Master Thesis Notes

Xiangwen Guan

From 4 February to 18 February

Contents

Wed, Feb 7

1 one-matrix integral

One-matrix model for a $N \times N$ Hermitian matrix $M = M^\dagger$ is defined by the following gauge transformation

$$M \to M' = UMU^\dagger, \quad U \in U(N). \tag{1}$$

Hermicity is kept under this transformation. It maybe useful also to write down the infinitesimal version

$$\delta_A M = i[A, M] \equiv \mathrm{ad}_A M, \quad A \in \mathfrak{u}(1) \tag{2}$$

where A is Hermitian.

Not all transformations in $U(N)$ will change the matrix. Note that a $U(1)$ subgroup $e^{i\theta} \in U(1)$ will leave the matrix invariant. For a general diagonal matrix, a subgroup $U(1) \times \cdots \times U(1)$ will leave it invariant; if one or more diagonal elements are equal, this subgroup will be enhanced to $U(N_1) \times \cdots \times U(N_k)$, $\sum_i N_i = N$.

A Gaussian distribution can be associated to each entry of M, for which the measure can be written as

$$\exp\left(-\frac{1}{2}\mathrm{Tr}M^2\right)[\mathrm{d}M].$$

The measure is gauge invariant, and can be understood as a measure on $\mathfrak{u}(1)$ on which $\mathfrak{u}(1)$ acts by ad.

This model can be modified by adding extra terms like $\mathrm{Tr}M^3, \mathrm{Tr}M^4, \cdots$ or like $(\mathrm{Tr}M^2)^2, \cdots$. We can study the modification by expanding around the original Gaussian model if the extra terms are small in some sense. However, it should be careful that such term has N dependence from trace. In large N-limit, N is another order counter.

Thu, Feb 8

2 reparameterization identity

Consider the following matrix integral

$$\int [\mathrm{d}\phi] e^{-\frac{N}{g}\mathrm{Tr}V(\phi)}. \tag{3}$$

and change the variable $\phi \to \phi' = \phi + \epsilon\phi^n$, where ϵ is small. The measure changes as $[\mathrm{d}\phi'] = [\mathrm{d}\phi](1 + \epsilon\sum_{k=0}^{n-1}\mathrm{tr}\phi^k\mathrm{tr}\phi^{n-k-1})$ to the first order. One can understand this by noting that taking derivative $\partial/\partial\phi$ of $\phi^n = \phi\cdots\phi$ will break the "string" into two parts, and the Jacobian determinant which becoming trace for small ϵ will take the trace of the two parts separately. and the exponent $V(\phi') = V(\phi) + \epsilon\phi^n V'(\phi)$. According to the identity

$$\int [\mathrm{d}\phi] e^{-\frac{N}{g}\mathrm{Tr}V(\phi)} = \int [\mathrm{d}\phi'] e^{-\frac{N}{g}\mathrm{Tr}V(\phi')}.$$

to the first order of ϵ

$$\left\langle \sum_{k=0}^{n-1} \mathrm{tr}\phi^k \mathrm{tr}\phi^{n-k-1} \right\rangle = \left\langle \frac{N}{g}\mathrm{tr}\phi^n V'(\phi) \right\rangle. \tag{4}$$

This implies that the double trace correlators may be written as single trace correlators.

Sat, Feb 10

3 IKKT

IKKT model ⚭ **IKKT96** can be obtained from dimensional reduction of $d = 10, \mathcal{N} = 2$ super Yang-Mills in the large-N limit.

In IKKT96, the model is written as

$$Z = \sum_{n=0}^{\infty} \int \mathrm{d}A\mathrm{d}\psi \mathrm{e}^{-S}, \tag{5}$$

$$S = \alpha \left\{ -\frac{1}{4}\mathrm{Tr}[A_\mu, A_\nu]^2 - \frac{1}{2}\mathrm{Tr}(\bar{\psi}\Gamma^\mu[A_\mu, \psi]) \right\} + \beta \mathrm{Tr}1. \tag{6}$$

They assume this model has a well behaved contitnuum limit such that the world-sheet action of superstring can be recovered. Then how can we study the Z? The sum diverges? Each term increases or decreases with n? Could we plot the n-dependence qualitatively? Could we compare the neighbour term? What's the necessary condition for the existence of the continuum limit?

The commutator $[A_\mu, A_\nu]^2$ may be easier to work with in the canonical basis of Hermitian matrices. Inside the Cartan subalgebra, the commutator is equal to zero. As for other commutators, one has the root system in mind: the commutators of matrices translate into the geometry of root vectors. The real dimension is $\dim \mathfrak{u}(n) = n^2$ with the Cartan subalgebra $\dim \mathfrak{h}(n) = n$.

The saddle point equation of resolvent method. Sun, Feb 11

$$\omega^2(z) + \frac{1}{g}V'(z)\omega(z) + \frac{1}{4g^2}R(z) = 0. \tag{7}$$

where $R(z)$ is an unknow polynomial of degree $l - 2$ when V is of degree l.

$$R(z) = 4g \int \mathrm{d}\lambda \rho(\lambda) \frac{V'(z) - V'(\lambda)}{z - \lambda}.$$

This equation is important because it tells us the singular part of $\omega_{\text{sing}}(z)$ has the form

$$2g\omega_{\text{sing}}(z) = \sqrt{(V'(z))^2 - R(z)}.$$

Therefore, there are branch cuts connecting the zeros of the polynomial $(V'(z))^2 - R(z)$. However, it's impossible to determine the zeros directly because $R(z)$ is unknown.

