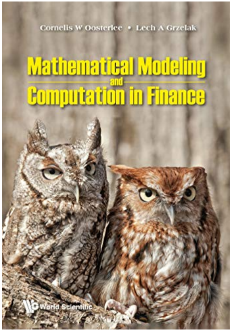
Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.



- Youtube Channel with courses can be found here.
- Slides and the codes can be found here.

List of content

Types of Hybrid Models

- The objective of this course is to create an xVA/HVaR framework that is defined in terms of *exposures*.
- Exposures represent a potential future value of a portfolio or a derivative. In practice portfolio may consist of contacts belonging to different asset classes, like: interest rates, stocks, foreign exchange, commodities, credit, inflation, etc.
- It is important, due to the netting effect, to price the portfolio while including all the risk factors.
- Hybrid models can be used for hybrid payoffs which have a limited sensitivity to the interest rate smile. However, these models are also important in the context of risk management, particularly for Credit Valuation Adjustment (CVA), where the implied volatility smile and the stochastic interest rate may have a prominent effect, for example on the Potential Future Exposure (PFE) quantity.

Motivation for Hybrid Models

- Movements in the interest rate market may have an influence on the behavior of stock prices, especially in the long run. This is taken into account in so-called *hybrid models*.
- Hybrid models can be expressed by a system of SDEs, for example for stock, volatility and interest rate, with a full correlation matrix.

$$\mathrm{d}\mathbf{X}(t) = \bar{\mu}(\mathbf{X}(t))\mathrm{d}t + \bar{\sigma}(\mathbf{X}(t))\mathrm{d}\mathbf{W}(t),$$

  with Brownian motions $\widetilde{\mathbf{W}}(\mathbf{t})$.
- By correlating these SDEs from the different asset classes, one can define the hybrid models.
- Even if each of the individual SDEs yields a closed-form solution, a nonzero correlation structure between the processes may cause difficulties for efficient valuation and calibration.
- Typically, however, a closed-form solution for hybrid models is not known, and a numerical approximation by means of a Monte Carlo (MC) simulation or a discretization of the corresponding PDEs has to be employed.

Motivation for Hybrid Models

- The speed of pricing European derivative products is crucial, especially for the calibration of the SDEs. Several theoretically attractive SDE models, that cannot fulfill the speed requirements, are not used in practice, because of the fact that the calibration stage is too time-consuming.
- Although hybrid models can relatively easily be defined, these models are only used, when they provide a satisfactory fit to market implied volatility structure and when it is possible to set a nonzero correlation structure among the processes from the different asset classes.
- Highly efficient valuation and calibration are mandatory. For this reason we focus to derive the characteristic function of the hybrid SDE system.
- With *a characteristic function* available, highly efficient pricing of European options may take place, for example, by the COS method. (See: Computational Finance Course.)

Black-Scholes Hull-White Model (BSHW)

- As a starting point, we extend the standard Black-Scholes model by the Hull-White short-rate model.
- Under the risk-neutral measure $\mathbb{Q}$, the dynamics of the model with $\mathbf{X}(t) = [S(t), r(t)]^{\mathrm{T}}$ are given by the following system of SDEs:

$$\begin{aligned} \mathrm{d}S(t) &= \boxed{r(t)} S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W_x(t), \quad S(t_0) = S_0 > 0, \\ \boxed{\mathrm{d}r(t)} &= \lambda(\theta(t) - r(t))\mathrm{d}t + \eta \mathrm{d}W_r(t), \qquad r(t_0) = r_0, \end{aligned}$$

  where $W_x(t)$ and $W_r(t)$ are two *correlated* Brownian motions with $\mathrm{d}W_x(t)\mathrm{d}W_r(t) = \rho_{x,r}\mathrm{d}t$, and $|\rho_{x,r}| < 1$.
- Parameters σ and η determine the volatility of equity and interest rate, respectively; $\theta(t)$ is a deterministic function and λ determines the speed of mean reversion.
- In practice, σ and η would be time-dependent functions obtained from a calibration routine.

The BSHW Model- Affinity and ChF

- After a transformation into log-coordinates, $X(t) = \log S(t)$, the model reads:

$$\begin{aligned} \mathrm{d}X(t) = &\quad (r(t) - 1/2\sigma^2)\mathrm{d}t + \sigma \mathrm{d}W_x(t), \\ \mathrm{d}r(t) = &\quad \lambda\left(\theta(t) - r(t)\right)\mathrm{d}t + \eta \mathrm{d}W_r(t). \end{aligned}$$

  it is easy to see that the model satisfies the affinity conditions, so that the corresponding characteristic function, $\phi_{\text{BSHW}}(u; t, T)$, can easily be derived. Details on ChF and Affine Models are covered in Comp. Finance Course- Lecture 6.

- For the state vector $\mathbf{X}(t) = [X(t), r(t)]^{\mathrm{T}}$, the discounted characteristic function, with $\mathbf{u} = [u, 0]^{\mathrm{T}}$ and $\tau := T - t$, reads:

$$\boxed{\phi_{\text{BSHW}}(u; t, T) = \exp\left(\bar{A}(u,\tau) + \bar{B}(u,\tau)X(t) + \bar{C}(u,\tau)r(t)\right),}$$

  with final condition, $\phi_{\text{BSHW}}(u; T, T) = \exp\left(iuX(T)\right)$.

- In the case of the BSHW model it is not necessary to use Fourier transform. One can still derive the solution for option pricing analytically (it will be presented later in this lecture).

