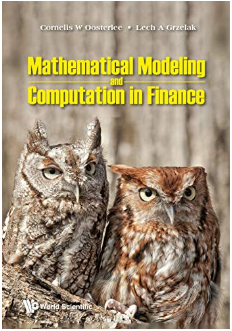
Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.



- Youtube Channel with courses can be found here.
- Slides and the codes can be found here.

List of content

A Bit of History

- The foreign exchange market has grown tremendously over several decades. It is well-known that before WWI the role of the exchange rates were taken by the commodity *gold*.
- Each country under the gold standard would express its currency in terms of the gold price, e.g., two countries A and B would quote their prices for gold in their local currencies as x and y, respectively, then the exchange rate would effectively be either x/y or y/x, depending on the country.
- The gold standard as the reference for the value of currencies came to an end in 1933, when president Franklin D. Roosevelt's administration restricted substantial private gold ownership essentially to jewelers.
- When the gold standard was abandoned, many countries pegged their exchange rates to the U.S. dollar, and the U.S dollar was in turn directly connected to the price of gold. Since 1971, the U.S. dollar is no longer fixed to the gold price and, as such, currencies are no longer linked to a commodity which can be bought, sold or stored.

A Bit of History

- Initially, governments attempted to prescribe the exchange rates themselves to improve a country's trading position, e.g. in a scenario where a country would set the exchange rate lower, relative to the other rates, it would improve the country's export by making the export relatively more affordable, while import from other countries would be relatively more expensive.
- On multiple occasions, however, such attempts have led to trade wars. On the other hand, for countries that depend on foreign debt, a lower exchange rate would imply higher installments to be paid, especially if loans are taken in a foreign currency.
- Currently, the major currency exchange rates are determined by supply and demand (i.e., *exchange rates are "floating"*). It is still common practice for countries and their central banks to "fine tune" the exchange rates by keeping gold reserves or foreign currencies, that are then known as foreign exchange reserves. They buy and sell these currencies to stabilize their own currency, when necessary.

How Fx Works?

- Let us consider the euro €, to be the domestic currency and the dollar \$, to be the foreign currency. We set the FX rate to $y_{\$}^{€}(t_0) =: y(t_0)$, expressed in units of the domestic currency, per unit of a foreign currency. If the exchange rate is $y_{\$}^{€}(t_0) = 0.85$ and we wish to exchange \$100 to euros, the calculation is given by:

$$\$100 \cdot y_{\$}^{€}(t_0) = \$100 \cdot 0.85 \frac{€}{\$} = €85.$$

  In this example, the amount \$100 is called the *notional in foreign currency*, which will be denoted by N_{f}. So, a notional in the foreign currency $\$N_f$ is equal to $\$N_f \cdot y(t)$ in the domestic currency (€).
- More generally, we will use the following notation

$$y(t) := y_{\mathrm{f}}^{\mathrm{d}}(t).$$

  We exchange a foreign amount of money to our base, domestic, currency.

Forward FX Contract

- One of the most liquidly traded FX products is the so-called "outright FX forward". This contract is an obligation for a physical exchange of funds at a future date at an agreed rate and there is no payment upfront.
- Such a forward contract is typically well-suited for hedging the foreign exchange risk at a single payment date (the so-called *"bullet payment"*), as opposed to a stream of FX payments.
- A stream of FX payments is usually covered by a contract which is called *an FX swap*. In practice, however, forwards are sometimes favored as they are better affordable, albeit less effective, hedging instruments than the swaps.

Forward FX Contract

- A *forward contract* is equivalent to borrowing and lending the same amount in two different currencies and converting the proceeds in the domestic currency.
- Since one borrows and lends the same amount, the initial value has to be zero, as it is usual for a forward contract when the forward price is equal to the forward exchange rate under no arbitrage.
- Let us consider a (static) replication strategy for an FX forward contract. We consider two currencies, € and \$, with the corresponding interest rates in their markets, indicated here by $r_{€}$ and $r_{\$}$, respectively.

Forward FX Contract

- An example is given of how an FX forward contract can be replicated. At time t_0 borrowing takes place in USD.
- The interest rate to be paid is $r_\$$, and subsequently this money is exchanged at the spot FX rate to EUR. After the money is exchanged, it is loaned in the EUR market, where it will grow with interest rate $r_€$.
- At the contract's maturity time T, the above transactions will be reversed.

Table: Replication strategy for an FX forward

time t_0	time T
borrow in \$: \$1	return: $\$1 \cdot \mathrm{e}^{r_\$ T}$
lend in € : $\$1 \cdot y(t_0)$	obtain: $\$1 \cdot y(t_0) \cdot \mathrm{e}^{r_€ T}$

Forward FX Contract

- The forward rate $y_F(t_0, T)$ is defined such that the contract value equals zero,

$$\$1 \cdot \mathrm{e}^{r_\$ T} \boxed{y_F(t_0, T)} = \$1 \cdot y(t_0) \cdot \mathrm{e}^{r_€ T},$$

and therefore,

$$\boxed{y_F(t_0, T) = y(t_0) \frac{\mathrm{e}^{r_€ T}}{\mathrm{e}^{r_\$ T}} =: y(t_0) \frac{P_\$(t_0, T)}{P_€(t_0, T)},}$$

where the foreign and domestic zero coupon bonds are defined as $P_\$(t, T) = \mathrm{e}^{-r_\$ T}$ and $P_€(t, T) = \mathrm{e}^{-r_€ T}$, respectively.

Forward FX Contract

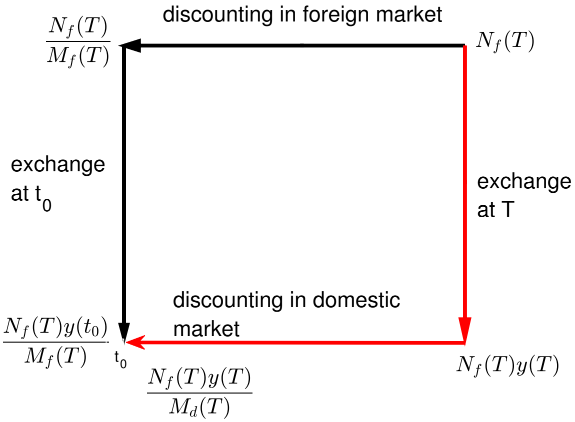


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Figure: Schematic representation of discounting foreign amount to domestic currency.

Forward FX Contract

- Another way to define the forward FX rate is to connect it to a *future FX rate which is seen from today's perspective*.
- Suppose we may obtain at a future time T some amount $N_{\mathrm{f}}(T)$ in a foreign currency. There are two possibilities to calculate today's value in the domestic currency d.
- On the one hand, the notional amount $N_{\mathrm{f}}(T)$ is discounted to today with the foreign currency, giving $N_{\mathrm{f}}(T)/M_{\mathrm{f}}(T)$; subsequenty this amount needs to be exchanged at today's (spot) exchange rate to the domestic currency, yielding,

$$\boxed{V^{\mathrm{FX}}(t_0) = \mathbb{E}^{\mathbb{Q}^{\mathrm{f}}}\left[y(t_0)\frac{N_{\mathrm{f}}(T)}{M_{\mathrm{f}}(T)}\middle|\mathcal{F}(t_0)\right] = y(t_0)P_f(t_0,T)N_f(T),}$$

where $\mathbb{Q}^{\mathrm{f}}$ denotes the foreign currency risk-neutral measure.

