Master Thesis Notes

Xiangwen Guan

update: February 16, 2024

Contents

1 continum limit

A critical point can be defined for matrix integral in the $N \to \infty$ imit. Consider the partition function $Z(g; N)$

$$\mathrm{e}^Z = \int [\mathrm{d}M] \mathrm{e}^{-\frac{N}{g} V[M]}. \tag{1}$$

and denote the leading order in $N \to \infty$ as $Z_0(g)$. This corresponds to the planar diagrams. The critical point g_* then is conventionally defined as the point where

$$Z_0(g) \sim (g - g_*)^{2-\gamma}.$$

The critical exponent γ is introduced in such a way to make contact with that of string susceptibility (in the contex of 2d gravity). This definition follows ✎ **Francesco92**.

The motivation for introducing the critical point is that at this point the matrix model is argued to generate the partition function of a continuous 2d surface. This can be proved order by order: the order in N labels the genus and the order in g labels the area. It is around the critical point that the "macroscopic" area contributes significantly to the partition function. However, this "order by order argument" leads to a non-perturbative statement which is skeptical... One should also note that Z is the ln of the matrix integral, or the matrix partition function. It's the free energy in the matrix model sense.

What's the value of γ? The typical value that is calculated in conformal field theory of 2d gravity is $\gamma = -1/m, m = 2, 3, \cdots$. The value m is related to the central charge. In these cases, g_* is the end point of branch cuts of $Z_0(g)$. Let's remark on the position of g_*. The matrix integral is usually well defined on the positive real axis $g > 0$. However, to locate the critical point, one needs to continue g

to the complex plane. The g_* generally locate at the extended regime. A typical example for quartic action $V(M) = \frac{1}{2}M^2 + \frac{1}{4}M^4$ will give a critical point $g_* = -\frac{1}{12}$. The calculation is sketched in **🔗 Francesco92**. I will review it in the following.

1.1 methods of calculating the critical point (the resolvent method)

It's possible to determine g_* by the resolvent method.

This method bases on the assumption that the $N \to \infty$ limit of matrix model can be studied according to a contitnuous function $\rho(\lambda)$. It's defined as

$$\rho(\lambda) := \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle.$$

$\langle \cdots \rangle$ denotes the average over the probability distribution; λ_i are the eigenvalues of the matrices in the ensemble. By definition the moment $\langle \frac{1}{N} \mathrm{Tr} M^k \rangle$ can be calculated by $\int \lambda^k \rho(\lambda) \mathrm{d}\lambda$.

I'd like to add here a remark on $\rho(\lambda)$. It's by no mean a complete characterization of the matrix model. In particular, it's impossible to rewrite the $\exp(-S[M])$ in terms of $\rho(\lambda)$. Interesting quantities, like "average level spacing" or trace product $\langle \mathrm{Tr} M^k \mathrm{Tr} M^j \rangle$, can not be calculated from $\rho(\lambda)$. A proper understanding of $\rho(\lambda)$ is rather than telling us how the probability distribution looks like, it tells us what's the "typical matrix" of the matrix model in $N \to \infty$.

$\rho(\lambda)$ is analized by an auxillary function $\omega(z)$, the average trace of the resolvent,

$$\omega(z) := \left\langle \mathrm{tr} \frac{1}{z - M} \right\rangle. \tag{2}$$

It turns out that $\omega(z)$ is the Stieltjes transformation of $\rho(\lambda)$. Reversely, $\rho(\lambda)$ is determined by the discontinuity of $\omega(z)$ accrossing the branch cut. The analyticity of $\omega(z)$ will be determined by the saddle point equation of $N \to \infty$, It turns out that the singular part of $\omega(z)$ has the form of branch cuts of square root. The end points of the branch cuts are essentially determined by the action $S[M]$. After solving for the $\omega(z)$, one can use the saddle point approximation to evaluate $Z_0(g)$, for which the "on shell action" is evaluated at $\rho(\lambda)$.

