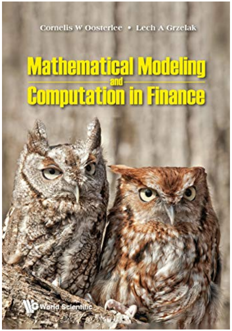
Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.



- Youtube Channel with courses can be found here.
- Slides and the codes can be found here.

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Libor Market Model

Background for the LMM

- Since its introduction by Brace, Gatarek, and Musiela and Jamshidian, the Libor market model (LMM) framework enjoys popularity among practitioners from the financial industry, mainly due to the fact that the model primitives can be directly related to observed products and quantities in the money market, e.g. the forward Libor rates and caplet implied Black volatilities.
- In this framework, closed-form solutions for caps and European swaptions (although not in the same formulation) can be obtained, when the LMM is based on the assumption that the discrete forward Libor rate follows a lognormal distribution, under its own numéraire.
- The *forward measure* pricing methodology, by Jamshidian (1987,1989), and Geman, Karoui and Rochet (1995), is convenient for the valuation of various interest rate contracts.

Background for the LMM

- The LMM modeling approach is also contained in the HJM framework. The fundamental techniques to price interest rate derivatives stem from the original work of Heath, Jarrow and Morton (HJM) in the late 1980s, which is considered to be a modern modeling framework for interest rates.
- Complex fixed-income products, such as various kinds of path-dependent swaptions, usually involve cash flows at different points in time. As the valuation of these products can not be decomposed into a sequence of independent payments, it is a challenging task to develop a model, which is mathematically consistent and agrees with the well–established market formulas like with variants of the Black-Scholes model for the pricing of interest rate derivatives.
- For this, more general LMM dynamics will be presented, that are different from the lognormal LMM dynamics.

Forward Libor rate

- The interest rate model of interest should provide arbitrage-free dynamics for all Libor rates and has to facilitate the pricing of caplets and floorlets in a similar fashion as the market conventional Black-Scholes formula, under the assumption of lognormality, while, on the other hand, we should be able to price complex interest rate derivatives, like swaptions.
- The model which satisfies these requirements is the well-known Libor market model (LMM), also known as the BGM model. Let us recall the definition of the Libor rate: For $t \leq T_{i-1}$, the definition of the *Libor rate* has been given by:

$$\begin{aligned}\ell(t; T_{i-1}, T_i) &= \frac{1}{\tau_i}\left(\frac{P(t, T_{i-1}) - P(t, T_i)}{P(t, T_i)}\right)\\ &= \frac{1}{\tau_i}\left(\frac{P(t, T_{i-1})}{P(t, T_i)} - 1\right),\end{aligned}$$

with $P(t, T_i)$ the ZCB which pays out at time T_i. We will use the short-hand notation $\ell_i(t) := \ell(t; T_{i-1}, T_i)$, for convenience.

Forward Libor rate

- The definition of the Libor rate can be connected to the HJM framework. The forward rate has been defined as,

$$P_f(t, T_1, T_2) = \frac{P(t, T_2)}{P(t, T_1)}.$$

- When assuming a simple compounding, we obtain,

$$P_f(t, T_1, T_2) = \frac{1}{1 + (T_2 - T_1)R(t, T_1, T_2)},$$

using $\tau_2 = T_2 - T_1$ and $\ell(t; T_1, T_2) := R(t, T_1, T_2)$, it follows that,

$$\frac{1}{1 + \tau_2 \ell(t; T_1, T_2)} = \frac{P(t, T_2)}{P(t, T_1)}.$$

- The Libor rates, $\ell(t_0; T_{i-1}, T_i)$, are forward rates that are thus defined by three moments in time, the present time t_0 at which the rate is modeled by an SDE, its "expiry date", T_{i-1} (which is also called the "fixing date" or the "reset date") when this Libor rate is fixed and thus known, and its maturity date T_i at which the Libor rate terminates, with $t_0 \leq T_{i-1} \leq T_i$.

Forward Libor rate

- Libor rate $\ell(t; T_{i-1}, T_i)$ is thus determined at its reset date T_{i-1}, after which the Libor rate is known and does not depend on any volatility anymore. The Libor rate $\ell(t_0; T_{i-1}, T_i)$ may be interpreted as a rate which will be accrued over the period $[T_{i-1}, T_i]$, but is observed "today" at t_0.
- Another element of a forward rate is the *accrual period*, denoted by $\tau_i = T_i - T_{i-1}$, which is the time between expiry and maturity date.
- In practice, the accrual period is not necessarily exactly the period between T_{i-1} and T_i, as typically an additional *"reset delay"* is involved. This reset delay is often equal to a few business days, and it depends on the currency for which the forward rate is calculated (in Euros, it is 2 business days).

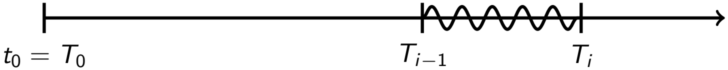

Figure: A forward rate, $\ell(t_0; T_{i-1}, T_i)$, visualized.

Forward Libor rate and its dynamics

- Another date which is important in the contract evaluations is the *pay delay*, τ^*, $T_p := T_i + \tau^*$, as in Figure below. The pay date of a contract typically is a few days after the maturity of the contract. However, τ^* is often neglected in the evaluations,

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\ell(T_{i-1}; T_{i-1}, T_i)}{M(T_i + \tau^*)}\right] \approx \mathbb{E}^{\mathbb{Q}}\left[\frac{\ell(T_{i-1}; T_{i-1}, T_i)}{M(T_i)}\right],$$

  as τ^* is typically only a few business days.

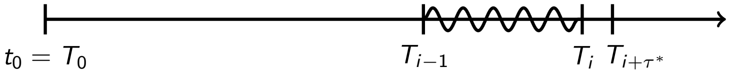

Figure: Forward rate $\ell(t_0; T_{i-1}, T_i)$ visualized with pay delay.

- In the case that the pay delay τ^* tends to be a long time period, a *convexity correction* needs to be taken into account.

