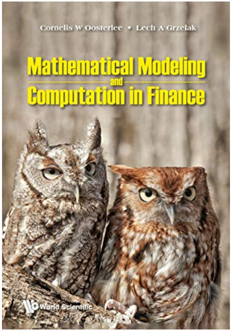
# Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.




- Youtube Channel with courses can be found here.
- Slides and the codes can be found here.

# List of content

# The solution of the Hull-White SDE

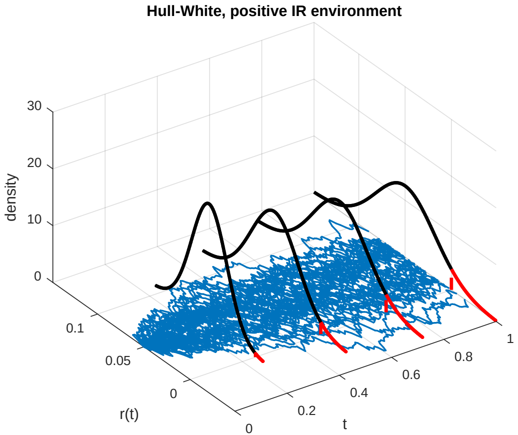



Figure: Monte Carlo paths for the Hull-White model.

# The solution of the Hull-White SDE

- To obtain the solution to the HW SDE, we apply Itô's lemma to a process $y(t) := \mathrm{e}^{\lambda t} r(t)$, i.e.

$$\mathrm{d}y(t) = \lambda y(t)\mathrm{d}t + \mathrm{e}^{\lambda t}\mathrm{d}r(t).$$

- After substitution of the dynamics of the HW process, we find:

$$\mathrm{d}y(t) = \lambda y(t)\mathrm{d}t + \mathrm{e}^{\lambda t}\left[\lambda\left(\theta(t) - r(t)\right)\mathrm{d}t + \eta\mathrm{d}W_r(t)\right],$$

  and arrive at the following system of equations:

$$\begin{cases} y(t) = \mathrm{e}^{\lambda t} r(t), \\ \mathrm{d}y(t) = \lambda\theta(t)\mathrm{e}^{\lambda t}\mathrm{d}t + \eta\mathrm{e}^{\lambda t}\mathrm{d}W_r(t). \end{cases}$$

# The solution of the Hull-White SDE

- The right-hand side of the $y(t)$-dynamics does not depend on state vector $y(t)$, and we can therefore determine the solution of $y(t)$ by integrating both sides of the SDE,

$$\int_0^t \mathrm{d}y(z) \quad = \quad \lambda \int_0^t \theta(z)\mathrm{e}^{\lambda z}\mathrm{d}z + \eta \int_0^t \mathrm{e}^{\lambda z}\mathrm{d}W_r(z),$$

which gives us,

$$y(t) \quad = \quad y(0) + \lambda \int_0^t \theta(z)\mathrm{e}^{\lambda z}\mathrm{d}z + \eta \int_0^t \mathrm{e}^{\lambda z}\mathrm{d}W_r(z).$$

By the definition of $y(t)$, with $y(0) = r_0$, the solution for process $r(t)$ reads:

$$\boxed{r(t) = \mathrm{e}^{-\lambda t} r_0 + \lambda \int_0^t \theta(z)\mathrm{e}^{-\lambda(t-z)}\mathrm{d}z + \eta \int_0^t \mathrm{e}^{-\lambda(t-z)}\mathrm{d}W_r(z).}$$

# The solution of the Hull-White SDE

- Interest rate $r(t)$ is thus *normally distributed* with

$$\mathbb{E}\left[r(t)|\mathcal{F}(t_0)\right] = r_0\mathrm{e}^{-\lambda t} + \lambda\int_0^t \theta(z)\mathrm{e}^{-\lambda(t-z)}\mathrm{d}z,$$

using $t_0 = 0$, and

$$\mathbb{V}\mathrm{ar}\left[r(t)|\mathcal{F}(t_0)\right] = \frac{\eta^2}{2\lambda}\left(1-\mathrm{e}^{-2\lambda t}\right).$$

- Moreover, for $\theta(t)$ constant, i.e., $\theta(t)\equiv\theta$ (in which case we deal with the *Vašiček model*), we have

$$\lim_{t\to\infty}\mathbb{E}\left[r(t)|\mathcal{F}(t_0)\right] = \theta.$$

This means that the first moment of the process converges to the mean reverting level $\theta$, for large values of $t$.

