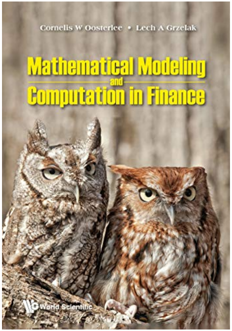
# Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.




- Youtube Channel with courses can be found here.
- Slides and the codes can be found here.

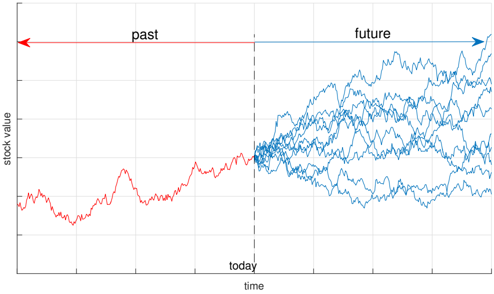
# List of content

# Filtration

- A stochastic process, $X(t)$, is a collection of random variables indexed by a *time* variable $t$.
- Suppose we have a set of calendar dates/days, $T_1, T_2, \ldots, T_m$. Up to *today*, we have observed certain state values of the stochastic process $X(t)$.
- The past is known, and we therefore "see" the historical asset path.
- For the future we do not know the precise path but we may simulate the future according to some asset price distribution.

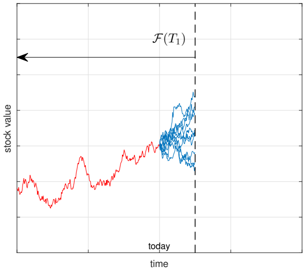
# Filtration

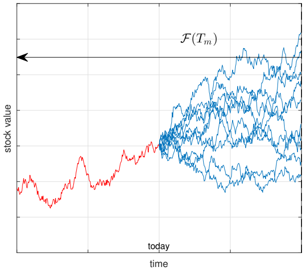
- Past and present in an asset price setting. We do not know the precise future asset path but we may simulate it according to some price distribution.



# Filtration

- Filtration figure, with $\mathcal{F}(t_0) \subseteq \mathcal{F}(T_1) \subseteq \mathcal{F}(T_2) \ldots \subseteq \mathcal{F}(T_m)$.
- When $X(t)$ is $\mathcal{F}(t_0)$ measurable this implies that at time $t_0$ the value of $X(t)$ is known. $X(T_1)$ is $\mathcal{F}(T_1)$ measurable, but $X(T_1)$ is a "future realization" which is not yet known at time $t_0$ ("today") and thus not $\mathcal{F}(t_0)$ measurable.



# Filtration

- Filtration and measure changes are a very powerful tool, but the wrong usage may lead to terrible mistakes.

# Filtration

- If we write that a process is $\mathcal{F}(T)$-measurable, we mean that at any time $t \leq T$, the realizations of this process are known. A simple example for this may be the market price of a stock and its historical values, i.e., we know the stock values up to today exactly, but we do not know any future values.
- We then say "the stock is today measurable". However, when we deal with an SDE model for the stock price, the value may be $T$ measurable, as we know the distribution for the period $T$ of a financial contract.
- A stochastic process $X(t)$, $t \geq 0$, is said to be adapted to the filtration $\mathcal{F}(t)$, if

$$\sigma(X(t)) \subseteq \mathcal{F}(t).$$

  By the term "adapted process" we mean that a stochastic process "cannot look into the future". In other words, for a stochastic process $X(t)$ its realizations (paths), $X(s)$ for $0 \leq s < t$, are known at time $s$ *but not yet at time* $t$.

# Examples of Filtration

- Examples of processes that are adapted to the filtration $\mathcal{F}(t)$ are:
  - $W(t)$ and $W^2(t) - t$, with $W(t)$ a Wiener process.
  - $\max\limits_{0\leq s\leq t} W(s)$ and $\max\limits_{0\leq s\leq t} W^2(s)$.
- Examples of processes that are NOT adapted to the filtration $\mathcal{F}(t)$ are:
  - $W(t+1)$,
  - $W(t) + W(T)$ for some $T > t$.

# Conditional Expectations

- Another important and useful concept is the concept of iterated expectations. The law of iterated expectations, also called the *tower property*, states that for any given random variable $X \in L^2$ (where $L^2$ indicates a so-called Hilbert space for which $\mathbb{E}[X^2(t)] < \infty$) , which is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, and for any sigma-field $\mathcal{G} \subseteq \mathcal{F}$, the following equality holds:

$$\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}]|\mathcal{G}], \quad \text{for} \quad \mathcal{G} \subseteq \mathcal{F}.$$

- If we consider another random variable $Y$, which is defined on the sigma-field $\mathcal{G}$, so that $\mathcal{G} \subseteq \mathcal{F}$, then the above equality can be written as