Dynamics of the Libor rate

- For Libor rate $\ell_i(t)$, we define the dynamics, as follows:

$$\mathrm{d}\ell_i(t) \quad = \quad \bar{\mu}_i^{\mathbb{P}}(t)\mathrm{d}t + \bar{\sigma}_i(t)\mathrm{d}W_i^{\mathbb{P}}(t), \quad \text{for} \quad i = 1 \ldots m,$$

  with a certain, possibly stochastic, volatility function $\bar{\sigma}_i(t)$ and with $W_i^{\mathbb{P}}(t)$ Brownian motion under measure $\mathbb{P}$, which can be correlated according to,

$$\mathrm{d}W_i^{\mathbb{P}}(t)\mathrm{d}W_j^{\mathbb{P}}(t) = \rho_{i,j}\mathrm{d}t.$$

  Correlation $\rho_{i,j}$ can also be time-dependent.

- We denote by $\mathbb{Q}^i$ the T_i-*forward measure*, associated with the ZCB $P(t, T_i)$ as the numéraire, and $\mathbb{E}^{T_i}$ is the corresponding expectation operator under this measure.

Dynamics of the Libor rate

- By the results of Harrison and Kreps, it is known that for a given arbitrage-free market, for any strictly positive numéraire financial product whose price is given by $g_1(t)$, there exists a measure for which $\frac{g_2(t)}{g_1(t)}$ is a martingale for all product prices $g_2(t)$. This implies that the following *martingale equality* holds:

$$\mathbb{E}^{T_i}\left[\frac{P(T_{i-1},T_{i-1})}{P(T_{i-1},T_i)}\middle|\mathcal{F}(t)\right] = \frac{P(t,T_{i-1})}{P(t,T_i)}.$$

- The left- and right-hand sides can be rewritten so that

$$\mathbb{E}^{T_i}\left[1+\tau_i\ell(T_{i-1};T_{i-1},T_i)\middle|\mathcal{F}(t)\right] = 1+\tau_i\ell(t;T_{i-1},T_i),$$

or,

$$\mathbb{E}^{T_i}\left[\ell(T_{i-1};T_{i-1},T_i)|\mathcal{F}(t)\right] = \ell(t;T_{i-1},T_i).$$

- This result can be explained by a simple analogy. Since $\frac{1}{P(T_{i-1},T_i)}$ is a martingale under the $\mathbb{Q}^i$-measure, the same holds true for Libor rate $\ell(t;T_{i-1},T_i)$

Dynamics of the Libor rate

- To see that $\ell(t; T_{i-1}, T_i)$ is indeed a martingale, we may look at the right-hand side, which represents the price of a traded asset (the spread between two zero-coupon bonds with nominal $\frac{1}{\tau_i}$).
- If we consider the measure $\mathbb{Q}^i$, the T_i-forward measure associated with numéraire $P(t, T_i)$, the forward rate, $\ell(t; T_{i-1}, T_i)$, should be a martingale, so it should be free of drift terms. This implies that it should be possible to transform SDE to,

$$\mathrm{d}\ell_i(t) = \bar{\sigma}_i(t)\mathrm{d}W_i^i(t), \quad \text{for } t < T_{i-1},$$

with $\bar{\mu}_i^i(t) = 0$, and $\bar{\sigma}_i(t)$ an instantaneous volatility of the forward rate $\ell(t; T_{i-1}, T_i)$. $W_i^i(t)$ is the Brownian motion of $\ell_i(t)$ under the T_i-forward measure. So, the subscript in W_i^i refers to the specific Libor rate and the superscript to the measure.

HJM and Libor Market Model

- We explicitly mention, however, that *only* the i-th Libor rate, $\ell_i(t)$, is a martingale under the $\mathbb{Q}^i$ forward measure. If we, for example, would represent the dynamics of Libor rate $\ell_i(t)$ under the T_j-forward measure (with $i \neq j$), the dynamics of $\ell_i(t)$ would be given by
$$\mathrm{d}\ell_i(t) = \bar{\mu}_i^j(t)\mathrm{d}t + \bar{\sigma}_i(t)\mathrm{d}W_i^j(t),$$
with a certain nonzero drift term $\bar{\mu}_i^j(t)$.
- In the lognormal Libor market model, the choice of volatility, $\bar{\sigma}_i(t)$, is given by
$$\bar{\sigma}_i(t) = \sigma_i(t)\ell_i(t),$$
with a time-dependent volatility parameter $\sigma_i(t)$.
- The dynamics of the *lognormal Libor rate* $\ell_i(t)$ then read:

$$\boxed{\frac{\mathrm{d}\ell_i(t)}{\ell_i(t)} = \sigma_i(t)\mathrm{d}W_i^i(t), \quad \text{for} \quad t < T_{i-1}.}$$

Change of measure

- The dynamics of the lognormal LMM can also be obtained by a particular choice of the HJM volatility $\bar{\eta}(t,T)$.
- By choosing $\bar{\eta}(t,z)=\eta(t)$, which does *not* depend on z, and is given by:

$$\bar{\eta}(t,z)=\eta(t)=\frac{\ell_i(t)\sigma_i(t)}{1+\tau_i\ell_i(t)},$$

- The dynamics of the Libor rate under the T_i-forward measure have been described by the SDE,

$$\mathrm{d}\ell_i(t)=\bar{\sigma}_i(t)\mathrm{d}W_i^i(t),\quad \text{for}\quad t<T_{i-1},$$

  where $\bar{\sigma}_i(t)$ is a certain state-dependent process.
- In this lecture, we will change measures and determine the dynamics of the Libor rate $\ell_i(t)$ *under the T_{i-1}-forward measure*. This result will be generalized to the dynamics *under any measure*, T_j, with $j\neq i$.

