

# Approximation

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# Approximation

Basis functions:  $\{\phi_0(t), \dots, \phi_n(t)\}$

$$p(x) = \sum_{j=0}^n c_j \phi_j(x) \quad (1)$$

$x = [x_0, \dots, x_m]$   $m > n$  (often  $m \gg n$ )

$$p(x_j) \approx f(x_j) \quad (2)$$

Approximation error  $E$ :

	Discrete	Continuous
$L^\infty$	$\min_{p \in \mathbb{P}_n} \max_{0 \leq j \leq m}  f(x_j) - p(x_j) $	$\min_{p \in \mathbb{P}_n} \max_{x \in [a, b]}  f(x) - p(x) $
$L^2$	$\min_{p \in \mathbb{P}_n} \sum_{j=0}^m  f(x_j) - p(x_j) ^2$	$\min_{p \in \mathbb{P}_n} \int_a^b  f(x) - p(x) ^2$

## Discrete least squares approximation

$$A = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_m & x_m^2 & \dots & x_m^n \end{bmatrix}, c = \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix}, y = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \dots \\ f(x_m) \end{bmatrix} \quad (3)$$

Normal equation:

$$A^T A c = A^T y \quad (4)$$

$$c = (A^T A)^{-1} A^T y = A^+ y \quad (5)$$

# Discrete least squares approximation

$$p(x) = \sum_{j=0}^n c_j x^j \quad (6)$$

$$S_k = \sum_{i=0}^m x_i^k, \quad k = 0, 1, \dots, 2n \quad (7)$$

$$T_k = \sum_{i=0}^m x_i^k y_i, \quad k = 0, 1, \dots, n \quad (8)$$

$$\begin{bmatrix} S_0 & S_1 & \dots & S_n \\ S_1 & S_2 & \dots & S_{n+1} \\ \dots & \dots & \dots & \dots \\ S_n & S_{n+1} & \dots & S_{2n} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix} = \begin{bmatrix} T_0 \\ T_1 \\ \dots \\ T_n \end{bmatrix} \quad (9)$$

## Continuous least squares approximation

$p_{\star}(t)$  to be optimal requires approximation error  $E = E(c_0, \dots, c_n)$  to satisfy

$$\frac{\partial E}{\partial c_0} = \dots = \frac{\partial E}{\partial c_n} = 0 \quad (10)$$

# Continuous least squares approximation

$$\langle f, g \rangle = \int_{-1}^1 w(x) f(x) g(x) dx, \quad (11)$$

$$\begin{bmatrix} \langle \phi_0, \phi_0 \rangle & \dots & \langle \phi_0, \phi_n \rangle \\ \langle \phi_1, \phi_0 \rangle & \dots & \langle \phi_1, \phi_n \rangle \\ \dots & \dots & \dots \\ \langle \phi_n, \phi_0 \rangle & \dots & \langle \phi_n, \phi_n \rangle \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle f, \phi_0 \rangle \\ \langle f, \phi_1 \rangle \\ \dots \\ \langle f, \phi_n \rangle \end{bmatrix} \quad (12)$$

# Continuous least squares approximation

For orthogonal polynomials  $\phi_0, \dots, \phi_n$  we have  $\langle \phi_i, \phi_j \rangle = 0$  for  $i \neq j$ .

$$\begin{bmatrix} \langle \phi_0, \phi_0 \rangle & 0 & \dots & 0 \\ 0 & \langle \phi_1, \phi_1 \rangle & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \langle \phi_n, \phi_n \rangle \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \dots \\ c_n \end{bmatrix} = \begin{bmatrix} \langle f, \phi_0 \rangle \\ \langle f, \phi_1 \rangle \\ \dots \\ \langle f, \phi_n \rangle \end{bmatrix} \quad (13)$$

$$c_k = \frac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} \quad (14)$$

$$p_\star = \sum_{k=0}^n c_k \phi_k = \sum_{k=0}^n \frac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} \phi_k \quad (15)$$

# Orthogonal polynomials

Name	Symbol	Interval	Weight function
Legendre	$P_n$	$[-1, 1]$	1
Chebyshev, 1st kind	$T_n$	$[-1, 1]$	$(1 - t^2)^{-1/2}$
Chebyshev, 2st kind	$U_n$	$[-1, 1]$	$(1 - t^2)^{1/2}$
Jacobi	$J_n$	$[-1, 1]$	$(1 - t)^\alpha (1 + t)^\beta$
Laguerre	$L_n$	$[0, \infty)$	$e^{-t}$
Hermite	$H_n$	$(-\infty, \infty)$	$e^{-t^2}$



# Chebyshev polynomials

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$$

$$T_k(x) = \cos(k \arccos(x))$$

$$T_0 = 1$$

$$T_1 = x$$

$$T_2 = 2x^2 - 1$$

$$T_3 = 4x^3 - 3x$$

$$T_4 = 8x^4 - 8x^2 + 1$$

$$T_5 = 16x^5 - 20x^3 + 5x$$

$$\langle T_j, T_j \rangle = \begin{cases} \pi & \text{if } j = 0 \\ \pi/2 & \text{if } j > 0 \end{cases}$$

# Legendre polynomials

$$P_k(x) = \frac{2k-1}{k} P_{k-1}(x) - \frac{k-1}{k} P_{k-2}(x)$$

$$P_0 = 1$$

$$P_1 = x$$

$$P_2 = (3x^2 - 1)/2$$

$$P_3 = (5x^3 - 3x)/2$$

$$P_4 = (35x^4 - 30x^2 + 3)/8$$

$$P_5 = (63x^5 - 70x^3 + 15x)/8$$

$$\langle P_j, P_j \rangle = \frac{2}{j+1}$$

# Orthogonal trigonometric functions

Functions	Interval	Weight
$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin(nx), \frac{1}{\sqrt{\pi}} \cos(nx), n = 1, 2 \dots$	$[-\pi, \pi]$	1
$\frac{2}{\sqrt{\pi}} \sin(nx), n = 1, 2 \dots$	$[0, \pi]$	1
$\frac{1}{\sqrt{\pi}}, \frac{2}{\sqrt{\pi}} \cos(nx), n = 1, 2 \dots$	$[0, \pi]$	1

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