Forward FX Contract

- On the other hand, one may exchange $N_{\mathrm{f}}(T)$ to the domestic currency at time T based on the exchange rate $y(T)$, and then the amount $N_{\mathrm{f}}(T)y(T)$ will be expressed in the domestic currency.
- This amount needs to be discounted to time t_0 with a domestic interest rate, i.e.,

$$\begin{aligned} V^{\mathrm{FX}}(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{y(T)N_{\mathrm{f}}(T)}{M_{\mathrm{d}}(T)}\middle|\mathcal{F}(t_0)\right] = N_{\mathrm{f}}(T)\mathbb{E}^{\mathbb{Q}}\left[\frac{y(T)}{M_{\mathrm{d}}(T)}\middle|\mathcal{F}(t_0)\right] \\ &= N_f(T)P_d(t_0,T)\mathbb{E}^{T,d}\left[y(T)\middle|\mathcal{F}(t_0)\right], \end{aligned}$$

where $\mathbb{Q}=\mathbb{Q}^{\mathrm{d}}$ denotes the commonly used domestic currency risk-neutral measure (as typically domestic currency is exchanged to foreign currency).

Forward FX Contract

- These two ways to calculate today's value of the contract should result in the same value, as otherwise an arbitrage opportunity would occur.
- Equating both equations we find, i.e.,

$$\boxed{y(t_0)P_f(t_0,T)N_f(T) = N_f(T)P_d(t_0,T)\mathbb{E}^{T,d}\left[y(T)\middle|\mathcal{F}(t_0)\right]}$$

gives,

$$\begin{aligned} y_F(t_0,T) := \mathbb{E}^{T,d}[y(T)|\mathcal{F}(t_0)] &= y(t_0)\frac{N_f(T)P_f(t_0,T)}{N_f(T)P_d(t_0,T)} \\ &= y(t_0)\frac{P_f(t_0,T)}{P_d(t_0,T)}. \end{aligned}$$

- In other words, the forward exchange rate $y_F(t_0,T)$ is defined as the expectation of the future exchange rate $y(T)$.

Basis Spreads

- The relation, between the forward and spot rates, does not hold exactly. There is a slight difference, which is called *basis spread*. These spreads indicate the preference of owning certain currency, especially in the times of the crisis.
- Cross-currency basis spreads typically widen during the crisis and afterwards, pointing to increased differences in the costs of currencies.
- This means that the forward relation does not hold, i.e.,

$$y_F(t_0,T) := \mathbb{E}^{T,d}[y(T)|\mathcal{F}(t_0)] \quad \neq \quad y(t_0)\frac{P_f(t_0,T)}{P_d(t_0,T)}.$$

- This *can be fixed* by introduction of the basis spreads:

$$\boxed{y_F(t_0,T) := \mathbb{E}^{T,d}[y(T)|\mathcal{F}(t_0)] = y(t_0)\frac{P_f(t_0,T)ZF_f(t_0,T)}{P_d(t_0,T)ZF_d(t_0,T)}.}$$

- The spread factors, $ZF_d(t_0,T)$ and $ZF_f(t_0,T)$, are bootstraps the same way as $P_d(t_0,T)$ and $P_f(t_0,T)$ are.

Cross-Currency Swaps

- A currency swap is the typical long-dated FX product, enables a corporate to transform a loan in one currency (the funding currency) into a loan in another currency.
- An FX swap, is similar to a standard Swap with a few exceptions:
  - Technically we should have different daycount fractions for the two currencies.
  - Most currency swaps involve an initial and a terminal exchange of principal amounts N^d and foreign N^f.
  - There are possible variants of the cross currency swaps: fixed-fixed, fixed-float, float-fixed and float-float.

Cross-Currency Swaps

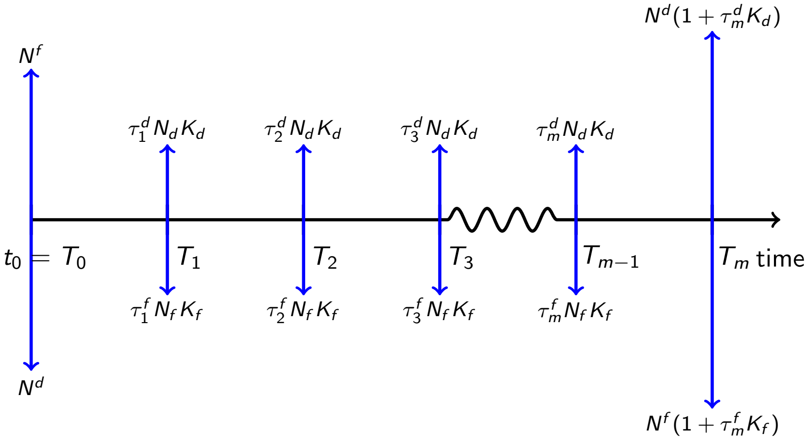


Figure: Cash flows for a xccy swap with $\tau_k = T_k - T_{k-1}$.

Cross-Currency Swaps

- The pricing is of fixed-fixed xCcy swap is the given by (we exclude the first payment)

$$
\begin{aligned}
V^d(t_0) \quad = \quad & \mathbb{E}^d\Big[\sum_{i=1}^{m-1}\frac{1}{M_d(T_i)}\Big(\tau_i^d N_d K_d - \tau_i^f N_f K_f y_f^d(T_i)\Big) \\
& + \frac{1}{M_d(T_m)}\Big(N^d(1+\tau_m^d K_d) - N^f(1+\tau_m^f K_f)y_f^d(T_m)\Big)\Big].
\end{aligned}
$$

- Since

$$
\begin{aligned}
\mathbb{E}^d[\frac{1^f \times y_f^d(T)}{M_d(T)}] \quad & = \quad P_d(t_0,T)\mathbb{E}^{T,d}[y_f^d(T)] \\
& = \quad P_d(t_0,T)y(t_0)\frac{P_f(t_0,T)}{P_d(t_0,T)} = y(t_0)P_f(t_0,T).
\end{aligned}
$$

- Then:

$$
\begin{aligned}
V^d(t_0) \quad = \quad & \sum_{i=1}^{m-1}\Big(P_d(t_0,T_i)\tau_i^d N_d K_d - \tau_i^f P_f(t_0,T_i) N_f K_f y_f^d(t_0)\Big) \\
& + \Big(P_d(t_0,T_m)N^d(1+\tau_m^d K_d) - P_f(t_0,T_m)N^f(1+\tau_m^f K_f)y_f^d(t_0)\Big).
\end{aligned}
$$