Let's illustrate the result for the quartic potential $V(\lambda) = \frac{1}{2}\lambda^2 + \frac{1}{4}\lambda^4$. There is only one branch cut $[-a, a]$, and the end point turns out to be

$$a^2(g) = \frac{2}{3}(-1 + \sqrt{1 + 12g}).$$

a decreases to 0 as g decreasing to 0. The matrix model is ill defined for $g \leq 0$, however the partition function can be defined by continuation. Then for $g < 0$, the branch cut is on the imaginary axis. Until $g = g_* = -\frac{1}{12}$, one expects a branch cut singularity, which turns out to be the critical point of $Z_0(g)$. I will simply cite

the equations that can be used to study the singularity of $Z_0(g)$

$$\frac{\partial}{\partial g}\left(g^3\frac{\partial Z_0(g)}{\partial g}\right) \equiv gu(g), \tag{3}$$

$$\frac{\partial gu(g)}{\partial g} = 2N^2 g\frac{\ln a(g)}{\partial g}. \tag{4}$$

By integrating one gets the leading singularity of $Z_0(g)$ is

$$Z_0(g) \sim -\frac{4}{15}N^2 x^{\frac{5}{2}} + \cdots \tag{5}$$

with $x = 1 - \frac{g}{g_c}$.

It's also possible to determine g_* with the orthogonal polynomial method. Assume that one finds the family of orthogonal polynomials with the normalization

$$\int(\mathrm{d}\lambda)e^{-\frac{N}{g}\mathrm{tr}V(\lambda)}p_n(\lambda)p_m(\lambda) = s_n\delta_{nm}. \tag{6}$$

To obtain the continuum limit, one must first take $N \to \infty$, then replacing $s_n/s_{n-1} = f(\xi), \xi = n/N$ with a continuous function. The matrix partition function $Z(g)$ then can be calculated from $f(\xi)$ through

$$\lim_{N\to\infty}\frac{1}{N^2}\ln Z(g) \sim \int \mathrm{d}\xi(1-\xi)\ln f(\xi). \tag{7}$$

We use $\sim$ to ignore additive constant which is irrelevent to the asymptotic $g \to g_*$. It turns out that around the critical point $g \approx g_*$, $f(\xi)$ should have the following scaling

$$f(\xi) - f_* \sim (g_* - g\xi)^{-\gamma}. \tag{8}$$

where f_* is some constant determined by the potential. This will give the expected scaling of $\ln Z(g)$.

The definition of g_* above based on $N \to \infty$. Is it possible to define it for finite N?

2   RG equation for continuum limit

In this section, I review the RG method proposed in ✎ **BZ92**. This is proposed as another way to find the critical point of continuum limit, so $N \to \infty$ is imposed in the calculation. They use an even potential as example $V(\lambda) = \frac{1}{2}\lambda^2 + \frac{g}{4}\lambda^4$, whose critical point is $g_* = -\frac{1}{12} < 0$. It's hard to make sense of the matrix integral with $g < 0$, so it should be understood as analytic continuation.

How does this method give the critical point? The critical point is defined as the point at where the matrix model free energy $Z_N(g)$ has a scaling law. Then this scaling law is formulated as a Callan-Symanzik like equation

$$\left[N\frac{\partial}{\partial N} - \beta(g)\frac{\partial}{\partial g} + \gamma(g)\right]Z_N(g) = r(g).$$

At the fix point $\beta(g_*) = 0$, the scaling law can be recovered. Such an equation will arise from the RG calculation, from which one could obtain the functions

3

$\beta(g), \gamma(g), r(g)$. Then one can find the point g_* where the free energy exhibits proper scaling law.

Another point is that an RG method could allow us to study more general matrix models, especially those without an eigenvalue representation. This application of the RG method appears in ✎ **KT13** in which the "Yang-Mills two-matrices model" is studied. This model is characterized by a term $[A_1, A_2]^2$ in the action. Such a term can arise from the Yang-Mills Lagrangian $F^2 = (\mathrm{d}A)^2$ by ignoring the derivative terms. The gauge group of this model is $U(N)$.

2.1 simpler example: the vector model

Integration could be studied using an RG spirit method, which is similar to the Wilson's RG scheme. However, there is no natural notion of energy associated with the integral.

