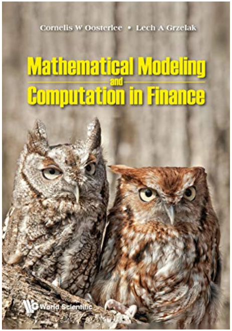
Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.



- Youtube Channel with courses can be found here.
- Slides and the codes can be found here.

List of content

Equilibrium vs. Term-Structure Models

- A no-arbitrage model is a model designed to be consistent with today's term structure of interest rates.
- The difference between equilibrium (endogenous) and non-arbitrage models (exogenous) is that today's term structure of interest rates is an output in an equilibrium model.
- In a no-arbitrage model, today's term structure of interest rates is an input. This means that we take the observed actual rates while constructing the model and estimate the unobserved rates.
- The HJM framework described a clear path from the equilibrium towards term-structure models.

Equilibrium vs. Term-Structure Models[1]

- Historically, equilibrium models start with assumptions about economic variables and derive a process for the short rate, which means that the current term structure of interest rates hence is an output rather than input in the model.
- Such models are also called *endogenous* term-structure models. The (instantaneous) short rate at time t is the rate that applies to an infinitesimally short period at time t.
- Some popular equilibrium models. Namely, the Vasicek:

$$\mathrm{d}r(t) = \lambda\left(\theta - r(t)\right)\mathrm{d}t + \eta \mathrm{d}W(t),$$

  and the Cox, Ingersoll and Ross (CIR) model:

$$\mathrm{d}r(t) = \lambda\left(\theta - r(t)\right)\mathrm{d}t + \gamma\sqrt{r(t)}\mathrm{d}W(t).$$

- These models are one-factor models, which have several shortcomings, e.g., the interest rates are perfectly correlated between different maturities.

[1] Great overview of the short-rate model can be found in book of Brigo-Mercurio: Interest Rate Models- Theory and Practice.

Equilibrium vs. Term-Structure Models

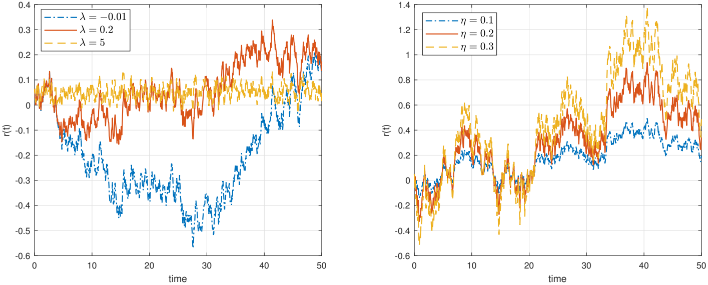


Figure: Within the Vasicek Model model context, the impact of variation of mean-reversion λ, and of the volatility parameter η on the Monte Carlo paths.

Equilibrium vs. Term-Structure Models

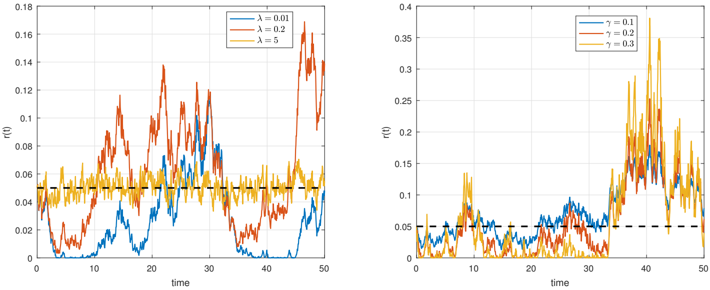


Figure: For the CIR model, the impact of variation of mean-reversion λ, and of the volatility parameter γ on the Monte Carlo paths.

The HJM framework

- The Heath-Jarrow-Morton framework represents a class of models that are derived by directly modeling the dynamics of instantaneous forward-rates.
- The framework constitutes the fundament for interest rate models as it provides an explicit relation between the volatility of the instantaneous forward rates and arbitrate-free drift.
- Both the standard short-rate and Libor Market models can be derived in the HJM framework however, in general.
- The HJM models are non-Markovian so only a number of models with a closed-form solution exists.

