Master Thesis Notes

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From 16 February to 1 March

Contents

The RG method of matrix model seems having the ability to apply to a complicate model, combining with the perturbative calculation method. The complication may arise from two aspects: 1. the matrix product usually generate complicate interaction; 2. the integration is hard to perform in general; However, our hope is not doing the calculation, but understand whether or not new interaction terms will arise in a qualitative level. Whether symmetries will prevent the action from generating the new terms.

Sat, Feb 17

2×2 matrix is the simplest matrix. It's far away from the large N limit, but some interesting properties may show already at this level.

It's not hard to write down the terms in $\mathrm{tr}(ABBA\cdots)$. If one writes down the matrix in the following form

$$A = \begin{pmatrix} a_1 & \alpha \\ \alpha^* & a_2 \end{pmatrix}.$$

Then the next step is to integrate out α, α^* and a_2. We don't need to actually do the integration; We only need to know what new interactions can be generated

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from the perturbation calculation. If possible, it's also important to know the coupling constants of the new interactions.

There are some typical terms that could appear in $\text{Tr}(A^4)$ and $\text{Tr}(A^3B)$

$$\alpha^*\alpha(a_1)^2\alpha^*\alpha a_1 a_2 \qquad\qquad \alpha^*\alpha(a_2)^2$$
$$(\alpha^*\beta + \beta^*\alpha)a_1 a_2\alpha^*\alpha(b_1 a_2 + b_2 a_1) \qquad \alpha^*\alpha(\alpha^*\beta + \beta^*\alpha) \qquad (1)$$

Performing the integration over α^*, α and a_2 perturbatively, the question is that what the correction to the first order. An easy way to think about the perturbative calculation is through the Wick contraction. It gives non-zero result only if the a_2 is of even order, and α^*, α could form pairs. For example, $\alpha^*\alpha(a_1)^2$ will give a term that is proportional to $(a_1)^2$, while $(\alpha^*\alpha)a_1 a_2$ will give zero. There is no essence difficulty in doing these calculations. Getting the correct combinatoric factors is important. The number of terms explodes in higher order, so maybe some patterns will be helpful to organize the calculation.

Another problem of the perturbative calculation is the choice of quadratic term. For example, a quadratic term in α has the form $\alpha\alpha^*(\cdots)$, where the $(\cdots)$ does not contain α. But $(\cdots)$ could contain other variables, The consequence is that the propagator does not have a fixed value. This may cause difficulty on the calculation, for example, giving a non-linear term in variables that need to be integrated out. The origin of those nonlinear terms is the interaction terms in the matrix integral. So it's possible to treat them perturbatively. However, if one is interested in the non-perturbative aspect, then it's necessary to use the exact propagator. It's possible to study the non-perturbative effect because one works with the large N limit. However our interest is not the large N limit, but the property at finite N. It seems like only perturbation effect is doable when N is finite. Then at certain order of g, the terms will involve different order of N. It's very important to difference our calculation with that was done in the literatures.

the interaction between A and B What kind of features that we want our model have? After integrating out one row and one column, there is no new interaction being generated. To understand under what situation this property will be satisfied, let's think about it order by order. Consider the following 2×2 matrix

$$A = \begin{pmatrix} a_1 & \alpha \\ \alpha^* & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & \beta \\ \beta^* & b_2 \end{pmatrix}.$$

The interaction terms that we are interested in are

$$g\text{Tr}(ABAB) \quad h\text{Tr}(AABB).$$

Looking at them, one expects that $a_1, (a_1)^2, (a_1)^3, (a_1)^4, \cdots$ (similar for b_1) can be generated by integrating out α, α^*, a_2. Now let's figure out which terms in the perturbative expansion will contribute to these "new interactions".

It maybe useful to keep track of the terms by drawing diagrams. For a matrix, let's draw a line with an arrow pointing from one end point to another. The end points indicate the row and column indices respectively. Matrix product is represented

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by joining two end points together, which means that they always take the same value. Therefore, the trace is represented by a loop. One also need to differentiate matrix A and B by using different kinds of lines.

What will be integrated out is the line with end points taking value in a certain row and column. In the case of 2×2 matrix, the value may be 2. The lines that being integrated out will be connected by a "propagator" diagramatically. For the off-diagonal elements α, α^*, the "propagator" is represented by double-line diagram; for the diagonal element a_1, it will be represented by a single line because the row and column indices taking the same value therefore can be regarded as a single point.

The first thing to note is that the odd terms $a_1, (a_1)^3, \cdots$ cannot be generated. We always have even number of A matrix, the terms that could contribute to the odd terms cannot be fully contracted. The next thing to consider is the $(a_1)^2$ term. To the first order of g, h, one gets from $g\mathrm{Tr}(ABAB)$ a term $g(a_1)^2(b_1)^2$ and from $h\mathrm{Tr}(AABB)$ a term $h(a_1)^2\beta^*\beta$. The term $h(a_1)^2\beta^*\beta$ will contribute to the modification of the propagator. Then let's consider the $(a_1)^4$ term. It will only appear to the second order of g, h. We have the following terms

$$\frac{g^2}{2}(a_1 b_1 a_1 b_1)(a_1 b_1 a_1 b_1), \quad \frac{h^2}{2}(a_1 a_1 \beta\beta^*)(a_1 a_1 \beta\beta^*), \quad gh(a_1 a_1 \beta\beta^*)(a_1 b_1 a_1 b_1).$$

How to understand these terms? Do they imply that new interactions will appear along the RG flow? No, for example, the term $(g^2/2)(a_1)^4(b_1)^4$ could be reproduced by $g\mathrm{Tr}(ABAB)$ interaction by expanding the exponent. Similarly the term $gh(a_1)^4(b_1)^2(\beta^*\beta)$ could be reproduced by $g\mathrm{Tr}(ABAB)$ and a proper modification of the quadratic term $\mathrm{Tr}A^2$. However, the second term $(h^2/2)(a_1 a_1 \beta\beta^*)(a_1 a_1 \beta\beta^*)$ will generate a new quartic term $(h^2/2)\mathrm{Tr}A^4$.

