The Application of Geometric Partial Differential Equation to Active Contour Model in Image Processing（勾配流方程式の画像処理における動的輪郭モデルへの応用）

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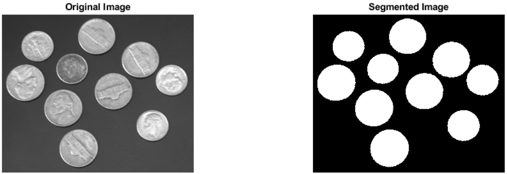
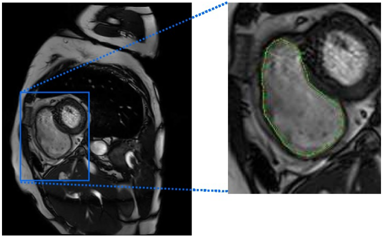
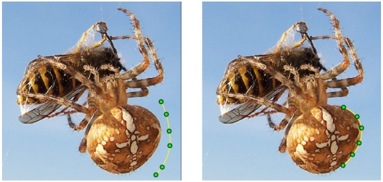
About Active Contour Model







The Classification of Active Contour Model

- Direct Method (e.g. Phase Field Approach [16])
- Indirect Method (e.g. Level Set Approach [15])

Each of these has merits and demerits, so it would be good to combine all these with good balance.
(The waves of deep learning [19] and 3D images are coming!!!!)

The Purpose of My Thesis

The main points of my thesis are as follows:

1. To improve the proof of the theorem on evolving plane curves.
2. To construct a better scheme for the Direct Method in image segmentation.
3. To construct a new scheme for the Indirect Method by making use of my Direct Method scheme.

Preliminary

Definition

Closed curve

$$\Gamma(t) := \left\{ \mathbf{x}(u,t) \in \mathbb{R}^2 | u \in [0,1] \right\}, t \geq 0$$

Decompose the time-evolving curve

$$\partial_t \mathbf{x} = \beta \mathbf{N} + \alpha \mathbf{T} \tag{2.1}$$

(where $\mathbf{N}$ is the inner normal vector and $\mathbf{T}$ is the tangential vector)

We consider the geometric equation

$$\beta = v(\mathbf{x}, k, \nu)$$

where $\mathbf{x} \in \Gamma(t)$, k is the curvature, ν is the tangential angle. α does not affect the geometry of the curve. [13]

Relative Local Length

Definition

Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be a nonnegative even function $\varphi(-k) = \varphi(k) > 0$ for $k \neq 0$ and be nondecreasing for $k > 0$.
This function is called a shape function.

Definition

Define a generalized relative local length adapted to the shape function φ:

$$r_\varphi(u,t) = \frac{g(u,t)\varphi(k(u,t))}{L^t\langle\varphi(k(\dot{,}t))\rangle}$$

where $g = |\partial_u \mathbf{x}|$ is the local length.

The Target Equations

In [3] [5], Ševčovič and Yazaki proved the local existence and uniqueness of a classical solution to the full system of

$$\begin{aligned}
\partial_t k &= \partial_s^2 \beta + \alpha \partial_s k + k^2 \beta \\
\partial_t \nu &= \beta' \partial_s^2 \nu + (\alpha + \beta'_\nu) \partial_s \nu + \nabla_{\mathbf{x}} \beta . \mathbf{T} \\
\partial_t \mathbf{x} &= w \partial_s^2 \mathbf{x} + \alpha \partial_s \mathbf{x} + F \mathbf{N} \\
\partial_t r_\varphi &= (r_\varphi - 1)(\kappa_1 + \kappa_2 \langle k \beta \rangle)
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
\partial_s &= \frac{1}{g} \partial_u \\
\beta(\mathbf{x}, k, \nu) &= w(\mathbf{x}, k, \nu) k + F(\mathbf{x}, \nu) \\
\mathbf{T} &= (\cos \nu, \sin \nu)
\end{aligned}$$

Theorem (Main Theorem)

$$E_\mu := c^{2\mu+\sigma} \times c_*^{2\mu+\sigma} \times (c^{2\mu+\sigma})^2 \times c^{1+\sigma} \text{ for } \mu = 0, 1/2, 1$$

where $c^{\mu+\sigma}$ is the little Hölder space and
$c_*^{2\mu+\sigma}(S^1) = \left\{\nu : \mathbb{R} \to \mathbb{R} : \mathbf{T} = (\cos\nu, \sin\nu) \in \left(c^{2\mu+\sigma}(S^1)\right)^2\right\}$.
Assume

