COVARIANT DESCRIPTION OF SUPERSTRINGS

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A Lorentz-covariant superstring action with ten-dimensional supersymmetry is presented. The action also possesses local supersymmetry that permits the choice of a light-cone gauge in which it reduces to one discussed previously.

In older string theories a light-cone-gauge formulation can be derived from a covariant and gauge-invariant one [1]. For example, free bosonic strings with coordinates $X^\mu(\sigma, \tau)$, where σ and τ parametrize the world-sheet of the string, can be described by the reparametrization-invariant action [2] [‡1]

$$S = -\frac{1}{2\pi}\int \mathrm{d}\sigma\, \mathrm{d}\tau \sqrt{-g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X_\mu\,. \tag{1}$$

The metric $g_{\alpha\beta}$ can be eliminated from the classical action by substituting the solution of its equation of motion thereby obtaining the Nambu form. It is also possible classically to choose a gauge with $g^{\alpha\beta} \propto \eta^{\alpha\beta}$ (we take $\eta^{\sigma\sigma} = -\eta^{\tau\tau} = 1$ and $\eta^{\sigma\tau} = \eta^{\tau\sigma} = 0$), provided the resulting action is supplemented by the constraints

$$(\partial_\tau X \pm \partial_\sigma X)^2 = 0\,. \tag{2}$$

Quantum mechanically, this analysis is only correct (after a certain amount of normal ordering) for $D=26$. At this stage enough gauge freedom still remains to make the light-cone gauge choice

$$X^+(\sigma, \tau) = x^+ + p^+\tau\,, \tag{3}$$

so that all the string dynamics are given by the transverse coordinates $X^i(\sigma, \tau)$ and described by the trivial action

$$S_{\mathrm{l.c.}} = -\frac{1}{2\pi}\int \mathrm{d}\sigma\, \mathrm{d}\tau\, \eta^{\alpha\beta}\, \partial_\alpha X^i\, \partial_\beta X^i\,. \tag{4}$$

Eq. (2) may then be regarded as determining X^- (up to a constant) in terms X^+ and X^i.

In the case of superstrings, we are in the peculiar position of knowing the analogue of eq. (4), but not of eq. (1). Namely, by adding Grassmann coordinates $\theta^{Aa}(\sigma, \tau)$, where $A = 1,2$ is a two-dimensional spinor index and a is D-dimensional spinor index of appropriate type, we were able to make a supersymmetrical generalization [3] of eq. (4). The most interesting case is $D = 10$, with θ^{1a} and θ^{2a} 32-component Majorana–Weyl spinors having either the same or opposite handedness. There are other possibilities for classical theories (as we will show), but they give rise to the same difficulties quantum mechanically as the $D \neq 26$ bosonic theory. The generalization of eq. (4) for superstrings is [3] [‡2]

$$S_{\mathrm{l.c.}} = \int \mathrm{d}\sigma\, \mathrm{d}\tau [-(2\pi)^{-1}\eta^{\alpha\beta}\, \partial_\alpha X^i\, \partial_\beta X^i + (\mathrm{i}p^+/\pi)\, \bar{\theta}\gamma^-\rho^\alpha\, \partial_\alpha\theta]\,. \tag{5}$$

The condition in eq. (3) must be supplemented by

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‡1 Indices μ, ν, ... are D-valued vector indices contracted with a flat Minkowski metric. In light-cone coordinates, $A \cdot B = A^iB^i - A^+B^- - A^-B^+$. The two-vector indices, α, β, ... take the values σ and τ. The string tension has been set equal to $1/\pi$. These conventions are the same as in refs. [3,4].

‡2 In this expression ρ^α_{AB} and γ^μ_{ab} are Dirac matrices satisfying $\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}$ and $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$. $\bar{\bar{\theta}}^{Aa} = \bar{\theta}^{Ba}\rho^0_{BA} = \theta^{Bb}\rho^0_{BA}\gamma^0_{ba}$. θ^{Aa}_{ba} is related to S^{Aa}, defined in ref. [3], by $\theta^{Aa} = (1/2\sqrt{p^+})\, S^{Aa}$.

