Solving nonlinear equations

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Convergence rate

Absolute error ε_k:

$$\varepsilon_k = |x_k - x_*|, \tag{1}$$

$$\lim_{k \to \infty} \frac{\varepsilon_{k+1}}{\varepsilon_k^r} = C \tag{2}$$

r – order of convergence (a.k.a. convergence rate)
C – rate of convergence, $C > 0$

Convergence

linear	$r = 1,\ 0 < C < 1$
superlinear	$r > 1$
quadratic	$r = 2$
cubic	$r = 3$

Convergence rate	Digits gained per iteration
linear	constant
superlinear	increasing
quadratic	doubled

Order of convergence

Method	Convergence	Order of convergence
bisection method	linear	1
Newton method	quadratic	2
secant method	superlinear	$\frac{1+\sqrt{5}}{2} \approx 1.618$
inverse quadratic interpolation	superlinear	1.839

Empirical convergence rate

$$\frac{\varepsilon_{k+1}}{\varepsilon_k^r} = \frac{\varepsilon_k}{\varepsilon_{k-1}^r} \tag{3}$$

$$\frac{\varepsilon_{k+1}}{\varepsilon_k}\left(\frac{\varepsilon_{k-1}}{\varepsilon_k}\right)^r = 1 \tag{4}$$

$$\ln\left(\frac{\varepsilon_{k-1}}{\varepsilon_k}\right)^r = \ln\frac{\varepsilon_k}{\varepsilon_{k+1}} \tag{5}$$

$$r = \frac{\ln\frac{\varepsilon_k}{\varepsilon_{k+1}}}{\ln\frac{\varepsilon_{k-1}}{\varepsilon_k}} \tag{6}$$

Iteration schemes

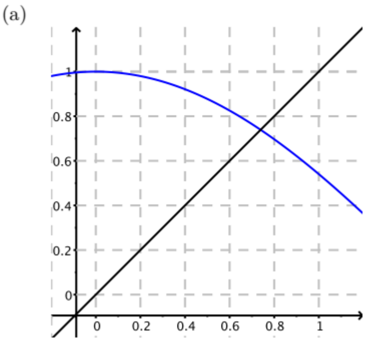
$$f(x) = 0 \tag{7}$$

$$x = g(x) \tag{8}$$

$$x_{k+1} = g(x_k) \tag{9}$$

Fixed-point iteration

Figure 2.2.1: Finding the fixed point of $\cos(x)$.



Newton's method

$$f(x) = 0 \tag{10}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{11}$$

Sufficient conditions for convergence

(1) $f \in C^2[a, b]$

- f, f', f'' continuous in $[a, b]$

(2) $f(a) \cdot f(b) < 0$

(3) f' and f'' do not change sign in $[a, b]$

(4) x_0 satisfies $f(x_0) \cdot f''(x_0) > 0$, where $x_0 = f(a)$ or $x_0 = f(b)$

Stopping critera

(1)

$$|x_k - x_{k-1}| < \epsilon$$

(2)

$$\frac{|x_k - x_{k-1}|}{|x_k|} < \epsilon$$

(3)

$$|f(x_k) < \epsilon|$$

(4) number of iterations

Implementation details

Bisection method

$$x_m = \frac{a+b}{2}$$

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Bisection method

$$x_m = \frac{a+b}{2}$$

$$x_m = a + \frac{b-a}{2}$$

Necessary condition

$$f(a)f(b) < 0$$

$$f(a)f(b)/|f(b)| < 0$$

Housholder's method

$$f(x) = 0 \tag{12}$$

$$x_{k+1} = x_k + d\,\frac{(1/f)^{(d-1)}(x_k)}{(1/f)^{(d)}(x_k)} \tag{13}$$

$d = 1$ – Netwon's method

$d = 2$ – Halley's method

References

[1] http://heath.cs.illinois.edu/scicomp/notes/cs450_chapt05.pdf