Stochastic process for FX

- Let us consider the task of pricing a European-type option on the FX rate $y(T)$, with payoff
$$\max(y(T)-K,0).$$
- For simplicity, we assume the foreign and domestic interest rates, r_f and r_d, respectively, to be constant. The stochastic process for the FX rate $y(t)$ is here defined as,
$$\mathrm{d}y(t) \;\;=\;\; \mu y(t)\mathrm{d}t+\sigma y(t)\mathrm{d}W^{\mathbb{P}}(t).$$
- These dynamics are under the real-world measure $\mathbb{P}$. To determine the dynamics *under the domestic risk-neutral measure*, the arbitrage-free principle is leading.
- A money-savings account in the foreign market, $M_\mathrm{f}(t)$, is expressed in the domestic currency by means of the exchange rate.
- Therefore, in the domestic market the foreign savings account is worth $y(t)M_\mathrm{f}(t)$.

FX process and martingales

- Any discounted asset in the domestic market needs to be a martingale, so that, $\chi(t) := y(t)\frac{M_\mathrm{f}(t)}{M_\mathrm{d}(t)}$, needs to a martingale under the domestic risk-neutral measure $\mathbb{Q}^\mathrm{d}$.
- By application of Itô's lemma, we find:

$$\begin{aligned}
\mathrm{d}\chi(t) &= (r_\mathrm{f}-r_\mathrm{d})\frac{M_\mathrm{f}(t)}{M_\mathrm{d}(t)}y(t)\mathrm{d}t+\frac{M_\mathrm{f}(t)}{M_\mathrm{d}(t)}\mathrm{d}y(t)\\
&= (r_\mathrm{f}-r_\mathrm{d})\frac{M_\mathrm{f}(t)}{M_\mathrm{d}(t)}y(t)\mathrm{d}t+\mu y(t)\frac{M_\mathrm{f}(t)}{M_\mathrm{d}(t)}\mathrm{d}t+\sigma\frac{M_\mathrm{f}(t)}{M_\mathrm{d}(t)}y(t)\mathrm{d}W^\mathbb{P}(t),
\end{aligned}$$

and thus

$$\boxed{\mathrm{d}\chi(t)=(r_\mathrm{f}-r_\mathrm{d})\,\chi(t)\mathrm{d}t+\mu\chi(t)\mathrm{d}t+\sigma\chi(t)\mathrm{d}W^\mathbb{P}(t).}$$

- The process $\chi(t)$ is not a martingale under the $\mathbb{P}$ measure.

FX process and martingales

- Process $\chi(t)$ will be a martingale if the governing dynamics are free of drift terms. This implies the following change of measure,

$$\mathrm{d}W_{\mathsf{d}}^{\mathbb{Q}}(t) = \frac{r_{\mathsf{f}} - r_{\mathsf{d}} + \mu}{\sigma}\mathrm{d}t + \mathrm{d}W^{\mathbb{P}}(t).$$

- The following dynamics of the FX process $y(t)$, under domestic risk-neutral measure $\mathbb{Q}^{\mathsf{d}}$, are obtained:

$$\begin{aligned}\mathrm{d}y(t) &= \mu y(t)\mathrm{d}t + \sigma y(t)\mathrm{d}W^{\mathbb{P}}(t)\\ &= \mu y(t)\mathrm{d}t + \sigma y(t)\mathrm{d}W_{\mathsf{d}}^{\mathbb{Q}}(t) - \sigma y(t)\frac{r_{\mathsf{f}} - r_{\mathsf{d}} + \mu}{\sigma}\mathrm{d}t.\end{aligned}$$

- After simplifications, the following dynamics for process $y(t)$ are obtained,

$$\boxed{\mathrm{d}y(t) = (r_{\mathsf{d}} - r_{\mathsf{f}})\, y(t)\mathrm{d}t + \sigma y(t)\mathrm{d}W_{\mathsf{d}}^{\mathbb{Q}}(t).}$$

Options on FX

- Note that $y(t)$ is a *lognormally distributed random variable*.
- With the FX process, which is defined under the domestic risk-neutral measure, the value of an FX call option, with constant interest rates, is thus given by,

$$\begin{aligned} V_c^{\text{FX}}(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M_{\text{d}}(t_0)}{M_{\text{d}}(T)}\max\left(y(T)-K,0\right)\Big|\mathcal{F}(t_0)\right] \\ &= \mathrm{e}^{-r_{\text{d}}(T-t_0)}\mathbb{E}^{\mathbb{Q}}\left[\max\left(y(T)-K,0\right)\Big|\mathcal{F}(t_0)\right] \\ &= \mathrm{e}^{-r_{\text{d}}(T-t_0)}\mathbb{E}^{\mathbb{Q}}\left[y(T)\mathbb{1}_{y(T)>K}\Big|\mathcal{F}(t_0)\right]-\mathrm{e}^{-r_{\text{d}}(T-t_0)}K\mathbb{Q}\left[y(T)>K\right], \end{aligned}$$

so that the solution reads,

$$\boxed{V_c^{\text{FX}}(t_0)=\mathrm{e}^{-r_{\text{f}}(T-t_0)}y(t_0)F_{\mathcal{N}(0,1)}(d_1)-K\mathrm{e}^{-r_{\text{d}}(T-t_0)}F_{\mathcal{N}(0,1)}(d_2),}$$

with

$$d_1 = \frac{\log\frac{y(t_0)}{K}+\left(r_{\text{d}}-r_{\text{f}}+\frac{1}{2}\sigma^2\right)(T-t_0)}{\sigma\sqrt{T-t_0}},\; d_2=d_1-\sigma\sqrt{T-t_0}.$$

Motivation for the Heston-FX model

- Here we discuss the FX Heston-type models, in which the interest rates are stochastic processes, that are correlated with the governing FX process.
- We first discuss the Heston FX model with Gaussian i.e. Hull-White, short-rate processes. In this model, a full matrix of correlations is prescribed.
- Short-rate interest rate models can typically provide a satisfactory fit to at-the-money interest rate products.
- In practice, the FX calibration is performed with an a-priori calibrated interest rate model.
- A highly efficient and fast model evaluation is required. We focus on the efficient evaluation for the vanilla FX options under this hybrid process, and assume that the parameters for the short-rate model have been determined.
- For plain vanilla European options on a whole *strip of strike prices*, the approximate hybrid model in this section can be evaluated in just milliseconds.

