The Application of Geometric Partial Differential Equation to Direct Method in Active Contour Model

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1 Introduction

These days, the theory of image segmentation is more and more important in many fields, e.g, increasing efficiency and correctness of medical diagnosis, image recognition by autonomous cars and so on. The issues behind this theory include the cases of noises, obscure contrast and computation complexity. In response to these problems, many researchers from various backgrounds develop various methods such as deep learning, edge detection by using filtering.

The purpose of this article is to advance the direct method in the geometric contour of active contour, which focuses on normal parts of time-evolving plane curve on $\mathbb{R}^2$ and tries to substitute suitable characters for the normal parts in order to develop the better algorithms describing the edge of the targeting object.

Together with "indirect method", which represents the motion of the targeting curve by surface and its 0-contour, this type of image segmentation technique is being now incorporated into the emerging technology of deep convolutional neural networks(CNN). [19]

The compositon of this article is as follows: First of all, Chapter 2 explains about the necessary knowledge of the main modeling. Nextly, Chapter 3 prove the existence and uniqueness of the partial differential equation frequently used in such this theory, which is significant in terms of the application of abstract mathematics to this modeling. Chapter 4 states specific application to this active contour modeling and suggests some resolutions to problems accompanied with this application.

2 Preliminary

2.1 Plain Curve

The main character of this paper is a time-dependent family of plane closed Jordan curves $\{\Gamma(t)\}_{t\in[0,T)}$. evolving in the direction of the inner normal $\mathbf{N}$ with a speed v.

More precisely, this paper denotes $\partial_x F = \partial F/\partial x$, $g = |\mathbf{x}| = \sqrt{\mathbf{x}.\mathbf{x}}$ where $\mathbf{x}.\mathbf{y}$ is the Euclidean inner product between $\mathbf{x}$ and $\mathbf{y}$. The unit tangent vector $\mathbf{T} = \partial_u\mathbf{x}/|\partial_u\mathbf{x}| = \partial_s\mathbf{x}$ where s is the arc-length parameter $ds = |\partial_u\mathbf{x}|du$, and the unit inward normal vector is uniquely determined by $\det(\mathbf{T},\mathbf{N}) = 1$. A signed curvature in the direction $\mathbf{N}$ is denoted by k. This is calculated by $k = \det(\partial_s\mathbf{x}, \partial_s^2\mathbf{x})$. Let ν be the angle of $\mathbf{T}$, that is $\mathbf{T} = (\cos\nu, \sin\nu)$ and $\mathbf{N} = (-\sin\nu, \cos\nu)$. This paper assumes $g > 0$

Then let formulate the main character of this paper as follows:

Definition 2.1.

$$\Gamma(t) := \left\{\mathbf{x}(u,t) \in \mathbb{R}^2 | u \in [0,1]\right\}, t \geq 0$$

Decompose the time-evolving curve

$$\partial_t\mathbf{x} = v\mathbf{N} + \alpha\mathbf{T} \tag{2.1}$$

Furthermore, the most important equation is the following geometric equation

$$v = \beta(\mathbf{x}, k, \nu) \tag{2.2}$$

where $\mathbf{x} \in \Gamma^t$, k is the curvature, ν is the tangential angle.
the local length $g = |\partial_u \mathbf{x}|$. The reason to pay attention to v is the fact: The tangential part α does not affect the geometry of the curve. [13] However, it is known that the choices of suitable α dramatically improve the results of numerical schemes. For the application to image segmentation, I will state the method called Curvature Adjusted Method proposed in [3] [4] [5]
The above equation is originated by Mullins in as the modeling of depicting grain boundries in the annealing metal, and nowadays is applied to various fields such as thermomechanics and material science. This is also linked to Kuramoto-Sivashinsky equation and used as flame/smoldering evolution equations. [21]

2.2 Curvature Adjusted Method in Preparation for numerical scheme

As mentioned above, the tangential part of $\partial_t \mathbf{x}$ does not affect the shape of the curve. So, extremely, it can be considered to set $\alpha = 0$. However, in terms of application, determining α disorderly easily leads to failures of numerical simulation.
This section is devoted to explaining about the so-called "curvature adjusted method" to prevent the failure by plot more grid points in positions which have larger curvature.
First of all, define the so-called relative local length:

Definition 2.2 (Relative Local Length)**.**

$$r(u,t) := \frac{g(u,t)}{L^t}, \quad u \in [0,1], \ t \in [0,T)$$

where $L(t)$ is the total length of $\Gamma(t)$:

$$L(t) = \int_{\Gamma(t)} ds = \int_0^1 g du, \quad t \in [0,T)$$

and $T > 0$ is the maximal time of existence of a solution.

Nextly, we define a function which "detects" the modulus $|k|$.

Definition 2.3 (Shape Function)**.** *Let $\phi : \mathbb{R} \to \mathbb{R}_+$ be a nonnegative even function $\phi(-k) = \phi(k) > 0$ for $k \neq 0$ and be nondecreasing for $k > 0$.*
This function is called a shape function.

[3] enumerates examples of the shape function: $\varphi(k) \equiv 1, \varphi(k) = |k|, \varphi(k) = 1 - \epsilon + \epsilon|k|(\epsilon \in [0,1]), \varphi(k) = \sqrt{\epsilon^2 + k^2}(|\epsilon| << 1)$
In the following application section, following the article [4], this paper adopts

the shape function $\varphi(k) = 1 - \epsilon + \epsilon\sqrt{1-\epsilon+\epsilon k^2}$.
Actually, only the relative local length r plays a critical role in plotting the grid points on the curve asymptotically uniformly. Combining the relative local length and the shape function, we introduce the following important indicator:

Definition 2.4 (Generalized Relative Local Length)**.** *Define a generalized relative local length adopted to the shape function φ:*

$$r_\varphi(u,t) := \frac{g(u,t)\varphi(k(u,t))}{L^t\langle\varphi(k(\dot{,}t))\rangle}$$

where the bracket $\langle F\rangle$ denotes the average of function $F(u,t)$ on the curve $\Gamma(t)$:

$$\langle F(\cdot,t)\rangle := \frac{1}{L(t)}\int_{\Gamma(t)} F ds = \frac{1}{L(t)}\int_0^1 F(u,t)g(u,t)du.$$

The intuitive explanation for the role of r_φ is as follows: Suppose N grid points $\{\mathbf{x}(u_i,t)\}_{i=1}^N$ are distributed on the curve $\Gamma(t)$ for $u_i = i/N, i = 1,\ldots,N$. Furthermore, suppose $r_\varphi(u,t) \equiv 1$ for all u at a time t.
Since the arc-length s is given by $s(u,t) = s(0,t) + \int_0^u g(u,t)du$, then

$$s(u_i,t) - s(u_{i-1},t) = \int_{u_{i-1}}^{u_i} g(u,t)du = L(t)\langle\varphi\rangle\int_{u_{i-1}}^{u_i}\frac{r_\varphi}{\varphi}du$$

Because $r_\varphi(u,t) \equiv 1$ for all u at a time t, if $|k|$ satisfies $\varphi(k) \lesseqgtr \langle\varphi(k)\rangle$ on the i-th interval $[u_{i-1},u_i]$, $s(u_i,t) - s(u_{i-1},t) \gtreqless L^t/N$ holds. In other words, with the help of the nondecreasingness of the shape function, if curvature k at the i-th point is higher that the average of k, then the distance between the point and the backward point is shorter, and vice versa. This crucial role of the generalized relative local length enables us to construct the desired curvature adjusted tangential redistribution.
Actually, the reality is that we make $r_\varphi(u,t) \to 1$,i.e.,

$$\theta(u,t) = \ln r_\varphi(u,t) \to 0$$

hold for all $u \in [0,1]$ as $t \to T$, rather than $r_\varphi(u,t) \equiv 1$, by constructing a suitable tangential velocity. Therefore, almost repeatedly, for t close to T, we can conclude if $|k|$ is above/below the average in the sense $\varphi(k) \gtreqless \langle\varphi(k)\rangle$, then $g \lesseqgtr L(t)$ holds, respectively. Distribution of grid points on corresponding subarcs is dense/sparse, respectively.
As a remark, if $\varphi(k) \equiv 1$, the generalized relative local length is $r_\varphi = r$. Then , this plotting is called the asymptotically uniform redistribution.
As mentioned above, it is inevitable for this redistribution to construct θ such that $\theta(u,t) = \ln r_\varphi(u,t) \to 0$ as $t \to T$. However, we have not still mentioned how we make θ go to 0. For this purpose, define the so-called relaxation function:

Definition 2.5 (Relaxation Function)**.** *Let θ be given in the previous way. Define $\omega \in L^1_{loc}[0,T)$ such that*

$$\int_0^T \omega(\tau)d\tau = \infty$$

and

$$\theta(u,t) = \ln((e^{\theta(u,0)} - 1)e^{-\int_0^t \omega(\tau)d\tau} + 1) \tag{2.3}$$

for all $u \in [0,1]$ *and* $t \in [0,T)$.

The previous equation (2.3) can be rewritten as an ODE:

$$\partial_t\theta(u,t) + \omega(t)(1 - e^{-\theta}) = 0. \tag{2.4}$$

ω does not have to be continuous, so this ODE might not necessarily seem to have uniqueness or existence of solution. But we just know the variation rate of θ, so there is no problem for the time being.

With regard to [5], one can choose the relaxation function $\omega(t)$ as follows:

$$\omega(t) = \kappa_1 - \kappa_2\partial_t \ln L(t) = \kappa_1 + \kappa_2\langle k\beta\rangle.$$

where $\kappa_1 \geq 0$ and $\kappa_2 \geq 0$ nonnegative constants, and from

$$\partial_t L(t) = \partial_t \int_0^1 g(u,t)du = \int_{\Gamma(t)} (-k\beta + \partial_s\alpha)ds = \int_{\Gamma(t)} -k\beta ds.$$

Then, finally the equation for the tangential velocity α can be derived as follows.

Differentiating $\theta = \ln g + \ln\phi - \ln\int_0^1 \phi g du$ with respect to t, and from the system of PDEs main in the next section,

$$\begin{aligned}\varphi(\partial_t\theta) &= -\varphi k\beta + \varphi(\partial_s\alpha) + \varphi'(\partial_s^2\beta) + \varphi'\alpha(\partial_s k) + \varphi' k^2\beta + \frac{\langle f\rangle}{\langle\varphi\rangle} \\ &= \partial_s(\varphi\alpha) + \frac{\varphi\langle f\rangle}{\langle\varphi\rangle} - f - \frac{\varphi}{L^t\langle\varphi\rangle}\int_0^1 \varphi'\alpha(\partial_s k) + \varphi(\partial_s\alpha)ds\end{aligned}$$

Therefore, we obtain the following desired equation:

$$\frac{\partial_s(\varphi\alpha)}{\varphi} = \frac{f}{\varphi} - \frac{\langle f\rangle}{\langle\varphi\rangle} + \omega(t)(r_\varphi^{-1} - 1),\ f = \varphi k\beta - (\partial_s^2\beta + k^2\beta)\varphi'_k, \tag{2.5}$$

Equation (2.5) is written in the form $\partial_s(\varphi(k)\alpha) = \mathcal{F}$, where

$$\mathcal{F} := f - \frac{\langle f\rangle}{\langle\varphi(k)\rangle}\varphi(k) + \omega(t)\varphi(k)(r_\varphi^{-1} - 1).$$

Since $\int_{\Gamma^t}\varphi(k)(r_\varphi^{-1} - 1)ds = \int_0^1(L(t)\langle\varphi(k)\rangle)g^{-1} - \varphi(k))gdu = 0$, we obtain $\langle\mathcal{F}\rangle = 0$, we obtain $\langle\mathcal{F}\rangle = 0$. Thus the equation $\partial_s(\varphi(k)\alpha) = \mathcal{F}$,since $\int_0^u \partial_u(\varphi(k)\alpha)du = \varphi(k)\alpha + C(t) = 0(C(t)$ is independent function from u), so we can set $\alpha(u,t) = -C(t)/\varphi(k)$. Furthermore, in order to construct a unique solution α, we can set the renormalization condtion for α:

$$\langle\varphi(k)\alpha\rangle = 0$$

. Thus, we obtain the desired α which satisfies the curvature adjusted tangential redistribution.

3 Consideration of the system of the main PDEs

We shall assume $\beta'_k(\mathbf{x}, k, \nu) > 0$. Then, we can express the normal velocity β in the following term:

$$\beta = w(\mathbf{x}, k, \nu)k + F(\mathbf{x}, \nu)$$

where $w(\mathbf{x}, k, \nu) > 0$ and $F(\mathbf{x}, \nu) = \beta(\mathbf{x}, 0, \nu)$. By β'_k, β'_ν and $\nabla_{\mathbf{x}}\beta$, we denote partial derivatives of β with respect to k and ν and $\nabla_{\mathbf{x}}\beta = (\partial_{x_1}\beta, \partial_{x_2}\beta)$ for $\mathbf{x} = (x_1, x_2)^T$.

In order to guarantee the upcoming technique of image segmentation, this section prove the following theorem firstly proposed by [20], and the improved version by Sevcovic and Yazaki in [5]. This theorem gives us the measurement to fix the normal part of the curve by proving the local existence and uniqueness of a classical solution to the full system of

$$\begin{aligned}
\partial_t k &= \partial_s^2\beta + \alpha\partial_s k + k^2\beta && (3.1)\\
\partial_t \nu &= \beta'\partial_s^2\nu + (\alpha + \beta'_\nu)\partial_s\nu + \nabla_{\mathbf{x}}\beta.\mathbf{T} && (3.2)\\
\partial_t g &= (-k\beta + \partial_s\alpha)g && (3.3)\\
\partial_t \mathbf{x} &= w\partial_s^2\mathbf{x} + \alpha\partial_s\mathbf{x} + F\mathbf{N} && (3.4)
\end{aligned}$$

in [20], or

$$\begin{aligned}
\partial_t k &= \partial_s^2\beta + \alpha\partial_s k + k^2\beta && (3.5)\\
\partial_t \nu &= \beta'\partial_s^2\nu + (\alpha + \beta'_\nu)\partial_s\nu + \nabla_{\mathbf{x}}\beta.\mathbf{T} && (3.6)\\
\partial_t r_\varphi &= (r_\phi - 1)(\kappa_1 + \kappa_2\langle k\beta\rangle), && (3.7)\\
\partial_t \mathbf{x} &= w\partial_s^2\mathbf{x} + \alpha\partial_s\mathbf{x} + F\mathbf{N} && (3.8)
\end{aligned}$$

in [3].

Considering the above two systems of PDEs are actually equivalent because

$$r_\varphi = \frac{g}{L}\frac{\varphi(k)}{\langle\varphi(k)\rangle}.$$

So, to determine g is equivalent to determine r_φ.

The importance to consider these equations derived from some given curve comes from the equivalence between the given curve and the equations. That is, this equivalence answers to the question what kind of β we can insert into (2.2).

First of all, we should develop the above system of PDEs by using Frenet's formula $\partial_s\mathbf{T} = k\mathbf{N}, \partial_s\mathbf{N} = -k\mathbf{T}$, commutation relation $\partial_t\partial_u = \partial_u\partial_t$ and $(\cos\nu, \sin\nu) = \partial_s\mathbf{x}$. As for g,

$$\begin{aligned}
\partial_t(g^2) &= 2g(\partial_t g) = 2g\frac{\partial_u\mathbf{x}\cdot\partial_t\partial_u\mathbf{x}}{g}\\
&= \mathbf{T}\cdot\partial_u(v\mathbf{N} + \alpha\mathbf{T})
\end{aligned}$$

$$\begin{aligned}
\partial_t g &= \frac{\mathbf{T}\cdot((\partial_u V)\mathbf{N} + V\partial_u\mathbf{N} + (\partial_u\alpha)\mathbf{T} + \alpha\partial_u\mathbf{T})}{2g} \\
&= \mathbf{T}\cdot(kV\mathbf{T} + (\partial_u\alpha)\mathbf{T}) \\
&= kV + \partial_s\alpha.
\end{aligned}$$

As for $\mathbf{x}$, from (2.1)

$$\begin{aligned}
\partial_t\mathbf{x} &= (wk+F)\mathbf{N} + \alpha\mathbf{T} \\
&= w\partial_s^2\mathbf{x} + \alpha\partial_s\mathbf{x} + F\mathbf{N} \quad \text{from Frenet's theorem}
\end{aligned}$$

As for ν, compute $\partial_t\mathbf{T}$, then

$$\begin{aligned}
\partial_t(g\mathbf{T}) &= \partial_t\partial_u\mathbf{x} \\
(\partial_t g)\mathbf{T} + g(\partial_t\mathbf{T}) &= \partial_u(\beta\mathbf{N} + \alpha\mathbf{T}) \\
&= (\partial_u\beta)\mathbf{N} + \beta(\partial_u\mathbf{N}) + (\partial_u\alpha)\mathbf{T} + \alpha(\partial_u\mathbf{T}) \\
\therefore \partial_t\mathbf{T} &= (\partial_s\beta + \alpha k)\mathbf{N}.
\end{aligned}$$

So, since $\mathbf{T} = (\cos\nu, \sin\nu)$,

$$\begin{aligned}
\partial_t\mathbf{T} &= (-\sin\nu, \cos\nu)\partial_t\nu \\
(\partial_s V - k\alpha)\mathbf{N} &= \mathbf{N}\partial_t\nu \\
\therefore \partial_t\nu &= \partial_s V + k\alpha.
\end{aligned}$$

In fact, the second system PDEs can be extended to more relax conditions, so this section adopt it and prove the theorem.
In order to formulate the desirable theorem, we need to arrange the convenient function space. This explanation is given to the next section.

3.1 Semigroup Thoery

This section is devoted to explaining about C_0 and analytic semigroup theory in order to complement the critical part of the upcoming proof.

3.1.1 C_0 Semigroup and infinitesimal generator

Definition 3.1 (Semigroup and infinitesimal generator). *X:Banach Space. A one parameter family $T(t), 0 \le t < \infty$ of bounded linear operators from X into X is a semigroup of bounded linear operator on X if*

1. *$T(0) = I$, (I is the identity operator on X)*

2. *$T(t+s) = T(t)T(s)$ for every $t, s \ge 0$.*

A semigroup of bounded linear operators, $T(t)$, is <u>*uniformly continuous*</u> *if*

$$\lim_{t\downarrow 0} ||T(t) - I|| = 0. \tag{3.9}$$

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} exists\} \tag{3.10}$$

and

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \left.\frac{d^+T(t)x}{dt}\right|_{t=0} \quad for\ x \in D(A) \tag{3.11}$$

is the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is the domain of A.

Definition 3.2 (Differentiabilty)**.** *Let $T(t)$ be a C_0 semigroup on a Banach space X. The semigroup $T(t)$ is called differentiable for $t > t_0$ if for every $x \in X, t \to T(t)x$ is differentiable for $t > t_0$. $T(t)$ is called differentiable if it is differentiable for $t > 0$.*

It would be necessary to remark the difference between A and "differentiability". The above definition of Ax in is in the range of $x \in D(A)$, but if $T(t)$ is differentiable, we can define $\lim_{t\downarrow 0} \frac{T(t)x-x}{t}$ for all $x \in X$.

Definition 3.3 (C_0 semigroup)**.** *A semigroup $T(t), 0 \le t < \infty$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if*

$$\lim_{t\downarrow 0} T(t)x = x \quad for\ every \quad x \in X. \tag{3.12}$$

strongly continuous semigroups of bounded linear operatorson X will be called a semigroup of class C_0 or simply C_0 semigroup.

Suppose that

$$\exists \omega \ge 1, M \ge 1\ \ s.t\ \|T(t)\| \le Me^{\omega t}\ for\ t \ge 1 \tag{3.13}$$

At this time, if $\omega = 0$, $T(t)$ is called uniformly bounded and if moreover $M = 1$ it is called a C_0 semigroup of contractions.

Actually, if $T(t)$ is a C_0 smigroup, then (3.13) automatically holds.

Theorem 3.1 (Hille-Yoshida)**.** *A linear (unbounded) operator A is the infinitesimal generator of a C_0 semigroup of contraction $T(t), t \ge$ iff*

1. *A is closed and $\overline{D(A)} = X$.*

2. The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda} \tag{3.14}$$

Furthermore, it is known that some of C_0 semigroup can be extended to complex domain, which is called "Analytic Semirgroup. In other words, the restriction of an analytic semigroup to the real axis is a C_0 semigroup. Therefore, anlytic semigroup might be viewed as something like Analytic continuation from C_0 semigroup.