- $\Phi_0 = (k_0, \nu_0, \mathbf{x}_0, r_{\varphi 0}) \in E_1$,
- $\varphi(k) > 0$ and $\beta = \tilde{\beta}(\mathbf{x}, k, \nu) + \mathcal{F}_\Gamma$, such that $\partial_s \mathcal{F}_\Gamma = 0$,
- $\tilde{\beta}$ and $\varphi(k)$ are C^3 smooth function, such that $\tilde{\beta}$ is 2π-periodic in ν and $\min_{\Gamma_0} \tilde{\beta}'_k(\mathbf{x}_0, k_0, \nu_0) > 0$,
- $\mathcal{F}_\Gamma$ is C^1 function from a neighborhood $\mathcal{O}_{1/2} \subset E_{1/2}$ of Φ_0 into $\mathbb{R}$.

$\Rightarrow$ There exists a unique solution
$\Phi = (k, \nu, \mathbf{x}, r_\varphi) \in C([0, T], E_1) \cap C^1([0, T], E_0)$ of Eqs (2.2) defined on some time interval $[0, T]$, $T > 0$.

Moreover...

If solving Eq (2.2), one can construct solutions by

$$\mathbf{x}(u,t) = \mathbf{x}^0(u) + \int_0^t (\tilde{\beta}\tilde{\vec{N}} + \tilde{\beta}\tilde{\vec{T}})d\tau \tag{2.3}$$

where $\tilde{\vec{N}} = (-\sin\tilde{\nu}, \cos\tilde{\nu})$, $\tilde{\vec{T}} = (\cos\tilde{\nu}, \sin\tilde{\nu})$, $\tilde{\beta} = \beta(\tilde{k},\tilde{\nu}), \alpha(\tilde{k},\tilde{\nu},\tilde{g})$

Theorem

Let $\tilde{\Phi} = (\tilde{k},\tilde{\nu},\tilde{g})^T$ be a classical solution of (2.2) such that $\tilde{k}, \tilde{\beta},$ *and* $\tilde{g}^{-1}\partial_u\tilde{\alpha}$ are bounded.
$\Rightarrow \mathbf{x} = \mathbf{x}(u,t)$ given by (2.3) satisfies
$|\partial_u\mathbf{x}| = \tilde{g}, k = \tilde{k}, \nu = \tilde{\nu}, \vec{N} = \tilde{\vec{N}}, \vec{T} = \tilde{\vec{T}},$ where $k, \nu, \vec{N}, \vec{T}$ represent curvature, tangent angle, and unit normal and tangent vectors of Γ^t.

1. This theorem guarantees the equivalence between Eq (2.2) and the geometry of (2.1).
2. This is proved in [3] by the theory of linear parabolic equations [11].

It's now time to prove!!!

The main PDEs are rewritten as

$$\partial_t \Phi = \mathcal{G}(\Phi), \quad \Phi_0 = \Phi(0).$$

Take $\Phi_0 \in \mathcal{O}_1 \subset E_1$ such that $r_\varphi > 0$ and $\tilde{\beta}(\mathbf{x}, k, \nu) > 0$ for $(k, \nu, \mathbf{x}, r_\varphi) \in \mathcal{O}_1$.
Then the mapping $\mathcal{G}$ is considered to be C^1 smooth function from $\mathcal{O}_1 \subset E_1$ to E_0.
The Fréchet derivative of $\mathcal{G}$ at $\bar{\Phi} = (\bar{k}, \bar{\nu}, \bar{\mathbf{x}}, \bar{r}_\varphi) \in \mathcal{O}_1$ has the form for $c(\mathbf{x}, k, \nu) = \beta(\mathbf{x}, k, \nu) - k$.

$$A = \mathcal{G}'(\bar{\Phi}) = A_1 + A_2, \text{ where } A_1 = \bar{D}\partial_u^2, \ A_2 = \bar{B}\partial_u + \bar{C}.$$

$$\bar{D} = \begin{pmatrix} \bar{g}^{-2}\tilde{\beta}'_k & 0 & 0 & 0 & 0 \\ 0 & \bar{g}^{-2}\tilde{\beta}'_k & 0 & 0 & 0 \\ 0 & 0 & \bar{g}^{-2} & 0 & 0 \\ 0 & 0 & 0 & \bar{g}^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\bar{B} = \begin{pmatrix} \partial_s \tilde{\beta}'_k - \tilde{\beta}'_k(\partial_u g)g^{-3} + \tilde{\beta}_\nu g^{-1} + \alpha g^{-1} & 0 & \partial_s \partial_x \tilde{\beta} & -k\partial_x \tilde{\beta} g^{-1} & 0 \\ 0 & -(\partial_u g)g^{-3} + \alpha g^{-1} & \partial_x \tilde{\beta} g^{-1} & \partial_y \beta g^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha g^{-1} - (\partial_u g)g^{-3} & -c & 0 \\ 0 & 0 & c & \alpha - (\partial_u g)g^{-3} & 0 \end{pmatrix}$$

$$\bar{C} = \begin{pmatrix} k^2\beta \\ 0 \\ 0 \\ 0 \\ (r_\varphi - 1)(\kappa_1 + \kappa_2\langle k\beta\rangle) \end{pmatrix}$$

