Institute for Applied Mathematics
Summer 2015
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"Stochastic Processes"

Exercise sheet 10

Hand in Tue 23.6.2015 during lecture break

Exercise 1 (Discrete-time martingales) *[5 Pts.]*

A discrete-time stochastic process $(X_n)_{n\in\mathbb{N}}$ is a martingale with respect to a filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$, if

(i) $(X_n)_n$ is adapted to $(\mathcal{F}_n)_n$ (i.e. X_n is $\mathcal{F}_n$-measurable),

(ii) $\mathbb{E}(|X_n|) < \infty$ for every $n \in \mathbb{N}$,

(iii) $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ for every $n \in \mathbb{N}$ (martingale property).

Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_n) = 0$. Define the random walk $S_n := X_0 + \cdots + X_n$ and let $\mathcal{F}_n := \sigma(X_0, \ldots, X_n)$ be its natural filtration.

a) Show that S_n is a martingale with respect to $\mathcal{F}_n$.

b) Assume additionally that every X_n has finite variance σ^2. Show that $M_n := S_n^2 - n\sigma^2$ is a martingale with respect to $\mathcal{F}_n$.

c) Now let $\tilde{X}_n$ be a sequence of i.i.d. random variables such that $\mathbb{E}(\tilde{X}_n) = 1$ for every $n \in \mathbb{N}$. Show that $M_0 := 1, M_n := \prod_{k=1}^n \tilde{X}_k$ is a martingale with respect to $\mathcal{F}_n := \sigma(\tilde{X}_1, \ldots, \tilde{X}_n)$.

Exercise 2 (Brownian motion) *[5 Pts.]*

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion.

a) Let $Z := \sup_{t\geq 0} B_t$. Show that for every $\lambda > 0$ the random variables λZ and Z have the same distribution and deduce that $\mathbb{P}(Z \in \{0, \infty\}) = 1$.

b) Show that $\mathbb{P}(Z = 0) \leq \mathbb{P}(B_1 \leq 0)\mathbb{P}(\sup_{t\geq 0}(B_{1+t} - B_1) = 0)$ and conclude that $\mathbb{P}(Z = 0) = 0$.

c) Show that $\mathbb{P}(\sup_{t\geq 0} B_t = +\infty, \inf_{t\geq 0} B_t = -\infty) = 1$

Exercise 3 (Another Brownian motion) *[5 Pts.]*

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and let Z be an independent random variable such that

$$\mathbb{P}(Z=1)=\mathbb{P}(Z=-1)=\frac{1}{2},$$

and let $t_* \in [0,\infty)$. Define $(W_t)_{t\geq 0}$ by

$$W_t := B_t \mathbb{1}_{\{t<t_*\}} + (B_{t_*} + Z(B_t - B_{t_*}))\mathbb{1}_{\{t\geq t_*\}}.$$

Show that W_t is a standard Brownian motion.

Exercise 4 (Brownian bridge) *[5 Pts.]*

A stochastic process $(X_t)_{t\in[0,1]}$ is called a Brownian bridge, if the following four properties hold:

(i) $X_0 = X_1 = 0$

(ii) For any $t_0 < t_1 < \cdots < t_n$ the random vector $(X_{t_0}, X_{t_1}, \ldots, X_{t_n})$ has a multivariate normal distribution with mean zero (i.e. X_t is a centered Gaussian process)

(iii) $\mathrm{Cov}(X_s, X_t) = s \wedge t - st$

(iv) The trajectories $t \mapsto X_t(\omega)$ are continuous for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and $(X_t)_{t\in[0,1]}$ be a Brownian bridge.

a) Show that $\tilde{X}_t := B_t - tB_1, t \in [0,1]$ is a Brownian bridge.

b) Let Z be a $\mathcal{N}(0,1)$-distributed random variable. Show that $\tilde{B}_t := X_t + tZ$ is a Brownian motion for $t \in [0,1]$.

c) Show that $\tilde{B}_t := (t+1)X_{\frac{t}{t+1}}$ is a Brownian motion for $t \in [0,\infty)$.

d) Given a standard Brownian motion $(B_t)_{t\geq 0}$, find a similar expression as in c) for the Brownian bridge.