Definition 3.4 (Analytic Semigroup)**.** *Let $\Delta = \{z : \varphi_1 < argz < \varphi_2, \varphi_1 < 0 < \varphi_2\}$ and for $z \in \Delta$ let $T(z)$ be a bounded linear operator. The family $T(z), z \in \Delta$ is an analytic semigroup in Δ if*

1. *$z \to T(z)$ is analytic in Δ*

2. *$T(0) = I$ and $\lim_{z\to 0, z\in\Delta} T(z)x = x$ for every $x \in X$.*

3. *$T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$.*

A semigroup $T(t)$ will be called analytic if it is analytic in some sector Δ containing the nonnegative real axis.

In connection to C_0 semigroup, there is a theorem that states what kind of C_0 semigroup can be extended to analytic semigroups.
In this paper, I will name tentatively this theorem "Analytic Extention Thoemre".

Theorem 3.2 (Analytic Extension Theoreom)**.** *Let $T(t)$ be a uniformly bounded C_0 semigroup. Let A be the infinitesimal generator of $T(t)$ and assume $0 \in \rho(A)$. The following statements are equivalent:*

1. *$T(t)$ can be extended to an analytic semigroup in an sector $\Delta_\delta = \{z : |argz| < \delta\}$ and $\|T(z)\|$ is uniformly bounded in every closed sub-sector $\Delta_{\delta'}, \delta' < \delta$, of Δ_δ.*

2. *There exists a constant C such that for every every $\alpha > 0, \tau \neq 0$*

$$\|R(\sigma + i\tau : A)\| \leq \frac{C}{|\tau|} \tag{3.15}$$

3. *There exists $0 < \delta < \pi/2$ and $M > 0$ such that*

$$\rho(A) \supset \Sigma = \{\lambda : |arg\lambda| < \frac{\pi}{2} + \delta\} \cup \{0\} \tag{3.16}$$

and

$$\|R(\lambda : A)\| \leq \frac{M}{|\lambda|} \quad for \quad \lambda \in \Sigma, \lambda \neq 0. \tag{3.17}$$

4. *$T(t)$ is differentiable for $t > 0$ and there is a constant C such that*

$$\|AT(t)\| \leq \frac{C}{t} \quad for \quad t > 0. \tag{3.18}$$

This paper will divide this section into two subsection, and in each subsection, we will prove Hille-Yoshda Theorem and Analytci Extention Theorem and add some comments about the relatied topics.

3.1.2 C_0 semigroup and Yoshida Hille Thorem

Theorem 3.3. *If A is the infinitesimal generator of a C_0 semigroup $T(t)$, then $D(A)$, the domain of A, is dense in X and A is a closed linear operator.*

We need the following lemma that states the relation between differentiation of $T(t)$ and A.

Lemma 3.1. *Let $T(t)$ be a C_0 semigroup and let A be its infinitesimal generator. Then*

1. *For $x \in X$,*

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)xds = T(t)x. \tag{3.19}$$

2. *For $x \in X$, $\int_0^t T(s)xds \in D(A)$ and*

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x. \tag{3.20}$$

3. *For $x \in D(A), T(t)x \in D(A)$ and*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax \tag{3.21}$$

4. *For $x \in D(A)$,*

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau \tag{3.22}$$

Proof. Part (1) directly follows from the continuity of $t \to T(t)x$.
To prove (2), let $x \in X$ and $h > 0$. Then,

$$\begin{aligned} \frac{T(h)-I}{h}\int_0^t T(s)xds &= \frac{1}{h}\int_0^t (T(s+h)x - T(s)x)ds \\ &= \frac{1}{h}\int_t^{t+h} T(s)xds - \frac{1}{h}\int_0^h T(s)xds \end{aligned}$$

and as $h \downarrow 0$ the right-hand side tends to $T(t)x - x$, which proves (2).

To prove (3), let $x \in D(A)$ and $h > 0$. Then

$$\frac{T(h)-I}{h}T(t)x = T(t)\left(\frac{T(h)-I}{h}\right)x \to T(t)Ax \quad \text{as} \quad h \downarrow 0. \tag{3.23}$$

Thus, $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$. (3.23) implies also that

$$\frac{d^+}{dt}T(t)x = AT(t)x = T(t)Ax.$$

To prove (3.21), we have to show that for $t > 0$, the left derivative of $T(t)x$ exists and equals $T(t)Ax$.

This follows from

$$\begin{aligned}
\lim_{h\downarrow 0}\left[\frac{T(t)x - T(t-h)x}{h} - T(t)Ax\right] &= \lim_{h\downarrow 0} T(t-h)\left[\frac{T(h)x - x}{h} - Ax\right] + \lim_{h\downarrow 0}(T(t-h)Ax - T(t)Ax) \\
&= \lim_{h\downarrow 0} T(t-h)\left[\frac{T(h)x - x}{h} - Ax\right] + \lim_{h\downarrow 0} T(t-h)(Ax - T(h)Ax)
\end{aligned}$$

Since $x \in D(A)$ and $\|T(t-h)\|$ is bounded on $0 \le h \le t$, and the strong continuity of $T(t)$, both terms on the right-hand side are zero. Therefore, the proof of (3) is done.

Part (4) is obtained by integration of (3.21) from s to t. □

Proof of Theorem 3.3. For every $x \in X$, set $x_t = 1/t\int_0^t T(s)xds$. By part (2) of Theorem 3.1, $x_t \in D(A)$ for $t > 0$ and by part (1) of the same theorem $x_t \to x$ as $t \downarrow 0$. Thus $\overline{D(A)}$ equals X. The linearlity of A is evident.

To prove the closedness, let $x_n \in D(A), x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. From part (4) of Theorem 3.1, we have

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n ds \tag{3.24}$$

The integrand on the RHS of (3.24) converges to $T(s)y$ uniformly on bounded intervals. ($T(.)x_n$ uniformly converges to $T(.)y$ as a function on $[0,t]$
Suppose $\forall\epsilon > 0, \exists N > 0, \forall n \ge N, \|Ax_n - y\| < \epsilon$, then $\forall\epsilon > 0, \exists N > 0, \forall n > N, \forall s \in [0,t], \|T(s)Ax_n - T(s)y\| \le \|T(s)\|\|Ax_n - T(s)y\| \le M\epsilon$) **Is the condition of bounded intervals necessary??**

Consequently, letting $n \to \infty$ in (3.24) yields

$$T(t)x - x = \int_0^t T(s)yds. \tag{3.25}$$

Dividing (3.25) by $t > 0$ and letting $t \downarrow 0$, using (1) of Theorem 3.1, that $x \in D(A)$ and $Ax = y$.

□

The necessity part of the proof of Theorem 3.1 can be explained by the application of Theorem 3.3 and Lemma 3.1. So, first of all, we will show this part of the proof.

Proof of Theorem 3.1. (Necessity)
If A is the infinitesimal generator of a C_0 semigroup then it is closed and $\overline{D(A)} = X$ by Corollary 3.3. For $\lambda > 0$ and $x \in X$ let

$$R(\lambda)x := \int_0^\infty e^{-\lambda t} T(t)x dt. \tag{3.26}$$

Since $t \to T(t)x$ is continuous and uniformly bounded the integral exist and defines a bounded linear operator $R(\lambda)$ satisfying

$$\|R(\lambda)x\| \leq \int_0^\infty e^{-\lambda t} \|T(t)x\| dt \leq \frac{1}{\lambda}\|x\|, \tag{3.27}$$

(since $T(t)$ is a C_0 semigroup of contractions.)

Furthermore, for $h > 0$

$$\begin{aligned} \frac{T(h)-I}{h} R(\lambda)x &= \frac{1}{h}\int_0^\infty e^{-\lambda t}(T(t+h)x - T(t)x)dt & (3.28) \\ &= \frac{e^{\lambda h}}{h}\int_h^\infty e^{-\lambda t} T(t)x dt - \frac{1}{h}\int_0^\infty e^{-\lambda t} T(t)x dt & (3.29) \\ &= \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t} T(t)x dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t} T(t)x dt & (3.30) \end{aligned}$$

As $h \downarrow 0$, the right-hand side of (3.28) converges to $\lambda R(\lambda)x - x$. This implies that for every $x \in X$ and $\lambda > 0, R(\lambda)x \in D(A)$ and $AR(\lambda) = \lambda R(\lambda) - I$, or

$$(\lambda I - A)R(\lambda) = I. \tag{3.31}$$

For $x \in D(A)$ we have

$$\begin{aligned} R(\lambda)Ax &= \int_0^\infty e^{-\lambda t} T(t)Ax dt = \int_0^\infty e^{-\lambda t} AT(t)x dt & (3.32) \\ &= A\left(\int_0^\infty e^{-\lambda t} T(t)x dt\right) = AR(\lambda)x & \\ & & (3.33) \end{aligned}$$

Here, we used Theorem 3.1(3) and the closedness of A. (Construct the approximation sequence of Riemann sum and improper integral) From (3.31) and (3.32), it follows that

$$R(\lambda)(\lambda I - A)x = x \quad for \quad x \in D(A). \tag{3.34}$$

Thus, $R(\lambda)$ is the inverse of $\lambda I - A$, it exists for all $\lambda > 0$. So, these prove the necessity.
(3.14) is from (3.27) $\square$

In order to prove the conditions are the sufficiency, we have to prove the following the lemma.

Lemma 3.2. *Let A satisfy the conditions 1,2 of Theorem* 3.1 *and let $R(\lambda : A) = (\lambda I - A)^{-1}$. Then*

$$\lim_{\lambda\to\infty} \lambda R(\lambda : A)x = x \quad for \quad x \in X \tag{3.35}$$

Proof. Suppose first that $x \in D(A)$. Then

$$\begin{aligned} \|\lambda R(\lambda : A)x - x\| &= \|(\lambda I - A)^{-1} (\lambda I - (\lambda I - A)x)\| \\ &= \|AR(\lambda : A)x\| \\ &= \|R(\lambda : A)Ax\| \le \frac{1}{\lambda}\|Ax\| \to 0 \;\; as \;\; \lambda \to \infty. \end{aligned}$$

But $D(A)$ is dense in X and $\|\lambda R(\lambda : A)\| \le 1$. Take $\forall x \in X, \{x_n\} \subset D(A), \lim_{n\to\infty} x_n = x$,

$$\|\lambda R(\lambda : A)x - x\| \le \|\lambda Rx - \lambda Rx_n\| + \|\lambda Rx_n - x_n\| + \|x_n - x\|$$

Therefore $\lambda R(\lambda : A)x \to x$ as $\lambda \to \infty$ for every $x \in X$. ☐

Definition 3.5 (Yoshida Approximation)**.** *For every $\lambda > 0$, the Yoshida Approximation of A by*

$$A_\lambda = \lambda AR(\lambda : A) = \lambda^2 R(\lambda : A) - \lambda I \tag{3.36}$$

Lemma 3.3. *Let A be satisfy the condition 1 and 2 of Theorem* 3.1*. If A_λ is the Yoshida approximation of A, then*

$$\lim_{\lambda\to\infty} A_\lambda x = Ax \quad for \quad x \in D(A). \tag{3.37}$$

Proof. For $x \in D(A)$, we have by Lemma 3.2 and the definition of A_λ that

$$\lim_{\lambda\to\infty} A_\lambda x = \lim_{\lambda\to\infty} \lambda R(\lambda : A)Ax = Ax.$$

☐

For the next lemma, I will prove the fact about "uniformly continuous semigroup".

Lemma 3.4. *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup iff A is a bounded linear operator.*

Proof. A: bounded linear operator on X.

$$T(t) := e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \tag{3.38}$$

This equation converges in norm for every $t \ge 0$ and defines a bounded linear operator $T(t)$. Since $(\sum_{n=0}^{\infty} a_k)(\sum_{n=0}^{\infty} b_k) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n-k}b_k$ generally holds,

$$\begin{aligned} T(t)T(s) &= \left(\sum_{n=0}^{\infty} \frac{(tA)^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(sA)^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(tA)^{n-k}}{(n-k)!}\frac{(sA)^k}{k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_n\mathrm{C}_k (sA)^k (tA)^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{((t+s)A)^n}{n!} \end{aligned}$$

Estimating the power series yields

$$\|T(t) - I\| \leq t\|A\|e^{t\|A\|}$$

and

$$\begin{aligned} \left\|\frac{T(t)-I}{t} - A\right\| &= \|(A + \frac{t}{2!}A^2 + \cdots) - A\| \\ &\leq \|A\|\|T(t) - I\| \end{aligned}$$

which imply that $T(t)$ is a uniformly continuous and A is its infinitesimal generator .

Conversely, let $T(t)$ be a unique continuous semigroup of bounded linear operators on X.Since $\|I - \rho^{-1}\int_0^\rho T(s)ds\| = \rho^{-1}\|\int_0^\rho I - T(s)ds\| \leq \sup_s \|I - T(s)\|$, we can take sufficiently small $\rho > 0$ such that $\|I - \rho^{-1}\int_0^\rho T(s)ds\| < 1$. This implies that $\rho^{-1}\int_0^\rho T(s)ds$ is invertible and $\int_0^\rho T(s)ds$ is invertible. (Since e.g. $\|A\| < 1 \Longleftarrow I - A$ invertible $\Longleftrightarrow (I - A)^{-1} = \sum_{n=0}^{\infty} A^n$ exists.

Now,

$$\begin{aligned} h^{-1}(T(h) - I)\int_0^\rho T(s)ds &= h^{-1}(\int_0^\rho T(s+h)ds - \int_0^\rho T(s)ds) \\ &= h^{-1}(\int_\rho^{\rho+h} T(s)ds - \int_0^h T(s)ds) \end{aligned}$$

and therefore

$$h^{-1}(T(h) - I) = \left(h^{-1}\int_\rho^{\rho+h} T(s)ds - h^{-1}\int_0^h T(s)ds\right)\left(\int_0^\rho T(s)ds\right)^{-1} \quad (3.39)$$

Letting $h \downarrow 0$ in (3.39) shows that $h^{-1}(T(h) - I)$ converges in norm and strongly to $(T(\rho) - I)(\int_0^\rho T(s)ds)^{-1}$ □

Lemma 3.5. *Let A satisfy the condition 1 and 2 of Theorem* 3.1*. If A_λ is the Yoshida approximation of A, then A_λ is the infinitesimal generator of a* *uniformly continuous semigroup of contraction* e^{tA_λ}. *Furthermore, for every* $x \in X, \lambda, \mu > 0$ *we have*

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \le t\|A_\lambda x - A_\mu x\| \tag{3.40}$$

Proof. From (3.36) it is clear that A_λ is a bounded linear operator and thus is the infinitesimal generator of a uniformly continuous semigroup e^{tA_λ} of bounded linear operators by Theorem 3.4. Also,

$$\|e^{tA_\lambda}\| = e^{-t\lambda}\|e^{t\lambda^2 R(\lambda:A)}\| \le e^{-t\lambda}e^{t\lambda^2\|R(\lambda:A)\|} = e^{t\lambda^2(\|R\|-\frac{1}{\lambda})} \le 1, \tag{3.41}$$

from the condition 1. Therefore, e^{tA_λ} is a semigroup of contractions. By the definitions, $e^{tA_\lambda}, e^{tA_\mu}, A_\lambda$ and A_μ commute with each other. Consequently,

$$\begin{aligned} \|e^{tA_\lambda}x - e^{tA_\mu}x\| &= \left\|\int_0^1 \frac{d}{ds}(e^{tsA_\lambda}e^{t(1-s)A_\mu}x)ds\right\| \\ &\le \int_0^1 t\|e^{tsA_\lambda}e^{t(1-s)A_\mu}(A_\lambda x - A_\mu x)\|ds \le t\|A_\lambda x - A_\mu x\| \end{aligned}$$

since e^{tA_λ} is a semigroup of contractions. □

Proof of Theorem 3.1. We will prove its sufficiency. Let $x \in D(A)$. Then

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \le t\|A_\lambda x - A_\mu x\| \le t\|A_\lambda x - Ax\| + t\|Ax - A_\mu x\|. \tag{3.42}$$

From (3.42) and Lemma 3.3 it follows that for $x \in D(A), e^{tA_\lambda}x$ converges as $\lambda \to \infty$ and the convergence is uniform on bounded intervals on t????. Since $D(A)$ is dense in X and $\|e^{tA_\lambda} \le 1$, we can write

$$T(t)x := \lim_{\lambda\to\infty} e^{tA_\lambda}x \quad for \forall x \in X. \tag{3.43}$$

The limit in (3.43) is again uniform on bounded intervals.???? From (3.43) the limit $T(t)$ satisfies the semigroup property, that $T(0) = I$ and that $\|T(t)\| \le 1$. Also since it is a uniform limit of the continuous functions $t \to e^{tA_\lambda}x$, $t \to T(t)x$ is continuous for $t \ge 0$. Thus $T(t)$ is a C_0 semigroup of contraction on X.
To conclude the proof, we will show that A is the infinitesimal generator of $T(t)$. Let $x \in D(A)$. Then using (3.43) and Theorem 3.1 3 we have

$$T(t)x - x = \lim_{\lambda\to\infty}(e^{tA_\lambda}x - x) = \lim_{\lambda\to\infty}\int_0^t e^{sA_\lambda}A_\lambda x ds = \int_0^t T(s)Axds \tag{3.44}$$

since the uniform convergence of $e^{tA_\lambda}A_\lambda x$ to $T(t)Ax$ on bounded intervals.

Let B be the infinitesimal generator of $T(t)$ and let $x \in D(A)$. Dividing (**??**) by $t > 0$ and letting $t \downarrow 0$.

$$Bx = \lim_{t\downarrow 0}\frac{T(t) - x}{t} = \lim_{t\downarrow 0}\frac{\int_0^t T(s)Axds}{t} = Ax$$

we see that $x \in D(B)$ and that $Bx = Ax$. Thus $B \supset A$

$$(B \supset A \iff for\ \forall x \in D(A), Ax = Bx,\ and\ D(B) \supset D(A))$$

Since B is the infinitesimal generator of $T(t)$, $1 \in \rho(B)$ from the necessary conditions.
On the other hand, by the assumption (2) $1 \in \rho(A)$.(So, $I - A$ and $I - B$ are bijective). Since $B \supset A, (I - B)D(A) = (I - A)D(A) = X$ which implies $D(B) = (I - B)^{-1}X = D(A)$ and therefore $A = B$.□ □

This Hille-Yoshida theorem and Theorem 3.3 motivates the following setting for defining C_0 semigroup class.

Definition 3.6. *Let $E = (E_1, E_0)$ be a pair of real Banach spaces for which E_1 is densely included by E_0.*

$$\mathcal{L}(E_1, E_0) = \{bounded\ linear\ operator\ from\ E_1\ to\ E_0\} \tag{3.45}$$

$$Gen(E) := \{A \in \mathcal{L}(E_1, E_0) : A\ is\ an\ infinitesimal\ generator\ of C_0\text{-}semigroup\} \tag{3.46}$$

(3.47)

Here, the Hille -Yoshida theorem is very helpful for better understanding of $Gen(E)$.

3.1.3 Analytic Extension Theorem

We have to organize a few theorems or lemmas, whose proofs will be sometimes skipped because the margin in this section is too narrow to contain.

Theorem 3.4. *Let A be a densely defined operator in X satisfying the following conditions.:*

1. *For some $0 < \delta < \pi/2, \rho(A) \supset \Sigma_\delta = \{\lambda : |\arg\lambda| < \pi/2 + \delta\} \cup \{0\}$.*

2. *There exists a constant M such that*

$$\|R(\lambda : A)\| \leq \frac{M}{|\lambda|} \quad for \quad \lambda \in \Sigma_\delta, \quad \lambda \neq 0 \tag{3.48}$$

Then A is the infinitesimal generator of a C_0 semigroup $T(t)$ satisfying $\|T(t)\| \leq C$ for some constant C. Moreover

$$T(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} R(\lambda : A) d\lambda \tag{3.49}$$

where Γ is a smooth curve in Σ_δ running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for $\pi/2 < \theta < \pi/2 + \delta$. The integral (3.49) converges for $t > 0$ in the uniform operator topology.

