Master Thesis Notes

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From 17 April to 24 April

Contents

Thu, Apr 18

Gaussian with interaction Canonical generator basis (Apr. 10). $\mathfrak{a} = (k, l)$, $[k, l]$ or (k, k). Fix $k < l$. $\mathfrak{a}' = (k, N), [k, N], k = 1, \cdots, N-1$. Normalize such that $\mathrm{Tr} T^{\mathfrak{a}} T^{\mathfrak{b}} = \delta^{\mathfrak{ab}}$.

$$\sum_{\mathfrak{a}'} (T^{\mathfrak{a}'})_{iN} (T^{\mathfrak{a}'})_{Nj} = \delta_{ij}.$$

$\mathfrak{a}' = (i, N), [i, N]$ contributes to the sum. Gaussian from bosonic action (Apr. 10)

$$\frac{\alpha}{2} \left[\mathcal{A}_\mu (\mathfrak{A}^2 \eta^{\mu\nu} + \mathfrak{A}^\mu \mathfrak{A}^\nu - 2\mathfrak{F}^{\mu\nu}) \mathcal{A}_\nu \right].$$

~~The simplest case $\mathfrak{F} = 0$ because $[A, A] = 0$. $\mathfrak{A}^2$ is diagonal because A is diagonal.~~
Consider $A \oplus a$, a is a number, the (N, N) matrix element; A is a general $(N-1) \times (N-1)$ matrix. Calculate the adjoint

$$\mathfrak{A} = \sum_{\mathfrak{c}} (A \oplus a)_{\mathfrak{c}} \mathrm{Tr}(T^{\mathfrak{a}} [T^{\mathfrak{c}}, T^{\mathfrak{b}}]).$$

$\mathfrak{c} \in \{(N, N), (i, i), (k, l), [k, l]\}$. First consider $\mathfrak{c} = (N, N)$. This contributes to the matrix element $(\mathfrak{a} = (i, N), \mathfrak{b} = [i, N])$ or $(\mathfrak{a} = [i, N], \mathfrak{b} = (i, N))$.

$$\boxed{[T^{(N,N)}, T^{(i,N)}] = iT^{[i,N]}, \quad [T^{(N,N)}, T^{[i,N]}] = -iT^{(i,N)}}$$

$\mathfrak{c} = (N, N), (i, i)$ will generate the matrix element

$$(\mathfrak{A})_{(i,N),[i,N]} = -(\mathfrak{A})_{[i,N],(i,N)} = i(a - A_{ii}).$$

There are matrix elements between different $(i,N),(j,N)$ (also $(i,N),[j,N]$ and $[i,N],[j,N]$)

$$(\mathfrak{A})_{(i,N),(j,N)} = \sum_{\mathfrak{c}} A_{\mathfrak{c}} \mathrm{Tr}(T^{(i,N)}[T^{\mathfrak{c}}, T^{(j,N)}]) = \frac{i}{\sqrt{2}} A_{[i,j]}.$$

$$(\mathfrak{A})_{(i,N),[j,N]} = \sum_{\mathfrak{c}} A_{\mathfrak{c}} \mathrm{Tr}(T^{(i,N)}[T^{\mathfrak{c}}, T^{[j,N]}]) = \frac{i}{\sqrt{2}} A_{(i,j)}.$$

$$(\mathfrak{A})_{[i,N],[j,N]} = \sum_{\mathfrak{c}} A_{\mathfrak{c}} \mathrm{Tr}(T^{[i,N]}[T^{\mathfrak{c}}, T^{[j,N]}]) = \frac{i}{\sqrt{2}} A_{[i,j]}.$$

The matrix elements $((i,j),(i,N))$ vanish because $A_{[j,N]} = 0$.

The matrix elements of $\mathfrak{A}$:

	(i,N)	$[i,N]$	(j,N)	$[j,N]$
(i,N)	0	$i(a-A_{ii})$	$\frac{i}{\sqrt{2}}A_{[ij]}$	$\frac{i}{\sqrt{2}}A_{(ij)}$
$[i,N]$	$-i(a-A_{ii})$	0	$-\frac{i}{\sqrt{2}}A_{(ij)}$	$\frac{i}{\sqrt{2}}A_{[ij]}$
(j,N)	$-\frac{i}{\sqrt{2}}A_{[ij]}$	$\frac{i}{\sqrt{2}}A_{(ij)}$	0	$i(a-A_{jj})$
$[j,N]$	$-\frac{i}{\sqrt{2}}A_{(ij)}$	$-\frac{i}{\sqrt{2}}A_{[ij]}$	$-i(a-A_{jj})$	0

square

	(i,N)	$[i,N]$	(j,N)	$[j,N]$
(i,N)	$(a-A_{ii})^2$	0	?	?
$[i,N]$	0	$(a-A_{ii})^2$	?	?
(j,N)	?	?	$(a-A_{jj})^2$	0
$[j,N]$	?	?	0	$(a-A_{jj})^2$

The simplest approximation $a - A_{ii} \approx a - A_{jj} \equiv d$, $A_{(ij)} \approx 0, A_{[ij]} \approx 0$ (classical background). The Gaussian from bosonic action for $\alpha_{\mu\mathfrak{a}'}$

$$\frac{\alpha}{2} \sum_{\mathfrak{a}'} \left[\alpha_{\mu\mathfrak{a}'} (d^2 \eta^{\mu\nu} + d^{\mu} d^{\nu}) \alpha_{\nu\mathfrak{a}'} \right] \tag{1}$$

We ignore d depending on $\mathfrak{a}'$ in the approximation.

Consider the interaction

$$\frac{1}{4} \mathrm{Tr}([T^{\mathfrak{a}'}, T^{\mathfrak{a}}][T^{\mathfrak{b}'}, T^{\mathfrak{b}}]) = \sum_{ijkl} T^{[\mathfrak{a}'}_{ij} T^{\mathfrak{a}]}_{jk} T^{[\mathfrak{b}'}_{kl} T^{\mathfrak{b}]}_{li}.$$

Because one of the index of $T^{\mathfrak{a}'}$ must be N, while no index of $T^{\mathfrak{a}}$ is N, only the following terms non-vanishing

$$-\frac{1}{4} \sum_{ijk} \left(T^{\mathfrak{a}'}_{iN} T^{\mathfrak{b}'}_{Nj} T^{\mathfrak{a}}_{ki} T^{\mathfrak{b}}_{jk} + T^{\mathfrak{a}'}_{Ni} T^{\mathfrak{a}}_{ik} T^{\mathfrak{b}}_{kj} T^{\mathfrak{b}'}_{jN} \right).$$