# Affinity of the Hull-White Model

- The Hull-White process belongs to the class of affine diffusion processes (See Computational Finance Course- Lecture 6).

$$\mathrm{d}r(t) = \lambda(\theta(t) - r(t))dt + \sigma \mathrm{d}W^{\mathbb{Q}}(t),$$

- Therefore we are able to determine its Discounted Characteristic function: of the spot interest rate $r(t)$ in the Hull-White model,

$$\phi_{\mathrm{HW}}(u; t, T) \quad := \quad \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(z)\mathrm{d}z + iur(T)} \middle| \mathcal{F}(t)\right],$$

- The Hull-White short-rate process can be decomposed as

$$\boxed{r(t) = \widetilde{r}(t) + \psi(t),}$$

where

$$\psi(t) = r_0 \mathrm{e}^{-\lambda t} + \lambda \int_0^t \theta(z) \mathrm{e}^{-\lambda(t-z)} \mathrm{d}z,$$

and

$$\boxed{\mathrm{d}\widetilde{r}(t) = -\lambda \widetilde{r}(t)\mathrm{d}t + \eta \mathrm{d}W_r(t), \text{with } \widetilde{r}_0 = 0.}$$

# Affinity of the Hull-White Model

- By the decomposition of $r(t)$, can be expressed as:

$$\begin{aligned}\phi_{\text{HW}}(u;t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{iu\psi(T)-\int_t^T\psi(z)\mathrm{d}z}\cdot\mathrm{e}^{iu\widetilde{r}(T)-\int_t^T\widetilde{r}(z)\mathrm{d}z}\Big|\mathcal{F}(t)\right]\\ &= \mathrm{e}^{iu\psi(T)-\int_t^T\psi(z)\mathrm{d}z}\cdot\phi_{\widetilde{\text{HW}}}(u;t,T),\end{aligned}$$

- Hence, the discounted characteristic function, which is written as $\phi_{\widetilde{\text{HW}}}(u;t,T)$, with $u\in\mathbb{C}$, for the affine short-rate model is of the following form:

$$\phi_{\widetilde{\text{HW}}}(u;t,T) = \exp\left(\bar{A}(u,\tau)+\bar{B}(u,\tau)\widetilde{r}(t)\right).$$

- Functions $\bar{A}(u,\tau)$ and $\bar{B}(u,\tau)$ for $\tau=T-t$, are known in the closed form.

# Affinity of the Hull-White Model

- Let us take another look at the ChF of the HW model:

$$\phi_{\text{HW}}(u;t,T) \quad := \quad \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(z)\mathrm{d}z + iur(T)} \middle| \mathcal{F}(t)\right].$$

- By setting $u=0$ we have a closed-form solution for the ZCB for the HW model.

$$P(t,T) = \phi_{\text{HW}}(0;t,T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(z)\mathrm{d}z} \middle| \mathcal{F}(t)\right].$$

- In general we can write:

$$\boxed{P(t,T) = \mathrm{e}^{A(0,\tau)+B(0,\tau)r(t)}, \quad \tau = T-t.}$$

- If a short rate process for interest rate does not allow for affine form, then when pricing of derivatives in e.g. xVA one needs to simulate $P(T_i,T_j)$ for $T_i > t_0$ which may require nested Monte Carlo simulation to compute conditional expectation:

$$P(T_i,T_j) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_{T_i}^{T_j} r(z)\mathrm{d}z} \middle| \mathcal{F}(T_i)\right].$$

# Affinity of the Hull-White Model

- Because the model belongs to the affine class of processes we can represent the ZCB as:

$$\boxed{P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(z)\mathrm{d}z}\big|\mathcal{F}(t)\right] = \mathrm{e}^{A(0,\tau)+B(0,\tau)r(t)}.}$$

- The essence in the formulation above lies in the fact, that it is no longer necessary to perform the conditional integration

$$\int_t^T r(z)\big|r(t).$$

- The evolution of the conditional integration is contained in functions $A(0,\tau)$ and $B(0,\tau)$, with $\tau = T - t$.
- The computation benefits of this representation are huge- especially in the context of the xVA or other framework that requires fast computations of linear interest rate products for different market scenarios.