Lemma 3.6. *Let $T(t)$ be a differentiable C_0 semigroup and let A be its infinitesimal generator . Then*

$$T^{(n)}(t) = \left(AT\left(\frac{t}{n}\right)\right)^n = \left(T'\left(\frac{t}{n}\right)\right)^n \quad n = 1, 2, \ldots \tag{3.50}$$

Proof. First of all, we will prove that for $t > nt_0.n = 1, 2, \ldots, T(t) : X \to D(A^n)$ and $T^{(n)}(t) = A^n T(t)$ is a bounded linear operator.

Start with $n = 1$. By our assumption $t \to T(t)x$ is differentiable for $t > t_0$ and all $x \in X$. Therefore, $T(t)x \in D(A)$ and $T'(t)x = AT(t)x$ for every $x \in X$ and $t > t_0$. Moreover, since A is closed and $T(t)$ is bounded, $AT(t)$ is closed. (Take $\{x_n\} \subset X, x \in X$. and $ATx_n \to y$. Then T is bounded, so $Tx_n \to Tx$. Furthermore, A is closed, so $ATx_n \to ATx$.) For $t > t_0$, $AT(t)$ is defined on all of X and therefore, by the closed theorem, it is a bounded linear operator.

So we will prove (3.50) by induction on n. As we prove as above, this proves $n = 1$. If (3.50) holds for n and $t \geq s$ then

$$T^{(n)}(t) = \left(AT\left(\frac{t}{n}\right)\right)^n = T(t-s)\left(AT\left(\frac{s}{n}\right)\right)^n. \tag{3.51}$$

Differentiating (3.51) with respect to t we find

$$T^{(n+1)}(t) = AT(t-s)\left(AT\left(\frac{s}{n}\right)\right)^n. \tag{3.52}$$

Substituting $s = nt/n + 1$ in (3.52),

$$T^{(n+1)}(t) = AT\left(\frac{1}{n+1}\right)\left(AT\left(\frac{t}{n+1}\right)\right)^n$$

yields the result for $n+1$. Substituting $s = nt/n + 1$ in (3.52) yields the result for $n + 1$. □

Lemma 3.7. *A is a linear operator, and $\lambda \in \rho(A)$, then its resolvent $\lambda \mapsto R(\lambda, A)$ is holomorphic and it satisfies*

$$\frac{d}{d\lambda}R(\lambda : A) = -R(\lambda : A)^2. \tag{3.53}$$

Proof. For $\lambda, \mu \in \rho(A)$,

$$\begin{aligned}
R(\lambda : A) - R(\mu : A) &= R(\lambda : A)(\mu I - A)R(\mu : A) - R(\lambda : A)(\lambda I - A)R(\mu : A) \\
&= (\mu - \lambda)R(\lambda : A)R(\mu : A) \\
&= -(\lambda - \mu)R(\lambda : A)R(\mu : A)
\end{aligned}$$

From this equation, for every $\lambda \in \rho(A), \lambda \to R(\lambda : A)$ is holomorphic, and

$$\frac{d}{d\lambda}R(\lambda : A) = -R(\lambda : A)^2. \tag{3.54}$$

$\square$

Theorem 3.5. *A linear operator A is the infinitesimal generator of a C_0 semi-group $T(t)$ satisfying $\|T(t)\| \leq Me^{\omega t}$, iff*

1. *A is closed and $D(A)$ is dense in X.*
2. *The resolvent set $\rho(A)$ of A satisfies that for all $\lambda \in \rho(A)$ $Re\lambda \in]\omega, \infty[$ and*

$$\|R(\lambda : A)^n\| \leq M/(Re\lambda - \omega)^n \quad for \quad Re\lambda > \omega, \quad n = 1, 2, \ldots \tag{3.55}$$

From now on, we move on to the main proof.

Proof of Theorem 3.2. (1) $\Rightarrow$ (2). Let $0 < \delta' < \delta$ be such that $\|T(z)\| \leq C_1$ for $z \in \bar{\Delta}_{\delta'} = \{z : |argz| \leq \delta'\}$. For $x \in X$ and $\sigma > 0$ we have

$$R(\sigma + i\tau : A)x = \int_0^\infty e^{-(\sigma+i\tau)t}T(t)xdt. \tag{3.56}$$

Then,

$$\begin{aligned}
C_2 \quad &: \quad \{t = \rho\cos\theta + is : s, 0 \to \rho\sin\theta\} \\
\|\int_{C_2} e^{-(\sigma+i\tau)t}T(t)xdt\| \quad &\leq \quad \int_0^{\rho\sin\theta} e^{-(\sigma+i\tau)(\rho\cos\theta+is)}T(\rho\cos\theta + is)xds\| \\
&\leq \quad e^{-\sigma\rho\cos\theta}\int_0^{\rho\sin\theta} C\|x\|ds \\
&= \quad e^{-\sigma\rho\cos\theta}\rho\sin\theta C\|x\| \to 0(\rho \to \infty)
\end{aligned}$$

From the analyticity and the uniform boundedness of $T(z)$ in $\bar{\Delta}_{\delta'}$, it follows that we can shift the path of integration in (3.56) from the positive real axis to any ray $\rho e^{i\theta}, 0 < \rho < \infty$ and $|\theta| \leq \delta'$.
For $\tau > 0$, shifting the path of integration to the ray $\rho e^{i\delta'}$ and estimating the resulting integral we find, taking $t = \rho(\cos\delta' + i\sin\delta'$

$$\begin{aligned}
\|R(\sigma + i\tau : A)x\| \quad &= \quad \|\int_0^\infty e^{-(\sigma+i\tau)\rho(\cos\delta'+i\sin\delta')}T(t)x(\cos\delta' + i\sin\delta')d\rho \\
&\leq \quad \int_0^\infty e^{-\rho(\sigma\cos\delta'+\tau\sin\delta')}C_1\|x\|d\rho \\
&\leq \quad \frac{C_1\|x\|}{\sigma\cos\delta' + \tau\sin\delta'} \leq \frac{C\|x\|}{\tau}.
\end{aligned}$$

Similaly for $\tau < 0$ we shift the path of integration to the ray $\rho e^{-i\delta'}$ and obtain $\|R(\sigma + i\tau : A)\| \leq -C/\tau$ and thus (3.15) holds.

(2) $\Rightarrow$ (3). Since A is by assumption the infinitesimal generator of a C_0 semigroup we have $\|R(\lambda : A)\| \leq M_1/Re\lambda$ for $Re\lambda > 0$ by the boundedness of $T(t)$. From (b), we have for $Re\lambda > 0, \|R(\lambda : A)\| \leq C/|Im\lambda|$ and therefore, $\|R(\lambda : A)\| \leq C_1/|\lambda|$ for $Re\lambda > 0$. Let $\sigma > 0$ and write the Taylor expansion for $R(\lambda : A)$ around $\lambda = \sigma + i\tau$, from (3.53)

$$R(\lambda : A) = \sum_{n=0}^{\infty} R(\sigma + i\tau : A)^{n+1}(\sigma + i\tau - \lambda)^n. \tag{3.57}$$

This series converges in $B(X) = \{\text{bounded operator from } X \text{ to } X\}$ for $\|R(\sigma + i\tau : A)\||\sigma + i\tau| \leq k < 1$. Choosing $\lambda = Re\lambda + i\tau$ in (3.57) and using (3.15) we see that the series converges uniformly in $B(X)$ for $|\sigma - Re\lambda| \leq k|\tau|/C$. (

$$\begin{aligned}
\|R(\lambda : A)\| &= \sum_{n=0}^{\infty} \frac{C}{|\tau|}(\sigma + i\tau - \lambda)^n \left(\frac{C^{n+1}}{|\tau|^{n+1}}\right) \\
&\leq \sum_{n=0}^{\infty} \frac{C^{n+1}}{|\tau|^{n+1}} \frac{k^n|\tau|^n}{C^n} = \sum_{n=0}^{\infty} \frac{C}{|\tau|}k^n)
\end{aligned}$$

Since $\sigma > 0$ and $k < 1$ are arbitrary, taking $\sigma \to 0$ and $k \to 1$, it follows that the set of all λ with $Re\lambda \leq 0$ satisfying $|Re\lambda|/|Im\lambda| < 1/C$ and in particular

$$\underline{\rho(A) \supset \{\lambda : |arg\lambda| \leq \frac{\pi}{2} + \delta\}} \tag{3.58}$$

where $\delta = k \arctan 1/C, 0 < k < 1$. Moreover, in this region

$$\|R(\lambda : A)\| \leq \frac{C}{1-k} \cdot \frac{1}{|\tau|} \leq \frac{\sqrt{C^2+1}}{(1-k)} \frac{1}{|\lambda|} = \frac{M}{|\lambda|} \tag{3.59}$$

The second inequality comes from

$$\begin{aligned}
C^2(1 + \left(\frac{Re\lambda}{Im\lambda}\right)^2) &< C^2 + 1 \\
C^2\left((Im\lambda)^2 + (Re\lambda)^2\right) &< \tau^2(C^2+1) \quad (\tau = Im\lambda) \\
C|\lambda| &\leq |\tau|\sqrt{C^2+1}.
\end{aligned}$$

Since by assumption $0 \in \rho(A)$, A satisfies (c).

(3) $\Rightarrow$ (4). If A satisfies (c), it follows from **Theorem** 3.4 that

$$T(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}R(\lambda : A)d\lambda \tag{3.60}$$

where Γ is the path composed from the two rays $\rho e^{i\theta}$ and $\rho e^{-i\theta}, 0 < \rho < \infty$ and $\pi/2 < \theta < \pi/2 + \delta$ Γ is orientated so that $Im\lambda$ increases along Γ. The integral (3.60) converges in $B(X)$ for $t > 0$. ちゃんとこれは示す！！ .
Differentiating (3.60) w.r.t t yields

$$T'(t) = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t} R(\lambda : A)d\lambda. \tag{3.61}$$

The integral (3.61) converges in $B(X)$ for every $t > 0$ since

$$\|T'(t)\| \leq \frac{1}{\pi}\int_0^\infty M e^{-\rho\cos\theta t}d\rho = \left(\frac{M}{\pi\cos\theta}\right)\frac{1}{t}. \tag{3.62}$$

Therefore, the formal differentiation of $T(t)$ is justified, $T(t)$ is differentiable for $t > 0$ and

$$\|AT(t)\| = \|T'(t)\| \leq C/t \quad for \quad t > 0. \tag{3.63}$$

(4) $\Rightarrow$ (1).Since $T(t)$ is differentiable for $t > 0$, it follows from **Lemma** 3.6 that $\|T^{(n)}(t)\| = \|T'(t/n)^n\| \leq \|T'(t/n)\|^n$. Using this fact and (3.63) and $n!e^n \geq n^n$ we have

$$\begin{aligned} \frac{1}{n!}\|T^{(n)}(t)\| &\leq \frac{e^n}{n^n}\|T'(t/n)\|^n && (3.64) \\ &\leq \left(\frac{Ce}{t}\right)^n. && (3.65) \end{aligned}$$

We consider now the power series

$$T(z) = T(t) + \sum_{n=1}^{\infty}\frac{T^{(n)}(t)}{n!}(z-t)^n. \tag{3.66}$$

This series converges uniformly in $B(X)$ for $|z - t| \leq k(t/eC)$ for every $k < 1$ since

$$\begin{aligned} \frac{1}{n!}\|T^{(n)}(t)\||z-t|^n &< \left(\frac{Ce}{t}\right)^n |z-t|^n < 1 \\ &\Rightarrow |z-t| < t/eC \end{aligned}$$

Therefore, $T(z)$ is analytic in $\Delta = \{z : |argz| < \arctan(1/Ce)\}$.
Since for real value of z, $T(z) = T(t)$, $T(z)$ extends $T(t)$ to the sector Δ. But the analyticity of $T(z)$, it follows that $T(z)$ satisfies the semigroup $(T(z_1)T(z_2) = T(z_1 + z_2)$ for $z \in \mathbb{R}^+)$. From 3.66 one sees that $T(z)x \to x$ as $z \to 0$ in Δ. Finally, reducing the sector Δ to every closed subsector $\bar{\Delta_\epsilon} = \{z : |argz| \leq \arctan(1/Ce) - \epsilon\}$ we see that $\|T(z)\|$ is uniformly bounded in $\bar{\Delta_\epsilon}$. Finished. $\square$

Thanks to this theorem, we can define the subspace of $Gen(E)$, which is very important for the main proof.

Definition 3.7.

$Hol(E) \;:=\; \{A \in Gen(E);$ *The associated semigroup* e^{tA} *is an analytic semigroup*$\}$

3.2 Interpolation Space and Trace Method

3.3 K-method

Definition 3.8 (Intermediate Space)**.** *$\subset$ denotes continuous embedding here. If A, B, C are Banach spaces such that*

$$A \subset B \subset C, \tag{3.67}$$

then B is called an intermediate space between A and C.

Let $Y \subset X$, and let $c > 0$ be such that

$$\|y\| \leq c\|y\|_Y, \quad \forall y \in Y.$$

We describe the construction of a family of intermediate spaces between X and Y, called real interpolation spaces, denoted by $(X,Y)_{\theta,p}, (X,Y)_\theta$ with $0 < \theta \leq 1, 1 \leq \infty$. We suppose $1/\infty = 0$.

Definition 3.9. *For every $x \in X$ and $t > 0$, set*

$$K(t,x,X,Y) = \inf_{x=a+ba\in X, b\in Y}(\|a\|_X + t\|b\|_Y) \tag{3.68}$$

If there is no danger of confusion, we shall write $K(t,x)$ instead of $K(t,x,X,Y)$.

From (3.68) it follows that for every $t > 0$ and $x \in X$,

1. $\min\{1,t\}K(1,x) \leq K(t,x) \leq \max\{1,t\}K(1,x)$

2. $K(t,x) \leq \|x\|_X$

Definition 3.10 (K-method)**.** *Let $0 < \theta \leq 1, 1 \leq p \leq \infty$, and set*

$$\begin{cases} (X,Y)_{\theta,p} = \{x \in X : t \mapsto t^{-\theta-1/p}K(t,x,X,Y) \in L^p(0,\infty)\}, \\ \|x\|_{(X,Y)_{\theta,p}} = \|t^{-\theta-1/p}K(t,x,X,Y)\|_{L^p(0,\infty)} \end{cases} \tag{3.69}$$

$$(X,Y)_\theta = \{x \in X : \lim_{t\to 0} t^{-\theta}K(t,x,X,Y) = 0\} \tag{3.70}$$

Since $t \mapsto K(t,x)$ is bounded, only the behavior at $t = 0$ of $t^{-\theta}K(t,x)$ plays a role in the definition of $(X,Y)_{\theta,p}$ ad $(X,Y)_\theta$. So, one could replace $(0,\infty)$ by any interval $(0,a)$ in the Definition **??**.

For $\theta = 1$, from the first equality (**??**) we obtain

$$(X,Y)_1 = (X,Y)_{1,p} = \{0\}, \quad p < \infty.$$

Therefore, we consider the cases $(\theta,p) \in (0,1) \times [1,\infty]$ and $(\theta,p) = (1,\infty)$.

If $X = Y$, then $K(t,x) = \min\{t,1\}\|x\|$. Therefore, $(X,X)_{\theta,p} = (X,X)_{1,\infty} = X$ for $0 < \theta < 1, 1 \le p \le \infty$, and

$$\|x\|_{(X,X)_{\theta,p}} = \left(\frac{1}{p\theta(1-\theta)}\right)^{1/p} \|x\|_X, 0 < \theta < 1, p < \infty,$$
$$\|x\|_{(X,X)_{\theta,\infty}} = \|x\|_X, 0 < \theta \le 1.$$

some inclusion properties.

Proposition 3.1. *For $0 < \theta < 1, 1 \le p_1 \le p_2 \le \infty$ we have*

$$Y \subset (X,Y)_{\theta,p_1} \subset (X,Y)_{\theta,p_2} \subset (X,Y)_\theta \subset (X,Y)_{\theta,\infty} \subset \bar{Y} \tag{3.71}$$

For $0 < \theta_1 < \theta_2 \le 1$ we have

$$(X,Y)_{\theta_2,\infty} \subset (X,Y)_{\theta_1,1}. \tag{3.72}$$

Proof. **From $K(t,x) \le \min\{c,t\}\|x\|_Y$ for every $x \in Y$ it follows that Y is continuously embedded in $(X,Y)_{1,\infty}$ and in $(X,Y)_{\theta,p}$ for $0 < \theta < 1, 1 \le p \le \infty$.**

$$\begin{aligned} K(t,x) &= \inf_{x=a+b, a\in X, b\in Y} (\|a\|_X + t\|b\|_Y) \\ \inf_{x=a+, a\in X, b\in Y} (c\|a\|_Y + t\|b\|_Y) &= \min\{c,t\}\|x\|_Y \end{aligned}$$

Let us show that $(X,Y)_{\theta,\infty}$ is contained in $\bar{Y}$ and it is continuously embedded in X. For $x \in (X,Y)_{\theta,\infty}$ and for every $n \in \mathbb{N}$ there are $a_n \in X, b_n \in Y$ such that $x = a_n + b_n$

$$n^\theta(\|a_n\|_X + \frac{1}{n}\|b_n\|_Y) \le 2\|x\|_{\theta,\infty},$$

since

$$\begin{aligned} \|x\|_{\theta,\infty} &= \sup_{t>0} |t^{-\theta}K(t,x)| \le n^\theta K(\frac{1}{n}, x) \\ &= n^\theta(\inf \|a\|_X + \frac{1}{n}\|b_n\|_Y), \end{aligned}$$

and $\|\cdot\|_X, \|\cdot\|_Y$ are continuous.

In particular, $\|x - b_n\|_X = \|a_n\|_X \le 2\|x\|_{\theta,\infty} n^{-\theta}$, so that the sequence $\{b_n\}$ goes to x in X as $n \to \infty$. So, $(X,Y)_{\theta,\infty}$ is contained in $\bar{Y}$. Moreover, from

$$\|x\|_X \le \|a\|_X + \|b\|_X \le \|a\|_X + \|b\|_Y, \quad if\ x = a + b,$$

we obtain

$$\|x\|_X \le K(c,x) \le c^\theta \|x\|_{\theta,\infty}, \quad \forall x \in (X,Y)_{\theta,\infty}.$$

The second inequality comes from

$$
\begin{aligned}
t^{-\theta}K(t,x) &= \inf t^{-\theta}\|a\|_X + t^{1-\theta}\|b\|_Y \\
c^{-\theta}K(c,x) &= \inf c^{-\theta}\|a\|_X + c^{1-\theta}\|b\|_Y
\end{aligned}
$$

so, $(X,Y)_{\theta,\infty}$ is continuously embedded in X.
The inclusion $(X,Y)_\theta \subset (X,Y)_{\theta,p}$ is trivial, since $K(\cdot,x)$ is bounded.