The BSHW Model- Affinity and ChF

- When the dynamics of state vector $\mathbf{X}(t)$ are affine, it can be shown that the discounted characteristic function (ChF) is of the following form:

$$\phi_{\mathbf{X}}(\mathbf{u};t,T)=\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s+i\mathbf{u}^{\mathrm{T}}\mathbf{X}(T)}\Big|\mathcal{F}(t)\right]=\mathrm{e}^{\bar{A}(\mathbf{u},\tau)+\bar{\mathbf{B}}^{\mathrm{T}}(\mathbf{u},\tau)\mathbf{X}(t)},$$

  where the expectation is taken under the risk-neutral measure $\mathbb{Q}$, with time lag, $\tau=T-t$, and $\bar{A}(\mathbf{u},0)=0$ and $\bar{B}(\mathbf{u},0)=i\mathbf{u}^{\mathrm{T}}$.
- The coefficients $\bar{A}:=\bar{A}(\mathbf{u},\tau)$ and $\bar{\mathbf{B}}^{\mathrm{T}}:=\bar{\mathbf{B}}^{\mathrm{T}}(\mathbf{u},\tau)$ in (8) satisfy the following complex-valued *Riccati* ordinary differential equations (ODEs),

$$\boxed{\begin{aligned}\frac{\mathrm{d}\bar{A}}{\mathrm{d}\tau}&=-r_0+\bar{\mathbf{B}}^{\mathrm{T}}a_0+\frac{1}{2}\bar{\mathbf{B}}^{\mathrm{T}}c_0\bar{\mathbf{B}},\\ \frac{\mathrm{d}\bar{\mathbf{B}}}{\mathrm{d}\tau}&=-r_1+a_1^{\mathrm{T}}\bar{\mathbf{B}}+\frac{1}{2}\bar{\mathbf{B}}^{\mathrm{T}}c_1\bar{\mathbf{B}}.\end{aligned}}$$

The BSHW Model- Affinity and ChF

- The functions $\bar{A}(u,\tau)$, $\bar{B}(u,\tau)$ and $\bar{C}(u,\tau)$ are found to be:

$$\begin{aligned}
\bar{B}(u,\tau) &= iu,\\
\bar{C}(u,\tau) &= \frac{1}{\lambda}(iu-1)(1-\mathrm{e}^{-\lambda\tau}),\\
\bar{A}(u,\tau) &= \frac{1}{2}\sigma^2 iu(iu-1)\tau + \frac{\rho_{x,r}\sigma\eta}{\lambda}iu(iu-1)\left(\tau+\frac{1}{\lambda}\left(\mathrm{e}^{-\lambda\tau}-1\right)\right)\\
&+ \frac{\eta^2}{4\lambda^3}(i+u)^2\left(3+\mathrm{e}^{-2\lambda\tau}-4\mathrm{e}^{-\lambda\tau}-2\lambda\tau\right)\\
&+ \lambda\int_0^\tau \theta(T-z)\bar{C}(u,z)\mathrm{d}z.
\end{aligned}$$

- The expression for $\bar{A}(u,\tau)$ contains an integral over the deterministic function $\theta(t)$, which may be calibrated to the current market interest rate yield. This integral can be determined analytically.

Pricing of European Call and Put Options

- The pricing problem of a European-style payoff function, $H(T,S)$, can be expressed as

$$\begin{aligned} V(t_0,S) &= M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}H(T,S)\Big|\mathcal{F}(t_0)\right] \\ &= M(t_0)\int_\Omega \frac{1}{M(T)}H(T,S)\mathrm{d}\mathbb{Q}, \end{aligned}$$

where $M(t_0)=1$.

- By Radon-Nikodym's derivative we have:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Q}^T}=\frac{M(T)}{M(t_0)}\frac{P(t_0,T)}{P(T,T)} \quad \text{thus} \quad \mathrm{d}\mathbb{Q}=\frac{M(T)}{M(t_0)}\frac{P(t_0,T)}{P(T,T)}\mathrm{d}\mathbb{Q}^T,$$

therefore:

$$\begin{aligned} V(t_0,S) &= M(t_0)\int_\Omega \frac{1}{M(T)}H(T,S)\mathrm{d}\mathbb{Q} \\ &= M(t_0)\int_\Omega \frac{1}{M(T)}H(T,S)\frac{M(T)}{M(t_0)}\frac{P(t_0,T)}{P(T,T)}\mathrm{d}\mathbb{Q}^T. \end{aligned}$$

Pricing of European Call and Put Options

- Which further simplifies to:

$$\begin{aligned} V(t_0,S) &= M(t_0)\int_\Omega \frac{1}{M(T)}H(T,S)\frac{M(T)}{M(t_0)}\frac{P(t_0,T)}{P(T,T)}\mathrm{d}\mathbb{Q}^T \\ &= P(t_0,T)\int_\Omega H(T,S)\mathrm{d}\mathbb{Q}^T \\ &= P(t_0,T)\mathbb{E}^T\left[H(T,S)\middle|\mathcal{F}(t_0)\right]. \end{aligned}$$

- Therefore:

$$M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}H(T,S)\middle|\mathcal{F}(t_0)\right] = P(t_0,T)\mathbb{E}^T\left[H(T,S)\middle|\mathcal{F}(t_0)\right].$$

- We still need to determine the dynamics of $S(t)$ under the $T-$ forward measure.

Pricing of European Call and Put Options

- Under the $\mathbb{Q}$-measure, the stock, discounted by the money-savings account, is a martingale, however, this is not the case under the T-forward measure.
- Under T-forward measure $\mathbb{Q}^T$, the numéraire is the zero-coupon bond $P(t,T)$ and the process $\frac{S(t)}{P(t,T)}$ thus has to be a martingale.
- With the *forward stock price*, defined as

$$\boxed{S_F(t,T) := \frac{S(t)}{P(t,T)}}.$$

- Since European-type payoffs only depend on a stock value at T we see that because $P(T,T)=1$ we have:

$$\boxed{S_F(T,T) = S(T)}.$$

- And we also know that $S_F(t,T)$ is a martingale under the $T-$forward measure

Pricing of European Call and Put Options

- By Itô's lemma, that

$$\begin{aligned}\mathrm{d}S_F(t,T) &= \frac{1}{P(t,T)}\mathrm{d}S(t) - \frac{S(t)}{P(t,T)}\mathrm{d}P(t,T) + \frac{S(t)}{P^3(t,T)}\left(\mathrm{d}P(t,T)\right)^2\\ &\quad -\frac{1}{P^2(t,T)}\mathrm{d}P(t,T)\mathrm{d}S(t),\end{aligned}$$

which can be simplified to

$$\boxed{\frac{\mathrm{d}S_F(t,T)}{S_F(t,T)} = \sigma\mathrm{d}W_x^T(t) - \eta\bar{B}_r(t,T)\mathrm{d}W_r^T(t).}$$

- Process $S_F(t,T)$ does not contain any $\mathrm{d}t$-terms, so $S_F(t,T)$ is a martingale under the $T-$forward measure, $\mathbb{Q}^T$.