The Maximal Regularity Class $M_h(E)$

For fully nonlinear equations, letting $\mathcal{O} \subset E_1$ be an open subset, f be $C^k(k = 1, 2 \ldots,)$ $f : \mathcal{O} \to E_0$, consider

$$\begin{aligned} x'(t) &= f(x(t)) \quad (0 \leq t \leq T) \\ x(0) &= x_0. \end{aligned}$$

Theorem

If the Fréchet derivative $df(x) \in \mathcal{L}(E_1, E_0)$ belongs to $\mathcal{M}_1(E)$ for all $x \in \mathcal{O}$, then the above problem has a unique solution $x \in C^1([0, T]; E_0) \cap C^0([0, T]; E_1)$ on some small interval.

As for A_1, consider

$$A_1 = A_1' \oplus 0 \quad s.t \;\; A_1(k,\nu,\mathbf{x},r_\varphi = 0) = (A_1'(k,\nu,\mathbf{x}),0)$$

for $E_\mu' = c^{2\mu+\sigma} \times c_*^{2\mu+\sigma} \times \left(c^{2\mu+\sigma}\right)^2$, $\mu = 0, 1/2, 1$.

$$g > 0 \text{ and } \tilde{\beta}_k'(\mathbf{x},k,\nu) > 0 \text{ for } (k,\nu,\mathbf{x},r_\varphi) \in \mathcal{O}_1.$$

$\Rightarrow$ A_1' is normally elliptic and by Theorem 20 in [31],

$$A_1' \in \mathcal{M}_1(E_1', E_0').$$

Since A_1 is an extension of A_1',

$$A_1 \in \mathcal{M}_1(E_1' \oplus c^{1+\sigma}, E_0' \oplus c^{1+\sigma}) = \mathcal{M}_1(E_1, E_0).$$

A_2 can be considered $A_2 \in \mathcal{L}(E_{1/2}, E_0)$.
From the interpolation inequality between c^σ and $c^{2+\sigma}$ by Corollary 24 in [25] and Young's inequality,

$$\begin{aligned} \|A_2\Phi\|_{E_0} \leq C_0\|\Phi\|_{E_{1/2}} &\leq C_0\|\Phi\|_{E_1}^{1/2}\|\Phi\|_{E_0}^{1/2} \\ &\leq C_0\left(\epsilon\|\Phi\|_{E_1} + \frac{\|\Phi\|_{E_0}}{4\epsilon}\right) \end{aligned}$$

where $C_0 > 0$ and $\epsilon > 0$.
$\Rightarrow$ A_2 has the relative zero norm.
From Theorem 17 in [14], $\mathcal{G}'(\Phi) = (A_1 + A_2)(\Phi) \in \mathcal{M}_1(E_1, E_0)$ for any $\Phi \in \mathcal{O}_1$.
The rest is by Theorem 15 in [14]. $\square$

Application to Direct Method in Image Segmentation

Classical curvature flow is the geometric equation, which shortens the total length of curves most rapidly. [8] [13]

$$\beta = k$$

we can consider $\beta = k + F$,

$$F(\mathbf{x}) = F_{max} - (F_{max} - F_{min})I(\mathbf{x}) \quad (\mathbf{x} \in \Omega,\ I \in [0,1])$$

where $F_{max} > 0, F_{min} < 0.$ [4]