Contraction between $\mathfrak{a}', \mathfrak{b}'$ gives

$$\sum_{\mathfrak{a}'\mathfrak{b}'} \delta_{\mathfrak{a}',\mathfrak{b}'}(\cdots) = -\frac{1}{2}\mathrm{Tr}(T^{\mathfrak{a}}T^{\mathfrak{b}}).$$

This result is applied to

$$-\frac{\alpha}{2}\mathrm{Tr}\left([\alpha_\mu, \mathcal{A}_\nu][\alpha^\mu, \mathcal{A}^\nu] + [\alpha_\mu, \mathcal{A}_\nu][\mathcal{A}^\mu, \alpha^\nu]\right)$$

$$= -\frac{\alpha}{2}\mathrm{Tr}([T^{\mathfrak{a}'}, T^{\mathfrak{a}}][T^{\mathfrak{b}'}, T^{\mathfrak{b}}])\alpha_{\mu\mathfrak{a}'}\alpha_{\nu\mathfrak{b}'}(\eta^{\mu\nu}\mathcal{A}_{\rho\mathfrak{a}}\mathcal{A}^\rho_{\mathfrak{b}} + \mathcal{A}^\nu_{\mathfrak{a}}\mathcal{A}^\mu_{\mathfrak{b}}).$$

Contract α

$$\alpha\mathrm{Tr}(T^{\mathfrak{a}}T^{\mathfrak{b}})(\Delta^{-1})_{\mu\nu}(\eta^{\mu\nu}\mathcal{A}_{\rho\mathfrak{a}}\mathcal{A}^\rho_{\mathfrak{b}} + \mathcal{A}^\nu_{\mathfrak{a}}\mathcal{A}^\mu_{\mathfrak{b}})$$
$$= \alpha(\Delta^{-1})_{\mu\nu}\mathrm{Tr}\left(\eta^{\mu\nu}\mathcal{A}_\rho\mathcal{A}^\rho + \mathcal{A}^\mu\mathcal{A}^\nu\right).$$

Δ^{-1} is the inverse of $\eta^{\mu\nu}d^2 + d^\mu d^\nu$.

$$-\frac{\alpha}{2}\mathrm{Tr}\left([\alpha_\rho, \mathcal{A}_\mu][\alpha_\sigma, \mathcal{A}_\nu]\right) \to \alpha(\Delta^{-1})_{\rho\sigma}\mathrm{Tr}\left(\mathcal{A}_\mu\mathcal{A}_\nu\right) \tag{2}$$

Note. there is no $\mathfrak{A}_{\mathfrak{a}\mathfrak{b}}$ dependence. What is d?

fermionic contribution the action

$$-\frac{\alpha}{2}\mathrm{Tr}(\overline{\psi}\Gamma^\mu[A_\mu, \psi]) = -\frac{\alpha}{2}\psi^\alpha_{\mathfrak{a}}(C\Gamma^\mu)_{\alpha\beta}(\mathfrak{A}_\mu)^{\mathfrak{a}\mathfrak{b}}\psi^\beta_{\mathfrak{b}} \tag{3}$$

$$\int[\mathrm{d}\psi]\mathrm{e}^{-\frac{\alpha}{2}\mathrm{Tr}(\overline{\psi}\Gamma^\mu[A_\mu,\psi])} = \mathrm{Pf}\left(-\frac{\alpha}{2}(C\Gamma^\mu)_{\alpha\beta}(\mathfrak{A}_\mu)^{\mathfrak{a}\mathfrak{b}}\right).$$

The matrix elements are labeled by $(\alpha, \mathfrak{a}), (\beta, \mathfrak{b})$.

The RG setting, background $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2$, $\mathfrak{A}_2$ in the directions $\mathfrak{a}', \mathfrak{b}'$.

$$\mathrm{Pf}\left(-\frac{\alpha}{2}(C\Gamma^\mu)_{\alpha\beta}(\mathfrak{A}_\mu)^{\mathfrak{a}\mathfrak{b}}\right) \approx \mathrm{Pf}\left(-\frac{\alpha}{2}(C\Gamma^\mu)_{\alpha\beta}(\mathfrak{A}_{1\mu})^{\mathfrak{a}\mathfrak{b}}\right)\mathrm{Pf}\left(-\frac{\alpha}{2}(C\Gamma^\mu)_{\alpha\beta}(\mathfrak{A}_{2\mu})^{\mathfrak{a}'\mathfrak{b}'}\right)$$

Factorization only works for background, not for fluctuation.

$$\tilde{\mathfrak{A}} = \sum_{\mathfrak{c}} \alpha_{\mathfrak{c}}\mathrm{Tr}(T^{\mathfrak{a}}[T^{\mathfrak{c}}, T^{\mathfrak{b}}])$$

No simple relation between $\mathrm{Pf}(A \otimes B)$ and $\mathrm{Pf}(A)$, $\mathrm{Pf}(B)$.

Write

$$M_{(\alpha,\mathfrak{a}),(\beta,\mathfrak{b})} \equiv (C\Gamma^\mu)_{\alpha,\beta} \otimes (\mathfrak{A}_\mu)_{\mathfrak{a},\mathfrak{b}}.$$

Use

$$\mathrm{Pf}\left(-\frac{\alpha}{2}M\right) = \pm\exp\left[\frac{1}{4}\mathrm{Tr}\ \log\left(-\frac{\alpha^2}{4}M^2\right)\right].$$

$$M = C\Gamma^\mu \otimes (\mathfrak{A}_{1\mu} \oplus \mathfrak{A}_{2\mu}).$$

$$M^2 = (C\Gamma^\mu C\Gamma^\nu) \otimes (\mathfrak{A}_{1\mu}\mathfrak{A}_{1\nu} \oplus \mathfrak{A}_{2\mu}\mathfrak{A}_{2\nu}).$$

$\mathfrak{A}_2$ for the RG directions. Factor out

$$\exp\left[\frac{1}{4}\mathrm{Tr}\ \log\left(-\frac{\alpha^2}{4}(C\Gamma^\mu\otimes\mathfrak{A}_{2\mu})^2\right)\right].$$

Background $(\mathfrak{A}_2)^2_{\mathfrak{a}'\mathfrak{b}'} = d^2\delta_{\mathfrak{a}'\mathfrak{b}'}$ (the approximation).

$$(C\Gamma^\mu\otimes\mathfrak{A}_{2\mu})^2 = \left(\sum_{\mu=2}^{9}(d_\mu)^2\right)\mathbb{1}\otimes\mathbb{1} + (\cdots?).$$

