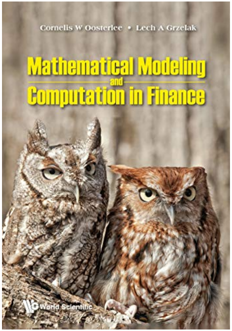
Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.



- Youtube Channel with courses can be found here.
- Slides and the codes can be found here.

List of content

Mortgages

- A mortgage is a long-term loan that is secured by a registered good. The two counterparties of a mortgage are the lender (the *mortgagee*), and the borrower (*mortgagor*).
- The nature of this collateral can largely differ, since mortgages could be used, for example, to buy real estates, ships or aircrafts.
- Mortgage cash-flows are NOT known with certainty (a client may default).
- The mortgages that we consider here (that are common products in the Netherlands) provide a mortgagor with two embedded options,
    - one regarding the choice of the mortgage's interest rate and the other as a possibility to deviate from the scheduled cash flows. The first option gives rise to so-called *pipeline risk*: the borrower has the opportunity to get a mortgage based on the lowest rate from the grace period, which usually is three months.
    - The second option generates the *prepayment risk*, and is our current interest.

Mortgages

- Mortgages come with an embedded option that gives the possibility to the mortgagor to redeem part of the debt in advance concerning the scheduled plan of amortization.
- When a mortgage is settled, the borrower obtains a precise schedule of payments that has to be followed. These cash flows guarantee that the borrower ultimately pays back the initial sum, i.e., the notional, plus an extra amount of money representing his/her cost of the loan (or, from the other perspective, the bank's profit).
- These payments are called *repayments*.
- On the contrary, *prepayments* are extra payments that the mortgagor can affect during the mortgage life, and they consist of deviations from the scheduled cash flows.

Mortgages

- This risk can be substantial, as a financial institution typically relies on long-term payment periods by the mortgagor, with corresponding interest rate payments.
- The prepayment of a large amount will reduce the bank's incoming cash flows and it may also reduce the contract duration.
- When a mortgagor signs a mortgage contract with a bank, then typically the bank sets up a deal with another financial institution to collect the lump sum for immediate payment to the client, and mirrors the cash flows that will occur with the mortgagee.
- When the contract details suddenly change, due to prepayment, this has a significant impact on the contracts in the context of this mortgage. When, a large number of clients of a bank suddenly pre-pay, significant pre-payment risk may result for the bank, which needs to be analyzed and hedged.

Mortgages

- The interest, which depends on interest rate, notional and duration of the loan, represents the profit that a bank generates from selling the mortgage. Prepayment will therefore result in a loss, because either the notional or the interest rate decreases, and also the loan duration may be reduced.
- Mortgages are classified according to the amortization plan that the notional follows along the duration of the contract. With all other characteristics equal (duration, initial notional, interest rate), the amortization plan influences the amount of interest raised during the lifetime of the mortgage and the way the notional is paid back.
- Different amortization plans also give rise to other impact on the prepayment rate. Two of the most common mortgae plans are *bullet mortgage* and *annuity mortgage*.

Types of Mortgages: Bullet

- The bullet is a straightforward mortgage, where the borrower receives N_0 at the time of settling, t_0, and the notional is fully redeemed at the end of the contract period, in one single payment. At the end of each payment period, only the interest part is paid to the loaner, so the notional remains constant until T_M,

$$N(T_i) = N_0 1_{\{T_i < T_M\}}.$$

- The installment $C(T_i)$ serves to pay the interest part which is due at time T_i, which is based on the notional of the loan, N_0, the interest rate K and the time span that the payment covers, i.e.,

$$C(T_i) = KN_0\tau_i,$$

  with $\tau_i = T_{i+1} - T_i$.
- The total amount of interest that a lender receives at the end of the contract equals $I = \sum_{i=1}^{M} KN_0\tau_i$.

Types of Mortgages: Bullet

- Assuming a constant CPR, Λ, and considering that in a bullet the repayments are equal to zero, the notional at time T_i is given by:

$$N(T_i) = (1-\Lambda)N(T_{i-1}) = (1-\Lambda)^2 N(T_{i-2}) = ... = (1-\Lambda)^i N_0,$$

with the total amount of interest,

$$I = \sum_{i=0}^{M-1} KN(T_i) = KN_0 \sum_{i=0}^{M-1} (1-\Lambda)^i = \frac{KN_0}{\Lambda}(1-(1-\Lambda)^M).$$

Types of Mortgages: Bullet

- Let us consider an example with: the maturity is 10 years, and the mortgage rate is 3%.
- We consider two cases, one without prepayments and another with $\Lambda = 12\%$.
- In the first case, the installment is constant, as this contract does not involve any repayment. In the other case, we see how the prepayments (represented by the cyan bars) are added to the contractual payments of the interest.
- As a consequence the notional diminishes in time, and so do the interest payments based on it.
- The final installment is therefore not as large as in the case of no prepayments. In the right-side graph, the effect of Λ on the remaining notional $N(t)$ is shown. Clearly, the higher the prepayment rate, the stronger the decay of the notional in time.

