Master Thesis Notes

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Wed, Feb 21

rethink the model In constructing the previous model, the way we introduce the matrix η is suspicious. Specifically $\delta^2\eta = 0$, but it should generate the unitary transformation $\delta^2\eta = [H, \eta]$. Remember that the idea behind the BRST construction is "taking the square root of gauge transformation". However, it seems impossible to satisfy $\delta^2 H = 0$ simultaneously. It seems like the only way to solve this problem is to introduce again an auxillary matrix such that $\delta H = [X, \eta]$. That is the gauge transformation is parameterized by the auxillary matrix X.

What's the motivation to introduce η in our model? I want to introduce a term $\mathrm{Tr}([\overline{Z}, Z]H)$ in the action, from doing the variation $\delta(\cdots)$. To obtaine H from δ, I introduce η such that $\delta\eta = H$. This is a bad move as having been said above: this breaks the unitary invariance when the model involving the matrix η.

one possible solution to the previous problem? If we introduce another auxillary matrix X to parameterize the unitary transformation, and modify $\delta H = [X, \eta]$ with $\delta X = 0$. (Remark. H, X, η should all be anti-hermitian to keep each term "real" under the hermitian conjugation †) The model build may still start from the commutator term

$$-\frac{1}{2}\mathrm{Tr}H^2 + ig\mathrm{Tr}\left([\overline{Z}, Z]H\right).$$

Then one wants to find δ-exact terms to generate these two terms. The candidates are

$$ig\delta\mathrm{Tr}\left([\overline{Z}, Z]\eta\right) - \frac{1}{2}\delta\mathrm{Tr}\left(\eta H\right)$$

Other terms are generated along

$$ig\mathrm{Tr}\left([\overline{\Psi}, Z]\eta + [\overline{Z}, \Psi]\eta\right) + \frac{1}{2}\mathrm{Tr}\left(\eta[X, \eta]\right).$$

Note that $(1/2)\mathrm{Tr}(\eta[X, \eta]) = \mathrm{Tr}(\eta X\eta)$ gives a quadratic term of η. To deal with the interaction terms $ig\mathrm{Tr}([\overline{Z}, Z]H)$ and $ig\mathrm{Tr}([\overline{\Psi}, Z]\eta + [\overline{Z}, \Psi]\eta)$, it's important to introduce quadratic term for $Z, \overline{Z}$ and $\Psi, \overline{\Psi}$. This is done by adding to following δ-exact term

$$i\kappa\delta\mathrm{Tr}\left(\overline{Z}\Psi - Z\overline{\Psi}\right) = 2i\kappa\mathrm{Tr}(\overline{\Psi}\Psi) - 2\kappa\epsilon\mathrm{Tr}(\overline{Z}Z) - 2i\kappa\mathrm{Tr}\left([\overline{Z}, Z]X\right).$$

Here X is not a dynamical matrix, and it should be anti-hermitian. In summary, let's collect all terms in the action

$$\begin{aligned} -\frac{1}{2}\mathrm{Tr}H^2 + \mathrm{Tr}\eta X\eta \\ +\frac{i}{2}\kappa\mathrm{Tr}\overline{\Psi}\Psi - \frac{1}{2}\kappa\epsilon\mathrm{Tr}\overline{Z}Z - \frac{i}{4}\kappa\mathrm{Tr}\left([\overline{Z}, Z]X\right) \\ +ig\mathrm{Tr}\left([\overline{Z}, Z]H\right) + ig\mathrm{Tr}\left([\overline{\Psi}, Z]\eta + [\overline{Z}, \Psi]\eta\right) \end{aligned}$$

It can be verified all the terms are real under † (It acts on the matrix, will not take $i \to -i$).

Compare to previous wrong model, it has several differences. First is the appearance of the X matrix. It it is taken to be zero, the model will become very similar to what we have studied previously, but with an additional term $ig\mathrm{Tr}\left([\overline{Z}, Z]H\right)$. However, the previous calculation is wrong because η should be anti-hermitian, rather than hermitian. This means that there is no θ variable to be integrated out; also h^* and h terms should have a relative minus sign. Let's list all terms that are relavent to the integration

$$\begin{aligned} -v^*A\beta + \beta^*Av + v^*B\alpha - \alpha^*Bv + \alpha^*H\beta - \beta^*H\alpha \\ x^*Ah - h^*Ax - y^*Bh + h^*By \\ \beta^*\psi h - h^*\psi\beta - \alpha^*\chi h + h^*\chi\alpha \\ -\alpha^*\eta x - x^*\eta\alpha + \beta^*\eta y + y^*\eta\beta \end{aligned}$$

It's interesting to take $X = iz\mathbb{1}$. z is a real constant to be determined later. One consequence is

$$\mathrm{Tr}\left(\eta X\eta\right) \sim -izh_i^*h_i.$$

Remember that $\eta_{(N+1),i} = -h_i^*,\ \eta_{i,(N+1)} = h_i$. Another consequence is

$$\frac{\kappa}{2}\operatorname{Tr}([A,B]X) \sim \frac{\kappa}{2} iz(\beta_i^*\alpha_i - \alpha_i^*\beta_i).$$

So the quadratic term for α, β is

$$-\frac{\kappa\epsilon}{2}(\alpha_i^*\alpha_i + \beta_i^*\beta_i) + \frac{i\kappa z}{2}(\beta_i^*\alpha_i - \alpha_i^*\beta_i).$$

This motivates us to redefine

$$\tilde{\alpha} = \frac{1}{\sqrt{2}}(\alpha + i\beta), \quad \tilde{\beta} = \frac{1}{\sqrt{2}}(\alpha - i\beta).$$

Then the quadratic term becomes

$$-\frac{\kappa}{2}\left[(\epsilon+z)\tilde{\alpha}_i^*\tilde{\alpha}_i + (\epsilon - z)\tilde{\beta}_i^*\tilde{\beta}_i\right].$$

The interaction of A, B with the "bosonic legs" in terms of $\tilde{\alpha}, \tilde{\beta}$ reads

$$\frac{1}{\sqrt{2}}\left(iv^*A\tilde{\alpha} + i\tilde{\alpha}^*Av - iv^*A\tilde{\beta} - i\tilde{\beta}^*Av + v^*B\tilde{\alpha} - \tilde{\alpha}^*Bv + v^*B\tilde{\beta} - \tilde{\beta}^*Bv\right).$$