$\log(-\cdots)$? For simplicity, assume $d_0 = 0$,

$$(C\Gamma^\mu\otimes\mathfrak{A}_{2\mu})^2 = \left(\sum_{\mu=1}^{9}(d_\mu)^2\right)\mathbb{1}\otimes\mathbb{1} \equiv d^2\mathbb{1}.$$

$\mathrm{Tr} = 16\cdot 2(N-1)$, for $\mathfrak{a}'\in\{(i,N),[i,N]\}$.

$$\exp\left[\frac{1}{4}\mathrm{Tr}\ \log\left(-\frac{\alpha^2}{4}(C\Gamma^\mu\otimes\mathfrak{A}_{2\mu})^2\right)\right] = \left(-\frac{\alpha^2}{4}d^2\right)^{8(N-1)}.$$

Question 1. *where is the scaling?*

Clarify M

$$M_{(\alpha,\mathfrak{a}),(\beta,\mathfrak{b})} \equiv (C\Gamma^\mu)_{\alpha,\beta}\otimes(\mathfrak{A}_\mu)_{\mathfrak{a},\mathfrak{b}}.$$

The inverse of M: M^{-1}? Diagonal background

$$\sum_{\mathfrak{c}}(\mathfrak{A}_\mu)_{\mathfrak{a}\mathfrak{c}}(\mathfrak{A}_\nu)_{\mathfrak{c}\mathfrak{b}} = \delta_{\mathfrak{a},\mathfrak{b}}d_{\mathfrak{a}\mu}d_{\mathfrak{a}\nu}.$$

Spinor part is not easy, but easy if $d_0 = 0$.

$$\{(C\Gamma^i),(C\Gamma^j)\} = 2\delta^{ij}\mathbb{1},\quad C\Gamma^0 = -\mathbb{1}$$
$$\{(C\Gamma^0),(C\Gamma^i)\} = -2(C\Gamma^i).$$

Define $\overline{C\Gamma}^\mu$

$$\overline{C\Gamma}^0 = -C\Gamma^0,\quad \overline{C\Gamma}^i = C\Gamma^i.$$

Reminiscent of Pauli 4-vector? If $d_0 = 0$, $M^{-1}\propto M$. $M^{-1}\propto\overline{C\Gamma}^\mu\otimes\mathfrak{A}_\mu$?

Fri, Apr 19

summarize the ingredients General assumptions: diagonal background A_μ; imagine $(A_\mu)_{ii}, i = 1,\cdots,N-1$ are small while $(A_\mu)_{NN}$ is large:

$$(A_\mu)_{NN} - (A_\mu)_{ii} \approx d_\mu,\quad (\mathfrak{A}_\mu)_{\mathfrak{a}'\mathfrak{b}'} \equiv (\mathfrak{A}_\mu)_{(i,N)[j,N]} = -(\mathfrak{A}_\mu)_{[j,N](i,N)} = id_\mu\delta_{ij}.$$

The idea is just to make the Gaussian term simple.

The bosonic part of IKKT gives

$$\frac{\alpha}{2}\sum_{\mathfrak{a}',\mathfrak{b}'}\alpha_{\mu\mathfrak{a}'}\left[(\mathfrak{A}^\rho\mathfrak{A}_\rho)_{\mathfrak{a}'\mathfrak{b}'}\eta^{\mu\nu} + (\mathfrak{A}^\mu\mathfrak{A}^\nu)_{\mathfrak{a}'\mathfrak{b}'}\right]\alpha_{\nu\mathfrak{b}'} \tag{4}$$

Define the quadratic matrix

$$(\Delta_{\mathrm{B}})_{\mathfrak{a}'\mathfrak{b}';\mu\nu} \equiv \frac{\alpha}{2}(d^2\eta_{\mu\nu} + d_\mu d_\nu)\delta_{\mathfrak{a}'\mathfrak{b}'} \tag{5}$$

A simplification: gauge fixing

$$\sum_{\mathfrak{b}'}(\mathfrak{A}^\mu)_{\mathfrak{a}'\mathfrak{b}'}\alpha_{\mu\mathfrak{b}'} = 0, \quad \delta_t\alpha_{\mu\mathfrak{a}'} = i\sum_{\mathfrak{b}'}(\mathfrak{A}_\mu)_{\mathfrak{a}'\mathfrak{b}'}t_{\mathfrak{b}'}.$$

In gauge fixing action, the bosonic part

$$\frac{\alpha}{2}\sum_{\mathfrak{a}',\mathfrak{b}'}\alpha_{\mu\mathfrak{a}'}\left[(\mathfrak{A}^\rho\mathfrak{A}_\rho)_{\mathfrak{a}'\mathfrak{b}'}\eta^{\mu\nu}\right]\alpha_{\nu\mathfrak{b}'} \tag{6}$$

$$(\Delta_{\mathrm{B}})_{\mathfrak{a}'\mathfrak{b}';\mu\nu} \equiv \frac{\alpha}{2}d^2\eta_{\mu\nu}\delta_{\mathfrak{a}'\mathfrak{b}'} \tag{7}$$

Then it's easy to inverse

$$(\Delta_{\mathrm{B}}^{-1})^{\mu\nu}_{\mathfrak{a}'\mathfrak{b}'} = \frac{2}{\alpha d^2}\eta^{\mu\nu}\delta_{\mathfrak{a}'\mathfrak{b}'} \tag{8}$$

The fermionic part of IKKT gives

$$-\frac{\alpha}{2}\sum_{\mathfrak{a}'\mathfrak{b}',\alpha\beta}\psi^\alpha_{\mathfrak{a}'}(C\Gamma^\mu)_{\alpha\beta}(\mathfrak{A}_\mu)_{\mathfrak{a}'\mathfrak{b}'}\psi^\beta_{\mathfrak{b}'} \tag{9}$$

$$(\Delta_{\mathrm{F}})_{\mathfrak{a}'\mathfrak{b}';\alpha\beta} \equiv -\frac{\alpha}{2}(C\Gamma^\mu)_{\alpha\beta}d_\mu J_{\mathfrak{a}'\mathfrak{b}'} \tag{10}$$

J forms a block diagonal form, each block reads

$$J_{(i,N),[i,N]} = -J_{[i,N],(i,N)} = i, \quad J_i = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad J = J_1 \oplus \cdots \oplus J_{N-1}.$$

