# Fourier Transform

By Dr. Zhenyu He and Yao Xu

Autumn 2010

# Outline

- Introduction
- Fourier Series
- Fourier Transform
  - Continuous Fourier Transform
  - Discrete Fourier Transform
- Disadvantage
- Conclusion

# Introduction

- The conception of Fourier transform is put forward by the French mathematician and physicist Mr. Jean Baptiste Joseph Fourier.
- In signal analysis and signal processing, Fourier transform is an important and basis technology.
- It can transform a signal in time domain to another in frequency domain.

# Mr. Jean Baptiste Joseph Fourier
# 1768～1830

# Fourier Series

- Definition:

  In mathematics, Fourier series decomposes any periodic function or periodic signal into the sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines.

- The family of functions $\{\cos n\omega t, \sin n\omega t\}$ constructs an orthonormal basis of $L^2[t-T/2, t+T/2]$ .

# Fourier Series

- It can be written as:

$$f(t) = \sum_{k} a_k e_k$$

- Where

$$a_k = \langle f(t), e_k(t) \rangle$$

- Also, more detailedly:

$$f(t) = \sum_{n=0}^{\infty} \left( a_n \cos n\omega_1 t + b_n \sin n\omega_1 t \right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left( a_n \cos n\omega_1 t + b_n \sin n\omega_1 t \right)$$

# Fourier Series

- The Fourier coefficients in the equation are:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt$$

# Fourier Series

The periodic function $f(t)$ should satisfy the Dirichilet condition:

- must have a finite number of extrema in any given interval;
- must have a finite number of discontinuities in any given interval;
- must be absolutely integrable over a period;
- must be bounded.

# Fourier Series

- Trigonometric functions are orthogonal to each other.

$$\int_{t-T/2}^{t+T/2} \sin(n\omega t)\sin(m\omega t)dt = \begin{cases} 0, & n \neq m \\ \dfrac{T}{2}, & n = m \end{cases}$$

$$\int_{t-T/2}^{t+T/2} \cos(n\omega t)\cos(m\omega t)dt = \begin{cases} 0, & n \neq m \\ \dfrac{T}{2}, & n = m \end{cases}$$

$$\int_{t-T/2}^{t+T/2} \sin(n\omega t)\cos(m\omega t)dt = 0,\ m,n = 0,1,\cdots,\infty.$$

# Fourier Series

- Another expression with trigonometric functions:

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega t + \phi_n)$$

$$f(t) = d_0 + \sum_{n=1}^{\infty} d_n \cos(n\omega t + \theta_n)$$

$$a_0 = c_0 = d_0$$

$$c_n = d_n = \sqrt{a_n^2 + b_n^2}$$

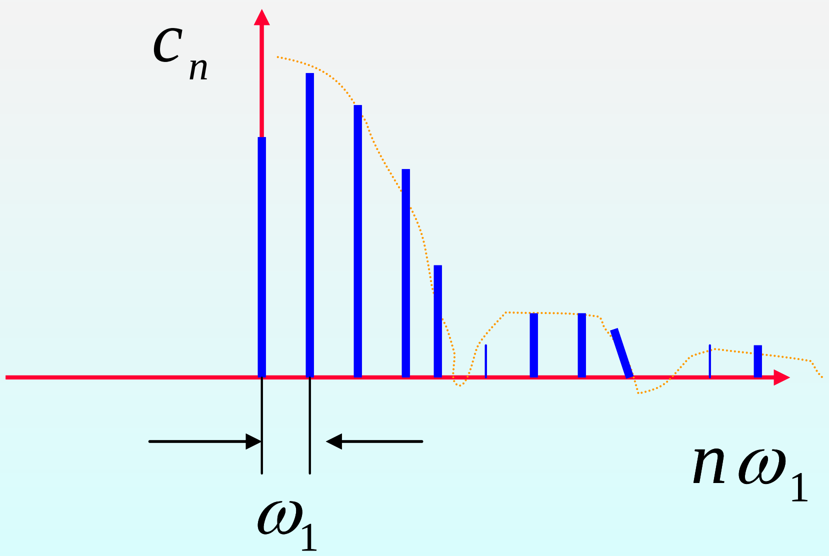
$$a_n = c_n \cos\phi_n = d_n \sin\theta_n$$