Pricing of European Call and Put Options

- We further simplify the SDE, recalling that for two correlated Brownian motions, $W_1(t)$ and $W_2(t)$, with correlation $\rho_{1,2}$ and positive constants a, b, the following equality holds in distributional sense,

$$aW_1(t) + bW_2(t) \stackrel{\mathrm{d}}{=} \sqrt{a^2 + b^2 + 2ab\rho_{1,2}}\,W_3(t),$$

  where $W_3(t)$ is also a Brownian motion.

- With this insight, the SDE can be rewritten as:

$$\boxed{\frac{\mathrm{d}S_F(t,T)}{S_F(t,T)} = \bar{\sigma}_F(t)\mathrm{d}W_F(t), \quad S_F(t_0,T) = \frac{S_0}{P(t_0,T)},}$$

  with $\bar{\sigma}_F(t) = \sqrt{\sigma^2 + \eta^2\bar{B}_r^2(t,T) - 2\rho_{x,r}\sigma\eta\bar{B}_r(t,T)}$.

Implied Volatility

- When generalizing the Black-Scholes model by means of stochastic interest rates, a natural question is how to calculate the implied volatilities within the Black-Scholes formula?
- In the standard procedure to determine the implied volatility, a constant interest rate is inserted in the formulas. Which r-value should be chosen when the interest rates are modeled by a stochastic process?
- The answers are related to the previous derivations. It was shown that the stock dynamics in the BSHW model can be written as,

$$\frac{\mathrm{d}S(t)}{S(t)} = r(t)\mathrm{d}t + \sigma \mathrm{d}W_x(t),$$

  with $r(t)$ governed by the Hull-White model.
- A measure transformation led to the dynamics that were free of drift terms for the forward stock $S_F(t,T) = S(t)/P(t,T)$.

$$\boxed{\frac{\mathrm{d}S_F(t,T)}{S_F(t,T)} = \bar{\sigma}_F(t)\mathrm{d}W_F(t).}$$

The BSHW Model

- The equality at final time T, $S(T) = S_F(T,T)$, implies that for the valuation of a contract with a fixed maturity time, the drift-free stochastic process $S_F(t,T)$ may be employed.
- For a payoff $H(T,S) = \max(S(T) - K, 0)$, we find

$$
\begin{aligned}
V(t_0, S) &= P(t_0,T)\mathbb{E}^T\left[\max(S_F(T,T) - K, 0) \middle| \mathcal{F}(t_0)\right] \\
&= S_{F,0}P(t_0,T)F_{\mathcal{N}}(d_1) - KP(t_0,T)F_{\mathcal{N}}(d_2),
\end{aligned}
$$

with

$$
d_1 = \frac{\log\left(\frac{S_{F,0}}{K}\right) + \frac{1}{2}\sigma_c^2(T-t_0)}{\sigma_c\sqrt{T-t_0}}, \quad d_2 = \frac{\log\left(\frac{S_{F,0}}{K}\right) - \frac{1}{2}\sigma_c^2(T-t_0)}{\sigma_c\sqrt{T-t_0}}.
$$

Furthermore, $S_{F,0} := \frac{S_0}{P(t_0,T)}$, and

$$
\boxed{\sigma_c^2 = \frac{1}{T-t_0}\int_{t_0}^{T}\bar{\sigma}_F^2(z)\mathrm{d}z.}
$$

- Parameter σ_c is often called the "effective parameter."

Implied Volatilities for the BSHW Model

- We analyze now possible shapes of the ATM implied volatilities for the BSHW Model.
- Clearly, a time-dependent volatility function is insufficient to generate implied volatility smiles, but it is sufficient to describe the implied volatility term structure, which can be observed in the interest rate market.
- The numerical experiment was performed with the following set of BSHW parameters $\sigma = 0.2$, $\lambda = 0.1$, $\eta = 0.01$ and $\rho_{x,r} = 0.3$. Each parameter is varied individually, keeping the remaining parameters fixed. All parameters have a significant effect on the model's implied ATM volatilities.

Implied Volatilities for the BSHW Model

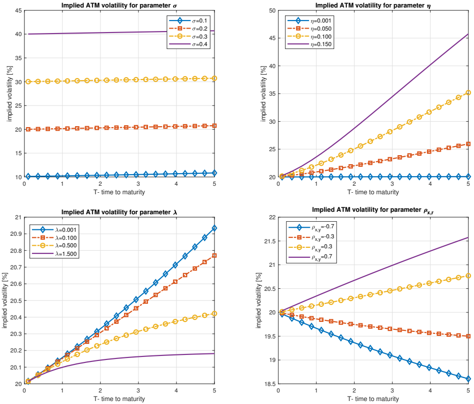


Figure: Implied volatility term structure in the Black-Scholes Hull-White model (equivalent to a Black-Scholes model with time-dependent volatility).

Schöbel-Zhu Hull-White Hybrid Model

- We present a first *stochastic volatility (SV) equity hybrid model*, which contains a *stochastic interest rate* process and a *full matrix of correlations* between the underlying Brownian motions. Also here particularly the Hull-White stochastic interest rate process is added to the SV model.
- For state vector $\mathbf{X}(t) = [S(t), r(t), \sigma(t)]^{\mathrm{T}}$, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and a filtration $\mathcal{F} = \{\mathcal{F}(t) : t \geq 0\}$, which satisfies the usual conditions.
- Under the risk-neutral measure $\mathbb{Q}$, consider a 3D system of stochastic differential equations, of the following form:

$$\boxed{\begin{aligned} \mathrm{d}S(t)/S(t) &= r(t)\mathrm{d}t + \sigma(t)\mathrm{d}W_x(t), \\ \mathrm{d}r(t) &= \lambda(\theta(t) - r(t))\mathrm{d}t + \eta\mathrm{d}W_r(t), \\ \mathrm{d}\sigma(t) &= \kappa\left(\bar{\sigma} - \sigma(t)\right)\mathrm{d}t + \gamma\mathrm{d}W_\sigma(t), \end{aligned}}$$