Let's take the "$O(N)$ vector model" as the example ✎ **Zinn14**

$$e^{Z_N} = \left(\frac{N}{2\pi}\right)^{N/2} \int \mathrm{d}^N \mathbf{x} e^{-NV(\mathbf{x}^2)}. \tag{9}$$

The normalization is chosen to normalize a Gaussian integral. The factor N before $V(\mathbf{x}^2)$ gives a "dimension" of V under the scaling of N. To implement the changing of N, one calculates Z_{N+1} by integrating out one component of the vector $\mathbf{x}$

$$e^{Z_{N+1}} = \left(\frac{N+1}{2\pi}\right)^{(N+1)/2} \int \mathrm{d}^N \mathbf{x} \int \mathrm{d}y e^{-(N+1)V(\mathbf{x}^2+y^2)} \tag{10}$$

It's impossible to integrate out y exactly in general, so one may use the saddle point approximation. For a generic $\mathbf{x}$, the integrand $e^{-(N+1)V(\mathbf{x}^2+y^2)}$ is localized at the critical value of y when $N \to \infty$. To the leading order, assuming that the saddle point is at $y = 0$, the approximation gives

$$e^{Z_{N+1}} = \left(\frac{N+1}{2\pi}\right)^{N/2} \int \mathrm{d}^N \mathbf{x} e^{-(N+1)V(\mathbf{x})-\frac{1}{2}\ln 2V'(\mathbf{x}^2)} \left(1 + O(\frac{1}{N})\right).$$

It should be noted that the saddle point approximation fail when $V'(\mathbf{x}^2) = 0$. The Gaussian potential $V(\mathbf{x}^2) = (1/2)\mathbf{x}^2$ will give $V'(\mathbf{x}^2) = 1/2$. In this case, the leading order vanishes.

It's not clear to what extent the subleading corrections will modify the scaling around the critical point; however, I guess they are not very important if the critical point by itself requires $N \to \infty$ (similar to the fact that phase transition requires thermodynamic limit). Due to various methods to solve $N \to \infty$ model, it deserves study; however, is it possible to obtain a matrix model that has scaling covariance for all N? This is a stringent requirement, but there is example of matrix model that is independent of N. For example, see the discussion of the Kontsevich integral in ✎ **Morozov94** below the equation (3.70).

Come back to the discussion of the vector model. The change $N \to N + 1$ induces

a change in $\mathbf{x}$ and V

$$\mathbf{x} \to \mathbf{x}\left(1 - \frac{1}{2N}(1+\gamma)\right), \tag{11}$$

$$V \to V + \delta V \tag{12}$$

such that the partition function scales as

$$Z_{N+1}(V) = -\frac{1}{2}\gamma + Z_N(V + \delta V) + O(\frac{1}{N}) \tag{13}$$

The $-\frac{1}{2}\gamma$ term comes form the scaling of the integration measure $\mathrm{d}^N\mathbf{x}$. The $V + \delta V$ comes from the integration over y, also the scaling of $\mathbf{x}$

$$N\delta V(\rho) = V(\rho) - (1+\gamma)\rho V'(\rho) + \frac{1}{2}\ln 2V'(\rho) + O(\frac{1}{N}).$$

the ρ is a shorthand notation for $\mathbf{x}^2$. The differential equation for the deformation of V along the RG flow reads then

$$\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}V(\rho, \lambda) = V(\rho, \lambda) - (1+\gamma(\lambda))\,\rho V'(\rho, \lambda) + \frac{1}{2}\ln 2V'(\rho, \lambda) \tag{14}$$

where λ is a continuous parameter $N \to \lambda N$. Note that γ is assumed to depend on N.

$V(\rho) = \frac{1}{2}\rho$ is a fix point with $\gamma = 0$, the Gaussian fixed point. It's not obvious in general how to solve the fix point equation due to the non-linear term $\ln V'$. One possible solution is

$$V(\rho) = \frac{1}{2}\ln \rho.$$

with $\gamma = -1$ such that the second term vanish. If plugging this back to the integral, one finds the integrand is $\rho^{-N/2}$. This cancels with the radius measure in spherical coordinate. Therefore, this potential corresponds to a trivial vector model.