It's possible to obtain a new operator $\operatorname{Tr}(AB)$ through the contraction. To the lowest order there are four terms that contribute to $\operatorname{Tr}(AB)$:

$$\frac{i}{2}\left(\tilde{\alpha}^*Avv^*B\tilde{\alpha} - v^*A\tilde{\alpha}\tilde{\alpha}^*Bv + v^*A\tilde{\beta}\tilde{\beta}^*Bv - \tilde{\beta}^*Avv^*B\tilde{\beta}\right).$$

We see that there is actually no $\operatorname{Tr}(AB)$ generated at this order. For the interaction of A, B with the "fermionic legs"

$$x^*Ah - h^*Ax - y^*Bh + h^*By.$$

There are two terms that contribute to the $\operatorname{Tr}(AB)$

$$x^*Ahh^*By + h^*Axy^*Bh.$$

The order of contraction is yx^* and y^*x. So there is also no contribution at this order.

(TODO: more discussions on the propagators $\alpha^*\alpha$ and $\beta^*\beta$, and possible cancellation or suppression of new interaction generations.)

Fri, Feb 23

A comparison between the "dimensional model" and the "dimensionless model

Let's start with a "dimensionful" model, whose partition function is given by

$$Z_N(g) = \int \exp\left[-N\left(\frac{1}{2}\operatorname{Tr} M^2 + \frac{g}{4}\operatorname{Tr} M^4\right)\right] \mathrm{d}M. \tag{1}$$

The scaling of the matrix M, and the coupling g under $N \to \lambda N$ is

$$M \to \lambda^{-1/2} M$$
$$g \to \lambda g$$

One could also write the same theory in a "dimensionless" way, by introducing $\tilde{M} = \sqrt{N} M$ and $\tilde{g} = N^{-1} g$. Then in terms of $\tilde{M}$ and $\tilde{g}$ the partition function reads

$$Z_N(g) = N^{N^2/2} \int \exp\left[-\left(\frac{1}{2}\operatorname{Tr}\tilde{M}^2 + \frac{\tilde{g}}{4}\operatorname{Tr}\tilde{M}^4\right)\right] \mathrm{d}\tilde{M}. \tag{2}$$

Starting from the "dimensionless model" (2), it's possible to recover the "dimensionful model" (1) by specifying the quadratic term $\frac{N}{2}\operatorname{Tr} M^2$ first.

the notions of scaling First is the scaling from the RG flow. Let's start by thinking about the RG calculation to the lowest order. This will just reproduce the classical scaling: $g \to \lambda g$ and $\tilde{g}$ is invariant. However, they are essentially different. In the language of QFT, one can think of N as a cut-off. Then the RG method allows us to relate theories with different cut-off such that they will produce the same results (the correlation functions). Specifically, to the lowest order, the coupling $g \to \lambda g$ with the change of cut-off $N \to \lambda N$. To the lowest order, these are exactly the same as the classical scaling. However, the higher order corrections are essential for the RG flow.

Conceptually, the RG flow keeps the theory invariant, while the classical scaling not: one knows that the matrix model has a non-trivial N dependence although they share the same form of action. The classical scaling is just a natural way to define how the matrix and coupling depending on the underlying scale such that the form of the action keeping the same. The classical scaling works like "zooming in" or "zooming out". However, the RG flow works like "coarse graining". There is no reason to believe that the "coarse graining" will give a similar result comparing to the "zooming" in general. The Gaussian model is a special example that they giving exactly the same result.

Now what about the notion of "scaling invariance"? In this case, it's more interesting to consider yet another scaling. Let's call it "dynamical scaling". The meaning is that only "dynamical variables" should be rescaled. The matrix is dynamical but the coupling constant is not. Therefore, this scaling should not be understood as a change of dimension; It's a symmetry of the action. The interesting thing about the RG flow is that the dynamical scaling invariance could emerge at certain critical point g_*. The existence of such a point $g_* \neq 0$ seems impossible by just looking at the action, because it is written in a form that only the classical scaling is obvious. It's impossible to obtain a dynamical scaling invariance along the classical scaling. While the RG flow could deviate from the classical scaling significantly at some points, along which the scaling of g could be frozen.

Start with the $N+1$-model, and decomposing the matrix M as

$$\begin{pmatrix} M & v \\ v^\dagger & \alpha \end{pmatrix}.$$

The action can be expanded as

$$S_{N+1}[M, v, v^\dagger, \alpha; g] = (N+1)\,\mathrm{Tr}\left(\frac{1}{2}M^2 + \frac{g}{4}M^4\right) + (N+1)\left(v^\dagger v + \frac{1}{2}\alpha^2\right) + g(N+1)\left(v^\dagger M^2 v + \alpha v^\dagger M v + \alpha^2 v^\dagger v + \frac{1}{2}(v^\dagger v)^2 + \frac{1}{4}\alpha^4\right). \tag{3}$$

To do the integration over $v, v^\dagger$ and α, one needs to first identify the quadratic term around which the interaction is treated perturbatively. For $v, v^\dagger$, the quadratic term is Sat, Feb 24

$$(N+1)v^\dagger\left(\mathbb{1} + gM^2\right)v.$$

For α, it's

$$\frac{N+1}{2}\alpha^2.$$

These leads to the following contraction rules

$$\left\langle v_i^\dagger v_j \right\rangle = \frac{1}{N+1}\left(\frac{1}{1+gM^2}\right)_{ij}$$
$$\langle \alpha\alpha \rangle = \frac{1}{N+1}$$

The interaction terms are

$$(N+1)\left(g\alpha v^\dagger M v + g\alpha^2 v^\dagger v + \frac{g}{2}(v^\dagger v)^2 + \frac{g}{4}\alpha^4\right).$$

First, at the zeroth order, the partition function has the form

$$Z_{N+1} \sim \int [\mathrm{d}M] \frac{1}{(N+1)^N \mathrm{Det}(\mathbb{1} + gM^2)} \mathrm{e}^{-(N+1)\,\mathrm{Tr}\left(\frac{1}{2}M^2 + \frac{g}{4}M^4\right)}.$$