Types of Mortgages: Bullet

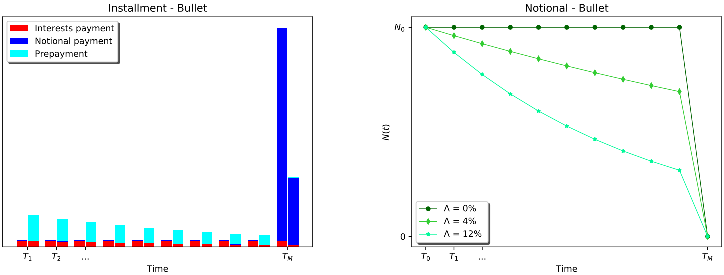


Figure: Bullet mortgage payment profile with $T_M = 10$ and $K = 3\%$ under different scenarios. Left: the installment composition in case $\Lambda = 0\%$ or $\Lambda = 12\%$. Right: outstanding notional in time under different levels of prepayment.

Types of Mortgages: Bullet

- Let us investigate the impact of prepayment on payment schedule.

time	left notional	prepayment	notional_repayment	interest part	installment
0	1,000,000	-	0	-	-
1	900,000	100,000	0	50,000	150,000
2	810,000	90,000	0	45,000	135,000
3	729,000	81,000	0	40,500	121,500
4	656,100	72,900	0	36,450	109,350
5	590,490	65,610	0	32,805	98,415
6	531,441	59,049	0	29,525	88,574
7	478,297	53,144	0	26,572	79,716
8	430,467	47,830	0	23,915	71,745
9	387,420	43,047	0	21,523	64,570
10	348,678	38,742	0	19,371	58,113
11	313,811	34,868	0	17,434	52,302
12	282,430	31,381	0	15,691	47,072
13	254,187	28,243	0	14,122	42,364
14	228,768	25,419	0	12,709	38,128
15	205,891	22,877	0	11,438	34,315
16	185,302	20,589	0	10,295	30,884
17	166,772	18,530	0	9,265	27,795
18	150,095	16,677	0	8,339	25,016
19	135,085	15,010	0	7,505	22,514
20	121,577	13,509	0	6,754	20,263
21	109,419	12,158	0	6,079	18,237
22	98,477	10,942	0	5,471	16,413
23	88,629	9,848	0	4,924	14,772
24	79,766	8,863	0	4,431	13,294
25	71,790	7,977	0	3,988	11,965
26	64,611	7,179	0	3,589	10,769
27	58,150	6,461	0	3,231	9,692
28	52,335	5,815	0	2,907	8,722
29	47,101	5,233	0	2,617	7,850
30	42,391	4,710	0	2,355	7,065
31	-	-	42391.2	2,120	44,511

Types of Mortgages: Annuity

- An annuity is a somewhat more involved contract, because, contrary to the bullet, it involves *repayments*,

$$C(T_i) = I(T_i) + Q(T_i).$$

- A repayment, $Q(T_i)$, diminishes the notional by the same amount, i.e.,

$$N(T_{i+1}) = N(T_i) - \Delta T_i Q(T_i) = N(T_i) - \Delta T_i \big(C(T_i) - I(T_i)\big).$$

- Keep in mind that interest rate payment, $I(T_i)$, does not decrease the mortgage notional (therefore it needs to be subtracted).
- When a first payment has taken place, one year after signing the contract, the annuity is called *ordinary annuity*, whereas, if the first amount was paid immediately, it would be an *annuity due*. Our focus is on the ordinary annuity and which we will simply call annuity, from now on.

Types of Mortgages: Annuity

- An essential characteristic of an annuity is that the installments, $C(T_i)$, are a fixed amount: $C(T_i) \equiv C$, with $i = 1, ..., M$, an equidistant partitioning of the time interval in years.
- The interest rate and the principal parts have to be balanced, so that the sum is constant at each payment date.
- Therefore, they will follow opposite trends, when the notional is progressively paid back, the interests computed on the notional will diminish.
- To calculate the correct installment amount C, we impose that the present value of all the future installments should be equal to the notional of the mortgage, i.e.,

$$\mathrm{An}(t_0; K) = \sum_{i=1}^{M} \frac{C}{(1+K)^{T_i}} = \frac{C}{(1+K)} \sum_{i=0}^{M-1} \frac{1}{(1+K)^{T_i}}.$$

