Master Thesis Notes

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Mon, Apr 15

background matrices configuration A_μ appears in the model only through $\mathfrak{A}_\mu$. A_μ are $N \times N$ Hermitian; $\mathfrak{A}_\mu$ are $N^2 \times N^2$ Hermitian, also antisymmetric. Consider the Cartan directions

$$H \equiv \sum_{\mathfrak{a} \in \text{Cart.}} H_\mathfrak{a} T^\mathfrak{a}, \quad .$$

1. If A_μ are diagonal (along the Cartan directions), then $\mathfrak{A}_\mu H = 0$ because Cartan directions commute.
2. For any A_μ, $\mathfrak{A}_\mu H$ has no component along the Cartan directions, but has components along the off-diagonal directions.
3. Diagonal A_μ gives zero modes, but can be lift by including non-diagonal elements.
4. $U(N)$ transformation can diagonalize one of A_μ, thus generates zero modes along certain spacetime direction.

Extremal configuration:

$$[A^\mu, [A_\mu, A_\nu]] = 0 \tag{1}$$

Or

$$\sum_\mathfrak{b} (\mathfrak{A}^\mu \mathfrak{A}_\mu)^{\mathfrak{a}\mathfrak{b}} (A_\nu)_\mathfrak{b} = 0.$$

Note: A_μ along the zero direction of $\mathfrak{A}^\mu \mathfrak{A}_\mu$. Or

$$([A^\mu, F_{\mu\nu}])^\mathfrak{a} = \sum_\mathfrak{b} (\mathfrak{A}^\mu)^{\mathfrak{a}\mathfrak{b}} (F_{\mu\nu})_\mathfrak{b} = -\sum_\mathfrak{b} (\mathfrak{F}_{\mu\nu})^{\mathfrak{a}\mathfrak{b}} (A^\mu)_\mathfrak{b} = 0.$$

The existence of zero modes is quite general. The e.o.m. (1) is $U(N)$ invariant.

"RG directions" in the background Decomposition

$$A \overset{\text{background}}{\to} A + \mathcal{A} \overset{\text{RG}}{\to} A + (\mathcal{A}_1 + \mathcal{A}_2).$$

$\mathcal{A}_1$ is effectively the $(N-1)\times(N-1)$ Hermitian; $\mathcal{A}_2$ is the integrated-out part.

The Gaussian term, assume diagonal A_μ (ignore zero modes temporarily)

$$\mathcal{A}_{\mu\mathfrak{a}}(\mathfrak{A}^2)^{\mathfrak{a}\mathfrak{b}}\eta^{\mu\nu}\mathcal{A}_{\nu\mathfrak{b}} \to \mathcal{A}_{\mu\mathfrak{a}}(\mathfrak{A}^2)^{\mathfrak{a}\mathfrak{b}}\eta^{\mu\nu}\mathcal{A}_{\nu\mathfrak{b}} + \alpha_{\mu\mathfrak{a}'}(\mathfrak{A}^2)^{\mathfrak{a}'\mathfrak{b}'}\eta^{\mu\nu}\alpha_{\nu\mathfrak{b}'}.$$

$$\mathfrak{a}', \mathfrak{b}' \in \{(N,i), [N,i] | i = 1, \cdots, N-1\}.$$

$$\mathfrak{A}_\mu\alpha_\nu = \sum_{\mathfrak{a}',\mathfrak{b}'}(\mathfrak{A}_\mu)_{\mathfrak{a}'}{}^{\mathfrak{b}'}\alpha_{\nu\mathfrak{b}'}T^{\mathfrak{a}'}.$$

The interaction terms, cubic part

$$-\frac{\alpha}{4}\mathrm{Tr}[A_\mu, A_\nu][A^\mu, A^\nu] \to -\alpha\mathrm{Tr}\left(\mathfrak{A}_{[\mu}\mathcal{A}_{\nu]}[\mathcal{A}^\mu, \mathcal{A}^\nu]\right) \tag{2}$$

The direction of $\mathcal{A}$: $\mathfrak{a}'$ or $\mathfrak{a}$. Commutators, schematically

$$[\mathfrak{a}', \mathfrak{a}'] \in \mathfrak{a}, \quad [\mathfrak{a}, \mathfrak{a}] \in \mathfrak{a}, \quad [\mathfrak{a}', \mathfrak{a}] \in \mathfrak{a}'.$$

Diagonal A: $\mathfrak{A}$ keeps the separation $\mathfrak{a}' \to \mathfrak{a}'$, $\mathfrak{a} \to \mathfrak{a}$.

Separate the RG direction

$$\begin{aligned} -\alpha\mathrm{Tr}\left(\mathfrak{A}_{[\mu}\mathcal{A}_{\nu]}[\mathcal{A}^\mu, \mathcal{A}^\nu]\right) &\to -\alpha\mathrm{Tr}\left(\mathfrak{A}_{[\mu}\mathcal{A}_{\nu]}[\mathcal{A}^\mu, \mathcal{A}^\nu]\right) \\ &-2\alpha\mathrm{Tr}\left(\mathfrak{A}_{[\mu}\alpha_{\nu]}[\alpha^\mu, \mathcal{A}^\nu]\right) - \alpha\mathrm{Tr}\left(\mathfrak{A}_{[\mu}\mathcal{A}_{\nu]}[\alpha^\mu, \alpha^\nu]\right) \end{aligned}$$

Indicies notation

$$-\alpha\mathrm{Tr}\left(\mathfrak{A}_{[\mu}\mathcal{A}_{\nu]}[\alpha^\mu, \alpha^\nu]\right) = -\alpha\sum_{\mathfrak{a}',\mathfrak{b}',\mathfrak{c},\mathfrak{d}}\mathrm{Tr}(T^{\mathfrak{c}}[T^{\mathfrak{a}'}, T^{\mathfrak{b}'}])\mathfrak{A}_{[\mu|\mathfrak{c}|}{}^{\mathfrak{d}}\mathcal{A}_{\nu]\mathfrak{d}}\alpha^\mu_{\mathfrak{a}'}\alpha^\nu_{\mathfrak{b}'}$$

$$-2\alpha\mathrm{Tr}\left(\mathfrak{A}_{[\mu}\alpha_{\nu]}[\alpha^\mu, \mathcal{A}^\nu]\right) = -2\alpha\sum_{\mathfrak{a}',\mathfrak{b}',\mathfrak{c},\mathfrak{d}'}\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{b}'}, T^{\mathfrak{c}}])\mathfrak{A}_{[\mu|\mathfrak{a}'|}{}^{\mathfrak{d}'}\alpha_{\nu]\mathfrak{d}'}\alpha^\mu_{\mathfrak{b}'}\mathcal{A}^\nu_{\mathfrak{c}}$$

