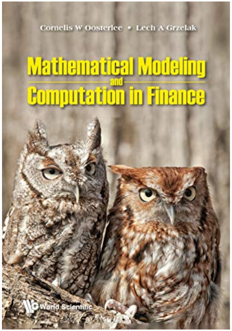
Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.



- Youtube Channel with courses can be found here.
- Slides and the codes can be found here.

List of content

Course road map

Value at Risk- Main Principle

- Value at risk (VaR) is a market risk measure (it measures potential loss associated with market fluctuations).
- VaR TRIES to provide a single number that would answer the following question "How much I can loose at most on this investment in a given period of time?"
- From the regulatory perspective, central banks require that every bank keeps enough capital to cover potential losses estimated based on VaR methodology.
- Calculation of VaR results in s a quantile associated with the potential loss,

$$VaR_\alpha(X) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}.$$

- If the 95% one-day VaR is 1 million, there is 95% confidence that over the next day the portfolio won't lose more than 1 million.

Value at Risk

- Depending on the portfolio composition, market data used and quantile level (confidence/significance level, α), VaR measures the overall risks associated with market movements.
- The VaR approach is attractive because it is easy to understand.
- It provides an estimate of the amount of capital needed to support a certain level of risk.
- Another advantage of this measure is the ability, to some extent, to incorporate the effects of portfolio diversification.
- The model parameters are standardized for capital requirement purposes and require banks to use a one-sided confidence interval of 99%.
- Regulators often require a holding period of 10 days and at least one year of historical data for the market risk factors.
- Although the model parameters are standardized, banks do not have to a particular approach to estimate VaR (missing data, interpolations etc.). In other words, banks may choose their approach towards VaR.

Value at Risk- Categories

There are four main categories in calculation of VaR:

- **Parametric VaR**: Under the assumption of normally distributed returns the estimation of the portfolio values are computed. It allows only for linear portfolios (strong assumptions regarding normality of returns!)
- **Monte-Carlo Simulation:** similarly to computations of exposures for xVA, a stochastic model is calibrated to historical data and a distribution of portfolio value is estimated.
- **Historical VaR, HVaR**: Historical data (running window) is used to asses the distribution of the portfolio.
- **Stressed Var, SVaR:** A stressed period of historical data is used to estimate the distribution of potential portfolio losses.

Coherent Risk Measures

- When quantifying risks associated with capital and potential losses there are number of principles that a good risk measure has:
- **Sub-additivity**:

$$\rho(X+Y) \leq \rho(X) + \rho(Y),$$

  which means that the overall risk of the portfolio does not exceed the sum of all individual risks. Subadditivity ensures that the portfolio diversification principle holds since a subadditive measure would always generate a lower risk measure for a diversified portfolio than a non–diversified one.
- **Monotonicity**:

$$\text{if } X \leq Y, \rho(X) \geq \rho(Y),$$

  which implies that if a value of asset X is less or equal the value of asset Y, then the risk of X should be greater than the risk of Y. In other words, the risk of good assets should be less of inferior assets.

Coherent Risk Measures

- **Positive homogenity**:

$$\rho(aX) = a\rho(X),$$

meaning that unit of measurement does not affect the risk measure.

- **Transaction invariance**:

$$\rho(X + a) = \rho(X) - a,$$

implying that if the cash amount a is added to the asset X, it offsets the corresponding risk associated with X.

- Value-at-Risk does NOT satisfy **sub-additivity** requirement.
- Violations of subadditivity can cause several problems for financial institutions. For example, a financial institution is employing a VaR measure without realizing it violates subadditivity, e.g., using VaR to rank investment choices or impose limits on traders. In this case, the financial institution is likely to assume too much risk or not hedge when needed.

