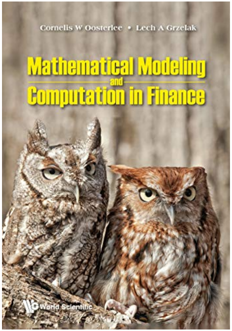
Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.



- Youtube Channel with courses can be found here.
- Slides and the codes can be found here.

Course road map

List of content

Introduction

- Until now we have been pricing of different types of derivatives assuming that the counterparty that bought/sold this derivative will meet her payment obligations.
- Let us now consider a simple case of an interest rate swap, $V^S(t_0)$, between two counterparties A and B. The interest rate swap contract is a simple exchange of fixed vs. float rate payments of the notional.
- Now, imagine a situation that interest rates have moved substantivally and that the counterparty A has significant financial obligations towards counterparty B.
- Depending on the financial situation of the counterparty A, it may well be that the counterparty may encounter some financial difficulties to meet the payments.
- This means that counterparty B may encounter some financial losses due to default of counterparty A. In practice, if the counterparty defaulted, the loss would be the replacement cost of the contract (i.e. the current market value).

Counterparty Credit Risk

- In financial jargon the situation described above is defined as the *Counterparty Credit Risk* (CCR).
- CCR is related to the situation when the counterparty will default prior to the expiration of the contract and will not make all the payments required by the contract.
- As a consequence of the CCR and its effect on the price of derivative we immediately conclude that a derivative contract with a defaultable counterparty is less worthy than a contract with a risk-free counterparty and that the lower the creditworthiness of the counterparty, the lower the market value of the contract.
- In the year 2007, a financial crisis occurred, which originated in the United States' credit and housing market, and spread around the world, from the financial markets into the real economy. Financial institutions with a high reputation went bankrupt or were bailed out, including the investment bank Lehman Brothers (founded in 1850).

Introduction

- In the worst times of that crisis, the bankruptcy of large financial institutions triggered a widespread propagation of so-called *default risk* through the financial network.
- This initiated a thorough review of the standards and methodologies for the valuation of financial derivatives. Policies, rules and regulations in the financial world changed drastically in the wake of that crisis.
- An important area of financial risk which required special attention referred to the CCR. This is the risk that a party of a financial contract is not able to fulfill the payment duties that are agreed upon in the contract, which is also known as *a default*.

Introduction

- Since then, the *probability of default* of the counterparty of a financial contract has been incorporated in the prices of financial derivatives, and thus plays a prominent role in the pricing context.
- Counterparties are charged an additional premium, which is added to the fair price of the derivative, due to the probability of default. This way the risk that the counterparty would miss payment obligations is compensated for the other party in the contract.
- The total amount of trades of complex, and thus risky, financial derivatives has significantly reduced in the wake of the financial crisis. The lack of confidence in the financial system may have resulted in a drastic reduction of complexity, simply because risk of basic financial products is easier to estimate, and also just to keep money in the pocket.

Exposures

- From the pricing perspective we immediately see that pricing under the risk-neutral measure is not sufficient to encapsulate a risk related to default of a counterparty. Typically, probabilities of a default for a counterparty can be either *implied* from the credit derivatives like Credit Default Swaps (CDS) or inferred from the rating agencies scores.
- The pricing of a derivative under a CCR is closely related to credit exposures which defines the loss in the event of a counterparty default.
- In mathematical terms the (positive) exposure, $E(t)$, is given as:

$$\boxed{E(t) = \max(V(t), 0), \quad V(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t)}{M(T)}H(T,\cdot)\middle|\mathcal{F}(t)\right],}$$

where $V(t)$ represents the value of a derivative at time t, and T is the maturity of the contract.

Potential Future Exposure

- Another measure which contributes to the valuation of CCR, especially in the context of risk management, is called the *Potential Future Exposure* (PFE).
- PFE is the maximum credit exposure calculated at some confidence level. In risk management and for setting trading limits to traders, PFE is often considered as the *worst-case scenario exposure*.
- PFE, at time t, i.e. $\text{PFE}(t_0, t)$, is defined as a quantile of the exposure $E(t)$,

$$\text{PFE}(t_0, t) = \inf\left\{x \in \mathbb{R} : p \leq F_{E(t)}(x)\right\},$$

where p is the significance level and $F_{E(t)}(x)$ is the CDF of the exposure at time t.

Types of Exposures

- In practice, portfolio of derivatives are large, containing from a few up to millions of deals. The exposures can be classified into three main categories
- "Contract-level exposures" which indicates an exposure calculated for one individual contract, just as discussed in earlier example. For a derivative with value $V_1(t)$ the contract-level exposure is simply given by:
$$E(t) = \max(V_1(t), 0).$$
- "Counterparty-level exposure" means that the exposure takes into account all the derivatives which are traded with one counterparty.

Types of Exposures

- "Netting" which entails offsetting the value of multiple trades between two, or more, counterparties. As above, for two contracts with the values $V_1(t)$ and $V_2(t)$ the netted exposure, $E_n(t)$, is given by:
$$E_n(t) = \max(V_1(t) + V_2(t), 0).$$
- We immediately see that since $\max(x + y, 0) < \max(x, 0) + \max(y, 0)$ the netting exposure is always lower than the counterparty-level exposure.
- From the exposure perspective we see that in order to reduce the risk to a counterparty the concept of netting is very beneficial.
- However, not all the trades can be used in the netting- the netting is only applicable to homogenous trades that can be legally netted (as specified in the ISDA master agreement).