Some infomation: A diagonal $\to$ $\mathfrak{A}$ has "almost diagonal" structure: (N,i), $[N,j]$-elements only for $i=j$. Trace of T depends on the choice of normalization. In both traces, if $\mathfrak{a}'$, $\mathfrak{b}'$ in the same 2×2 block: $i=j$, $\mathfrak{c}$ must be the Cartan directions (i,i) or (N,N). We ignore the fluctuation along those directions?

The quartic interaction...

Tue, Apr 16

quartic interaction Schematically

$$\begin{aligned} -\frac{\alpha}{4}\mathrm{Tr}[A_\mu, A_\nu][A^\mu, A^\nu] &\to \mathrm{Tr}[\mathfrak{a}, \mathfrak{a}][\mathfrak{a}, \mathfrak{a}] \\ +\mathrm{Tr}[\mathfrak{a}', \mathfrak{a}'][\mathfrak{a}, \mathfrak{a}] &+ \mathrm{Tr}[\mathfrak{a}', \mathfrak{a}][\mathfrak{a}', \mathfrak{a}] + \mathrm{Tr}[\mathfrak{a}', \mathfrak{a}'][\mathfrak{a}', \mathfrak{a}'] \end{aligned}$$

the quartic interaction terms

$$\begin{aligned} &-\frac{\alpha}{4}\mathrm{Tr}[\mathcal{A}_\mu, \mathcal{A}_\nu][\mathcal{A}^\mu, \mathcal{A}^\nu] - \frac{\alpha}{4}\mathrm{Tr}[\alpha_\mu, \alpha_\nu][\alpha^\mu, \alpha^\nu] \\ -\frac{\alpha}{2}\mathrm{Tr}[\alpha_\mu, \alpha_\nu][\mathcal{A}^\mu, \mathcal{A}^\nu] &- \frac{\alpha}{2}\mathrm{Tr}\left([\alpha_\mu, \mathcal{A}_\nu][\alpha^\mu, \mathcal{A}^\nu] + [\alpha_\mu, \mathcal{A}_\nu][\mathcal{A}^\mu, \alpha^\nu]\right) \end{aligned}$$

$\mathfrak{u}(N)$ index $\mathfrak{a}, \mathfrak{a}'$ structure:

$$\mathrm{Tr}([T^{\mathfrak{a}'}, T^{\mathfrak{b}'}][T^{\mathfrak{a}}, T^{\mathfrak{b}}]), \quad \mathrm{Tr}([T^{\mathfrak{a}'}, T^{\mathfrak{a}}][T^{\mathfrak{b}'}, T^{\mathfrak{b}}]), \quad \mathrm{Tr}([T^{\mathfrak{a}'}, T^{\mathfrak{b}'}][T^{\mathfrak{c}'}, T^{\mathfrak{d}'}]).$$

$$-\frac{\alpha}{2}\mathrm{Tr}[\alpha_\mu, \alpha_\nu][\mathcal{A}^\mu, \mathcal{A}^\nu] = -\frac{\alpha}{2}\mathrm{Tr}([T^{\mathfrak{a}'}, T^{\mathfrak{b}'}][T^{\mathfrak{a}}, T^{\mathfrak{b}}])\alpha_{\mu\mathfrak{a}'}\alpha_{\nu\mathfrak{b}'}\mathcal{A}^\mu_{\mathfrak{a}}\mathcal{A}^\nu_{\mathfrak{b}}.$$

$$\begin{aligned}
&-\frac{\alpha}{2}\mathrm{Tr}\left([\alpha_\mu, \mathcal{A}_\nu][\alpha^\mu, \mathcal{A}^\nu] + [\alpha_\mu, \mathcal{A}_\nu][\mathcal{A}^\mu, \alpha^\nu]\right) \\
&= -\frac{\alpha}{2}\mathrm{Tr}([T^{\mathfrak{a}'}, T^{\mathfrak{a}}][T^{\mathfrak{b}'}, T^{\mathfrak{b}}])\alpha_{\mu\mathfrak{a}'}\alpha_{\nu\mathfrak{b}'}(\eta^{\mu\nu}\mathcal{A}_{\rho\mathfrak{a}}\mathcal{A}^\rho_{\mathfrak{b}} + \mathcal{A}^\nu_{\mathfrak{a}}\mathcal{A}^\mu_{\mathfrak{b}})
\end{aligned}$$

$$-\frac{\alpha}{4}\mathrm{Tr}[\alpha_\mu, \alpha_\nu][\alpha^\mu, \alpha^\nu] = -\frac{\alpha}{4}\mathrm{Tr}([T^{\mathfrak{a}'}, T^{\mathfrak{b}'}][T^{\mathfrak{c}'}, T^{\mathfrak{d}'}])\alpha_{\mu\mathfrak{a}'}\alpha_{\nu\mathfrak{b}'}\alpha^\mu_{\mathfrak{c}'}\alpha^\nu_{\mathfrak{d}'}$$

Maybe useful formula:

$$\mathrm{Tr}\left([T^{\mathfrak{a}'}, T^{\mathfrak{a}'}][T^{\mathfrak{a}}, T^{b}]\right) = 0.$$

$$\sum_{\mathfrak{a}'}\mathrm{Tr}\left([T^{\mathfrak{a}'}, T^{\mathfrak{a}}][T^{\mathfrak{a}'}, T^{b}]\right) \propto \mathrm{Tr}T^{\mathfrak{a}}T^{\mathfrak{b}}.$$

what's the coefficient?