The inverse

$$(\Delta_{\mathrm{F}}^{-1})^{\alpha\beta}_{\mathfrak{a}'\mathfrak{b}'} = -\frac{2}{\alpha d^2}(\overline{C\Gamma}^\mu)^{\alpha\beta}d_\mu J_{\mathfrak{a}'\mathfrak{b}'} \tag{11}$$

$$\overline{C\Gamma}^0 = -C\Gamma^0, \quad \overline{C\Gamma}^i = C\Gamma^i, \quad i = 1, \cdots, 9.$$

$$(C\Gamma^\mu)(\overline{C\Gamma}^\nu) = \eta^{\mu\nu}\mathbb{1}.$$

Contractions

$$\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{ij}(T^{\mathfrak{b}'})_{kl}\delta_{\mathfrak{a}'\mathfrak{b}'}.$$

$$\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{Ni}(T^{\mathfrak{b}'})_{Nj}\delta_{\mathfrak{a}'\mathfrak{b}'} = 0, \quad \sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{iN}(T^{\mathfrak{b}'})_{jN}\delta_{\mathfrak{a}'\mathfrak{b}'} = 0.$$

$$\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{iN}(T^{\mathfrak{b}'})_{Nj}\delta_{\mathfrak{a}'\mathfrak{b}'} = \sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{Ni}(T^{\mathfrak{b}'})_{jN}\delta_{\mathfrak{a}'\mathfrak{b}'} = 2\delta_{ij}.$$

$$\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{ij}(T^{\mathfrak{b}'})_{kl}J_{\mathfrak{a}'\mathfrak{b}'}.$$

$$\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{Ni}(T^{\mathfrak{b}'})_{Nj}J_{\mathfrak{a}'\mathfrak{b}'}=0,\quad \sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{iN}(T^{\mathfrak{b}'})_{jN}J_{\mathfrak{a}'\mathfrak{b}'}=0.$$

$$\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{iN}(T^{\mathfrak{b}'})_{Nj}J_{\mathfrak{a}'\mathfrak{b}'}=-\sum_{\mathfrak{a}'\mathfrak{b}'}(T^{\mathfrak{a}'})_{Ni}(T^{\mathfrak{b}'})_{jN}J_{\mathfrak{a}'\mathfrak{b}'}=2\delta_{ij}.$$

fermionic interaction

$$-\frac{\alpha}{2}\psi^{\alpha}_{\mathfrak{a}'}(C\Gamma^{\mu})_{\alpha\beta}\psi^{\beta}_{\mathfrak{b}'}\mathcal{A}_{\mu\mathfrak{a}}\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{a}},T^{\mathfrak{b}'}])$$

square

$$\frac{\alpha^2}{4}\psi^{\alpha}_{\mathfrak{a}'}(C\Gamma^{\mu})_{\alpha\beta}\psi^{\beta}_{\mathfrak{b}'}\psi^{\gamma}_{\mathfrak{c}'}(C\Gamma^{\nu})_{\gamma\delta}\psi^{\delta}_{\mathfrak{d}'}\mathcal{A}_{\mu\mathfrak{a}}\mathcal{A}_{\nu\mathfrak{b}}\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{a}},T^{\mathfrak{b}'}])\mathrm{Tr}(T^{\mathfrak{c}'}[T^{\mathfrak{b}},T^{\mathfrak{d}'}])$$

Contraction 1. $\mathfrak{a}'-\mathfrak{d}'$ and $\mathfrak{b}'-\mathfrak{c}'$

$$\frac{\alpha^2}{4}(\Delta_{\mathrm{F}}^{-1})^{\alpha\delta}_{\mathfrak{a}'\mathfrak{d}'}(\Delta_{\mathrm{F}}^{-1})^{\beta\gamma}_{\mathfrak{b}'\mathfrak{c}'}(C\Gamma^{\mu})_{\alpha\beta}(C\Gamma^{\nu})_{\gamma\delta}\mathcal{A}_{\mu\mathfrak{a}}\mathcal{A}_{\nu\mathfrak{b}}\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{a}},T^{\mathfrak{b}'}])\mathrm{Tr}(T^{\mathfrak{c}'}[T^{\mathfrak{b}},T^{\mathfrak{d}'}]).$$

$$=-\frac{4\times 16}{d^4}\mathrm{Tr}(T^{\mathfrak{a}}T^{\mathfrak{b}})d^{\mu}d^{\nu}\mathcal{A}_{\mu\mathfrak{a}}\mathcal{A}_{\nu\mathfrak{b}}=-\frac{64}{d^4}d^{\mu}d^{\nu}\mathrm{Tr}(\mathcal{A}_{\mu}\mathcal{A}_{\nu}).$$

Contraction 2 $\mathfrak{a}'-\mathfrak{c}'$ and $\mathfrak{b}'-\mathfrak{d}'$ gives the same.

In the contraction 1.

$$\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{a}},T^{\mathfrak{b}'}])\mathrm{Tr}(T^{\mathfrak{c}'}[T^{\mathfrak{b}},T^{\mathfrak{d}'}])\to -4\mathrm{Tr}(T^{\mathfrak{a}}T^{\mathfrak{b}})$$

Cubic interaction

Sat, Apr 20

$$\mathrm{Tr}[A_{\mu},A_{\nu}][A^{\mu},A^{\nu}]\to 4\mathrm{Tr}[A_{\mu},\mathscr{A}_{\nu}][\mathscr{A}^{\mu},\mathscr{A}^{\nu}].$$

$$4\sum_{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}}(\mathfrak{A}_{\mu})_{\mathfrak{a}\mathfrak{b}}\mathscr{A}_{\nu\mathfrak{b}}\mathscr{A}^{\mu}_{\mathfrak{c}}\mathscr{A}^{\nu}_{\mathfrak{d}}\mathrm{Tr}(T^{\mathfrak{a}}[T^{\mathfrak{c}},T^{\mathfrak{d}}]).$$

$$4\sum_{\mathfrak{a}',\mathfrak{b}',\mathfrak{c}',\mathfrak{d}'}(\mathfrak{A}_{\mu})_{\mathfrak{a}'\mathfrak{b}'}\alpha_{\nu\mathfrak{b}'}\alpha^{\mu}_{\mathfrak{c}'}\alpha^{\nu}_{\mathfrak{d}'}\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{c}'},T^{\mathfrak{d}'}]).$$

Fermionic action

$$\mathrm{Tr}(\overline{\psi}\Gamma^{\mu}[A_{\mu},\psi])=\mathrm{Tr}(\psi^{\alpha}(C\Gamma^{\mu})_{\alpha\beta}[A_{\mu},\psi^{\beta}]).$$

fermionic background

$$\psi+\varphi.$$

with the direct sum bosonic background

$$\sum_{\mathfrak{a}'\mathfrak{b}'}\varphi^{\alpha}_{\mathfrak{a}'}(C\Gamma^{\mu})_{\alpha\beta}(\mathfrak{A}_{\mu})_{\mathfrak{a}'\mathfrak{b}'}\varphi^{\beta}_{\mathfrak{b}'}+\sum_{\mathfrak{a}'\mathfrak{b}'\mathfrak{c}'}\varphi^{\alpha}_{\mathfrak{a}'}(C\Gamma^{\mu})_{\alpha\beta}\alpha_{\mu\mathfrak{b}'}\varphi^{\beta}_{\mathfrak{c}'}\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{b}'},T^{\mathfrak{c}'}]).$$