The $\frac{1}{(N+1)^N}$ factor will be compensated by a rescaling of the effective partition function. The determinant can be re-exponented as

$$\frac{1}{\mathrm{Det}(\mathbb{1} + gM^2)} = \mathrm{e}^{-\,\mathrm{Tr}\ln(1+gM^2)}.$$

This leads to the following "effective action"

$$S_{\text{eff}} = (N+1)\,\mathrm{Tr}\left(\frac{1}{2}M^2 + \frac{g}{4}M^4\right) + \mathrm{Tr}\ln(1+gM^2). \tag{4}$$

Then at the order g, one could write the partition function Z_{N+1} as

$$Z_{N+1} \sim -g(N+1)\int [\mathrm{d}M] \left\langle \alpha v^\dagger M v + \alpha^2 v^\dagger v + \frac{1}{2}(v^\dagger v)^2 + \frac{1}{4}\alpha^4 \right\rangle \mathrm{e}^{-S_{\text{eff}}[M]}.$$

The first term will not contribute, while other terms giving

$$\left\langle \alpha^2 v^\dagger v \right\rangle = \frac{1}{(N+1)^2} \operatorname{Tr}\left(\frac{1}{\mathbb{1}+gM^2}\right)$$

$$\frac{1}{2}\left\langle (v^\dagger v)^2 \right\rangle = \frac{1}{2(N+1)^2}\left[\operatorname{Tr}\left(\frac{1}{\mathbb{1}+gM^2}\right)\right]^2 + \frac{1}{2(N+1)^2}\operatorname{Tr}\left(\frac{1}{\mathbb{1}+gM^2}\right)^2$$

$$\frac{1}{4}\left\langle \alpha^4 \right\rangle = \frac{3}{4(N+1)}.$$

Let's study the case where g is small and keep only the first order of g. The "effective action" becomes

$$S_{\text{eff}}[M] = (N+1)\operatorname{Tr}\left(\frac{1}{2}M^2 + \frac{g}{4}M^4\right) + g\operatorname{Tr} M^2.$$

The corrections from the interactions to the partition function are

$$Z_{N+1} \sim -g(N+1)\int [\mathrm{d}M]\left(\frac{3N}{2(N+1)^2} + \frac{N^2}{2(N+1)^2} + \frac{3}{4(N+1)}\right)\mathrm{e}^{-S_{\text{eff}}[M]}.$$

Re-exponentiate these terms will give us a new action, which could reproduce the correlation functions calculated from $S_{N+1}[M]$ to the first order of g,

$$S'_{\text{eff}}[M] = (N+1)\operatorname{Tr}\left(\frac{1}{2}M^2 + \frac{g}{4}M^4\right) + g\operatorname{Tr} M^2 + \frac{3N}{2(N+1)}g + \frac{N^2}{2(N+1)}g + \frac{3}{4}g.$$

A feature of the RG flow procedure is the appearance of terms with different orders of N.

To put the effective action in the canonical form, one defines

$$M' = \left(1 + \frac{1+2g}{N}\right)^{\frac{1}{2}} M,$$

$$g' = \frac{N(N+1)}{(N+1+2g)^2} g$$

To the first order of g

$$M' \approx \left(\frac{N+1}{N}\right)^{\frac{1}{2}}\left(1 + \frac{g}{N+1}\right) M,$$

$$g' \approx \left(\frac{N}{N+1}\right)\left(1 - \frac{4g}{N+1}\right) g$$

such that

$$S'_{\text{eff}}[M] = N\operatorname{Tr}\left(\frac{1}{2}M'^2 + \frac{g'}{4}M'^4\right) + \cdots \tag{5}$$

To get a feeling about what kind of terms could be generated through this RG procedure, let's calculate the decomposition of $\frac{1}{6}\mathrm{Tr}M^6$: Sun, Feb 25

$$v^\dagger M^4 v + (v^\dagger v)(v^\dagger M^2 v) + \frac{1}{2}(v^\dagger M v)(v^\dagger M v) + \frac{1}{3}(v^\dagger v)^3$$
$$+\alpha^2 v^\dagger M^2 v + \frac{3}{2}\alpha^2 (v^\dagger v)^2 + \alpha^4 v^\dagger v.$$

Looking at the first line (terms without α), one could think about the possible contractions. The first term is quadratic in v, while other terms should be treated by perturbation theory. It's important to note that, for the second term, if contracting $(v^\dagger v)$ one gets

$$\frac{1}{N+1}\mathrm{Tr}\left(\frac{1}{\mathbb{1}+\cdots}\right).$$

where $\cdots$ coming from the quadratic terms like gM^2. The leading order in g would be

$$\frac{1}{N+1}\mathrm{Tr}\mathbb{1} = \frac{N}{N+1}.$$

One would say that this term has the N-dimension 0. This is inconsistent with the dimension of $(v^\dagger v)$, which is the same as $\mathrm{Tr}M^2$. The reason is that $\mathrm{Tr}\mathbb{1}$ has dimension 1. The dimension of Tr should also be taken into account correctly.

To the zeroth order of g, contraction of the first line gives

$$\frac{N}{(N+1)^2}\mathrm{Tr}M^2 + \frac{3}{2(N+1)^2}\mathrm{Tr}M^2 + \frac{1}{2(N+1)^2}\left(\mathrm{Tr}M\right)^2 + \frac{1}{3}\left(\frac{N}{N+1}\right)^3.$$

The first term has the "wrong" dimension because of the $v^\dagger v$ contraction. When N is large, however, it gives the leading contribution to $\mathrm{Tr}M^2$. The N-dimension essentially tells us how the terms behave when changing N. This is an assumption we made for our theory.