# Affinity of the Hull-White Model

- The complete formulation of the ZCB under the Hull-White mode reads
$$P(t,T)=\mathrm{e}^{A(0,\tau)+B(0,\tau)r(t)},$$
- where functions $A(0,\tau)$ and $B(0,\tau)$ for $\tau=T-t$ are given by:

$$\begin{aligned} A(0,\tau) &= -\frac{\eta^2}{4\lambda^3}\left(3+\mathrm{e}^{-2\lambda\tau}-4\mathrm{e}^{-\lambda\tau}-2\lambda\tau\right)+\lambda\int_0^\tau\theta(T-z)B(0,\tau)\mathrm{d}z,\\ B(0,\tau) &= -\frac{1}{\lambda}\left(1-\mathrm{e}^{-\lambda(T-t)}\right). \end{aligned}$$

- Ultimately, the formulation above can be further simplified and the integration over $\theta(t)$ can be avoided. Then the ZCB will explicitly depend on ZCBs from the market.
- The formulation above is crucial in later part of this course where will discuss xVA.

python™

# A Yield curve

- A yield curve is one of the most important elements needed to determine present value of future cash flows. For a given yield curve one is able to provide the forward rates of other interest rate derivatives, like swaps.
- The main concept of the yield curve is to map the market quotes of liquid instruments to some, one, unified curve which would represents the expectation of the future rates.
- A yield curve is constructed from a discrete set of instruments, $\pi$, which are mapped to a discrete set of input nodes of the curve, $\Omega_{yc}$.
- The yield curve is often viewed as a leading indicator, providing an early warning on the likely direction of a country's economy – for example, the yield curve has historically become inverted 12-18 months before a recession.
- Yield curves reflect not only interest rate expectations, but investors' attitude to risk and their need for different maturities of bond.

# A Yield Curve

- A yield curve can be represented by a set of the nodes:

$$\Omega_{yc} = \{(t_1, df(t_1)), (t_2, df(t_2)), \ldots, (t_n, df(t_N))\}, \tag{1}$$

where the discount factor $df(t_i)$ is defined as:

$$df(t_i) := P(t_0, t_i) = \mathbb{E}^{\mathbb{Q}}\left[1 \times \mathrm{e}^{-\int_{t_0}^{t_i} r(s)\mathrm{d}s} \Big| \mathcal{F}(t_0)\right].$$

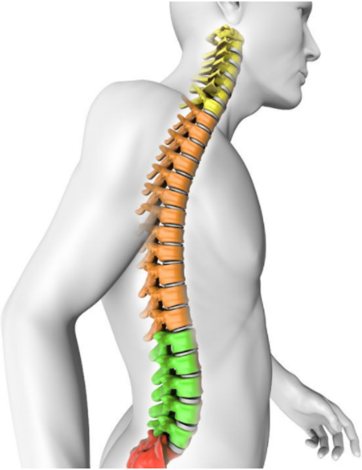
- Since discount factors, contrary to the short rate, $r(t)$, are deterministic they can be expressed using simply compounded rate $r_i$: $df(t_i) = \mathrm{e}^{-r_i t_i}$.
- Mathematically, a yield curve is a function which maps a discrete set of zero rates into real numbers,

$$f_{yc} : \Omega_{yc} \to \mathbb{R}.$$

- Typically the set of the points $\Omega_{yc}$ in (1) is called the set of *spine points* of the yield curve. In essence the spine points are directly implied from the calibration instruments. The points in between the spine points can be calculated using some interpolation routine.

# Yield Implied from the Hull-White Model

- Yield curve spine points together with an interpolation function define the yield curve.



