Master Thesis

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Abstract

The continuum limit of $U(N)$-invariant one-matrix model is reviewed using different methods. The emphasis is on a renormalization group (RG) inspired method, the matrix RG, which enables us to interpret the continuum limit as a fixed point of the RG flow. This method is applied to the IKKT matrix model to study the scaling behavior of the matrices and the 0d Yang-Mills coupling. The IKKT matrix model is a low energy effective action of multiple $D(-1)$-branes, and the scaling along the RG flow is understood as corresponding to the dilaton profile of the supergravity solution. The possibility to have a conformal matrix model, constructed by adding interactions to the IKKT matrix model, as an example of the AdS/CFT correspondence for the D(-1)/D7 brane system is discussed. This is motivated by a similar construction for the D1/D5 brane system. Along the way, I will also review some classical results of random matrix theory and their applications in physics, in particular, how geometry could emerge from matrix model.

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1 Introduction and motivation

Geometry has its roots in people's intuition of how the world looks: all objects have shapes and lengths, and in principle, should be infinitely divisible. To describe the geometry of a physical object numerically, one needs to assume a standard ruler for measuring length, and vice versa. Apart from the ruler used in geometry, there are other "ruler" in physics: clocks for time, and balances for mass. Length, time, and mass constitute the physical aspects of Newton's mechanics, with the rulers implicitly assumed. This implies the following assumptions in the theory: 1. The rulers are not dynamic: they do not play a role in the equations of motion (eom). 2. The rulers have no fundamental meaning: a change of rulers merely signifies a change of dimensions; the theory should remain invariant.

However, the rulers have a more fundamental meaning in modern physics, in the sense that they are often replaced by real physical objects. A central theme in theoretical physics research is to develop theories about fundamental particles. Photon is the best example. The complete theory of photon is quantum electrodynamics, which marriages the classical Maxwell theory for electromagnetic wave with the quantum mechanics for atoms. The Maxwell theory requires that the speed of light c is a constant in vacuum. The quantum mechanics requires that the ratio between the photon energy and frequency is the Planck constant $\hbar = E/\nu$. These two constants are therefore understood as intrinsic to the photon. Unlike the resistance of resistors, we believe that all photons have the same value of c and $\hbar$ in nature. It's therefore possible to use photon as a physical ruler to replace the clock and balance. The clock is adjusted with respect to the ruler (for length) such that the numerical value of $c = 1$; the balance is adjusted with respect to the clock such that the numerical value of $\hbar = 1$.

Length is still measured by an arbitrary ruler: there is no fundamental particle that can be used to provide an alternative to the ruler. A theory of gravity, if it could be described by a fundamental object and consistent with the theory of photon, would provide such a thing. The strength of gravity is controlled by the Newton constant G. It defines a basic length scale, the Planck scale, if $\hbar = c = 1$:

$$l_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}} = \sqrt{G}$$

If the Newton constant G, like c and $\hbar$ for photon, is an intrinsic constant for the graviton, it should be possible to use the graviton to measure length. However, it's hard to imagine that, in a classical picture of geometry, using a point particle to measure length. It is easier to imagine using a string, with certain intrinsic length scale l_{s}, to measure the length. An arbitrary string is not less arbitrary than an imagined ruler, but at least it has dynamics. The interesting things happen when one requires that c and $\hbar$ are also intrinsic to the string, that is, considering a relativistic quantum string. String theory is such a theory. It is a surprise that a theory of gravity, along many other things, follows automatically. This will give us a real dimensionless description of the physical world, without assuming any arbitrary ruler.

In this thesis, I will consider a class of models without assuming any ruler, clock and balance, in another word, dimensionless. They are matrix models, which can be understood as a physical model defined on a point, therefore no underlying geometry. With so little assumption, it is hard to imagine how they can be physically relevant. The key locates at matrix size N and the number of matrices, which could be large enough to include interesting physical ingredients. There are also some motivations to use matrix, instead of number and vector. Many physical observables are formulated by matrices, especially in quantum mechanics. Even some properties of continuous geometry could be formulated

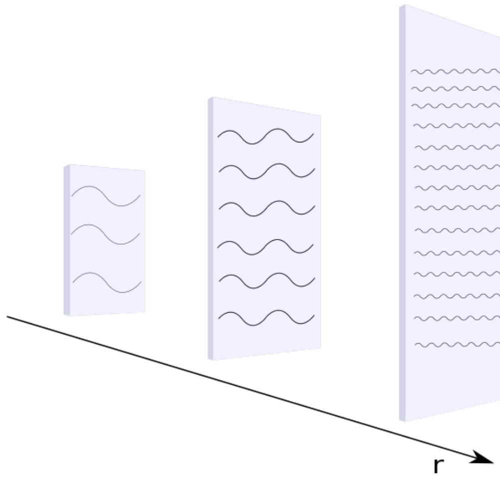


Figure 1: An illustration of the radial coordinate emerges as the energy scale of the field theory living on the D3-brane. The wavelength illustrates the energy scale.

approximately by matrices and their algebraic structures. It is therefore worthy to develop methods to understand the matrix models on their own right, without caring too much about the physical interpretation. However, our motivation has its origin in string theory.

The AdS/CFT correspondence is an important result of string theory. It reshapes our understanding of quantum gravity in the AdS space and conformal field theory in the flat space. The well established example can be written as $\mathrm{AdS}_5/\mathrm{CFT}_4$. The gravity theory lives in a five dimensional geometry with constant curvature AdS_5 along with other five compact dimensions forming a sphere S^5; The conformal field theory CFT_4 on the other hand living in a four dimensional flat space-time. One of the most remarkable features is that the two theories have different dimensions. One extra dimension is the radial coordinate of the AdS_5, which corresponds to the energy scale of the CFT_4. This is an example of the notion of the emergence of geometry. Not only the radial coordinate, the five-dimensional sphere S^5 can also find its correspondence in CFT_4. In particular, the oscillation modes of S^5 can be mapped to certain multiplets that transforming under the so-called R-symmetry group of CFT_4. Therefore, AdS/CFT gives a concrete realization of how geometry could emerge from a quantum system.

The idea of emergence of geometry is fascinating, and will deepen our understanding of interplay between geometry and quantum mechanics. The simplest quantum mechanics system is the matrix model, where the space and time are not assumed. In string theory, there is a natural candidate matrix model that could serve as the field theory part of AdS/CFT correspondence. Although, there is no well-established example of the AdS/CFT correspondence for a matrix model, it is interesting to explore this case. If such correspondence can be established, it means that in principle, a ten-dimensional geometry, including both space and time, could emerge from a quantum mechanic system defined on a point.

An important feature of the AdS/CFT duality is that the field theory has conformal symmetry. This means that the field theory dynamics is independent of the energy scale at which we probe the theory. As being said, the energy scale corresponds the the radial coordinate of the AdS geometry. This scale invariance manifests itself as a constant scalar field (dilaton field) $\phi(r) = \phi_0$ as the geometric solution. As we will see later, the solution is simpler when the dilaton field is constant. That means the AdS/CFT duality is easy to understand when the field theory has conformal symmetry. One obstacle when trying to study the matrix model is that there is no available notion of conformal symmetry for it.

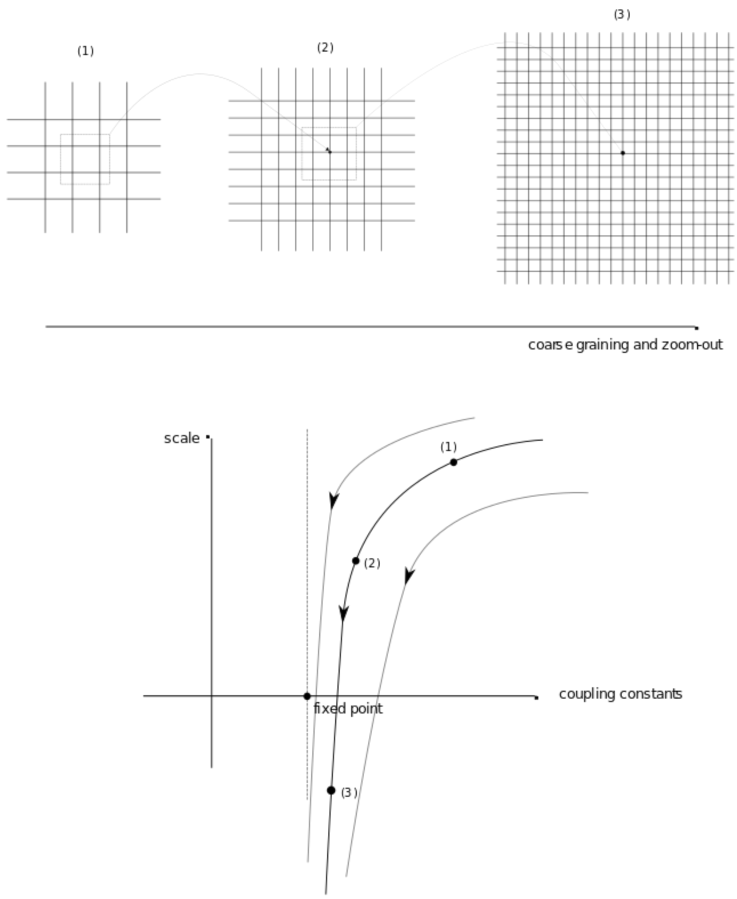


Figure 2: An illustration for the coarse graining. A (infinite) lattice is shown above. At each step, some lattices sites are assembled into a single lattice site. At the same time, we zoom out the picture to remind that the length scale is changed. The picture below shows the idea of how the system flows to a fixed point along the coarse graining.

Therefore, our expectation is that a notion of conformal matrix model would pave the way to establish the AdS/CFT duality.

Matrix model is a QFT defined on a point, therefore conformal symmetry can not be realized as a transformation of certain geometry. Actually, conformal symmetry often appears when there is a scale invariance, and the scale does not necessarily mean the energy scale. For example, we know that conformal symmetry could emerge in a statistical system when it approaches to certain critical points. Near the critical point, long-range interaction dominates the dynamical behavior of the system, where the correlation function exhibits a power-law behavior over distance. This is a characteristic of scale invariance. One important method to study the critical point is the renormalization group (RG) method. The change of scale is implemented by changing the number of degrees of freedom (dof) in such a way that the physical predictions keep unchanged. This change of scale induces a flow of the parameters that control the theory, which correctly take into account the effects of those integrated out dofs. Then the scale invariance will emerge at the fixed point of such flows. The usual conformal field theory can also be understood as the fixed points of RG flows that induced by changing energy scales.

As we will show in this thesis, the RG method can also be applied to general matrix models, therefore defining a notion of changing scale. We expect that such a study will lead us to a notion of conformal matrix model, saying that **conformal matrix model is the fixed point of the renormalization group flow.** In particular, we will implement the RG method by integrating out the matrix elements, which can be understood as a coarse graining procedure of the matrix model. Because the dofs are discrete in natural, the RG flow is discrete.

2 Dp-branes, supergravity and the AdS/CFT duality

The classical example of AdS/CFT is about D3-brane, which has three spatial dimensions, and the worldvolume has four dimensions including time. Before explaining what is special about D3-brane, let's remark on the dimension of the space-time and general Dp-brane. A consistent string theory requires that the space-time dimension is 10 [19]. One could in principle consider Dp-branes with $p = -1, 0, 1, \cdots, 9$. D9-brane can be understood as the space-time itself. The rest of branes separate into two classes: p are odd or even, according to the types of strings (IIB or IIA) that they can emit or absorb. In this thesis, I will solely focus on the case when p is odd $p = -1, 1, 3, 5, 7$, because the construction that we will use can not be applied to the even cases. Now, let's start with $p = 3$.

$\mathrm{AdS}_5 \times \mathrm{S}^5/\mathrm{CFT}_4$ The CFT_4 denotes the effective worldvolume theory of D3-branes. It is a four dimensional Yang-Mills theory, similar to the theory that describes the real world gluons. But there is an important difference. Gluon has self-interactions, and this interaction has a strong dependence on the energy scale. In the daily life, the self-interaction is so significant such that no one can see the free gluons. However, if one uses high energetic probes to detect them, they will be similar to photon, which has no self-interaction. The situation is different for the D3-brane, which has many supersymmetries. The strings moving on it will not only excite the gluons, but also their supersymmetric partners, which does not exist in the real world. These supersymmetric partner will help to control the difference between the daily life and high energy scale. It turns out that the theory will be exactly the same in all scales, therefore a conformal field theory (CFT).

To construct Yang-Mills theory with $N \times N$ matrix-valued A_μ, one needs to take N D3-branes and put them very close to each other. Their distances r should be small compare to the string scale $r/l_s \to 0$, such that they are coincident for the strings stretched between them. This ensures that the gauge group is $U(N)$. This is possible because it turns out that the attractive and repulsive interactions between D3-branes balance each other, which is again a result of supersymmetry [19]. One may worry about the $r/l_s \to 0$ condition because as we said, the effective theory only works when the strings are small. However, there is an interesting uncertainty principle [12] which ensures that the small transversal scale r will correspond to a large length scale on the worldvolume. The effective theory is therefore valid for such configurations.

The $\mathrm{AdS}_5 \times \mathrm{S}^5$ is the geometry near those D3-branes. It is separated into two parts, AdS_5 is the direction along the worldvolume with the transversal direction r. The sphere S^5 has the same curvature radius R as the AdS_5 [16]. A surprise result is that $R/l_s \propto N^{1/4}$, which means that more D3-branes will lead to flatter geometry. Another particular feature of this geometry is that the dilaton field is constant $\Phi(r) = \Phi_0$. This means that the string coupling $g_s = \mathrm{e}^{\Phi_0}$ [19] is constant everywhere in the background. One could also change its value by shifting the dilaton field, this will not modify the equation of motion. As we will review, this feature is consistent with the fact that the coupling of Yang-Mills theory is independent of the energy scale.

non-conformal branes This is not the case for other Dp-brane ($p \neq 3$) solutions. We will in particular look at the cases D1 and D(-1), where the dilaton field will diverge as moving toward the horizon. This corresponds to the positive mass dimension of the Yang-Mills coupling in these cases. The effective dimensionless couplings will diverge as moving toward the low energy limit.

D1 and D(-1) branes are the source of gauge fields $(C_2)_{\mu\nu}$ and C_0 in the bulk theory, the type IIB supergravity. This situation is similar to the case where the electric charge is the source of the gauge field A_μ. The D5 and D7 branes could also be the source of them, and they are called the magnetic counterpart of D1 and D(-1). When serving as the "non-conformal" holographic backgrounds, unlike the D1 case, they are not well-understood because the geometry solutions are singular in these cases [13]. (D(-1) is also not well understood, for example see [18].) We will include D5 and D7, but only wrapping them over certain torus with the number of non-compact directions the same as the D1 and D(-1) respectively. Then, by combining them with the D1 and D(-1)-branes in a certain way, we will construct the so-called D1/D5 and D(-1)/D7 system. The idea of such construction is that there is a "electric-magnetic duality" that relating the Dp-D$(6-p)$ pairs. This duality could lead to vacuum solution of string theory with constant dilaton field. The intuitive picture of this is that dilaton field diverges near D1 and D(-1)-branes, but vanishes near D5 and D7-branes. These two effects balance to each other, leading to a constant dilaton field.

D1/D5 system The AdS/CFT duality for the D1/D5 system was already discussed in the original paper [16]. The vacuum solution of supergravity has the form $\mathrm{AdS}_3 \times S^3 \times \mathbb{T}^4$, where four spatial directions of D5-branes are along a four dimensional torus $\mathbb{T}^4$ and two others share with D1. The constant dilaton field can be understood intuitively as a result of the "electric-magnetic duality". For the gauge field C_2, D1 acts like an electric source, and D5 acts like a magnetic sources. It is possible to tune the charges Q_1 and Q_5 change the value of string coupling. From the 2d field theory side, the situation is realized by a duality between electric and magnetic coupling. This is a strong-weak duality, which means that when one coupling is strong, another is weak. They are both running with energy scales, but in an opposite way. It's then possible to construct a scale invariant theory by adding interactions between these two theories. The result is the so-called D1/D5 CFT.

D(-1)/D7 system and conformal matrix model One expects that a similar construction could give a D(-1)/D7 system, and give an example of $\mathrm{AdS}_1/\mathrm{CFT}_0$ where a geometry emerges from a matrix model. A vacuum solution of the form $\mathrm{AdS}_1 \times S^1 \times \mathbb{T}^8$ was found in [1]. An additional C_4 gauge field is required to stabilize the solution, which can be physically understood as adding D3-branes to the system. A further study of the flow equation of the supergravity fields on the AdS_1 space confirms the existence of a constant dilaton field solution [Seb]. An ambiguity in the choice of supersymmetry transformation is also clarified in the solution. In these works, the dual CFT_0 was conjectured to be the matrix model constructed in [3]. This conjecture was motivated by the similarity with the D1/D5 case: there are two IKKT matrix models, one for the D(-1)-branes, another for the D7-branes wrapped on the $\mathbb{T}^8$, with strings stretched between them. In this thesis, we try to explore the idea that the scale invariance property of the matrix model is captured by the scaling of the matrix size $N \to \lambda N$.

2.1 open string description: DBI action

Open strings with endpoints on a Dp-brane excite massless modes. The most well-established low-energy effective field theory for these massless modes is for the case of a single bosonic Dp-brane, described by the Dirac-Born-Infeld (DBI) action [19]:

$$S_{\mathrm{D}p} = -T_p \int (\mathrm{d}^{p+1}\xi) \mathrm{e}^{-\phi} \sqrt{-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}, \tag{1}$$

where $\{\xi^a\},\ a = 0, \ldots, p$ are the coordinates on the brane. The spacetime fields $G_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$ are projected onto the brane as follows:

$$G_{ab}(\xi) = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}(X(\xi)), \tag{2}$$

$$B_{ab}(\xi) = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu}(X(\xi)). \tag{3}$$

Here, $F_{ab} = \partial_a A_b - \partial_b A_a$ is the gauge field strength for the massless modes. Note that α' has the length dimension $[\mathrm{L}^2]$, making the combination $2\pi\alpha' F_{ab}$ dimensionless. T_p is the "bare" tension of the brane. If there is a constant dilaton field $\phi = \phi_0$ in the background, the physical tension is modified as $\tau_p \equiv T_p \mathrm{e}^{-\phi_0}$. The meaning of this action becomes clear when one expands the determinant.

To simplify the discussion, we will ignore the NS-NS field $B_{\mu\nu}$, focusing solely on how gravity interacts with the Dp-brane. First, let us factor out the determinant of the metric $\det G$

$$\det(G + 2\pi\alpha' F) = (\det G)\det(\mathbb{1} + 2\pi\alpha' G^{-1} F)$$

The expansion is obtained by applying the following formula:

$$\det(\mathbb{1} + M) = \exp\left(\mathrm{tr}M - \frac{1}{2}\mathrm{tr}M^2 + \frac{1}{3}\mathrm{tr}M^3 - \frac{1}{4}\mathrm{tr}M^4 + \cdots\right)$$

The linear term vanishes, $\mathrm{tr}(G^{-1}F) = 0$, because the metric in symmetric and the gauge strength is antisymmetric. The quadratic term will give us:

$$\mathrm{tr}(G^{-1}FG^{-1}F) = F^{ab}F_{ba}$$

Therefore, at the leading order, we have

$$\det(G + 2\pi\alpha' F) = (\det G)\left(1 - \frac{1}{2}(2\pi\alpha')^2 F^{ab}F_{ba} + \cdots\right) \tag{4}$$

In summary, the expansion form of the DBI action is

$$S_{\mathrm{D}_p} = -T_p \int \mathrm{d}^{p+1}\xi \mathrm{e}^{-\phi}\sqrt{-G} - \frac{(2\pi\alpha')^2}{4} T_p \int \mathrm{d}^{p+1}\xi \mathrm{e}^{-\phi}\sqrt{-G}F^{ab}F_{ab} + \cdots \tag{5}$$

The first term represents the volume of the p-brane. The second term has the same form as the Maxwell action.

and one could identify the Yang-Mills coupling as

$$\mathrm{g}_{\mathrm{YM}}^2 = \frac{1}{(2\pi\alpha')^2 \tau_p}$$

In later applications in AdS/CFT, there's a desire to generalize the action to multiple Dp-branes. When the branes are close to each other, the open strings stretched between different branes effectively become massless. The gauge group $U(1)$ is promoted to $U(N)$ in the case of N coincident branes. Therefore, one faces the problem of generalizing the Dirac-Born-Infeld (DBI) action to include non-Abelian gauge fields. One simple proposal is to require the determinant to also calculate over the gauge group matrices. In the expansion form, the Maxwell action is generalized to the Yang-Mills action:

$$-\frac{(2\pi\alpha')^2}{4} T_p N \int \mathrm{d}^{p+1}\xi \mathrm{e}^{-\phi}\sqrt{-G}\mathrm{tr}(F^{ab}F_{ab}) \tag{6}$$

where

$$F_{ab} = \partial_a A_b - \partial_b A_a - i[A_a, A_b]$$

However, another issue arises when considering multiple coincident branes. When writing down the DBI action for a single brane, the embedding coordinates $X^\mu(\xi)$ is used to project the space-time fields onto the brane. However, in the case of N coincident branes, $X^\mu(\xi)$ become $N \times N$ matrices. These are called the collective coordinates of the branes. The diagonal elements could be understood as the usual embedding coordinates of the corresponding branes, while the off-diagonal elements signal the non-commutativity of the Dp-brane geometry [22]. There is no simple generalization of the DBI action to describe this non-commutative geometry.