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$$\gamma^+\theta^A = 0 . \tag{6}$$

It is then straightforward to show that S is invariant under the global supersymmetry transformations

$$\delta_\eta X^i = 2\bar{\eta}\gamma^i\theta , \tag{7a}$$

$$\delta_\eta\theta = (i/2p^+)[\gamma_- \gamma_\mu p^\alpha \eta]\, \partial_\alpha X^\mu . \tag{7b}$$

This is a lot of symmetry, because η^{Aa} has 32 independent real components (for $D = 10$). Rules for Lorentz transformations were also found in ref. [3], and the formalism was used to construct interacting theories in the light-cone gauge [4,5]. The purpose of this letter is to present a generalization of eq. (1) from which eq. (5) can be derived.

A covariant action principle for massless superparticles is known [6]. It contains a number of useful clues for the superstring problem so we first describe it briefly. The action is

$$S_{\text{particle}} = -\frac{1}{2}\int V^{-1}\Big(\dot{X}^\mu - i\sum_{A=1}^{N} \bar{\theta}^A\gamma^\mu\dot{\theta}^A\Big)^2 \mathrm{d}\tau, \tag{8}$$

where V may be regarded as the square-root of a one by one metric tensor. This expression has local one-dimensional reparametrization invariance. It also has global super-Poincaré invariance including N-extended supersymmetry

$$\delta_\epsilon\theta^A = \epsilon^A , \tag{9a}$$

$$\delta_\epsilon X^\mu = i\sum_{A=1}^{N} \bar{\epsilon}^A\gamma^\mu\theta^A , \tag{9b}$$

$$\delta_\epsilon V = 0 . \tag{9c}$$

(Henceforth, summation of repeated indices is understood.) It was found in ref. [6] that half the components of θ^A drop out of the equations of motion so that it is possible to impose eq. (6). Siegel has shown [7] that the reason for this is that the action defined in eq. (8) also has local supersymmetry:

$$\delta_\kappa X^\mu = -\bar{\theta}^A\gamma^\mu\gamma\cdot p\kappa^A , \tag{10a}$$

$$\delta_\kappa\theta^A = i\gamma\cdot p\kappa^A , \tag{10b}$$

$$\delta_\kappa V = 4V\dot{\bar{\theta}}^A\kappa^A , \tag{10c}$$

where

$$p^\mu = \dot{X}^\mu - i\bar{\theta}^A\gamma^\mu\dot{\theta}^A . \tag{11}$$

It also has an additional local bosonic symmetry

$$\delta_\lambda\theta^A = \lambda\dot{\theta}^A , \tag{12a}$$

$$\delta_\lambda X^\mu = i\bar{\theta}^A\gamma^\mu\delta_\lambda\theta^A , \tag{12b}$$

$$\delta_\lambda V = 0 . \tag{12c}$$

As written, the proof of these symmetries requires the Majorana property for θ, but if $\bar{\theta}\gamma^\mu\dot{\theta}$ is replaced by $\frac{1}{2}(\bar{\theta}\gamma_\mu\dot{\theta} - \dot{\bar{\theta}}\gamma_\mu\theta)$, it can be generalized to other cases.

In view of these results, we expect a covariant superstring action to have both global ϵ and local κ supersymmetries generalizing eqs. (9) and (10). If it exists, the global η transformations of eq. (7) should be understood as a linear combination of an ϵ transformation, an induced local κ transformation, and an induced ξ^α reparametrization. The induced transformations are determined by requiring that the gauge conditions of eqs. (3) and (6) are preserved. By studying this connection in detail we have discovered that in the superstring case the two global supersymmetries must be interpreted as a pair of world-sheet scalars, whereas the two local supersymmetries must form components of a world-sheet vector. This is rather astonishing in as much as the combined η transformation of the light-cone-gauge action is a world-sheet spinor! Once this is understood, it is not difficult to discover the result for the covariant superstring action:

