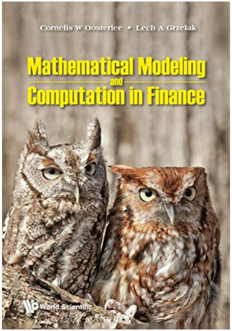
Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.



- Youtube Channel with courses can be found here.
- Slides and the codes can be found here.

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Simple Compounded Forward Rate

- Let us assume there are two counterparties, A and B, where counterparty A will pay to counterparty B 1€ at time T_1 and at time T_2 counterparty A will receive back 1€ and will also receive the interest rate K over the accrual time $T_2 - T_1$.

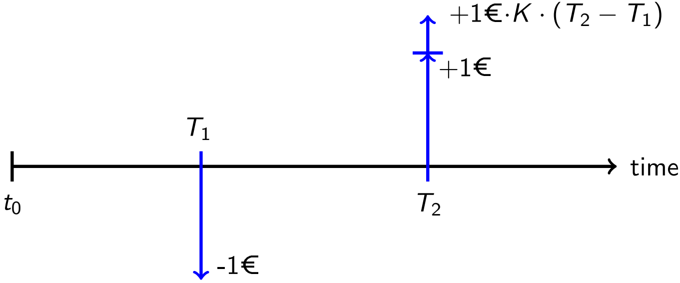


Figure: Cash flows between two counterparties.

Simple Compounded Forward Rate

- The fair value of this contract is calculated as follows,

$$
\begin{aligned}
V(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{-1}{M(T_1)} + \frac{1+K\cdot(T_2-T_1)}{M(T_2)}\Big|\mathcal{F}(t_0)\right] \\
&= -P(t_0,T_1) + (1+(T_2-T_1)\cdot K)P(t_0,T_2).
\end{aligned}
$$

- The interest rate K for which the value of this contract at time t_0 equals 0, meaning no payments at inception, is given by

$$
K = \frac{1}{(T_2-T_1)}\left(\frac{P(t_0,T_1)}{P(t_0,T_2)} - 1\right).
$$

- Generally, the fair rate K for interbank lending with trade date t, starting date T_{k-1} and maturity date T_k with tenor $\tau_k = T_k - T_{k-1}$, and denoted by $K \equiv \ell_k(t) := \ell(t; T_{k-1}, T_k)$, equals,

$$
\boxed{\ell(t;T_{k-1},T_k) = \frac{1}{\tau_k}\left(\frac{P(t,T_{k-1})}{P(t,T_k)} - 1\right).}
$$

Simple Compounded Forward Rate

- We deal with a trading time horizon $[0, T^*]$ and a set of times $\{T_k, i = 1, \ldots, m\}, m \in \mathbb{N}$, such that $0 \leq T_0 < T_k < \cdots < T_m \leq T^*$, and define a so-called *tenor*, $\tau_k := T_k - T_{k-1}$.

Definition (Simple compounded forward Libor rate)

For a given tenor τ_k, and the risk-free ZCB maturing at time T_k with a nominal of 1 (unit of currency), $P(t, T_k)$, the Libor forward rate $\ell(t; T_{k-1}, T_k)$ for a period $[T_{k-1}, T_k]$ is defined as,

$$\boxed{\ell_k(t) \equiv \ell(t; T_{k-1}, T_k) = \frac{1}{\tau_k} \frac{P(t, T_{k-1}) - P(t, T_k)}{P(t, T_k)}.}$$

For notational convenience, we use $\ell_k(t) \equiv \ell(t; T_{k-1}, T_k)$, so that $\ell_k(T_{k-1}) = \ell(T_{k-1}; T_{k-1}, T_k)$.

FRA

- In the interest rate market, it is common to fix an interest rate at a future time T_{k-1} which is then accrued over a future time period $[T_{k-1}, T_k]$. For this period $[T_{k-1}, T_k]$, the two parties agree to *exchange a fixed rate K* for a payment of the (floating) Libor rate, which is observed at time T_{k-1}. Typically, these payments are exchanged at time T_k (which is often two business days after time T_{k-1}).
- The payoff of the FRA contract at time T_{k-1} is given by:

$$V^{\text{FRA}}(T_{k-1}) = H^{\text{FRA}}(T_{k-1}) = \frac{\tau_k\left(\ell(T_{k-1}; T_{k-1}, T_k) - K\right)}{1 + \tau_k \ell(T_{k-1}; T_{k-1}, T_k)},$$

  with the tenor $\tau_k = T_k - T_{k-1}$ and we assume a unit notional amount, $N = 1$.