The HJM framework


- 1. Equilibrium models.
- 2. Short-rate models in the HJM Framework.
- 3. Market Models models in the HJM Framework.

Instantaneous Forward Rate

Definition (Forward and instantaneous forward rates)

Suppose that at time t we enter into a forward contract to deliver at time S a bond that will mature at time T. Let the forward price of the bond be denoted by $P(t, S, T)$. At the same time, a zero-coupon bond, $P(t, S)$, that matures at time S is purchased. Further, a bond, $P(t, T)$, that matures at time T is also bought. Assuming no-arbitrate and market completeness the following equality has to hold:

$$P(t, T) = P(t, S)P(t, S, T).$$

Now, we define the implied forward rate, $R(t, S, T)$, at time t for the period $[S, T]$ as:

$$P(t, S, T) = \exp\left(-(T - S)R(t, S, T)\right).$$

Instantaneous Forward Rate

Definition (Forward and instantaneous forward rates cont.)

By equating the two equations we find:

$$\mathrm{e}^{-(T-S)R(t,S,T)} = \frac{P(t,T)}{P(t,S)},$$

which reads the forward rate $R(t,S,T)$ to be given by:

$$R(t,S,T) = -\frac{\log P(t,T) - \log P(t,S)}{T-S}.$$

By setting the limit $T - S \rightarrow 0$ we arrive at the definition of the instantaneous forward rate

$$\boxed{f(t,T) \stackrel{\text{def}}{=} \lim_{S\rightarrow T} R(t,S,T) = -\frac{\partial}{\partial T}\log P(t,T).}$$

Instantaneous Forward Rate

- In the HJM framework the dynamics of the instantaneous forward rates, $f(t, T)$, is analyzed.
- We start with an assumption that for a certain, fixed, maturity $T \geq 0$, the instantaneous forward rate $f(t, T)$ under real-world measure $\mathbb{P}$ is driven by the following dynamics:

$$\mathrm{d}f(t, T) = \alpha^{\mathbb{P}}(t, T)dt + \sigma(t, T)\mathrm{d}W^{\mathbb{P}}(t), \quad f(0, T) = f_0(T),$$

  for any time $t < T$, with a corresponding drift $\alpha^{\mathbb{P}}(t, T)$.
- Under this model we also define a money-savings account as:

$$\boxed{M(t) = \exp\left(\int_0^t r(s)\mathrm{d}s\right) \equiv \exp\left(\int_0^t f(s, s)\mathrm{d}s\right).}$$

Arbitrage-free HJM

- As we see, under the HJM framework the short rate $r(t)$, is defined as the limit of the instantaneous forward rate $r(t) \equiv f(t,t)$. The zero-coupon bond, $P(t,T)$, with maturity T, follows:

$$\begin{aligned} P(t,T) &= M(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}\cdot 1\Big|\mathcal{F}(t)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\mathrm{d}s\right)\cdot 1\Big|\mathcal{F}(t)\right]. \end{aligned}$$

- What are the tradables in this market and what quantities are the martingales ?
- How to find $\alpha^{\mathbb{Q}}(t,T)$ in

$$\mathrm{d}f(t,T) = \alpha^{\mathbb{Q}}(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}W^{\mathbb{Q}}(t), \quad f(0,T) = f_0(T), ?$$

Arbitrage-free HJM cont.

- Although the ZCB $P(t,T)$ can be priced as an expectation its value can be directly related to today's yield curve via:

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T).$$

- From above we can easily determine the following relation:

$$P(t,T) = \exp\left(-\int_t^T f(t,s)\mathrm{d}s\right),$$

- Let us start with deriving the dynamics of discounted ZCB:

$$\mathrm{d}\left(\frac{P(t,T)}{M(t)}\right) = \mathrm{d}\left[\exp\left(-\int_t^T f(t,s)\mathrm{d}s - \int_0^t r(s)\mathrm{d}s\right)\right] \stackrel{\text{def}}{=} \mathrm{d}Z(t).$$

- After a lot of derivations for $\mathrm{d}Z(t)$ and setting the drift to zero we find.

Arbitrage-free HJM cont.

