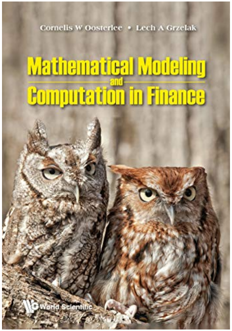
Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.



- Youtube Channel with courses can be found here.
- Slides and the codes can be found here.

List of content

A yield curve

- A yield curve is one of the most important elements needed to determine present value of future cash flows. For a given yield curve one is able to provide the forward rates of other interest rate derivatives, like swaps.
- The main concept of the yield curve is to map the market quotes of liquid instruments to some, one, unified curve which would represents the expectation of the future rates.
- A yield curve is constructed from a discrete set of instruments, π, which are mapped to a discrete set of input nodes of the curve, Ω_{yc}.
- Typically, the set π will contain cash instruments, swaps, forward-rate agreements (FRAs), Futures, and possibly, depending on market, many others interest rate instruments and Ω_{yc} will consist of discount factors.

A Yield Curve

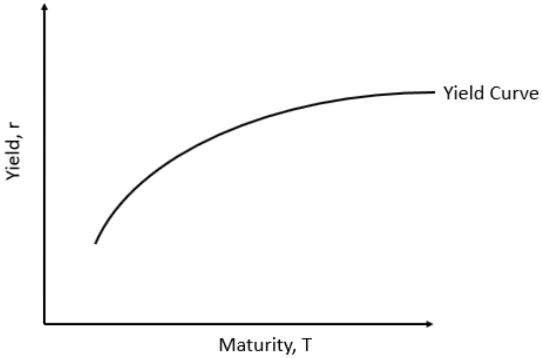
- The yield curve is often viewed as an insightful indicator, providing an early warning on the likely direction of a country's economy. Usually, the yield curve has historically become inverted 1y - 1.5y years before a recession.
- The slope and shape of the curve reflect investors expectations about future interest rates. It is often considered as a self-fulfilling prophecy.
- It's essential to remember limitations on what can be gleaned from looking at the yield curve.
- Yield curves reflect interest rate expectations and investors' attitudes to risk and their need for different maturities of a bond.

A Yield Curve

- The relationship between yields with different maturities is called the term structure of interest rates.
- The yield curve on US Treasury bond instruments is used to serve as a benchmark for pricing bonds and to set yields in other sectors of the debt market.
- The government bonds, like US Treasury bonds, are viewed as default-free, and they have the highest liquidity. Situation may vary depending on a country and amount of debt denominated in foreign currencies.
- Risk-Premium: A spread between yields (corporate vs. government), reflects the additional risks that investor faces when acquiring a security.

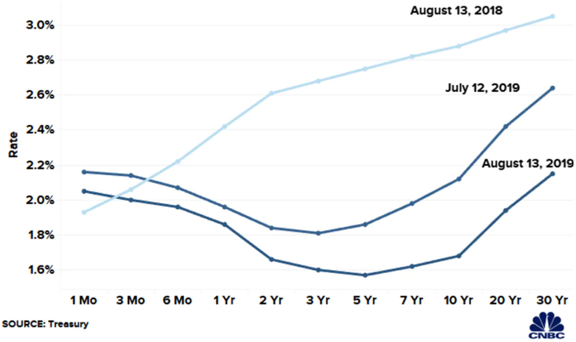
A Shape of a Yield Curve- Normal

- Normal - The market expects the economy to function at an average growth rate: No significant inflation or available capital changes. Investors who risk their money for more extended periods expect higher yields in return. (e.g. Dec 1984 – middle of a longest postwar expansion in the US; GDP growth rates at a steady 2-5%.)



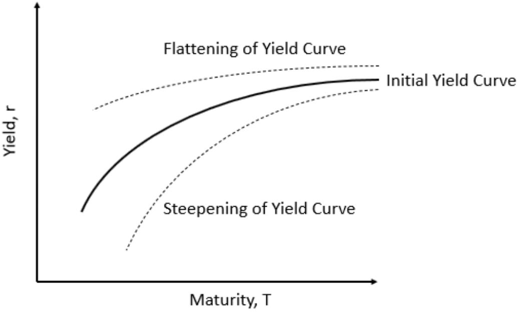
A Shape of a Yield Curve- Inverted

- Inverted- The market expects the economy to slow down and interest rates to drop in the future. Long term investors want to take the opportunity to lock in interest rates before they fall even further.
  - Recession in the early 1980s. Has become permanent in the UK due to excess demand from pension funds.
  - Repo crisis in October 2019 (pre Covid event).



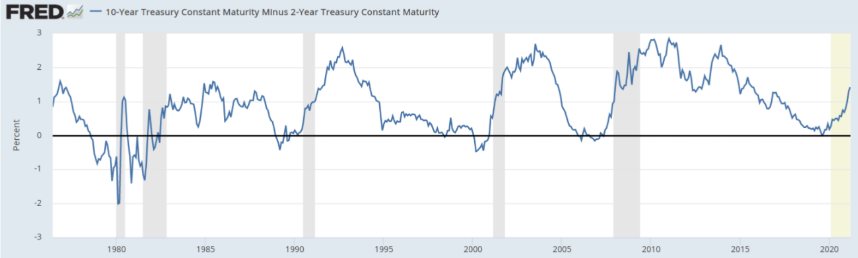
A Shape of a Yield Curve- Steep

- Steep - Long-term bond holders expect the economy to improve quickly in the future. Long-term investors fear being locked into low interest rates so therefore demand greater compensation more quickly than the more liquid short-term rate holders.