Let us show that $(X,Y)_{\theta,p}$ is contained in $(X,Y)_\theta$ and it is continuously embedded in $(X,Y)_{\theta,\infty}$ for $p<\infty$. Note that $K(\cdot,x)$ satisfies

$$K(t,x) \le \frac{t}{s}K(s,x) \ for \ x \in X, 0<s<t.$$

Therefore, for each $x \in (X,Y)_{\theta,p}$ and $t>0$

$$
\begin{aligned}
t^{1-\theta}K(t,x) &= [(1-\theta)p]^{1/p}\left(\int_0^t s^{(1-\theta)p-1}ds\right)^{1/p} K(t,x) \\
&= [(1-\theta)p]^{1/p}\left(\int_0^t s^{(1-\theta)p-1}K(t,x)^p ds\right)^{1/p} \\
&\le [(1-\theta)p]^{1/p}\left(\int_0^t s^{(1-\theta)p-1}\frac{t^p}{s^p}K(s,x)^p ds\right)^{1/p} \\
&\le [(1-\theta)p]^{1/p}\left(\int_0^t s^{-\theta p-1}t^p K(s,x)^p ds\right)^{1/p},
\end{aligned}
$$

so that

$$t^{-\theta}K(t,x) \le [(1-\theta)p]^{1/p}\left(\int_0^t s^{-\theta p-1}K(s,x)^p ds\right)^{1/p}$$

Letting $t\to 0$ it follows that $x \in (X,Y)_\theta$. The same inequality yields

$$\|x\|_{\theta,\infty} \le [(1-\theta)p]^{1/p}\|x\|_{\theta,p} \tag{3.73}$$

The next goal is that $(X,Y)_{\theta,p_1} \subset (X,Y)_{\theta,p_2}$ for $p_2<p_1$. For $x \in (X,Y)_{\theta,p_1}$ we have

$$
\begin{aligned}
\|x\|_{\theta,p_2} &= \left(\int_0^\infty t^{-\theta p_2-1}K(t,x)^{p_2}dt\right)^{1/p_2} = \left(\int_0^\infty t^{-\theta p_1-1}t^{\theta p_1-\theta p_2}K(t,x)^{p_1}K(t,x)^{-p_1+p_2}dt\right)^{1/p_2} \\
&\le \left(\int_0^\infty t^{-\theta p_1-1}K(t,x)^{p_1}dt\right)^{1/p_2}\left(\sup_{t>0} t^{-\theta}K(t,x)\right)^{(p_2-p_1)/p_1} \\
&= (\|x\|_{\theta,p_1})^{p_1/p_2}(\|x\|_{\theta,\infty})^{1-p_1/p_2},
\end{aligned}
$$

and using (3.73) we find

$$
\begin{aligned}
\|x\|_{\theta,p_2} &\leq (\|x\|_{\theta,p_1})^{p_1/p_2}\left(((1-\theta)p_1)^{1/p_1}\|x\|_{\theta,p_1}\right)^{1-p_1/p_2} \\
&= [(1-\theta)p_1]^{1/p_1-1/p_2}\|x\|_{\theta,p_1}
\end{aligned}
$$

Let us prove that (3.72) holds. If $0<\theta_1<\theta_2\leq 1$ and $x\in(X,Y)_{\theta_2,\infty}$, we have

$$
\begin{aligned}
\|x\|_{\theta_1,1} &= \int_0^1 t^{-\theta_1-1}K(t,x)dt+\int_1^\infty t^{-\theta_1-1}K(t,x)dt \\
&\leq \int_0^1 t^{-\theta_1-1}\|x\|_{\theta_2,\infty}t^{\theta_2}dt+\int_1^\infty t^{-\theta_1-1}\|x\|_X dt \\
&\leq \frac{1}{\theta_2-\theta_1}\|x\|_{\theta_2,\infty}+\frac{1}{\theta_1}\|x\|_X \qquad (3.74)
\end{aligned}
$$

The statement is proved.

□

Proposition 3.2. *$(X,Y)_{\theta,p}$ is a Banach space.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X,Y)_{\theta,p}$. Due to the continuous embedding of $(X,Y)_{\theta,p}$ in X, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X too, so that it converges to $x\in X$.

Let us estimate $\|x_n-x\|_{\theta,p}$. Fix $\epsilon>0$, and let $\|x_n-x_m\|_{\theta,p}\leq\epsilon$ for $n,m\geq n_\epsilon$. Since $y\mapsto K(t,y)$ is a norm, for every $n,m\in\mathbb{N}$ and $t>0$ we have

$$t^{-\theta}K(t,x_n-x)\leq t^{-\theta}K(t,x_n-x_m)+t^{-\theta}\|x_n-x\|_X \qquad (3.75)$$

since (**??**).

Let $p=\infty$. Then for every $t>0$ and $n.m\geq n_\epsilon$

$$t^{-\theta}K(t,x_n-x)\leq\epsilon+t^{-\theta}\|x_m-x\|_X. \qquad (3.76)$$

from the definition of $\|\cdot\|_{\theta,p}$.

Letting $m\to\infty$, we find $t^{-\theta}K(t,x_n-x)\leq\epsilon$ for every $t>0$. This implies that $x\in(X,Y)_{\theta,\infty}$ and that $x_n\to x$ in $(X,Y)_{\theta,\infty}$.

Consider the case $p<\infty$. Then as the definition of the norm,

$$\|x_n-x\|_{\theta,p}=\lim_{\delta\to 0}\left(\int_\delta^{1/\delta}t^{-\theta p-1}K(t,x_n-x)^p dt\right)^{1/p}$$

Due again to (3.75)**, for every** $\delta\in(0,1)$ **and** $n,m\geq n_\epsilon$**,**

$$
\begin{aligned}
\left(\int_\delta^{1/\delta} t^{-\theta p-1}K(t,x_n-x)^p dt\right)^{1/p} &\leq \|x_n-x_m\|_{\theta,p}+\|x_m-x\|_X\left(\int_\delta^{1/\delta} t^{-\theta p-1}dt\right)^{1/p}. \\
&\leq \epsilon+\|x_m-x\|_X\left(\frac{1}{\theta p\delta^{\theta p}}\right)^{1/p}.
\end{aligned}
$$

これはなぜか？？？？

Letting $m\to\infty$ and then $\delta\to 0$（この操作をするために δ を用いた。）we get $x\in(X,Y)_{\theta,p}$ and $x_n\to x$ in $(X,Y)_{\theta,p}$. □

Corollary 3.1. *For $0<\theta\leq 1$, $(X,Y)_\theta$ is a Banach space, endowed with the norm of $(X,Y)_{\theta,\infty}$.*

Proof. $(X,Y)_\theta$ is a closed subspace of $(X,Y)_{\theta,\infty}$. Since $(X,Y)_{\theta,\infty}$ is complete, then also $(X,Y)_\theta$ is complete. □

Proposition 3.3. *Let X_1,X_2,Y_1,Y_2 be Banach spaces, such that Y_i is continuously embedded in X_i, for $i=1,2$. If $T\in L(X_1,X_2)\cap L(Y_1,Y_2)$, then $T\in L((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p})\cap L((X_1,Y_1)_\theta,(X_2,Y_2)_\theta)$ for every $\theta\in(0,1)$ and $p\in[1,\infty]$, and for $(\theta,p)=(1,\infty)$. Moreover,*

$$
\|T\|_{L((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p})}\leq\left(\|T\|_{L(X_1,X_2)}\right)^{1-\theta}\left(\|T\|_{L(Y_1,Y_2)}\right)^{\theta}. \tag{3.77}
$$

Proof. Since the case $T=0$ is trivial, so we assume $T\neq 0$. Let $x\in(X_1,Y_1)_{\theta,p}$: then for every $a\in X_1, b\in Y_1$ such that $x=a+b$ and for every $t>0$ we have

$$
\begin{aligned}
\|Ta\|_{X_2}+t\|Tb\|_{Y_2} &\leq \|T\|_{L(X_1,X_2)}\|a\|_{X_2}+t\|T\|_{L(Y_1,Y_2)}\|b\|_{Y_2} \\
&\leq \|T\|_{L(X_1,X_2)}\left(\|a\|_{X_1}+t\frac{\|T\|_{L(Y_1,Y_2)}}{\|T\|_{L(X_1,X_2)}}\|b\|_{Y_1}\right),
\end{aligned}
$$

so that

$$
K(t,Tx,X_2,Y_2)\leq\|T\|_{L(X_1,X_2)}K\left(t\frac{\|T\|_{L(Y_1,Y_2)}}{\|T\|_{L(X_1,X_2)}},x.X_1,Y_1\right). \tag{3.78}
$$

Setting $s=t\frac{\|T\|_{L(Y_1,Y_2)}}{\|T\|_{L(X_1,X_2)}}$, we get $Tx\in(X_2,Y_2)_{\theta,p}$, and

$$
\begin{aligned}
\|Tx\|_{(X_2,Y_2)_{\theta,p}} &= \int_0^\infty t^{-\theta p-1}K(t,Tx)^p dt \\
&\le \int_0^\infty t^{-\theta p-1}\|T\|^p_{L(X_1,X_2)}K\left(t\frac{\|T\|_{L(Y_1,Y_2)}}{\|T\|_{L(X_1,X_2)}}, x.X_1,Y_1\right)^p dt \\
&= \int_0^\infty \left(\frac{\|T\|_{L(X_1,X_2)}s}{\|T\|_{L(Y_1,Y_2)}}\right)^{-\theta p-1}\|T\|^p_{L(X_1,X_2)}K(s,x)^p\frac{\|T\|_{L(X_1,X_2)}s}{\|T\|_{L(Y_1,Y_2)}}ds \\
&= \|T\|_{L(X_1,X_2)}\left(\frac{\|T\|_{L(Y_1,Y_2)}}{\|T\|_{L(X_1,X_2)}}\right)^\theta \|x\|_{(X_1,Y_1)_{\theta,p}}.
\end{aligned}
$$

and (3.77) follows. From (3.78) it follows that

$$
\lim_{t\to 0} t^{-\theta}K(t,x,X_1,Y_1) = 0 \Longrightarrow \lim_{t\to 0} t^{-\theta}K(t,Tx,X_2,Y_2) = 0, \tag{3.79}
$$

that is, T maps $(X_1,Y_1)_\theta$ into $(X_2,Y_2)_\theta$ □

Corollary 3.2. *For $0 < \theta < 1, 1 \le p \le \infty$ and for $(\theta,p) = (1,\infty)$ there is $c(\theta,p)$ such that*

$$
\|y\|_{(X,Y)_{\theta,p}} \le c(\theta,p)\|y\|_X^{1-\theta}\|y\|_Y^\theta \quad \forall y \in Y. \tag{3.80}
$$

Proof. $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, according to the fact that X is a real or a complex Banach space. Take $y \in Y$, and define $T : \mathbb{K} \to X$, by $T(\lambda) = \lambda y$ for each $\lambda \in \mathbb{K}$. Then $\|T\|_{L(\mathbb{K},X)} = \|y\|_X, \|T\|_{L(\mathbb{K},Y)} = \|y\|_Y$, and $\|T\|_{L(\mathbb{K},(X,Y)_{\theta,p})} = \|y\|_{(X,Y)_{\theta,p}}$. Since clearly $(\mathbb{K},\mathbb{K})_{\theta,p} = \mathbb{K}$, the statement holds. □

Since for $y \in (X,Y)_\theta$, $\|T\|_\theta \le \|T\|_{\theta,\infty}$ as $(X,Y)_\theta$ is a closed subspace of $(X,Y)_{\theta,p}$. So this statement holds for $(X,Y)_\theta$.

3.4 The Trace Method

Under this condition, we can construct their interpolation space, denoted by $(X,Y)_{\theta,p}, (X,Y)_\theta$ for $0 < \theta \le 1, 1 \le p \le \infty$. Actually, there are a few methods to construct the interpolation space, e.g., K-method, or J-method. In this section, for the convenience, this section is dedicated to explaining about "Trace Method".

Definition 3.11. *For $0 \le \theta < 1$ and $1 \le p \le \infty$ set*

$$
\begin{aligned}
V(p,\theta,Y,X) &= \{u : \mathbb{R}^+ \to X : t \mapsto u_\theta(t) = t^{\theta-1/p}u(t) \in L^p(0,+\infty;Y), \\
t &\mapsto v_\theta(t) = t^{\theta-1/p}u'(t) \in L^p(0,\infty;X)\}, \qquad (3.81) \\
\|u\|_{V(p,\theta,Y,X)} &= \|u\|_{L^p(0,\infty;Y)} + \|v_\theta\|_{L^p(0,\infty;X)}. \qquad (3.82)
\end{aligned}
$$

Moreover, for $p=\infty$ we define a subspace of $V(\infty,\theta,Y,X)$, by

$$V_0(\infty,\theta,Y,X)=\{u\in V(\infty,\theta,Y,X):\lim_{t\to 0}\|t^\theta u(t)\|_X=\lim_{t\to 0}t^\theta u'(t)\|_Y=0\} \tag{3.83}$$

Theorem 3.6 (the Hardy-Young inequality)**.** *For positive measurable function $\varphi:(0,a)\to\mathbb{R}, 0<a\leq\infty$, and $\alpha>0, p\geq 1$, the inequalities*

1.
$$\int_0^a t^{-\alpha p}\left(\int_0^t \varphi(s)\frac{ds}{s}\right)^p\frac{dt}{t}\leq\frac{1}{\alpha^p}\int_0^a s^{-\alpha p}\varphi(s)^p\frac{ds}{s},$$

2.
$$\int_0^a t^{\alpha p}\left(\int_t^a \varphi(s)\frac{ds}{s}\right)^p\frac{dt}{t}\leq\frac{1}{\alpha^p}\int_0^a s^{\alpha p}\varphi(s)^p\frac{ds}{s},$$

hold.

Corollary 3.3. *Let u be a function such that $t\mapsto u_\theta(t)=t^{\theta-1/p}u(t)$ belongs to $L^p(0,a;X)$, with $0<a\leq\infty, 0<\theta<1$ and $1\leq p\leq\infty$. Then the mean value*

$$v(t)=\frac{1}{t}\int_0^t u(s)ds,\quad t>0 \tag{3.84}$$

has the same property, and setting $v_\theta(t)=t^{\theta-1}v(t)$ we have

$$\|v_\theta\|_{L^p(0,a;X)}\leq\frac{1}{1-\theta}\|u_\theta\|_{L^p(0,a;X)} \tag{3.85}$$

Proof.

$$\begin{aligned}\int_0^a |v(t)|^p dt &= \int_0^a t^{-(1-\frac{1}{p})p}\left(\int_0^t su(s)\frac{ds}{s}\right)^p dt\\ &\leq \frac{1}{\left(1-\frac{1}{p}\right)^p}\int_0^a s^{1-p}(su(s))^p\frac{ds}{s}\\ &= \frac{1}{\left(1-\frac{1}{p}\right)^p}\int_0^a u(s)^p ds<\infty\end{aligned}$$

Therefore, $v(t)\in L^p(0,a;X)$.
Moreover,

$$\begin{aligned}
\|v_\theta(t)\|^p_{L^p(0,a;X)} &= \int_0^a t^{p\theta} v(t)^p \frac{dt}{t} \\
&= \int_0^a t^{-(1-\theta)p} \left(\int_0^t s u(s) \frac{ds}{s} \right)^p \frac{dt}{t}
\end{aligned}$$

$$\frac{1}{(1-\theta)^p} \int_0^a s^{-(1-\theta)p} (su(s))^p \frac{ds}{s}$$

$$= \frac{1}{(1-\theta)^p} \int_0^a (u_\theta(t))^p ds$$

□

Then, we can show the important theorem in the trace method.

Theorem 3.7. *For $(\theta, p) \in]0,1[\times [1,\infty] \cup \{(1,\infty)\}$, $(X,Y)_{\theta,p}$ constructed by the other real interpolation method coincides with the set of the traces at $t = 0$ of the functions on $V(p, 1-\theta, Y, X)$, Furthermore, the norm*

$$\|x\|^T_{\theta,p} = \inf\{\|u\|_{V(p,1-\theta,Y,X)} : x = u(0), u \in V(p, 1-\theta, Y, X)\} \tag{3.86}$$

is an equivalent norm in $(X,Y)_{\theta,p}$. Moreover, for $0 < \theta < 1$, $(X,Y)_\theta$ is the set of the trace at $t = 0$ of the function in $V_0(\infty, 1-\theta, Y, X)$.

Proof. Let $x \in (X,Y)_{\theta,p}$. For everyn $n \in \mathbf{N}$ let a_n, b_n be such that $a_n + b_n = x$, and

$$\|a_n\|_X + \frac{1}{n}\|b_n\|_Y \le 2K(\frac{1}{n}, x). \tag{3.87}$$

For $t > 0$ set

$$u(t) = \sum_{n=1}^{\infty} b_{n+1} \chi_{]\frac{1}{n+1},\frac{1}{n}]}(t) = \sum_{n=1}^{\infty} (x - a_{n+1}) \chi_{]\frac{1}{n+1},\frac{1}{n}]}(t),$$

where χ_I is the characteristic function of I, and

$$v(t) = \frac{1}{t} \int_0^t u(s) ds. \tag{3.88}$$

Since $(X,Y)_{\theta,p} \subset (X,Y)_{\theta,\infty}$, then by the definition of $(X,Y)_{\theta,\infty}$, $\lim_{t\to 0} K(t,x) = 0$.

In particular, from 3.87, $x = \lim_{n\to\infty} b_n$, so that $x = \lim_{t\to 0} u(t) = \lim_{t\to 0} v(t)$. Moreover,

$$\underline{\|t^{1-\theta}u(t)\|_Y \le t^{1-\theta} \sum_{n=1}^{\infty} \chi_{]\frac{1}{n+1},\frac{1}{n}]}(t) 2(n+1) K(1/(n+1), x) \le 4t^{-\theta} K(t,x)} \tag{3.89}$$

so that $t^{1-\theta-1/p}u(t) \in L^p(0,\infty;Y)$. By Corollary 3.3, $t^{1-\theta-1/p}v(t) \in L^p(0,\infty;Y)$, and

$$\underline{\|t^{1-\theta-1/p}v\|_{L^p(0,\infty;Y)} \leq 4\theta^{-1}\|x\|_{\theta,p}}. \tag{3.90}$$

The reason why the second inequality holds is that suppose $1/(n+1) < t \leq 1/n$, then

$$t\chi_{]\frac{1}{n+1},\frac{1}{n}]}(t)2(n+1)K(1/(n+1),x) \leq \frac{1}{n}2(n+1)K(1/(n+1),x) \leq 4K(t,x)$$

since $K(t,x)$ is monotonically increasing.

On the other hand,

$$v(t) = x - \frac{1}{t}\int_0^t \sum_{n=1}^{\infty} \chi_{]\frac{1}{n+1},\frac{1}{n}}(s)a_{n+1}ds \tag{3.91}$$

so that v is differentiable almost everywhere with values in X(see χ_I). So denote $g(t) = \sum_{n=1}^{\infty} \chi_{]\frac{1}{n+1},\frac{1}{n}}(s)a_{n+1}$, and

$$v'(t) = \frac{1}{t^2}\int_0^t g(s)ds - \frac{1}{t}g(t).$$

$g(t)$ can be evaluated in a similar way as (3.89):

$$\|g(t)\|_X \leq t^{-\theta}\sum_{n=1}^{\infty} \chi_{]\frac{1}{n+1},1/n}(t)2K(1/(n+1),x) \leq 2K(t,x). \tag{3.92}$$

And as for $t^{-\theta}g(t)$, suppose $1/(n+1) < t \leq 1/n$,

$$t^{-\theta}\sum_{n=1}^{\infty} \chi_{]\frac{1}{n+1},\frac{1}{n}}(s)a_{n+1} \quad \leq \quad \left(\frac{1}{n+1}\right)^{-\theta} a_{n+1} \leq t^{-\theta}2K(t,x).$$

Thus it follows that

$$\|t^{1-\theta}v'(t)\| \leq t^{-\theta}\sup_{0<s<t}\|g(s)\| + \|t^{-\theta}g(t)\| \leq 4t^{-\theta}K(t,x). \tag{3.93}$$

Then $t^{1-\theta-1/p}v'(t) \in L^p(0,\infty;X)$, and

$$\underline{\|t^{1-\theta-1/p}v'\|_{L^p(0,\infty;X)} \leq 4\|x\|_{\theta,p}}$$

Therefore, x is the **trace** at $t=0$ of a function $v \in V(p, 1-\theta, Y, X)$, and

$$\begin{aligned}
\|x\|_{\theta,p}^T &\leq \|u_{1-\theta}\|_{L^p} + \|u'_{1-\theta}\|_{L^p}(\text{ for some } u \in V(p,1-\theta,Y,X), u(0)=x) \\
&\leq \|t^{1-\theta-1/p}u\|_{L^p} + \|t^{1-\theta-1/p}u'\|_{L^p} \\
&\leq 4\theta^{-1}\|x\|_{\theta,p} + 4\|x\|_{\theta,p}
\end{aligned}$$

If $x \in (X,Y)_\theta$, then by (3.89), $\lim_{t\to 0} t^{1-\theta}\|u(t)\|_Y = 0$, so that $\lim_{t\to 0} t^{1-\theta}\|v(t)\|_Y = 0$. By (3.93), $\lim_{t\to 0} t^{-\theta}\|g(t)\|_X = 0$, so that $\lim_{t\to 0} t^{1-\theta}\|v'(t)\|_X = 0$. Then $v \in V_0(\infty, 1-\theta, Y, X)$.