In general, it turns out that it's easier to solve the equation for V'. Actually, if one define a function $R(\rho)$

$$R(\rho) \equiv \frac{1}{2V'(\rho)}.$$

Then the fix point equation will simply be

$$\gamma R + RR' - (1+\gamma)\rho R' = 0 \tag{15}$$

The $\gamma = -1$ case gives $R' = 1$, which corresponding to the solution $V(\rho) = \frac{1}{2}\ln \rho$. Then how to obtain the solution of, for example $\gamma = -2$?

Note that the finite scaling of $\mathbf{x}$ is

$$\mathbf{x} \to \mathbf{x}\left(1 + \frac{1+\gamma}{2}\ln \lambda\right).$$

Instead of power of λ, it scales as $\ln \lambda$.

2.2 a matrix integral calulation

Similar calculation is done in the following references: ... I'll present a slightly different method which is natural to consider.

The model under consideration is the one-matrix ϕ^4 model

$$\zeta_N(g) = \int d\phi_N \exp(-S_N[\phi_N, g]), \tag{16}$$

with an action

$$S_N[\phi_N, g] = \text{Tr}(\frac{1}{2}\phi_N^2 + \frac{g}{4}\phi_N^4). \tag{17}$$

Start with the rank $N+1$ model, and decomposing the matrix ϕ_{N+1} as

$$\phi_{N+1} = \begin{pmatrix} \phi_N & v \\ v^\dagger & \alpha \end{pmatrix}. \tag{18}$$

Then the action can be expanded as

$$S_{N+1}[\phi_{N+1}, g] = \text{Tr}\left(\frac{1}{2}\phi_N^2 + \frac{g}{4}\phi_N^4\right) + v^\dagger v + \frac{1}{2}\alpha^2$$
$$+ g\left(v^\dagger \phi_N^2 v + \alpha v^\dagger \phi_N v + \alpha^2 v^\dagger v + \frac{1}{2}(v^\dagger v)^2 + \frac{1}{4}\alpha^4\right). \tag{19}$$

Now the trace is over $N \times N$ matrix.

Recall that the matrix model is $U(N)$-invariant; it may be useful to first gauge away some variables in v, a. Let's consider how the gauge transformation acting on ϕ_N, v and a. The infinitesimal transformation reads

$$\delta_t \phi_N = i(vt^\dagger - tv^\dagger)$$
$$\delta_t v = i(\phi_N - \alpha 1)t$$
$$\delta_t v^\dagger = it^\dagger(\alpha 1 - \phi_N)$$
$$\delta_t \alpha = i(v^\dagger t - t^\dagger v) \tag{20}$$

where t ($N \times 1$ vector) is the components of the following generator

$$T = \begin{pmatrix} 0 & t \\ t^\dagger & 0 \end{pmatrix}.$$

It's possible to choose t to gauge away v if $\phi_N - \alpha 1$ is not degenerate. So let's impose the gauge fixing condition as $v = 0, v^\dagger = 0$.

How to implement this condition in the integral? Essentially what we should do is to change the integration variable from v to t. t is understood as the parameter for the coset $U(N+1)/U(N)$ through exponent. Around infinitesimal neighbour of $v = 0$, $v(t)$ is a linear function given by the infinitesimal gauge transformation

$$v(t) = -i(\phi_N - \alpha 1)t.$$

Then the "coordinate transformation" will give a Jacobian that is proportional to $\det(\phi_N - \alpha 1)$. The same factor will be obtained from $v^\dagger$. Therefore we get

$$[\det(\phi_N - \alpha 1)]^2$$

in the integrand, which can be re-exponentiate to give a term in the action

$$S_{N+1}[\phi_{N+1}, g] = \text{Tr}\left(\frac{1}{2}\phi_N^2 + \frac{g}{4}\phi_N^4\right) + \frac{1}{2}\alpha^2 + \frac{g}{4}\alpha^4 - 2\text{Tr}\ln(\phi_N - \alpha 1). \quad (21)$$

Note that this formula is equivalent to the eigenvalue representation if we diagonalize ϕ_N.