Lemma (HJM no arbitrage drift condition)

For the instantaneous forward rates given by following SDE:

$$\mathrm{d}f(t,T) = \alpha^{\mathbb{Q}}(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}W^{\mathbb{Q}}(t),$$

the no-arbitrage drift condition is given by

$$\alpha^{\mathbb{Q}}(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)\mathrm{d}s.$$

Proof.

The proof easily follows by deriving the dynamics of $Z(t)$ and equating all the drift terms to zero. □

Short-Rate dynamics under HJM

- By using the relation that $f(t,t)=r(t)$ and by integrating the SDEs we obtain the short rate dynamics under the HJM framework of the form:

$$f(t,T)=f(0,T)+\int_0^t \alpha^{\mathbb{Q}}(s,T)\mathrm{d}s+\int_0^t \sigma(s,T)\mathrm{d}W^{\mathbb{Q}}(s),$$

which for time $T=t$ simply becomes:

$$r(t)\equiv f(t,t)=f(0,t)+\int_0^t \alpha^{\mathbb{Q}}(s,t)\mathrm{d}s+\int_0^t \sigma(s,t)\mathrm{d}W^{\mathbb{Q}}(s),$$

- Now by applying Leibniz integral rule [2] we obtain following short rate dynamics:

$$\begin{aligned}\mathrm{d}r(t) \quad = \quad & \left[\frac{\partial}{\partial t}f(0,t)+\alpha^{\mathbb{Q}}(t,t)+\int_0^t \frac{\partial}{\partial t}\alpha^{\mathbb{Q}}(s,t)\mathrm{d}s\right. \\ & \left.+\int_0^t \frac{\partial}{\partial t}\sigma(s,t)\mathrm{d}W^{\mathbb{Q}}(s)\right]\mathrm{d}t+\sigma(t,t)\mathrm{d}W^{\mathbb{Q}}(t).\end{aligned}$$

[2]

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\mathrm{d}x=f(b,\alpha)\frac{\partial}{\partial\alpha}b(\alpha)-f(a,\alpha)\frac{\partial}{\partial\alpha}a(\alpha)+\int_{a(\alpha)}^{b(\alpha)}\frac{\partial}{\partial\alpha}f(x,\alpha)\mathrm{d}x$$

Ho-Lee Model

- We specify a certain form of a volatility $\sigma(t,T)$ for the instantaneous forward rate $f(t,T)$ and determine the resulting short-rate dynamics. The first, and the simplest possibility is to consider $\sigma(t,T)$ to be constant, i.e.:

$$\boxed{\sigma(t,T)=\sigma.}$$

- From previous derivations we find:

$$\alpha^{\mathbb{Q}}(t,T)=\sigma\int_t^T\sigma\mathrm{d}s=\sigma^2(T-t).$$

- This can be used in Equation for the short-rate dynamics, i.e.:

$$\begin{aligned}\mathrm{d}r(t) \quad = \quad & \left[\frac{\partial}{\partial t}f(0,t)+\alpha^{\mathbb{Q}}(t,t)+\int_0^t\frac{\partial}{\partial t}\alpha^{\mathbb{Q}}(s,t)\mathrm{d}s\right.\\ & \left.+\int_0^t\frac{\partial}{\partial t}\sigma(s,t)\mathrm{d}W^{\mathbb{Q}}(s)\right]\mathrm{d}t+\sigma(t,t)\mathrm{d}W^{\mathbb{Q}}(t).\end{aligned}$$

Ho-Lee Model

- Therefore the short-rate dynamics under the HJM model with $\sigma(t,T)=\sigma$ is given by:

$$\mathrm{d}r(t)=\left(\frac{\partial}{\partial t}f(0,t)+\sigma^2 t\right)\mathrm{d}t+\sigma\mathrm{d}W^{\mathbb{Q}}(t).$$

- By setting $\theta(t)=\dfrac{\partial}{\partial t}f(0,t)+\sigma^2 t$, we arrive at:

$$\mathrm{d}r(t)=\theta(t)\mathrm{d}t+\sigma\mathrm{d}W^{\mathbb{Q}}(t),$$

  which is well-recognized as the Ho-Lee short-rate model.
- Now we can compute ZCBs $P(t,T)$ using this model

$$P(t,T)=\mathbb{E}\left[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}\middle|\mathcal{F}(t)\right]=\mathrm{e}^{A(t,T)+B(t,T)r(t)}.$$

- Functions $A(t,T)$ and $B(t,T)$ will be presented later.