FRA

- Using the definition of the Libor rate $\ell(T_{k-1}; T_{k-1}, T_k)$, we can directly connect the denominator with the ZCB $P(T_{k-1}, T_k)$, as follows,

$$P(T_{k-1}, T_k) = \frac{1}{1 + \tau_k \ell(T_{k-1}; T_{k-1}, T_k)},$$

so that the FRA's payoff function can also be expressed as,

$$V^{\mathrm{FRA}}(T_{k-1}) = \tau_k P(T_{k-1}, T_k) \left(\ell(T_{k-1}; T_{k-1}, T_k) - K \right).$$

- The current price of the FRA contract is determined by,

$$\begin{aligned} V^{\mathrm{FRA}}(t_0) &= \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{M(T_{k-1})} \tau_k P(T_{k-1}, T_k) \left(\ell(T_{k-1}; T_{k-1}, T_k) - K \right) \Big| \mathcal{F}(t_0) \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[\frac{1 - P(T_{k-1}, T_k)}{M(T_{k-1})} - \tau_k K \frac{P(T_{k-1}, T_k)}{M(T_{k-1})} \Big| \mathcal{F}(t_0) \right], \end{aligned}$$

since $M(t_0) = 1$.

FRA

- Since the bonds $P(T_{k-1}, T_k)$ are traded assets, the discounted bonds should be *martingales*, which gives us,

$$\begin{aligned} V^{\text{FRA}}(t_0) &= P(t_0, T_{k-1}) - P(t_0, T_k) - \tau_k K P(t_0, T_k) \\ &= \tau_k P(t_0, T_k)\left(\ell(t_0; T_{k-1}, T_k) - K\right). \end{aligned}$$

  Note that, by definition,

$$P(t_0, T_{k-1}) := \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_{k-1})}\Big|\mathcal{F}(t_0)\right].$$

- Most commonly, the FRAs are traded at zero value, which implies that for the fixed rate we should have, $K = \ell(t_0; T_{k-1}, T_k)$.

Floating Rate Note

- Another heavily traded product is the floating rate note (FRN). Given the floating Libor rate, $\ell_k(T_k) := \ell(T_{k-1}; T_{k-1}, T_k)$, and a notional amount N, the FRN is an instrument with coupon payments, which is defined as,

$$V_k^{\text{FRN}}(T_k) = \begin{cases} N\tau_k \ell(T_{k-1}; T_{k-1}, T_k), & i \in \{1, 2, \ldots, m-1\}, \\ N\tau_m \ell(T_{m-1}; T_{m-1}, T_m) + N, & i = m. \end{cases}$$

- The bond thus consists of a sum of payments, see also Figure 2, and each cash flow can be priced separately, as

$$\begin{aligned} V_k^{\text{FRN}}(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_k)} V_k^{\text{FRN}}(T_k) \middle| \mathcal{F}(t_0)\right] \\ &= P(t_0, T_k)\mathbb{E}^{T_k}\left[V_k^{\text{FRN}}(T_k) \middle| \mathcal{F}(t_0)\right], \end{aligned}$$

by a measure change, from the risk-neutral measure $\mathbb{Q}$ to the T_k forward measure.

Floating Rate Note

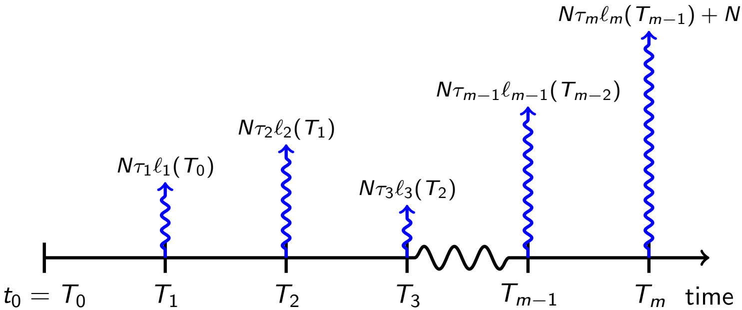


Figure: Cash flows for a floating rate note with $\ell_k(T_{k-1}) := \ell(T_{k-1}; T_{k-1}, T_k)$ and each coupon defined as $N\tau_k\ell(T_{k-1}; T_{k-1}, T_k)$, $\tau_k = T_k - T_{k-1}$.

Floating Rate Note

- What can we say about $\ell(T_{k-1}; T_{k-1}, T_k)$?
- By definition the Libor rate is given by:

$$\ell(t; T_{k-1}, T_k) = \frac{1}{\tau_k} \frac{P(t, T_{k-1}) - P(t, T_k)}{P(t, T_k)} \quad \text{thus} \ldots$$

- Lets calculate the expectation under the $\mathbb{Q}^k$ forward measure:

$$\mathbb{E}^{T_k}\left[\ell(T_{k-1}; T_{k-1}, T_k) | \mathcal{F}(t)\right] = \frac{1}{\tau_k} \mathbb{E}^{T_k}\left[\frac{P(T_{k-1}, T_{k-1}) - P(T_{k-1}, T_k)}{P(T_{k-1}, T_k)} | \mathcal{F}(t)\right],$$

- Since the RHS is a martingale we find:

$$\begin{aligned} \mathbb{E}^{T_k}\left[\ell(T_{k-1}; T_{k-1}, T_k) | \mathcal{F}(t)\right] &= \frac{1}{\tau_k} \frac{P(t, T_{k-1}) - P(t, T_k)}{P(t, T_k)} \\ &= \ell(t; T_{k-1}, T_k). \end{aligned}$$