Also consider

$$\sum_{\mathfrak{a}\mathfrak{b}'\mathfrak{c}}\psi^{\alpha}_{\mathfrak{a}}(C\Gamma^{\mu})_{\alpha\beta}\alpha_{\mu\mathfrak{b}'}\psi^{\beta}_{\mathfrak{c}}\mathrm{Tr}(T^{\mathfrak{a}}[T^{\mathfrak{b}'},T^{\mathfrak{c}}]).$$

Show that it's vanish. Also consider

$$\mathrm{Tr}\psi^{\alpha}(C\Gamma^{\mu})_{\alpha\beta}[\alpha_{\mu},\varphi^{\beta}]+\mathrm{Tr}\varphi^{\alpha}(C\Gamma^{\mu})_{\alpha\beta}[\alpha_{\mu},\psi^{\beta}]=-2\mathrm{Tr}\varphi^{\alpha}(C\Gamma^{\mu})_{\alpha\beta}[\psi^{\beta},\alpha_{\mu}].$$

Similarly, define the adjoint matrix

$$\Psi_{\mathfrak{ab}}\equiv\sum_{\mathfrak{c}}\psi_{\mathfrak{c}}\mathrm{Tr}(T^{\mathfrak{a}}[T^{\mathfrak{c}},T^{\mathfrak{b}}]).$$

The product and trace of the $C\Gamma^{\mu}$ matrices Mon, Apr 22

$$\mathrm{Tr}(C\Gamma^{\mu}\overline{C\Gamma}^{\nu})=\eta^{\mu\nu}\mathrm{Tr}\mathbb{1}.$$

This is checked by considering three cases separately: $\mu=i,\nu=j$; $\mu=0,\nu=i$ or $\mu=i,\nu=0$; $\mu=0,\nu=0$. Also use $\mathrm{Tr}(C\Gamma^{i})=0$.

Use $\{C\Gamma^{i},C\Gamma^{j}\}=2\delta^{ij}\mathbb{1}$, we can prove that (cyclic property of trace)

$$\mathrm{Tr}(C\Gamma^{1}C\Gamma^{2})=0,\quad \mathrm{Tr}(C\Gamma^{1}C\Gamma^{2}C\Gamma^{3}C\Gamma^{4})=0.$$

Is it the case? $\mathrm{Tr}(C\Gamma^{1}C\Gamma^{2}C\Gamma^{3})=0$.

a contraction Consider the following contraction

$$\left\langle\varphi^{\alpha}_{\mathfrak{a}}(C\Gamma^{\mu})_{\alpha\beta}\mathcal{A}_{\mu\mathfrak{ab}}\varphi^{\beta}_{\mathfrak{b}}\varphi^{\gamma}_{\mathfrak{c}}(C\Gamma^{\mu})_{\gamma\delta}\mathcal{A}_{\mu\mathfrak{cd}}\varphi^{\delta}_{\mathfrak{d}}\right\rangle.$$

There are three possible contractions

$$\begin{aligned}&(\Delta^{-1})^{\alpha\beta}_{\mathfrak{ab}}(C\Gamma^{\mu})_{\alpha\beta}\mathcal{A}_{\mu\mathfrak{ab}}(\Delta^{-1})^{\gamma\delta}_{\mathfrak{cd}}(C\Gamma^{\nu})_{\gamma\delta}\mathcal{A}_{\nu\mathfrak{cd}}\\+&(\Delta^{-1})^{\alpha\delta}_{\mathfrak{ad}}(C\Gamma^{\mu})_{\alpha\beta}\mathcal{A}_{\mu\mathfrak{ab}}(\Delta^{-1})^{\beta\gamma}_{\mathfrak{bc}}(C\Gamma^{\nu})_{\gamma\delta}\mathcal{A}_{\nu\mathfrak{cd}}\\-&(\Delta^{-1})^{\alpha\gamma}_{\mathfrak{ac}}(C\Gamma^{\mu})_{\alpha\beta}\mathcal{A}_{\mu\mathfrak{ab}}(\Delta^{-1})^{\beta\delta}_{\mathfrak{bd}}(C\Gamma^{\nu})_{\gamma\delta}\mathcal{A}_{\nu\mathfrak{cd}}\end{aligned}$$

The sign is obtained by comparing the "reverse order of Δ^{-1}": $\mathfrak{dcba}$ for the first line to the order in $\langle\cdots\rangle$: $\mathfrak{abcd}$. First deal with $\mathfrak{a}$-indices: $(\Delta^{-1})_{\mathfrak{ab}}$ include $J_{\mathfrak{ab}}$. The first term gives $\mathrm{Tr}(\mathcal{A}_{\mu}J)\mathrm{Tr}(\mathcal{A}_{\nu}J)$; the second and third terms give $-2\mathrm{Tr}(\mathcal{A}_{\mu}\mathcal{A}_{\nu})$. The spinor part: the first term

$$\mathrm{Tr}(C\Gamma^{\mu}\overline{C\Gamma}^{\rho})\mathrm{Tr}(C\Gamma^{\nu}\overline{C\Gamma}^{\sigma}).$$

The second term and the third term are the same

$$\mathrm{Tr}(C\Gamma^{\mu}\overline{C\Gamma}^{\rho}C\Gamma^{\nu}\overline{C\Gamma}^{\sigma}).$$

(Note that $C\Gamma^{\mu}$ is symmetric) The contraction reads then

$$\mathrm{Tr}(C\Gamma^{\mu}\overline{C\Gamma}^{\rho})\mathrm{Tr}(C\Gamma^{\nu}\overline{C\Gamma}^{\sigma})\mathrm{Tr}(\mathcal{A}_{\mu}J)\mathrm{Tr}(\mathcal{A}_{\nu}J)a_{\rho}a_{\sigma}.$$

$$-2\mathrm{Tr}(C\Gamma^{\mu}\overline{C\Gamma}^{\rho}C\Gamma^{\nu}\overline{C\Gamma}^{\sigma})\mathrm{Tr}(\mathcal{A}_{\mu}\mathcal{A}_{\nu})a_{\rho}a_{\sigma}.$$

Don't forget to multiply $\frac{g^4}{a^4}$ from propagator and $\frac{1}{8g^4}$ from expansion of the interaction.