Conversely, let x be the trace of a function $u \in V(p, 1-\theta, Y, X)$. Then

$$x = x - u(t) + u(t) = -\int_0^t u'(s)ds + u(t) \quad \forall t > 0, \tag{3.94}$$

so that

$$\begin{aligned} t^{-\theta}K(t,x) &\leq t^{-\theta}\|x\|_X \\ &\leq \underline{t^{1-\theta}\|\frac{1}{t}\int_0^t u'(s)ds\|_X + t^{1-\theta}\|u(t)\|_Y} \end{aligned} \tag{3.95}$$

From Corollary 3.3, $t^{-\theta-1/p}K(t,p)$ belongs to $L^p(0,\infty)$, so that $x \in (X,Y)_{\theta,p}$, and

$$\underline{\|x\|_{\theta,p} \leq \frac{1}{\theta}\|x\|^T_{\theta,p}.} \tag{3.96}$$

If x is the trace of a function $u \in V_0(\infty, 1-\theta, Y, X)$, then, by (3.95), $\lim_{t\to 0} t^{-\theta}K(t,x) = 0$, so that $x \in (X$

$\square$

Thanks to this theorem, we can identify the interpolation space by the K-method with the one by the trace method. So, let us denote generally such space as $(X,Y)_{\theta,p}$ for $1 \leq p < \infty$ or $(X,Y)_\theta$ for $p = \infty$

3.5 The Maximal Regularity Class and Its Application

Let $E = (E_1, E_0)$ be a pair of real Banach spaces for which E_1 is densely included by E_0 in this section, In the semigroup theory section, I will define the operator space $Gen(E)$ and $Hol(E)$ in (3.6) and (3.7).
In this section, I will discuss the maximal regularity class, which is a very useful subspace of $Gen(E)$ and $Hol(E)$ for proving the existence and uniqueness of solutions of nonlinear inhomogeneous equations.

$$\mathcal{L}(E_1, E_0) = \{\text{bounded linear operator from } E_1 \text{ to } E_0\} \tag{3.97}$$

As stated in the previous sections, we are already ready for defining the following operator spaces, especially from Hille-Yoshida Theorem (3.1) and Theorem 3.2.

$$\begin{aligned} Gen(E) &:= \{A \in \mathcal{L}(E_1, E_0) : A \text{ is an infinitesimal generator of} C_0\text{-semigroup}\} \\ Hol(E) &:= \{A \in Gen(E);\ \text{The associated semigroup } e^{tA} \text{ is an analytic semigroup}\} \end{aligned}$$

Given $E = (E_1, E_0)$, we want to construct the so-called "continuous interpolation spaces" $E_\theta, \theta \in [0,1]$.

The procedure is as follows. First of all, for $\theta = 1$, we put

$$\begin{aligned} X_1 &:= C([0,1]; E_0) \\ Y_1 &:= C([0,1]; E_1) \cap C^1([0,1]; E_0). \end{aligned}$$

In this time, we can construct the interpolation space by the trace method introduced in the previous section.
In particular,

$$\begin{aligned} X_\theta(E) &:= \left\{ u \in C((0,1]; E_0); \lim_{t\to 0} \|t^{1-\theta} u(t)\|_{E_0} = 0 \right\} \\ Y_\theta(E) &:= \left\{ u \in C((0,1]; E_1) \cap C^1((0,1]; E_0); \lim_{t\to 0} t^{1-\theta}(\|u'(t)\|_{E_0} + \|u(t)\|_{E_1}) = 0 \right\} \end{aligned}$$

And each norm of these spaces can be set as:

$$\begin{aligned} \|u\|_{X_\theta(E)} &= \sup_{0<t\le 1} \|t^{1-\theta} u(t)\|_{E_0} \\ \|u\|_{Y_\theta(E)} &= \sup_{0<t\le 1} \{\|t^{1-\theta} u(t)\|_{E_1} + \|t^{1-\theta} u'(t)\|_{E_0}\}. \end{aligned}$$

For simplicity, denote $X_\theta(E)$ and $Y_\theta(E)$ as X and Y.

From the definition of Y, we can show $\|u'(t)\|_{E_0} \le Ct^{1-\theta}$ for $C > 0$. Otherwise,

$$\lim_{t\to 0} t^{1-\theta}(\|u'(t)\|_{E_0} + \|u(t)\|_{E_1}) \quad > \quad \lim_{t\to 0}(C + t^{1-\theta}\|u\|_{E_0}) > 0.$$

This is a contradiction.

In this time, $u(0)$ is well-defined by $u(0) = u(1) - \int_0^1 u'(\tau) d\tau$. Then we can define the Banach space.

Definition 3.12. *For $\theta \in (0,1)$,*

$$E_\theta := \{u(0); u \in Y_\theta(E)\}. \quad \|x\|_{E_\theta} = \inf\{\|u\|_Y | x = u(0), u \in Y\} \tag{3.98}$$

Nextly, we will define the most important linear operator space for the main proof.

3.5.1 Linear Case

Write the differential operator from Y to X:

$$\partial_t : Y \to X, \quad (\partial_t u)(t) = u'(t).$$

$\|\partial_t u\|_X = \sup_{0<t\le 1} \|t^{1-\theta} u(t)\|_{E_0} < \infty$ because of the definition of Y.

Then we proceed to the definition:

Definition 3.13 (Maximal Regularity Class)**.** *Given $A \in \mathcal{L}(E_1, E_0)$. Then define the operator $\hat{A} : Y \to X \oplus E_\theta$ by*

$$\hat{A}u = (\partial_t u(t) - Au(t), u(0)).$$

In this situation, the class $\mathcal{M}_\theta(E)$ is defined as:

$$\mathcal{M}_\theta(E) := \{A \in Hol(E) | \hat{A} \text{ is an isomorphism between} Y \text{ and } X \oplus E_\theta\} \quad (3.99)$$

This class is called the maximal regularity class.
For $A \in \mathcal{M}_\theta(E)$, $E = (E_1, E_0)$ is called maximal parabolic regularity pair.

in other words, $\mathcal{M}_\theta(E)$ contains generators of analytic semigroups A for which

$$x'(t) - Ax(t) = f(t) \quad (0 < t \leq 1), \quad (3.100)$$
$$x(0) = x_0 \quad (3.101)$$

has a unique solution $x \in X$ for any $f \in X$ and $x_0 \in E_\theta$.

The solution of (3.100), if it exists, is given by the "variation of contants formula",

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-\tau)A} f(\tau) d\tau \quad (3.102)$$

as an E_0-valued function.

We will state some of the basic properties of $\mathcal{M}_\theta(E)$. Since the invertible elements of $\mathcal{L}(Y, X \oplus E_\theta)$ form an the continuity of the mapping $A \to \hat{A}$ implies

Lemma 3.8. *$\mathcal{M}_\theta(E)$ is an open subset of $Hol(E)$, and hence also an open subset of $\mathcal{L}(E_1, E_0)$.*

Lemma 3.9. *Denote $Y^{(0)} = \{u \in Y | u(0) = 0\}$*
If $A \in Hol(E)$, then $A \in \mathcal{M}_\theta(E)$ if and only if $\partial_t - A : Y^{(0)} \to X$ is an isomorphism.

Proof. For any $x_0 \in E_0$, there is a $y \in Y$ such that $y(0) = x_0$. If we rewrite $x(t) = y(t) + z(t)$, (3.100) is equivalent to $z'(t) - Az(t) = f(t) + y'(t) - Ay(t), z(0) = 0$. Since $\lim_{t \Downarrow 0} \|y'(t) - Ay(t)\|_{E_0} \leq \lim_{t \Downarrow 0} t^{1-\theta}(\|y'(t)\|_{E_0} + \|Ay(t)\|_{E_0}) = 0 (\because y \in Y)$, $y'(t) - Ay(t) \in X$, this problem has a solution by assumption. This show sufficiency: The necessity is clear. □

If we replace the interval $[0, 1]$ in the definition of X and Y by some other interval $[0, T], (T > 0)$, we obtain X_T and Y_T. In the same way as $\mathcal{M}_\theta$, we define the new class $\mathcal{M}_{\theta,T}(E)$. But actually, this new definition does not give us anything new.

Lemma 3.10. *1. $\mathcal{M}_{\theta,T}(E) = \mathcal{M}_\theta(E)$ for all $T > 0$.*

2. $\mathcal{M}_\theta(E) \subset \mathcal{M}_1(E)$ for all $\theta \leq 1$.

Proof. Let $A \in \mathcal{M}_{\theta,T}$ be given. Then $x'(t) - Ax(t) = w(t)(0 < t \leq T), x(0) = 0$ has a unique solution $x \in Y^{(0)}_{\theta,T}$ for any $w \in X_{\theta,T}$. The same problem is also solvable on any smaller interval $[0, T'] \leq [0, T]$. Since $A \in Hol(E)$, we also have unique solvability on any interval. Hence $\mathcal{M}_{\theta,T'} \subset \mathcal{M}_{\theta,T}$ whenever $T' \leq T$.

Nextly, we show that $\mathcal{M}_{\theta,T} \subset \mathcal{M}_{1,T'}$ for any $T' < T$. Suppose $A \in \mathcal{M}_{\theta,T}$ and $f \in C([0, T'], E_0)$ be given. Then our goal is to show that (3.100) has a unique solution $x \in Y^{(0)}_{\theta,T'}$. If $f(t)$ were constant, say $f(t) \equiv f_* \in E_0$, then the solution is given by $x(t) = \int_0^t e^{\tau A} f_* d\tau$, which belongs to $Y_{1,T} = C([0, 1]; E_1) \cap C^1([0, 1]; E_0)$. Thus it is justifiable to assume that $f(0) = 0$. Define $g \in C([0, T]; E_0)$ by

$$g(t) = \begin{cases} 0 & (0 \leq t \leq T - T'), \\ f(t - T + T') & (T - T' \leq t \leq T). \end{cases} \tag{3.103}$$

Since $f(t)$ is defined at $t = 0$, we have $g \in X_{\theta,T} = \left\{u \in C((0, 1]; E_0) \,|\, \lim_{t\Downarrow 0} \|t^{1-\theta} u(t)\|_{E_0} = 0\right\}$. S, there is a solution $y \in Y^{(0)}_{\theta,T}$ of $y'(t) - Ay(t) = g(t), y(0) = 0$. By uniqueness of the solution, this solution vanishes for $t \leq T - T'$, and $x(t) = y(t - T + T')$ is the desired solution of (3.100). Thus $\mathcal{M}_{\theta,T}(E) \subset \mathcal{M}_{1,T'}(E)$ for $T' < T$.

We conclude the proof by showing that $\mathcal{M}_{\theta,T'} \subset \mathcal{M}_{\theta,T}$ whenever $T' < T$. Let $A \in \mathcal{M}_{\theta,T'}$, and $f \in X_{\theta,T}$ be given. Firstly, solve the equation $x(t)$ on the interval $[0, T']$ since $A \in \mathcal{M}_{\theta,T'}$.

Then we choose a $T_1 < T'$ such that $n = (T - T')/T_1$ is an integer. Since $A \in \mathcal{M}_{\theta,T'} \subset \mathcal{M}_{1,T_1}$, by solving $x'(t) = Ax(t) + f(t)$ on $[T', T' + T_1]$ with initial value $x(T')$, we can ext
なぎ目のところで微分不可能の場合があるのでは？？ 少し定義域をずらす可能性を考えてもよいのでは？？） By repeating this process, we can extend $x(t)$ to $[0, T' + kT]$ for $k = 2, 3, \ldots, n$. So we get $x \in Y_{\theta,T}$. $\square$

From here, denote $J_{A,T}$ as the inverse of the operator $\partial_t - A$, i.e., $J_{A,T} : X_{\theta,T} \to Y^{(0)}_{\theta,T}$ and it satisfies $J_{A,T}(\partial_t - A) = (\partial_t - A)J_{A,T} = I$.

Lemma 3.11. *The operator norm of $J_{A,T}$ is a nondecreasing function of T.*

Proof. By the variation of contants formula, $J_{A,T} f(t) = \int_0^t e^{(t-\tau)A} f(\tau) d\tau, (0 \leq t \leq T)$. So we can write $J_{A,T'}$ as $\rho \circ J_{A,T} \circ \epsilon$, where $\rho : Y^{(0)}_{\theta,T} \to Y^{(0)}_{\theta,T'}$ is the restriction operator, and $\epsilon : X_{\theta,T'} \to X_{\theta,T}$ is an extension operator. If we define $\epsilon(f)(t) = f(t)$ for $t \leq T'$, and $\epsilon(f)(t) = (T'/t)^{1-\theta} f(T')$ for $T' \leq t \leq T$, then ϵ is a contraction, and since ρ is also a contraction one obtains $\|J_{A,T'}\| \leq \|J_{A,T}\| <$ if $T' \leq T$. $\square$

3.5.2 A Perturbation for $\mathcal{M}_\theta(E)$

If $A \in Hol(E)$ and $B : E_1 \to E_0$ is a "relatively bounded perturbation with relative bound zero", then $A + B$ also belongs to $Hol(E)$. We will discuss the case for $\mathcal{M}_\theta(E)$.

Definition 3.14 (Relative Norm)**.** *The relative norm of a bounded operator $B : E_1 \to E_0$ (with respect to the Banach couple $E = (E_1, E_0)$) is the infimum over $C \in \mathbb{R}$ such that, for any $\epsilon > 0$, there is a $k_\epsilon > 0$ with $\|Bx\|_{E_0} \leq (C + \epsilon)\|x\|_{E_1} + k_\epsilon\|x\|_{E_0}$.*

E.g. 3.1. *If the operator B satisfies an inequality like $\|B\|_{E_0} \leq C\|x\|_{E_1}^{\lambda}\|x\|_{E_0}^{1-\lambda}$ for all $x \in E_1$ and some constant $\lambda \in (0, 1)$, then it follows from the identity*

$$x^\lambda y^{1-\lambda} = \inf_{s>0}[\lambda s^{1-\lambda} x + (1-\lambda)s^\lambda y].$$

This equation comes from Young's inequality and its equality condition. Therefore,

$$\|Bx\|_{E_0} \leq C(\lambda s^{1-\lambda}\|x\|_{E_1} + (1-\lambda)s^\lambda\|x\|_{E_0})(s > 0) \tag{3.104}$$

B has relative norm zero.

Lemma 3.12. *If $B \in \mathcal{L}(E_1, E_0)$ has relative norm zero, then for any $A \in \mathcal{M}_\theta(E)$ $A + B \in \mathcal{M}_\theta(E)$.*

Proof. □

3.5.3 Fully Nonlinear Case

For fully nonlinear cases, letting $\mathcal{O} \subset E_1$ be an open subset, f be $C^k(k = 1, 2 \ldots,)$ $f : \mathcal{O} \to E_0$,set

$$x'(t) = f(x(t))(0 \leq t \leq T) \tag{3.105}$$
$$x(0) = x_0. \tag{3.106}$$
$$\tag{3.107}$$

A strict solution $x(t)$ of (3.105) is in $C^1([0, T]; E_0) \cap C^0([0, T]; E_1)$.

Theorem 3.8. *If the Frechet derivative $df(x) \in \mathcal{L}(E_1, E_0)$ belongs to $\mathcal{M}_1(E)$ for all $x \in \mathcal{O}$, then the above has a unique solution $x \in C^1([0, T]; E_0) \cap C^0([0, T]; E_1)$ on some small interval.*

Proof. By writing $x(t) = x_0 + x_1(t)$, without loss of generality, suppose $x_0 = 0$. Expand f in a Taylor's series about $x = 0$: $f(x) = f_0 + Ax + r(x)$, where $A = df(0) \in \mathcal{M}_1(E)$, and $r \in C^k(\mathcal{O}; E_0)$ satisfies $r(0) = 0, dr(0) = 0$. Recall that the spaces X and $Y^{(0)}$ in the case $\theta = 1$ were defined by

$$X = C([0, T]; E_0),$$
$$Y^{(0)} = \{x \in C^1([0, T]; E_0) \cap C^0([0, T]; E_1) | x(0) = 0\},$$

and that $J_{A,T} = (\partial_t - A)^{-1} : X \to Y^{(0)}$ is well-defined and bounded since $A \in \mathcal{M}_1(E)$. Moreover, if we let $T \leq 1$, then the norm of $J_{A,T}$ is bounded uniformly in T, say by $M < \infty$.

The initial value problem (3.105) is equivalent to

$$x = J_{A,T}(r(x) + f_0), \tag{3.108}$$

which can be solved by a contraction mapping argument.

That is, let $x^{(0)} = 0$ and defiine $x^{(n+1)} = J_{A,T}(r(x^{(n)}) + f_0)$ inductively.Define $\Phi : x \mapsto J_{A,T}(r(x) + f_0)$. Since $dr(0) = 0$, there is an $\epsilon > 0$ such that $B_\epsilon(0) \subset \mathcal{O}$ and $\|dr(x)\| \leq M/2$ on $B_\epsilon(0)$.

Consider the Banach space $W = C^0([0,T]; E_1)$. Then on $W_\epsilon = \{x \in W | \|x\|_W \leq \epsilon\}$,

□

3.6 Little Hölder Space, Differential Operators, And Maximal Regularity Class

3.6.1 Periodic Functions Over $\mathbb{R}$

$\mathbb{T} = [-\pi, \pi]$ where π and $-\pi$ are identified: Denote $\{\pi, -\pi\}$. Endow $\mathbb{T}$ with the metric topology τ by

$$d_{\mathbb{T}} : \mathbb{T} \times \mathbb{T} \to \mathbb{R} \quad d_{\mathbb{T}}(x, y) := |x - y| \wedge (2\pi - |x - y|), \quad where\ a \wedge b := \min(a, b).$$

$$\mathbb{B}_{\mathbb{T}}(x, \epsilon) := \{y \in \mathbb{T} : d_{\mathbb{T}}(x, y)\epsilon\} \quad \epsilon > 0, x \in \mathbb{T}.$$

3.6.2 Regularity on $\mathbb{T}$

Given $f : \mathbb{T} \to \mathbb{C}$, we define its periodic extension

$$\tilde{f}(x) := f(x - 2\pi k) \quad for \quad x \in [\pi(2k-1), \pi(2k+1)], \quad k \in \mathbb{Z}$$

Denote by ϕ the periodic operator, bijective froom $\mathbb{C}^{\mathbb{T}}$ to $(\mathbb{C}^{\mathbb{R}})_{per} = \{2\pi \text{ periodic function on } \mathbb{R}\}$.

Denote $C_{per}(\mathbb{R})$ and $C^k_{per}(\mathbb{R})$ the spaces of 2π-periodic functions over $\mathbb{R}$. which are continuous and k-times continuously differentiable. Each are Banach spaces with norms:

$$\|f\|_{C(\mathbb{R})} := \sup_{x \in \mathbb{R}} |f(x)| \tag{3.109}$$

$$\|f\|_{C^k(\mathbb{R})} := \sum_{j=0}^{k} \|f^{(j)}\|_{C(\mathbb{R})}. \tag{3.110}$$

For $\alpha \in (0,1)$ and $k \in \mathbf{N}_0$ define the seminorm $[\dot{]}_{\alpha,\mathbb{R}}$ the space of H$\acute{}$older continuous functions $C^{k+\alpha}_{per}(\mathbb{R})$ to be those functions $f \in C^k_{per}(\mathbb{R})$ such that

$$[f^{(k)}]_{\alpha,\mathbb{R}} := \sup_{x,y\in\mathbb{R},x\neq y} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x-y|^\alpha} < \infty.$$

And $C^{k+\alpha}_{per}(\mathbb{R})$ is a Banach space with the norm

$$\|f\|_{C^{k+\alpha}(\mathbb{R})} := \|f\|_{C^k(\mathbb{R})} + [f^{(k)}]_{\alpha,\mathbb{R}}. \tag{3.111}$$

For simplicity, given $\theta \in \mathbb{R}_+$, we define $C^\theta_{per}(\mathbb{R}) := C^{\lfloor\theta\rfloor+\{\theta\}}_{per}(\mathbb{R})$, where $\lfloor\theta\rfloor$ is the largest interger not exceeding θ and $\{\theta\} := \theta - \lfloor\theta\rfloor$

For $\theta \in \mathbb{R}_+$,

$$C^\theta(\mathbb{T}) := \{f \in \mathbb{C}^\mathbb{T} : \phi(f) \in C^\theta_{per}(\mathbb{R})\} \quad \text{with} \quad \|f\|_{C^\theta(\mathbb{T})} := \|\phi(f)\|_{C^\theta(\mathbb{R})}. \tag{3.112}$$

$C^\theta(\mathbb{T})$ is a Banach space and ϕ is a linear isometric isomorphism from $C^\theta(\mathbb{T})$ to $C^\theta_{per}(\mathbb{R})$. Furthermore, if $\theta \geq 1$ and $f \in C^\theta(\mathbb{T})$, we define the derivative $f' \in C^\mathbb{T}$ by $f' := \phi^{-1}(\phi(f)') = \left(\frac{d}{dx}\tilde{f}\right)|_\mathbb{T}$.