3.1 review the resolvent method

In the resolvent method, instead of $\rho(\lambda)$, the central object under study is the resolvent $\omega(z), z \in \mathbb{C}\backslash\mathbb{R}$

$$\omega(z) = \int \frac{\rho(\lambda)\mathrm{d}\lambda}{\lambda - z}.$$

Then the basic equations of $\omega(z)$ follow from the saddle point equation. One of them simply says that

$$\omega(\lambda + i0) + \omega(\lambda - i0) = -\frac{1}{g}V'(\lambda) \quad \lambda \in \mathbb{R}. \tag{8}$$

This combining with the analyticity of $\omega(\lambda)$ will give us an expression of $\omega(z)$, $z \in \mathbb{C}$. The simplest assumption is $\omega(z)$ has one branch cut at $[a_1, a_2]$, which corresponding to $\rho(\lambda)$ supported on $[a_1, a_2]$. It turns out that the form

$$\omega(z) = \frac{\sqrt{(z-a_1)(z-a_2)}}{2\pi g} \int_{a_1}^{a_2} \frac{\mathrm{d}\lambda}{\lambda - z} \frac{V'(\lambda)}{\sqrt{(a_2-\lambda)(\lambda-a_1)}}.$$

satisfies all the requirement for the one cut solution.

a_1, a_2 can be determined by the asymptotic $\omega(z) \sim -1/z$. The integral form simplies for $|z| \to \infty$, the leading piece $O(1)$ should vanish

$$\frac{1}{2\pi g} \int_{a_1}^{a_2} \frac{V'(\lambda)\mathrm{d}\lambda}{\sqrt{(a_2-\lambda)(\lambda-a_1)}} = 0.$$

The $O(1/z)$ piece should give the correct coefficient

$$\int_{a_1}^{a_2} \mathrm{d}\lambda \frac{\lambda V'(\lambda)}{\sqrt{(a_2-\lambda)(\lambda-a_1)}} = 2\pi g.$$

The integral can be simplified by changing the variables

$$\lambda = z + \frac{1}{2}(a_1 + a_2) + \frac{(a_1-a_2)^2}{16z}.$$

Then the integral path can be written as a circle around origin for $z \in \mathbb{C}$

$$\oint \frac{\mathrm{d}z}{2\pi i z} V'(\lambda(z)) = 0 \tag{9}$$

$$\oint \frac{\mathrm{d}z}{2\pi i} V'(\lambda(z)) = g. \tag{10}$$

There are some properties can be read directly from above equations. If $V(\lambda)$ is even, $V'(\lambda)$ is odd. Then the first equation is always satisfied by $a_1 + a_2 = 0$. This means that one can always find a solution whose branch cut is symmetric $[-a, a]$. If $a_1 = a_2$, $\lambda = z + \frac{1}{2}(a_1 + a_2)$, the second equation implies that $g = 0$. Therefore it seems like not possible to obtain $g_* \neq 0$ for one cut solution.

4 motivation letter

I've been lucky with my educations, learning not just the knowledge and ideas in physics, but also how to enjoy learning from people who have genuine passion in physics. Sometimes, I'd like to dream about being in Plato's Academy, thinking about our knowledge of nature and sharing those ideas with others. I find that people today study physics in a similar way. That's my motivation for continuing my physics career.

What are my interests in general? Quite generally, I'm interested in using the tools and ideas from quantum field theory (QFT) to understand physical models. One of the intriguing aspects may be that how QFT offering people insights on the dynamics of a quantum system with numerous degrees of freedom. The spectrum of applications of QFT is also wide: particle physics, string theory, and even mathematics. Many deep physical ideas and mathematical structures can be formulated in the language of QFT. It's definitely one of the most interesting subjects to study.

Prof. Skenderis' research interests me, especially those about CFT. The initial success of QFT based on its perturbative structure, however many nowadays researches focus on the elusive non-perturbative structure. One highlight is the AdS/CFT. It not only provide a feasible way to understand quantum gravity, but also allowing people to rethink the problems in QFT from the geometry point of view. I want to learn more about QFTs with special symmetries, they show interesting structures while allowing people to apply various methods to study them. They may appear to be unrealistic, but there may be some universalities that pertaining to the real world. Skenderis's researches involve discussions on those toy model QFTs, and look at them from various sides. The central role of CFT is obvious, which surprisingly related to a bunch of physics interests. CFT manifests itself in terms of the energy-momentum tensor, the study of the energy-momentum tensor of a lattice field theory is interesting. It's interesting to see how the stress tensor is formulated in the lattice theory, and how the lattice regularization modifies the property of the stress tensor. Is it possible to have a lattice regularization of CFT? I love to think about these questions.

I will also give a brief statement about my bachelor and master thesis topics. The bachelor thesis is about the linear perturbation theory of black hole solution in pure general relativity. I was fascinated by the interrelation between three properties of the theory: separability, D-type curvature and Killing spinor. The physics solution surprised me by its rich structure, which somewhat is hidden behind the complex metric.

In master thesis, I'm exploring possible scale invariant matrix models using the idea of renormalization group. The physics motivation is a D-instanton system in which the partition function being expected to be conformal invariant. Without an intrinsic notion of spacetime, the scaling of couplings can only arise from the scaling of matrix rank: a typical example is the double scaling limit. With a similar spirit, we wonder whether the couplings in the D-instanton system are scale invariant. This point of view is not well explored, especially for a supersymmetric system. Rum trying to gain some insights on this problem.

REFERENCES

Prof. Toine Van Proeyen
Emeritus Professor, Institute for Theoretical Physics, KU Leuven, Celestijnenlaan 200D, B-3001 Leuven, Belgium
E-mail: antoine.vanproeyen@kuleuven.be
Prof. Thomas Van Riet
Associate Professor, Institute for Theoretical Physics, KU Leuven, Celestijnenlaan 200D, B-3001 Leuven, Belgium
E-mail: thomas.vanriet@kuleuven.be

Mon, Feb 12

5 an RG study of integral

Integration could be studied using an RG spirit method, which is similar to the Wilson's RG scheme. However, there is no natural notion of energy associated with the integral.