Schöbel-Zhu Hull-White Hybrid Model

- $W_k(t)$ with $k = \{x, r, \sigma\}$ are correlated Wiener processes, also governed by the instantaneous correlation matrix:

$$\mathbf{C} := \begin{bmatrix} 1 & \rho_{x,\sigma} & \rho_{x,r} \\ \rho_{\sigma,x} & 1 & \rho_{\sigma,r} \\ \rho_{r,x} & \rho_{r,\sigma} & 1 \end{bmatrix}.$$

- The plain Schöbel-Zhu model is a particular case of the original Heston model: for $\bar{\sigma} = 0$, the Schöbel-Zhu model equals the Heston model in which $\kappa^H = 2\kappa$, $\bar{\sigma}^H = \gamma^2/2\kappa$, and $\gamma^H = 2\gamma$. This relation gives a direct connection between their discounted characteristic functions.
- Finally, if we set $r(t)$ constant in the system plus zero correlations, the model collapses to the standard Black-Scholes model.

Affinity of the SZHW Model

- By extending the state vector with another, latent stochastic variable, defined by $v(t) := \sigma^2(t)$, and using $X(t) = \log S(t)$, we obtain the following 4D system of SDEs,

$$
\begin{cases}
\mathrm{d}X(t) = & \left(\widetilde{r}(t) + \psi(t) - \dfrac{1}{2}v(t)\right)\mathrm{d}t + & \sqrt{v(t)}\mathrm{d}W_x(t), \\
\mathrm{d}\widetilde{r}(t) = & -\lambda\widetilde{r}(t)\mathrm{d}t + & \eta\mathrm{d}W_r(t), \\
\mathrm{d}v(t) = & \left(-2v(t)\kappa + 2\kappa\bar{\sigma}\sigma(t) + \gamma^2\right)\mathrm{d}t + & 2\gamma\sqrt{v(t)}\mathrm{d}W_\sigma(t), \\
\mathrm{d}\sigma(t) = & \kappa(\bar{\sigma} - \sigma(t))\mathrm{d}t + & \gamma\mathrm{d}W_\sigma(t),
\end{cases}
$$

  where we used $r(t) = \widetilde{r}(t) + \psi(t)$ and where $\theta(t)$ is included in $\psi(t)$.
- After using the transformation: $v(t) := \sigma^2(t)$, and using $X(t) = \log S(t)$ the model becomes affine.

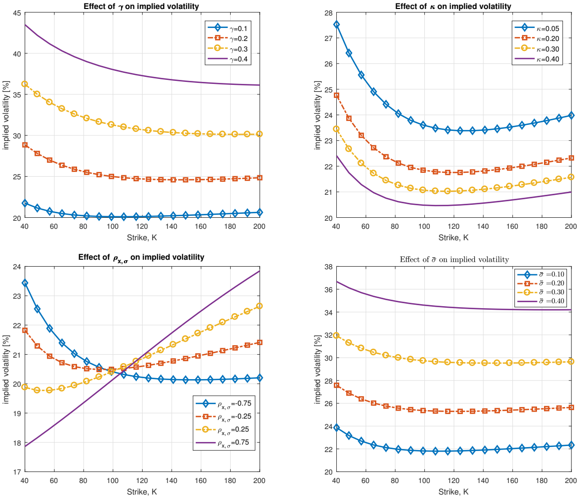
Schöbel-Zhu Hull-White Hybrid Model

- The inst. covariance matrix is given by:

$$\bar{\boldsymbol{\sigma}}(\mathbf{X}(t))\bar{\boldsymbol{\sigma}}(\mathbf{X}(t))^{\mathrm{T}} = \begin{bmatrix} v(t) & \rho_{x,r}\eta\sigma(t) & 2\rho_{x,\sigma}v(t) & \rho_{x,\sigma}\gamma\sigma(t) \\ * & \eta^2 & 2\rho_{r,\sigma}\gamma\eta\sigma(t) & \rho_{r,\sigma}\eta\gamma \\ * & * & 4\gamma^2 v(t) & 2\gamma^2\sigma(t) \\ * & * & * & \gamma^2 \end{bmatrix}_{(4\times 4)}.$$

- The model belongs to affine class of processes and we are able to determine its characteristic function.

Implied volatility for the SZHW model



Diversification product

- Hybrid products in strategic trading are so-called diversification products.
- A simple example is a portfolio with two assets: a stock with a high risk and high return and a bond with a low risk and low return.
- If one introduces an equity component in a pure bond portfolio the expected return will increase. However, because of a nonperfect correlation between these two assets also a risk reduction is expected.
- If the percentage of the equity in the portfolio is increased, it eventually starts to dominate and the risk may increase with a higher impact for a low or negative correlation.
- An example is a financial product, defined in the following way:

$$V^d(t_0, S, r) = \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T)}\max\left(0, \omega_d \cdot \frac{S(T)}{S_0} + (1-\omega_d)\cdot\frac{P(T,T_1)}{P(t_0,T_1)}\right)\Big|\mathcal{F}(t_0)\right],$$

where $S(T)$ is the underlying asset at time T, $M(t_0)=1$, $P(t,T)$ is the zero-coupon bond, ω_d represents a percentage ratio.