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]], \quad \text{for} \quad \sigma(Y) \subseteq \sigma(X).$$

# Conditional Expectations

- Assuming that both random variables, $X$ and $Y$, are continuous on $\mathbb{R}$ and are defined on the same sigma-field, we can prove the equality given above, as follows

$$\begin{aligned}\mathbb{E}[\mathbb{E}[Y|X]] &= \int_{\mathbb{R}} \mathbb{E}[Y|X=x] f_X(x)\mathrm{d}x\\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} y f_{Y|X}(y|x)\mathrm{d}y\right) f_X(x)\mathrm{d}x.\end{aligned}$$

- By the definition of the conditional density, i.e. $f_{Y|X}(y|x) = f_{Y,X}(y,x)/f_X(x)$, we have:

$$\begin{aligned}\mathbb{E}[\mathbb{E}[Y|X]] &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} y \frac{f_{Y,X}(y,x)}{f_X(x)}\mathrm{d}y\right) f_X(x)\mathrm{d}x\\ &= \int_{\mathbb{R}} y \left(\int_{\mathbb{R}} f_{Y,X}(y,x)\mathrm{d}x\right)\mathrm{d}y\\ &= \int_{\mathbb{R}} y f_Y(y)\mathrm{d}y \stackrel{\text{def}}{=} \mathbb{E}[Y].\end{aligned}$$

# Conditional Expectations

- Let us consider today as $t_0$ and future time $T_1 > t_0$. We take a well-defined process $X(t)$.
- Is the expectation stochastic or deterministic?

$$\mathbb{E}[X(T_1)|\mathcal{F}(T_1)] = X(T_1).$$

- The process $X(t)$, at time $T_1$ is measurable wrt filtration $\mathcal{F}(T_1)$ but it is a stochastic quantity as $T_1$ is in the future.

# Python Programming of Conditional Expectations

- Show, by means of a Monte Carlo simulation, that

a. $\mathbb{E}[W(t)|\mathcal{F}(t_0)] = W(t_0)$ for $t_0 = 0$,

b. $\mathbb{E}[W(t)|\mathcal{F}(s)] = W(s)$, with $s < t$. Note that this latter exercise requires Monte Carlo *sub-simulations*.

python™

# Application of Conditional Expectations

- We present another basic and often used application of the tower property of the expectation in finance. In this example we assume the following SDE for a stock price,

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \boxed{J}S(t)\mathrm{d}W^{\mathbb{Q}}(t),$$

  where $J$ represents a certain *stochastic volatility* random variable which has, for example, a lognormal distribution.
- After standard calculations, we obtain the following solution for $S(T)$, given by:

$$S(T) = S_0 \exp\left(\left(r - \frac{1}{2}J^2\right)T + JW^{\mathbb{Q}}(T)\right).$$

# Application of Conditional Expectations

- Since the stock, $S(t)$, contains $J$, it is nontrivial to determine a closed-form solution for the value of a European option.
- A possible solution for the pricing problem is to use the *tower property of iterated expectations*, to determine the European option prices, conditioned on "realizations" of the volatility process $J$.
- By the tower property of expectations, using $\mathbb{E} = \mathbb{E}^{\mathbb{Q}}$, the European call value can be reformulated as a discounted expectation with[1]:

$$\mathbb{E}\left[\max\left(S(T)-K,0\right)\Big|\mathcal{F}(t_0)\right] = \mathbb{E}\left[\mathbb{E}\left[\max\left(S(T)-K,0\right)\Big|J=j\right]\Big|\mathcal{F}(t_0)\right].$$

[1] The discount term $M(T)$ is omitted, only the expectation is displayed to save some space. The interest rates in this model are constant and do not influence the final result.

# Application of Conditional Expectations

- Conditioned on the realizations of the variance process, the calculation of the inner expectation is equivalent to the Black-Scholes solution with a time-dependent volatility, i.e. for given realizations of $Y(t),\ t_0 \leq t \leq T$, the asset value $S(T)$, is given by:

$$S(T) = S(t_0)\exp\left(\left(r - \frac{1}{2}j^2\right)(T - t_0) + j(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t_0))\right),$$

- The solution of the inner expectation is then given by:

$$\mathbb{E}\left[\max\left(S(T) - K, 0\right)\Big| J = j\right] = S(t_0)\mathrm{e}^{r(T-t_0)}F_{\mathcal{N}(0,1)}(d_1) - KF_{\mathcal{N}(0,1)}(d_2),$$

with

$$d_1 = \frac{\log\frac{S(t_0)}{K} + (r + \frac{1}{2}j^2)(T - t_0)}{j\sqrt{T - t_0}}, \quad d_2 = d_1 - j\sqrt{T - t_0},$$

$F_{\mathcal{N}(0,1)}$ being the standard normal cumulative distribution function.