Let's take the "$O(N)$ vector model" as the example ✎ **Zinn-Justin14**

$$e^{Z_N} = \left(\frac{N}{2\pi}\right)^{N/2} \int d^N\mathbf{x} e^{-NV(x^2)}. \tag{11}$$

The normalization is chosen to normalize a Gaussian integral. The factor N before $V(\mathbf{x}^2)$ gives a "dimension" of V under the scaling of N. To implement the changing of N, one calculates Z_{N+1} by integrating out one component of the vector $\mathbf{x}$

$$e^{Z_{N+1}} = \left(\frac{N+1}{2\pi}\right)^{(N+1)/2} \int d^N\mathbf{x} \int dy e^{-(N+1)V(x^2+y^2)} \tag{12}$$

It's impossible to integrate out y exactly in general, so one may use the saddle point approximation. For a generic $\mathbf{x}$, the integrand $e^{-(N+1)V(x^2+y^2)}$ is localized at the critical value of y when $N \to \infty$. To the leading order, assuming that the saddle point is at $y = 0$,the approximation gives

$$e^{Z_{N+1}} = \left(\frac{N+1}{2\pi}\right)^{N/2} \int d^N\mathbf{x} e^{-(N+1)V(x)-\frac{1}{2}\ln 2V'(x^2)} \left(1 + O(\frac{1}{N})\right).$$

It should be noted that the saddle point approximation fail when $V'(\mathbf{x}^2) = 0$. The Gaussian potential $V(\mathbf{x}^2) = (1/2)\mathbf{x}^2$ will give $V'(\mathbf{x}^2) = 1/2$. In this case, the leading order vanishes.

I'm not clear to what extent the subleading corrections will modify the scaling around the critical point; however, I guess they are not very important if the critical point by itself requires $N \to \infty$ (similar to the fact that phase transition requires thermodynamic limit). Due to various methods to solve $N \to \infty$ model, it deserves study; however, is it possible to obtain a matrix model that has scaling covariance for all N? This is a stringent requirement, but there is example of matrix model that is independent of N. For example, see the discussion of the Kontsevich integral in ✎ **Morozov94** below the equation (3.70).

Come back to the discussion of the vector model. The change $N \to N + 1$ induces a change in $\mathbf{x}$ and V

$$\mathbf{x} \to \mathbf{x}\left(1 - \frac{1}{2N}(1+\gamma)\right), \tag{13}$$

$$V \to V + \delta V \tag{14}$$

such that the partition function scales as

$$Z_{N+1}(V) = -\frac{1}{2}\gamma + Z_N(V + \delta V) + O(\frac{1}{N}) \tag{15}$$

The $-\frac{1}{2}\gamma$ term comes form the scaling of the integration measure $d^N\mathbf{x}$. The $V + \delta V$ comes from the integration over y, also the scaling of $\mathbf{x}$

$$N\delta V(\rho) = V(\rho) - (1+\gamma)\rho V'(\rho) + \frac{1}{2}\ln 2V'(\rho) + O(\frac{1}{N}).$$

the ρ is a shorthand notation for $\mathbf{x}^2$. The differential equation for the deformation of V along the RG flow reads then

$$\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} V(\rho, \lambda) = V(\rho, \lambda) - (1 + \gamma(\lambda))\, \rho V'(\rho, \lambda) + \frac{1}{2} \ln 2V'(\rho, \lambda) \qquad (16)$$

where λ is a continuous parameter $N \to \lambda N$. Note that γ is assumed to depend on N.

$V(\rho) = \frac{1}{2}\rho$ is a fix point with $\gamma = 0$, the Gaussian fixed point. It's not obvious in general how to solve the fix point equation due to the non-linear term $\ln V'$. One possible solution is

$$V(\rho) = \frac{1}{2} \ln \rho.$$

with $\gamma = -1$ such that the second term vanish. If plugging this back to the integral, one finds the integrand is $\rho^{-N/2}$. This cancels with the radius measure in spherical coordinate. Therefore, this potential corresponds to a trivial vector model.

In general, it turns out that it's easier to solve the equation for V'. Actually, if one define a function $R(\rho)$

$$R(\rho) \equiv \frac{1}{2V'(rho)}.$$

Then the fix point equation will simply be

$$\gamma R + RR' - (1+\gamma)\rho R' = 0 \qquad (17)$$

The $\gamma = -1$ case gives $R' = 1$, which corresponding to the solution $V(\rho) = \frac{1}{2}\ln \rho$. Then how to obtain the solution of, for example $\gamma = -2$?

Note that the finite scaling of $\mathbf{x}$ is

$$\mathbf{x} \to \mathbf{x} \left(1 + \frac{1+\gamma}{2} \ln \lambda\right).$$

Instead of power of λ, it scales as $\ln \lambda$.

Tue, Feb 13

6 matrix integral from N to N-1

The model under consideration is the one-matrix ϕ^4 model

$$\zeta_N(g) = \int \mathrm{d}\phi_N \exp(-S_N[\phi_N, g]), \qquad (18)$$

with an action

$$S_N[\phi_N, g] = \mathrm{Tr}(\frac{1}{2}\phi_N^2 + \frac{g}{4}\phi_N^4). \qquad (19)$$

Start with the rank $N+1$ model, and decomposing the matrix ϕ_{N+1} as

$$\phi_{N+1} = \begin{pmatrix} \phi_N & v \\ v^\dagger & \alpha \end{pmatrix}. \qquad (20)$$

Then the action can be expanded as

$$
\begin{aligned}
S_{N+1}[\phi_{N+1}, g] =& \mathrm{Tr}\left(\frac{1}{2}\phi_N^2 + \frac{g}{4}\phi_N^4\right) + v^\dagger v + \frac{1}{2}\alpha^2 \\
&+ g\left(v^\dagger \phi_N^2 v + \alpha v^\dagger \phi_N v + \alpha^2 v^\dagger v + \frac{1}{2}(v^\dagger v)^2 + \frac{1}{4}\alpha^4\right). \quad (21)
\end{aligned}
$$

Now the trace is over $N \times N$ matrix.