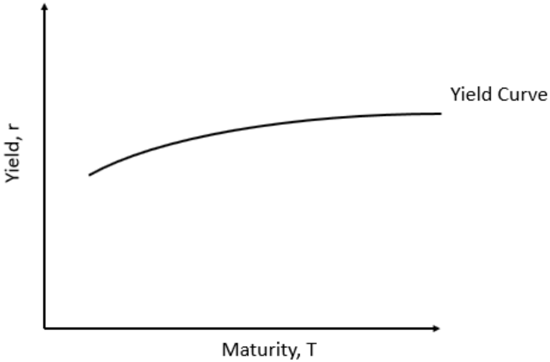
A Shape of a Yield Curve- Steep

- Steepening of the yield curve can be measured as a difference between Constant Maturity Swaps with different maturity, e.g., $10y - 2y$.



A Shape of a Yield Curve- Flat

- Flat - The market is at the point of inflection, preceding either a recession or an economic pick-up. (e.g. Nov 1989 – the curve flattened. The economy was in recession by 1991.)



Yield curve control (YCC)

- The dynamics of the Yield Curve is not solely driven by the market investors, often central banks play an active role in steering short and long term rates.
- Central banks steer the economy by raising or lowering very short-term interest rates, such as the rate that banks earn on their overnight deposits.
- Under yield curve control (YCC), central banks can target some longer-term rates and pledge to buy enough long-term bonds to keep the rate from rising above its target. To some extend that may have positive impact on the mortgages, but it is damaging for pensions and banks.
- Effectiveness of the central banks can be significantly limited in the periods of increasing inflationary pressures.

A yield curve

- A yield curve can be represented by a set of the nodes:

$$\Omega_{yc} = \{(t_1, df(t_1)), (t_2, df(t_2)), \ldots, (t_n, df(t_N))\},$$

where the discount factor $df(t_i)$ is defined as:

$$df(t_i) := P(t_0, t_i) = \mathbb{E}^{\mathbb{Q}}\left[1 \times \mathrm{e}^{-\int_{t_0}^{t_i} r(s)\mathrm{d}s} \middle| \mathcal{F}(t_0)\right].$$

- Since discount factors, contrary to the short rate, $r(t)$, are deterministic they can be expressed using simply compounded rate r_i: $df(t_i) = \mathrm{e}^{-r_i t_i}$.
- Mathematically, a yield curve is a function which maps a discrete set of zero rates into real numbers,

$$f_{yc} : \Omega_{yc} \to \mathbb{R}.$$

Typically the set of the points Ω_{yc} is called the set of *spine points* of the yield curve. In essence the spine points are directly implied from the calibration instruments. The points in between the spine points can be calculated using some interpolation routine.

A yield curve

- A combination of the spine points and the interpolation scheme used to construct a yield curve is crucial not only for pricing but also for hedging purposes and risk exposures, e.g. once a price of an interest rate product is determined the sensitivities of this product to the instruments used to construct the yield curve would be used for a proper hedge and risk management.
- The main criteria to use in judging a curve construction and its interpolation are:
  - The yield curve should be able to price back the instruments which are used to construct it.
  - The forward rates implied from the curve should be continuous.
  - The interpolation used to construct a curve should be as local as possible, i.e.: a small change in the node of the curve should not affect the values which are “far away”.
  - The hedge should be local too.

A yield curve

- Every calibration instrument which is of plain vanilla type (does not involve volatility) is a function of spine points, i.e.: the i'th instrument is defined as: $V_i(t_0) := V_i(t_0, \Omega_{yc})$, on the other hand the product is assumed to be quoted in the market with the quote rate indicated by q_i.
- In the calibration process we search for the set of spine discount factors

  $$\mathbf{df} := [df_1, df_2, \ldots, df_N]^{\mathrm{T}} := [df(t_1), df(t_2), \ldots, df(t_N)]^{\mathrm{T}},$$

  for which

  $$pv_i := V_i(t_0, \mathbf{df}) - q_i = 0.$$

  where pv_i stands for the present value of a derivative minus the market quote.
- Thus, in essence we search for a vector of $df's$ for which:

  $$||pv||_L = 0.$$

Solving the inverse pricing model function

How to find implied volatility?

The BS pricing function BS does not have a closed-form solution for its inverse $g_\sigma(\cdot)$. Instead, a root finding technique is used to solve the equation:

$$BS(\sigma_{impl}, r, T, K, S_0) - V_C^{mkt} = 0.$$

There are many ways to solve this equation, one of the most popular method are methods of "Newton" and "Brent"[1]. Since the options prices can move very quickly, it is often important to use the most efficient method when calculating implied volatilities.

