N_{-i} completely orthonormal systems $e_1, e_2, \ldots, e_m, e_1', e_2', \ldots, e_m'$. Set the element $\tilde{z} = \sum_{k=1}^{m} c_k e_k' \in N_{-i}$ in correspondence to the element $z = \sum_{k=1}^{m} c_k e_k \in N_i$. Evidently, this correspondence is isometrically isomorphic and generates an operator U, isometrically mapping all of N_i onto all of N_{-i}. It is possible to take N_i and N_{-i}, respectively, as the subspaces T_i and T_{-i},

$$D(B) = D(A) + N_i + U(N_i).$$

The operator B has deficiency indices (0, 0) and, consequently, is a self-adjoint extension of a symmetric operator A. This extension is an infinite set. In fact, to the element $z = \sum_{k=1}^{m} c_k e_k$ there can be set in correspondence the element $\tilde{z}(\tau) = \sum_{k=1}^{m} c_k e^{i\tau} e_k'$, τ real, and a more general element $\tilde{z}(\tau_1, \tau_2, \ldots, \tau_m) = \sum_{k=1}^{m} c_k e^{i\tau_k} e_k'$.

Thus, we obtain a continuum of isometric operators $U_{\tau_1, \tau_2 \ldots \tau_m}$ and corresponding to this a continuum of self-adjoint extensions.

Let A have deficiency indices (m_1, m_2) which are finite and not equal, say $m_1 > m_2$. Choosing m_2 as the first element of an orthonormal basis in N_i and denoting by T_i the subspace spanned by them, we take all of N_{-i} as T_{-i}. Then

$$D(B) = D(A) + T_i + U(T_i) = D(A) + T_i + N_{-i}.$$

The symmetric operator B has deficiency indices $(m_1 - m_2, 0)$ and does not admit further extension, either symmetric or self-adjoint. This symmetric operator, one of whose deficiency indices is zero and the other is different from zero, is said to be **maximal**. A self-adjoint operator, if both of its deficiency indices are zero, is sometimes said to be **hypermaximal**. If A has deficiency indices (m, ∞) or (∞, m) then in a way similar to that above, it is possible to construct an extension of A upto a maximal operator; a self-adjoint extension of A does not exist.

Finally, suppose that an operator A has deficiency indices (∞, ∞). Consider the case of a separable space when $(\aleph_0, \aleph_0)$ are deficiency indices. This operator admits maximal as well as hypermaximal extensions. Let T_i be a countable deficiency subspace of N_i. The isometrically isomorphic mapping of T_i onto N_{-i} leads to a maximal operator B with domain $D(B) = D(A) + T_i + N_{-i}$ and the deficiency indices $(m, 0)$, where m can be a finite or infinite number depending upon the choice of T_i. The isometrically isomorphic mapping of N_i onto all of N_{-i} leads to the self-adjoint extension B of the operator A.

Thus, *every symmetric but not self-adjoint operator A admits maximal or self-adjoint or even other extensions.* There exists a continuum of distinct maximal or self-adjoint extensions of A.

The examples of extensions of symmetric operators shall be derived in the next seetion.

7.8. SPECTRAL EXPANSION OF NON-BOUNDED SELF-ADJOINT OPERATORS. FUNCTIONS OF SELF-ADJOINT OPERATORS

THE INTEGRAL representation obtained in the foregoing for bounded self-adjoint operators extends also to non-bounded self-adjoint operators. Following F. RIESZ and E. R. LORCH, the extension of this method is accomplished by reducing non-bounded operators, to a sequence of bounded operators.

7.81. The Stieltjes Integral. Let E_λ, $-\infty < \lambda < \infty$, be a **resolution of the identity**, that is, a family of projection operators depending on the real parameter λ and having the following properties:

(i) $E_\lambda \leqslant E_\mu$ or, equivalently, $E_\lambda E_\mu = E_\lambda$, for $\lambda < \mu$;
(ii) $E_{\lambda-0} = E_\lambda$, that is, E_λ is continuous from the left with respect to λ;
(iii) $E_{-\infty} = 0$, $E_{+\infty} = E$.

Further, let $f(\lambda)$ (bounded or unbounded) be a complex-valued function defined on the interval $(-\infty, \infty)$ and uniformly continuous there.

Partition $(-\infty, \infty)$ into the half-open subintervals $\Delta_k = [\lambda_k, \mu_k)$ and consider the series

$$\sum_{k=-\infty}^{\infty} f(\nu_k)\, E(\Delta_k)\, x, \qquad \lambda_k \leqslant \nu_k < \mu_k. \tag{1}$$

This series is composed of mutually orthogonal members and for its convergence it is necessary and sufficient that the series

$$\sum_{k=-\infty}^{\infty} |f(\nu_k)|^2\, \| E(\Delta_k)\, x \|^2 = \sum_{k=-\infty}^{\infty} |f(\nu_k)|^2\, (E(\Delta_k)\, x, x) \tag{2}$$

be convergent.

The latter series represents integral sums of the STIELTJES integral

$$\int_{-\infty}^{\infty} |f(\lambda)|^2 \, d(E_\lambda x, x) \tag{3}$$

and converges for any partitions of the interval $(-\infty, \infty)$ iff this integral converges. Denote by $D(f)$ a set of elements $x \in H$, for which the series (2) or, equivalently the integral (3), converges.

Let α be an arbitrary positive number. Consider the half-open interval $\Delta_\alpha = [-\alpha, \alpha)$ on which the function $|f(\lambda)|$ is bounded and hence

$$\int_{-\alpha}^{\alpha} |f(\lambda)|^2 \, d(E_\lambda x, x) < \infty$$

for every $x \in H$. However,

$$\int_{-\alpha}^{\alpha} |f(\lambda)|^2 \, d(E_\lambda x, x) = \int_{-\infty}^{\infty} |f(\lambda)|^2 \, d(E_\lambda E(\Delta_\alpha) x, E(\Delta_\alpha) x).$$

Thus, the elements of the form $E(\Delta_\alpha) x$, α and x arbitrary, belong to $D(f)$.

Since $E(\Delta_\alpha) x \to x$ as $\alpha \to \infty$, it follows that the set $\{E(\Delta_\alpha) x\}$ and, what is more, $D(f)$, are everywhere dense in H. It is easy to remark that $D(f)$ is a linear manifold.

Select $x \in D(f)$ and consider the sums (1) corresponding, respectively, to the half-open subintervals Δ_k' and Δ_k'' of $(-\infty, \infty)$, where

$$\max(\mu_k' - \lambda_k') \leqslant \delta \quad \text{and} \quad \max(\mu_k'' - \lambda_k'') \leqslant \delta.$$

Then, making use of the property of additivity and orthogonality of $E(\Delta)$, it is routine to calculate that

$$\left\| \sum_{k=-\infty}^{\infty} f(\nu_k') E(\Delta_k') x - \sum_{k=-\infty}^{\infty} f(\nu_k'') E(\Delta_k'') x \right\|^2$$

$$\leqslant \omega^2 \{(E_\infty x, x) - (E_{-\infty} x, x)\} = \omega^2 (x, x), \tag{4}$$

$$\omega = \sup_{|\lambda - \mu| \leqslant \delta} |f(\lambda) - f(\mu)| \,.$$

Given a sequence of contracting partitions of $(-\infty, \infty)$, such that

$$\delta_n = \max |\mu_k^{(n)} - \lambda_k^{(n)}| \to 0,$$

and let $\{s_n\}$ be a sequence of the sums of the series (1), corresponding to these partitions. By (4) the sequence $\{s_n\}$ satisfies CAUCHY's condition

$$\| s_{n+p} - s_n \|^2 \leqslant \omega_n^2 (x, x) \to 0 \quad \text{as } n \to \infty, \quad p > 0$$

and, consequently, converges to some limit $s \in H$. This limit, denoted by $\int_{-\infty}^{\infty} f(\lambda)\, dE_\lambda x$, is called the **Stieltjes integral** of the function $f(\lambda)$ with respect to E_λ.

The STIELTJES integral

$$\int_{-\infty}^{\infty} f(\lambda)\, dE_\lambda x \tag{5}$$

represents, evidently, a certain linear operator S defined on a linear manifold $D(S) = D(f)$,

$$S = \int_{-\infty}^{\infty} f(\lambda)\, dE_\lambda x. \tag{6}$$

The formula (5) also implies that

$$(Sx, y) = \int_{-\infty}^{\infty} f(\lambda)\, d(E_\lambda x, y) \tag{7}$$

for every $x \in D(S)$ and $y \in H$.

It is plain that the definition of the STIELTJES integral deduced above extends to the function $f(\lambda)$, which is piecewise uniformly continuous, that is, continuous everywhere excluding a finite number of jump points and is uniformly continuous in the interval of continuity.

From (7) it follows that

$$(Sx, x) = \int_{-\infty}^{\infty} f(\lambda)\, d(E_\lambda x, x),$$

and if $f(\lambda)$ is a real function, then (Sx, x) is also real and, hence the STIELTJES integral of real functions defines a symmetric operator.

Show that S is also a self-adjoint operator. Since $D(S) \subset D(S^*)$ in virtue of S being symmetric, it remains to prove the inverse inclusion.

Let x be an arbitrary element in H and $\Delta_n = [-n, n)$. Then, $x_n = E(\Delta_n)x \in D(S)$. Denote by L_n the subspace of the projection operator $E(\Delta_n)$. Evidently, S and $E(\Delta_n)$ commute. Then,

$$Sx_n = SE(\Delta_n)\, x = E(\Delta_n)\, Sx \in L_n.$$

Now, let y be an arbitrary element in $D(S^*)$. Since $y_n = E(\Delta_n)y \in D(S)$ and $D(S) \subset D(S^*)$, hence $y_n \in D(S^*)$ and $S^*y_n = Sy_n \in L_n$. Let $z_n = y - y_n$, to receive $z_n \in D(S^*)$ and $z_n \perp L_n$. If $\tilde{x}_n$ is arbitrary in L_n, then

$$(S^*z_n, \tilde{x}_n) = (z_n, S\tilde{x}_n) = 0,$$

since $S\tilde{x}_n \in L_n$. Consequently, $S^*z_n \perp L_n$. Hence,

$$\| S^*y \|^2 = \| S^*y_n \|^2 + \| S^*z_n \|^2 \geqslant \| S^*y_n \|^2 = \| Sy_n \|^2.$$

However, $$\| Sy_n \|^2 = \int_{-n}^{n} [f(\lambda)]^2\, d(E_\lambda y, y).$$

Thus,
$$\int_{-n}^{n} [f(\lambda)]^2 \; d(E_\lambda y, y) \leqslant \| S^*y \|^2$$

for every n, implying
$$\int_{-\infty}^{\infty} [f(\lambda)]^2 \; d(E_\lambda y, y) < \infty,$$

that is, $y \in D(S)$. The inclusion $D(S^*) \subseteq D(S)$ is proved and together with this it is also established that the operator S is self-adjoint.

7.82. The Two Lemmas. Lemma 1. *Let $H_1, H_2, \ldots, H_n, \ldots$, be a sequence of pairwise orthogonal subspaces of a Hilbert space, whose orthogonal sum coincides with H. Denote by x_n the projection of an element x on the subspace H_n. Further, let $A_1, A_2, \ldots, A_n, \ldots$, be a sequence of orthogonal self-adjoint operators, defined, respectively, on $H_1, H_2, \ldots, H_n, \ldots$ and mapping these subspaces into themselves.*

Then, in H there exists a unique self-adjoint operator A, which coincides with A_n on every H_n. Its domain $D(A)$ consists of exactly those $x \in H$, for which the series
$$\sum_{n=1}^{\infty} \| A_n x_n \|^2 \tag{8}$$

converges. For these $x \in D(A)$,
$$Ax = \sum_{n=1}^{\infty} A_n x_n. \tag{9}$$

Proof. Denote by $D(A)$ the set of those $x \in H$ for, which (8) converges. $D(A)$ is a linear manifold. Let $x, y \in D(A)$. Then, for any complex α and β,
$$\begin{aligned}\sum_{n=1}^{\infty} \| A_n(\alpha x + \beta y)_n \|^2 &= \sum_{n=1}^{\infty} \| \alpha A_n x_n + \beta A_n y_n \|^2 \\ &\leqslant \sum_{n=1}^{\infty} (\| A_n(\alpha x_n) \| + \| A_n(\beta y_n) \|)^2 \\ &\leqslant C \sum_{n=1}^{\infty} (\| A_n x_n \|^2 + \| A_n y_n \|^2) < \infty,\end{aligned}$$

where C depends only on α and β.

$D(A)$ is everywhere dense in H, since it consists of all elements of the form $\sum_{k=1}^{n} x_k$. $x_k \in H_k$, $k = 1, 2, \ldots, n$.

Let the operator A be defined on $D(A)$ by means of (9). The series on the right-hand side of this equality converges, since

$$\left\| \sum_{k=n+1}^{n+p} A_k x_k \right\|^2 = \sum_{k=n+1}^{n+p} \| A_k x_k \|^2 \to 0 \quad \text{as } n \to \infty, \quad p > 0,$$

owing to the mutual orthogonality of the elements $A_k x_k$. The operator A defined by (9) is, evidently, linear. Furthermore, for $x, y \in D(A)$,

$$(Ax, y) = \left(\sum_{n=1}^{\infty} A_n x_n, \sum_{k=1}^{\infty} y_k \right) = \sum_{n=1}^{\infty} (A_n x_n, y_n)$$

$$= \sum_{n=1}^{\infty} (x_n, A_n y_n) = \left(\sum_{k=1}^{\infty} x_k, \sum_{n=1}^{\infty} A_n y_n \right) = (x, Ay)$$

and, consequently, A is a symmetric operator. Hence, there is an adjoint operator A^* and $A^* \supset A$.

It is sought to establish the inverse inclusion. Let $y \in D(A^*)$, then

$$(x, A^* y) = (Ax, y) = \sum_{k=1}^{\infty} (A_k x_k, y_k) = \sum_{k=1}^{\infty} (x_k, A_k y_k)$$

for every $x \in D(A)$. Select for x an arbitrary element $z_n \in H_n$, to receive $(z_n, A^* y) = (z_n, A_n y_n)$, that is,

$$P_{H_n}(A^* y) = P_{H_n}(A_n y_n) = A_n y_n.$$

Hence,
$$\sum_{n=1}^{\infty} \| A_n y_n \|^2 = \sum_{n=1}^{\infty} \| P_{H_n}(A^* y) \|^2 = \| A^* y \|^2 < \infty,$$

implying also that $y \in D(A)$ and $A^* y = Ay$.

It remains to prove the uniqueness of the operator A. If B be any other such operator, then, firstly,

$$B\left(\sum_{k=1}^{n} x_k \right) = \sum_{k=1}^{n} B x_k = \sum_{k=1}^{n} A_k x_k = A\left(\sum_{k=1}^{n} x_k \right),$$

that is, both the operators coincide on a finite sum of the form $\sum_{k=1}^{n} x_k$. Now, if $x \in D(A)$, then $\sum_{k=1}^{n} x_k \to x$

$$B\left(\sum_{k=1}^{n} x_k \right) = \sum_{k=1}^{n} A_k x_k \to y = Ax,$$

and since B, being a self-adjoint operator, is closed, hence $x \in D(B)$ and $Bx = y = Ax$. Thus, $B \supset A$. On the other hand, the passage to an adjoint operator in this inclusion, yields $A \subset B$. Consequently, $A = B$. ∎

LEMMA 2. *For every self-adjoint operator A there exist two bounded self-adjoint operators B and C, such that*

(i) $R(B) \subset D(A)$, $\quad R(C) \subset D(A)$;
(ii) $0 \leqslant B \leqslant E$, $\| C \| \leqslant 1$; $Bx = 0$ *implies* $x = 0$;
(iii) $C = AB$;
(iv) *C and B permute or commute with A.*

Select the bounded operators $R_i = (A - iE)^{-1}$ and $R_{-i} = (A + iE)^{-1}$, mapping H onto $D(A)$ one-one. Note also that $R_i^* = R_{-i}$, $R_-^* = R_i$. Set

$$B = \frac{1}{2i}(R_i - R_{-i}), \qquad C = \frac{1}{2}(R_i + R_{-i}). \tag{10}$$

It is obvious that B and C are bounded, self-adjoint and satisfy as well the property (i). It follows from (10) that

$$R_i = C + iB, \qquad R_{-i} = C - iB. \tag{11}$$

Hence,
$$(A - iE)(C + iB) = (A - iE) R_i = E,$$
$$(A + iE)(C - iB) = (A + iE) R_{-i} = E,$$

or, removing the parentheses,

$$(AC + B) + i(AB - C) = E, \quad (AC + B) - i(AB - C) = E,$$

whence, by addition and subtraction,

$$AC + B = E, \qquad AB = C, \tag{12}$$

and property (iii) is also proved.

Since R_i and R_{-i} commute with A, among themselves and with any bounded operator commuting with A†, property (iv), evidently, holds for B and C.

Finally, it remains to establish property (ii) for B and C. Simple calculations yield $\| (A - iE) x \|^2 \geqslant \| x \|^2$ for $x \in D(A)$. Set $(A - iE) x = y$, to receive $x = R_i y$ and $\| R_i y \| \leqslant \| y \|$ for every $y \in H$. Consequently, $\| R_i \| \leqslant 1$.

Similarly, $\| R_{-i} \| \leqslant 1$, implying

$$\| B \| \leqslant 1, \qquad \| C \| \leqslant 1. \tag{13}$$

Further, if both sides of the first equality in (12) are multiplied by B from right, then

$$B = B^2 + ACB = B^2 + CAB = B^2 + C^2 \geqslant 0. \tag{14}$$

From (13) and (14) it follows that $0 \leqslant (Bx, x) \leqslant (x, x)$, that is, $0 \leqslant B \leqslant E$. Finally, let $Bx = 0$, getting also $Cx = ABx = 0$, implying

$$x = Ex = (B + AC)\, x = 0,$$

† If $AB = BA$, then $R_i B = R_i B (A - iE) R_i = R_i (A - iE) B R_i = B R_i$.

and the lemma is completely proved.

7.83. Integral representation of operators. Let A be a non-bounded self-adjoint operator and $\mathcal{E}_\lambda$ the spectral function of the bounded operator B, constructed in Lemma 2. Since $0 \leqslant (Bx, x) \leqslant (x, x)$, the spectrum of this operator lies on $[0, 1]$. Further, since $Bx = 0$ implies $x = 0$, $\lambda = 0$ is not an eigenvalue of the operator B and hence $\mathcal{E}_\lambda$ is continuous at the point $\lambda = 0$; consequently,

$$\mathcal{E}_{\lambda+0} = \mathcal{E}_0 = 0.$$

Let H_n be the subspaces onto which the operator $\mathcal{E}(\Delta_n)$ projects; $\Delta_n = \left[\frac{1}{n+1}, \frac{1}{n}\right)$ for $n \geqslant 2$ and $\Delta_1 = \left[\frac{1}{2}, 1+\varepsilon\right)$, ε any positive number. The spaces H_n are pairwise orthogonal, and since

$$\sum_{n=1}^{\infty} \mathcal{E}(\Delta_n) = \lim_{n\to\infty}\left(\mathcal{E}_{1+\varepsilon} - \mathcal{E}_{\frac{1}{n+1}}\right) = E - 0 = E,$$

the orthogonal sum of H_n yields all of the space H. Introduce the function

$$\varphi_n(\lambda) = \begin{cases} \dfrac{1}{\lambda} & \text{for } \dfrac{1}{n+1} \leqslant \lambda < \dfrac{1}{n} \\ 0 & \text{outside } \left[\dfrac{1}{n+1}, \dfrac{1}{n}\right), \end{cases}$$

and the operator

$$\varphi_n(B) = \int_0^{1+\varepsilon} \varphi_n(\lambda)\, d\mathcal{E}_\lambda = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{1}{\lambda}\, d\mathcal{E}_\lambda = \mathcal{E}(\Delta_n)\, \varphi_n(B).$$

Evidently, we have

$$B\varphi_n(B) = \varphi_n(B)\, B = \int_{\frac{1}{n+1}}^{\frac{1}{n}} d\mathcal{E}_\lambda = \mathcal{E}(\Delta_n).$$

Hence, for every $x \in H_n$,

$$x = \mathcal{E}(\Delta_n)\, x = B\varphi_n(B)\, x = Bz \in D(A),$$

and, furthermore,

$$Ax = AB\varphi_n(B)x = C\varphi_n(B)x = \varphi_n(B)Cx = \mathcal{E}(\Delta_n)\, \varphi_n(B)Cx.$$

By inspection of the last equality, the operator A on H_n is a bounded self-adjoint operator of A_n, which carries H_n into itself. Let $E_\lambda^{(n)}$ be a spectral function of A_n, defined by

$$A_n = \int_{a_n}^{b_n} \lambda\, dE_\lambda^{(n)}.$$

By Lemma 1, there is a self-adjoint operator E_λ, $-\infty < \lambda < +\infty$ which coincides with $E_\lambda^{(n)}$ on every H_n. Let x_n be a projection of the element x on H_n. Since

$$\sum_{n=1}^{\infty} \left\| E_\lambda^{(n)} x \right\|^2 \leqslant \sum_{n=1}^{\infty} \| x_n \|^2 \leqslant \| x \|^2, \tag{15}$$

hence the series $\sum_{n=1}^{\infty} \| E_\lambda^{(n)} x \|^2$ converges for every $x \in H$, the operator

$$E_\lambda x = \sum_{n=1}^{\infty} E_\lambda^{(n)} x \tag{16}$$

is completely defined and is, consequently, a bounded self-adjoint operator.

The orthogonality of H_n implies for this operator that

$$(E_\lambda x, y) = \sum_{n=1}^{\infty} (E_\lambda^{(n)} x_n, y_n),$$

$$\| E_\lambda x \|^2 = \sum_{n=1}^{\infty} \| E_\lambda^{(n)} x \|^2 = \sum_{n=1}^{\infty} (E_\lambda^{(n)} x_n, x_n).$$

From (16) it follows that

$$E_\lambda E_\mu x = E_\lambda \left(\sum_{n=1}^{\infty} E_\mu^{(n)} x_n \right) = \sum_{n=1}^{\infty} E_\lambda E_\mu^{(n)} x_n$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_\lambda^{(m)} E_\mu^{(n)} x = \sum_{n=1}^{\infty} E_\lambda^{(n)} E_\mu^{(n)} x = \sum_{n=1}^{\infty} E_\lambda^{(n)} x = E_\lambda x$$

for $\lambda < \mu$ and, analogously, $E_\mu E_\lambda x = E_\lambda x$.

It thereby follows, in particular, that $E_\lambda^2 = E_\lambda$, that is, E_λ is a projection operator. Further, for $\nu < \lambda$,

$$\| E_\lambda x - E_\nu x \|^2 = \sum_{n=1}^{\infty} \| E_\lambda^{(n)} x_n - E_\nu^{(n)} x_n \|^2$$

$$= \sum_{n=1}^{N} \| E_\lambda^{(n)} x_n - E_\nu^{(n)} x_n \|^2 + \sum_{n=N+1}^{\infty} \| E_\lambda^{(n)} x_n - E_\nu^{(n)} x_n \|^2.$$

Since $\| E_\lambda^{(n)} x_n - E_\nu^{(n)} x_n \| \leqslant 2 \| x_n \|$ for any n, hence

$$\| E_\lambda x - E_\nu x \|^2 \leqslant \sum_{n=1}^{N} \| E_\lambda^{(n)} x_n - E_\nu^{(n)} x_n \|^2 + 2 \sum_{n=N+1}^{\infty} \| x_n \|^2.$$

Given $\varepsilon > 0$. First, select N so large that

$$2 \sum_{n=N+1}^{\infty} \| x_n \|^2 < \frac{\varepsilon^2}{2}.$$

Then, $E_\lambda^{(n)}$ being continuous from the left, ν can be chosen so close to λ that

$$\sum_{n=1}^{N} \| E_\lambda^{(n)} x_n - E_\nu^{(n)} x_n \|^2 < \frac{\varepsilon^2}{2}.$$

For such ν we then have $\| E_\lambda x - E_\nu x \|^2 < \varepsilon^2$, that is, $E_\nu x \to E_\lambda x$, as $\nu \to \lambda$ and for $\nu < \lambda$, proving the continuity of E_λ from the left. Similarly, it can be verified that

$$E_\lambda x \to 0 \quad \text{as} \quad \lambda \to -\infty, \qquad E_\lambda x \to x \quad \text{as} \quad \lambda \to +\infty.$$

Consequently, E_λ is a resolution of the identity.

With the aid of this E_λ we construct the STIELTJES integral

$$\tilde{A} = \int_{-\infty}^{\infty} \lambda \, dE_\lambda,$$

which, as indicated above, defines a self-adjoint operator.

Let $x \in H_n$, then $E_\lambda x = E_\lambda^{(n)} x$ and, consequently,

$$\int_{-\infty}^{\infty} \lambda^2 \, d(E_\lambda x, x) = \int_{a_n}^{b_n} \lambda^2 \, d(E_\lambda^{(n)} x, x) < \infty.$$

Thus, $\tilde{A}x$ exists and

$$\tilde{A}x = \int_{-\infty}^{\infty} \lambda \, dE_\lambda x = \int_{a_n}^{b_n} \lambda \, dE_\lambda^{(n)} x = A_n x.$$

Consequently, $\tilde{A}$ coincides with A_n on every H_n. On the other hand, A also coincides with A_n on every H_n, and since there is just one such operator, hence $\tilde{A} = A$. Thus,

$$Ax = \tilde{A}x = \int_{-\infty}^{\infty} \lambda \, dE_\lambda x.$$

This gives the integral representation of a non-bounded self-adjoint operator.

The domain $D(A)$ of the operator A consists of exactly those elements $x \in H$, for which $\int_{-\infty}^{\infty} \lambda^2 \, d(E_\lambda x, x) < \infty$. It can also be shown that a resolution of the identity for the operator A is uniquely defined.

7.84. Functions of the operator. The construction of functions of self-adjoint bounded operators, dealt with in the foregoing, can be extended also to the construction of functions of self-adjoint non-bounded operators with the only distinction that here the *additive* and *multiplicative* properties of the correspondence between the functions of a real variable and the operator become somewhat complex and restricted.

Thus, let A be a non-bounded self-adjoint operator with domain $D(A)$ and let E_λ be a resolution of the identity generated by this operator. For an arbitrary function $f(\lambda)$, piecewise uniformly continuous on $(-\infty, \infty)$, construct the operator

$$Bx = \int_{-\infty}^{\infty} f(\lambda)\, dE_\lambda\, x$$

with domain $D(B)$, consisting of those $x \in H$ which satisfy

$$\int_{-\infty}^{\infty} | f(\lambda) |^2 d\,(E_\lambda\, x, x) < +\infty.$$

As seen above, $D(B)$ is everywhere dense in H and if $f(\lambda)$ is real, then B is a self-adjoint operator. The operator B† is called a **function of the operator** A and denoted by $f(A)$:

$$f(A) = \int_{-\infty}^{\infty} f(\lambda)\, dE_\lambda\, x.$$

Given an operator $f(A)$, with a function $f(\lambda)$ piecewise uniformly continuous on $(-\infty, \infty)$. The domain of $f(A)$ is denoted by $D\{f(A)\}$. Then, $E(\Delta_n)\, x \in D\{f(A)\}$ for arbitrary n and every $x \in H$, as evidenced on p. 247.

However, it is then clear that also $E(\Delta)\, x \in D\{f(A)\}$ for every $\Delta = [\alpha, \beta)$, if α and β are finite, and every $x \in H$.

†It is possible to construct [29] a broader class of the operator functions. Namely, a spectral function E_λ $(-\infty < \lambda < \infty)$ of the operator A generates functions of the interval $E(\Delta)$. This process in analogy to the procedure adopted in the theory of functions of real variables, admits extension to an operator of measure $E(M)$ of a linear point set M. This measure $E(M)$ is defined on a certain class of sets called A-measurable, including all Borel-sets. After defining the class of measurable sets, it is customary to define A-measurable functions and construct an operator by the LEBESGUE-STIELTJES integral

$$f(A) = \int_{-\infty}^{\infty} f(\lambda)\, dE_\lambda,$$

first for bounded and then for non-bounded functions. The domain of $f(A)$, which is a function of the operator A only in the sense indicated, again consists of just those x which satisfy

$$\int_{-\infty}^{\infty} |f(\lambda)|^2\, d(E_\lambda\, x, x) < \infty.$$

The last integral is also understood in the LEBESGUE-STIELTJES sense [29].

Such general functions of an operator are, however, beyond the scope of this text.

Let $x \in D\{f(A)\}$. From the inequality

$$\int_{-\infty}^{\infty} |f(\lambda)|^2 \, d(E_\lambda E(\Delta) x, \; E(\Delta) x) = \int_{\Delta} |f(\lambda)|^2 \, d(E_\lambda x, x)$$

$$\leqslant \int_{-\infty}^{\infty} |f(\lambda)|^2 \, d(E_\lambda x, x),$$

valid for any (finite or infinite) $\Delta = [\alpha, \beta)$, it is inferred that $E(\Delta) x \in D\{f(A)\}$. Since the STIELTJES integral is the limit of integral sums, it follows that

$$f(A) E(\Delta) x = \int_{-\infty}^{\infty} f(\lambda) \, d[E_\lambda (E(\Delta) x)]$$

$$= E(\Delta) \int_{-\infty}^{\infty} f(\lambda) \, dE_\lambda x = E(\Delta) f(A).$$

Thus, $E(\Delta)$ commutes with $f(A)$ for any Δ.

Let $k \neq 0$ be any (real or complex) number. Since the integral $\int_{-\infty}^{\infty} |k|^2 |f(\lambda)|^2 d(E_\lambda x, x)$ converges for exactly those x, for which $\int_{-\infty}^{\infty} |f(\lambda)|^2 d(E_\lambda x, x)$ converges, the domain of $f(A)$ coincides with that of the operator $(kf)(A)$ and

$$(kf)(A) x = kf(A) x.$$

Let $f_1(\lambda)$ and $f_2(\lambda)$ be a pair of functions piecewise uniformly continuous on $(-\infty, \infty)$. If $x \in D\{f_1(A)\} \cap D\{f_2(A)\}$, then

$$\left[\int_{-\infty}^{\infty} |f_1(\lambda) + f_2(\lambda)|^2 \, d(E_\lambda x, x) \right]^{1/2}$$

$$\leqslant \left[\int_{-\infty}^{\infty} |f_1(\lambda)|^2 \, d(E_\lambda x, x) \right]^{1/2} + \left[\int_{-\infty}^{\infty} |f_2(\lambda)|^2 d(E_\lambda x, x) \right]^{1/2},$$

and consequently, $x \in D\{(f_1 + f_2)(A)\}$. Thus,

$$f_1(A) + f_2(A) \subset (f_1 + f_2)(A). \tag{17}$$

It is sought to clarify when the sign of equality holds in (17).

Since $[f_1(\lambda) + f_2(\lambda)] - f_2(\lambda) = f_1(\lambda)$, hence

$$D\{(f_1 + f_2)(A)\} \cap D\{f_2(A)\} \subset D\{f_1(A)\}.$$

From (17) it follows that

$$D\{(f_1 + f_2)(A)\} \cap D\{f_2(A)\} \subseteq D\{f_1(A)\} \cap D\{f_2(A)\}$$
$$\subset D\{(f_1 + f_2)(A)\},$$

These inclusions imply that

$$D\{(f_1+f_2)(A)\} \cap D\{f_2(A)\} = D\{f_1(A)\} \cap D\{f_2(A)\}.$$

Analogously,

$$D\{(f_1+f_2)(A)\} \cap D\{f_1(A)\} = D\{f_1(A)\} \cap D\{f_2(A)\}.$$

Consequently, equality holds in (17) if either of the underlying inclusions is true:

$$D\{(f_1+f_2)(A)\} \subset D\{f_1(A)\} \quad \text{or} \quad D\{(f_1+f_2)(A)\} \subset D\{f_2(A)\}.$$

This occurs, for example, if either of the operators $f_1(A)$ and $f_2(A)$ is bounded.

Again, let $f_1(\lambda)$ and $f_2(\lambda)$ be piecewise uniformly continuous on a real line. Let $x \in D\{f_1(A)f_2(A)\}$. This implies that $x \in D\{f_2(A)\}$ and $f_2(A)x \in D\{f_1(A)\}$. The latter inclusion signifies that

$$\int_{-\infty}^{\infty} |f_1(\lambda)|^2 \, d(E_\lambda f_2(A)x, f_2(A)x) < \infty. \tag{18}$$

However,

$$(E_\lambda f_2(A)x, f_2(A)x) = \| E_\lambda f_2(A)x \|^2 = \int_{-\infty}^{\lambda} |f_2(\mu)|^2 \, d(E_\mu x, x).$$

Hence, $d(E_\lambda f_2(A)x, f_2(A)x) = |f_2(\lambda)|^2 \, d(E_\lambda x, x)$, and (18) reduces to

$$\int_{-\infty}^{\infty} |f_1(\lambda) f_2(\lambda)|^2 \, d(E_\lambda x, x) < \infty.$$

Thereupon, it follows that $x \in D\{(f_1 f_2)(A)\}$. Thus,

$$D\{f_1(A)f_2(A)\} \subset D\{(f_1 f_2)(A)\} = D\{(f_2 f_1)(A)\}, \tag{19}$$

that is,

$$f_1(A)f_2(A) \subset (f_1 f_2)(A). \tag{20}$$

It is sought to make explicit as to when the sign of equality holds in (20). Let $x \in D\{(f_1 f_2)(A)\}$ and $x \in D\{f_2(A)\}$. Then

$$\int_{-\infty}^{\infty} |f_1(\lambda)|^2 \, d(E_\lambda f_2(A)x, f_2(A)x) = \int_{-\infty}^{\infty} |f_1(\lambda) f_2(\lambda)|^2 \, d(E_\lambda x, x) < \infty,$$

implying $f_2(A)x \in D\{f_1(A)\}$ and, consequently, $x \in D\{f_1(A)f_2(A)\}$, leading to the inclusion

$$D\{f_2(A)\} \cap D\{(f_1 f_2)(A)\} \subset D\{f_1(A)f_2(A)\}. \tag{21}$$

Thence, by inspection of (19),

$$D\{f_2(A)\} \cap D\{(f_1 f_2)(A)\} \subset D\{f_1(A)f_2(A)\} \subset D\{(f_1 f_2)(A)\}.$$

From these inclusions it follows that: In order that equality holds in (20), it is necessary and sufficient that $D\{(f_1 f_2)(A)\} \subset D\{f_2(A)\}$.

Consider the case $f_1(\lambda) = f_2(\lambda) = f(\lambda)$.

Since $f(\lambda)$ is bounded on every finite interval, the divergence of the integral

$$\int_{-\infty}^{\infty} |f(\lambda)|^n \, d(E_\lambda x, x) \tag{22}$$

can occur only if $|f(\lambda)$ increases infinitely as $\lambda| \to \infty$. However, since $|f(\lambda)|^{n-1}$ grows less rapidly than $|f(\lambda)|^n$, the convergence of the integral (22) implies the convergence of the integral

$$\int_{-\infty}^{\infty} |f(\lambda)|^{n-1} \, d(E_\lambda x, x),$$

signifying that $D\{(f^n)(A)\} \subset D\{(f^{n-1})(A)\}$. Thereupon, $[f^{n-1}(A)]f(A) = (f^n)(A)$ and, consequently, $[f(A)]^n = (f^n)(A)$, that is,

$$[f(A)]^n = \int_{-\infty}^{\infty} [f(\lambda)]^n \, dE_\lambda .$$

It is sought to determine $f(A)^*$, the adjoint operator of $f(A)$. If $f(\lambda)$ is real then, as is known, $f(A)$ is a self-adjoint operator.