Recall that the matrix model is $U(N)$-invariant; it may be useful to first gauge away some variables in v, a. Let's consider how the gauge transformation acting on ϕ_N, v and a. The infinitesimal transformation reads

$$
\begin{aligned}
\delta_t \phi_N &= i(vt^\dagger - tv^\dagger) \\
\delta_t v &= i(\phi_N - \alpha \mathbb{1})t \\
\delta_t v^\dagger &= it^\dagger(\alpha \mathbb{1} - \phi_N) \\
\delta_t \alpha &= i(v^\dagger t - t^\dagger v)
\end{aligned} \quad (22)
$$

where t ($N \times 1$ vector) is the components of the following generator

$$
T = \begin{pmatrix} 0 & t \\ t^\dagger & 0 \end{pmatrix}.
$$

It's possible to choose t to gauge away v if $\phi_N - \alpha \mathbb{1}$ is not degenerate. So let's impose the gauge fixing condition as $v = 0, v^\dagger = 0$.

How to implement this condition in the integral? Essentially what we should do is to change the integration variable from v to t. t is understood as the parameter for the coset $U(N+1)/U(N)$ through exponent. Around infinitesimal neighbour of $v = 0$, $v(t)$ is a linear function given by the infinitesimal gauge transformation

$$
v(t) = -i(\phi_N - \alpha \mathbb{1})t.
$$

Then the "coordinate transformation" will give a Jacobian that is proportional to $\det(\phi_N - \alpha \mathbb{1})$. The same factor will be obtained from $v^\dagger$. Therefore we get

$$
[\det(\phi_N - \alpha \mathbb{1})]^2
$$

in the integrand, which can be re-exponentiate to give a term in the action

$$
S_{N+1}[\phi_{N+1}, g] = \mathrm{Tr}\left(\frac{1}{2}\phi_N^2 + \frac{g}{4}\phi_N^4\right) + \frac{1}{2}\alpha^2 + \frac{g}{4}\alpha^4 - 2\mathrm{Tr}\ln(\phi_N - \alpha \mathbb{1}). \quad (23)
$$

Note that this formula is equivalent to the eigenvalue representation if we diagonalize ϕ_N.

However, the above calculation based on the linearized version of gauge transformation. The full gauge transformation is more complicate. We can proceed as following. Consider a $(N+1) \times (N+1)$-matrix $\phi_{N+1}(\phi_N, 0, a)$ with $v = 0$. Then do a gauge transformation, parameterized by t, on this matrix $\phi_{N+1} \to \tilde{\phi}_{N+1} = U(t)\phi_{N+1}U^\dagger(t)$. The idea is that the integration over $\tilde{\phi}_{N+1}$ can be replaced with an integration over ϕ_{N+1} and t, with a proper Jacobian that taking into account the functional dependence $\tilde{\phi}_{N+1}(\phi_{N+1}, t)$. In the exponent, because of the gauge invariance, one can replace $\tilde{\phi}$ directly with ϕ. To write down the Jacobian, one needs to solve for the $\tilde{v}(t)$, which is fixed by the condition $v = 0$. It turns out that the t-dependence of the Jacobian can be factorized out, so the t-integral can be performed independently. This factorization property also implies that the linearized result giving the exact Jacobian for the ϕ factor, although we don't know the t-factor.

In summary, we have

$$\int \mathrm{d}\phi_N \mathrm{d}\alpha [\det(\phi_N - \alpha \mathbb{1})]^2 \mathrm{e}^{-\frac{1}{2}\alpha^2 - \frac{g}{4}\alpha^4 - S_N[\phi_N, g]} \tag{24}$$

or

$$\int \mathrm{d}\phi_N \mathrm{d}\alpha \mathrm{e}^{-\frac{1}{2}\alpha^2 - \frac{g}{4}\alpha^4 - S_N[\phi_N, g] + 2\mathrm{Tr}\ln(\phi_N - \alpha\mathbb{1})} \tag{25}$$

The α integral is difficult to understand. But I try to study it as follows.

The determinant is a characteristic polynomial $\det(\alpha\mathbb{1} - \phi_N) \equiv p_N(\alpha)$ of ϕ_N in α with coefficients given by the following formula

$$p_N(\alpha) = \sum_{k=0}^{N} \alpha^{N-k}(-1)^k \mathrm{Tr}\left(\wedge^k \phi_N\right), \tag{26}$$

with

$$\mathrm{Tr}\left(\wedge^k \phi\right) = \frac{1}{k!}\det \begin{vmatrix} \mathrm{Tr}\phi & k-1 & 0 & \cdots & 0 \\ \mathrm{Tr}\phi^2 & \mathrm{Tr}\phi & k-2 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \mathrm{Tr}\phi^{k-1} & \mathrm{Tr}\phi^{k-2} & & \ddots & 1 \\ \mathrm{Tr}\phi^k & \mathrm{Tr}\phi^{k-1} & \mathrm{Tr}\phi^{k-2} & \cdots & \mathrm{Tr}\phi \end{vmatrix}$$

The leading term of $p_N(\alpha)$ is α^N, and the last term is $(-1)^N \det\phi_N$. Also, let's write down first few terms to get a feeling

$$p_N(\alpha) = \alpha^N - \alpha^{N-1}\mathrm{Tr}\phi + \frac{1}{2}\alpha^{N-2}\left[(\mathrm{Tr}\phi)^2 - \mathrm{Tr}\phi^2\right] - \frac{1}{6}\alpha^{N-3}\left[(\mathrm{Tr}\phi)^3 - 3\mathrm{Tr}\phi\mathrm{Tr}\phi^2 + \mathrm{Tr}\phi^3\right] + \cdots$$

If we square it as in the integrand, we get

$$\begin{aligned} p_N^2(\alpha) =& \alpha^{2N} - 2\alpha^{2N-1}\mathrm{Tr}\phi + \alpha^{2N-2}\left[2(\mathrm{Tr}\phi)^2 - \mathrm{Tr}\phi^2\right] \\ & - \alpha^{2N-3}\left[\frac{4}{3}(\mathrm{Tr}\phi)^3 - 2(\mathrm{Tr}\phi)(\mathrm{Tr}\phi^2) + \frac{1}{3}\mathrm{Tr}\phi^3\right]\cdots \end{aligned} \tag{27}$$

One can only consider even terms if the potential is even. For each term, α is decoupled from ϕ, therefore can be integrated out.