[1] http://en.wikipedia.org/wiki/Brent's_method

Method of Newton

- One starts with an initial guess which is reasonably close to the true root.
- The function is then approximated by its tangent line, and one computes the x-intercept of this tangent line.
  Suppose $g : [a, b] \rightarrow \mathbb{R}$ is a differentiable function.
- From basic calculus we have:

$$g'(x_n) = \frac{g(x_n) - 0}{x_n - x_{n+1}} = \frac{0 - g(x_n)}{x_{n+1} - x_n},\; n = 0, 1, \ldots$$

which gives us the iteration:

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)},\; n = 0, 1, \ldots$$

with some arbitrary initial value x_0. In the case of BS we have:

$$\sigma_{n+1} = \sigma_n - \frac{BS(\sigma_n, \cdot) - V_C^{mkt}}{\frac{\partial BS(\sigma_n, \cdot)}{\partial \sigma_n}},\; n = 0, 1, \ldots.$$

Implied Volatility Shapes

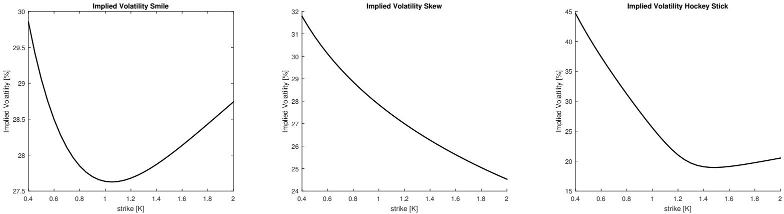


Figure: Typical implied volatility shapes: a smile, a skew and the so-called hockey stick. The hockey stick can be seen as a combination of the implied volatility smile and the skew.

A yield curve

- Similar problem we have seen earlier, where we have considered the problem of finding implied volatilities[2].
- Now the situation is a bit more complicated as typically the yield curve is constructed from multiple instruments at the same time, thus the calibration problem becomes multidimensional.
- We take $\mathbf{pv} = [pv_1, pv_2, \dots, pv_N]^{\mathrm{T}}$ and and to determine the optimal spine points we solve $\mathbf{pv} = 0$ with the standard Newton-Raphson technique:

$$\begin{aligned}\mathbf{pv}(\mathbf{df} + \mathbf{\Delta df}) &= \mathbf{pv}(\mathbf{df}) + \frac{\partial \mathbf{pv}(\mathbf{df})}{\partial \mathbf{df}} \mathbf{\Delta df} + O(\mathbf{\Delta df}^2) \\ &=: \mathbf{pv}(\mathbf{df}) + \mathbf{J}(\mathbf{df}) \mathbf{\Delta df} + O(\mathbf{\Delta df}^2),\end{aligned}$$

where Δdf, is determined on the gradient descent.

[2] More Details in Lecture 4 of Computational Finance Course

A yield curve

- Which in matrix notation is equivalent with:

$$\begin{bmatrix} pv_1(\mathbf{df}+\mathbf{\Delta df}) \\ pv_2(\mathbf{df}+\mathbf{\Delta df}) \\ \vdots \\ pv_N(\mathbf{df}+\mathbf{\Delta df}) \end{bmatrix} = \begin{bmatrix} pv_1(\mathbf{df}) \\ pv_2(\mathbf{df}) \\ \vdots \\ pv_N(\mathbf{df}) \end{bmatrix} + \begin{bmatrix} \frac{\partial pv_1}{\partial df_1} & \frac{\partial pv_1}{\partial df_2} & \dots & \frac{\partial pv_1}{\partial df_N} \\ \frac{\partial pv_2}{\partial df_1} & \frac{\partial pv_2}{\partial df_2} & \dots & \frac{\partial pv_2}{\partial df_N} \\ \vdots & \ddots & \vdots & \\ \frac{\partial pv_N}{\partial df_1} & \frac{\partial pv_N}{\partial df_2} & \dots & \frac{\partial pv_N}{\partial df_N} \end{bmatrix} \begin{bmatrix} \Delta df_1 \\ \Delta df_2 \\ \vdots \\ \Delta df_N \end{bmatrix} + O(\mathbf{\Delta df}^2).$$

A yield curve

- So clearly for each individual pv_i we have:

$$pv_i(\mathbf{df} + \mathbf{\Delta df}) = pv_i(\mathbf{df}) + \sum_{j=1}^{N} \frac{\partial pv_i}{\partial df_j} \Delta df_j + O(\mathbf{\Delta df}^2).$$

- Now, we ignore the higher order terms and let for every i, $pv_i(\mathbf{df} + \mathbf{\Delta df}) = 0$ (i.e.: $\mathbf{df} + \mathbf{\Delta df}$ is the zero-crossing of the tangent), and get:

$$0 = pv_i(\mathbf{df}) + \sum_{j=1}^{N} \frac{\partial pv_i}{\partial df_j} \Delta df_j + O(\mathbf{\Delta df}^2).$$

- Solving this linear equation system for $\mathbf{\Delta df}$, we get:

$$\mathbf{\Delta df} = -\mathbf{J}^{-1}(\mathbf{df})\mathbf{pv}(\mathbf{df}).$$

- Finally, if we introduce a superscript (n) indicating the iteration we have:

$$\boxed{\mathbf{df}^{(n+1)} = \mathbf{df}^{(n)} - \mathbf{J}^{-1}(\mathbf{df}^{(n)})\mathbf{pv}(\mathbf{df}^{n}).}$$

A yield curve: how many spine points?