Randon-Nikodym derivative

- As explained in detail in Lecture 2, to perform a measure transformation the dynamics of the Radon-Nikodym derivative $\lambda_i^{i-1}(t)$, for a measure change from the T_i-forward to the T_{i-1}-forward measure, are needed, i.e.
- The Randon-Nikodym derivative is given by:

$$\lambda_i^{i-1}(t) \quad = \quad \frac{\mathrm{d}\mathbb{Q}^{i-1}}{\mathrm{d}\mathbb{Q}^{i}}\Big|_{\mathcal{F}(t)} := \frac{P(t,T_{i-1})}{P(t_0,T_{i-1})}\frac{P(t_0,T_i)}{P(t,T_i)}.$$

- By the definition of the Libor rate, the Radon-Nikodym derivative can be expressed in terms of $\ell_i(t)$, as

$$\lambda_i^{i-1}(t) \quad = \quad \frac{P(t_0,T_i)}{P(t_0,T_{i-1})}\left(\tau_i\ell_i(t)+1\right),$$

with the corresponding dynamics given by:

$$\mathrm{d}\lambda_i^{i-1}(t) \quad = \quad \frac{P(t_0,T_i)}{P(t_0,T_{i-1})}\tau_i\mathrm{d}\ell_i(t).$$

Randon-Nikodym derivative

- With $\mathrm{d}\ell_i(t)$, and by using, the dynamics of $\lambda_i^{i-1}(t)$ are given by:

$$\mathrm{d}\lambda_i^{i-1}(t) \quad = \quad \lambda_i^{i-1}(t)\frac{\tau_i\bar{\sigma}_i(t)}{\tau_i\ell_i(t)+1}\mathrm{d}W_i^i(t).$$

- Based on the *Girsanov Theorem*, the dynamics of $\lambda_i^{i-1}(t)$ define the following measure transformation, i.e.

$$\boxed{\mathrm{d}W_i^{i-1}(t) = -\frac{\tau_i\bar{\sigma}_i(t)}{\tau_i\ell_i(t)+1}\mathrm{d}t + \mathrm{d}W_i^i(t).}$$

- So, the dynamics of Libor rate $\ell_i(t)$, under the T_{i-1}-forward measure, read:

$$\begin{aligned}\mathrm{d}\ell_i(t) \quad &= \quad \bar{\sigma}_i(t)\mathrm{d}W_i^i(t)\\ &= \quad \bar{\sigma}_i(t)\frac{\tau_i\bar{\sigma}_i(t)}{\tau_i\ell_i(t)+1}\mathrm{d}t + \bar{\sigma}_i(t)\mathrm{d}W_i^{i-1}(t),\end{aligned}$$

with some volatility function $\bar{\sigma}_i(t)$.

Measure transformation

- This formula can be used for other "one time-step measure changes" as well.
- In particular, when moving from the T_i- to the T_{i+1}-measure, the following measure transformation is required,

$$\boxed{\mathrm{d}W_i^i(t) = -\frac{\tau_{i+1}\bar{\sigma}_{i+1}(t)}{\tau_{i+1}\ell_{i+1}(t)+1}\mathrm{d}t + \mathrm{d}W_i^{i+1}(t),}$$

  using with i and $i+1$.
- The dynamics of the Libor rate $\ell_i(t)$ under the T_{i+1}-forward measure can be determined accordingly.
- Function $\bar{\sigma}_i(t)$ may contain a stochastic volatility term, correlations, or the Libor rate itself (as in the lognormal case).
- Two standard formulations to express the dynamics of the Libor market model are particularly practical. Their differences are found in the measure under which the models are presented.
- Commonly, the LMM is derived under either the *spot* or the *terminal* measure.

The LMM under the terminal measure

- Based on the transition between the forward measures of two consecutive points in time, T_i and T_{i+1}, in we perform a one time-step recursion, and find for the relation between T_i and T_{i+2}:

$$\begin{aligned}\mathrm{d}W_i^i(t) &= -\frac{\tau_{i+1}\bar{\sigma}_{i+1}(t)}{\tau_{i+1}\ell_{i+1}(t)+1}\mathrm{d}t + \left(-\frac{\tau_{i+2}\bar{\sigma}_{i+2}(t)}{\tau_{i+2}\ell_{i+2}(t)+1}\mathrm{d}t + \mathrm{d}W_i^{i+2}(t)\right)\\ &= -\sum_{k=i+1}^{i+2}\frac{\tau_k\bar{\sigma}_k(t)}{\tau_k\ell_k(t)+1}\mathrm{d}t + \mathrm{d}W_i^{i+2}(t).\end{aligned}$$

- This result can be generalized to the terminal measure T_m, as follows,

$$\boxed{\mathrm{d}W_i^i(t) = -\sum_{k=i+1}^{m}\frac{\tau_k\bar{\sigma}_k(t)}{\tau_k\ell_k(t)+1}\mathrm{d}t + \mathrm{d}W_i^m(t),}$$

implying the following dynamics for Libor rate $\ell_i(t)$ under the T_m-forward measure, for any $i < m$,

The LMM under the spot measure

- The main problem with market models is that they *do not provide* a continuous time dynamics for any bond in the tenor structure.
- The well-known, continuously re-balanced, money-savings account, in terms of the instantaneous short rate, is given by $\mathrm{d}M(t) = r(t)M(t)\mathrm{d}t$, with $M(t_0) = 1$. The use of a continuous account as a numéraire in the Libor market model, however, does not fit well to the discrete set-up of the Libor market model.
- Let us consider a discrete tenor structure $\mathcal{T}$ and the Libor rates $\ell_i(t)$. A numéraire for the Libor model should preferably be based on a preassigned maturity and on the tenor structure.

The LMM spot measure

- We may use a *discretely re-balanced money-savings account*, where re-balancing takes place at predefined maturity dates, based on the following strategy,
  - At time t_0, we start with 1 currency unit, and we buy $\frac{1}{P(0,T_1)}$ T_1-bonds.
  - At time T_1, we receive the amount $\frac{1}{P(0,T_1)}$, as the owned bonds all pay out 1, and we buy the amount of $\frac{1}{P(0,T_1)} \cdot \frac{1}{P(T_1,T_2)}$ T_2-bonds.
  - At time T_2, we thus receive the amount $\frac{1}{P(0,T_1)P(T_1,T_2)}$, and buy $\ldots$, etc.
- This strategy shows that, between the time points T_0 and T_{i+1}, the spot-Libor rate portfolio contains the following amount of T_{i+1}-bonds:

$$\prod_{k=1}^{i+1} \frac{1}{P(T_{k-1}, T_k)}.$$

The LMM spot measure

- With $\bar{m}(t) := \min\left(i : t \leq T_i\right)$, as the next reset moment, the value of the portfolio at time t is found from the following definition:

Definition (Spot-Libor measure)

The spot-Libor measure, $\mathbb{Q}^{\bar{m}(t)}$, is associated to a money-savings account, which is defined as

$$M(t) := \frac{P\left(t, T_{\bar{m}(t)}\right)}{\prod_{k=1}^{\bar{m}(t)} P(T_{k-1}, T_k)},$$

with $\bar{m}(t) = \min\left(i : t \leq T_i\right)$.