Is it possible to cancel $(h^2/2)\mathrm{Tr}A^4$ by adding new interactions to the model? One natural idea is to replace the B^2 in $\mathrm{Tr}(AABB)$ by a Grassmann valued matrix such that the integration will get an extra minus sign. Schematically we may add a term $h\mathrm{Tr}(AA\Psi\overline{\Psi})$ with its conjugation. Then we can look at the term $(h^2/2)(a_1 a_1 \psi\overline{\psi})(a_1 a_1 \psi\overline{\psi})$. The contraction of $\psi, \overline{\psi}$ will give $-(h^2/2)a_1^4$. Whether this cancellation could continue to higher order. Whether there is a symmetry reason behind this cancellation?

What if we take $h = 0$, that is only considering the interaction $g\mathrm{Tr}(ABAB)$? Still the $\mathrm{Tr}A^4$ could be generated, but at higher order g^4. For example, consider the contraction of the following term

$$\frac{g^4}{4!}(a_1\beta a_2\beta^*)(a_1\beta a_2\beta^*)(a_1\beta a_2\beta^*)(a_1\beta a_2\beta^*).$$

Also, it will also generate the "double trace" term $\mathrm{Tr}A^2\mathrm{Tr}A^2$, which may not be able to be absorbed into the quadratic term $\mathrm{Tr}A^2$.

Discussion of symmetries What's the reason behind the generation of these new terms? Comparing $\mathrm{Tr}(AABB)$ and $\mathrm{Tr}(A^4) + \mathrm{Tr}(B^4)$, one finds that the first term is invariant under the rescaling $A \to \lambda A, B \to \lambda^{-1}B$, while the second term

not. The reason behind the broken of rescaling invariance is the presence of the quadratic term $\text{Tr}(A^2), \text{Tr}(B^2)$. What if there is no quadratic term? Then it's hard to do the perturbative calculation. But in some interesting cases, the quadratic term may have no influence on the model (they can always be shifted away). How this can be the case?

Another thing that may be worth to note is that the interaction $\text{Tr}([A, B]^2), \text{Tr}([A, B]^4), \cdots$ is invariant under "translation" $A \to A + \lambda \mathbb{1}, B \to B + \mu \mathbb{1}$. Reversly, the commutator arises naturally if requiring the "translation invariance". However, the quadratic term also breaks this invariance. If one wants to explore the consequence of these symmetries, it's important to get around the quadratic term.

Let's try to introduce two fermionic matrices ψ, χ. It's convenient to define

$$Z = A + iB, \quad \Psi = \psi + i\chi \tag{2}$$

Also denote $\overline{Z} = A - iB, \overline{\Psi} = \psi - i\chi$. Now I will write down an action that having a BRST-like symmetry. To do this, one needs to introduce two auxillary matrices H (bosonic) and η (fermionic). The BRST-like symmetry for H, η is

$$\delta\eta = H, \quad \delta H = 0.$$

For Z, Ψ we will define

$$\delta Z = \Psi, \quad \delta\Psi = [H, Z] + i\epsilon Z.$$

where ϵ is an arbitrary real parameter. The idea behind this definition is that the square δ^2 will give a usual symmetry of matrix models

$$\delta^2 Z = [H, Z] + i\epsilon Z, \quad \delta^2 \Psi = [H, \Psi] + i\epsilon \Psi.$$

The commutator $[H, \cdot]$ is the generator of the unitary transformation. The $i\epsilon$ term generate the rotation symmetry

$$\delta^2 A = -\epsilon B, \quad \delta^2 B = \epsilon A.$$

To construct an action with desired symmetries: unitary, translation, rotation and BRST-like, we write

$$S[H, \eta, Z, \Psi] = -\frac{1}{2}\text{Tr}H^2 + g\text{Tr}\left[\delta(i[\overline{Z}, Z]\eta)\right] \tag{3}$$

The second term is δ-exact, so $\delta S = 0$ by definition. Expanding the action of δ, one gets

$$-\frac{1}{2}\text{Tr}H^2 + ig\text{Tr}([\overline{Z}, Z]H) + ig\text{Tr}([\overline{\Psi}, Z]\eta + [\overline{Z}, \Psi]\eta) \tag{4}$$

The rotation $Z \to e^{i\theta}Z, \Psi \to e^{i\theta}\Psi$ and translation $Z \to Z + \lambda \mathbb{1}$ symmetries are satisfied.

One can also consider adding the following δ-exact term to the action

$$i\delta\left(\text{Tr}(\overline{Z}\Psi - Z\overline{\Psi})\right) = 2i\text{Tr}(\overline{\Psi}\Psi) - 2\epsilon\text{Tr}(\overline{Z}Z) - 2i\text{Tr}([\overline{Z}, Z]H).$$

This term breaks the translation symmetry. As a consequence, the quadratic terms are generated.

how the BRST structure could be helpful? Let's test that whether the BRST structure is restrictive enough such that no new interaction could be generated from the RG calculation.