Example of Netting

- To illustrate the netting principle, consider two counterparties with only two transactions. Deal 1 has a value of €100 and Deal 2 has a loss of -€50.
- In the case of a counterparty default, the other party is obliged to pay any remaining financial obligations, and in the case of a positive trade value the party would only obtain a certain percentage of the full amount recovered.
- Let us assume for the recovery rate $R_c = 40\%$.
- It can be observed that the netting may have a significant effect on the value of the portfolio, in the case of a default of the counterparty.

	scenario without netting	scenario with netting
deal 1	$40\% \times €100 = €40$	$€100 = €100$
deal 2	-€50=-€50	-€50=-€50
total	-€10	$40\% \times €50 = €20$

Expected Discounted Exposures

- Given the exposure profiles $E(t)$, we wish to calculate the *expected (positive) exposure*, which is defined as follows,

$$\boxed{\mathrm{EE}(t_0,t)=\mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(t)}E(t)\Big|\mathcal{F}(t_0)\right],}$$

  where $E(T)$ is the positive exposure and $M(T)$ is the money-savings account; $\mathrm{d}M(t)=r(t)M(t)\mathrm{d}t$.
- The concept of expected exposure is particularly important for the computation of the so-called *credit value adjustment (CVA)* and CVA, BCVA, FVA,..., that also will be discussed.
- With the (positive) exposure, $E(t)$, given as:

$$\boxed{E(t)=\max(V(t),0),\quad V(t)=\mathbb{E}^{\mathbb{Q}}\left[\frac{M(t)}{M(T)}H(T,\cdot)\Big|\mathcal{F}(t)\right],}$$

  where $V(t)$ represents the value of a derivative at time t, and T is the maturity of the contract.

Potential Future Exposure

- Before the appearance in the Basel II accords, the concepts EE and PFE had already emerged and were commonly used as representative metrics for credit exposure.
- EE thus represents the *average* expected loss in the future, while PFE may manifest the *worst* exposure given a certain confidence level. These two quantities indicate the loss from both a pricing and risk management perspective, respectively.
- There has been a debate on the computation of PFE, whether to compute it under the real-world or the risk-neutral measure.
- It is argued that PFE should be computed based on simulations under the real-world measure, reflecting the future developments in the market realistically, from a risk management perspective.

Expected Exposure for a Stock

- Let us consider a contract which will pay at time T the value of some stock $H(T,S)=S(T)$. The value of the contract today is simply:

$$V(t_0)=M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)}{M(T)}\Big|\mathcal{F}(t_0)\right]=S(t_0),$$

since the discounted stock process $S(t)$ is a martingale under $\mathbb{Q}$.

- By definition the expected exposure at time t is given by:

$$EE(t) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(t)}\max\left(V(t),0\right)\Big|\mathcal{F}(t_0)\right].$$

- After substitution gives us:

$$\begin{aligned} EE(t) \quad &= \quad \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(t)}\max\left(M(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)}{M(T)}\Big|\mathcal{F}(t)\right],0\right)\Big|\mathcal{F}(t_0)\right] \\ &= \quad M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(t)}\max\left(S(t),0\right)\Big|\mathcal{F}(t_0)\right]. \end{aligned}$$

- We see that the exposure for a payoff which pays a stock $S(T)$ at some future time T is equivalent with a call option on the stock with $K=0$.

Expected Exposures for an IR Swap

- Now we derive the exposure profile for an interest rate swap. The today's value on an interest rate swap with notional $N = 1$ is given by (more details in Lecture 5 and Lecture 7):

$$V^S(t_0) = \sum_{k=m+1}^{n} \tau_k P(t, T_k) \left(\ell(t, T_{k-1}, T_k) - K\right)$$

- By definition the expected exposure at time t is given by:

$$\begin{aligned} EE(t) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(t)} \max\left(V^S(t), 0\right) \Big| \mathcal{F}(t_0)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(t)} \max\left[\sum_{k=m+1}^{n} \tau_k P(t, T_k) \left(\ell(t, T_{k-1}, T_k) - K\right), 0\right] \Big| \mathcal{F}(t_0)\right], \end{aligned}$$

  for $t \geq T_m$.

- We immediately recognize that expected exposures $EE(t)$ is equivalent with a value of swaption $V^{\text{Swpt}}(t, K)$ with strike K.

$$EE(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(t)} V^{\text{Swpt}}(t, K) \Big| \mathcal{F}(t_0)\right], \quad M(t_0) = 1.$$

Expected Exposures for an FX Swap

- An FX swap is a financial derivative in which two parties exchange a fixed for a floating FX rate. One party receives a floating FX rate, $y(T)$, and the other party pays the fixed rate K. The amount to be paid is based on some foreign notional amount N_{f}.
- At the maturity date T, the value of the FX swap contract, in the domestic currency, is given by,

$$V^{\mathrm{d}}(T) = N_{\mathrm{f}}\left(y(T) - K\right),$$

(remember that $y(t) := y_{\mathrm{f}}^{\mathrm{d}}(t)$)
- The FX swap value at time t is equal to:

$$V^{\mathrm{d}}(t) = M_{\mathrm{d}}(t)\mathbb{E}^{\mathbb{Q},\mathrm{d}}\left[\frac{N_{\mathrm{f}}}{M_{\mathrm{d}}(T)}\left(y(T) - K\right)\Big|\mathcal{F}(t)\right].$$