However, the above calculation based on the linearized version of gauge transformation. The full gauge transformation is more complicate. We can proceed as following. Consider a $(N + 1) \times (N + 1)$-matrix $\phi_{N+1}(\phi_N, 0, a)$ with $v = 0$. Then do a gauge transformation, parameterized by t, on this matrix $\phi_{N+1} \to \tilde{\phi}_{N+1} = U(t)\phi_{N+1}U^\dagger(t)$. The idea is that the integration over $\tilde{\phi}_{N+1}$ can be replaced with an integration over ϕ_{N+1} and t, with a proper Jacobian that taking into account the functional dependence $\tilde{\phi}_{N+1}(\phi_{N+1}, t)$. In the exponent, because of the gauge invariance, one can replace $\tilde{\phi}$ directly with ϕ. To write down the Jacobian, one needs to solve for the $\tilde{v}(t)$, which is fixed by the condition $v = 0$. It turns out that the t-dependence of the Jacobian can be factorized out, so the t-integral can be performed independently. This factorization property also implies that the linearized result giving the exact Jacobian for the ϕ factor, although we don't know the t-factor.

In summary, we have

$$\int d\phi_N d\alpha [\det(\phi_N - \alpha 1)]^2 e^{-\frac{1}{2}\alpha^2 - \frac{g}{4}\alpha^4 - S_N[\phi_N, g]} \quad (22)$$

or

$$\int d\phi_N d\alpha\, e^{-\frac{1}{2}\alpha^2 - \frac{g}{4}\alpha^4 - S_N[\phi_N, g] + 2\text{Tr}\ln(\phi_N - \alpha 1)} \quad (23)$$

The α integral is difficult to understand. But I try to study it as follows.

The determinant is a characteristic polynomial $\det(\alpha 1 - \phi_N) \equiv p_N(\alpha)$ of ϕ_N in α with coefficients given by the following formula

$$p_N(\alpha) = \sum_{k=0}^{N} \alpha^{N-k}(-1)^k \text{Tr}\left(\wedge^k \phi_N\right), \quad (24)$$

with

$$\text{Tr}\left(\wedge^k \phi\right) = \frac{1}{k!}\det\begin{vmatrix} \text{Tr}\phi & k-1 & 0 & \cdots & 0 \\ \text{Tr}\phi^2 & \text{Tr}\phi & k-2 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \text{Tr}\phi^{k-1} & \text{Tr}\phi^{k-2} & & \ddots & 1 \\ \text{Tr}\phi^k & \text{Tr}\phi^{k-1} & \text{Tr}\phi^{k-2} & \cdots & \text{Tr}\phi \end{vmatrix}$$

The leading term of $p_N(\alpha)$ is α^N, and the last term is $(-1)^N \det\phi_N$. Also, let's write down first few terms to get a feeling

$$p_N(\alpha) = \alpha^N - \alpha^{N-1}\text{Tr}\phi + \frac{1}{2}\alpha^{N-2}\left[(\text{Tr}\phi)^2 - \text{Tr}\phi^2\right] - \frac{1}{6}\alpha^{N-3}\left[(\text{Tr}\phi)^3 - 3\text{Tr}\phi\text{Tr}\phi^2 + \text{Tr}\phi^3\right] + \cdots$$

If we square it as in the integrand, we get

$$p_N^2(\alpha) = \alpha^{2N} - 2\alpha^{2N-1}\mathrm{Tr}\phi + \alpha^{2N-2}\left[2(\mathrm{Tr}\phi)^2 - \mathrm{Tr}\phi^2\right]$$
$$- \alpha^{2N-3}\left[\frac{4}{3}(\mathrm{Tr}\phi)^3 - 2(\mathrm{Tr}\phi)(\mathrm{Tr}\phi^2) + \frac{1}{3}\mathrm{Tr}\phi^3\right]\cdots \qquad (25)$$

One can only consider even terms if the potential is even. For each term, α is decoupled from ϕ, therefore can be integrated out.