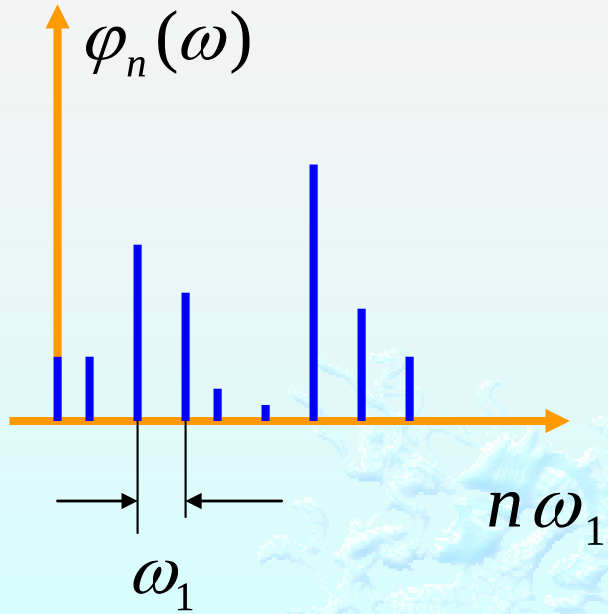
$$b_n = -c_n \sin\phi_n = d_n \cos\theta_n$$

$$\tan\theta_n = \frac{a_n}{b_n}, \tan\phi_n = -\frac{b_n}{a_n}$$

# Frequency Spectrum

- The spectral lines of periodic signal only appear at the frequencies that are integral times of the basis frequency.



# Complex Exponential Series

- The family of trigonometric functions can be written in the form of Euler's function:

$$\left\{ e^{jn\omega t} \right\}_{n=0,\pm1,\pm2,\cdots}$$

- The Fourier series can be written as:

$$f(t) = \sum_{-\infty}^{\infty} F(n\omega) e^{jn\omega t}$$

- where

$$F(0) = a_0$$

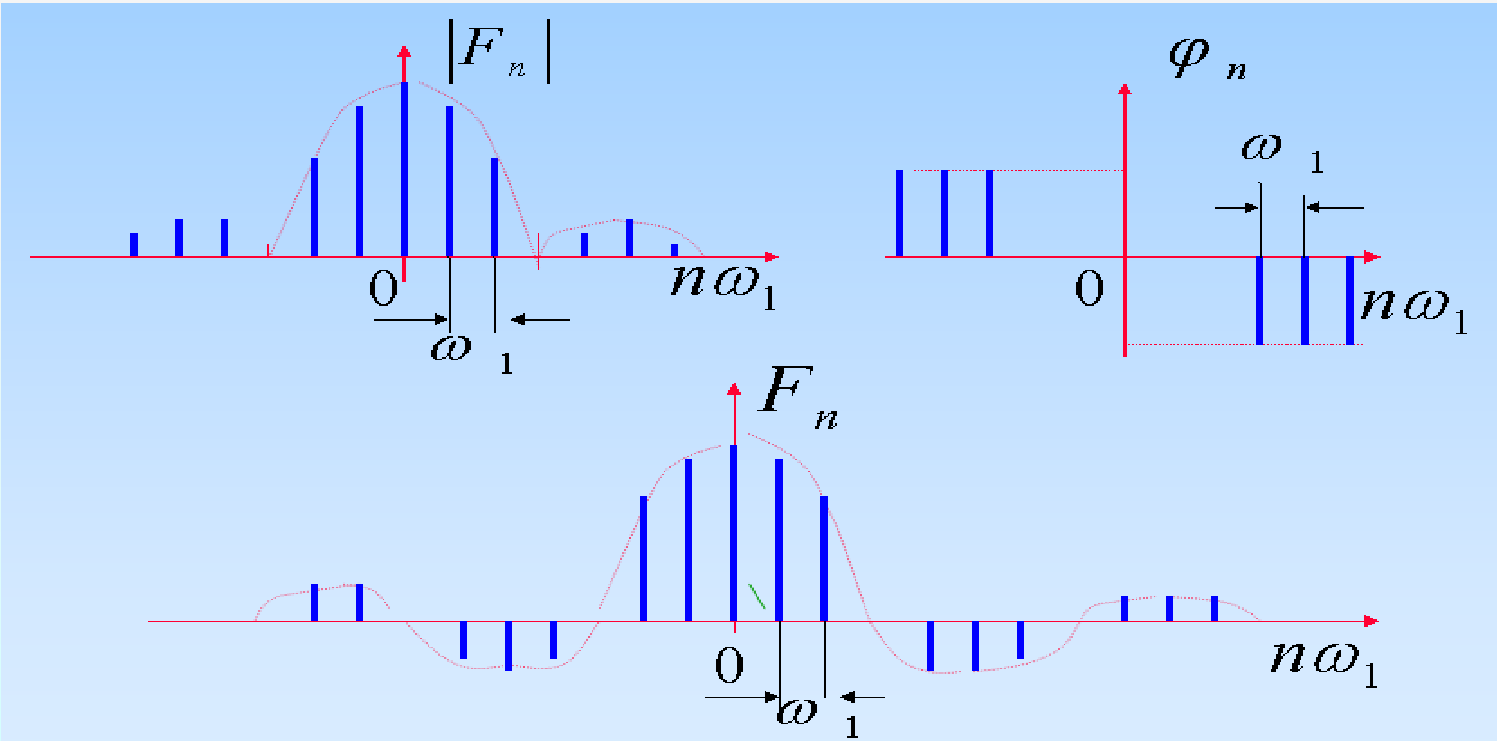
$$F(n\omega) = \frac{1}{2}(a_n - jb_n)$$