Types of Mortgages: Annuity

- Therefore,

$$\mathrm{An}(t_0;K)=\frac{C}{K}\left(1-\frac{1}{(1+K)^{T_M}}\right)\equiv N_0,$$

so,

$$\boxed{C=\frac{KN_0}{1-(1+K)^{-T_M}}.}$$

- With this, we derive the interest rate payment, $I(T_i)=KN(T_i)$, and the principal payment, $Q(T_i)=C(T_i)-I(T_i)$.
- Then the prepayment mount is given by

$$P(T_i)=\Lambda(N(T_i)-Q(T_i)),$$

- Then the outstanding notional, $N(T_{i+1})$ is given by:

$$\begin{aligned}N(T_{i+1}) &= N(T_i)-(Q(T_i)+P(T_i))\\ &= N(T_i)-(C(T_i)-I(T_i)+\Lambda(N(T_i)-Q(T_i)))\\ &= N(T_i)-\Big(C(T_i)-KN(T_i)+\Lambda\times(N(T_i)-C(T_i)-KN(T_i))\Big).\end{aligned}$$

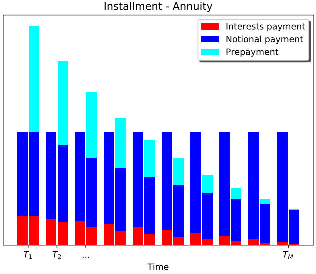
Types of Mortgages: Annuity

- The prepayment rate $\Lambda(T_i)$ at time T_i can also be interpreted as a reformulation of the interest payment $I(T_{i+1})$ and the installment $C(T_{i+1})$.
- When a mortgagor decides to prepay, the installment for the remaining dates is rebalanced according to the updated outstanding notional.
- Consequently, $C(T_i)$ becomes a time-dependent quantity,

$$C(T_i) = \frac{KN(T_i)}{1-(1+K)^{-(T_M-T_i)}}.$$

- Let us now compare the coupon magnitude for $\Lambda = 0\%$ and $\Lambda = 12\%$ and analyze the impact of varying prepayment levels on the outstanding notional is shown. The larger the Λ, the greater the rate of reduction.

Types of Mortgages: Annuity



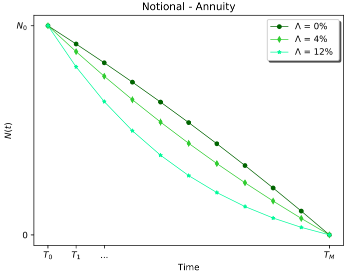


Figure: annuity with $T_M = 10$ and $K = 3\%$ under different scenarios. Left: the installment composition in case $\Lambda = 0\%$ or $\Lambda = 12\%$. Right: the outstanding notional in time under different levels of prepayment.

Python experiment for Annuity

- Let us investigate the impact of prepayment on payment schedule.

time	left notional	prepayment	notional_repayment	interest part	installment
0	1,000,000	-	0	-	-
1	887,281	98,587	14132.1	50,000	64,132
2	786,534	87,393	13354.9	44,364	57,719
3	696,522	77,391	12620.3	39,327	51,947
4	616,136	68,460	11926.2	34,826	46,752
5	544,379	60,487	11270.3	30,807	42,077
6	480,356	53,373	10650.4	27,219	37,869
7	423,262	47,029	10064.6	24,018	34,082
8	372,376	41,375	9511.08	21,163	30,674
9	327,049	36,339	8987.97	18,619	27,607
10	286,700	31,856	8493.64	16,353	24,846
11	250,806	27,867	8026.49	14,335	22,362
12	218,899	24,322	7585.03	12,540	20,125
13	190,558	21,173	7167.85	10,945	18,113
14	165,406	18,378	6773.62	9,528	16,302
15	143,104	15,901	6401.07	8,270	14,671
16	123,350	13,706	6049.01	7,155	13,204
17	105,870	11,763	5716.32	6,167	11,884
18	90,422	10,047	5401.92	5,294	10,695
19	76,785	8,532	5104.81	4,521	9,626
20	64,765	7,196	4824.05	3,839	8,663
21	54,186	6,021	4558.73	3,238	7,797
22	44,890	4,988	4308	2,709	7,017
23	36,737	4,082	4071.06	2,244	6,316
24	29,601	3,289	3847.15	1,837	5,684
25	23,369	2,597	3635.56	1,480	5,116
26	17,940	1,993	3435.6	1,168	4,604
27	13,224	1,469	3246.64	897	4,144
28	9,140	1,016	3068.08	661	3,729
29	5,617	624	2899.33	457	3,356
30	2,589	288	2739.87	281	3,021
31	0	0	2589.18	129	2,719

python™

Prepayment determinants

- There are multiple reasons for prepayment that are very different in nature. Research on variables that significantly influence the prepayment rate is very rich.
- The choice of the variables is of course impacted by the available prepayment data, however, there appears to be agreement about a specific driver, i.e., the refinancing incentive.
- The *Refinancing Incentive* is the primary driver in any prepayment model. The refinancing incentive occurs when a borrower observes a lower rate than the rate on her own mortgage.
- Mortgage rates are fluctuating quantities that are quoted by different financial institutions, and they depend on the type of mortgage, the contract's maturity, and often on the type of house that is used as the collateral.
- Other well-known drivers include the mortgage age, the month of the year, and the burn-out, all of which have natural explanation. Based on the available data, other variables may be age of the client, housing turnover, default, bank account amount.