- Let us consider the case where we wish to build a yield curve based on two swaps: 2y and 5y.
- Assuming frequency of 1y for both swaps we encounter under-defined system of equations, i.e.,
- We take every payment of a swap as a spine point $\mathbf{df} = [df_1, df_2, df_3, df_4, df_5]^{\mathrm{T}}$:

$$\begin{bmatrix} pv_{2y} \\ pv_{5y} \end{bmatrix} = \begin{bmatrix} \psi(df_{1y}, df_{2y}) \\ \psi(df_{1y}, df_{2y}, df_{3y}, df_{4y}, df_{5y}) \end{bmatrix},$$

and

$$\mathbf{J} := \begin{bmatrix} \frac{\partial pv_{2y}}{\partial df_{1y}} & \frac{\partial pv_{2y}}{\partial df_{2y}} & \frac{\partial pv_{2y}}{\partial df_{3y}} & \frac{\partial pv_{2y}}{\partial df_{4y}} & \frac{\partial pv_{2y}}{\partial df_{5y}} \\ \frac{\partial pv_{5y}}{\partial df_{1y}} & \frac{\partial pv_{5y}}{\partial df_{2y}} & \frac{\partial pv_{5y}}{\partial df_{3y}} & \frac{\partial pv_{5y}}{\partial df_{4y}} & \frac{\partial pv_{5y}}{\partial df_{5y}} \end{bmatrix}.$$

- Since we have more unknowns than prices (equations) the system is under determined. This could be fixed by adding more market instruments, but it is consider impractical as we wish to consider only liquid market instruments.
- As a solution of this problem, we always consider only the last payments as the spine points and rely on interpolation.

A yield curve construction- example

- We consider now the construction of the yield curve given two instruments: 1) a Forward Rate Agreement (FRA), 2) and a swap agreement.
- It is a rather trivial curve as it only involves two instruments. In practice the number of the instruments for the curve construction varies from a few to tens depending on the market and its liquidity.
- The present value of a FRA contract is given by:

$$V^{FRA}(t_0) = P(t_0, T_s)\left(\frac{\tau(L(t_0, T_f, T_m) - K_1)}{1 + \tau L(t_0, T_f, T_m)}\right),$$

with $\tau = T_m - T_f$ and T_s, T_f, T_m being the settlement date, the fixing date and the maturity date respectively. For simplification reasons we set $T_s = T_f = t_0$ and $T_m = 1$ giving us:

$$V^{FRA}(t_0) = \frac{L(t_0, t_0, T_m) - K_1}{1 + \tau L(t_0, t_0, T_m)}.$$

So the pv_1 is given by:

$$pv_1 = \frac{L(t_0, t_0, 1) - K_1}{1 + L(t_0, t_0, 1)} - q_1 = 0.$$

A yield curve construction- example

- By definition of the Libor rate, $L(t_0, T_1, T_2) = \frac{1}{T_2 - T_1}\left(\frac{P(t_0,T_1)-P(t_0,T_2)}{P(t_0,T_2)}\right)$ we can express the pv_1 in terms of spine discount factors reads:

$$pv_1 = \frac{\left(\frac{1}{df_1} - 1\right) - K_1}{1 + \left(\frac{1}{df_1} - 1\right)} - q_1 = 0,$$

which further simplifies to:

$$pv_1 = 1 - (1 + K_1)df_1 - q_1 = 0.$$

- And the swap has the following present value:

$$V^S(t_0) = P(t_0, T_m) - P(t_0, T_n) - K_2 \sum_{k=m+1}^{n} \tau_k P(t_0, T_k).$$

For simplicity we take $T_m = 1$ and $T_n = 2$ which gives the following pv_2 in terms of the spine discount factors df_1 and df_2:

$$pv_2 = df_1 - df_2 - K_2 df_2 - q_2 = 0.$$

A yield curve construction- example

- Having the results we are able to perform the calibration of the yield curve spine discount factors $\mathbf{df} = [df_1, df_2]^{\mathrm{T}}$:

$$\begin{bmatrix} pv_1 \\ pv_2 \end{bmatrix} = \begin{bmatrix} 1 - (1 + K_1)df_1 - q_1 \\ df_1 - df_2 - K_2 df_2 - q_2 \end{bmatrix},$$

and

$$\mathbf{J} := \begin{bmatrix} \frac{\partial pv_1}{\partial df_1} & \frac{\partial pv_1}{\partial df_2} \\ \frac{\partial pv_2}{\partial df_1} & \frac{\partial pv_2}{\partial df_2} \end{bmatrix} = \begin{bmatrix} -(1 + K_1) & 0 \\ 1 & -(1 + K_2) \end{bmatrix},$$

with the inverse of the Jacobian:

$$\mathbf{J}^{-1} = \frac{1}{(1 + K_1)(1 + K_2)} \begin{bmatrix} -(1 + K_2) & 0 \\ -1 & -(1 + K_2) \end{bmatrix}.$$

- Finally, given the quotes from the market q_1, q_2 and the corresponding strikes K_1 and K_2 one can obtain optimal, spine discount factors df_1 and df_2.
- Note that typically in the market the strikes K_1 and K_2 are such that $q_1 = 0$ and $q_2 = 0$ at the inception, i.e., the most common contracts are such that the value at the start of the contract is zero.