- This definition shows that, in essence, the money-savings account under the Libor market model is governed by *the next zero-coupon bond*, plus the amount of money accumulated so far.

The LMM spot measure

- The dynamics of Libor rate $\ell_i(t)$, under the Libor spot measure, still needs to be determined. From Definition 1, it is known that, for a given time t, the money-savings account only depends on the volatility of the *next* zero-coupon bond, $P(t, T_{\bar{m}(t)})$, while the zero-coupon bonds $P(T_{j-1}, T_j)$, for $j \in \{1, \dots, \bar{m}(t)\}$ are supposed to be known, i.e. deterministic, and do not affect the volatility of $M(t)$.
- This defines the following Radon-Nikodym derivative for changing the measures, from the $T_{\bar{n}(t)}$-forward, with $\bar{n}(t) = \bar{m}(t) + 1$, to the spot-Libor measure $\mathbb{Q}^M$, generated by the money-savings account $M(t)$ (from Definition 1):

$$\lambda^M_{\bar{n}(t)}(t) = \frac{\mathrm{d}\mathbb{Q}^M}{\mathrm{d}\mathbb{Q}^{\bar{n}(t)}}\Big|_{\mathcal{F}(t)} = \frac{M(t)}{M(t_0)} \frac{P\left(t_0, T_{\bar{n}(t)}\right)}{P\left(t, T_{\bar{n}(t)}\right)}.$$

LMM spot measure

- With the definition of the money-savings account under the Libor market model, this gives

$$\lambda^M_{\bar{n}(t)}(t) = \frac{\mathrm{d}\mathbb{Q}^M}{\mathrm{d}\mathbb{Q}^{\bar{n}(t)}}\Big|_{\mathcal{F}(t)} = \frac{P\left(t, T_{\bar{m}(t)}\right)}{P\left(t, T_{\bar{n}(t)}\right)} \frac{P\left(t_0, T_{\bar{n}(t)}\right)}{M(t_0)} \prod_{k=1}^{\bar{m}(t)} \frac{1}{P(T_{k-1}, T_k)}$$

$$= \frac{P\left(t, T_{\bar{m}(t)}\right)}{P\left(t, T_{\bar{n}(t)}\right)} \cdot \bar{P}$$

- From Girsanov's theorem, we may conclude that the measure transformation is defined by:

$$\boxed{\mathrm{d}W^M_{\bar{n}(t)}(t) = -\frac{\tau_{\bar{n}(t)}\bar{\sigma}_{\bar{n}(t)}(t)}{\tau_{\bar{n}(t)}\ell_{\bar{n}(t)}(t)+1}\mathrm{d}t + \mathrm{d}W^{\bar{n}(t)}_{\bar{n}(t)}(t).}$$

- The change of measure, from the $T_{\bar{n}(t)}$-forward to the $\mathbb{Q}^M$ measure, gives rise to a similar transformation as when changing forward measures from T_{i-1} to T_i. This is a consequence of the fact that the spot-Libor measure is, in essence, a forward measure associated with a zero-coupon bond, $P\left(t, T_{\bar{m}(t)}\right)$, with $\bar{m}(t)$ the nearest reset date.

Dynamics under the spot measure

- We end up with the following dynamics for the Libor rate $\ell_i(t)$, $i > \bar{m}(t)$, under the spot-Libor measure $\mathbb{Q}^M$:

$$\boxed{\mathrm{d}\ell_i(t) = \bar{\sigma}_i(t) \sum_{k=\bar{m}(t)+1}^{i} \frac{\tau_k \bar{\sigma}_k(t)}{\tau_k \ell_k(t) + 1} \mathrm{d}t + \bar{\sigma}_i(t) \mathrm{d}W_i^M(t).}$$

  In the lognormal Libor market model, the choice of volatility, $\bar{\sigma}_i(t)$, is given by

$$\bar{\sigma}_i(t) = \sigma_i(t)\ell_i(t),$$

  with a time-dependent volatility parameter $\sigma_i(t)$.

- The formulation above is generic so that function $\bar{\sigma}_i(t)$ may also involve stochastic volatility process.

The Heston model for Libor rates

- As the general dynamics for the Libor rate, Heston-type dynamics are considered for each Libor rate with a nonzero correlation between the forward rates and the stochastic volatility. *This model, however, is not well-suited for accurately and efficiently modeling Libor rate dynamics, as can be seen below.*
- For this, we first consider the Libor rate $\ell_i(t)$, which, under its own measure, is governed by the following dynamics,

$$\left\{\begin{array}{lcl} \mathrm{d}\ell_i(t) & = & \sigma_i\ell_i(t)\sqrt{\nu(t)}\mathrm{d}W_i^i(t), \\ \mathrm{d}\nu(t) & = & \lambda(\nu_0-\nu(t))dt+\eta\sqrt{\nu(t)}\mathrm{d}W_\nu^i(t), \end{array}\right.$$

with the correlation structure given by:

$$\mathrm{d}W_i^i(t)\mathrm{d}W_l^i(t)=\rho_{i,l}\mathrm{d}t,\quad \mathrm{d}W_i^i(t)\mathrm{d}W_\nu^i(t)=\rho_{i,\nu}\mathrm{d}t.$$

The Heston model for Libor rates

- We will discuss a measure change for $\ell_i(t)$. In terms of independent Brownian motions we have

$$\begin{cases} \mathrm{d}\ell_i(t) = \sigma_i \ell_i(t)\sqrt{\nu(t)}\mathrm{d}\widetilde{W}_i^i(t), \\ \mathrm{d}\nu(t) = \lambda(\nu_0 - \nu(t))dt + \eta\sqrt{\nu(t)}\left(\rho_{i,\nu}\mathrm{d}\widetilde{W}_i^i(t) + \sqrt{1-\rho_{i,\nu}^2}\mathrm{d}\widetilde{W}_\nu^i(t)\right). \end{cases}$$

- From the measure change from the T_i- to the T_{i-1}-forward measure gives rise to the following adjustment of the Brownian motion,

$$\mathrm{d}\widetilde{W}_i^i(t) = \frac{\tau_i \bar{\sigma}_i(t,\ell)}{\tau_i \ell_i(t) + 1}\mathrm{d}t + \mathrm{d}\widetilde{W}_i^{i-1}(t),$$

  with $\bar{\sigma}_i(t,\ell) = \sigma_i \ell_i(t)\sqrt{\nu(t)}$.