Expected Exposures for an FX Swap

- From the definition of the FX forward rate, we have:

$$y_F(t,T) = y(t)\frac{P_\mathrm{f}(t,T)}{P_\mathrm{d}(t,T)}.$$

- By the change of measure, from the risk-neutral measure $\mathbb{Q}^\mathrm{d}$ to the forward measure $\mathbb{Q}^{T,\mathrm{d}}$, it follows that,

$$\begin{aligned} V^\mathrm{d}(t) &= M_\mathrm{d}(t)\mathbb{E}^{T,\mathrm{d}}\left[\frac{N_\mathrm{f}}{M_\mathrm{d}(T)}\frac{M_\mathrm{d}(T)}{M_\mathrm{d}(t)}\frac{P_\mathrm{d}(t,T)}{P_\mathrm{d}(T,T)}\left(y(T)-K\right)\Big|\mathcal{F}(t)\right] \\ &= P_\mathrm{d}(t,T)N_\mathrm{f}\mathbb{E}^{T,\mathrm{d}}\left[\left(y(T)-K\right)\Big|\mathcal{F}(t)\right]. \end{aligned}$$

- Using the definition of the forward FX rate, the following holds true,

$$\mathbb{E}^{T,\mathrm{d}}\left[y(T)\Big|\mathcal{F}(t)\right] = y(t)\frac{P_\mathrm{f}(t,T)}{P_\mathrm{d}(t,T)} =: \mathrm{FX}(t,T),$$

and therefore, we have,

$$\begin{aligned} V^\mathrm{d}(t) &= P_\mathrm{d}(t,T)N_\mathrm{f}\mathbb{E}^{T,\mathrm{d}}\left[\left(y(T)-K\right)\Big|\mathcal{F}(t)\right] \\ &= P_\mathrm{d}(t,T)N_\mathrm{f}\left(\mathrm{FX}(t,T)-K\right). \end{aligned}$$

Expected Exposures for an FX Swap

- The expected exposure, in domestic currency, is given by:

$$\begin{aligned}\mathrm{EE}^{\mathrm{d}}(t_0,t) &= \mathbb{E}^{\mathbb{Q},\mathrm{d}}\left[\frac{M_{\mathrm{d}}(t_0)}{M_{\mathrm{d}}(t)}\max(V^{\mathrm{d}}(t),0)\Big|\mathcal{F}(t_0)\right]\\ &= N_{\mathrm{f}}\mathbb{E}^{\mathbb{Q},\mathrm{d}}\left[\frac{M_{\mathrm{d}}(t_0)}{M_{\mathrm{d}}(t)}P_{\mathrm{d}}(t,T)\max\left(\mathrm{FX}(t,T)-K,0\right)\Big|\mathcal{F}(t_0)\right].\end{aligned}$$

- Inserting the definition of the ZCB, $P_{\mathrm{d}}(t,T)=\mathbb{E}^{\mathbb{Q},d}\left[M(t)/M(T)\Big|\mathcal{F}(t)\right]$, gives,

$$\mathrm{EE}^{\mathrm{d}}(t_0,t)=N_{\mathrm{f}}\mathbb{E}^{\mathbb{Q},\mathrm{d}}\left[\frac{M_{\mathrm{d}}(t_0)}{M_{\mathrm{d}}(t)}\mathbb{E}^{\mathbb{Q},d}\left[\frac{M_{\mathrm{d}}(t)}{M_{\mathrm{d}}(T)}\Big|\mathcal{F}(t)\right]\max\left(\mathrm{FX}(t,T)-K,0\right)\Big|\mathcal{F}(t_0)\right].$$

- By means of the tower property of expectations, $\mathbb{E}\left[X\cdot\mathbb{E}\left[Y|\mathcal{F}(t)\right]|\mathcal{F}(t_0)\right]=\mathbb{E}\left[X\cdot Y|\mathcal{F}(t_0)\right]$, we obtain (see Lecture 2),

$$\begin{aligned}\mathrm{EE}^{\mathrm{d}}(t_0,t) &= N_{\mathrm{f}}\mathbb{E}^{\mathbb{Q},\mathrm{d}}\left[\frac{M_{\mathrm{d}}(t_0)}{M_{\mathrm{d}}(T)}\max\left(\mathrm{FX}(t,T)-K,0\right)\Big|\mathcal{F}(t_0)\right]\\ &= N_{\mathrm{f}}P_{\mathrm{d}}(t_0,T)\mathbb{E}^{T,\mathrm{d}}\left[\max\left(\mathrm{FX}(t,T)-K,0\right)\Big|\mathcal{F}(t_0)\right],\end{aligned}$$

because $\mathrm{FX}(t,T)$ is a martingale under the domestic T-forward measure.

Expected Exposures for an FX Swap

- Now, assuming that the forward rate $\mathrm{FX}(t, T)$, under the domestic T-forward measure, follows a lognormal process,

$$\mathrm{dFX}(t, T) = \bar{\sigma}\mathrm{FX}(t, T)\mathrm{d}W^{T,\mathrm{d}}(t),$$

  the price of a European option equals the expected exposure, and is given by:

$$\boxed{\mathrm{EE}^{\mathrm{d}}(t_0, t) := V(t_0) = P_{\mathrm{d}}(t_0, T)N_{\mathrm{f}}\left(\mathrm{FX}(t_0, T)\Phi(d_1) - K\Phi(d_2)\right),}$$

$$d_1 = \frac{\log\left(\frac{\mathrm{FX}(t_0, T)}{K}\right) + \frac{1}{2}\sigma^2(t - t_0)}{\sigma\sqrt{t - t_0}}, \quad d_2 = d_1 - \sigma\sqrt{t - t_0},$$

  which is similar to the pricing of caplets (see Lecture 5, and Lecture 10).
- Expected exposure for an FX swap can thus be interpreted as a European option on an FX forward rate.