Python experiment

- We take now some recent data from the US Department of the treasury `https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield` and build a yield curve.

Date	1 Mo	2 Mo	3 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
06/01/21	0.01	0.01	0.02	0.04	0.04	0.16	0.31	0.81	1.28	1.62	2.22	2.30
06/02/21	0.01	0.01	0.02	0.04	0.05	0.13	0.30	0.80	1.26	1.59	2.21	2.28
06/03/21	0.00	0.01	0.02	0.04	0.04	0.16	0.34	0.84	1.30	1.63	2.22	2.30
06/04/21	0.01	0.02	0.02	0.04	0.05	0.14	0.32	0.78	1.23	1.56	2.16	2.24
06/07/21	0.01	0.02	0.02	0.04	0.05	0.16	0.33	0.79	1.24	1.57	2.17	2.25
06/08/21	0.01	0.02	0.02	0.04	0.05	0.14	0.32	0.77	1.20	1.53	2.13	2.21
06/09/21	0.01	0.02	0.03	0.04	0.05	0.16	0.31	0.75	1.17	1.50	2.10	2.17
06/10/21	0.01	0.02	0.03	0.04	0.05	0.14	0.30	0.73	1.14	1.45	2.07	2.15
06/11/21	0.01	0.02	0.03	0.04	0.05	0.16	0.31	0.76	1.16	1.47	2.08	2.15
06/14/21	0.01	0.02	0.03	0.05	0.05	0.16	0.33	0.80	1.20	1.51	2.12	2.19
06/15/21	0.02	0.02	0.03	0.05	0.08	0.16	0.34	0.79	1.21	1.51	2.12	2.20
06/16/21	0.04	0.04	0.04	0.06	0.08	0.21	0.41	0.89	1.29	1.57	2.13	2.20

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Yield Curve Interpolations

- There are many different types of yield curve interpolations. The most common are:
  - Linear on the log rates
  - Linear on discount factors
  - Quadratic splines
  - Cubic splines
  - B-splines
  - etc.
- An overview of many possible choices for YC interpolation is presented in "*Methods for Constructing a Yield Curve" by P. Hagan and G. West.*
- *The decision whether a local or global interpolation should be used often depends on the smoothness requirement and hedging.*

Linear and Log-Linear Interpolation

- The most straightforward approach for interpolating between ZCBs, df_i, is by using a linear interpolation on discount factors, i.e.,

$$df(t) = \frac{t - t_i}{t_{i+1} - t_i} df(t_{i+1}) + \frac{t_{i+1} - t}{t_{i+1} - t_i} df(t_i).$$

- Since $df(t)$ is not differentiable, it may cause problems with computations, e.g., $\theta(t)$ which is a function of $f(0,t) = -\frac{\partial}{\partial t} \log P(0,t)$.
- Another, simple interpolation, is a linear interpolation in the log of ZCBs, i.e.,

$$df(t) = \exp\left[\frac{t - t_i}{t_{i+1} - t_i} \log df(t_{i+1}) + \frac{t_{i+1} - t}{t_{i+1} - t_i} \log df(t_i)\right].$$

- Log-Linear interpolation implies a linear interpolation in the forward rates, $f(0,t)$.

Localness of an Interpolation Method

- Let us consider a case where we "change" the input at the spine point t_i.
- Then the most desired setting is that the yield curve is only affected locally, i.e., $[t_i - \epsilon, t_i + \epsilon]$ for $\epsilon \to 0$.
- Preferred choices are:
    - Monotone piecewise cubic method.
    - Linear interpolation.
- Less attractive are splines.
- Another very aspect of choosing of the interpolation routine is the aspect of "localized hedge".
- If we, for example, consider a $5y$ swap, a global interpolation could suggest a hedge with $30y$ swap. This, typically wouldn't be a desired choice of a trader.