If $f(\lambda) = u(\lambda) + iv(\lambda)$ is a complex-valued function, bounded on $(-\infty, \infty)$, then by what has been proved,

$$f(A)^* = [u(A) + iv(A)]^* = u(A)^* - iv(A)^* = u(A) - iv(A) = \bar{f}(A),$$

where $\bar{f}(\lambda)$ is a function, complex-conjugate to $f(\lambda)$. If $f(\lambda)$ is not bounded, it assumes the form

$$f(\lambda) = |f(\lambda)| \, e^{i \arg f(\lambda)} = g(\lambda)\, h(\lambda).$$

Here $g(\lambda)$ is real, $|h(\lambda)| = 1$ and the domains $f(A)$ and $g(A)$, evidently, coincide. The operator $g(A)$ is self-adjoint and $h(A)$ is bounded. Hence,

$$f(A)^* = [g(A)\, h(A)]^* = h(A)^*\, g(A) = \bar{h}(A)\, g(A) = \bar{f}(A).$$

Suppose that T is a linear bounded operator that commutes with A. Then, T commutes with $R_\lambda = (A - \lambda E)^{-1}$ for every regular (λ) and, consequently, commutes with the operator

$$B = (1/2i)(R_i - R_{-i}).$$

The commutativity of T and B, in turn, implies that T commutes with every bounded function $f(B)$, in particular with the spectral function $\mathcal{E}_\lambda$ of this operator and with the function $\varphi_n(B)$ introduced above. This last comutativity implies that the subspace H_n reduces T. Hence, $A_n Tx = ATx = TAx = TA_n x$ for $x \in H_n$, that is, A_n and T commute on H_n. However, then, T

commutes with $E_\lambda^{(n)}$, a spectral function of the operator A_n and since the spectral function E_λ of the operator A is representable in the form

$$E_\lambda x = \sum_{n=1}^{\infty} E_\lambda^{(n)} x,$$

where the series converges for every $x \in H$, hence $TE_\lambda - E_\lambda T$. From T commuting with E_λ, it follows that T commutes with any bounded function $f(A)$.

Finally, if $f(A)$ is a non-bounded function, then put $f_n(A) = f(A) \chi_n(A)$, $\chi_n(\lambda)$ a characteristic function on the half-open interval $[-n, n)$. For any $x \in D\{f(A)\}$, we have

$$f_n(A)\, Tx = Tf_n(A)\, x. \tag{23}$$

Since $x \in D\{f(A)\}$, $f_n(A)\, x \to f(A)\, x$ as $n \to \infty$. Consequently, $Tf_n(A)\, x \to Tf(A)\, x$. However, then, as $n \to \infty$ the left-hand side of (23) tends to the limit $Tf(A)\, x$, implying that

$$Tx \in D\{f(A)\} \qquad \text{and} \qquad f(A)\, Tx = Tf(A)x.$$

Thus, every function of the operator A commutes with A. In the case of separable HILBERT spaces, this is a characteristic property of operator functions. Hence,

THEOREM. *In order that a closed operator B with a dense domain is a function of a self-adjoint operator A, it is necessary and sufficient that B commutes with A.*

For the proof of this theorem, see [29], for instance.

Let λ be a complex number or a point on the real line, in some neighbourhood (α, β) of which E_μ is constant. Put $\varphi(\mu) = 1/(\mu-\lambda)$, $-\infty < \mu < \infty$ in the former case and

$$\varphi(\mu) = \begin{cases} 1/(\mu-\lambda) & \text{outside} \quad (\alpha, \beta), \\ 0 & \text{if} \quad \mu \in (\alpha, \beta), \end{cases}$$

in the latter case, then $\varphi(\mu)$ is bounded and uniformly continuous on the entire real line. Hence, $\varphi(A)$ is a bounded operator and, consequently,

$$(A-\lambda E)\, \varphi(A) = \varphi(A)\, (A-\lambda E)$$
$$= \int_{-\infty}^{\infty} (\mu-\lambda) \frac{1}{\mu-\lambda}\, dE_\mu = \int_{-\infty}^{\infty} dE_\mu = E.$$

Again, it is found that complex points and points of a real line, in whose neighbourhood E_μ is constant, are regular points and the resolvent takes the form

$$R_\lambda = \int_{-\infty}^{\infty} \frac{dE_\mu}{\mu-\lambda}.$$

Conversely, let R_λ exist for real λ. Then, repeating the reasonings of Theorem 5 of Sec. 5, the spectral function E_μ is found to be constant in some neighbourhood of λ.

Finally, as also in the case of bounded self-adjoint operators, it can be shown that for λ_0 to be an eigenvalue of the operator, it is necessary and sufficient that λ_0 is a jump point of a resolution of the identity E_λ of this operator.

Reverting to the resolvent, we first have:

(i) If $R_\lambda x = 0$, then $x = (A-\lambda E)\, R_\lambda x = (A-\lambda E)\, 0 = 0$.
Further, the rules of operation on operator functions yield:

(ii) $R_\lambda^* = R_{\bar\lambda}$;

(iii) $$R_\lambda - R_\mu = \int_{-\infty}^{\infty} \frac{dE_\eta}{\eta-\lambda} - \int_{-\infty}^{\infty} \frac{dE_\eta}{\eta-\mu}$$

$$= \int_{-\infty}^{\infty} \frac{\lambda-\mu}{(\eta-\lambda)\,(\eta-\mu)}\, dE_\eta$$

$$= (\lambda-\mu) \int_{-\infty}^{\infty} \frac{dE_\eta}{\eta-\lambda} \int_{-\infty}^{\infty} \frac{dE_\eta}{\eta-\mu} = (\lambda-\mu)\, R_\lambda\, R_\mu\, .$$

We obtain the so-called **functional Hilbert equation** for resolvents.

Thus, the resolvent of a self-adjoint operator has the properties (i) thro' (iii). The converse is also true, namely: Given a family of linear bounded operators depending on a complex parameter λ and having the properties:

(i) $R_\lambda x = 0$ implies $x = 0$;

(ii) $R_\lambda^* = R_{\bar\lambda}$;

(iii) $R_\lambda - R_\mu = (\lambda-\mu)\, R_\lambda\, R_\mu$.

Then, there exists a self-adjoint (bounded or non-bounded) operator A, for which R_λ is a family of resolvents.

For proof, see [28], for instance.

Finally, it is remarked that relying on the functional equation of resolvents, it can be shown that a resolvent is an analytic function of the parameter λ, that is, in a neighbourhood of the regular point λ_0, the resolvent can be expanded in a power series of $(\lambda-\lambda_0)$ which converges with respect to operator norm [28].

7.9 EXAMPLES OF NON-BOUNDED OPERATORS

7.91. Operator of multiplication by independent variable. An operator of multiplication by the independent variable in the space $L_2(-\infty, \infty)$ serves as an example of non-bounded operators. Let $D(A)$ be a manifold of the function $x(t)$ square-integrable on $(-\infty, \infty)$ and such that

$$\int_{-\infty}^{\infty} t^2 \mid x(t) \mid^2 dt < \infty.$$

By inspection $D(A)$ is a linear manifold, everywhere dense in $L_2(-\infty, \infty)$, since it contains bounded functions, each of which vanishes outside of some interval $[a, b]$, $(\mid a \mid, \mid b \mid < \infty)$. The operator A is defined on this manifold by the equation $Ax = tx(t)$. Since

$$(Ax, x) = \int_{-\infty}^{\infty} tx(t)\,\bar{x}(t)\,dt = \int_{-\infty}^{\infty} t \mid x(t) \mid^2 dt$$

is real, A is a symmetric operator.

To show that A is a self-adjoint operator, let $y(t) \in D(A^*)$ and $x(t)$ be arbitrary square-integrable functions, which vanish for $\mid t \mid > n$. Then $x(t) \in D$, and $(Ax, y) = (x, y^*)$, or

$$\int_{-n}^{n} tx(t)\,\bar{y}(t)\,dt = \int_{-n}^{n} x(t)\,\overline{ty(t)}\,dt = \int_{-n}^{n} x(t)\,\overline{y^*(t)}\,dt,$$

whence

$$\int_{-n}^{n} x(t)\,\overline{[y^*(t)-ty(t)]}\,dt = 0.$$

$x(t)$ being arbitrary, $y^*(t) - ty(t) = 0$ a.e. on $[-n, n]$ for any fixed n and, consequently, a.e. also on $(-\infty, \infty)$. Since $y^*(t) \in L_2(-\infty, \infty)$, $ty(t) \in L_2$ $(-\infty, \infty)$, that is, $y(t) \in D(A)$. Thus, $D(A^*) \subset D(A)$ and consequently, $D(A^*) = D(A)$, proving that A is a self-adjoint operator.

The operator A does not have any eigenvalue, because, if $Ax = \sigma x$, then, $(t-\sigma)\,x(t) = 0$, whence $x(t) = 0$ a.e. on $(-\infty, \infty)$.

On the other hand, every real number σ is a spectrum point, which can be verified by reiterating the reasonings of Chap. 7.4, with regard to the operator of multiplication by the independent variable in the space $L_2[0, 1]$. Thus, the operator A has a purely continuous spectrum, filling in the entire real axis.

The resolvent of the operator A is defined by

$$R_\lambda x = \frac{1}{t-\lambda}\,x(t).$$

Thereupon,

$$(R_\lambda x, x) = \int_{-\infty}^{\infty} \frac{\mid x(t) \mid^2}{t-\lambda}\,dt = \int_{-\infty}^{\infty} \frac{1}{t-\lambda}\,d\varphi(t),$$

where $\varphi(t) = \int_{-\infty}^{t} |x(\tau)|^2 d\tau$. On the other hand,

$$(R_\lambda x, x) = \int_{-\infty}^{\infty} \frac{d(E_\mu x, x)}{\mu-\lambda} = \int_{-\infty}^{\infty} \frac{1}{\mu-\lambda} d\rho(\mu).$$

Equate both the expressions of $(R_\lambda x, x)$, to receive

$$\int_{-\infty}^{\infty} \frac{d\varphi(\xi)}{\xi-\lambda} = \int_{-\infty}^{\infty} \frac{d\rho(\xi)}{\xi-\lambda}$$

for all non-real λ. In virtue of the STIELTJES inversion formula and taking note of the continuity of $\varphi(\xi)$ and $\rho(\xi)$ it thereby follows [28] that $\rho(\xi) = \varphi(\xi)$, that is, $(E_\lambda x, x) = \int_{-\infty}^{\lambda} |x(t)|^2 dt$. Thereupon,

$$(E(\Delta) x, x) = \int_{\Delta} |x(t)|^2 dt = \int_{-\infty}^{\infty} \chi_\Delta(t) |x(t)|^2 dt,$$

$\chi_\Delta(t)$ a characteristic function of the interval Δ. Thus, $E(\Delta) x = \chi_\Delta(t) x(t)$ for any interval Δ.

The integral representation of the operator A takes the form

$$Ax = \int_{-\infty}^{\infty} \lambda \, dE_\lambda x = \int_{-\infty}^{\infty} \lambda d[\chi_\lambda(t) x(t)] = tx(t).$$

Here $\chi_\lambda(t)$ is a characteristic function of the interval $(-\infty, \lambda)$ and the STIELTJES integral degenerates into a subinterval expressed uniquely at the jump points of the integrating functions.

The operator function $F(A)$, corresponding to a function $f(\lambda)$ of real variables has, evidently, the form

$$F(A) x = F(t) x(t).$$

7.92. Differential operators. The differential operators constitute another example of non-bounded operators.

In a HILBERT space $L_2(a, b)$, where a and b are finite or infinite numbers, we consider the **differential operator** A defined by

$$A = i(d/dt).$$

(i) *Finite interval.* Suppose that a and b are finite, say $a = 0$ and $b = 1$. Assume that the domain $D(A)$ of the operator under consideration consists of absolutely continuous functions, having square-integrable derivatives and satisfying the boundary conditions

$$x(0) = x(1) = 0. \tag{1}$$

Then, integrating by parts, it is routine to verify that

$$(Ax, y) = \int_0^1 i \frac{dx(t)}{dt} \bar{y}(t)\, dt = \int_0^1 x(t) \overline{\left(i \frac{dy(t)}{dt} \right)}\, dt = (x, Ay)$$

for every $x, y \in D(A)$, that is, A is a symmetric operator.

(ii) *Semi-axis.* Now, let $a = 0$, $b = \infty$. Associate to $D(A)$ the function $x(t)$ which is square-integrable on $[0, \infty)$, has square-integrable derivative $[dx(t)]/dt$ there and satisfies the boundary condition $x(0) = 0$. It is required to show that in this case, we have also $x(\infty) = 0$.

Since $x(t)$ and $[dx(t)]/dt$ are square-integrable, $x(t)\,(\overline{[dx(t)]/dt})$ is integrable on $[0, \infty)$ and we can write

$$|x(t)|^2 = |x(0)|^2 + \int_0^t x(\tau) \overline{\frac{dx(\tau)}{d\tau}}\, d\tau + \int_0^t \bar{x}(\tau) \frac{dx(\tau)}{d\tau}\, d\tau.$$

The right-hand side of this equality tends to a finite limit as $t \to \infty$, so that there exists $\lim\limits_{t \to \infty} |x(t)| < \infty$. Because of the integrability of $|x(t)|^2$ on $[0, \infty)$, this limit can only be zero. Thus, we have

$$x(0) = x(\infty) = 0. \tag{2}$$

The formula

$$\int_0^n i \frac{dx(t)}{dt} \overline{y(t)}\, dt = ix(n)\, \overline{y(n)} + \int_0^n x(t) \overline{\left(i \frac{dy(t)}{dt} \right)}\, dt$$

as $n \to \infty$, implies in the limit $(Ax, y) = (x, Ay)$, that is, in case (ii) also A is symmetric.

(iii) *The whole real axis.* For $a = -\infty$ and $b = \infty$, we associate to $D(A)$ all functions square-integrable on $(-\infty, \infty)$ and having square-integrable derivatives on this interval. As above it can be verified that these assumptions imply that

$$x(-\infty) = x(\infty) = 0, \tag{3}$$

and here too A is a symmetric operator.

It is still necessary to show that $D(A)$ is everwhere dense in $L_2(a, b)$. Let an interval (α, β) coinciding with $(0, 1)$ in case of a finite interval be equal to $(0, \beta)$, where β is any finite number in case (ii) and any finite interval in case (iii). If $y(t)$ is a function in $L_2(a, b)$, orthogonal to $D(A)$, then treating $x(t)$ as any function of $D(A)$ in case (i) and any function of $D(A)$, vanishing outside of (α, β) in the remaining cases (ii) and (iii), we get

$$0 = (x, y) = \int_\alpha^\beta x(t)\, \overline{y(t)}\, dt = -\int_\alpha^\beta \frac{dx(t)}{dt} Y(t)\, dt,$$

where $Y(t)$ is the primitive of $y(t)$. Since $x(t)$ can be taken for any function, which is continuous on (α, β) and vanishes at the ends of this interval, it follows from the well-known lemma of calculus of variations applied to the continuous function $Y(t)$, that $Y(t) = \text{const}$ and, consequently, $y(t) \equiv 0$ on (α, β) and, hence, everywhere on (a, b). So, $A = i\,(d/dt)$ is a symmetric operator in the cases (i) thro' (iii).

It is sought to find the adjoint operator A^*. Let $y \in D(A^*)$. Then

$$(Ax, y) = \int_a^b i\,\frac{dx(t)}{dt}\,\overline{y(t)}\,dt = \int_a^b x(t)\,\overline{y^*(t)}\,dt = (x, y^*)$$

for every $x \in D(A)$. Select $x(t)$ for any function of $D(A)$ vanishing outside of (α, β). The preceding equality yields

$$\int_\alpha^\beta i\,\frac{dx(t)}{dt}\,\overline{y(t)}\,dt = \int_\alpha^\beta x(t)\,\overline{y^*(t)}\,dt.$$

Integrate by parts the right-hand side of this equality, to receive

$$\int_\alpha^\beta i\,\frac{dx(t)}{dt}\,\overline{y(t)}\,dt = -\int_\alpha^\beta \frac{dx(t)}{dt}\,\overline{Y^*(t)}\,dt, \tag{4}$$

where $Y^*(t) = \int_0^t y^*(\tau)\,d\tau$ is the primitive of $y^*(t)$. Eq. (4) reduces to

$$\int_\alpha^\beta \frac{dx(t)}{dt}\,\overline{[iy(t) - Y^*(t)]}\,dt = 0.$$

Thereupon, it again follows that

$$iy(t) - Y^*(t) = \text{const} \tag{5}$$

on (α, β) and, hence, everywhere on (a, b), that is, the function $y(t)$ is square-integrable on (a, b), having square-integrable derivatives on this interval. From (5) it follows that

$$y^*(t) = \frac{d}{dt}\,[Y^*(t)] = i\,\frac{dy(t)}{dt}, \quad \text{that is, } A^*y = i\,\frac{dy}{dt}$$

for every $y \in D(A^*)$. Conversely, if $y(t)$ is a function with properties indicated, then, integrating by parts,

$$\int_\alpha^\beta i\,\frac{dx(t)}{dt}\,\overline{y(t)}\,dt = \int_\alpha^\beta x(t)\,\overline{\left(i\,\frac{dy(t)}{dt}\right)}\,dt = \int_\alpha^\beta x(t)\,\overline{y^*(t)}\,dt,$$

and taking the limit in cases (ii) and (iii) as $\beta \to \infty$ or $\alpha \to -\infty$, $\beta \to \infty$, we get

$$(Ax, y) = (x, y^*)$$

that is, $y \in D(A^*)$.

Thus, $D(A^*)$ consists of functions square-integrable on (a, b) and having square-integrable derivatives on this interval.

Recalling the definition of $D(A)$, it is seen that in the cases (i) and (ii), $D(A^*)$ is wider than $D(A)$, but in case (iii), $D(A^*) = D(A)$. Consequently, in case (iii), A is a self-adjoint (or hypermaximal) operator.

It is desired to show that in cases (i) and (ii) A is a closed operator, necessitating to exhibit that $A = A^{**}$. Since $A^{**} \subset A^*$, A^{**} is again a differential operator on its domain $D(A^{**})$. Let $x(t) \in D(A^{**})$. For any function $y(t) \in D(A^*)$ in case (i) and any function of $D(A^*)$ vanishing outside of (α, β) in case (ii), integration by parts, yields

$$(A^{**}x, y) = \int_\alpha^\beta i \frac{dx(t)}{dt} \overline{y(t)}\, dt$$

$$= [x(\beta)\overline{y(\beta)} - x(\alpha)\overline{y(\alpha)}]\, i + \int_\alpha^\beta x(t)\overline{\left(i \frac{dy(t)}{dt}\right)}\, dt.$$

On the other hand,

$$(A^{**}x, y) = (x, A^*y) = \int_\alpha^\beta x(t)\overline{\left(i \frac{dy(t)}{dt}\right)}\, dt.$$

By comparing these expressions,

$$x(\beta)\overline{y(\beta)} - x(\alpha)\overline{y(\alpha)} = 0,$$

whence $x(1)\overline{y(1)} - x(0)\overline{y(0)} = 0$ in case (i) and $x(0)\overline{y(0)} = 0$ in case (ii) in the limit as $\beta \to \infty$. Since $y(0)$ and $y(1)$ can be chosen arbitrarily, the last equality is possible only if $x(0) = x(1) = 0$. However, then, $x(t) \in D_A$, proving that $D(A^{**}) \subset D(A)$. Consequently, $D(A) = D(A^{**})$.

It is now sought to determine the deficiency indices of the operator A. The equation $A^*x = ix$ in the present case takes the form

$$i \frac{dx}{dt} = ix,$$

and has a unique solution $x(t) = ce^t$, to within linear dependence. Similarly, $A^*x = -ix$ has a unique solution $x(t) = ce^{-t}$. In the case of a finite interval, both the solutions belong to the space $L_2(a, b)$, so that both the subspaces N_i and N_{-i} are one-dimensional and the operator has deficiency indices (1, 1). In case (ii), only the solution ce^{-t} of the latter equation belongs to the space $L_2[0, \infty)$, the subspace N_i consists of only null elements and the operator A has deficiency indices (0, 1). Consequently, in case (ii), A is a maximal symmetric operator and has no self-adjoint extension.

We construct in case (i) all self-adjoint extensions of A. The subspaces N_i and N_{-i} are, respectively, spanned by the elements e^t and e^{-t}.

Since

$$\| e^t \| = \left(\int_0^1 e^{2t}\, dt \right)^{1/2} = \sqrt{\frac{e^2-1}{2}},$$

and

$$\| e^{-t} \| = \left(\int_0^1 e^{-2t}\, dt \right)^{1/2} = \frac{1}{e} \sqrt{\frac{e^2-1}{2}},$$

e^t and e^{1-t} have an identical norm. Set in correspondence to e^t the element $e^{i\tau} e^{1-t}$, τ an arbitrary real number. For every τ on a linear manifold D_τ, consisting of the functions of the form

$$y(t) = x(t) + ce^t + ce^{i\tau} e^{1-t}, \tag{6}$$

$x(t) \in D(A)$, an operator A_τ is defined by

$$A_\tau y = Ax + ice^t - ice^{i\tau} e^{1-t}.$$

This is also a self-adjoint extension of A. The domain D_τ of A can be assigned boundary conditions. In fact, if $y(t)$ is representable in the form (6), then

$$y(0) = c + ce^{1+i\tau} = c(1+e^{1+i\tau}), \quad y(1) = ce + ce^{i\tau} = c(e^{i\tau}+e).$$

Thereupon,

$$\frac{y(0)}{y(1)} = \frac{1+e^{1+i\tau}}{e^{i\tau}+e}.$$

Since the primitive $\zeta = \dfrac{1+ez}{z+e}$ carries a unit circle of a complex plane into itself uniquely, one can write

$$\frac{1+e^{1+i\tau}}{e^{i\tau}+e} = e^{i\sigma},$$

σ real. Hence $y(0) = e^{i\sigma} y(1)$. Conversely, if this condition is satisfied, then $y(t)$ has form (6). In fact, let $y(0) = e^{i\sigma} y(1) = [(1 + e^{1+i\tau})/(e^{i\tau} + e)]\, y(1)$. Set

$$c = \frac{y(0)}{1+e^{1+i\tau}} = \frac{y(1)}{e^{i\tau}+e},$$

and let $x(t) = y(t) - c(e^t+e^{i\tau} e^{1-t})$. Then,

$$x(0) = y(0) - c(1+e^{1+i\tau}) = y(0) - \frac{y(0)}{1+e^{1+i\tau}}(1+e^{1+i\tau}) = y(0) - y(0) = 0,$$

and, similarly, it is verifiable that $x(1) = 0$. Consequently,

$$y(t) = x(t) + c(e^t+e^{i\tau} e^{1-t}),$$

$x(t) \in D(A)$, that is, $y(t)$ has the form (6), and we are through.

Thus, the domain D_τ of the self-adjoint extensions A_τ of A consists of exactly those functions $y(t)$ of the space $L_2\,[0, 1]$, which have square-integrable derivatives on [0, 1] and for which

$$y(0) = e^{i\sigma}\, y(1), \quad e^{i\sigma} = \frac{1+e^{1+i\tau}}{e^{i\tau}+e}.$$

The assignment of different values to the parameter τ, yields a continuum of distinct self-adjoint extensions of the operator A. It is now required to determine the spectrum of A_τ. The eigenfunction of the operator is the solution of the boundary value problem

$$i\frac{dx}{dt} = \lambda x, \quad \lambda \text{ real}$$

$$x(0) = e^{i\sigma}\, x(1). \tag{7}$$

The function $e^{-i\lambda t}$ is the solution, which satisfies the condition $1 = e^{i(\sigma-\lambda)}$, whence $\sigma - \lambda = 2k\pi$, or $\lambda_k = \sigma - 2k\pi$. Consequently, the eigenfunctions are given by

$$x_k(t) = e^{-i\lambda_k t} = e^{-i\sigma t} \cdot e^{2k\pi i t}.$$

These eigenfunctions are, evidently, normalized. All points of the real axis, distinct from λ_k, are regular points of the operator A_τ. In fact, the general solution of the equation

$$i\frac{dx}{dt} - \lambda x = y$$

has the form
$$x = e^{-i\lambda t}\left(c - i\int_0^t e^{i\lambda\xi} y(\xi)\, d\xi\right),$$

and it is necessary only to prove the feasibility of selecting a constant c such that the condition $x(0) = e^{i\sigma}\, x(1)$ is satisfied. This leads to the equation

$$c = e^{i\sigma}\left\{e^{-i\lambda}\left(c - i\int_0^1 e^{i\lambda\xi}\, y(\xi)\, d\xi\right)\right\},$$

which is, evidently, solvable if $e^{i(\sigma-\lambda)} \neq 1$ (that is, if $\lambda \neq \lambda_k$). Thus, A_τ has a pure point spectrum

$$x = \int_{-\infty}^{\infty} dE_\lambda\, x = e \sum_{n=-\infty}^{\infty} E_{\lambda_n} x = e^{-i\sigma t} \sum_{n=-\infty}^{\infty} c_n e^{2\pi n i t}, \tag{8}$$

$$A_\tau x = \int_{-\infty}^{\infty} \lambda dE_\lambda\, x = e^{-i\sigma t} \sum_{n=-\infty}^{\infty} \lambda_n c_n e^{2\pi n i t}, \tag{9}$$

where $c_n = (x, x_n)$. The functions of the operator A_τ have the form

$$F(A_\tau)x = e^{-i\sigma t} \sum_{n=-\infty}^{\infty} F(\lambda_n)\, c_n e^{2\pi n i t}. \tag{10}$$

In particular, the resolvent R_λ admits the expansion

$$R_\lambda x = e^{-i\sigma t} \sum_{n=-\infty}^{\infty} \frac{\epsilon_n}{\sigma - 2n\pi - \lambda} e^{2\pi nit}. \tag{11}$$

For $\sigma = 0$, the formula (8) gives the expansion of a square-integrable function in the usual FOURIER series.

Reverting to the infinite interval $(-\infty, \infty)$, let us again consider a self-adjoint operator $A = i\,(d/dt)$ in $L_2\,(-\infty, \infty)$. We now show that this differential operator is unitarily equivalent to an operator of multiplication by the independent variable in $L_2\,(-\infty, \infty)$. For this, the unitary equivalence between the operators A and B has the prescription : There exists a unitary operator U, such that $UD(A) = D(B)$ [consequently, also $D(A) = U^{-1}\,D(B)$] and $UAU^{-1}\,x = Bx$ for every $x \in D(B)$.

For establishing a unitary equivalence between the operators $A = i(d/dt)$ and $B = t$, we draw upon the well-known theorem due to PLANCHEREL (for details, see [35], for instance).

Let $x(t)$ be a (real- or complex-valued) function of the space $L_2\,(-\infty, \infty)$. Set

$$y(t, a) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x(\tau)\, e^{it\tau}\, d\tau.$$

Then, as $a \to \infty$, $y(t, a)$ converges in mean on $(-\infty, \infty)$ to some function $y(t) \in L_2\,(-\infty, \infty)$, and the function

$$x(t, a) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} y(\tau)\, e^{-it\tau}\, d\tau$$

converges in mean to $x(t)$.

The functions $x(t)$ and $y(t)$ are connected also by the formulae

$$y(t) = \frac{1}{\sqrt{2\pi}} \frac{d}{dt} \int_{-\infty}^{\infty} x(\tau)\, \frac{e^{it\tau} - 1}{i\tau}\, d\tau,$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \frac{d}{dt} \int_{-\infty}^{\infty} y(\tau)\, \frac{e^{-it\tau} - 1}{i\tau}\, d\tau.$$

In addition, $\| x(t) \| = \| y(t) \|$. The interval $(-a, a)$ in the limit formulae can be replaced by $(-a, b)$, where $a, b \to \infty$ independently of each other.

Now, let U be an operator in $L_2\,(-\infty, \infty)$, defined by

$$Ux = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(\tau)\, e^{it\tau}\, d\tau,$$

where the integral is understood as a limit in the mean of integrals with respect to a finite interval. By PLANCHEREL's theorem, the operator U maps $L_2(-\infty, \infty)$ onto itself one-one with preservation of the norm, that is, it is a unitary operator. The inverse operator has the form

$$U^{-1}x = U^*x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(\tau)\, e^{-it\tau}\, d\tau.$$

Let $x(t)$ be an arbitrary function of $D(A)$, that is, $x(t) \in L_2(-\infty, \infty)$ and suppose that there exists $[dx(t)/dt] \in L_2(-\infty, \infty)$. Then†

$$y(t) = Ux = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(\tau)\, e^{it\tau}\, d\tau = \text{l.i.m}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(\tau)\, e^{it\tau}\, d\tau$$

$$= \frac{1}{\sqrt{2\pi}}\, \text{l.i.m} \left[x(\tau)\, \frac{e^{it\tau}}{it} \Bigg|_{-a}^{a} - \frac{1}{it} \int_{-a}^{a} \frac{dx(\tau)}{d\tau}\, e^{it\tau}\, d\tau \right]$$

$$= \frac{1}{\sqrt{2\pi}}\, \frac{1}{it}\, \text{l.i.m}\, [\, x(a)\, e^{iat} - x(-a)\, e^{-iat}\,]$$

$$- \frac{1}{\sqrt{2\pi}}\, \frac{1}{it}\, \text{l.i.m} \int_{-a}^{a} \frac{dx(\tau)}{d\tau}\, e^{it\tau}\, d\tau.$$

Since $x(t)$ and dx/dt belong to $L_2(-\infty, \infty)$, $x(a)$ and $x(-a) \to 0$ as $a \to \infty$ and, consequently, the limit of the first member on the right-hand side of the preceding equality is zero. Together with $dx(t)/dt$, the FOURIER transform of this function also belongs to $L_2(-\infty, \infty)$. Hence,

$$ty(t) = -\frac{1}{i\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dx(\tau)}{d\tau}\, e^{it\tau}\, d\tau = z(t) \in L_2(-\infty, \infty).$$

Consequently, $y(t) \in L_2(-\infty, \infty)$, $ty(t) \in L_2(-\infty, \infty)$, that is, $y(t) \in D(B)$.

Conversely, let $x(t) \in D(B)$. Then,

$$\int_a^b |\, x(t)\, |\, dt \leqslant \left(\int_a^b t^{-2}\, dt \right)^{1/2} \left(\int_a^b |\, tx(t)\, |^2\, dt \right)^{1/2} < \infty$$

for every (finite or infinite) interval not containing the point $t = 0$ and, consequently, $x(t)$ is absolutely integrable on $(-\infty, \infty)$. Hence, the integral

$$y(t) = U^{-1}x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(\tau)\, e^{-it\tau}\, d\tau$$

†The letters l.i.m stand for the limit in the mean.

is absolutely convergent. Introduce the function

$$z(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tau x(\tau)\, e^{-it\tau}\, d\tau = \frac{1}{\sqrt{2\pi}} \frac{d}{dt} \int_{-\infty}^{\infty} \tau x(\tau)\, \frac{e^{-it\tau} - 1}{-i\tau}\, d\tau,$$

to receive

$$\int_0^t z(\tau)\, d\tau = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(\tau)\,(e^{-it\tau} - 1)\, d\tau + c = i\,[y(t) - y(0)] + c. \tag{12}$$

Note that the improper integral appearing in (12) converges absolutely and uniformly in t on the real line. From (12) it follows that

$$y(t) = \frac{1}{i} \int_0^t z(\tau)\, d\tau + c, \tag{13}$$

implying that $dy(t)/dt$ exists and belongs to $L_2(-\infty, \infty)$, that is, $y(t) \in D(A)$.

Thus, it is established that $Ux \in D(B)$ for $x \in D(A)$ and $U^{-1}x \in D(A)$ for $x \in D(B)$.

Let $x \in D(B)$. Then, by (13),

$$U^{-1}x = y(t) = \frac{1}{i} \int_0^t z(\tau)\, d\tau + c.$$

Consequently,

$$AU^{-1}x = i\frac{dy}{dt} = z(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tau x(\tau)\, e^{-it\tau}\, d\tau = U^{-1}Bx.$$

Thereupon, $UAU^{-1} = Bx$, which proves the unitary equivalence of A and B.

Suppose that A and B are any unitarily equivalent operators, $UAU^{-1} = B$.

If E_λ, $-\infty < \lambda < \infty$, is a spectral function of A, then, evidently, $\tilde{E}_\lambda = UE_\lambda U^{-1}$ is also a resolution of the identity. Let $\tilde{B}$ be a self-adjoint operator constructed by means of $\tilde{E}_\lambda$. Then,

$$\tilde{B}x = \int_{-\infty}^{\infty} \lambda d\, \tilde{E}_\lambda x = \lim \sum_{k=-\infty}^{\infty} \lambda_k\, \tilde{E}(\Delta_k)\, x$$

$$= \lim \sum_{k=-\infty}^{\infty} \lambda_k\, UE(\Delta_k)\, U^{-1} x = U\left(\lim \sum_{k=-\infty}^{\infty} \lambda_k\, E(\Delta_k)\right) U^{-1} x$$

$$= U\left(\int_{-\infty}^{\infty} \lambda\, dE_\lambda\right) U^{-1} x = UAU^{-1} x = Bx.$$

Thus, $\tilde{E}_\lambda = UE_\lambda U^{-1}$ is a spectral function of B, unitarily equivalent to A.

Thereupon, it follows, in particular, that the spectra of A and B coincide. In the present concrete example, if $A = i\,(d/dt)$, $B = t$, then it is found that A has a pure continuous spectrum, filling in the entire real axis.

A function of the operator A can be expressed as

$$F(A)x = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) \left\{ \int_{-\infty}^{\infty} x(\tau)\, e^{i\xi\tau}\, d\tau \right\} e^{-it\xi}\, d\xi$$

$$= U^{-1}\, F(B)\, Ux;$$

in particular, a spectral function of A is representable in the form

$$E_\lambda\, x = \frac{1}{2\pi} \int_{-\infty}^{\lambda} \left\{ \int_{-\infty}^{\infty} x(\tau)\, e^{i\xi\tau}\, d\tau \right\} e^{-it\xi}\, d\xi.$$

The unitary equivalence between the differential operator and the operator of multiplication by the independent invariable, established in the foregoing, is not exclusive for differential operators. It is found that for every self-adjoint operator, there exist spaces decomposable into pairwise orthogonal invariant subspaces, on each of which the operators are regarded unitarily equivalent (more precisely, isomorphic) to the operator of multiplication by the independent variable in a space of the functions square-integrable on $(-\infty, \infty)$ with some weight $\rho(t)$.

CHAPTER 8

SOME PROBLEMS OF DIFFERENTIAL AND INTEGRAL CALCULUS IN NORMED LINEAR SPACES

THE PRESENT chapter deals with the operations of differentiation and integration in the normed linear spaces and some of their applications.

8.1 DIFFERENTIATION AND INTEGRATION OF ABSTRACT FUNCTIONS OF REAL VARIABLES

LET E be a normed linear space and R a set of points on the real line. The operator $x = x(t)$, generally nonlinear, which maps R into E, will be hereafter called an **abstract function of real variable** t. Such functions are frequently encountered in analysis and its applications. It will suffice for our purpose to present here some functions of real arguments in differential geometry, one-parameter collection of solutions of differential equations, a family of operators (for instance, integral operators) depending on the parameter, and so on. For ease of exposition, it will be assumed in what follows that R is the interval $[a, b]$ of the real line.

Evidently, abstract functions of real variables admit addition and multiplication by a real number, that is, form a linear space.

A function $x(t)$ is said to be **continuous at the point** t_0 of $[a, b]$ if for every $\varepsilon > 0$ there is a $\delta = \delta(t_0, \varepsilon)$, such that $\| x(t)-x(t_0) \| < \varepsilon$ for $t \in [a, b]$ and $| t-t_0 | < \delta$.

Functions continuous at every point of $[a, b]$ are, as usual, said to be **continuous on this interval.**

It is evident that operations of addition of functions and their multiplication by a number do not exhaust the class of continuous functions.

A function $x(t)$ is said to be **uniformly continuous** on $[a, b]$, if for every $\varepsilon > 0$ it is possible to define δ, such that $\| x(t_1) - x(t_2) \| < \varepsilon$ for $t_1, t_2 \in [a, b]$ and $| t_1 - t_2 | < \delta$, irrespective of the position of points t_1 and t_2 on $[a, b]$.