An alternative route to obtain the effective action of N coincident Dp-branes takes into account the aforementioned features. One could start from the 10d, $\mathcal{N} = 1$ Super-Yang-Mills theory with the gauge group $U(N)$. The idea is to project this theory onto the worldvolume of Dp-branes. The gauge fields $A_\mu(x), \mu = 0, \cdots, 9$ are separated into those are longitudinal to the worldvolume $A_a(\xi), a = 0, \cdots, p$, and those are transverse $A_i(\xi) \equiv \phi_i(\xi), i = p+1, \cdots, 9$. The 10d gauge fields in the transverse directions then become scalar fields in the worldvolume field theory. These scalar fields are the collective coordinates of the multiple Dp-branes. The choice of the 10d, $\mathcal{N} = 1$ super-Yang-Mills theory is motivated by the number of supersymmetries, which matches with that of Dp-branes.

2.2 closed string description of Dp-branes: the supergravity solution

There is no direct way to obtain a field theory from string theory by considering short strings $l_s \to 0$. However, the worldsheet theory with the assumption of embedding in a flat space-time will imply some important properties of the space-time physics. In particular, the space-time supersymmetries are required by the massless spectrum of the type II superstring theory [19]. In particular, the supercharges that generate the supersymmetries Q^1_α, Q^2_α are two Majorana-Weyl spinors in $d = 10$. According to their relative chirality, we have type IIA superstring (opposite chirality) and type IIB superstring (same chirality). There are in total 32 real components of the supercharges. This huge amount of supersymmetries will only leave us with two kinds of gravity theory in ten dimensions, namely the type IIA and type IIB supergravity.

In the context of AdS/CFT correspondence, we are interested in finding the vacuum solution of type IIB supergravity. In particular, we will look for the simplest class of solutions where half of the supersymmetries 16 are preserved. This is the same situation as the Dp-branes. Actually, the solution will be interpreted as the geometry of Dp-branes. For this purpose, we can focus only on the bosonic sector of the type IIB supergravity. The field contents are: the metric $g_{\mu\nu}$, the dilaton field ϕ and the $(p+1)$-form gauge field C_{p+1} with their field strengths $F_{p+2} = \mathrm{d}C_{p+1}$ which is understood as the electric field of Dp-branes.

type IIB supergravity The bosonic part of type IIB supergravity is given by the following action [19]

$$\begin{aligned} S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x(-g)^{1/2} \Bigg\{ e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{2}|H_3|^2 \right] \\ -\frac{1}{2}|F_3 - C_0 \wedge H_3|^2 - \frac{1}{2}|F_1|^2 - \frac{1}{4}|F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3|^2 \Bigg\} \\ -\frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3. \end{aligned} \tag{7}$$

$H_3 = \mathrm{d}B_2$ comes from the NS-NS sector; $F_1 = \mathrm{d}C_0$, $F_3 = \mathrm{d}C_2$ and $F_5 = \mathrm{d}C_4$ comes from the R-R sector. For $p = -1, 3, 5$, the truncated version has the general form

$$S = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x(-g)^{1/2} \left[\mathrm{e}^{-2\phi}(R + 4(\partial\phi)^2) - \frac{1}{2}|F_{p+2}|^2 \right] \tag{8}$$

string frame and Einstein's frame Comparing to the Einstein-Hilbert action $S \propto \int \sqrt{-g}R$, there is a coupling between the Ricci scalar and the dilaton field ϕ. This is natural from the string perspective, because the dilaton field controls the coupling strength between strings. [1] When being constant, one expect it appears in front of the gravity action. However, there is an ambiguity in the metric when the dilaton field is not simply constant. One can use the dilaton field to redefine the metric, and the form of the action could change after this redefinition. A particular redefinition will lead us to the standard Einstein-Hilbert action.

Let's show explicitly how to get the standard form. First parameterize the Weyl rescaling by a function $\omega(x)$, which will later be chosen to get the standard action

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = \mathrm{e}^{2\omega(x)} g_{\mu\nu}(x)$$

$$g^{\mu\nu}(x) \to \tilde{g}^{\mu\nu}(x) = \mathrm{e}^{-2\omega(x)} g^{\mu\nu}(x)$$

The Ricci scalar is not covariant under this transformation, but has additional non-homogeneous terms

$$R = \mathrm{e}^{-2\omega} \left(\tilde{R} - 18\nabla^2\omega - 72\partial_\mu\omega\partial^\mu\omega \right)$$

One should be careful about which metric we use to contract the Lorentz indices μ, ν. In the above formula, we are still using the initial metric g in the rhs. Using the relations above, one could write everything in (8) in terms of $\tilde{g}$, including all the contractions

$$S = \frac{1}{(2\pi)^7 l_s^8} \int \mathrm{d}^{10}x \sqrt{-\tilde{g}} \Bigg\{ \mathrm{e}^{-2\phi-8\omega} \left[\tilde{R} - 18\nabla^2\omega - 72(\partial\omega)^2 + 4(\partial\phi)^2 \right] \\ - \frac{1}{2}\mathrm{e}^{2(p-3)\omega}|F_{p+2}|^2 \Bigg\}$$

An observation is that $p = 3$ is special because the gauge field F_{p+2}^2 term will not change under any rescaling. To decouple the Ricci scalar and the dilaton field, one must choose

$$\omega(x) = -\frac{1}{4}\phi(x)$$

One arrives at the following action

$$S = \frac{1}{(2\pi)^7 l_s^8} \int \mathrm{d}^{10}x \sqrt{-\tilde{g}} \left[\tilde{R} - \frac{1}{2}(\partial\phi)^2 - \frac{9}{2}\nabla^2\phi - \frac{1}{2}\mathrm{e}^{-\frac{1}{2}(p-3)\phi}|F_{p+2}|^2 \right] \tag{9}$$

Only in the $p = 3$ case it is possible to decouple the dilaton field completely with other fields.

[1] One may has question about why the dilaton field does not couple to the F_{p+2}^2 in the same way if its a universal string coupling. Actually, one can bring it into the same form by redefining the gauge field [19]. The reason for the form people normally use is that the Bianchi idenity $\mathrm{d}F = 0$ will be satisfied.

a discussion on F_{p+2} We will not write down all the equations, but only the one for the F_{p+2}, because the solution we will give bases on an important ansatz for F_{p+2}. The equation of motion is derived by varying with respect the gauge field A_{p+1}. Let's recall the definition of the exterior derivative $F = \mathrm{d}A$:

$$(F_{p+2})_{\mu_1\cdots\mu_{p+2}} = (\mathrm{d}A_{p+1})_{\mu_1\cdots\mu_{p+2}} = (p+2)\nabla_{[\mu_1}(A_{p+1})_{\mu_2\cdots\mu_{p+2}]}$$

The variation of the action is

$$\delta S = \frac{1}{(2\pi)^7 l_s^8 (p+2)!}\int \mathrm{d}^{10}x\sqrt{-g}\nabla^{\mu}(\mathrm{e}^{-\frac{1}{2}(p-3)\phi}F_{\mu\mu_1\cdots\mu_{p+1}})\delta A^{\mu_1\cdots\mu_{p+1}}$$

where the possible matter source $\int j_{\mu_1\cdots\mu_{p+1}}A^{\mu_1\cdots\mu_{p+1}}\mathrm{d}^{p+1}x$ and boundary terms are ignored. The eom reads

$$\nabla^{\mu}(\mathrm{e}^{-\frac{1}{2}(p-3)\phi}F_{\mu\mu_1\cdots\mu_{p+1}}) = 0 \tag{10}$$

The relation between the equation of motion and the Bianchi identity $\mathrm{d}F = 0$ will become clear if we use the Hodge start operator $*$. The definition of $*$ is

$$[(*F)_{8-p}]^{\mu_1\cdots\mu_{8-p}} = \frac{1}{(p+2)!}\varepsilon^{\mu_1\cdots\mu_{8-p}\mu_{9-p}\cdots\mu_{10}}(F_{p+2})_{\mu_{9-p}\cdots\mu_{10}}$$

One can show that the eom is equivalent to

$$*\mathrm{d}(\mathrm{e}^{-\frac{1}{2}(p-3)\phi} * F) = 0 \tag{11}$$

One can rewrite the equations using a $(8-p)$-form G

$$G \equiv \mathrm{e}^{-\frac{1}{2}(p-3)\phi} * F \tag{12}$$

which is defined such that (11) gives the Bianchi identity of G

$$\mathrm{d}G = 0$$

The eom of G follows from the Bianchi identity of F

$$*\mathrm{d}(\mathrm{e}^{\frac{1}{2}(p-3)\phi} * G) = \nabla^{\mu}(\mathrm{e}^{\frac{1}{2}(p-3)\phi}G_{\mu\mu_1\cdots\mu_{7-p}}) = 0$$

One could derive these equations from the following action (compare to (9))

$$S = \frac{1}{(2\pi)^7 l_s^8}\int \mathrm{d}^{10}x\sqrt{-\tilde{g}}\left[\cdots - \frac{1}{2}\mathrm{e}^{\frac{1}{2}(p-3)\phi}|G_{8-p}|^2\right] \tag{13}$$

An important observation is that when p-brane and $(6-p)$-brane exist simultaneously, one can always write the action only using $(p+2)$-form or $(8-p)$-form.

the ansatz Follow the discussion in [10], we will separate between the longitudinal directions $x^i, i = 1, \cdots p$ and the transversal directions (t, r, Ω_{8-p}). The longitudinal directions are along the spatial directions of the p-branes. We will assume that every slices that are parallel to the p-branes (fix the transversal directions), the solution metric is flat up to a scale factor

$$\mathrm{d}x_{||}^2 \propto \sum_{i=1}^{p}\mathrm{d}x^i\mathrm{d}x^i$$

For the transversal part, we are looking for a spherically symmetric solution. The p-branes are a point in the (t, r, Ω_{8-p})-coordinate, therefore we are trying to find a solution that is similar to the Reissner-Nordstrom solution. The ansatz for the gauge strength F_{p+2} is written using the dual G_{8-p} [10]

$$G_{8-p} = Q\epsilon_{8-p} \tag{14}$$

where Q is the charge of the p-branes and ϵ_{8-p} is the volume form of a unit S^{8-p}.

the solutions with $p = 3$ The simplest case is $p = 3$, where one can consistently set the dilaton field to constant $\phi(x) = \phi_0$ because it does not couple to any other fields. String theory requires that the 5-form F_5 to be self-dual $F_5 = *F_5 = G_5$. The solution given in [10] is

$$\mathrm{d}s^2 = -\frac{f_+(r)}{\sqrt{f_-(r)}}\mathrm{d}t^2 + \sqrt{f_-(r)}\sum_{i=1}^{3}\mathrm{d}x^i\mathrm{d}x^i + \frac{1}{f_-(r)f_+(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2$$
$$F = 2Q(\epsilon_5 + *\epsilon_5)$$

with

$$f_\pm(r) = 1 - \frac{r_\pm^4}{r^4}, \quad Q = r_+^2 r_-^2$$

Like the case of the Reissner-Nordstrom solution, we are interested in the extremal case $r_+ = r_- = Q^{\frac{1}{4}}$

$$\mathrm{d}s^2 = \left(1 - \frac{Q}{r^4}\right)^{\frac{1}{2}}\left(-\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^i\mathrm{d}x^i\right) + \left(1 - \frac{Q}{r^4}\right)^{-2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2 \tag{15}$$

To study the metric near $r = Q$, we need to introduce a new coordinate variable ω (the following discussion about the near horizon geometry refers to [8])

$$C\omega \equiv \left(1 - \frac{Q}{r^4}\right)^{\frac{1}{4}} \tag{16}$$

C is a constant to be chosen later. The inverse transformation is

$$r = \left(\frac{Q}{1-(C\omega)^4}\right)^{\frac{1}{4}} \tag{17}$$

The metric becomes

$$\mathrm{d}s^2 = C^2\omega^2\left(-\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^i\mathrm{d}x^i\right) + \frac{\sqrt{Q}}{[1-(C\omega)^4]^{\frac{5}{2}}}\frac{\mathrm{d}\omega^2}{\omega^2} + \frac{\sqrt{Q}}{[1-(C\omega)^4]^{\frac{1}{2}}}\mathrm{d}\Omega_5^2 \tag{18}$$

If one choose $C^2 = \sqrt{Q}$, and consider $\omega \to 0$, the metric has the form $\mathrm{AdS}_5 \times \mathrm{S}^5$

$$C^{-2}\mathrm{d}s^2 = \omega^2\left(-\mathrm{d}t^2 + \sum_{i=1}^{3}\mathrm{d}x^i\mathrm{d}x^i\right) + \frac{\mathrm{d}\omega^2}{\omega^2} + \mathrm{d}\Omega_5^2 \tag{19}$$

the embedding of AdS Let's consider embedding AdS_5 into a 6d flat space-time $(T, X^i, X_\pm)$ through the following equation

$$T^2 - \sum_{i=1}^{3}X^iX^i + X_+X_- = R^2 \tag{20}$$

where R is the radius of AdS_5. The metric of the flat space-time is

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \sum_{i=1}^{3}\mathrm{d}X^i\mathrm{d}X^i - \mathrm{d}X_-\mathrm{d}X_+ \tag{21}$$

Let's identity the coordinates as

$$\omega = X_-, \quad t\omega = T, \quad x^i\omega = X^i$$

The constraint then requires that

$$X_+ = \frac{R^2 + \omega^2(\sum_i x^i x^i - t^2)}{\omega}$$

The metric in the coordinate (t, x^i, ω) is

$$\mathrm{d}s^2 = \omega^2(-\mathrm{d}t^2 + \sum_i \mathrm{d}x^i \mathrm{d}x^i) + R^2 \frac{\mathrm{d}\omega^2}{\omega^2} \tag{22}$$

Therefore, in the above solution, we identity the AdS radius as

$$R = C = Q^{\frac{1}{4}}$$

2.3 field theory limit and the AdS/CFT correspondence

In the previous sections, we examined the Dirac-Born-Infeld (DBI) action and the supergravity action. In the limit as $\alpha' \to 0$, the DBI action leads us to a certain super-Yang-Mills theory on the worldvolume of Dp-branes. Meanwhile, the supergravity action provides vacuum solutions that describe the geometry of Dp-branes. String theory offers insights into the connections between these two descriptions, particularly regarding how to match the string coupling g_s with the Yang-Mills coupling g_{YM}.

We will write down the supergravity action schematically as

$$S_{\text{sugrav}} = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-g} \mathrm{e}^{-2\phi} R + \cdots$$

where the terms that involving the form fields are ignored. We also write the world-volume field theory schematically as

$$S_{\text{SYM}} = \frac{(2\pi\alpha')^2}{4} T_p \int \mathrm{d}^{p+1}\xi \mathrm{e}^{-\phi} \sqrt{-G} \mathrm{Tr} F^2 + \cdots$$

where the scalar fields and supersymmetric partners are ignored.

Consider the case where the dilaton field is constant $\phi = \phi_0$. We have a well-defined notion of string coupling in this background

$$g_s = \mathrm{e}^{\phi_0} \tag{23}$$

It should be noted there is always a freedom to shift the dilaton field $\phi \to \phi + c$, or define the g_s with an additional multiplication factor $g_s = A\mathrm{e}^{\phi}$. This definition fixes one of these two ambiguities. In [19], the other ambiguity is fixed by defining g_s as the ratio of tensions

$$g_s \equiv \frac{\tau_{\text{F1}}}{\tau_{\text{D1}}} = \frac{\kappa}{8\pi^{7/2}\alpha'^2} \tag{24}$$

where $\tau_{\text{F1}} = 1/(2\pi\alpha')$ is the tension of the fundamental string, and τ_{D1} is the tension of the D-string. κ is the physical gravity coupling $\kappa_{10}\mathrm{e}^{\phi_0}$. This quantity is obtained by matching the worldsheet theory (α') calculation of D1-brane scattering by those using the supergravity (κ).

The tension of a general Dp-brane is related to the gravity coupling κ by the following relations [19]

$$\tau_p^2 = \frac{\pi}{\kappa^2}(4\pi^2\alpha')^{3-p} \tag{25}$$

Tension has the meaning of energy density, one can check this formula has the correct dimension. This relation is the physical reason for relating the gravity coupling with the Yang-Mills coupling for the worldvolume field theory. Let us clarify this point. One can use the definition of the string coupling (24) to rewrite the supergravity action as

$$\frac{1}{(2\pi)^7(\alpha')^4 g_s^2}\int \mathrm{d}^{10}x\sqrt{-g}R+\cdots$$

We replace the τ_p in the DBI action with g_s

$$\frac{\pi^2}{(2\pi)^p}(\alpha')^{\frac{3-p}{2}}\frac{1}{g_s}\int \mathrm{d}^{p+1}\xi\sqrt{-G}\mathrm{Tr}F^2+\cdots$$

Compare this to the standard form of Yang-Mills action, we are led to the following identification

$$g_{\mathrm{YM}}^2=g_s(\alpha')^{\frac{p-3}{2}}(2\pi)^{p-2} \tag{26}$$

Dp-branes carry the R-R charge, therefore sourcing the R-R field strength $F_{p+2}=\mathrm{d}C_{p+1}$. The R-R gauge field C_{p+1} couples to the Dp-branes through the following action [19]

$$\mu_p\int C_{p+1} \tag{27}$$

where μ_p is the charge density. This charge density is equal to the bare tension $\mu_p=T_p$ [19]. This implies that it is independent of the string coupling

$$\mu_p=T_p=\frac{1}{2\pi}(\alpha')^{-\frac{p+1}{2}}$$

When there are N coincident Dp-branes, the charge density should be amplified to $N\mu_p$. When looking at the supergravity description of Dp-branes, we use the ansatz $*F=Q\epsilon_{8-p}$. The question then is how this Q relates to the charge density $N\mu_p$.

The AdS/CFT is set up in the field theory limit [16]. The physical situation is to consider N D3-branes that are close to each other. The scale of their separations is r. The field theory limit is obtained by considering

$$\alpha'\to 0,\quad U\equiv\frac{r}{\alpha'}=\text{fixed} \tag{28}$$

$\alpha'\to 0$ means that the string becomes small. This is also called the decoupling limit where the low energy theory of the D3-branes decoupled from the bulk theory. U represents the energy scale of the strings stretched between D3-branes. We keep it fixed means that we are bringing the $U(N)$ D3-branes on top of each others $r\to 0$. The worldvolume theory then becomes $U(N)$ super-Yang-Mills theory, for which U is the energy scale of this field theory. The string coupling g_s is kept fixed in this limit. This also means that we are keeping the Yang-Mills coupling fixed because

$$g_{\mathrm{YM}}^2=2\pi g_s$$

The situation is more complicate for $p\neq 3$ because the relation (26) depends on α'. It is impossible to keep g_{YM} and g_s fixed at the same time. In the field theory limit, one keeps the Yang-Mills coupling fixed [13]

$$\alpha'\to 0,\quad U=\frac{r}{\alpha'}=\text{fixed},\quad g_{\mathrm{YM}}^2N=\text{fixed} \tag{29}$$

For $p<3$, this means that $g_sN\to 0$; while for $p>3$, this means that $g_sN\to\infty$. However, this discussion bases on a constant dilaton field. As we have seen, the dilaton field diverges when $r \to 0$ in the case of $p < 3$. This seems to suggest that the field theory is not well-defined in these cases.

Following the discussion in [13], let us rewrite the supergravity solution of Dp-branes using the physical parameters in string theory. The metric solution has the form

$$\mathrm{d}s^2 = H_p^{-1/2}(U)\mathrm{d}s^2(E^{p,1}) + H_p^{1/2}(U)\mathrm{d}s^2(E^{9-p}) \tag{30}$$

with the harmonic function

$$H_p(U) = 1 + \frac{D_p g_{\mathrm{YM}}^2 N}{\alpha'^2 U^{7-p}} \tag{31}$$

The notation $E^{p,1}$ indicates the longitudinal directions of the Dp-brane worldvolume; while E^{9-p} denotes the transverse directions. The dilaton field solution is

$$\mathrm{e}^{\phi} = g_s H_p(U)^{\frac{3-p}{4}} \tag{32}$$

with the g_s factor included to satisfy $\mathrm{e}^{\phi(U)} \to g_s,\ U \to \infty$. The solution for the $(p+1)$-form gauge field C_{p+1} is

$$C_{0\cdots p} = g_s^{-1}(H_p(U)^{-1} - 1) \tag{33}$$

which is vanish when $U \to \infty$. This gauge field is along the directions $E^{p,1}$, for which the Dp-brane is an electric source. The normalization is chosen such that

$$\int_{S^{8-p}} *\mathrm{d}C_{p+1} = N \tag{34}$$

where S^{8-p} is a sphere in the transverse space E^{9-p}.