Let's get a closer look at the action

$$-\frac{1}{2}\text{Tr}H^2 + ig\text{Tr}\left([\overline{Z}, Z]H\right) + ig\text{Tr}\left([\overline{\Psi}, Z]\eta + [\overline{Z}, \Psi]\eta\right).$$

Let's assume that g is small and try to do the perturbative calculation. However, it's not possible because there is no quadratic term. To solve this, let's deform the action by adding the following term

$$i\kappa\delta\left(\text{Tr}(\overline{Z}\Psi - Z\overline{\Psi})\right) = 2i\kappa\text{Tr}\left(\overline{\Psi}\Psi\right) - 2\kappa\epsilon\text{Tr}\left(\overline{Z}Z\right) - 2i\kappa\text{Tr}\left([\overline{Z}, Z]H\right).$$

Because its δ-exact, the deformation is δ-invariant. κ is also a small parameter. To get the usual quadratic term for Z, we choose $\epsilon = 1/4\kappa$. Also note that the action can be simplified if $\kappa = g/2$ because the term $\text{Tr}\left([\overline{Z}, Z]H\right)$ being cancelled. Let's keep this choice, then the action reads

$$-\frac{1}{2}\text{Tr}H^2 - \frac{1}{2}\text{Tr}(\overline{Z}Z) + ig\text{Tr}(\overline{\Psi}\Psi) + ig\text{Tr}\left([\overline{\Psi}, Z]\eta + [\overline{Z}, \Psi]\eta\right).$$

Or getting back to the matrices A, B, ψ, χ

$$-\frac{1}{2}\text{Tr}H^2 - \frac{1}{2}\text{Tr}A^2 - \frac{1}{2}\text{Tr}B^2 - 2g\text{Tr}(\psi\chi) - 2g\text{Tr}\left([\psi, B]\eta - [\chi, A]\eta\right).$$

Let's work with 2×2 matrix and use the following parameterization

$$A = \begin{pmatrix} A & \alpha \\ \alpha^* & a \end{pmatrix}, \quad B = \begin{pmatrix} B & \beta \\ \beta^* & b \end{pmatrix}$$

$$\psi = \begin{pmatrix} \psi & y \\ y^* & \varphi \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi & x \\ x^* & \iota \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta & h \\ h^* & \theta \end{pmatrix}$$

For the $(1, 1)$ components, we use the same letter to indicate that they will not be integrated. Then the quadratic terms have the form

$$-\frac{1}{2}A^2 - \frac{1}{2}B^2 - \frac{1}{2}a^2 - \frac{1}{2}b^2 - \alpha^*\alpha - \beta^*\beta$$
$$-2g\psi\chi - 2gyx^* - 2gxy^* - 2g\varphi\iota$$

The interaction term $-2g\text{Tr}([\psi, B]\eta - [\chi, A]\eta)$ contains

$$-2g\psi\beta h^* + 2g\beta^*\psi h - 2gy^*Bh + 2gByh^* - 2gy\beta^*\eta + 2g\beta y^*\eta$$
$$+2g\chi\alpha h^* - 2g\alpha^*\chi h + 2gx^*Ah - 2gAxh^* + 2gx\alpha^*\eta - 2g\alpha x^*\eta$$
$$-2gy^*bh + 2gbyh^* - 2g\varphi\beta h^* + 2g\beta^*\varphi h - 2gy^*\beta\theta + 2g\beta^*y\theta$$
$$+2gx^*ah - 2gaxh^* + 2g\iota\alpha h^* - 2g\alpha^*\iota h + 2gx^*\alpha\theta - 2g\alpha^*x\theta$$

There is an additional term that only involving the $(1, 1)$-component. It has the same form with the initial action $-2g\text{Tr}([\psi, B]\eta - [\chi, A]\eta)$.

For the Grassmann integral, to get non-vanishing result, one needs to saturate the measure $\mathrm{d}y\mathrm{d}y^*\mathrm{d}\varphi\mathrm{d}x\mathrm{d}x^*\mathrm{d}\iota\mathrm{d}h\mathrm{d}h^*\mathrm{d}\theta$. Because there is no quadratic term of η, the non-vanishing terms must include $hh^*\theta$. For other variables, if they appear, they must form the pairs

$$a^2, \quad b^2, \quad \alpha^*\alpha, \quad \beta^*\beta, \quad yx^*, \quad xy^*, \quad \varphi\iota.$$

For fermionic variables $yx^*, xy^*, \varphi\iota$, they must not appear more than one time. These are the basic constraints to get non-vanishing integral result.

First note that h, h^*, θ appear in each term once. Therefore, to get non zero result, one needs at least the cubic term

$$-\frac{4}{3}g^3 \left[\mathrm{Tr}([\psi, B]\eta - [\chi, A]\eta)\right]^3.$$

Let's focus on the terms that containing these combinations: $\psi B, \psi A, \chi B, \chi A$. One would expect that all of them being vanish because they have an odd number of fermionic variables. For $\psi B, \chi A$, to fully contract the variables, one needs terms containing $x\beta^*, y\alpha^*$. There is no such term. For ψA, one selects the following terms

$$(-\psi\beta h^*)(x^*Ah)(\beta^*y\theta) + (\beta^*\psi h)(-Axh^*)(-y^*\beta\theta).$$

By contracting hh^*, xy^*, yx^* one finds that the result being zero. Similar for χB. So it's in accordance with the expectation. (TODO: analyze other terms like $A^2, AB, B^2, \psi\chi$)