susy, Ward identity?, BRST? Formula

$$\begin{aligned}
[\bar{\epsilon}\Gamma_\mu\psi, \psi] = [\bar{\epsilon}_\alpha(\Gamma_\mu)^\alpha{}_\beta\psi^\beta, \psi^\gamma] = \bar{\epsilon}_\alpha\{(\Gamma_\mu)^\alpha{}_\beta\psi^\beta, \psi^\gamma\} \\
+\{\bar{\epsilon}_\alpha, \psi^\gamma\}(\Gamma_\mu)^\alpha{}_\beta\psi^\beta
\end{aligned}$$

The last term is zero if $\bar{\epsilon}_\alpha \propto \mathbb{1}$.

$$\begin{aligned}
\mathrm{Tr}(\overline{\psi}\Gamma^\mu[\bar{\epsilon}\Gamma_\mu\psi, \psi]) &= \mathrm{Tr}\left(\overline{\psi}_\delta(\Gamma^\mu)^\delta{}_\gamma[\bar{\epsilon}_\alpha(\Gamma_\mu)^\alpha{}_\beta\psi^\beta, \psi^\gamma]\right) \\
= \mathrm{Tr}&\left(\overline{\psi}_\delta(\Gamma^\mu)^\delta{}_\gamma\bar{\epsilon}_\alpha\{(\Gamma_\mu)^\alpha{}_\beta\psi^\beta, \psi^\gamma\} - \overline{\psi}_\delta(\Gamma^\mu)^\delta{}_\gamma\{\bar{\epsilon}_\alpha, \psi^\gamma\}(\Gamma_\mu)^\alpha{}_\beta\psi^\beta\right) \\
= -\mathrm{Tr}&\left(\bar{\epsilon}_\alpha\big\{(\Gamma_\mu)^\alpha{}_\beta\psi^\beta, (\Gamma^\mu)^\delta{}_\gamma\psi^\gamma\big\}\overline{\psi}_\delta - \big\{\bar{\epsilon}_\alpha, (\Gamma^\mu)^\delta{}_\gamma\psi^\gamma\big\}(\Gamma_\mu)^\alpha{}_\beta\psi^\beta\overline{\psi}_\delta\right)
\end{aligned}$$

The first term

$$-\mathrm{Tr}\left(\bar{\epsilon}_\alpha\big\{(\Gamma_\mu)^\alpha{}_\beta\psi^\beta, (\Gamma^\mu)^\delta{}_\gamma\psi^\gamma\big\}\overline{\psi}_\delta\right) = -\mathrm{Tr}\left(\epsilon^\alpha\big\{(C\Gamma_\mu)_{\alpha\beta}\psi^\beta, (C\Gamma^\mu)_{\delta\gamma}\psi^\gamma\big\}\psi^\delta\right).$$

The last term

$$\mathrm{Tr}\left(\big\{\bar{\epsilon}_\alpha, (\Gamma^\mu)^\delta{}_\gamma\psi^\gamma\big\}(\Gamma_\mu)^\alpha{}_\beta\psi^\beta\overline{\psi}_\delta\right) = \mathrm{Tr}\left(\epsilon^\alpha\left[(C\Gamma^\mu)_{\delta\gamma}\psi^\gamma, (C\Gamma_\mu)_{\alpha\beta}\psi^\beta\psi^\delta\right]\right).$$

Add them to get

$$\mathrm{Tr}(\overline{\psi}\Gamma^\mu[\bar{\epsilon}\Gamma_\mu\psi, \psi]) = -2(C\Gamma_\mu)_{\alpha\beta}(C\Gamma^\mu)_{\delta\gamma}\mathrm{Tr}(\epsilon^\alpha\psi^\beta\psi^\delta\psi^\gamma).$$

Vanishes if $\epsilon \propto \mathbb{1}$. Non-vanishing part

$$3\mathrm{Tr}(\epsilon^\alpha\psi^\beta\psi^\delta\psi^\gamma) = \mathrm{Tr}(\epsilon^\alpha\psi^\beta\psi^\delta\psi^\gamma) + (\text{cyclic } \beta,\delta,\gamma).$$
$$+\mathrm{Tr}(\{\epsilon^\alpha,\psi^\beta\}\psi^{(\delta}\psi^{\gamma)}) - \mathrm{Tr}(\{\epsilon^\alpha,\psi^{(\gamma}\}\psi^{|\beta|}\psi^{\delta)}).$$

First part vanishes, second part is symmetrized for (γ,δ).

Γ-matrix $(\Gamma_\mu)^\alpha{}_\beta$: $(C\Gamma_\mu)_{\alpha\beta} = C_{\alpha\gamma}(\Gamma_\mu)^\gamma{}_\beta$. Spinor product: $\overline{\psi}_\alpha(\Gamma_\mu)^\alpha{}_\beta\psi^\beta$

$$= \psi^\alpha(C\Gamma_\mu)_{\alpha\beta}\psi^\beta.$$

$(C\Gamma_\mu)_{\alpha\beta}$ is symmetric. In our case:

$$(C\Gamma_\mu)_{\alpha\beta}(C\Gamma^\mu)_{\gamma\delta} + (C\Gamma_\mu)_{\alpha\gamma}(C\Gamma^\mu)_{\delta\beta} + (C\Gamma_\mu)_{\alpha\delta}(C\Gamma^\mu)_{\beta\gamma} = 0$$

Check by looking at the explicit $C\Gamma^\mu$ in MW basis. Procedure: fix γ,δ (MW directions). Only one μ is non-zero (remarkable). Choose another two directions α,β with that μ. Then two vector directions μ,ν in the formula. Locate the matrix elements.

Understand the property of $\Gamma^\mu \otimes \Gamma_\mu$...