Diversification product

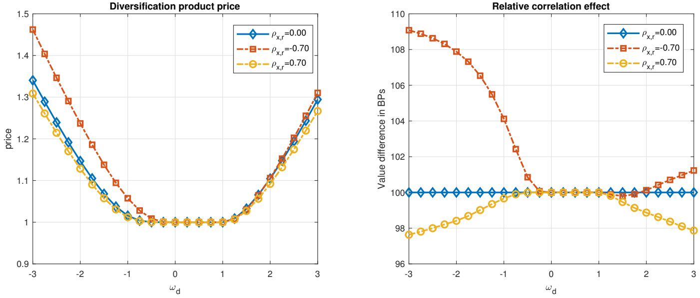


Figure: Left: Pricing of a diversification hybrid product, with $T = 9$ and $T_1 = 10$, under different correlations $\rho_{x,r}$. Right: Price differences with respect to the model with $\rho_{x,r} = 0\%$ expressed in basis points (BPs) for different correlations $\rho_{x,r}$.

python™

Heston Hull-White Hybrid Model

- We extend the Heston model state vector by a stochastic interest rate process, i.e., $\mathbf{X}(t) = [S(t), v(t), r(t)]^{\mathrm{T}}$. In particular, we add the Hull-White (HW) interest rate. The HHW model is presented in the following way, under the $\mathbb{Q}$-measure:

$$\begin{cases} \mathrm{d}S(t)/S(t) = r(t)\mathrm{d}t + \sqrt{v(t)}\mathrm{d}W_x(t), & S(0) > 0, \\ \mathrm{d}v(t) = \kappa(\bar{v} - v(t))\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v(t), & v(0) > 0, \\ \mathrm{d}r(t) = \lambda(\theta(t) - r(t))\mathrm{d}t + \eta\mathrm{d}W_r(t), & r(0) \in \mathbb{R}, \end{cases}$$

- For the HHW model the correlations are given by $\rho_{x,v}$, $\rho_{x,r}$, $\rho_{v,r}$.
- The model is not in the affine form, not even for $X(t) = \log S(t)$. The *symmetric* instantaneous covariance matrix is given by

$$\boxed{\bar{\boldsymbol{\sigma}}(\mathbf{X}(t))\bar{\boldsymbol{\sigma}}(\mathbf{X}(t))^{\mathrm{T}} = \begin{bmatrix} v(t) & \rho_{x,v}\gamma v(t) & \rho_{x,r}\eta\sqrt{v(t)} \\ * & \gamma^2 v(t) & \rho_{r,v}\gamma\eta\sqrt{v(t)} \\ * & * & \eta^2 \end{bmatrix}_{(3\times 3)}.}$$

Heston Hull-White Hybrid Model

- Since the full-scale HHW model is not affine, it is not possible to directly derive a characteristic function. We therefore *linearize* the Heston hybrid model, to provide an approximation.
- An approximation for the term $\eta\rho_{x,r}\sqrt{v(t)}$ in the inst. covariance matrix is found by *replacing it by its expectation*, i.e.,

$$\boxed{\eta\rho_{x,r}\sqrt{v(t)} \approx \eta\rho_{x,r}\mathbb{E}\left[\sqrt{v(t)}\right].}$$

- Expectation and variance for CIR-type process] For a given time $t > 0$, the expectation and variance of $\sqrt{v(t)}$, where $v(t)$ is a CIR-type process are given by

$$\mathbb{E}\left[\sqrt{v(t)}|\mathcal{F}(0)\right] = \sqrt{2\bar{c}(t,0)}\mathrm{e}^{-\bar{\kappa}(t,0)/2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\bar{\kappa}(t,0)}{2}\right)^k\frac{\Gamma\left(\frac{1+\delta}{2}+k\right)}{\Gamma(\frac{\delta}{2}+k)}.$$

Heston Hull-White Hybrid Model

- And the variance of $\sqrt{v(t)}$ is given by:

$$\begin{aligned}\mathbb{V}\text{ar}\left[\sqrt{v(t)}|\mathcal{F}(0)\right] &= \bar{c}(t,0)(\delta+\bar{\kappa}(t,0))\\ &- 2\bar{c}(t,0)\mathrm{e}^{-\bar{\kappa}(t,0)}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\bar{\kappa}(t,0)}{2}\right)^k\frac{\Gamma\left(\frac{1+\delta}{2}+k\right)}{\Gamma\left(\frac{\delta}{2}+k\right)}\right)^2,\end{aligned}$$

where

$$\bar{c}(t,0)=\frac{1}{4\kappa}\gamma^2(1-\mathrm{e}^{-\kappa t}),\quad \delta=\frac{4\kappa\bar{v}}{\gamma^2},\quad \bar{\kappa}(t,0)=\frac{4\kappa v(0)\mathrm{e}^{-\kappa t}}{\gamma^2(1-\mathrm{e}^{-\kappa t})},$$

with $\Gamma(k)$ being the Gamma function.

Delta Method for Approximating $\mathbb{E}[\sqrt{v(t)}]$

- In order to find a first-order approximation to $\mathbb{E}[\sqrt{v(t)}]$, the so-called *delta method* may be applied, which states that a function $g(X)$ can be approximated by a first-order Taylor expansion at $\mathbb{E}[X]$, for a given random variable X with expectation $\mathbb{E}[X]$ and variance $\mathbb{V}\mathrm{ar}[X]$.
- Assuming function g to be sufficiently smooth and the first two moments of X to exist, a first-order Taylor expansion gives,

$$g(X) \approx g(\mathbb{E}[X]) + (X - \mathbb{E}[X])\frac{\partial g}{\partial X}(\mathbb{E}[X]).$$

- Since the variance of $g(X)$ can be approximated by the variance of the right-hand side we have

$$\begin{aligned}\mathbb{V}\mathrm{ar}[g(X)] \quad &\approx \quad \mathbb{V}\mathrm{ar}\left[g(\mathbb{E}[X]) + (X - \mathbb{E}[X])\frac{\partial g}{\partial X}(\mathbb{E}[X])\right]\\ &= \quad \left(\frac{\partial g}{\partial X}(\mathbb{E}[X])\right)^2 \mathbb{V}\mathrm{ar}[X].\end{aligned}$$

Delta Method for Approximating $\mathbb{E}[\sqrt{v(t)}]$

- Using this result for function $g(v(t)) = \sqrt{v(t)}$, we find

$$\mathbb{V}\text{ar}\left[\sqrt{v(t)}\right] \approx \left(\frac{1}{2}\frac{1}{\sqrt{\mathbb{E}[v(t)]}}\right)^2 \mathbb{V}\text{ar}\left[v(t)\right] = \frac{1}{4}\frac{\mathbb{V}\text{ar}\left[v(t)\right]}{\mathbb{E}\left[v(t)\right]}.$$

