Notes on
Supergravity solutions for Dp/D(6-p) bound states

Sébastien Reymond, Thomas Van Riet, Mario Trigiante

Instituut voor Theoretische Fysica, K.U. Leuven,
Celestijnenlaan 200D, B-3001 Leuven, Belgium

Abstract

Near horizons of Dp-branes with $p \neq 3$ are singular with a running dilaton. One expects that a bound state of Dp branes with their magnetic cousins, D$(6-p)$ branes stabilises the dilaton such that an AdS factor appears in the near horizon region. We construct these explicit (partially smeared) brane bound state solutions whose horizons indeed lead to a chain of AdS vacua of the form $AdS_{p+2} \times S^{p+2} \times \mathbb{T}^{6-2p}$, unless $p = 2$ which is more complicated. The solutions with $p = 0, 2$ are non-SUSY. For $p = -1, 1, 3$ the bound states are supersymmetric with the cases $p = 1, 3$ being well-known examples already. On the other hand, the bound state of a D(-1)/D7 brane in supergravity was only hinted upon recently in [1] where the "AdS$_1$ vacuum" was constructed using a new alternative gravitino variation in Euclidean IIB. We confirm that this new variation gives rise to first-order BPS equations that solve the same second-order EOMS and we construct the solution that flows to the "AdS$_1$ vacuum".

1 Introduction

Supergravity p-brane solutions have been pivotal in our understanding of string theory and holography. Yet many basic questions about supergravity p-branes remain unanswered. For instance, what are the solutions corresponding to bound states of branes? Such bound states are only known when they preserve supersymmetry and even then the solutions are incomplete since the branes are smeared over some directions [2, 3]. Consider for instance the well-studied $D1-D5$ bound state, where the $D1$ extends along the $D5$ worldvolume:

$$\begin{array}{ll} D1 & \times\times-------- \\ D5 & \times\times\times\times\times\times---- \end{array} \tag{1.1}$$

Here a cross denotes a worldvolume direction and a bar a transversal direction. The first cross is time and we label it with zero and then we have 9 spatial directions. The metric for the known solution, in 10d string frame, is given by

$$ds^2_{10} = \frac{1}{\sqrt{H_1 H_5}}\left(-dx_0^2 + dx_1^2\right) + \sqrt{H_1 H_5}\left(dr^2 + r^2 d\Omega_3^2\right) + \sqrt{\frac{H_1}{H_5}}\left(dx_6^2 + dx_7^2 + dx_8^2 + dx_9^2\right) , \tag{1.2}$$

where the harmonic functions are defined as

$$H_{1,5}(r) = 1 + \frac{|Q_{1,5}|}{r^2} . \tag{1.3}$$

If we put $Q_1 = 0$ we get the $D5$ brane solution, but if we put $Q_5 = 0$ we find the $D1$ solution smeared over the directions the $D5$ did not share with the $D1$. This is common to all solutions for BPS bound states [4]. Nonetheless string theory informs us that a supersymmetric $Dm - Dn$ bound state can exist when the number of mixed boundary conditions (Dirichlet-Neumann) for the string equals a multiple of 4. Clearly this is the case for the $D1-D5$ bound state. Another example would the following intersection of $D3$ branes:

$$\begin{array}{ll} D3 & \times\times\times\times------ \\ D3' & \times\times--\times\times---- \end{array} \tag{1.4}$$

The solution for this bound state can similarly be written by inserting the harmonics in the metric in the right places but the $D3$ stack will be smeared over the directions 4 and 5 and the $D3'$ stack over the directions 2 and 3.[1] Surprisingly no solutions are known for which this smearing is absent, although it is believed, from a string theory perspective, localised solutions must exist.