Let us take a closer look at the dilaton profile for $p = 1$:

$$\mathrm{e}^{\phi} = g_s \left(1 + \frac{D_1 g_{\mathrm{YM}}^2 N}{\alpha'^2 U^6}\right)^{\frac{1}{2}} \overset{\alpha' \to 0}{\Longrightarrow} g_s \frac{\sqrt{D_1 g_{\mathrm{YM}}^2 N}}{\alpha' U^3}$$

Recall that keeping $g_{\mathrm{YM}}^2 N$ in the field theory limit requires that $g_s N \to 0$ because of the relation

$$g_{\mathrm{YM}}^2 N = \frac{g_s N}{2\pi\alpha'} \tag{35}$$

One should rewrite the dilaton profile using the finite parameters

$$\mathrm{e}^{\phi} = 2\pi \sqrt{\frac{D_1 g_{\mathrm{YM}}^6 N}{U^6}} \tag{36}$$

Note that here the g_{YM} should be understood as the Yang-Mills coupling for a D1-brane placing at infinity $U \to \infty$. However, it get modified at finite U because of the dilaton profile. Let us define the effective dimensionless Yang-Mills coupling g_1 for D1-brane as that follows from the DBI action

$$g_1^2 \equiv \frac{\mathrm{e}^{\phi}}{2\pi\alpha'} U^{-2} N \tag{37}$$

where U is the energy scale of the theory. Another definition is

$$g_1^2 \equiv g_{\mathrm{YM}}^2 U^{-2} N \tag{38}$$

They coincident when $U \to \infty$ and $\mathrm{e}^{\phi} \to g_s$ before taking the field theory limit $\alpha' \to 0$. Although the first one seems more reasonable from the Dp-brane point of view, it diverges when $\alpha' \to 0$. The reason is that, in the field theory limit, finite dilaton profile can only be obtained by sending $g_s N \to 0$ at the same time. Using (38) one has

$$\mathrm{e}^{\phi} = 2\pi D_1 \frac{g_1^3}{N} \tag{39}$$

while (37) gives

$$\mathrm{e}^{\phi} = 2\pi\alpha' U^2 \frac{g_1^2}{N} \tag{40}$$

and at the same time

$$g_1^2 = \sqrt{D_1}\frac{1}{U^5\alpha'}(g_{\mathrm{YM}}^2 N)^{\frac{3}{2}} \tag{41}$$

2.4 Dp/D$(6-p)$ bound states solutions

As mentioned in the last section, the field theory limit is complicate when the dilaton field is not constant. For Dp-branes with $p < 3$, the dilaton field diverges as moving closer to the horizon. This implies that we can not trust the supergravity solution near the horizon. In this section, we will review how the Dp/D$(6-p)$ system stabilizes the dilaton profile, which will lead to a well-behaved vacuum solution. The intuition is that the Dp-branes source the R-R flux F_{p+2} electrically, while D$(6-p)$-branes source it magnetically. They will contribute to the gradient of dilaton field in an opposite way. If a stable solution can be found, it is possible to obtain a constant dilaton field.

When considering the configurations of two different kinds of D-branes, one usually uses the following notation

D1	X	X	–	–	–	–	–	–	–	–
D5	X	X	X	X	X	X	–	–	–	–

The first line denotes a D1-brane, where the "X" labels the longitudinal directions, and the "-" labels the transverse directions. The first slot is understood as the time direction. The second line denotes a D5-brane. It shares one direction with the D1-brane, while four other directions extend along the transverse directions of D1-brane. These are called the mixed directions. In this example, there are four mixed directions, which is the situation of the D1/D5 bound state.

The metric of D1/D5 bound state in string frame is [Matthias]

$$\mathrm{d}s^2 = \frac{1}{\sqrt{H_1 H_5}}(-\mathrm{d}t^2 + \mathrm{d}x_1^2) + \sqrt{\frac{H_1}{H_5}}(\mathrm{d}x_2^2 + \cdots + \mathrm{d}x_5^2) + \sqrt{H_1 H_5}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2) \tag{42}$$

where

$$H_1(r) = 1 + \frac{Q_1}{r^2}, \quad H_5(r) = 1 + \frac{Q_5}{r^2}$$

Q_1 and Q_5 are the charges of the D1-brane and D5-brane respectively. Note that in the above solution, the longitudinal directions are associated with the factor $H^{-1/2}$, while the transverse directions are associated with the factor $H^{1/2}$. Here we also assume that the coordinates $(x_2, \cdots, x_5)$ are periodic $x_i \sim x_i + 2\pi$. The near horizon geometry $r \to 0$ has the form $\mathrm{AdS}_3 \times S^3 \times \mathbb{T}^4$, where the D5-branes wrapped over the torus $\mathbb{T}^4$, and the D1-branes are smeared over it

$$\mathrm{d}s^2 \to \frac{r^2}{\sqrt{Q_1 Q_5}}(-\mathrm{d}t^2 + \mathrm{d}x_1^2) + \frac{\sqrt{Q_1 Q_5}}{r^2}\mathrm{d}r^2 + \sqrt{Q_1 Q_5}\mathrm{d}\Omega_3^2 + \sqrt{\frac{Q_1}{Q_5}}(\mathrm{d}x_2^2 + \cdots + \mathrm{d}x_5^2) \tag{43}$$

The radius of the AdS geometry is

$$R = (Q_1 Q_5)^{\frac{1}{4}}$$

The flux solution is [Matthias]

$$F_3 = \mathrm{d}H_1^{-1} \wedge \mathrm{d}t \wedge \mathrm{d}x^1 + \mathrm{e}^{-\phi} * (\mathrm{d}H_5^{-1} \wedge \mathrm{d}t \wedge \cdots \wedge \mathrm{d}x^5) \tag{44}$$

The first part comes from D1-branes, and the second part comes from the D5-branes. Note that the $\mathrm{e}^{-\phi}*$ is consistent with the definition of G_{8-p} in (12) with $p = 5$. The dilaton field is

$$\mathrm{e}^{2\phi} = \frac{H_1}{H_5} \tag{45}$$

It approaches a constant Q_1/Q_5 in the near horizon limit $r \to 0$.

The D(-1)/D7 bound state is more complicate. Except for the F_1 flux, one also needs to include a F_5 flux [1]. This flux is essential if one wants to obtain a non-trivial solution of the type $\mathrm{AdS}_1 \times \mathrm{S}^1 \times \mathbb{T}^8$. Another point is that D(-1)-brane is a solution of Euclidean IIB supergravity, in which the F_5 flux is imaginary self-dual $*F_5 = \pm i F_5$. The D7-brane is wrapped over the $\mathbb{T}^8$, and D(-1)-brane is smeared over it. The orientation of the branes and flux has the form

D(-1)	–	–	–	–	–	–	–	–	–	–
D7	–	–	X	X	X	X	X	X	X	X
D3	–	–	X	X	X	X	–	–	–	–
D3'	–	–	–	–	–	–	X	X	X	X

where D3 and D3' source the imaginary self-dual F_5 flux. The directions of D3-branes are assigned such that there are 8 supersymmetries left in this system.

The solution is obtained by starting from the following string frame metric ansatz [Seba]

$$\mathrm{d}s^2 = L_y^2(y)\mathrm{d}y^2 + L_x^2(y)\mathrm{d}x^2 + L_1^2(y)\sum_{i=1}^{4}(\mathrm{d}\theta^i)^2 + L_5^2(y)\sum_{i=5}^{8}(\mathrm{d}\theta^i)^2 \tag{46}$$

where x and θ^i are periodic coordinates $x \sim x + 2\pi,\ \theta^i \sim \theta^i + 2\pi$. y is the coordinate for $\mathrm{AdS}_1 \simeq \mathbb{R}$. For the F_1 flux

$$F_1 = \alpha(y)\mathrm{d}x + i\beta(y)\mathrm{d}y \tag{47}$$

where α, β are assumed to be real. The factor i comes from the Wick rotation of the Euclidean time. Sometimes, it is easier to use the dual version of the second piece

$$F_9 \equiv *(i\beta(y)\mathrm{d}y) = \beta L_x L_y^{-1} L_1^4 L_5^4 \mathrm{d}x \wedge \mathrm{d}\theta^1 \wedge \cdots \wedge \mathrm{d}\theta^8 \equiv \tilde{\beta}\mathrm{d}x \wedge \mathrm{d}\theta^{1\cdots 8}$$

and F_1 only retain the first piece

$$F_1 = \alpha(y)\mathrm{d}x$$

For the F_5 flux

$$F_5 = (1 - i*)\mathcal{F}, \quad \mathcal{F} = \mathrm{d}\theta^{1234} \wedge (\gamma(y)\mathrm{d}x + \delta(y)\mathrm{d}y) \tag{48}$$

Again, for the component along the Euclidean time y, we prefer to use the dual version

$$F_5 = (1 - i*)\left(\gamma \mathrm{d}x \wedge \mathrm{d}\theta^{1234} + i\delta \frac{L_x}{L_y}\left(\frac{L_5}{L_1}\right)^4 \mathrm{d}x \wedge \mathrm{d}\theta^{5678}\right)$$

Also introduce the notation

$$\tilde{\delta} = \delta \frac{L_x}{L_y} \left(\frac{L_5}{L_1} \right)^4$$

The metric in Einstein frame is denoted as

$$\mathrm{d}s^2 = M_y^2(y)\mathrm{d}y^2 + M_x^2(y)\mathrm{d}x^2 + M_1^2(y) \sum_{i=1}^{4} (\mathrm{d}\theta^i)^2 + M_5^2(y) \sum_{i=5}^{8} (\mathrm{d}\theta^i)^2 \tag{49}$$

It is related with the string frame metric as usual

$$M^2 = \mathrm{e}^{-\frac{1}{2}\phi} L^2$$

The method for finding the supergravity solution is to first reduce the 10d action to a 1d effective action by using above ansatz [Seba]. Schematically, the action has two parts

$$S = S_{\mathrm{kin}}[M, M', \phi'] + V[M, \phi, \alpha, \tilde{\beta}, \gamma, \tilde{\delta}]$$

The first part is the kinetic term for the metric and dilaton field. The fluxes come into the second part which is an effective potential. However, not all solutions of the 1d effective action are the solutions of the original 10d equation of motion. It turns out that one needs to impose an additional condition on the flux

$$i\alpha\tilde{\beta} + \gamma\tilde{\delta} = 0 \tag{50}$$

along with the 1d action. Instead of trying to solve the eom directly, one could reduce the second order equations to first order by using supersymmetry. The requirement of the supersymmetry transformation vanishes on the vacuum solution will impose a set of first order equations (Killing spinor equations) on the metric and dilaton field. An ambiguity in Euclidean supersymmetry transformation can be fixed by requiring the Killing spinor equations reproduce the original equations from the effective action [Seba]. Then one can solve the first order equations to obtain the following solutions

$$M_y^2 = \mathrm{e}^{8\omega u} M_x^2, \quad M_x^2 = A_u \mathrm{e}^{-\frac{7}{4}\omega u} \cosh^2(\sqrt{\eta_d \alpha \tilde{\beta}} \mathrm{e}^{\omega u} y + C_\phi) \sin(2\sqrt{\eta_d \alpha \tilde{\beta}} \mathrm{e}^{\omega u} y + 2C_\chi)$$

$$M_5^2 = \mathrm{e}^{\omega u} M_1^{-2}, \quad M_1^2 = \mathrm{e}^{\frac{1}{2}\omega u} \left(-\eta_d \eta_p \frac{\gamma}{\sqrt{\eta_d \alpha \tilde{\beta}}} \tan(\sqrt{\eta_d \alpha \tilde{\beta}} \mathrm{e}^{\omega u} y + C_\chi) \right)^{-\frac{1}{2}}$$

$$\mathrm{e}^\phi = -\eta_d \sqrt{\eta_d \frac{\tilde{\beta}}{\alpha}} \mathrm{e}^{-\omega u} \tanh(\sqrt{\eta_d \alpha \tilde{\beta}} \mathrm{e}^{\omega u} y + C_\phi)$$

$\omega = (57/4)^{-1/4}$, and u, A_u, C_ϕ, C_χ are integration constants. $\eta_p, \eta_d = \pm 1$ are the sign choices for the projections of the supersymmetry parameter.

One can find a solution with constant dilaton field $\phi = \phi_0$ by sending the integration constant $C_\phi \to \infty$

$$\mathrm{e}^{\phi_0} = -\eta_d \sqrt{\eta_d \frac{\tilde{\beta}}{\alpha}} \mathrm{e}^{-\omega u}$$

Also note that the

$$-\mathrm{e}^{-\omega u} (\eta_d \alpha \tilde{\beta})^{-1/2} C_\chi < y < \mathrm{e}^{-\omega u} (\eta_d \alpha \tilde{\beta})^{-1/2} (\frac{\pi}{2} - C_\chi)$$

One can choose $0 < C_\chi < \pi/2$ and $u \to -\infty$ to extend this range to the whole real line. However, the dilaton field will blow up $\mathrm{e}^{\phi_0} \to \infty$.

2.5 the IKKT model as effective action of D(-1) branes

The effective action of N coincident D(-1)-branes are described by reducing the 10d, $\mathcal{N}=1$ super-Yang-Mills theory to 0d. The action can be found in [11] as

$$S[A,\psi;g_0,N,a] = -\frac{Na^4}{g_0^2}\left(\frac{1}{4}\mathrm{Tr}[A_\mu,A_\nu]^2 + \frac{1}{2}\mathrm{Tr}(\overline{\psi}\Gamma^\mu[A_\mu,\psi])\right) \tag{51}$$

$A_\mu, \mu = 0,\cdots,9$ are $N\times N$ (traceless) Hermitian matrices; ψ are Majorana-Weyl spinors in 10d (has 16 real components) with each component being a $N\times N$ Grassmann-valued Hermitian matrix. a is a cut-off length for the "eigenvalues" of A_μ

$$-\frac{\pi}{a} \leq \text{eigenvalues of } A_\mu < \frac{\pi}{a}$$

When we talk about the eigenvalues, we have the classical minimum $[A_\mu, A_\nu]=0$ in mind, where A_μ can be simultaneouly diagonalized. Then the a^4 factor in front of the action can be understood as keep the scaling invariance

$$A \to \lambda A, \quad a \to \lambda^{-1} a, \quad \psi \to \lambda^{3/2}\psi$$

The N factor appears because this action comes from the large-N reduction of ten-dimensional super-Yang-Mills theory [11]. The ten-dimensional theory has a classical mass scale [11] $1/g_0^{\frac{1}{3}}a$, which is used to figure out the double scaling limit of the matrix model

$$a\to 0, \quad g_0 \sim a^{-3} \to \infty, \quad N \sim a^{-10} \to \infty$$

where the prefactor is matched with that found in the DBI action

$$\frac{Na^4}{g_0^2} = 2\pi^3 \frac{1}{\alpha'^2 g_s} \tag{52}$$

which is kept fixed while taking the double scaling limit.

Let us ignore the complication of the double scaling limit, and simply use the action

$$S[A,\psi;g] = -\frac{1}{g^2}\left(\frac{1}{4}\mathrm{Tr}[A_\mu,A_\nu]^2 + \frac{1}{2}\mathrm{Tr}(\overline{\psi}\Gamma^\mu[A_\mu,\psi])\right) \tag{53}$$

An easy thing to note is that this action is invariant under the shift of identity matrices $A_\mu \to A_\mu + a_\mu \mathbb{1}$. This physically corresponds to the shift of the center-of-mass coordinates of the D-instantons. The traceless condition is usually imposed to fixing this gauge redundancy.

Supersymmetry transformations are given by

$$\delta_\epsilon \psi = \frac{i}{2}[A_\mu, A_\nu]\Gamma^{\mu\nu}\epsilon, \tag{54}$$

$$\delta_\epsilon A_\mu = i\bar{\epsilon}\Gamma_\mu \psi. \tag{55}$$

where the factor i is introduced to keep the Hermicity. ϵ are MW spinors whose components are not matrices but simply Grassmann numbers. It is interesting to check this supersymmetry. First the $\delta_\epsilon A$ transformation for the second term vanishes by itself

$$\mathrm{Tr}\overline{\psi}\Gamma^\mu[\delta A_\mu,\psi] = \mathrm{Tr}\overline{\psi}\Gamma^\mu[(\bar{\epsilon}\Gamma_\mu\psi),\psi] = 0$$

One important point in this derivation is to use the following identity of the MW spinors [9]

$$\gamma_\mu\psi_1\overline{\psi}_2\gamma^\mu\psi_3 + \gamma_\mu\psi_2\overline{\psi}_3\gamma^\mu\psi_1 + \gamma_\mu\psi_3\overline{\psi}_1\gamma^\mu\psi_2 = 0 \tag{56}$$

This identity is not straightforward to verify, but can be checked explicitly if working in a particular basis. (Appendix)

The δA of the first term will cancel with the $\delta\psi$ of the second term: the first variation

$$\mathrm{Tr}[\bar{\epsilon}\Gamma_\mu\psi, A_\nu][A^\mu, A^\nu] = \mathrm{Tr}\epsilon^\alpha(C\Gamma_\mu)_{\alpha\beta}\psi^\beta[A_\nu,[A^\mu,A^\nu]]$$

will cancel with the second variation

$$\mathrm{Tr}\left([A_\mu, A_\nu](\Gamma^{\mu\nu})^\gamma{}_\alpha\epsilon^\alpha(C\Gamma^\rho)_{\gamma\beta}[A_\rho,\psi^\beta]\right) + \mathrm{Tr}\left(\psi^\beta(C\Gamma^\rho)_{\beta\gamma}[A_\rho,[A_\mu,A_\nu](\Gamma^{\mu\nu})^\gamma{}_\alpha\epsilon^\alpha]\right)$$

Here I have written out the spinor indices α,β,γ to make sure that the spinor product will not be confused with the matrix product. The C-matrix in front of Γ^μ is for the Majorana conjugation $(\overline{\psi})_\beta = \psi^\alpha C_{\alpha\beta}$. One can simplifty the second variation to

$$(C\Gamma^\rho\Gamma^{\mu\nu})_{\beta\alpha}\mathrm{Tr}(\epsilon^\alpha[A_\rho,\psi^\beta][A_\mu,A_\nu]) = \left[(C\Gamma^{\mu\nu\rho})_{\beta\alpha} + 2\eta^{\rho[\mu}(C\Gamma^{\nu]})_{\beta\alpha}\right]\mathrm{Tr}(\epsilon^\alpha\psi^\beta[A_\rho,[A_\nu,A_\mu]])$$

The $(C\Gamma^{\mu\nu\rho})$ part vanishes due to the Jacobi identity

$$[A_\rho,[A_\mu,A_\nu]] + (\text{cyclic } \rho,\mu\nu) = 0$$

while the second part has the same form as the first variation

$$2\mathrm{Tr}\epsilon^\alpha(C\Gamma_\mu)_{\beta\alpha}\psi^\beta[A_\nu,[A^\mu,A^\nu]]$$

Above, I sketch the main identities without keeping track of possible minus signs and the factor of 2.

As we have mentioned, the classical solution of this model is given by the commutative matrices (here we assume the fermionic background vanishes $\psi = 0$)

$$[A_\mu, A_\nu] = 0 \tag{57}$$

However, the eom obtained by extremize the variation δA allows more general situation

$$[A^\mu,[A_\mu,A_\nu]] = 0 \tag{58}$$

For example, one could consider the configuration that gives

$$[A_\mu, A_\nu] \propto \mathbb{1}$$

Although this identity can not be realized by finite matrices.

3 Random matrix theory and its applications

3.1 an overview of matrix model and its applications

Quantum mechanics is a theory of matrices, more precisely, operators in Hilbert space. The Hilbert space is usually an infinite dimensional vector space, so the operators are infinite dimensional matrices. The major question in quantum mechanics is to solve the Hamiltonian operator to obtain the energy spectrum of the system. This is not an easy task, but even more complicate if one does not know the Hamiltonian precisely. This is the situation people face when trying to understand the structure of the energy level of complex nuclei. An idea for solving this problem is to consider all possible Hamiltonians, therefore a statistical ensemble of matrices, based on the knowledge of symmetry principles of nucleus interactions [21]. The hope is that the statistics of energy levels will follow from such an ensemble of Hamiltonians. This initiates the applications of random matrix theory in physics.

one-matrix model The idea of random Hamiltonian is realized by a class of matrix models with one matrix. The matrix models in this thesis will not be interpreted as models of random Hamiltonian, but the one-matrix model is still an ideal starting point for testing the RG method.