To simpilfy the result, the following equation maybe useful

$$\begin{aligned} 2\left\langle \mathrm{Tr}\phi^2 \right\rangle + \left\langle (\mathrm{Tr}\phi)^2 \right\rangle &= \left\langle \mathrm{Tr}(\phi^4 + g\phi^6) \right\rangle \\ 2\left\langle \mathrm{Tr}\phi^3 \right\rangle + 2\left\langle (\mathrm{Tr}\phi)(\mathrm{Tr}\phi^2) \right\rangle &= \left\langle \mathrm{Tr}(\phi^5 + g\phi^7) \right\rangle \end{aligned} \tag{28}$$

The $\langle\rangle$ indicates the matrix integral. These equations follow from changing the integral variable $\phi \to \phi + \epsilon\phi^{3,4}$, with infinitesimal ϵ. The left hand side comes from the change of measure; while the right hand side follows from the change of the action. For example, the α^{2N-2} term becomes

$$\alpha^{2N-2}\left[\mathrm{Tr}(-5\phi^2+2\phi^4+2g\phi^6)\right].$$

As for the integration over α

$$\int(\mathrm{d}\alpha)\alpha^{2N-k}\mathrm{e}^{-\frac{1}{2}\alpha^2-\frac{g}{4}\alpha^4}.$$

The same trick allows us to derive that

$$(2N-k)\langle\alpha^{2N-k-1}\rangle=\langle\alpha^{2N-k+1}\rangle+g\langle\alpha^{2N-k+3}\rangle.$$

For example $k=1$

$$\langle\alpha^{2N-2}\rangle=\frac{1}{2N-1}(\langle\alpha^{2N}\rangle+g\langle\alpha^{2N+2}\rangle).$$

$k=3$

$$\langle\alpha^{2N-4}\rangle=\frac{1}{2N-3}(\langle\alpha^{2N-2}\rangle+g\langle\alpha^{2N}\rangle).$$

For small g, it's reasonable that we only keep the $\alpha^{2N},\alpha^{2N-2}$ terms in large N limit. For a general g, it's not clear that whether or not $\alpha^{2N-2},\alpha^{2N-4}$ are in the same order or not.

Let's try to study the small g limit

$$r_k(g)\equiv\int\alpha^{2(N-k)}\mathrm{e}^{-\frac{1}{2}\alpha^2-\frac{g}{4}\alpha^4}\mathrm{d}\alpha \tag{29}$$

to the order of g we have

$$r_k(g)\sim\sqrt{2\pi}(2N-2k-1)!!-\frac{g}{4}\sqrt{2\pi}(2N-2k+3)!!+O(g^2)$$

In the large N limit

$$(2N-2k-1)!!\sim\sqrt{2}\left(\frac{2N-2k-1}{e}\right)^{N-k}$$

so we have

$$r_k(g)\sim 2\sqrt{\pi}\left(\frac{2N-2k-1}{e}\right)^{N-k}-\frac{g}{2}\sqrt{\pi}\left(\frac{2N-2k+3}{e}\right)^{N-k+1}$$

Let's factor out a k independent factor $2\sqrt{\pi}[(2N+3)/e]^{N+1}$ such that

$$\tilde{r}_0(g)\sim\frac{e^3}{2N}-\frac{g}{4}$$

$$\tilde{r}_1(g)\sim-\frac{e^2g}{8}\frac{1}{N}+O(\frac{1}{N^2})$$

$$\tilde{r}_2(g)\sim O(\frac{1}{N^2})$$

Then the asymptotic behavior of the integrand is

$$-\frac{g}{4}+\frac{e^3}{2N}-\frac{e^2g}{8N}\left[2(\mathrm{Tr}\phi)^2-\mathrm{Tr}\phi^2\right]+O(\frac{1}{N^2})$$

Thu, Feb 15

6.1 a nonlinear RG equation for matrix model

This part is a review of [**?**]. Consider the following matrix integral

$$Z_N(g_j) = \int \mathrm{d}\phi \exp[-N\mathrm{Tr}V(\phi)], \tag{30}$$

where

$$V(\phi) = \sum_{k\geq 1} \frac{g_k}{k}\phi^k.$$

The free energy is then defined as

$$F(N, g_j) = -\frac{1}{N^2}\log\left(\frac{Z_N(g_j)}{Z_N(g_2 = 1, \text{others} = 0)}\right). \tag{31}$$

Later, the RG equation will be of the form

$$\left[N\frac{\partial}{\partial N} + 2\right] F(N, g) = G\left(g, \frac{\partial F}{\partial g}\right) + O(\frac{1}{N}) \tag{32}$$

The normalization $1/N^2$ of $f(N, g)$ gives the factor 2 in the above RG equation, which is designed to reproduce the scaling law of string partition function [**?**].

Starting from $Z_{N+1}(g)$, perform the integration over the last row and column, then we obtain an induced matrix model with rank N. The partition function of the induced model is the same as $Z_{N+1}(g)$. In [**?**], the integration gives

$$Z_{N+1}(g) = \left(\frac{\pi}{N+1}\right)^N \int \mathrm{d}\phi \exp[-(N+1)\mathrm{tr}V(\phi)]$$
$$\cdot \int \mathrm{d}\alpha \exp[-(N+1)V(\alpha) - \mathrm{tr}\ \log(\mathbb{1} + g(\phi + \alpha\mathbb{1}))]$$

This is different from $Z_N(g)$, and the RG equation formulate the difference. The α integral is performed by the saddle point method. Denote the saddle point value $\alpha_s = \alpha_s(g, \phi)$, which is determined by the following saddle point equation

$$\alpha_s + g\alpha_s^2 + \frac{g}{N}\mathrm{tr}\frac{1}{\mathbb{1} + g(\phi + \alpha_s\mathbb{1})} = 0. \tag{33}$$

At this point α_s is not solved in terms of g and ϕ, but latter we can solve it with the help of resolvent and loop equations.

The factorization property of large-N limit: for $U(N)$-invariants $\mathcal{O}, \mathcal{O}'$, we have

$$\langle \mathcal{O}\mathcal{O}' \rangle = \langle \mathcal{O} \rangle \langle \mathcal{O}' \rangle + O(N^{-2}) \tag{34}$$

can be used to put the average $\langle \cdots \rangle$ inside any polynomial functions of gauge invariant quantities. This leads to a great simplification which enables us to derive an RG equation.