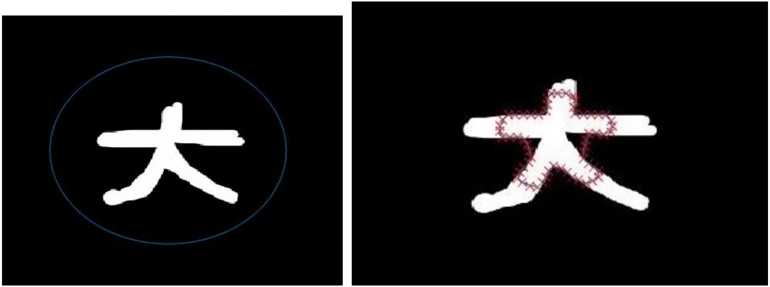


Figure: 大

1st Weakness of the Scheme

Weakness to Noise

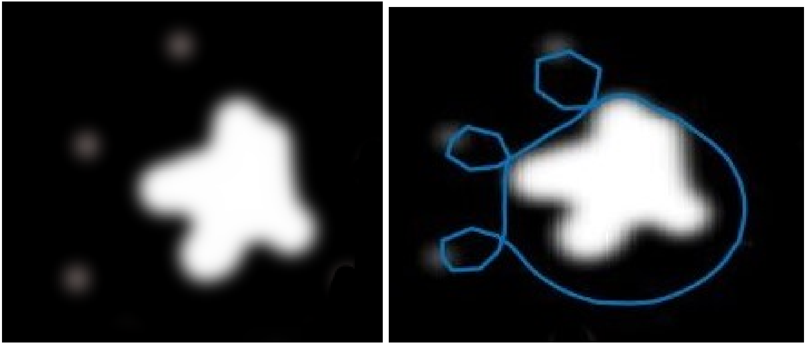


Figure: Effect of noise on curve evolution

2nd Weakness of the Scheme

Weakness to more than 2 components

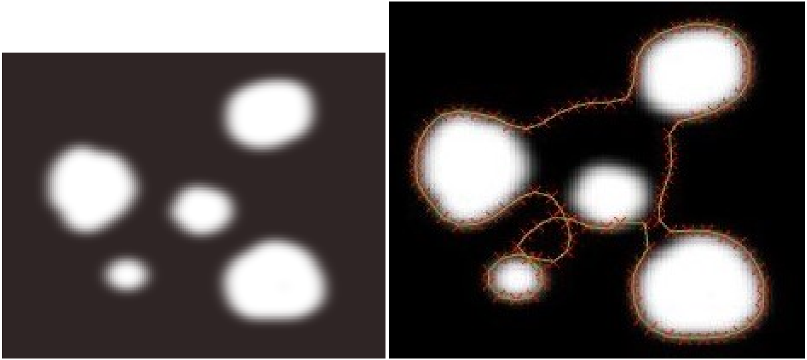


Figure: Dealing with several components

Approach to 1st Weakness

I introduce "Sigmoid Function", which is used in neural biology and machine leaning as activation function, into the external force F. That is, for $a > 0, x \in [0, 1]$,

$$F(x, a) = \frac{(F_{max} - F_{min})}{2} tanh(-a(x - 0.5)) + \left(F_{max} - \frac{F_{max} - F_{min}}{2} \right)$$

or defining $s(x, a) = 1/(1 + e^{a(x-0.5)})$,

$$F(x, a) = F_{max} - (F_{max} - F_{min}) \frac{s(x, a) - s(0, a)}{s(1, a) - s(0, a)} \tag{4.1}$$

a is called "gain" in the field of neural biology, and plays a role in overcoming noise.

Practice is Everything

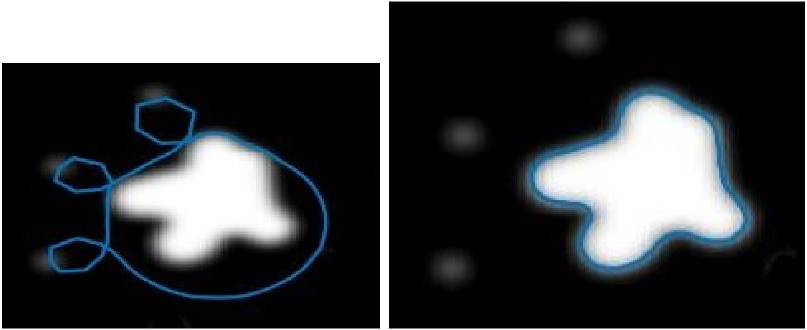


Figure: Newly proposed scheme with $F_{max} = 7, F_{min} = -7.5, a = 6$

Approach to 2nd Weakness

Compared to Direct Method, Indirect Method can deal with multiple components.