Consider a random $N \times N$ matrix M, with certain symmetry property. The symmetry property could be real orthogonal, Hermitian, symplectic. For a quantum mechanics system with unitary time evolution, one should take Hermitian Hamiltonians. The Hermitian property for the matrix elements is $M_{ij} = M^*_{ji}$. A random M means that each matrix elements are random variables. A simple, and powerful assumption of these random variables is that they are statistically independent. This, together with the unitary assumption, will lead to the following specific form of probability distribution (see reference therein [6])

$$\exp\left(-\frac{\alpha}{2}\mathrm{Tr}M^2\right)$$

It is remarkable that the Gaussian distribution could follow from those general assumptions. However, this assumption has no physical motivation. This is the reason for Dyson [6] to consider an alternative model. We will also relax the statistically independence condition, but keep the unitary invariance, in the following discussions. The unitary invariance is important for deriving the statistical properties of the eigenvalues of M. It is therefore very interesting to explore these different theories and try to discover the links between them. However, the matrix models discussed later will not be understood as the random Hamiltonians. They are either of no real physical meaning, or just labeling the configurations of certain objects (D-instantons). Here, I will introduce this class of matrix models by starting with their QFT counterparts, then only focusing on constant field profile, therefore reducing it to zero dimension.

matrix model from 0d QFT The well-established example of QFT with matrix-valued field is the gauge field theory, or Yang-Mills theory. The formalism of this theory is fairly intricate because one needs to engineer a space of matrices on each point of a smooth geometry such that the resulting structure is simple under the gauge transformation. However, in this thesis, there is no smooth geometry, but just a point. A crude description of the theory is sufficient.

From physics point of view, the Yang-Mills theory could describe the interactions between quarks. Except for being particles that carry masses and electric charges, their states are also labeled by vectors in a N dimensional vector space. The assumption underlying the

Yang-Mills theory is that any unitary transformation U on that vector space should not change the physics. When it comes to the field theory, the unitary transformations are promoted to a space-dependent transformation $U(x)$. They are $N \times N$ unitary matrices parameterized by x. For the interactions, it's important to compare quarks at different points in a way that is invariant under arbitrary $U(x)$. This requires us to introduce the gauge connection (or gauge potential) $A_\mu(x)$. They are $N \times N$ Hermitian matrices, which are the infinitesimal version of unitary matrices. The index $\mu = 1, \cdots, d$ labels the directions of the underlying geometry with dimension d. They generate infinitesimal unitary transformations along certain directions, which allows us to compare quarks at different positions. They have peculiar transformation properties to keep the theory gauge invariant. Intuitively, they are rulers for the quarks. Physically, they are dynamical, and have the name gluons.

In summary, the basic ingredients of a Yang-Mills theory is quark and gluon. Quark is vector-valued matter field and gluon is matrix-valued interaction field. For our interests in matrix model, it's sufficient to focus only on the gluon part $A_\mu(x)$. If one consider a constant gluon field A_μ, the Yang-Mills theory will degenerate to a matrix model for $N \times N$ Hermitian matrices. The number of matrices is d. It has the name of Yang-Mills matrix model. A comprehensive discussion appears in [20]. In particular, it's interesting to study how does it depend on d and N [14, 15].

This connection between QFT and matrix model suggests that no dynamic information is left for the matrix model. We will not expect the matrix model could help us to understand the QFT. Rather, we will take the matrix model as starting point, from which the dynamics of QFT could emerge through some unknown mechanisms.

There are many applications of matrix model to study physical phenomena, ranging from condensed matter theory to the energy spectrum of QCD. Here is a collection of interesting applications [2]. I will review those are relevant for the thesis.

as a model of two dimensional quantum gravity Gravity is a theory of geometry, which should work presumably in any dimension. The fundamental variable of this theory is the metric field $g(x)$, which measures the lengths and angles around every points of the geometry. But this variable depends on directions: the lengths are more stretched in some directions than others. It is possible to construct a quantity out of the metric to describe the curvature of the geometry, but does not depend on directions. One simple choice is called the Ricci scalar $R(g(x))$. A gravity theory then can be built by assigning a number $S[R(g)]$ to every possible geometries, which roughly speaking, is the overall curvature. The classical gravity theory requires that the shape $g(x)$ of the geometry should minimize the number $S[R(g)]$. In four dimensions, this gives the Einstein's theory of gravity without matter fields and cosmological constant.

A naive understanding of quantum gravity is to consider all possible geometries, include those away from the classical solutions. They form a statistical ensemble, with relative probability given by the Boltzmann factor $\exp(-S)$.[2] This statistical model of smooth geometries $g(x)$ is difficult because it is hard to define a measure on the space of all possible geometries $\{g(x)\}$, even around the classical geometries. However, the problem is more tractable in two dimensions. This does not describe our real world, but some lessons can be learned from it.

Matrix model provides a method to study the two dimensional quantum gravity. I will introduce this method by following the review [7]. One simplification in two dimensions is

[2]Strictly speaking, this is not quantum gravity, but a statistical model of random geometry. At the level of current discussion, I will take them as the same.

that $S[R(g)]$ can only take discrete values, which means that it is a topological number. One could denote this number as $S[R(g)] = \gamma\chi$, where γ is a fixed parameter, and χ is the Euler number. For example, $\chi = 2$ for a sphere or disk; $\chi = 0$ for a cylinder or torus. More complicate geometries have $\chi < 0$. Therefore, the two dimensional classical gravity is topological. However, the statistical ensemble is non-trivial: one must consider of all possible shapes, and "sum" over them. Without the knowledge of $\{g(x)\}$ space, it is impossible to define the "sum". One idea is to consider discrete surfaces, which could be understood as a triangulation of the smooth surfaces. It is easier to define the sum over discrete surfaces, then the smooth surfaces can be obtained by taking certain limit.

There is an indirect connection between the matrix model and the discrete surfaces. The ensemble of matrices gives a partition function $Z(g; N)$, where g is a parameter of the model, and N is the size of the matrices. In the class of the one-matrix model discussed above, this partition function could be calculated by the following series

$$Z(g; N) = N^2 Z_0(g) + Z_1(g) + N^{-2} Z_2(g) + \cdots \tag{59}$$

Each term has a diagrammatic interpretation, which corresponds to some discrete surfaces with a Euler number χ. This Euler number is dictated by the power of N^χ of each term. All possible triangulations should contribute to each $Z_h(g)$, therefore, the matrix model defines a sum over all discrete surfaces.

However, to obtain the smooth surfaces, one needs to take certain "continuous limit". For the matrix model, this limit has the form $N \to \infty,\ g \to g_*$. The $N \to \infty$ limit is technical, and has no intuitive meaning. One way to understand is that it is the limit where the asymptotic expansion (59) works. Another way to understand is that it is similar to the thermodynamic limit, which is a necessary condition for phase transitions. g_* could be understood as a critical point where the smooth surfaces dominate the sum. $Z_h(g) \sim (g - g_*)^{2-\gamma_h}$ has simple scaling behavior around the critical point, which again is a reminiscent of phase transition. γ_h is a character number of the matrix model, whose calculation is essential for understanding how does smooth surfaces emerge from it. It turns out that γ_h has a linear dependence on the Euler number χ, which implies that one could take $N \to \infty$ and $g \to g_*$ simultaneously at certain rate such that all terms in (59) have a uniform scaling behavior. Therefore, in this regime, smooth surfaces with arbitrary topologies emerge at the same rate.

There are some methods to study the continuum limit, but only when the model is simple enough for an exact solution. People want to develop a universal method that could be applied to more complicate models, although maybe not exact. This is the original motivation for developing an RG method [4]. We will use this method to understand matrix models, but in a different spirit.

as a constructive definition of string theory This is the original motivation for proposing the IKKT matrix model [11], although we will use another interpretation. Before explaining the meaning of "constructive definition", we must point out that a complete definition of string theory, which in principle contains all string physics, is not available.

It is easy to define the motion of string in a classical space-time. The trajectory is prescribed by a world-sheet (surface) embedded in the space-time, which is just an extension of the world-line of particles. However, two difficulties arise if one wants to define a quantum theory. First, one can not keep the geometry picture of a world-sheet embedded in the space-time, just like the geometric picture of particle trajectory is not well-defined. People usually choose to keep the geometry of the world-sheet, then the embedding space-time coordinates become quantum fields on the surface. So extracting the space-time physics from a quantum string is not straightforward.

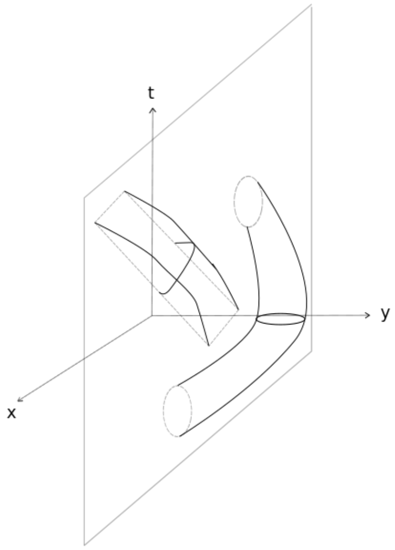
The second difficulty relates with the interactions between strings. The understanding of interactions between particles is promoted by QFT, because it generates a set of world-line diagrams that can be interpreted as the interaction processes. Unfortunately, we do not have a similar model to generate the world-sheet diagrams for string interactions. Instead, one must consider each world-sheet geometry separately, and then sum over them. This is the approach of perturbative string theory. This is similar to the difficulty of two-dimensional quantum gravity discussed above, but in this case, we also have quantum fields on the surfaces. There is no available matrix model to generate the geometries together with the quantum fields.

The understanding of perturbative string bases on some mathematical techniques to study quantum fields on surfaces, however those techniques require the surface theory has enough symmetries related with the surface geometry (superconformal symmetry). It turns out that one can also develop other surface theories, which are equivalent to each other as theories of classical string. The one that is essential for the IKKT matrix model is the Green-Schwarz action [9]. Although there is no superconformal structure, therefore hard to define a quantum theory through those techniques, it manifests certain space-time symmetries. The space-time symmetries are essential for understanding the implications of string theory to the real world. It is therefore interesting to seek for an alternative way to "quantize" the theory. The IKKT matrix model provides a "formal quantization"[3] of the Green-Schwarz action, in the sense that the fields on the surface $X^\mu(\sigma, \tau)$ are mapped to Hermitian matrices A^μ with some requirements to keep the structures of the theory, in particular symmetries. In the last part of [11], this matrix model is related to a Yang-Mills theory reducing to a point. This serves as an evidence for the IKKT matrix model is well-defined as a quantum theory. The conjecture is that this model defines string theory, and recover the Green-Schwarz theory in certain classical limit.

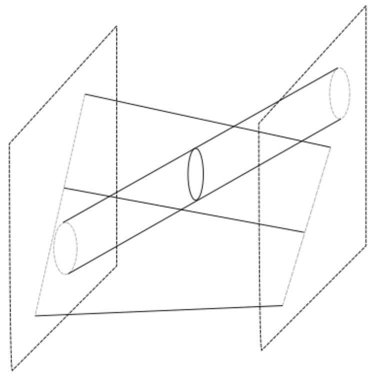
space-time physics from the IKKT matrix model It is natural to ask what is the physical significance of the IKKT matrix model: if there is only a point underlying this model, how could we explain the interactions between physical objects separated by certain distance? If it is a model of string theory, can we find graviton in its spectrum? Unlike the field theory or string theory, there is no oscillation mode. So the spectrum can not be understood as the set of oscillation modes. To understand how matrix model could give rise to space-time physics, we take a closer look.

As a matter of fact, any question that is asked with a geometric object in mind can not be formulated in matrix model directly, with the exception that the geometric object is just a point embedded into the space-time. The space-time coordinates of this point are "quantized" to matrices A^μ (non-commutative), therefore no question related with the classical space-time geometry can be asked directly. This situation is actually the same as in quantum mechanics, except there is also no time parameter here. In quantum mechanics, we formally introduce the eigenstates of the position operators $\hat{x}|x\rangle = x|x\rangle$, with the eigenvalues are the coordinates of the position. Therefore, if all the $N \times N$ matrices A^μ are diagonal (therefore $[A^\mu, A^\nu] = 0$), it is natural to interpret them as the coordinates of a "lattice" with N sites embedded into an imagined space-time. This could provide a classical geometry for asking the space-time questions, but we also remark on the general

[3]Quantization is a formal procedure to define a quantum theory out of a classical theory that is in general both mathematically ill-defined and physically meaningless. In particular, quantum theory is physically more fundamental than classical theory. However, one could still take the result quantum theory seriously, but its relation with the classical theory becomes clear only in certain limit regime.



(a) the Dirichlet boundary condition



(b) interactions between two D-strings

non-commutative matrices.

This "lattice" is not an absolute background, but dynamical in the sense that all possible diagonal matrices, actually all possible matrices, will contribute to the statistical ensemble. However, because a potential energy $\mathrm{Tr}[A, A]^2$ appears in the IKKT matrix model, which is minimized only if $[A^\mu, A^\nu] = 0$, one could approximate the off-diagonal matrix elements as small fluctuations around the minimum of the potential. As we will see in the thesis, these fluctuations will lead to an effective potential energy between the diagonal elements $V(a_1^\mu, \cdots, a_N^\mu)$. In this way, we could ask about the interactions between these "lattice sites". It turns out certain extended objects (strings, branes etc.) can also be formulated by considering the non-commutative configuration $[A^\mu, A^\nu] \neq 0$. This is not easy to understand, so I will leave it to the later sections. It turns out that the IKKT matrix model will give a long-range interaction between extended objects with the same polarization structure as the gravity field $g_{\mu\nu}$ [11]. This is the evidence that the IKKT matrix model will give a space-time physics containing gravity.

as an effective theory of D(-1)-branes A family of extended objects called the Dirichlet branes (D-branes) could exist in string theory. They are essential for the AdS/CFT correspondence. One member of the family could give a concrete physical meaning for the IKKT matrix model. Therefore, it is worthy of a short introduction here.

We have discussed that the worldsheet formulation defines the perturbative string theory, and the worldsheet geometry describes the dynamics of strings. The worldsheet geometry, for example a strip, could have boundaries. Therefore, boundary conditions must be specified when defining the quantum field theory on it. One possible boundary condition is the Dirichlet boundary condition that requires some fields have constant values at the boundary. The boundary condition acquires a geometric interpretation when one looks at it from the classical space-time point of view. Remember that the fields are interpreted as the embedding coordinates, therefore the Dirichlet conditions suggest there is an "imagined wall" to which the boundaries of the worldsheets (or the end points of the strings) are attached. Although in the picture, this wall has one spatial dimension, fixed at a certain value of the y-coordinate, in general, it can be any dimension and has any orientation. There is also nothing could stop us from assigning different conditions to different boundaries, for example, we could have two separated walls.

If one slices the space-time into constant times, the picture has interesting explanations. The walls are actually one-dimensional static strings, from now on we will call them D1-branes, or D-strings. Some strings are stretching between the D1-branes. Some strings are emitted by the D1-branes, propagating in the space-time, and re-absorbed by the D1-branes later. There are two things that are noteworthy in this picture. The strings play the role of messengers between D1-branes, therefore inducing interactions between them. This is an important reason for taking these "walls" as physical objects. However, there is a seeming asymmetry in this picture: both the D-string and the string are one-dimensional objects, but one is propagating and another is static. In the worldsheet picture, the asymmetry is not a surprise because the D-strings are boundary conditions. But from the space-time picture, this is quite uncomfortable. It turns out that there is indeed a relation between them, which can only be understood by formulating string theory as a space-time theory.

The way to describe the space-time dynamics of string theory is to zoom out the picture such that strings become small $l_s/R \to 0$ comparing to the curvature of the space-time. They then look like usual particle excitation in quantum field theory. This is called the low energy limit, or the effective field theory description, of string theory. The details of the string oscillations are lost, but the center-of-mass motion, which is quantized to massless particles, is kept. Therefore, the long-range interactions meditated by strings should be described in such an effective description. The finer structure of the string would be taken into account by including sub-leading corrections. The effective field theory is the main tool to understand the long-range interactions of D-branes.

It is useful to separate the effective theory into two sectors: one describes the strings moving on the D-branes (the world-volume theory); another describes the strings moving freely in the space-time (the bulk theory). These two sectors have interactions, which corresponds to the picture that some strings pinch off, and be absorbed by the D-branes.

We use the D-string as an example because it is easy to visualize. When it comes to the D(-1)-brane (or D-instanton), we need to replace the "walls" by points. The strings then attach to the points, and describe the interactions between D-instantons. The IKKT matrix model turns out to be the effective theory for those strings [22], where the matrix elements A_{ij} labels the string stretched between the i-th and j-th D-instantons; the diagonal element A_{ii} is understood as the space-time coordinate of the i-th D-instanton. The interactions between D-instantons separated by a long-distance calculated using the IKKT matrix model agrees with the result from the bulk theory [5].

3.2 Gaussian unitary ensemble

3.3 eigenvalue statistics

A procedure that is similar to gauge fixing leads to the eigenvalue representation. Any Hermitian matrix can be diagonalized by certain unitary matrix

$$M \to D = UMU^\dagger = \mathrm{diag}(\lambda_1, \cdots, \lambda_N). \tag{60}$$

The space of Hermitian matrices can be decomposed into equivalent classes under the gauge group $U(N)$, where the diagonal elements are chosen to be the representative elements. Then we try to decompose the integration into two parts:

$$\int [\mathrm{d}M] \to \int [\mathrm{d}D] \int_{U(N)} [\mathrm{d}U]$$

One part is an integration over all diagonal (of course real) matrices, the other part is the gauge group. This is analogous to the change of integration variables, and the Jacobian is worked out by Faddeev-Popov method.

To get the desired decomposition, we need to work out the Jacobian for the change of coordinate $M \to (D, U)$. The coordinate transformation is implicitly given by the equation

$$M - U^\dagger D U = 0. \tag{61}$$

Assume that $[\mathrm{d}U]$ is the Haar measure on $U(N)$, then we define the Faddeev-Popov measure $\Delta^2(M)$ by

$$1 = \Delta^2(M) \int_{U(N)} [\mathrm{d}U] \int [\mathrm{d}D] \delta(M - U^\dagger D U), \tag{62}$$

where $[\mathrm{d}D]$ written in component form is $\prod_i \mathrm{d}\lambda_i$. Here we can understand the δ-functional gives a distribution in (U, D) space for a certain M. However, it's not simply factored out as $\delta(U - U(M))\delta(D - D(M))$, so the integral over (U, D) space gives a notrivial M dependence. Then insert the identity to the matrix integral

$$\int [\mathrm{d}M] = \int [\mathrm{d}M] \Delta^2(M) \int [\mathrm{d}U] \int [\mathrm{d}D] \delta(M - U^\dagger D U).$$

We can then interchange the order of integration to first perform $\int [\mathrm{d}M]$, which is easy because the δ-functional simply gives $M = U^\dagger D U$,

$$\int [\mathrm{d}M] = \int [\mathrm{d}U] \int [\mathrm{d}D] \Delta^2(U^\dagger D U).$$

The measure $\Delta(M)$ is by definition gauge invariant: $\Delta^2(U^\dagger D U) = \Delta^2(D)$. Then the integral over the gauge group produce a factor Ω_N that is propotional to the $\mathrm{vol}(U(N))$. The result is that the integration can be performed over eigenvalues with the Faddeev-Popov measure:

$$\int [\mathrm{d}M] = \Omega_N \int \left(\prod_{i=1}^N \mathrm{d}\lambda_i \right) \Delta^2(\lambda). \tag{63}$$

To evaluate the $\Delta^2(\lambda)$, let's look at the definition (62). Take $M = D$, rename the integrated D as D', then dropping the $\int [\mathrm{d}D']$ by using the $\delta(D - U^\dagger D' U) = \delta(D' - UDU^\dagger)$. We can equivalently separate the equation $D' - UDU^\dagger = 0$ into two equations

$$D' = D,$$
$$UDU^\dagger = D.$$

That is $\delta(D' - UDU^\dagger) = \delta(D' - D)\delta(D - UDU^\dagger)$. Integrate $\mathrm{d}D'$ to get

$$\Delta^{-2}(D) = \int_{U(N)} [\mathrm{d}U]\delta(D - UDU^\dagger).$$

This allows us to expand $U(N)$ near the identity $U(N) = 1 + \varepsilon A + O(\varepsilon^2)$, where A is anti-Hermitian. Then $\delta(D - UDU^\dagger) = \delta(\varepsilon[A, D])$. Use the component form of the measure

$$[\mathrm{d}U] = \varepsilon \prod_{1 \leq i < j \leq N} [\mathrm{d}\Re A_{ij}][\mathrm{d}\Im A_{ij}].$$

The δ-functional takes the form

$$\delta(\varepsilon A_{ij}(\lambda_i - \lambda_j)) = \frac{1}{\varepsilon} \frac{1}{(\lambda_i - \lambda_j)^2} \delta(\Re A_{ij})\delta(\Im A_{ij}).$$

This gives the so called Vandermonde determinant

$$\Delta^2(\lambda) = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2. \tag{64}$$

Now we can write the partition function as

$$Z = \frac{\Omega_N}{\mathrm{vol}(U(N))} \int \left(\prod_{i=1}^{N} \mathrm{d}\lambda_i \right) \Delta^2(\lambda) e^{-\frac{1}{g_s} W(\lambda)}. \tag{65}$$

3.4 the orthogonal polynomial method

orthogonal polynomial [7]. Physicists use the orthogonal polynomials as a basis for writing down the wave functions of a quantum particle moving in 1d. It's possible and also simple because the physical observables, especially the energy, have definite value for each state. As for the matrix model, the use of orthogonal polynomial is more like a mathematical construction, rather than a part of physical theory. But for the convenience of a physics mind, also interesting by itself, let's first try to give a physical interpretation.