- The Brownian motion $\widetilde{W}_\nu^i(t)$ of the variance process $\nu(t)$ is assumed to be independent of the Libor rates, which implies that this Brownian motion is also invariant under measure changes, i.e. $\mathrm{d}\widetilde{W}_\nu^i(t) = \mathrm{d}\widetilde{W}_\nu^{i-1}(t)$.

The Heston model for Libor rates

- The dynamics of the Libor rate $\ell_i(t)$, under the $\mathbb{Q}^{i-1}$-forward measure, now read

$$\mathrm{d}\ell_i(t) = \sigma_i\ell_i(t)\sqrt{\nu(t)}\left(\frac{\tau_i\bar{\sigma}_i(t,\ell)}{\tau_i\ell_i(t)+1}\mathrm{d}t + \mathrm{d}\widetilde{W}_i^{i-1}(t)\right),$$

and the dynamics for the variance process are given by,

$$\begin{aligned}\mathrm{d}\nu(t) &= \lambda(\nu_0-\nu(t))\mathrm{d}t\\ &+ \eta\sqrt{\nu(t)}\left[\rho_{i,\nu}\left(\frac{\tau_i\bar{\sigma}_i(t,\ell)}{\tau_i\ell_i(t)+1}\mathrm{d}t + \mathrm{d}\widetilde{W}_i^{i-1}(t)\right) + \sqrt{1-\rho_{i,\nu}^2}\mathrm{d}\widetilde{W}_\nu^{i-1}(t)\right].\end{aligned}$$

- After rewriting, we find:

$$\begin{aligned}\mathrm{d}\nu(t) &= \lambda\left(\nu_0-\nu(t)+\rho_{i,\nu}\frac{\eta}{\lambda}\frac{\tau_i\sigma_i\ell_i(t)}{\tau_i\ell_i(t)+1}\nu(t)\right)\mathrm{d}t\\ &+ \eta\sqrt{\nu(t)}\left[\rho_{i,\nu}\mathrm{d}\widetilde{W}_i^{i-1}(t)+\sqrt{1-\rho_{i,\nu}^2}\mathrm{d}\widetilde{W}_\nu^{i-1}(t)\right].\end{aligned}$$

The Heston model for Libor rates

- The derivation above shows that a measure change which is applied to the Heston-type Libor model will affect the drift term of the Libor rate ℓ_i, *as well as the variance process* $\nu(t)$.
- The Libor rates and stochastic volatility process are correlated, which has a significant impact on the dynamics of the processes under a measure change.
- When the original definition of the volatility process changes, we cannot easily use the Fourier-based pricing methods for the efficient pricing of caplets and swaptions. With *parameter freezing techniques* an approximate model may be derived, which is to a square-root type model.
- Here, the correlation parameter between the Libor rate and the volatility is set *equal to zero*. Because the variance process is now not correlated to the Libor rates, a measure change will not affect the form of the dynamics.

The Displaced Diffusion model for Libor rates

- This leads to the following *displaced diffusion* version,

$$\begin{cases} \mathrm{d}\ell_i(t) &= \sigma_i\left(\beta\ell_i(t)+(1-\beta)\ell_i(t_0)\right)\sqrt{\nu(t)}\mathrm{d}W_i^i(t), \\ \mathrm{d}\nu(t) &= \lambda(\nu_0-\nu(t))\mathrm{d}t+\eta\sqrt{\nu(t)}\mathrm{d}W_\nu(t), \end{cases}$$

with the correlations,

$$\mathrm{d}W_i^i(t)\mathrm{d}W_l^i(t)=\rho_{il}\mathrm{d}t,\ \ \mathrm{d}W_i^i\mathrm{d}W_\nu(t)=0.$$

- Typically, a model with zero correlation is able to generate different volatility smile patterns, but it cannot represent skew-shaped implied volatilities. The present model is however also able to generate an implied volatility skew shape, due to the displacement parameter β, and the stochastic volatility process.
- The model is called *the Displaced Diffusion Stochastic Volatility Libor Market Model*, or SV-LMM for short.

The Displaced Diffusion model for Libor rates

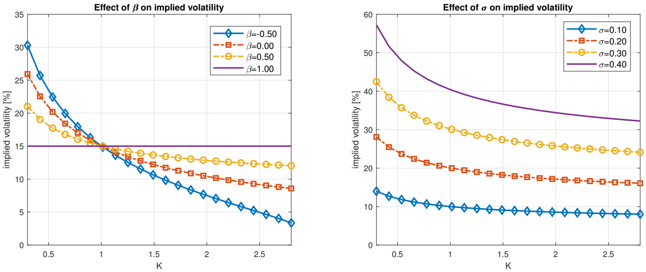

Figure: The effect of β and σ on the implied volatilities under the DD model with $\ell_i(t_0) = 1$, $T = 2$. In the left-hand figure, $\sigma = 0.15$; in the right-hand figure, $\beta = 0.5$.

python™

The Displaced Diffusion and the Heston Model

- The Libor rate dynamics can be written as,

$$\mathrm{d}\ell_i(t) \;=\; \beta\sigma_i\left(\ell_i(t)+(1-\beta)\frac{\ell_i(t_0)}{\beta}\right)\sqrt{\nu(t)}\mathrm{d}W_i^i(t),$$

which is equivalent to,

$$\frac{\mathrm{d}\ell_i(t)}{\ell_i(t)+(1-\beta)\frac{\ell_i(t_0)}{\beta}} \;=\; \beta\sigma_i\sqrt{\nu(t)}\mathrm{d}W_i^i(t).$$

- Since $\mathrm{d}(\ell_i(t)+a)=\mathrm{d}\ell_i(t)$, for constant values of a, the process equals,

$$\frac{\mathrm{d}\left(\ell_i(t)+(1-\beta)\frac{\ell_i(t_0)}{\beta}\right)}{\ell_i(t)+(1-\beta)\frac{\ell_i(t_0)}{\beta}} \;=\; \beta\sigma_i\sqrt{\nu(t)}\mathrm{d}W_i^i(t).$$