However, from the definition of the variance, we also have

$$\mathbb{V}\text{ar}\left[\sqrt{v(t)}\right] = \mathbb{E}[v(t)] - \left(\mathbb{E}\left[\sqrt{v(t)}\right]\right)^2,$$

and combining Equations above gives the approximation:

$$\mathbb{E}\left[\sqrt{v(t)}\right] \approx \sqrt{\mathbb{E}[v(t)] - \frac{1}{4}\frac{\mathbb{V}\text{ar}[v(t)]}{\mathbb{E}[v(t)]}}.$$

Delta Method for Approximating $\mathbb{E}[\sqrt{v(t)}]$

- As $v(t)$ is a square-root process,

$$v(t) = v(0)\mathrm{e}^{-\kappa t} + \bar{v}(1-\mathrm{e}^{-\kappa t}) + \gamma\int_0^t \mathrm{e}^{\kappa(z-t)}\sqrt{v(z)}\mathrm{d}W_v(z).$$

  The expectation $\mathbb{E}\left[v(t)|\mathcal{F}(0)\right] = \bar{c}(t,0)(\delta+\bar{\kappa}(t,0))$ and the variance, $\mathbb{V}\mathrm{ar}[v(t)|\mathcal{F}(0)] = \bar{c}^2(t,0)(2\delta+4\bar{\kappa}(t,0))$.

- The expectation $\mathbb{E}\left[\sqrt{v(t)}\right]$ can be approximated by

$$\mathbb{E}\left[\sqrt{v(t)}|\mathcal{F}(0)\right] \approx \sqrt{\bar{c}(t,0)(\bar{\kappa}(t,0)-1)+\bar{c}(t,0)\delta+\frac{\bar{c}(t,0)\delta}{2(\delta+\bar{\kappa}(t,0))}}$$

  with $\bar{c}(t,0)$, δ, and $\bar{\kappa}(t,0)$ given earlier, and where κ, $\bar{v}$, γ and $v(0)$ are the model parameters.

- Once the expectation $\mathbb{E}\left[\sqrt{v(t)}\right]$ is known we are able to derive the corresponding ChF.

Monte Carlo for hybrid models

- We consider the Monte Carlo simulation of the following general hybrid system of SDEs:

$$
\begin{aligned}
\mathrm{d}S(t)/S(t) &= r(t)\mathrm{d}t + b_x(t, v(t))\mathrm{d}W_x(t),\\
\mathrm{d}v(t) &= a_v(t, v(t))\mathrm{d}t + b_v(t, v(t))\mathrm{d}W_v(t),\\
\mathrm{d}r(t) &= a_r(t, r(t))\mathrm{d}t + b_r(t, r(t))\mathrm{d}W_r(t),
\end{aligned}
$$

  with $S(t_0) > 0$, $v(t_0) > 0$, $r(t_0) \in \mathbb{R}$, $\mathrm{d}W_x(t)\mathrm{d}W_v(t) = \rho_{x,v}\mathrm{d}t$, $\mathrm{d}W_x(t)\mathrm{d}W_r(t) = \rho_{x,r}\mathrm{d}t$, $\mathrm{d}W_r(t)\mathrm{d}W_v(t) = 0$, and functions $b_x(\cdot)$, $a_v(\cdot)$, $b_v(\cdot)$, $a_r(\cdot)$ and $b_r(\cdot)$ satisfying the usual growth conditions.
- If we wish to make, next to Monte Carlo simulation, use of a characteristic function and thus Fourier techniques for pricing, the number of choices for the model parameters is limited.
- Staying with the class of affine models would require that squares of the volatility coefficients and all possible combinations, $b_i^2(t, v(t))$, $b_i(t, v(t))b_j(t, v(t))$, $i \neq j$, $i, j \in \{x, v, r\}$, are linear in the state variables.

Euler discretization

- We first define a time grid, $t_i = i\frac{T}{m}$, with $i = 0, \ldots, m$, and $\Delta t_i = t_{i+1} - t_i$. A very basic strategy to simulate the system is by means of an Euler discretization. This gives rise, using $X(t) = \log(S(t))$, to the following, discrete system:

$$
\begin{aligned}
x_{i+1} &= x_i + \left[r_i - \frac{1}{2}b_x^2(t_i, v_i)\right]\Delta t_i + b_x(t_i, v_i)\sqrt{\Delta t_i}Z_x,\\
v_{i+1} &= v_i + a_v(t_i, v_i)\Delta t_i + b_v(t_i, v_i)\sqrt{\Delta t_i}Z_v,\\
r_{i+1} &= r_i + a_r(t_i, r_i)\Delta t_i + b_r(t_i, r_i)\sqrt{\Delta t_i}Z_r,
\end{aligned}
$$

  with $Z_i \sim \mathcal{N}(0, 1)$, that are correlated as the Brownian motions.
- Although the discrete system is commonly used for pricing by Monte Carlo simulation, issues may be encountered when one of the processes is of CIR-type, like in the Heston model.
- The Euler disretization scheme may lead to negative values, making the Monte Carlo simulation results unrealistic.

Euler discretization

- A formulation in terms of the independent Brownian motions, based on the Cholesky decomposition of correlation matrix $\mathbf{C}$, as $\mathbf{C} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$, and

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & \rho_{x,r} \\ 0 & 1 & \rho_{x,v} \\ \rho_{x,r} & \rho_{x,v} & 1 \end{bmatrix}, \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \rho_{x,r} & \rho_{x,v} & \sqrt{1-\rho_{x,r}^2-\rho_{x,v}^2} \end{bmatrix},$$

reformulates, for $X(t) = \log S(t)$, the system, as:

$$\begin{aligned} \mathrm{d}r(t) &= a_r(t,r(t))\mathrm{d}t + b_r(t,r(t))\mathrm{d}\widetilde{W}_r(t), \\ \mathrm{d}v(t) &= a_v(t,v(t))\mathrm{d}t + b_v(t,v(t))\mathrm{d}\widetilde{W}_v(t), \end{aligned}$$

and the log-stock dynamics,

$$\begin{aligned} \mathrm{d}X(t) &= \left(r(t) - \frac{1}{2}b_x^2(t,v(t))\right)\mathrm{d}t + \rho_{x,r}b_x(t,v(t))\mathrm{d}\widetilde{W}_r(t) \\ &+ \rho_{x,v}b_x(t,v(t))\mathrm{d}\widetilde{W}_v(t) + \sqrt{1-\rho_{x,r}^2-\rho_{x,v}^2}\,b_x(t,v(t))\mathrm{d}\widetilde{W}_x(t). \end{aligned}$$