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-V(\lambda)} P_n(\lambda) P_m(\lambda) \mathrm{d}\lambda = s_n \delta_{nm} \tag{66}$$

This orthogonal condition is not normalized to 1. s_n is the normalization constant fixed by requiring the leading term of $P_n(\lambda) \sim \lambda^n$. This integration makes sense only if $V(\lambda)$ goes to infinity when $\lambda \to \pm\infty$.

The eigenvalues of the random matrices can be viewed as labeling the "position" of particles $\# = N$ in a fictitious 1d space. (random point process?) There is an external potential, given by the function $V(\lambda)$, under which the particles tend to "move towards" each other (higher probability for them to be close). The Vandermonde determinant has the interesting interpretation as the repulsion force between the particles.... Recall that the electric potential $\varphi(r) \sim r^{2-D}$ for $D > 2$, and $\varphi(r) \sim \ln r$ for $D = 2$, where D is the space dimension. The Vandermonde determinant provides a ln-like repulsion potential, but we are imagining some 1D space. Is this an in-consistency?....

the "positions" of "charged particles" (number $= N$) under an "external potential". The matrix integral is a partition function of this statistical system. [what's the polynomial corresponds? physical states of the N-particles? The configuration space of the N-particles is much larger than the number of polynomials. How does this reduction happen? Because we are only interested in partition function so this trick really work. What about other observables? even correlation functions? It seems important that the observables are symmetric under interchaning the "particles".]

If we are given a partition function [17]

$$Z = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{2\pi} \Delta(\lambda) e^{-\frac{1}{g_s} \sum_{i=1}^{N} W(\lambda_i)}, \tag{67}$$

the desired polynomials $p_n(\lambda), 0 \le n \le N-1$ is normalized such that $p_n(\lambda) = \lambda^n + \cdots$ and orthorgonal in the sense that

$$\int p_n(\lambda) p_m(\lambda) e^{-\frac{1}{g_s} W(\lambda)} \frac{\mathrm{d}\lambda}{2\pi} = h_n \delta_{nm}. \tag{68}$$

[rem. it maybe interesting to think about how $p_n(\lambda)$ depends on g_s?] [edit. unify the notations: especially Z or e^Z, and normalizations]

Let's look at the product $\prod_{i<j}(\lambda_i - \lambda_j)$ which can be written as the determinant

$$\prod_{i<j}(\lambda_i - \lambda_j) = \det \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{N-1} \end{pmatrix} = \det p_{j-1}(\lambda_i).$$

Then the Vandermonde determinant is given by

$$\Delta(\lambda) = [\det p_{j-1}(\lambda_i)]^2 .$$

In terms of the orthogonality, we obtain a simple result ($N!$ is canceled by permutations)

$$Z = \prod_{i=0}^{N-1} h_i \equiv h_0^N \prod_{i=1}^{N} r_i^{N-i}, \quad r_k = \frac{h_k}{h_{k-1}}. \tag{69}$$

some trials. It's remarkable that the problem of integration is transformed to a problem of production. By taking the log, it becomes a problem of summation

$$F \equiv \ln Z = \sum_{i=0}^{N-1} \ln h_i$$

At this stage, it maybe interesting to think about how h_i depends on i? One typical dependence is (Gaussian example below, but how general it is?)

$$h_n \propto n! a^n$$

Can we make sense of the sum

$$\sum_{n=0}^{N-1} \ln(n! a^n)$$

when $N \to \infty$. One may think about write $n!$ in terms of an integral, then use some tricks to interchange integration and summation. But the ln is an obstruction. Or one may write it as

$$\sum_{n=0}^{N-1} \ln(n!) + n\ln a = \text{const.} + (\sum_{n=0}^{N-1} n)\ln a$$

rem. By using the orthogonal polynomial, we factorize the λ_i dependence. Then the integration can be done for each λ_i. rem. This is similar (?) to do the Fourier transformation for $(\partial\phi)^2$.

example. Gaussian model and Hermite polynomial For the Gaussian matrix model, the orthogonal polynomials are the Hermite polynomials $H_n(x)$, which can be defined as

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}. \tag{70}$$

The orthogonality reads

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} \mathrm{d}x = \sqrt{\pi} 2^n n! \delta_{n,m}. \tag{71}$$

By changing the variable $x \to x' = x/\sqrt{2g_s}$, we obtain the desired orthogonality (68)

$$\int_{-\infty}^{\infty} H_n(\frac{x}{\sqrt{2g_s}}) H_m(\frac{x}{\sqrt{2g_s}}) e^{-\frac{x^2}{2g_s}} \frac{\mathrm{d}x}{2\pi} = \sqrt{\frac{g_s}{2\pi}} 2^n n! \delta_{n,m}.$$

While the leading coefficients of $H_n(x/\sqrt{2g_s})$ is $(2/g_s)^{n/2}$, so the desired polynomials are

$$p_n(\lambda) = \left(\frac{g_s}{2}\right)^{n/2} H_n(\frac{\lambda}{\sqrt{2g_s}}), \quad h_n = \sqrt{\frac{g_s}{2\pi}} n! g_s^n. \tag{72}$$

The partition function then can be written as

$$Z = \prod_{i=0}^{N-1} h_i = g_s^{N^2/2} (2\pi)^{N/2} 1!2! \cdots (N-1)!. \tag{73}$$

Not identical to previous result, something goes wrong.

recursion relation of r_n [17]. For a general potential $g_p \neq 0, p \geq 3$, the orthogonal polynomial is hard to find. However, a recursion relation of the polynomials can be used to compute the partition function. The general form of the recursion relation is (can check by the orthogonality (68))

$$(\lambda + s_n)p_n(\lambda) = p_{n+1}(\lambda) + \frac{h_n}{h_{n-1}}p_{n-1}(\lambda). \tag{74}$$

This can be understood as by multiplying λ to $p_n(\lambda)$, we obtain a linear combination of $p_{n-1}(\lambda), p_n(\lambda)$ and $p_{n+1}(\lambda)$

$$\lambda \begin{pmatrix} \vdots \\ p_{n-1} \\ p_n \\ p_{n+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \cdots & 0 & r_{n-1} & -s_{n-1} & 1 & 0 & \cdots & & & \\ & \cdots & 0 & r_n & -s_n & 1 & 0 & \cdots & & \\ & & \cdots & 0 & r_{n+1} & -s_{n+1} & 1 & 0 & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ p_{n-1} \\ p_n \\ p_{n+1} \\ \vdots \end{pmatrix}$$

or formally

$$\lambda p_n(\lambda) = B_{nm} p_m(\lambda). \tag{75}$$

The starting point is $n = 0$, which gives $\lambda = B_{00} + B_{01}p_1(\lambda) = B_{00} + B_{01}(\lambda + s_0)$. We see that $B_{01} = 1$ and $B_{00} = -s_0 = \langle\lambda\rangle$. In the case of an even potential $s_0 = 0$. Actually, for an even potential, all $s_n = 0$, which can be obtained by considering $\langle \lambda p_n^2 \rangle$.

Then the idea is that the orthogonality (68) will give equations for B_{nm} in terms of the potential $W(\lambda)$. An equation can be obtained by considering

$$\int p_n(\lambda)p_n(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(e^{-\frac{1}{g_s}W(\lambda)}\right)\frac{\mathrm{d}\lambda}{2\pi}.$$

On one hand, integrate by part to get (where $\cdots$ contains terms with highest order λ^{n-2})

$$-2\int (np_{n-1}(\lambda) + \cdots)p_n(\lambda)e^{-\frac{1}{g_s}W(\lambda)}\frac{\mathrm{d}\lambda}{2\pi} = 0.$$

On the other hand, take the derivative to get

$$-\frac{1}{g_s}\int p_n(\lambda)p_n(\lambda)W'(\lambda)e^{-\frac{1}{g_s}W(\lambda)}\frac{\mathrm{d}\lambda}{2\pi} = -\frac{h_n}{g_s}W'(B)_{nn}.$$

Comparing the two results we get

$$W'(B)_{nn} = 0. \tag{76}$$

Another equation can be obtained similarly, by considering

$$\int p_{n-1}(\lambda)p_n(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(e^{-\frac{1}{g_s}W(\lambda)}\right)\frac{\mathrm{d}\lambda}{2\pi}.$$

We get

$$W'(B)_{n,n-1} = r_n W'(B)_{n-1,n} = ng_s. \tag{77}$$

In the Gaussian model $W'(B) = B$, then these equations reproduce the result from Hermite polynomials (72). We can try to consider a quartic potential

$$W(\lambda) = \frac{g_2}{2}\lambda^2 + \frac{g_4}{4}\lambda^4.$$

Because this is an even potential, $s_n = 0$ gives $W'(B)_{n,n} = 0$. From $W'(\lambda) = g_2\lambda + g_4\lambda^3$, the equation reads

$$W'(B)_{n,n-1} = r_n\left[g_2 + g_4(r_{n-1} + r_n + r_{n+1})\right] = ng_s,$$

starting with $r_0 = 0, r_1 = \langle\lambda^2\rangle$. In principle, this recursion relation gives all $r_n, 0 \leq n \leq N-1$.

3.5 the continuum limit

What is a well-defined $N \to \infty$ limit? What is an ill-defined $N \to \infty$ limit?

Continuum limit is not just $N \to \infty$ limit, but also with a tuning of coupling constant [7]

$$Z_0(g) \sim \sum_n n^{\gamma-3}(g/g_c)^n \sim (g_c - g)^{2-\gamma} \tag{78}$$

$$\langle A \rangle = \langle n \rangle = \frac{\partial}{\partial g} \ln Z_0(g) \sim \frac{1}{g - g_c}$$

g_c is the critical value of the coupling constant, at which the continuum limit can be taken. To obtain its value, we need to solve the model. γ as critical exponent, is also obtained by solving the model. What happens in the limit $N \to \infty$ when $g \neq g_c$? No emergent geometry? At least, there is no simple scaling behavior of $Z_0(g)$. The sum over n is infinite, and it's unrelated to N. In the geometric interpretation, the number n denote the number of vertices insertion. If $g < g_c$, then one imagine the number of vertices insertion has certain cut-off (?) above which the contribution to $Z_0(g)$ is not significant. The continuum limit is in essence when this cut-off becomes infinity. $Z_0(g)$ is the free energy (of partition function) of the matrix model with the number of handle equal to zero.

Why interested in $N \to \infty$ limit: the emergent of geometry. (But if "conformal", geometry at finite N, or no geometry emerge at all?)

How to take $N \to \infty$ limit? First do the integration, then taking the limit. Do the integration: could think about large N but finite.

3.6 ribbon diagram and 2d quantum gravity

4 Matrix RG method and its application to IKKT model

[the idea of the intro is to give a big picture of the RG method applied to matrix model: what and why? Also as a disclaimer where the deficiency of my understanding should be mentioned. Make a difference with the motivation in literature.]

This section is the main part of this work, with the hope of developing a notion of "conformal matrix model". I will try to study different matrix models in the spirit of Wilsonian renormalization group, which in essence introducing a notion of "effective matrix model".

The meaning of "effective" is, for a model defined by $n \times n$ matrices, we think about it as a model of certain sub-matrices of larger matrices, say $N \times N, N > n$. The model effectively describe the probability distribution of the sub-matrices, which should be a marginal distribution from the larger matrix model.

First, the RG method applied to matrix model is an old subject [references here]. There people hope that this method could give a qualitative understanding of critical points of those models for which other methods are not available. The physics motivation is that the critical points could describe a continuum physics corresponding to certain models of 2d quantum gravity. In this context, people seems to have in mind the analog between the matrix models and statistical systems. The large N-limit is similar to the thermodynamic limit, where some non-trivial critical points could emerge.

However, our point of view on matrix models will be more align with QFT. The continuum limit is just a limit of a *single* model, not a limit where *other* models could emerge. Just as in QFT, the RG method is understood as a way to *define* the continuum limit, which corresponds to take the energy cut-off $\Lambda \to \infty$. [refer to Skinner's note] Let's remark that the physical interpretation of continuum limit is different in statistical physics and QFT: in renormalizable QFT, $\Lambda \to \infty$ is taken because the cut-off is unphysical, and observables should not depend on Λ. Or let's say in this way: one tunes the coupling constants with Λ in such a way that physics is independent of Λ. In the following, we will interpret N as the cut-off scale of a matrix model. [However, the crucial difference is that N is discrete while Λ is continuous.]

Now let's remark on some technical problems. One natural concern is that there is no natural sub-matrix for the matrix models that we are interesting in due to the $U(N)$ gauge symmetry. By performing a $U(N)$ rotation, one could always mix certain sub-matrix with its ambient. That means that any naive attempt to define an "effective" model will only leave a subgroup of $U(N)$ manifest. [4] It's easy to reject the method because it breaks the gauge symmetry, at least make it implicit. However, the same thing happens in the usual RG method, where the gauge symmetry is the diffeomorphism group, and it's broken once we introduce a cut-off scale.

A natural sub-matrix is the one that restricting on a subspace of vector space which the whole matrix acting on. There are infinitely many subspaces, and there is no natural way to difference them. This high degeneracy can be lifted by introducing a "background matrix", then those subspaces are differed by their eigenvalues on this background matrix. This method also makes the analog of the usual RG method, where different degrees of freedom are labeled by their energy scales. However, this is "ad hoc" from the matrix model point of view because one can not prefer a particular background over another. Let's take it as a provisional strategy to make calculation easier.

[4] There maybe a way to take into account all the possibilities of sub-matrices, and $U(N)$ is realized as a transformation among these possibilities. But I have no idea of how to do that.

[continue...]

4.1 the Wilsonian RG flow

[the idea of this section is to seheory with fixed cut-offt up the basic concepts of RG flow to be used later in matrix model]

[Q. By integrating out high energy modes, one lower the cut-off Λ. However, the RG flow is for a theory with fixed cut-off. This is contradictory? How do we keep the information of the cut-off along the RG flow? How to difference the cut-off scale and the scale we are looking at in Wilsonian RG?]

[try. 1d field on a circle, energy scale?]

$$S[\phi; R, m] = \int_{S^1} \left((\frac{\mathrm{d}\phi}{\mathrm{d}x})^2 + m^2\phi^2(x) \right) \mathrm{d}x$$

Assume that the radius is R: it comes into play by the requirement $x \sim x + 2\pi R$. Then it's natural to write $z = \mathrm{e}^{ix/R}$, $z \in S^1 \subset \mathbb{C}$. We take x having energy dimension $[x] = -1$, then z is dimensionless.

$$\mathrm{d}z = i\frac{z}{R}\mathrm{d}x, \quad \frac{\mathrm{d}}{\mathrm{d}x} = i\frac{z}{R}\frac{\mathrm{d}}{\mathrm{d}z}$$

Now define $\varphi : \mathbb{C} \to \mathbb{R}$ such that $\varphi(z)|_{z\in S^1} = \phi(x)$. (I'm not sure whether it's legitimate, but for simplicity, let's work on $\varphi(z)$, holomorphic except at $z = 0$.)

$$S[\varphi; R, m] \equiv -i\frac{R}{z} \int_{S^1} \left(-\frac{z^2}{R^2}(\frac{\mathrm{d}\varphi}{\mathrm{d}z})^2 + m^2\varphi^2(z) \right) \mathrm{d}z$$

Expand $\varphi(z)$ in terms of the power of z

$$\varphi(z) = \sum_{k=-\infty}^{+\infty} \varphi_k z^k, \quad \partial_z\varphi(z) = \sum_{k=-\infty}^{+\infty} k\varphi_k z^{k-1}$$

Then collect z^0 term in $\varphi^2(z)$: $\sum_{k=-\infty}^{+\infty} \varphi_k\varphi_{-k}$; and z^{-2} term in $(\partial\varphi)^2(z)$: $-\sum_{k=-\infty}^{+\infty} k^2\varphi_k\varphi_{-k}$. We have S is proportional (where is the 2π factor...)

$$S \propto \sum_{k=-\infty}^{\infty} \left(\frac{k^2}{R}\varphi_k\varphi_{-k} + m^2\varphi_k\varphi_{-k} \right)$$

We keep S dimensionless, then $[\varphi] = \frac{1}{2}[R] = -\frac{1}{2}$, and $[m] = 1$. To understand how this theory changes with scale, we should work with dimensionless quantities (to justify....) Define $\tilde{\varphi} \equiv (R)^{-1/2}\varphi$, which is dimensionless, and $m_0^2 \equiv m^2R$ which is also dimensionless.

$$S \propto \sum_{k=-\infty}^{+\infty} (k^2 + m_0^2)\, \tilde{\varphi}_k\tilde{\varphi}_{-k}$$

In physics, it is the dimensionless quantity that could be given a value.

What's the RG running in this context? Set a cut-off $k^2 \leq K$. Note here k and K are dimensionless. The $K-1$ modes contribution is

$$[(K-1)^2 + m_0^2]\, \tilde{\varphi}_{K-1}\tilde{\varphi}_{K-1}$$

The rescaling is to define $\tilde{\varphi}' = \frac{K}{K-1}\tilde{\varphi}$ such that the above contribution becomes

$$(K^2 + m_0'^2)\tilde{\varphi}'\tilde{\varphi}', \quad m_0' = m_0 \cdot \frac{K-1}{K}.$$

It's interesting to see that, as we go to low energy regime, m_0 becomes smaller. [This is counter-intuitive, we would imagine the effective mass become larger as we lower the energy?]

4.2 matrix RG of one-matrix model

[the idea of this section is to apply the RG method to the simplest case, maybe also solved some (conceptual and technical) problems appearing later in this simple context...]

The one-matrix model for a $N \times N$ Hermitian matrix $M = M^\dagger$ is featured by the following gauge transformation

$$M \to M' = UMU^\dagger, \quad U \in U(N) \tag{79}$$

It maybe useful also to write down the infinitesimal version

$$\delta_t M = i[t, M] \equiv \mathrm{ad}(t)M, \quad t \in \mathfrak{u}(N) \tag{80}$$

where $U = \mathrm{e}^{it}$. As we have seen, this gauge symmetry plays an essential role in solving the one-matrix model by diagonalizing the random matrices. Let's also clarify a point that may seem not important at this point: although M and t are both Hermitian matrices, they play different roles in the model. [Q. Permutation between rows and columns is a subgroup? some remarks ... $SU(N) = U(N)/U(1)$, Permutation = symmetry groups S_N, real gauge group $SU(N)/S_N$?]

Not all transformations in $U(N)$ will change the matrix. A $U(1)$ subgroup $\mathrm{e}^{i\theta} \in U(1)$ will leave any matrix invariant. So the gauge group that acts non-trivially on the matrices should be $SU(N) = U(N)/U(1)$. [Why for the Dp-brane action, we say it's $U(N)$ gauge theory rather than $SU(N)$? Then $U(1)$ factor corresponds to the center-of-mass coordinate of the branes. The gauge generator = matrices appearing in the model, not the case here.] For a general diagonal matrix, the subgroup $U(1) \times \cdots \times U(1)$ will leave it invariant; if one or more diagonal elements are equal, this subgroup will be enhanced. [to edit]

4.2.1 the Gaussian

the Gaussian matrix model - recap. The simplest matrix model partition function is proportional to the following integral

$$Z_N(\alpha) \propto \int_{N\times N} \mathrm{e}^{-\frac{\alpha}{2}\mathrm{Tr}M^2}[\mathrm{d}M] \equiv I_N(\alpha). \tag{81}$$

Before any other things, one first notes that $I_N(\alpha) \propto \alpha^{-N^2/2}$ by simply rescaling $M \to (\alpha)^{-1/2}M$. This also works for the expectation values $\langle M^k\rangle \propto \alpha^{-k/2}$. Roughly speaking, the value of α determines the scale of M.