Exposure generation for IR Swap

- As an experiment let us now consider an interest rate swap where the price, for notional N is defined as:

$$V^S(t) = N \sum_{k=m+1}^{n} \tau_k P(t, T_k) \left(\ell(t, T_{k-1}, T_k) - K\right),$$

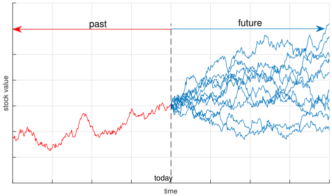
  where $\tau_k = T_k - T_{k-1}$ and the dates $T_{m+1}, \ldots T_n$ are the payment dates.
- As already discussed previously the value of a swap today (at time t_0) is completely determined by the zero-coupon bonds $P(t_0, T_k)$ for $k = m, m+1, \ldots, n$ which are observable in the market, therefore as such we don't need a model for pricing:

$$V^{\text{Swap}}_{m,n}(t) = N\left[P(t, T_m) - P(t, T_n)\right] - KN \sum_{k=m+1}^{n} \tau_k . P(t_0, T_k).$$

- On the other hand, if we wish to calculate the value at any future time $t > t_0$ we need to choose a stochastic model which will include the uncertainty related to the future.

Exposure generation for IR Swap

- Past and present in an asset price setting. We do not know the precise future asset path but we may simulate it according to some price distribution.



- The pricing of a swap does not depend on t_0 but on a time t.

$$V^S(t) = N \sum_{k=m+1}^{n} \tau_k P(t, T_k) \left(\ell(t, T_{k-1}, T_k) - K \right) \mathbb{1}_{T_k > t},$$

- Important: In the evaluation, don't include "past" cash flows, but only "future" flows.

Exposure generation for IR Swap

- Given the dynamics of the Hull-White model we have:

$$\mathrm{d}r(t) = \lambda(\theta(t) - r(t))\mathrm{d}t + \eta \mathrm{d}W^{\mathbb{Q}}(t),$$

with the term-structure parameter

$$\theta(t) \quad = \quad f(0,t) + \frac{1}{\lambda}\frac{\partial}{\partial t} f(0,t) + \frac{\eta^2}{2\lambda^2}\left(1 - \mathrm{e}^{-2\lambda t}\right),$$

and

$$\boxed{P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(z)\mathrm{d}z}\Big|\mathcal{F}(t)\right] = \mathrm{e}^{\bar{A}_r(t,T)+\bar{B}_r(t,T)r(t)},}$$

where,

$$\begin{aligned}
\bar{B}_r(t,T) \quad &= \quad \frac{1}{\lambda}\left(\mathrm{e}^{-\lambda(T-t)} - 1\right), \\
\bar{A}_r(t,T) \quad &= \quad \lambda \int_t^T \theta(t)\bar{B}_r(t,T)\mathrm{d}t \\
&\quad + \frac{\eta^2}{4\lambda^3}\left[\mathrm{e}^{-2\lambda(T-t)}\left(4\mathrm{e}^{\lambda(T-t)} - 1\right) - 3\right] + \frac{\eta^2(T-t)}{2\lambda^2}.
\end{aligned}$$

Expected Exposures

- Given the exposure profiles $E(t)$ we are able to calculate the expected (positive) exposures which are defined as follows:

$$EE(T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T)}E(T)\Big|\mathcal{F}(t_0)\right],$$

where $E(T)$ is the positive exposure and $M(T)$ is the money savings account defined as

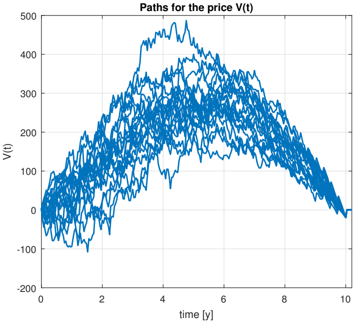
$$\mathrm{d}M(t) = r(t)M(t)\mathrm{d}t$$

with $r(t)$ being the Hull-White short-rate process.

- And exposures are computet via:

$$\boxed{E(t) = \max(V(t), 0), \quad V(t) = \mathbb{E}^{\mathbb{Q}}\left[H(T,\cdot)\Big|\mathcal{F}(t)\right],}$$

Computations of Expected Exposures (EEs)



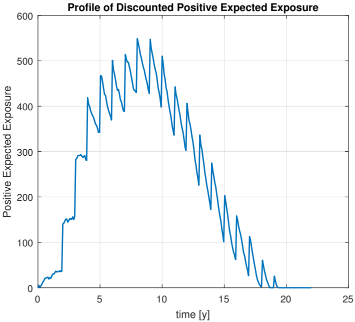


Figure: Value of a swap in time $V^S(t)$ and a expected positive exposure profile $EE(t)$.