The susy transformation of $\mathrm{Tr}[A_\mu,A_\nu][A^\mu,A^\nu]$, consider

$$\mathrm{Tr}[\bar\epsilon\Gamma_\mu\psi, A_\nu][A^\mu,A^\nu] = \mathrm{Tr}\epsilon^\alpha(C\Gamma_\mu)_{\alpha\beta}\psi^\beta[A_\nu,[A^\mu,A^\nu]].$$

Susy vary ψ in $\mathrm{Tr}\overline{\psi}\Gamma^\mu[A_\mu,\psi]$

$$\begin{aligned}\mathrm{Tr}(\psi^\alpha(C\Gamma^\mu)_{\alpha\beta}[A_\mu,\psi^\beta]) \to{}& \mathrm{Tr}\left([A_\mu,A_\nu](\Gamma^{\mu\nu})^\gamma{}_\alpha\epsilon^\alpha(C\Gamma^\rho)_{\gamma\beta}[A_\rho,\psi^\beta]\right)\\ &+\mathrm{Tr}(\psi^\beta(C\Gamma^\rho)_{\beta\gamma}[A_\rho,[A_\mu,A_\nu](\Gamma^{\mu\nu})^\gamma{}_\alpha\epsilon^\alpha])\\ &= 2(C\Gamma^\rho)_{\beta\gamma}(\Gamma^{\mu\nu})^\gamma{}_\alpha\mathrm{Tr}(\epsilon^\alpha[A_\rho,\psi^\beta][A_\mu,A_\nu])\\ &= 2(C\Gamma^\rho\Gamma^{\mu\nu})_{\beta\alpha}\mathrm{Tr}(\epsilon^\alpha[A_\rho,\psi^\beta][A_\mu,A_\nu])\end{aligned}$$

$$\begin{aligned}2(C\Gamma^\rho\Gamma^{\mu\nu})_{\beta\alpha}\mathrm{Tr}(\epsilon^\alpha[A_\rho,\psi^\beta][A_\mu,A_\nu]) ={}& \left[2(C\Gamma^{\mu\nu\rho})_{\beta\alpha} + 4\eta^{\rho[\mu}(C\Gamma^{\nu]})_{\beta\alpha}\right]\\ &\cdot\left[\mathrm{Tr}(\epsilon^\alpha[A_\rho,\psi^\beta[A_\mu,A_\nu]]) + \mathrm{Tr}(\epsilon^\alpha\psi^\beta[A_\rho,[A_\nu,A_\mu]])\right]\end{aligned}$$

Use the Jacobi identity

$$[A_\rho,[A_\mu,A_\nu]] + (\text{cyclic } \rho,\mu\nu) = 0$$

to prove

$$(C\Gamma^{\mu\nu\rho})\mathrm{Tr}(\cdots[A_\rho,[A_\nu,A_\mu]]) = 0.$$

The last term

$$4\eta^{\rho[\mu}(C\Gamma^{\nu]})_{\beta\alpha}\mathrm{Tr}(\epsilon^\alpha\psi^\beta[A_\rho,[A_\nu,A_\mu]])$$

will cancel with $\mathrm{Tr}[\bar\epsilon\Gamma_\mu\psi, A_\nu][A^\mu,A^\nu]$.

Left with (vanish when $\epsilon \propto \mathbb{1}$)

$$\left[2(C\Gamma^{\mu\nu\rho})_{\beta\alpha} + 4\eta^{\rho[\mu}(C\Gamma^{\nu]})_{\beta\alpha}\right] \cdot \mathrm{Tr}(\epsilon^\alpha[A_\rho, \psi^\beta[A_\mu, A_\nu]]).$$

...

Thu, Apr 18

Gaussian with interaction Canonical generator basis (Apr. 10). $\mathfrak{a} = (k, l)$, $[k, l]$ or (k, k). Fix $k < l$. $\mathfrak{a}' = (k, N), [k, N], k = 1, \cdots, N-1$. Normalize such that $\mathrm{Tr} T^\mathfrak{a} T^\mathfrak{b} = \delta^{\mathfrak{ab}}$.

$$\sum_{\mathfrak{a}'} (T^{\mathfrak{a}'})_{iN} (T^{\mathfrak{a}'})_{Nj} = \delta_{ij}.$$

$\mathfrak{a}' = (i, N), [i, N]$ contributes to the sum. Gaussian from bosonic action (Apr. 10)

$$\frac{\alpha}{2}\left[\mathcal{A}_\mu(\mathfrak{A}^2\eta^{\mu\nu} + \mathfrak{A}^\mu\mathfrak{A}^\nu - 2\mathfrak{F}^{\mu\nu})\mathcal{A}_\nu\right].$$

~~The simplest case $\mathfrak{F} = 0$ because $[A, A] = 0$. $\mathfrak{A}^2$ is diagonal because A is diagonal.~~
Consider $A \oplus a$, a is a number, the (N, N) matrix element; A is a general $(N-1) \times (N-1)$ matrix. Calculate the adjoint

$$\mathfrak{A} = \sum_\mathfrak{c} (A \oplus a)_\mathfrak{c} \mathrm{Tr}(T^\mathfrak{a}[T^\mathfrak{c}, T^\mathfrak{b}]).$$

$\mathfrak{c} \in \{(N, N), (i, i), (k, l), [k, l]\}$. First consider $\mathfrak{c} = (N, N)$. This contributes to the matrix element ($\mathfrak{a} = (i, N), \mathfrak{b} = [i, N]$) or ($\mathfrak{a} = [i, N], \mathfrak{b} = (i, N)$).

$$[T^{(N,N)}, T^{(i,N)}] = iT^{[i,N]}, \quad [T^{(N,N)}, T^{[i,N]}] = -iT^{(i,N)}$$

$\mathfrak{c} = (N, N), (i, i)$ will generate the matrix element

$$(\mathfrak{A})_{(i,N),[i,N]} = -(\mathfrak{A})_{[i,N],(i,N)} = i(a - A_{ii}).$$

There are matrix elements between different $(i, N), (j, N)$ (also $(i, N), [j, N]$ and $[i, N], [j, N]$)

$$(\mathfrak{A})_{(i,N),(j,N)} = \sum_\mathfrak{c} A_\mathfrak{c} \mathrm{Tr}(T^{(i,N)}[T^\mathfrak{c}, T^{(j,N)}]) = \frac{i}{\sqrt{2}} A_{[i,j]}.$$

$$(\mathfrak{A})_{(i,N),[j,N]} = \sum_\mathfrak{c} A_\mathfrak{c} \mathrm{Tr}(T^{(i,N)}[T^\mathfrak{c}, T^{[j,N]}]) = \frac{i}{\sqrt{2}} A_{(i,j)}.$$

$$(\mathfrak{A})_{[i,N],[j,N]} = \sum_\mathfrak{c} A_\mathfrak{c} \mathrm{Tr}(T^{[i,N]}[T^\mathfrak{c}, T^{[j,N]}]) = \frac{i}{\sqrt{2}} A_{[i,j]}.$$

The matrix elements $((i, j), (i, N))$ vanish because $A_{[j,N]} = 0$.