It's possible to obtain the following terms

$$32\mathrm{Tr}(\mathcal{A}^2)a^2, \quad -64\mathrm{Tr}(\mathcal{A}_\mu\mathcal{A}_\nu)a^\mu a^\nu, \quad (16\mathrm{Tr}(\mathcal{A}_\mu)a^\mu)^2.$$

Multiply the $\frac{1}{8a^4}$

$$\frac{4}{a^2}\mathrm{Tr}(\mathcal{A}^2), \quad -\frac{8}{a^4}\mathrm{Tr}(\mathcal{A}_\mu\mathcal{A}_\nu)a^\mu a^\nu, \quad \frac{32}{a^4}(\mathrm{Tr}(\mathcal{A}_\mu J)a^\mu)^2.$$

tech remark. Because $C\Gamma^0 = -\mathbb{1}$, $\mathrm{Tr}(C\Gamma^0 C\Gamma^0) = \mathrm{Tr}\mathbb{1}$. This seems to lead the wrong sign in the time direction (compare to $\mathrm{Tr}(C\Gamma^i C\Gamma^j) = \delta^{ij}\mathbb{1}$). However, an extra minus is provided by $C\Gamma^0 C\Gamma^i = C\Gamma^i C\Gamma^0$ (compare to $C\Gamma^i C\Gamma^j = -C\Gamma^j C\Gamma^i$, $i \neq j$).

Simpler contractions

$$-\frac{1}{2g^2}\left\langle \alpha_\mu(2a_\rho\mathcal{A}^\rho J\eta^{\mu\nu} + \mathcal{A}^2\eta^{\mu\nu} - 2\mathcal{F}^{\mu\nu})\alpha_\nu \right\rangle = -\frac{5}{a^2}\mathrm{Tr}(2a_\rho\mathcal{A}^\rho J + \mathcal{A}^2).$$

$$\frac{1}{2g^2}\left\langle \varphi^\alpha(C\Gamma^\mu)_{\alpha\beta}\mathcal{A}_\mu\varphi^\beta \right\rangle = \frac{8}{a^2}a^\rho\mathrm{Tr}(A_\rho J).$$

use $\mathrm{Tr}(\overline{C\Gamma}^\mu C\Gamma^\nu) = 16\eta^{\mu\nu}$.

$$\frac{1}{g^2}\left\langle b(2a_\rho\mathcal{A}^\rho J + \mathcal{A}^2) \right\rangle = \frac{1}{a^2}\mathrm{Tr}(2a_\rho\mathcal{A}^\rho J + \mathcal{A}^2).$$

Some cancellations... Also the $\mathrm{Tr}(\mathcal{A}_\mu\mathcal{A}_\nu)a^\mu a^\nu$ is canceled from the bosonic and ghost quadratic term expansion.

Notations Tue, Apr 23

$$(\Sigma_{(\alpha)})^{\mu\nu}_{\mathfrak{a}\mathfrak{b}} \equiv \left[2a_\rho(\mathcal{A}^\rho J)_{\mathfrak{a}\mathfrak{b}} + \mathcal{A}^2_{\mathfrak{a}\mathfrak{b}}\right]\eta^{\mu\nu} - 2\mathcal{F}^{\mu\nu}_{\mathfrak{a}\mathfrak{b}}.$$

note. this term couples to the fluctuations $\alpha_{mu\mathfrak{a}}$ through the form $\alpha_{\mu\mathfrak{a}}(\Sigma_{(\alpha)})^{\mu\nu}_{\mathfrak{a}\mathfrak{b}}\alpha_{\nu\mathfrak{b}}$. Also

$$(\Sigma_{(\varphi)})_{\alpha\mathfrak{a};\beta\mathfrak{b}} \equiv (C\Gamma^\mu)_{\alpha\beta}\mathcal{A}_{\mu\mathfrak{a}\mathfrak{b}}.$$

$$(\Sigma_{(\mathrm{g})})_{\mathfrak{a}\mathfrak{b}} \equiv 2a_\mu(\mathcal{A}^\mu J)_{\mathfrak{a}\mathfrak{b}} + \mathcal{A}^2_{\mathfrak{a}\mathfrak{b}}.$$

Maybe simply use number to denote the index: Σ_{12} for example.

The action to expand

$$\begin{aligned} -S = &-\frac{1}{2}\alpha(\Delta_{\mathrm{B}})\alpha + \frac{1}{2}\varphi(\Delta_{\mathrm{F}})\varphi - b(\Delta_{\mathrm{G}})c \\ &-\frac{1}{2g^2}\alpha(\Sigma_{(\alpha)})\alpha + \frac{1}{2g^2}\varphi(\Sigma_{(\varphi)})\varphi - \frac{1}{g^2}b(\Sigma_{(\mathrm{g})})c \\ &-\frac{1}{g^2}\varphi(C\Gamma^\mu)\Psi\alpha_\mu + \frac{1}{4g^2}\mathrm{Tr}([\alpha_\mu,\alpha_\nu][\alpha^\mu,\alpha^\nu]) \end{aligned} \tag{12}$$

First consider the contribution getting from the first two lines: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$, x-term

$$-\frac{1}{2g^2}\alpha_1\Sigma_{(\alpha)1,2}\alpha_2 + \frac{1}{2g^2}\varphi_1\Sigma_{(\varphi)1,2}\varphi_2 - \frac{1}{g^2}b_1\Sigma_{(\mathrm{g})1,2}c_2.$$

Contract by the propagators:

$$-\frac{1}{2g^2}\Sigma_{(\alpha)1,2}(\Delta_{\mathrm{B}}^{-1})_{2,1}+\frac{1}{2g^2}\Sigma_{(\varphi)1,2}(\Delta_{\mathrm{F}}^{-1})_{2,1}-\frac{1}{g^2}\Sigma_{(\mathrm{g})1,2}(-\Delta_{\mathrm{g}}^{-1})_{2,1}.$$

$\frac{1}{2}x^2$-term

$$\begin{aligned}&\frac{1}{8g^4}\alpha_1\Sigma_{(\alpha)1,2}\alpha_2\alpha_3\Sigma_{(\alpha)3,4}\alpha_4+\frac{1}{8g^4}\varphi_1\Sigma_{(\varphi)1,2}\varphi_2\varphi_3\Sigma_{(\varphi)3,4}\varphi_4\\&+\frac{1}{2g^4}b_1\Sigma_{(\mathrm{g})1,2}c_2b_3\Sigma_{(\mathrm{g})3,4}c_4-\frac{1}{4g^4}\alpha_1\Sigma_{(\alpha)1,2}\alpha_2\varphi_3\Sigma_{(\varphi)3,4}\varphi_4\\&-\frac{1}{2g^4}\varphi_1\Sigma_{(\varphi)1,2}\varphi_2b_3\Sigma_{(\mathrm{g})3,4}c_4+\frac{1}{2g^4}\alpha_1\Sigma_{(\alpha)1,2}\alpha_2b_3\Sigma_{(\mathrm{g})3,4}c_4\end{aligned}$$

