# Error analysis

Marcin Kuta

# Accuracy and precision

**Accuracy** refers to the error of an approximate quantity

**Precision** is the accuracy with which the basic arithmetic operations are performed

- Accuracy and precision are the same for scalar computation $c = a * b$
- Accuracy can be much worse than precision in the solution of a linear system of equations
- Accuracy is not limited by precision. Arithmetic of a given precision can be used to simulate arithmetic of arbitrarily high precision.

# Computational error

**Numerical error** Błąd numeryczny

**Truncation error** Błąd metody

# Errors

$$\hat{f}(x) = \hat{y}$$

Forward error:

$$\Delta y = \hat{y} - y \tag{1}$$

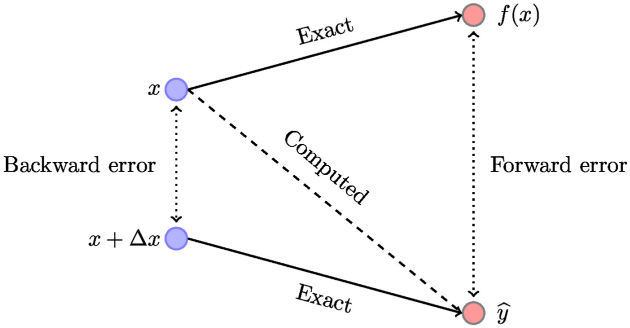
$$\hat{f}(x) = f(\hat{x})$$

Backward error:

$$\Delta x = \hat{x} - x \tag{2}$$

forward error $\lesssim$ condition number $\times$ backward error

# Errors




Source: [4]

# Conditioning

$$\text{cond} = \frac{\Delta y / y}{\Delta x / x} \tag{3}$$

$$\text{cond} = \frac{x f'(x)}{f(x)} \tag{4}$$

# Stability

- Forward stability

  Forward error and bacward error are of similar magnitude
  Stabilność numeryczna

- Backward stability

  Specific to problems where numerical errors are the dominant form of errors
  Poprawność numeryczna

# Errors in numerical derivatives

$$f(x+h) = f(x) + f'(x) \cdot h + f''(\theta) \cdot \frac{h^2}{2}, \quad \theta \in [x, x+h] \qquad (5)$$

$$\frac{f(x+h) - f(x)}{h} = f'(x) + f''(\theta) \cdot \frac{h}{2}, \quad \theta \in [x, x+h] \qquad (6)$$

$$\underbrace{E(h)}_{\substack{\text{computational}\\ \text{error}}} \le \underbrace{\frac{Mh}{2}}_{\substack{\text{truncation}\\ \text{error}\\ \\ =\text{błąd}\\ \text{metody}}} + \underbrace{\frac{2\epsilon}{h}}_{\substack{\text{rounding}\\ \text{error}\\ \\ =\text{błąd}\\ \text{numeryczny}}} \qquad (7)$$

$$h_{\text{min}} = 2\sqrt{\epsilon/M} \qquad (8)$$

# Errors in numerical derivatives

$$f(x+h) = f(x) + f'(x) \cdot h + f''(x) \cdot \frac{h^2}{2} + f''''(\theta_1) \cdot \frac{h^3}{6}, \quad \theta_1 \in [x, x+h]$$

$$f(x-h) = f(x) - f'(x) \cdot h + f''(x) \cdot \frac{h^2}{2} - f''''(\theta_2) \cdot \frac{h^3}{6}, \quad \theta_2 \in [x-h, x]$$

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{f'''(\theta_1) + f'''(\theta_2)}{2} \cdot \frac{h^2}{6} \tag{9}$$

$$|f(t)| \leq M \text{ for all } t \in [x-h, x+h] \tag{10}$$

# Errors in numerical derivatives

$$\underbrace{E(h)}_{\substack{\text{computational}\\ \text{error}}} \leq \underbrace{\frac{Mh^2}{6}}_{\substack{\text{truncation}\\ \text{error}\\ \\ =\text{błąd}\\ \text{metody}}} + \underbrace{\frac{\epsilon}{h}}_{\substack{\text{rounding}\\ \text{error}\\ \\ =\text{błąd}\\ \text{numeryczny}}} \tag{11}$$

$$h_{\text{min}} = \sqrt[3]{3\epsilon/M} \tag{12}$$

# References I


[1] Michael T. Heath,
Scientific Computing. An Introductory Survey, 2nd Edition,
Chapter 1: Scientific Computing
2002

[2] Michael T. Heath,
Chapter 1: Scientific Computing
http://heath.cs.illinois.edu/scicomp/notes/cs450_chapt01.pdf

[3] https://pythonnumericalmethods.berkeley.edu/notebooks/chapter09.00-Representation-of-Numbers.html

[4] Nicholas Higham
What Is Backward Error?

# References II


[5] Nicholas Higham
Accuracy and Stability of Numerical Algorithms
2002

[6] https://pl.wikipedia.org/wiki/IEEE_754