To simpilfy the result, the following equation maybe useful

$$2\left\langle \mathrm{Tr}\phi^2\right\rangle + \left\langle(\mathrm{Tr}\phi)^2\right\rangle = \left\langle \mathrm{Tr}(\phi^4 + g\phi^6)\right\rangle$$
$$2\left\langle \mathrm{Tr}\phi^3\right\rangle + 2\left\langle(\mathrm{Tr}\phi)(\mathrm{Tr}\phi^2)\right\rangle = \left\langle \mathrm{Tr}(\phi^5 + g\phi^7)\right\rangle \qquad (26)$$

The $\langle\rangle$ indicates the matrix integral. These equations follow from changing the integral variable $\phi \to \phi + \epsilon\phi^{3,4}$, with infinitesimal ϵ. The left hand side comes from the change of measure; while the right hand side follows from the change of the action. For example, the α^{2N-2} term becomes

$$\alpha^{2N-2}\left[\mathrm{Tr}(-5\phi^2 + 2\phi^4 + 2g\phi^6)\right].$$

As for the integration over α

$$\int(\mathrm{d}\alpha)\alpha^{2N-k}e^{-\frac{1}{2}\alpha^2 - \frac{g}{4}\alpha^4}.$$

The same trick allows us to derive that

$$(2N-k)\langle\alpha^{2N-k-1}\rangle = \langle\alpha^{2N-k+1}\rangle + g\langle\alpha^{2N-k+3}\rangle.$$

For example $k = 1$

$$\langle\alpha^{2N-2}\rangle = \frac{1}{2N-1}(\langle\alpha^{2N}\rangle + g\langle\alpha^{2N+2}\rangle).$$

$k = 3$

$$\langle\alpha^{2N-4}\rangle = \frac{1}{2N-3}(\langle\alpha^{2N-2}\rangle + g\langle\alpha^{2N}\rangle).$$

For small g, it's reasonable that we only keep the $\alpha^{2N}, \alpha^{2N-2}$ terms in large N limit. For a general g, it's not clear that whether or not $\alpha^{2N-2}, \alpha^{2N-4}$ are in the same order or not.

2.3 confusing remarks on RG flow

Intuition of RG flow starts from statistical model, but is also applied in studying various QFT. Lattice spacing is a natural scale in statistical model, which allows us to define a family of models with different scales but related with the same observables. Classical field theory, on the other hand, as a continuum theory, is not defined with a particular scale: a continuum model is defined naturally in all scales. However, one can still define a family of models by rescaling various ingredients: x, ϕ, g etc. There is a natural way to do that such that the functional form of

the action doesn't change along the rescaling. This way of rescaling gives the notion of "engineer dimension" of those ingredients. Without any additional input, there maybe ambiguous in how to rescale: for example, if one ingredient (or a certain combination of two ingredients) is homogeneous, then one can give it an arbitrary engineer dimension. The most common input is like: the action should be dimensionless (in natural unit), but more may necessary to eliminate the ambiguity. But such rescaling does not lead to any interesting RG flow: no coarse-graining is involved because a continuum model is defined naturally in all scales. Instead, the rescaling invariance plays a more important role. When it comes to QFT, the theory is usually defined in the form of path integral of classical field theory. This is the formalism where the analogs with a statistical model is intriguing, which makes a nontrivial notion of RG flow possible while the rescaling invariance still present in the action. The invariant property in QFT is usually formulated in terms of the Ward identity, while the coarse-graining procedure naturally appears when one realizes that a regulator is necessary to make sense the path integral measure. Usually, regulator will introduce a particular scale in the theory, just like what happens in statistical model. A difference maybe, one doesn't think the cut-off scale is physical in QFT, but it may be taken as essential in a statistical model (a part of definition of the model). Fortunately, RG flow method doesn't care this philosophical aspect of that scale, but just ask what happens if a coarse-graining procedure taking us from one scale to a nearby another scale. Moreover, if there exist a fixed point, then one may relieve: a certain continuum theory is stable along the coarse-graining.

3   related references

In this section I will give a short remark on references. The idea stems from ✎ **BZ92**. A more systematic way to extend the calculation to subleading orders was presented at ✎ **Aya93**. The idea of using the reparameterization identity to reduce the number of coupling constants in the flow equation appears in ✎ **HIS93**; A more compact flow equation that realizes this idea is given in a following paper ✎ **HINS94-a**. A similar discussion that involving a two-matrix model is presented in ✎ **HINS94-b**.