Ho-Lee Model: Python Exercise

Python Exercise:

- Define $P_{mrkt}(t,T) = \exp(-r(T-t))$ (it can be much more involved or implied from the market), for some r, calculate $f(t,T)$ and use it for simulating $r(t)$.
- Consider the Ho-Lee model with a freely chosen parameter σ.
- It is important to properly choose the initial value for the process $r(t)$:

$$r(0) = f(0,0) \approx -\frac{\partial \log P_{mrkt}(0,\epsilon)}{\partial \epsilon}, \text{ for } \epsilon \to 0.$$

- Using Monte Carlo Paths calculate

$$P_{model}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s} \middle| \mathcal{F}(t) \right].$$

- Are $P_{mrkt}(0,T)$ and $P_{model}(0,T)$ the same for all T ?

Ho-Lee Model: Python Exercise

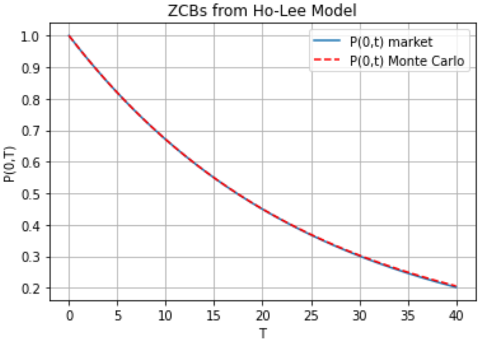


Figure: Comparison of the ZCBs from the Ho-Lee model vs. Market $P(0, T)$ for different T.

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Hull-White Model

- Let us now consider a short-rate model generated by the HJM volatility given by:

$$\boxed{\sigma(t,T)=\sigma\cdot\mathrm{e}^{-\lambda(T-t)}\text{ with }\lambda>0.}$$

As before, the short-rate dynamics under HJM arbitrage free assumptions is given by:

$$\begin{aligned}\mathrm{d}r(t) &= \left[\frac{\partial}{\partial t}f(0,t)+\alpha^{\mathbb{Q}}(t,t)+\int_0^t\frac{\partial}{\partial t}\alpha^{\mathbb{Q}}(s,t)\mathrm{d}s\right.\\ &\quad\left.+\int_0^t\frac{\partial}{\partial t}\sigma(s,t)\mathrm{d}W^{\mathbb{Q}}(s)\right]\mathrm{d}t+\sigma(t,t)\mathrm{d}W^{\mathbb{Q}}(t).\end{aligned}$$

- By Lemma we find:

$$\alpha^{\mathbb{Q}}(s,t) = \sigma\mathrm{e}^{-\lambda(t-s)}\int_s^t\sigma\mathrm{e}^{-\lambda(u-t)}\mathrm{d}u=-\frac{\sigma^2}{\lambda}\mathrm{e}^{-\lambda(t-s)}\left(\mathrm{e}^{-\lambda(t-s)}-1\right),$$

which implies that $\alpha^{\mathbb{Q}}(t,t)=0$.

Hull-White Model

- The remaining terms are as follows:

$$\int_0^t \frac{\partial}{\partial t}\alpha^{\mathbb{Q}}(s,t)\mathrm{d}s \quad = \quad \frac{\sigma^2}{\lambda}\mathrm{e}^{-2\lambda t}(\mathrm{e}^{\lambda t}-1),$$

  and

$$\frac{\partial}{\partial t}\sigma(s,t) \quad = \quad -\lambda\sigma\mathrm{e}^{-\lambda(t-s)} = -\lambda\sigma(s,t),$$

  with $\sigma(t,t)=\sigma$.