It's easy to understand that $\left[\mathrm{Tr}([\psi, B]\eta - [\chi, A]\eta)\right]^5$ always vanishing because one of the h, h^*, θ, η will appear twice. So one only need to consider the quartic term

$$\frac{2}{3}g^4 \left[\mathrm{Tr}([\psi, B]\eta - [\chi, A]\eta)\right]^4.$$

Out of all possible combinations, it may be interestring to consider those containing $\psi\eta$ and $\chi\eta$ first. To find the terms that containing $\psi\eta$, one first select $(-\psi\beta h^*)$ or $(\beta^*\psi h)$. Let's take the first choice as an example. Then one needs to add a term that containing h but not containing A, B, χ because we only focus on $\psi\eta$. The possibilities are the following

$$-y^*bh, \quad x^*ah, \quad \beta^*\varphi h, \quad -\alpha^*\iota h.$$

Each of them containing one of a, b, φ, ι. To get non-vanishing result, one needs another term that only containing one of them. But there is no such term: they either contain $\varphi b, \iota a$ or containing none of them. Therefore, $\psi\eta$ will not be generated through the perturbative calculation.

It's also easy to see that the terms like $A\psi\chi$ will not be generated because h, h^* would be over-saturate. Then it's interesting to consider $\psi A\eta$ which does not appear in the original action. For example, one starts by selecting the two terms

$$(-\psi\beta h^*)(x^*Ah).$$

Then one must select the term $(-y\beta^*\eta)$ that containing η. Because otherwise β, x^* cannot be contracted. However, then one must include also an extra θ term

to saturate $d\theta$. Any of them will contain one of $\alpha, \beta, \varphi, \iota, a, b$ that having no partner to contract. Therefore, no $\psi A \eta$ could be generated.

let's consider another possible new term $\psi \chi \eta$. It contains three fermionic variables, therefore being expected to vanish. Let's verify it explicitly. The following terms would contribute to $\psi \chi \eta$

$$(-\psi \beta h^*)(-\alpha^* \chi h)\left[(-y\beta^*\eta)(x^*\alpha\theta) + (-\alpha x^*\eta)(\beta^* y\theta)\right].$$

$$(\beta^* \psi h)(\chi \alpha h^*)\left[(\beta y^*\eta)(\alpha^* x\theta) + (x\alpha^*\eta)(-y^*\beta\theta)\right].$$

It's straightforward to check they vanish because the order of y, x^* and y^*, x are different in the two terms.

It turns out that nothing could be generated from integrating out those variables. Then the question is that whether the same thing hold for general N?

Extend the previous analysis to other cases. It's clear that the fermionic matrix η plays a special role in this model. It has no "partner", so the only way to get non-vanishing result is to saturate the measure. In the case of 2×2 matrix, this forces the first non-vanishing perturbative expansion starting from the cubic term (interactions)^3. Similarly, for $N \times N$ matrix, the first non-trivial contribution would start from $(\cdots)^{2N-1}$.

There is one natrual question to think about. Given an $N \times N$ fermionic matrix ψ, one may wonder what's the largest power $\mathrm{Tr}\psi^k$ for non-zero result. It's obvious that if $k > N^2$, then it must vanish because there are only N^2 elements in ψ. But what if $k = N^2$? If N is even, the $\mathrm{Tr}\psi^{N^2} = 0$ according to the cyclic identity. (TODO: I don't know how to prove it's non-vanishing for odd N. And how to prove that it's non-vanishing for $\mathrm{Tr}\psi^{N^2-1}$ with even N.)

(TODO: There is an error in the analyze below: the matrix product arising from the contraction of fermionic variables should not be understood in the usual sense. The same contraction, for example between x_1, y_1^*, can not happen twice.)

For a general N, one needs to saturate $dh_1 dh_1^* \cdots dh_{N-1} dh_{N-1}^* d\theta$. The first case that one may consider is that all the hs are contributed by $x^* A h$ and $h^* A x$. The only way to contract these two terms is by adding $\beta^* \eta y$ and $y^* \eta \beta$ in the following way

$$(h^* A x)(y^* \eta \beta)(\beta^* \eta y)(x^* A h).$$

Then one expects that the term that could be generated containing the elements of the matrix $A\eta\eta A$. For example, a typical term could look like

$$(A\eta\eta A)_{1\sigma(1)} \cdots (A\eta\eta A)_{(N-1)\sigma(N-1)}.$$

where σ is a certain permutation. However, one still need to saturate the fermionic variable θ. That is, one must replace one of the η above by θ. This always accompanies with a same term with opposite sign $(A\eta\theta A)$ and $(A\theta\eta A)$. Or one can expect that by realizing there must be an odd number of fermionic variables. So there is no term that containing purely A or B could be generated.

At this stage one may wonder that how is it possible to get only one θ in the perturbative expansion while getting non-vanishing result? This leads us to consider the coupling between the bosonic and fermionic matrices. Let's start with $A\psi$, the two structures x^*Ah and $h^*\psi\beta$ can be contracted by using $\beta^*\eta y$ or $\beta^*\theta y$.

$$(h^*\psi\beta)(\beta^*\eta y)(x^*Ah).$$

Then one expect that the elements of matrix $\psi\eta A$ will appear. However, don't forget that whenever such a term exist, there is a corresponding possibility: replacing it with the following

$$(h^*Ax)(y^*\eta\beta)(\beta^*\psi h).$$

Note that they are conjugate to each other

$$[(h^*\psi\beta)(\beta^*\eta y)(x^*Ah)]^\dagger = -(h^*Ax)(y^*\eta\beta)(\beta^*\psi h).$$

The minus sign arises from the interchange between fermionic numbers in each bracket $(\cdots)$. (TODO: So if the action is self-conjugate, one expects that the terms involving $A\eta\psi$ will vanish. But it's not clear that the action is self-conjugate.)