LEMMA 1. *A function $x = x(t)$, $t \in [a, b]$, $x \in E$, which is continuous on $[a, b]$, is uniformly continuous on this interval.*

Consider a real-valued function $\varphi(t, \tau)$ defined in the square $a \leqslant t, \tau \leqslant b$ by $\varphi(t, \tau) = \| x(t) - x(\tau) \|$. It is plain that $\varphi(t, \tau)$ is continuous on this square and is, consequently, also uniformly continuous there. Hence, for every $\varepsilon > 0$ there exists $\delta > 0$, such that

$$| \varphi(t_1-\tau_1) - \varphi(t_2, \tau_2) | < \varepsilon, \tag{1}$$

for $|t_1-t_2| < \delta$, $|\tau_1-\tau_2 < \delta$ and irrespective of the location of points (t_1, τ_1) and (t_2, τ_2) in the square $a \leqslant t, \tau \leqslant b$. Set $\tau_1 = t_2$, $\tau_2 = t_2$, with $|t_1-t_2| < \delta$. Noting that, $\varphi(t_2, t_2) = \| x(t_2)-x(t_2) \| = 0$, it follows from (1) that $|\varphi(t_1, t_2)| = \| x(t_1)-x(t_2) \| < \varepsilon$ for $|t_1-t_2| < \delta$, and the uniform continuity of $x(t)$ is proved.

It is trivial to verify that $x(t)$, continuous on $[a, b]$, is bounded on this interval (that is, the set of values of the function $x(t)$, $t \in [a, b]$, is a bounded set of the space E).

8.11. Differentiation. Consider a function $x = x(t)$, $x \in E$ and $t \in [a, b]$.

The derivative $x'(t)$ of the function $x(t)$ is defined by

$$\frac{dx(t)}{dt} = x'(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t)-x(t)}{\Delta t}, \tag{2}$$

if the limit on the right-hand side exists in the sense of convergence in the space E. Thus,

$$x'(t) = \frac{x(t+\Delta t)-x(t)}{\Delta t} - \alpha(t, \Delta t),$$

where $\alpha(t, \Delta t) \to 0$ as $\Delta t \to 0$. Hence,

$$x(t+\Delta t) - x(t) = x'(t)\,\Delta t + \alpha(t, \Delta t)\,\Delta t. \tag{3}$$

As $\Delta t \to 0$, the right-hand side of (3) tends to 0, implying that if $x(t)$ has a derivative at a point t, then $x(t)$ is continuous at this point.

The following properties of differentiation operation can be conveniently established:

(i) $[x(t)+y(t)]' = x'(t) + y'(t)$;

(ii) $[(\lambda x)-(t)]' = \lambda x'(t)$ for every number λ;

(iii) Let a left-hand (right-hand) multiplication by the element $y \in E_y$ be defined for the element $x \in E_x$. Let this be continuous and distributive with respect to addition and commutative with respect to scalar multiplication. Then,

$$[yx(t)]' = yx'(t), \tag{4}$$

that is, the constant factor can be taken out from under the differentiation sign.

Properties (i) and (ii) are evident, and (iii) follows from the continuity and distributivity of left-hand multiplication, thus

$$\begin{aligned} \frac{d}{dt}[yx(t)] &= \lim_{\Delta t \to 0} \frac{yx(t+\Delta t)-yx(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} y\,\frac{x(t+\Delta t)-x(t)}{\Delta t} \\ &= y \lim_{\Delta t \to 0} \frac{x(t+\Delta t)-x(t)}{\Delta t} = y\,\frac{dx(t)}{dt}. \end{aligned}$$

Similarly, $$[x(t)\, y]' = x'(t)\, y. \tag{5}$$

Example. Let $x = x(t)$, $x \in E$, and let A be an operator in $(E_x \to E_y)$. Then,

$$[Ax(t)]' = Ax'(t). \tag{6}$$

Let $A = A(t) \in (E_x \to E_y)$ and $x \in Ex$. Then,

$$[A(t)\, x]' = A'(t)\, x. \tag{7}$$

In particular, for a linear functional,

$$\{f[x(t)]\}' = f[x'(t)], \tag{8}$$

$$[f(t)\, x]' = f'(t)\, x. \tag{9}$$

8.11.1. Derivatives of higher order. It is now sought to define derivatives of higher order. It is possible to give two definitions of the n-th derivative $x^{(n)}(t)$ at the point t of a function $x = x(t)$, similar to those known for ordinary functions of real variables.†

(i) Let $$\Delta^n_{\Delta t}(x)t = \sum_{k=0}^{n} (-1)^{n-k} C^k_n\, x(t + k\,\Delta t),$$

C^k_n a binomial coefficient. $\Delta^n_{\Delta t}\, x(t)$ is the n-**th difference** of $x(t)$ at the point t. Correspondingly, $\delta^n_{\Delta t}\, x(t) = \Delta^n_{\Delta t}\, x\left(t - \frac{n}{2}\,\Delta t\right)$ is called the n-**th central difference.**

The expression

$$x^{(n)}(t) = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^n}\, \delta^n_{\Delta t}\, x(t), \tag{10}$$

subject to the assumption that this limit exists, is called the n-**th difference derivative** of the function $x(t)$ at the point t.

If the limit in (10) is uniform in the neighbourhood of every point t, then $x^{(n)}(t)$ is called the **uniform** n-**th difference derivative.**

(ii) The n-**th derivative** $x^{(n)}(t)_0$ is defined by n-times successive differentiation of the function $x(t)$, namely

$$x'(t)_0 = (d/dt)\, x(t), \quad x''(t)_0 = (d/dt)\, x'(t)_0, \ldots, x^{(n)}(t)_0 = (d/dt)\, x^{(n-1)}(t)_0.$$

THEOREM 1. *If there exists a continuous n-th derivative $x^{(n)}(t)_0$ in the neighbourhood of a point t, then the uniform n-th difference-derivative $x^{(n)}(t)$ also exists in this neighbourhood, and*

$$x^{(n)}(t) = x^{(n)}(t)_0. \tag{10*}$$

Conversely, *if there exists the uniformly continuous, uniform difference-derivative $x^{(n)}(t)$ in the neighbourhood of a point t, then the n-th derivative $x^{(n)}(t)_0$ also exists in this neighbourhood, and* (10*) *again holds.*

†See, Appendix IV.

These assertions hold also for functions whose range is in the set of numbers.† The transition to abstract functions is effected by a method frequently applied in functional analysis. It is desired to carry out this transition for the former hypothesis.

For any linear functional $f \in E^*$, $\varphi(t) = f[x(t)]$ is a function with the domain of definition and range in the set of numbers, and in view of (8),

$$\begin{aligned} f[x'(t)_0] &= \{f[x(t)]\}' &&= \varphi'(t), \\ f[x''(t)_0] &= \{f[x'(t)_0]\}' &&= \varphi''(t), \\ &\ldots\ldots\ldots\ldots \\ f[x^{(n)}(t)_0] & &&= \varphi^{(n)}(t). \end{aligned}$$

Furthermore,

$$f\left[\frac{1}{(\Delta t)^n}\,\delta^n_{\Delta t}\,x(t)\right] = \frac{1}{(\Delta t)^n}\sum_{k=0}^{n}(-1)^{n-k}\,C_n^k\,\varphi\left(t+\left(k-\frac{n}{2}\right)\Delta t\right)$$

$$= \frac{1}{(\Delta t)^n}\,\delta^n_{\Delta t}\,\varphi(t) = \varphi^{(n)}(t+\theta\Delta t) = f[x^{(n)}(t+\theta\Delta t)_0]\dagger\dagger,$$

where $-n/2 < \theta < n/2$. Since, by hypothesis, $x^{(n)}(t)_0$ is continuous in a neighbourhood of the point t,

$$\| x^{(n)}(t+\theta\,\Delta t)_0 - x^n(t)_0 \| \leqslant \varepsilon_{\Delta t},$$

where $\varepsilon_{\Delta t} \to 0$ as $\Delta t \to 0$ uniformly in a neighbourhood of t. Thereupon,

$$\left| f\left[\frac{1}{(\Delta t)^n}\delta^n_{\Delta t}\,x(t)\right] - f[x^{(n)}(t)_0] \right| \leqslant \varepsilon_{\Delta t}\,\|f\|\,. \tag{11}$$

Ineq. (11) holds for any $f \in E^*$; thus,

$$\left\| \frac{1}{(\Delta t)^n}\,\delta^n_{\Delta t}\,x(t) - x^{(n)}(t)_0 \right\| \leqslant \varepsilon_{\Delta t},$$

and consequently, $x^n(t)_0 = \lim\limits_{\Delta t \to 0} \dfrac{1}{(\Delta t)^n}\delta^n_{\Delta t}\,x(t)$, where the convergence to the limit is uniform in a neighbourhood of every t, giving the required proof.

8.11.2. Partial derivatives. With the objective of introducing the notion of the **partial derivative** of an abstract function, consider a function of n real variables $t_1, t_2, \ldots, t_n$ with the range lying in a normed linear space E: $x = x(t_1, t_2, \ldots, t_n) \in E$. It is possible to regard $t_1, t_2, \ldots, t_n$ as components of the n-dimensional vector $\bar{t} = \sum\limits_{i=1}^{n} t_i\,e_i$, where the e_i are orthonormal basis vectors, that is, n-dimensional pairwise orthogonal unit vectors.

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† See, Appendix IV.

†† See, Appendix IV.

To define the n-th partial difference-derivative at the point

$$\bar{t}_0 = \sum_{i=1}^{n} t_i^0 e_i \text{ as } \frac{\partial^n}{\partial t_1 \partial t_2 \ldots \partial t_n} x(t_1, t_2, \ldots, t_n),$$

we form the *n-th partial difference*

$$\Delta^n_{t_1, \ldots, t_n; \Delta t} x(\bar{t}_0) = \sum_{i_1, i_2, \ldots, i_k} (-1)^{n-k} x[\bar{t}_0 + \Delta t\, (e_{i_1} + e_{i_2} + \ldots + e_{i_k})];$$

here $(i_1, i_2, \ldots, i_k)$, $0 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$ is a subset of $(1, 2, \ldots, n)$, and sum is taken over all such subsets. The *n-th central partial difference* at the point $\bar{t}_0$, $\delta^n_{t_1, t_2, \ldots, t_n; \Delta t}\, x(\bar{t}_0)$ is the n-th partial difference, taken at the point

$$\bar{\tau}_0 = \bar{t}_0 - \tfrac{1}{2} \Delta t \sum_{i=1}^{n} e_i.$$

Thus,

$$\delta^n_{t_1, t_2, \ldots, t_n; \Delta t}\, x(\bar{t}_0) = \Delta^n_{t_1, t_2, \ldots, t_n; \Delta t}\, x\left(\bar{t}_0 - \frac{1}{2} \Delta t \sum_{i=1}^{n} e_i\right).$$

Then the limit of $\dfrac{1}{(\Delta t)^n} \delta^n_{t_1, t_2, \ldots, t_n; \Delta t}\, x(\bar{t}_0)$ as $\Delta t \to 0$, if existing, is called the **n-th partial difference derivative**

$$\frac{\partial^n}{\partial t_1 \partial t_2 \ldots \partial t_n} x(t_1^0, t_2^0, \ldots, t_n^0)$$

of the function $x(\bar{t})$ at the point $\bar{t}_0 = \left(t_1^0, t_2^0, \ldots, t_n^0\right)$.

Similarly, the **n-th partial derivative** can be defined. It is obtained through successive differentiation of the function $x(t_1, t_2, \ldots, t_n)$ with respect to $t_{k_n}, t_{k_{n-1}}, \ldots, t_{k_1}$, where $k_1, k_2, \ldots, k_n$ is an arbitrary permutation of the indices $1, 2, \ldots, n$. It must be naturally assumed that the successive derivatives

$$\frac{\partial}{\partial t_{k_n}} x(t_1, t_2, \ldots, t_n), \frac{\partial}{\partial t_{k_{n-1}}}\left(\frac{\partial}{\partial t_{k_n}} x(t_1, t_2, \ldots, t_n)\right), \ldots, \frac{\partial}{\partial t_{k_1}} \times \left\{\frac{\partial}{\partial t_{k_2}}\left[\ldots \frac{\partial}{\partial t_{k_n}} x(t_1, t_2, \ldots, t_n) \ldots\right]\right\}$$

exist in a neighbourhood of $\bar{t}_0$.

THEOREM 2. *If an n-th partial derivative of the function $x(\bar{t})$ exists in a neighbourhood of the point $\bar{t}_0 = (t_1^0, t_2^0, \ldots, t_n^0)$ and if this derivative is continuous at $\bar{t}_0$, then the n-th partial difference derivative also exists at $\bar{t}_0$. Furthermore, both the derivatives coincide.*

PROOF. Let f be an arbitrary linear functional in E^*. Then,

$$\varphi(t_1, t_2, \ldots, t_n) = f[x(t_1, t_2, \ldots, t_n)]$$

is a function of $t_1, t_2, \ldots, t_n$ whose range is in the set of numbers. Hence†

$$\frac{1}{(\Delta t)^n} \delta^n_{t_1, t_2, \ldots, t_n; \Delta t} \varphi(t_1^0, \ldots, t_n^0)$$
$$= \frac{\partial}{\partial t_{k_1}}\left\{ \ldots \frac{\partial}{\partial t_{k_n}} \varphi(t_1^0 + \theta_1 \Delta t, \ldots, t_n^0 + \theta_n \Delta t) \ldots \right\},$$
$$0 < \theta_i < 1, \qquad i = 1, 2, \ldots, n,$$

for an arbitrary permutation $k_1, k_2, \ldots, k_n$ of the indices $1, 2, \ldots, n$. Consequently,

$$f\left[\frac{1}{(\Delta t)^n} \delta^n_{t_1, t_2, \ldots, t_n; \Delta t} x(t_1^0, t_2^0, \ldots, t_n^0)\right]$$
$$= \frac{1}{(\Delta t)^n} \delta^n_{t_1, t_2, \ldots, t_n; \Delta t} \varphi(t_1^0, t_2^0, \ldots, t_n^0)$$
$$= \frac{\partial}{\partial t_{k_1}}\left\{ \ldots \frac{\partial}{\partial t_{k_n}} \varphi(t_1^0 + \theta_1 \Delta t, \ldots, t_n^0 + \theta_n \Delta t) \ldots \right\}$$
$$= f\left[\frac{\partial}{\partial t_{k_1}}\left\{ \ldots \frac{\partial}{\partial t_{k_n}} x(t_1^0 + \theta_1 \Delta t \ldots) \ldots \right\}\right],$$

since the linear functional f can be taken out of the differentiation symbol.

By hypothesis, the n-th partial derivative of $x(t)$ is continuous at the point $\bar{t}_0$. Hence,

$$\left\| \frac{\partial}{\partial t_{k_1}}\left\{ \ldots \frac{\partial}{\partial t_{k_n}} x(t_1^0 + \theta_1 \Delta t, \ldots, t_n^0 + \theta_n \Delta t) \ldots \right\} - \frac{\partial}{\partial t_{k_1}}\left\{ \ldots \frac{\partial}{\partial t_{k_n}} x(t_1^0, \ldots, t_n^0) \ldots \right\} \right\| \leqslant \varepsilon_{\Delta t},$$

where $\varepsilon_{\Delta t} \to 0$ as $\Delta t \to 0$. Consequently,

$$\left| f\left[\frac{1}{(\Delta t)^n} \delta^n_{t_1, t_2, \ldots, t_n; \Delta t} x(\bar{t}_0)\right] - f\left[\frac{\partial}{\partial t_{k_1}}\left\{ \ldots \frac{\partial}{\partial t_{k_n}} x(\bar{t}_0) \ldots \right\}\right] \right|$$
$$= \left| f\left[\frac{\partial}{\partial t_{k_1}}\left\{ \ldots \frac{\partial}{\partial t_{k_n}} x(t_1^0 + \theta_1 \Delta t, \ldots, t_n^0 + \theta_n \Delta t) \ldots \right\}\right] - f\left[\frac{\partial}{\partial t_{k_1}}\left\{ \ldots \frac{\partial}{\partial t_{k_n}} x(t_1^0, \ldots, t_n^0) \ldots \right\}\right] \right|$$
$$\leqslant \|f\| \left\| \frac{\partial}{\partial t_{k_1}}\left\{ \ldots \frac{\partial}{\partial t_{k_n}} x(t_1^0 + \theta_1 \Delta t, \ldots, t_n^0 + \theta_n \Delta t) \ldots \right\} - \frac{\partial}{\partial t_{k_1}}\left\{ \ldots \frac{\partial}{\partial t_{k_n}} x(t_1^0, \ldots, t_n^0) \ldots \right\} \right\| \leqslant \|f\| \varepsilon_{\Delta t}.$$

† See Appendix IV.

This inequality holds for every $f \in E^*$. Consequently,

$$\left\| \frac{1}{(\Delta t)^n} \delta^n_{t_1, t_2, \ldots, t_n; \Delta t} x(\bar{t}_0) - \frac{\partial}{\partial t_{k_1}} \left\{ \ldots \frac{\partial}{\partial t_{k_n}} x(\bar{t}_0) \ldots \right\} \right\| \leqslant \varepsilon_{\Delta t},$$

implying that

$$\frac{1}{(\Delta t)^n} \delta^n_{t_1, t_2, \ldots, t_n; \Delta t} x(\bar{t}_0) \to \frac{\partial}{\partial t_{k_1}} \left\{ \ldots \frac{\partial}{\partial t_{k_n}} x(\bar{t}_0) \ldots \right\}$$

as $\Delta t \to 0$, which proves the existence of the limit of the expression

$$\frac{1}{(\Delta t)^n} \delta^n_{t_1, t_2, \ldots, t_n; \Delta t} x(\bar{t}_0)$$

and its equality with the n-th partial derivative.

COROLLARY. *Two n-th partial derivatives corresponding to two different permutations of the indices* $1, 2, \ldots, n$, *coincide at the points where both are continuous, that is, the n-th partial derivative does not depend on the order of the differentiation.*

In fact, the equality

$$\frac{\partial}{\partial t_{k_1}} \left\{ \ldots \frac{\partial}{\partial t_{k_n}} x(t_1, t_2, \ldots, t_n) \ldots \right\}$$
$$= \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^n} \delta^n_{t_1, t_2, \ldots, t_n; \Delta t} x(t_1, \ldots, t_n)$$

holds in this case, though its right-hand side does not depend on the permutation $k_1, k_2, \ldots, k_n$.

In the sequel, when we speak of the n-th partial derivative defined in some region, its continuity in that region shall be implied without any explicit mention to that effect. Thus, both the definitions of this derivative are one and the same, and we denote it by the symbol

$$\frac{\partial^n}{\partial t_1 \partial t_2 \ldots \partial t_n} x(t_1, \ldots, t_n).$$

8.12. Integration. The STIELTJES-type abstract integrals were considered in the foregoing while dealing with the spectral theory. However, the integrable function there happened to be usually a real- or complex valued function of real variables and the integrating function an abstract function. It is now intended to consider the opposite position, that is, the integrable function is here an abstract function, and the integration is carried out over real variables. Just as in the spectral theory, the discussions will be confined here to the RIEMANN integral of continuous functions.

Let $x = x(t)$, $a \leqslant t \leqslant b$, $x \in E$, be an abstract function. Consider all possible partitions $A = [t_0, t_1, \ldots, t_n]$ of the closed interval $[a, b]$ into closed subintervals $[t_i, t_{i+1}]$ with a set of numbers $\{t_k\}_0^n$ or points such that $t_0 = a < t_1 < t_2 < \ldots < t_n = b$.

The partition $B = [s_0, s_1, \ldots, s_m]$ is called a **refinement** of $A = [t_0, t_1, \ldots, t_n]$, if each of the intervals $[s_k, s_{k+1}]$ is a part of one of the intervals $[t_i, t_{i+1}]$. Thus, in the partition B, each of the intervals $[t_i, t_{i+1}]$ of A is again partitioned into $[s_k, s_{k+1}]$ of B. If every interval $[t_i, t_{i+1}]$ of A has a length which does not exceed a positive number δ, $t_{i+1} - t_i \leqslant \delta$, then A is called a **δ-partition** of the interval $[a, b]$ and denoted by A_δ.

The sum

$$S[A, x(t)] = \sum_{i=0}^{n-1} x(t_i)(t_{i+1}-t_i) \tag{12}$$

is called an **integral sum** of the function $x(t)$ with respect to the partition $A = [t_0, t_1, \ldots, t_n]$.

Consider a function $x(t)$, $t \in [a, b]$, $x \in E$, where E is a complete space and, consequently, a sequence $\{A_{\delta_n}\}$ of δ_n-partitions of the interval $[a, b]$, such that $\delta_n \to 0$ as $n \to \infty$. Construct an integral sum $S[A_{\delta_n}, x(t)]$. If this sum tends to a limit $S = \lim_n S[A_{\delta_n}, x(t)]$, as $n \to \infty$ and independent of the choice of A_{δ_n}, then this limit is called the **Riemann integral** of the function $x(t)$ on $[a, b]$ and is denoted as $\int_a^b x(t)\, dt$.

THEOREM 3. *If $x(t)$ is continuous on $[a, b]$, then the Riemann integral $\int_a^b x(t)\, dt$ exists.*

The proof of this theorem rests on the upcoming two lemmas.

LEMMA 2. *If the partition B of the interval $[a, b]$ is a refinement of the δ-partition $A = A_\delta$ of $[a, b]$, then*

$$\| S[A, x(t)] - S[B, x(t)] \| \leqslant \omega(\delta)(b-a), \tag{13}$$

where,

$$\omega(\delta) = \sup_{|\tau_1 - \tau_2| \leqslant \delta} \| x(\tau_1) - x(\tau_2) \|. \tag{14}$$

PROOF. If the point t belongs to the interval $[t_i, t_{i+1}]$ of the δ-partition A, then $|t - t_i| \leqslant |t_{i+1} - t_i| \leqslant \delta$. Thus, because of (14),

$$\| x(t) - x(t_i) \| \leqslant \omega(\delta). \tag{15}$$

Let n and m be the numbers of intervals $[t_i, t_{i+1}]$ of A and $[s_j, s_{j+1}]$ of B, respectively. Since B is a refinement of A, each of the points t_i, $i = 1, 2, \ldots, n$, coincides with one of the points s_j, $s_{k_i} = t_i$ (here $0 = k_0 < k_1 < \ldots < k_n = m$, $m > n$). Thus, the interval $[t_i, t_{i+1}]$ of A is partitioned into $k_{i+1} - k_i$ intervals $[s_j, s_{j+1}]$ of B, $(j = k_i, k_i + 1, \ldots, k_{i+1} - 1)$. For these j, by (15), we have

$$\| x(s_j) - x(t_i) \| \leqslant \omega(\delta).$$

Hence

$$S[A, x(t)] = \sum_{i=0}^{n-1} x(t_i)\,(t_{i+1} - t_i) = \sum_{i=0}^{n-1} x(t_i) \sum_{j=k_i}^{k_{i+1}-1} (s_{j+1}-s_j),$$

$$S[B, x(t)] = \sum_{j=0}^{n-1} x(s_j)\,(s_{j+1}-s_j) = \sum_{i=0}^{n-1} \sum_{j=k_i}^{k_{i+1}-1} x(s_j)\,(s_{j+1}-s_j),$$

$$\begin{aligned} \| S[A, x(t)] - S[B, x(t)] \| \\ &= \left\| \sum_{i=0}^{n-1} \sum_{j=k_i}^{k_{i+1}-1} [x(t_i) - x(s_j)]\,(s_{j+1}-s_j) \right\| \\ &\leqslant \sum_{i=0}^{n-1} \sum_{j=k_i}^{k_{i+1}-1} \| x(t_i) - x(s_j) \|\,(s_{j+1}-s_j) \\ &\leqslant \omega(\delta) \sum_{i=0}^{n-1} \sum_{j=k_i}^{k_{i+1}-1} [s_{j+1}-s_j] = \omega(\delta)\,(b-a), \end{aligned} \tag{16}$$

giving also the required proof.

LEMMA 2. *Let A_δ and A_ε be, respectively, arbitrary δ- and ε-partitions of the interval* $[a, b]$. *Then,*

$$\| S[A_\delta, x(t)] - S[A_\varepsilon, x(t)] \| \leqslant [\omega(\delta) + \omega(\varepsilon)]\,(b-a).$$

PROOF. In fact, it is always possible to choose a partition B of the interval $[a, b]$, which is a refinement simultaneously of both A_δ and A_ε. Then, because of (13),

$$\| S[A_\delta, x(t)] - S[B, x(t)] \| \leqslant \omega(\delta)\,(b-a),$$

and

$$\| S[A_\varepsilon, x(t)] - S[B, x(t)] \| \leqslant \omega(\varepsilon)\,(b-a),$$

whence

$$\| S[A^\delta, x(t)] - S[A_\varepsilon, x(t)] \| \leqslant [\omega(\delta) + \omega(\varepsilon)]\,(b-a). \quad \blacksquare$$

Now, the stage is set to prove the theorem. Consider a sequence $\{A_{\delta_n}\}$ of δ_n-partitions of the interval $[a, b]$, where $\delta_n \to 0$ as $n \to \infty$. Then, by Lemma 2, the inequality

$$\| S[A_{\delta_n}, x(t)] - S[A_{\delta_{n+p}}, x(t)] \| \leqslant [\omega(\delta_n) + \omega(\delta_{n+p})]\,(b-a)$$

holds for the corresponding integral sums $S[A_{\delta_n}, x(t)]$; owing to the uniform continuity of the function $x(t)$, the right-hand side of this inequality tends to zero as $n \to \infty$, Thus, the sequence $\{S[A_{\delta_n}, x(t)]\}$ is a CAUCHY sequence. However, since $S[A_{\delta_n}, x(t)] \in E$ and E is a complete space, this sequence has a limit S in E.

Now let $\{A_{\varepsilon_n}\}$ be another sequence of ε_n-partitions of the interval $[a, b]$, where $\varepsilon_n \to 0$. In view of the foregoing, $\{S[A_{\varepsilon_n}, x(t)]\}$ converges to a limit S_ε as $n \to \infty$. It is to be shown that $S_\varepsilon = S$.

For this purpose, combine both the partition sequences into a sequence $A_{\delta_1}, A_{\varepsilon_1}, A_{\delta_2}, A_{\varepsilon_2}, \ldots$ The integral sums $S[A_{\delta_1}, x(t)]$, $S[A_{\varepsilon_1}, x(t)], \ldots$ corresponding to this sequence, form a convergent sequence and its limit $S_{\delta+\varepsilon}$ is equal to the limits S and S_ε of subsequences $\{S[A_{\delta_n}, x(t)]\}$ and $\{S[A_{\varepsilon_n}, x(t)]\}$. Consequently, $S = S_\varepsilon = S_{\delta+\varepsilon}$. ■

The definition of RIEMANN integral would thus appear to be completely justifiable only for continuous functions. It is deficient for other functions, as even very simple functions turn out to be non-integrable.

The operator S is called the **integral** of $x(t)$ over $[a, b]$ and is denoted by $\int_a^b x(t)\, dt$.

8.12.1. Properties of the integral:

(i) $\int_a^b [x(t) + y(t)]dt = \int_a^b x(t)\, dt + \int_a^b y(t)\, dt$, that is, the integral of a sum of functions is equal to the sum of the integrals of the summands;

(ii) If c is any point between a and b, then

$$\int_a^b x(t)\, dt = \int_a^c x(t)\, dt + \int_c^b x(t)\, dt;$$

(iii) $\int_a^b \lambda x(t)\, dt = \lambda \int_a^b x(t)\, dt$, for a fixed scalar λ.

The properties (i) to (iii) are obvious.

(iv)
$$\left\| \int_a^b x(t)\, dt \right\| \leqslant \int_a^b \| x(t) \|\, dt. \tag{17}$$

In fact, for a partition $A_\delta = (t_0, t_1, \ldots, t_n)$, we have

$$\| S[A_\delta, x(t)] \| = \left\| \sum_{i=0}^{n-1} x(t_i)\,(t_{i+1} - t_i) \right\|$$
$$\leqslant \sum_{i=0}^{n-1} \| x(t_i) \|\,(t_{i+1} - t_i) = S(A_\delta, \| x(t) \|). \tag{18}$$

As $\delta \to 0$, the integral sums tend, respectively, to the integrals

$$\int_a^b x(t)\, dt \quad \text{and} \quad \int_a^b \| x(t) \|\, dt,$$

and Ineq. (18) turns into (17). ■

(v) If a right (or left) continuous multiplication by the elements $y \in E_y$ is defined for the elements $x \in E_x$ (see p. 273), then

$$\int_a^b x(t)\, y\, dt = \int_a^b x(t)\, dt\, y \tag{19}$$

(the constant factor y is taken outside the sign of integral).
In fact, for a δ-partition $A_\delta = [t_0, t_1, \ldots, t_n]$, we have

$$S[A_\delta, x(t)\, y] = \sum_{i=0}^{n-1} x(t_i)\, y(t_{i+1} - t_i)$$

$$= \left(\sum_{i=0}^{n-1} x(t_i)\,(t_{i+1} - t_i) \right) y = S[A_\delta, x(t)]\, y. \tag{20}$$

As $\delta \to 0$, Eq. (20) turns into (19).
Similarly, for a left multiplication, the equality

$$\int_a^b yx(t)\, dt = y \int_a^b x(t)\, dt \tag{21}$$

holds.

Examples. 1. If A is a linear operator mapping E_x into E_y, and if $x(t) \in E_x$, then

$$\int_a^b Ax(t)\, dt = A \int_a^b x(t)\, dt.$$

2. If $A = A(t)$ is an operator in $(E_x \to E_y)$, which depends continuously on t, and if $\in x\, E_x$, then

$$\int_a^b A(t)\, x\, dt = \left(\int_a^b A(t)\, dt \right) x.$$

In the first case the operator A is a constant left-hand factor and in the second, x is a constant right-hand factor.

In particular, if f is a linear functional in E^*, then

$$f\left(\int_a^b x(t)\, dt \right) = \int_a^b f[x(t)]\, dt, \qquad \int_a^b f(t)\, x\, dt = \left(\int_a^b f(t)\, dt \right) x. \tag{22}$$

(vi) If the function $x = x(t)$ with $x \in E$ and $t \in [a, b]$, has a continuous derivative with respect to t, $x'(t) = (d/dt)\, x(t)$, then

$$\int_a^b x'(t)\, dt = x(b) - x(a). \tag{23}$$

In fact, for any linear functional f, by (22), we have

$$f\left(\int_a^b x'(t)\,dt\right) = \int_a^b f[x'(t)]\,dt = \int_a^b \frac{d}{dt} f[x(t)]\,dt.$$

However, $f[x(t)]$ is a continuous function of t whose range is in the set of numbers. Hence,

$$\int_a^b \frac{d}{dt} f[x(t)]\,dt = f[x(b)] - f[x(a)]$$

holds for it. Thus,

$$f\left(\int_a^b x'(t)\,dt\right) = f[x(b)] - f[x(a)]. \tag{24}$$

Eq. (24) holds for any linear functional $f \in E^*$. However, then, Eq. (23) also remains valid, as was required to prove.

Finally, consider the integrals with variable upper limits. Let $x(t)$ be continuous on $[a, b]$ and let $a < t < b$. Then, there exists

$$y(t) = \int_a^t x(\tau)\,d\tau,$$

where $y(t)$ is some abstract function of (t).

(vii) An integral with a variable upper limit of a continuous abstract function $x(t)$ is a differentiable abstract function of upper bound and

$$\left(\int_a^t x(\tau)\,d\tau\right)' = x(t).$$

Evidently,
$$y(t+\Delta t) - y(t) = \int_t^{t+\Delta t} x(\tau)\,d\tau,$$

and
$$x(t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} x(t)\,d\tau.$$

Thereupon,
$$\frac{y(t+\Delta t) - y(t)}{\Delta t} - x(t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} [x(\tau) - x(t)]\,d\tau.$$

Given an arbitrary $\varepsilon > 0$. There is a $\delta > 0$, such that $\| x(t_1) - x(t_2) \| < \varepsilon$ for $| t_1 - t_2 | < \delta$. However, if $| \Delta t | < \delta$, then

$$\left\| \frac{y(t+\Delta t) - y(t)}{\Delta t} - x(t) \right\| \leqslant \frac{1}{|\Delta t|} \left| \int_t^{t+\Delta t} \| x(\tau) - x(t) \|\,d\tau \right|$$

$$< \varepsilon \frac{1}{|\Delta t|} \left| \int_t^{t+\Delta t} d\tau \right| = \varepsilon.$$

The last inequality signifies that

$$y'(t) = \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}$$

exists and that $\quad y'(t) = x(t).$

8.13. Solution of differential equations. Consider the differential equation

$$\frac{dx}{dt} = f(t, x), \tag{25}$$

where x and $f(t, x)$ are elements of a complete normed space E and $t \in [a, b]$. Assume that $f(t, x)$ is continuous in t and as a function of x satisfies the LIPSCHITZ condition

$$\| f(t, x_1) - f(t, x_2) \| \leqslant L \| x_1 - x_2 \|. \tag{26}$$

Let $C^E[a, b]$ denote the space of all continuous functions $x(t)$, $t \in [a, b]$ and $x(t) \in E$. Introduce a norm in $C^E[a, b]$ by putting $\| x \|_C = \max_t \| x(t) \|$.

Together with E, $C^E[a, b]$ is also a complete space. This can be easily proved in analogy with the particular case $E = R$ and $C^E[a, b] = C[0, 1]$.

Together with (25), consider the equation

$$x(t) = x_0 + \int_{t_0}^{t} f[\tau, x(\tau)]\, d\tau, \qquad (a \leqslant t_0 \leqslant t \leqslant t_0 + \delta \leqslant b). \tag{27}$$

Denote by $A_t(x)$ the right-hand side of Eq. (27). $A_t(x)$ is an operator which transforms $x = x(t) \in C^E[t_0, t_0 + \delta]$ into some new element of the same space. We get

$$\| A_t[x(t)] - A_t[y(t)] \| = \left\| \int_{t_0}^{t} [f(\tau, x(\tau)) - f(\tau, y(\tau))]\, d\tau \right\|$$

$$\leqslant \int_{t_0}^{t_0+\delta} \| f[\tau, x(\tau)] - f[\tau, y(\tau)] \|\, d\tau.$$

Thence, in view of (26),

$$\| A(x) - A(y) \|_C \leqslant L \int_{t_0}^{t_0+\delta} \| x(\tau) - y(\tau) \|\, d\tau$$

$$\leqslant L\delta \max_t \| x(t) - y(t) \| = L\delta \| x - y \|_C . \tag{28}$$

If $L\delta < 1$, then by (28), A defines a contraction mapping of the space $C^E[t_0, t_0 + \delta]$ into itself and, consequently, Eq. (27) *has* a unique solution (see Chap. 8.7).

Eq. (27) is equivalent to Eq. (25) for the initial value $x(t_0) = x_0$. Consequently, Eq. (25) has a unique solution on the interval $[t_0, t_0 + \delta]$, such that $x(t_0) = x_0$.

In particular, this equation has a unique solution $x(t)$ on the interval $[a, a + \delta]$ for any initial value $x(a) = x_0$. $x(t)$ can be extended to the entire interval $[a, b]$. In fact, if $a + \delta < b$ and $x(a + \delta) = x_1$, then we construct by repeating this process, a solution on the interval $[a + \delta, a + 2\delta]$ with the initial value x_1, and so on.

Examples. 1. If E is the n-dimensional linear space, we get the well-known existence theorem for a system of n differential equations.

2. If E is one of the spaces l_p, c, m, ... then we obtain the existence theorem for a solution of corresponding classes of infinite systems of differential equations.

8.2 THE DIFFERENCE SCHEME AND THE THEOREM OF P. LAX

WHILE DERIVING approximate solutions of the boundary value problems in mathematical physics by the finite-difference method, a situation is encountered where in some cases the convergence process fails when the difference of independent variables arbitrarily tends to zero. In addition, during the course of solving a difference boundary value problem, in the process of successive calculations of the values of unknown functions the accumulation of errors at lattice points may become so large that the substitution of approximate solutions of difference equations for the solutions of differential equations is rendered impossible. Thus, for solving the CAUCHY problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

by the difference $u_{t\bar{t}} = u_{x\bar{x}}$, where $u_{t\bar{t}}$ and $u_{x\bar{x}}$ are second order symmetric differences in corresponding variables, we have the convergence only if the difference ratio of independent variables $\Delta t/\Delta x$ does not exceed unity. Exactly in the same way, replacing the boundary value problem for the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

$$u(x, 0) = \varphi(x), \qquad 0 \leqslant x \leqslant l, \qquad u(0, t) = u(l, t) = 0$$

by the difference scheme

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{(\Delta x)^2},$$

$$u_m^0 = \varphi(m\,\Delta x), \ \ u_0^n = u_M^n = 0 \qquad \begin{pmatrix} m = 1, 2, \ldots, M, \\ n = 0, 1, 2, \ldots, \end{pmatrix},$$

where $u_k^n = u(k\,\Delta x, n\,\Delta t)$, and successively determining u_m^{n+1} in terms of u_m^n, we obtain for $(2\Delta x/(\Delta x)^2) > 1$ such large errors that the application of the indicated scheme becomes infeasible. It is to be established that these two occurrences are closely related to each other. The results achieved by P. LAX [31] in this direction are given below for simple cases.