Computations of Expected Exposures (EEs)

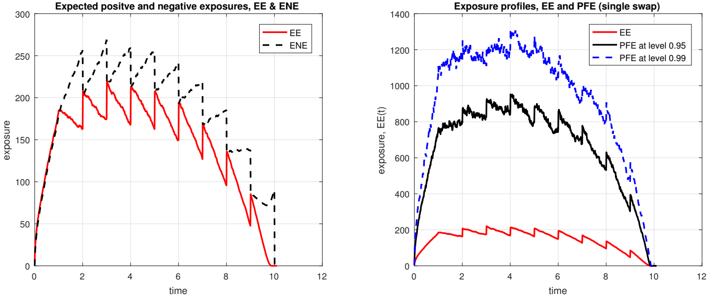


Figure: Left: the (positive) Expected Exposure, EE, and the Expected Negative Exposure, ENE, which is defined as $\mathrm{ENE}(t) = \max(-V(t), 0)$; Right: The EE and two PFE quantities, with levels, 0.95 and 0.99, respectively.

Portfolio composition

- The system of SDEs for the FX model, under the risk-neutral measure of a base currency, reads:

$$
\begin{aligned}
\mathrm{d}y_k(t) &= \left(r_b(t) - r_k(t)\right) y_k(t)\mathrm{d}t + \sigma_{y,k} y_k(t)\mathrm{d}W_k^y(t),\\
\mathrm{d}r_b &= \lambda_b\left(\theta_b(t) - r_b(t)\right)\mathrm{d}t + \eta_b \mathrm{d}W^b(t),\\
\mathrm{d}r_k &= \lambda_k\left(\hat{\theta}_k(t) - r_k(t)\right)\mathrm{d}t + \eta_k \mathrm{d}W_k(t),
\end{aligned}
$$

for $k = 1, \ldots, d_c$ indicating foreign currencies, where we consider a full matrix of correlations with the following correlation coefficients,

$$
\mathrm{d}W_k^y(t)\mathrm{d}W^b(t) = \rho_{k,b}^y \mathrm{d}t, \quad \mathrm{d}W_k^y(t)\mathrm{d}W_j(t) = \rho_{k,j}^y \mathrm{d}t, \quad \mathrm{d}W^b(t)\mathrm{d}W_k(t) = \rho_k \mathrm{d}t,
$$

and where the convexity adjusted term structure for foreign short rate processes is given by $\hat{\theta}_k(t) = \theta_k(t) - \eta_k \sigma_{y,k} \rho_{k,k}^y$ with adjusted $\theta(t)$.

Portfolio composition

- All the linear interest rate products can be expressed as linear combinations of zero-coupon bonds.
- The representation of the system allows for large time step Monte Carlo simulation with the following solution for each FX process:

$$y_k(t) = y_k(s)\exp\left(\int_s^t \left(r_b(z) - r_k(z) - \frac{1}{2}\sigma_{y,k}^2\right)\mathrm{d}t + \sigma_{y,k}(W_k^y(t) - W_k^y(s))\right),$$

  and large-time steps for $r_b(t)$ and $r_k(t)$.
- A cross-currency portfolio consisting of swaps in d_c different currencies is then given by:

$$V(t,\mathbf{X}(t)) = \sum_{i=1}^{M_b} V_i^b(t, r_b(t)) + \sum_{k=1}^{d_c} y_k^b(t) \sum_{i=1}^{M_k} V_i^k(t, r_k(t)),$$

  with the following state vector,
  $\mathbf{X}(t) = [r_b(t), y_1^b(t), \ldots, y_{d_c}^b(t), r_1(t), \ldots, r_{d_c}(t)]^{\mathrm{T}}$, and where the first sum indicates the sum over all swaps under the base currency, the second term indicates the summation over all swaps under foreign currencies that are *exchanged* with $y_k^b(t)$ to the base currency.

Unilateral Credit Value Adjustment

- We show now how to derive an adjustment to risk-free pricing.
- We assume that transactions are seen from the point of view of the safe investor (us), namely the company facing counterparty risk and secondly we are default-free, this means that we cannot default.
- We denote by $V^D(t, T)$ to be a discounted payoff between t and T, subject to counterparty default risk and by $V(t, T)$ the analogous quantity when counterparty risk is not considered (risk-free).

Unilateral Credit Value Adjustment

- If the default of a counterparty happens after the final payment of derivative $\tau > T$ the value at time t is simply

$$1_{\tau > T} V(t, T).$$

- If the default occurs before the maturity time $\tau < T$:
  1. we receive/pay all the payments until the default time:

  $$1_{\tau \leq T} V(t, \tau),$$

  2. depending on the counterparty, we may be able to recover some of the future payments, assuming the recovery fraction to be R the value yields:

  $$1_{\tau \leq T} R \max(V(\tau, T), 0),$$

  3. on the other hand, if we owe the money to the counterparty that has defaulted we cannot keep the money but we need to pay it completely back:

  $$1_{\tau \leq T} \min(V(\tau, T), 0).$$

Unilateral Credit Value Adjustment

- Thus, when including all the components a price of a “risky” derivative is given by:

$$\begin{aligned} V^D(t,T) &= \mathbb{E}^{\mathbb{Q}}\Big[1_{\tau>T}V(t,T)+1_{\tau\leq T}V(t,\tau) \\ &\quad +\frac{1}{M(\tau)}1_{\tau\leq T}R\max(V(\tau,T),0) \\ &\quad +\frac{1}{M(\tau)}1_{\tau\leq T}\min(V(\tau,T),0)\Big|\mathcal{F}(t)\Big]. \end{aligned}$$

- Since $x=\max(x,0)+\min(x,0)$ the simplified equation reads:

$$\begin{aligned} V^D(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[1_{\tau>T}V(t,T)+1_{\tau\leq T}V(t,\tau)+\frac{1}{M(\tau)}1_{\tau\leq T}V(\tau,T)\right. \\ &\quad \left.+\frac{1}{M(\tau)}1_{\tau\leq T}(R-1)\max(V(\tau,T),0)\Big|\mathcal{F}(t)\right]. \end{aligned}$$