Refinancing Incentive

- The interest rate incentive is one of the reasons to prepay. A suitable definition of this is based on the questions:
  - Which quantity is a suitable surrogate for the market rate offered at time t?
  - And how do people compare this rate with the one they have in their contract, and do they decide to change?
- A reasonable benchmark for the price of a mortgage will be a swap rate, $S_{t,T}(t)$, which matches the maturity and the frequency of the payments of the mortgage, where a spread is added to protect against an liquidity risk.
- In fact, banks derive the at-the-money mortgage rate for new clients themselves by starting from the present levels of the swap rates.

Refinancing Incentive

- The initial mortgage rate is indicated here by K, while the new mortgage rate that could be found in the market at time t for a mortgage with maturity T is defined by,

$$\kappa(t;T,\zeta) := S_{t,T}(t) + \zeta,$$

  where ζ denotes a deterministic spread which covers liquidity risk and the profit for the bank.
- People do not always act rationally, meaning that they do not prepay when the incentive do to so occurs, and mortgagors may prepay when it is not optimal.
- If rationality were a fact, the "reaction function" would be represented by the step function.

Refinancing Incentive

- The "new" mortgage rate is defined by:

$$\boxed{\kappa(t;T,\zeta) := S_{t,T}(t)+\zeta.}$$

- There are two main approaches for choosing a functional form for the incentive. The first one is based on the difference between the rates:

$$\epsilon(t) = K - \kappa(t),$$

which is the form that we will use. Clearly, the smaller the market rate, the greater the difference, with for an at-the-money mortgage $\epsilon(t) = \epsilon^* = 0$.

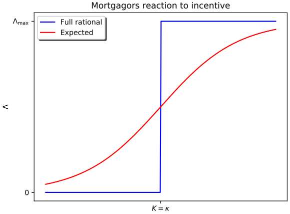
- An alternative is using the ratio, i.e.,

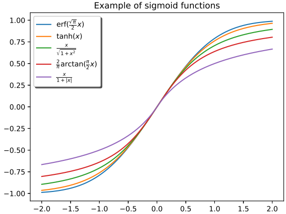
$$\epsilon(t) = \frac{K}{\kappa(t)},$$

which is related to the evaluation of the present value of an annuity per Euro of the monthly payments.

Refinancing Incentive

- The S-shape defines a class of functions, called the "sigmoid" functions.
- This smoother function mitigates the rationality and allows for non-rational behaviour, possibly including a reaction time to the incentive.
- An "S-shape" graph as the red line is well-known from the literature, and appears more realistic than the step function to model prepayment behaviour.





Mortgage Portfolio and Amortizing Swap

- In general, when $\Lambda = \Lambda(T_i)$, we need to explicitly state the dependence on the repayment scheme at each time. So,

$$\boxed{N(T_i) = N(T_{i-1}) \cdot \Psi(\Lambda(T_i)),}$$

- Function $\Psi(\cdot)$ defined as follows:

$$\Psi = \begin{cases} 1 - \Lambda(T_i) & \text{(Bullet)}, \\ 1 + \dfrac{K(\Lambda(T_i) - 1)}{1 - (1+K)^{-(T_M - T_{i-1})}} + K - \Lambda(T_i) \cdot (K+1) & \text{(Annuity)}. \end{cases}$$

with $i = 1, ..., M$ and $N(T_0) = N_0$.

Constant Prepayment Rate function

- We start with a basic test case, assuming that the prepayment rate is a deterministic function of time, $\Lambda \equiv \Lambda(t)$.
- This simplification implies that we only need to analyze two "levels", i.e.,