- The dynamics for $r(t)$ is therefore given by:

$$\mathrm{d}r(t) = \frac{\partial}{\partial t}f(0,t) + \int_0^t \frac{\partial}{\partial t}\alpha^{\mathbb{Q}}(s,t)\mathrm{d}s - \lambda\boxed{\int_0^t \sigma(s,t)\mathrm{d}W^{\mathbb{Q}}(s)} + \sigma\mathrm{d}W^{\mathbb{Q}}(t).$$

Hull-White Model

- We see that Brownian motion $\mathrm{d}W^{\mathbb{Q}}(t)$ is present in two terms. In order to find explicitly the solution for the integral $\int_0^t \sigma(s,t)\mathrm{d}W^{\mathbb{Q}}(s)$ we can use definition of the short-rate which yields:

$$r(t) = f(0,t) + \int_0^t \alpha^{\mathbb{Q}}(s,t)\mathrm{d}s + \int_0^t \sigma(s,t)\mathrm{d}W^{\mathbb{Q}}(s),$$

therefore the first integral can be determined via:

$$\int_0^t \sigma(s,t)\mathrm{d}W^{\mathbb{Q}}(s) = r(t) - f(0,t) - \int_0^t \alpha^{\mathbb{Q}}(s,t)\mathrm{d}s.$$

As:

$$\int_0^t \alpha^{\mathbb{Q}}(s,t)\mathrm{d}s = \frac{\sigma^2}{2\lambda^2}\mathrm{e}^{-2\lambda t}\left(\mathrm{e}^{\lambda t} - 1\right)^2,$$

we obtain the following dynamics for process $r(t)$:

$$\boxed{\mathrm{d}r(t) = \lambda\left(\frac{1}{\lambda}\frac{\partial}{\partial t}f(0,t) + f(0,t) + \frac{\sigma^2}{2\lambda^2}\left(1 - \mathrm{e}^{-2\lambda t}\right) - r(t)\right)\mathrm{d}t + \sigma\mathrm{d}W^{\mathbb{Q}}(t).}$$

Hull-White Model

- So finally, by taking:

$$\theta(t) = \frac{1}{\lambda}\frac{\partial}{\partial t}f(0,t) + f(0,t) + \frac{\sigma^2}{2\lambda^2}\left(1-\mathrm{e}^{-2\lambda t}\right),$$

the dynamics of the process $r(t)$ yields:

$$\mathrm{d}r(t) = \lambda(\theta(t) - r(t))dt + \sigma \mathrm{d}W^{\mathbb{Q}}(t),$$

which can be easily recognized as the Hull-White short rate process.

- It is important to properly choose the initial value for the process $r(t)$:

$$r(0) = f(0,0) \approx -\frac{\partial \log P(0,\epsilon)}{\partial \epsilon}, \;\; \text{for} \;\; \epsilon \to 0.$$

Simulation of the Hull-White Model

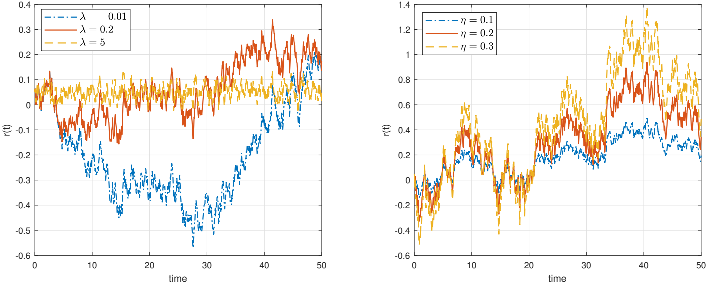

Figure: Within the HW model context, the impact of variation of mean-reversion λ, and of the volatility parameter η on the Monte Carlo paths.

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Hull-White Model: Python Exercise

Python Exercise:

- Define $P_{mrkt}(t,T) = \exp(-r(T-t))$ (it can be much more involved or implied from the market), for some r, calculate $f(t,T)$ and use it for simulating $r(t)$.
- Consider the Hull-White model with a freely chosen parameters λ and σ.
- It is important to properly choose the initial value for the process $r(t)$:

$$r(0) = f(0,0) \approx -\frac{\partial \log P_{mrkt}(0,\epsilon)}{\partial \epsilon}, \quad \text{for} \quad \epsilon \to 0.$$

- Using Monte Carlo Paths calculate

$$P_{model}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}\middle|\mathcal{F}(t)\right].$$

- Are $P_{mrkt}(0,T)$ and $P_{model}(0,T)$ the same for all T ?