$$S = \frac{1}{\pi}\int \mathrm{d}\sigma\, \mathrm{d}\tau (L_1 + L_2) , \tag{13}$$

$$L_1 = -\tfrac{1}{2}\sqrt{-g}\, g^{\alpha\beta}\Pi^\mu_\alpha\Pi_{\mu\beta} , \tag{14a}$$

$$L_2 = -i\epsilon^{\alpha\beta}\partial_\alpha X^\mu[\bar{\theta}^1\gamma_\mu\partial_\beta\theta^1 - \bar{\theta}^2\gamma_\mu\partial_\beta\theta^2] + \epsilon^{\alpha\beta}\bar{\theta}^1\gamma^\mu\partial_\alpha\theta^1\bar{\theta}^2\gamma_\mu\partial_\beta\theta^2 , \tag{14b}$$

where

$$\Pi^\mu_\alpha = \partial_\alpha X^\mu - i\bar{\theta}^A\gamma^\mu\partial_\alpha\theta^A \tag{15}$$

and $\epsilon^{\alpha\beta}$ is the two-dimensional Levi-Civita tensor density. L_1 has been considered previously by a number of people (P. di Vecchia, M. Roček, B. Julia, L. Brink, W. Siegel [8] and ourselves). It is a straightforward extension of eqs. (1) and (8) and has the global supersymmetry of eq. (9). However, it does not possess local supersymmetry. L_2 drops out in a bosonic or a one-dimensional (point particle) restriction, and it is also invariant (up to a total derivative) under the global ϵ transformations of eq. (9). The proof requires the identity

$$\gamma_\mu \psi_1 \bar{\psi}_2 \gamma^\mu \psi_3 + \gamma_\mu \psi_2 \bar{\psi}_3 \gamma^\mu \psi_1 + \gamma_\mu \psi_3 \bar{\psi}_1 \gamma^\mu \psi_2 = 0 . \tag{16}$$

This is precisely the identity that arises in the proof of supersymmetry for super Yang–Mills theories [9]. Therefore, at the classical level, the possibilities for superstring actions are in one-to-one correspondence with super Yang–Mills theories: they are $D = 10$ with Majorana–Weyl spinors, $D = 6$ with Weyl spinors, $D = 4$ with Majorana spinors, and $D = 3$ with Majorana spinors, as well as their dimensionally-reduced forms. However, quantum-mechanically only the $D = 10$ theory is consistent, unless a Polyakov-type interpretation [10] is possible in other cases.

It is convenient to define projection operators

$$P_\pm^{\alpha\beta} = \tfrac{1}{2}(g^{\alpha\beta} \pm \epsilon^{\alpha\beta}/\sqrt{-g}) . \tag{17}$$

These satisfy a number of obvious identities as well as the less obvious one

$$P_\mp^{\alpha\gamma} P_\mp^{\beta\delta} = P_\mp^{\beta\gamma} P_\mp^{\alpha\delta} . \tag{18}$$

The local supersymmetry transformations are then given by

$$\delta_\kappa \theta^1 = 2\mathrm{i}\Pi_\alpha^\mu \gamma_\mu P_-^{\alpha\beta} \kappa_\beta^1 , \tag{19a}$$

$$\delta_\kappa \theta^2 = 2\mathrm{i}\Pi_\alpha^\mu \gamma_\mu P_+^{\alpha\beta} \kappa_\beta^2 , \tag{19b}$$

$$\delta_\kappa X^\mu = \mathrm{i}\bar{\theta}^A \gamma^\mu \delta_\kappa \theta^A , \tag{19c}$$

$$\delta_\kappa(\sqrt{-g}\, g^{\alpha\beta}) = -16\sqrt{-g} \times (P_-^{\alpha\gamma} P_-^{\beta\delta} \bar{\kappa}_\delta^1 \partial_\gamma \theta^1 + P_+^{\alpha\gamma} P_+^{\beta\delta} \bar{\kappa}_\delta^2 \partial_\gamma \theta^2) . \tag{19d}$$