The classical curvature flow $\beta = k$ in $\mathbb{R}^3$ corresponds to mean curvature flow, that is, if one surface is represented by $u = u(\mathbf{x}, t)$,

$$\frac{\partial_t u}{|\nabla u|} = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$$

Approach to 2nd Weakness

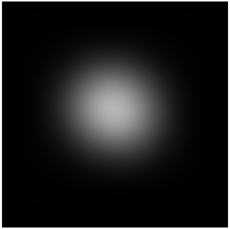
Therefore, adding F to mean curvature flow,

$$\frac{\partial_t u}{|\nabla u|} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + F(I(\mathbf{x}), a),$$

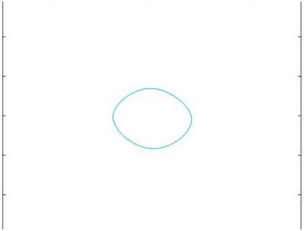
where $I(\mathbf{x})$ is image matrix.
I recommend to use the second F (4.1), that is,

$$\frac{\partial_t u}{|\nabla u|} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + F_{max} - (F_{max} - F_{min})\frac{s(I(\mathbf{x}), a) - s(0, a)}{s(1, a) - s(0, a)}. \quad (4.2)$$

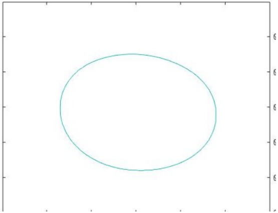
The (local) existence of the solution of this PDE can be proved in viscosity solution. Furthermore, its numerical scheme can be implemented by the viscosity solution [23] and the use of signed distance function [29].



(a) An image with a blurred boundary



(b) $F_{max} = 5, F_{min} = -1$



(c) $F_{max} = 5, F_{min} = -100$

Comparison with Other Indirect method

The most classical Indirect Method is

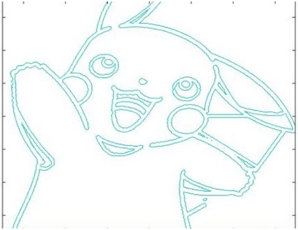
$$\frac{\partial_t u}{|\nabla u|} = \frac{1}{1 + |\nabla G_\sigma * I(\mathbf{x})|^2}$$

where I is an image, G_σ is the Gaussian kernel with standard deviation σ. Furthermore, let's try the "standard force"

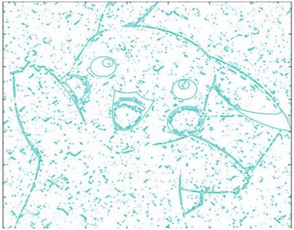
$$\frac{\partial_t u}{|\nabla u|} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + F_{max} - (F_{max} - F_{min})I(\mathbf{x})$$

and the case of not using signed distance function.

(a) Original image



(b) Classical model



(c) Standard force

(d) Sigmoid force

Strong Point of My Scheme

1. Possible to stop on the target boundary, compared to the nonzero property of edge detection function $\frac{1}{1+|\nabla G_\sigma * I(\mathbf{x})|^2}$ in the classical indirect method.
2. Allows to adjust the tightness of the contour.
3. Is able to deal with complicated shapes of edges.
4. Is intuitive and easy to compute.

Thank you for listening!!!

Introduction to Interpolation spaces 1

Definition

X, Y Banach spaces and $Y \subset X$. For $0 \le \theta < 1$ and $1 \le p \le \infty$ set

$$\begin{aligned} V(p,\theta,Y,X) &= \{u : \mathbb{R}^+ \to X : t \mapsto u_\theta(t) = t^{\theta-1/p}u(t) \in L^p(0,+\infty;Y), \\ t &\mapsto v_\theta(t) = t^{\theta-1/p}u'(t) \in L^p(0,\infty;X)\}, \\ \|u\|_{V(p,\theta,Y,X)} &= \|u\|_{L^p(0,\infty;Y)} + \|v_\theta\|_{L^p(0,\infty;X)}. \end{aligned}$$

Moreover, for $p = \infty$ we define a subspace of $V(\infty,\theta,Y,X)$, by

$$V_0(\infty,\theta,Y,X) = \{u \in V(\infty,\theta,Y,X) : \lim_{t\to 0}\|t^\theta u(t)\|_X = \lim_{t\to 0}\|t^\theta u'(t)\|_Y = 0\}$$

Introduction to Interpolation Spaces 2

Theorem (Trace Method)

For $(\theta, p) \in]0,1[\times[1,\infty] \cup \{(1,\infty)\}$, *the Banach space* $(X,Y)_{\theta,p}$ *constructed by the set of the traces at* $t=0$ *of the functions on* $V(p, 1-\theta, Y, X)$, *and its norm is defined by*

$$\|x\|^{T}_{\theta,p} = \inf\{\|u\|_{V(p,1-\theta,Y,X)} : x = u(0), u \in V(p, 1-\theta, Y, X)\}.$$