In QFT, the measure $[\mathrm{d}M]$ is usually normalized such that $Z_N(\alpha) = 1$. This can be done explicitly for the Gaussian model. It's also interesting to note that how the normalization depends on α. Let's start by using

$$[\mathrm{d}M] = \prod_{i=1}^N \mathrm{d}M_{ii} \prod_{i<j} \mathrm{d}M_{ij}\mathrm{d}M_{ij}^*$$

The component form of the integral is

$$I_N(\alpha) \equiv \int \mathrm{e}^{-\frac{\alpha}{2}\sum_{i=1}^N (M_{ii})^2 - \alpha\sum_{i<j}|M_{ij}|^2} \left(\prod_{i=1}^N \mathrm{d}M_{ii}\right)\left(\prod_{i<j}\mathrm{d}M_{ij}\mathrm{d}M_{ij}^*\right) \tag{82}$$

The Gaussian integral can be calculated directly $\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi}$:[5]

$$I_N(\alpha) \equiv \int_{N\times N} \mathrm{e}^{-\frac{\alpha}{2}\mathrm{Tr}M^2}[\mathrm{d}M] = \left(\frac{2\pi}{\alpha}\right)^{N^2/2} \tag{83}$$

[5] Here is a note on the integral over complex number $z = x + iy$, $x, y \in \mathbb{R}$: $\int_{\mathbb{C}} \mathrm{d}z\mathrm{d}z^*(\cdots) \equiv 2\int_{-\infty}^{+\infty} \mathrm{d}x\mathrm{d}y(\cdots)$

This result is easy to understand: each matrix element contributes to a factor of $2\pi/\alpha$.

To study the expectation values $\langle\cdots\rangle$, it's convenient to use the generating functional $Z_N[J;\alpha]$:

$$Z_N[J;\alpha] \equiv \int \mathrm{e}^{-\frac{\alpha}{2}\mathrm{Tr}M^2+\mathrm{Tr}JM}[\mathrm{d}M] \tag{84}$$

J is an arbitrary $N\times N$ matrix, not necessarily Hermitian. To calculate this integral, rewrite

$$-\frac{\alpha}{2}\mathrm{Tr}M^2+\mathrm{Tr}JM = -\frac{\alpha}{2}\mathrm{Tr}(M-J/\alpha)^2+\frac{1}{2\alpha}\mathrm{Tr}J^2$$

The first part will just reproduce the result of $I_N(\alpha)$:

$$Z_N[J;\alpha] = \left(\frac{2\pi}{\alpha}\right)^{N^2/2}\exp\left(\frac{1}{2\alpha}\mathrm{Tr}J^2\right) \tag{85}$$

Then, for example, one could calculate

$$\langle\mathrm{Tr}M^2\rangle = \sum_{i,j=1}^{N}\frac{\partial}{\partial J_{ij}}\frac{\partial}{\partial J_{ji}}Z_N[J;\alpha]\Big|_{J=0} = \left(\frac{2\pi}{\alpha}\right)^{N^2/2}\cdot\frac{N^2}{\alpha}$$

Note that the N^2 factor comes from the sum $\sum_{i,j}$. Or the normalized one

$$\langle\mathrm{Tr}M^2\rangle_{\mathrm{norm.}} = \sum_{i,j=1}^{N}\frac{\partial}{\partial J_{ij}}\frac{\partial}{\partial J_{ji}}\ln Z_N[J;\alpha]Z_N[J;\alpha]\Big|_{J=0} = \frac{N^2}{\alpha}$$

[These are simple results, showing how different quantities depend on α, N. Can we make connection with the RG method?][to edit]

implement the RG method to the Gaussian model. We will study the normalized matrix model integral:

$$Z_N(\alpha) = \left(\frac{\alpha}{2\pi}\right)^{N^2/2}\int \mathrm{e}^{-\frac{\alpha}{2}\mathrm{Tr}M^2}[\mathrm{d}M]. \tag{86}$$

To apply the RG method, the matrix elements to be integrated out are $M(N,i), M(i,N), i = 1,\cdots,N-1$. Then the result will be an "effective matrix model" for the sub-matrix $M(i,j),\ i,j=1,\cdots,N-1$. For the Gaussian integral, this can be done exactly.

To simplify the notation, let's use $n\equiv N-1$. The RG calculation based on the following decomposition of M_N:

$$\begin{pmatrix} M_{N|n} & \mu \\ \mu^\dagger & m \end{pmatrix}.$$

We will apply the same decomposition to other matrix models in the following sections. Then the action becomes

$$-\frac{\alpha}{2}\mathrm{Tr}_N M^2 = -\frac{\alpha}{2}\mathrm{Tr}_n M^2 - \alpha\mu^\dagger\mu - \frac{\alpha}{2}m^2$$

The integration over $\mu, \mu^\dagger$ and m is of course Gaussian. Let's write it explicitly,

$$Z_N(\alpha) = \left(\frac{\alpha}{2\pi}\right)^{N^2/2}\int_{n\times n}[\mathrm{d}M]\int\prod_{i=1}^{N}[\mathrm{d}\mu_i][\mathrm{d}\mu_i^*][\mathrm{d}m]\mathrm{e}^{-\frac{\alpha}{2}\mathrm{Tr}_n M^2-\alpha\sum_{i=1}^{n}\mu_i^*\mu_i-\frac{\alpha}{2}m^2}.$$

After the integration, one finds exactly the same partition function of the $n \times n$ matrix model:

$$Z_N(\alpha) = \left(\frac{\alpha}{2\pi}\right)^{n^2/2} \int_{n\times n} \mathrm{e}^{-\frac{\alpha}{2}\mathrm{Tr}_n M^2}[\mathrm{d}M] = Z_n(\alpha)$$

This is a trivial calculation, but let's remark on a particular point that is important for the RG method. In the above calculation, α is taken as a "dimensionless" parameter (means having a well-defined value), and $\mathrm{Tr}M^2$ is taken as a "marginal operator" (Here I use the language in QFT). However, if one considers the following form

$$\alpha = N^x \alpha_0$$

with $x \in \mathbb{R}$, and take α_0 as the "dimensionless" parameter, one needs to do some rescalings after the integration to compare the result with the $n \times n$ model.

Let's use the notation $\zeta_N(\alpha_0) \equiv Z_N(\alpha) = Z_N(N^x\alpha_0)$. Just plug $\alpha = N^x\alpha_0$ into the above result

$$\zeta_N(\alpha_0) = \left(\frac{\alpha_0 N^x}{2\pi}\right)^{n^2/2} \int_{n\times n} \mathrm{e}^{-\frac{\alpha_0 N^x}{2}\mathrm{Tr}_n M^2}[\mathrm{d}M]$$

However, $\zeta_N(\alpha_0) \neq \zeta_n(\alpha_0)$ if $x \neq 0$

$$\zeta_n(\alpha_0) = \left(\frac{\alpha_0 n^x}{2\pi}\right)^{n^2/2} \int_{n\times n} \mathrm{e}^{-\frac{\alpha_0 n^x}{2}\mathrm{Tr}_n M^2}[\mathrm{d}M]$$

There are two equivalent rescalings that could write $\zeta_N(\alpha_0)$ as an "effective $n \times n$ model": One may rescale $\alpha_0 \to \alpha_0'$

$$\alpha_0' = \alpha_0 \left(\frac{N}{n}\right)^x$$

Or one could rescale $M \to M'$

$$M' = M\left(\frac{n}{N}\right)^x$$

If one uses the language of QFT: the first take could be interpreted as the classical scaling dimension of α_0; the second take could be interpreted as the wave function renormalization of M.[6] [to edit]

a remark about the trace. There is also an ambiguity concerning the trace Tr_N. The subscript indicates that it acts on $N \times N$ matrices. In particular, $\mathrm{Tr}_N \mathbb{1}_{N\times N} = N$. However, in some cases, one may not want the result depending on N, instead, one could use a normalized trace $\mathrm{Tr}' \equiv N^{-1}\mathrm{Tr}_N$, $\forall N$ such that $\mathrm{Tr}'\mathbb{1} = 1$. If we use Tr' in the above calculation, effectively, we replace $x \to x + 1$.

4.2.2 the quartic

quartic interaction Now let's include a coupling constant g into the model and study how the scaling properties get modified. In principle, away from the Gaussian fixed point, one should include all possible interactions with an infinite set of couplings $\{g_1, g_2, \cdots\}$. But let's start the analysis by just adding a single trace quartic term $\mathrm{Tr}M^4$, and think about it as a small perturbation around the Gaussian fixed point. Let's also think about g as a coupling with certain N-dimension. Introduce the corresponding "dimensionless" coupling g_0 as $g = g_0 N^y$. We mimic here the discussion of RG in QFT.

[6] Again, these are two equivalent points of view for this simple model, but please do not mix them together.

The model is

$$Z_N(\alpha,g) \propto \int_{N\times N} \mathrm{e}^{-\frac{\alpha}{2}\mathrm{Tr}M^2-\frac{g}{4}\mathrm{Tr}M^4}[\mathrm{d}M]. \tag{87}$$

Unlike the Gaussian case, we don't know how to do the integration exactly. We will leave the normalization undetermined.

classical scaling. The action keeps the same under the following simple scaling

$$\begin{aligned} M &\to \lambda M \\ \alpha &\to \lambda^{-2}\alpha \\ g &\to \lambda^{-4}g \end{aligned}$$

In the following, we will interpret such scaling being induced by rescaling N. For example $N \to n = \lambda N$, then from $\alpha = \alpha_0 N^x$ and $g = g_0 N^y$, one gets

$$\begin{aligned} \alpha(N) &\to \alpha(n) = \lambda^x \alpha(N) \\ g(N) &\to g(n) = \lambda^y g(N) \end{aligned}$$

The classical scaling is defined by requiring the action being invariant:

$$\frac{\alpha(N)}{2}\mathrm{Tr}_N M_N^2 + \frac{g(N)}{4}\mathrm{Tr}_N M_N^4 = \frac{\alpha(n)}{2}\mathrm{Tr}_n M_n^2 + \frac{g(n)}{4}\mathrm{Tr}_n M_n^4$$

This requires that M scales in the following way with a constraint on x, y

$$M_n = \lambda^{-x/2} M_N, \quad y = 2x$$

This scaling does not make sense because the lhs and the rhs are matrices with different sizes. We should instead write

$$M_n = \lambda^{-x/2} M_{N|n}$$

The rhs is then understood as restricting the $N \times N$ matrix to a $n \times n$ sub-matrix in any way.

In the following, I will discuss the conceptual aspect of the RG calculation. It's important to give a clear explanation of the notion of scaling. There are two crucial steps in the RG calculation: integrating out certain degrees of freedom and taking the action back to the pre-defined form. The first step is to make sure that the physical content of theory unchanged. While the second step is to describe the effective theory according to the parameters that defining it.

three scalings [to edit] First is the scaling from the RG flow. Let's start by thinking about the RG calculation to the lowest order. This will just reproduce the classical scaling: $g \to \lambda g$ and $\tilde{g}$ is invariant. However, they are essentially different. In the language of QFT, one can think of N as a cut-off. Then the RG method allows us to relate theories with different cut-off such that they will produce the same results (the correlation functions). Specifically, to the lowest order, the coupling $g \to \lambda g$ with the change of cut-off $N \to \lambda N$. To the lowest order, these are exactly the same as the classical scaling. However, the higher order corrections are essential for the RG flow.

Conceptually, the RG flow keeps the theory invariant, while the classical scaling not: one knows that the matrix model has a non-trivial N dependence although they share the same form of action. The classical scaling is just a natural way to define how the matrix and coupling depending on the underlying scale such that the form of the action keeping the same. The classical scaling works like "zooming in" or "zooming out". However, the RG flow works like "coarse graining". There is no reason to believe that the "coarse graining" will give a similar result comparing to the "zooming" in general. The Gaussian model is a special example that they giving exactly the same result.

Now what about the notion of "scaling invariance"? In this case, it's more interesting to consider yet another scaling. Let's call it "dynamical scaling". The meaning is that only "dynamical variables" should be rescaled. The matrix is dynamical but the coupling constant is not. Therefore, this scaling should not be understood as a change of dimension; It's a symmetry of the action. The interesting thing about the RG flow is that the dynamical scaling invariance could emerge at certain critical point g_*. The existence of such a point $g_* \neq 0$ seems impossible by just looking at the action, because it is written in a form that only the classical scaling is obvious. It's impossible to obtain a dynamical scaling invariance along the classical scaling. While the RG flow could deviate from the classical scaling significantly at some points, along which the scaling of g could be frozen.

the calculation for the quartic model Now let's implement the RG for the quartic model (87). Start with the N-model, and decomposing the matrix M_N in the same way

$$\begin{pmatrix} M_{N|n} & \mu \\ \mu^\dagger & m \end{pmatrix}$$

The action can be expanded as

$$\begin{aligned} S_N[M_{N|n}, \mu, \mu^\dagger, m; \alpha, g] = & \operatorname{Tr}_n \left(\frac{\alpha}{2} M_{N|n}^2 + \frac{g}{4} M_{N|n}^4 \right) + \alpha \left(\mu^\dagger \mu + \frac{1}{2} m^2 \right) \\ & + g \left(\mu^\dagger M_{N|n}^2 \mu + m \mu^\dagger M_{N|n} \mu + m^2 \mu^\dagger \mu + \frac{1}{2} (\mu^\dagger \mu)^2 + \frac{1}{4} m^4 \right). \end{aligned} \tag{88}$$

The notation $M_{N|n}$ is to emphasize that it's a sub-matrix of M_N. It's different from M_n, which is the random matrix for the "effective" n-model. Because the main part of the calculation dealing with $M_{N|n}$, we will just use M to denote it.

We want to do the integration over $\mu, \mu^\dagger$ and m. The method is to expand the second line of S_N from the exponential function $\mathrm{e}^x = 1 + x + \frac{1}{2}x^2 + \cdots$ to reduce the integral to a Gaussian one. This is the simplest method, and can be applied to more complicate cases.

The Gaussian measure is given by

$$\alpha \mu^\dagger \mu + \frac{\alpha}{2} m^2$$

The Wick contraction rules are then

$$\langle \mu_i^* \mu_j \rangle = \frac{1}{\alpha} \delta_{ij}$$
$$\langle m^2 \rangle = \frac{1}{\alpha}$$

The terms that we will expand from the exponential function are

$$g \left(\mu^\dagger M^2 \mu + m \mu^\dagger M \mu + m^2 \mu^\dagger \mu + \frac{1}{2} (\mu^\dagger \mu)^2 + \frac{1}{4} m^4 \right)$$

In the expansion, we will break this interaction term into three parts, and use r, s, t to label their power respectively

$$\sum_{r,s,t=0}^{\infty} \frac{(-1)^{r+s+t} g^{r+s+t}}{r!s!t!} \left(\mu^\dagger M^2 \mu \right)^r \left(m \mu^\dagger M \mu \right)^s \left(\frac{1}{2} (\mu^\dagger \mu)^2 + m^2 \mu^\dagger \mu + \frac{1}{4} m^2 \right)^t \tag{89}$$

This makes it easy to do a power counting.

the "effective operators" The result of the integration has the following general form

$$\int_{N\times N} \mathrm{e}^{-S_N[M_N;\alpha(N),g(N)]}[\mathrm{d}M_N] = \int_{n\times n} \sum_i c_i \mathcal{O}_i[M_{N|n}]\mathrm{e}^{-S_n[M_{N|n};\alpha(N),g(N)]}[\mathrm{d}M_{N|n}] \quad (90)$$

The left hand side is the starting point; the right hand side is an integration over the $n \times n$ sub-matrix $M_{N|n}$. The action S_n has the same form as S_N, just replacing $M_N \to M_{N|n}$. The $\sum_i c_i\mathcal{O}_i$ terms come from applying the contraction rules to the expansion (89). c_i depends on the parameters α, g, and will also have explicit dependence on N

$$\sum_i c_i(\alpha(N), g(N), N)\mathcal{O}_i[M_{N|n}]$$

Presumably the $\mathcal{O}_i[M]$ will have the most general form,

$$\mathcal{O}_i[M] = [\mathrm{Tr}(M^{k_1})]^{l_1} \cdots [\mathrm{Tr}(M^{k_p})]^{l_p}, \quad k_1 > \cdots > k_p$$

Let's denote

$$|i| \equiv k_1 l_1 + \cdots + k_p l_p$$

This counts the number of M. The calculation of $c_i(\alpha, g, N)$ is just a combinatorial problem.

All possible gauge invariant terms could be generated. However, symmetry could protect some terms from generation: for example, if all the interaction terms are even (symmetry transformation $M \to -M$), then $|i| = 2r + s$ must also be even. So s must be even. This is the case that we will consider in the following.

It's simple to do a power counting in this case. For each $g(\mu^\dagger M^2 \mu)$ term, one need to contract one $\mu^\dagger, \mu$-pair. Therefore one get a coefficient g/α. The term $g(m\mu^\dagger M \mu)$ appears an even number of time s. The contraction of $g^2(m\mu^\dagger M\mu)^2$ will give a coefficient (g^2/α^3). The last piece of interaction (labeled by the power t) will give a coefficient g/α^2. Then one can summary this power counting as

$$c_i(\alpha, g, N) = \left(\frac{g}{\alpha}\right)^{|i|/2} \sum_{2r+s=|i|} \frac{(-1)^{r+s}}{r!s!} \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} \left(\frac{g}{\alpha^2}\right)^{t+s/2} (a_i)_{t+s/2} \quad (91)$$

One would like to have some remarks. The first factor $(g/\alpha)^{|i|/2}$ basically counts the number of M in $\mathcal{O}_i[M]$. The restriction $2r + s = |i|$ gives the correct number of M. Because the t-terms do not contain M, there is no restriction on its sum. But increasing t will increase the power of another factor g/α^2. So the series is essentially an asymptotic series of $g/\alpha^2 \to 0$. A little complication is g/α^2 also get contribution form the s-term. We need to calculate the coefficients $(a_i)_{t+s/2}$, for which the label corresponding to the power of the factor g/α and g/α^2 respectively.

The "effective operators" are obtained by re-exponentiating these terms

$$c_0 + \sum_{i,|i|\neq 0} c_i\mathcal{O}_i = \mathrm{e}^{d_0 + \sum_{j,|j|\neq 0} d_j \mathcal{O}_j}$$

Here on both sides, I separate the M-independent term $\mathrm{Tr}\mathbb{1}$ term ($|i| = 0$), c_0 and d_0. We want to match the coefficients c_i and d_i

$$c_0 + \sum_i c_i \mathcal{O}_i = \mathrm{e}^{d_0}(1 + \sum_j d_j\mathcal{O}_j + \frac{1}{2}(\sum_j d_j \mathcal{O}_j)^2 + \cdots)$$

It's easy to match between c_0 and d_0

$$c_0 = \mathrm{e}^{d_0}$$

The next easy term to match is the single trace operator $\mathrm{Tr}M^k$. Let's label the corresponding coefficients as $c_{(k)}$ and $d_{(k)}$. On the right hand side, the single trace operators only appear from the linear term. This gives the identification

$$c_{(k)} = c_0 d_{(k)} = \mathrm{e}^{d_0} d_{(k)}$$

For multi-trace operators, the matching is not straightforward. For example, the single trace terms $d_{(k)}\mathcal{O}_{(k)}[M]$ will also contribute to the multi-trace part of the lhs.

Then the RG flow is obtained by requiring

$$-S_n[M_n; \alpha(n), g(n)] = -S_n[M_{N|n}; \alpha(N), g(N)] + \sum_i d_i \mathcal{O}_i(M_{N|n}) \tag{92}$$

This will give the relation between $\alpha(N), g(N), M_{N|n}$ and $\alpha(n), g(n), M_n$ (the RG flow). The fact that $\sum_i d_i \mathcal{O}_i$ contains all possible operators requires a consistent calculation to include all such terms in the action S_N from the start. However, let's start by an inconsistent calculation to get a feeling of what kind of terms we would get.

There are essentially two expansions: one is controlled by $r + s/2 \equiv |i|/2$, which we will call it the "effective operator expansion"; another is controlled by $t + s/2$, which we will call it the "loop expansions". Let's first list the effective operator expansion up to $|i| = 6$ (only even terms)

$|i| = 0$:

$$\mathrm{Tr}\mathbb{1}$$

$|i| = 2$:

$$\mathrm{Tr}M^2, \quad (\mathrm{Tr}M)^2$$

$|i| = 4$:

$$\mathrm{Tr}M^4, \quad (\mathrm{Tr}M^2)^2, \quad \mathrm{Tr}M^2(\mathrm{Tr}M)^2, \quad (\mathrm{Tr}M)^4$$

$|i| = 6$:

$$\mathrm{Tr}M^6, \quad \mathrm{Tr}M^5\mathrm{Tr}M, \quad \mathrm{Tr}M^4\mathrm{Tr}M^2, \quad \mathrm{Tr}M^4(\mathrm{Tr}M)^2$$
$$(\mathrm{Tr}M^3)^2, \quad \mathrm{Tr}M^3\mathrm{Tr}M^2\mathrm{Tr}M, \quad \mathrm{Tr}M^3(\mathrm{Tr}M)^3$$
$$(\mathrm{Tr}M^2)^3, \quad (\mathrm{Tr}M^2)^2(\mathrm{Tr}M)^2, \quad \mathrm{Tr}M^2(\mathrm{Tr}M)^4, \quad (\mathrm{Tr}M)^6$$

For each effective operator, one can calculate the loop expansion by Wick contractions.