To derive the RG equation, we can start by considering the ratio

$$\frac{Z_{N+1}(g)}{Z_N(g)} = \cdots .$$

The above consideration enables us to express the right hand side as an exponential function

$$\frac{Z_{N+1}(g)}{Z_N(g)} = \left(\frac{\pi}{N+1}\right)^N \exp[-\langle \mathrm{tr}V(\phi)\rangle - NV(\langle\alpha_s\rangle) - \langle \mathrm{tr}\log(\mathbb{1}+g(\phi+\alpha_s\mathbb{1})) + O(N^0)\rangle]$$

This, combined with the definition of free energy, gives the following prototype of the RG equation

$$\left[N\frac{\partial}{\partial N}+2\right]F(N,g) = -\frac{1}{2}+\langle\frac{1}{N}\mathrm{tr}V(\phi)\rangle+V(\langle\alpha_s\rangle)+\langle\frac{1}{N}\mathrm{tr}\log(\mathbb{1}+g(\phi+\langle\alpha_s\rangle\mathbb{1}))\rangle+O(\frac{1}{N}). \tag{35}$$

The right hand side is a function of $\langle\cdots\rangle$ in the rank N model, which can be generated by the derivatives of free energy. $\langle\alpha_s\rangle$ is understood as $\langle\alpha_s(g,\phi)\rangle$, for which the concrete form still need to be solved by resolvent method. Although it's not obvious at this step, the terms we retain on the right hand side is of order $O(N^0)$. This will be verified by the loop equations which essentially relates different correlators

$$\langle\frac{1}{N}\mathrm{tr}\phi^i\rangle,$$

there will be no other N factors appear in the loop equation, which indicates all such correlators are of the same N order. For $\langle\alpha_s\rangle$, it turns out that, from again loop equations,

$$\langle\alpha_s\rangle = \langle\frac{1}{N}\mathrm{tr}\phi\rangle$$

This fact is consistent with how α appears in the matrix decomposition.

Let's write down the nonlinear RG equation we obtain through this procedure. To be specific, [**?**] takes the cubic potential

$$\left(N\frac{\partial}{\partial N}+2\right)F(N,g) = G(g,\frac{\partial F}{\partial g})+O(\frac{1}{N}) \tag{36}$$

with

$$G(g,a) = -\frac{g}{2}a+\frac{1}{2}\bar\alpha(g,a)^2+\frac{g}{3}\bar\alpha(g,a)^3+\log(1+g\bar\alpha(g,a))+\int_{-\infty}^{-1/g-\bar\alpha(g,a)}\mathrm{d}z\left(W(z,g,a)-\frac{1}{z}\right)$$

$$\bar\alpha(g,a) = -g+3g^2a \equiv \langle\alpha_s\rangle$$

$$W(z,g,a) = \frac{1}{2}\left(z+gz^2-\sqrt{(z+gz^2)^2-4(1+gz-g^2+3g^3a)}\right)$$

$W(z,g,a)$ is the average value of the resolvent

$$W(z,g,a) = \langle\frac{1}{N}\mathrm{tr}\frac{1}{z\mathbb{1}-\phi}\rangle$$

and a is the notation for

$$a_j = \frac{1}{j}\langle\frac{1}{N}\mathrm{tr}\phi^j\rangle = \frac{\partial F}{\partial g_j}$$

The resolvent here can be understood as a way to encode all higher interactions $\mathrm{tr}\phi^k$ contribution to the RG equation in terms of what appears in the original potential $V(\phi)$.

The information of critical point g_* and the critical exponent γ_0 can be extracted from the RG equation. In [?], g_*, γ_0 is characterized by the singular scaling behavior of the free energy. The leading order of free energy in large N expansion is $O(N^0)$, so let's denote it as $F^0(g)$. Then $F^0(g)$ is assumed to contain an analytic part and non-analytic part around the critical point g_*,

$$F^0(g) = \sum_{k=0}^{\infty} a_k (g - g_*)^k + \sum_{k=0}^{\infty} b_k (g - g_*)^{k+2-\gamma_0}, \quad \gamma_0 \notin \mathbb{Z}. \tag{37}$$

The right hand side of the RG equation depends on g and $\partial F^0(g)/\partial g$. Let's denote at the critical point $\partial F^0(g)/\partial g|_{g=g_*} = a_1$. Then $G(g, a)$ is assumed to have the following expansion form

$$G(g, a) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \beta_{n,k} (g - g_*)^k (a - a_1)^n. \tag{38}$$

Then the idea is to compare the coefficients of various powers of $g - g_*$ on both sides. But the problem is that there are non-analytic parts in $F^0(g)$, which does not appear in $G(g, a)$ To be continue.......

6.2 notes

A notion of **canonical dimension** of coupling constants, like $g_k \mathrm{Tr}\phi^k$, is defined in [?]. In the vicinity of the Gaussian fixed point, we expect $g_k, k \geq 2$ has the following scaling

$$g_k \sim N^{-d_k + \mathcal{O}(g_2, g_3, \cdots)}$$

then d_k is called the canonical dimension. Let's discuss how this scaling is obtained. Consider the following partition function

$$Z_{N+1}(g_3, g_4, \cdots) = \int [\mathrm{d}\phi] \exp\left(-\frac{1}{2}\mathrm{Tr}\phi_{N+1}^2 - \frac{g_3}{3}\mathrm{Tr}\phi_{N+1}^3 - \frac{g_4}{4}\mathrm{Tr}\phi_{N+1}^4 + \cdots \right).$$

Following the conventional method, the terms involving $v, v^\dagger$ appear in the potential

$$\exp\left\{ \cdots - v^\dagger v - g_3 v^\dagger (\phi_N + \alpha \mathbb{1}) v - g_4 \left[v^\dagger (\phi_N^2 + \alpha \phi_N + \alpha^2 \mathbb{1}) v + \frac{1}{2} (v^\dagger v)^2 \right] + \cdots \right\}$$