Floating Rate Note

- Now, $\ell(T_{k-1}; T_{k-1}, T_k)$ is a traded quantity, so it should again be a martingale, and therefore we have:

$$\mathbb{E}^{T_k}\left[\ell(T_{k-1}; T_{k-1}, T_k)\middle|\mathcal{F}(t_0)\right] = \ell(t_0; T_{k-1}, T_k),$$

resulting in,

$$\mathbb{E}^{T_k}\left[V_k^{\text{FRN}}(T_k)\middle|\mathcal{F}(t_0)\right] = \begin{cases} N\tau_k\ell(t_0; T_{k-1}, T_k), & i \in \{1, 2, \ldots, m-1\}, \\ N\tau_m\ell(t_0; T_{m-1}, T_m) + N, & i = m. \end{cases}$$

- Which can be used in the final pricing equation:

$$V_k^{\text{FRN}}(t_0) \quad = \quad P(t_0, T_k)\mathbb{E}^{T_k}\left[V_k^{\text{FRN}}(T_k)\middle|\mathcal{F}(t_0)\right].$$

Swaps

- In financial jargon two terms related to interest rate swaps are often used, *interest rate swap payer* and *interest rate swap receiver*. Those terms are used to distinguish between cases when one receives a float leg and pays the fixed leg (swap payer) and the case when one is paying the float leg and receives the fixed leg (swap receiver).
- For notational convenience, we use $\ell_k(t) \equiv \ell(t, T_{k-1}, T_k)$, so that $\ell_k(T_{k-1}) = \ell(T_{k-1}, T_{k-1}, T_k)$.
- If today's date is indicated by t_0 the payoff of the interest rate swap (payer) is given by:

$$V(T_m, \dots, T_n) = \sum_{k=m+1}^{n} \tau_k \boxed{N} \Big(\ell(T_{k-1}, T_{k-1}, T_k) - K\Big),$$

- We will often use the short-hand notation $\ell(T_{k-1}, T_k) := \ell(T_{k-1}, T_{k-1}, T_k)$ which indicates that the libor rate is determined (expires) at time T_{k-1}. N stands for the notional.

Swaps

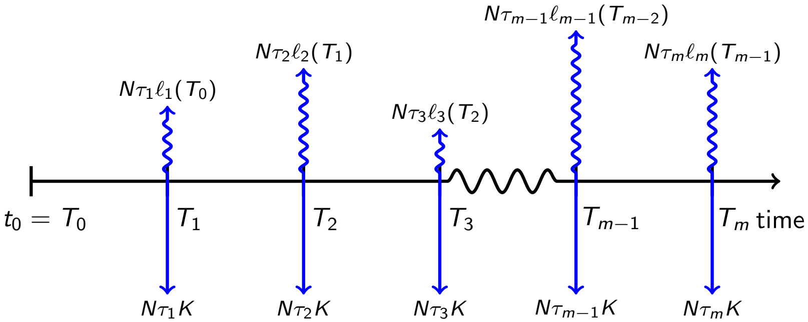


Figure: Cash flows for a swap with $\ell_k(T_{k-1}) := \ell(T_{k-1}; T_{k-1}, T_k)$, $\tau_k = T_k - T_{k-1}$.

Swaps

- In order to determine today's value of the swap we need to evaluate the corresponding expectation of the discounted future cash-flows, i.e., each payment which takes place at times $T_{m+1}, \dots, T_n$ needs to be discounted back to today,

$$V^{\text{Swap}}_{m,n}(t_0) = M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\sum_{k=m+1}^{n} \frac{1}{M(T_k)}\tau_k\big(\ell(T_{k-1}, T_k) - K\big)\Big|\mathcal{F}(t_0)\right].$$

- By applying a measure change, from the risk-free measure, $\mathbb{Q}$, to the T_k-forward measure, $\mathbb{Q}^{T_k}$, we find:

$$\begin{aligned} V^{\text{Swap}}_{m,n}(t_0) &= \sum_{k=m+1}^{n} \tau_k P(t_0, T_k)\Big(\mathbb{E}^{T_k}\left[\ell(T_{k-1}, T_k)\big|\mathcal{F}(t_0)\right] - K\Big) \\ &= \sum_{k=m+1}^{n} \tau_k P(t_0, T_k)\Big(\ell(t_0, T_{k-1}, T_k) - K\Big), \end{aligned}$$

where the last step is due to the observation that the libor rate $\ell(t, T_{k-1}, T_k)$, under its own natural measure $\mathbb{Q}^{T_k}$ is a martingale, i.e. $\mathbb{E}^{T_k}[\ell(t, T_{k-1}, T_k)|\mathcal{F}(t_0)] = \ell(t_0, T_{k-1}, T_k)$.