# Yield Implied from the Hull-White Model

- Yield is simply computed from

$$P(0,T)=\mathrm{e}^{-rT}\quad\rightarrow\quad r=-\frac{\log P(0,T)}{T}.$$

- When computing yield based on the simulated the Hull-White model it is a mistake to associate

$$\int_0^T r(s)\mathrm{d}s,$$

with the yield at time $T$.
- In order to compute yield based on the Hull-White (or any other model) we need firstly to compute the associated ZCBs, i.e.:

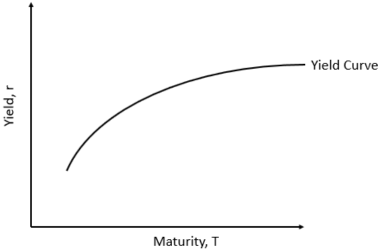
$$P(t,T)=\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}\Big|\mathcal{F}(t_0)\right]=\mathrm{e}^{A(t,T)+B(t,T)r(t_0)}=:\mathrm{e}^{-r(T-t)}.$$

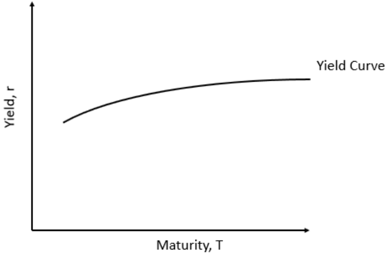
- Thus, the continuously compounded rate $r$ yields:

$$r=-\frac{1}{T-t}(A(t,T)+B(t,T)r(t)).$$

# A Shape of a Yield Curve- Normal

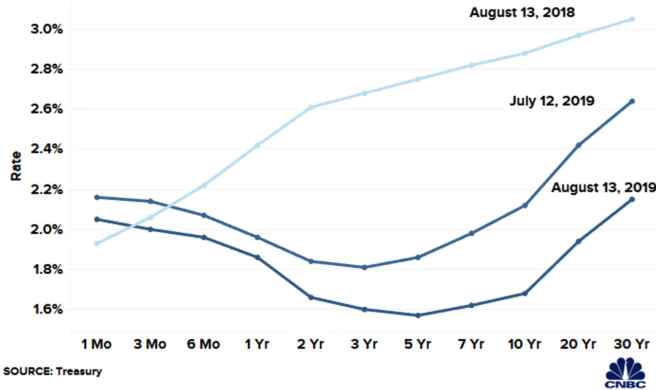
- There are many different shapes of a yield curve possible:
  - Normal: the market expects the economy to function at an average growth rate: No significant inflation or available capital changes.
  - Flat: the market is at the point of inflection, preceding either a recession or an economic pick-up.



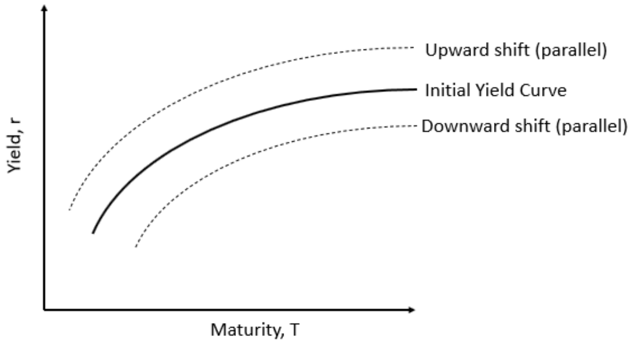
# A Shape of a Yield Curve- Inverted

- Inverted: the market expects the economy to slow down and interest rates to drop in the future. Long term investors want to take the opportunity to lock in interest rates before they fall even further.



# A Shape of a Yield Curve- Shift

- Parallel shift of a yield curve: it happens when bonds with different maturity rates experience the same change in interest rate at the same time. Often this kind of shift is performed in risk analysis.
- For investors who buy bonds and hold them to maturity, parallel shifts in the yield curve aren't meaningful in a practical sense, as they will have no effect on the ultimate cash flow.



# Dynamics of a Yield Curve

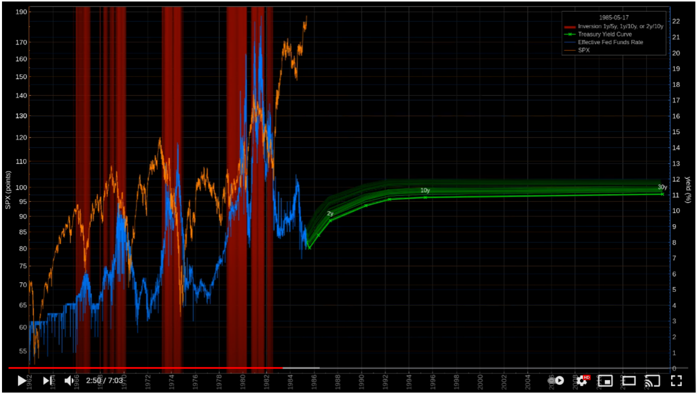



Figure: Animation of the US Treasury Yield Curve with Inversions from 1962-01-01 to 2019-04-01 (Timo Bingmann)