κ^1 and κ^2 may have either the same or opposite $D = 10$ chirality, as required to match up with the choices made for θ^1 and θ^2. Because of the projections $P_\pm$, only one linear combination of components of the two-vector contributes in each case. The local bosonic transformation generalizing eq. (12) is

$$\delta_\lambda \theta^1 = \sqrt{-g}\, P_-^{\alpha\beta} \partial_\beta \theta^1 \lambda_\alpha , \tag{20a}$$

$$\delta_\lambda \theta^2 = \sqrt{-g}\, P_+^{\alpha\beta} \partial_\beta \theta^2 \lambda_\alpha , \tag{20b}$$

$$\delta_\lambda X^\mu = \mathrm{i}\bar{\theta}^A \gamma^\mu \delta_\lambda \theta^A , \tag{20c}$$

$$\delta_\lambda(\sqrt{-g}\, g^{\alpha\beta}) = 0 . \tag{20d}$$

If κ^α (with upper index) and λ_α (with lower index) are regarded as the fundamental parameters, then the metric tensor occurs in al formulas only through the unimodular combination $\sqrt{-g}\, g^{\alpha\beta}$. The key to proving the invariance of S under κ and λ transformations is to note that when δX^μ and $\delta\theta^A$ are related as in eqs. (19c) and (20c), one has (using eq. (16))

$$\delta S = \pi^{-1} \int \mathrm{d}\sigma\, \mathrm{d}\tau [-\tfrac{1}{2}\delta(\sqrt{-g}\, g^{\alpha\beta})\, \Pi_\alpha^\mu \Pi_{\mu\beta} - 4\mathrm{i}\sqrt{-g}\, \Pi_\alpha^\mu (P_-^{\alpha\beta} \partial_\beta \bar{\theta}^1 \gamma_\mu \delta\theta^1 + P_+^{\alpha\beta} \partial_\beta \bar{\theta}^2 \gamma_\mu \delta\theta^2)] . \tag{21}$$

The equations of motion following from the covariant superstring action are

$$\Pi_\alpha^\mu \Pi_{\mu\beta} = \tfrac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \Pi_\gamma^\mu \Pi_{\mu\delta} , \tag{22a}$$

$$\gamma_\mu \Pi_\alpha^\mu P_-^{\alpha\beta} \partial_\beta \theta^1 = 0 , \quad \gamma_\mu \Pi_\alpha^\mu P_+^{\alpha\beta} \partial_\beta \theta^2 = 0 , \tag{22b,c}$$

$$\partial_\alpha [\sqrt{-g}(g^{\alpha\beta} \partial_\beta X^\mu - 2\mathrm{i}P_-^{\alpha\beta} \bar{\theta}^1 \gamma_\mu \partial_\beta \theta^1 - 2\mathrm{i}P_+^{\alpha\beta} \bar{\theta}^2 \gamma_\mu \partial_\beta \theta^2)] = 0 . \tag{22d}$$

These equations must be supplemented by boundary conditions which are described below.

It is easy to see that by taking $g^{\alpha\beta} \propto \eta^{\alpha\beta}$ and imposing the gauge conditions of eqs. (3) and (6) these equations reduce to the field equations of the light-cone action (5), supplemented by the constraints

$$(\Pi_\tau \pm \Pi_\sigma)^2 = 0 , \tag{23}$$

generalizing eq. (2). It is less trivial to show, but nonetheless true, that the local reparametrization and supersymmetry invariances allow these gauge choices to be made. An intermediate possibility is to use only the reparametrization symmetry to make the covariant gauge choice

$$\sqrt{-g}\, g^{\alpha\beta} = \eta^{\alpha\beta} . \tag{24}$$

In this case the theory is described by the action of eqs. (13)–(15), with the substitution of eq. (24) and the constraints of eq. (23).