$|i| = 0$ **case** We have $r = s = 0$.

$$c_{(0)} = \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} \left(\frac{g}{\alpha^2}\right)^t (a_{(0)})_t$$

Let's list the result up to $t = 2$

$$(a_{(0)})_0 = 1$$
$$(a_{(0)})_1 = \frac{1}{2}N^2 + \frac{3}{2}N + \frac{3}{4}$$
$$(a_{(0)})_2 = \frac{1}{4}N^4 + \frac{5}{2}N^3 + 9N^2 + \frac{59}{4}N + \frac{105}{16}$$

One can write the following partial general form

$$(a_{(0)})_t = \frac{1}{2^t}N^{2t} + (\frac{1}{2^t}\binom{2t}{2} + \frac{t}{2^{t-1}})N^{2t-1} + \cdots + (?)N + \frac{(4t-1)!!}{4^t}$$

$|i| = 2$ **case.** Let's only calculate the coefficient of TrM^2, $c_{(2)}$ up to $t = 1$. One must have $r = 1, s = 0$ or $r = 0, s = 2$

$$c_{(2)} = -\left(\frac{g}{\alpha}\right)\sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\left(\frac{g}{\alpha^2}\right)^t (a_{(2)}^{(1)})_t + \frac{1}{2}\left(\frac{g}{\alpha}\right)\sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\left(\frac{g}{\alpha^2}\right)^{t+1} (a_{(2)}^{(2)})_{t+1}$$

There are two terms due to the two possibilities of r, s.

$$(a_{(2)}^{(1)})_0 = 1$$

$$(a_{(2)}^{(1)})_1 = \frac{1}{2}N^2 + \frac{5}{2}N + \frac{11}{4}$$

$$(a_{(2)}^{(2)})_1 = 1$$

$|i| = 4$ **case.** Let's only calculate the coefficient of TrM^4, $c_{(4)}$ up to $t = 1$. One has the possibilities $r = 2, s = 0$, $r = 1, s = 2$ or $r = 0, s = 4$

$$c_{(4)} = \frac{1}{2}\left(\frac{g}{\alpha}\right)^2\sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\left(\frac{g}{\alpha^2}\right)^t (a_{(4)}^{(1)})_t - \frac{1}{2}\left(\frac{g}{\alpha}\right)^2\sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\left(\frac{g}{\alpha^2}\right)^{t+1} (a_{(4)}^{(2)})_{t+1}$$
$$+\frac{1}{4!}\left(\frac{g}{\alpha}\right)^2\sum_{t=0}^{\infty}\frac{(-1)^t}{t!}\left(\frac{g}{\alpha^2}\right)^{t+2} (a_{(4)}^{(3)})_{t+2}$$

The second line starts from $(g/\alpha^2)^2$, which can be ignored up to the first two orders.

$$(a_{(4)}^{(1)})_0 = 1$$

$$(a_{(4)}^{(1)})_1 = \frac{1}{2}N^2 + 3N + \frac{23}{4}$$

$$(a_{(4)}^{(2)})_1 = 2$$

summary. To summary the results, we will only look up to $t = 1$ order.

$$c_{(0)} = 1 - \left(\frac{1}{2}N^2 + \frac{3}{2}N + \frac{3}{4}\right)\left(\frac{g}{\alpha^2}\right) + O(\left(\frac{g}{\alpha^2}\right)^2) \tag{93}$$

$$c_{(2)} = \left(\frac{g}{\alpha}\right)\left[-1 + \left(\frac{1}{2}N^2 + \frac{5}{2}N + \frac{9}{4}\right)\left(\frac{g}{\alpha^2}\right) + O(\left(\frac{g}{\alpha^2}\right)^2)\right] \tag{94}$$

$$c_{(4)} = \left(\frac{g}{\alpha}\right)^2\left[\frac{1}{2} - \left(\frac{1}{4}N^2 + \frac{3}{2}N + \frac{31}{8}\right)\left(\frac{g}{\alpha^2}\right) + O(\left(\frac{g}{\alpha^2}\right)^2)\right] \tag{95}$$

The corresponding $d_{(0)}, d_{(2)}, d_{(4)}$ to the same order are

$$d_{(0)} = -\left(\frac{1}{2}N^2 + \frac{3}{2}N + \frac{3}{4}\right)\left(\frac{g}{\alpha^2}\right) + O(\left(\frac{g}{\alpha^2}\right)^2) \tag{96}$$

$$d_{(2)} = \frac{c_{(2)}}{c_{(0)}} = \left(\frac{g}{\alpha}\right)\left[-1 + \left(N + \frac{3}{2}\right)\left(\frac{g}{\alpha^2}\right) + O(\left(\frac{g}{\alpha^2}\right)^2)\right] \tag{97}$$

$$d_{(4)} = \frac{c_{(4)}}{c_{(0)}} = \frac{1}{2}\left(\frac{g}{\alpha}\right)^2\left[1 - \left(\frac{3}{2}N + 7\right)\left(\frac{g}{\alpha^2}\right) + O(\left(\frac{g}{\alpha^2}\right)^2)\right] \tag{98}$$

It's interesting to see that N^2 terms disappear in ds. The reason is that they all come from "disconnected contraction", therefore will not contribute once being re-exponentiated.

the effective action. We start from the action

$$S_N = \frac{\alpha(N)}{2}\mathrm{Tr}_N M_N^2 + \frac{g(N)}{4}\mathrm{Tr}_N M_N^4 \tag{99}$$

then integrating out the Nth column and the Nth row to get an "effective action"

$$S_n = \left(\frac{\alpha(N)}{2} - d_{(2)}\right)\mathrm{Tr}_n M^2 + \left(\frac{g(N)}{4} - d_{(4)}\right)\mathrm{Tr}_n M^4 + \cdots \tag{100}$$

The next step is to figure out $\alpha(n), g(n)$ and M_n such that

$$S_n = \frac{\alpha(n)}{2}\mathrm{Tr}_n M_n^2 + \frac{g(n)}{4}\mathrm{Tr}_n M_n^4 + \cdots \tag{101}$$

Let's first recall that $\alpha = \alpha_0 N^x$ and $g = g_0 N^y$, with also $M = M_0 N^z$, with the constraint $y = 2x = -4z$. The classical scaling invariance ensures that α_0, g_0, M_0 do not change while rescaling N. However, this assumption does not hold because of $d_{(2)}, d_{(4)}$. Let's modify them by adding the N-dependence

$$\begin{aligned}
\alpha(N) &= \alpha_0(N) N^x, \\
g(N) &= g_0(N) N^y, \\
M_N &= M_0(N) N^z
\end{aligned}$$

$t = 0$ order is the easiest one, with $d_{(2)} = -g/\alpha$ and $d_{(4)} = (1/2)(g/\alpha)^2$. For S_n we have

$$S_n = \frac{\alpha_0}{2} N^{x+2z}\left(1 + \frac{2g_0}{\alpha_0^2} N^{y-2x}\right)\mathrm{Tr} M_0^2 + \frac{g_0}{4} N^{y+4z}\left(1 - \frac{2g_0}{\alpha_0^2} N^{y-2x}\right)\mathrm{Tr} M_0^4$$

We can keep the $y = 2x = -4z = $ const. at this level

$$S_n = \frac{\alpha_0}{2}\left(1 + \frac{2g_0}{\alpha_0^2}\right)\mathrm{Tr} M_0^2 + \frac{g_0}{4}\left(1 - \frac{2g_0}{\alpha_0^2}\right)\mathrm{Tr} M_0^4$$

One obtains the following relations

$$M_0^2(n) = \left(1 + 2\frac{g_0(N)}{\alpha_0^2}\right) M_0^2(N) \tag{102}$$

$$g_0(n) = \left(1 - 6\frac{g_0(N)}{\alpha_0^2}\right) g_0(N) \tag{103}$$

These are discrete maps, with the fix point $g_0 = 0$. The next order $t = 1$ will add terms with N-dependence

$$\begin{aligned}
S_n = &\frac{\alpha_0}{2}\left[1 + \frac{2g_0}{\alpha_0^2} - (2N+3)\left(\frac{g_0}{\alpha_0^2}\right)^2\right]\mathrm{Tr} M_0^2 \\
&+ \frac{g_0}{4}\left[1 - \frac{2g_0}{\alpha_0^2} + (3N+14)\left(\frac{g_0}{\alpha_0^2}\right)^2\right]\mathrm{Tr} M_0^4
\end{aligned}$$

Still, the only fix point is $g_0 = 0$. The appearance of N makes it difficult to interpret this result in the large N limit. We are not going to this regime. Let's think about N a finite number, and require $g_0 N$ to be small to make sense of the perturbative expansion.

4.2.3 more interactions, but only leading order

There is a proliferation of terms as one going to higher powers. The hope is that the situation can be simplified by taking certain limit, and focus only on the leading order terms. To figure out which terms are in the leading order, we will check one more example $\mathrm{Tr}M^6$.

Let's consider the interaction term with coupling constant λ

$$\frac{\lambda}{6}\mathrm{Tr}M^6$$

By carefully calculate the matrix product under the matrix RG decomposition $M_N \to M_{N|n}$ we have the following term

$$\mu^\dagger M^4\mu + m\mu^\dagger M^3\mu + (\mu^\dagger\mu)(\mu^\dagger M^2\mu) + \frac{1}{2}(\mu^\dagger M\mu)(\mu^\dagger M\mu) + m^2\mu^\dagger M^2\mu$$
$$+2m(\mu^\dagger\mu)(\mu^\dagger M\mu) + m^3\mu^\dagger M\mu + \frac{1}{3}(\mu^\dagger\mu)^3 + \frac{3}{2}m^2(\mu^\dagger\mu)^2 + m^4\mu^\dagger\mu + \frac{1}{6}m^6$$

According to the number of $\mu, \mu^\dagger$ and m in each term, the contraction will give a certain power of α^{-1}. Let's assume again the N-dimension as $\alpha = \alpha_0 N^x$ and $M = M_0 N^{-x/2}$. The the classical scaling invariance (dimension analysis) requires that $\lambda = \lambda_0 N^{3x}$. There will be two power counting parameters. One is

$$\left(\frac{\lambda}{\alpha}\right)^{1/4} = \left(\frac{\lambda_0}{\alpha_0}\right)^{1/4} N^{x/2}$$

which is dimensionful, and counts the number of M in the corresponding terms. Another is dimensionless

$$\frac{\lambda}{\alpha^3} = \frac{\lambda_0}{\alpha_0^3}$$

which counts the number of contractions (loops). In this case, the leading order means that

$$\frac{\lambda_0}{\alpha_0^3} \to 0$$

Let's try to give more details. Consider the following term that is obtained by expanding the exponential function

$$\left[\mu^\dagger M^4\mu\right]^r \left[m\mu^\dagger M^3\mu\right]^s \left[(\mu^\dagger\mu)(\mu^\dagger M^2\mu) + \frac{1}{2}(\mu^\dagger M\mu)(\mu^\dagger M\mu) + m^2(\mu^\dagger M^2\mu)\right]^p$$
$$\left[2m(\mu^\dagger\mu)(\mu^\dagger M\mu) + m^3\mu^\dagger M\mu\right]^q \left[\frac{1}{3}(\mu^\dagger\mu)^3 + \frac{3}{2}m^2(\mu^\dagger\mu)^2 + m^4\mu^\dagger\mu + \frac{1}{6}m^6\right]^t$$

We have organize terms in terms of the number of M. The power counting goes as follows.

$$\mu^\dagger M^4\mu : \frac{\lambda}{\alpha}$$

This will be the leading order contribution. The followings are sub-leading. I list them just for completeness.

$$(m\mu^\dagger M^3\mu)^2 : \frac{\lambda^2}{\alpha^3} = \frac{\lambda}{\alpha}\cdot\frac{\lambda}{\alpha^2}$$

$$(\mu^\dagger\mu)(\mu^\dagger M^2\mu) + \frac{1}{2}(\mu^\dagger M\mu)(\mu^\dagger M\mu) + m^2(\mu^\dagger M^2\mu) : \frac{\lambda}{\alpha^2}$$

$$[2m(\mu^\dagger\mu)(\mu^\dagger M\mu)+m^3\mu^\dagger M\mu]^2: \ \frac{\lambda^2}{\alpha^5}=\frac{\lambda}{\alpha^2}\cdot\frac{\lambda}{\alpha^3}$$
$$\frac{1}{3}(\mu^\dagger\mu)^3+\frac{3}{2}m^2(\mu^\dagger\mu)^2+m^4\mu^\dagger\mu+\frac{1}{6}m^6: \ \frac{\lambda}{\alpha^3}$$

One may notice that the factors do not organize into the two kinds of combinations λ/α and λ/α^3: there is also a factor λ/α^2. However, if we write out the entire factor corresponding to the above term (r,s,p,q,t)

$$\left(\frac{\lambda}{\alpha}\right)^{r+s/2}\left(\frac{\lambda}{\alpha^2}\right)^{p+s/2+q/2}\left(\frac{\lambda}{\alpha^3}\right)^{t+q/2}$$

The number of M in this expansion is given by the number $|i|=4r+3s+2p+q$, which is related to the power counting

$$4r+3s+2p+q=4\times\left(r+\frac{s}{2}\right)+2\times\left(p+\frac{s}{2}+\frac{q}{2}\right)$$

This suggests us to rewrite

$$\left(\frac{\lambda}{\alpha}\right)^{r+s/2}\left(\frac{\lambda}{\alpha^2}\right)^{p+s/2+q/2}\left(\frac{\lambda}{\alpha^3}\right)^{t+q/2}=\left(\frac{\lambda}{\alpha}\right)^{|i|/4}\left(\frac{\lambda}{\alpha^3}\right)^{t+\frac{3}{4}q+\frac{1}{2}p+\frac{1}{4}s}$$

Then we get perfectly the two power counting parameters.

However, here we are only considering the expansion of $\mathrm{Tr}M^6$. There can be terms that mix $\mathrm{Tr}M^4$ with it. As in the last section, let's denote the coupling constant of the quartic term as g. Then the factor that counts the number of M is

$$\left(\frac{g}{\alpha}\right)^{|i_1|/2}\left(\frac{\lambda}{\alpha}\right)^{|i_2|/4}$$

with $|i_1|+|i_2|=|i|$. To compare terms with different $(|i_1|,|i_2|)$, one should make assumption on the relative order between g,λ. For example, if they are in the same order, then $|i_1|=|i|$ dominates over other therms.

In the following, we will only consider interactions that having the form $\mathrm{Tr}M^{2k}$. Multi-trace interactions could be generated along the RG flow, and they are not even sub-leading as we will see. However, we will prove that they will disappear once we do the "re-exponentiation".

At the last part, I will construct a leading order "conformal matrix model" in the sense that the dimensionless coupling constant g_0 does not change $g_0(n)=g_0(N)$ in the leading order. This model is obtained by adding higher power terms in the action to cancel the modification of previous terms. For example, the modification of $\mathrm{Tr}M^4$ term calculated previously can be canceled by a proper $\mathrm{Tr}M^6$ term. This is possible only when, for example, $\mathrm{Tr}M^6$ does not contribute to the modification of $\mathrm{Tr}M^2$. This property is ensured in the leading order.

a leading order conformal matrix model We know that the Gaussian model $S=\frac{\alpha}{2}\mathrm{Tr}M^2$ in conformal in the sense that α_0, M_0 is N-independent. Once we add a quartic term $\frac{g}{4}\mathrm{Tr}M^4$, even in the leading order, the $M_0(N), g_0(N)$ are dependent on N. Can we cancel the modification by including higher power terms?

Let's start by writing down the following action

$$S=\frac{\alpha}{2}\mathrm{Tr}M^2+\frac{g_2}{4}\mathrm{Tr}M^4+\frac{g_4}{6}\mathrm{Tr}M^6+\frac{g_6}{8}\mathrm{Tr}M^8+\cdots \tag{104}$$

In the leading order, one could only consider the following expansion

$$\frac{(-g_2)^{r_2}(-g_4)^{r_4}(-g_6)^{r_6}\cdots}{(r_2)!(r_4)!(r_6)!\cdots}(\mu^\dagger M^2\mu)^{r_2}(\mu^\dagger M^4\mu)^{r_4}(\mu^\dagger M^6\mu)^{r_6}\cdots$$

and

$$|i| = 2r_2 + 4r_4 + 6r_6 + \cdots$$

First note that for $|i| = 2$, the only possibility is $r_2 = 1$. It's impossible to cancel the effective operator $\mathrm{Tr}M^2$, but it only reproduces the wave function renormalization that we have already calculated for the quartic interaction:

$$\frac{\alpha}{2}\left(1+\frac{2g_2}{\alpha^2}\right)\mathrm{Tr}M^2$$

We need introduce a "wave function renormalization" (QFT language)

$$M_0^2(n) = \left(1+\frac{2(g_2)_0}{\alpha_0^2}\right)M_0^2$$

The subscript 0 is used to denote the dimensionless parameters. This will lead already to a part of the modification of higher power interactions $\mathrm{Tr}M^{2k}$

$$\frac{g_{2k-2}}{2k}\mathrm{Tr}M^{2k} = \left(\frac{g_{2k-2}}{2k} - 2k\frac{g_2}{\alpha^2}\right)\mathrm{Tr}M_n^{2k}$$

Then $|i| = 4$ for the effective operator $\mathrm{Tr}M^4$. There are two possibilities: $r_2 = 2, r_4 = 0$ or $r_2 = 0, r_4 = 1$. The cancellation condition gives the following equation

$$\frac{1}{2}\frac{(g_2)^2}{\alpha^2} - \frac{g_4}{\alpha} = 4\frac{g_2}{\alpha^2}$$

The lhs comes from contractions, while the rhs comes from the wave function renormalization. For the $\mathrm{Tr}M^6$, $|i| = 6$, we have three possibilities $r_2 = 3, r_4 = 0, r_6 = 0$ or $r_2 = 1, r_4 = 1, r_6 = 0$, or $r_2 = 0, r_4 = 0, r_6 = 1$.

$$-\frac{1}{3}\frac{(g_2)^3}{\alpha^3} + \frac{g_2}{\alpha}\frac{g_4}{\alpha} - \frac{g_6}{\alpha} = 6\frac{g_2}{\alpha^2}$$

One last calculation for $\mathrm{Tr}M^8$. The cancellation condition is

$$\frac{1}{4}\frac{(g_2)^4}{\alpha^4} - \frac{(g_2)^2}{\alpha^2}\frac{g_4}{\alpha} + \frac{g_2}{\alpha}\frac{g_6}{\alpha} + \frac{1}{2}\frac{(g_4)^2}{\alpha^2} - \frac{g_8}{\alpha} = 8\frac{g_2}{\alpha^2}$$

g_2 is the only free coupling constant. Let's express g_4, g_6, g_8 in terms of it.

$$g_4 = \frac{1}{2\alpha}\left[(g_2)^2 - 8g_2\right]$$

$$g_6 = -\frac{6}{\alpha}g_2 + \frac{1}{6\alpha^2}\left[(g_2)^3 - 24(g_2)^2\right]$$

Let's write the action. To simplify the notation, $g \equiv g_2$

$$S = \frac{\alpha}{2}\mathrm{Tr}M^2 + \frac{g}{4}\mathrm{Tr}M^4 + \frac{1}{12\alpha}\left(g^2 - 8g\right)\mathrm{Tr}M^6 + \left(\frac{1}{48\alpha^2}g^3 - \frac{1}{2\alpha^2}g^2 - \frac{3}{4\alpha}g\right)\mathrm{Tr}M^8 + \cdots$$

Let's recall that the leading order approximation is valid in the following limit

$$\frac{g}{\alpha^2}\to 0,\quad \frac{g_4}{\alpha^3}\to 0,\quad \frac{g_6}{\alpha^4}\to 0,\cdots$$

If one check it with the solution we obtained, one finds that the anomalous g_2 term (arise because the wave function renormalization) requires also

$$\frac{g}{\alpha^3} \to 0, \quad \frac{g_2}{\alpha^5} \to 0$$

These requirements can be satisfied by the following limit

$$\alpha \to \infty, \quad \frac{g}{\alpha} \equiv \lambda = \text{finite} \tag{105}$$

Then it's natural to factor out α in the action, and keep only finite terms

$$\alpha^{-1} S = \frac{1}{2}\mathrm{Tr}M^2 + \frac{\lambda}{4}\mathrm{Tr}M^4 + \frac{\lambda^2}{12}\mathrm{Tr}M^6 + \frac{\lambda^3}{48}\mathrm{Tr}M^8 + \cdots \tag{106}$$

4.3 matrix RG of a "superfield" matrix model

premier This model is interesting because it's secretly Gaussian. "superfield" is a trick used by physicists to construct and study supersymmetric field theory. Usual fields are promoted to the "superfield" which depends also on some Grassmann coordinates, along with the usual coordinates, form a superspace. Then supersymmetry could be realized as a coordinate transformation of the "superspace".