Introduce the negative frequency

$$F(-n\omega) = \frac{1}{2}(a_n + jb_n)$$

# Complex Exponential Series

- The frequency spectrogram of periodic complex exponential signal is show as follows:



# Complex Exponential Series

- The coefficients of the complex Fourier series will be:

$$F(n\omega) = F_n = \frac{1}{T}\int_{t_0 - T/2}^{t_0 + T/2} f(t)e^{-jn\omega t}dt$$

# Fourier Series

- Fourier series can only expand the periodic function, the integral domain is one period of the function;
- For non-periodic function, we should expand the integral domain to the whole field, i.e. $T \rightarrow \infty$, then we get the Fourier transform.

# Fourier Transform

- For any signal (periodic or not) $\forall f(t) \in L^2(R)$ , the Fourier transform is defined as:

$$F(t) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi xt} dx$$

$$j^2 = -1$$

- The inverse Fourier transform is:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{j2\pi xt} dx$$

# Definition

- Continuous FT
  - 1-D Case

$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi ut} dt$$

$$j^2 = -1$$

  - 2-D Case

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

# Definition

- Discrete FT
  - 1-D Case

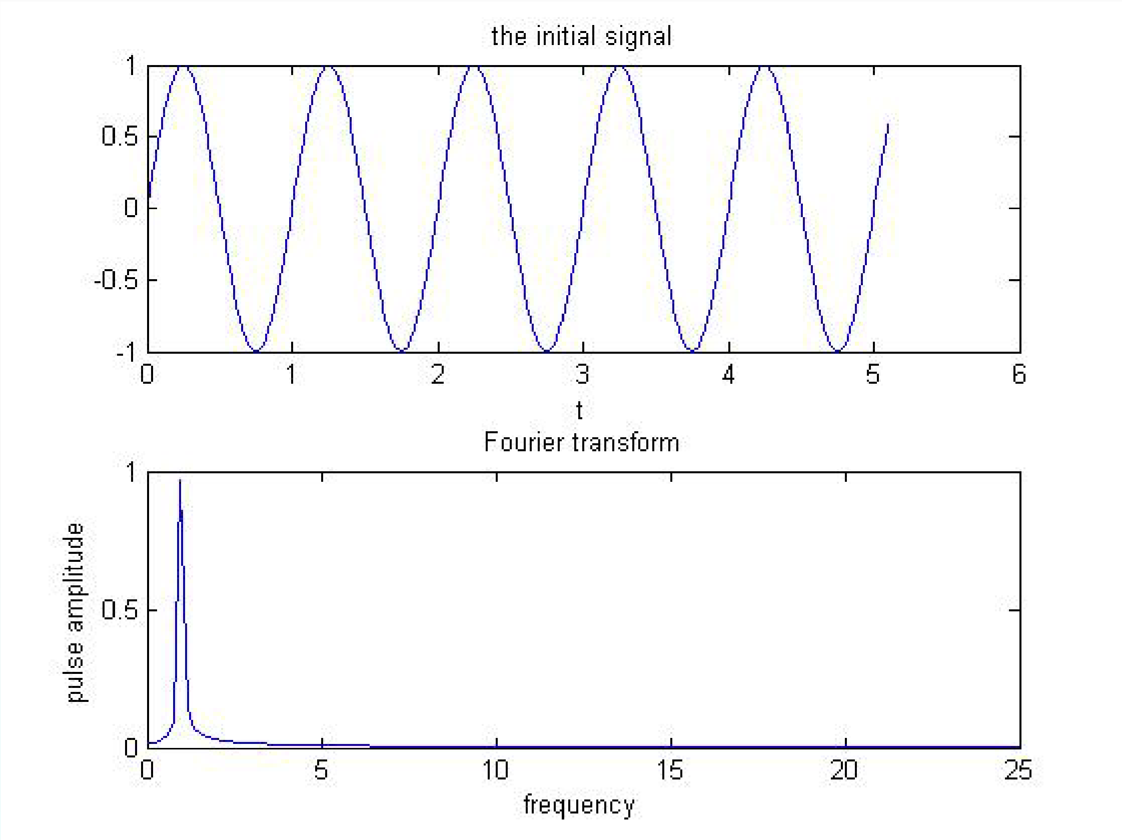
$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/N} \qquad u = 0,1,\ldots, N-1.$$