Stochastic interest rates

- The model describes the spot FX, $y(t)$, which is expressed in units of the domestic currency, per unit of a foreign currency.
- The analysis starts with the specification of the underlying short-rate processes, $r_\mathrm{d}(t)$ and $r_\mathrm{f}(t)$, under their spot measures, i.e. the $\mathbb{Q}-$domestic and $\mathbb{Q}^\mathrm{f}-$foreign measures.
- They are governed by the Hull-White one-factor model,

$$\begin{aligned}\mathrm{d}r_\mathrm{d}(t) &= \lambda_\mathrm{d}(\theta_\mathrm{d}(t) - r_\mathrm{d}(t))\mathrm{d}t + \eta_\mathrm{d}\mathrm{d}W_\mathrm{d}^{\mathbb{Q}}(t),\\ \mathrm{d}r_\mathrm{f}(t) &= \lambda_\mathrm{f}(\theta_\mathrm{f}(t) - r_\mathrm{f}(t))\mathrm{d}t + \eta_\mathrm{f}\mathrm{d}W_\mathrm{f}^{\mathbb{Q}^\mathrm{f}}(t),\end{aligned}$$

  where $W_\mathrm{d}^{\mathbb{Q}}(t)$ and $W_\mathrm{f}^{\mathbb{Q}^\mathrm{f}}(t)$ are Brownian motions under $\mathbb{Q}$ and $\mathbb{Q}^\mathrm{f}$, respectively.
- Parameters λ_d, λ_f determine the speed of mean reversion to the time-dependent term structure functions $\theta_\mathrm{d}(t)$, $\theta_\mathrm{f}(t)$, and the parameters η_d, η_f are the respective volatility coefficients.

Stochastic interest rates

- These processes, under the appropriate measures, are linear in their state variables, so that, for a given maturity T, with $0 \leq t \leq T$, the ZCBs are of the following form:

$$\begin{aligned} P_{\mathrm{d}}(t,T) &= \exp\left(\bar{A}_{\mathrm{d}}(t,T) + \bar{B}_{\mathrm{d}}(t,T) r_{\mathrm{d}}(t)\right), \\ P_{\mathrm{f}}(t,T) &= \exp\left(\bar{A}_{\mathrm{f}}(t,T) + \bar{B}_{\mathrm{f}}(t,T) r_{\mathrm{f}}(t)\right), \end{aligned}$$

  with $\bar{A}_{\mathrm{d}}(t,T)$, $\bar{A}_{\mathrm{f}}(t,T)$ and $\bar{B}_{\mathrm{d}}(t,T)$, $\bar{B}_{\mathrm{f}}(t,T)$ analytically known functions.

- The money market accounts in this model are respectively given by,

$$\mathrm{d}M_{\mathrm{d}}(t) = r_{\mathrm{d}}(t) M_{\mathrm{d}}(t)\mathrm{d}t, \qquad \mathrm{d}M_{\mathrm{f}}(t) = r_{\mathrm{f}}(t) M_{\mathrm{f}}(t)\mathrm{d}t.$$

Stochastic interest rates for the FX

- Using the Heath-Jarrow-Morton arbitrage-free argument, as in Lecture 3, the dynamics for the ZCBs, under their own measures generated by the respective money-savings accounts, are known and given by the following result:
- ZCB dynamics under the risk-neutral measure] The risk-free dynamics of the ZCBs $P_\mathrm{d}(t,T)$ and $P_\mathrm{f}(t,T)$, with maturity time T are given by:

$$\begin{cases} \dfrac{\mathrm{d}P_\mathrm{d}(t,T)}{P_\mathrm{d}(t,T)} = r_\mathrm{d}(t)\mathrm{d}t - \left(\int_t^T \bar{\sigma}_\mathrm{d}(t,z)\mathrm{d}z\right)\mathrm{d}W_\mathrm{d}^{\mathbb{Q}}(t), \\ \dfrac{\mathrm{d}P_\mathrm{f}(t,T)}{P_\mathrm{f}(t,T)} = r_\mathrm{f}(t)\mathrm{d}t - \left(\int_t^T \bar{\sigma}_\mathrm{f}(t,z)\mathrm{d}z\right)\mathrm{d}W_\mathrm{f}^{\mathbb{Q}^\mathrm{f}}(t), \end{cases}$$

where $\bar{\sigma}_\mathrm{d}(t,T)$, $\bar{\sigma}_\mathrm{f}(t,T)$ are the volatility functions

Stochastic interest rates for the FX

- The spot rates at time t are defined by $r_{\rm d}(t) \equiv f^r_{\rm d}(t,t)$ and $r_{\rm f}(t) \equiv f^r_{\rm f}(t,t)$.
- The Hull-White short-rate processes $r_{\rm d}(t)$ and $r_{\rm f}(t)$ are then obtained, as well as the term structure functions, $\theta_{\rm d}(t)$, and $\theta_{\rm f}(t)$, that are expressed in terms of instantaneous forward rates, are also known.
- The choice of the specific volatility determines the dynamics of the ZCBs, as

$$\begin{aligned}
\mathrm{d}P_{\rm d}(t,T) &= r_{\rm d}(t)P_{\rm d}(t,T)\mathrm{d}t + \eta_{\rm d}\bar{B}_{\rm d}(t,T)P_{\rm d}(t,T)\mathrm{d}W_{\rm d}^{\mathbb{Q}}(t),\\
\mathrm{d}P_{\rm f}(t,T) &= r_{\rm f}(t)P_{\rm f}(t,T)\mathrm{d}t + \eta_{\rm f}\bar{B}_{\rm f}(t,T)P_{\rm f}(t,T)\mathrm{d}W_{\rm f}^{\mathbb{Q}^{\rm f}}(t),
\end{aligned}$$

with $\bar{B}_{\rm d}(t,T)$ and $\bar{B}_{\rm f}(t,T)$, given by:

$$\bar{B}_{\rm d}(t,T) = \frac{1}{\lambda_{\rm d}}\left(\mathrm{e}^{-\lambda_{\rm d}(T-t)}-1\right),\quad \bar{B}_{\rm f}(t,T) = \frac{1}{\lambda_{\rm f}}\left(\mathrm{e}^{-\lambda_{\rm f}(T-t)}-1\right).$$

Stochastic interest rates for the FX

- The FX-HHW model, with all the processes defined *under the domestic risk-neutral measure* $\mathbb{Q}$, is of the following form:

$$\begin{aligned}
\mathrm{d}y(t)/y(t) &= (r_\mathrm{d}(t) - r_\mathrm{f}(t))\,\mathrm{d}t + \sqrt{v(t)}\mathrm{d}W_y^{\mathbb{Q}}(t),\\
\mathrm{d}v(t) &= \kappa(\bar{v} - v(t))\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v^{\mathbb{Q}}(t),\\
\mathrm{d}r_\mathrm{d}(t) &= \lambda_\mathrm{d}(\theta_\mathrm{d}(t) - r_\mathrm{d}(t))\mathrm{d}t + \eta_\mathrm{d}\mathrm{d}W_\mathrm{d}^{\mathbb{Q}}(t),\\
\mathrm{d}r_\mathrm{f}(t) &= \left(\lambda_\mathrm{f}(\theta_\mathrm{f}(t) - r_\mathrm{f}(t)) - \eta_\mathrm{f}\rho_{y,\mathrm{f}}\sqrt{v(t)}\right)\mathrm{d}t + \eta_\mathrm{f}\mathrm{d}W_\mathrm{f}^{\mathbb{Q}}(t),
\end{aligned}$$

  with $y(t_0) > 0$, $v(t_0) > 0$.
- The parameters κ, λ_d, and λ_f determine the speed of mean reversion of the latter three processes, their long-term mean is given by $\bar{v}$, $\theta_\mathrm{d}(t)$, $\theta_\mathrm{f}(t)$, respectively.
- The volatility coefficients for the processes $r_\mathrm{d}(t)$ and $r_\mathrm{f}(t)$ are given by η_d and η_f, and the volatility-of-variance parameter for process $v(t)$ is γ.