One crucial idea for analyzing the matrix model is to reduce the number of variables from N^2 to N. This can be done because of the symmetry of the model. What's the physical insight follows from this reduction? If one interpret the matrix as a Hamiltonian, the reduction is essentially chosing the energy eigen-basis for the physical states. The symmetry is just the fact that the physical observables do not depend on the choice of basis. However, we would like to take about the matrix model arising in the string theory. There, rather than an operator on the space of physical states, the matrix itself represents the physical states: the Chan-Paton degrees of freedom of open string. The number of the Hermitian matrices matches the number of the Chan-Paton degrees of freedom. Wed, Feb 28

The question: imagine a scaling matrix model, in the sense that the matrix M and coupling g scale with N in a simple way. Along this scaling, the physical observables are kept invariant. If such a scaling phenomenon exists, what can we say about the physical observables around such critical point? Thu, Feb 29

It's not clear that what's the meaning of conformal invariance for a matrix model, but we have the motivation to believe that there exists such a notion. From the point view of AdS/CFT, the matching between the conformal group, for example in $4d$ the conformal group is $SO(4,2)$, and the AdS isometry group, for example in $5d$ the isometry group is $O(4,2)$. However, such a "group argument" does not make sense for AdS1/CFT0. Fri, Mar 1

Let's review that how the AdS/CFT is understood in string theory. The central object is the D-brane whose dynamics can be described by an effective action of field theory. The fields live on the world-volume of D-brane: there are massless scalar fields and vector fields. Their appearance is understood as the massless excitation of open strings with end points on the D-brane. The form of the action is obtained by matching with the calculation from the perturbative string theory. This field theory will become the CFT side.

Question 1. *But why the field theory of D-branes must be a CFT? Does this fact relate to the dynamics of the D-branes? Or is there a "string theory" argument for the appearance of conformal invariance?*

Answer 1. *Because the open string excitation is massless when the two end points locate on the coincident D-branes.*

D-branes carry the R-R charges, which means that it will couple to the higher form gauge fields. Therefore they can be understood physically as the sources of those gauge fields. The D(-1)-brane couples to the 0-form potential in the IIB theory C_0. The energy of that coupling is given by the value of C_0 at the point where the D(-1)-brane locating. Therefore the D(-1)-brane is properly interpreted as an instanton.

We are interested in the D-instanton, a natural question is that how the D-instantons interact with themselves and other objects like higher dimensional D-branes. This question could be answered in the perturbative string theory. One calculates the string amplitudes for exchanging the closed string states (graviton, dilaton and the R-R state) between the D-branes. The fundamental physical properties of the D-branes (the tension and the R-R charge) could be related to the fundamental constants in string theory.

Question 2. *When discussing the interaction and action of D-branes, we have in mind a picture where the D-brane locating at a particular space-time position. In this way, rather than taking them as a fundamental degrees of freedom, we regard them as absolute objects (like black hole in GR). In such a setting, how should we understand the physics under the D-brane action? A clearer name seems to be the "action of D-brane fluctuation", and the fluctuation has its origin at the open string oscillation. Or maybe one should call it the "open string action restricted on the D-branes"?*

There is an interesting explanation of the collective coordinates X^μ of the D-branes: the Goldstone bosons for the spontaneous translation symmetry breaking. But why it's called the "spontaneous"?

Idea 1. *Why not trying to look further at the "Ward identity" of the matrix model and think about the possible implications on our calculations?*

To include supersymmetry in the matrix model, we are forced to consider the Grassmann-valued matrix. This leads us to think about how to build a matrix upon Sun, Mar 3 the Grassmann algebra. In particular, think about how the supersymmetry transformation is realized as a transformation inside the Grassmann algerba (interchange odd and even elements).

First, one should difference between the elements of Grassmann algebra and the Grassmann variables. The later are the generators of the Grassmann algebra. The differential and integration are defined for the Grassmann variables. Then

Question 3. *The fermionic matrix has entries of the Grassmann variables or the general odd elements of the Grassmann algebra?*

It seems natural to allow general odd elements and take the number of the Grassmann variables to infinity.

Question 4. *How to define the integration over such a Grassmann-valued matrix?*

The supersymmetry of a matrix model may imply the vanishing of some correlation functions. This may be regarded as the matrix analog of the Ward identity. An example is the superfield model mentioned in ✎ **Ple96**. The reason behind the vanishing is the cancellation between bosonic and fermionic degrees of freedom. However, the prove given in the article based on the diagonalization of the bosonic matrix. One would like to see the cancellation in a direct way: how the cancellation works when calculating the effective action. The first step would be a careful study of the action under a parameterization of the model.

One problem of the RG calculation is that the susy transformation of the reduced matrix involving the variables that being integrated out. This implies that the parameterization is not done in a way that the susy is preserved. To avoid the problem, maybe we should use the formalism where the susy is realized linearly.

Idea 2. *Do the RG calculation for a model where the supersymmetry is realized linearly.*

The supersymmetry Ward identity gives a relation between correlation functions

$$\frac{1}{2}\left\langle \mathrm{Tr}V'(\phi)\phi^{n-1}\right\rangle = \sum_{a+b=n-2}\left\langle \mathrm{Tr}\phi^a\psi\phi^b\overline{\psi}\right\rangle . \tag{6}$$

This identity maybe used to reduce the result of the calculation.

Problem 1. *Try to derive this Ward identity.*

Problem 2. *How does the measure* $\mathrm{d}\phi\mathrm{d}\psi\mathrm{d}\overline{\psi}$ *transforms under the susy?*

First note that the super-determinant must involve $\bar{\epsilon}\epsilon$ such that the integral is bosonic. So the measure is invariant to the first order of $\epsilon, \bar{\epsilon}$. The "superfield" $\Phi = \phi + \overline{\psi}\theta + \bar{\theta}\psi + \theta\bar{\theta}F$ could be the starting point for deriving the Ward identity. Calculate Φ^n

$$
\begin{aligned}
\Phi^n &= \left(\phi + \overline{\psi}\theta + \bar{\theta}\psi + \theta\bar{\theta}F\right)^n \\
&= \phi^n + \left(\sum_{a+b=n-1} \phi^a\overline{\psi}\phi^b\right)\theta + \bar{\theta}\left(\sum_{a+b=n-1} \phi^a\psi\phi^b\right) \\
&+ \theta\bar{\theta}\left(\sum_{a+b+c=n-2} \phi^a\overline{\psi}\phi^b\psi\phi^c + \sum_{a+b=n-1} \phi^a F\phi^b\right).
\end{aligned}
$$

This leads us to consider the following reparameterization (induced by Φ^n, this reparameterization should be supersymmetric.)

$$
\begin{aligned}
&\phi \to \phi' = \phi + \varepsilon\phi^n \\
&\overline{\psi} \to \overline{\psi}' = \overline{\psi} + \varepsilon \sum_{a+b=n-1} \phi^a\overline{\psi}\phi^b \\
&\psi \to \psi' = \psi + \varepsilon \sum_{a+b=n-1} \phi^a\psi\phi^b \\
&F \to F' = F + \varepsilon\left(\sum_{a+b+c=n-2} \phi^a\overline{\psi}\phi^b\psi\phi^c + \sum_{a+b=n-1} \phi^a F\phi^b\right)
\end{aligned}
$$

Consider how the measure $\mathrm{d}\phi\mathrm{d}\overline{\psi}\mathrm{d}\psi\mathrm{d}F$ changes to the first order of ε. The usual coordinate transformation formula detJac is generalized to that involving Grassmann variables BerJac. The Berezinian satisfies the formula

$$
\mathrm{Ber}\left(\mathbb{1} + \varepsilon M\right) = 1 + \varepsilon \mathrm{STr} M. \tag{7}
$$

Here the STr is the super-trace. It can be proved that there is no change of the measure.