Unilateral Credit Value Adjustment

- We immediately note that first three terms in the expression above yield:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\tau>T}V(t,T)+\mathbb{1}_{\tau\leq T}V(t,\tau)+\frac{1}{M(\tau)}\mathbb{1}_{\tau\leq T}V(\tau,T)\Big|\mathcal{F}(t)\right]& \\
=\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\tau>T}V(t,T)+\mathbb{1}_{\tau\leq T}V(t,T)\Big|\mathcal{F}(t)\right]& \\
=V(t,T)\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\tau>T}+\mathbb{1}_{\tau\leq T}\Big|\mathcal{F}(t)\right]& \\
=V(t,T).&
\end{aligned}
$$

- So, assuming the recovery rate R to be constant, the value of the risky asset now yields:

$$
\begin{aligned}
V^D(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[V(t,T)+\frac{1}{M(\tau)}\mathbb{1}_{\tau\leq T}(R-1)\max(V(\tau,T),0)\Big|\mathcal{F}(t)\right]\\
&= V(t,T)+\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(\tau)}\mathbb{1}_{\tau\leq T}(R-1)\max(V(\tau,T),0)\Big|\mathcal{F}(t)\right]\\
&= V(t,T)-(1-R)\,\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(\tau)}\mathbb{1}_{\tau\leq T}\max(V(\tau,T),0)\Big|\mathcal{F}(t)\right]\\
&=: V(t,T)-CVA(t,T),
\end{aligned}
$$

Unilateral Credit Value Adjustment

- CVA formula:

$$\boxed{CVA(t,T) = (1-R)\,\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(\tau)}1_{\tau\leq T}\max(V(\tau,T),0)\Big|\mathcal{F}(t)\right].}$$

- Since $CVA(t,T) \geq 0$ we conclude that from our perspective a risk-free derivative is more valuable than a derivative which carries a risk of counterparty default. As a general conclusion we see:

$$\boxed{\text{Risky Derivative} = \text{Risk-Free Derivative - CVA.}}$$

- With CVA having the interpretation of the price of counterparty risk (expected loss due to counterparty default in the future). Note, that in the industry for R being the recovery rate, $(1-R)$ is named as loss-given-default (LGD).

Approximations in calculation of CVA

- From the definition of CVA we see that in order to calculate the CVA charge we need to have a joint distribution between default time τ and the exposure $V(t,T)$ at time t. Using the tower property of the expectation we find:

$$
\begin{aligned}
CVA(t,T) &= (1-R)\,\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(\tau)}1_{\tau\leq T}\max(V(\tau,T),0)\Big|\mathcal{F}(t)\right]\\
&= (1-R)\,\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(\tau)}1_{\tau\leq T}\max(V(\tau,T),0)\Big|\mathcal{F}(\tau)\right]\Big|\mathcal{F}(t)\right]\\
&= (1-R)\,\mathbb{E}^{\mathbb{Q}}\left[1_{\tau\leq T}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(\tau)}\max(V(\tau,T),0)\Big|\mathcal{F}(\tau)\right]\Big|\mathcal{F}(t)\right].
\end{aligned}
$$

- Default probability is correlated to exposures, it is related to the so-called Wrong-Way-Risk and Right-Way-Risk.
- In order to “impose” the correlation between default probabilities and exposures, one can consider a stochastic process for the hazard rate or impose the correlation via copulas.

Approximations in calculation of CVA

- Assume independence between default time τ and exposure $V(\tau,T)$ we derive the following approximation:

$$\begin{aligned} CVA(t,T) &= (1-R)\int_t^T \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(\tau)}\max(V(\tau,T),0)\Big|\tau=s\right] f_\tau(s)\mathrm{d}s \\ &= (1-R)\int_t^T \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(\tau)}\max(V(\tau,T),0)\Big|\tau=s\right] \mathrm{d}F_\tau(s) \\ &\approx (1-R)\sum_{i=1}^{M} EE(T_i)q(T_{i-1},T_i), \end{aligned}$$

- with expected positive exposures given by

$$EE(T_i)=\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_i)}\max(V(T_i,T),0)\Big|\mathcal{F}(t)\right],$$

and where probability of a default in $(T_{i-1},T_i]$ is given by:

$$q(T_{i-1},T_i) = F_\tau(T_i)-F_\tau(T_{i-1})=\mathbb{E}\left[1_{T_{i-1}<\tau\leq T_i}\right].$$

Approximations in calculation of CVA

- As the final result we conclude that the CVA charge can be approximated as:

$$\boxed{\textbf{CVA} \approx \underbrace{\text{LGD}}_{\text{loss given default}} \times \underbrace{\text{PD}}_{\text{probability of default}} \times \underbrace{\text{EE}}_{\text{expected positive exposure}}.}$$

- From the computational perspective the most numerically expensive element is the evaluation of *EEs*.