- YouTube video: `https://www.youtube.com/watch?v=GY6tJykKQjE`

# Yield Curve Dynamics for the HW Model

- Let us now perform an experiment in which we will observe the dynamics of the continuously compounded rates $r(t, T)$.
- The objective here is to simulate short rate process $r(t)$ and compute continuously compounded rate $r(t, T)$ on each stochastic realization $r_i$.
- Note that the process $r(t)$ depends on function $\theta(t)$ which is implied from today's yield curve.
- The approach described above is often used in risk computations, where for each simulated market scenario a yield curve is generated.
- In the experiment we choose maturity $T = 10y$, a set of model parameters for the Hull-White model and observe rate $r(10y, T)$ for $T \in [10y, 50y]$.

# Yield Curve & parallel shifts under the HW model

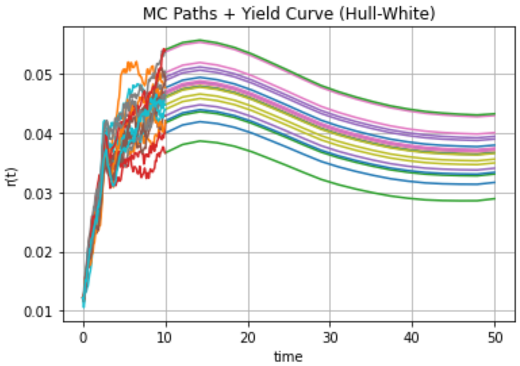



Figure: Dynamics of the yield curve for random market scenarios under the 1F Hull-White model.

# Limitations of the HW Model

- Although the Hull-White model is a very elegant model that allows fitting the entire yield curve but it does not allow fitting the entire forward curve (as the initial condition).
- The model has only 2 parameters to fit the whole volatility structure which is impossible.
- The model has only 1 stochastic factor for describing the whole multi-dimensional dynamics of the multi-curve.
- Under Affine 1 factor models ZCBs at a given point are perfectly correlated. Let us consider two ZCBs for maturity $T_1$ and $T_2$, $T_1 \neq T_2$:

$$P(t,T_1) = \mathbb{E}\left[\mathrm{e}^{-\int_t^{T_1} r(s)\mathrm{d}s}\Big|\mathcal{F}(t)\right], \quad P(t,T_2) = \mathbb{E}\left[\mathrm{e}^{-\int_t^{T_2} r(s)\mathrm{d}s}\Big|\mathcal{F}(t)\right].$$

- We see that both models share the same paths in the interval $[t, T_1]$, however, because of the affine form we have:

$$P(t,T_1) = \mathrm{e}^{A(t,T_1)+B(t,T_1)r(t)}, \quad P(t,T_2) = \mathrm{e}^{A(t,T_2)+B(t,T_2)r(t)}.$$

# Hull-White Model and Correlation between Forwards

- Now, if we look at the continuously compounded rates, we have:

$$r(t,T_1) = -\frac{1}{T_1-t}\left(A(t,T_1)+B(t,T_1)r(t)\right),$$
$$r(t,T_2) = -\frac{1}{T_2-t}\left(A(t,T_2)+B(t,T_2)r(t)\right),$$

  therefore

$$\mathrm{corr}(r(t,T_1), r(t,T_2)) = 1.$$

- This result implied that all the ZCBs are fully correlated at a given time $t$. This is an unrealistic assumption as the yield curve is dynamic and all the instruments are correlated, but not with correlation 1.

# Gaussian 2F Model

- Under the risk-free measure $\mathbb{Q}$ the 2D Hull-White model is defined as:

$$\begin{aligned} \mathrm{d}r(t) &= (\theta(t) + u(t) - ar(t))\,\mathrm{d}t + \sigma_1 \mathrm{d}W_1(t), \\ \mathrm{d}u(t) &= -bu(t)\mathrm{d}t + \sigma_2 \mathrm{d}W_2(t), \end{aligned}$$

with correlation $\mathrm{d}W_1(t)\mathrm{d}W_2(t) = \bar{\rho}\mathrm{d}t$.