Expected Shortfall

- In response to the lack of coherence for the VaR risk measure, several alternatives have been proposed. Of these, the most common is expected shortfall.
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- Expected shortfal is defined based on the results from the VaR calculation, i.e.,

$$ES_\alpha(X) = \mathbb{E}\left[X | X < VaR_\alpha(X)\right],$$

with

$$VaR_\alpha(X) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}.$$

Simulation algorithm

- First, we should find the risk factors that effect the return of a portfolio. We utilize the following equation to express the relationship between value of asset and risk factors:

$$V(t_0, \mathbf{X}), \quad \mathbf{X}(t) = [X_1(t), X_2(t), \dots, X_n(t)]^{\mathrm{T}},$$

  where each $X_i(t)$ represents a risk factor that affects present value of our portfolio.
- The objective is NOT to evaluate the portfolio with historical data, but to evaluate current portfolio while taking account possible, historical, market movements.
- Market movements can be 1 day, 10 days, or even longer depending on the risk management (and regulator).
- The VaR evaluation rely on the so-called market scenarios defined simply as increments of the risk factors in time:

$$\Delta\mathbf{X}(s) = \mathbf{X}(s) - \mathbf{X}(s - \Delta t),$$

  where s is the time.

Simulation algorithm

- Then, we are able to estimate the PnL profile for a portfolio, assuming that historical data, movements, are good predictor for the future, i.e.:

$$\begin{aligned} P\&L(t) &= V(t_0, \mathbf{X}(t_0)) - V(t_0, \mathbf{X}(t_0) + \Delta\mathbf{X}(s)), \\ \Delta\mathbf{X}(s) &= \mathbf{X}(s) - \mathbf{X}(s - \Delta t), \end{aligned}$$

- Note that in a portfolio consistent of simple derivatives like spot products, e.g., stocks. The VaR calculation is very straightforward as we only need to "adjust" the spot values of assets with proper scenarios.
- However, in the case of interest rate products, the situation becomes much more complicated as every change in interest rate product would require re-build of market objects, like Yield Curves. -This case we will discuss in the numerical experiment.

Python Experiment

- Let us now perform a Var Experiment with 160 historical Yield Curves from `https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yieldYear&year=2021`
- Details on the YC construction are in Lecture 6.

Resource Center

Home » Resource Center » Data and Charts Center » Interest Rate Statistics » TextView

Daily Treasury Yield Curve Rates

Get updates to this content.

XML These data are also available in XML format by clicking on the XML icon.

XSD The schema for the XML is available in XSD format by clicking on the XSD icon.

If you are having trouble viewing the above XML in your browser, click here.

To access interest rate data in the legacy XML format and the corresponding XSD schema, click here.

Select type of Interest Rate Data

Daily Treasury Yield Curve Rates Go

Select Time Period

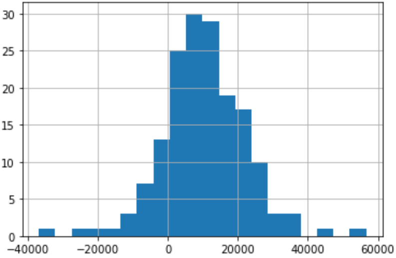
2021 Go

Date	1 Mo	2 Mo	3 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
01/04/21	0.09	0.09	0.09	0.09	0.10	0.11	0.16	0.36	0.64	0.93	1.46	1.66
01/05/21	0.08	0.09	0.09	0.09	0.10	0.13	0.17	0.38	0.66	0.96	1.49	1.70
01/06/21	0.09	0.09	0.09	0.09	0.11	0.14	0.20	0.43	0.74	1.04	1.60	1.81
01/07/21	0.09	0.09	0.09	0.09	0.11	0.14	0.22	0.46	0.78	1.08	1.64	1.85
01/08/21	0.08	0.08	0.08	0.09	0.10	0.14	0.24	0.49	0.81	1.13	1.67	1.87
01/11/21	0.09	0.08	0.08	0.10	0.10	0.14	0.22	0.50	0.84	1.15	1.68	1.88
01/12/21	0.09	0.08	0.09	0.09	0.11	0.14	0.23	0.50	0.83	1.15	1.68	1.88
01/13/21	0.09	0.08	0.09	0.10	0.12	0.14	0.22	0.48	0.80	1.10	1.63	1.82
01/14/21	0.09	0.09	0.09	0.09	0.10	0.16	0.23	0.49	0.82	1.15	1.69	1.88
01/15/21	0.08	0.09	0.09	0.10	0.10	0.13	0.20	0.46	0.78	1.11	1.66	1.85
01/19/21	0.07	0.09	0.09	0.11	0.10	0.14	0.21	0.45	0.78	1.10	1.65	1.84
01/20/21	0.08	0.08	0.08	0.10	0.10	0.13	0.19	0.45	0.78	1.10	1.65	1.84
01/21/21	0.07	0.09	0.09	0.09	0.10	0.13	0.19	0.45	0.79	1.12	1.68	1.87