BCVA

- In the unilateral CVA case, as discussed in the previous section, we have seen a generalization of the concept of risk-free derivatives pricing to a situation where a counterparty may default.
- However, there may be a *symmetry problem* in the logic followed there. Since in any financial transaction there are (at least) two counterparties, investor I and counterparty C, the CVA charge may differ, and may depend on the perspective. An investor will calculate the price of the derivative as,

$$\bar{V}_I(t_0, T) = \bar{V}(t_0, T) - \mathrm{CVA}_I(t_0, T),$$

  with $\mathrm{CVA}_I(t_0, T)$ the charge computed by assuming that the counterparty may default.
- On the other hand, counterparty C would calculate the price of the derivative as,

$$\bar{V}_C(t_0, T) = \bar{V}(t_0, T) - \mathrm{CVA}_C(t_0, T).$$

BCVA

- In general, $\mathrm{CVA}_I(t_0, T) \neq \mathrm{CVA}_C(t_0, T)$, because the two counterparties may be subject to a different credit risk value, so that the adjusted value which is calculated by investor I is not the opposite of the adjusted value calculated by counterparty C. In other words, the two parties may *not agree* on the adjusted price of the risky derivative.
- The above reasoning motivates to use a *generalized version of the unilateral CVA case*, which is called *bilateral CVA* (BCVA).
- In BCVA, the fact that both counterparties may default is included in the modeling.
- When deriving the price of a "risky asset", we do not only include the CVA component, but also a component which accounts for the risk associated with the *own* default risk (which is known under the name *"debt value adjustment"*, $\mathrm{DVA}(t_0, T)$).

BCVA

- Depending on the perspective, the $\mathrm{CVA}(t_0, T)$ charge for the investor is equivalent to the $\mathrm{DVA}(t_0, T)$ for the counterparty, i.e., $\mathrm{CVA}_C(t_0, T) = \mathrm{DVA}_I(t_0, T)$ and $\mathrm{CVA}_I(t_0, T) = \mathrm{DVA}_C(t_0, T)$.
- For the two counterparties, we would have the following pricing equations:

$$\begin{aligned}\bar{V}_I(t_0, T) &= \bar{V}(t_0, T) - \mathrm{CVA}_I(t_0, T),\\ \bar{V}_C(t_0, T) &= \bar{V}(t_0, T) - \mathrm{CVA}_C(t_0, T).\end{aligned}$$

- Since the investor and the counterparty should agree on the price, the condition $\bar{V}_I(t_0, T) = \bar{V}_C(t_0, T)$ should be imposed, which implies the following adjustments,

$$\begin{aligned}\bar{V}_I(t_0, T) &= \bar{V}(t_0, T) - \mathrm{CVA}_I(t_0, T) + \mathrm{CVA}_C(t_0, T)\\ &=: \bar{V}(t_0, T) - \mathrm{CVA}_I(t_0, T) + \mathrm{DVA}_I(t_0, T),\end{aligned}$$

and, equivalently, for the counterparty,

$$\begin{aligned}\bar{V}_C(t_0, T) &= \bar{V}(t_0, T) - \mathrm{CVA}_C(t_0, T) + \mathrm{CVA}_I(t_0, T)\\ &=: \bar{V}(t_0, T) - \mathrm{CVA}_C(t_0, T) + \mathrm{DVA}_C(t_0, T).\end{aligned}$$

BCVA

- This means that $\mathrm{DVA}(t_0, T)$ is given by:

$$\mathrm{DVA}(t, T) = (1 - R_c)\,\mathbb{E}^{\mathbb{Q}}\left[\frac{M(t)}{M(\hat{t}_D)}\mathbb{1}_{\hat{t}_D \le T}\max(-\bar{V}(\hat{t}_D, T), 0)\Big|\mathcal{F}(t)\right],$$

where $\hat{t}_D$ indicates the investor's own default time.

- Here, the *Negative Exposure* is the negative part of the default-free contract value.
- Modeling CVA and DVA is commonly referred to as *bilateral CVA* (BCVA), which is defined, as follows,

$$\mathrm{BCVA}(t, T) = \mathrm{CVA}(t, T) - \mathrm{DVA}(t, T).$$

- Note that $\mathrm{DVA}(t, T)$ is a positive quantity, and it can therefore be seen as a *profit upon self-default*, which is a problematic aspect.

Funding Value Adjustment

- Once the simulated exposures are available other risk measures can be computed. One of them is the so-called Funding Value Adjustment (FVA) which takes into account the costs of funding (borrowing money) into account.
- The computation of FVA (Funding Cost Adjustment (FCA) and Funding Benefit Adjustment) depends on *positive exposures*, $\max(V(t),0)$, and *negative exposures* $\max(-V(t),0)$.
- Assuming no WWR and independence between the funding/borrowing rate and exposures we the pricing formula for FVA is given as follows:

$$FCA(t_0) = \int_0^T \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(t)}\max(V(t),0)\Big|\mathcal{F}(t_0)\right]}_{EE(t)} s_B(t)\mathbb{P}[t<\tau_I]\mathbb{P}[t<\tau_C]\mathrm{d}t,$$

where $s_B(t)$ indicated the funding spread when derivative business requires funding.

Funding Value Adjustment

- Funding Benefit Adjustment, FBA, is given by:

$$FBA(t_0) = \int_0^T \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(t)}\max(-V(t),0)\Big|\mathcal{F}(t_0)\right]}_{NEE(t)} s_L(t)\mathbb{P}[t<\tau_I]\mathbb{P}[t<\tau_C]\mathrm{d}t,$$

  where $s_L(t)$- is the funding business when derivative business generates funding.