Let $x = x(t)$, $0 \leqslant t \leqslant T$, be an abstract function with values in the BANACH space E and let A be a linear operator acting in this space with an everywhere dense domain $D(A)$. The operator A can be also non-bounded. Further, let x_0 be a fixed element of E. Now, consider the problem

$$\frac{dx}{dt} = Ax, \quad 0 \leqslant t \leqslant T, \tag{1}$$

$$x(0) = x_0. \tag{2}$$

For solving this problem, the abstract function $x(t)$ is regarded such that

(i) $x(t) \in D(A)$ for all $t \in [0, T]$;
(ii) the difference ratio $[x(t + h) - x(t)] / h$, as $h \to 0$, converges uniformly to $x'(t)$ on the entire interval $[0, T]$;
(iii) $x(t)$ satisfies Eq. (1) and the initial condition (2).

For some x_0, suppose that there exists, also a unique solution for the problem (1)—(2). It is trivial to verify that the set $D(R)$ of such x_0 is a linear manifold. For every t, to the element $x_0 \in D(R)$ there corresponds uniquely the element $x(t)$, satisfying the conditions (i)—(iii) and leads to the operator $R_0(t)$, defined by $x(t) = R_0(t)\,x_0$, $R_0(0)\,x_0 = x_0$ $(0 \leqslant t \leqslant T)$, giving a solution of the problem under consideration. By the linearity of A, $R_0(t)$ is also linear for all t. In addition, assume that $R_c(t)\,x \to x$ as $t \to 0, x \in D(R)$. It is said that the problem is **correctly formulated (well posed)**, if $D(R)$ is everywhere dense in E and a family of operators $R_0(t)$ is uniformly bounded in t on $D(R)$. It is clear that this definition of a correctly formulated problem is equivalent to the customary one in mathematical physics.

The assumptions made on the operator $R_0(t)$ for every t can be extended by continuity to a linear operator $R(t)$, defined on the entire space E. Designate the abstract function $x(t) = R(t)\,x_0$ as a generalized solution of the problem (1)—(2), defined by the initial value $x_0 \in D(R)$. Note that a family of operators $R(t)$ is also uniformly bounded on the segment $[0, T]$. In addition, $R(t) \to I$ in the sense of weak convergence as $t \to +0$.

Let us make one more assumption: assume that if, starting from the initial value x_0 defining the solution $x(t)$ at the point $t = t_0$ and next, from $x(t_0)$ as the initial value, there exists $x(t)$ when $t > t_0$, then the results for the newly found $x(t)$ also are just the same as starting directly from x_0. Applying this to the operator $R(t)$, it signifies that

$$R(t - t_0)\,R(t_0) = R(t),$$

or, in other words,
$$R(t_1)\,R(t_2) = R(t_1 + t_2) \tag{3}$$
for any t_1, $t_2 > 0$.

The formulated assumptions, satisfied in most of the cases, find important applications and stem from the principle of scientific determinism, which asserts that the succeeding state of a physical system is completely defined by its preceding state.

It is possible to determine the type of conditions which must be imposed on the operator A in order that Eq. (3) holds. Concerning this, see [12].

Now, introduce the concept of the finite-difference approximation of the problem. Let $x_1, x_2, \ldots, x_N$ be a system of points in the space E, which are taken as approximate values of the function $x(t)$. Thus, set $x_n \approx x(n\,\Delta t)$, $n = 1, 2, \ldots, N$; $N\,\Delta t = T$. Assume that for defining these points we have some operator equation which is regarded for simplicity as connecting only two points with the adjacent indices. Since x_n is an approximate value of $x(n\,\Delta t)$, it is natural to assume that the operator entering the equation and connecting x_n and x_{n+1}, depends on Δt. In addition, assume that the equation is solvable in x_{n+1}. This leads to the recurrence relation
$$x_{n+1} = C(\Delta t)\,x_n, \quad n = 0, 1, \ldots, N-1, \quad (x_0 \text{ given}), \tag{4}$$
were $C(\Delta t)$ is a bounded linear operator, called **finite-difference approximation** of the problem (1)—(2). Since
$$\frac{x[(n+1)\,\Delta t] - x(n\,\Delta t)}{\Delta t} = \frac{x_{n+1} - x_n}{\Delta t}$$
$$= \frac{C(\Delta t)\,x_n - x_n}{\Delta t} = \frac{C(\Delta t) - I}{\Delta t}\,x_n,$$
$(C(\Delta t) - I)/\Delta t$ must approximate dx/dt. However, on the other hand, by (1), $dx/dt = Ax$ and, consequently, in order that (4) really approximates the boundary value problem, it is necessary that the expression $(C(\Delta t) - I)/\Delta t$ in a certain sense approximates the operator A. It is accordingly said that the finite-difference approximation (4) of the problem (1) — (2) satisfies the **consistency condition,** if
$$\left\| \left(\frac{C(\Delta t) - I}{\Delta t} - A \right) x(t) \right\| \to 0$$
as $\Delta t \to 0$ uniformly in t, $0 \leqslant t \leqslant T$, on some set L of the exact solution $x(t)$, and the set of initial values x_0 corresponding to the solution $x(t) \in L$, lies everywhere dense in E.

Let
$$x_{k+1} = C(\Delta t)\,x_k, \quad k = 0, 1, 2 \ldots, N-1, \quad (x_0 \text{ given}) \tag{4}$$
be a finite-difference approximation of the boundary value problem (1) — (2). Apply (4) successively for $k = 0, 1, \ldots, n-1$, to receive
$$x_n = [C(\Delta t)]^n\,x_0. \tag{5}$$

Since this point x_n is an approximate value for $t = n\,\Delta t$, of the exact solution $x(t) = R(t)\,x_0$, this permits the introduction of the following definition: the finite-difference approximation (4) of the problem (1) — (2) is said to be **convergent**, if for any sequence $\{\Delta_k t\}$ tending to zero and any $x_0 \in E$,

$$\| [C(\Delta_k t)^{n_k} x_0 - R(t)\, x_0 \| \to 0$$

as $k \to \infty$ and $n_k \Delta_k t \to t$, $0 \leqslant t \leqslant T$.

A finite-difference approximation is said to be **stable**, if for any sequence $\{\Delta_k t\}$ tending to zero, the set of operators

$$[C(\Delta_k t)]^n, \quad n = 1, 2, \ldots; \ 0 \leqslant n\,\Delta_k t \leqslant T,$$

s norm-bounded in an operator space. If a finite-difference approximation is stable, then all the approximate values x_n of an exact solution $x(n\,\Delta t)$ are totally bounded for every fixed initial element x_0.

The stage is set for formulating the upcoming basic theorem.

THEOREM (P. LAX). *Given a well-posed problem* (1) — (2) *and let its finite-difference approximation* (4) *satisfy the consistency condition. For the finite-difference approximation to be convergent, it is necessary and sufficient that it is stable.*

Necessity. Let $\{[C(\Delta_j t)]^n\}$, where $\Delta_j t \to 0$, $0 \leqslant n\,\Delta_j t \leqslant T$, be an unbounded set of the space $(E \to E)$. By Remarks appended to the BANACH-STEINHAUS theorem, there exists a subsequence $\{[C(\Delta_k t)]^{n_k}\}$ and an element x_0, such that $\{[C(\Delta_k t]^{n_k} x_0\}$ is an unbounded sequence of elements in E.

Since $0 \leqslant n_k\,\Delta_k t \leqslant T$, it is possible to extract from the sequence $\{n_k\,\Delta_k t\}$ a subsequence $\{n_{k_i}\,\Delta_{k_i}\, t\}$, which converges to some number $t_0 \in [0, T]$. Consequently, we have

$$\| [\, C(\Delta_{k_i}\, t)]^{n_{k_i}} x_0 \| \to \infty, \qquad n_{k_i}\,\Delta_{k_i}\, t \to t_0.$$

On the other hand, by the hypothesis of the convergence of the approximation,

$$\| [C\,(\Delta_{k_i}\, t)]^{n_{k_i}} x_0 - R(t_0)\, x_0 \| \to 0,$$

whence
$$\| [\, C(\Delta_{k_i}\, t)]^{n_{k_i}}\, x_0 \| \to \| R(t_0)\, x_0 \| = c,$$

where c is a finite number, a contradiction. Consequently, $\{[C(\Delta_j t)]^n\}$ is a bounded set in an operator space, proving the necessity.

Sufficiency. Let $x(t) = R(t)x_0$ be one of the exact solutions of the boundary value problem, which belongs to a manifold L, entering in the consistency definition. Then, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$\left\| \left(\frac{C(\Delta t) - I}{\Delta t} - A \right) x(t) \right\| < \frac{\varepsilon}{2}$$

for $0 < \Delta t < \delta_1$ and every t in $[0, T]$. Furthermore, by the definition of exact solution,

$$\left\| \left(\frac{x(t + \Delta t) - x(t)}{\Delta t} - Ax(t) \right\| < \frac{\varepsilon}{2}$$

for $0 < \Delta t < \delta_2$ and uniformly in $[0, T]$, or, noticing that $x(t + \Delta t) = R(t + \Delta t)\, x_0 = R(\Delta t)\, R(t)\, x_0 = R(\Delta t)\, x(t)$,

$$\left\| \left(\frac{R(\Delta t) - I}{\Delta t} - A \right) x(t) \right\| < \frac{\varepsilon}{2},$$

for $0 < \Delta t < \delta_2$ and uniformly in $[0, T]$.

Thereupon, for $0 < \Delta t < \delta = \min(\delta_1, \delta_2)$,

$$\| [C(\Delta t) - R(\Delta t)]\, x(t) \| < \varepsilon\, \Delta t, \tag{6}$$

and, in addition, directly for all t in $[0, T]$.

Let x_0 be the initial element defining the solution $x(t)$ under consideration. Set

$$z_k = \{[\, C(\Delta_k t)]^{n_k} - R(n_k\, \Delta_k t)\}\, x_0,$$

where $\{\Delta_k t\}$ is a sequence, convergent to zero, and $n_k\, \Delta_k t \to t$. Simple calculations aided with the equality

$$R(t_1)\, R(t_2) = R(t_1 + t_2), \qquad t_1, t_2 > 0$$

yield

$$\begin{aligned} z_k = \sum_{m=0}^{n_k - 1} & \{[\, C(\Delta_k t)]^{m+1}\, R[(n_k - (m+1))\, \Delta_k t] \\ & - [\, C(\Delta_k t)]^m\, R[(n_k - m)\, \Delta_k t]\}\, x_0 \\ = \sum_{m=0}^{n_k - 1} & [C(\Delta_k t)]^m \{C(\Delta_k t) - R(\Delta_k t)\}\, R[(n_k - (m+1))\, \Delta_k t]\, x_0. \end{aligned}$$

Thence,

$$\begin{aligned} \| z_k \| \leqslant \sum_{m=0}^{n_k - 1} & \| C(\Delta_k t) \|^m \| \{ C(\Delta_k t) - R(\Delta_k t)\} \\ & \times R[(n_k - (m+1))\, \Delta_k t]\, x_0 \|. \end{aligned} \tag{7}$$

The approximation being stable by assumption, there is a constant K such that

$$\| C(\Delta_k t) \|^m \leqslant K. \tag{8}$$

Then, (6) and (8) adapted to (7), yield

$$\| z_k \| \leqslant \sum_{m=0}^{n_k - 1} K\varepsilon\, \Delta_k t = K\varepsilon n_k\, \Delta_k t \leqslant K\varepsilon T.$$

Since ε can be chosen arbitrarily, $\| z_n \| \to 0$ as $\Delta_k t \to 0$, $n_k\, \Delta_k t \to t$. Furthermore,

$$\begin{aligned} \| \{[\, C(\Delta_k t)]^{n_k} - R(t)\}\, x_0 \| & \leqslant \| \{\, [\, C(\Delta_k t)]^{n_k} - R(n_k\, \Delta_k t)\}\, x_0 \| \\ & \quad + \| [\, R(n_k\, \Delta_k t) - R(t)]\, x_0 \| \\ & = \| z_k \| + \| [\, R(n_k\, \Delta_k t) - R(t)]\, x_0 \|. \end{aligned} \tag{9}$$

Consider the last summand, to receive

$$R(n_k \Delta_k t) - R(t) = \begin{cases} [R(\tau) - I]\, R(t) \text{ for } \tau = n_k \Delta_k t - t > 0, \\ -\ [R(\tau) - I]\, R(\tilde{t}) \text{ for } \tau = t - n_k \Delta_k t > 0, \end{cases}$$

$$\tilde{t} = n_k \Delta_k t.$$

In both the cases, $[R(\tau) - I] \to 0$ as $n_k \Delta_k t \to t$, and $R(t)$ or $R(\tilde{t})$ is bounded. Hence, for $t < n_k \Delta_k t$,

$$\| [R(n_k \Delta_k t) - R(t)]\, x_0 \| \leqslant \| R(t) \| \; \| [R(\tau) - I]\, x_0 \| < \varepsilon M, \tag{10}$$

where $\varepsilon \to 0$ as $\tau \to 0$, and $M = \sup\limits_t \| R(t) \|$. A similar estimate holds for $t > n_k \Delta_k t$. Consequently, (9) and (10) imply

$$\| \{ [C(\Delta_k t)]^{n_k} - R(t)\}\, x_0 \| \leqslant \| z_k \| + \varepsilon M,$$

and since both the terms on the right-hand side can be made arbitrarily small as $n_k \Delta_k t \to t$, hence

$$\{ C[(\Delta_k t)]^{n_k} - R(t) \}\, x_0 \to 0$$

at x_0, which is the intial value of the solution of the manifold L. If x is an arbitrary element in E, we can write

$$\{[C(\Delta_k t)]^{n_k} - R(t)\}\, x$$
$$= \{[C(\Delta_k t)]^{n_k} - R(t)\}\, x_0 + [C(\Delta_k t)]^{n_k} (x - x_0) - R(t)\,(x - x_0),$$

and it is easy see that all the three terms on the right-hand side can be made as small as desired, the first term by what has just been proved and the remaining two because of the element x_0 lying everywhere dense in E and the sets $\{[C(\Delta_k t)]^{n_k}\}$ and $\{R(t)\}$ being bounded. Thus,

$$\{[C(\Delta_k t)]^{n_k} - R(t)\}\, x \to 0$$

as $\Delta_k t \to 0$, $n_k \Delta_k t \to t$ everywhere in E, proving the sufficiency.

An example of the application of the theorem of P. LAX is the solution by finite-difference method of the CAUCHY problem for the heat conduction equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial \xi^2}, \qquad 0 \leqslant \xi \leqslant a,\ 0 \leqslant t \leqslant T, \tag{11}$$

$$u(0, t) = u(a, t) = 0, \quad u(\xi, 0) = \varphi(\xi).$$

As a basic BANACH space, we take the space $C_0\,[0, a]$ of functions, continuous on the segment $[0, a]$ and vanishing at the end-points of this segment. Then, a continuous function $u(\xi, t)$ of two variables, where $u(0, t) = u(a, t) = 0$, can be regarded as a one-parametric family $u_t(\xi) = x(t)$ of elements in $C_0\,[0, a]$, and the boundary value problem (11) can be expressed in the form

$$dx/dt = Ax, \quad x(0) = \varphi, \tag{12}$$

where the differential operator A, $A = c^2 \, (d^2/d\xi^2)$ defined on $D(A)$ is a twice continuously differentiable function of $C_0\,[0, a]$, vanishing for $\xi = 0$ and $\xi = a$.

As an approximating boundary value problem, choose the system

$$\frac{u_j^n - u_j^{n-1}}{\Delta t} = c^2 \, \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta\xi)^2}, \tag{13}$$

$j = 1, 2, \ldots, J - 1, n = 1, 2, \ldots, N, J\,\Delta\xi = a, N\,\Delta t = T$, and the boundary conditions

$$u_0^{(n)} = u_j^{(n)} = 0, \quad u_j^{(0)} = \varphi\,(j\,\Delta\xi). \tag{14}$$

The solutions of system (13) are initially determined only at the lattice point, that is, at a point of the form $(j\,\Delta\xi, n\,\Delta t)$. By linear interpolation, they are extended to all the rest of the points of the rectangle $0 \leqslant \xi \leqslant a$, $0 \leqslant t \leqslant T$ and denoted by $\tilde{u}(\xi, t)$. Again, consider $\tilde{u}(\xi, n\Delta t)$ as the elements x_n in $C_0\,[0, a]$ and regard them approximate values of the solution $x(t)$ at the point $t = n\Delta t$. Assuming $\Delta\xi$ and Δt to be independent and regarding that $\Delta\xi = g(\Delta t)$, where $g\,(\alpha) \to 0$ as $\alpha \to 0$, the approximating boundary value problem can be expresed in the form of the recurrence formulae

$$x_{n+1} = C(\Delta t)\, x_n, \quad x_0 = \varphi, \tag{15}$$
$$n = 0, 1, 2, \ldots, N - 1.$$

The problem (11) and, consequently, also (12), as is known, are well posed. If it is further remarked that for sufficiently smooth functions the difference ratio converges to the derivatives uniformly in the rectangle $0 \leqslant \xi \leqslant a$, $0 \leqslant t \leqslant T$, then for such functions,

$$\left\| \left(\frac{C(\Delta t) - I}{\Delta t} - A \right) x(t) \right\| \to 0$$

as $\Delta t \to 0$ and, consequently, the consistency condition is also satisfied.

It is to be shown that for the solution of the difference boundary value problem the **extremum principle** holds : *The greatest (least) value of a solution in the interior points of a lattice cannot be larger (smaller) than the greatest (least) value of the solution at the boundary points.*

For proof, assume the contrary. Let $\mu = u_{j_0}^{n_0}$ be the maximal value of the solution, taken in the interior points and also assume that n_0 and j_0 are the least values of the indices n and j, verifying $u_j^n = \mu$. Writing Eq. (13) in terms of these values of indices, we get

$$\frac{u_{j_0}^{n_0} - u_{j_0}^{n_0-1}}{\Delta t} = c^2 \, \frac{u_{j_0+1}^{n_0} - 2u_{j_0}^{n_0} + u_{j_0-1}^{n_0}}{(\Delta\xi)^2}.$$

However, this equality is impossible, since its left-hand side because of

$u_{j_0}^{n_0} > u_{j_0}^{n_0-1}$ is positive and the right-hand side because of $u_{j_0}^{n_0} \geqslant u_{j_0+1}^{n_0}$ and $u_{j_0}^{n_0} > u_{j_0-1}^{n_0}$ is negative. The contradiction obtained proves the extremum principle.

The extremum principle implies that the function $\tilde{u}(\xi, t)$ takes the greatest and least values on the boundary of the rectangle $0 \leqslant \xi \leqslant a, 0 \leqslant t \leqslant T$.

Now, if Eq. (15) is written in the form

$$x_n = [C(\Delta t)]^n \varphi,$$

then by what has just been asserted, we have

$$\sup_{\xi} | x_n(\xi) | = \| [C(\Delta t)]^n \varphi \| \leqslant \sup_{\xi} | \varphi(\xi) | = \| \varphi \|,$$

whence $\| [C(\Delta t)]^n \| \leqslant 1$ for any Δt and n. Consequently, the approximation is stable and thereupon it follows from the theorem of P. Lax that the solution of difference boundary value problem converges to the solution of boundary value problem for differential equations.

8.3 DIFFERENTIAL OF AN ABSTRACT FUNCTION

Let E_x and E_y by normed linear spaces and let $y = f(x)$ be an abstract function with domain E_x and range E_y.

In analogy with the definition of the differential of a function of a finite number of variables, let us introduce two definitions of the **differential of an abstract function.**

8.31. Strong differential (Fréchet differential). Let h be an arbitrary element in E_x, and assume that there is a linear operator $l \in (E_x \to E_y)$ (depending, in general, on x), such that

$$f(x+h) - f(x) = lh + \omega(x, h), \tag{1}$$

where

$$\frac{\| \omega(x, h) \|}{\| h \|} \to 0 \text{ as } \| h \| \to 0. \tag{2}$$

In this case lh is called the **strong differential** (or the **Fréchet differential) of the function** $f(x)$ **at the point** x **for the increment** h. It is denoted by $df(x, h)$ or $df(x)\, h$.

The linear operator l which, in general, depends on x is also denoted by $f'(x)$. Then,

$$df(x, h) = f'(x)\, h, \quad f'(x) \in (E_x \to E_y). \tag{3}$$

The operator $f'(x)$ can be regarded as a function of x, defined on a point set $\{x\} \subset E_x$, where $f(x)$ is differentiable, with values in $(E_x \to E_y)$.

$f'(x)$ is called the **first Fréchet derivative (differential) of the function** $f(x)$ **at the point** x. Eq. (1) can be written in the form

$$f(x + h) - f(x) = f'(x)\, h + o(\| h \|). \tag{4}$$

The first summand on the right-hand of this equality is a linear function of h, approximating $f(x+h) - f(x)$ to within an order higher than $\| h \|$.

8.32. Weak differential (Gâteux differential). The expression

$$Df(x, h) = \frac{d}{dt} f(x + th) |_{t=0} = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

is called the **directional derivative** or the **weak (Gâteux) differential** of the function $f(x)$ at the point x; here it is assumed that the limit appearing on the right-hand of equality, and understood in the sense of norm convergence, exists.†

Examples. 1. Let $E_x = E_y = C[a, b]$, and let

$$f(x) = \int_a^b K(t, s)\, g[s, x(s)]\, ds,$$

the kernel $K(t, s)$ continuous in the square $a \leqslant t, s \leqslant b$, and let $g(u, v)$ be a function of two variables, defined and continuous in the regions $a \leqslant u \leqslant b, -\infty < v < +\infty$. Then, $f(x)$ is an abstract function, defined on $C[a, b]$, with values in the same space.

Assume that $g(u, v)$ is not only continuous but has also the partial derivative $g'_v(u, v)$ uniformly continuous in $a \leqslant u \leqslant b, -\infty < v < +\infty$. Then, $f(x)$ is a strongly differentiable function. In fact, for any function $h(s) \in C[a, b]$ we have

$$f(x + h) - f(x) = \int_a^b K(t, s)\, g\, [s, x(s) + h(s)]\, ds$$

$$- \int_a^b K(t, s)\, g\, [s, x(s)]\, ds$$

$$= \int_a^b K(t, s)\, [g\, (s, x(s) + h(s)) - g\, (s, x(s))]\, ds.$$

By LAGRANGE's theorem,

$$g\, [s, x(s) + h(s)] - g\, [s, x(s)] = g'_v\, [s, x(s) + \theta(s)\, h(s)]\, h(s),$$

where $0 \leqslant \theta(s) \leqslant 1$. Furthermore,

$$g'_v\, [s, x(s) + \theta(s)\, h(s)] = g'_v\, [s, x(s)] + \alpha[s, x(s), \theta(s)\, h(s)]††,$$

where as $\| h \| \to 0$, that is, as $h(s) \to 0$ uniformly on $[a, b]$, $\alpha\, [s, x(s), \theta(s)\, h(s)] \to 0$ also uniformly on $[a, b]$, since the functions continuous in closed bounded domains $a \leqslant s \leqslant b$,

† Sometimes $\lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$ is also called a **weak differential** if the limit is understood in the sense of weak convergence of the elements. Also note that the GÂTEUX differential is homogeneous but its additivity is not presupposed.

†† $\alpha(s, x, u) = g'_v(s, x + u) - g'_v(s, x)$.

$|x| \leqslant c_1, |u| \leqslant c_2$, are uniformly continuous there. Hence

$$f(x+h) - f(x) = \int_a^b K(t, s)\, g_v'[s, x(s)]\, h(s)\, ds$$

$$+ \int_a^b K(t, s)\, \alpha\, [s, x(s), \theta(s)\, h(s)]\, h(s)\, ds$$

$$= lh + \omega(x, h),$$

where
$$lh = \int_a^b K(t, s)\, g_v'[s, x(s)]\, h(s)\, ds,$$

and
$$\omega(x, h) = \int_a^b K(t, s)\, \alpha\, [s, x(s), \theta(s)\, h(s)]\, h(s)\, ds.$$

Also
$$\|\omega(x, h)\| = \max_t \left| \int_a^b K(t, s)\, \alpha\, [s, x(s), \theta(s)\, h(s)]\, h(s)\, ds \right|$$

$$\leqslant \max_{t, s} |K(t, s)|\, \|\alpha\, [s, x(s), \theta(s)\, h(s)]\|\, (b-a)\, \|h\|.$$

$$= c\, \|\alpha\, [s, x(s), \theta(s)\, h(s)]\|\, \|h\|,$$

and, hence,
$$\frac{\|\omega(x, h)\|}{\|h\|} \leqslant c\, \|\alpha\, [s, x(s), \theta(s)\, h(s)]\| \to 0$$

as $\|h\| \to 0$.

Consequently, $f(x)$ is Fréchet-differentiable, and

$$df(x, h) = \int_a^b K(t, s)\, g_v'[s, x(s)]\, h(s)\, ds.$$

2. Consider in the space $C^1[a, b]$ continuously differentiable functions $y(t)$, $a \leqslant t \leqslant b$, with the norm

$$\|y\| = \max_t (|y(t)|, |y'(t)|),$$

and the functional of a simple problem of the calculus of variations

$$J\{y\} = \int_a^b F[t, y(t), y'(t)]\, dt.$$

The definitions of both the Fréchet and (weak) Gâteux differentials then correspond the usual definitions of the variation.

We can similarly define other functionals of the calculus of variations. The notion of the differential of an abstract function naturally has its origin in the calculus of variations.

Theorem 1. *If the Fréchet differential $df(x, h)$ exists, then the weak differential $Df(x, h)$ also exists, and $Df(x, h) = df(x, h)$.*

PROOF. In fact,

$$f(x+th) - f(x) = df(x, th) + \omega(x, th) = t\, df(x, h) + \omega(x, th).$$

By (2), $\| \omega(x, th) \| = o(\| th \|) = o(| t | \, \| h \|) = o(t)$ to an order higher than t as $t \to 0$. Hence,

$$\frac{f(x+th)-f(x)}{t} = df(x, h) + \frac{\omega(x, th)}{t} \to df(x, h)$$

as $t \to 0$. Thus,

$$Df(x, h) = \lim_{t\to 0} \frac{f(x+th)-f(x)}{t} = df(x, h),$$

whereby it is proved that the weak differential exists and is equal to the FRÉCHET differential.

The definition of $Df(x, h)$ does not demand it to be linear in h. However, if it is so, then $Df(x, h) = Lh = f'(x)_c\, h, f'(x)_c$ a linear operator in $h, f'(x)_c \in (E_x \to E_y)$. Then, $f'(x)_c$ is called the **weak derivative of the function** $f(x)$ **at the point** x.

THEOREM 2. *If the weak differential $Df(x, h)$ exists in the sphere $\| x - x_0 \| < r$ and if it is uniformly continuous in x and continuous in h, then the Fréchet differential $df(x, h)$ exists there and $df(x, h) = Df(x, h)$.*

PROOF. In fact, for $\| h \| < r(x)$, where the number $r(x)$, the radius of the spherical neighbourhood of the point x, belongs to the sphere $\| x - x_0 \| < r$, the weak differential $Df(x_t, h)$ exists at all the points $x_t = x + th, 0 \leqslant t \leqslant 1$. Since

$$Df(x_t, h) = \lim_{\Delta t\to 0} \frac{f(x_t+\Delta th)-f(x_t)}{\Delta t},$$

and
$$x_t + \Delta th = x + (t + \Delta t)\, h = x_{t+\Delta t},$$

so,
$$Df(x_t, h) = \lim_{\Delta t\to 0} \frac{f(x_{t+\Delta t})-f(x_t)}{\Delta t} = \frac{d}{dt} f(x_t)$$
$$= \frac{d}{dt} f(x+th).$$

The requirement is to show the additivity of $Df(x, h)$ with respect to h:

$$Df(x, h_1 + h_2) = Df(x, h_1) + Df(x, h_2), \tag{5}$$

It may be remarked in the first place that by hypothesis the function $Df(x_t, h) = (d/dt) f(x+th)$ is continuous, implying that

$$f(x+th_1) - f(x) = \int_0^t \frac{d}{d\tau} f(x+\tau h_1)\, d\tau$$
$$= \int_0^t Df(x+\tau h_1, h_1)\, d\tau = tDf(x, h_1) + \omega_1, \tag{6}$$

where $$\omega_1 = \int_0^t [Df(x + \tau h_1, h_1) - Df(x, h_1)]\, d\tau.$$

Similarly, $$f[x + t(h_1 + h_2)] - f(x) = tDf(x, h_1 + h_2) + \omega_2, \tag{7}$$

where $$\omega_2 = \int_0^t [Df(x + \tau(h_1 + h_2), h_1 + h_2) - Df(x, h_1 + h_2)]\, d\tau,$$

and also $$f[x + t(h_1 + h_2)] - f(x + th_1) = tDf(x, h_2) + \omega_3, \tag{8}$$

where $$\omega_3 = \int_0^t [Df(x + th_1 + \tau h_2, h_2) - Df(x, h_2)]\, d\tau.$$

Since $Df(x, h)$ is continuous in x, it follows that

$$\| Df(x + \tau h_1, h_1) - Df(x, h_1) \| < \varepsilon/3,$$
$$\| Df(x + \tau(h_1 + h_2), h_1 + h_2) - Df(x, h_1 + h_2) \| < \varepsilon/3,$$
$$\| Df(x + th_1 + \tau h_2, h_2) - Df(x, h_2) \| < \varepsilon/3,$$

for an arbitrary $\varepsilon > 0$, sufficiently small $t > 0$ and $0 \leqslant \tau \leqslant t$. Hence

$$\| \omega_1 \| = \left\| \int_0^t [Df(x + \tau h_1, h_1) - Df(x, h_1)]\, d\tau \right\| < \frac{\varepsilon}{3} t,$$

and, similarly,

$$\| \omega_2 \| < (\varepsilon/3)\, t, \qquad \| \omega_3 \| < (\varepsilon/3)\, t.$$

From (6), (7) and (8) it follows that

$$\begin{aligned} 0 &= [f(x + th_1) - f(x)] + [f(x + t(h_1 + h_2)) - f(x + th_1)] \\ &\qquad - [f(x + t(h_1 + h_2)) - f(x)] \\ &= t\,[Df(x, h_1) + Df(x, h_2) - Df(x, h_1 + h_2)] + \omega_1 + \omega_2 - \omega_3. \end{aligned}$$

Thence, $Df(x, h_1) + Df(x, h_2) - Df(x, h_1 + h_2) = (1/t)(\omega_1 + \omega_2 - \omega_3)$, and, consequently,

$$\|Df(x, h_1) + Df(x, h_2) - Df(x, h_1 + h_2)\| \leqslant (1/t)(\| \omega_1 \| + \| \omega_2 \| + \| \omega_3 \|) < \varepsilon.$$

Since ε is arbitrary by choice,

$$\| Df(x, h_1) + Df(x, h_2) - Df(x, h_1 + h_2) \| = 0,$$

and, consequently, (5) is proved.

In addition, since $Df(x, h)$ is continuous in h, it is a linear and bounded operator in h: $Df(x, h) = f'(x)_c\, h$. Since $Df(x, h) = f'(x)_c\, h$ is uniformly continuous in x, hence so is $f'(x)_c$.

Now, it is required to show that

$$f(x + h) - f(x) = f'(x)_c\, h + o(\| h \|). \tag{4'}$$

Then, $Df(x, h)$ as the principal linear part of the increment $f(x+h) - f(x)$ in h coincides with $df(x, h)$, so that

$$f(x+h) - f(x) = \int_0^1 \frac{d}{dt} f(x+th)\, dt = \int_0^1 Df(x+th, h)\, dt$$

$$= \left(\int_0^1 f'(x+th)_c\, dt \right) h = f'(x_c)\, h + \omega, \tag{9}$$

where
$$\omega = \left(\int_0^1 [f'(x+th)_c - f''(x)_c]\, dt \right) h.$$

By the uniform continuity of $f'(x)_c$, for $0 \leqslant t \leqslant 1$ we have

$$\| f'(x+th)_c - f'(x)_c \| \leqslant \alpha\, (\| h \|) \to 0 \quad \text{as} \quad \| h \| \to 0.$$

Thence,
$$\| \omega \| \leqslant \left\| \int_0^1 [f'(x+th)_c - f'(x)_c]\, dt \right\| \| h \|$$

$$\leqslant \int_0^1 \| f'(x+th)_c - f'(x)_c \|\, dt\, \| h \| \leqslant \alpha\, (\| h \|)\, \| h \| \tag{9'}$$

and (4′) is proved. Thus,

$$Df(x, h) = df(x, h) \quad \text{and} \quad f'(x)_c = f'(x). \qquad \blacksquare$$

It shall always be assumed hereafter that $Df(x, h) = df(x, h)$ for all differentiable functions f under consideration.

REMARK. If the inequality $\| f'(x) \| \leqslant L$ holds in the sphere $\| x - x_0 \| < r$, and if x_1, x_2 belong to this sphere, then

$$\| f(x_1) - f(x_2) \| \leqslant L \| x_1 - x_2 \|.$$

In fact, since the convexity of the sphere $\| x - x_0 \| < r$ together with x_1 and x_2 of this sphere belongs to the entire segment $\{x_t\}$, where

$$x_t = (1-t)\, x_1 + t x_2 = t\,(x_2 - x_1) + x_1, \quad 0 \leqslant t \leqslant 1,$$

hence $\| f'(x_t) \| \leqslant L$ and since $f(x_2) - f(x_1) = \int_0^1 f'(x_t)\,(x_2 - x_1)\, dt$, it follows that

$$\| f(x_2) - f(x_1) \| \leqslant \int_0^1 \| f'(x_t) \| \, \| x_2 - x_1 \|\, dt \leqslant L \| x_2 - x_1 \|.$$

8.4 THEOREM OF INVERSE OPERATOR. NEWTON'S METHOD

MAKING USE of the notion of the derivative of an operator, it is possible to prove the local existence theorem for inverse operators, analogous to the existence theorem for the functions, inverse to monotone functions, having nonvanishing derivatives.

THEOREM 1. *Let $y = f(x)$ be an operator, defined in some neighbourhood of the point x_0 of the space E_x, and mapping this neighbourhood into E_y. Assume that*

(i) $f(x_0) = y_0$;

(ii) *The derivative $f'(x)$ exists in the considered neighbourhood of x_0, and it is bounded and continuous there;*

(iii) $[f'(x_0)]^{-1}$ *exists.*

Then in some neighbourhood of the point y_0 there exists an inverse operator $x = f^{-1}(y)$, it takes the value x_0 at the point y_0 and is continuous in the neighbourhood of y_0.

PROOF. Consider the equation

$$x = A(x;\ y), \tag{1}$$

where
$$A(x;\ y) = x - [f'(x_0)]^{-1}\,(f(x) - y), \tag{2}$$

and y plays the role of a parameter.

It is trivial to verify that if, for a given $y \in E_y$, Eq. (1) has a solution x, then $f(x) = y$, and conversely. For proving the existence of a solution of Eq. (1), an appeal is sought to be made to the principle of contraction mappings.