Continuity, differetiablity and H$\acute{}$older continuity can be defined intrinsically on $\mathbb{T}$.

We denote by $d^\alpha_\mathbb{T}(\dot{,})$ the quantity $d_\mathbb{T}(\dot{,})^\alpha$.

Proposition 3.4. *Let $f \in \mathbb{C}^\mathbb{T}$, then*

1. *$f \in C(\mathbb{T})$ iff f is continuous in the metric topology τ.*

2. *for $\alpha \in (0,1)$, $f \in C^\alpha(\mathbb{T})$ iff $[f]_{\alpha,\mathbb{T}} := \sup_{x,y\in\mathbb{T}x\neq y} \frac{|f(x)-f(y)|}{d^\alpha_\mathbb{T}(x,y)} < \infty$.*
   Moreover, $[f]_{\alpha,\mathbb{T}} = [\tilde{f}]_{\alpha,\mathbb{R}}$ in this case.

3. *if $f \in C^1(\mathbb{T})$ and $x, y \in \mathbb{T}$, then $|f(x) - f(y)| \leq \|f'\|_{C(\mathbb{T})} d_\mathbb{T}(x,y)$.*

Proof. □

Definition 3.15 (little-H$\acute{}$lder space)**.** *For $\theta \in \mathbb{R}_+\backslash\mathbb{Z}$, define the periodic little-H$\acute{}$lder spaces over $\mathbb{R}$ as*

$$h^\theta_{per}(\mathbb{R}) := \left\{ f \in C^\theta_{per}(\mathbb{R}) : \lim_{\delta\to 0} \sup_{x,y\in\mathbb{R},0<|x-y|<\delta} \frac{|f^{\lfloor\theta\rfloor}(x) - f^{\lfloor\theta\rfloor}(y)|}{|x-y|^{\{\theta\}}} = 0 \right\} \tag{3.113}$$

Define $h^\theta(\mathbb{T}) := \{f \in \mathbb{C}^\mathbb{T} : \phi(f) \in h^\theta_{per}(\mathbb{R})\}$ for $\theta \in \mathbb{R}_+\backslash\mathbb{Z}$. By Proposition 3.4*, its equivalent definition is*

$$h^{\theta}(\mathbb{T}) := \left\{ f \in C^{\theta}(\mathbb{T}) : \lim_{x,y\in\mathbb{T}0<d_{\mathbb{T}}(x,y)<\delta} \frac{|f^{\lfloor\theta\rfloor}(x) - f^{\lfloor\theta\rfloor}(x)|}{d_{\mathbb{T}}^{\{\theta\}}(x,y)} = 0 \right\} \tag{3.114}$$

Then $h^{\theta}_{per}(\mathbb{R})$ is a closed subspace of $C^{\theta}_{per}(\mathbb{R})$ and a Banach space with $\|\dot{\|}_{C^{\theta}(\mathbb{R})}$.

The little-H$\acute{}$older spaces are, in fact, Banach algebras, in both the periodic and non-periodic settings.

If E and F are Banach spaces, we say that E is continuously embedded in F, denoted $E \hookrightarrow F$, if there exists a continuous injective operator $i : E \to F$. Moreover, we say E is densely included in F, denoted $E \hookrightarrow^{d} F$, if $i(E) \subset F$ is dense. Further, let $(\cdot,\cdot)_{\eta} := (\cdot,\cdot)^{0}_{\eta,\infty}$ denote the continuous interpolation functor with $\eta \in (0,1)$

Proposition 3.5. *1. For $\theta \in \mathbb{R}_{+}\backslash\mathbb{Z}$ and $\sigma \in (\theta,\infty]$, $h^{\theta}(\mathbb{T})$ is the closure of $C^{\sigma}(\mathbb{T})$ in $(C^{\theta}(\mathbb{T}), \|\cdot\|_{C^{\theta}(\mathbb{T})})$. Hence, $h^{\sigma}(\mathbb{T}) \hookrightarrow^{d} h^{\theta}(\mathbb{T})$ for $\sigma \in (\theta,\infty)\backslash\mathbb{Z}$.*

2. For $\theta_1, \theta_2 \in \mathbb{R}_{+}/\mathbb{Z}$ with $\theta_2 \geq \theta_1$, it follows that $(h^{\theta_1}(\mathbb{T}), h^{\theta_2}(\mathbb{T}))_{\eta} = h^{\eta\theta_2+(1-\eta)\theta_1}(\mathbb{T})$, provided $(\eta\theta_2 + (1-\eta)\theta_1) \notin \mathbb{Z}$.

Proof. □

3.6.3 Periodic Besov Spaces

Let $\mathcal{D}(\mathbb{T})$ denote the space $C^{\infty}(\mathbb{T})$ equipped with the locally convex topology generated by the family of semi-norms $\|f\|_k := \|f^{(k)}\|_{C(\mathbb{T})}$, for $k \in \mathbf{N}_0$.
Define the space of periodic distributions $\mathcal{D}'(\mathbb{T}) := (\mathcal{D}(\mathbb{T}))^{*}$, the set of all bounded linear functional on $\mathcal{D}(\mathbb{T})$.

Denote by $e_k\ [x \mapsto e^{ikx}] : \mathbb{T} \to \mathbb{C}$, then $e_k \in \mathcal{D}(\mathbb{T})$ for $k \in \mathbb{Z}$.
For $T \in \mathcal{D}'(\mathbb{T})$, define the *Fourier coefficients* $\hat{T}(k) := \langle T, e_{-k}\rangle$, where $\langle\cdot,\cdot\rangle : \mathcal{D}'(\mathbb{T}) \times \mathcal{D}(\mathbb{T}) \to \mathbb{C}$ is the duality pairing.

Every test function $\varphi \in \mathcal{D}(\mathbb{T})$ can be identified with $T_{\varphi} \in \mathcal{D}'(\mathbb{T})$ defined by $\langle T_{\varphi}\rangle := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \varphi(x)\psi(x)ds, \psi \in \mathcal{D}(\mathbb{T})$. Then the Fourier coefficient of T_{φ} coincide with the usual Fourier coefficient for $\varphi \in \mathcal{D}(\mathbb{T})$, namely

$$\hat{T}_{\varphi}(k) = \hat{\varphi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \varphi(x) e^{-ikx} dx.$$

In fact, we have the Fourier series representation

$$f = \sum_{k\in\mathbb{Z}} \hat{f}(k) e_k \quad for \quad f \in \mathcal{D}'(\mathbb{T}) \tag{3.115}$$

Let $\mathcal{S}(\mathbb{R})$ be the Shwartz space on $\mathbb{R}$ and $\mathcal{S}'(\mathbb{R})$ the space of tempered distribution on $\mathbb{R}$.
Furthermore, let $\Phi(\mathbb{R})$ denote the collection of all systems $(\varphi_j)_{j\in\mathbf{N}} \subset \mathcal{S}(\mathbb{R})$ satisfying

1. $\mathrm{supp}\varphi_0 \subset [-2,2]$, $\mathrm{supp}\varphi_j \subset [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}]$, $j \geq 1$.

2. $\sum_{j\in\mathbf{N}_0} \varphi_j(x) = 1, \quad x \in \mathbb{R}$.

3. $\forall l \in \mathbf{N}_0, \exists Cl > 0$ so that $\sup_{j\in\mathbf{N}_0} 2^{l_j} \|\varphi_j^{(l)}\|_{C(\mathbb{R})} \leq C_l$.

Now let $1 \leq p, q \leq \infty, s \in \mathbb{R}$ be fixed parameters and $\varphi = (\varphi_j) \in \Psi(\mathbb{R})$. For $f \in \mathcal{D}'(\mathbb{T}), j \in \mathbf{N}_0$, the series $\sum_{k\in\mathbb{Z}} \varphi_j(k)\hat{f}(k)e_k$ has only finitely many nonzero terms, by compactness of the support, and it follows that $\sum_{k\in\mathbb{Z}} \varphi_j(k)\hat{f}(k)e_k \in L_p(\mathbb{T})$. The norm on $L_p(\mathbb{T})$ is given by

$$\|g\|_p = \begin{cases} \left(\frac{1}{2\pi}\int_{\mathbb{T}} |g(x)|^p dx\right)^{1/p} & 1 \leq p < \infty \\ \mathrm{esssup}_{x\in\mathbb{T}} |g(x)| & p = \infty \end{cases} \tag{3.116}$$

Definition 3.16 (Periodic Besov Space)**.**

$$B^{s,\varphi}_{p,q}(\mathbb{T}) := \left\{ f \in \mathcal{D}'(\mathbb{T}) : \left(2^{sj} \left\| \sum_{k\in\mathbb{Z}} \varphi_j(k)\hat{f}(k)e_k \right\|_p \right)_{j\in\mathbf{N}_0} \in l^q(\mathbf{N}_0) \right\} \tag{3.117}$$

And the norm is given as follows:

$$\|f\|_{B^{s,\varphi}_{p,q}} := \begin{cases} \left(\sum_{j\in\mathbf{N}_0} 2^{sjq} \left\| \sum_{k\in\mathbb{Z}} \varphi_j(k)\hat{f}(k)e_k \right\|_p^q \right)^{1/q} & q < \infty \\ \sup_{j\in\mathbf{N}_0} 2^{sj} \left\| \sum_{k\in\mathbb{Z}} \varphi_j(k)\hat{f}(k)e_k \right\|_p & q = \infty \end{cases} \tag{3.118}$$

This space is a Banach space and it is known that $(B^{s,\varphi}_{p,q}, \|\cdot\|_{B^{s,\varphi}})$ is equivalent to $(B^{s,\psi}_{p,q}, \|\cdot\|_{B^{s,\psi}})$ for two systems $\varphi, \psi \in \Psi(\mathbb{R})$. So we drop the reference the particular system $\varphi \in \Psi(\mathbb{R})$.

Proposition 3.6. *For $s \in \mathbb{R}_+ \backslash \mathbb{Z}$, it holds that $B^s_{\infty,\infty}(\mathbb{T}) = C^s(\mathbb{T})$.*

Remark 3.1. *Suppose X is a normed space. For $u_0 \in X, f_1, \ldots, f_n \in X^*, \epsilon > 0$, define*

$$U(u_0; f_1, \ldots, f_n : \epsilon) := \{u \in X | |f_f(u - u_0)| < \epsilon, k = 1, \ldots, n\}$$

By using this,

$$\begin{aligned}\mathcal{U}(u_0) &= \{U(u_0; f_1, \dots, f_n; \epsilon, n \in \mathbf{N} f_1, \dots, f_n \in X^* \epsilon > 0\} \\ \mathcal{U} &= \bigcup_{u_0 \in X} \mathcal{U}(u_0)\end{aligned}$$

Actually, $\mathcal{U}$ satisfies the following fundamental systems of neighborhood.

1. $\mathcal{U} \neq \emptyset$

2. $U \in \mathcal{U}(u_0) \Longrightarrow u_0 \in U.$

3. $U, V \in \mathcal{U}(u_0) \Longrightarrow \exists W \in \mathcal{U}(u_0), \quad W \subset U \cap V$

4. $U \in \mathcal{U}(u_0) \Longrightarrow$ *for* $\forall v \in U, \quad \exists W \in \mathcal{U}(v), \quad$ *s.t* $\; W \subset U$.

5. $u_0 \neq u_1 \Longrightarrow \exists U \in \mathcal{U}(u_0), \exists V \in \mathcal{U}(u_1) \quad$ *s.t* $\; U \cap V = \emptyset$

The topology induced by $\mathcal{U}$ is called weak topology of X.

Generally, X is called locally convex linear topological space if it satisfies

1. *X is a topological space.*

2. *$(X, \mathcal{U})$ is a topological space.*

3. *$(u, v) \mapsto u + v$ and $(\alpha, u) \mapsto \alpha u$ is continuous with respect to the topology of $\mathcal{U}$*

4. *$\mathcal{U}$ is a Hausdorff space.*

Remark 3.2. *Even if X is not supposed to be norm space, if given the family of seminorm $\{p_\alpha\}_{\alpha \in A}$, we can construct the normed space. For every finite set $B \subset A$ and every $\epsilon > 0$, define*

$$U_{B,\epsilon}(y) := \{x \in X : p_\alpha(x - y) < \epsilon(\forall b \in B)\}$$

If we set this family as the family of neighborhood, then this induced topology is also locally convex.

Remark 3.3 (Schwartz Space)**.**

$$p_{\alpha,\beta}(\varphi) = \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta \varphi(x)|$$

Then the Schwarz space is defined as

$$\mathcal{S}(\mathbb{R}) = \{\varphi \in C^\infty(\mathbb{R}) : p_{\alpha,\beta}(\varphi) < \infty\}.$$

The tempered distribution T is defined as, if $\lim_{m\to\infty} p_{\alpha,\beta}(\varphi) = 0$, then $\lim_{m\to\infty} T(\varphi_m) = 0$ for all α, β.

3.6.4 A Fourier Multiplier Theorem

For $1 \leq p \leq \infty$, we define the Sobolev space $W_p^1(\mathbb{T}) := \{f \in L_p(\mathbb{T}) : f' \in L_p(\mathbb{T})\}$ with $\|f\|_{W_p^1} := \|f\|_p + \|f'\|_p$.

Theorem 3.9 (Fourier Multiplier Theorem)**.** *Let $r, s \in \mathbb{R}_+$ and $1 \leq p, q \leq \infty$. Suppose that $(M_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ is a sequence such that*

$$\begin{aligned} s_1 &:= \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s}|M_k| < \infty \\ s_2 &:= \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s+1}|M_{k+1} - M_k| < \infty \end{aligned} \tag{3.119}$$

$$\tag{3.120}$$

Then the Fourier multiplier with symbol $(M_k)_{k\in\mathbb{Z}}$ is a continuous mapping from $B^s_{p.q}(\mathbb{T})$ to $B^r_{p,q}(\mathbb{T})$, namely

$$T : \left[\sum_{k\in\mathbb{Z}} \hat{f}(k)e_k \mapsto \sum_{k\in\mathbb{Z}} M_k\hat{f}(k)e_k\right] \in \mathcal{L}(B^s_{p,q}(\mathbb{T}), B^r_{p,q}(\mathbb{T})).$$

The space involved are of Fourier type 2 . $\mathcal{F}L_1(\mathbb{R},\mathbb{C})$ is the space of compactly supported distributions whose Fourier transform is in L_1

Lemma 3.13. *Let $1 \leq p \leq \infty$ and let $m \in C_c(\mathbb{R},\mathbb{C}) \cap \mathcal{F}L_1(\mathbb{R},\mathbb{C})$. Then*

$$\left\|\sum_{k\in\mathbb{Z}} m(k)\hat{f}(k)e_k\right\|_p \leq C_p\eta_2(m)\left\|\sum_{k\in\mathbb{Z}} \hat{f}(k)e_k\right\|_p \tag{3.121}$$

holds whenever $f \in L_p(\mathbb{T})$ is a trigonometric polynomial, where C_p is a constant depending only on p, and $\eta_2(m) := \inf\{\|m(a\cdot)\|_{W_2^1} : a > 0\}$.

Proof of Theorem 3.13. Fix $(M_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ satisfying (3.119) and $s, r \in \mathbb{R}, 1 \leq p, q \leq \infty$ and $\varphi := \{\varphi_j\}_{j\geq 0} \in \Phi(\mathbb{R})$.

To see the boundness of T, it will suffice to show that there exists $C > 0$ such that

$$\left\|\sum_{k\in\mathbb{Z}} \left(2^{(r-s)j}M_k\right)\varphi_j(k)\hat{f}e_k\right\|_p \leq C\left\|\sum_{k\in\mathbb{Z}} \varphi_j(k)\hat{f}(k)e_k\right\|_p \tag{3.122}$$

holds uniformly for $f \in B^s_{p,q}(\mathbb{T})$ and $j \geq 0$.

Remark 3.4. *This is because firstly $\sum_{k\in\mathbb{Z}} \hat{f}(k)e_k \in B^s_{p,q}$, then*

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \hat{f}(k)e_k &= \sum_{j\in\mathbf{N}_0} \|\sum_{k\in\mathbb{Z}} \hat{f}(k)\varphi_j(k)e_k\|_p^q < \infty \\ &\iff (\forall j \in \mathbf{N}_0)\|\sum_{k\in\mathbb{Z}} \hat{f}(k)\varphi_j(k)e_k\|_p^q < \infty \text{ (since finite sum)}\end{aligned}$$

In order for T to be bound operator, for all combination $\sum_{k\in\mathbb{Z}} \hat{f}(k)e_k$, $\|T(\sum_{j\in\mathbf{N}_0}\sum_{k\in\mathbb{Z}} \hat{f}(k)\varphi_j(k)e_k)\| = \|\sum\sum M_k\hat{f}(k)\varphi_j(k)e_k\|_{B^s_{p,q}(\mathbb{T})} < \infty$. By the definition of it norm,

$$\begin{aligned}\sum_{j\in\mathbf{N}_0} 2^{sjq}\|\sum_{k\in\mathbb{Z}}\varphi_j(k)\sqrt{2\pi}M_k\hat{f}(k)e_k\|_p^q < \infty &\iff \sum_{j\in\mathbf{N}_0}\|\sum_{k\in\mathbb{Z}}2^{rj}\varphi_j(k)\hat{f}(k)M_ke_k\|_p^q < \infty \\ &\iff \forall j\in\mathbf{N}_0 \;\; 2^{-sjq}\|\sum_{k\in\mathbb{Z}}2^{rj}\varphi_j(k)\hat{f}(k)M_ke_k\|_p^q < \infty\end{aligned}$$

Therefore, (3.122) *is crucial to prove this theorem.*

For $j \geq 1$, define $m_j : \mathbb{R} \to \mathbb{C}$ by $m_j(x) = 0$ if $|x| \geq 2^{j+2}$ or $|x| \leq 2^{j-2}$, $m_j(k) = 2^{(r-s)j}M_k$ for $k \in \mathbb{Z}$ with $2^{j-1} \leq |k| \leq 2^{j+1}$, and m_j is affine on $[k, k+1]$ for all $k \in \mathbb{Z}$. We define m_0 in a similar manner, where $m_0(x) = 0$ if $|x| \geq 2, m_0(k) = M_k$ for $-1 \leq |k| \leq 1$, and m_0 is affine on every interval $[k, k+1], k \in \mathbb{Z}$.

One can see that $m_j \in C_c(\mathbb{R}) \cap \mathcal{F}L^1(\mathbb{R})$ and by compactness of suppφ_j, $\sum_{k\in\mathbb{Z}}\varphi_j(k)\hat{f}(k)e_k$ is a trigonometric poynomial, for $j \geq 0$. Hence, by Lemma 3.13, for $j \geq 1$,

$$\begin{aligned}\left\|\sum_{k\in\mathbb{Z}}\left(2^{(r-s)j}M_k\right)\varphi_j(k)\hat{f}(k)e_k\right\|_p &= \left\|\sum_{2^{j-1}\leq|k|\leq 2^{j+1}} m_j(k)\varphi_j(k)\hat{f}(k)e_k\right\|_p \\ &\leq C_p\eta_2(m_j)\left\|\sum_{k\in\mathbb{Z}}\varphi_j(k)\hat{f}(k)e_k\right\|_p \\ &\leq C_p\|m_j(2^j\cdot)\|_{W_2^1}\left\|\sum_{k\in\mathbb{Z}}\varphi_j(k)\hat{f}(k)e_k\right\|_p\end{aligned}$$

The first equality comes from the definition of φ_j.