- Under a rectangular or trapezoidal discretization we have:

$$\begin{aligned} FCA(t_0) \;\approx\; & \sum_{i=1}^{N}\Big(\omega_1 EE(t_{i-1})\mathbb{P}[t_{i-1}<\tau_I]\mathbb{P}[t_{i-1}<\tau_C] \\ & +\omega_2 EE(t_i)\mathbb{P}[t_i<\tau_I]\mathbb{P}[t_i<\tau_C]\Big) \\ & \times\Big(L^{borrow}(t_0,T_{i-1},T_i)-L^{fund}(t_0,T_{i-1},T_i)\Big)\Delta t \end{aligned}$$

with $\Delta t = t_i - t_{i-1}$, $\omega_1=\omega_2=0.5$ for trapezoidal integration and $\omega_1 = 0$ and $\omega_2 = 1$ for rectangular integration.

Attributions

- From risk management purposes it is important to see which trades contribute the most to the overall CVA per counterparty (or netset). Risk management needs to understand what trades are the most "risky" from the counterparty risk perspective.
- If a new trade is added to a portfolio it is important to see the impact of this trade (and netting effects) on the CVA.
- CVA is not additive, i.e.,

$$\mathrm{CVA}_I(t_0, T) \neq \sum_{i=1}^{N} \mathrm{CVA}_I^i(t_0, T),$$

where $\mathrm{CVA}_I^i(t_0, T)$ indicates CVA for counterparty I and trade i.

- If we wish to see a trade's impact on the overall CVA we can perform so-called "incremental CVA" that would require two runs of the CVA (with and without).

Attributions

- In the industry the most common method to determine individual trades attributions is the so-called "Euler allocation process".
- The technique relies on function's f homogeneity, i.e., a function $f:\mathbb{R}^n\rightarrow\mathbb{R}$ is said to be homogeneous of degree k if $f(a\mathbf{x})=a^kf(\mathbf{x})$ for some constant a.
- Suppose $f:\mathbb{R}^n\rightarrow\mathbb{R}$ is continuously differentiable in R^n. Then f is homogeneous of degree k if and only if:

$$kf(\mathbf{x})=\sum_{i=1}^{n}\frac{\partial f(\mathbf{x})}{\partial x_i}x_i,$$

for $\mathbf{x}=[x_1,\dots,x_n]^{\mathrm{T}}$, which trivially for $k=1$ becomes:

$$f(\mathbf{x})=\sum_{i=1}^{n}x_i\frac{\partial f(\mathbf{x})}{\partial x_i}.$$

Attributions

- Based on Euler's allocation process the idea is to find $\widehat{\mathrm{CVA}}_I^i(t_0, T)$ such that,

$$\mathrm{CVA}_I(t_0, T) = \sum_{i=1}^{N} \widehat{\mathrm{CVA}}_I^i(t_0, T),$$

so that the additivity is preserved at the trade level.

- Since the main driver for xVA is exposure, we can focus on the additivity of exposures:

$$\begin{aligned} EE(T) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T)} \max\left(\sum_{i=1}^{N} V_i(T), 0\right) \Big| \mathcal{F}(t_0)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T)} \max\left(\sum_{i=1}^{N} \alpha_i x_i, 0\right) \Big| \mathcal{F}(t_0)\right] =: f(\alpha) \end{aligned}$$

for $x_i = V_i(T)/\alpha_i$.

Attributions

- Using Euler's theorem we have:

$$EE(T) = f(\alpha) = \sum_{i=1}^{N} \alpha_i \frac{\partial f(\alpha)}{\partial \alpha_i}.$$

- The formulation above is numerically beneficial as it requires "shocking" the PVs of each individual trade and it does not require to re-running of the whole Monte Carlo Pricing.

Summary

- Introduction and Basics of CVA
- Exposures and Potential Future Exposure
- Expected Exposures
- Expected Exposures and Closed Form Solutions
- Generation of Exposures with Python (1D Case)
- Exposure Generation for Portfolio of Assets
- Unilateral Credit Value Adjustment (CVA)
- Approximations in Calculation of CVA
- Bilateral Credit Value Adjustment (BCVA)
- Funding Value Adjustment (FVA)
- Trade Attributions in (B)CVA
- Summary of the Lecture + Homework

Homework Exercises

- **Exercise**
- Consider an economy driven by the BSHW hybrid model:

$$
\begin{aligned}
\mathrm{d}S_1(t) &= r(t)S_1(t)\mathrm{d}t + \sigma_1 S_1(t)\mathrm{d}W_1(t),\\
\mathrm{d}S_2(t) &= r(t)S_2(t)\mathrm{d}t + \sigma_2 S_2(t)\mathrm{d}W_2(t),\\
\mathrm{d}S_3(t) &= r(t)S_3(t)\mathrm{d}t + \sigma_3 S_3(t)\mathrm{d}W_3(t),\\
\mathrm{d}r(t) &= \lambda(\theta(t) - r(t))\mathrm{d}t + \eta\mathrm{d}W_r(t).
\end{aligned}
$$

- Assume full matrix of correlations and realistic set of model parameters.
- Consider a portfolio consisting of:
  - 10 of $S_1(t)$, 20 of $S_2(t)$ and 5 of $S_3(t)$of correlated stocks.
  - 10 IR Swaps with different configurations (some starting today, some in the future).
  - 5 of call options with freely chosen strikes and maturities.

Homework Exercises

- Calculate:
  - Expected Exposure profile
  - PFE with quantile $\alpha = 0.07$
  - Given the probability of default equal to 0.05 and recovery rate $R = 0.25$.
  - By means of simulation calculate the value of the portfolio and CVA. Discuss the results.
- What derivative shall be added to the portfolio (see: Netting) so that the CVA charge would be reduced at most?

Homework Exercises

- **Exercise**
- Confirm, in means of a numerical experiment, that Expected Exposures of a Swap are equivalent with pricing of a Swaption.