# Application of Conditional Expectations

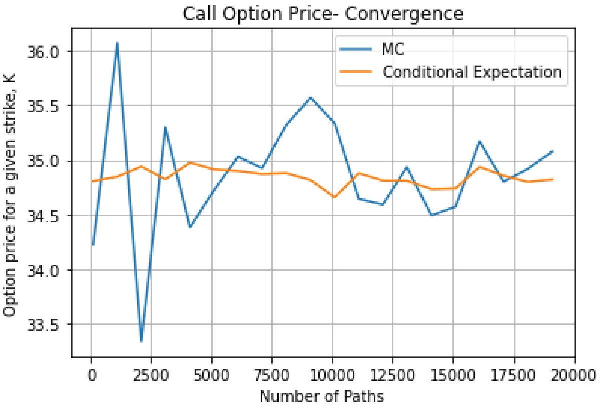
- We can substitute these results into main equation, giving:

$$
\begin{aligned}
\mathbb{E}\left[\max\left(S(T)-K,0\right)\right] &= \mathbb{E}\left[S(t_0)\mathrm{e}^{r(T-t_0)}F_{\mathcal{N}(0,1)}(d_1)-KF_{\mathcal{N}(0,1)}(d_2)\right]\\
&= S(t_0)\mathrm{e}^{r(T-t_0)}\mathbb{E}\left[F_{\mathcal{N}(0,1)}(d_1)\right]-K\mathbb{E}\left[F_{\mathcal{N}(0,1)}(d_2)\right].
\end{aligned}
$$

- The option pricing problem under these nontrivial asset dynamics has been transformed into the calculation of an expectation of a normal CDF. The difficult part of this expectation is that both arguments of the CDF, $d_1$ and $d_2$, are functions of $j$, which itself is a function of $J$.
- One possibility to deal with the expectation is to use Monte-Carlo simulation. What are the benefits of the conditional expectation if we still need to perform Monte Carlo simulation?
- Practical examples of such conditional expectations are presented in book in:
  - Book, Section 3.2.3 (stochastic volatility).
  - Book, Section 12.2.1 (pricing of caplets).

# Application of Conditional Expectations

- Let us analyze the impact of the conditional expectation on the convergence of Monte Carlo simulation.




- 

# Numéraire

- When dealing with involved systems of SDEs, it is sometimes possible to reduce the complexity of the pricing problem by an appropriate measure transformation.
- In Financial Mathematical numéraire is a tradable entity in terms of whose price the relative prices of all other tradables are expressed.
- Under the appropriate numéraire, processes may become martingales. Working with martingales is typically favorable as these processes are free of drift terms
- Keep in mind that although a process which is free of drift terms may still have an involved volatility structure, it is considered to be simpler to work with.
- Are currencies a numéraire? Is Bitcoin a numéraire?

# Numéraire

- Consider $X(t)$ to be a "tradable asset", three measures and the corresponding martingale property are as follows.
  - Risk-neutral measure is associated with the money-savings account, $M(t)$, as the numéraire,

$$\mathrm{d}X(t) = \bar{\mu}^{\mathbb{Q}}(t)\mathrm{d}t + \bar{\sigma}(t)\mathrm{d}W^{\mathbb{Q}}(t) \Longrightarrow \mathbb{E}^{\mathbb{Q}}\left[\frac{X(t)}{M(t)}\middle|\mathcal{F}(t_0)\right] = \frac{X(t_0)}{M(t_0)}.$$

  - Forward measure is associated with the ZCB, $P(t,T)$, as the numéraire, will be discussed later in this course,

$$\mathrm{d}X(t) = \bar{\mu}^{T}(t)\mathrm{d}t + \bar{\sigma}(t)\mathrm{d}W^{T}(t) \Longrightarrow \mathbb{E}^{T}\left[\frac{X(t)}{P(t,T)}\middle|\mathcal{F}(t_0)\right] = \frac{X(t_0)}{P(t_0,T)}.$$

  - Stock measure is associated with the stock, $S(t)$, as the numéraire,

$$\mathrm{d}X(t) = \bar{\mu}^{S}(t)\mathrm{d}t + \bar{\sigma}(t)\mathrm{d}W^{S}(t) \Longrightarrow \mathbb{E}^{S}\left[\frac{X(t)}{S(t)}\middle|\mathcal{F}(t_0)\right] = \frac{X(t_0)}{S(t_0)}.$$

- Is Foreign Exchange (FX) a good candidate to be a numéraire?