For $y \in E_y$ fixed, we have

$$A'(x;\ y) = I - [f'(x_0)]^{-1} f'(x) = [f'(x_0)]^{-1}\,[f'(x_0) - f(x)],$$

whence, for $\| x - x_0 \| \leqslant r$

$$\| A'(x;\ y) \| \leqslant \| [f'(x_0)]^{-1} \|\, \| f'(x_0) - f(x) \| \leqslant q(r),$$

where $q(r) \to 0$ as $r \to 0$, $f'(x)$ being continuous by hypothesis. Hence, the operator $A(x;\ y)$ with respect to the variable x satisfies the LIPSCHITZ condition

$$\| A(x_1;\ y) - A(x_2;\ y) \| \leqslant q(r)\, \| x_1 - x_2 \|, \quad x_1, x_2 \in \bar{S}(x_0, r). \tag{3}$$

To estimate the difference $A(x_0;\ y) - x_0$, we have

$$\begin{aligned} \| A(x_0;\ y) - x_0 \| &= \| [f'(x_0)]^{-1}\,(f(x_0) - y) \| \\ &= \| [f'(x_0)]^{-1}\,(y - y_0) \| \leqslant \| [f'(x_0)]^{-1} \|\, \| y - y_0 \|, \end{aligned}$$

that is,
$$\| A(x_0;\ y) - x_0 \| \leqslant \| [f'(x_0)]^{-1} \|\, \| y - y_0 \|. \tag{4}$$

Furthermore, (3) and (4) imply

$$\begin{aligned} \| A(x;y) - x_0 \| &\leqslant \| A(x;\ y) - A(x_0;\ y) \| + \| A(x_0;\ y) - x_0 \| \\ &\leqslant q(r)\, \| x - x_0 \| + \| [f'(x_0)]^{-1} \|\, \| y - y_0 \|. \end{aligned}$$

Now, choose r_x such that $q = q(r_x) < 1$ and consider y in the sphere $\| y - y_0 \| < r_y = (1 - q)\, r_x / \| [f'(x_0)]^{-1} \|$. Then, the preceding inequality yields

$$\| A(x; y) - x_0 \| \leqslant q \| x - x_0 \| + \frac{1-q}{\| [f'(x_0)]^{-1} \|} \| [f'(x_0)]^{-1} \| r_x \leqslant r_x,$$

and, consequently, the operator A maps the sphere $\| x - x_0 \| \leqslant r_x$ into itself, satisfying the principle of contraction mappings. Hence every y, $\| y - y_0 \| < r_y$ corresponds uniquely to x, such that $\| x - x_0 \| < r_x$ and $f(x) = y$. By the same token, the inverse operator $x = \varphi(y)$ is defined on the sphere $\| y - y_0 \| < r_y$ with the range $\| x - x_0 \| < r_x$. It is plain that $\varphi(y_0) = x_0$.

Ineq. (3) implies that

$$\begin{aligned} \| \varphi(y_1) - \varphi(y_2) \| &= \| x_1 - x_2 \| = \| A(x_1; y_1) - A(x_2; y_2) \| \\ &\leqslant \| A(x_1; y_1) - A(x_1; y_2) \| + \| A(x_1; y_2) - A(x_2; y_2) \| \\ &\leqslant \| [f'(x_0)]^{-1} \| \, \| y_1 - y_2) \| + q \| x_1 - x_2 \|. \end{aligned}$$

Thence, $(1 - q) \| \varphi(y_1) - \varphi(y_2) \| \leqslant \| [f'(x_0)]^{-1} \| \, \| y_1 - y_2 \|$,

or,
$$\| \varphi(y_1) - \varphi(y_2) \| \leqslant \frac{\| [f'(x_0)]^{-1} \|}{1-q} \| y_1 - y_2 \|, \tag{5}$$

that is, the inverse operator $\varphi(y)$ satisfies a LIPSCHITZ condition in the sphere $\| y - y_0 \| < r_y$ and is, consequently, continuous. ■

By the principle of contraction mappings, the inverse operator $\varphi(y)$ can be obtained as the limit of a sequence of operators $\varphi_n(y)$, defined by the rule

$$\varphi_0(y) = x_0,$$

$$\varphi_n(y) = A(\varphi_{n-1}(y); y) \quad (\| y - y_0 \| < r_y, \; n = 1, 2, \ldots). \tag{6}$$

Since $A(x; y)$ is continuous in the collection of variables, it can be shown by induction that all the successive approximations $\varphi_n(y)$ are continuous functions of y.

Furthermore, the estimate

$$\begin{aligned} \| \varphi(y) - \varphi_n(y) \| &\leqslant \frac{q^n}{1-q} \| \varphi_1(y) - \varphi_0(y) \| \\ &\leqslant \frac{q^n}{1-q} \| [f'(x_0)]^{-1} \| \, \| f(x_0) - y \| \end{aligned}$$

exhibits that the convergence of the sequence $\varphi_n(y)$ to a limit operator $\varphi(y)$ is uniform in the sphere $\| y - y_0 \| < r_y$.

Example. In the space $C[0, 1]$ consider the non-linear integral equation

$$x(t) - \int_0^1 K[t, s, x(s)] \, ds = y(t), \tag{7}$$

where the kernel $K(t, s, u)$ is continuous in the strips $0 \leqslant t, s \leqslant 1$, $-\infty < u < +\infty$ and has there the continuous derivative $K'_u(t, s, u)$. In addition, assume that

(a) $K(t, s, 0) \equiv 0$ and $K'_u(t, s, 0) \not\equiv 0$;

(*b*) 1 is not the eigenvalue of the kernel $K'_u(t, s, 0)$, that is, the linear integral equation

$$z(t) - \int_0^1 K'_u(t, s, 0)\, z(s)\, ds = 0$$

has no non-zero solution.

Writing Eq. (7) in the form

$$f(x) = y, \tag{8}$$

it is trivial to verify that

(i) $f(0) = 0$;

(ii) The derivative $f'(x)$ exists in the neighbourhood of zero and has the form

$$f'(x)\, h = h(t) - \int_0^1 K'_u[t, s, x(s)]\, h(s)\, ds;$$

thus, it is bounded and continuous in this neighbourhood.

(iii) In virtue of (*b*), $[f'(0)]^{-1}$ exists.

Then by the theorem just proved, Eq. (7) has for sufficiently small right side $y(t)$ a unique solution, which can be realized by the successive approximation method.

8.41. Newton's method. NEWTON's method for solving operator equations furnishes another example of the application of the notion of the derivatives of abstract functions. As is known, for the scalar equation $f(x) = 0$, NEWTON's method consists in finding a sequence of approximate solutions by the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Subject to certain conditions imposed on the function $f(x)$ and its derivatives, it has been established that the approximate solutions x_n converge to a finite limit and this limit is a solution of the equation.

It was shown by L. KANTOROVICH that NEWTON'S method can be extended to operator equations. It is proposed to consider this extension. For the simplicity of proof, conditions of more rigorous nature are assumed to be satisfied.†

Thus, given the equation

$$f(x) = 0, \tag{9}$$

$f(x)$ an abstract function defined on a BANACH space E_x with values in a BANACH space E_y. Assume that in some sphere $S(x_0, r)$, whose center x_0 is taken as an approximate value of the solution of Eq. (1), the function $f(x)$ is strongly differentiable and its derivative $f'(x)$ satisfies the LIPSCHITZ condition

$$\| f'(x) - f'(\xi) \| \leqslant L \| x - \xi \|. \tag{10}$$

In adition, if $[f'(x)]^{-1}$ exists, then, in analogy to the scalar case, it is possible to construct successive approximations by the formula

$$x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n).$$

†For a detailed discussion and more stringent restrictions, see [15].

This formula is, however, inconvenient in that it entails determining successively the inverse operators, $[f'(x_n)]^{-1}$, that is, as a matter of fact, involves solving the linear operator equations $f'(x_n)\,h = g$. In order to circumvent the indicated difficulty, L. KANTOROVICH introduced a modification to NEWTON'S method, by which successive approximations are realized by the formula

$$x_{n+1} = x_n - [f'(x_0)]^{-1} f(x_n), \tag{11}$$

where one and the same inverse operator appears for every n.

It is proposed to draw upon only the **modified Newton's formula.**

Introduce the following constants

$$M_0 = \| [f'(x_0)]^{-1} \|, \quad \eta_0 = \| [f'(x_0)]^{-1} f(x_0) \|.$$

THEOREM 2. *If $h_0 = M_0\eta_0 L \leqslant \frac{1}{4}$ and t_0 is a smaller root of the equation $h_0 t^2 - t + 1 = 0$, then in the sphere $\| x - x_0 \| \leqslant t_0\eta_0$, the equation $f(x) = 0$ has a unique solution x^* and the successive approximations x_n determined by* (3), *converge to this solution.*

Consider the operator

$$Ax = x - [f'(x_0)]^{-1} f(x).$$

This operaor carries the sphere $\| x - x_0 \| \leqslant t_0\eta_0$ into itself. In fact,

$$\begin{aligned} Ax - x_0 &= x - x_0 - [f'(x_0)]^{-1} f(x) \\ &= [f'(x_0)]^{-1} \{f'(x_0)(x - x_0) - f(x) + f(x_0)\} - [f'(x_0)]^{-1} f(x_0). \end{aligned}$$

Thence

$$\begin{aligned} \| Ax - x_0 \| \leqslant \| [f'(x_0)]^{-1} \| \, \| f(x) - f(x_0) - f'(x_0)(x - x_0) \| \\ + \| [f'(x_0)]^{-1} f(x_0) \|, \end{aligned}$$

that is,

$$\| Ax - x_0 \| \leqslant M_0 \| f(x) - f(x_0) - f'(x_0)(x - x_0) \| + \eta_0.$$

Consider the function $\varphi(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$, to receive†

$$\varphi'(x) = f'(x) - f'(x_0).$$

Let $x \in S(x_0, t_0\eta_0)$. Then,

$$\| \varphi'(x) \| = \| f'(x) - f'(x_0) \| \leqslant L \| x - x_0 \| \leqslant Lt_0\eta_0.$$

Hence, $\| \varphi(x) \| = \| \varphi(x) - \varphi(x_0) \| \leqslant L\, t_0\eta_0 \| x - x_0 \| \leqslant L(t_0\eta_0)^2$. Thus, if $x \in S(x_0, t_0\eta_0)$, then

$$\| Ax - x_0 \| \leqslant M_0 Lt_0^2 \eta_0^2 + \eta_0 = \eta_0 (M_0L\eta_0 t_0^2 + 1) = \eta_0(\eta_0 t_0^2 + 1) = \eta_0 t_0,$$

that is, the operator A maps the sphere $\| x - x_0 \| \leqslant t_0\eta_0$ into itself.

It is to be shown that the operator A satisfies the conditions of the principle of contraction mappings in this sphere.

†The derivative of a linear operator $Ux = f'(x_0)\,x$ is equal to $f'(x_0)$.

For $x \in S(x_0, t_0\eta_0)$, we have

$$A'(x) = I - [f'(x_0)]^{-1} f'(x) = [f'(x_0)]^{-1} [f'(x_0) - f'(x)],$$

and, hence,

$$\| A'(x) \| \leqslant M_0 \| f'(x) - f'(x_0) \| \leqslant M_0 L \| x - x_0 \| \leqslant M_0 L \eta_0 t_0.$$

Since t_0 is a smaller root of the equation $h_0 t^2 - t + 1 = 0$, it follows that

$$t_0 = \frac{1 - \sqrt{1 - 4h_0}}{2h_0}.$$

Consequently,

$$\| A'(x) \| \leqslant M_0 L\eta_0 t_0 = h_0 \frac{1 - \sqrt{1 - 4h_0}}{2h_0} = \frac{1 - \sqrt{1 - 4h_0}}{2} < q < 1,$$

whence

$$\| Ax - A\xi \| \leqslant q \| x - \xi \|, \tag{12}$$

as we sought to exhibit.

Thus, the operator A mapping the sphere $S(x_0, t_0\eta_0)$ into itself satisfies the conditions of the principle of contraction mappings and, hence, has there a unique fixed point x^*, admitting

$$x^* = x^* - [f'(x_0)]^{-1} f(x^*),$$

that is, x^* is a solution of Eq. (9),

$$f(x^*) = 0.$$

The point x^* is the limit of the successive approximations

$$x_{n+1} = Ax_n = x_n - [f'(x_0)]^{-1} f(x_n), \tag{11}$$

and the theorem is completely proved.

REMARK 1. The condition $h_0 \leqslant \frac{1}{4}$ can be realized at the expense of the initial approximation x_0 being sufficiently close to the solution.

REMARK 2. The rate of convergence of the successive approximations in the modified NEWTON's method is defined by

$$\| x_n - x^* \| \leqslant \frac{q^n}{1 - q} \| [f'(x_0)]^{-1} f(x_0) \|,$$

which can be easily obtained from (12). If, however, the original (non-modified) NEWTON's method is considered, then the convergence rate will be higher, namely

$$\| x_n - x^* \| \leqslant \frac{1}{2^{n-1}} (2h_0)^{2^n - 1} \eta_0 .$$

For greater details, see [14].

Example. Consider the non-linear HAMMERSTEIN integral equation

$$x(t) - \int_0^1 K(t, s)\, g[s, x(s)]\, ds = 0. \tag{13}$$

The kernel $K(t, s)$ is continuous in the collection of variables in the square $0 \leqslant t, s \leqslant 1$. The function $g(s, u)$ is continuous in the collection of variables in the region $0 \leqslant s \leqslant 1$, $-\infty < u < +\infty$ and has a continuous derivative $g'_u(s, u)$, satisfying with respect to the second variable the LIPSCHITZ condition

$$|g'_u(s, u_1) - g'_u(s, u_2)| \leqslant \lambda |u_1 - u_2|.$$

Then, the abstract function $f(x) = x(t) - \int_0^1 K(t, s)\, g[s, x(s)]\, ds$ maps the space $C[0, 1]$ into itself and is strongly differentiable; moreover,

$$f'(x)\, h = h(t) - \int_0^1 K(t, s)\, g'_u[s, x(s)]\, h(s)\, ds,$$

and the derivative $f'(x)$ also satisfies the LIPSCHITZ condition

$$\|f'(x) - f'(\xi)\| \leqslant a\lambda \|x - \xi\|,$$

with

$$a = \sup_{t,\, s} |K(t, s)|.$$

Thereupon, if 1 is not an eigenvalue of the linear integral equation

$$h(t) - \lambda \int_0^1 K(t, s)\, g'_u[s, x_0(s)]\, h(s)\, ds = 0, \tag{14}$$

and, if

$$M_0 = \sup_t \int_0^1 |R_0(t, s, 1)|\, ds,$$

where $R_0(t, s, \lambda)$ is the resolvent of Eq. (14), then, provided that

$$h_0 = LM_0\eta_0 \leqslant \tfrac{1}{4},$$

with

$$\eta_0 = \sup_t \int_0^1 |R_0(t, s, 1)| \left| x_0(s) - \int_0^1 K(s, \sigma) g[\sigma, x_0(\sigma)]\, d\sigma \right| ds,$$

NEWTON's method is applicable to Eq. (13) for the convergence of the initial approximation $x_0(t)$ to a solution of this equation.

8.5 HOMOGENEOUS FORMS AND POLYNOMIALS

8.51. Multiplication of elements. We frequently encounter a multiplication, in which factors and product are elements of different spaces.

Consider three normed linear spaces E_x, E_y and E_z and introduce a **multiplication** associating an element $z = xy \in E_z$ with any two arbitrary elements $x \in E_x$ and $y \in E_y$. This multiplication has the following properties:

(i) $(x_1 + x_2)y = x_1 y + x_2 y$;
(ii) $x(y_1 + y_2) = xy_1 + xy_2$; and
(iii) $x_n y_n \to x_0 y_0$ as $x_n \to x_0$ and $y_n \to y_0$.

The multiplication xy is an additive and continuous operation with respect to y for x fixed, that is, it is a linear operator with domain E_y and range

E_z. Similarly, the multiplication xy is a linear operation for fixed y, mapping E_x into E_z. Thereupon, it follows that

$$(\lambda x)\, y = \lambda\, (xy) \quad \text{and} \quad x\, (\lambda y) = \lambda\, (xy).$$

Examples. 1. Let A and B be linear operators mapping, respectively, E_x into E_y and E_y into E_z. Then, BA is a linear operator transforming E_x into E_z.

2 If $x\,(t)$ is a function in $L_{p_1}\,[0, 1]$ and $y\,(t)$ in $L_{p_2}\,[0, 1]$, then their product in the usual sense, $z(t) = x(t)\, y(t)$, is an element in $L_q[0, 1]$, where $1/q = (1/p_1) + (1/p_2)$.

3. The inner product of two elements x and y of a real HILBERT space can be regarded as the product of x and $y : x \in H, y \in H, xy \in R = (-\infty, \infty)$.

4. If $x \in E_x$ and A is a linear operator in $(E_x \to E_y)$, then Ax can be regarded as the product of A and x:

$$A \in (E_x \to E_y) = E_A\,, \quad x \in E_x, \quad Ax \in E_y.$$

It is easy to verify that all the indicated properties of the multiplication are satisfied in the above examples.

THEOREM. *If E_x, E_y and E_z are normed linear spaces and the product $xy = z$ is defined ($x \in E_x$, $y \in E_y$, $z \in E_z$), then there is a positive constant M, such that*

$$\| xy \| \leqslant M \| x \| \; \| y \| \quad (x \in E_x, y \in E_y).$$

PROOF. Assume the contrary. Then, for every natural number n, there exist the elements $x_n \in E_x$ and $y_n \in E_y$, such that

$$\| x_n\, y_n \| > n^2 \| x_n \| \, \| y_n \|.$$

Hence, put

$$x_n^0 = \frac{1}{n \| x_n \|}\, x_n, \qquad y_n^0 = \frac{1}{n \| y_n \|}\, y_n,$$

to receive,

$$\| x_n^0\, y_n^0 \| = \frac{1}{n^2}\, \frac{1}{\| x_n \| \, \| y_n \|}\, \| x_n y_n \| > 1.$$

On the other hand,

$$\| x_n^0 \| = \frac{1}{n}\,, \qquad \| y_n^0 \| = \frac{1}{n}\,.$$

Thus, x_n^0 and y_n^0 tend to zero; however, the norm $\| x_n^0\, y_n^0 \|$ remains greater than 1, contradicting the continuity of multiplication.

8.52. n-ary linear forms. Let $E_1, E_2, \ldots, E_n, E$ be normed linear spaces. A function $a\,(h_1, h_2, \ldots, h_n) \in E$, $h_i \in E_i$, linear in each of the variables $h_1, h_2, \ldots, h_n$ is called an **n-ary linear form.**

We agree to write $a\,(h_1, h_2, \ldots, h_n) = ah_1\, h_2 \ldots h_n$. The number†

$$\| a \| = \sup \frac{\| ah_1\, h_2 \ldots h_n \|}{\| h_1 \| \, \| h_2 \| \ldots \| h_n \|} \tag{1}$$

†By the BANACH-STEINHAUS theorem n-ary linear forms, continuous in every variable, are continuous in the collection of variables.

denotes the **norm** $\| a \|$ of the form $ah_1h_2 \ldots h_n$. Evidently,

$$\| ah_1 h_2 \ldots h_n \| \leqslant \| a \| \|h_1 \| \| h_2 \| \ldots \| h_n \|.$$

If
$$ah_1 h_2 \ldots h_n = bh_1 h_2 \ldots h_n$$

for any $h_1 \in E_1$, $h_2 \in E_2, \ldots, h_n \in E_n$, then $a = b$. The collection of the forms $ah_1h_2 \ldots h_n$ with $h_i \in E_i$ and $ah_1h_2 \ldots h_n \in E$ forms a normed linear space, if the sum and the multiplication of the form by a scalar, are understood in the usual sense, and the norm $\| a \|$ is defined by Eq. (1). For this, it is possible to interpret the form $ah_1h_2 \ldots h_{n-1}$ as a linear operator acting from E_n into E, that is, $ah_1h_2 \ldots h_{n-1} \in (E_n \to E)$; the form $ah_1h_2 \ldots h_{n-2}$ as a linear operator acting from E_{n-1} into the space $(E_n \to E)$, that is, $ah_1h_2 \ldots h_{n-2} \in [E_{n-2} \to (E_n \to E)]$, and so on; finally,

$$a \in [E_1 \to (E_2 \to \ldots (E_n \to E) \ldots)].$$

Then, the form $ah_1h_2, \ldots, h_n$ can be regarded as a product of the elements $a, h_1, h_2, \ldots, h_n$ successively formed from left to right:

$$ah_1h_2 \ldots h_n = (\ldots [(ah_1) h_2] \ldots) h_n.$$

A form is said to be **symmetric**, if $E_1 = E_2 = \ldots = E_n = E$ and $ah_1h_2 \ldots \ldots h_n = ah_{i_1} h_{i_2} \ldots h_{i_n}$, where $(i_1, i_2, \ldots, i_n)$ is any permutation of the indices $1, 2, \ldots, n$.

The bilinear form (Ax, y), where A is a self-adjoint operator in the real HILBERT space, is an example of a symmetric form.

A symmetric form

$$Sah_1h_2 \ldots h_n = \frac{1}{n!} \sum_{(i_1, i_2, \ldots, i_n)} ah_{i_1} h_{i_2} \ldots h_{i_n}, \tag{2}$$

can be associated with every form $ah_1h_2 \ldots h_n$, where sum is taken over all permutations $(i_1, i_2, \ldots, i_n)$ of the indices $1, 2, \ldots, n$. The form $Sah_1h_2 \ldots \ldots h_n$ is, evidently, symmetric. If $ah_1h_2 \ldots h_n$ is symmetric, then

$$Sah_1h_2 \ldots h_n = ah_1h_2 \ldots h_n.$$

The form $ahh \ldots h$, obtained from the n-ary linear symmetric form $ah_1h_2 \ldots h_n$ for $h_1 = h_2 = \ldots = h_n = h$ is called an n-**ary homogeneous form**. A binary homogeneous form is called **quadratic** (see Chap. 7.1).

This admits the abbreviated notation

$$ah^n = ahh \ldots h.$$

8.53. Properties of n-ary homogeneous linear forms:

(i) $a\,(th)^n = t^n\, ah^n$;

(ii) If a is a symmetric n-ary linear form, then the product $ah_1h_2 \ldots h_n$ is distributive and commutative with respect to every pair of factors. Thus, for example, by the polynomial law, $a(t_1h_1 + t_2h_2 + \ldots + t_kh_k)^n$ we obtain the sum of k terms,

$$a(t_1h_1 + t_2h_2 + \ldots + t_kh_k)^n = \sum_{n_1+n_2+\ldots+n_k=n,\ n_i \geqslant 0} \frac{n!}{n_1! \ldots n_k!} t_1^{n_1} \ldots t_k^{n_k}\, ah_1^{n_1} \ldots h_k^{n_k}. \quad (3)$$

(iii) For symmetric n-ary linear form,

$$\frac{\partial^n}{\partial t_1 \partial t_2 \ldots \partial t_n} a(t_1h_1 + \ldots + t_nh_n)^n = n!\, ah_1h_2 \ldots h_n. \quad (4)$$

In fact, the operator $\partial^n/\partial t_1 \partial t_2 \ldots \partial t_n$ causes all terms of the sum on the right-hand side of (3) to vanish, except the summand containing all the factors $t_1, t_2, \ldots, t_n$, that is, the term with $n_1 = n_2 = \ldots = n_n = 1$. This is equal to $n!\, t_1t_2 \ldots t_nah_1h_2 \ldots h_n$ and, consequently, (4) holds.

(iv) If the equality

$$ah^n = bh^n \quad (5)$$

holds for any $h \in E_x$, then the corresponding n-ary linear forms, assumed to be symmetric, coincide

$$ah_1h_2 \ldots h_n = bh_1h_2 \ldots h_n \quad (6)$$

for every $h_1, h_2, \ldots, h_n \in E_x$, that is, $a = b$.
In fact, by (5), $a(t_1h_1 + t_2h_2 + \ldots + t_nh_n)^n = b(t_1h_1 + t_2h_2 + \ldots + t_nh_n)^n$ for any $h_1, h_2, \ldots, h_n$, whence

$$\frac{\partial^n}{\partial t_1 \partial t_2 \ldots \partial t_n} a(t_1h_1 + \ldots + t_nh_n)^n = \frac{\partial^n}{\partial t_1 \partial t_2 \ldots \partial t_n} b(t_1h_1 + \ldots + t_nh_n)^n$$

and (4) implies (6).

(v) Given a linear operator $A \in (E_y \to E_z)$ and an n-ary homogeneous form ah^n, $h \in E_x$ and $ah^n \in E_y$, then $A(ah)^n$ is also an n-ary homogeneous form.
In fact, $ah_1 h_2 \ldots h_n$ is a symmetric n-ary linear form. Since $A(ah_1 h_2 \ldots h_n)$ depends linearly on every $h_i \in E_x$, so $A(ah_1h_2 \ldots h_n)$ is an n-ary form linear in $h_1, h_2, \ldots, h_n$ with range E_z. This form is also symmetric. Hence, $A(ah^n)$ is an n-ary homogeneous form.

(vi) The product $(a_n h^n)(b_m h^m)$ of n-ary (m-ary) homogeneous forms $a_n h^n$ and $b_m h^m$ is an $(n+m)$-ary homogeneous form. In fact, the product of corresponding n-ary and m-ary linear forms $(a_n h_1 h_2 \ldots h_n)$ $(b_m h_{n+1} h_{n+2} \ldots h_{n+m})$ is an $(n+m)$-ary linear form.
Symmetrize this, to receive a symmetric $(n+m)$-ary linear form

$$c_{n+m} h_1 h_2 \ldots h_{n+m} = S[(a_n h_1h_2 \ldots h_n)(b_m h_{n+1} h_{n+2} \ldots h_{n+m})].$$

Then, put $h_1 = h_2 = \ldots = h_{n+m} = h$, leading to the $(n+m)$-ary homogeneous form $c_{n+m} h^{n+m}$, which because of (2), is equal to $(a_n h^n)$ $(b_m h^m)$.

8.54. Polynomials. A sum $\sum_{k=0}^{n} a_kh^k$ of homogeneous forms, where $h \in E_x$ and $a_n \neq 0$, and all the a_kh^k are elements of the same space E_y, is called an **n-th degree polynomial** in h.

Some simple properties of the polynomials are:

(i) If $y = P_n(h) = \sum_{k=0}^{n} a_k h^k \in E_y$ and $z = Q_m(h) = \sum_{l=0}^{m} b_l h^l \in E_z$ are, respectively, the n-th degree and m-th degree polynomials in h and if a product of any pair of elements in E_y and E_z is defined, then

$$yz = P_n(h)\, Q_m(h)$$

is an $(n + m)$-th degree polynomial in h.
Indeed, by the distributivity of multiplication

$$yz = \left(\sum_{k=0}^{n} a_k h^k\right)\left(\sum_{l=0}^{n} b_l h^l\right) = \sum_{k=0}^{n}\sum_{l=0}^{m} (a_k h^k)\,(b_l h^l),$$

every term $(a_k h^k)\,(b_l h^l)$, because of the property (vi) of homogeneous forms, is a $(k + l) \leqslant (n + m)$-ary form. Thus, all of the product yz is a polynomial in h of degree not higher than $(n + m)$. ■

(ii) Let $P_n(h)$ and $Q_m(g)$ be polynomials of n-th degree in h and of m-th degree in g, respectively. Then, $P_n(h) = P_n\,[Q_m(g)]$ is a polynomial in g of degree not higher than nm for $h = Q_m(g)$.
This is now proved by induction on n. Let $n = 1$, that is, $P_1(h) = a_1 h + a_0$, a_1 a linear operator relative to h. If

$$Q_m(g) = \sum_{l=0}^{m} b_l g^l,$$

then, by the property (v) of homogeneous forms,

$$a_1 h + a_0 = a_1 Q_m(g) + a_0 = \sum_{l=0}^{m} a_1\, b_l g^l + a_0$$

is a polynomial in g of degree not higher than m. Thus, the statement is true for $n = 1$.
Now, let this hold for all polynomials $P_k(h)$ in h of degree $k \leqslant n - 1$.
Consider an n-th degree polynomial $P_n(h) = \sum_{k=0}^{n} a_k h^k$ in h, to receive

$$P_n(h) = P_{n-1}(h) + a_n h^n = P_{n-1}(h) + (a_n h^{n-1})\, h,$$

where $P_{n-1}(h)$ is a polynomial of degree $n - 1$ in h. If $h = Q_m(g)$, then by assumption,

$$P_{n-1}(h) = R_{(n-1)m}(g) \quad \text{and} \quad a_n h^{n-1} = R^*_{(n-1)m}(g),$$

where $R_{(n-1)m}\,(g)$ and $R^*_{(n-1)m}(g)$ are polynomials in h of degree not exceeding $(n-1)m$. Thence,

$$P_n(h) = R_{(n-1)m}{}'(g) + R^*_{(n-1)m}\,(g)\, Q_m(g).$$

By the property (i) for polynomials, a product of two polynomials in g, $R^*_{(n-1)m}(g)\, Q_m(g)$, of degree not higher than $(n-1)\, m$ and m, respectively, is a polynomial $R_{nm}(g)$ in g of degree not higher than nm. Thus,

$$P_n(h) = P_n[Q_m(g)] = R_{(n-1)m}(g) + R_{mn}(g) = T_{nm}(g),$$

where $T_{nm}(g)$ is a polynomial of degree not higher than nm in g. ■

8.6 DIFFERENTIALS AND DERIVATIVES OF HIGHER ORDER

8.61. Notations. Let E_x and E_y be normed linear spaces and let $y = f(x)$ be an abstract function with domain E_x and range E_y.

Let $x \in E_x$, $y_1 = y_1(x) \in E_y$ and $y_2 = y_2(x) \in E_y$. The relation

$$y_1(x) \overset{x}{\underset{n}{=}} y_2(x)$$

signifies that $\| y_1(x) - y_2(x) \| = o(\| x \|^n)$ [$y_1(x)$ equals $y_2(x)$ to within magnitudes of order higher than n in comparison to $\| x \|$].

It is possible to establish the following statements.

(i) If $y_1(x) \overset{x}{\underset{n}{=}} y_2(x)$ and $y_2(x) \overset{x}{\underset{n}{=}} y_3(x)$, then $y_1(x) \overset{x}{\underset{n}{=}} y_3(x)$.

(ii) If $y_1(x) \overset{x}{\underset{n}{=}} y_2(x)$, $x = f(\xi)$ and $\| \xi \| = O(\| x \|)$, then $y_1 \overset{\xi}{\underset{n}{=}} y_2$.

(iii) If $P(h)$ and $Q(h)$ are polynomials in h with equal coefficients for the exponents $h_1, h_2, \ldots, h_n$, that is, if

$$P(h) - Q(h) = \sum_{k=n+1}^{p} a_k h^k,$$

then,

$$P(h) \overset{h}{\underset{n}{=}} Q(h).$$

The multiplication appearing in the last formula is assumed to have a meaning.

(iv) If $\| A(x) \|$ is bounded in a neighbourhood of $x = 0$ and $y_1(x) \overset{x}{\underset{n}{=}} y_2(x)$,

then $$A(x)\, y_1(x) \overset{x}{\underset{n}{=}} A(x)\, y_2(x).$$

Analogously, $$y_1(x)\, A(x) \overset{x}{\underset{n}{=}} y_2(x)\, A(x).$$

(v) If $y(x) \overset{x}{\underset{n}{=}} y_1(x)$, $z(x) \overset{x}{\underset{n}{=}} z_1(x)$, then

$$y(x)\, z(x) \overset{x}{\underset{n}{=}} y_1(x)\, z_1(x).$$

In fact, by (iv),

$$y(x)\, z(x) \overset{x}{\underset{n}{=}} y(x)\, z_1(x) \overset{x}{\underset{n}{=}} y_1(x)\, z_1(x).$$

Thereupon, (i) implies (v).

8.62. Taylor's formula. We had defined the FRÉCHET differential of the first order by considering an approximation of the function $f(x+h)$ by a first degree polynomial in h.

Suppose that there exists the polynomial

$$P_n(h) = a_1 h + a_2 h^2 + \ldots + a_n h^n$$

of degree n in h, such that $f(x+h) - f(x) \overset{h}{\underset{n}{=}} P_n(h)$, (1)

that is, let $f(x+h) - f(x) = P_n(h) + \omega_n(x, h)$,

where $\| \omega_n(x, h) \| \leqslant \varepsilon(\|h\|)\, \|h\|^n$, $\varepsilon(\|h\|) \to 0$ as $\|h\| \to 0$. (2)

The polynomial $P_n(h)$ is called **Taylor's sum** of degree n for the function $f(x+h)$, its n-th term multiplied by $n!$ is called the n-th **Fréchet differential of the function** $f(x)$ and the function $f(x)$ itself is said to be **n-times differentiable at the point** x.

Denote the n-th FRÉCHET differential by $d^n f(x, h)$, to receive

$$d^n f(x, h) = n!\, a_n h^n.$$

The symmetric n-ary linear form corresponding to $d^n f(x, h)$ is $d^n f(x; h_1, h_2 \ldots \ldots h_n) = n!\, a_n h_1 \ldots h_n$. This linear form $n! a_n$ is called the **n-th Fréchet derivative of thc function** $f(x)$ **at the point** x and is denoted by $f^{(n)}(x)$. Thus, $d^n f(x, h) = f^n(x) h^n$. The formula (1) takes the form

$$f(x+h) - f(x) \overset{h}{\underset{n}{=}} f'(x)\, h + \frac{1}{2!} f''(x)\, h^2 + \ldots + \frac{1}{n!} f^n(x)\, h^n.$$

Now, note that $d^n f(x, h) = \dfrac{d^n}{dt^n} f(x+th)\,|_{t=0}$. (3)

In fact, (1) implies

$$f(x+th) = f(x) + \sum_{k=1}^{n-1} t^k a_k h^k + t^n a_n h^n + \omega(x; th),$$

where $$\lim_{t \to 0} \frac{\| \omega(x; th) \|}{t^n} = 0.$$

Thus, $$\frac{d^n}{dt^n} \left(\sum_{k=1}^{n-1} t^k a_k h^k \right) = 0, \qquad \frac{d^n}{dt^n} (t^n a_n h^n) = n!\, a_n h^n,$$

and, finally,

$$\frac{d^n}{dt^n}\,\omega(x;th)\Big|_{t=0} = \lim_{\Delta t\to 0} \frac{\delta^n_{\Delta t}\,\omega(x;th)}{(\Delta t)^n}\Big|_{t=0}$$

$$= \lim_{\Delta t\to 0} \frac{1}{(\Delta t)^n} \sum_{k=0}^{n} (-1)^{n-k} C_n^k\,\omega\left(x;\left(k-\frac{n}{2}\right)\Delta th\right).$$

By (2),

$$\frac{\omega\left(x;\left(k-\frac{n}{2}\right)\Delta th\right)}{(\Delta t)^n} \to 0$$

as $\Delta t \to 0$, whence, $\frac{d^n}{dt^n}\,\omega(x;th)\,|_{t=0} = 0$. In this case,

$$d^n f(x,h) = n!\, a_n h^n = \frac{d^n}{dt^n} f(x+th)\Big|_{t=0}.$$

If $d^n f(x,h)$ exists in a certain region, and if the relation (2) is also uniformly satisfied there for x, then $d^n f(x,h)$ is said to be a **uniform Fréchet differential.** In this case, the uniform difference derivative appears on the right-hand side of (3).