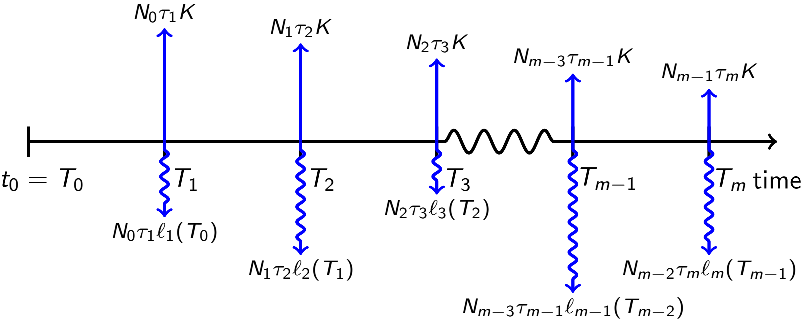
$$
\begin{aligned}
V_{\mathrm{AS}}(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{M} \tau_i \frac{N(T_{i-1})}{M(T_i)} \left(K - L(T_{i-1}; T_{i-1}, T_i)\right) \Big| \mathcal{F}(t_0)\right], \\
N(T_i) &= N(T_{i-1}) \cdot \Psi(\Lambda(T_i)).
\end{aligned}
$$

- Because there is no need to further define the prepayment rate, the pricing and hedging of a mortgage is linked to amortizing swap (AS). In this setting, we will have an analytic solution for the price of the AS.

Mortgage Portfolio and Amortizing Swap

- For mortgages without prepayment, the pricing is linked to amortizing swaps (AS):

$$V_{\mathrm{AS}}(t_0) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{M}\tau_i\frac{N(T_{i-1})}{M(T_i)}\cdot\left(K-\ell(T_{i-1};T_{i-1},T_i)\right)\Big|\mathcal{F}(t_0)\right].$$



Constant Prepayment Rate function

- The price of the AS, with a time-dependent notional, is given by,

$$
\begin{aligned}
V_{\mathrm{AS}}(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{M}\tau_i\frac{N(T_{i-1};\Lambda(T_{i-1}))}{M(T_i)}\cdot\left(K-L(T_{i-1};T_{i-1},T_i)\right)\Big|\mathcal{F}(t_0)\right]\\
&= \sum_{i=1}^{M}\tau_i P(t_0,T_i)N(T_{i-1})\left(K-\mathbb{E}^{T_i}\left[L(T_{i-1};T_{i-1},T_i)\Big|\mathcal{F}(t_0)\right]\right)\\
&= \sum_{i=1}^{M}N(T_{i-1})\left[P(t_0,T_i)(\tau_i K+1)-P(t_0,T_{i-1})\right].
\end{aligned}
$$

- By letting the notional N to depend on interest rates, $r(t)$, we will establish a clear link from a market simulation to the impact of prepayments on a mortgage portfolio evaluation.
- The stochastic prepayment rate, Λ, we no longer deal with a standard Amortizing Swap, bit Index Amortizing Swap (IAS).

Mortgage Portfolio and Amortizing Swap

- Important to keep in mind that once the notional is stochastic, the change of measure may NOT help us!
- Libor rate $\ell_k(t)$ is a martingale under the T_k-forward measure.

$$\boxed{\mathbb{E}^{T_k}\left[\ell(T_{k-1};T_{k-1},T_k)\Big|\mathcal{F}(t_0)\right]=\ell(t_0;T_{k-1},T_k).}$$

- However:

$$\boxed{\mathbb{E}^{T_k}\left[\ell^2(T_{k-1};T_{k-1},T_k)\Big|\mathcal{F}(t_0)\right]\neq\ell^2(t_0;T_{k-1},T_k).}$$

- Under T_k-forward measure libor ℓ_k is a martingale, i.e.,

$$\mathrm{d}\ell_k(t)=\sigma_k\ell_k(t)\mathrm{d}W^k(t),$$

however:

$$\mathrm{d}\ell_k^2(t) = \boxed{\sigma_k^2\ell_k^2(t)\mathrm{d}t}+2\sigma_k\ell_k^2(t)\mathrm{d}W^k(t).$$

- Therefore $\ell_k^2(t)$ is not a martingale under T_k-forward measure.

Mortgage Portfolio and Index Amortizing Swap

- From a interest-rate risk perspective, only fixed-rate loans contain prepayment risk, because loans with a variable rate will pay a coupon that is adjusted to the market rates.
- The mismatch between the pre-agreed mortgage rate and the at-the-market rate would be zero, because, effectively, $K \equiv \kappa(t)$ for each time point t. The fixed rate loans are therefore usually hedged to reduce the interest rate risk, and connecting mortgages with interest rate swaps achieves this.
- However, the interest rate swaps are not contractually linked to the mortgage loan. Thus, prepayments may cause a misalignment of the cash flow of the hedge.

Mortgage Portfolio and Index Amortizing Swap

- An Index Amortizing Swap (IAS) is an over-the-counter interest rate swap that it is a combination of a plain vanilla interest rate swap and, partially, a swaption.
- Amortizing swaps with a deterministic amortization scheme are commonly traded instruments. The peculiarity of the IAS, which makes it "hybrid", is that its amortization scheme is predetermined only as a function of a specific interest rate.
- Therefore, the IAS effectively is like an option on that rate. The dependency of the IAS on the interest rate level is the critical aspect in the prepayment replication.
- As an IAS and a mortgage portfolio share essentially the same embedded optionality, enables a tailor-made construction of the former as an optimal representation of the latter.