# Girsanov Theorem

## Theorem (The Girsanov Theorem)

*Let us consider the following stochastic process for $X(t)$:*

$$\mathrm{d}X(t) = \mu^A(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W^A(t), \quad X(0) = x_0,$$

*where the Brownian motion $\mathrm{d}W^A(t)$ is defined under the $\mathbb{Q}^A$ measure, $\mu^A(X(t))$ and $\sigma(X(t))$ satisfy Lipschitz continuity condition. For a drift $\mu^B(X(t))$ for which the ratio $Y(t) = \frac{\mu^B(X(t))-\mu^A(X(t))}{\sigma(X(t))}$, we define the measure $\mathbb{Q}^B$ by:*

$$\left.\frac{\mathrm{d}\mathbb{Q}^B}{\mathrm{d}\mathbb{Q}^A}\right|_{\mathcal{F}(t)} = \exp\left(-\frac{1}{2}\int_0^t Y^2(s)\mathrm{d}s + \int_0^t Y(s)\mathrm{d}W^A(t)\right).$$

*Then measure $\mathbb{Q}^B$ is equivalent measure to $\mathbb{Q}^A$. The process $W^B(t)$ defined by: $\mathrm{d}W^B(t) = -Y(t)\mathrm{d}t + \mathrm{d}W^A(t)$, is a Brownian motion under $\mathbb{Q}^B$ and the process for $X(t)$ under the $\mathbb{Q}^B$ measure is given by:*

$$\mathrm{d}X(t) = \mu^B(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}W^B(t), \quad X(0) = x_0.$$

# From $\mathbb{P}$ to $\mathbb{Q}$ in the Black-Scholes model

- Recall that in the Black-Scholes model the stock is driven under the real-world $\mathbb{P}$ measure by $\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{P}}(t)$, with constant parameters $\mu$ and $\sigma$.
- Under the risk-neutral $\mathbb{Q}$ measure the stock divided by the numéraire (which was the money-savings account, $\mathrm{d}M(t) = rM(t)\mathrm{d}t$) is a martingale.
- Here, we will relate these results to the concept of measure transformation. Applying Itô's lemma to $S(t)/M(t)$ with the stock gives us:

$$\mathrm{d}\frac{S(t)}{M(t)} = \frac{1}{M(t)}\mathrm{d}S(t) - \frac{S(t)}{M^2(t)}\mathrm{d}M(t) + \frac{1}{M^3(t)}(\mathrm{d}M(t))^2 - \frac{1}{M^2(t)}\mathrm{d}S(t)\mathrm{d}M(t).$$

- Remember that by the Itô's table, in the second chapter, the third and the fourth terms will vanish, so that we should consider only

$$\mathrm{d}\frac{S(t)}{M(t)} = \frac{1}{M(t)}\mathrm{d}S(t) - \frac{S(t)}{M^2(t)}\mathrm{d}M(t).$$

# From $\mathbb{P}$ to $\mathbb{Q}$ in the Black-Scholes model

- By substitution of the dynamics of $S(t)$ and $M(t)$, we further obtain:

$$\begin{aligned}\mathrm{d}\frac{S(t)}{M(t)} &= \frac{1}{M(t)}\left(\mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{P}}(t)\right) - rM(t)\frac{S(t)}{M^2(t)}\mathrm{d}t\\ &= \mu\frac{S(t)}{M(t)}\mathrm{d}t + \sigma\frac{S(t)}{M(t)}\mathrm{d}W^{\mathbb{P}}(t) - r\frac{S(t)}{M(t)}\mathrm{d}t.\end{aligned}$$

- It is known that $S(t)/M(t)$ is a martingale and by Girsanov Theorem we know that Itô integrals are martingales when they do not contain any drift terms.
- Moreover we also know that the volatility is not affected by a change of measure. Therefore, under the risk-neutral measure, the ratio $S(t)/M(t)$ should be of the following form:

$$\mathrm{d}\frac{S(t)}{M(t)} = \sigma\frac{S(t)}{M(t)}\mathrm{d}W^{\mathbb{Q}}(t),$$

# From $\mathbb{P}$ to $\mathbb{Q}$ in the Black-Scholes model

- By equating both equations for $\mathrm{d}S(t)/M(t)$, we find:

$$\mu\frac{S(t)}{M(t)}\mathrm{d}t+\sigma\frac{S(t)}{M(t)}\mathrm{d}W^{\mathbb{P}}(t)-r\frac{S(t)}{M(t)}\mathrm{d}t=\sigma\frac{S(t)}{M(t)}\mathrm{d}W^{\mathbb{Q}}(t),$$

  which is equivalent (as both $S(t)$ and $M(t)$ are positive) to

$$\mathrm{d}W^{\mathbb{Q}}(t)=\frac{\mu-r}{\sigma}\mathrm{d}t+\mathrm{d}W^{\mathbb{P}}(t).$$

- Equation above determines the measure transformation from the real-world $\mathbb{P}$ measure, to the risk-neutral $\mathbb{Q}$-measure, under the Black-Scholes model. So, we have,

$$\begin{aligned}\mathrm{d}S(t) &= \mu S(t)\mathrm{d}t+\sigma S(t)\mathrm{d}W^{\mathbb{P}}(t)\\ &= \mu S(t)\mathrm{d}t+\sigma S(t)\left(\mathrm{d}W^{\mathbb{Q}}(t)-\frac{\mu-r}{\sigma}\mathrm{d}t\right)\\ &= rS(t)\mathrm{d}t+\sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t).\end{aligned}$$