Now the following terms arise from the variation of the action

$$
\begin{aligned}
&\varepsilon \mathrm{Tr}[V''(\phi)\phi^n F] - 2\varepsilon \sum_{a+b=n-1} \mathrm{Tr}(\phi^a F\phi^b F) + \varepsilon \sum_{a+b=n-1} \mathrm{Tr}[V'(\phi)\phi^a F\phi^b] \\
&-2\varepsilon \sum_{a+b+c=n-2} \mathrm{Tr}(\phi^a\overline{\psi}\phi^b\psi\phi^c F) + \varepsilon \sum_{a+b+c=n-2} \mathrm{Tr}[V'(\phi)\phi^a\overline{\psi}\phi^b\psi\phi^c] \\
&+\varepsilon \sum_{k=0}^{\infty} k g_k \sum_{a+b=k-2} \Big[a\mathrm{Tr}(\phi^{a+n-1}\overline{\psi}\phi^b\psi) + b\mathrm{Tr}(\phi^a\overline{\psi}\phi^{b+n-1}\psi) \\
&\qquad +2 \sum_{c+d=n-1} \mathrm{Tr}(\phi^{a+c}\overline{\psi}\phi^{b+d}\psi)\Big]
\end{aligned}
$$

The second line will vanish after integrating over F. The first line will give $-2\varepsilon \sum_{a+b=n-1} \mathrm{Tr}\phi^a \mathrm{Tr}\phi^b + \frac{\varepsilon}{2}\mathrm{Tr}[V''(\phi)\phi^n V'(\phi)]$.

Remark 1. *The above calculation gives a fairly complicate identity. It's also not clear whether it will be useful or not. Also, this does not lead to the Ward identity.*

Study the RG flow of the "superfield" matrix model. This model is mentioned in **Ple96** Mon, Mar 4

The matrix-valude superfield is constructed as

$$\Phi = \phi + \overline{\psi}\theta + \overline{\theta}\psi + \theta\overline{\theta}F. \tag{8}$$

$\theta, \overline{\theta}$ are the coordinates of the superspace. ϕ, F are bosonic $N \times N$ matrix, and we assume them to be Hermitian. $\psi, \overline{\psi}$ are fermionic $N \times N$ matrix, whose entries are Grassmann variables. Recall that the complex conjugate of the product of Grassmann variables is defined as $(\xi\eta)^* = \eta^*\xi^*$. To keep Φ Hermitian $\Phi^\dagger = \Phi$, we require that

$$\psi^\dagger = \overline{\psi}, \quad \theta^* = \overline{\theta}.$$

The measure over the superfield will be the usual Berezian integral of the matrix entries

$$\mathrm{d}\Phi = \mathrm{d}\phi \mathrm{d}F \mathrm{d}\overline{\psi}\mathrm{d}\psi. \tag{9}$$

The supersymmetric action can be constructed as

$$S[\Phi] = \mathrm{Tr}\int \mathrm{d}\overline{\theta}\mathrm{d}\theta \left\{ -D_\theta \Phi D_{\overline{\theta}} \Phi + \sum_{k=0}^{\infty} g_k \Phi^k \right\}. \tag{10}$$

The super-derivative acts from the right. The matrix model partition function is

$$Z_\Phi[g_k] = \int [\mathrm{d}\Phi] \exp\left(-NS[\Phi]\right). \tag{11}$$

We have $D_\theta \Phi = \overline{\psi} - \overline{\theta}F$ and $D_{\overline{\theta}}\Phi = -\psi + \theta F$. Their product will contribute to the action only if the measure $\mathrm{d}\theta\mathrm{d}\overline{\theta}$ is saturated. This will give a term $-\mathrm{Tr}F^2$ in the action. We calculate the Φ^k term as

$$\begin{aligned}
\Phi^k &= \left(\phi + \overline{\psi}\theta + \overline{\theta}\psi + \theta\overline{\theta}F\right)^k \\
&= \phi^k + \left(\sum_{a+b=k-1} \phi^a \overline{\psi} \phi^b\right)\theta + \overline{\theta}\left(\sum_{a+b=k-1} \phi^a \psi \phi^b\right) \\
&+ \theta\overline{\theta}\left(\sum_{a+b+c=k-2} \phi^a \overline{\psi}\phi^b\psi\phi^c + \sum_{a+b=k-1} \phi^a F \phi^b\right).
\end{aligned}$$

By taking trace and keeping only the $\theta\overline{\theta}$ term, we get in the action a term

$$\mathrm{Tr}[V'(\phi)F] + \sum_{k=0}^{\infty} k g_k \sum_{a+b=k-2} \mathrm{Tr}(\phi^a\overline{\psi}\phi^b\psi).$$

Let's assume a quartic potential $V(\phi) = \frac{1}{2}\phi^2 + \frac{g}{4}\phi^4$,

$$\begin{aligned}
S[\Phi] = &-\mathrm{Tr}F^2 + \mathrm{Tr}(\phi F) + g\mathrm{Tr}(\phi^3 F) \\
&+ \mathrm{Tr}(\overline{\psi}\psi) + g\left[\mathrm{Tr}(\phi^2\overline{\psi}\psi) + \mathrm{Tr}(\phi\overline{\psi}\phi\psi) + \mathrm{Tr}(\overline{\psi}\phi^2\psi)\right].
\end{aligned}$$