The matrix elements of $\mathfrak{A}$:

square

	(i,N)	$[i,N]$	(j,N)	$[j,N]$
(i,N)	0	$i(a-A_{ii})$	$\frac{i}{\sqrt{2}}A_{[ij]}$	$\frac{i}{\sqrt{2}}A_{(ij)}$
$[i,N]$	$-i(a-A_{ii})$	0	$-\frac{i}{\sqrt{2}}A_{(ij)}$	$\frac{i}{\sqrt{2}}A_{[ij]}$
(j,N)	$-\frac{i}{\sqrt{2}}A_{[ij]}$	$\frac{i}{\sqrt{2}}A_{(ij)}$	0	$i(a-A_{jj})$
$[j,N]$	$-\frac{i}{\sqrt{2}}A_{(ij)}$	$-\frac{i}{\sqrt{2}}A_{[ij]}$	$-i(a-A_{jj})$	0

	(i,N)	$[i,N]$	(j,N)	$[j,N]$
(i,N)	$(a-A_{ii})^2$	0	?	?
$[i,N]$	0	$(a-A_{ii})^2$	?	?
(j,N)	?	?	$(a-A_{jj})^2$	0
$[j,N]$	?	?	0	$(a-A_{jj})^2$

The simplest approximation $a - A_{ii} \approx a - A_{jj} \equiv d$, $A_{(ij)} \approx 0, A_{[ij]} \approx 0$ (classical background). The Gaussian from bosonic action for $\alpha_{\mu\mathfrak{a}'}$

$$\frac{\alpha}{2}\sum_{\mathfrak{a}'}\left[\alpha_{\mu\mathfrak{a}'}(d^2\eta^{\mu\nu} + d^\mu d^\nu)\alpha_{\nu\mathfrak{a}'}\right] \tag{3}$$

We ignore d depending on $\mathfrak{a}'$ in the approximation.

Consider the interaction

$$\frac{1}{4}\mathrm{Tr}([T^{\mathfrak{a}'}, T^{\mathfrak{a}}][T^{\mathfrak{b}'}, T^{\mathfrak{b}}]) = \sum_{ijkl} T^{[\mathfrak{a}'}_{ij} T^{\mathfrak{a}]}_{jk} T^{[\mathfrak{b}'}_{kl} T^{\mathfrak{b}]}_{li}.$$

Because one of the index of $T^{\mathfrak{a}'}$ must be N, while no index of $T^{\mathfrak{a}}$ is N, only the following terms non-vanishing

$$-\frac{1}{4}\sum_{ijk}\left(T^{\mathfrak{a}'}_{iN}T^{\mathfrak{b}'}_{Nj}T^{\mathfrak{a}}_{ki}T^{\mathfrak{b}}_{jk} + T^{\mathfrak{a}'}_{Ni}T^{\mathfrak{a}}_{ik}T^{\mathfrak{b}}_{kj}T^{\mathfrak{b}'}_{jN}\right).$$

Contraction between $\mathfrak{a}', \mathfrak{b}'$ gives

$$\sum_{\mathfrak{a}'\mathfrak{b}'}\delta_{\mathfrak{a}',\mathfrak{b}'}(\cdots) = -\frac{1}{2}\mathrm{Tr}(T^{\mathfrak{a}}T^{\mathfrak{b}}).$$

This result is applied to

$$\begin{aligned}&-\frac{\alpha}{2}\mathrm{Tr}\left([\alpha_\mu, \mathcal{A}_\nu][\alpha^\mu, \mathcal{A}^\nu] + [\alpha_\mu, \mathcal{A}_\nu][\mathcal{A}^\mu, \alpha^\nu]\right)\\ =&-\frac{\alpha}{2}\mathrm{Tr}([T^{\mathfrak{a}'}, T^{\mathfrak{a}}][T^{\mathfrak{b}'}, T^{\mathfrak{b}}])\alpha_{\mu\mathfrak{a}'}\alpha_{\nu\mathfrak{b}'}(\eta^{\mu\nu}\mathcal{A}_{\rho\mathfrak{a}}\mathcal{A}^\rho_{\mathfrak{b}} + \mathcal{A}^\nu_{\mathfrak{a}}\mathcal{A}^\mu_{\mathfrak{b}}).\end{aligned}$$

Contract α

$$\begin{aligned}&\alpha\mathrm{Tr}(T^{\mathfrak{a}}T^{\mathfrak{b}})(\Delta^{-1})_{\mu\nu}(\eta^{\mu\nu}\mathcal{A}_{\rho\mathfrak{a}}\mathcal{A}^\rho_{\mathfrak{b}} + \mathcal{A}^\nu_{\mathfrak{a}}\mathcal{A}^\mu_{\mathfrak{b}})\\ &= \alpha(\Delta^{-1})_{\mu\nu}\mathrm{Tr}\left(\eta^{\mu\nu}\mathcal{A}_\rho\mathcal{A}^\rho + \mathcal{A}^\mu\mathcal{A}^\nu\right).\end{aligned}$$

Δ^{-1} is the inverse of $\eta^{\mu\nu}d^2 + d^\mu d^\nu$.

$$-\frac{\alpha}{2}\mathrm{Tr}\left([\alpha_\rho, \mathcal{A}_\mu][\alpha_\sigma, \mathcal{A}_\nu]\right) \to \alpha(\Delta^{-1})_{\rho\sigma}\mathrm{Tr}\left(\mathcal{A}_\mu\mathcal{A}_\nu\right) \tag{4}$$

Note. there is no $\mathfrak{A}_{\mathfrak{a}\mathfrak{b}}$ dependence. What is d?

fermionic contribution the action

$$-\frac{\alpha}{2}\mathrm{Tr}(\overline{\psi}\Gamma^{\mu}[A_{\mu},\psi]) = -\frac{\alpha}{2}\psi^{\alpha}_{\mathfrak{a}}(C\Gamma^{\mu})_{\alpha\beta}(\mathfrak{A}_{\mu})^{\mathfrak{a}\mathfrak{b}}\psi^{\beta}_{\mathfrak{b}} \tag{5}$$

$$\int[\mathrm{d}\psi]\mathrm{e}^{-\frac{\alpha}{2}\mathrm{Tr}(\overline{\psi}\Gamma^{\mu}[A_{\mu},\psi])} = \mathrm{Pf}\left(-\frac{\alpha}{2}(C\Gamma^{\mu})_{\alpha\beta}(\mathfrak{A}_{\mu})^{\mathfrak{a}\mathfrak{b}}\right).$$

The matrix elements are labeled by $(\alpha,\mathfrak{a}),(\beta,\mathfrak{b})$.