Moreover, for $0<\theta<1$, $(X,Y)_{\theta}$ *is the set of the trace at* $t=0$ *of the function in* $V_0(\infty, 1-\theta, Y, X)$, *and its norm is defined as above.*

$(X,Y)_{\theta}$ is a closed subspace of $(X,Y)_{\theta,p}$

Introduction to Interpolation Spaces 3

Theorem (Interpolation Inequality)

For $0 < \theta < 1, 1 \leq p \leq \infty$ *and for* $(\theta, p) = (1, \infty)$ *there is* $c(\theta, p)$ *such that*

$$\|y\|_{(X,Y)_{\theta,p}} \leq c(\theta, p)\|y\|_X^{1-\theta}\|y\|_Y^{\theta} \quad \forall y \in Y.$$

Semigroup Theory in Real Version

Definition (Semigroup and infinitesimal generator)

X:Banach Space. $\{T(t)\}_{0\le t<\infty}$ of bounded linear operators from X into X is called a semigroup of bounded linear operator on X if

1. $T(0) = I$, *(I is the identity operator on X)*
2. $T(t+s) = T(t)T(s)$ *for every* $t, s \ge 0$.

The linear operator A defined by

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \left.\frac{d^+T(t)x}{dt}\right|_{t=0} \quad \text{for } x \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is the domain of A.

Semigroup Theory in Real Version 2

Definition (C_0 semigroup)

A semigroup $T(t), 0 \leq t < \infty$ of bounded linear operators on X is a C_0 semigroup if

$$\lim_{t \downarrow 0} T(t)x = x \quad \text{for every} \quad x \in X.$$

Suppose that

$$\exists \omega \geq 1, M \geq 1 \text{ s.t } \|T(t)\| \leq Me^{\omega t} \text{ for } t \geq 1$$

At this time, if $\omega = 0$, $T(t)$ is called uniformly bounded and if moreover $M = 1$ it is called a C_0 semigroup of contractions.

Hille-Yoshida Theorem

Theorem (Hille-Yoshida)

A linear (unbounded) operator A is the infinitesimal generator of a C_0 semigroup of contraction $T(t), t \geq$ iff

1. *A is closed and $\overline{D(A)} = X$.*
2. *The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$*

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}$$

Semigroup in Complex Version

Definition (Analytic Semigroup)

Let $\Delta = \{z : \varphi_1 < argz < \varphi_2, \varphi_1 < 0 < \varphi_2\}$ and for $z \in \Delta$ let $T(z)$ be a bounded linear operator. The family $T(z), z \in \Delta$ is an analytic semigroup in Δ if

1. *$z \to T(z)$ is analytic in Δ*
2. *$T(0) = I$ and $\lim_{z\to 0, z\in\Delta} T(z)x = x$ for every $x \in X$.*
3. *$T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$.*

A semigroup $T(t)$ will be called analytic if it is analytic in some sector Δ containing the nonnegative real axis.

There is a theorem that states the necessary and sufficient condition for C_0 semigroup to be analytic semigroup.

Maximal Parabolic Regularity Class

Let $E = (E_1, E_0)$ be a pair of real Banach spaces for which E_1 is densely included by E_0.

$$\mathcal{L}(E_1, E_0) = \{\text{bounded linear operator from } E_1 \text{ to } E_0\}$$

.

From Hille-Yoshida's theorem and Analytic Extension Theorem, we can define

$$Gen(E) := \{A \in \mathcal{L}(E_1, E_0) : A \text{ generates an } C_0\text{-semigroup}\}$$

$$Hol(E) := \{A \in Gen(E); e^{tA} \text{ is an analytic semigroup}\}$$

Define the following spaces:

$$\begin{aligned} X_1 &:= C([0,1]; E_0) \\ Y_1 &:= C([0,1]; E_1) \cap C^1([0,1]; E_0). \end{aligned}$$

$$X_\theta(E) := \left\{ u \in C((0,1]; E_0);\ \lim_{t\to 0} \|t^{1-\theta} u(t)\|_{E_0} = 0 \right\}$$

$$Y_\theta(E) := \left\{ u \in C((0,1]; E_1) \cap C^1((0,1]; E_0);\ \lim_{t\to 0} t^{1-\theta}(\|u'(t)\|_{E_0} + \|u(t)\|_{E_1}) = 0 \right\}$$

where $\theta \in (0,1)$

From the trace method of interpolation theory

$$E_\theta := \{u(0); u \in Y_\theta(E)\}.$$

Given $A \in \mathcal{L}(E_1, E_0)$, define $\hat{A} : Y_\theta(E) \to X_\theta(E) \oplus E_\theta$ by

$$\hat{A}u = (\partial_t u(t) - Au(t), u(0)).$$

Definition

Define the maximal regularity class

$$\mathcal{M}_\theta(E) := \{A \in Hol(E); \hat{A} \text{ is an isomorphism between } Y \text{ and } X \oplus E_\theta\}$$

For $A \in \mathcal{M}_\theta(E)$, $E = (E_1, E_0)$ is called maximal parabolic regularity pair.