Stochastic interest rates for the FX

- Under the domestic-spot measure, the drift in the short-rate process $r_{\mathrm{f}}(t)$ gives rise to an additional term, i.e., $-\eta_{\mathrm{f}}\rho_{y,\mathrm{f}}\sqrt{v(t)}$, which is often called the *quanto correction term*.
- The r_{f}-process is thus not a HW process anymore under this measure. When the volatility of the FX model is assumed to be deterministic, then r_{f} will follow an HW process.
- Another simplification is based on the assumption $\rho_{y,\mathrm{f}} = 0$, but this is not realistic in practice.
- The additional term in the r_{f}-process ensures the existence of martingales under the domestic spot measure, for the following prices (for more discussion:

$$\boxed{\chi_1(t) := y(t)\frac{M_{\mathrm{f}}(t)}{M_{\mathrm{d}}(t)} \quad \text{and} \quad \chi_2(t) := y(t)\frac{P_{\mathrm{f}}(t,T)}{M_{\mathrm{d}}(t)},}$$

where $P_{\mathrm{f}}(t,T)$ is the foreign zero-coupon bond, and the money-savings accounts $M_{\mathrm{d}}(t)$ and $M_{\mathrm{f}}(t)$.

Stochastic interest rates for the FX

- To confirm that the processes $\chi_1(t)$ and $\chi_2(t)$ are martingales, one may apply the Itô product rule, which gives:

$$\begin{aligned}
\mathrm{d}\chi_1(t)/\chi_1(t) &= \sqrt{v(t)}\mathrm{d}W_y^{\mathbb{Q}}(t),\\
\mathrm{d}\chi_2(t)/\chi_2(t) &= \sqrt{v(t)}\mathrm{d}W_y^{\mathbb{Q}}(t) + \eta_\mathrm{f}B_\mathrm{f}(t,T)\mathrm{d}W_\mathrm{f}^{\mathbb{Q}}(t).
\end{aligned}$$

- The change of dynamics of the underlying processes, from the foreign-spot to the domestic-spot measure, also influences the dynamics for the associated bonds, that, under the domestic risk-neutral measure $\mathbb{Q}$, with the money-savings account as the numéraire, have the following representations,

$$\begin{cases}
\dfrac{\mathrm{d}P_\mathrm{d}(t,T)}{P_\mathrm{d}(t,T)} = r_\mathrm{d}(t)\mathrm{d}t + \eta_\mathrm{d}\bar{B}_\mathrm{d}(t,T)\mathrm{d}W_\mathrm{d}^{\mathbb{Q}}(t),\\
\dfrac{\mathrm{d}P_\mathrm{f}(t,T)}{P_\mathrm{f}(t,T)} = \left(r_\mathrm{f}(t) - \rho_{y,\mathrm{f}}\eta_\mathrm{f}\bar{B}_\mathrm{f}(t,T)\sqrt{v(t)}\right)\mathrm{d}t + \eta_\mathrm{f}\bar{B}_\mathrm{f}(t,T)\mathrm{d}W_\mathrm{f}^{\mathbb{Q}}(t).
\end{cases}$$

Pricing of FX options

- To perform a calibration of the model, we need to price basic options on the FX rate, that are denoted by $V_c^{\text{FX}}(t)$, highly efficiently for a given state vector $\mathbf{X}(t) = [y(t), v(t), r_\text{d}(t), r_\text{f}(t)]^\text{T}$,
- $$\boxed{V_c^{\text{FX}}(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{M_\text{d}(t)}{M_\text{d}(T)}\max(y(T)-K,0)\Big|\mathcal{F}(t)\right],}$$

  with
  $$M_\text{d}(t) = \exp\left(\int_0^t r_\text{d}(z)\text{d}z\right).$$
- To reduce the complexity of the pricing problem, *a transformation* is performed, from the domestic spot measure, which is generated by the money-savings account in the domestic market $M_\text{d}(t)$, to the domestic *forward FX measure*, where the numéraire is the domestic zero-coupon bond $P_\text{d}(t, T)$.

Pricing of FX options

- The forward rate is given by,

$$\boxed{FX(t,T) = y(t)\frac{P_{\mathrm{f}}(t,T)}{P_{\mathrm{d}}(t,T)},}$$

  where $FX(t,T)$ represents the *forward exchange rate under the domestic T-forward measure*, and $y(t)$ stands for foreign exchange rate under the domestic spot measure.

- By switching from the domestic risk-neutral measure $\mathbb{Q}$, to the domestic T-forward measure $\mathbb{Q}^T$, the process of discounting will be decoupled from taking the expectation, i.e.

$$V_c^{\mathrm{FX}}(t) = P_{\mathrm{d}}(t,T)\mathbb{E}^T\left[\max\left(FX(T,T)-K,0\right)|\mathcal{F}(t)\right].$$

Pricing of FX options

- FX-HHW model dynamics under the $\mathbb{Q}^T$ measure Under the T-forward domestic measure, the model is governed by the following dynamics:

$$\frac{\mathrm{d}FX(t,T)}{FX(t,T)} = \sqrt{v(t)}\mathrm{d}W_y^T(t) - \eta_\mathrm{d}\bar{B}_\mathrm{d}(t,T)\mathrm{d}W_\mathrm{d}^T(t) + \eta_\mathrm{f}\bar{B}_\mathrm{f}(t,T)\mathrm{d}W_\mathrm{f}^T(t),$$

where

$$\begin{aligned}
\mathrm{d}v(t) &= \left(\kappa(\bar{v} - v(t)) + \gamma\rho_{v,\mathrm{d}}\eta_\mathrm{d}\bar{B}_\mathrm{d}(t,T)\sqrt{v(t)}\right)\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v^T(t),\\
\mathrm{d}r_\mathrm{d}(t) &= \left(\lambda_\mathrm{d}(\theta_\mathrm{d}(t) - r_\mathrm{d}(t)) + \eta_\mathrm{d}^2\bar{B}_\mathrm{d}(t,T)\right)\mathrm{d}t + \eta_\mathrm{d}\mathrm{d}W_\mathrm{d}^T(t),\\
\mathrm{d}r_\mathrm{f}(t) &= \left(\lambda_\mathrm{f}(\theta_\mathrm{f}(t) - r_\mathrm{f}(t)) - \eta_\mathrm{f}\rho_{y,\mathrm{f}}\sqrt{v(t)} + \eta_\mathrm{d}\eta_\mathrm{f}\rho_{\mathrm{d},\mathrm{f}}\bar{B}_\mathrm{d}(t,T)\right)\mathrm{d}t +\\
&\quad \eta_\mathrm{f}\mathrm{d}W_\mathrm{f}^T(t),
\end{aligned}$$

with a full matrix of correlations.