Euler discretization

- One may wonder whether the expression under the square root may become negative. In such a case, however, the correlation matrix is not positive definite, and is therefore not a valid correlation matrix.
- Integration over a time interval $[t_i, t_{i+1}]$, gives,

$$
\begin{aligned}
x_{i+1} &= x_i + \int_{t_i}^{t_{i+1}} \left(r(z) - \frac{1}{2} b_x^2(z, v(z)) \right) \mathrm{d}z \\
&+ \rho_{x,r} \int_{t_i}^{t_{i+1}} b_x(z, v(z)) \mathrm{d}\widetilde{W}_r(z) + \rho_{x,v} \int_{t_i}^{t_{i+1}} b_x(z, v(z)) \mathrm{d}\widetilde{W}_v(z) \\
&+ \sqrt{1 - \rho_{x,r}^2 - \rho_{x,v}^2} \int_{t_i}^{t_{i+1}} b_x(z, v(z)) \mathrm{d}\widetilde{W}_x(z).
\end{aligned}
$$

- The resulting representation for x_{i+1} forms the basis for a general Monte Carlo simulation procedure.

Monte Carlo for the HHW model

- By specific choices for the functions, well-known SDE systems can be recognized. The Heston-Hull-White model is defined with the following set:

$$
\begin{aligned}
b_x(t, v(t)) &= \sqrt{v(t)}, \\
a_v(t, v(t)) &= \kappa(\bar{v} - v(t)), \qquad b_v(t, v(t)) = \gamma\sqrt{v(t)}, \\
a_r(t, r(t)) &= \lambda(\theta(t) - r(t)), \qquad b_r(t, r(t)) = \eta,
\end{aligned}
$$

- The Monte Carlo technique for the standard Heston model presented can directly be employed for the discretization of x_{i+1}, i.e.,

$$
\begin{aligned}
x_{i+1} &= x_i + \int_{t_i}^{t_{i+1}} \left(r(z) - \frac{1}{2} v(z) \right) \mathrm{d}z + \rho_{x,r} \int_{t_i}^{t_{i+1}} \sqrt{v(z)} \mathrm{d}\widetilde{W}_r(z) \\
&+ \frac{\rho_{x,v}}{\gamma} \left(v_{i+1} - v_i - \kappa\bar{v}\Delta t_i + \kappa \int_{t_i}^{t_{i+1}} v(z)\mathrm{d}z \right) \\
&+ \sqrt{1 - \rho_{x,r}^2 - \rho_{x,v}^2} \int_{t_i}^{t_{i+1}} \sqrt{v(z)} \mathrm{d}\widetilde{W}_x(z).
\end{aligned}
$$

- More details on the Almost Exact Simulation of the Heston model can be found in Comp. Finance Course: Lecture 10.

Monte Carlo for the HHW model

- Collecting the different terms of the discretization for x_i gives:

$$\begin{aligned} x_{i+1} &= x_i + k_0 + k_1 \int_{t_i}^{t_{i+1}} v(z)\mathrm{d}z + \int_{t_i}^{t_{i+1}} r(z)\mathrm{d}z + k_2\big(v_{i+1} - v_i\big) \\ &+ \rho_{xr} \int_{t_i}^{t_{i+1}} \sqrt{v(z)}\mathrm{d}\widetilde{W}_r(z) + k_3 \int_{t_i}^{t_{i+1}} \sqrt{v(z)}\mathrm{d}\widetilde{W}_x(z), \end{aligned}$$

with

$$k_0 = -\frac{\rho_{x,v}}{\gamma}\kappa\bar{v}\Delta t_i, \quad k_1 = \kappa k_2 - \frac{1}{2}, \quad k_2 = \frac{\rho_{x,v}}{\gamma}, \quad k_3 = \sqrt{1 - \rho_{x,r}^2 - \rho_{x,v}^2}.$$

- The integrals in the discretization above are difficult to determine analytically. We therefore apply the Euler discretization scheme, resulting in the following approximation:

$$\begin{aligned} x_{i+1} &\approx x_i + k_0 + k_1 v_i \Delta t_i + r_i \Delta t_i + k_2\big(v_{i+1} - v_i\big) \\ &+ \rho_{x,r}\sqrt{v_i}\left(\widetilde{W}_r(t_i) - \widetilde{W}_r(t_i)\right) + k_3\sqrt{v_i}\left(\widetilde{W}_x(t_{i+1}) - \widetilde{W}_x(t_i)\right). \end{aligned}$$

Monte Carlo for the HHW model

- After simplifications we find:

$$\begin{aligned} x_{i+1} &\approx x_i + k_0 + k_1 v_i \Delta t_i + r_i \Delta t_i + k_2 \big(v_{i+1} - v_i \big) \\ &+ \rho_{xr} \sqrt{v_i \Delta t_i} \widetilde{Z}_r + k_3 \sqrt{v_i \Delta t_i} \widetilde{Z}_x, \end{aligned}$$

with

$$k_0 = -\frac{\rho_{x,v}}{\gamma} \kappa \bar{v} \Delta t_i, \quad k_1 = \kappa k_2 - \frac{1}{2}, \quad k_2 = \frac{\rho_{x,v}}{\gamma}, \quad k_3 = \sqrt{1 - \rho_{x,r}^2 - \rho_{x,v}^2}.$$