The Property of $M_h(E)$

For fully nonlinear cases, letting $\mathcal{O} \subset E_1$ be an open subset, f be $C^k(k = 1, 2 \ldots,)$ $f : \mathcal{O} \to E_0$, set

$$\begin{aligned} x'(t) &= f(x(t))(0 \le t \le T) \\ x(0) &= x_0. \end{aligned}$$

Theorem

If the Frechet derivative $df(x) \in \mathcal{L}(E_1, E_0)$ belongs to $\mathcal{M}_1(E)$ for all $x \in \mathcal{O}$, then the above has a unique solution $x \in C^1([0, T]; E_0) \cap C^0([0, T]; E_1)$ on some small interval.

The Property of $M_h(E)$ 2

Definition

The relative norm of a bounded operator $A : E_1 \to E_0$ is the infimum over all $C \in \mathbb{R}$ such that, for any $\epsilon > 0$, there exists $k_\epsilon > 0$ with $\|A\|_{E_0} \leq (C + \epsilon)\|x\|_{E_1} + k_\epsilon \|x\|_{E_0}$

Theorem (Perturbation Result)

If $B \in \mathcal{L}(E_1, E_0)$ has relative norm zero, then for any $A \in \mathcal{M}_\theta(E)$ $A + B \in \mathcal{M}_\theta(E)$.

The Property of $M_h(E)$ 3

A Banach couple $E = (E_1, E_0)$ and Banach space F.
Construct $E \oplus F = (E_1 \oplus F, E_2 \oplus F)$ and $A' \in \mathcal{L}(E_1 \oplus F, E_0 \oplus F)$ as an extension of $A \in \mathcal{L}(E_1, E_0)$ such that $A'(x \oplus 0) = (Ax \oplus 0)$ for $x \in E_1$.

Theorem

Let $A' \in \mathcal{L}(E_1 \oplus F, E_0 \oplus F)$ be an extension of $A \in \mathcal{L}(E_1, E_0)$. Then

$$A \in \mathcal{M}_\theta(E) \iff A' \in \mathcal{M}_\theta(E \oplus F)$$

Elliptic Operator on Little Hölder Space

Definition (Little Hölder Space)

For $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}_+ \backslash \mathbb{Z}$, the little Hölder space $c^{m,\alpha}(S^1)$ is defined as the closure of $C^\infty(S^1)$ in the topology of the Hölder space $C^{m+\alpha}(S^1)$.

E : Banach space.
Fix $\{b_k : k = 0, \ldots, 2m\} \subset c^\alpha(S^1, \mathcal{L}(E))$. Consider the differential operator $\mathcal{A}$, $D = i\partial_x$

$$\mathcal{A}u(x) := \mathcal{A}(x, D)u(x) := \sum_{k=0}^{2m} b_k(x)(D^k u)(x) = \sum_{k=0}^{2m} i^k b_k(x)u^{(k)}(x), \quad x \in S^1$$

$\mathcal{A}$ maps $c^{2m+\alpha}(S^1)$ to $c^\alpha(S^1)$.
Define principal symbol of $\mathcal{A}$, $\sigma\mathcal{A} : S^1 \times \mathbb{R} \to \mathcal{L}(E)$ such that
$\sigma\mathcal{A}(x, \xi) := \xi^{2m} b_{2m}(x)$.

Elliptic Operator on Little Hölder Space 2

If E is finite-dimensional, we say $\mathcal{A}$ is a normally elliptic operator on S^1 if there exists $0 < r < R$ such that

$$\sigma(\sigma(\mathcal{A}(x,\xi)) \subset \{z \in \mathbb{C} : \mathrm{Re}\, z \geq r\} \cap \{z \in \mathbb{C} : |z| \leq R\}$$

for $x \in S^1$ and $|\xi| = 1$.

Theorem

A Banach space E, $\alpha \in \mathbb{R}_+ \backslash \mathbb{Z}, m \in \mathbb{N}, \theta \in (0,1]$ and $J = [0, T]$, for $T > 0$ arbitrary. Suppose the operator $\mathcal{A} := \mathcal{A}(\cdot, D) = \sum_{k \leq 2m} b_k(\cdot) D^k$, with coefficient $b_k \in c^\alpha(S^1, \mathcal{L}(E))$ is normally elliptic. Then $\mathcal{A} \in \mathcal{M}_\theta(c^{2m+\alpha}(S^1), c^\alpha(S^1))$. [31]

We need Fourier multiplier theorem and Analytic semigroup theory for this proof.