The Displaced Diffusion and the Heston Model

- By another process, $\jmath_i(t)$, which is defined as,

$$\jmath_i(t) := \ell_i(t) + (1-\beta)\frac{\ell_i(t_0)}{\beta},$$

with $\jmath_i(t_0) = \ell_i(t_0)/\beta$, it is easy to see that, with β and $\ell_i(t_0)$ constant,

$$\mathrm{d}\jmath_i(t) = \mathrm{d}\ell_i(t).$$

- The dynamics for $\jmath_i(t)$ are then given by:

$$\frac{\mathrm{d}\jmath_i(t)}{\jmath_i(t)} = \beta\sigma_i\sqrt{\nu(t)}\mathrm{d}W_i^i(t) = \sqrt{\beta^2\sigma_i^2\nu(t)}\mathrm{d}W_i^i(t).$$

- By defining $\hat{\nu}(t) := \beta^2\sigma_i^2\nu(t)$, the process for $\jmath_i(t)$ can be written as

$$\frac{\mathrm{d}\jmath_i(t)}{\jmath_i(t)} = \sqrt{\hat{\nu}(t)}\mathrm{d}W_i^i(t).$$

The Displaced Diffusion and the Heston Model

- The final step is the derivation of the dynamics for $\hat{\nu}(t)$. By Itô's lemma , we find:

$$\begin{aligned}\mathrm{d}\hat{\nu}(t) &= \beta^2\sigma_i^2\mathrm{d}\nu(t)\\ &= \beta^2\sigma_i^2\lambda(\nu_0-\nu(t))\mathrm{d}t+\beta^2\sigma_i^2\eta\sqrt{\nu(t)}\mathrm{d}W_\nu(t)\\ &= \lambda(\beta^2\sigma_i^2\nu_0-\beta^2\sigma_i^2\nu(t))\mathrm{d}t+\beta\sigma_i\eta\sqrt{\beta^2\sigma_i^2\nu(t)}\mathrm{d}W_\nu(t),\end{aligned}$$

  which gives us,

$$\mathrm{d}\hat{\nu}(t) = \lambda(\hat{\nu}_0-\hat{\nu}(t))\mathrm{d}t+\hat{\eta}\sqrt{\hat{\nu}(t)}\mathrm{d}W_\nu(t),$$

  with $\hat{\nu}_0=\beta^2\sigma_i^2\nu_0$ and $\hat{\eta}=\beta\sigma_i\eta$.
- The resulting dynamics for $\jmath_i(t)$ then read:

$$\begin{cases}\dfrac{\mathrm{d}\jmath_i(t)}{\jmath_i(t)} = \sqrt{\hat{\nu}(t)}\mathrm{d}W_i^i(t),\\ \mathrm{d}\hat{\nu}(t) = \lambda(\hat{\nu}_0-\hat{\nu}(t))\mathrm{d}t+\hat{\eta}\sqrt{\hat{\nu}(t)}\mathrm{d}W_\nu(t),\end{cases}$$

  which resemble the dynamics of a standard Heston model, with appropriately shifted parameters.
- This implies that the model can be calibrated like the standard Heston model.

Simplifying drift in the LMM

- Often, in order to simplify the complicated dynamics of the LMM, practicioners perform the so-called *freezing* technique which simply projects stochastic quantities on the initial value, i.e., $\ell_i(t) \approx \ell_i(t_0)$.

$$
\begin{aligned}
\mathrm{d}\ell_i(t) &= \sigma_i(t)\ell_i(t)\sum_{k=\bar{m}(t)+1}^{i}\frac{\tau_k\sigma_k(t)\ell_k(t)}{\tau_k\ell_k(t)+1}\mathrm{d}t + \sigma_i(t)\ell_i(t)\mathrm{d}W_i^M(t)\\
&\approx \sigma_i(t)\ell_i(t_0)\sum_{k=\bar{m}(t)+1}^{i}\frac{\tau_k\sigma_k(t)\boxed{\ell_k(t_0)}}{\tau_k\boxed{\ell_k(t_0)}+1}\mathrm{d}t + \sigma_i(t)\ell_i(t)\mathrm{d}W_i^M(t).
\end{aligned}
$$

- Freezing of the libors often facilitates analytical solution at the cost of the quality.

Simplifying drift in the LMM

- Freezing approximation may work well, but only under very strict conditions, i.e., *flat volatility term structure*.
- Enforcing of the libor freezing may significantly limit the purpose of the LMM model.

Definition of convexity

- The term *convexity* is often used in finance. The concept of convexity adjustment is required for all asset classes when there are payment delays or when the moments of payment do not correspond to the payment date of the numéraire.
- Generally, if we have a maturity date T but payment takes place at time $T+\tau^*$, convexity has to be taken into account. The higher the uncertainty in the market (high volatility) the more pronounced the effect of the convexity will become.
- Let us consider a basic interest rate payoff function, which pays a percentage of a notional N, and the percentage paid will be determined by the Libor rate $\ell(T_{i-1};T_{i-1},T_i)$, at time T_i. The price of such a contract is given by:

$$\begin{aligned}V(t_0) &= N\cdot M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{\ell(T_{i-1};T_{i-1},T_i)}{M(T_i)}\right]\\ &= N\cdot P(t_0,T_i)\mathbb{E}^{T_i}\left[\ell(T_{i-1};T_{i-1},T_i)\right].\end{aligned}$$

Convexity and measure change

- Since $\ell(T_{i-1}; T_{i-1}, T_i)$ is a martingale under the T_i-forward measure, we have,

$$V(t_0) \quad = \quad N \cdot P(t_0, T_i)\ell(t_0; T_{i-1}, T_i).$$

- Suppose now that we consider the same contract, however, the payment will take place at some earlier time $T_{i-1} < T_i$. The current value of the contract is then given by:

$$V(t_0) \quad = \quad N \cdot M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{\ell(T_{i-1}; T_{i-1}, T_i)}{M(T_{i-1})}\right].$$