The followings are all possible results of contraction

$$A\eta\eta A, \quad B\eta\eta B, \quad AB$$
$$A\eta\psi, \quad B\eta\chi$$
$$\psi\psi, \quad \chi\chi, \quad \psi\eta\eta\chi$$

These forms can be read from the structure of the interaction. A general term in the perturbative expansion will be a multiplication of matrix elements M_{ij} with M choosing from the above matrices and i and j run over all possible values from 1 to $N-1$. This prescription is to ensure the saturation of h. For the saturation of θ, one can replace one η with θ. But this replacement will lead to cancellation as above. Therefore, it is understood that the fermionic matrix η plays an essential role in all the cancelation.

rethink the model   In constructing the previous model, the way we introduce the matrix η is suspicious. Specifically $\delta^2\eta = 0$, but it should generate the unitary transformation $\delta^2\eta = [H, \eta]$. Remember that the idea behind the BRST construction is "taking the square root of gauge transformation". However, it seems impossible to satisfy $\delta^2 H = 0$ simultaneously. It seems like the only way to solve this problem is to introduce again an auxillary matrix such that $\delta H = [X, \eta]$. That is the gauge transformation is parameterized by the auxillary matrix X.

What's the motivation to introduce η in our model? I want to introduce a term $\text{Tr}([\overline{Z}, Z]H)$ in the action, from doing the variation $\delta(\cdots)$. To obtaine H from δ, I introduce η such that $\delta\eta = H$. This is a bad move as having been said above: this breaks the unitary invariance when the model involving the matrix η.

one possible solution to the previous problem?   If we introduce another auxillary matrix X to parameterize the unitary transformation, and modify $\delta H = [X, \eta]$ with $\delta X = 0$. (Remark. H, X, η should all be anti-hermitian to keep each term

"real" under the hermitian conjugation †) The model build may still start from the commutator term

$$-\frac{1}{2}\mathrm{Tr}H^2 + ig\mathrm{Tr}\left([\overline{Z}, Z]H\right).$$

Then one wants to find δ-exact terms to generate these two terms. The candidates are

$$ig\delta\mathrm{Tr}\left([\overline{Z}, Z]\eta\right) - \frac{1}{2}\delta\mathrm{Tr}\left(\eta H\right)$$

Other terms are generated along

$$ig\mathrm{Tr}\left([\overline{\Psi}, Z]\eta + [\overline{Z}, \Psi]\eta\right) + \frac{1}{2}\mathrm{Tr}\left(\eta[X, \eta]\right).$$

Note that $(1/2)\mathrm{Tr}(\eta[X, \eta]) = \mathrm{Tr}(\eta X\eta)$ gives a quadratic term of η. To deal with the interaction terms $ig\mathrm{Tr}([\overline{Z}, Z]H)$ and $ig\mathrm{Tr}([\overline{\Psi}, Z]\eta + [\overline{Z}, \Psi]\eta)$, it's important to introduce quadratic term for $Z, \overline{Z}$ and $\Psi, \overline{\Psi}$. This is done by adding to following δ-exact term

$$i\kappa\delta\mathrm{Tr}\left(\overline{Z}\Psi - Z\overline{\Psi}\right) = 2i\kappa\mathrm{Tr}(\overline{\Psi}\Psi) - 2\kappa\epsilon\mathrm{Tr}(\overline{Z}Z) - 2i\kappa\mathrm{Tr}\left([\overline{Z}, Z]X\right).$$

Here X is not a dynamical matrix, and it should be anti-hermitian. In summary, let's collect all terms in the action

$$-\frac{1}{2}\mathrm{Tr}H^2 + \mathrm{Tr}\eta X\eta$$
$$+\frac{i}{2}\kappa\mathrm{Tr}\overline{\Psi}\Psi - \frac{1}{2}\kappa\epsilon\mathrm{Tr}\overline{Z}Z - \frac{i}{4}\kappa\mathrm{Tr}\left([\overline{Z}, Z]X\right)$$
$$+ig\mathrm{Tr}\left([\overline{Z}, Z]H\right) + ig\mathrm{Tr}\left([\overline{\Psi}, Z]\eta + [\overline{Z}, \Psi]\eta\right)$$

It can be verified all the terms are real under † (It acts on the matrix, will not take $i \rightarrow -i$).

Compare to previous wrong model, it has several differences. First is the appearance of the X matrix. It it is taken to be zero, the model will become very similar to what we have studied previously, but with an additional term $ig\mathrm{Tr}\left([\overline{Z}, Z]H\right)$. However, the previous calculation is wrong because η should be anti-hermitian, rather than hermitian. This means that there is no θ variable to be integrated out; also h^* and h terms should have a relative minus sign. Let's list all terms that are relevent to the integration

$$-v^*A\beta + \beta^*Av + v^*B\alpha - \alpha^*Bv + \alpha^*H\beta - \beta^*H\alpha$$
$$x^*Ah - h^*Ax - y^*Bh + h^*By$$
$$\beta^*\psi h - h^*\psi\beta - \alpha^*\chi h + h^*\chi\alpha$$
$$-\alpha^*\eta x - x^*\eta\alpha + \beta^*\eta y + y^*\eta\beta$$