Pricing of FX options

- Now, we analyze the numerical errors resulting from the various approximations.
- In a set-up, the interest rate curves are modeled by ZCBs, using $t_0 = 0$, $P_{\rm d}(0,T) = \exp(-0.02T)$ and $P_{\rm f}(0,T) = \exp(-0.05T)$.
- Furthermore,

$$\eta_{\rm d} = 0.7\%,\ \ \eta_{\rm f} = 1.2\%,\ \ \lambda_{\rm d} = 1\%,\ \ \lambda_{\rm f} = 5\%.$$

  Model parameters that do not satisfy the Feller condition are prescribed,

$$\kappa = 0.5,\ \ \gamma = 0.3,\ \ \bar{v} = 0.1,\ \ v(0) = 0.1.$$

- The correlation structure is given by:

$$\begin{pmatrix} 1 & \rho_{y,v} & \rho_{y,\rm d} & \rho_{y,\rm f} \\ \rho_{y,v} & 1 & \rho_{v,\rm d} & \rho_{v,\rm f} \\ \rho_{y,\rm d} & \rho_{v,\rm d} & 1 & \rho_{\rm d,f} \\ \rho_{y,\rm f} & \rho_{v,\rm f} & \rho_{\rm d,f} & 1 \end{pmatrix} = \begin{pmatrix} 100\% & -40\% & -15\% & -15\% \\ -40\% & 100\% & 30\% & 30\% \\ -15\% & 30\% & 100\% & 25\% \\ -15\% & 30\% & 25\% & 100\% \end{pmatrix}.$$

- A number of FX options with many expiry times and strike prices, using two different pricing methods for the FX-HHW model, are priced.

Pricing of FX options

- The first method is the plain Monte Carlo method, with 50.000 paths and $20 \cdot T_i$ steps, for the full-scale FX-HHW model.
- In the second pricing method, the characteristic function, which is based on the approximations is used. Efficient pricing of plain vanilla products is then done by means of the COS method.
- The experiments are set up with expiry times given by $T_1, \dots, T_{10}$, and the strike prices are computed by the formula:

$$\begin{aligned} K_n(T_i) &= FX^{T_i}(0,T)\exp\left(0.1c_n\sqrt{T_i}\right), \quad \text{with} \\ c_n &= \{-1.5,\ -1.0,\ -0.5,\ 0,\ 0.5,\ 1.0,\ 1.5\}, \end{aligned}$$

  and $FX^{T_i}(0,T)$ with $y(0) = 1.35$, the initial spot FX rate (like in dollar \$ per euro €) is set to 1.35.
- Formula for the strike prices is convenient, since for $n = 4$, strike prices $K_4(T_i)$, with $i = 1, \dots, 10$, are equal to the forward FX rates for time T_i.

Pricing of FX options

T_i	$K_1(T_i)$	$K_2(T_i)$	$K_3(T_i)$	$K_4(T_i)$	$K_5(T_i)$	$K_6(T_i)$	$K_7(T_i)$
6m	1.1961	1.2391	1.2837	1.3299	1.3778	1.4273	1.4787
1y	1.1276	1.1854	1.2462	1.3101	1.3773	1.4479	1.5221
5y	0.8309	0.9291	1.0390	1.1620	1.2994	1.4531	1.6250
10y	0.6224	0.7290	0.8538	1.0001	1.1714	1.3721	1.6071
20y	0.3788	0.4737	0.5924	0.7409	0.9265	1.1587	1.4491
30y	0.2414	0.3174	0.4174	0.5489	0.7218	0.9492	1.2482

Table: Expiries and strike prices of FX options used in the FX-HHW model, with $y(0) = 1.35$.

- python™

Introduction to inflation

- Definition of inflation from a 1913 edition of Webster's Dictionary, the same year the Federal Reserve was created.

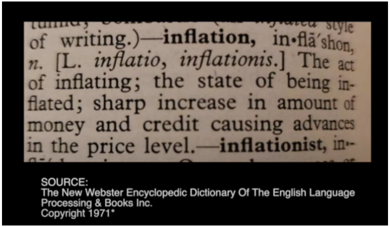
In*fla"tion (?), *n.* [L. *inflatio*: cf. F. *inflation*.] **1.** The act or process of inflating, or the state of being inflated, as with air or gas; distention; expansion; enlargement. *Boyle*.

2. The state of being puffed up, as with pride; conceit; vanity. *B. Jonson*.

3. Undue expansion or increase, from overissue; -- said of currency. [U.S.]

In*fla"tion*ist, *n.* One who favors an increased or very large issue of paper money. [U.S.]

- Definition of inflation from a 1971 edition of Webster's Dictionary, it does mention rising prices, but only as a cause of inflation – an increase in the amount of money and credit


of writing.)—**inflation,** in•flā′shon, *n.* [L. *inflatio, inflationis.*] The act of inflating; the state of being inflated; sharp increase in amount of money and credit causing advances in the price level.—**inflationist,** in-

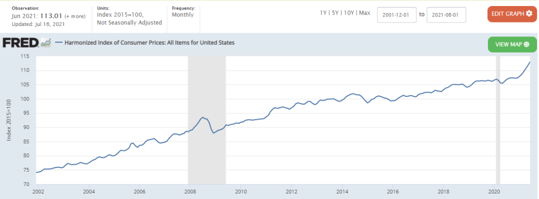
SOURCE:
The New Webster Encyclopedic Dictionary Of The English Language
Processing & Books Inc.
Copyright 1971*

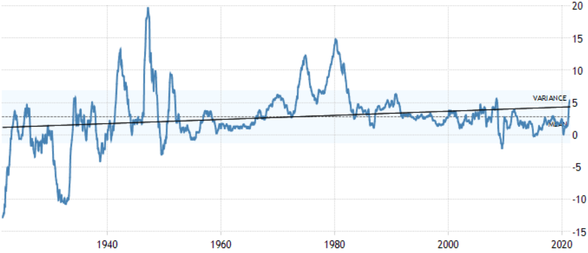
Introduction to inflation

- These days inflation is defined in terms of the percentage increments of a reference index, the (Harmonized) Consumer Price Index (HICP and CPI), which is a representative basket of goods and services.
- The Harmonised Index of Consumer Prices (HICP) differs from the US CPI in two primary aspects. First, the HICP attempts to incorporate rural consumers into the sample while the US maintains a survey strictly based on the urban population. Secondly, the HICP also differs from the US CPI as it excludes so-called *owner-occupied* housing from its scope.
- It is therefore difficult to compare inflation figures from different economies.