Hence, it suffices to show that $\{\|m_j(2^j\cdot)\|_{W_2^1}\}_{j\geq 1}$ is uniformly bounded. From supp$m_j \subset [\frac{1}{4}, 4]$ and the bounds

$$\sup_{x\in\mathbb{R}}|m_j(2^j x)| \le \sup_{2^{j-1}\le|k|\le 2^{j+1}} 2^{(r-s)j}|M_k| \le \sup_{2^{j-1}\le|k|\le 2^{j+1}} \left(\frac{2^{(r-s)j}}{|k|^{r-s}}\right) s_1 \le 2^{|r-s|}s_1$$

$$\sup_{2^{j-1}\le|p|\le 2^{j+1}} 2^{(r-s+1)j}|M_{p+1}-M_p| \le \sup_{2^{j-1}\le|p|\le 2^{j+1}} \left(\frac{2^{(r-s+1)j}}{|p|^{(r-s+1)}}\right) s_2 \le 2^{|r-s+1|}s_2.$$

Then $W_2^1(\mathbb{T})$ norms can be bounded explicitly for all $j \ge 0$, for all $j \ge 0$, and it follows that the operator norm of T can be bounded in terms of the constants s_1 and s_2. $\square$

3.6.5 Ellipticity And Generation Of Analytic Semigroups

$D := i\frac{d}{dx}$ over $\mathbb{T}$ and let $m \in \mathbf{N}$.
Fix a collection $\{b_k : k = 0, \ldots, 2m\} \subset h^\alpha(\mathbb{T})$ of coefficient functions and $\mathcal{A}$, acting on $h^{2m+\alpha}(\mathbb{T})$, defined by

$$\mathcal{A}u(x) := \mathcal{A}(x,D)u(x) := \sum_{k=0}^{2m} b_k(x)(D^k u)(x) = \sum_{k=0}^{2m} i^k b_k(x)u^{(k)}(x), \quad x \in \mathbb{T}$$

By the embedding property Proposition 3.5(1) and the fact that $h^\alpha(\mathbb{T})$ is a Banach algebra, it follows that $\mathcal{A}$ maps $h^{2m+\alpha}(\mathbb{T})$ into $h^\alpha(\mathbb{T})$.

Remark 3.5. *I will show the special case, $u^{(2m)}(x) \in h^\alpha(\mathbb{T})$. Since $u \in h^{2m+\alpha}(\mathbb{T})$, by Proposition 3.5, since there exists $\sigma \in (2m+\alpha, \infty], h^{2m+\alpha}(\mathbb{T}) = \overline{C^\sigma(\mathbb{T})}$ in $(C^{2m+\alpha}, \|\cdot\|_{C^{2m+\alpha}})$, there exists $\{u_l\}_l \subset C^\sigma(\mathbb{T}) \implies \lim_{l\to\infty}\|u - u_l\|_{C^{2m+\alpha}} = 0$.*

$$\lim \|u-u_l\|_{C^{2m+\alpha}} = \lim\left(\sum_j^{2m} \sup|u^{(j)} - u_l^{(j)}| + \sup_{x\ne y}\frac{|u^{(2m)} - u_l^{(2m)}(y)|}{|x-y|^\alpha}\right) = 0$$

On the other hand, $u^{(2m)}(x) \in h^\alpha(\mathbb{T})$ means there exists $\{v_l\}_l \subset C^\sigma(\mathbb{T})$ such that $\lim_{l\to\infty}\|u^{(2m)} - v_l\|_{C^\alpha(\mathbb{T})} = 0$.
Therefore

$$\lim \|u^{(2m)} - v_l\|_{C^\alpha(\mathbb{T})} = \lim\left(\sup_{x\in\mathbb{R}}|u^{(k)}(x) - v_l(x)| + \sup_{x\ne y}\frac{|u^{(k)}(x) - v_l(y)|}{|x-y|^\alpha}\right) = 0$$

Let $\{v_l\} = \{u_l^{(k)}\}$, then $0 = \lim_{l\to\infty}\|u-u_l\|_{C^{2m+\alpha}} \ge \lim_{l\to\infty}\|u^{(k)} - u_l^{(k)}\|$. So it shows.

Now, denote by $\sigma\mathcal{A} : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$ the *principal symbol* of $\mathcal{A}$, defined by $\sigma\mathcal{A}(x, \xi) := b_{2m}(x)\xi^{2m}$. Then we say that $\mathcal{A}$ is a *uniformly elliptic* operator on $\mathbb{T}$ if there exists a constant $c_1 > 0$ such that

$$Re(\sigma\mathcal{A}(x, \xi)) \geq c_1 \quad \textit{for all} \quad x \in \mathbb{T}, |\xi| = 1. \tag{3.123}$$

If b_{2m} is simply a $\mathbb{R}$-valued function, we see that uniform ellipticity is equivalent to $b_{2m}(x) \geq c_1$ for all $x \in \mathbb{T}$. If b_{2m} in $\mathbb{C}\backslash\mathbb{R}$, this is equivalent to $b_{2m}(\mathbb{T}) \subset \{z \in \mathbb{C} : Rez \geq c_1\}$.

b_{2m} continuous on $\mathbb{T}$, there exists some constant $c_2 > 0$ for $\|b_{2m}\|_{C(\mathbb{T})} \leq c_2$.

E_0 and E_1 are Banach space with $E_1 \hookrightarrow E_0$, we denote by $\mathcal{H}(E_1, E_0)$ the collection of $A \in \mathcal{L}(E_1, E_0)$ such that $-A$ is the infinitesimal generator of an analytic semigroup on E_0, with $D(A) = E_1$.
Moreover, given $\kappa \geq 1, \omega > 0$, we denote bu $\mathcal{H}(E_1, E_0, \kappa, \omega)$ the set of linear operators $A : E_1 \to E_0$, closed in E_0, such that $\omega + A \in \mathcal{L}_{isom}(E_1, E_0)$ and

$$\kappa^{-1} \leq \frac{\|(\lambda + A)x\|_0}{|\lambda|\|x\|_0 + \|x\|_1} \leq \kappa, \quad x \in E_1\backslash\{0\}, Re\lambda \geq \omega.$$

Then it is known that $\mathcal{H}(E_1, E_0) = \bigcup_{\kappa\geq 1 \omega>0} \mathcal{H}(E_1, E_0, \kappa, \omega)$..

Theorem 3.10. *Let $m \in \mathbf{N}, \alpha \in \mathbb{R}_+\backslash\mathbb{Z}$, and consider the differential operator $\mathcal{A}_b := bD^{2m}$ with constant coefficient $b \in \mathbb{C}$. If $\mathcal{A}_b$ is uniformly elliptic, with constant $c_1 > 0$, and $c_2 \geq c_1 > 0$ is chosen so that $|b| \leq c_2$, then $-\mathcal{A}_b$ generates a analytic semigroup on $h^\alpha(\mathbb{T})$ with domain $h^{2m+\alpha}(\mathbb{T})$. Moreover, for any $\omega > 0$, there exists $\kappa = \kappa(\omega, c_1, c_2, m)$ such that*

$$\mathcal{A}_b \in \mathcal{H}(h^{2m+\alpha}, h^\alpha, \kappa(\omega, c_1, c_2, m), \omega).$$

Lemma 3.14. *Suppose $T \in \mathcal{L}(C^{k+\alpha}(\mathbb{T}), C^{l+\alpha}(\mathbb{T}))$ such that $T(C^{k+r}(\mathbb{T})) \subset C^{l+r}(\mathbb{T})$, for $k, l \in \mathbf{N}_0, \alpha \in \mathbb{R}_+\backslash\mathbb{Z}$ and $r > \alpha$. Then $T \in \mathcal{L}(h^{k+\alpha}(\mathbb{T}), h^{l+\alpha}(\mathbb{T}))$.*

Proof. Firstly, for $T \in \mathcal{L}(C^{k+\alpha}(\mathbb{T}), C^{l+\alpha}(\mathbb{T}))$, $\forall x \in h^{l+\alpha}(\mathbb{T}), \exists\{x_n\} \subset C^{k+\alpha}(\mathbb{T})$ such that $x = \lim_{n\to\infty} x_n$. Then $Tx = \lim_{n\to\infty} Tx_n < \infty$. So, $T \in \mathcal{L}(h^{k+\alpha}(\mathbb{T}), C^{l+\alpha}(\mathbb{T}))$.

So, we only have to prove $T(h^{k+\alpha}(\mathbb{T})) \subset h^{l+\alpha}(\mathbb{T})$. Let $f \in h^{k+\alpha}(\mathbb{T})$ and we can find $(f_j)_j \subset C^{k+r}(\mathbb{T})$ such that $f_j \to f$ in $\|\cdot\|_{C^{k+\alpha}}$. Then $Tf_j \to Tf$ in $\|\cdot\|_{C^{l+\alpha}}$ by $T \in \mathcal{L}(C^{k+\alpha}(\mathbb{T}), C^{l+\alpha}(\mathbb{T}))$ and $Tf_j \in C^{l+r}(\mathbb{T})$ for $j \in \mathbf{N}$. Therefore, $Tf \in \overline{C^{l+r}(\mathbb{T})}^{\|\cdot\|_{C^{l+\alpha}}} = h^{l+\alpha}$. The lemma is proved. $\square$

Proof of Theorem 3.10. Fix $\alpha \in \mathbb{R}_+\backslash\mathbb{Z}, \omega > 0$ and $b \in \mathbb{C}$ as indicated. We assume $b \in \sum(c_1, c_2) := \{z \in \mathbb{C} : Rez \geq c_1\} \cap \{z \in \mathbb{C} : |z| \leq c_2\}$. First, we realize the operator $-\mathcal{A}_b$ as a Fourier multiplier. Since, $e_k = e^{ikx}$ and a_k constant,

$$\mathcal{A}_b\left(\sum_{k\in\mathbb{Z}} a_k e_k\right) = \sum_{k\in\mathbb{Z}} b(i)^{2m} a_k (ik)^{2m} e_k = \sum_{k\in\mathbb{Z}} bk^{2m} a_k e_k,$$

we see that $-\mathcal{A}_b$ is associated with the multiplier symbol $(M_k)_k := (-bk^{2m})_k$. We prove the theorem by showing the following two facts.

1. $(\lambda + \mathcal{A}_b) \in \mathcal{L}_{isom}(h^{2m+\alpha}(\mathbb{T}), h^{\alpha}(\mathbb{T}))$ for $Re\lambda \geq \omega$, i.e.,

$$\rho(-\mathcal{A}_b) \supset \{\lambda \in \mathbb{C} : Re\lambda \geq \omega\}$$

   Moreover, the set $\{\|(\lambda + \mathcal{A}_b)^{-1}\|_{\mathcal{L}(h^{\alpha}, h^{2m+\alpha})} : Re\lambda \geq \omega\}$ is uniformly bounded by some $M_1 = M_1(\omega, c_1, c_2, m) < \infty$.

2. $\lambda(\lambda + \mathcal{A}_b)^{-1} \in \mathcal{L}(h^{\alpha}(\mathbb{T}))$ for $Re\lambda \geq \omega$, and there is an upper bound $M_2 = M_2(\omega, c_1, c_2, m) < \infty$ for the set $\{|\lambda| \|(\lambda + \mathcal{A}_b)^{-1}\|_{\mathcal{L}(h^{\alpha}(\mathbb{T}))} : Re\lambda \geq \omega\}$

Step 1

Since $C^{2m+\sigma}(\mathbb{T}) \hookrightarrow C^{\sigma}(\mathbb{T}), (\lambda + \mathcal{A}_b) \in \mathcal{L}(C^{2m+\sigma}(\mathbb{T}), C^{\sigma}(\mathbb{T}))$ for arbitrary $\sigma \in \mathbb{R}_+$. In particular, we see that

$$\|(\lambda + \mathcal{A}_b)f\|_{C^{\sigma}} \leq |\lambda| \|f\|_{C^{\sigma}} + |b| \|f^{(2m)}\|_{C^{\sigma}} \leq (c(\sigma)|\lambda| + c_2)\|f\|_{C^{2m+\sigma}}. \quad (3.124)$$

where $c(\sigma) > 0$ is the embedding constant, i.e., $\|f\|_{C^{\sigma}} \leq c(\sigma)\|f\|_{C^{2m+\sigma}}$ for all $f \in C^{2m+\sigma}(\mathbb{T})$, and

$$\begin{aligned} \|f^{(2m)}\|_{C^{\sigma}} &= \|f^{(2m)}\|_{C^{\lfloor \sigma \rfloor}} + [f^{(2m+\lfloor \sigma \rfloor)}]_{\{\sigma\},\mathbb{R}} \\ \|f\|_{C^{2m+\sigma}} &= \|f\|_{C^{2m+\lfloor \sigma \rfloor}} + [f^{(2m+\lfloor \sigma \rfloor)}]_{\{\sigma\},\mathbb{R}} \end{aligned}$$

So, comparing these equations, define c_2. We will show the invertibility of $(\lambda + \mathcal{A}_b$. In particular, we will use Theorem 3.9 and the fact $B^{\sigma}_{\infty,\infty}(\mathbb{T}) = C^{\sigma}(\mathbb{T})$ for $\sigma \in \mathbb{R}_+ \backslash \mathbb{Z}$. Let $Re\lambda \geq \omega (> 0)$ and consider $\left(\tilde{M}_k(\lambda)\right)_k := \left(\frac{1}{\lambda + bl^{2m}}\right)_k$, which we will show satisfies (3.119), with $r = 2m + \sigma$ and $s = \sigma$. Then $r - s = 2m$ and we have,

$$\begin{aligned} |k|^{2m} |\tilde{M}_k(\lambda)| &= \frac{k^{2m}}{|\lambda + bk^{2m}|} \leq \frac{k^{2m}}{Rebk^{2m}} = \frac{1}{Reb} \quad \text{for} \quad k \in \mathbb{Z}\backslash\{0\} \\ &\Longrightarrow \quad s_1 := \sup_{k \in \mathbb{Z}\backslash\{0\}} |k|^{r-s} |\tilde{M}_k(\lambda)| \leq \frac{1}{c_1} < \infty \quad \text{since uniform ellipticity of } b. \end{aligned}$$

and

$$\begin{aligned} |k|^{2m+1} |\tilde{M}_{k+1}(\lambda) - \tilde{M}_k(\lambda)| &= |k|^{2m+1} \left| \frac{1}{\lambda + b(k+1)^{2m}} - \frac{1}{\lambda + bk^{2m}} \right| \\ &= \frac{|k|^{2m}}{|\lambda + b(k+1)^{2m}|} \frac{|k|^{2m}}{|\lambda + bk^{2m}} \frac{|b|(k+1)^{2m} - k^{2m}}{|k^{2m-1}} \\ &\leq \frac{|k|^{2m}}{|\lambda + b(k+1)^{2m}|} \frac{|b|}{Reb} \frac{|(k+1)^{2m} - k^{2m}}{|k|^{2m-1}} \end{aligned}$$

If $k=-1$, then this last term is equal to $\frac{1}{|\lambda|}(\frac{|b|}{|\lambda|Reb}????)$, whici is majorized by $C\frac{1}{\omega}$ for some constant C.
For all other $k \in \mathbb{Z}\backslash\{0\}$, we eliminate dependence on λ, as in step 1, so that we have

$$
\begin{aligned}
s_2 &:= \sup_{k\in\mathbb{Z}\backslash\{0\}} |k|^{r-s+1}\left|\tilde{M}_{k+1}(\lambda)-\tilde{M}_k(\lambda)\right| \\
&\leq \left(\frac{1}{\omega}\vee\frac{c_2}{c_1^2}\right)\sup_{k\in\mathbb{Z}\backslash\{-1\}}\left(\frac{|k|^{2m}}{|k+1|^{2m}}\sum_{j=0}^{2m-1}\binom{2m}{j}|k|^{j-2m+1}\right)<\infty
\end{aligned}
$$

This is because, carefully, since $(k+1)^{2m}-k^{2m}=\sum_{j=0}^{2m-1}\binom{2m}{j}k^j$,

$$
\frac{|(k+1)^{2m}-k^{2m}|}{|k|^{2m-1}} \leq \sum_{j=0}^{2m-1}\binom{2m}{j}|k|^{j-2m+1}
$$

and the fact that $Re\lambda \geq \omega > 0$ and the uniform ellipticity of b.

Hence, by Theorem 3.9, we have $R(\lambda)\in\mathcal{L}(B^r_{p,q}(\mathbb{T}),B^{r+2m}_{p,q})$ for any $1\leq p,q\leq\infty$ and $r\in\mathbb{R}_+$, where $R(\lambda)$ is the operator associated with $\left(\tilde{M}_k(\lambda)\right)_k$.

Letting $p=q=\infty$ and $r=\sigma$, we see that $R(\lambda)\in\mathcal{L}(C^\sigma(\mathbb{T}),C^{2m+\sigma}(\mathbb{T}))$. Furthermore, denote $C^{2m+\sigma}(\mathbb{T})\ni f=\sum_{k\in\mathbb{Z}}a_ke_k$ and $C^\sigma(\mathbb{T})\ni g=\sum_{k\in\mathbb{Z}}b_ke_k$, and

$$
\begin{aligned}
R(\lambda)(\lambda+\mathcal{A}_b)f &= R(\lambda)\left(\sum\lambda a_ke_k+\sum bk^{2m}a_ke_k\right) \\
&= \sum\frac{1}{\lambda+bk^{2m}}(\lambda+bk^{2m})a_ke_k=f \\
(\lambda+\mathcal{A}_b)R(\lambda)g &= \sum b_k(\lambda+\mathcal{A}_b)\tilde{M}_k(\lambda)e_k \\
&= \sum b_k(\lambda+\mathcal{A}_b)\frac{1}{\lambda+bk^{2m}}e_k \\
&= \sum\frac{\lambda}{\lambda+bk^{2m}}b_ke_k+\sum\frac{bk^{2m}}{\lambda+bk^{2m}}b_ke_k \\
&= \sum b_ke_k=g
\end{aligned}
$$

since $\mathcal{A}_be_k=bD^{2m}e_k=b(i\frac{d}{dx})^{2m}e^{ikx}=bi^{2m}(ik)^{2m}e_k=bk^{2m}e_k$. Therefore, $R(\lambda)=(\lambda+\mathcal{A}_b)^{-1}$ and

$$
(\lambda+\mathcal{A}_b)\in\mathcal{L}_{isom}(C^{2m+\sigma}(\mathbb{T}),C^\sigma(\mathbb{T}))\quad for\quad Re\lambda\geq\omega,\sigma\in\mathbb{R}_+\backslash\mathbb{Z}. \tag{3.125}
$$

The rest is by Lemma 3.2.

Step 2

Fix $Re\lambda \geq \omega$ and $\lambda(\lambda+\mathcal{A}_b)^{-1}$ has the associated multiplier symbol $\left(\frac{\lambda}{\lambda+bk^{2m}}\right)$. From Step 1, $(\lambda+\mathcal{A}_b)^{-1}$ is a well-defined operator from $h^{\alpha}(\mathbb{T})$ into $h^{2m+\alpha}(\mathbb{T})$. By Theorem **??** (1), we can regard $(\lambda+\mathcal{A}_b)^{-1}$ as an operator from $h^{\alpha}(\mathbb{T})$ into itself.

From Lemma 3.14 and Theorem 3.9, we take $r = s = \sigma$ and $p = q = \infty$. We will show s_1 and s_2 can be bounded independent of $Re\lambda \geq \omega$.

We can find $\theta = \theta(c_1, c_2)$ such that $\sum(c_1, c_2) := \{z \in \mathbb{C} : Rez \geq c_1\} \cap \{z \in \mathbb{C} : |z| \geq c_2\} \subset S_\theta := \{z \in \mathbb{C} : |argz| < \theta\}$. Moreover, there exists a constant $C(\theta)$ such that $|\lambda + z| \geq |\lambda|/C(\theta)$ for all $z \in S_\theta \cup \{0\}$, $Re\lambda > 0$ since $\theta < \frac{\pi}{2}$ $(Rez > 0)$. In particular, we have

$$s_1 = \sup_{k\in\mathbb{Z}\backslash\{0\}} \frac{|\lambda|}{|\lambda + bk^{2m}|} \leq C(\theta) \quad forall \quad Re\lambda \geq \omega. \tag{3.126}$$

Now, considering s_2, we have the bound

$$\begin{aligned}
|k|\left|\frac{|\lambda|}{\lambda+b(k+1)^{2m}} - \frac{|\lambda|}{\lambda+bk^{2m}}\right| &= \frac{|\lambda|}{|\lambda+b(k+1)^{2m}|}\frac{k^{2m}}{|\lambda+bk^{2m}}\frac{|b||(k+1)^{2m}-k^{2m}}{|k|^{2m-1}} \\
&\leq C(\theta)\frac{k^{2m}}{Rebk^{2m}}\frac{|b|((k+1)^{2m}-k^{2m}}{|k|^{2m-1}} \\
&\leq C(\theta)\frac{c_2}{c_1}\frac{(k+1)^{2m}-k^{2m}}{|k|^{2m-1}},
\end{aligned}$$

for $k \in \mathbb{Z}\backslash\{0\}$. Hence, from the similar computation to Step 1,

$$s_2 \leq \left(C(\theta)\frac{c_2}{c_1}\right)\sup_{k\in\mathbf{N}}\left(\sum_{j=0}^{2m-1}\binom{2m}{j}k^{j-2m+1}\right) < \infty$$

again uniformly in $Re\lambda \geq \omega$. Now we see that $\lambda(\lambda+\mathcal{A}_b)^{-1} \in \mathcal{L}(C^{\sigma}(\mathbb{T}), C^{2m+\sigma}(\mathbb{T}))$ holds by Theorem 3.9 for $\lambda \geq \omega, \sigma \in \mathbb{R}_+\backslash\mathbb{Z}$. The Step 2 holds by Lemma 3.14 and we fix a constant $M_2 = M_2(\omega, c_1, c_2, m) < \infty$ such that $\|(\lambda+\mathcal{A}_b)^{-1}\|_{\mathcal{L}(h^{\alpha})} \leq M_2/|\lambda|$ holds uniformly for $Re\lambda \geq \omega$ and $b \in \sum(c_1, c_2)$.