# Example: Stock as the Numéraire

- Assuming Black-Scholes market, with the money-savings account $M(t)$ and standard lognormal process for $S(t)$:

$$\begin{aligned} \mathrm{d}S(t) &= rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t) \\ \mathrm{d}M(t) &= rM(t)\mathrm{d}t. \end{aligned}$$

- We want to determine the price of the following derivative:

$$V(t_0) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}\max(S^2(T) - S(T)K, 0)\Big|\mathcal{F}(t_0)\right].$$

# Example: Stock as the Numéraire

- We perform the measure transformation, i.e.: we change the numeraire from the money-savings account to the stock. We define the following Radon-Nikodym derivative:

$$\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}} = \frac{S(T)}{S(t_0)}\frac{M(t_0)}{M(T)}.$$

- So the expectation becomes:

$$\begin{aligned}
V(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}\max(S^2(T) - S(T)K, 0)\middle|\mathcal{F}(t_0)\right] \\
&= \int_\Omega \frac{1}{M}\max(S^2 - SK, 0)\mathrm{d}\mathbb{Q} \\
&= \int_\Omega \frac{1}{M}\max(S^2 - SK, 0)\frac{S(t_0)}{S(T)}\frac{M(T)}{M(t_0)}\mathrm{d}\mathbb{Q}^S \\
&= \mathbb{E}^S\left[\frac{1}{M(T)}\max(S^2(T) - S(T)K, 0)\frac{S(t_0)}{S(T)}\frac{M(T)}{M(t_0)}\middle|\mathcal{F}(t_0)\right]
\end{aligned}$$

# Stock as the Numéraire

So:

$$
\begin{aligned}
V(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}\max(S^2(T)-S(T)K,0)\big|\mathcal{F}(t_0)\right]\\
&= \mathbb{E}^{S}\left[\frac{1}{M(T)}\max(S^2(T)-S(T)K,0)\frac{S(t_0)}{S(T)}\frac{M(T)}{M(t_0)}\big|\mathcal{F}(t_0)\right]\\
&= \mathbb{E}^{S}\left[\max(S(T)-K,0)\frac{S(t_0)}{M(t_0)}\big|\mathcal{F}(t_0)\right].
\end{aligned}
$$

- How to find the dynamics of $S(t)$ under the $\mathbb{Q}^S$ measure?
- Which quantities are martingales?
- What is the discounting process?
- Does it make any sense?

# Stock as the Numéraire

- Under the $\mathbb{Q}^S$ measure we discount the future values not with the money-savings account but with the Stock account.
- Thus $\mathrm{d}\frac{M(t)}{S(t)}$ needs to be a martingale, i.e.: we expect the following dynamics:

$$\mathrm{d}\frac{M(t)}{S(t)} = 0 \cdot \mathrm{d}t + \boxed{?}\,\mathrm{d}W^S(t).$$

By Itô's lemma we have:

$$\begin{aligned}
\mathrm{d}\frac{M(t)}{S(t)} &= -\frac{1}{S^2(t)}M(t)\left(rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t)\right) + \frac{1}{S}rM(t)\mathrm{d}t + \frac{1}{S^3}M\sigma^2S^2\mathrm{d}t\\
&= -\frac{M(t)}{S(t)}\left(r\mathrm{d}t + \sigma\mathrm{d}W^{\mathbb{Q}}(t)\right) + r\frac{M(t)}{S(t)}\mathrm{d}t + \frac{M(t)}{S(t)}\sigma^2\mathrm{d}t.
\end{aligned}$$

# Stock as the Numéraire

Which finally gives us:

$$\begin{aligned}\mathrm{d}\frac{M(t)}{S(t)} &= -\frac{M(t)}{S(t)}\left(r\mathrm{d}t+\sigma\mathrm{d}W^{\mathbb{Q}}(t)\right)+r\frac{M(t)}{S(t)}\mathrm{d}t+\frac{M(t)}{S(t)}\sigma^2\mathrm{d}t\\ &= \frac{M(t)}{S(t)}\sigma^2\mathrm{d}t-\sigma\frac{M(t)}{S(t)}\mathrm{d}W^{\mathbb{Q}}(t).\end{aligned}$$

- It is not a martingale... what to do?
- We know that $M(t)/S(t)$ is a martingale therefore we know that under the $\mathbb{Q}^S$ measure the process is driftless, this implies that we can deduct the measure-transform Brownian motion from the process above:

$$\frac{M(t)}{S(t)}\sigma^2\mathrm{d}t-\sigma\frac{M(t)}{S(t)}\mathrm{d}W^{\mathbb{Q}}(t)=-\sigma\frac{M(t)}{S(t)}\mathrm{d}W^S(t),$$