It's interesting to take $X = iz\mathbb{1}$. z is a real constant to be determined later. One consequence is

$$\mathrm{Tr}\left(\eta X\eta\right) \sim -izh_i^*h_i.$$

Remember that $\eta_{(N+1),i} = -h_i^*$, $\eta_{i,(N+1)} = h_i$. Another consequence is

$$\frac{\kappa}{2}\text{Tr}\left([A,B]X\right) \sim \frac{\kappa}{2}iz(\beta_i^*\alpha_i - \alpha_i^*\beta_i).$$

So the quadratic term for α, β is

$$-\frac{\kappa\epsilon}{2}(\alpha_i^*\alpha_i + \beta_i^*\beta_i) + \frac{i\kappa z}{2}(\beta_i^*\alpha_i - \alpha_i^*\beta_i).$$

This motivates us to redefine

$$\tilde{\alpha} = \frac{1}{\sqrt{2}}(\alpha + i\beta), \quad \tilde{\beta} = \frac{1}{\sqrt{2}}(\alpha - i\beta).$$

Then the quadratic term becomes

$$-\frac{\kappa}{2}\left[(\epsilon + z)\tilde{\alpha}_i^*\tilde{\alpha}_i + (\epsilon - z)\tilde{\beta}_i^*\tilde{\beta}_i\right].$$

The interaction of A, B with the "bosonic legs" in terms of $\tilde{\alpha}, \tilde{\beta}$ reads

$$\frac{1}{\sqrt{2}}\left(iv^*A\tilde{\alpha} + i\tilde{\alpha}^*Av - iv^*A\tilde{\beta} - i\tilde{\beta}^*Av + v^*B\tilde{\alpha} - \tilde{\alpha}^*Bv + v^*B\tilde{\beta} - \tilde{\beta}^*Bv\right).$$

It's possible to obtain a new operator $\text{Tr}(AB)$ through the contraction. To the lowest order there are four terms that contribute to $\text{Tr}(AB)$:

$$\frac{i}{2}\left(\tilde{\alpha}^*Avv^*B\tilde{\alpha} - v^*A\tilde{\alpha}\tilde{\alpha}^*Bv + v^*A\tilde{\beta}\tilde{\beta}^*Bv - \tilde{\beta}^*Avv^*B\tilde{\beta}\right).$$

We see that there is actually no $\text{Tr}(AB)$ generated at this order. For the interaction of A, B with the "fermionic legs"

$$x^*Ah - h^*Ax - y^*Bh + h^*By.$$

There are two terms that contribute to the $\text{Tr}(AB)$

$$x^*Ahh^*By + h^*Axy^*Bh.$$

The order of contraction is yx^* and y^*x. So there is also no contribution at this order.

(TODO: more discussions on the propagators $\alpha^*\alpha$ and $\beta^*\beta$, and possible cancellation or suppression of new interaction generations.)

A comparison between the "dimensional model" and the "dimensionless model"
Let's start with a "dimensionful" model, whose partition function is given by

$$Z_N(g) = \int \exp\left[-N\left(\frac{1}{2}\text{Tr}\,M^2 + \frac{g}{4}\text{Tr}\,M^4\right)\right]dM. \tag{5}$$

The scaling of the matrix M, and the coupling g under $N \to \lambda N$ is

$$M \to \lambda^{-1/2}M$$
$$g \to \lambda g$$

One could also write the same theory in a "dimensionless" way, by introducing $\tilde{M} = \sqrt{N}M$ and $\tilde{g} = N^{-1}g$. Then in terms of $\tilde{M}$ and $\tilde{g}$ the partition function reads

$$Z_N(g) = N^{N^2/2} \int \exp\left[-\left(\frac{1}{2}\operatorname{Tr}\tilde{M}^2 + \frac{\tilde{g}}{4}\operatorname{Tr}\tilde{M}^4\right)\right] \mathrm{d}\tilde{M}. \tag{6}$$

Starting from the "dimensionless model" (6), it's possible to recover the "dimensionful model" (5) by specifying the quadratic term $\frac{N}{2}\operatorname{Tr}M^2$ first.

the notions of scaling   First is the scaling from the RG flow. Let's start by thinking about the RG calculation to the lowest order. This will just reproduce the classical scaling: $g \to \lambda g$ and $\tilde{g}$ is invariant. However, they are essentially different. In the language of QFT, one can think of N as a cut-off. Then the RG method allows us to relate theories with different cut-off such that they will produce the same results (the correlation functions). Specifically, to the lowest order, the coupling $g \to \lambda g$ with the change of cut-off $N \to \lambda N$. To the lowest order, these are exactly the same as the classical scaling. However, the higher order corrections are essential for the RG flow.

Conceptually, the RG flow keeps the theory invariant, while the classical scaling not: one knows that the matrix model has a non-trivial N dependence although they share the same form of action. The classical scaling is just a natural way to define how the matrix and coupling depending on the underlying scale such that the form of the action keeping the same. The classical scaling works like "zooming in" or "zooming out". However, the RG flow works like "coarse graining". There is no reason to believe that the "coarse graining" will give a similar result comparing to the "zooming" in general. The Gaussian model is a special example that they giving exactly the same result.

Now what about the notion of "scaling invariance"? In this case, it's more interesting to consider yet another scaling. Let's call it "dynamical scaling". The meaning is that only "dynamical variables" should be rescaled. The matrix is dynamical but the coupling constant is not. Therefore, this scaling should not be understood as a change of dimension; It's a symmetry of the action. The interesting thing about the RG flow is that the dynamical scaling invariance could emerge at certain critical point g_*. The existence of such a point $g_* \neq 0$ seems impossible by just looking at the action, because it is written in a form that only the classical scaling is obvious. It's impossible to obtain a dynamical scaling invariance along the classical scaling. While the RG flow could deviate from the classical scaling significantly at some points, along which the scaling of g could be frozen.