Now, in order to give another definition of the n-th differential : Let the first differential $df(x,h) = f'(x)\,h$ exist in a neighbourhood of the point x. The function $f'(x)$ and, hence, also $df(x,h)$ can, in turn, be differentiable with respect to x. Thus, we arrive at the **second differential** $d\,[df(x,h), h_1] = d\,[f'(x)h, h_1] = df'(x,h_1)h$. If we write

$$df'(x,h_1) = f''(x)_0\, h_1,$$

$f''(x)_0$ is called the **second derivative,** so that

$$d\,[df(x,h)h_1] = f''(x)_0\, h_1 h = \frac{d}{dt_1}\, df(x+t_1h_1, h)\,|_{t_1=0}$$

$$= \frac{\partial}{\partial t_1}\left[\frac{\partial}{\partial t} f(x+th+t_1h_1)\Big|_{t=0}\right]\Big|_{t_1=0}$$

$$= \frac{\partial^2}{\partial t_1\,\partial t} f(x+th+t_1h_1)\Big|_{t_1=t=0}.$$

The n-**th differential** is similarly defined under the assumption that the $(n-1)$-th differential

$$d\{\,d[\ldots(df(x,h_1))h_2\ldots]h_{n-1}\} = f^{n-1}(x)_0\, h_{n-1}\, h_{n-2}\ldots h_1$$

exists and that it is differentiable as a function of x, that is, $f^{(n-1)}(x)_0$ is differentiable. Finally,

$$d\,(d\,\{d\,[\ldots df(x,h_1)h_2\ldots]\,h_{n-1}\}h_n) = d\,(f^{n-1}(x)_0\, h_{n-1}\, h_{n-2}\ldots h_1, h_n)$$

$$= df^{(n-1)}(x,h_n)h_{n-1}\, h_{n-2}\ldots h_1. \qquad (4)$$

If we write $df^{(n-1)}(x,h_n) = f^n(x)_0 h_n$, we get the n-**th derivative** $f^n(x)_0$, and

$$d(d\{d[\ldots df(x, h_1)h_2\ldots]h_{n-1}\}h_n) = f^{(n)}(x)_0\, h_n\, h_{n-1} \ldots h_1.$$

Eq. (4) assumes the form

$$d^n f(x;\ h_n, h_{n-1}, \ldots, h_1) = f^{(n)}(x)_0\, h_n\, h_{n-1} \ldots h_1$$

$$= \frac{\partial_n}{\partial t_1\, \partial t_2 \ldots \partial t_n} f\left(x + \sum_{i=1}^{n} t_i h_i\right)\Bigg|_{t_1=t_2=\ldots=t_n=0}.$$

Put $h_1 = h_2 = \ldots = h_n = h$, to receive

$$d^n f(x, h)_0 = f^{(n)}(x)_0\, h^n = \frac{d^n}{dt^n} f(x + th)\Big|_{t=0}. \tag{5}$$

Since uniformly continuous n-th difference derivatives coincide consecutively (see, p. 274), (3) and (5) imply the statement: *If an n-th uniform Fréchet differential $d^n f(x, h)$ exists in a domain G and is continuous in x, then the n-th differential $d^n f(x, h)_0$ also exists in G, and*

$$d^n f(x, h)_0 = d^n f(x, h), \quad \text{or } f^{(n)}(x)_0 = f^{(x)}x.$$

Conversely, *suppose that there exists the n-th differential $d^n f(x, h)_0 = f^{(n)}(x)_0 h^n$ and that $f^n(x)_0$ is uniformly continuous in x in a certain region G. Then, the n-th uniform Fréchet differential also exists in G and it coincides with the n-th differential in G.*

Proof. It is intended to prove this assertion by induction on n. For $n = 1$, this statement is trivial. Let this be true for $n - 1$. Since $f^{(n)}(x)_0 = [f'(x)_0]^{(n-1)}$, it follows that

$$f'(x + h)_0 = f'(x)_0 + f''(x)_0\, h + \frac{1}{2!} f'''(x)_0 h^2$$

$$+ \ldots + \frac{1}{(n-1)!} f^{(n)}(x)_0\, h^{n-1} + \omega(x;\ h),$$

where $\| \omega(x;\ h) \| \leqslant \varepsilon_{n-1}(\| h \|)\, \| h \|^{n-1}$, $\varepsilon_{n-1}(u) \to 0$ as $u \to 0$.

Thereupon, for $0 \leqslant t \leqslant 1$,

$$f'(x + th)_0 = f'(x)_0 + tf''(x)_0\, h + \frac{1}{2!} t^2 f'''(x)_0\, h^2$$

$$+ \ldots + \frac{1}{(n-1)!} t^{n-1} f^{(n)}(x)_0\, h^{n-1} + \omega(x;\ th),$$

where $\| \omega(x;\ th) \| \leqslant \varepsilon_{n-1}(t \| h \|) t^{n-1} \| h \|^{n-1} \leqslant \varepsilon_{n-1}(t \| h \|)\, \| h \|^{n-1}$. Hence,

$$f(x + h) - f(x) = \int_0^1 f'(x + th)_0\, h\, dt$$

$$= \int_0^1 \Big\{ f'(x)_0 + tf''(x)_0 h + \frac{1}{2!} t^2 f'''(x)_0\, h^2$$

$$+ \ldots + \frac{1}{(n-1)!} t^{n-1} f^{(n)}(x)_0\, h^{n-1} \Big\} h\, dt + R_n,$$

where $$R_n = \int_0^1 \omega(x;\ th)h\,dt.$$

Thereupon,

$$f(x+h) = f(x) + f'(x)_0\,h + \frac{1}{2!}\,f''(x)_0\,h^2 + \frac{1}{3!}\,f'''(x)_0\,h^3 + \ldots + \frac{1}{n!}\,f^{(n)}(x)_0\,h^n + R_n,$$

where $\| R_n \| \leqslant \int_0^1 \| \omega(x;\ th) \|\,\| h \|\,dt \leqslant \varepsilon_n(\| h \|)\,\| h \|^n$, $\varepsilon_n(u) \to 0$ as $u \to 0$.

Thus, the sum on the right-hand side of the above expression for $f(x+h)$ is a TAYLOR'S sum for the function $f(x)$, and $f^{(n)}(x)_0\,h^n = d^n f(x, h)$, that is,

$$d^n f(x, h)_0 = d^n f(x, h).$$

■

In the sequel, the n-th derivative and the n-th differential will be taken in the FRÉCHET sense. Now, examine the n-th derivative of a **composite** function and a product.

(i) *Let $y = \varphi(x)$, $z = \psi(y)$, $x \in E_x$, $y \in E_y$, $z \in E_z$; then $z = f(x)$, where $f(x) = \psi[\varphi(x)]$. Further, let $y_0 = \varphi(x_0)$ and $z_0 = \psi(y_0) = f(x_0)$. If $\varphi(x)$ and $\psi(y)$ are n-times differentiable at the points x_0 and y_0 respectively, then $f(x)$ is also n-times differentiable at x_0.*

PROOF. In fact, by assumption, there exists in E_x an n-th degree polynomial, $P_n(h)$, such that

$$\varphi(x_0 + h) - \varphi(x_0) \underset{n}{\overset{h}{=}} P_n(h).$$

On the other hand, an n-th degree polynomial $Q_n(g)$ is defined in E_y, such that

$$\psi(y_0 + g) - \psi(y_0) \underset{n}{\overset{g}{=}} Q_n(g).$$

In particular, for $g = \varphi(x_0 + h) - \varphi(x_0)$ and, consequently, for $\varphi(x_0+h) = \varphi(x_0) + g = y_0 + g$, we have

$$f(x_0 + h) - f(x_0) = \psi\,[\varphi(x_0) + h] - \psi\,[\varphi(x_0)] \underset{n}{\overset{g}{=}} Q_n(g). \tag{6}$$

However, $$g = \varphi(x_0 + h) - \varphi(x_0) \underset{n}{\overset{h}{=}} P_n(h);$$

hence,

$$Q_n(g) \underset{n}{\overset{h}{=}} Q_n[P_n(h)].$$

In view of the property (ii) for polynomials, $Q_n\,[P_n(h)]$ is a polynomial in h. This polynomial can be approximated to desired exactness by the n-th degree polynomial $R_n(h)$ in h, which is a segment of $Q_n\,[P_n(h)]$:

$$Q_n(g) \underset{n}{\overset{h}{=}} Q_n\,[P_n(h)] \underset{n}{\overset{h}{=}} R_n(h). \tag{7}$$

Further, since the function φ is differentiable at the point x_0, hence $\| g \| = \| \varphi(x_0+h) - \varphi(x_0) \| = O(\| h \|)$. Thus, the symbol $\overset{g}{\underset{n}{=}}$ can be replaced by $\overset{h}{\underset{n}{=}}$, and Eq. (6) assumes the form $f(x_0 + h) - f(x_0) \overset{h}{\underset{n}{=}} Q_n(g)$. Thereupon, (7) implies that

$$f(x_0+h) - f(x_0) \overset{h}{\underset{n}{=}} R_n(h). \tag{8}$$

Since the polynomial $R_n(h)$ satisfies (8), the assertion is proved.

If $\varphi(x)$ and $\psi(x)$ are n-times continuously differentiable with respect to x, then $f(x) = \psi[\varphi(x)]$ also has this property. In this case, the coefficients of the polynomials $P_n(h)$ and $Q_n(h)$ are continuous functions of x. Thus, the coefficients of $Q_n[P_n(h)]$ are continuous in x and, consequently, the coefficients of the polynomial $R_n(h)$ are also continuous functions of h.

(ii) *Let $x \in E_x$, $y = f(x) \in E_y$, $z = \varphi(x) \in E_z$, and let the product $u \in E_u$ be defined for the elements $y \in E_y$ and $z \in E_z$. If $f(x)$ and $\varphi(x)$ are n-times continuously differentiable functions of x, then $F(x) = f(x)\,\varphi(x)$ is also n-times continuously differentiable with respect to x.*

PROOF. In fact,

$$f(x+h) \overset{h}{\underset{n}{=}} f(x) + P_n(h), \qquad \varphi(x+h) \overset{h}{\underset{n}{=}} \varphi(x) + Q_n(h),$$

where $P_n(h)$ and $Q_n(h)$ are n-th degree polynomials in h:

$$P_n(h) = \sum_{k=1}^{n} a_k(x)\, h^k, \quad Q_n(h) = \sum_{k=1}^{n} b_k(x)\, h^k,$$

the coefficients $a_k(x)$ and $b_k(x)$ being continuous functions of x. Thereupon,

$$f(x+h)\,\varphi(x+h) \overset{h}{\underset{n}{=}} f(x)\,\varphi(x) + [f(x)\,Q_n(h) + P_n(h)\,\varphi(x) + P_n(h)\,Q_n(h)].$$

The expression inside brackets is a polynomial with coefficients which are continuous functions of x. Neglect the terms which have h in power higher than n, to receive the polynomial $R_n(h) = \sum_{k=1}^{n} c_k(x)\, h^k$ with coefficients which are continuous functions of x:

$$f(x)\,Q_n(h) + \varphi(x)\,P_n(h) + P_n(h)\,Q_n(h) \overset{h}{\underset{n}{=}} R_n(h).$$

Hence,
$$f(x+h)\,\varphi(x+h) \overset{h}{\underset{n}{=}} f(x)\,\varphi(x) + R_n(h). \tag{9}$$

This relation shows that $F(x) = f(x)\,\varphi(x)$ is an n-times continuously differentiable function of x.

Finally, note that for functionals defined in complex linear spaces, the existence of the first differential in the neighbourhood of some point implies the existence of all differentials of higher order in this neighbourhood and also that a functional can be represented by a *Taylor's series.*

8.7 DIFFERENTIATION OF FUNCTIONS OF TWO VARIABLES

EXAMINE A function $\varphi(x, y)$ of two variables $x \in E_x$, $y \in E_y$ with $\varphi(x, y) \in E_z$. It is possible to regard (x, y) as an element of the direct sum $E_x \oplus E_y$ of the spaces E_x and E_y. The function $\varphi(x, y)$ is said to be n-**times differentiable at the point** (x_0, y_0), if

$$\varphi(x_0 + h, y_0 + g) - \varphi(x_0, y_0) \overset{(h,g)}{\underset{n}{=}} a_1(h, g) + a_2(h, g)^2 + \ldots + a_n(h, g)^n.$$

Here $a_k(h, g)^k$ are k-th degree homogeneous forms in the element $(h, g) \in E_x \oplus E_y$. Obviously, $a_k(h, g)^k$ is a sum of k-linear forms of the type $a_k h_1 h_2 \ldots h_k$, where every h_i is equal to h or g.

THEOREM. *If $\varphi(x, y)$ and $y = f(x)$ are n-times differentiable at (x_0, y_0) and x_0, respectively, with $y_0 = f(x_0)$, then $\varphi[x, f(x)]$ is a function of x and is n-times differentiable at $x = x_0$.*

PROOF. If $g = f(x_0 + h) - f(x_0)$, then $f(x_0 + h) = y_0 + g$, and

$$g \overset{h}{\underset{n}{=}} P_n(h) = a_1 h + a_2 h^2 + \ldots + a_n h^n. \tag{1}$$

Thereupon $\| g \| = O(\| h \|)$, implying that

$$\| (h, g) \| = \| h \| + \| g \| = O(\| h \|). \tag{2}$$

Since $\varphi(x, y)$ is an n-times differentiable function, hence

$$\varphi(x_0 + h, y_0 + g) - \varphi(x_0, y_0) \overset{h}{\underset{n}{=}} a_1(h, g) + a_2(h, g)^2 + \ldots + a_n(h, g)^n, \tag{3}$$

because, by (2), the symbol $\overset{(h,g)}{\underset{n}{=}}$ can be substituted for $\overset{h}{\underset{n}{=}}$ in (3). However, since (1) yields

$$a_k (h, g)^k \overset{h}{\underset{n}{=}} a_k [h, P_n(h)]^k,$$

hence, $$\varphi (x_0 + h, y_0 + g) - \varphi (x_0, y_0) \overset{h}{\underset{n}{=}} \sum_{k=1}^{n} a_k [h, P_n(h)]^k. \tag{4}$$

Here, $a_k [h, P_n(h)]^k$ is a sum of terms of the form $a_k h_1 h_2 \ldots h_k$, where every h_i is equal to h or to $P_n(h)$. Hence $a_k [h, P_n(h)]^k$ and, consequently, the entire sum in (4) is a polynomial in h. Omit the terms of degree higher than n, to receive an n-th degree polynomial $R_n(h)$ in h, satisfying

$$R_n(h) \overset{h}{\underset{n}{=}} \sum_{k=1}^{n} a_k [h, P_n(h)]^k.$$

Thus, $$\varphi(x_0 + h, y_0 + g) - \varphi(x_0, y_0) \underset{n}{\overset{h}{=}} R_n(h). \qquad ■$$

Now, to introduce the notion of the **partial derivatives** of a function of two variables, we have

$$d\varphi\,[(x_0, y_0); (h, g)] = a_1(h, g) = a_1 h + a_2 g,$$

$$d^2\varphi\,[(x_0, y_0); (h, g)] = a_2(h, g)^2 = a_{11}h^2 + a_{12}hg + a_{21}gh + a_{22}g^2,$$

and so on. Introduce the notations

$$\begin{aligned} a_1 &= \varphi'_x(x_0, y_0), & a_2 &= \varphi'_y(x_0, y_0),\\ a_{11} &= \varphi''_{xx}(x_0, y_0), & a_{12} &= \varphi''_{xy}(x_0, y_0),\\ a_{21} &= \varphi''_{yx}(x_0, y_0), & a_{22} &= \varphi''_{yy}(x_0, y_0), \end{aligned}$$

to receive

$$a_1 h = \varphi'_x h = \frac{d}{dt}\varphi(x_0 + th, y_0)\,|_{t=0},$$

$$a_2 g = \varphi'_y g = \frac{d}{dt}\varphi(x_0, y_0 + tg)\,|_{t=0},$$

$$a_{11}h^2 = \varphi''_{xx}h^2 = \frac{d^2}{dt^2}\varphi(x_0 + th, y_0)\,|_{t=0},$$

$$a_{12}hg = \varphi''_{xy}hg = \frac{\partial^2}{\partial t_1 \partial t_2}\varphi(x_0 + t_1 h, y_0 + t_2 g)\,|_{t_1=t_2=0},$$

$$a_{21}gh = \varphi''_{yx}gh = \frac{\partial^2}{\partial t_1 \partial t_2}\varphi(x_0 + t_1 h, y_0 + t_2 g)\,|_{t_1=t_2=0},$$

$$a_{22}g^2 = \varphi''_{yy}g^2 = \frac{d^2}{dt^2}\varphi(x_0, y_0 + tg)\,|_{t=0}, \ldots$$

If $\varphi(x, y)$ has a second differential, continuous in (x, y), then $a_{12}hg = a_{21}gh$, since the operators $\partial/\partial t_1$ and $\partial/\partial t_2$ commute if $(\partial^2/\partial t_1 \partial t_2)\,\varphi(x_0 + t_1 h, y_0 + t_2 g)$ is a continuous function of t_1 and t_2.

8.8 THEOREMS ON IMPLICIT FUNCTIONS

CONSIDER THE direct sum $E_x \oplus E_y$ of the spaces E_x and E_y as well as an operator $\varphi(x, y)$, transforming $E_x \oplus E_y$ into E_z: $x \in E_x$, $y \in E_y$, $z = \varphi(x, y) \in E_z$. Assume that

(i) $\varphi(x_0, y_0) = 0$; (1)

(ii) $\varphi(x, y)$ is continuous in a neighbourhood of the point (x_0, y_0);

(iii) $\varphi(x, y)$ has a continuous derivative $\varphi'_y(x, y)$ in a neighbourhood of the point (x_0, y_0), and $[\varphi'_y(x_0, y_0)]^{-1}$ exists.

THEOREM 1. *If the assumptions* (i)—(iii) *are satisfied, then there exist positive constants* δ *and* ε *and an operator* $y = f(x)$, $x \in E_x$, $y \in E_y$, *defined in the neighbourhood* $\| x - x_0 \| < \delta$ *of* x_0, *and such that the equation*

$$y = f(x), \tag{2}$$

in some neighbourhood of x_0, *is equivalent to*

$$\varphi(x, y) = 0, \tag{3}$$

that is, every element (x, y) *with* $\| x - x_0 \| < \delta$ *which satisfies* (2) *also satisfies* (3); *conversely, every pair of elements* (x, y) *satisfying Eq.* (3) *for* $\| x - x_0 \| < \delta$, $\| y - y_0 \| \leqslant \varepsilon$ *also satisfies Eq.* (2). *The operator* $f(x)$ *is continuous in* x, *and* $f(x_0) = y_0$.

PROOF. Equation (3) is equivalent to

$$y = A(x, y), \tag{4}$$

where the operator $A(x, y)$ is defined by

$$A(x, y) = y - [\varphi'_y(x_0, y_0)]^{-1}\, \varphi(x, y). \tag{5}$$

We apply the principle of contraction mappings for proving the existence and uniqueness of the solution of Eq. (4).

Since

$$\begin{aligned}\frac{\partial A(x, y)}{\partial y} &= I - [\, \varphi'_y(x_0, y_0)\,]^{-1}\varphi'_y(x, y) \\ &= [\, \varphi'_y(x_0, y_0)]^{-1}\, \{\, \varphi'_y(x_0, y_0) - \varphi'_y(x, y)\},\end{aligned}$$

it follows that

$$\left\| \frac{\partial A(x, y)}{\partial y} \right\| \leqslant q(r) \qquad (\| x - x_0 \| \leqslant r, \quad \| y - y_0 \| \leqslant r),$$

where
$$q(r) \to 0 \quad \text{as} \quad r \to 0 \tag{6}$$

because of $\varphi'_y(x, y)$ being continuous by hypothesis.

Hence, $A(x, y)$ satisfies in y the LIPSCHITZ condition

$$\| A(x, y_1) - A(x, y_2) \| \leqslant q(r) \, \| y_1 - y_2 \|, \tag{7}$$

$$\| x - x_0 \| \leqslant r, \quad \| y_i - y_0 \| \leqslant r, \quad i = 1, 2.$$

Furthermore,

$$\| A(x, y_0) - y_0 \| \leqslant \| [\varphi'_y(x_0, y_0)]^{-1} \| \, \| \varphi(x, y_0) \| \leqslant p(r)$$
$$(\| x - x_0 \| \leqslant r),$$

where
$$p(r) \to 0 \quad \text{as} \quad r \to 0 \tag{8}$$

by the continuity of $\varphi(x, y)$ and the condition $\varphi(x_0, y_0) = 0$.

Select a number $\varepsilon > 0$ so small that $q(\varepsilon) = q < 1$, permissible by (6). Ineq. (7) implies that for $\|x - x_0\| \leqslant \varepsilon$, the operator $A(x, y)$ on the sphere $\| y - y_0 \| \leqslant \varepsilon$ of the space E_y is an operator of contraction. Now, choose $\delta \leqslant \varepsilon$ so small that

$$p(\delta) \leqslant (1 - q)\, \varepsilon.$$

Then, for $\| x - x_0 \| \leqslant \delta$, $A(x, y)$ maps the sphere $\| y - y_0 \| \leqslant \varepsilon$ into itself and Eq. (4) has a unique solution in $\| y - y_0 \| \leqslant \varepsilon$. Denoting this solution by $y = f(x)$, it is clear that $y_0 = f(x_0)$.

For completing the proof of the theorem, it remains to exhibit that $f(x)$ is a continuous operator. We have $f(x) = A[x, f(x)]$, whence

$$\begin{aligned}\|f(x) - f(x_0)\| &\leqslant \|A[x, f(x)] - A[x, f(x_0)]\| \\ &\quad + \|A[x, f(x_0)] - A[x_0, f(x_0)]\| \\ &\leqslant q\,\|f(x) - f(x_0)\| + \|[\varphi'_y(x_0, y_0)]^{-1}\|\,\|\varphi(x, y_0)\|,\end{aligned}$$

and, consequently,

$$f(x) - f(x_0)\| \leqslant \frac{1}{1-q}\,\|[\varphi'_y(x_0, y_0)]^{-1}\|\,\varphi(x, y_0)\|. \tag{9}$$

It is seen that $f(x)$ is continuous at the point x_0. Similarly, the continuity is proved at other points of the neighbourhood $\| x - x_0 \| \leqslant \delta$. ■

REMARK 1. By the principle of contraction mappings, the operator $f(x)$ can be taken as the limit of a sequence of operators $y = f_k(x)$, $\| x - x_0 \| \leqslant \delta$, $\| f_k(x) - y_0 \| \leqslant \varepsilon$, defined by

$$f_0(x) \equiv y_0,$$

$$f_k(x) = f_{k-1}(x) - [\varphi'_y(x_0, y_0)]^{-1}\,\varphi[x, f_{k-1}(x)], \quad k = 1, 2, \ldots \tag{10}$$

In addition, the underlying estimate for the convergence rate

$$\|f(x) - f_k(x)\| \leqslant \frac{q^k}{1-q}\,\|[\varphi'_y(x_0, y_0)]^{-1}\|\,\|\varphi(x, y_0)\| \tag{11}$$

holds.

REMARK 2. If it is further known that in the neighbourhood considered, $\varphi'_x(x, y)$ exists and is bounded, $\| \varphi'_x(x, y)\| \leqslant \alpha$, then the estimate

$$\|f(x) - f(x_0)\| \leqslant c_1\,\|x - x_0\| \tag{12}$$

remains valid. Indeed, in this case

$$\|\varphi(x, y_0)\| = \|\varphi(x, y_0) - \varphi(x_0, y_0)\| \leqslant \alpha\,\|x - x_0\|,$$

and (9) gives the desired result.

REMARK 3. Let $\varphi'_x(x, y)$ be bounded, $\| \varphi'_x(x, y) \| \leqslant \alpha$ and let $\varphi'_y(x, y)$ satisfy the inequality

$$\|\varphi'_y(x, y) - \varphi'_y(x_0\, y_0)\| \leqslant a\,\|x - x_0\| + b\,\|y - y_0\|. \tag{13}$$

Then, a faster convergence rate given by the estimate

$$\|f(x) - f_k(x)\| \leqslant c_2\,\|x - x_0\|^k\,\|\varphi(x, y_0)\| \tag{14}$$

holds.

THEOREM 2. *In the hypotheses of Theorem* 1, *let* $\varphi(x, y)$ *be an n-times differentiable function in some neighbourhood of the points* (x_0, y_0) *of* $E_x \oplus E_y$. *Then, the operator* $y = f(x)$ *is also an n-times differentiable function of* x *in the* δ*-neighbourhood of the point* x_0.

PROOF. To start with, assume that $\varphi(x, y)$ is a polynomial, defined by

$$\varphi(x, y) = \sum_{k=1}^{n} a_k (x - x_0, y - y_0)^k.$$

Put $x - x_0 = h$, $y - y_0 = u$, $f(x) = y_0 + u(h)$. Then the operator $u(h)$ can be regarded as the limit of successive approximations

$$u_0(h) = 0$$
$$u_k(h) = u_{k-1}(h) - B^{-1}\varphi[x_0 + h, y_0 + u_{k-1}(h)],$$
$$B = \varphi_y'(x_0, y_0).$$

Since the substitution of a polynomial into a polynomial is again a polynomial, it follows by using inductive method that all the $u_k(h)$ are polynomials in h.

In the given case, all the first-order and second-order partial derivatives of the function $\varphi(x, y)$ are continuous and, consequently, are also bounded in some neighbourhood of (x_0, y_0). Hence (see, Remark 3),

$$\| u(h) - u_n(h) \| \leqslant c_2 \| h \|^n \| \varphi(x_0 + h, y_0) \|.$$

Since $u_n(h)$ is a polynomial and $\| \varphi(x_0 + h, y_0) \| \to 0$ as $\| h \| \to 0$, the last inequality means that $u(h)$ is n-times differentiable at the point $h = 0$. Consequently, $y = f(x)$ is n-times differentiable at $x = x_0$.

Passing on to the general case, we receive by hypothesis,

$$\varphi(x, y) = \tilde{\varphi}(x, y) + (\| x - x_0 \| + \| y - y_0 \|)^n \omega(x - x_0, y - y_0),$$

where
$$\tilde{\varphi}(x, y) = \sum_{k=1}^{n} a_k(x - x_0, y - y_0)^k,$$

and
$$\| \omega(x - x_0, y - y_0) \| \to 0 \text{ as } \| x - x_0 \|, \| y - y_0 \| \to 0.$$

Since the function $\tilde{\varphi}(x, y)$ satisfies all hypotheses of Theorem 1 and is a polynomial, there is an n-times differentiable operator $\tilde{f}(x)$, satisfying $\tilde{\varphi}[x, \tilde{f}(x)] = 0$.

The identities

$$f(x) = B^{-1}[Bf(x) - \varphi(x, f(x))],$$
$$\tilde{f}(x) = B^{-1}[B\tilde{f}(x) - \tilde{\varphi}(x, \tilde{f}(x))]$$

imply that

$$\begin{aligned}\| f(x) - \tilde{f}(x) \| \leqslant \| B^{-1} \| \| B[f(x) - \tilde{f}(x)] - \varphi[x, f(x)] \\ + \varphi[x, \tilde{f}(x)] - \varphi[x, \tilde{f}(x)] + \tilde{\varphi}[x, f(x)] \| \\ \leqslant \| B^{-1} \| \| B[f(x) - \tilde{f}(x)] - [\varphi(x, f(x)) - \varphi(x, \tilde{f}(x))] \| \\ + \| B^{-1} \| \| \varphi[x, \tilde{f}(x)] - \tilde{\varphi}[x, \tilde{f}(x)] \|. \qquad (15)\end{aligned}$$

To introduce the operator $B(x, y, \tilde{y})$, put

$$\varphi(x, y) - \varphi(x, \tilde{y}) = \int_0^1 \varphi'_y [(x, \tilde{y}) + t(y - \tilde{y})]\, dt(y - \tilde{y})$$
$$\equiv B(x, y, \tilde{y})\,(y - \tilde{y}).$$

It is trivial to verify that $B(x, y, \tilde{y})$ is continuous in the collection of variables and $B(x, y, \tilde{y}) \to B$ as $x \to x_0$, $y, \tilde{y} \to y_0$. By the continuity of $f(x)$ and $\tilde{f}(x)$, $\delta_0 > 0$ can be found, such that

$$\| B^{-1} \| \, \| B - B[x, f(x), \tilde{f}(x)] \| \leqslant q < 1 \quad \text{for } \| x - x_0 \| \leqslant \delta_0. \tag{16}$$

Then, Ineq. (15) yields

$$\| f(x) - \tilde{f}(x) \| \leqslant \frac{1}{1 - q} \| B^{-1} \| \, \| \varphi [x, \tilde{f}(x)] - \tilde{\varphi} [x, \tilde{f}(x)] \|,$$

and, consequently, since $\| \tilde{f}(x) - y_0 \| \leqslant c_1 \| x - x_0 \|$ (see, Remark 2),

$$\begin{aligned} \| f(x) - \tilde{f}(x) \| &\leqslant c \| \varphi [x, \tilde{f}(x)] - \tilde{\varphi} [x, \tilde{f}(x)] \| \\ &\leqslant c\,(\| x - x_0 \| + \| \tilde{f}(x) - y_0 \|)^n \, \| \omega (x - x_0, \tilde{f}(x) - y_0) \| \\ &\leqslant c\,(1 + c_1)^n \, \| x - x_0 \|^n \, \| \omega (x - x_0, \tilde{f}(x) - y_0) \|. \end{aligned}$$

It is plain that
$$f(x) \overset{x - x_0}{\underset{n}{=\!=}} \tilde{f}(x). \tag{17}$$

Since the operator $\tilde{f}(x)$ is n-times differentiable at x_0, (17) implies that $f(x)$ is also so.

Similar reasonings exhibit the differentiability of $f(x)$ at the rest of the points of the sphere $\| x - x_0 \| < \delta$.

Finally, it remains to show that

$$df(x_0, h) = - [\varphi'_y (x_0, y_0)]^{-1} \varphi'_x (x_0, y_0) h, \tag{18}$$

that is,
$$f'(x_0) = - [\varphi'_y (x_0, y_0)]^{-1} \varphi'_x (x_0, y_0).$$

In fact, by definition $u_1(h)$ is an operator (a polynomial in h), such that

$$df(x_0, h) = d\tilde{f}(x_0, h) \overset{h}{\underset{1}{=}} u_1(h).$$

Since
$$\varphi(x_0 + h, y_0) = \varphi(x_0 + h, y_0) - \varphi(x_0, y_0) \overset{h}{\underset{1}{=}} \varphi'_x (x_0, y_0) h,$$

it follows that

$$u_1(h) = - [\varphi'_y (x_0, y_0)]^{-1} \varphi(x_0 + h, y_0) \overset{h}{\underset{1}{=}} - [\varphi'_y (x_0, y_0)]^{-1} \varphi'_x (x_0, y_0) h,$$

which implies (18).

8.9 APPLICATIONS OF IMPLICIT FUNCTION THEOREM

8.91. Dependence of the solution of the equation. Consider the space $C^1 [G; E]$ of functions $f(x)$ with domain G of the space E and range

$E : x \in G, f(x) \in E$. Let the function $f(x)$ be differentiable, and $f(x)$, $f'(x)$ be continuous with a bounded norm. Let $\|f\| = \sup(\|f(x)\| + \|f'(x)\|)$. Then $C^1[G;E]$ is a normed linear space.

Now, examine the equation

$$f(x) = 0, \quad x \in G, \quad f \in C^1[G;E].$$

Assume that $f_0(x_0) = 0$ for some $x_0 \in G$ and that the operator $f_0'(x_0) \in (E \to E)$ has an inverse. Then, the upcoming theorem holds.

THEOREM 1. *There exist constants* $\delta > 0$, $\varepsilon > 0$, *such that the equation* $f(x) = 0$ *has a solution* $x = x_0 + \Delta x$, $\|\Delta x\| < \varepsilon$, *for any* $f \in C^1[G;E]$ *with* $\|f - f_0\| < \delta$, *and if* $f \to f_0$, *then* $\Delta x \to 0$.

PROOF. In fact, regard $f(x)$ to be a function of $x \in G$ and let $f \in C^1[G;E]$: $f(x) = \Phi(f, x)$. The construction of the function $\Phi(f, x)$ implies that $\Phi(f, x)$ and $\Phi_x'(f, x)$ are continuous. Then $\Phi_x'(f, x) = f'(x)$. By hypothesis $f_0(x_0) = 0$ and $[f_0'(x_0)]^{-1}$ exists. This means that $\Phi(f_0, x_0) = 0$ and $[\Phi_x'(f_0, x_0)]^{-1}$ exists. The theorem of implicit functions implies that for some $\delta > 0$ and $\varepsilon > 0$, the equation $\Phi(f_0 + \Delta f, x_0 + \Delta x) = 0$ has a solution $x_0 + \Delta x$ whenever $\|\Delta f\| < \delta$, that is, $f(x) = 0$, $f = f_0 + \Delta f$, $x = x_0 + \Delta x$, and also $\|\Delta x\| < \varepsilon$; moreover, if $\delta \to 0$, then $\|\Delta x\| \to 0$, proving the theorem.

REMARK. For $\Phi(f, x) = f(x)$, there exists the differential $\Phi_f'(f, x)\,\Delta f = \Delta f$. The existence of $\Phi_f'(f, x)$ implies that x is a differentiable function of f, and

$$\Delta x \stackrel{\Delta f}{\underset{1}{=}} - [\Phi_x'(f_0, x_0)]^{-1}\,\Phi_f'(f_0, x_0)\,\Delta f = -[f'(x_0)]^{-1}\,\Delta f.$$

The right side of this equation is a uniform differential of the function $x = \varphi(f)$ for $x = x_0$, that is, $d\varphi(f_0, \Delta f)$.

8.92. Application to eigenvectors. Consider the direct sum $H_1 = H \oplus R$, H a real HILBERT space and R the real line; let $\{x, t\}$ denote the elements of H_1, $x \in H$ and $t \in R$. Let f be a non-linear operator in $(H_1 \to H_1)$:

$$f(A; x, t) = \{y, \tau), \qquad x \in H \text{ and } t \in R$$

where $y = Ax - tx \in H$, $\tau = (x, x) - 1$ and A is a completely continuous linear self-adjoint operator in $(H \to H)$.

Thus, the equation $f(A; x, t) = 0$ has the form

$$Ax - tx = 0, \quad (x, x) = 1,$$

that is, t is an eigenvalue and x the corresponding normalized eigenvector of A. If $\{x, t\}$ is given an increment $\{\Delta x, \Delta t\}$, then $f(A; x, t)$ changes to $\{A\,\Delta x - t\Delta x - \Delta t x - \Delta t \Delta x, 2(x, \Delta x) + (\Delta x, \Delta x)\}$. The principal linear part of this increment is

$$d_{H_1} f(A; x, t; \Delta x, \Delta t) = \{A\Delta x - t\Delta x - \Delta t x, 2(x, \Delta x)\}.$$

Consequently, $f'(A; x, t)$ is a linear operator in $(H_1 \to H_1)$. It transforms $\{\Delta x, \Delta t\}$ into $\{A\Delta x - t\Delta x - x\Delta t, 2(x, \Delta x)\}$.

If t_0 is a simple eigenvalue of the operator A_0† and x_0 is the corresponding eigenvector, then there is an inverse operator

$$[f'_{H1}(A_0; x_0, t_0)]^{-1},$$

that is, for any $\{y, \tau\} \in H_1$ the equation

$$A_0\Delta x - t_0\Delta x - x_0\Delta t = y, \quad 2(x_0, \Delta x) = \tau \tag{1}$$

has a solution $\{\Delta x, \Delta t\}$, and this solution is unique.

In fact, $\Delta x = ax_0 + (\Delta x)_1$ where $a = (\Delta x, x_0)$ and $[(\Delta x)_1, x_0)] = 0$. Analogously $y = bx_0 + y_1$, where $b = (y, x_0)$ and $(y_1, x_0) = 0$. Further, since $(A_0 - t_0E)x_0 = 0$,

$$(A_0\Delta x - t_0\Delta x) - x_6\Delta t = (A_0 - t_0E)(\Delta x)_1 - x_0\Delta t,$$

and $[(A_0 - t_0E)(\Delta x)_1, x_0] = [(A_0 - t_0E)x_0, (\Delta x)_1] = 0$, whence (1) implies

$$(A_0 - t_0E)(\Delta x)_1 = y_1, \tag{2}$$

$$\Delta t = -b = -(y, x_0), \quad 2a = \tau. \tag{3}$$

Since the right side of (2) is orthogonal to x_0 (that is, orthogonal to all the eigenvectors belonging to the eigenvalue t_0), this equation has a unique solution

$$(\Delta x)_1 = (A_0 - t_0E)_1^{-1} y_1 = (A_0 - t_0E)_1^{-1}[y - (y, x_0)x_0],$$

orthogonal to x_0, where $(A_0 - t_0E)_1$ denotes the operator $A_0 - t_0E$ on the subspace of the elements orthogonal to x_0.

Thus, Eq. (1) has, for every $\{y, \tau\} \in H_1$, a solution

$$\Delta x = \frac{\tau}{2} x_0 + (A_0 - t_0E)_1^{-1}[y - (y, x_0)x_0], \qquad \Delta t = -(y, x_0).$$

This implies the existence of the operator $[f'(A_0; x_0, t_0)]^{-1}$.

By Theorem 1, the constants $\delta > 0$ and $\varepsilon > 0$ are found, such that for $\|\Delta A\| < \delta$, there exist an eigenvalue $t_0 + \Delta t$ and a normalized eigenvector $x_0 + \Delta x$ for the operator $A = A_0 + \Delta A$:

$$[(A_0 + \Delta A) - (t + \Delta t)E](x_0 + \Delta x) = 0, \qquad (x_0 + \Delta x, x_0 + \Delta x) = 1,$$

where such eigenvalue and eigenvector are defined uniquely for $\|\Delta x\| + |\Delta t < \varepsilon$.