- which gives:

$$\mathrm{d}W^{\mathbb{Q}}(t)-\sigma\mathrm{d}t=\mathrm{d}W^S(t).$$

# Stock as the Numéraire

- Therefore after the substitution we find:

$$\begin{aligned}\mathrm{d}\frac{M(t)}{S(t)} &= \frac{M(t)}{S(t)}\sigma^2\mathrm{d}t - \sigma\frac{M(t)}{S(t)}\mathrm{d}W^{\mathbb{Q}}(t)\\ &= -\sigma\frac{M(t)}{S(t)}\mathrm{d}W^S(t),\end{aligned}$$

  with the following measure transformation:

$$\mathrm{d}W^{\mathbb{Q}}(t) - \sigma\mathrm{d}t = \mathrm{d}W^S(t).$$

- So now we know how to derive the dynamics of the stock process $S(t)$ under the $\mathbb{Q}^S$ measure:

$$\begin{aligned}\mathrm{d}S(t) &= rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t)\\ &= rS(t)\mathrm{d}t + \sigma S(t)\left(\mathrm{d}W^S(t) + \sigma\mathrm{d}t\right)\\ &= \left(r+\sigma^2\right)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^S(t).\end{aligned}$$

# Stock as the Numéraire

- What is the distribution of $S(T)$ under the $\mathbb{Q}^S$?
- It is still a log-normal distribution, but with adjusted drift parameter.

Thus, finally we can simply evaluate the expectation:

$$V(t) = S_0\mathbb{E}^{\boxed{S}}\left[\max(S(T) - K, 0)\right],$$

just like in the standard Black-Scholes case with stock $S(t)$ driven by the following SDE:

$$\mathrm{d}S(t) \quad = \quad \left(r + \sigma^2\right) S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\boxed{S}}(t).$$

# 2D stock processes

Let us consider the Black-Scholes model under the risk-neutral measure, $\mathbb{Q}$, where the dynamics of two stocks are driven by the following system of the SDEs:

$$\begin{aligned}
\mathrm{d}S_1(t) &= rS_1(t)\mathrm{d}t + \sigma_1 S_1(t)\mathrm{d}W_1^{\mathbb{Q}}(t),\\
\mathrm{d}S_2(t) &= rS_2(t)\mathrm{d}t + \sigma_2 S_2(t)\mathrm{d}W_2^{\mathbb{Q}}(t),
\end{aligned}$$

with $r$, $\sigma_1$, $\sigma_2$ constant and the correlation between Brownian motions: $\mathrm{d}W_1^{\mathbb{Q}}(t)\mathrm{d}W_2^{\mathbb{Q}}(t) = \rho\mathrm{d}t$. The money-savings account, $M(t)$, is given by $\mathrm{d}M(t) = rM(t)\mathrm{d}t$.

How to value the following payoff:

$$V(t_0, S_1, S_2) = M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}S_1(T)1_{S_2(T)>K}\middle|\mathcal{F}(t_0)\right],$$

for some $K \geq 0$.

# 2D stock processes

- We see that the derivative product presented above resembles a modified asset-or-nothing digital with stocks $S_1(T)$ and $S_2(T)$. This product pays-off amount $S_1(T)$ only is stock $S_2(T)$ is above certain prespecified limit $K$. This product exposes the holder to a correlation risk between stock $S_1(T)$ and $S_2(T)$.
- Let us now engage the change of measure technique presented in the previous section and instead of considering the money-savings account, $M(t)$, as a numéraire we consider the stock, $S_1(t)$, as the new numéraire.
- Application of the Radon-Nikodym technique gives us the following relation:

$$\frac{\mathrm{d}\mathbb{Q}^{S_1}}{\mathrm{d}\mathbb{Q}}\bigg|_{\mathcal{F}(T)} = \frac{S_1(T)M(t_0)}{S_1(t_0)M(T)},$$

where $\mathbb{Q}^S$ represents the measure where the stock, $S_1(t)$, is a numéraire and $\mathbb{Q}$ stands for the standard risk-free measure.

# 2D stock processes

- Under the new numéraire the value of the derivative becomes:

$$\begin{aligned} V(t_0) &= M(t_0)\mathbb{E}^{S_1}\left[\frac{S_1(T)}{M(T)}1_{S_2(T)>K}\frac{S_1(t_0)M(T)}{S_1(T)M(t_0)}\middle|\mathcal{F}(t_0)\right] \\ &= S_1(t_0)\mathbb{E}^{S_1}\left[1_{S_2(T)>K}\middle|\mathcal{F}(t_0)\right] \\ &= S_1(t_0)\mathbb{Q}^{S_1}[S_2(T)>K]. \end{aligned}$$

- Now, in order to determine the probability of the stock $S_2(T)$ to be larger than the strike, $K$, we need to determine the stock dynamics under the measure $\mathbb{Q}^{S_1}$.
- The new, $\mathbb{Q}^{S_1}$, measure is associated the the stock $S_1(T)$ being the numérare.
- Under the new stock-measure we need to ensure that all the market underlying processes are martingales.