  - 2-D Case

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)} dxdy$$

$$u = 0,1,\ldots, M-1; \;\; v = 0,1,\ldots, N-1$$

# DEMO (1-D)

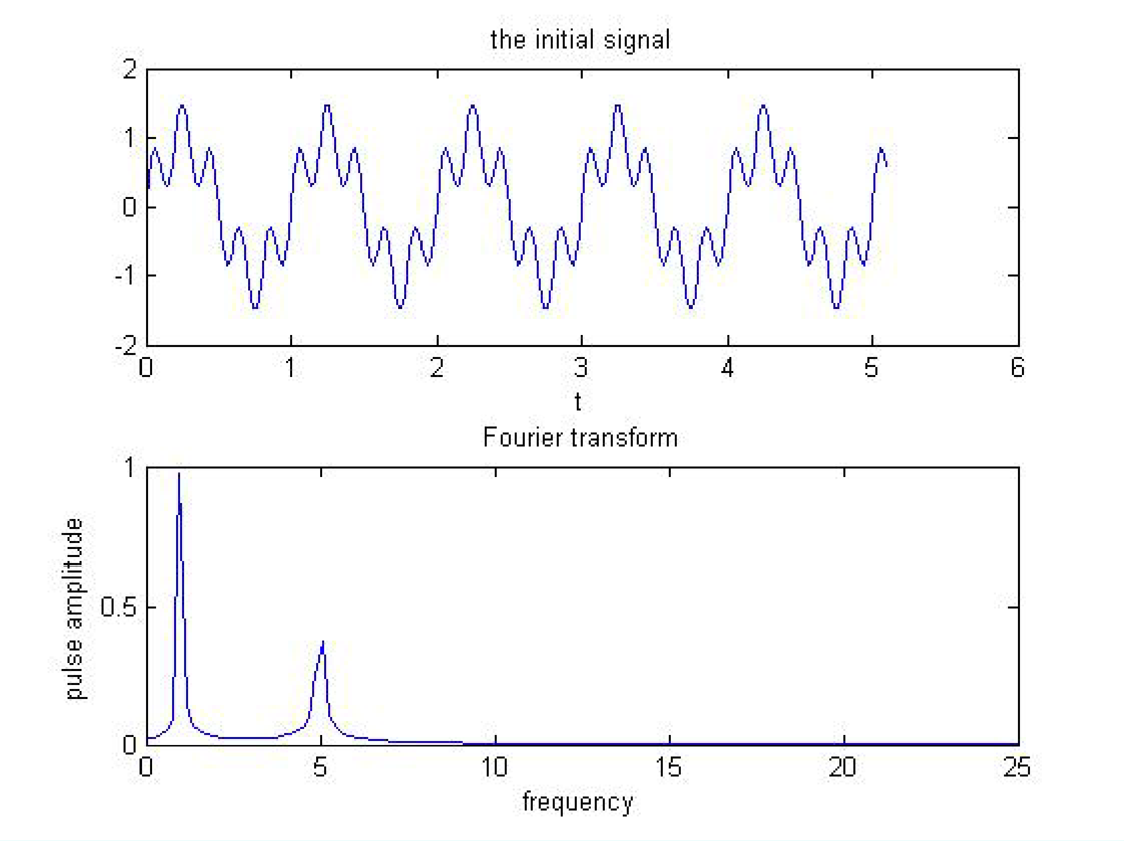



From the figure we see that the initial signal is a pulse signal. FT can find its frequency easily.