- It is possible, to re-formulate the model and present it in affine form:

$$\begin{aligned} r(t) &= x(t) + y(t) + \varphi(t), \\ \mathrm{d}x(t) &= -ax(t)\mathrm{d}t + \sigma_1 \mathrm{d}W_1(t), \quad x(0) = 0, \\ \mathrm{d}y(t) &= -bx(t)\mathrm{d}t + \sigma_2 \mathrm{d}W_2(t), \quad y(0) = 0, \end{aligned}$$

with correlation $\mathrm{d}W_1(t)\mathrm{d}W_2(t) = \rho\mathrm{d}t$,

- where

$$\varphi(t) = r_0 \mathrm{e}^{-at} + \int_0^t \theta(s)\mathrm{e}^{-a(t-s)}\mathrm{d}s.$$

- Note that converting between these two representations the parameters change.

# Gaussian 2F Model

- The corresponding zero-coupon bond price is known analytically and it reads:

$$
\begin{aligned}
P(t,T) &= \frac{P_M(0,T)}{P_M(0,t)}\exp\left[\frac{1}{2}\left(V(t,T)-V(0,T)+V(0,t)\right)+\psi(t,T)\right],\\
\psi(t,T) &= -\frac{1}{a}\left(1-\mathrm{e}^{-a(T-t)}\right)\boxed{x(t)}-\frac{1}{b}\left(1-\mathrm{e}^{-b(T-t)}\right)\boxed{y(t)}.
\end{aligned}
$$

- with variance $V(t,T)$ given by:

$$
\begin{aligned}
V(t,T) &= \frac{\sigma^2}{a^2}\left(T-t+\frac{2}{a}\mathrm{e}^{-a(T-t)}-\frac{1}{2a}\mathrm{e}^{-2a(T-t)}-\frac{3}{2a}\right)\\
&+ \frac{\eta^2}{b^2}\left(T-t+\frac{2}{b}\mathrm{e}^{-b(T-t)}-\frac{1}{2b}\mathrm{e}^{-2b(T-t)}-\frac{3}{2b}\right)\\
&+ 2\rho\frac{\sigma\eta}{ab}\left(T-t+\frac{1}{a}\left(\mathrm{e}^{-a(T-t)}-1\right)+\frac{1}{b}\left(\mathrm{e}^{-b(T-t)}-1\right)\right)\\
&- \frac{2\rho\sigma\eta}{ab(a+b)}\left(\mathrm{e}^{-(a+b)(T-t)}-1\right).
\end{aligned}
$$

# Gaussian 2F Model

- The Gaussian 2F model is defined as a sum of two mean-reverting processes and a term structure function $\varphi(t)$

$$r(t) \quad = \quad x(t) + y(t) + \varphi(t),$$

- where $\varphi(t)$ is given by:

$$\begin{aligned}\varphi(t) \quad &= \quad f(0,t) + \frac{\sigma^2}{2a^2}\left(1-\mathrm{e}^{-at}\right)^2 + \frac{\eta^2}{2b^2}\left(1-\mathrm{e}^{-bt}\right)^2 \\ &+ \quad \rho\frac{\sigma\eta}{ab}\left(1-\mathrm{e}^{-at}\right)\left(1-\mathrm{e}^{-bt}\right),\end{aligned}$$

- with the instantaneous forward rate $f(0,t)$ defined as:

$$f(0,t) = -\frac{\partial \log P_M(0,t)}{\partial t},$$

and where $P_M(0,t)$ indicates a zero-coupon bond computed from the market data.

# Gaussian 2F Model

- The Gaussian 2F model is defined as a sum of two mean-reverting processes and a term structure function $\varphi(t)$

$$r(t) \quad = \quad x(t) + y(t) + \varphi(t),$$

- A natural interpretation of these variables is that $x(t)$controls the levels of the rates, while $y(t)$ controls the steepness of the forward curve.
- The correlation coefficient $\rho$ is typically large and negative, $\rho < -90\%$, indicating that steepening curve moves tend to correlate negatively with parallel move of a forward curve.