The first approximation $\{\Delta_1 x, \Delta_1 t\}$ of $\{\Delta x, \Delta t\}$ is given by the equations

$$A_0\Delta_1 x - t_0\Delta_1 x - \Delta_1 t x_0 + \Delta A x_0 = 0, \quad (x_0, \Delta_1 x) = 0.$$

Multiply first of these equations scalarly by x_0 and note that

$$[(A_0 - t_0E)\Delta_1 x, x_0] = [(A_0 - t_0E)x_0, \Delta_1 x] = 0,$$

†That is, an eigenvalue of multiplicity 1.

to receive $\Delta_1 t = (\Delta A x_0, x_0)$. Furthermore,

$$(A_0 - t_0 E)\, \Delta_1 x = \Delta_1 t x_0 - \Delta A x_0. \tag{4}$$

Equation (4) always has a solution, for its right side is orthogonal to x_0:

$$(\Delta_1 t x_0, x_0) - (\Delta A x_0, x_0) = \Delta_1 t - (\Delta A x_0, x_0) = 0.$$

Hence,
$$\Delta_1 x = (A_0 - t_0 E)_1^{-1}\, [\Delta_1 t x_0 - \Delta A x_0].$$

8.93. Equations depending on parameters. THEOREM 2. *Let $y = y(t, x)$ be a function af the element $x \in E$ and the numerical parameter t. Let its range lie in E. Further, let $y(t, x)$ be n-times differentiable with respect to t and x. Suppose also that the equation $y(t_0, x) = 0$ has a solution $x = x_0$ for $t = t_0$, and the operator $[y'_x(t_0, x_0)]^{-1}$ exists. Then, there exist constants $\delta > 0$ and $\varepsilon > 0$, such that for $|t - t_0| < \delta$ the equation*

$$y(t, x) = 0 \tag{5}$$

has a uniquc solution $x = x(t)$, such that $\| x(t) - x_0 \| < \varepsilon$. This solution $x(t)$, as a function of t, is n-times differentiable.

This theorem is immediate from the implicit function theorem.

Example. Let $A(t)$ be a completely continuous linear operator in $(H \to H)$ and, being a function of t, let it be n-times differentiable. Let the operator $A(t_0)$ have a simple eigenvalue λ_0 with the corresponding normalized eigenvector x_0:

$$A(t_0)\, x_0 - \lambda_0\, x_0 = 0.$$

Consider the direct sum $H_1 = H \oplus R$ and define a function $\Phi(t; x, \lambda)$ with

$$\{x, \lambda\} \in H \oplus R, \quad \Phi(t; x, \lambda) \in H \oplus R$$

as
$$\Phi(t; x, \lambda) = \{A(t)\, x - \lambda x, (x, x) - 1\}.$$

For the normalized eigenvector $x(t)$ and the eigenvalue $\lambda(t)$ of $A(t)$, this equation has the form

$$\Phi(t; x. \lambda) = 0.$$

Since $A(t)$ is n-times differentiable with respect to t, so is also $\Phi(t; x, \lambda)$. Just as in the previous section, it can be shown that $[(\partial/\partial\{x, \lambda\})\ \Phi(t_0; x_0, \lambda_0)]^{-1}$ exists. However, then Theorem 2 can be applied and, consequently, the equation, $\Phi(t; x, \lambda) = 0$, that is, the equation

$$A(t)\, x - \lambda\, x = 0, \ (x, x) = 1 \tag{6}$$

has the solution $\{x(t), \lambda(t)\}$ and it is an n-times differentiable function with respect to the parameter t.

8.94. Variational equations. By the hypothesis of Theorem 2, the solution $x(t)$ of Eq. (5) is a differentiable function of t. The derivative $x'(t)$ of $x(t)$ with respect to t for $t = t_0$ is called the **variation**: $\delta x = \left.\dfrac{dx}{dt}\right|_{t = t_0}$. Similarly, for $y = y(t, x)$, $\delta y = \left.\dfrac{\partial y}{\partial t}\right|_{t=t_0}$.

The equation $y[t, x(t)] = 0$ is an identity, and differentiating it with respect to t yields

$$\frac{\partial y\,[t, x(t)]}{\partial t} + \frac{\partial y\,[t, x(t)]}{\partial x}\, x'(t) = 0.$$

For $t = t_0$, we have $$\delta y + y'_x(t_0, x_0)\,\delta x = 0. \tag{7}$$

Eq. (7) is called the **variational equation** for the original equation $y(t, x) = 0$.

Since $[y'_x(t_0, x_0)]^{-1}$ exists, it follows that

$$\delta x = -\,[y'_x(t_0, x_0)]^{-1}\,\delta y. \tag{8}$$

For example, $(A - \lambda E)\,\Delta x + \Delta A x - \Delta\lambda x = 0$ can be regarded as a variational equation for (6), if the elements ΔA, Δx, $\Delta\lambda$ are replaced by δA, δx, $\delta\lambda$:

$$\delta x = -\,(A-\lambda E)_1^{-1}\,(\delta A x-\delta\lambda x), \qquad \delta\lambda = (\delta A x,\, x).$$

9.95. Applications to differential equations. Recall the differential equation

$$\frac{dx}{dt} = f(t, x), \quad \text{with the initial condition } x(0) = x_0, \tag{9}$$

where $f(t, x)$ and x are elements in E. Eq. (9) is equivalent to the integral equation

$$x(t) - x_0 - \int_0^t f[\tau, x(\tau)]\,d\tau = 0. \tag{10}$$

Let $F[x_0, x(t)]$ denote the left side of Eq. (10). The functions $x(t)$ belong to $C_1^E[0, 1]$† and $F[x_0, x(t)]$ is an operator mapping the direct sum

$$E \oplus C_1^E[0, 1] \tag{10'}$$

into $C_1^E[0, 1]$. If $f(t, x)$ is n-times differentiable with respect to x, and if $\partial^n f(t, x)/\partial x^n$ is continuous in (t, x), then $x(t) - \int_0^t f[\tau, x(\tau)]\,d\tau$ is an n-times differentiable operator from $C_1^E[0, 1]$ into $C_1^E[0, 1]$. Hence, $F[x_0, x(t)]$ is an operator n-times differentiable with respect to $x(t)$. Since, x_0 occurs in F as a separate term, F is an n-times differentiable function in the direct sum (10′).

If $x = x(t)$ takes the increment $\Delta x = \Delta x(t)$, then

$$F'_x\,\Delta x = \Delta x(t) - \int_0^t f'_x[\tau, x(\tau)]\,\Delta x(\tau)\,d\tau \tag{11}$$

is the principal linear part of the increment of $F(x_0, x)$ with respect to Δx. The right-hand side of (11) is an operator from $C_1^E[0, 1]$ into $C_1^E[0, 1]$ and has an inverse, because for every $y(t) \in C_1^E[0, 1]$, the equation

$$F'_x\,\Delta x = y(t), \quad \text{or} \quad \Delta x(t) - \int_0^t f'_x[\tau, x(\tau)]\,\Delta x(\tau)\,d\tau = y(t)$$

† C_1^E is the set of all the continuously differentiable functions $x(t)$ with $t \in [0, 1]$ and $x(t) \in E$.

is equivalent to the differential equation in $\Delta x(t)$:

$$\frac{d\,\Delta x(t)}{dt} = f'_x[t, x(t)]\,\Delta x(t) + y'(t) \quad \text{and} \quad \Delta x(0) = y(0).$$

The last equation has by the existence theorem, a unique solution (the right side is linear with respect to $\Delta x(t)$ and, consequently, a LIPSCHITZ condition is automatically satisfied relative to Δx). This solution represents the inverse operator

$$\Delta x(t) = \Delta x = [F'_x]^{-1}\,y(t).$$

The hypotheses of the implicit function theorem are satisfied.

The solution $x = x(t)$ of Eq. (9) can be regarded as a function of the initial value $x_0 : x = x(t, x_0)$, $x(t, x_0)$ being n-times differentiable with respect to x_0.

In particular, if E is an n-dimensional space, then the continuous differentiability theorem gives the dependence of the solution on the initial data.

8.10 TANGENT MANIFOLDS

8.101. The particular case. Let the function $\varphi(x)$ map a BANACH space E_x into a BANACH space $E_y : x \in E_x$, $\varphi(x) \in E_y$. Consider the collection $\mathcal{M}$ of the points satisfying the equation $\varphi(x) = 0$. Further, put $\varphi(x_0) = 0$, that is, $x_0 \in \mathcal{M}$, and let the function $\varphi(x)$ be continuously differentiable at a neighbourhood of the point x_0, that is, let

$$\varphi(x_0 + h) \overset{h}{\underset{1}{=}} \varphi'(x_0)\,h.$$

If the operator $\varphi'(x_0) \in (E_x \to E_y)$ maps the space E_x onto the entire space E_y, then the point x_0 is said to be **regular.**

Hereafter, x_0 will always be treated regular. The collection of the elements $h \in E$, satisfying $\varphi'(x_0)\,h = 0$, will be denoted by T_0. T_0 is a subspace of E.

The collection of the elements $x_0 + h$, $h \in T_0$, is called the **linear manifold** T_{x_0} **tangent to** $\mathcal{M}$ at the point x_0.

Consider first the case when E_x is the direct sum of T_0 and some subspace T_ξ. Thus, every element $x \in E_x$ has the form $x = h + \xi$ with $h \in T_0$, $\xi \in T_\xi$.

The linear operator $\varphi'(x_0)$ maps T_ξ onto the entire space E_y. In fact, $\varphi'(x_0)$ maps E_x onto the whole of E_y, implying thereby that for every $y \in E_y$, there is an element $x \in E_x$, such that

$$\varphi'(x_0)\,x = y.$$

However, $x = h + \xi$ with $h \in T_0$, $\xi \in T_\xi$ and $\varphi'(x_0)\,h = 0$. Hence,

$$\varphi'(x_0)\xi = \varphi'(x_0)x = y.$$

Define a linear operator $A \in (T_\xi \to E_y)$ by $A\xi = \varphi'(x_0)\,\xi$. By what has just been proved, A maps T_ξ onto the whole of E_y. Besides, if $\xi, \xi_1 \in T_\xi$, and if $A\xi = A\xi_1$, then $\xi = \xi_1$. In fact, let $A(\xi - \xi_1) = 0$, that is, $\varphi'(x_0)(\xi - \xi_1) = 0$. Thence, $\xi - \xi_1 \in T_0$. However, since E_x is the direct sum of T_0 and T_ξ, hence $\xi - \xi_1 = 0$, $\xi = \xi_1$.

By the BANACH theorem, the operator A has an inverse linear operator A^{-1}.

THEOREM 1. *If E_x is the direct sum of the subspaces T_0 and T_ξ, then there is a topological (that is, one-one and mutually continuous) mapping, transforming a neighbourhood of x_0 in the manifold $\mathcal{M}$ and a neighbourhood of the same element x_0 in the linear tangent manifold T_{x_0} into each other, where the corresponding points have a distance which from higher order onwards is less than their distance from the contact point x_0.*

PROOF. In the neighbourhood of x_0, the element x has the form

$$x = x_0 + h + \xi, \qquad h \in T_0, \ \xi \in T_\xi.$$

The defining equation of $\mathcal{M}$ can be written in the form

$$\Phi(h, \xi) = \varphi(x_0 + h + \xi) = 0. \tag{1}$$

For $h = \xi = 0$, $\Phi(0, 0) = 0$ too. Further, the partial differential of the function $\Phi(h, \xi)$ corresponding to the increment $\Delta\xi$, has the form

$$\Phi'_\xi(0, 0)\,\Delta\xi = \varphi'(x_0)\Delta\xi = A\,\Delta\xi \tag{2}$$

for $h = \xi = 0$. The operator $A = \Phi'_\xi(0, 0)$ has an inverse. Hence, by the implicit function theorem, in a neighbourhood of $h = 0$, $\xi = 0$, Eq. (1) is equivalent to $\xi = \psi(h)$, $\psi(h)$ a differentiable function, satisfying the condition $\psi(0) = 0$. Thus, every point $x \in \mathcal{M}$ in a neighbourhood of x_0 has the form

$$x = x_0 + h + \psi(h), \quad h \in T_0, \quad \psi(h) \in T_\xi.$$

Consequently, a mapping is formed which maps one-one and continuously (that is, topologically) a point $\bar{x} = x_0 + h$ in a neighbourhood of x_0 in T_{x_0} onto a point $x = x_0 + h + \psi(h)$ in a neighbourdood of x_0 in $\mathcal{M}$. The equality

$$\Phi'_h(0, 0)\,h + \Phi'_\xi(0, 0)\ \psi'(0)\,h = \Phi'_h(0, 0)\,h + A\psi'(0)\,h = 0$$

implies $\quad d\psi(0, h) = \psi'(0)h = -A^{-1}\,\Phi'_h(0, 0)h = -A^{-1}\,\varphi'(x_0)\,h = 0.$

Hence $$\psi(h) \overset{h}{\underset{1}{=}} \psi(0) + \psi'(0)h = 0,$$

that is, $\| \psi(h) \| = o(\| h \|)$. However, $\| \psi(h) \|$ is the distance between the point $x_0 + h$ of the linear tangent manifold T_{x_0} and the corresponding point $x = x_0 + h + \xi(h)$ of $\mathcal{M}$. This distance is smaller than first order in $\| h \|$ or $\| h + \psi(h) \|$, that is, than the distance from the point $x_0 + h$ or $x_0 + h + \psi(h)$ to the contact point x_0. ■

8.102. The general case. In the general case, it is not possible to verify the existence of a subspace T_ξ, such that $E_x = T_0 \oplus T_\xi$. However, Theorem 1 remains true even in the general case in a rather weaker form, as will be seen in what follows.

Let us form the quotient space E_x/T_0 of the cosets with respect to T_0 (see p. 41). Every element T of the quotient space E_x/T_0 is some set of elements of E_x. If $x_1, x_2 \in T$, then $x_1 - x_2 \in T_0$, that is, $\varphi'(x_0)(x_1 - x_2) = 0$. Hence,

$$\varphi'(x_0)\,x_1 = \varphi'(x_0)\,x_2.$$

The operator $\varphi'(x_0) \in (E_x \to E_y)$ transforms every pair of elements x_1 and x_2 of T into one and the same element of E_y. Conversely, if

$$\varphi'(x_0)\, x_1 = \varphi'(x_0)\, x_2,$$

then $\qquad \varphi'(x_0)\,(x_1 - x_2) = 0, \qquad$ that is, $x_1 - x_2 \in T_0$,

and, hence, x_1 and x_2 belong to one and the same set $T \in E_x/T_0$. Consequently, the operator $\varphi'(x_0)$ generates a linear operator A, mapping E_x/T_0 into E_y, namely if $T \in E_x/T_0$, then

$$AT = \varphi'(x_0)\, x,$$

where x is any point of T. By these considerations, AT does not depend on the choice of $x \in T$.

Let y be an arbitrary point of E_y. By assumption, the operator $\varphi'(x_0)$ maps E_x onto the entire space E_y. Hence, there is an element $x \in E_x$, such that $\varphi'(x_0)\, x = y$. However, x belongs to a certain $T \in E_x/T_0$. By definition, $AT = \varphi'(x_0)\, h = y$. Thus, the operator A has an inverse A^{-1}. By the BANACH theorem, the operator A^{-1} is also bounded.

THEOREM 2. *A point $\bar{x}$ of the tangent manifold T_{x_0} can be associated with every point x of the manifold $\mathcal{M}$; conversely, a point x of the manifold $\mathcal{M}$ can be associated with every point $\bar{x}$ of T_{x_0}, such that the distance between corresponding points of higher order becomes smaller than their distance from the point of contact x_0 (this correspondence is, generally speaking, not one-one).*

The proof of Theorem 2 is rather a variant of the proof of implicit function theorem and this theorem itself is a direct extension of the latter.

PROOF. Let $h \in T_0$. Construct a sequence $\{T_n\}$ of elements of E_x/T_0 and a sequence $\{\xi_n\}$ of points, $\xi_n \in T_n$. Let $\xi_0 = 0 \in T_0$ and let all T_i, ξ_i for $i = 1, 2, \ldots, n-1$, be constructed. Then, T_n and ξ_n are defined thus:

$$T_n = T_{n-1} - A^{-1}\, \varphi(x_0 + h + \xi_{n-1}). \tag{3}$$

Further, select any point ξ_n on T_n, such that

$$\| \xi_n - \xi_{n-1} \| \leqslant 2\, \| T_n - T_{n-1} \|.$$

Such a choice is plausible, since

$$\| T_n - T_{n-1} \| = \inf_{\xi \in T_n} \| \xi - \xi_{n-1} \|.$$

Since $\xi_{n-1} \in T_{n-1}$, by the definition of the operator A we have

$$AT_{n-1} = \varphi'(x_0)\, \xi_{n-1}.$$

Hence (3) can be written in the form

$$T_n = -A^{-1}[\,\varphi(x_0 + h + \xi_{n-1}) - \varphi'(x_0)\, \xi_{n-1}\,].$$

Since, $\qquad T_{n-1} = -A^{-1}[\,\varphi(x_0 + h + \xi_{n-2}) - \varphi'(x_0)\, \xi_{n-2}\,],$

hence $\qquad T_n - T_{n-1} = -A^{-1}[\,\varphi(x_0 + h + \xi_{n-1}) - \varphi(x_0 + h + \xi_{n-2}) - \varphi'(x_0)\,(\xi_{n-1} - \xi_{n-2})\,].$

Put $\xi_t = \xi_{n-2} + t(\xi_{n-1} - \xi_{n-2})$, to receive

$$\varphi(x_0 + h + \xi_{n-1}) - \varphi(x_0 + h + \xi_{n-2}) = \int_0^1 \varphi'(x_0 + h + \xi_t)\, dt\, (\xi_{n-1} - \xi_{n-2}).$$

Hence,

$$T_n - T_{n-1} = -A^{-1} \int_0^1 [\varphi'(x_0 + h + \xi_t) - \varphi'(x_0)]\, dt\, (\xi_{n-1} - \xi_{n-2}). \tag{4}$$

Let $\quad \| h \| \leqslant r, \quad \| \xi_{n-1} \| \leqslant r, \quad \| \xi_{n-2} \| \leqslant r.$

Then, $\| \xi_t \| \leqslant r$ and also $\| h + \xi_t \| \leqslant 2r$. By the continuity of $\varphi'(x)$ at the point x_0, there is a number ε_r, $\varepsilon_r \to 0$, as $r \to 0$ for every $r > 0$, such that

$$\| \varphi'(x) - \varphi'(x_0) \| \leqslant \varepsilon_r \qquad \text{for } \| x - x_0 \| \leqslant 2r.$$

Hence, $\| \varphi'(x_0 + h + \xi_t) - \varphi'(x_0) \| \leqslant \varepsilon_r$, and (4) implies that

$$\| T_n - T_{n-1} \| \leqslant \| A^{-1} \| \int_0^1 \| \varphi'(x_0 + h + \xi_t) - \varphi'(x_0) \|\, dt\, \| \xi_{n-1} - \xi_{n-2} \|$$

$$\leqslant \| A^{-1} \|\, \varepsilon_r\, \| \xi_{n-1} - \xi_{n-2} \|,$$

$$\| \xi_n - \xi_{n-1} \| \leqslant 2 \| T_n - T_{n-1} \| \leqslant 2 \| A^{-1} \|\, \varepsilon_r\, \| \xi_{n-1} - \xi_{n-2} \|.$$

For sufficiently small r, $2 \| A^{-1} \| \varepsilon_r \leqslant \frac{1}{2}$, thus also $\| \xi_n - \xi_{n-1} \| \leqslant \frac{1}{2} \| \xi_{n-1} - \xi_{n-2} \|$. Now, let $\| h \| = r$; if $\| \xi_i \| \leqslant r$, $i = 1, 2, \ldots, n-1$, then

$$\| \xi_n - \xi_{n-1} \| \leqslant \frac{1}{2} \| \xi_{n-1} - \xi_{n-2} \| \leqslant \ldots \leqslant \frac{1}{2^{n-1}} \| \xi_1 - \xi_0 \| = \frac{1}{2^{n-1}} \| \xi_1 \|,$$

and

$$\begin{aligned} \| \xi_n \| &= \| \xi_1 + (\xi_2 - \xi_1) + \ldots + (\xi_n - \xi_{n-1}) \| \\ &\leqslant \| \xi_1 \| + \| \xi_2 - \xi_1 \| + \ldots + \| \xi_n - \xi_{n-1} \| \\ &\leqslant \| \xi_1 \| \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}} \right) \leqslant 2 \| \xi_1 \|. \end{aligned}$$

Since $\xi_0 = \theta$, hence $T_1 = -A^{-1} \varphi(x_0 + h)$, and

$$\| \xi_1 \| \leqslant 2 \| T_1 \| \leqslant 2 \| A^{-1} \| \| \varphi(x_0 + h) \|.$$

Further, $\quad \varphi(x_0 + h) = \varphi(x_0) + \varphi'(x_0) h + \varepsilon(h), \quad \varepsilon(h) = o(\| h \|).$

However, then, $\varphi(x_0) = 0$, and $h \in T_0$. Consequently, $\varphi'(x_0) h = 0$; hence

$$\varphi(x_0 + h) = \varepsilon(h).$$

Thereupon,

$$\| \xi_1 \| \leqslant 2 \| A^{-1} \| \| \varepsilon(h) \|. \tag{5}$$

For sufficiently small $r > 0$ and $\| h \| \leqslant r$,

$$\| \varepsilon(h) \| \leqslant \frac{1}{4 \| A^{-1} \|} \| h \|.$$

Hence, $$\| \xi_1 \| \leqslant \tfrac{1}{2} \| h \| \leqslant \tfrac{1}{2} r,$$

and therefore, $\| \xi_n \| \leqslant r$. Hence for all n,

$$\| \xi_n - \xi_{n-1} \| \leqslant \frac{1}{2} \| \xi_{n-1} - \xi_{n-2} \|.$$

Therefore, the sequence $\{\xi_n\}$ converges to an element $\xi \in E_x$ with $\| \xi \| \leqslant \| h \|$, and, what is more, by (5),

$$\| \xi \| \leqslant 2 \| \xi_1 \| \leqslant 4 \|A^{-1} \| \| \varepsilon (h) \|. \tag{6}$$

Similarly, $T_n \in E_x/T_0$ converges to $T \in E_x/T_0$, and $\xi \in T$.

As $n \to \infty$, $\xi_n \to \xi$ and $T_n \to T$, Eq. (3) reduces to $T = T - A^{-1} \varphi (x_0 + h + \xi)$ or, $A^{-1} \varphi (x_0 + h + \xi) = 0$, or $\varphi (x_0 + h + \xi) = 0$. Consequently,

$$x_0 + h + \xi \in \mathcal{M}.$$

Associate $x_0 + h \in T_0$ with this point. Ineq. (6) evidences that

$$\| \xi \| = o (\| h \|),$$

that is, the distance $\| \xi \|$ between the point $x_0 + h \in T_0$ and the corresponding point $x_0 + h + \xi \in \mathcal{M}$ becomes from higher order onwards less than the distance $\| \xi \|$ from the point of contact x_0.

Now, let $x = x_0 + u$ belong to $\mathcal{M}$, that is,

$$\varphi (x_0 + u) = \varphi (x_0) = 0.$$

We have $\varphi (x_0 + u) - \varphi (x_0) = \varphi' (x_0) \; u + \varepsilon (u) = 0$, with $\varepsilon (u) = o(\| u \|)$. Thereupon,

$$\varphi' (x_0) \, u = - \varepsilon (u).$$

Denote by T those elements of E_x/T_0, which belong to u. Then,

$$\varphi' (x_0) \, u = AT,$$

whence $AT = - \varepsilon (u)$, and $T = - A^{-1} \varepsilon (u)$, $\| T \| \leqslant \| A^{-1} \| \; \| \varepsilon (u) \|$.

Among the elements of the space E_x which belong to T, there is an element ξ, such that

$$\| \xi \| \leqslant 2 \| T \| \leqslant 2 \| A^{-1} \| \| \varepsilon (u) \|.$$

Since $\xi \in T$ and $u \in T$, it follows that

$$u - \xi \in T_0, \quad x_0 + u - \xi \in Tx_0.$$

Associate $x = x_0 + u$ with this point, to receive the estimate

$$\| \xi \| = o (\| u \|),$$

which holds for the distance $\| \xi \|$ between these points. The theorem is completely proved.

8.103. Locally linear space. The idea of a **locally linear space** is connected with the notion of the tangent manifold. It finds important applications in a number of research fields.

Consider two metric spaces X and Y. Let φ be a topological mapping of X onto Y, and let $\varphi(x) \in Y$ correspond to the point $x_0 \in X$. The mapping φ is called **almost isometric at the point** $x_0 \in X$, if the distance between any pair of elements x_1 and x_2 of X is related to the distance of their images in Y by the inequality

$$\rho(x_1, x_2)(1 - \varepsilon) \leqslant \rho[\varphi(x_1), \varphi(x_2)] \leqslant \rho(x_1, x_2)(1 - \varepsilon),$$

where ε tends to zero together with $\rho(x_1, x_0) + \rho(x_2, x_0)$.

Example. Let $\mathcal{M}$ be a manifold in E_x, defined by the equation $\psi(x)' = 0$, T_{x_0} a linear tangent manifold belonging to $\mathcal{M}$ at the regular point $x_0 \in \mathcal{M}$ and let $E_x = T_0 \oplus T_\xi$, where T_0 is the collection of elements h, for which $\psi'(x_0)\, h = 0$. For some $r > 0$, associate the element $x_0 + h + \xi(h) \in \mathcal{M}$ with every element $x_0 + h \in T_{x_0}$, $\| h \| \leqslant r$ (see Theorem 1). We obtain a topological mapping χ of the neighbourhood of the point x_0 in T_{x_0} onto a neighbourhood of x_0 in $\mathcal{M}$.

The mapping χ is almost isometric at the point $x_0 \in T_0$ (or $x_0 \in \mathcal{M}$). In fact, let $x_1, x_2 \in T_{x_0}$, $x_i = x_0 + h_i$, $\| h_i \| \leqslant r$; then $\chi(x_1) = x_0 + h_1 + \xi(h_1) \in \mathcal{M}$ corresponds to x_1, and $\chi(x_2) = x_0 + h_2 + \xi(h_2) \in \mathcal{M}$ to x_2. We have $\chi(x_1) - \chi(x_2) = h_1 - h_2 + [\xi(h_1) - \xi(h_2)]$, whence

$$\| h_1 - h_2 \| - \| \xi(h_1) - \xi(h_2) \| \leqslant \| \chi(x_1) - \chi(x_2) \| \leqslant \| h_1 - h_2 \| + \| \xi(h_1) - \xi(h_2) \|. \tag{7}$$

The function $\xi(h)$ has a continuous derivative $\xi'(h)$ with $\xi'(0) = 0$. Hence, $\| \xi'(h) \| \leqslant \varepsilon_r$ for $\| h \| \leqslant r$, where $\varepsilon_r \to 0$ as $r \to 0$. Furthermore,

$$\left\| \xi(h_1) - \xi(h_2) \right\| = \left\| \int_0^1 \xi'[h_2 + t(h_1 - h_2)]\, dt (h_1 - h_2) \right\| \leqslant \int_0^1 \left\| \xi'[h_2 + t(h_1 - h_2)] \right\| dt \left\| h_1 - h_2 \right\|.$$

If $\| h_1 \| + \| h_2 \| \leqslant r$, then $\| h_1 \| \leqslant r$ and $\| h_2 \| \leqslant r$. Consequently, for $0 \leqslant t \leqslant 1$,

$$\| h_2 + t(h_1 - h_2) \| \leqslant r \quad \text{and} \quad \| \xi(h_1) - \xi(h_2) \| \leqslant \varepsilon_r \| h_1 - h_2 \|.$$

Hence, (7) takes the form

$$\| x_1 - x_2 \| (1 - \varepsilon_r) \leqslant \| \chi(x_1) - \chi(x_2) \| \leqslant \| x_1 - x_2 \| (1 + \varepsilon_r),$$

proving that our mapping is almost isometric.

We now define the locally linear space.

Given a metric space X. If every sufficiently small neighbourhood of an arbitrary point $x \in X$ can be mapped almost isometrically onto a neighbourhood of 0 of a Banach space, then X is called **locally linear**.

The above example shows that in a *Banach space every manifold, all of whose points are regular, forms a locally linear space.*

The notion of a differential can be extended to functions which are defined on locally linear spaces. The spaces of admissible functions of various classical variational problems are locally linear. The variations of the functionals considered there form examples of differentials of functions in locally linear spaces.

8.11 EXTREMA

THE OBJECT of this section is to consider the applications of some of the notions introduced above to the problems of the calculus of variations.

Let $f(x)$ be a functional defined in the space E_x; $f(x)$ has at $x_0 \in E_x$ a **minimum** (or **maximum**) if $f(x) \geqslant f(x_0)$ [or $f(x) \leqslant f(x_0)$] for all points x in a neighbourhood of x_0. Minima and maxima are called **extrema.**

THEOREM 1. *If x_0 is an extremal point of the functional $f(x)$ and if $f(x)$ is differentiable at this point, $df(x_0, h) = f'(x_0)h$, then $f'(x_0) = 0$, that is, $df(x_0, h) = 0$ for every $h \in E_x$.*

PROOF. In fact,

$$f'(x_0)\, h = \frac{d}{dt} f(x_0 + th)\,|_{t=0}.$$

However, $f(x_0 + th)$ is a function of t and attains its extremum for $t = 0$. Hence,

$$df(x_0, h) = f'(x_0)\, h = \frac{d}{dt} f(x_0 + th)\,|_{t=0} = 0.$$

Since h is an arbitrary element of E_x, the theorem is proved.

It is now sought to determine the conditions for extrema. Let $\varphi(x)$ be a function with domain E_x and range E_y, $x \in E_x$, $\varphi(x) \in E_y$. Further, let $f(x)$ be a functional defined on E_x.

The functional $f(x)$ achieves at x_0 its *minimum* (or *maximum*) *with the additional condition* $\varphi(x_0) = 0$, if $f(x) \geqslant f(x_0)$ [or $f(x) \leqslant f(x_0)$] for all x in some neighbourhood of x_0, for which $\varphi(x) = 0$.

THEOREM 2. *If x_0 at which the functional $f(x)$ attains its minimum under the condition $\varphi(x) = 0$, is a regular point of the manifold $\varphi(x) = 0$, then there is a linear functional l defined on the space E_y, $l \in E_y^*$, such that for the functional $F(x) = f(x) - l\varphi(x)$, we have*

$$F'(x_0) = 0, \quad \textit{that is, } dF(x_0, h) = 0$$

for every $h \in E$.

PROOF. First it is to be shown that $df(x_0, h) = 0$ for all h, defines a linear tangent manifold at the point x_0, that is, for all $h \in T_0$. In fact, let $h \in T_0$ and $df(x_0, h) = c \neq 0$. By Theorem 2 of Chap. 8.10, for any t, to the point $x_0 + th$ there corresponds a point $x_0 + th + u(t)$ of the manifold $\varphi(x) = 0$, such that $\| u(t) \|$ is of order higher than t. By the definition of the differential, we have

$$\begin{aligned} f[x_0 + th + u(t)] &= f(x_0) + df[x_0, th + u(t)] + \omega(t) \\ &= f(x_0) + f'(x_0)\, th + f'(x_0)\, u(t) + \omega(t) \\ &= f(x_0) + ct + f'(x_0)\, u(t) + \omega(t). \end{aligned}$$

As $t \to 0$, $f'(x_0)\, u(t) + \omega(t)$ tends to zero of order higher than ct. Thus, the sign of the difference

$$f[x_0 + th + u(t)] - f(x_0)$$

coincides with that of ct, so that t and this difference change their signs simultaneously. However, x cannot be an extremum of $f(x)$. Consequently, the assumption that $c \neq 0$ is false, as was required to prove.

Thus, under the hypotheses of the theorem $df(x_0, h) = 0$ for all h, for which $\varphi'(x_0)\, h = 0$, that is, $d\varphi\,(x_0, h) = 0$. This implies that

$$df(x_0, h_1) = df(x_0, h_2),$$

if h_1 and h_2 belong to the same coset $T \in E_x/T_0$. Introduce the functional $\chi(T) = df(x_0, h)$, $h \in T$ arbitrary, to receive

$$\chi(T) \mid = \mid df(x_0, h) \mid = \mid f'(x_0)\, h \mid \leqslant \| f'(x_0) \| \, \| h \|,$$

whence, by taking the greatest lower bound on the right-hand side with respect to $h \in T$, we get

$$\mid \chi(T) \mid \leqslant \| f'(x_0) \| \, \| T \|.$$

Consequently, $\chi(T)$ is a linear functional defined on E_x/T_0. On the other hand,

$$T = [\varphi'(x_0)]^{-1}\, y,$$

where y is an element of E_y, such that $\varphi'(x_0)\, h = y$, for every $h \in T$ (see p. 326). Thus,

$$df(x, h) = \chi(T) = \chi\{[\varphi'(x_0)]^{-1}\, y\} = l(y).$$

Since $y = \varphi'(x_0)\, h = d\varphi\,(x_0, h)$, then

$$df(x, h) = l\, d\varphi\,(x_0, h).$$

Thereupon, put $F(x) = f(x) - l\varphi(x)$, to receive $dF(x_0, h) = 0$ for all $h \in E_x$, giving also the required proof.

Example. *An isoperimetric problem.* We seek an extremum of the functional $f(x)$ under the conditions $\varphi_i(x) = 0$, $i = 1, 2, \ldots, n$, where $f(x)$, $\varphi_i(x)$ are defined on E. We regard the φ_i to be the components of an n-dimensional vector $\bar{\varphi}$ and denote by $\bar{\varphi}(x)$ a vector with the components $\varphi_i(x)$, $i = 1, 2, \ldots, n$. Suppose that an extremum is attained at the point $x_0 \in E$. By Theorem 2, there is a linear functional $l(\varphi)$, such that $dF(x_0, h) = 0$ for $F(x) = f(x) - l\bar{\varphi}(x)$. However, since $l\bar{\varphi} = \sum_{i=1}^{n} \lambda_i \varphi_i$ in the n-dimensional space of vectors φ, where λ_i are constants, hence

$$F(x) = f(x) - \sum_{i=1}^{n} \lambda_i\, \varphi_i(x).$$

Consequently, at the extremum point x_0, we get

$$f'(x_0) - \sum_{i=1}^{n} \lambda_i\, \varphi_i'(x_0) = 0.$$

This gives LAGRANGE's multiplier rule.

APPENDICES

APPENDIX 1. INEQUALITIES

I.11. The class L_p, $p > 1$. A function $x(t)$, defined and measurable on $[0, 1]$, is said to **belong to the class** $L_p(0, 1)$ or, in other words, is said to be **p-th power summable,** if

$$\int_0^1 | x(t) |^p \, dt < \infty.$$

Here, the integral is understood in the LEBESGUE sense and p is some positive number.

In what follows, it shall be assumed that $p \geqslant 1$. If $p = 1$, then we obtain a class of summable functions, denoted by $L(0, 1)$.

THEOREM. *If* $x(t) \in L_p\ (0, 1)$ *and* $y(t) \in L_p\ (0, 1)$, *then also* $x(t) + y(t) \in L_p\ (0, 1)$, *that is the sum of two functions in* $L_p\ (0, 1)$ *is again a function in* $L_p\ (0, 1)$.

PROOF. Select a pair of numbers a and b. Then

$$| a + b | \leqslant | a | + | b | .$$

Consider the two cases

(i) $| a | > | b |$, then $| a + b | \leqslant 2 | a |$, and
$| a + b |^p \leqslant 2^p | a |^p \leqslant 2^p(| a |^p + | b |^p)$;

(ii) $| b | \geqslant | a |$, then $| a + b |^p \leqslant 2^p | b |^p \leqslant 2^p (| a |^p + | b |^p)$.