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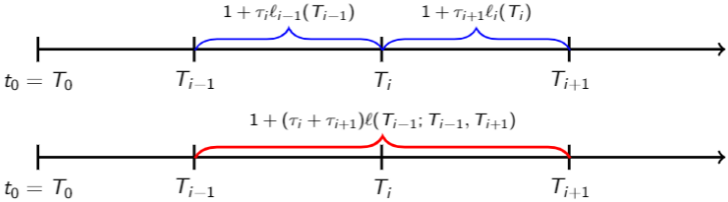
Motivation for Multi-Curve

- The market uses multiple curves when pricing financial products, which we will explain below.
- We use an equally spaced tenor structure, $0 \leq T_0 < T_1 < \cdots < T_{m-1} < T_m$, where the spacing between dates T_{i-1} and T_i is typically related to a product's payment structure, like 3 or 6 months periods, depending on the currency.
- Consider an investment over a period $[T_{i-1}, T_{i+1}]$ having two different tenor structures, i.e. with $\tau_i = T_i - T_{i-1}$ and $\hat{\tau}_i = \tau_i + \tau_{i-1} \equiv T_i - T_{i-2}$.
- Basically, we then have two possible strategies. Under the first tenor structure, we can invest for $[T_{i-1}, T_i]$ and re-invest for $[T_i, T_{i+1}]$, while under the second structure we invest over the entire period $[T_{i-1}, T_{i+1}]$.

Motivation for Multi-Curves

- Based on arbitrage arguments, the strategy involving re-investment should give exactly the same result as one investment over the whole period, so that the following equality should hold See the book for details (Chapter 14).



- In practice however, there may be a difference between these two strategies. The difference is linked to the credit risk.

Multi-curves and market practice

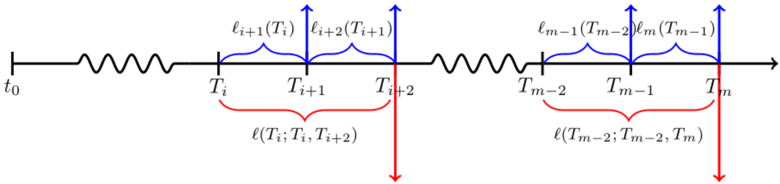
- Arbitrage-free pricing implies that if we trade in a swap with two float legs with different payment frequency, the value of this derivative should equal zero, i.e.

$$V^S(t_0) = N \cdot M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\sum_{k_1=i+1}^{m_1}\frac{\tau_{k_1}\ell_{k_1}(T_{k_1-1})}{M(T_{k_1})} - \sum_{k_2=i+1}^{m_2}\frac{\tau_{k_2}\ell_{k_2}(T_{k_2-1})}{M(T_{k_2})}\right] = 0,$$

  with index k_1 corresponding to payments at $\{T_1, T_2, \ldots, T_{m-1}, T_m\}$ and index k_2 to less frequent payments, at $\{T_2, T_4, \ldots, T_m\}$.

Multi-curves and market practice

- This financial derivative is commonly known as a *basis swap*, which is a floating-floating interest rate swap. In the case of a *Euribor basis swap* there can be 3 month and 12 month Euribor cash flow exchanges.



Multi-curves and market practice

- From a *risk management perspective*, however, it is safer to receive frequent payments, as a counterparty *may default* in between payments, and, in the case of frequent payments, there is less money lost if a default occurs.
- Until the financial crisis from 2007/2008, the basis spreads based on different tenors, were negligible [3].
- These days, interest rate instruments with different tenors are nowadays characterized by different liquidity and credit risk premia, which is reflected in non-zero basis spread values.
- Before the mortgage crisis, discounting was based on a single curve which was used for all tenors and also for discounting.
- The existence of nonzero basis spreads in the market after the mortgage crisis essentially implies that, when modeling interest rate forwards, we need to distinguish forward rates with different tenor structure (with different frequencies).

[3] The reported differences for 3 month versus 6 month EUR basis swaps were up to 0.01% (between 2004-2007).

Multi-curves and market practice

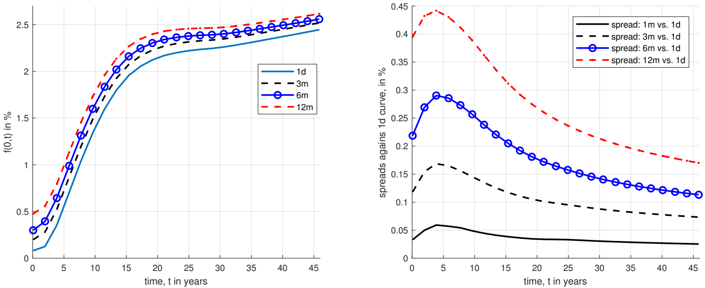


Figure: Left: forward rates corresponding to different tenor curves, 1d, 3m, 6m and 12m; Right: spreads between different curves against the 1d curve.

Multi-curves and market practice

- It is therefore market practice to construct for each tenor a different forward curve.
- Each curve is based on a specific selection of interest rate derivatives, that are homogeneous in their tenor (typically 1 month, 3 months, 6 months, 12 months).
- On the other hand, financial derivatives involving different tenors should be discounted with a unique discount curve.
- An optimal choice for discounting is a curve which carries the smallest possible credit risk, which suggests that the discounting curve should correspond to the curve with the shortest tenor available on the market (which is typically 1 day).
- In the Euro zone this is the so-called EONIA (EURO OverNight Index Average) and in the US it is the Fed fund (US Federal Reserve overnight rate).
- Notice that, under the multiple curves framework, well-known (single curve) no-arbitrage relations are not valid anymore.