- When changing measures, to the T_{i-1}-forward measure, we work with the following Radon-Nikodym derivative:

$$\frac{\mathrm{d}\mathbb{Q}^{T_{i-1}}}{\mathrm{d}\mathbb{Q}}\Big|_{\mathcal{F}(T_{i-1})} = \frac{P(T_{i-1}, T_{i-1})}{P(t_0, T_{i-1})}\frac{M(t_0)}{M(T_{i-1})},$$

so that,

$$V(t_0) \quad = \quad N \cdot M(t_0)\mathbb{E}^{T_{i-1}}\left[\frac{P(t_0, T_{i-1})}{P(T_{i-1}, T_{i-1})}\frac{M(T_{i-1})}{M(t_0)}\frac{\ell(T_{i-1}; T_{i-1}, T_i)}{M(T_{i-1})}\right].$$

Convexity and measure change

- Therefore,

$$V(t_0) \;=\; N \cdot P(t_0, T_{i-1})\mathbb{E}^{T_{i-1}}\left[\ell(T_{i-1}; T_{i-1}, T_i)\right].$$

- Although the Libor rate $\ell(T_{i-1}; T_{i-1}, T_i)$ is a martingale under the T_i-forward measure, *it is however not a martingale* under the T_{i-1} forward measure, i.e.,

$$\mathbb{E}^{T_{i-1}}\left[\ell(T_{i-1}; T_{i-1}, T_i)\right] \neq \mathbb{E}^{T_i}\left[\ell(T_{i-1}; T_{i-1}, T_i)\right] = \ell(t_0; T_{i-1}, T_i).$$

- The difference between these two expectations is commonly referred to as a *convexity*.
- By the change of measure technique, we can simplify the expressions above, to some extent. By changing to the T_i-forward measure, we find:

$$\frac{\mathrm{d}\mathbb{Q}^i}{\mathrm{d}\mathbb{Q}^{i-1}}\bigg|_{\mathcal{F}(T_{i-1})} = \frac{P(T_{i-1}, T_i)}{P(t_0, T_i)}\frac{P(t_0, T_{i-1})}{P(T_{i-1}, T_{i-1})}.$$

Convexity and measure change

- So that the value of the derivative is equal to:

$$
\begin{aligned}
V(t_0) &= N \cdot P(t_0, T_{i-1})\mathbb{E}^{T_{i-1}}\left[\ell(T_{i-1}; T_{i-1}, T_i)\right] \\
&= N \cdot P(t_0, T_{i-1})\mathbb{E}^{T_i}\left[\ell(T_{i-1}; T_{i-1}, T_i)\frac{P(t_0, T_i)}{P(T_{i-1}, T_i)}\frac{P(T_{i-1}, T_{i-1})}{P(t_0, T_{i-1})}\right] \\
&= N \cdot \mathbb{E}^{T_i}\left[\ell(T_{i-1}; T_{i-1}, T_i)\frac{P(t_0, T_i)}{P(T_{i-1}, T_i)}\right].
\end{aligned}
$$

- It can be written as:

$$
\begin{aligned}
V(t_0) &= N \cdot \mathbb{E}^{T_i}\left[\ell(T_{i-1}; T_{i-1}, T_i)|\mathcal{F}(t_0)\right] \\
&+ N \cdot \mathbb{E}^{T_i}\left[\ell(T_{i-1}; T_{i-1}, T_i)\left(\frac{P(t_0, T_i)}{P(T_{i-1}, T_i)} - 1\right)\right].
\end{aligned}
$$

The last equality holds by simply adding and subtracting $\ell(T_{i-1}; T_{i-1}, T_i)$.

Convexity and measure change

- After further simplifications, we find,

$$V(t_0) \quad = \quad N \cdot \left(\ell(t_0; T_{i-1}, T_i) + \mathrm{cc}(T_{i-1}, T_i)\right),$$

with the *convexity correction*, $\mathrm{cc}(T_{i-1}, T_i)$, given by:

$$\mathrm{cc}(T_{i-1}, T_i) = \mathbb{E}^{T_i}\left[\ell(T_{i-1}; T_{i-1}, T_i)\left(\frac{P(t_0, T_i)}{P(T_{i-1}, T_i)} - 1\right)\right].$$

and the convexity correction reads:

$$\mathrm{cc}(T_{i-1}, T_i) \quad = \quad P(t_0, T_i)\mathbb{E}^{T_i}\left[\frac{\ell(T_{i-1}; T_{i-1}, T_i)}{P(T_{i-1}, T_i)}\right] - \ell(t_0; T_{i-1}, T_i).$$

- From the definition of the Libor rate $\ell(T_{i-1}; T_{i-1}, T_i)$, we know,

$$P(T_{i-1}, T_i) = \frac{1}{1 + \tau_i \ell(T_{i-1}; T_{i-1}, T_i)} =: \frac{1}{1 + \tau_i \ell_i(T_{i-1})}.$$

Convexity and measure change

- The expectation can be written as:

$$\mathbb{E}^{T_i}\left[\frac{\ell_i(T_{i-1})}{P(T_{i-1},T_i)}\right] = \ell_i(t_0) + \tau_i \mathbb{E}^{T_i}\left[\ell_i^2(T_{i-1})\right].$$

It is important to note that although $\ell_i(T_{i-1})$ is a martingale under the T_i-forward measure, the quantity $\ell_i^2(T_{i-1})$ is *not* a martingale under the same measure! To see this, let us consider the dynamics

$$\mathrm{d}\ell_i(t) = \sigma \ell_i(t)\mathrm{d}W_i^i(t),$$

and apply Itô's Lemma to $\ell_i^2(t)$. This will give us,

$$\mathrm{d}\ell_i^2(t) = \frac{1}{2}\sigma^2\ell_i^2(t)\mathrm{d}t + 2\sigma\ell_i^2(t)\mathrm{d}W_i^i(t).$$

- This SDE has a drift term, so it is not a martingale.