The RG setting, background $\mathfrak{A}=\mathfrak{A}_1\oplus\mathfrak{A}_2$, $\mathfrak{A}_2$ in the directions $\mathfrak{a}',\mathfrak{b}'$.

$$\mathrm{Pf}\left(-\frac{\alpha}{2}(C\Gamma^{\mu})_{\alpha\beta}(\mathfrak{A}_{\mu})^{\mathfrak{a}\mathfrak{b}}\right) \approx \mathrm{Pf}\left(-\frac{\alpha}{2}(C\Gamma^{\mu})_{\alpha\beta}(\mathfrak{A}_{1\mu})^{\mathfrak{a}\mathfrak{b}}\right)\mathrm{Pf}\left(-\frac{\alpha}{2}(C\Gamma^{\mu})_{\alpha\beta}(\mathfrak{A}_{2\mu})^{\mathfrak{a}'\mathfrak{b}'}\right)$$

Factorization only works for background, not for fluctuation.

$$\tilde{\mathfrak{A}} = \sum_{\mathfrak{c}}\alpha_{\mathfrak{c}}\mathrm{Tr}(T^{\mathfrak{a}}[T^{\mathfrak{c}},T^{\mathfrak{b}}])$$

No simple relation between $\mathrm{Pf}(A\otimes B)$ and $\mathrm{Pf}(A)$, $\mathrm{Pf}(B)$.

Write

$$M_{(\alpha,\mathfrak{a}),(\beta,\mathfrak{b})} \equiv (C\Gamma^{\mu})_{\alpha,\beta}\otimes(\mathfrak{A}_{\mu})_{\mathfrak{a},\mathfrak{b}}.$$

Use

$$\mathrm{Pf}\left(-\frac{\alpha}{2}M\right) = \pm\exp\left[\frac{1}{4}\mathrm{Tr}\ \log\left(-\frac{\alpha^2}{4}M^2\right)\right].$$

$$M = C\Gamma^{\mu}\otimes(\mathfrak{A}_{1\mu}\oplus\mathfrak{A}_{2\mu}).$$

$$M^2 = (C\Gamma^{\mu}C\Gamma^{\nu})\otimes(\mathfrak{A}_{1\mu}\mathfrak{A}_{1\nu}\oplus\mathfrak{A}_{2\mu}\mathfrak{A}_{2\nu}).$$

$\mathfrak{A}_2$ for the RG directions. Factor out

$$\exp\left[\frac{1}{4}\mathrm{Tr}\ \log\left(-\frac{\alpha^2}{4}(C\Gamma^{\mu}\otimes\mathfrak{A}_{2\mu})^2\right)\right].$$

Background $(\mathfrak{A}_2)^2_{\mathfrak{a}'\mathfrak{b}'} = d^2\delta_{\mathfrak{a}'\mathfrak{b}'}$ (the approximation).

$$(C\Gamma^{\mu}\otimes\mathfrak{A}_{2\mu})^2 = \left(\sum_{\mu=2}^{9}(d_{\mu})^2\right)\mathbb{1}\otimes\mathbb{1} + (\cdots?).$$

$\log(-\cdots)$? For simplicity, assume $d_0=0$,

$$(C\Gamma^{\mu}\otimes\mathfrak{A}_{2\mu})^2 = \left(\sum_{\mu=1}^{9}(d_{\mu})^2\right)\mathbb{1}\otimes\mathbb{1} \equiv d^2\mathbb{1}.$$

$\mathrm{Tr} = 16\cdot 2(N-1)$, for $\mathfrak{a}'\in\{(i,N),[i,N]\}$.

$$\exp\left[\frac{1}{4}\mathrm{Tr}\ \log\left(-\frac{\alpha^2}{4}(C\Gamma^{\mu}\otimes\mathfrak{A}_{2\mu})^2\right)\right] = \left(-\frac{\alpha^2}{4}d^2\right)^{8(N-1)}.$$

Question 1. *where is the scaling?*

Clarify M

$$M_{(\alpha,\mathfrak{a}),(\beta,\mathfrak{b})} \equiv (C\Gamma^{\mu})_{\alpha,\beta} \otimes (\mathfrak{A}_{\mu})_{\mathfrak{a},\mathfrak{b}}.$$

The inverse of M: M^{-1}? Diagonal background

$$\sum_{\mathfrak{c}} (\mathfrak{A}_{\mu})_{\mathfrak{a}\mathfrak{c}} (\mathfrak{A}_{\nu})_{\mathfrak{c}\mathfrak{b}} = \delta_{\mathfrak{a},\mathfrak{b}} d_{\mathfrak{a}\mu} d_{\mathfrak{a}\nu}.$$

Spinor part is not easy, but easy if $d_0 = 0$.

$$\{(C\Gamma^{i}), (C\Gamma^{j})\} = 2\delta^{ij}\mathbb{1}, \quad C\Gamma^{0} = -\mathbb{1}$$

$$\{(C\Gamma^{0}), (C\Gamma^{i})\} = -2(C\Gamma^{i}).$$

Define $\overline{C\Gamma}^{\mu}$

$$\overline{C\Gamma}^{0} = -C\Gamma^{0}, \quad \overline{C\Gamma}^{i} = C\Gamma^{i}.$$

Reminiscent of Pauli 4-vector? If $d_0 = 0$, $M^{-1} \propto M$. $M^{-1} \propto \overline{C\Gamma}^{\mu} \otimes \mathfrak{A}_{\mu}$?