To the first order of the coupling constant, the integral over $v, v^\dagger$ gives terms like

$$-g_3 \mathrm{tr}(\phi_N + \alpha \mathbb{1}) - g_4 \mathrm{tr}(\phi_N^2 + \alpha \phi_N + \alpha^2 \mathbb{1}) + \cdots$$

If we re-exponentiate them, there will be new quadratic terms like $-g_4 \mathrm{tr}\phi^2$. To keep the normalization of the quadratic term, we need to rescale ϕ_N as

$$\phi_N' = (1 + g_4)\phi_N$$

In all of these calculations, we keep only to the first order of g_4. Therefore, at the Gaussian fix point $g_4 = 0$, there is no scaling of ϕ_N. Look at the coefficient of $\mathrm{Tr}(\phi_N')^4$, define it as $g_4'/4$, g_4' is equal to

$$g_4' = g_4(1 - 4g_4)$$

In the theory, written in this way (with no factor N in the action), all couplings g and the matrix ϕ are dimensionless, if we take N as the only dimensional scale. To be continue.......

Sat, Feb 17

The RG method of matrix model seems having the ability to apply to a complicate model, combining with the perturbative calculation method. The complication may arise from two aspects: 1. the matrix product usually generate complicate interaction; 2. the integration is hard to perform in general; However, our hope is not doing the calculation, but understand whether or not new interaction terms will arise in a qualitative level. Whether symmetries will prevent the action from generating the new terms.

2×2 matrix is the simplest matrix. It's far away from the large N limit, but some interesting properties may show already at this level.

It's not hard to write down the terms in $\mathrm{tr}(ABBA\cdots)$. If one writes down the matrix in the following form

$$A = \begin{pmatrix} a_1 & \alpha \\ \alpha^* & a_2 \end{pmatrix}.$$

Then the next step is to integrate out α, α^* and a_2. We don't need to actually do the integration; We only need to know what new interactions can be generated from the perturbation calculation. If possible, it's also important to know the coupling constants of the new interactions.

There are some typical terms that could appear in $\mathrm{Tr}(A^4)$ and $\mathrm{Tr}(A^3B)$

$$\begin{array}{cc} \alpha^*\alpha(a_1)^2\alpha^*\alpha a_1 a_2 & \alpha^*\alpha(a_2)^2 \\ (\alpha^*\beta+\beta^*\alpha)a_1a_2\alpha^*\alpha(b_1a_2+b_2a_1) & \alpha^*\alpha(\alpha^*\beta+\beta^*\alpha) \end{array} \tag{39}$$

Performing the integration over α^*, α and a_2 perturbatively, the question is that what the correction to the first order. An easy way to think about the perturbative calculation is through the Wick contraction. It gives non-zero result only if the a_2 is of even order, and α^*, α could form pairs. For example, $\alpha^*\alpha(a_1)^2$ will give a term that is proportional to $(a_1)^2$, while $(\alpha^*\alpha)a_1a_2$ will give zero. There is no essence difficulty in doing these calculations. Getting the correct combinatoric factors is important. The number of terms explodes in higher order, so maybe some patterns will be helpful to organize the calculation.

Another problem of the perturbative calculation is the choice of quadratic term. For example, a quadratic term in α has the form $\alpha\alpha^*(\cdots)$, where the $(\cdots)$ does not contain α. But $(\cdots)$ could contain other variables, The consequence is that the propogator does not have a fixed value. This may cause difficulty on the calculation, for example, giving a non-linear term in variables that need to be integrated out. The origin of those nonlinear terms is the interaction terms in the matrix integral. So it's possible to treat them perturbatively. However, if one is interested in the non-perturbative aspect, then it's necessary to use the exact propagator. It's possible to study the non-perturbative effect because one works with the large N limit. However our interest is not the large N limit, but the property at finite N. It seems like only perturbation effect is doable when N is finite. Then at certain order of g, the terms will involve different order of N. It's very important to difference our calculation with that was done in the literatures.

Sun, Feb 18

the interaction between A and B What kind of features that we want our model have? After integrating out one row and one column, there is no new interaction being generated. To understand under what situation this property will be satisfied, let's think about it order by order. Consider the following 2×2 matrix

$$A = \begin{pmatrix} a_1 & \alpha \\ \alpha^* & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & \beta \\ \beta^* & b_2 \end{pmatrix}.$$

The interaction terms that we are interested in are

$$g\mathrm{Tr}(ABAB) \quad h\mathrm{Tr}(AABB).$$

Looking at them, one expects that $a_1, (a_1)^2, (a_1)^3, (a_1)^4, \cdots$ (similar for b_1) can be generated by integrating out α, α^*, a_2. Now let's figure out which terms in the perturbative expansion will contribute to these "new interactions".

It maybe useful to keep track of the terms by drawing diagrams. For a matrix, let's draw a line with an arrow pointing from one end point to another. The end points indicate the row and column indices respectively. Matrix product is represented by joining two end points together, which means that they always take the same value. Therefore, the trace is represented by a loop. One also need to differentiate matrix A and B by using different kinds of lines.

What will be integrated out is the line with end points taking value in a certain row and column. In the case of 2×2 matrix, the value may be 2. The lines that being integrated out will be connected by a "propagator" diagramatically. For the off-diagonal elements α, α^*, the "propagator" is represented by double-line diagram; for the diagonal element a_1, it will be represented by a single line because the row and column indices taking the same value therefore can be regarded as a single point.