Swaps

- We can simplify the swap pricing equation, as follows:

$$V^{\text{Swap}}_{m,n}(t_0)=\sum_{k=m+1}^{n}\tau_k P(t_0,T_k)\ell(t_0,T_{k-1},T_k)-K\sum_{k=m+1}^{n}\tau_k P(t_0,T_k).$$

- The first summation can be further simplified, using the definition of the libor rate,

$$\begin{aligned}\sum_{k=m+1}^{n}\tau_k P(t_0,T_k)\ell(t_0,T_{k-1},T_k) &= \sum_{k=m+1}^{n}\tau_k P(t_0,T_k)\left[\frac{P(t_0,T_{k-1})-P(t_0,T_k)}{\tau_k P(t_0,T_k)}\right]\\ &= \sum_{k=m+1}^{n} P(t_0,T_{k-1})-P(t_0,T_k)\\ &= P(t_0,T_m)-P(t_0,T_n),\end{aligned}$$

where in the last step the telescopic summation was recognized.

- The price of the swap is as follows:

$$V^{\text{Swap}}_{m,n}(t_0)=\left[P(t_0,T_m)-P(t_0,T_n)\right]-K\sum_{k=m+1}^{n}\tau_k P(t_0,T_k).$$

Swaps

- Let us define the "*annuity factor*", which is also called the "*present value of a basis point*" as,

$$A_{m,n}(t) \stackrel{\text{def}}{=} \sum_{k=m+1}^{n} \tau_k P(t, T_k).$$

  In fact, annuity $A_{m,n}(t)$ is nothing but a linear combination of zero-coupon bonds. As each of these zero-coupon bonds is a tradable asset the linear combination, and therefore the annuity, is it as well.
- Therefore, we can consider the annuity function as a numéraire in derivatives pricing, which will be helpful later in this section.
- Typically, interest rate swaps are considered perfect interest rate instruments where two parties can hedge their particular exposures.
- Standard practice in determining the strike K for interest rate swaps is to choose it in such a way that the value of the swap at initial time t_0 equals zero.

Swaps

- By setting the value of the swap to zero, entering into such a deal is *for free*. Moreover, the strike value for which the swap equals zero is called *swap rate* and is indicated by $S_{m,n}(t_0)$.
- By equating to zero we find,

$$S_{m,n}(t_0) = \frac{P(t_0, T_m) - P(t_0, T_n)}{\sum_{k=m+1}^{n} \tau_k P(t_0, T_k)} = \frac{P(t_0, T_m) - P(t_0, T_n)}{A_{m,n}(t_0)},$$

which, alternatively, by not simplifying the first summation we can write,

$$S_{m,n}(t_0) = \frac{1}{A_{m,n}(t_0)} \sum_{k=m+1}^{n} \tau_k P(t_0, T_k)\ell_k(t_0) = \sum_{k=m+1}^{n} \omega_k(t_0)\ell_k(t_0),$$

with $\omega_k(t_0) = \tau_k P(t_0, T_k) A_{m,n}(t_0)$ and $\ell_k(t_0) := \ell(t_0, T_{k-1}, T_k)$.

Swaps

- Now, we can express the value of the swap as:

$$V_{m,n}^{\text{Swap}}(t_0) \;=\; A_{m,n}(t_0)\big(S_{m,n}(t_0) - K\big).$$

  This compact representation is handy as all the components have a particular meaning: the annuity $A_{m,n}(t_0)$ represents the present value of a basis point of the swap, $S_{m,n}(t)$ is the swap rate and K is the strike price. Obviously for $K = S_{m,n}(t_0)$ today's value of this swap is equal to zero.
- **What model assumptions do we need to price a Swap?**
- **REMARK**: In order to price a basic interest rate swap the pricing can be done without assumptions about the underlying model. The pricing can be simply done by using interest rate instruments available in the market.

Swaps- Types of Notionals

- So far we discussed Swaps with constant notional, N,:

$$V(T_m,\dots,T_n)=\sum_{k=m+1}^{n}\tau_k\boxed{N}\big(\ell(T_{k-1},T_k)-K\big),$$

- More complicated notional structures are possible:

$$V(T_m,\dots,T_n)=\sum_{k=m+1}^{n}\tau_k\boxed{N_k}\big(\ell(T_{k-1},T_k)-K\big).$$

  - Amortizing notional: $N_1>N_2>N_3>\dots>N_m$ (typically associated with mortgages).
  - Accreting notional: $N_1<N_2<N_3<\dots<N_m$.
  - Roller-coaster notional: $N_1<N_2<\dots<N_k>N_{k+1}>N_{k+2}>\dots>N_m$.
  - Stochastic notional: notional depends on the underlying risk factor, i.e.:

$$N_k(f(\ell(T_{k-1},T_k))).$$

Swaps- Types of Notionals

- Important to keep in mind that once the notional is stochastic, the change of measure may NOT help us!
- Libor rate $\ell_k(t)$ is a martingale under the T_k-forward measure.

$$\boxed{\mathbb{E}^{T_k}\left[\ell(T_{k-1};T_{k-1},T_k)\middle|\mathcal{F}(t_0)\right]=\ell(t_0;T_{k-1},T_k).}$$

- However:

$$\boxed{\mathbb{E}^{T_k}\left[\ell^2(T_{k-1};T_{k-1},T_k)\middle|\mathcal{F}(t_0)\right]\neq\ell^2(t_0;T_{k-1},T_k).}$$