- For the dynamics of the interest rate process,

$$\mathrm{d}r(t) = \lambda(\theta(t) - r(t))\mathrm{d}t + \eta \mathrm{d}\widetilde{W}_r(t),$$

the Euler discretization gives rise to the following approximation:

$$r_{i+1} \approx r_i + \lambda \theta(t_i) \Delta t_i - \lambda r_i \Delta t_i + \eta \sqrt{\Delta t_i} \widetilde{Z}_r.$$

Monte Carlo for the HHW model

- Summarizing, the Monte Carlo scheme for the HHW model can be discretized as follows:

$$
\begin{aligned}
v_{i+1} &= \bar{c}(t_{i+1},t_i)\chi^2(\delta,\bar{\kappa}(t_{i+1},t_i)),\\
r_{i+1} &\approx r_i+\lambda\theta(t_i)\Delta t_i-\lambda r_i\Delta t_i+\eta\sqrt{\Delta t_i}\widetilde{Z}_r,\\
x_{i+1} &\approx x_i+k_0+(k_1\Delta t_i+k_2)\,v_i+r_i\Delta t_i+k_2 v_{i+1}+\sqrt{v_i\Delta t_i}\big(\rho_{x,r}\widetilde{Z}_r+k_3\widetilde{Z}_x\big),
\end{aligned}
$$

with,

$$
k_0=-\frac{\rho_{x,v}}{\gamma}\kappa\bar{v}\Delta t_i,\quad k_1=\kappa k_2-\frac{1}{2},\quad k_2=\frac{\rho_{x,v}}{\gamma},\quad k_3=\sqrt{1-\rho_{x,r}^2-\rho_{x,v}^2},
$$

and

$$
\begin{aligned}
\bar{c}(t_{i+1},t_i) &= \frac{\gamma^2}{4\kappa}\big(1-\mathrm{e}^{-\kappa(t_{i+1}-t_i)}\big),\quad \delta=\frac{4\kappa\bar{v}}{\gamma^2},\\
\bar{\kappa}(t_{i+1},t_i) &= \frac{4\kappa\mathrm{e}^{-\kappa(t_{i+1}-t_i)}}{\gamma^2\big(1-\mathrm{e}^{-\kappa(t_{i+1}-t_i)}\big)}v_i,
\end{aligned}
$$

with $\widetilde{Z}_r,\widetilde{Z}_x$ independent $\mathcal{N}(0,1)$ random variables; $\chi^2(\delta,\bar{\kappa}(t_{i+1},t_i))$ a random variable with noncentral chi-squared distribution with δ degrees of freedom and noncentrality parameter $\bar{\kappa}(t_{i+1},t_i)$.

Monte Carlo for the HHW model

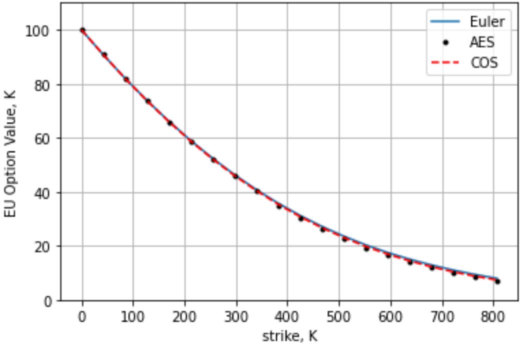


Figure: The Heston-Hull-White model: COS method vs. Monte Carlo.

Summary

- Hybrid Models for xVA and VaR
- The Black-Scholes Hull-White Model
- Implied Volatility for Models with Stochastic Interest Rates
- Stochastic Vol Models with Stochastic Interest Rates
- Example of a Hybrid Payoff: Diversification Product
- The Heston Hull-White Hybrid Model
- Monte Carlo for Hybrid Models
- Monte Carlo for the Heston-Hull-White Model
- Summary of the Lecture + Homework

Homework Exercises

- **Exercise 1 (Exercise 13.8 from the book)**
- The dynamics for the stock, $S(t)$, in the so-called Heston-CIR model read:

$$\begin{cases} \mathrm{d}S(t)/S(t) = & r(t)\mathrm{d}t + & \sqrt{v(t)}\mathrm{d}W_x^{\mathbb{Q}}(t), & S(0) > 0 \\ \mathrm{d}v(t) = & \kappa(\bar{v} - v(t))\mathrm{d}t + & \gamma\sqrt{v(t)}\mathrm{d}W_v^{\mathbb{Q}}(t), & v(0) > 0, \\ \mathrm{d}r(t) = & \lambda(\theta(t) - r(t))\mathrm{d}t + & \eta\sqrt{r(t)}\mathrm{d}W_r^{\mathbb{Q}}(t), & r(0) > 0, \end{cases}$$

with $\mathrm{d}W_x^{\mathbb{Q}}(t)\mathrm{d}W_v^{\mathbb{Q}}(t) = \rho_{x,v}\mathrm{d}t$, $\mathrm{d}W_x^{\mathbb{Q}}(t)\mathrm{d}W_r^{\mathbb{Q}}(t) = \rho_{x,r}\mathrm{d}t$ and $\mathrm{d}W_v^{\mathbb{Q}}(t)\mathrm{d}W_r^{\mathbb{Q}}(t) = 0$.
Assume that the nonaffine term in the pricing PDE, $\mathbf{\Sigma}_{(1,3)}$, can be approximated, as

$$\mathbf{\Sigma}_{(1,3)} \approx \eta\rho_{x,r}\mathbb{E}\left[\sqrt{r(t)}\sqrt{v(t)}\right] \overset{indep}{=} \eta\rho_{x,r}\mathbb{E}\left[\sqrt{r(t)}\right] \cdot \mathbb{E}\left[\sqrt{v(t)}\right].$$

a. Determine the characteristic function for the log-stock, $X(t) = \log S(t)$, and the corresponding Riccati ODEs
b. Find the solution for these Riccati ODEs.

Homework Exercises

- **Exercise 2 (Exercise 13.10 from the book)**
- In this exercise we focus on the SZHW model.
  a. Develop a Monte Carlo Euler simulation for the SZHW model.
  b. Check the convergence of the Monte Carlo simulation by decreasing the time step Δt and by increasing the number of Monte Carlo paths.
  c. Derive the characteristic function of the SZHW model.