Although it's easy to do the integral over F, we will not do that. Because it will lead to a non-linear supersymmetry transformation rule. However, let's define $F' = F - \frac{1}{2}\phi$ and rewrite the action in terms of F such that a quadratic term in ϕ will appear in the action

$$
\begin{aligned}
S[\Phi] = &-\mathrm{Tr}F'^2 + \frac{1}{4}\mathrm{Tr}\phi^2 + \frac{g}{2}\mathrm{Tr}\phi^4 + g\mathrm{Tr}(\phi^3 F') \\
&+ \mathrm{Tr}(\overline{\psi}\psi) + g\left[\mathrm{Tr}(\phi^2\overline{\psi}\psi) + \mathrm{Tr}(\phi\overline{\psi}\phi\psi) + \mathrm{Tr}(\overline{\psi}\phi^2\psi)\right].
\end{aligned}
$$

The supersymmetry variation reads

$$
\begin{aligned}
\delta\phi &= \bar{\varepsilon}\psi + \overline{\psi}\varepsilon \\
\delta\psi &= -\varepsilon(F' + \frac{1}{2}\phi) \\
\delta\overline{\psi} &= -\bar{\varepsilon}(F' + \frac{1}{2}\phi) \\
\delta F' &= -\frac{1}{2}(\bar{\varepsilon}\psi + \overline{\psi}\varepsilon)
\end{aligned}
$$

Think about applying the RG strategy on this action. If we denote $\phi_{i,N} = v_i, \phi_{N,i} = v_i^*$, the following term will appear

$$
gv^\dagger F'\phi v + gv^\dagger \phi F' v
$$

from the interaction $g\mathrm{Tr}(\phi^3 F')$. It will contribute a $\mathrm{Tr}(\phi F')$ term to the effective action. Why the supersymmetry cancellation does not happen here? This is because the supersymmetry transformation acts on all matrix entries simultaneously: it does not realize solely on the variables that being integrated out.

Idea 3. *Study the effect of a "local" supersymmetry transformation: that is, those only act on a part of the matrix.*

Idea 4. *Carry out the calculation anyway, to see what you get.*

For the RG strategy, let's decompose the matrix

$$
F'_0 = \begin{pmatrix} F' & f \\ f^\dagger & a \end{pmatrix}, \quad \phi_0 = \begin{pmatrix} \phi & v \\ v^\dagger & \alpha \end{pmatrix}
$$
$$
\psi_0 = \begin{pmatrix} \psi & \chi \\ \omega^\dagger & \beta \end{pmatrix}, \quad \overline{\psi}_0 = \begin{pmatrix} \overline{\psi} & \omega \\ \chi^\dagger & \overline{\beta} \end{pmatrix}
$$

The interactions decompose correspondingly

$$
\begin{aligned}
\frac{g}{2}\mathrm{Tr}\phi_0^4 &= \frac{g}{2}\mathrm{Tr}\phi^4 + 2gv^\dagger\phi^2 v + g(v^\dagger v)^2 + 2g\alpha v^\dagger\phi v + 2g\alpha^2 v^\dagger v + \frac{g}{2}\alpha^4 \\
g\mathrm{Tr}(\phi_0^3 F_0') &= g\mathrm{Tr}(\phi^3 F') + g(v^\dagger\phi^2 f + f^\dagger\phi^2 v) + g(v^\dagger\phi F' v + v^\dagger F'\phi v) \\
&\quad + g(\alpha v^\dagger\phi f + \alpha f^\dagger\phi v) + gav^\dagger\phi v + g\alpha v^\dagger F' v + g(v^\dagger v v^\dagger f + v^\dagger v f^\dagger v) \\
&\quad + g(\alpha^2 v^\dagger f + \alpha^2 f^\dagger v) + 2ga\alpha v^\dagger v + ga\alpha^3 \\
g\mathrm{Tr}(\phi_0^2\overline{\psi}_0\psi_0 + \overline{\psi}_0\phi_0^2\psi_0) &= g\mathrm{Tr}(\phi^2\overline{\psi}\psi + \overline{\psi}\phi^2\psi) + g(v^\dagger\overline{\psi}\psi v + v^\dagger\psi\overline{\psi}v) - g(\omega^\dagger\phi^2\omega - \chi^\dagger\phi^2\chi) \\
&\quad + g(\chi^\dagger\psi\phi v + v^\dagger\phi\overline{\psi}\chi - v^\dagger\phi\psi\omega - \omega^\dagger\overline{\psi}\phi v) \\
&\quad + g(\alpha\chi^\dagger\psi v - \alpha v^\dagger\psi\omega + \alpha v^\dagger\overline{\psi}\chi - \alpha\omega^\dagger\overline{\psi}v) \\
&\quad + g(\overline{\beta}\omega^\dagger\phi v + \overline{\beta}v^\dagger\phi\chi - \beta v^\dagger\phi\omega - \beta\chi^\dagger\phi v) \\
&\quad + g(v^\dagger\omega\omega^\dagger v - v^\dagger\chi\chi^\dagger v + v^\dagger v\chi^\dagger\chi - v^\dagger v\omega^\dagger\omega) \\
&\quad + g(\alpha\overline{\beta}\omega^\dagger v + \alpha\overline{\beta}v^\dagger\chi + v^\dagger\omega\beta\alpha + \chi^\dagger v\alpha\beta) \\
&\quad + g(2v^\dagger v\overline{\beta}\beta + \alpha^2\chi^\dagger\chi - \alpha^2\omega^\dagger\omega) + g\alpha^2\overline{\beta}\beta \\
g\mathrm{Tr}(\phi_0\overline{\psi}_0\phi_0\psi_0) &= g\mathrm{Tr}(\phi\overline{\psi}\phi\psi) + g(v^\dagger\overline{\psi}\phi\chi + \chi^\dagger\phi\psi v - v^\dagger\psi\phi\omega - \omega^\dagger\phi\overline{\psi}v) \\
&\quad + g(\alpha\chi^\dagger\phi\chi - \alpha\omega^\dagger\phi\omega + \overline{\beta}v^\dagger\psi v - \beta v^\dagger\overline{\psi}v) + g(\chi^\dagger v\omega^\dagger v + v^\dagger\omega v^\dagger\chi) \\
&\quad + g(\alpha\overline{\beta}\omega^\dagger v - \alpha\beta v^\dagger\omega + \alpha\overline{\beta}v^\dagger\chi - \alpha\beta\chi^\dagger v) + g\alpha^2\overline{\beta}\beta.
\end{aligned}
$$