The first thing to note is that the odd terms $a_1, (a_1)^3, \cdots$ cannot be generated. We always have even number of A matrix, the terms that could contribute to the odd terms cannot be fully contracted. The next thing to consider is the $(a_1)^2$ term. To the first order of g, h, one gets from $g\mathrm{Tr}(ABAB)$ a term $g(a_1)^2(b_1)^2$ and from $h\mathrm{Tr}(AABB)$ a term $h(a_1)^2\beta^*\beta$. The term $h(a_1)^2\beta^*\beta$ will contribute to the modification of the propagator. Then let's consider the $(a_1)^4$ term. It will only appear to the second order of g, h. We have the following terms

$$\frac{g^2}{2}(a_1 b_1 a_1 b_1)(a_1 b_1 a_1 b_1), \quad \frac{h^2}{2}(a_1 a_1 \beta\beta^*)(a_1 a_1 \beta\beta^*), \quad gh(a_1 a_1 \beta\beta^*)(a_1 b_1 a_1 b_1).$$

How to understand these terms? Do they imply that new interactions will appear along the RG flow? No, for example, the term $(g^2/2)(a_1)^4(b_1)^4$ could be reproduced by $g\mathrm{Tr}(ABAB)$ interaction by expanding the exponent. Similarly the term $gh(a_1)^4(b_1)^2(\beta^*\beta)$ could be reproduced by $g\mathrm{Tr}(ABAB)$ and a proper modification of the quadratic term $\mathrm{Tr}A^2$. However, the second term $(h^2/2)(a_1 a_1 \beta\beta^*)(a_1 a_1 \beta\beta^*)$ will generate a new quartic term $(h^2/2)\mathrm{Tr}A^4$.

Is it possible to cancel $(h^2/2)\mathrm{Tr}A^4$ by adding new interactions to the model? One natural idea is to replace the B^2 in $\mathrm{Tr}(AABB)$ by a Grassmann valued matrix such that the integration will get an extra minus sign. Schematically we may add a term $h\mathrm{Tr}(AA\Psi\overline{\Psi})$ with its conjugation. Then we can look at the term $(h^2/2)(a_1a_1\psi\overline{\psi})(a_1a_1\psi\overline{\psi})$. The contraction of $\psi, \overline{\psi}$ will give $-(h^2/2)a_1^4$. Whether this cancellation could continue to higher order. Whether there is a symmetry reason behind this cancellation?

What if we take $h = 0$, that is only considering the interaction $g\mathrm{Tr}(ABAB)$? Still the $\mathrm{Tr}A^4$ could be generated, but at higher order g^4. For example, consider the contraction of the following term

$$\frac{g^4}{4!}(a_1\beta a_2\beta^*)(a_1\beta a_2\beta^*)(a_1\beta a_2\beta^*)(a_1\beta a_2\beta^*).$$

Also, it will also generate the "double trace" term $\mathrm{Tr}A^2\mathrm{Tr}A^2$, which may not be able to be absorbed into the quadratic term $\mathrm{Tr}A^2$.

Discussion of symmetries What's the reason behind the generation of these new terms? Comparing $\mathrm{Tr}(AABB)$ and $\mathrm{Tr}(A^4) + \mathrm{Tr}(B^4)$, one finds that the first term is invariant under the rescaling $A \to \lambda A, B \to \lambda^{-1}B$, while the second term not. The reason behind the broken of rescaling invariance is the presence of the quadratic term $\mathrm{Tr}(A^2), \mathrm{Tr}(B^2)$. What if there is no quadratic term? Then it's hard to do the perturbative calculation. But in some interesting cases, the quadratic term may have no influence on the model (they can always be shifted away). How this can be the case?

Another thing that may be worth to note is that the interaction $\mathrm{Tr}([A, B]^2), \mathrm{Tr}([A, B]^4), \cdots$ is invariant under "translation" $A \to A + \lambda\mathbb{1}, B \to B + \mu\mathbb{1}$. Reversly, the commutator arises naturally if requiring the "translation invariance". However, the quadratic term also breaks this invariance. If one wants to explore the consequence of these symmetries, it's important to get around the quadratic term.

Let's try to introduce two fermionic matrices ψ, χ. It's convenient to define

$$Z = A + iB, \quad \Psi = \psi + i\chi \tag{40}$$

Also denote $\overline{Z} = A - iB, \overline{\Psi} = \psi - i\chi$. Now I will write down an action that having a BRST-like symmetry. To do this, one needs to introduce two auxillary matrices H (bosonic) and η (fermionic). The BRST-like symmetry for H, η is

$$\delta\eta = H, \quad \delta H = 0.$$

For Z, Ψ we will define

$$\delta Z = \Psi, \quad \delta\Psi = [H, Z] + i\epsilon Z.$$

where ϵ is an arbitrary real parameter. The idea behind this definition is that the square δ^2 will give a usual symmetry of matrix models

$$\delta^2 Z = [H, Z] + i\epsilon Z, \quad \delta^2\Psi = [H, \Psi] + i\epsilon\Psi.$$

The commutator $[H, \cdot]$ is the generator of the unitary transformation. The $i\epsilon$ term generate the rotation symmetry

$$\delta^2 A = -\epsilon B, \quad \delta^2 B = \epsilon A.$$

To construct an action with desired symmetries: unitary, translation, rotation and BRST-like, we write

$$S[H, \eta, Z, \Psi] = -\frac{1}{2}\mathrm{Tr} H^2 + g\mathrm{Tr}\left[\delta(i[\overline{Z}, Z]\eta)\right] \tag{41}$$

The second term is δ-exact, so $\delta S = 0$ by definition. Expanding the action of δ, one gets

$$-\frac{1}{2}\mathrm{Tr} H^2 + ig\mathrm{Tr}([\overline{Z}, Z]H) + ig\mathrm{Tr}([\overline{\Psi}, Z]\eta + [\overline{Z}, \Psi]\eta) \tag{42}$$

The rotation $Z \to \mathrm{e}^{i\theta} Z, \Psi \to \mathrm{e}^{i\theta}\Psi$ and translation $Z \to Z + \lambda\mathbb{1}$ symmetries are satisfied.

One can also consider adding the following δ-exact term to the action

$$i\delta\left(\mathrm{Tr}(\overline{Z}\Psi - Z\overline{\Psi})\right) = 2i\mathrm{Tr}(\overline{\Psi}\Psi) - 2\epsilon\mathrm{Tr}(\overline{Z}Z) - 2i\mathrm{Tr}([\overline{Z}, Z]H).$$

This term breaks the translation symmetry. As a consequence, the quadratic terms are generated.