Python Experiment

- Details on the YC construction are in Lecture 6.

```
Current Portfolio PV is  10750.800765106782
(H)VaR for alpha =  0.05  is equal to= -7756.312549662641
P&L which < VaR_alpha = [-19802.37722068 -14158.57896468  -8273.91500205 -36874.13684742
 -25216.56363424 -11263.84225496  -7905.6896605  -11227.49644044
 -11291.53900846]
Expected shortfal =  -16223.793225937394
```



Re-gridding of the YC

- In order to perform historical VaR computation we need to have a consistent set of market instruments over the whole time-horizon of VaR calculations.
- In practice however, it often happens that the market instruments change, i.e., a new instruments may be introduced or removed. E.g. a new $5Y$ swap instrument is introduced and quotes are available. How to fill in the gap in historical data?
- On such occasions the missing data can be “implied” from existing instruments.
- Often, risk management, has a fixed set of market instruments for interest rates, like for example $1Y, 2Y, 5Y, 10Y, 30Y$. This means that all the yield curves need to be “projected” on these set of instruments.
- The instruments of interest can be simply implied from existing instruments, or additional criteria, like sensitivity can be taken into account.

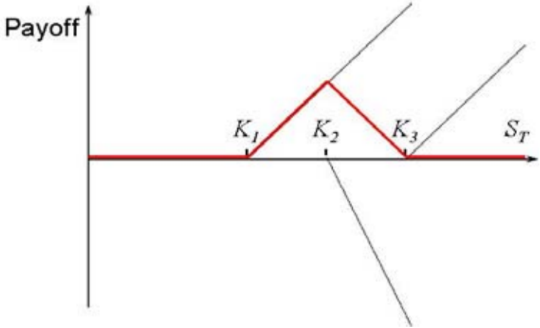
Potential issues with volatilities

- Another aspect that needs to be taken into account is associated with volatilities.
- Although handling of ATM volatilities seems straightforward it may happen that the shock size that is large enough implies negative volatility.
- Additionally, when computing historical shocks and applying them to implied volatility surface, arbitrage may be generated.

Types of arbitrage in the volatility objects

We distinguish two types of arbitrage in the volatility objects

- Calendar arbitrage: $C(T_1, K) > C(T_2, K)$, for $T_1 < T_2$ and where C is a call option and K is a strike.
- Butterfly arbitrage $C(T, K_1) - 2C(T, K_2) + C(T, K_3) < 0$ for $K_1 < K_2 < K_3$.



Butterfly Arbitrage

- Without loss of generality we can assume that $K_3 - K_2 = K_2 - K_1 =: \delta_K$ thus since $\delta_K > 0$ we have:

$$\frac{C(K+\delta_K) - 2C(K) + C(K-\delta_K)}{\delta_K^2} \approx \boxed{\frac{\partial^2 C(K)}{\partial K^2}} \quad \text{for} \quad \delta_K \to 0.$$

- A call price is given by:

$$C(K) = \int_{\mathbb{R}} \max(x-K,0) f_S(x)\mathrm{d}x,$$

so by differentiation we find the following relation:

$$\boxed{\frac{\partial^2 C(K)}{\partial K^2} = f_S(K)}.$$

- So the presence of the butterfly arbitrage is equivalent with assigning negative probabilities to stock's movements.
- The elimination of the butterfly arbitrage is equivalent with ensuring that probability density is nonnegative and it integrates to unit.