Introduction to inflation

- CPI index and Inflation for the US

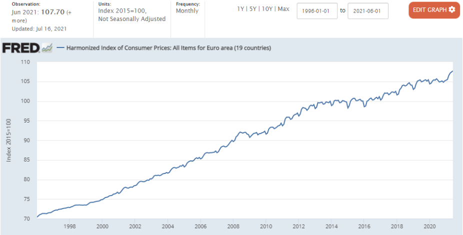




SOURCE: TRADINGECONOMICS.COM | U.S. BUREAU OF LABOR STATISTICS

Introduction to inflation

- CPI index and Inflation for the EU



Consumer Basket

Table A. Percent changes in CPI for All Urban Consumers (CPI-U): U.S. city average

	Seasonally adjusted changes from preceding month							Un-adjusted 12-mos. ended Jul. 2021
	Jan. 2021	Feb. 2021	Mar. 2021	Apr. 2021	May 2021	Jun. 2021	Jul. 2021	
All items	0.3	0.4	0.6	0.8	0.6	0.9	0.5	5.4
Food	0.1	0.2	0.1	0.4	0.4	0.8	0.7	3.4
Food at home	-0.1	0.3	0.1	0.4	0.4	0.8	0.7	2.6
Food away from home[1]	0.3	0.1	0.1	0.3	0.6	0.7	0.8	4.6
Energy	3.5	3.9	5.0	-0.1	0.0	1.5	1.6	23.8
Energy commodities	7.3	6.6	8.9	-1.4	-0.6	2.6	2.3	41.2
Gasoline (all types)	7.4	6.4	9.1	-1.4	-0.7	2.5	2.4	41.8
Fuel oil[1]	5.4	9.9	3.2	-3.2	2.1	2.9	0.6	39.1
Energy services	-0.3	0.9	0.6	1.5	0.7	0.2	0.8	7.2
Electricity	-0.2	0.7	0.0	1.2	0.3	-0.3	0.4	4.0
Utility (piped) gas service	-0.4	1.6	2.5	2.4	1.7	1.7	2.2	19.0
All items less food and energy	0.0	0.1	0.3	0.9	0.7	0.9	0.3	4.3
Commodities less food and energy commodities	0.1	-0.2	0.1	2.0	1.8	2.2	0.5	8.5
New vehicles	-0.5	0.0	0.0	0.5	1.6	2.0	1.7	6.4
Used cars and trucks	-0.9	-0.9	0.5	10.0	7.3	10.5	0.2	41.7
Apparel	2.2	-0.7	-0.3	0.3	1.2	0.7	0.0	4.2
Medical care commodities[1]	-0.1	-0.7	0.1	0.6	0.0	-0.4	0.2	-2.1
Services less energy services	0.0	0.2	0.4	0.5	0.4	0.4	0.3	2.9
Shelter	0.1	0.2	0.3	0.4	0.3	0.5	0.4	2.8
Transportation services	-0.3	-0.1	1.8	2.9	1.5	1.5	-1.1	6.4
Medical care services	0.5	0.5	0.1	0.0	-0.1	0.0	0.3	0.8

[1] Not seasonally adjusted.

Estimation of inflation

- It is important to keep in mind that handling of inflation is a subject to "adjustments", e.g.,
  - Hedonic effect: How much product quality has changed? The impact of the quality improvement is subtracted from overall price increase.
  - Substitution: A phenomena when prices for items change relative to one another.
  - Owner equivalent rent: Housing units are not in the CPI market basket in the US CPI, as it is considered as capital and not consumption. Instead *shelter* impact is incorporated. For an owner-occupied unit, the cost of shelter is the implicit rent that owner-occupants would need to pay if they were renting their homes.
- Since the basket of products used in inflation calculation varies in time, the inflation figures are not comparable.
- CPI is often considered to be a lagging indicator of inflation. Increases in the price of commodities are a better indicator of current inflation because inflation initially affects commodity prices.

Inflation and FX analogy

- It is common to model inflation with the so-called foreign currency analogy, according to which real rates are viewed as interest rates in the real (i.e. foreign) economy, and the CPI is interpreted as the exchange rate between the nominal (i.e. domestic) and real "currencies".
- Pension funds, (life) insurance companies and banks deal with these inflation dependent derivatives. Pension funds are, for example, interested in the conditional (future) indexation of pension rights, which can be viewed as an exotic derivative depending on the inflation level.
- It is clear that there is a connection between the nominal and real interest rates on the market and the break-even inflation rate [1]. Therefore, the use of stochastic nominal and real interest rates is crucial for an accurate inflation pricing model.

[1] The break-even inflation rate is the yield spread between nominal and inflation-linked bonds and is a fundamental indicator of inflation expectations

Inflation Swaps

- Inflation swap consists of two legs: *inflation leg* and *fixed leg*

$$V(T) = N\left(\frac{I(T)}{I(t_0)} - (1+K)^{\alpha}\right),$$

  with $I(T)$ indicating the CPI index at time T and where K indicates the fixed rate.
- Because inflation is quoted on a monthly basis, often the payoff above involves interpolation between neighbouring months. Typically, inflation indexes are either interpolated linearly or a piece-wise constant interpolation is used.
- It is important to remember that inflation is often delayed, with a month or two so the fixing published on a given date reflects the level of price valid earlier.
- In the literature, inflation is often modelled similarly as the FX market where the CPI, *exchanges* nominal (foreign) to real (domestic) currencies.

Inflation Forward and Swap

- Following similar strategy as for the FX forward the forward inflation can be derived and it is given by:

$$I_F(t,T) := I(t)\frac{P_r(t,T)}{P_n(t,T)} = \mathbb{E}^{T,n}[I(T)|\mathcal{F}(t)].$$

- On the other hand the discounted inflation swap reads:

$$\begin{aligned} V(t_0) &= N\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}\left(\frac{I(T)}{I(t_0)} - [(1+K)^{\alpha}]\right)\right] \\ &= NP_n(t_0,T)\mathbb{E}^{T,n}\left[\left(\frac{I(T)}{I(t_0)} - [(1+K)^{\alpha}]\right)\right] \\ &= NP_n(t_0,T)\left(\frac{I_F(t,T)}{I(t_0)} - [(1+K)^{\alpha}]\right). \end{aligned}$$

Inflation Forward and Swap

- By setting the value of the Inflation Swap to zero we find:

$$V(t_0) \quad = \quad NP_n(t_0, T)\left(\frac{I_F(t, T)}{I(t_0)} - (1 + K)^{\alpha}\right) = 0.$$

- Thus,

$$I_F(t, T) = I(t_0)[(1 + K)^{\alpha}].$$

- By combining with the definition of the inflation forward,

$$I_F(t, T) = I(t_0)[(1 + K)^{\alpha}] = I(t)\frac{P_r(t, T)}{P_n(t, T)},$$

so finally:

$$(1 + K)^{\alpha} P_n(t, T) = P_r(t, T),$$

which indicates the *compensation* that needs to be included in the nominal to equal the real ZCB.