Start with the $N+1$-model, and decomposing the matrix M as

$$\begin{pmatrix} M & v \\ v^\dagger & \alpha \end{pmatrix}.$$

The action can be expanded as

$$S_{N+1}[M, v, v^\dagger, \alpha; g] = (N+1)\operatorname{Tr}\left(\frac{1}{2}M^2 + \frac{g}{4}M^4\right) + (N+1)\left(v^\dagger v + \frac{1}{2}\alpha^2\right)$$
$$+ g(N+1)\left(v^\dagger M^2 v + \alpha v^\dagger M v + \alpha^2 v^\dagger v + \frac{1}{2}(v^\dagger v)^2 + \frac{1}{4}\alpha^4\right).$$
(7)

To do the integration over v, $v^\dagger$ and α, one needs to first identify the quadratic
term around which the interaction is treated perturbatively. For v, $v^\dagger$, the quadratic
term is

$$(N+1)v^\dagger\left(\mathbb{1} + gM^2\right)v.$$

For α, it's

$$\frac{N+1}{2}\alpha^2.$$

These leads to the following contraction rules

$$\left\langle v_i^\dagger v_j \right\rangle = \frac{1}{N+1}\left(\frac{1}{1+gM^2}\right)_{ij}$$

$$\langle\alpha\alpha\rangle = \frac{1}{N+1}$$

The interaction terms are

$$(N+1)\left(g\alpha v^\dagger M v + g\alpha^2 v^\dagger v + \frac{g}{2}(v^\dagger v)^2 + \frac{g}{4}\alpha^4\right).$$

First, at the zeroth order, the partition function has the form

$$Z_{N+1} \sim \int [\mathrm{d}M]\frac{1}{(N+1)^N \operatorname{Det}(\mathbb{1} + gM^2)}e^{-(N+1)\operatorname{Tr}\left(\frac{1}{2}M^2 + \frac{g}{4}M^4\right)}.$$

The $\frac{1}{(N+1)^N}$ factor will be compensated by a rescaling of the effective partition
function. The determinant can be re-exponented as

$$\frac{1}{\operatorname{Det}(\mathbb{1} + gM^2)} = e^{-\operatorname{Tr}\ln(1+gM^2)}.$$

This leads to the following "effective action"

$$S_{\mathrm{eff}} = (N+1)\operatorname{Tr}\left(\frac{1}{2}M^2 + \frac{g}{4}M^4\right) + \operatorname{Tr}\ln(1 + gM^2).$$
(8)

Then at the order g, one could write the partition function Z_{N+1} as

$$Z_{N+1} \sim -g(N+1)\int[\mathrm{d}M]\left\langle \alpha v^\dagger M v + \alpha^2 v^\dagger v + \frac{1}{2}(v^\dagger v)^2 + \frac{1}{4}\alpha^4 \right\rangle e^{-S_{\mathrm{eff}}[M]}.$$

The first term will not contribute, while other terms giving

$$\langle \alpha^2 v^\dagger v \rangle = \frac{1}{(N+1)^2} \, \text{Tr} \left(\frac{1}{\mathbb{1} + gM^2} \right)$$

$$\frac{1}{2} \langle (v^\dagger v)^2 \rangle = \frac{1}{2(N+1)^2} \left[\text{Tr} \left(\frac{1}{\mathbb{1} + gM^2} \right) \right]^2 + \frac{1}{2(N+1)^2} \, \text{Tr} \left(\frac{1}{\mathbb{1} + gM^2} \right)^2$$

$$\frac{1}{4} \langle \alpha^4 \rangle = \frac{3}{4(N+1)}.$$

Let's study the case where g is small and keep only the first order of g. The "effective action" becomes

$$S_{\text{eff}}[M] = (N+1) \, \text{Tr} \left(\frac{1}{2} M^2 + \frac{g}{4} M^4 \right) + g \, \text{Tr} \, M^2.$$

The corrections from the interactions to the partition function are

$$Z_{N+1} \sim -g(N+1) \int [dM] \left(\frac{3N}{2(N+1)^2} + \frac{N^2}{2(N+1)^2} + \frac{3}{4(N+1)} \right) e^{-S_{\text{eff}}[M]}.$$

Re-exponentiate these terms will give us a new action, which could reproduce the correlation functions calculated from $S_{N+1}[M]$ to the first order of g,

$$S'_{\text{eff}}[M] = (N+1) \, \text{Tr} \left(\frac{1}{2} M^2 + \frac{g}{4} M^4 \right) + g \, \text{Tr} \, M^2 + \frac{3N}{2(N+1)} g + \frac{N^2}{2(N+1)} g + \frac{3}{4} g.$$

A feature of the RG flow procedure is the appearance of terms with different orders of N.