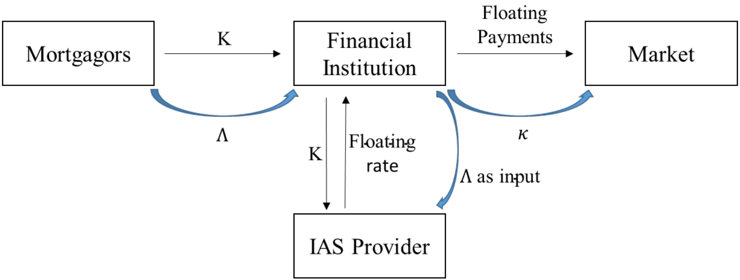
Mortgage Portfolio and Index Amortizing Swap

- We want to establish a link between the value of the mortgage portfolio and the level of the interest rates in the market (leaving aside other possible drivers for prepayment).
- The main idea behind the pricing model is to replicate the mortgage portfolio by an Index Amortizing Swap (IAS), whose notional will depend on the characteristics of the contract, the prepayment rate, and the interest rates in the market.

$$V_{\mathrm{IAS}}(t_0) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{M} \tau_i \frac{N(T_{i-1};\Lambda(T_{i-1}))}{M(T_i)} \cdot \left(K - L(T_{i-1};T_{i-1},T_i)\right)\Big|\mathcal{F}(t_0)\right].$$

- Even if our model considers only the Refinancing Incentive as the driver of prepayments, this will not lead to a fully rational model because a smooth functional form for Λ will be preferred over a step-function.
- Our objective is to evaluate a variation of an Index Amortizing Swap (IAS) where the notional is based on prepayment rate.
- In order to use IAS we define a link between the (stochastic) amortizing plan of the swap and the probability of prepayment.

Mortgage Portfolio and Index Amortizing Swap



Mortgage Portfolio and Index Amortizing Swap

- A challenge is the modeling of the notional of the IAS, which embodies the (possibly stochastic) amortization via an involved function of the type of mortgage, the history of prepayments, the prepayment drivers, and the prepayments' functional form.
- The starting point for the evaluation of the IAS is the following expectation,

$$V_{\text{IAS}}(t_0) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{M} \tau_i \frac{N(T_{i-1};\Lambda(T_{i-1}))}{M(T_i)} \cdot \big(K - L(T_{i-1};T_{i-1},T_i)\big)\Big|\mathcal{F}(t_0)\right].$$

- The notional of the IAS at time T_i is assumed to be the notional of the mortgage at time T_{i-1}. The prepayment rate, $\Lambda(T_i)$, is denoted as one of the dependent arguments in the notional N. K is here also the mortgage rate that at time t_0 was the at-the-money rate.

Mortgage Portfolio and Amortizing Swap

- The assumption we make here is that the prepayment rate is only determined by the Refinancing Incentive (RI), i.e.,

$$\Lambda = \Lambda(\mathrm{RI}(T_i)).$$

- Subsequently, we need to choose a functional form for the Refinancing Incentive.
- The Refinancing Incentive (RI) is written in the following form:

$$\mathrm{RI}(T_i) = \begin{cases} \Lambda_{\max} 1_{\{\epsilon(T_i) > \epsilon^*\}} & \text{Optimal, Rational,} \\ \alpha_1 + \alpha_2 \left(1 + \mathrm{e}^{\alpha_3 \epsilon(T_i) + \alpha_4}\right)^{-1} & \text{Realistic, Human Behaviour.} \end{cases}$$

Mortgage Portfolio and Amortizing Swap

- The complete formulation for the evaluation of a (Dutch) mortgage portfolio. The following equations summarise the model in compact form:
- Index Amortizing Swap:

$$V_{\text{IAS}}(t_0) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{M}\tau_i\frac{N(T_{i-1})}{M(T_i)}\left(K - L(T_{i-1};T_{i-1},T_i)\right)\Big|\mathcal{F}(t_0)\right],$$

- Mortgage type:

$$N(T_i) = N(T_{i-1})\cdot\Psi(\Lambda(T_i)),$$

- Prepayment driver:

$$\Lambda(T_i) = \text{RI}(T_i;\epsilon(T_i)),$$

- Market conditions:

$$\epsilon(T_i) = K - \kappa(T_i).$$

Incentive's impact on notional profiles: Python experiment

- Now, let us investigate the impact of the stochasticity on the mortgage notional, depending on different incentive functions.