Multi-curves and market practice

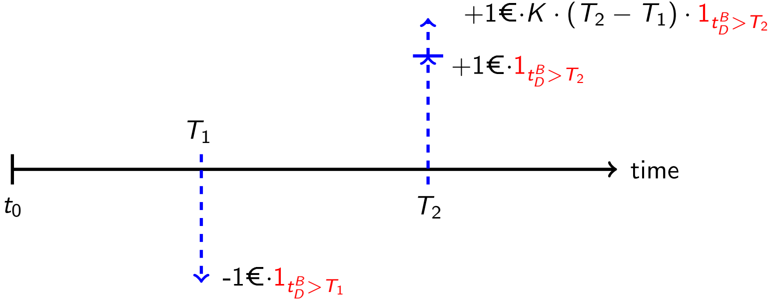
- Let us denote discounting by subscript "dc" and forecasting by "fc", then we have,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\frac{\ell(T_{i-1};T_{i-1},T_i)}{M(T_i)}\right] &= P_{\mathrm{fc}}(t_0,T_i)\mathbb{E}^{T_i}\left[\ell(T_{i-1};T_{i-1},T_i)\right] \\
&\neq P_{\mathrm{dc}}(t_0,T_i)\mathbb{E}^{T_i}\left[\ell(T_{i-1};T_{i-1},T_i)\right].
\end{aligned}
$$

- In other words, the measure-change machinery for changing measures from the spot to the forward measure does not comply with the separation of discounting and forecasting. Clearly, extended versions of the Libor rate and measure change are needed to address curve separation.

Multi-curves and market practice

- We introduce the *risk of default* into the lending transactions.
- Let t_D^B be a random variable indicating the first time probability of default of counterparty B.
- So, at time t_0, two counterparties agree to make a transaction, if counterparty B didn't go in default prior to time T_1, counterparty A will lend at time T_1 an amount of 1€ to B and, in the case of no default at T_2, counterparty B will return 1€ at time T_2 with the additional interest.



Multi-curves and market practice

- Since the payments at times T_1 and T_2 are uncertain, we need to incorporate this information while calculating the fair value of the trade. Assuming independence between the time to default, t_D^B, and the interest rates, we find,

$$
\begin{aligned}
V(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{-1}{M(T_1)}1_{t_D^B>T_1}+\frac{1}{M(T_2)}\left(1+K\cdot(T_2-T_1)\right)1_{t_D^B>T_2}\middle|\mathcal{F}(t_0)\right]\\
&= -P(t_0,T_1)\mathbb{E}^{\mathbb{Q}}\left[1_{t_D^B>T_1}\middle|\mathcal{F}(t_0)\right]\\
&\quad +P(t_0,T_2)\left(1+K\cdot(T_2-T_1)\right)\mathbb{E}^{\mathbb{Q}}\left[1_{t_D^B>T_2}\middle|\mathcal{F}(t_0)\right].
\end{aligned}
$$

- The expectations above are now associated with a survival probability, i.e. for $i=1,2$,

$$
\mathbb{E}^{\mathbb{Q}}\left[1_{t_D^B>T_i}\middle|\mathcal{F}(t_0)\right]=\mathbb{Q}\left[t_D^B>T_i\right]=1-F_{t_D^B}(T_i)=:\mathrm{e}^{-\int_{t_0}^{T_i}h(s)\mathrm{d}s},
$$

  where $h(s)$ represents the deterministic *hazard rate*.

- Typically, the hazard rate for a counterparty can be determined based on credit derivatives, like *Credit Default Swaps, (CDS)*.

Multi-curves and market practice

- The option value $V(t_0)$ can be written as,

$$V(t_0) = P(t_0, T_2)\left(1 + K \cdot (T_2 - T_1)\right) D(t_0, T_2) - P(t_0, T_1) D(t_0, T_1),$$

with $D(t_0, T_i) = \mathrm{e}^{-\int_{t_0}^{T_i} h(s)\mathrm{d}s}$.

- The fair value K, for which the contract equals 0 at the inception time t_0, i.e. $V(t_0) = 0$, is given by,

$$\boxed{K = \frac{1}{(T_2 - T_1)}\left(\frac{P(t_0, T_1)}{P(t_0, T_2)}\frac{D(t_0, T_1)}{D(t_0, T_2)} - 1\right).}$$

- As in the single curve setting, the *fair* strike value at which two counterparties agree to exchange funds is called the Libor rate and it is denoted by $\hat{\ell}_i(t) := \hat{\ell}(t; T_{i-1}, T_i)$.
- In this setting, however, the rate $\hat{\ell}_i(t)$ depends on the creditworthiness of the counterparty and on the tenor $\tau_i = T_i - T_{i-1}$,

$$\boxed{\hat{\ell}(t; T_{i-1}, T_i) = \frac{1}{\tau_i}\left(\frac{P(t_0, T_{i-1})}{P(t_0, T_i)}\frac{D(t_0, T_{i-1})}{D(t_0, T_i)} - 1\right).}$$