Introduction to Interpolation Spaces

Definition

X, Y Banach spaces and $Y \subset X$. For every $x \in X$ and $t > 0$, set

$$K(t, x, X, Y) = \inf_{x=a+b \ a\in X, b\in Y} (\|a\|_X + t\|b\|_Y)$$

Definition (K-method)

Let $0 < \theta \leq 1, 1 \leq p \leq \infty$, and set

$$\begin{cases} (X, Y)_{\theta,p} = \{x \in X : t \mapsto t^{-\theta-1/p} K(t, x, X, Y) \in L^p(0, \infty)\}, \\ \|x\|_{(X,Y)_{\theta,p}} = \|t^{-\theta-1/p} K(t, x, X, Y)\|_{L^p(0,\infty)} \end{cases}$$

$$(X, Y)_\theta = \{x \in X : \lim_{t\to 0} t^{-\theta} K(t, x, X, Y) = 0\}$$

Introduction to Interpolation Spaces 2

Theorem

Let X_1, X_2, Y_1, Y_2 be Banach spaces, such that Y_i is continuously embedded in X_i, for $i = 1, 2$. If $T \in L(X_1, X_2) \cap L(Y_1, Y_2)$, then $T \in L((X_1, Y_1)_{\theta,p}, (X_2, Y_2)_{\theta,p}) \cap L((X_1, Y_1)_\theta, (X_2, Y_2)_\theta)$ for every $\theta \in (0, 1)$ and $p \in [1, \infty]$, and for $(\theta, p) = (1, \infty)$. Moreover,

$$\|T\|_{L((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p})} \leq \left(\|T\|_{L(X_1,X_2)}\right)^{1-\theta}\left(\|T\|_{L(Y_1,Y_2)}\right)^{\theta}.$$

Theorem (Interpolation Inequality)

For $0 < \theta < 1, 1 \leq p \leq \infty$ and for $(\theta, p) = (1, \infty)$ there is $c(\theta, p)$ such that

$$\|y\|_{(X,Y)_{\theta,p}} \leq c(\theta, p)\|y\|_X^{1-\theta}\|y\|_Y^{\theta} \quad \forall y \in Y.$$

The Diagram Shown in the Above Theorem

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\quad T \quad} & X_2 \\
\cup\Big\uparrow & & \cup\Big\uparrow \\
Y_1 & \xrightarrow{\quad T \quad} & Y_2
\end{array}
$$

$$(\cdot,\cdot)_{\theta,p} \Downarrow \ (\cdot,\cdot)_{\theta}$$

$$
\begin{array}{ccc}
(X_1, Y_1)_{\theta,p} & \xrightarrow{\quad T \quad} & (X_2, Y_2)_{\theta,p} \\
\cup\Big\uparrow & & \cup\Big\uparrow \\
(X_1, Y_1)_{\theta} & \xrightarrow{\quad T \quad} & (X_2, Y_2)_{\theta}
\end{array}
$$

Theorem (Analytic Extension Theorem)

Let $T(t)$ be a uniformly bounded C_0 semigroup. Let A be the infinitesimal generator of $T(t)$ and assume $0 \in \rho(A)$. The following statements are equivalent:

1. *$T(t)$ can be extended to an analytic semigroup in an sector $\Delta_\delta = \{z : |argz| < \delta\}$ and $\|T(z)\|$ is uniformly bounded in every closed sub-sector $\Delta_{\delta'}, \delta' < \delta$, of Δ_δ.*
2. *There exists a constant C such that for every every $\alpha > 0, \tau \neq 0$*
$$\|R(\sigma + i\tau : A)\| \leq \frac{C}{|\tau|}$$
3. *There exists $0 < \delta < \pi/2$ and $M > 0$ such that*
$$\rho(A) \supset \Sigma = \{\lambda : |arg\lambda| < \frac{\pi}{2} + \delta\} \cup \{0\}$$
and
$$\|R(\lambda : A)\| \leq \frac{M}{|\lambda|} \quad for \quad \lambda \in \Sigma, \lambda \neq 0.$$
4. *$T(t)$ is differentiable for $t > 0$ and there is a constant C such that*
$$\|AT(t)\| \leq \frac{C}{t} \quad for \quad t > 0.$$

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