Consider only “connected” contractions

$$\begin{aligned}&\frac{1}{8g^4}\left[\Sigma_{(\alpha)1,2}(\Delta_{\mathrm{B}}^{-1})_{2,3}\Sigma_{(\alpha)3,4}(\Delta_{\mathrm{B}}^{-1})_{4,1}+\Sigma_{(\alpha)1,2}(\Delta_{\mathrm{B}}^{-1})_{1,3}\Sigma_{(\alpha)3,4}(\Delta_{\mathrm{B}}^{-1})_{4,2}\right]\\&\frac{1}{8g^4}\left[\Sigma_{(\varphi)1,2}(\Delta_{\mathrm{F}}^{-1})_{3,2}\Sigma_{(\varphi)3,4}(\Delta_{\mathrm{F}}^{-1})_{4,1}+\Sigma_{(\varphi)1,2}(\Delta_{\mathrm{F}}^{-1})_{1,3}\Sigma_{(\varphi)3,4}(\Delta_{\mathrm{F}}^{-1})_{4,2}\right]\\&\qquad\qquad\frac{1}{2g^4}\left[-\Sigma_{(\mathrm{g})1,2}(-\Delta_{\mathrm{g}}^{-1})_{2,3}\Sigma_{(\mathrm{g})3,4}(-\Delta_{\mathrm{g}}^{-1})_{4,1}\right]\end{aligned}$$

Note: how to think about the order and the sign? First order the contraction pair $\langle(x_1y_1)(x_2y_2)\cdots\rangle$. Imagine couple them with sources in the action $u_1x_1+v_1y_1+u_2x_2+v_2y_2+\cdots$. Then $\langle(x_1y_1)(x_2y_2)\cdots\rangle$ is obtained by taking the derivatives $\partial_{u_1}\partial_{v_1}\partial_{u_2}\partial_{v_2}\cdots$ to the generating functional. The plus or minus sign of the source term doesn't matter because we always have an even number of them. The propagator appears from the same action (rewriting) having the form $x_i\Delta_{ij}y_j\ \to\ v_i(\Delta^{-1})_{ij}u_j$. Be careful about the position of u,v. So acting the derivative we get $(\Delta^{-1})_{v_1u_1}(\Delta^{-1})_{v_2u_2}\cdots$. The only sign in this procedure comes from the initial ordering.

$\frac{1}{6}x^3$-term (for simplification, only show the terms leading to some connected contractions) (also, the last line of the action is ignored)

$$\begin{aligned}&-\frac{1}{48g^6}\alpha_1\Sigma_{(\alpha)1,2}\alpha_2\alpha_3\Sigma_{(\alpha)3,4}\alpha_4\alpha_5\Sigma_{(\alpha)5,6}\alpha_6\\&\frac{1}{48g^6}\varphi_1\Sigma_{(\varphi)1,2}\varphi_2\varphi_3\Sigma_{(\varphi)3,4}\varphi_4\varphi_5\Sigma_{(\varphi)5,6}\varphi_6\\&\quad-\frac{1}{6g^6}b_1\Sigma_{(\mathrm{g})1,2}c_2b_3\Sigma_{(\mathrm{g})3,4}c_4b_5\Sigma_{(\mathrm{g})5,6}c_6\end{aligned}$$

Contractions

$$(23)(45)(16)\text{ with all possible interchanges }(1\leftrightarrow 2)(3\leftrightarrow 4)(5\leftrightarrow 6).$$

For the ghost part, only one possibility: all interchanges happen at once.

$$\begin{aligned}-\frac{1}{48g^6}&\Sigma_{(\alpha)1,2}(\Delta_{\mathrm{B}}^{-1})_{2,3}\Sigma_{(\alpha)3,4}(\Delta_{\mathrm{B}}^{-1})_{4,5}\Sigma_{(\alpha)5,6}(\Delta_{\mathrm{B}}^{-1})_{6,1}\\ \frac{1}{48g^6}&\Sigma_{(\varphi)1,2}(\Delta_{\mathrm{F}}^{-1})_{3,2}\Sigma_{(\varphi)3,4}(\Delta_{\mathrm{F}}^{-1})_{5,4}\Sigma_{(\varphi)5,6}(\Delta_{\mathrm{F}}^{-1})_{6,1}\\ -\frac{1}{6g^6}&\Sigma_{(\mathrm{g})1,2}(-\Delta_{\mathrm{g}}^{-1})_{2,3}\Sigma_{(\mathrm{g})3,4}(-\Delta_{\mathrm{g}}^{-1})_{4,5}\Sigma_{(\mathrm{g})5,6}(-\Delta_{\mathrm{g}}^{-1})_{6,1}\\ &+\text{all possible interchanges }\cdots\end{aligned}$$

$\frac{1}{24}x^4$-term (for simplification, only show the terms leading to some connected contractions) (also, the last line of the action is ignored)

$$\begin{aligned}&\frac{1}{384g^8}\alpha_1\Sigma_{(\alpha)1,2}\alpha_2\alpha_3\Sigma_{(\alpha)3,4}\alpha_4\alpha_5\Sigma_{(\alpha)5,6}\alpha_6\alpha_7\Sigma_{(\alpha)7,8}\alpha_8\\ &\frac{1}{384g^6}\varphi_1\Sigma_{(\varphi)1,2}\varphi_2\varphi_3\Sigma_{(\varphi)3,4}\varphi_4\varphi_5\Sigma_{(\varphi)5,6}\varphi_6\varphi_7\Sigma_{(\varphi)7,8}\varphi_8\\ &\frac{1}{24g^6}b_1\Sigma_{(\mathrm{g})1,2}c_2b_3\Sigma_{(\mathrm{g})3,4}c_4b_5\Sigma_{(\mathrm{g})5,6}c_6b_7\Sigma_{(\mathrm{g})7,8}c_8\end{aligned}$$

More contraction possibilities... number $= 6(3)\times 2^4 = 96(48)$. (todo. find a systematic way to consider all connected contractions)

Further calculation for the fermionic x^3-term Contractions:

$$\frac{8}{48g^6}\Sigma_{12}\Delta_{23}\Sigma_{34}\Delta_{45}\Sigma_{56}\Delta_{61},\quad \Delta=\overline{C\Gamma}\cdot aJ,\ \Sigma=C\Gamma\cdot\mathcal{A}.$$

next:

$$\begin{aligned}&\frac{1}{6a^6}\mathrm{Tr}(\Gamma^1\overline{\Gamma}^2\Gamma^3\overline{\Gamma}^4\Gamma^5\overline{\Gamma}^6)a^2a^4a^6\mathrm{Tr}(\mathcal{A}^1\mathcal{A}^3\mathcal{A}^5J).\\ &=\frac{32}{3a^6}\mathrm{Tr}(a\cdot\mathcal{A}a\cdot\mathcal{A}a\cdot\mathcal{A}J)-\frac{8}{a^4}\mathrm{Tr}(a\cdot\mathcal{A}\mathcal{A}^2J).\end{aligned}$$

Something wrong or not?