- Under T_k-forward measure libor ℓ_k is a martingale, i.e.,

$$\mathrm{d}\ell_k(t)=\sigma_k\ell_k(t)\mathrm{d}W^k(t),$$

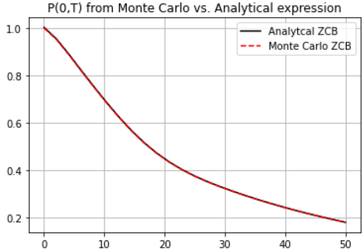
however:

$$\mathrm{d}\ell_k^2(t) \quad = \quad \boxed{\sigma_k^2\ell_k^2(t)\mathrm{d}t}+2\sigma_k\ell_k^2(t)\mathrm{d}W^k(t).$$

- Therefore $\ell_k^2(t)$ is not a martingale under T_k-forward measure.

Swaps- Python Exercise

- In this experiment we price swaps using ZCBs, $P(0, T)$, and the Hull-White model.
- We also look into "par-swaps".



Zero-Coupon Bond Under The Hull-White model

- For a ZCB, $P(0,T)$, its value can be directly related to today's yield curve via:
$$f^r(0,T) = -\frac{\partial}{\partial T}\log P(0,T).$$
- Therefore ZCB is directly related to the instantaneous forward rates $f^r(t,T)$, via the relation:
$$P(t,T) = \exp\left(-\int_t^T f^r(t,z)\mathrm{d}z\right).$$
- Therefore he ZCB is given by
$$P(t,T) = \exp(Z(t,T)), \quad \text{with} \quad Z(t,T) := -\int_t^T f^r(t,s)\mathrm{d}s.$$
- By Itô's lemma the dynamics of $P(t,T)$ are then given by:
$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = \mathrm{d}Z(t,T) + \frac{1}{2}(\mathrm{d}Z(t,T))^2.$$

Zero-Coupon Bond Under The Hull-White model

- After simplification we obtain the following result for the dynamics of the ZCB under the HJM framework.
- The risk-free dynamics of the ZCB are given by:

$$\mathrm{d}P(t,T) = r(t)P(t,T)\mathrm{d}t - P(t,T)\left(\int_t^T \bar{\eta}(z,T)\mathrm{d}z\right)\mathrm{d}W^{\mathbb{Q}}(t).$$

- For the Hull-White model, the HJM volatility is given by,

$$\bar{\eta}(t,T) = \eta \mathrm{e}^{-\lambda(T-t)}.$$

- This implies the following dynamics for the ZCB:

$$\begin{aligned}\frac{\mathrm{d}P(t,T)}{P(t,T)} &= r(t)\mathrm{d}t - \left(\int_t^T \eta \mathrm{e}^{-\lambda(T-z)}\mathrm{d}z\right)\mathrm{d}W^{\mathbb{Q}}(t) \\ &= r(t)\mathrm{d}t + \eta \bar{B}_r(t,T)\mathrm{d}W^{\mathbb{Q}}(t),\end{aligned}$$

with $\bar{B}_r(t,T) = \frac{1}{\lambda}\left(\mathrm{e}^{-\lambda(T-t)} - 1\right).$

Hull-White model under the T-Forward Measure

- We discuss the dynamics of the Hull-White short-rate model under the $T-$forward measure.
- The advantages of the short-rate model under the changed measure will become clear in particular in the context of equity hybrid models, or stochastic discounting.
- To change between measures, from $\mathbb{Q}$ governed by the money-savings account $M(t)$, to the T-forward measure $\mathbb{Q}^T$, implied by the zero-coupon bond $P(t,T)$, we make use again of the Radon-Nikodym derivative,

$$\lambda_{\mathbb{Q}}^{T}(t) = \frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}}\Big|_{\mathcal{F}(t)} = \frac{P(t,T)}{P(t_0,T)}\frac{M(t_0)}{M(t)}, \quad \text{for} \quad t > t_0.$$

Hull-White model under the T-Forward Measure

- Based on the dynamics for the ZCB and the definition of the money-savings account $M(t)$, the dynamics of $\lambda_{\mathbb{Q}}^{T}(t)$ read,

$$\begin{aligned}\mathrm{d}\lambda_{\mathbb{Q}}^{T}(t) &= \frac{M(t_0)}{P(t_0,T)}\left(\frac{1}{M(t)}\mathrm{d}P(t,T) - \frac{P(t,T)}{M^2(t)}\mathrm{d}M(t)\right)\\ &= \frac{M(t_0)}{P(t_0,T)}\frac{P(t,T)}{M(t)}\eta\bar{B}_r(t,T)\mathrm{d}W_r^{\mathbb{Q}}(t).\end{aligned}$$

  with $\bar{B}_r(t,T) = \frac{1}{\lambda}\left(\mathrm{e}^{-\lambda(T-t)}-1\right)$.
- Using the definition of $\lambda_{\mathbb{Q}}^{T}(t)$, we thus find,

$$\frac{\mathrm{d}\lambda_{\mathbb{Q}}^{T}(t)}{\lambda_{\mathbb{Q}}^{T}(t)} = \eta\bar{B}_r(t,T)\mathrm{d}W_r^{\mathbb{Q}}(t).$$

