Formúlublað - Stærðfræði 2

Heildarafleiða falls $f : \mathbb{R}^n \to \mathbb{R}$ í vigrinum $\mathbf{x}$ er kallaður **stigull** og er gefin með

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) & \frac{\partial f}{\partial x_2}(\mathbf{x}) & \dots & \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

og önnur afleiða f í $\mathbf{x} \in \mathbb{R}^n$ er gefin með *Hesse-fylki* f í $\mathbf{x} \in \mathbb{R}^n$,

$$H = \begin{pmatrix} \nabla \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \nabla \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \nabla \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{x}) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(\mathbf{x}) \end{pmatrix}$$

Heildarafleiða falls $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ í vigrinum $\mathbf{x}$ er **Jacobi-fylkið**

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \frac{\partial f_m}{\partial x_2}(\mathbf{x}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{pmatrix}.$$

Stefnuafleiða falls $f : \mathbb{R}^n \to \mathbb{R}$ í punktinum $\mathbf{x}$ í stefnu einingavigursins $\mathbf{u}$ er

$$D_{\mathbf{u}} f(\mathbf{x}) = \mathbf{u} \bullet \nabla f(\mathbf{x}).$$

Keðjureglan fyrir föll $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ og $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^p$. Ef $\mathbf{f}$ er diffranlegt í $\mathbf{a} \in \mathbb{R}^n$ og $\mathbf{g}$ er diffranlegt í $\mathbf{f}(\mathbf{a}) \in \mathbb{R}^m$, þá er samskeytta fallið $\mathbf{g} \circ \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^p$, $(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$ diffranlegt í $\mathbf{a} \in \mathbb{R}^n$ og heildarafleiðan er

$$[D(\mathbf{g} \circ \mathbf{f})](\mathbf{a}) = [D\mathbf{g}(\mathbf{f}(\mathbf{a}))][D\mathbf{f}(\mathbf{a})].$$

Útgildi falla

Fall $f : \mathbb{R}^n \to \mathbb{R}$ hefur útgildi í $\mathbf{x} \in \mathbb{R}^n$ ef $\nabla f(\mathbf{x}) = \mathbf{0}$. Nú gildir:

(i) Ef Hesse-fylki f í $\mathbf{x}$ hefur öll eigingildi > 0, þá hefur f staðbundið lággildi í $\mathbf{x}$.

(ii) Ef Hesse-fylki f í $\mathbf{x}$ hefur öll eigingildi < 0, þá hefur f staðbundið hágildi í $\mathbf{x}$.

(iii) Ef Hesse-fylki f í $\mathbf{x}$ hefur a.m.k. eitt eigingildi < 0 og a.m.k. eitt eigingildi > 0, þá hefur f hvorki staðbundið hágildi né staðbundið lággildi í $\mathbf{x}$. (hér er $\mathbf{x}$ kallað *söðulpunktur* (e. saddle point) f).

Snertiplan

Fall $z = f(x, y)$ hefur snertiplanið

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$

í punktinum (a, b).

Línuleg nálgun

Besta línulega nálgun við fallið $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ í nágrenni $\mathbf{z}$ er

$$\mathbf{y}(\mathbf{x}) = [D\mathbf{f}(\mathbf{z})]\,(\mathbf{x} - \mathbf{z}) + \mathbf{f}(\mathbf{z})$$

þar sem $D\mathbf{f}$ er heildarafleiða fallsins $\mathbf{f}$.

Bogalengd

Lengd ferilsins $\mathcal{C}$ í $\mathbb{R}^n$ sem er stikaður með $\mathbf{r} : [a, b] \to \mathbb{R}^n$ er

$$|\mathcal{C}| = \int_a^b \|\mathbf{r}'(t)\| dt.$$

Ferilheildi

Heildi fallsins $f : \mathbb{R}^n \to \mathbb{R}$ eftir ferlinum $\mathcal{C}$ í $\mathbb{R}^n$ sem er stikaður með $\mathbf{r} : [a, b] \to \mathbb{R}^n$ er

$$\int_{\mathcal{C}} f(\mathbf{x}) ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| dt$$