By Step 1 and 2, from **Pazy Theorem 5.2**, $-\mathcal{A}_b$ generates an analytic semigroup. Moreover, if we choose $\kappa = \kappa(c_1, c_2, \omega) \geq 2(M_1 \vee M_2) \vee (1 \vee c_2)$ it holds that

$$\kappa^{-1} \leq \frac{\|\lambda+\mathcal{A}_b)f\|_{h^{\alpha}(\mathbb{T})}}{|\lambda|\|f\|_{h^{\alpha}(\mathbb{T})} + \|f\|_{h^{2m+\alpha}(\mathbb{T})}} \leq \kappa, \quad f \in h^{2m+\alpha}(\mathbb{T})\backslash\{0\}, Re\lambda \geq \omega.$$

Hence, $\mathcal{A}_b \in \mathcal{H}(h^{\alpha}(\mathbb{T}), h^{2m+\alpha}(\mathbb{T}), \kappa, \omega)$, as claimed. $\square$

3.7 Formulation of The Theorem And Its Proof

By applying the long preparation so far, let's finish the main proof of local existence and uniqueness of (3.5) in terms of classical solution.

First of all, we can start rewriting (3.5) abstractly as

$$\partial_t \Phi = \mathcal{G}(\Phi), \quad \Phi(0) = \Phi_0 \tag{3.127}$$

where $\mathcal{G}(\Phi) = G(\Phi, \alpha(\Phi))$ and $\alpha = \alpha(\Phi)$ is the unique solution defined as adjusted tangential velocity.

The outline of this proof is as follows. By lengthy computation, we will find the Fréchet derivative $\mathcal{G}'(\bar{\Phi}) \in \mathcal{L}(E_1, E_0)$ is decomposed into A_1+A_2 where A_1 is the second derivative part and A_2 is the first and non derivative part. Actually, the pair of the little H$\acute{}$older spaces $(c^\sigma, c^{2+\sigma})$ is the maximal regularity pair explained in the above section, for $A = -\partial_u^2$. Furthermore, the linear operator A_2 can be regarded as "perturbation", so $A_1 + A_2$ is also in maximal regularity class. Then by applying Theorem 3.8, we can obtain the desired result.

First of all, define the suitable Banach spaces.

Definition 3.17. *Let $c^{2\mu+\sigma} = c^{2\mu+\sigma}(S^1)$, be a function space, which is the closure of $C^\infty(S^1)$ in $C^{2\mu+\sigma}(S^1)$ for $0 < \sigma < 1, \mu = 0, 1/2, 1$. This space is called "little H$\acute{}$older space.*

Then, predicting to utilize the interpolation spaces and semigroup theory, define the following parametrized spaces:

$$E_\mu = c^{2\mu+\sigma} \times c_*^{2\mu+\sigma} \times c^{1+\sigma} \times (c^{2\mu+\sigma})^2 \quad for \quad \mu = 0, 1/2, 1, \tag{3.128}$$

where the Banach manifold

$$c_*^{2\mu+\sigma}(S^1) = \{\nu : \mathbb{R} \to \mathbb{R}, \mathbf{T} = (\cos\nu, \sin\nu) \in (c^{2\mu+\sigma}(S^1))^2\} \tag{3.129}$$

Actually, from the viewpoint of interpolation theory, it is known that for $\theta_1, \theta_2 \in \mathbb{R}^+ \backslash \mathbb{Z}$ with $\theta_2 \geq \theta_1$, it follows that $(c^{\theta_1}(S^1), c^{\theta_2}(S^1))_\mu = c^{\mu\theta_2+(1-\mu)\theta_1}(S^1)$, provided $(\mu\theta_2+(1-\mu)\theta_1) \notin \mathbb{Z}$. Following this notation, confirm that $(E_0, E_1)_{1/2} = E_{1/2}$.

Henceforce, assume that $\varphi = \varphi(k)$ and $\beta = \beta(\mathbf{x}, k, \nu)$ are at least C^3 smooth functions such that $\varphi(k) > 0$ and β is a 2π-periodic function in ν.
By $\mathcal{O}_{\frac{1}{2}} \subset E_{\frac{1}{2}}$, we denote a bounded open subset $r_\varphi > 0$ for any $(k, \nu, r_\varphi, \mathbf{x}) \in \mathcal{O}_{\frac{1}{2}}$.

Lemma 3.15. *Let $\alpha = \alpha(\Phi)$ be the tangential velocity function as a unique solution to* **??** *satisfying $\langle \varphi(k)\alpha \rangle = 0$, where $\Phi = (k, \nu, r_\varphi, x) \in \mathcal{O}_{\frac{1}{2}}$. Then $\alpha \in C^1(\mathcal{O}_{\frac{1}{2}}, c^\sigma(S^1))$.*

Proof. □

It is time to formulate the theorem.

Theorem 3.11. *Assume $\Phi_0 = (k_0, \nu_0, r_{\phi 0}, \mathbf{x}_0) \in E_1$, where k_0 is the curvature, ν_0 is the tangential vector, and $r_{\phi 0} > 0$ is the ϕ-adjusted relative local length of an initial regular curve $\Gamma^0 = image(\mathbf{x}_0)$. Assume $\phi(k) > 0$ and $\beta = \tilde{\beta}(\mathbf{x}, k, \nu) + \mathcal{F}_\Gamma$, where $\tilde{\beta} : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ are C^3 smooth functions of their arguments such that $\tilde{\beta}$ is a 2π-periodic function in the ν variable and $\min_{\Gamma_0} \tilde{\beta}'_k(\mathbf{x}_0, k_0, \nu_0) > 0$. The nonlocal part of the normal velocity $\mathcal{F}_\Gamma$ is assumed to be a C^1 smooth function from a neighborhood $\mathcal{O}_{1/2} \subset E_{1/2}$ of Φ_0 into $\mathbb{R}$, that is, $\mathcal{F}_\Gamma \in C^1(\mathcal{O}_{1/2}, \mathbb{R})$.*
Then, there exists a unique solution $\Phi = (k, \nu, r_\phi, \mathbf{x}) \in C([0,T], E_1) \cap C^1([0,T], E_0)$ of the governing system of Eqs (3.5) defined on some time interval $[0,T], T > 0$.

Proof. □

Theorem 3.12. *Let $\tilde{\Phi} = (\tilde{k}, \tilde{\nu}, \tilde{g})^T$ be a classical solution of (3.5) such that the quantities $\tilde{k}, \tilde{\beta}$, and $\tilde{g}^{-1}\partial_u \tilde{\alpha}$ are bounded. Then $x = x(u,t)$ given by (**??**) satisfies $|\partial_u x| = \tilde{g}, k = \tilde{k}, \nu = \tilde{\nu}, \vec{N} = \tilde{\vec{N}}, \vec{T} = \tilde{\vec{T}}$, where $k, \nu, \vec{N}, \vec{T}$ represent the curvature, the tangent angle, and the unit normal and tangent vectors of Γ^t.*

Proof. □

This theorem guarantees the equivalence between Eq (**??**) and the geometry of 2.1 and is proved in [3] by the theory of linear parabolic equation. [11]

Actually, the Frechet derivative of $\mathcal{G}$ at $\bar{\Phi} = (\bar{k}, \bar{\nu}, \bar{r}_\phi, \bar{\mathbf{x}}) \in \mathcal{O}_1$ has the form

$$A = \mathcal{G}'(\bar{\Phi}) = A_1 + A_2, \text{ where } A_1 = \bar{D}\partial_u^2,\ A_2 = \bar{B}\partial_u + \bar{C}.$$

$$D = \begin{pmatrix} \bar{g}^{-2}\tilde{\beta}'_k & 0 & 0 & 0 & 0 \\ 0 & \bar{g}^{-2}\tilde{\beta}'_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{g}^{-2} & 0 \\ 0 & 0 & 0 & 0 & \bar{g}^{-2} \end{pmatrix}$$

$$B = \begin{pmatrix} \partial_s \tilde{\beta}'_k - \tilde{\beta}'_k(\partial_u g)g^{-3} + \tilde{\beta}_\nu g^{-1} + \alpha g^{-1} & 0 & 0 & \partial_s \partial_x \tilde{\beta} & -k\partial_x \tilde{\beta} g^{-1} \\ 0 & -(\partial_u g)g^{-3} + \alpha g^{-1} & 0 & \partial_x \tilde{\beta} g^{-1} & \partial_y \beta g^{-1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha g^{-1} - (\partial_u g)g^{-3} & -c \\ 0 & 0 & 0 & c & \alpha - (\partial_u g)g^{-3} \end{pmatrix}$$

$$C = \begin{pmatrix} k^2\beta \\ 0 \\ (r_\phi - 1)(\kappa_1 + \kappa_2\langle k\beta \rangle) \\ 0 \end{pmatrix}$$

By complicated computation, we can show that $A_1 \in \mathcal{M}_1(E)$ and A_2 has relative norm zero. Therefore, by the theorem above mentioned, $A_1 + A_2 \in \mathcal{M}_1(E)$.

Again by the theorem, it shows the existence and uniqueness of the solution of (3.5).

4 Application to the Image Segmentation

4.1 Direct Method

Classical curvature flow is the geometric equation, which shortens the total length of curves most rapidly. [8] [13]

$$\beta = k \tag{4.1}$$

This equation can be characterized as a gradient flow.

We can consider $\beta = k + F$,

$$F(\mathbf{x}) = F_{max} - (F_{max} - F_{min})I(\mathbf{x})(\mathbf{x} \in \Omega) \tag{4.2}$$

where $F_{max} > 0, F_{min} < 0$. [4]

This modeling has the advantage of low computational complexity compared to Snake method and indirect method. Furthermore, this modeling could overcome some slight noises, because the curve's curvature would increase due to the change of its shape.

On the other hand, this one has the disadvantage of topological change and strong noises. Especially, for example, if the noise is so clear that this one can be regarded as another object, then the "Dislocation Dynamics" proposed in [1] [2] can be applied to this modeling. However, even in the case where this dislocation dynamics is used, if the noise was not enough clear to stop the curve moving, then the splited curve finally goes to its singularity and it would blow up. [13]

By enlarging F_{max} and F_{min}, we can sharpen the contour of the target object whose contour is obscure. But, the small objects, sometimes regarded as noises, whose color are as light as the colors the curve would overcome. So, even if you use dislocation dynamics in your algorithm, its split curve finally would shrink to one point and blow up. In this process. we want the new external force that make the curve overcome such obstacles.

As for the strong noise,

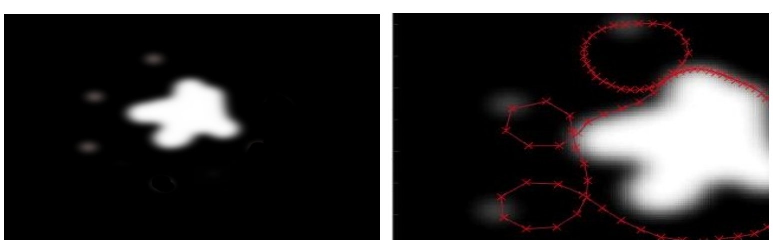

Figure 1: Curve Intersection By Noise

I : $m \times n$ image matrix. By allowing the adequate rotation, we assume $m \leq n$.

4.1.1 Changing The External Force F

Fixing the suitable F_{max} and F_{min} is important to overcome noises. However, it has a limitation since (4.2) is the linear form, and if F_{max} is raised up, then the power to overcome noises would increase, but the targeting curve would try to twine around the noises more strongly. This reciprocity is inevitable as long as F is represented by linear form.

Therefore, this paper proposes to use the following external force F:

$$F(\mathbf{x}) = \left(\frac{F_{max} - F_{min}}{2}\right) \tanh\left(-\alpha\left(\frac{I(\mathbf{x})}{255} - 0.5\right)\right) + \left(F_{max} - \frac{F_{max} - F_{min}}{2}\right) \tag{4.3}$$

where F_{max}, F_{min} are defined in the previous way, and α is a positive constant. Actually α is a characteristic constant which empowers the curve to overcome the noises. In other words, the larger the α is, then the more power to get over the noises is.

From my rule of thumb, α might be $3 \leq \alpha \leq 6$.

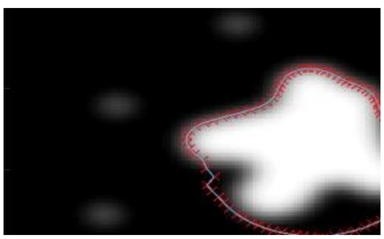

Figure 2: $F_{max} = 30, F_{min} = -35, \alpha = 3.5$

4.2 Indirect Method

Indirect method is the approach depicting the motion of curve by contour of surface. From the beginning and still today, direct method has the disadvantage toward protrusions or topological changes. On the other hand, with help of the emergence of viscosity solution, indirect method is now reported to have the stability and irrelevancy with topology.

However, it had been also reported that since the edge detection function g explained below is never zero, the curve eventually pass through the boundary. [27]

Actually, while $\phi(x, y, t)$ represents for surface, we want $\phi(x, y, t) = 0$ on $(x, y) \in \Omega \subset \mathbb{R}^2$ to stand for the curve used for image segmentation. Then, evolving this surface, if $\partial_t \mathbf{x} = F \frac{\nabla_{\mathbf{x}} \phi}{|\nabla \phi|}$ where F is a speed function,

$$
\begin{aligned}
\partial_t\left(\phi(x(t),y(t),t)\right) &= \partial_t\phi(x,y,t)+\nabla_{\mathbf{x}}\phi(x,y,t)\cdot\partial_t\mathbf{x}=0\\
&\Longleftrightarrow \partial_t\phi(x,y,t)+F\nabla_{\mathbf{x}}\phi(x,y,t)\cdot\frac{\nabla_{\mathbf{x}}\phi}{|\nabla\phi|}=0\\
&\Longleftrightarrow \partial_t\phi+F|\nabla\phi|=0
\end{aligned}
$$

Then, classical edge detection function is, say,

$$
g=\frac{1}{1+|\nabla G_\sigma * I|^2} \tag{4.4}
$$

where I is an image, G_σ is the Gaussian kernel with standard deviation σ. For the beginner of image processing, the term $\nabla G_\sigma * I$ means that blurring the image by Gaussian for removing noises, then it measures the difference of I for detecting the edge of object.

Then if $F=g$, we can construct the classical level set method. Historically, evolving this idea or sometimes working out different concepts, many prominent researchers propose various schemes. For example,

$$
\begin{cases}
\partial_t\phi=|\nabla\phi|\left(-\nu+\frac{\nu}{(M_1-M_2)}(|\nabla G_\sigma * u_0|-M_2)\right) & (0,\infty)\times\mathbb{R}^2\\
\phi(x,y,0)=\phi_0(x,y) & \{0\}\times\mathbb{R}^2
\end{cases}
$$

where ν is a constant, M_1 and M_2 are the maximum and minimum of $|\nabla G_\sigma * u_0|$.

And from the geodesic example, consider the minimization problem:

$$
\inf_C 2\int_0^1 |C'(s)|\cdot g(|\nabla u_0(C(s)|)ds.
$$

This is based on the geodesic on the riemannian metric $g(|\nabla u_0(C(s)|)ds$.

All these adopt the stopping function g, so they can not escape the problem of stopping the curve on the boundary. Then, [28] proposed “active contour without edge”. That is, we consider the minimization problem:

$$
\inf_{c_1,c_2,C} F(c_1,c_2,C)
$$

where

$$
\begin{aligned}
F(c_1,c_2,C)=\mu\cdot\mathrm{Length}(C)+\nu\cdot\mathrm{Area}(inside(C)) &+ \lambda_1\int_{inside(C)}|u_0(x,y)-c_1|^2dxdy\\
&+ \lambda_2\int_{outside(C)}|u_0(x,y)-c_2|^2dxdy,
\end{aligned}
$$

$\mu \geq 0, \nu \geq 0, \lambda_1, \lambda_2 > 0$.

From the view point of computation, the simpler the equation is, the better the algorithm would be. Based on that, I will propose active contour model without edge, by following the direct method approach in the previous section.

4.3 Numerical Scheme

My scheme is mainly based on the mean curvature flow, which is the 3-dimentional classical curvature flow, i.e.,

$$\frac{\partial_t \phi}{|\nabla \phi|} = \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right). \tag{4.5}$$

However, there is a singularity at $|\nabla \phi| = 0$, so we have to think out some new numerical scheme.

Ordinarily, there is mainly two ways to compute (4.5). [29] One is to use "signed distance function". That is, if we define Ω^+ on which $\phi(x) > 0$, Ω^- on which $\phi(x) < 0$ and $\partial\Omega$ is the boundary of Ω^+ or Ω^-, then *a singed distance function* ϕ is defined

$$\phi(\mathbf{x}) = \begin{cases} d(\mathbf{x}) & (\mathbf{x} \in \Omega^+) \\ 0 & (\mathbf{x} \in \partial\Omega) \\ -d(\mathbf{x}) & (\mathbf{x} \in \Omega^-) \end{cases}$$

The merit of the use of a signed distance function is $|\nabla \phi| = 0$ a.e. So, by reconstructing sign distance function for a certain time, we can obtain stable numerical scheme.

The second one is to use viscosity solution. It is important to theoretically adopt viscosity solution since this theory guarantees the uniqueness and existence of the targeting PDE under the initial 0-contour condition, which is a little more tricky than other PDE theories. Furthermore, viscosity solution gives us approximation method, which gives global solution. So, switching its initial condition such as sign distance function is not necessary. However, this method does not hold for every case, so we need to check for every case.

Fortunately, Chen,Giga, Hitaka, and Honma studies the numerical scheme of mean curvature flow from the perspective of viscosity solution. Then by borrowing the idea of (4.3), I constructed the following equation.

$$\frac{\partial_t u(\mathbf{x}, t)}{|\nabla u(\mathbf{x}, t)|} = \operatorname{div}\left(\frac{\nabla u(\mathbf{x}, t)}{|\nabla u(\mathbf{x}, t)|}\right) - F(\mathbf{x}) \tag{4.6}$$

and

$$F(\mathbf{x}) = \left(\frac{F_{max} - F_{min}}{2}\right) \tanh\left(-\alpha\left(\frac{I(\mathbf{x})}{255} - 0.5\right)\right) + \left(F_{max} - \frac{F_{max} - F_{min}}{2}\right) \tag{4.7}$$

where the constants α, F_{max}, F_{min} would be determined as (4.3).

Let's compare the classical scheme (4.4) and my scheme.
First of all, I choose the following image as the test image which has a complicated form

Figure 3: Original Image

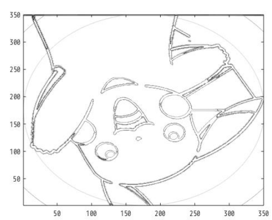

Figure 4: Based on (4.4)

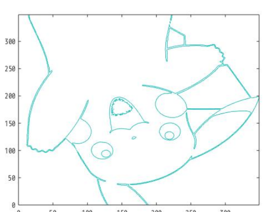

Figure 5: Based on (4.6)

If you look closely at Figure 4 and Figure 5, Figure 5 is seemingly much clear than Figure 4.

5 Conclusion

This paper reviews geometric direct method of image segmentation and proposes a new external force F in a normal part of the plane curve in order to overcome noises, then apply this force term to indirect method. In contrast to indirect method which represents the plane curve by 0-contour of surface, direct method has the advantage of flexibility to choose what kind of topological change or how large noises we should ignore.

Therefore, even if there emerge new techniques of image segmentation, it is still valuable to further explore the suitable F in terms of such advantage of direct method.

斎藤先生のパラドックス　古典解と粘性解の乖離はないか？？ 応用数学者としての務めである最後に曲面を追いかけてを紹介して、終了

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