# 2D stock processes

- As in the current model we consider the market with only three assets: money-savings account, $M(t)$, the stock, $S_1(t)$, and the stock $S_2(t)$ the drift for stocks $S_1(t)$ and $S_2(t)$ need to be adjusted in the way that $M(t)/S_1(t)$ and $S_2(t)/S_1(t)$ are martingales.
- By Itô's lemma we have:

$$\begin{aligned}\mathrm{d}\left(\frac{M(t)}{S_1(t)}\right) &= \frac{1}{S_1(t)}\mathrm{d}M(t)-\frac{M(t)}{S_1^2(t)}\mathrm{d}S_1(t)+\frac{M(t)}{S_1^3(t)}(\mathrm{d}S_1(t))^2\\ &= \frac{M(t)}{S_1(t)}\left(\sigma_1^2\mathrm{d}t-\sigma_1\mathrm{d}W_1^{\mathbb{Q}}(t)\right).\end{aligned}$$

- This implies that the following measure transformation:

$$\mathrm{d}W_1^{\mathbb{Q}}(t)=\mathrm{d}W_1^{S_1}(t)+\sigma_1\mathrm{d}t,$$

# 2D stock processes

- Therefore the stock dynamics under the $\mathbb{Q}^{S_1}$ measure is given by:

$$\begin{aligned}\frac{\mathrm{d}S_1(t)}{S_1(t)} &= r\mathrm{d}t+\sigma_1\left(\mathrm{d}W_1^{S_1}(t)+\sigma_1\mathrm{d}t\right)\\ &= \left(r+\sigma_1^2\right)\mathrm{d}t+\sigma_1\mathrm{d}W_1^{S_1}(t).\end{aligned}$$

- For the second stock we have:

$$\mathrm{d}\left(\frac{S_2(t)}{S_1(t)}\right)\Big/\left(\frac{S_2(t)}{S_1(t)}\right) = \left(\sigma_1^2-\rho\sigma_1\sigma_2\right)\mathrm{d}t+\sigma_2\mathrm{d}W_2^{\boxed{\mathbb{Q}}}(t)-\sigma_1\mathrm{d}W_1^{\boxed{\mathbb{Q}}}(t).$$

- Therefore, we have:

$$\mathrm{d}\left(\frac{S_2(t)}{S_1(t)}\right)\Big/\left(\frac{S_2(t)}{S_1(t)}\right) = -\rho\sigma_1\sigma_2\mathrm{d}t+\sigma_2\mathrm{d}W_2^{\boxed{\mathbb{Q}}}(t)-\sigma_1\mathrm{d}W_1^{\boxed{S_1}}(t).$$

# 2D stock processes

- The expression above implies the following change of measure for the Brownian motion in the second stock process, $\mathrm{d}W_2^{\mathbb{Q}}(t)$,

$$\mathrm{d}W_2^{\mathbb{Q}}(t) = \mathrm{d}W_2^{S_1}(t) + \rho\sigma_1 \mathrm{d}t.$$

- Finally, the model under the stock-measure $\mathrm{d}W^{S_1}(t)$ is given by:

$$\begin{array}{rcrcl}
\mathrm{d}S_1(t) & = & (r+\sigma_1^2)S_1(t)\mathrm{d}t & + & \sigma_1 S_1(t)\mathrm{d}W_1^{S_1}(t),\\
\mathrm{d}S_2(t) & = & (r+\rho\sigma_1\sigma_2)S_2(t)\mathrm{d}t & + & \sigma_2 S_2(t)\mathrm{d}W_2^{S_1}(t),\\
\mathrm{d}M(t) & = & rM(t)\mathrm{d}t, & &
\end{array}$$

- Returning to our main problem, in order to calculate $\mathbb{Q}^{S_1}[S_2(T) > K]$, one can easily use the fact that the stock process $S_2(T)$ under the $\mathbb{Q}^{S_1}$-measure has the following solution:

$$S_2(T) = S_2(t_0)\exp\left[\left(r+\rho\sigma_1\sigma_2 - \frac{1}{2}\sigma_2^2\right)(T-t_0) + \sigma_2\left(W_2^{S_1}(T) - W_2^{S_1}(t_0)\right)\right],$$

which can be easily recognized as a $\boxed{\text{lognormal distribution.}}$

# The T-Forward Measure

- A zero-coupon bond is a contract with price $P(t,T)$, at time $t<T$, to deliver at time $T$, $P(T,T)=€1$.