Study the RG flow of the "superfield" matrix model. This model is mentioned in ✎ **Ple96**

the "superfield" Φ The matrix-valude superfield is constructed as

$$\Phi = \phi + \overline{\psi}\theta + \overline{\theta}\psi + \theta\overline{\theta}F. \tag{107}$$

$\theta, \overline{\theta}$ are the coordinates of the superspace. ϕ, F are bosonic $N \times N$ matrices, and we assume them to be Hermitian. $\psi, \overline{\psi}$ are "fermionic" $N \times N$ matrices, whose entries are Grassmann variables. They are not spinors, of course. Because $\theta\theta = 0$, $\overline{\theta\theta} = 0$ and $\theta\overline{\theta} = -\overline{\theta}\theta$, no other term could be written down.

Both ψ and θ are Grassman-valued, but they are independent.

Let's clarify the meaning of "Grassmann-valude" Hermitian matrix.

reality. One should think about real Grassmann directions $\xi^* = \xi(?)$ Recall that the complex conjugate of the product of Grassmann variables is defined as $(\xi\eta)^* = \eta^*\xi^*$. To keep Φ Hermitian $\Phi^\dagger = \Phi$, we require that

$$\psi^\dagger = \overline{\psi}, \quad \theta^* = \overline{\theta}.$$

(which one is correct? keep the generators real, or conjugate to each other?) (Hermitian conjugation only acts on the $\mathfrak{u}(N)$ generators? Or we should also think about the complex conjugation of the Grassmann variables also? An important criterion is to keep the action real under conjugation?)

The measure over the superfield will be the usual Berezian integral of the matrix entries

$$\mathrm{d}\Phi = \mathrm{d}\phi\mathrm{d}F\mathrm{d}\overline{\psi}\mathrm{d}\psi. \tag{108}$$

The supersymmetric action can be constructed as

$$S[\Phi] = \mathrm{Tr}\int \mathrm{d}\overline{\theta}\mathrm{d}\theta \left\{ -D_\theta\Phi D_{\overline{\theta}}\Phi + \sum_{k=0}^{\infty} g_k\Phi^k \right\}. \tag{109}$$

The super-derivative acts from the right. The matrix model partition function is

$$Z_\Phi[g_k] = \int [\mathrm{d}\Phi] \exp\left(-NS[\Phi]\right). \tag{110}$$

We have $D_\theta \Phi = \overline{\psi} - \overline{\theta} F$ and $D_{\overline{\theta}}\Phi = -\psi + \theta F$. Their product will contribute to the action only if the measure $\mathrm{d}\theta \mathrm{d}\overline{\theta}$ is saturated. This will give a term $-\mathrm{Tr} F^2$ in the action. We calculate the Φ^k term as

$$\begin{aligned}
\Phi^k &= \left(\phi + \overline{\psi}\theta + \overline{\theta}\psi + \theta\overline{\theta}F\right)^k \\
&= \phi^k + \left(\sum_{a+b=k-1} \phi^a \overline{\psi} \phi^b\right)\theta + \overline{\theta}\left(\sum_{a+b=k-1} \phi^a \psi \phi^b\right) \\
&+ \theta\overline{\theta}\left(\sum_{a+b+c=k-2} \phi^a \overline{\psi}\phi^b \psi \phi^c + \sum_{a+b=k-1} \phi^a F \phi^b\right).
\end{aligned}$$

By taking trace and keeping only the $\theta\overline{\theta}$ term, we get in the action a term

$$\mathrm{Tr}[V'(\phi)F] + \sum_{k=0}^{\infty} k g_k \sum_{a+b=k-2} \mathrm{Tr}(\phi^a \overline{\psi}\phi^b \psi).$$

Let's assume a quartic potential $V(\phi) = \frac{1}{2}\phi^2 + \frac{g}{4}\phi^4$,

$$\begin{aligned}
S[\Phi] = &-\mathrm{Tr} F^2 + \mathrm{Tr}(\phi F) + g \mathrm{Tr}(\phi^3 F) \\
&+ \mathrm{Tr}(\overline{\psi}\psi) + g\left[\mathrm{Tr}(\phi^2\overline{\psi}\psi) + \mathrm{Tr}(\phi\overline{\psi}\phi\psi) + \mathrm{Tr}(\overline{\psi}\phi^2\psi)\right].
\end{aligned}$$

Although it's easy to do the integral over F, we will not do that. Because it will lead to a non-linear supersymmetry transformation rule. However, let's define $F' = F - \frac{1}{2}\phi$ and rewrite the action in terms of F such that a quadratic term in ϕ will appear in the action

$$\begin{aligned}
S[\Phi] = &-\mathrm{Tr} F'^2 + \frac{1}{4}\mathrm{Tr}\phi^2 + \frac{g}{2}\mathrm{Tr}\phi^4 + g\mathrm{Tr}(\phi^3 F') \\
&+ \mathrm{Tr}(\overline{\psi}\psi) + g\left[\mathrm{Tr}(\phi^2\overline{\psi}\psi) + \mathrm{Tr}(\phi\overline{\psi}\phi\psi) + \mathrm{Tr}(\overline{\psi}\phi^2\psi)\right].
\end{aligned}$$

The supersymmetry variation reads

$$\begin{aligned}
\delta\phi &= \overline{\varepsilon}\psi + \overline{\psi}\varepsilon \\
\delta\psi &= -\varepsilon(F' + \frac{1}{2}\phi) \\
\delta\overline{\psi} &= -\overline{\varepsilon}(F' + \frac{1}{2}\phi) \\
\delta F' &= -\frac{1}{2}(\overline{\varepsilon}\psi + \overline{\psi}\varepsilon)
\end{aligned}$$

4.4 matrix RG of Yang-Mills matrix model

The Yang-Mills matrix model action

$$S_{\mathrm{Euc}}[A; g] = -\frac{1}{g^2}\mathrm{Tr}\left([A^\mu, A^\nu][A_\mu, A_\nu]\right) \tag{111}$$

Euclidean signature is simple: the minus is chosen such that $S \geq 0$. For Lorentz signature, let's rewrite using $\{A_0, A_i\}$ (Hermitian),

$$S_{\mathrm{Lor}}[A; g] = \frac{2}{g^2}\mathrm{Tr}\left([A_0, A_i][A_0, A_i]\right) - \frac{1}{g^2}\mathrm{Tr}\left([A_i, A_j][A_i, A_j]\right) \tag{112}$$

The second term is positive definite, while the first term is negative definite.

The partition function

$$Z_{\text{Euc}}(g;N) = \int_{N\times N} \mathrm{e}^{-S_{\text{Euc}}[A;g]}[\mathrm{d}A] \tag{113}$$

In Lorentzian signature [reference here]

$$Z_{\text{Lor}}(g;N) = \int_{N\times N} \mathrm{e}^{-iS_{\text{Lor}}[A;g]}[\mathrm{d}A] \tag{114}$$

The integrand in the Lorentz case is oscillating. The integral is not necessarily converge. One could regularize it by introducing an imaginary part to the action [reference here]

$$S_{\text{Lor}}[A;g] \to S_\varepsilon[A;g] \equiv -\frac{1}{g^2}\text{Tr}\left([A^\mu, A^\nu][A_\mu, A_\nu] + i\varepsilon \sum_\mu A_\mu A_\mu\right) \tag{115}$$

The term $\sum_\mu A_\mu A_\mu$ breaks the Lorentz invariance, but is positive definite. The integral

$$Z_\varepsilon(g;N) = \int_{N\times N} \mathrm{e}^{-iS_\varepsilon[A;g]}[\mathrm{d}A] \tag{116}$$

is convergent then.

One can always absorb g into A in S_{Euc} and S_{Lor}.

$$Z(g;N) \propto g^{N^2 D/2} Z(1;N)$$

In the regularized Z_ε, we need rescale $\tilde{\varepsilon} = \varepsilon/g$

$$Z_\varepsilon(g;N) \propto g^{N^2 D/2} Z_{\tilde{\varepsilon}}(1;N)$$

This suggests that there is no intrinsic scale in this model. Although the energy dimension of the Yang-Mills coupling is expected to be

$$[g] = \frac{4-D}{2}$$

which is $[g] = 2$ when $D = 0$.

We should also be careful about the rescaling argument. The underlying assumption is that the integral region extend to the whole matrix space. In practical calculation, the integral is often ill defined, so one should introduce a regulator. This introduces a scale by hand. So let's still keep the parameter g in the calculation.

Again, we will induce the scaling by changing N. Define

$$g = g_0 N^x, \quad A = A_0 N^y, \quad x = 2y \tag{117}$$

(or if $\text{Tr} = N\text{Tr}_0$, then $2x = 4y + 1$)

the RG scheme The general scheme is to decompose the matrix into background and fluctuations (subscript to denote the size of the matrix)

$$A_N \longrightarrow A_N + \mathscr{A}_N \tag{118}$$

To do the matrix RG calculation, we don't need to decompose the whole matrix, but only the Nth row and Nth column. That is, for the background part, we will assume the direct sum structure $A_N = A_{N|n} \oplus a$, taking A to be a general $n \times n$ matrix with $n = N - 1$. The result of matrix RG will be an effective theory of A_n, which presumably a rescaling of $A_{N|n}$. a is a parameter we introduce by hand. It turns out that a is a regulator of the matrix RG.

To treat the commutator, it's easier to work with the adjoint matrix. We will introduce a basis $T^{\mathfrak{a}}$ of the $N \times N$ Hermitian matrices. The basic requirement is the orthonormal condition

$$\mathrm{Tr}(T^{\mathfrak{a}} T^{\mathfrak{b}}) = \delta^{\mathfrak{ab}}. \tag{119}$$

We don't need to care about whether it's an upper or lower index. The index $\mathfrak{a}$ could take N^2 value, if with traceless condition, $N^2 - 1$ value. But for the RG calculation, we only need to focus on a subset. I will first discuss this point in detail.

We will be very explicit about the basis $T^{\mathfrak{a}}$. Introducing another label

$$\mathfrak{a} \equiv (k,l) \text{ or } [k,l] \text{ or } (k,k), \quad k < l \in \{1, \cdots, N\}$$

The example of 2×2 Hermitian matrices will be

$$T^{(1,2)} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T^{[1,2]} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad T^{(1,1)} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

To avoid duplication, we restrict $k < l$. However, it's still useful to keep in mind that

$$T^{(k,l)} = T^{(l,k)}, \quad T^{[k,l]} = -T^{[l,k]}$$

a_μ is related to the background eigenvalue distribution, for which we don't anything. However, the eigenvalue distribution will determine the shape of integral over a_μ.

sketch the β-function Schematically, the RG procedure will lead to the "effective operators"

$$\int_{a_-}^{a_+} (\mathrm{d}^D a)(a^{-k}) \mathrm{Tr} A^k \sim (a_+^{D-k} - a_-^{D-k}) \mathrm{Tr} A^k$$

We assume that the Nth eigenvalue a_μ living in a spherical shell. However, except for simplicity, there is no other reason to consider this situation. The a^{-k} factor is a general result of the RG method. $a^{-k}\mathrm{Tr}A^k$ is dimensionless in the sense that $A \to \lambda A$ with $a \to \lambda a$.

It's natural to write the Yang-Mills action as

$$\frac{a^{D-4}}{g^2} \mathrm{Tr}[A, A]^2 \tag{120}$$

such that the $\beta(g^{-2})$ is independent of the cut-off a_+. When $D = 10$, we get

$$\frac{a^6}{g^2} \mathrm{Tr}[A, A]^2$$

I don't know whether there is something about it. But one remark is that the a^{D-4} factor is still based on the assumption of the eigenvalue distribution.

Now let's write down $\beta(g^{-2})$.

For $\mathrm{Tr}\mathcal{O}$ we get from one-loop integration

$$a^{-k} \mathrm{Tr} \mathcal{O}$$

We are considering an asymptotic series of $g \to 0$.

$$a_+^{D-k} g^l \mathrm{Tr}\mathcal{O} + (a_+^{D-k} - a_-^{D-k}) P(g) \mathrm{Tr}\mathcal{O} \equiv a_+^{D-k} \tilde{g}^l \mathrm{Tr}\mathcal{O}$$

$P(g)$ must start with $O(1)$ if $\mathrm{Tr}\mathcal{O}$ can be generated in one-loop already. (disclaimer. I ignore possible wave function renormalization $\tilde{O}$. I don't know what is l. I keep in mind just the special case $\mathcal{O} = [A, A]^2$, $k = 4$ and $l = -2$.) Rewrite the equation as

$$g^l - \tilde{g}^l = \left(\frac{a_-}{a_+}\right)^{D-k} - 1$$

Introduce a small $\varepsilon \equiv \mathrm{d}\mu/\mu \equiv 1 - a_-/a_+ > 0$

$$\beta(g^l) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left(g^l - \tilde{g}^l\right) = k - D$$

The example I have in mind is

$$\beta(g^{-2}) = 4 - D$$

Here D is the dimension of the target space. $D = 4$ is special. $D = 10$, $\beta(g^{-2}) = -6$.

This formalism is strange, because we do not expect that the scaling depends on the dimension of the target space.

($k \geq 4$ for Yang-Mills matrix models? Super-Yang-Mills $k \geq 8$? the same as $D = 10$ one loop?)

the integral over diagonal elements

$$\int_0^{a_+} \mathrm{d}a_1 \int_{a_1}^{a_+} \mathrm{d}a_2 \cdots \int_{a_{N-2}}^{a_+} \mathrm{d}a_{N-1} \int_{a_{N-1}}^{a_+} \mathrm{d}a_N$$

We order $a_1 < a_2 < \cdots < a_{N-1} < a_N$. (In this way, the gauge group is reduced to $U(N)/\mathbb{Z}_N$.)

There is a confusion about the expansion parameter. First note that we expand before the integration over a_i. Then we would recognize the expansion parameters as $(a_i/a_N)^2 < 1$, which can be very close to 1. In this sense, the expansion may not be valid. Another concern is that, we should not use variables as expansion parameters. There is an "ad hoc" way to get around with these two confusions, saying that

$$\int_0^{a_-} \mathrm{d}a_1 \int_{a_1}^{a_-} \mathrm{d}a_2 \cdots \int_{a_{N-2}}^{a_-} \mathrm{d}a_{N-1} \int_{a_{N-1}}^{a_+} \mathrm{d}a_N$$

Then the expansion parameter would be $(a_-/a_+)^2$. We can first "regularize" it (to make our expansion good) by assuming $a_- \ll a_+$. The true result would be recovered after taking the limit $a_- \to a_+$.

Few notes. We only care about the integral over a_N, so a_{N-1} is fixed. The integral regime of a_N is from a_{N-1} to a_+, not a_- to a_+. This seems not very important for the validity of the expansion because $a_{N-1} < a_-$. But a_{N-1} can be very small: $a_{N-1} \ll a_-$. Furthermore, this part is the major contribution to $\int \mathrm{d}a_N$.

4.5 matrix RG of IKKT matrix model

We would go for a perturbative analysis of the RG flow.

Background with fluctuation

$$A_N \longrightarrow A_N + \mathscr{A}_N \tag{121}$$

Expand the action. The bosonic part of the action:

$$\begin{aligned}\mathrm{Tr}[A_\mu, A_\nu][A^\mu, A^\nu] \to \mathrm{Tr}[A_\mu, A_\nu][A^\mu, A^\nu] + 4\underbrace{\mathrm{Tr}[\mathscr{A}_\mu, A_\nu][A^\mu, A^\nu]}_{\text{tadpole}} \\ + 2\underbrace{\mathrm{Tr}\left([A_\mu, \mathscr{A}_\nu][A^\mu, \mathscr{A}^\nu] - [A_\mu, \mathscr{A}^\mu][A_\nu, \mathscr{A}^\nu] + 2[A_\mu, A_\nu][\mathscr{A}^\mu, \mathscr{A}^\nu]\right)}_{\text{quadratic}} \\ + 4\underbrace{\mathrm{Tr}[A_\mu, \mathscr{A}_\nu][\mathscr{A}^\mu, \mathscr{A}^\nu]}_{\text{cubic}} + \underbrace{\mathrm{Tr}[\mathscr{A}_\mu, \mathscr{A}_\nu][\mathscr{A}^\mu, \mathscr{A}^\nu]}_{\text{self-interaction}}\end{aligned}$$

note. The quadratic terms are organized in terms of the Lorentz indices contraction: within background and fluctuations, or between them. The second term is isolated from the third term, which will vanish after gauge fixing. The fermionic part of the action

$$\begin{aligned}-\frac{1}{2g^2}\mathrm{Tr}(\overline{\psi}\Gamma^\mu[A_\mu, \psi]) \to -\frac{1}{2g^2}\mathrm{Tr}(\overline{\psi}\Gamma^\mu[A_\mu, \psi]) \underbrace{-\frac{1}{2g^2}\mathrm{Tr}(\overline{\psi}\Gamma^\mu[\mathscr{A}_\mu, \psi]) - \frac{1}{g^2}\mathrm{Tr}(\overline{\varphi}\Gamma^\mu[A_\mu, \psi])}_{\text{tadpole}} \\ \underbrace{-\frac{1}{2g^2}\mathrm{Tr}(\overline{\varphi}\Gamma^\mu[A_\mu, \varphi]) + \frac{1}{g^2}\mathrm{Tr}(\overline{\varphi}\Gamma^\mu[\psi, \mathscr{A}_\mu])}_{\text{quadratic}} - \frac{1}{2g^2}\mathrm{Tr}(\overline{\varphi}\Gamma^\mu[\mathscr{A}_\mu, \varphi])\end{aligned}$$

gauge fixing. The gauge fixing condition

$$\mathcal{G}(\mathscr{A}) = i[A_\mu, \mathscr{A}^\mu] = 0 \tag{122}$$

i makes $\mathcal{G}(\mathscr{A})$ Hermitian. Infinitesimal gauge transformation parameterized by $t \in \mathfrak{u}(N)$

$$A + \mathscr{A} \to A + \mathscr{A} + i[A + \mathscr{A}, t] \tag{123}$$

Keep the background invariant

$$\mathscr{A} \to \mathscr{A} + i[A + \mathscr{A}, t] \tag{124}$$

Faddeev-Popov method: consider a family of $\mathcal{G}^\omega(\mathscr{A}) = \mathcal{G}(\mathscr{A}) + \omega$, where $\omega \in \mathfrak{u}(N)$. We will impose the conditions $\mathcal{G}^\omega = 0$ through the Dirac δ-function in the matrix integral. The following identity is used

$$1 = \int_{U(N)} [\mathrm{d}U]\delta(\mathcal{G}^\omega(\mathscr{A}_U))\det\left(\frac{\partial \mathcal{G}^\omega(\mathscr{A}_U)}{\partial U}\right) \tag{125}$$

the determinant? homogeneous gauge orbit? the volume $\mathrm{vol}(U(N))$? Note that it's independent of ω. $U = \mathrm{e}^{-it}, \quad U(\cdots)U^{-1}$.

On the decomposition

$$\begin{aligned}\delta_t \alpha_{\mu\mathfrak{a}'} = i([A_\mu + \alpha_\mu, t])_{\mathfrak{a}'} = i\sum_{\mathfrak{b}'}(\mathfrak{A}_\mu)_{\mathfrak{a}'\mathfrak{b}'}t_{\mathfrak{b}'} + i([\alpha_\mu, t])_{\mathfrak{a}'} = i\sum_{\mathfrak{b}'}(\mathfrak{A}_\mu)_{\mathfrak{a}'\mathfrak{b}'}t_{\mathfrak{b}'} \\ \delta_t \varphi^\alpha_{\mathfrak{a}'} = i\sum_{\mathfrak{b}'}(\Psi^\alpha)_{\mathfrak{a}'\mathfrak{b}'}t_{\mathfrak{b}'} + \text{(another term depends on } b\text{...)}\end{aligned}$$

$i([\alpha_\mu, t])_{\mathfrak{a}'} = 0$ if α and t has no component along the (N, N)-direction.

The matrix elements $\mathfrak{A}_{\mathfrak{a}'\mathfrak{b}'}$ are pure imaginary.

Gauge fixing function (rewrite)

$$\mathcal{G}^{\omega}_{\mathfrak{a}'}(\alpha_{(1-it)}) = i\left[\mathfrak{A}_\mu(\alpha^\mu + i\mathfrak{A}^\mu t)\right]_{\mathfrak{a}'} + \omega_{\mathfrak{a}'} = i(\mathfrak{A}_\mu \alpha^\mu)_{\mathfrak{a}'} - (\mathfrak{A}^2 t)_{\mathfrak{a}'} + \omega_{\mathfrak{a}'}$$

$$\frac{\partial \mathcal{G}^{\omega}_{\mathfrak{a}'}}{\partial(1-it)_{\mathfrak{b}'}} = -i(\mathfrak{A}^2)_{\mathfrak{a}'\mathfrak{b}'}$$

pure imaginary, anti-hermitian, symmetric.

Write ghosts, and $\mathrm{e}^{-\frac{\omega^2}{2\xi}}$ in the action, integrate over ω. Choose ξ to cancel the term... Should we worry about the normalization of ghosts kinetic term? No

5 Speculations and future works

5.1 supersymmetry cancellation to all loops?

5.2 D1/D5 superconformal field theory

5.3 D(-1)/D7 matrix model is conformal?

5.4 emergence of geometry from matrix model?

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