To the first order of g, the non-vanishing contributions are

$$2gv^\dagger\phi^2 v + g(v^\dagger\phi F' v + v^\dagger F'\phi v) - g(\omega^\dagger\phi^2\omega - \chi^\dagger\phi^2\chi).$$

The quadratic terms in the exponential are $-\frac{1}{2}(v^\dagger v)$ and $-\chi^\dagger\chi + \omega^\dagger\omega$. Then the effective interaction to the first order of g is

$$\frac{2g}{N}\mathrm{Tr}\phi^2 + \frac{4g}{N}\mathrm{Tr}(F'\phi).$$

The effective action becomes

$$S_{\text{eff}} = -\mathrm{Tr}F'^2 + \left(\frac{1}{4} + \frac{2g}{N}\right)\mathrm{Tr}\phi^2 + \frac{g}{2}\mathrm{Tr}\phi^4 + \frac{4g}{N}\mathrm{Tr}F'\phi + g\mathrm{Tr}\phi^3 F' + \cdots \quad (12)$$

where $\cdots$ are those involving $\overline{\psi}, \psi$, which does not change to this order. The new term $\mathrm{Tr}F'\phi$ could be absorbed by a redefinition of F'

$$\tilde{F}' = F' - \frac{2g}{N}\phi.$$

Then one could write the effective action as

$$S_{\text{eff}} = -\mathrm{Tr}\tilde{F}'^2 + \left(\frac{1}{4} + \frac{2g}{N}\right)\mathrm{Tr}\phi^2 + \frac{g}{2}\mathrm{Tr}\phi^4 + g\mathrm{Tr}\phi^3\tilde{F}' + \cdots \quad (13)$$

where the correction of the order g^2 is ignored. Is it possible to take it back into the original form by rescaling g, ϕ? It's impossible without a rescaling of $\tilde{F}'$. But then the term $-\mathrm{Tr}\tilde{F}'$ is modified. In this sense, the RG flow is even not well-defined.

Idea 5. *There are two ways through which the supersymmetry could help us in the calculation: 1. build up a model in which the supersymmetry is realized "locally"; 2. develop another RG method which is compatible with the global supersymmetry;*

We focus on susy because we believe it's a crucial ingredient to build a conformal matrix model.

notes about ✎ **BFFGLM21**: Tue, Mar 5

The paper consider a D(-1)/D7-brane system in the Type II B string theory with a magnetic flux along the world-volume of the D7-brane. There are $(N+M)$ D7-branes stacked along the first 8 directions $\mu = 1, \dots, 8$. A constant background field is introduced on the first N D7-branes in the following way

$$2\pi\alpha' F^{(0)} = \begin{pmatrix} F_1^{(0)} & 0 & 0 & 0 \\ 0 & F_2^{(0)} & 0 & 0 \\ 0 & 0 & F_3^{(0)} & 0 \\ 0 & 0 & 0 & F_4^{(0)} \end{pmatrix} \mathbb{1}_{N\times N}.$$

With $F_i^{(0)} = \begin{pmatrix} 0 & +f_1 \\ -f_1 & 0 \end{pmatrix}$. The initial gauge symmetry group $U(N+M)$ is broken to $U(N)\times U(M)$ by this constant background. It also breaks the Lorentz invariance in the 8-dimensional space in general.

The first N D7-branes are labeled by D7; while the M D7-branes are labeled by D7'.

Then a stack of k D(-1)-branes are added to study some non-perturbative features in the effective theory defined on the world-volume of the D7 and D7' branes. Now we study the physical states of the open strings stretching between two D(-1)-branes. Because they have Dirichlet/Dirchlet boundary conditions in all ten directions they do not carry any momentum and describe non-propagating degrees of freedom.

For the open string, the fermionic world-sheet fields $\psi^\mu(z), \tilde{\psi}^\mu(\bar{z})$ have two sectors: Neveu-Schwarz and Ramond. To be specific, in the coordinate $(\sigma_1, \sigma_2), 0 \le \sigma_1 \le \pi$ the possible boundary conditions are

$$\psi^\mu(0,\sigma_2) = \exp(2\pi i\nu)\tilde{\psi}^\mu(0,\sigma_2) \quad \psi^\mu(\pi,\sigma_2) = \exp(2\pi i\nu')\tilde{\psi}^\mu(\pi,\sigma_2). \tag{14}$$

where ν, ν' take the values 0 and $\frac{1}{2}$. However, ν' can be set to zero $\nu' = 0$ by a redefinition of $\tilde{\psi}^\mu \to \exp(-2\pi i\nu')\tilde{\psi}^\mu$. Therefore, there are two sectors: $\nu = 0$ for Ramond and $\nu = \frac{1}{2}$ For Neveu-Schwarz.

The physical states vertex operators are written down in the following ways. The space-time indices are $I = 1, 2, 3, 4$ for the 8-dimensions in four complex coordinates. The remaining 2-dimensions are just labeled by different letters.

The physical states in the NS sector under consideration are those excited by one fermionic oscillator. The vertex operators are written as

$$V_{B^I} = \frac{B^I}{\sqrt{2}}\overline{\Psi}_I(z)\mathrm{e}^{-\varphi(z)}, \quad V_{\overline{B}_I} = \frac{\overline{B}_I}{\sqrt{2}}\Psi^I(z)\mathrm{e}^{-\varphi(z)}$$
$$V_\xi = \xi\overline{\Psi}_5(z)\mathrm{e}^{-\varphi(z)}, \quad V_\chi = \chi\Psi^5(z)\mathrm{e}^{-\varphi(z)}.$$

Ψs are the fermionic string coordinates; the $\mathrm{e}^{-\varphi(z)}$ is the bosonization of the superconformal ghost operator $\delta(\gamma)$. This form of the vertex operator is called in the (-1)-picture. These vertex operators are conformal fields of dimension 1, and possess an even F-charge; thus they are preserved by the GSO projection.