Monte Carlo simulation for VaR

- Similarly, as for the Historical VaR calculations, we VaR based on Monte Carlo relies on simulated scenarios. However, the scenarios will be now generated from a stochastic model.
- The stochastic processes be either defined under the real-world measure $\mathbb{P}$ or under the risk neutral measure $\mathbb{Q}$, depending whether the models will be calibrated to historical data or market instruments like options.
- The evaluation algorithm is as follows:
  - Define a state-vector $\mathbf{S}(t) = [X_1(t), \ldots, X_n(t)]^{\mathrm{T}}$ for every risk factor X_i, and the corresponding system of SDEs:

$$\mathrm{d}\mathbf{S}(t) = \mu(\mathbf{S}(t))\mathrm{d}t + \sigma(\mathbf{S}(t))\mathrm{d}\mathbf{W}(t),$$

  - Define the *holding period*, Δt.
  - Define a change in risk-factor, $\Delta S(t)$, over period Δt.
  - Price your portfolio for the initial state $V(S(t_0))$ and *new* state, $V(S(t_0) + \Delta S(t))$.
  - *P&L* is then defined as a difference in portfolio evolution over Δt: $V(S(t_0) + \Delta S(t)) - V(S(t_0))$.

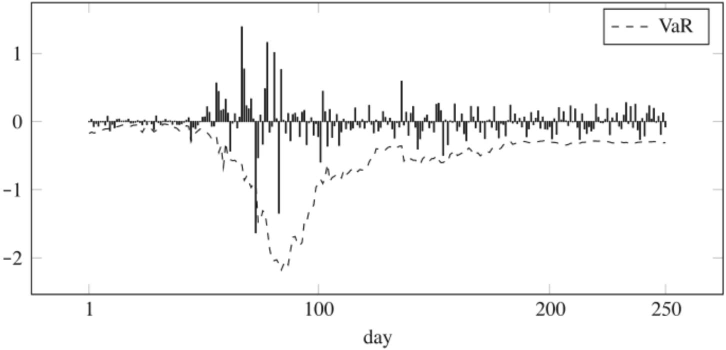
Monte Carlo simulation for VaR

- Revaluing the portfolio in each scenario is considered to be a computationally expensive task.
- Each revaluation may be very time-consuming, especially for a large portfolio consisting of complex derivative securities.
- Often, individual instruments require the execution of numerical pricing routines or even separate, "nested", Monte Carlo pricing.
- The time required to revalue a portfolio is the limiting factor in determining the number of scenarios that can be generated.

Backtesting

- Backtesting is a process where we measure the predicting power of VaR.
- The main idea is to count the number of events when *realized* P&L breaches the VaR limit predicted a day before. The measurement takes place over a certain period of time, usually one year or 250 business days (depending on the regulator).
- Clearly, backtesting isn't able to predict volatile events as it is backward looking statistical measure.
- It can be, to some extend, improved if VaR computation take place based on Monte Carlo simulation calibrated to *forward looking* market instruments like volatilities.
- As an example we can consider an asset like FX, $y(t)$, or stock, $S(t)$. In the VaR calculations only historical data will be used, however if we choose stochastic processes for these assets (and calibrate to option market), VaR measure may be more aligned with market expectations of possible fluctuations.

VaR and Backtesting



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[1] *"Hands-on-Value-at-Risk and Expected Shortfall" by M. Auer, Spinger, 2018.*

Summary

- Value at Risk (VaR), Stressed VaR (SVaR)
- Coherent Risk Measures
- Expected Shortfall
- Historical VaR (HVar) and Python Experiment
- Missing Data, Arbitrage and Re-Gridding
- VaR Computation with Monte Carlo
- Backtesting
- Summary of the Lecture + Homework

Homework Exercises

The solutions for the homework can be find at
https://github.com/LechGrzelak/QuantFinanceBook

- **Exercise**
- Extend the portfolio used in the Historical Value-At-Risk calculations so that additional risk factor will be taken into account, e.g., stock or FX.
- Download historical data for the chosen risk factors and perform VaR and ES computations.
- Think how can you reduce the variance of VaR by adding some additional, diversifying derivatives.