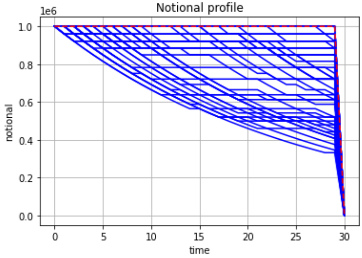


Figure: Profiles of the notional in time. Left: Bullet mortgage, Right: Annuity notional.

Stochastic Prepayment Rate function

- We will use non-linear financial instruments to achieve improved hedge performance. Consider an annuity with initial notional $N(0) = N_0$, maturity time $T_M = 2$ and the fixed rate K.
- Moreover, assume we are using the optimal, rational functional form for the Refinancing Incentive.
- This example model enables us to find a closed-form solution for the price of the IAS. Starting from the expectation formulation, with $M = 2$,

$$V_{\text{IAS}}(t_0) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{2} \tau_i \frac{N(T_{i-1})}{M(T_i)} \left(K - L(T_{i-1}; T_{i-1}, T_i)\right) \Big| \mathcal{F}(t_0)\right].$$

- The only possible prepayment may take place at T_2, so the randomness is in $N(T_1)$, and the first payment is deterministic, i.e.,

$$C_1 = P(t_0, T_1)N(t_0)\big(K - L(t_0; t_0, T_1)\big).$$

Stochastic Prepayment Rate function

- Since in this setting, the prepayment rate follows a step-function, we can assess the possible notional values at time T_1, being N^{Up} and N^{Low}
- For $L(T_1) := L(T_1; T_1, T_2)$, it follows that,

$$S_{T_1,T_2}(T_1) = L(T_1),$$

so that the notional at time T_1 can be written as

$$\begin{aligned} N(T_1) &= N^{\text{Up}} 1_{\{K<L(T_1)\}} + N^{\text{Low}} 1_{\{K>L(T_1)\}} \\ &= N^{\text{Up}} - \left(N^{\text{Up}} - N^{\text{Low}}\right) 1_{\{K>L(T_1)\}}. \end{aligned}$$

- Therefore, the second payment C_2 is equal to,

$$C_2 = M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{N(T_1)}{M(T_2)}\tau_2\big(K - L(T_1)\big)\right].$$

- Thus,

$$C_2 = \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_2)}N^{\text{Up}}\big(K - L(T_1)\big)\right] - \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_2)}\left(N^{\text{Up}} - N^{\text{Low}}\right)\big(K - L(T_1)\big)^{+}\right]$$

Stochastic Prepayment Rate function

- Finally, we find:

$$C_2 = N^{\mathrm{Up}}\left((K+1)P(t_0,T_2) - P(t_0,T_1)\right) - \left(N^{\mathrm{Up}} - N^{\mathrm{Low}}\right) V_{\mathrm{Floorlet}}(t_0;T_1,T_2).$$

- With all payments in an explicit form, the final price of the instrument is found to be,

$$\begin{aligned} V_{\mathrm{IAS}}(t_0) &= C_1 + C_2 \\ &= V_{\mathrm{AS}}(t_0) - \left(N^{\mathrm{Up}} - N^{\mathrm{Low}}\right) V_{\mathrm{Floorlet}}(t_0;T_1,T_2). \end{aligned}$$

- This price redefines the IAS, as a combination of an Amortizing Swap and a Floorlet. This is an interesting insight, because it enables us to separate a linear component from a non-linear one, suggesting that a long position in swaps plus a short position in an option on the refinancing driver better replicates the IAS.
- Purchasing a mortgage can thus be seen as entering a long position in a swap, while the prepayments effectively reduce the notional of the mortgage. Thus, the option to prepay would equal an option to enter a swap, i.e., a swaption.

Pipeline Risk

- Except for the prepayment option, the customer in Dutch mortgages receives an additional embedded option, i.e., the client can pay either the mortgage rate stated in his quotation, T_0, or the mortgage rate for new mortgages when he signs the contract, T_1.
- The interest rate risk associated with this option is called pipeline risk.
- Pipeline risk can be characterized by the risk that interest rates increase over a short period.
- The main challenge for pipeline risk is that it is hedged using swaptions. This implies that one can only hedge the current pipeline risk. As new mortgages enter the pipeline, one needs to adjust our hedge. This ongoing process requires us to calculate the price of the hedge frequently.

Pipeline Risk

- Assuming that the new mortgage rate is given by

$$\kappa(T_q) := S_{T_s,T}(T_q) + \zeta,$$

the client, at the settlement date, may also choose:

$$\kappa(T_s) := S_{T_s,T}(T_s) + \zeta.$$

- This implies that the rate that client will choose, κ_F, is given by:

$$\begin{aligned}\kappa_F &= \kappa(T_q) \times 1_{S(T_q)<S(T_s)} + \kappa(T_s) \times 1_{S(T_q)\geq S(T_s)}\\ &= \kappa(T_q) \times 1_{S(T_q)<S(T_s)} + \Big(\kappa(T_s) - \kappa(T_q) + \kappa(T_q)\Big) \times 1_{S(T_q)\geq S(T_s)},\end{aligned}$$

for $S(T_q) := S_{T_s,T}(T_q)$ and $S(T_s) := S_{T_s,T}(T_s)$.