Multi-curves and market practice

- Practitioners often associate the hazard rate $h(s)$ in the definition of $D(t_0, T_i)$ as a *spread between the risk-free and the unsecured rate* for different tenors.
- This implies that $D(t_0, T_i)$ can be interpreted as a discount factor. We see that in order to determine a fair price of the *unsecured* Libor rate $\hat{\ell}_i(t)$, a *secured*, risk-free, curve is needed, from which the ZCB $P(t_0, T_i)$ should be computed.
- Given that for each tenor structure, $\tau_i = T_i - T_{i-1}$, a basis spread $D_{\tau_i}(t_0, T_i)$ exists, we define a new, unsecured ZCB $P_{\tau_i}(t_0, T_i)$, as follows,

$$\boxed{P_{\tau_i}(t_0, T_i) := P(t_0, T_i) \cdot D_{\tau_i}(t_0, T_i).}$$

- Equation above also indicates how to construct a curve associated with a particular tenor τ_i, i.e. start with the estimation of the risk-free curve (the curve of the shortest tenor, often 1 day), once this *base curve* is established a function $D_{\tau_i}(t_0, T_i)$, corresponding to a particular tenor structure, is determined based on suitable market instruments

Valuation in a multiple Curves setting

- We rewrite the risky Libor rate, as

$$\hat{\ell}(t; T_{i-1}, T_i) = \frac{1}{\tau_i}\left(\frac{P_{\tau_i}(t, T_{i-1})}{P_{\tau_i}(t, T_i)} - 1\right).$$

  With the description above, pricing of an interest rate product given a risk-free curve $P(t_0, T_i)$ and a risky curve $P_{\tau_i}(t_0, T_i)$ can be defined.
- When pricing interest rate derivatives, the risk of default should be taken into account properly.
- When a risky Libor rate $\hat{\ell}(t; T_{i-1}, T_i)$ is determined, it is important to discount the future cash flows that depend on this Libor rate. Usually, discounting is based on the so-called *risk-free rate*, however, as discussed, a rate which relates to a risky bond is not representative for the risk-free rate.

Multi-curves and market practice

- After the crisis, the consensus has been made that the best approximation for the risk-free rate is the overnight index swap rate (OIS).
- The contractual agreements of the *overnight index swap* are that a counterparty pays a fixed rate and receives the daily-compounded overnight rate.
- By combining the concepts of the risky Libor rate and OIS discounting, for the pricing of a basic interest rate derivative, like a (payer/receiver) interest rate swap.
- Under the T_k-forward measure $\mathbb{Q}^{T_k}$, we find,

$$\boxed{V^{\text{PS,RS}}(t_0) = \bar{\alpha} \cdot N \sum_{k=m+1}^{n} \tau_k P(t_0, T_k)\left(\mathbb{E}^{T_k}\left[\hat{\ell}_k(T_{k-1})\middle|\mathcal{F}(t_0)\right] - K\right),}$$

where $P(t_0, T_k)$ is now the risk-free bond corresponding to the overnight rate.

Multi-curves and market practice

- The current market consensus is to approximate the above expectation by the forward rate $\hat{\ell}_k(t_0)$, i.e.,

$$V^{\text{PS,RS}}(t_0) \quad \approx \quad \bar{\alpha} \cdot N \sum_{k=m+1}^{n} \tau_k P(t_0, T_k)\Big(\hat{\ell}_k(t_0) - K\Big),$$

  where in the last step of the derivation the following approximation,

$$\mathbb{E}^{T_k}[\hat{\ell}_k(t)|\mathcal{F}(t_0)] \approx \hat{\ell}_k(t_0).$$

- Clearly, this is merely an approximation, which is not based on a proper measure transformation.

Multi-Curve and circular dependency

- Because of some market instruments (e.g. EONIA) are insufficient liquid for maturities longer than 1Y the construction of the discount curve may be challenging.
- There are however other market instruments that can be used to build a discount curve, e.g., basis swaps instruments like OIS-3M. These instruments are liquid and can be used to build the curve.
- Since the building of the discount curve also relies on construction of the 3M curve we encounter the circular dependency between the discount curve and the forward curves.
- This implies that in order determine these curves, the calibration need to be done "at once" as we cannot perform curve building one-by-one any longer.

Multi-Curve and Implementation in Python

- Let us now extend the previous experiment and add additional curve, the forward curve. In the experiment the curve building will be made sequentially, i.e., first we calibrate the discount curve and later build a forward curve with the discount curve used as a fixed input for pricing swaps.

Summary

- Yield Curve and its Dynamics
- Mathematical Formulation
- From Implied Volatilities to Building of YC
- Spine Points and Optimization Routine
- Analytical Example of YC Construction
- Python Experiment
- Different Interpolations and Impact on Hedging
- Introduction to Multi-Curves
- Multi-Curves and Connection to Default Probabilities
- Python Experiment for Multi-Curves
- Summary of the Lecture + Homework

Homework Exercises

The solutions for the homework can be find at
https://github.com/LechGrzelak/QuantFinanceBook

- **Exercise**
- In the YC construction we have used so far only swaps. Extend the model and include FRAs and Floating Rate Notes.
- Extend the multi-curve framework and implement a curve building algorithm where the 6M curve will rely on 3M curve and the discount curve.