- This representation gives us the Girsanov kernel, which enables us to change between measures, from the risk-neutral measure $\mathbb{Q}$ to the T-forward measure $\mathbb{Q}^T$, i.e.

$$\mathrm{d}W_r^{T}(t) = -\eta\bar{B}_r(t,T)\mathrm{d}t + \mathrm{d}W_r^{\mathbb{Q}}(t).$$

Hull-White model under the T-Forward Measure

- The measure transformation defines the following short-rate dynamics under the $\mathbb{Q}^T$ measure,

$$\begin{aligned}\mathrm{d}r(t) &= \lambda(\theta(t) - r(t))\mathrm{d}t + \eta \mathrm{d}W_r^{\mathbb{Q}}(t)\\ &= \lambda\left(\theta(t) + \frac{\eta^2}{\lambda}\bar{B}_r(t,T) - r(t)\right)\mathrm{d}t + \eta\mathrm{d}W_r^T(t).\end{aligned}$$

- The process in can further be rewritten into

$$\boxed{\mathrm{d}r(t) = \lambda\left(\hat{\theta}(t,T) - r(t)\right)\mathrm{d}t + \eta\mathrm{d}W_r^T(t),}$$

with

$$\hat{\theta}(t,T) = \theta(t) + \frac{\eta^2}{\lambda}\bar{B}_r(t,T), \quad \text{where} \quad \bar{B}_r(t,T) = \frac{1}{\lambda}\left(\mathrm{e}^{-\lambda(T-t)} - 1\right).$$

Options on Zero-Coupon Bond

- We discuss a European option with expiry at time T, on a zero-coupon bond $P(T, T_S)$ with *maturity time* T_S, $T < T_S$. Although this product is very basic and does not require extensive analysis, it is important to discuss the product, as it forms an important building block for pricing *swaption products*, in the follow-up Lecture.
- A European-style option is defined by the following equation:

$$V^{\text{ZCB}}(t_0, T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T)}\max\left(\bar{\alpha}(P(T, T_S) - K), 0\right)\middle|\mathcal{F}(t_0)\right],$$

with $\bar{\alpha} = 1$ for a call and $\bar{\alpha} = -1$ for a put option, strike price K and $\mathrm{d}M(t) = r(t)M(t)\mathrm{d}t$. By a measure change, from the risk-free to the T-forward measure, the pricing equation is given by,

$$V^{\text{ZCB}}(t_0, T) = P(t_0, T)\mathbb{E}^{T}\left[\max\left(\bar{\alpha}(P(T, T_S) - K), 0\right)\middle|\mathcal{F}(t_0)\right].$$

Options on Zero-Coupon Bond

- When dealing with an *affine* short-rate model $r(t)$, the zero-coupon bond $P(T, T_S)$ is an exponential function, and the pricing equation can be expressed as:

$$\begin{aligned} V^{\text{ZCB}}(t_0, T) &= P(t_0, T)\mathbb{E}^T\left[\max\left(\bar{\alpha}\left(\mathrm{e}^{\bar{A}_r(\tau)+\bar{B}_r(\tau)r(T)} - K\right), 0\right)\Big|\mathcal{F}(t_0)\right] \\ &= P(t_0, T)\mathrm{e}^{\bar{A}_r(\tau)}\mathbb{E}^T\left[\max\left(\bar{\alpha}\left(\mathrm{e}^{\bar{B}_r(\tau)r(T)} - \hat{K}\right), 0\right)\Big|\mathcal{F}(t_0)\right], \end{aligned}$$

  with $\tau = T_S - T$, $\hat{K} = K\mathrm{e}^{-\bar{A}_r(\tau)}$, and $r(T)$ is the short-rate process at time T under T-forward measure $\mathbb{Q}^T$.

- Functions $\bar{A}_r(\tau)$ and $\bar{B}_r(\tau)$ are given by

$$\begin{aligned} \bar{A}_r(\tau) &:= \lambda\int_0^\tau \theta(T_S - z)\bar{B}_r(z)\mathrm{d}z + \frac{\eta^2}{4\lambda^3}\left(\mathrm{e}^{-2\lambda\tau}(4\mathrm{e}^{\lambda\tau} - 1) - 3\right) + \frac{\eta^2}{2\lambda^2}\tau, \\ \bar{B}_r(\tau) &:= \frac{1}{\lambda}\left(\mathrm{e}^{-\lambda\tau} - 1\right). \end{aligned}$$