Pipeline Risk

- This can be further simplified:

$$\begin{aligned}
\kappa_F &= \kappa(T_q)1_{S(T_q)<S(T_s)} + \Big(\kappa(T_s) - \kappa(T_q) + \kappa(T_q)\Big)1_{S(T_q)\geq S(T_s)}\\
&= \Big(\kappa(T_q)1_{S(T_q)<S(T_s)} + \kappa(T_q)1_{S(T_q)\geq S(T_s)}\Big) + \Big(\kappa(T_s) - \kappa(T_q)\Big)1_{S(T_q)\geq S(T_s)}\\
&= \kappa(T_q) + \Big(\kappa(T_s) - \kappa(T_q)\Big)1_{S(T_q)\geq S(T_s)}\\
&= \kappa(T_q) - \Big(\kappa(T_q) - \kappa(T_s)\Big)1_{S(T_q)\geq S(T_s)}.
\end{aligned}$$

- Further we get:

$$\begin{aligned}
\kappa_F &= \kappa(T_q) - \Big(\kappa(T_q) - \kappa(T_s)\Big)1_{S(T_q)\geq S(T_s)}\\
&= \kappa(T_q) - \max\Big(\kappa(T_q) - \kappa(T_s), 0\Big).
\end{aligned}$$

- The fair value of the contract at "quotation" time T_q is then given by:

$$V(T_q) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_q)}\kappa(T_q) - \frac{1}{M(T_s)}\max\Big(\kappa(T_q) - \kappa(T_s), 0\Big)\Big|\mathcal{F}(T_q)\right]$$

Pipeline Risk

- The fair value of the contract at "quotation" time T_q is then given by:

$$V(T_q) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_q)}\kappa(T_q) - \frac{1}{M(T_s)}\max\Big(\kappa(T_q) - \kappa(T_s), 0\Big)\Big|\mathcal{F}(T_q)\right].$$

- This can be further decomposed to:

$$V(T_q) = \kappa(T_q) - \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_s)}\max\Big(\kappa(T_q) - \kappa(T_s), 0\Big)\Big|\mathcal{F}(T_q)\right].$$

- Since $\kappa(T_q) := S_{T_s,T}(T_q) + \zeta$ the first term is simply a swap and the second term represents a swaption.
- Pricing of swaption can be calculated by Black's formula given the volatility σ obtained from the implied volatility surface.

Summary

- Introduction to Mortgage Contracts
- Bullet Mortgage
- Bullet Mortgage: Python Experiment
- Annuity Mortgage
- Annuity Mortgage: Python Experiment
- Prepayment determinants
- CPR: Constant Prepayment Rate
- Index Amortizing Swap
- Inclusion of Refinancing Incentive
- Stochastic Prepayment: Python Experiment
- Stochastic Prepayment and Swaptions
- Pipeline Risk
- Summary of the Lecture + Homework

Homework Exercises

- **Exercise 1**
- In slide 35 "stochastic notional" profiles were presented. Use these profiles to price a mortgage using:

$$V_{\mathrm{IAS}}(t_0) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{M}\tau_i\frac{N(T_{i-1};\Lambda(T_{i-1}))}{M(T_i)}\cdot\left(K - L(T_{i-1};T_{i-1},T_i)\right)\middle|\mathcal{F}(t_0)\right].$$

Discuss the impact of:

- Different incentive functions;
- Rational vs. Irrational cases;

on the value of $V_{IAS}(t_0)$.

Homework Exercises

- **Exercise 2**
- In this lecture we have shown that:

$$\mathbb{E}^{T_k}\left[\ell^2(T_{k-1};T_{k-1},T_k)\middle|\mathcal{F}(t_0)\right] \neq \ell^2(t_0;T_{k-1},T_k).$$

- Compute the expectation above and determine ξ such that:

$$\boxed{\mathbb{E}^{T_k}\left[\ell^2(T_{k-1};T_{k-1},T_k)\middle|\mathcal{F}(t_0)\right] = \ell^2(t_0;T_{k-1},T_k)+\xi.}$$

- Discuss the magnitude of ξ depending on the volatility and time $T_i - T_{i-1}$.
- This problem is called the "convexity adjustment/correction" that will be discussed later in this course.

Homework Exercises

- **Exercise 3**
- Extend Python code from slide 17 (annuity mortgage) so that the prepayments happen only a few times during the mortgage contract, e.g. 5 times (currently the prepayments take place on every period).