To put the effective action in the canonical form, one defines

$$M' = \left(1 + \frac{1+2g}{N} \right)^{\frac{1}{2}} M,$$

$$g' = \frac{N(N+1)}{(N+1+2g)^2} g$$

To the first order of g

$$M' \approx \left(\frac{N+1}{N} \right)^{\frac{1}{2}} \left(1 + \frac{g}{N+1} \right) M,$$

$$g' \approx \left(\frac{N}{N+1} \right) \left(1 - \frac{4g}{N+1} \right) g$$

such that

$$S'_{\text{eff}}[M] = N \, \text{Tr} \left(\frac{1}{2} M'^2 + \frac{g'}{4} M'^4 \right) + \cdots \qquad (9)$$

To get a feeling about what kind of terms could be generated through this RG procedure, let's calculate the decomposition of $\frac{1}{6} \text{Tr} M^6$:

$$v^\dagger M^4 v + (v^\dagger v)(v^\dagger M^2 v) + \frac{1}{2} (v^\dagger M v)(v^\dagger M v) + \frac{1}{3} (v^\dagger v)^3$$

$$+ \alpha^2 v^\dagger M^2 v + \frac{3}{2} \alpha^2 (v^\dagger v)^2 + \alpha^4 v^\dagger v.$$

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Looking at the first line (terms without α), one could think about the possible contractions. The first term is quadratic in v, while other terms should be treated by perturbation theory. It's important to note that, for the second term, if contracting $(v^\dagger v)$ one gets

$$\frac{1}{N+1}\mathrm{Tr}\left(\frac{1}{\mathbb{1}+\cdots}\right).$$

where $\cdots$ coming from the quadratic terms like gM^2. The leading order in g would be

$$\frac{1}{N+1}\mathrm{Tr}\mathbb{1} = \frac{N}{N+1}.$$

One would say that this term has the N-dimension 0. This is inconsistent with the dimension of $(v^\dagger v)$, which is the same as $\mathrm{Tr}M^2$. The reason is that $\mathrm{Tr}\mathbb{1}$ has dimension 1. The dimension of Tr should also be taken into account correctly.

To the zeroth order of g, contraction of the first line gives

$$\frac{N}{(N+1)^2}\mathrm{Tr}M^2 + \frac{3}{2(N+1)^2}\mathrm{Tr}M^2 + \frac{1}{2(N+1)^2}\left(\mathrm{Tr}M\right)^2 + \frac{1}{3}\left(\frac{N}{N+1}\right)^3.$$

The first term has the "wrong" dimension because of the $v^\dagger v$ contraction. When N is large, however, it gives the leading contribution to $\mathrm{Tr}M^2$. The N-dimension essentially tells us how the terms behave when changing N. This is an assumption we made for our theory.

One crucial idea for analyzing the matrix model is to reduce the number of variables from N^2 to N. This can be done because of the symmetry of the model. What's the physical insight follows from this reduction? If one interpret the matrix as a Hamiltonian, the reduction is essentially chosing the energy eigen-basis for the physical states. The symmetry is just the fact that the physical observables do not depend on the choice of basis. However, we would like to take about the matrix model arising in the string theory. There, rather than an operator on the space of physical states, the matrix itself represents the physical states: the Chan-Paton degrees of freedom of open string. The number of the Hermitian matrices matches the number of the Chan-Paton degrees of freedom.

The question: imagine a scaling matrix model, in the sense that the matrix M and coupling g scale with N in a simple way. Along this scaling, the physical observables are kept invariant. If such a scaling phenomenon exists, what can we say about the physical observables around such critical point?

It's not clear that what's the meaning of conformal invariance for a matrix model, but we have the motivation to believe that there exists such a notion. From the point view of AdS/CFT, the matching between the conformal group, for example in $4d$ the conformal group is $SO(4,2)$, and the AdS isometry group, for example in $5d$ the isometry group is $O(4,2)$. However, such a "group argument" does not make sense for AdS1/CFT0.

Let's review that how the AdS/CFT is understood in string theory. The central object is the D-brane whose dynamics can be described by an effective action of field theory. The fields live on the world-volume of D-brane: there are massless scalar fields and vector fields. Their appearance is understood as the massless

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excitation of open strings with end points on the D-brane. The form of the action is obtained by matching with the calculation from the perturbative string theory. This field theory will become the CFT side.

Question 1. *But why the field theory of D-branes must be a CFT? Does this fact relate to the dynamics of the D-branes? Or is there a "string theory" argument for the appearance of conformal invariance?*

Answer 1. *Because the open string excitation is massless when the two end points locate on the coincident D-branes.*

D-branes carry the R-R charges, which means that it will couple to the higher form gauge fields. Therefore they can be understood physically as the sources of those gauge fields. The D(-1)-brane couples to the 0-form potential in the IIB theory C_0. The energy of that coupling is given by the value of C_0 at the point where the D(-1)-brane locating. Therefore the D(-1)-brane is properly interpreted as an instanton.

We are interested in the D-instanton, a natural question is that how the D-instantons interact with themselves and other objects like higher dimensional D-branes. This question could be answered in the perturbative string theory. One calculates the string amplitudes for exchanging the closed string states (graviton, dilaton and the R-R state) between the D-branes. The fundamental physical properties of the D-branes (the tension and the R-R charge) could be related to the fundamental constants in string theory.

Question 2. *When discussing the interaction and action of D-branes, we have in mind a picture where the D-brane locating at a particular space-time position. In this way, rather than taking them as a fundamental degrees of freedom, we regard them as absolute objects (like black hole in GR). In such a setting, how should we understand the physics under the D-brane action? A clearer name seems to be the "action of D-brane fluctuation", and the fluctuation has its origin at the open string oscillation. Or maybe one should call it the "open string action restricted on the D-branes"?*

There is an interesting explanation of the collective coordinates X^μ of the D-branes: the Goldstone bosons for the spontaneous translation symmetry breaking. But why it's called the "spontaneous"?

Idea 1. *Why not trying to look further at the "Ward identity" of the matrix model and think about the possible implications on our calculations?*