Options on Zero-Coupon Bond

- The pricing equation is given by,

$$\begin{aligned}\frac{V_c^{\mathrm{ZCB}}(t_0,T)}{P(t_0,T)} &= \exp\left(\bar{A}_r(\tau)\right)\Bigg[\exp\left(\frac{1}{2}\bar{B}_r^2(\tau)v_r^2(T)+\bar{B}_r(\tau)\mu_r(T)\right)F_{\mathcal{N}(0,1)}(d_1)\\ &- \hat{K}F_{\mathcal{N}(0,1)}(d_2)\Bigg],\end{aligned}$$

  with $\tau = T_S - T$, $d_1 = a - \bar{B}_r(\tau)v_r(T)$, $d_2 = d_1 + \bar{B}_r(\tau)v_r(T)$, and $\hat{K} = K\mathrm{e}^{-\bar{A}_r(\tau)}$,

- with

$$a = \frac{\log\hat{K} - \bar{B}_r(\tau)\mu_r(T)}{\bar{B}_r(\tau)v_r(T)}.$$

- $\mu_r(T)$ and $v_r(T)$ are given by:

$$\begin{aligned}\mu_r(T) &= r_0\mathrm{e}^{-\lambda(T-t_0)} + \lambda\int_{t_0}^{T}\hat{\theta}(z,T)\mathrm{e}^{-\lambda(T-z)}\mathrm{d}z,\\ v_r^2(T) &= \frac{\eta^2}{2\lambda}\left(1-\mathrm{e}^{-2\lambda(T-t_0)}\right),\end{aligned}$$

Options on Zero-Coupon Bond

- In this experiment, under the Hull-White model, we will price an option on a ZCB:

$$V^{\mathrm{ZCB}}(t_0,T) \;=\; \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T)}\max\left(\bar{\alpha}(P(T,T_S)-K),0\right)\Big|\mathcal{F}(t_0)\right].$$

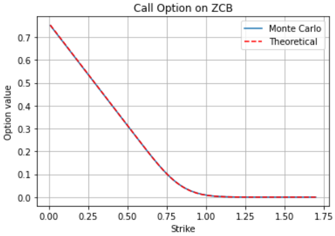


Figure: Call Option on a ZCB

Caplets/Floorlets

Definition (Caplet/Floorlet)

Given two future time points, $T_{k-1} < T_k$, with $\tau_k = T_k - T_{k-1}$, the T_{k-1}-caplet/floorlet with rate K_k and nominal amount N_k is a contract which pays, at time T_k, the amount,

$$\begin{aligned} V_k^{\text{CPL}}(T_k) &= \tau_k N_k \max\left(\ell_k(T_{k-1}) - K_k, 0\right), \\ V_k^{\text{FL}}(T_k) &= \tau_k N_k \max\left(K_k - \ell_k(T_{k-1}), 0\right), \end{aligned}$$

with $\ell_k(T_{k-1}) := \ell(T_{k-1}; T_{k-1}, T_k)$.

- How would you price such options? What model dynamics can be used?

Caplets/Floorlets

- Caplets and floorlets are basically the European options on the interest rate, accumulated from time T_{n-1} until T_n. It is important to know that the contract of this derivative pays at time T_n, while the rate is reset already at time T_{n-1}. So, at time T_{n-1} the rate which will be used for payment at time T_n is known. We should model Libor $\ell_n(t)$ as a stochastic quantity which, in financial terminology, *dies* at time T_{n-1}.
- The caplet represents a European call option on the Libor rate, whereas a floorlet is a put option.
- The caplet price under the T_k-forward measure is given by:

$$
\begin{aligned}
V_k^{\text{CPL}}(t) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{N_k\tau_k}{M(T_k)}\max\left(\ell_k(T_{k-1})-K_k,0\right)\Big|\mathcal{F}(t)\right]\\
&= N_k\tau_k P(t,T_k)\mathbb{E}^{T_k}\left[\max\left(\ell_k(T_{k-1})-K_k,0\right)\Big|\mathcal{F}(t)\right].
\end{aligned}
$$

Caplets/Floorlets

- The floorlet price under the T_k-forward measure is given by:

$$\begin{aligned} V_k^{\text{FL}}(t) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{N_k\tau_k}{M(T_k)}\max\left(K_k-\ell_k(T_{k-1}),0\right)\Big|\mathcal{F}(t)\right] \\ &= N_k\tau_k P(t,T_k)\mathbb{E}^{T_k}\left[\max\left(K_k-\ell_k(T_{k-1}),0\right)\Big|\mathcal{F}(t)\right]. \end{aligned}$$

- Now, assuming that the libor rate follows a lognormal distribution, i.e.:

$$\boxed{\mathrm{d}\ell(t;T_{k-1},T_k)=\sigma_k\ell(t;T_{k-1},T_k)\mathrm{d}W^k(t).}$$

- If we consider Black's model dynamics to value this option, then the value of caplet k is given by:

$$\begin{aligned} \mathrm{Caplet}_k(t_0) &= N_k\tau_k P(t_0,T_k)\left[\ell(t_0;T_{k-1},T_k)N(d_1)-K_kN(d_2)\right], \text{ with} \\ d_1 &= \frac{\log\left(\frac{\ell(t_0;T_{k-1},T_k)}{K}\right)+\frac{1}{2}\sigma_k^2(T_k-t_0)}{\sigma_k\sqrt{(T_k-t_0)}}, \\ d_2 &= d_1-\sigma_k\sqrt{T_k-t_0}. \end{aligned}$$