Figure: Cash flow for a zero-coupon bond, $P(t,T)$, with the payment at time $T$.

# The T-Forward Measure

- Whenever we are dealing with stochastic discounting we may benefit from the so-called $T$-forward measure associated with the Zero Coupon Bond (ZCB), $P(t_0, T)$.
- A basic interest rate product is the zero-coupon bond, $P(t, T)$, which pays 1 currency unit at maturity time $T$, i.e. $P(T, T) = 1$. We are interested in its value at a time $t < T$.
- The *fundamental theorem of asset pricing* states that the price at time $t$ of any contingent claim with payoff, $H(T)$, is given by:

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(z)dz} H(T) \middle| \mathcal{F}(t)\right],$$

  where the expectation is taken under the risk-neutral measure $\mathbb{Q}$.
- The price of a zero-coupon bond at time $t$ with maturity $T$ is thus given by:

$$P(t, T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(z)\mathrm{d}z} \middle| \mathcal{F}(t)\right],$$

  since $H(T) = V(T) = P(T, T) \equiv 1$.

# The T-Forward Measure

- Since $P(t,T)$ is a tradable asset it can be used as a numéraire.
- A change of measure, from the risk-neutral measure $\mathbb{Q}$ which is implied by the money-savings account $M(t)$, to a measure implied by the zero-coupon bond $P(t,T)$, with $t_0 < t < T$, requires the following Radon-Nikodym derivative:

$$\boxed{\lambda_{\mathbb{Q}}^{T}(t) = \frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}}\Big|_{\mathcal{F}(t)} = \frac{P(T,T)}{P(t_0,T)}\frac{M(t_0)}{M(T)}.}$$

- Therefore if we consider a payoff with stochastic discounting we have the following measure transformation:

$$\begin{aligned} V(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T)}H(T)\Big|\mathcal{F}(t_0)\right] \\ &= \mathbb{E}^{T}\left[\frac{M(t_0)}{M(T)}H(T)\frac{P(t_0,T)}{P(T,T)}\frac{M(T)}{M(t_0)}\Big|\mathcal{F}(t_0)\right] \\ &= P(t_0,T)\mathbb{E}^{T}\left[H(T)\Big|\mathcal{F}(t_0)\right]. \end{aligned}$$

# Summary

- Filtration
- Conditional Expectations
- Conditional Expectations in Python
- Option Pricing Using Conditional Expectation
- Convergence Experiment in Python
- Concept of Numeraire
- From P to Q in the Black-Scholes Model
- Change of Numeraire: Stock Measure
- Change of Numeraire: Dimension Reduction
- The T-Forward Measure
- Summary of the Lecture + Homework

# Exercises

The solutions for the homework can be find at https://github.com/LechGrzelak/QuantFinanceBook

- **Exercise 1.9**
  Show that, for a continuously, differentiable function $g(t)$, the process

$$X(t) = g(t)W(t) - \int_0^t \frac{\mathrm{d}g(z)}{\mathrm{d}z} W(z)\mathrm{d}z,$$

is a martingale, and subsequently show that

$$\mathbb{E}[\mathrm{e}^{2t}W(t)] = \mathbb{E}\left[\int_0^t 2\mathrm{e}^{2z}W(z)\mathrm{d}z\right].$$

# Exercises

- **Exercise 2.1**
  Apply Itô's lemma to find:
  a. The dynamics of process $g(t) = S^2(t)$, where $S(t)$ follows a log-normal Brownian motion given by:
  $$\frac{\mathrm{d}S(t)}{S(t)} = \mu \mathrm{d}t + \sigma \mathrm{d}W(t),$$
  with constant parameters $\mu$, $\sigma$ and Wiener process $W(t)$.
  b. The dynamics for $g(t) = 2^{W(t)}$, where $W(t)$ is a standard Brownian motion. Is this a martingale?

# Exercises

- **Exercise from the Lecture**
  As discussed in this lecture, compute the value of the following derivative, using Monte Carlo and compare to analytical expression based on the measure change:

$$V(t_0, S_1, S_2) = M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}S_1(T)1_{S_2(T)>K}\middle|\mathcal{F}(t_0)\right],$$

  for some $K \geq 0$.
- Consider $T = 5$, $S_1(t_0) = 1$, $S_2(t_0) = 2$, $r = 0.05$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$ and $\rho = 0.7$.
- Perform pricing for a range of strikes, $K$, from 0.1 to 5 and plot the results $(K, V(t_0))$.

# Exercises

- **Exercise from the Lecture**
  Adjust the Python code for conditional expectation (slide 17) to properly handle discounting and pricing of put options.
- **Exercise from the Lecture**
  Perform pricing from slide 24: Monte Carlo simulation under measure $\mathbb{Q}$ vs. analytical expression under measure $\mathbb{Q}^S$.