Thus, we always have $| a + b |^p \leqslant 2^p (| a |^p + | b |^p)$. Now, put $a = x(t)$, $b = y(t)$, to receive

$$| x(t) + y(t) |^p \leqslant 2^p (| x(t) |^p + | y(t) |^p).$$

Since $\displaystyle\int_0^1 | x(t) |^p \, dt < \infty$ and $\displaystyle\int_0^1 | y(t) |^p \, dt < \infty$,

hence, also $\displaystyle\int_0^1 | x(t) + y(t) |^p \, dt < \infty.$ ■

Given a set of numerical sequences $x = \{\xi_i\}$, such that $\sum_{i=1}^{\infty} |\xi_i |^p < \infty$. We denote this set by l_p. Just as in the preceding case, it can be verified that if $x \in l_p$ and $y \in l_p$ then $x + y \in l_p$, that is, if

$$x = \{\xi_i\}, \quad y = \{\eta_i\}, \quad \sum_{i=1}^{\infty} | \xi_i |^p < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} | \eta_i |^p < \infty,$$

then
$$\sum_{i=1}^{\infty} | \xi_i + \eta_i |^p < \infty.$$

I.12. Hölder's inequality. The well-known CAUCHY BUNYAKOVSKII-SCHWARZ (CBS) inequality† finds wide applications in a variety of mathematical investigations. However, a more general inequality due to HÖLDER is now sought to be proved.

Consider the function $\tau = t^\alpha$ with $\alpha > 0$. Thus, $\tau' = \alpha t^{\alpha-1} > 0$ for $t > 0$ and, therefore, $\tau = t^\alpha$ is an increasing function for the positive t. For these t, the single-valued function $t = \tau^{1/\alpha}$ is defined.

To represent the function $\tau = t^\alpha$ graphically, choose two real positive numbers ξ and η, mark the points corresponding to them on t- and τ-axis respectively, and draw straight lines parallel to axes through these points.

We obtain two triangles (Fig. 6) with curvilinear boundary, whose areas are given by

$$S_1 = \frac{\xi^{\alpha+1}}{\alpha + 1} \quad \text{and} \quad S_2 = \frac{\eta^{\frac{1}{\alpha}+1}}{\frac{1}{\alpha} + 1}.$$

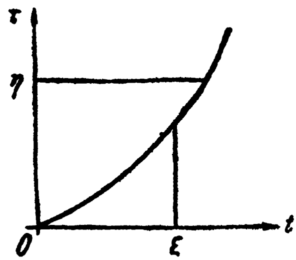

Fig. 6.

On the other hand, evidently, $S_1 + S_2 \geqslant \xi\eta$, where the equality holds only for $\eta = \xi^\alpha$. Thus,

$$\xi\eta \leqslant \frac{\xi^{\alpha+1}}{\alpha + 1} + \frac{\eta^{\frac{1}{\alpha}+1}}{\frac{1}{\alpha} + 1}.$$

Put $\alpha + 1 = p$ and $\frac{1}{\alpha} + 1 = q$. Then p and q are connected by

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{1}$$

† This well-known inequality was established in 1821 for E_n by CAUCHY who, no doubt, used a terminology different from that of the theory of linear spaces. It was extended to integrals in 1859 by V. YA. BUNYAKOVSKII and in 1884 by H. SCHWARZ.

Such numbers p and q are called mutually **conjugate**. Obviously, $q > 1$ for $p > 1$. Thus,

$$\xi\eta \leqslant \frac{\xi^p}{p} + \frac{\eta^q}{q} \tag{2}$$

for any ξ and η and for all pairs of conjugate numbers p and q.

Take a pair of functions: $x(t) \in L_p(0, 1)$ and $y(t) \in L_p(0, 1)$, and put

$$\xi = \frac{|x(t)|}{\left(\int_0^1 |x(t)|^p \, dt\right)^{1/p}} \quad \text{and} \quad \eta = \frac{|y(t)|}{\left(\int_0^1 |y(t)|^q \, dt\right)^{1/q}}.$$

Insert these values in (2), to receive

$$\frac{|x(t)|\,|y(t)|}{\left(\int_0^1 |x(t)|^p \, dt\right)^{1/p} \left(\int_0^1 |y(t)|^q \, dt\right)^{1/q}} \leqslant \frac{|x(t)|^p}{p\int_0^1 |x(t)|^p \, dt} + \frac{|y(t)|^q}{q\int_0^1 |y(t)|^q \, dt}.$$

The functions on the right-hand side are summable. Hence, the left-hand side is also integrable. By integrating, we obtain

$$\frac{\int_0^1 |x(t)|\,|y(t)|\,dt}{\left(\int_0^1 |x(t)|^p \, dt\right)^{1/p} \left(\int_0^1 |y(t)|^q \, dt\right)^{1/q}} \leqslant \frac{1}{p} + \frac{1}{q} = 1,$$

or,
$$\int_0^1 |x(t)\,y(t)|\,dt \leqslant \left(\int_0^1 |x(t)|^p \, dt\right)^{1/p} \left(\int_0^1 |y(t)|^q \, dt\right)^{1/q}. \tag{3}$$

This is **Hölder's inequality for integrals.** It represents the CBS inequality in the particular case when $p = q = 2$.

Now, let $x = \{\xi_i\}$, $y = \{\eta_i\}$ with $x \in l_p$, $y \in l_q$. In Ineq. (2), we put

$$\xi = \frac{|\xi_i|}{\left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p}} \quad \text{and} \quad \eta = \frac{|\eta_i|}{\left(\sum_{i=1}^{\infty} |\eta_i|^q\right)^{1/q}},$$

to receive

$$\frac{|\xi_i|\,|\eta_i|}{\left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |\eta_i|^q\right)^{1/q}} \leqslant \frac{|\xi_i|^p}{p\sum_{i=1}^{\infty} |\xi_i|^p} + \frac{|\eta_i|^q}{q\sum_{i=1}^{\infty} |\eta_i|^q}.$$

Extending summation over all i, we get **Hölder's inequality for sums**

$$\sum_{i=1}^{\infty} |\xi_i \eta_i| \leqslant \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p} \left(\sum_{i=1}^{\infty} |\eta_i|^q\right)^{1/q}, \tag{4}$$

which also becomes the CBS inequality for sums when $p = q = 2$.

I.13. Minkowski's inequality. *Let $x(t)$ and $y(t)$ belong to $L_p(0, 1)$. Then, show that Minkowski's inequality*

$$\left(\int_0^1 |x(t) + y(t)|^p \, dt\right)^{1/p} \leqslant \left(\int_0^1 |x(t)|^p \, dt\right)^{1/p} + \left(\int_0^1 |y(t)|^p \, dt\right)^{1/p} \tag{5}$$

holds.

PROOF. First note that if $z(t) \in L_p(0, 1)$, then $|z(t)|^{p-1} \in L_q(0, 1)$. In fact,

$$(|z(t)|^{p-1})^q = |z(t)|^{(p-1)\frac{p}{p-1}} = |z(t)|^p,$$

which also implies that $(|z(t)|^{p-1})^q$ is an integrable function. Further, consider the integral

$$\int_0^1 |x(t) + y(t)|^p \, dt.$$

On applying twice HÖLDER'S inequality to the functions $|x(t) + y(t)|^{p-1} \in L_q(0, 1)$ and $x(t) \in L_p(0, 1)$, or $y(t) \in L_p(0, 1)$, we get

$$\begin{aligned}
\int_0^1 |x(t) + y(t)|^p \, dt &\leqslant \int_0^1 |x(t) + y(t)|^{p-1} |x(t)| \, dt \\
&\quad + \int_0^1 |x(t) + y(t)|^{p-1} |y(t)| \, dt \\
&\leqslant \left(\int_0^1 |x(t) + y(t)|^{(p-1)q} \, dt\right)^{1/q} \left(\int_0^1 |x(t)|^p \, dt\right)^{1/p} \\
&\quad + \left(\int_0^1 |x(t) + y(t)|^{(p-1)q} \, dt\right)^{1/q} \left(\int_0^1 |y(t)|^p \, dt\right)^{1/p} \\
&= \left(\int_0^1 |x(t) + y(t)|^p \, dt\right)^{1/q} \left[\left(\int_0^1 |x(t)|^p \, dt\right)^{1/p} + \left(\int_0^1 |y(t)|^p \, dt\right)^{1/p}\right].
\end{aligned}$$

Dividing both the sides by

$$\left(\int_0^1 |x(t) + y(t)|^p \, dt\right)^{1/q},$$

and noting that $1 - \frac{1}{q} = \frac{1}{p}$, we get **Minkowski's inequality (5) for integrals.**

Now, let $x = \{\xi_i\} \in l_p$ and $y = \{\eta_i\} \in l_p$. We get

$$\left(\sum_{i=1}^{\infty} |\xi_i + \eta_i|^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |\eta_i|^p\right)^{1/p}. \tag{6}$$

As in the preceding case, we show that $z = \{\zeta_i\} \in l_p$ implies $z' = \{|\zeta_i|^{p-1}\} \in l_p$.

We examine
$$\sum_{i=1}^{\infty} |\xi_i + \eta_i|^p.$$

On applying twice HÖLDER's inequality to the sequences

$$\{|\xi_i + \eta_i|^{p-1}\} \in l_p \text{ and } \{\xi_i\} \in l_p, \text{ correspondingly } \{\eta_i\} \in l_p,$$

we get

$$\sum_{i=1}^{\infty} |\xi_i + \eta_i|^p \leqslant \sum_{i=1}^{\infty} |\xi_i + \eta_i|^{p-1} |\xi_i| + \sum_{i=1}^{\infty} |\xi_i + \eta_i|^{p-1} |\eta_i|$$
$$\leqslant \left(\sum_{i=1}^{\infty} |\xi_i + \eta_i|^{(p-1)q}\right)^{1/q} \left[\left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |\eta_i|^p\right)^{1/p}\right]$$
$$= \left(\sum_{i=1}^{\infty} |\xi_i + \eta_i|^p\right)^{1/q} \left[\left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} |\eta_i|^p\right)^{1/p}\right].$$

Thereupon, dividing both the sides by

$$\left(\sum_{i=1}^{\infty} |\xi_i + \eta_i|^p\right)^{1/p}$$

and noting that $1 - \frac{1}{q} = \frac{1}{p}$, we obtain **Minkowski's inequality (6) for sums.**

Finally, we note that the sign of equality holds in (5) and (6), only if

$$y(t) = kx(t) \text{ with } k > 0$$

and, correspondingly $\eta_i = k\xi_i$ with $k > 0$ for $i = 1, 2, 3, \ldots$ almost everywhere on [0, 1].

All the inequalities obtained easily carry over to functions of several independent variables.

APPENDIX II. CONTINUITY IN THE MEAN OF FUNCTIONS OF THE CLASS $L_p(G)$

LET US denote by $L_p(G)$ the class of functions $\varphi(x, y)$ defined in a plane of the domain G, which are p-th power summable, $p \geqslant 1$. The norm $\|\varphi\| = (\int\int_G |\varphi^p| \, dx \, dy)^{1/p}$ can be introduced in $L_p(G)$. The properties of the class and space $L_p[0, 1]$, HÖLDER's inequality, MINKOWSKI's inequality, completeness etc. directly carry over to $L_p(G)$.

Let us prove the upcoming theorem, which is usually not stated in texts on the theory of functions of a real variable.

THEOREM. *Every function* $\varphi(x, y) \in L_p(G)$ *is continuous in the mean, that is, for any* $\varepsilon > 0$ *there can be found* $\delta > 0$, *such that*

$$\left(\iint_G |\varphi(x + h, y + k) - \varphi(x, y)|^p \, dx \, dy\right)^{1/p} < \varepsilon,$$

whenever $\sqrt{h^2 + k^2} < \delta$.

Moreover, if the point $(x + h, y + k)$ *falls outside of* G, *then*

$$\varphi(x + h, y + k) = 0.$$

PROOF. Let H_ρ be a boundary of the plane consisting of the points of G, apart from it by a distance not exceeding ρ, and let $G_\rho = G \backslash H_\rho$.

Let ρ be so small that mes $(H_\rho) < \eta$, η some pre-assigned positive number (the boundary of G is regarded sufficiently smooth). Since the function $\varphi(x, y) \in L_p(G_\rho)$ is summable on G_ρ, by LUZIN's theorem there is a closed set $F_\eta^1 \subset G_\rho$, such that the function $\varphi(x, y)$ is continuous on F_η^1 and mes $(G_\rho \diagdown F^1) < \eta$. Further, evidently, mes $(G \diagdown F_\eta^1) < 2\eta$.

Let h and k satisfy the condition $\sqrt{h^2 + k^2} < \rho$. For fixed h and k, satisfying this condition, we denote by F_η^2 the point set of the form $(x - h, y - k)$, where (x, y) run through F_η^1. It is clear that $F_\eta^2 \subset G$, the set F_η^2 is closed, and since this is obtained from the displacement of F_η^1 by the vector $\bar{l} = (-h, -k)$, mes $(F_\eta^2) =$ mes (F_η^1). Hence, mes $(G \diagdown F_\eta^2) < 2\eta$, too.

Finally, let $F_\eta = F_\eta^1 \cap F_\eta^2$. Then, F_η is a closed set and $\varphi(x, y)$ is continuous on F_η; consequently, $\varphi(x, y)$ is also uniformly continuous on F_η. Moreover,

$$\text{mes}\,(G \diagdown F_\eta) = \text{mes}\,[(G \diagdown F_\eta^1) \cup (G \diagdown F_\eta^2)]$$

$$\leqslant \text{mes}\,(G \diagdown F_\eta^1 + \text{mes}\,(G \diagdown F_\eta^2) < 4\eta.$$

Now, we assume η to be so small that for $\varepsilon > 0$ given,

$$\left(\iint_E |\varphi(x, y)|^p\right)^{1/p} < \frac{\varepsilon}{4}, \tag{1}$$

whenever $E \subset G$ and mes $(E) < 4\eta$.

Let us evaluate the integral

$$\left(\iint_G |\varphi(x+h, y+k) - \varphi(x, y)|^p \, dx\, dy\right)^{1/p}.$$

We have

$$\left(\iint_G |\varphi(x+h, y+k) - \varphi(x, y)|^p \, dx\, dy\right)^{1/p}$$
$$\leqslant \left(\iint_{F_\eta} |\varphi(x+h, y+k) - \varphi(x, y)|^p \, dx\, dy\right)^{1/p}$$
$$+ \left(\iint_{G\setminus F_\eta} |\varphi(x+h, y+k)|^p \, dx\, dy\right)^{1/p}$$
$$+ \left(\iint_{G\setminus F_\eta} |\varphi(x, y)|^p \, dx\, dy\right)^{1/p}.$$

By (1),

$$\left(\iint_{G\setminus F_\eta} |\varphi(x+h, y+k)|^p \, dx\, dy\right)^{1/p} + \left(\iint_{G\setminus F_\eta} |\varphi(x, y)|^p \, dx\, dy\right)^{1/p}$$
$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \tag{2}$$

Further, let us take $\delta < \rho$ to be so small that, for $\sqrt{h^2 + k^2} < \delta$,

$$|\varphi(x+h, y+k) - \varphi(x, y)| < \varepsilon/2\,[\text{mes}\,(G)]^{1/p}$$

uniformly on F_η. Then,

$$\left(\iint_{F_\eta} |\varphi(x+h, y+k) - \varphi(x, y)|^p \, dx\, dy\right)^{1/p} < \frac{\varepsilon}{2}. \tag{3}$$

From (2) and (3) it follows that

$$\iint_G |\varphi(x+h, y+k) - \varphi(x, y)|^p \, dx\, dy\Big)^{1/p} < \varepsilon,$$

whenever $\sqrt{h^2 + k^2} < \delta$. ■

APPENDIX III. BOLYA-BROUWER THEOREM

IT IS intended to prove here the BOLYA-BROUWER theorem on the **existence** of a fixed point for a continuous mapping of a closed convex body of an n-dimensional Euclidean space into itself. This theorem is extensively used in functional analysis for establishing the existence of a solution of an operator equation. Since all the closed convex bodies of an n-dimensional Euclidean space are homomorphic to each other, it is sufficient to prove the BOLYA-BROUWER theorem for continuous mapping of an n-dimensionel simplex into itself.† We shall draw heavily on the proofs of this theorem, due to B. KNASTER, K. KURATOWSKI and S. MAZURKIEWICZ.

Consider an n-dimensional **simplex** s_0 and denote by the points $x_0, x_1, \ldots, x_n$ the **vertices** of the simplex. Denote any k-dimensional **face** (or **edge**) of the simplex $(0 \leqslant k \leqslant n)$ by $(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$, where the x_{i_m}, $m = 0, 1, \ldots, k$, is the union of all vertices of this face. Let the simplex s_0 be simplicially divided into the simplexes s. To each vertex x of the simplexes s we assign an integer $\varphi(x)$ as follows.

Consider a face of the least dimension of the basic simplex s_0, which contains the point x. Let this face be $(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$. Set $\varphi(x)$ equal to one of the indices $i_0, i_1, \ldots, i_k$. For example, if x coincides with the vertex x_i of s_0, then $\varphi(x) = i$; if x lies on a one-dimensional face (x_i, x_j) coinciding with none of its vertices, then $\varphi(x)$ can be put equal to the number i or j, and so on and so forth. Finally, if x lies inside of s_0 (not belonging to any of the k-dimensional faces $k = 0, 1, \ldots, n - 1$), then $\varphi(x)$ can be equated to any of the $n + 1$ numbers $0, 1, 2, \ldots, n$. $\varphi(x)$ is called the **normal function of vertices.**

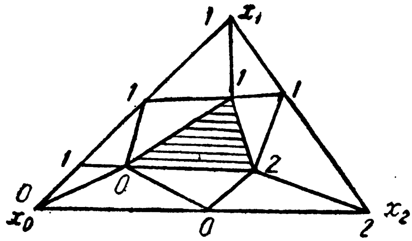

Fig. 7.

The simplex s of the given subdivisions is said to be **representative,** if $n + 1$ distinct numbers $0, 1, 2, \ldots, n$ are associated with its vertices.

Figure 7 depicts subdivisions of a two-dimensional simplex, the association of the vertices of the simplex with the subdivisions numbered 0, 1, 2, being as indicated above. The shaded triangle is the representative simplex.

† For the topological notion arising here see [30].

LEMMA 1. (Due to E. SPERNER). *Howsoever simplicially a simplex s_0 is divided and howsoever the normal function of vertices $\varphi(x)$ is defined on the vertices of the simplexes of this subdivision, there always exist representative simplexes and, in addition, their number is odd.*

The proof is derived by induction. For $n = 0$, when the simplex shrinks to a point, the theorem is trivial. Assuming the theorem to be true for an $(n - 1)$-dimensional simplex, it is to be established for an n-dimensional simplex.

Given simplicial subdivisions of an n-dimensional simplex s_0 and let the normal function of vertices $\varphi(x)$ be defined on the vertices x of the s-subdivisions of the simplex. We shall designate the $(n - 1)$-dimensional face of the subdivisions of simplex as the representative $(n - 1)$-dimensional face, on whose n vertices the function $\psi(x)$ takes the values $0, 1, \ldots, n - 1$. The number of $(n - 1)$-dimensional representative faces of the s-subdivisions of the simplex will be denoted by $\alpha(s)$.

The following three cases are possible.

(i) The function $\varphi(x)$ on the vertices of the simplex s_1 takes all the $n + 1$ values $0, 1, 2, \ldots, n$, that is, s_1 is a representative simplex and it contains a unique representative $(n - 1)$-dimensional face, just opposite to x, satisfying $\varphi(x) = n$. Thereupon $\alpha(s_1) = 1$, and

$$\sum \alpha(s_1) = \rho_n, \tag{1}$$

where ρ_n is the number of representative n-dimensional simplexes; the sum on the left-hand side of (1) is taken over all thc representative simplexes.

(ii) On the vertices of a non-representative simplex s_0 the function $\varphi(x)$ takes n value $0, 1, 2, \ldots, n - 1$. One of these values must be taken by it twice. Consequently, s_2 has two representative $(n - 1)$-dimensional faces, $\alpha(s_2) = 2$.

(iii) On the vertices of the simplex s_3, $\varphi(x)$ assumes one of the values $0, 1, 2, \ldots, n - 1$; consequently, $\alpha(s_3) = 0$. Thereupon,

$$\sum \alpha(s) \equiv \sum \alpha(s_1) \qquad (\text{mod } 2). \tag{2}$$

The sum on the left-hand side is taken over all n-dimensional subdivisions of the simplex s, and on the right-hand side over all representatives of n-dimensional subdivisions of the simplex s_1.

According to another evaluation of $(n - 1)$-dimensional representative faces, the following two cases are possible.

(*a*) The representative face falls inside the basic simplex s_0; it is common to two simplexes of the subdivision and in the sum $\Sigma\, \alpha(s)$ it is counted twice.

(*b*) The representative face falls on the boundary of s_0. From the definition of such a face and the function $\varphi(x)$ it follows that it can exist only on an $(n - 1)$-dimensional face $(x_0, x_1, \ldots, x_{n-1})$ of the basic simplex. Denote by ρ_{n-1} the number of $(n - 1)$-dimensional representative faces lying on $(x_0, x_1, \ldots, x_{n-1})$.

We have
$$\sum \alpha(s) \equiv \rho_{n-1} \pmod 2. \tag{3}$$

From (1), (2) and (3) it follows that
$$\rho_n \equiv \rho_{n-1} \pmod 2.$$

However, for an $(n - 1)$-dimensional simplex, ρ_{n-1} is odd by the hypotheses of the lemma. Consequently, ρ_n is odd and, hence, distinct fiom zero.

The lemma is completely proved.

LEMMA 2. *Let a simplex s_0 cover $n + 1$ closed sets $F_0, F_1, \ldots, F_n$, such that each of its k-dimensional faces $(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$ covers the sets F_{i_0}, $F_{i_1}, \ldots, F_{i_k}$. Subject to this condition, there exists in s_0 a point, which belongs to all the $n + 1$ sets F_i, $i = 0, 1, \ldots, n$.*

Divide s_0 simplicially and define the function $\varphi(x)$ on the vertices x of the simplexes of the subdivision: consider the face $(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$, $0 \leqslant k \leqslant n$, of the least dimension, inclusive of the point x. This point falls in one of the sets $F_{i_0}, F_{i_1}, \ldots, F_{i_k}$, which covers $(x_{i_0}, x_{i_1}, \ldots, x_{i_h})$. Take $\varphi(x)$ to be equal to the index of such of the sets as contains x (or to each of such indices, if the point falls in some of the sets $F_{i_0}, F_{i_1}, \ldots, F_{i_k}$). It is plain tbat $\varphi(x)$ is a normal function of vertices.

By SPERNER's lemma, among the simplexes of the given subdivision, there must exit a representative simplex s_1. On its vertex x, the function $\varphi(x)$ takes all $n + 1$ values $0, 1, \ldots, n$, that is, the vertices of s_1 belong to $n + 1$ distinct sets F_i.

Let us carry out simplicial subdivision of s_0 into still smaller simplexes. Assume that the diameter of the m-th simplex of the subdivision does not exceed δ_m, where $\delta_m \to 0$ as $m \to \infty$. Consider a sequence of representative subdivisions $s_1, s_2, \ldots s_m \ldots$ 1st, 2nd, $\ldots$, mth $\ldots$ of the simplex. By the compactness of s_0, the set of vertices of the simplex $\{s_m\}$ has the limit point x^*. Choose an arbitrary $\delta > 0$ and consider the simplex s_m, satisfying $\delta_m < \delta/2$. In a sphere of radius $\delta/2$ with centre at x^*, there falls at least one vertex of one of the simplexes s_m and, consequently, in a sphere of radius δ around x^* all $n + 1$ vertices of such a simplex fall. Since the vertices of s_m belong to $n + 1$ distinct sets $F_0, F_1, \ldots, F_n$, hence in any δ-neighbourhood of x^* there exists the set F_i of all points, $i = 0, 1, \ldots, n$. Consequently, x^* is the limit point for all the F_i, and since F_i is closed, x^* belongs to all the F_i, $i = 0, 1, \ldots, n$.

THEOREM (BOLYA-BROUWER). *For every continuous mapping $f(x)$ of an n-dimensional simplex s into itself, there exists a fixed point of this mapping, that is, a point $x^* \in s$, such that*
$$f(x^*) = x^*.$$

Introduce on s the *barycentric coordinates*

$$\mu_0, \mu_1, \ldots, \mu_n, \quad \sum_{i=0}^{n} \mu_i = 1.$$

For the point s, all the $\mu_i \geqslant 0$. Under the mapping f, suppose that the point $x(\mu_0, \mu_1, \ldots, \mu_n) \in s$ carries over into the point $y(\nu_0, \nu_1, \ldots, \nu_n) \in s$, $y = f(x)$. Again $\sum_{i=0}^{n} \nu_i = 1, \nu_i \geqslant 0, i = 1, 2, \ldots, n$. Let the point $x(\mu_0, \mu_1, \ldots, \mu_n)$ lie on the face $(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$, $0 \leqslant k \leqslant n$. The coordinates μ_j of the point x for $j \neq i_0, i_1, \ldots, i_k$ are equal to zero.

Since

$$1 = \mu_{i_0} + \mu_{i_1} + \ldots + \mu_{i_k} = \sum_{i=0}^{n} \nu_i \geqslant \nu_{i_0} + \nu_{i_1} + \ldots + \nu_{i_k},$$

it is not possible to satisfy simultaneously the inequalities

$$\mu_{i_0} < \nu_{i_0}, \mu_{i_1} < \nu_{i_1}, \ldots, \mu_{i_k} < \nu_{i_k},$$

and we have $\mu_{i_r} \geqslant \nu_{i_r}$ for at least one of these coordinates. Hence, if F_i is the set of points, whose coordinates μ_i do not increase under the mapping f, then every point x of the face $(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$ covers one of the sets $F_{i_0}, F_{i_1}, \ldots, F_{i_k}$.

The set F_i satisfies all hypotheses of the preceding lemma.† Hence, there exists on s the point $x^*(\mu_0^*, \mu_1^*, \ldots, \mu_n^*)$, which belongs to all of these sets. Under the mapping f, none of the coordinates increases, and if $f(x^*) = y^*(\nu_0^*, \nu_1^*, \ldots, \nu_n^*)$, then

$$\mu_i^* \geqslant \nu_i^*, \qquad i = 0, 1, \ldots, n. \tag{4}$$

From (4) and the properties of barycentric coordinates, it follows that

$$1 = \sum_{i=0}^{n} \mu_i^* \geqslant \sum_{i=0}^{n} \nu_i^* = 1. \tag{5}$$

Now, (4) and (5) imply that

$$\mu_i^* = \nu_i^*, \qquad i = 0, 1, \ldots, n,$$

that is, $f(x^*) = x^*$ and, consequently, x^* is a fixed point of the mapping.

The Bolya-Brouwer theorem is proved.

Corollary. *Under a continuous mapping of a bounded closed convex body S of an n-dimensional Banach space E into itself, there exists a fixed point.*

Let $e_1, e_2, \ldots, e_n$ be a basis in E. Set the point $\tilde{x} = \{\xi_1, \xi_2, \ldots, \xi_n\} \in E_n$

† The continuity of f implies F_i to be closed.

in correspondence to the element $x = \xi_1 e_1 + \xi_2 e_2 + \ldots + \xi_n e_n$, where E_n is the n-dimensional Euclidean space. This correspondence is isometric and isomorphic, carrying a closed convex set $S \subset E$ into a closed convex set $\tilde{S} \subset E_n$. Let f be a continuous mapping of S into itself.

Thus, $\tilde{f} = \varphi f \varphi^{-1}$ is a continuous mapping of $\tilde{S}$ into itself. By the BOLYA-BROUWER theorem, there exists a fixed point $\tilde{x}^*$ of this mapping, defined by

$$\varphi f \varphi^{-1}(\tilde{x}^*) = \tilde{x}^*.$$

However, then,

$$f \varphi^{-1}(\tilde{x}^*) = \varphi^{-1}(\tilde{x}^*),$$

and $x^* = \varphi^{-1}(\tilde{x}^*)$ is a fixed point for the mapping f.

APPENDIX IV. TWO DEFINITIONS OF THE n-th DERIVATIVE OF A FUNCTION OF REAL VARIABLES

THERE ARE two definitions for the n-th derivative of a function $x(t)$ at the point t.

(i) We introduce the notation

$$\delta^n_{\Delta t} x(t) = \sum_{k=0}^{n} (-1)^{n-k} C^k_n \, x\left(t + \left(k - \frac{n}{2}\right) \Delta t\right)$$

and call $\delta^n_{\Delta t}\, x(t)$ a **central difference of n-th order** of $x(t)$ at the point t. Then, we put

$$x^{(n)}(t) = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^n} \delta^n_{\Delta t}\, x(t)$$

under the assumption that this limit exists.

If the right-hand side of this equality tends uniformly to $x^{(n)}(t)$ on the interval $a \leqslant t \leqslant b$, then $x^{(n)}(t)$ is called **uniform difference derivative of order n.**

(ii) We now define the n-th derivative of the function $x(t)$ and denote it by $x^{(n)}(t)_0$. $x^{(n)}(t)_0$ is defined through successive n-times differentiations of the function $x(t)$ under the assumption that all preceding derivatives $x'(t)_0$, $x''(t)_0, \ldots, x^{(n-1)}(t)_0$ are defined in a neighbourhood of the point t.

Let $x^{(n)}(t)_0$ be defined and continuous on the interval $a \leqslant t \leqslant b$. Then, $x^{(n)}(t)$ also exists, and

$$x^{(n)}(t) = x^{(n)}(t)_0.$$

PROOF. As is directly observed,

$$\frac{1}{(\Delta t)^n} \delta^n_{\Delta t}\, x(t) = x^{(n)}(t + \theta \Delta t)_0, \qquad -\frac{n}{2} < \theta < \frac{n}{2}.$$

Since the right-hand side converges uniformly to $x^{(n)}(t)_0$ as $\Delta t \to 0$, it follows that

$$x^{(n)}(t) = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^n} \delta^n_{\Delta t}\, x(t) = x^{(n)}(t)_0 .$$

It is possible to prove the converse also: *The existence of the continuous uniform difference derivative $x^{(n)}(t)$ on some interval implies the existence of the n-th derivative $x^{(n)}(t)_0$ there, and*

$$x^{(n)}(t)_0 = x^{(n)}(t).$$

We prove this for $n = 2$. Let the function $x(t)$ have a continuous second difference derivative $x''(t)$ on the interval $[0, 1]$. We introduce a function $y(t)$

$$y(t) = \int_0^t \int_0^{\tau_1} x''(\tau)\, d\tau\, d\tau_1.$$

The second derivative $y''(t)_0$ is equal to the integrand $x''(t)$. But since $y''(t)_0 = x''(t)$ is continuous, it coincides with the second difference derivative: $y''(t)_0 = y''(t)$. Thus $[y(t) - x(t)]'' = 0$. We now show that the difference $\alpha(t) = y(t) - x(t)$ can be only a linear function: $x(t) = y(t) + a + bt$. Since the function $y(t) + a + bt$ has a second derivative $y''(t)_0 = x''(t)_0$, it follows that $y''(t)_0 = x''(t)$.

Now, let the identity $\alpha''(t) = 0$ hold on the interval $[0, 1]$. Set $\alpha_1(t) = \alpha(t) - \{\alpha(0) + [\alpha(1) - \alpha(0)]\, t\}$, to receive $\alpha_1(0) = \alpha_1(1) = 0$ and $\alpha_1''(t) \equiv 0$.

Let ε be an arbitrary positive number. Consider the function

$$\beta(t) = \alpha_1(t) + \varepsilon t^2.$$

Since $(\varepsilon t^2)'' = 2\varepsilon > 0$, hence $\beta''(t) = 2\varepsilon > 0$, also $\beta(t) \leqslant \varepsilon$ for $0 \leqslant t \leqslant 1$. In fact, were $\beta(t) > \varepsilon$ for some $t \in [0, 1]$, then the maximum of $\beta(t)$ would be greater than ε. But since $\beta(0) = 0$ and $\beta(1) = \varepsilon$, this maximum is attained at an interior point t_0 of $[0, 1]$. The second central difference at t_0 is

$$\Delta^2_{\Delta t}\, \beta(t_0) = \beta(t_0 + \Delta t) - 2\beta(t_0) + \beta(t_0 - \Delta t) \leqslant 0,$$

because $\beta(t_0) \geqslant \beta(t_0 \pm \Delta t)$. Hence $\beta''(t_0) = \lim\limits_{\Delta t \to 0} \dfrac{1}{(\Delta t)^2}\, \Delta^2_{\Delta t}\, \beta(t_0) \leqslant 0$, a contradiction to the assumption $\beta''(t_0) > \varepsilon > 0$. Consequently, $\beta(t) \leqslant \varepsilon$, $0 \leqslant t \leqslant 1$. But, then,

$$\alpha_1(t) = \beta(t) - \varepsilon t^2 \leqslant \beta(t) \leqslant \varepsilon,$$

for $0 \leqslant t \leqslant 1$. The inequality $\alpha_1(t) \geqslant -\varepsilon$ also can be similarly proved. Thus, for any $\varepsilon > 0$,

$$-\varepsilon \leqslant \alpha_1(t) \leqslant \varepsilon.$$

Hence $\alpha_1(t) \equiv 0$, which means that $\alpha(t) = \alpha(0) + [\alpha(1) - \alpha(0)]\, t = a + bt$, as was required to prove.

Finally, we note that for a function $x(t_1, t_2, \ldots, t_n)$ of n real variables, the equality

$$\Delta^{(n)}_{t_1, t_2, \ldots, t_n;\, \Delta t}\, x(t_1, t_2, \ldots, t_n)$$
$$= (\Delta t)^n \left\{\frac{\partial}{\partial t_{k_1}} \left[\cdots \frac{\partial}{\partial t_{k_n}} x(t_1 + \theta_1 \Delta t, \ldots, t_n + \theta_n \Delta t) \cdots \right]\right\} \tag{1}$$

holds, where

$$\Delta^{(n)}_{t_1, t_2, \ldots, t_n;\, \Delta t}\, x(t_1, \ldots, t_n) = \sum_{i_1, i_2, \ldots, i_k} (-1)^{n-k}\, x(\tau_1, \tau_2, \ldots, \tau_n),$$

$\tau_i = t_i + \Delta t$ for $i = i_1, i_2, \ldots, i_k$ and $\tau_i = t_i$ for the remaining i and sum is taken over all subsets $(i_1, i_2, \ldots, i_k)$, $0 \leqslant i_1 \leqslant i_2 \ldots \leqslant i_k \leqslant n$ of the set $(1, 2, \ldots, n)$. Here as also in the sequel, θ_i are the numbers between 0 and 1.

We shall prove equality (1) by induction. For $n = 1$, it leads to the mean-value theorem. Assume that (1) is true for partial differences of order $n - 1$. Then,

$$\Delta^{(n)}_{t_1, t_2, \ldots, t_n; \Delta t}\, x(t_1, t_2, \ldots, t_n)$$

$$= \Delta^{n-1}_{t_1, \ldots, t_{k_1}-1, t_{k_1}+1, \ldots, t_n; \Delta t}\, x(t_1, \ldots, t_{k_1} + \Delta t, \ldots, t_n)$$

$$- \Delta^{n-1}_{t_1, \ldots, t_{k_1}-1, t_{k_1}+1, \ldots, t_n; \Delta t}\, x(t_1, \ldots, t_{k_1}, \ldots, t_n)$$

$$= \Delta t \frac{\partial}{\partial t_{k_1}} [\Delta^{n-1}_{t_1, \ldots, t_{k_2}-1, t_{k_1}+1, \ldots, t_n; \Delta t}$$

$$\times\, x(t_1, \ldots, t_{k_1} + \theta_{k_1} \Delta t, \ldots, t_n)]. \qquad (2)$$

(The last equality is obtained by applying the mean-value theorem). However, by induction, we get

$$\Delta^{n-1}_{t_1, \ldots, t_{k_1}-1, t_{k_1}+1, \ldots, t_n; \Delta t}\, x(t_1, \ldots, t_{k_1} + \theta_{k_1} \Delta t, \ldots, t_n)$$

$$= \frac{\partial}{\partial t_{k_2}} \left\{ \ldots \frac{\partial}{\partial t_{k_n}} x(t_1 + \theta_1 \Delta t, \ldots, t_n + \theta_n \Delta t) \ldots \right\} (\Delta t)^{n-1}.$$

Thereupon, (2) implies (1).

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*In Russian.

INDEX OF SYMBOLS

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TABLE OF BASIC PROPERTIES OF IMPORTANT FUNCTIONAL SPACES

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