# Gaussian 2F Model and Correlation Between Forwards

- Similarly as for the Hull-White model we can compute the continuously compounded interest rate $r(t,T_1)$ and $r(t,T_2)$.
- Since the model also belongs to the affine class of processes we have:

$$P(t,T_1)=\mathrm{e}^{A(t,T_1)+B(t,T_1)x(t)+C(t,T_1)y(t)},\quad P(t,T_2)=\mathrm{e}^{A(t,T_2)+B(t,T_2)x(t)+C(t,T_2)y(t)}.$$

- Now, if we look at the continuously compounded rates, we have:

$$r(t,T_1)=-\frac{1}{T_1-t}\left(A(t,T_1)+B(t,T_1)x(t)+C(t,T_1)y(t)\right),$$
$$r(t,T_2)=-\frac{1}{T_2-t}\left(A(t,T_2)+B(t,T_2)x(t)+C(t,T_2)y(t)\right),$$

therefore

$$\begin{aligned}\mathrm{corr}(r(t,T_1),r(t,T_2)) &= \mathrm{corr}(b_1x(t)+c_1y(t),b_2x(t)+c_2y(t))\\ &\neq 1.\end{aligned}$$

# Gaussian 2F Model- Experiment

- As under the Hull-White model (1F), let us now perform an experiment in which we will observe the dynamics of the continuously compounded rates $r(t, T)$.
- The objective here is to simulate short rate process $r(t)$ and compute continuously compounded rate $r(t, T)$ on each stochastic realization $r_i$.
- Since $x(t)$ controls the levels of the rates, while $y(t)$ controls the steepness of the forward curve in this experiment we will focus on the impact of the second factor on the dynamics of the compounded rate, $r(t, T)$.

# Gaussian 2F Model

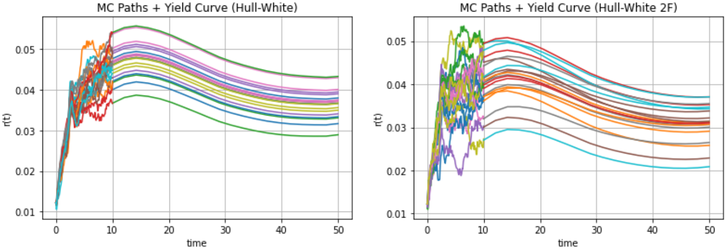



Figure: Dynamics of the yield curve for random market scenarios under the 1F Hull-White model.

# Summary

- Exact Solution for the HW Model
- Affinity of the Hull-White Model
- Brief Introduction to Yield Curves
- Limitations of the 1Factor Model and Yield Curve Dynamics
- Gaussian 2F Model
- Summary of the Lecture + Homework

# Homework Exercises

The solutions for the homework can be find at
https://github.com/LechGrzelak/QuantFinanceBook

- **Exercise 11.1**
  Show that for any stochastic process $r(t)$ with the dynamics given by:

$$\mathrm{d}r(t) = \lambda(\hat{\theta}(t, T_2) - r(t))\mathrm{d}t + \eta \mathrm{d}W_r^{T_2}(t), \quad r(t_0) = r_0,$$

  and

$$\hat{\theta}(t, T_2) = \theta(t) + \frac{\eta^2}{\lambda}\bar{B}_r(T_2 - t),$$

  for any time $t$, the mean and variance are given by,

$$\begin{aligned}\mathbb{E}^{T_2}\left[r(t)|\mathcal{F}(t_0)\right] &= r_0\mathrm{e}^{-\lambda t} + \lambda\int_0^t \hat{\theta}(z, T_2)\mathrm{e}^{-\lambda(t-z)}\mathrm{d}z,\\ \mathbb{V}\mathrm{ar}^{T_2}\left[r(t)|\mathcal{F}(t_0)\right] &= \frac{\eta^2}{2\lambda}\left(1 - \mathrm{e}^{-2\lambda t}\right).\end{aligned}$$

# Homework Exercises

- Show that the moment generating function (the Laplace transform) is of the following, closed form:

$$\begin{aligned}\mathbb{E}^{T_2}\left[\mathrm{e}^{ur(t)}\middle|\mathcal{F}(t_0)\right] &= \exp\left[u\left(r_0\mathrm{e}^{-\lambda t}+\lambda\int_0^t\hat{\theta}(z,T_2)\mathrm{e}^{-\lambda(t-z)}\mathrm{d}z\right)\right.\\ &+ \left.\frac{1}{2}u^2\frac{\eta^2}{2\lambda}\left(1-\mathrm{e}^{-2\lambda t}\right)\right].\end{aligned}$$