Problem 3. *The vertex operators are objects that are defined for the world-sheet CFT. The "moduli" B, ξ, χ determine the strength of the vertex operators. The effective action is written down in terms of these "moduli".*

The structure of the effective action is obtained by matching the calculation of the scattering amplitudes using the vertex operators. However, the world-sheet point of view seems quite different from the effective action point of view. Is it worth to review the world-sheet calculation to get some understanding of the structure of the effective action? Maybe first describing the scattering process in string theory?

In the R sector, the only physical state is the vacuum. (What's the meaning of vacuum here?)

Wed, Mar 6

TODO 1. *Learn something about the D1/D5 CFT.*

Such a D5/D1 system is described in ✎ **Mal97** as the following: Consider IIB string theory compactified on M^4 (where $M^4 = T^4$or$K3$) to six spacetime dimensions. Introduce a D5-brane with four dimensions wrapping on M^4 giving a string in six dimensions. Consider a system with Q_5 D5-branes and Q_1 D1-branes, where the D1-branes are parallel to the string in six dimensions arising from the D5-branes. This system is described at low energies by a 1+1 dimensional (4,4) superconformal field theory.

Problem 4. *Why such a D-brane system has the (4,4) supersymmetry? Why the low energy field theory is conformal?*

TODO 2. *Read* ✎ **Wit95** *for the CFT description of p-branes.*

The symmetry of this system is discussed as follows ✎ **AGMOO99**. The D1-branes along one non-compact direction; The D5-branes extend in the same direction; and they are coincident in that direction. The unbroken Lorentz symmetry of this configuration is $SO(1,1) \times SO(4)$ $SO(1,1)$ corresponds to boosts along the string, and $SO(4)$ is the group of rotations in the four non-compact directions transverse to the string. This configuration also preserves eight supersymmetries, actually $\mathcal{N} = (4, 4)$ supersymmetry. We decompose the supercharges into left and right moving spinors of $SO(1, 1)$.

What's the conformal field theory on this system? ✎ **AGMOO99**. The conformal field theory is the IR fixed point of the field theory living on the D1-D5 branes. The symmetry of this theory before going into the IR fixed point is the $\mathcal{N} = (4, 4)$ supersymmetry. This amount of supersymmetry is equivalent to $\mathcal{N} = 2$ in $d = 4$. There is a vector multiplet and a hypermultiplet. In two dimensions both multiplets have the same propagating degrees of freedom, four scalars and four fermions, but they have different properties under the $SU(2)_L \times SU(2)_R$ global R-symmetry. Under this group, the scalars in the hypermultiplets are in the trivial representation, while the scalars in the vector multiplet are in the $(\mathbf{2}, \mathbf{2})$. On the fermions these global symmetries act chirally. The left moving vector multiplet fermions are in the $(\mathbf{1}, \mathbf{2})$ and the left moving hypermultiplet fermions are in the $(\mathbf{2}, \mathbf{1})$. The right moving fermions have similar properties with $SU(2)_L \leftrightarrow SU(2)_R$. The theory can have a Coulomb branch where the scalars in the vector multiplets have expectation values, and a Higgs branch where the scalars in the hypermultiplets have expectation values.

Question 5. *A better understanding of the above discussion.*

- *the supermultiplet structure*
- *the global R-symmetry structure*
- *the IR fixed point*
- *the Coulomb branch and the Higgs branch*

How to interpret this CFT? ✎ **AGMOO99**. The D1-branes can be viewed as instantons of the low energy SYM on the D5-brane. These instantons live on M^4 and are translationally invariant along the time and the x_5 direction, where x_5 is the non-compact direction along the D5-branes. This instanton configuration has moduli, which are the parameters that parameterize a continuous family of solutions. All these solutions have the same energy. Small fluctuation of this configuration (at low energy) are described by fluctuations of the instanton moduli. So the low energy dynamics is given by a 1+1 dimensional sigma model whose target space is the instanton moduli space.

Question 6. *Some questions related to the above paragraph:*

- *What is the instanton of the SYM on D5-brane?*
- *Why the D1-brane can be viewed as those instantons?*
- *What's the meaning of the fluctuation of the moduli?*
- *What's the meaning of a 2d sigma model with the moduli target space?*
- *Is this sigma model the 2d SCFT discussed above?*

TODO 3. *Read the papers that study the D1-D5 system:* % **Cos98**, % **HW97-a**, % **HW97-b**

More on D5-brane bound state: a bound state of two D5-branes wrapped on $S^1 \times T^4$ with coordinates $x^1, \cdots, x^5$. % **Cos98** Each D5-brane has winding number N_i along S^1, p_i along the x^2-direction and $\overline{p}_i$ along the x^4-direction. Thus, the worldvolume fields take values on the $U(N_1 p_1 \overline{p}_1 + N_2 p_2 \overline{p}_2)$ Lie algebra. To get a non-trivial D5-brane configuration we turn on the worldvolume gauge field such that the corresponding field strength is diagonal and self-dual on T^4. The non-vanishing components are taken to be (assume $\tan\theta_1 > \tan\theta_2$)

$$G^0_{23} = G^0_{45} = \frac{1}{2\pi\alpha'} \mathrm{diag}\left(\tan\theta_1, \cdots, \tan\theta_1, \tan\theta_2, \cdots, \tan\theta_2\right).$$

where

$$\frac{1}{2\pi\alpha'}\tan\theta_i = \frac{2\pi}{L_2 L_3}\frac{q_i}{p_i} = \frac{2\pi}{L_4 L_5}\frac{\overline{q}_i}{p_i}.$$

with $q_i, \overline{q}_i$ are integers, and $L_{\hat{\alpha}} = 2\pi R_{\hat{\alpha}}$ the length of each T^4 circles.

Question 7. *People are discussing the D5-brane configuration on a compact manifold, with certain background gauge field.*

- *Why the background gauge field is necessary for the discussion?*
- *What is the G^0 field? Why the number of components are related to the D5-brane winding number?*
- *What's special about the value of* $\tan\theta$*?*

The background gauge field breaks the gauge invariance

$$U(N_1 p_1 \overline{p}_1 + N_2 p_2 \overline{p}_2) \to U(N_1 p_1 \overline{p}_1) \otimes U(N_2 p_2 \overline{p}_2).$$

Because the branes are wrapped along the x^1, x^2, x^4 directions, the gauge invariance is further broken to $U(1)^{\cdots}$.