A signal consisting of one frequqency

$f(t)=\sin(2\pi t)$

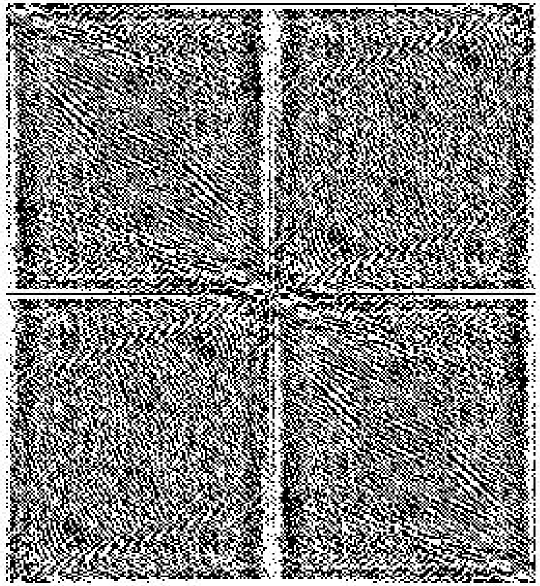
# DEMO (1-D)




From the figure we see that FT can transform the signal in time domain to signal in frequency domain. FT obviously finds the two waves of different frequencies.

A signal consisting of two frequqencies
f(t)=sin(2*pi*t)+0.5*sin(2*pi*5*t)

# DEMO (2-D)



a) the initial image    b) the frequency distribution

From above figure we see that FT can transform an image in time domain to an image in frequency domain.

# Properties Fourier Transform

- Symmetry and Linearity（对称性和线性叠加性）
- Conjugation（奇偶虚实性）
- Scaling （尺度变换特性）
- Time Translation and Frequency Translation（时移特性和频移特性）
- 微分和积分特性

# Symmetry

- IF

$$F(\omega) = FT\left[f(t)\right]$$

- THEN

$$FT\left[F(t)\right] = 2\pi f(-\omega)$$

# Linearity

- IF

$$FT\left[f_i(t)\right] = F_i(\omega)$$

- THEN

$$FT\left[\sum_{i=1}^{n} a_i f_i(t)\right] = \sum_{i=1}^{n} a_i F_i(\omega)$$

# Conjugation

- Whenever $f(t)$ is real function or complex function, the equations below are right:

$$FT\left[f(t)\right] = F(\omega)$$

$$FT\left[f(-t)\right] = F(-\omega)$$

$$FT\left[f^*(t)\right] = F^*(-\omega)$$

$$FT\left[f^*(-t)\right] = F^*(\omega)$$

# Scaling

- IF

$$FT\left[f(t)\right] = F(\omega)$$

- THEN

$$FT\left[f(at)\right] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

# Time Translation

- IF

$$FT\left[f(t)\right]=F(\omega)$$

- THEN

$$FT\left[f(t-t_0)\right]=F(\omega)e^{-j\omega t_0}$$

# Frequency Translation

- IF

$$FT\left[f(t)\right]=F(\omega)$$

- THEN

$$FT\left[f(t)e^{j\omega_0 t}\right]=F(\omega-\omega_0)$$

# Differential

- IF

$$FT\left[f(t)\right] = F(\omega)$$

- THEN

$$FT\left[\frac{df(t)}{dt}\right] = j\omega F(\omega)$$

$$FT\left[\frac{d^n f(t)}{dt^n}\right] = (j\omega)^n F(\omega)$$

# Integral

- IF

$$\omega = 0,\ \left|\frac{F(\omega)}{\omega}\right| < \infty \text{ or } F(0) = 0$$

$$FT\left[f(t)\right] = F(\omega)$$

- THEN

$$FT\left[\int_{-\infty}^{t} f(\tau)d\tau\right] = \frac{F(\omega)}{j\omega}$$

# Integral

- IF

$$FT\left[f(t)\right] = F(\omega)$$

$$F(0) \neq 0$$

- THEN

$$FT\left[\int_{-\infty}^{t} f(\tau)d\tau\right] = \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

# Fourier Transform

Analysis Fourier transform with physical sense:

- $F(\omega)$ is a concept of density function;
- $F(\omega)$ is a continuous spectrum;
- $F(\omega)$ includes all frequency components from 0 to infinite high frequency;
- Each frequency component is not harmonic with another.

# DISADVANTAGE

- FT uses sine basis function which expands the whole time domain, not a specific moment.
- So FT can not find the location of the specific signal.
- FT has a limitation that can only analyze the global frequency distribution.