Fri, Apr 19

summarize the ingredients General assumptions: diagonal background A_{μ}; imagine $(A_{\mu})_{ii}$, $i = 1, \cdots, N-1$ are small while $(A_{\mu})_{NN}$ is large:

$$(A_{\mu})_{NN} - (A_{\mu})_{ii} \approx d_{\mu}, \quad (\mathfrak{A}_{\mu})_{\mathfrak{a}'\mathfrak{b}'} \equiv (\mathfrak{A}_{\mu})_{(i,N)[j,N]} = -(\mathfrak{A}_{\mu})_{[j,N](i,N)} = i d_{\mu} \delta_{ij}.$$

The idea is just to make the Gaussian term simple.

The bosonic part of IKKT gives

$$\frac{\alpha}{2} \sum_{\mathfrak{a}',\mathfrak{b}'} \alpha_{\mu\mathfrak{a}'} \left[(\mathfrak{A}^{\rho}\mathfrak{A}_{\rho})_{\mathfrak{a}'\mathfrak{b}'} \eta^{\mu\nu} + (\mathfrak{A}^{\mu}\mathfrak{A}^{\nu})_{\mathfrak{a}'\mathfrak{b}'} \right] \alpha_{\nu\mathfrak{b}'} \tag{6}$$

Define the quadratic matrix

$$(\Delta_{\mathrm{B}})_{\mathfrak{a}'\mathfrak{b}';\mu\nu} \equiv \frac{\alpha}{2} (d^2 \eta_{\mu\nu} + d_{\mu} d_{\nu}) \delta_{\mathfrak{a}'\mathfrak{b}'} \tag{7}$$

A simplification: gauge fixing

$$\sum_{\mathfrak{b}'} (\mathfrak{A}^{\mu})_{\mathfrak{a}'\mathfrak{b}'} \alpha_{\mu\mathfrak{b}'} = 0, \quad \delta_t \alpha_{\mu\mathfrak{a}'} = i \sum_{\mathfrak{b}'} (\mathfrak{A}_{\mu})_{\mathfrak{a}'\mathfrak{b}'} t_{\mathfrak{b}'}.$$

In gauge fixing action, the bosonic part

$$\frac{\alpha}{2} \sum_{\mathfrak{a}',\mathfrak{b}'} \alpha_{\mu\mathfrak{a}'} \left[(\mathfrak{A}^{\rho}\mathfrak{A}_{\rho})_{\mathfrak{a}'\mathfrak{b}'} \eta^{\mu\nu} \right] \alpha_{\nu\mathfrak{b}'} \tag{8}$$

$$(\Delta_{\mathrm{B}})_{\mathfrak{a}'\mathfrak{b}';\mu\nu} \equiv \frac{\alpha}{2} d^2 \eta_{\mu\nu} \delta_{\mathfrak{a}'\mathfrak{b}'} \tag{9}$$

Then it's easy to inverse

$$(\Delta_{\mathrm{B}}^{-1})^{\mu\nu}_{\mathfrak{a}'\mathfrak{b}'} = \frac{2}{\alpha d^2} \eta^{\mu\nu} \delta_{\mathfrak{a}'\mathfrak{b}'} \tag{10}$$

The fermionic part of IKKT gives

$$-\frac{\alpha}{2}\sum_{\mathfrak{a}'\mathfrak{b}',\alpha\beta}\psi^{\alpha}_{\mathfrak{a}'}(C\Gamma^{\mu})_{\alpha\beta}(\mathfrak{A}_{\mu})_{\mathfrak{a}'\mathfrak{b}'}\psi^{\beta}_{\mathfrak{b}'} \tag{11}$$

$$(\Delta_{\mathrm{F}})_{\mathfrak{a}'\mathfrak{b}';\alpha\beta} \equiv -\frac{\alpha}{2}(C\Gamma^{\mu})_{\alpha\beta}d_{\mu}J_{\mathfrak{a}'\mathfrak{b}'} \tag{12}$$

J forms a block diagonal form, each block reads

$$J_{(i,N),[i,N]} = -J_{[i,N],(i,N)} = i, \quad J_i = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad J = J_1 \oplus \cdots \oplus J_{N-1}.$$

The inverse

$$(\Delta_{\mathrm{F}}^{-1})^{\alpha\beta}_{\mathfrak{a}'\mathfrak{b}'} = -\frac{2}{\alpha d^2}(\overline{C\Gamma}^{\mu})^{\alpha\beta}d_{\mu}J_{\mathfrak{a}'\mathfrak{b}'} \tag{13}$$

$$\overline{C\Gamma}^0 = -C\Gamma^0, \quad \overline{C\Gamma}^i = C\Gamma^i, \quad i = 1, \cdots, 9.$$

$$(C\Gamma^{\mu})(\overline{C\Gamma}^{\nu}) = \eta^{\mu\nu}\mathbb{1}.$$

Contractions

$$\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{ij}(T^{\mathfrak{b}'})_{kl}\delta_{\mathfrak{a}'\mathfrak{b}'}.$$

$$\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{Ni}(T^{\mathfrak{b}'})_{Nj}\delta_{\mathfrak{a}'\mathfrak{b}'} = 0, \quad \sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{iN}(T^{\mathfrak{b}'})_{jN}\delta_{\mathfrak{a}'\mathfrak{b}'} = 0.$$

$$\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{iN}(T^{\mathfrak{b}'})_{Nj}\delta_{\mathfrak{a}'\mathfrak{b}'} = \sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{Ni}(T^{\mathfrak{b}'})_{jN}\delta_{\mathfrak{a}'\mathfrak{b}'} = 2\delta_{ij}.$$

$$\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{ij}(T^{\mathfrak{b}'})_{kl}J_{\mathfrak{a}'\mathfrak{b}'}.$$

$$\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{Ni}(T^{\mathfrak{b}'})_{Nj}J_{\mathfrak{a}'\mathfrak{b}'} = 0, \quad \sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{iN}(T^{\mathfrak{b}'})_{jN}J_{\mathfrak{a}'\mathfrak{b}'} = 0.$$

$$\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{iN}(T^{\mathfrak{b}'})_{Nj}J_{\mathfrak{a}'\mathfrak{b}'} = -\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{Ni}(T^{\mathfrak{b}'})_{jN}J_{\mathfrak{a}'\mathfrak{b}'} = 2\delta_{ij}.$$

fermionic interaction

$$-\frac{\alpha}{2}\psi^{\alpha}_{\mathfrak{a}'}(C\Gamma^{\mu})_{\alpha\beta}\psi^{\beta}_{\mathfrak{b}'}\mathcal{A}_{\mu\mathfrak{a}}\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{a}}, T^{\mathfrak{b}'}])$$

square

$$\frac{\alpha^2}{4}\psi^{\alpha}_{\mathfrak{a}'}(C\Gamma^{\mu})_{\alpha\beta}\psi^{\beta}_{\mathfrak{b}'}\psi^{\gamma}_{\mathfrak{c}'}(C\Gamma^{\nu})_{\gamma\delta}\psi^{\delta}_{\mathfrak{d}'}\mathcal{A}_{\mu\mathfrak{a}}\mathcal{A}_{\nu\mathfrak{b}}\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{a}}, T^{\mathfrak{b}'}])\mathrm{Tr}(T^{\mathfrak{c}'}[T^{\mathfrak{b}}, T^{\mathfrak{d}'}])$$