Black-Scholes HW model for inflation

- The BSHW Inflation model, with all the processes defined *under the nominal risk-neutral measure* $\mathbb{Q}$, is of the following form:

$$\begin{aligned}
\mathrm{d}y(t)/y(t) &= (r_\mathrm{n}(t) - r_\mathrm{r}(t))\,\mathrm{d}t + \sigma_I \mathrm{d}W_y^{\mathbb{Q}}(t),\\
\mathrm{d}r_\mathrm{n}(t) &= \lambda_\mathrm{n}(\theta_\mathrm{n}(t) - r_\mathrm{n}(t))\mathrm{d}t + \eta_\mathrm{n}\mathrm{d}W_\mathrm{n}^{\mathbb{Q}}(t),\\
\mathrm{d}r_\mathrm{r}(t) &= (\lambda_\mathrm{r}(\theta_\mathrm{r}(t) - r_\mathrm{r}(t)) - \eta_\mathrm{r}\rho_{y,\mathrm{r}}\sigma_I)\,\mathrm{d}t + \eta_\mathrm{r}\mathrm{d}W_\mathrm{r}^{\mathbb{Q}}(t),
\end{aligned}$$

- The price of an inflation indexed cap (or floor) is defined as:

$$V(t_0) = M_n(t_0)\mathbb{E}^{\mathbb{Q}_n}\left[\frac{1}{M_n(T)}\max\left(\frac{I(T)}{I(t_0)} - (1+K)^\alpha, 0\right)\right]$$

- The pricing, after the measure change, is performed as under the BSHW hybrid model.

Black-Scholes HW model for inflation

- The price of a YoY (Year-On-Year) inflation caplet/floorlet option starting at time T_{k-1} and maturing at time T_k, written on the inflation index is given by:

$$V(t_0) = M_n(t_0)\mathbb{E}^{\mathbb{Q}_n}\left[\frac{1}{M_n(T_k)}\max\left(\frac{I(T_k)}{I(T_{k-1})} - (1+K)^{\alpha}, 0\right)\right].$$

- By measure change from $\mathbb{Q}_n$ to T_k-forward measure we find:

$$V(t_0) = P_n(t_0, T_k)\mathbb{E}^{T_k}\left[\max\left(\frac{I(T_k)}{I(T_{k-1})} - (1+K)^{\alpha}, 0\right)\right].$$

- By the definition of the forward inflation rate: $I_F(t, T) = I(t)\frac{P_r(t,T)}{P_n(t,T)}$, we further find:

$$V(t_0) = P_n(t_0, T_k)\mathbb{E}^{T_k}\left[\max\left(\frac{I_F(T_k, T_k)}{I_F(T_{k-1}, T_r)}\frac{P_r(T_{k-1}, T_k)}{P_n(T_{k-1}, T_k)} - (1+K)^{\alpha}, 0\right)\right].$$

Black-Scholes HW model for inflation

- By setting

$$X(T_{k-1},T_k)=\frac{I_F(T_k,T_k)}{I_F(T_{k-1},T_r)}\frac{P_r(T_{k-1},T_k)}{P_n(T_{k-1},T_k)},$$

- we perform the log-transformation:

$$\begin{aligned}\log X(T_{k-1},T_k) &= \log I_F(T_k,T_k)-\log I_F(T_{k-1},T_r)\\ &\quad+\log P_r(T_{k-1},T_k)-\log P_n(T_{k-1},T_k).\end{aligned}$$

- Now, we derive the forward ChF:

$$\phi_{YoY}(u,\log X(T_{k-1},T_k))=\mathbb{E}^{T_k}\left[\exp\left(iu\log X(T_{k-1},T_k)\right)\right].$$

- The pricing of YoY options is equivalent with pricing of the so-called forward-start options (Computational Finance Course: Lecture 12.)

Summary

- Introduction to Foreign Exchange
- Forward FX Contract
- Cross-Currency Swaps
- Pricing of FX Options, the Black-Scholes Case
- The Heston FX Model
- Pricing of FX Options with Stochastic Interest Rates
- Introduction to Inflation
- Pricing of Inflation Forwards and Swaps
- Modeling of Inflation with SDEs
- Summary of the Lecture + Homework

Homework Exercises

- **Exercise 1**
- Finalize derivations for the ChF for the YoY inflation.
- Implement evaluation with the COS method and compare to the Monte Carlo results.
- What approximations do we need to extend the BSHW model for inflation to the Heston SV model?

Homework Exercises

- **Exercise 15.3 from the book**
  Under the domestic T_i-forward measure, we deal with the following FX log-normal dynamics, $y_F(t, T_i)$,

$$\mathrm{d}y_F(t, T_i) = \sigma(t) y_F(t, T_i) \mathrm{d}W_y^F(t, T_i), \quad y_F(0, T_i) = 1.$$

- Show that for the expected ATM payoff at time T_i given by

$$\mathbb{E}\left[(y_F(T_i, T_i) - 1)^+\right] = \mathbb{E}\left[g(x)\right],$$

  the function $g(x)$ is given by,

$$g(x) := 2F_{\mathcal{N}(0,1)}\left(\frac{1}{2}\sqrt{x}\right) - 1, \quad x := \int_0^{T_i} \sigma^2(t) \mathrm{d}t,$$

  where, as usual, $F_{\mathcal{N}(0,1)}(x)$ denotes the standard normal cumulative distribution function.
- Implement a Monte Carlo method and compare the numerical results with the analytic result obtained.

Homework Exercises

- **Exercise 15.6 and 15.7 from the book**
  A change of measure, from the domestic-spot to the domestic $T-$forward measure, requires a change of numéraire, from the money-savings account $M_d(t)$ to the zero-coupon bond $P_d(t, T)$.
- Show that under the $T-$forward domestic measure the process reads,

$$\mathrm{d}y_F(t,T)/y_F(t,T) = \sigma_y \mathrm{d}W_y^T(t) - \eta_d \bar{B}_d(t,T)\mathrm{d}W_d^T(t) + \eta_f \bar{B}_f(t,T)\mathrm{d}W_f^T(t).$$

  with $\bar{B}_d(t, T)$, $\bar{B}_f(t, T)$ and remaining parameters as defined in the lecture.
- Determine the value for $\hat{\sigma}_y(t, T)$ for which the forward FX process, $y_F(t, T)$, will be equal in distribution to $\hat{y}_F(t, T)$, which is defined as,

$$\mathrm{d}\hat{y}_F(t,T)/\hat{y}_F(t,T) = \hat{\sigma}_y(t,T)\mathrm{d}W_*^T(t),$$

  where $W_*^T(t)$ is independent of the other stochastic processes.