Convexity and measure change

- We have different choices to determine the expectation at the right-hand side in the above expression.
- A basic choice for the Libor rate is then to define it as a lognormal process, as follows

$$\mathrm{d}\ell_i(t) = \sigma \ell_i(t) \mathrm{d}W_i^i(t),$$

with the solution given by, using $t_0 = 0$,

$$\ell_i(T_{i-1}) = \ell_i(t_0) \mathrm{e}^{-\frac{1}{2}\sigma^2 T_{i-1} + \sigma W_i^i(T_{i-1})}.$$

Taking the expectation of the squared Libor rate gives us,

$$\mathbb{E}^{T_i}\left[\ell^2(T_{i-1})\right] = \ell^2(t_0)\mathrm{e}^{-\sigma^2 T_{i-1}} \mathbb{E}^{T_i}\left[\mathrm{e}^{2\sigma W_i^i(T_{i-1})}\right] = \ell^2(t_0)\mathrm{e}^{\sigma^2 T_{i-1}}.$$

- We then find,

$$\mathrm{cc}(T_{i-1}, T_i) = P(t_0, T_i)\left(\ell_i(t_0) + \tau_i \ell_i^2(t_0)\mathrm{e}^{\sigma^2 T_{i-1}}\right) - \ell_i(t_0),$$

Convexity and measure change

In Figure below the convexity is affected by the volatility is presented.

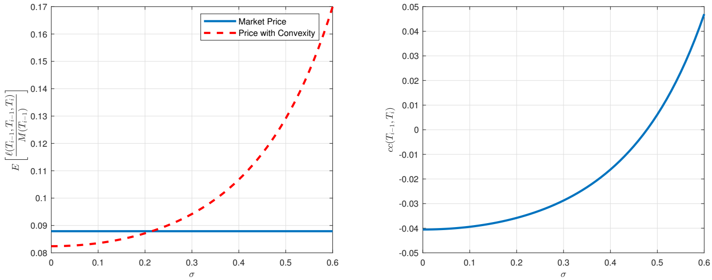

Figure: The effect of the convexity adjustment $cc(T_{i-1}, T_i)$, as a function of the volatility σ. Left: the effect of the convexity on the derivatives price; Right: the effect of the volatility on the convexity.

Breeden-Litzenberger method

- In practice, however, this specification is not optimal, as it is not clear how to specify the parameter σ in the dynamics. One may consider to choose σ to be the caplet ATM volatility, however this is not a unique choice.
- A more reliable approach for the calculation of the second moment of the Libor rate $\ell_i(T_{i-1})$ is to use all strike prices available in the market. This can be done by employing *the Breeden-Litzenberger method*, i.e.,

$$
\begin{aligned}
\mathbb{E}^{T_i}\left[\ell_i^2(T_{i-1})\right] &= \ell_i^2(t_0) + 2\int_0^{\ell_i(t_0)} V_p(t_0,\ell_i(t_0);y,T_{i-1})\mathrm{d}y \\
&+ 2\int_{\ell_i(t_0)}^{\infty} V_c(t_0,\ell_i(t_0);y,T_{i-1})\mathrm{d}y,
\end{aligned}
$$

where $V_p(t_0,\ell_i(t_0);y,T)$ and $V_c(t_0,\ell_i(t_0);y,T)$ are values (without discounting) of put and call options (i.e., floorlets $V_i^{\mathrm{FL}}(t_0)$ and caplets $V_i^{\mathrm{CPL}}(t_0)$) on the rate $\ell_i(T_{i-1})$ with strike price y.

Summary

- A bit of History
- Libor Market Model Specifications
- Libor Rate Dynamics, from $\mathbb{P} \to \mathbb{Q}^T$
- Lognormal Libor Market Model and Measure Changes
- LMM Under the Terminal and Spot Measures
- Stochastic Volatility LMM
- Smile and Skew in the LMM (Displaced Diffusion)
- Freezing Technique
- Convexity Correction
- Convexity and Inclusion of Volatility Smile and Skew
- Summary of the Lecture + Homework

Homework Exercises

The solutions for the homework can be find at
https://github.com/LechGrzelak/QuantFinanceBook

- **Exercise 14.4 from the book**
  "OIS" is an abbreviation which stands for "overnight index swaps". In the standard setting, this overnight index is based on a specified published index of the daily overnight rates. The time of the coupon payments of the OIS may range from 1 week up to 2 years. We denote the daily rate by $\ell(t, T_{i-1}, T_i)$, with $\tau_i = T_i - T_{i-1} = 1d$ (letter "d" indicates "day") show that the daily geometric averaging (often called *geometric compounding*) of the rate over some period $[T_0, T_m]$ equals the forward rate over the same time horizon, i.e., show that,

$$\prod_{i=1}^{m}(1+\tau_i\ell(t, T_{i-1}, T_i)) - 1 = (T_m - T_0)\ell(t, T_0, T_m).$$

Homework Exercises

- **Exercise 14.7 from the book**
  Consider the so-called *"double-Heston model"* for the forward rate $\ell_i(t) := \ell(t, T_{i-1}, T_i)$, which is described by the following dynamics,

$$\begin{aligned}
\mathrm{d}\ell_i(t)/\ell_i(t) &= \sqrt{v_1(t)}\mathrm{d}W^i_{i,1}(t) + \sqrt{v_2(t)}\mathrm{d}W^i_{i,2}(t),\\
\mathrm{d}v_1(t) &= (\bar{v}_1 - v_1(t))\,\mathrm{d}t + \sqrt{v_1(t)}\mathrm{d}W_{v,1}(t),\\
\mathrm{d}v_2(t) &= (\bar{v}_2 - v_2(t))\,\mathrm{d}t + \sqrt{v_2(t)}\mathrm{d}W_{v,2}(t),
\end{aligned}$$

  with all the correlation parameters equal to 0.
  Find the parameters of the process $\xi(t)$, such that the model can be reformulated into the following model,

$$\begin{aligned}
\mathrm{d}\ell_i(t)/\ell_i(t) &= \sqrt{\xi(t)}\mathrm{d}W^i_{*}(t),\\
\mathrm{d}\xi(t) &= \left(\bar{\xi} - \xi(t)\right)\mathrm{d}t + \sqrt{\xi(t)}\mathrm{d}W_{\#}(t).
\end{aligned}$$

Homework Exercises

- **Exercise**
- For derivations of the convexity adjustments we have used a simple lognormal process

$$\mathrm{d}\ell_i(t) = \sigma\ell_i(t)\mathrm{d}W_i^i(t).$$

Derive the convexity correction (as presented in Slide 43) for the following dynamics:

$$\mathrm{d}\ell_i(t) = \sigma\left(\ell_i(t) + \theta\right)\mathrm{d}W_i^i(t).$$