The conserved supercurrents, associated with the global ϵ supersymmetries, may be constructed by standard methods. In the gauge of eq. (24) one obtains

$$Q^{1\alpha} = i\gamma_\mu\theta^1(P_+^{\alpha\beta}\Pi_\beta^\mu + \tfrac{2}{3}i\epsilon^{\alpha\beta}\bar{\theta}^1\gamma^\mu\partial_\beta\theta^1)\,, \tag{25a}$$

$$Q^{2\alpha} = i\gamma_\mu\theta^2(P_-^{\alpha\beta}\Pi_\beta^\mu - \tfrac{2}{3}i\epsilon^{\alpha\beta}\bar{\theta}^2\gamma^\mu\partial_\beta\theta^2)\,. \tag{25b}$$

Similarly, associated with ten-dimensional Poincaré transformations there are conserved currents

$$P_\mu^\alpha = \partial^\alpha X_\mu - 2i(P_-^{\alpha\beta}\bar{\theta}^1\gamma_\mu\partial_\beta\theta^1 + P_+^{\alpha\beta}\bar{\theta}^2\gamma_\mu\partial_\beta\theta^2)\,, \tag{26}$$

$$\begin{aligned} J_{\mu\nu}^\alpha &= X_\mu P_\nu^\alpha - X_\nu P_\mu^\alpha \\ &\quad - i\bar{\theta}^1\gamma_{\mu\nu\rho}\theta^1(P_+^{\alpha\beta}\Pi_\beta^\rho + \tfrac{1}{2}i\epsilon^{\alpha\beta}\bar{\theta}^1\gamma^\rho\partial_\beta\theta^1) \\ &\quad - i\bar{\theta}^2\gamma_{\mu\nu\rho}\theta^2(P_+^{\alpha\beta}\Pi_\beta^\rho - \tfrac{1}{2}i\epsilon^{\alpha\beta}\bar{\theta}^2\gamma^\rho\partial_\beta\theta^2)\,. \end{aligned} \tag{27}$$

Regarded as a two-dimensional field theory, the covariant superstring action of eq. (13) has a number of interesting features, even at the classical level. First of all, since it reduces to eq. (5) in the light-cone gauge it is completely soluble. It has local supersymmetry, and yet it contains no gravitino fields, no two-dimensional spinors, and no zweibeins. In fact, the local supersymmetries transform as two-dimensional vectors. The arbitrary parameter N of the superparticle action in eq. (8) must be set equal to two for the superstring extension, but θ^1 and θ^2 may still have their chiralities chosen independently. Therefore the covariant superstring action is applicable to the chiral theories (type I with one supersymmetry and type IIB with two supersymmetries of the same handedness) as well as to the nonchiral one (type IIA which has two supersymmetries of opposite handedness). We also note that the global SO(2) symmetry of L_1 that rotates θ^1 into θ^2 in the chiral theories is broken by L_2.

In the case of type I open strings the action must be supplemented by the boundary conditions

$$\Pi_\mu^\sigma = 0 \quad \text{and} \quad \theta^1 = \theta^2 \text{ at the ends}\,. \tag{28}$$

There is then only one global supersymmetry, since compatibility of eqs. (28) and (9a) requires that $\epsilon^1 = \epsilon^2$. At the same time, the local symmetries are restricted to ones for which κ^σ, λ^σ, and ξ^σ vanish at the ends of the string. Theories IIA and IIB involve closed strings only, and their only boundary condition is periodicity in σ. The most bizarre feature, mentioned earlier, is that the scalar global supersymmetries and vector local supersymmetries of the covariant action combine to give a spinorial supersymmetry of the light-cone-gauge action. Also, the scalar fields θ^1 and θ^2 of the covariant action become spinors in the light-cone gauge. It may be interesting to investigate whether these occurrences can have higher-dimensional analogues. The existence of this covariant superstring action may make it possible to formulate covariant Feynman rules for superstring interactions.

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