Contraction 1. $\mathfrak{a}' - \mathfrak{d}'$ and $\mathfrak{b}' - \mathfrak{c}'$

$$\frac{\alpha^2}{4}(\Delta_{\mathrm{F}}^{-1})^{\alpha\delta}_{\mathfrak{a}'\mathfrak{d}'}(\Delta_{\mathrm{F}}^{-1})^{\beta\gamma}_{\mathfrak{b}'\mathfrak{c}'}(C\Gamma^{\mu})_{\alpha\beta}(C\Gamma^{\nu})_{\gamma\delta}\mathcal{A}_{\mu\mathfrak{a}}\mathcal{A}_{\nu\mathfrak{b}}\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{a}}, T^{\mathfrak{b}'}])\mathrm{Tr}(T^{\mathfrak{c}'}[T^{\mathfrak{b}}, T^{\mathfrak{d}'}]).$$

$$= -\frac{4\times 16}{d^4}\mathrm{Tr}(T^{\mathfrak{a}}T^{\mathfrak{b}})d^{\mu}d^{\nu}\mathcal{A}_{\mu\mathfrak{a}}\mathcal{A}_{\nu\mathfrak{b}} = -\frac{64}{d^4}d^{\mu}d^{\nu}\mathrm{Tr}(\mathcal{A}_{\mu}\mathcal{A}_{\nu}).$$

Contraction 2 $\mathfrak{a}' - \mathfrak{c}'$ and $\mathfrak{b}' - \mathfrak{d}'$ gives the same.

In the contraction 1.

$$\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{a}}, T^{\mathfrak{b}'}])\mathrm{Tr}(T^{\mathfrak{c}'}[T^{\mathfrak{b}}, T^{\mathfrak{d}'}]) \to -4\mathrm{Tr}(T^{\mathfrak{a}}T^{\mathfrak{b}})$$

Cubic interaction

Sat, Apr 20

$$\mathrm{Tr}[A_\mu, A_\nu][A^\mu, A^\nu] \to 4\mathrm{Tr}[A_\mu, \mathscr{A}_\nu][\mathscr{A}^\mu, \mathscr{A}^\nu].$$

$$4\sum_{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}} (\mathfrak{A}_\mu)_{\mathfrak{a}\mathfrak{b}}\mathscr{A}_{\nu\mathfrak{b}}\mathscr{A}^\mu_{\mathfrak{c}}\mathscr{A}^\nu_{\mathfrak{d}}\mathrm{Tr}(T^{\mathfrak{a}}[T^{\mathfrak{c}}, T^{\mathfrak{d}}]).$$

$$4\sum_{\mathfrak{a}',\mathfrak{b}',\mathfrak{c}',\mathfrak{d}'} (\mathfrak{A}_\mu)_{\mathfrak{a}'\mathfrak{b}'}\alpha_{\nu\mathfrak{b}'}\alpha^\mu_{\mathfrak{c}'}\alpha^\nu_{\mathfrak{d}'}\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{c}'}, T^{\mathfrak{d}'}]).$$

Fermionic action

$$\mathrm{Tr}(\overline{\psi}\Gamma^\mu[A_\mu, \psi]) = \mathrm{Tr}(\psi^\alpha(C\Gamma^\mu)_{\alpha\beta}[A_\mu, \psi^\beta]).$$

fermionic background

$$\psi + \varphi.$$

with the direct sum bosonic background

$$\sum_{\mathfrak{a}'\mathfrak{b}'}\varphi^\alpha_{\mathfrak{a}'}(C\Gamma^\mu)_{\alpha\beta}(\mathfrak{A}_\mu)_{\mathfrak{a}'\mathfrak{b}'}\varphi^\beta_{\mathfrak{b}'} + \sum_{\mathfrak{a}'\mathfrak{b}'\mathfrak{c}'}\varphi^\alpha_{\mathfrak{a}'}(C\Gamma^\mu)_{\alpha\beta}\alpha_{\mu\mathfrak{b}'}\varphi^\beta_{\mathfrak{c}'}\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{b}'}, T^{\mathfrak{c}'}]).$$

Also consider

$$\sum_{\mathfrak{a}\mathfrak{b}'\mathfrak{c}}\psi^\alpha_{\mathfrak{a}}(C\Gamma^\mu)_{\alpha\beta}\alpha_{\mu\mathfrak{b}'}\psi^\beta_{\mathfrak{c}}\mathrm{Tr}(T^{\mathfrak{a}}[T^{\mathfrak{b}'}, T^{\mathfrak{c}}]).$$

Show that it's vanish. Also consider

$$\mathrm{Tr}\psi^\alpha(C\Gamma^\mu)_{\alpha\beta}[\alpha_\mu, \varphi^\beta] + \mathrm{Tr}\varphi^\alpha(C\Gamma^\mu)_{\alpha\beta}[\alpha_\mu, \psi^\beta] = -2\mathrm{Tr}\varphi^\alpha(C\Gamma^\mu)_{\alpha\beta}[\psi^\beta, \alpha_\mu].$$

Similarly, define the adjoint matrix

$$\Psi_{\mathfrak{a}\mathfrak{b}} \equiv \sum_{\mathfrak{c}}\psi_{\mathfrak{c}}\mathrm{Tr}(T^{\mathfrak{a}}[T^{\mathfrak{c}}, T^{\mathfrak{b}}]).$$