Caps and Floors

-
- An interest rate cap is designed to provide insurance for a holder with a loan on a floating rate, against the floating rate rising above a pre-defined level, i.e. cap-rate K. A cap can be decomposed as the sum of a number of basic contracts, called caplets, that are defined as follows:
- Caps and Floors are simply the sums of caplets and floorlets.
- If we consider the lognormal dynamics for the libor $\ell(t, T_{k-1}, T_k)$ the price of a cap is simply given by:

$$
\begin{aligned}
V_{\text{cap}}(t) &= \sum_{k=1}^{m} \mathrm{Caplet}_k(t), \\
V_{\text{floor}}(t) &= \sum_{k=1}^{m} \mathrm{Floorlet}_k(t),
\end{aligned}
$$

with $\mathrm{Caplet}_k(t)$ defined in the previous slide.

Caplets/Floorlets

- The price of a caplet, with a strike price K, is given by:

$$\begin{aligned} V^{\text{CPL}}(t_0) &= N\tau_k \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_k)}\max\left(\ell_k(T_{k-1})-K,0\right)\Big|\mathcal{F}(t_0)\right] \\ &= N\tau_k P(t_0,T_k)\mathbb{E}^{T_k}\left[\max\left(\ell_k(T_{k-1})-K,0\right)\Big|\mathcal{F}(t_0)\right]. \end{aligned}$$

- By the definition of the Libor rate, the (scaled) caplet valuation formula can be written as,

$$\begin{aligned} \frac{V^{\text{CPL}}(t_0)}{P(t_0,T_k)} &= N\tau_k \mathbb{E}^{T_k}\left[\max\left(\frac{1}{\tau_k}\left(\frac{1}{P(T_{k-1},T_i)}-1\right)-K,0\right)\Big|\mathcal{F}(t_0)\right] \\ &= N\cdot\mathbb{E}^{T_k}\left[\max\left(\mathrm{e}^{-\bar{A}_r(\tau_k)-\bar{B}_r(\tau_k)r(T_{k-1})}-1-\tau_k K,0\right)\Big|\mathcal{F}(t_0)\right] \\ &= N\cdot\mathrm{e}^{-\bar{A}_r(\tau_k)}\mathbb{E}^{T_k}\left[\max\left(\mathrm{e}^{-\bar{B}_r(\tau_k)r(T_{k-1})}-\hat{K},0\right)\Big|\mathcal{F}(t_0)\right], \end{aligned}$$

with $\hat{K}=(1+\tau_k K)\mathrm{e}^{\bar{A}_r(\tau_k)}$.

- Alternative, based on the tower property, derivations can be found in the book on page 381.

Pricing of Caplets/Floorlets under the HW model

- In this experiment, under the Hull-White model, we will price a caplet (pricing of a floorlet is chosen as homework), defined as:

$$V^{\mathrm{CPL}}(t_0) \quad = \quad N\tau_k \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_k)}\max\left(\ell_k(T_{k-1})-K,0\right)\Big|\mathcal{F}(t_0)\right]$$

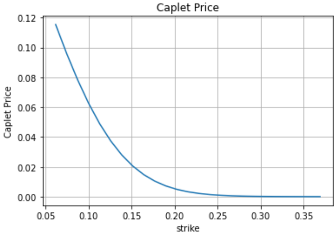


Figure: Caplet price as a function of strike, K.

Summary

- Simple Compounded Forward Rate
- Forward Rate Agreement
- Floating Rate Note
- Interest Rate Swap
- The Hull-White model under the T-Forward Measure
- Options on Zero-Coupon Bond
- Caplets and Floorlets
- Pricing of Caplets/Floorlets Under the HW Model
- Homework

Homework Exercises

The solutions for the homework can be find at
https://github.com/LechGrzelak/QuantFinanceBook

- **Exercise**
- In the lecture and the numerical experiments we have discussed the pricing of caplets in some detail- extend the Python code to also price floorlets, i.e.,

$$
\begin{aligned}
V^{\mathrm{FL}}(t_0) &= N\tau_k \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_k)}\max\left(K-\ell_k(T_{k-1}),0\right)\Big|\mathcal{F}(t_0)\right] \\
&= N\tau_k P(t_0,T_k)\mathbb{E}^{T_k}\left[\max\left(K-\ell_k(T_{k-1}),0\right)\Big|\mathcal{F}(t_0)\right].
\end{aligned}
$$

- Compare analytical expression with Monte Carlo results.

Homework Exercises

- **Exercise**
  In the case pricing of caplets under the lognormal assumption we have used the following dynamics:

$$\mathrm{d}\ell(t; T_{k-1}, T_k) = \sigma_k \ell(t; T_{k-1}, T_k)\mathrm{d}W^k(t).$$

- Introduce a shift ζ, i.e., $\tilde{\ell}_k(t) = \ell_k(t) + \zeta$ and derive Black's formula for option pricing. Confirm your results analytically.