Hull-White Model: Python Exercise

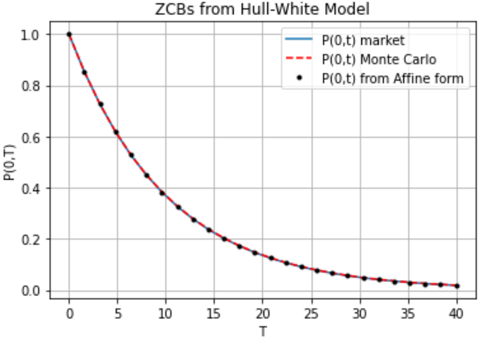

Figure: Comparison of the ZCBs from the Hull-White model vs. Market $P(0, T)$ for different T.

Summary

- Equilibrium vs. Term-Structure Models
- The HJM Framework
- Arbitrage Free Conditions under HJM
- Ho-Lee Model and Python Simulation
- Hull-White Model
- Hull-White Model and Simulation in Python
- Summary of the Lecture + Homework

Homework Exercises

The solutions for the homework can be find at
https://github.com/LechGrzelak/QuantFinanceBook

- **Exercise**
  Consider the "exponential-Vasicek" model given by the following system of equations:

$$\begin{aligned} r(t) &= \mathrm{e}^{y(t)}, \\ \mathrm{d}y(t) &= (\theta - a y(t))\mathrm{d}t + \sigma \mathrm{d}W(t) \quad y(t_0) = y_0. \end{aligned}$$

- Show that the dynamics for $r(t)$ yields:

$$\mathrm{d}r(t) = r(t)\left(\theta + \frac{\sigma^2}{2} - a \log r(t)\right)\mathrm{d}t + \sigma r(t)\mathrm{d}W(t).$$

- Show that

$$\lim_{t\to\infty} \mathbb{E}[r(t)] = \mathrm{e}^{\frac{\theta}{a} + \frac{\sigma^2}{4a}}.$$

Summary of the Lecture + Homework

- **Exercise 11.9**

  Consider the Vašiček short-rate model,

  $$\mathrm{d}r(t) = \lambda\left(\theta - r(t)\right)\mathrm{d}t + \eta \mathrm{d}W(t),$$

  with parameters $\lambda = 0.05$, $\theta = 0.02$ and $\eta = 0.1$ and initial rate $r(t_0) = 0$. At time t_0, we wish to hedge a position in a 10y zero-coupon bond, $P(t_0, 10y)$, using two other bonds, $P(t_0, 1y)$ and $P(t_0, 20y)$.

  a. Determine two weights, ω_1 and ω_2, such that $\omega_1 + \omega_2 = 1$ and $\omega_1 P(t_0, 1y) + \omega_2 P(t_0, 20y) = P(t_0, 10y)$.
  b. Perform a minimum variance hedge and determine the weights ω_1 and ω_2, such that

  $$\mathbb{V}\mathrm{ar}\left[\int_0^{10y} \omega_1 P(t, 1y) + \omega_2 P(t, 20y)\mathrm{d}t\right] = \mathbb{V}\mathrm{ar}\left[\int_0^{10y} P(t, 10y)\mathrm{d}t\right],$$

  while $\omega_1 + \omega_2 = 1$. What can be said about this type of hedge compared to the point addressed?
  c. Change the measure, to the $T = 10y$-forward measure, and, for the given weights determined earlier, check whether the variance of the estimator increases.

Summary of the Lecture + Homework

- **Exercise**
- The "exponential-Vasicek" model does not allow for negative interest rates. In order to "fix" that problem we introduce a so-called shift parameter ζ,

$$\tilde{r}(t) = r(t) - \zeta, \quad \zeta \in \mathbb{R}^{+}.$$

- Find the dynamics for $\tilde{r}(t)$, simulate Monte Carlo paths and compare to $r(t)$.
- Discuss the impact of shift on ZCBs, $P(t_0, T)$.