Consider the adjoint matrix $\mathcal{A}_{\mathfrak{a}'\mathfrak{b}'}$: the definition is Wed, Apr 24

$$\mathcal{A}_{\mathfrak{a}'\mathfrak{b}'}=\sum_{\mathfrak{c}}A_{\mathfrak{c}}\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{c}},T^{\mathfrak{b}'}]).$$

Try to write the following structures

$$\mathrm{Tr}(\mathcal{A}^2),\quad \mathrm{Tr}(\mathcal{A}^4),\quad \mathrm{Tr}(\mathcal{F}^2),$$

in terms of A. The basic identity to use is

$$\sum_{\mathfrak{a}'}\mathrm{Tr}(T^{\mathfrak{b}'}[T^{\mathfrak{e}},T^{\mathfrak{a}'}])\mathrm{Tr}(T^{\mathfrak{a}'}[T^{\mathfrak{f}},T^{\mathfrak{c}'}])=\mathrm{Tr}(T^{\mathfrak{b}'}T^{\mathfrak{e}}T^{\mathfrak{f}}T^{\mathfrak{c}'})+\mathrm{Tr}(T^{\mathfrak{e}}T^{\mathfrak{b}'}T^{\mathfrak{c}'}T^{\mathfrak{f}}).$$

One can derive then

$$\sum_{\mathfrak{a}'} \mathcal{A}^{\mu}_{\mathfrak{b}'\mathfrak{a}'} \mathcal{A}^{\nu}_{\mathfrak{a}'\mathfrak{c}'} = \mathrm{Tr}(T^{\mathfrak{b}'} A^{\mu} A^{\nu} T^{\mathfrak{c}'}) + \mathrm{Tr}(T^{\mathfrak{c}'} A^{\nu} A^{\mu} T^{\mathfrak{b}'}).$$

$$\sum_{\mathfrak{a}'\mathfrak{b}'} \mathcal{A}^{\mu}_{\mathfrak{b}'\mathfrak{a}'} \mathcal{A}^{\nu}_{\mathfrak{a}'\mathfrak{c}'} \mathcal{A}^{\rho}_{\mathfrak{c}'\mathfrak{d}'} = \mathrm{Tr}(T^{\mathfrak{b}'} A^{\mu} A^{\nu} A^{\rho} T^{\mathfrak{d}'}) - \mathrm{Tr}(T^{\mathfrak{d}'} A^{\rho} A^{\nu} A^{\mu} T^{\mathfrak{b}'}).$$

One can recogonize the pattern.

Therefore

$$\mathrm{Tr}(\mathcal{A}^2) = 2\mathrm{Tr}(A^2), \quad \mathrm{Tr}(\mathcal{A}^2\mathcal{A}^2) = 2\mathrm{Tr}(A^2A^2), \quad \mathrm{Tr}(\mathcal{F}^2) = 2\mathrm{Tr}(F^2).$$

Stupid identity, but lhs over the adjoint repr, rhs over the matrix elements.

It's desirable to also calculate

$$\frac{1}{384g^6}(\varphi\Sigma\varphi)(\varphi\Sigma\varphi)(\varphi\Sigma\varphi)(\varphi\Sigma\varphi).$$

First consider contractions: $6 \times 4 \times 2 = 48$. They are the same sign? Reduce to one term or few terms. Write down the Σ and the propagators Δ. Do the matrix product and the trace of $C\Gamma$...

conceptual discussion of renormalization The basic question asked by the RG flow is: how does the theory change with scale.

Scale is assumed as a part of definition of a theory. The number of degrees of freedom depends on the scale. Physically, we only have access to a part of dof, but assuming there are more. That is to say: the definition scale Λ (cut-off) is larger than the observational scale μ. What we are really asking is *not* how the theory depending on the Λ, *but* how the theory depending on the μ.

A theory in physics sometimes appears in the form of a path integral. It's by natural a proability distribution of the assumed dof. In this sense, an effective theory can be translated to a marginal pdf in math. A marginal pdf can be very different from the original pdf: we do not expect to write them in a similar form. But Gaussian is an exception. Then it seems reasonable to perturb the Gaussian, and study the marginal pdf.

One problem is: certain perturb may generate other perturb in marginal. So if the starting point is already taken as marginal, the consistency requires to include those "derived perturb". If there is an infinite amount of them...

Problem 1. *An assumption of the matrix RG: off diag. $\ll$ diag.? This is an assumption for the perturbative calculation. I don't know how the formulate this approximation properly because the matrix integral in any case get contribution from all possible matrix configurations. We are restricting to a particular integration domain, which don't have a definite mathematical definition.*

Problem 2. *Look at this term from the expansion (although it vanishes due to cancellation)*

$$\frac{1}{a^2}\mathrm{Tr}A^2$$

Assume that (as one can always do) a is much larger than the diagonal elements of A. (assume A is diagonal dominates) This means that

$$\frac{1}{a^2}\mathrm{Tr}A^2 \sim N^x, \quad x < 1$$

because the number of diagonal entries of A^2 that is comparable to a^2 is smaller than N. We need to compare it with $\frac{1}{g^2}\mathrm{Tr}A^2$. However, there is no $a > A$ to normalize the elements of A smaller than 1. We want this term is much larger than $\frac{1}{a^2}$ term whatever A is. Is it possible? Define $g^2N = t$, so $\frac{N}{t}\mathrm{Tr}A^2$ dominates when $N \to \infty$. Only in this limit, our calculation makes sense. It's always confusing we are studying N-flow in $N \to \infty$ limit. It maybe understood as n-flow: $n \equiv \frac{N}{N_0}$ in $N \to \infty, N_0 \to \infty$ limit.