When the number of mixed directions is 8, such as for the would-be $D0-D8$ bound state one has to be careful. The natural Ansatz, with the harmonic functions in the right places does not solve the EOMs. This is interesting since it points to some non-trivial physics. Indeed a dual version of the Hannany-Witten effect [5, 6] suggests that a fundamental string (denoted F1) is created that stretches between the $D0$ and the $D8$. Indeed, once one searches for solutions of that form

$$\begin{array}{ll} D0 & \times--------- \\ F1 & \times---------\times \\ D8 & \times\times\times\times\times\times\times\times\times- \end{array} \tag{1.5}$$

[1] T-duality along a spatial direction changes a cross for a bar. So if we T-dualise along the directions 2 and 3 we find the $D1-D5$ solution. T-duality clearly preserves the number of directions with mixed boundary conditions. In this case the D5 is smeared over $2, 3, 4$ and 5.

solutions can be found [7, 8, 9, 10].

In this paper we are interested in bound states of the form $Dp - D(6-p)$ for $p = -1, 0, 1, 2, 3$ because such solutions are expected to have smooth horizons leading to potential AdS/CFT dualities. This is known to be the case for $p = 1, 3$. To see this, note that Dp branes are electrically charged under a F_{p+2} field strength, and $D(6-p)$ branes are charged magnetically under the same field strength. So, these branes are solutions to the following bosonic truncation of IIA/B supergravity in 10 dimensions (in Einstein frame):

$$S = \int \sqrt{|g|}\left(\mathcal{R} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}\tfrac{1}{(p+2)!}e^{\frac{(3-p)}{2}\phi}F_{p+2}^2\right) . \tag{1.6}$$

A Dp brane with $p \neq 3$ has a singular horizon and a running dilaton near the horizon. This can be seen from the dilaton equation:

$$\nabla\partial\phi = \tfrac{3-p}{(p+2)!4}e^{\frac{(3-p)}{2}\phi}F_{p+2}^2 . \tag{1.7}$$

Clearly any electric charge leads to a non-zero and negative F^2 and thus a dilaton gradient, which diverges near the would-be horizon. Magnetic charges create a positive F^2 on the right hand side of the dilaton equation so when both electric and magnetic charges are present the F^2 can vanish and the dilaton be constant. This happens for the $D1 - D5$ system. A zero F^2 at the horizon is then a consequence of the fluxes being (anti)-self dual in the directions along which the D1 brane is not smeared: the near horizon is $AdS_3 \times S^3 \times R^4$, or equally valid $AdS_3 \times S^3 \times T^4$. The 3-form flux is self dual along $AdS_3 \times S^3$, explaining why the total F^2 vanishes. Since the D5 branes are smeared along the R^4 or T^4 directions, it makes physically more sense to think of the T^4 solution since then the smearing can be thought of as a standard Kaluza-Klein coarse graining procedure. The solution with the T^4 factor can be considered as a dyonic string solution in six dimensions.

It was already pointed out in [1] that similar reasonings applies to all dyonic objects: p-branes $2(p+2)$ dimensions. For even p this leads to $D0 - D6$ bound states

$$\begin{array}{ll} D0 & \times - - - - - - - - - \\ D6 & \times - - - \times \times \times \times \times \times \end{array} \tag{1.8}$$

where the last six directions can be taken to be a T^6 and then we have a dyonic black hole (particle) in 4d with horizon $AdS_2 \times S^2$. We can also have the $D2 - D4$ bound states

$$\begin{array}{ll} D2 & \times \times \times - - - - - - - \\ D4 & \times \times \times - - - - - \times \times \end{array} \tag{1.9}$$

Here the D2s are smeared over the last two directions, along which also the D4 branes extend. Again, if we take these to be periodic, so a T^2, the smearing can be justified as coarse graining at the solution is a dyonic membrane in 8d with horizon $AdS_4 \times S^4$. For odd values of p we have the well-known 1/2 BPS dyonic D3 solution in IIB with its fully BPS $AdS_5 \times S^5$. For $p = 1$ we have the dyonic strings in 6d and for $p = -1$ one would expect dyonic instantons in 2d with an $AdS_1 \times S^1$ horizon. The latter solution was constructed in [1].

The goal of this paper is simple: for all cases not studied earlier in the literature: $D0-D6$, $D2-D4$ and $D(-1) - D7$ we wish to provide the brane solution as for now only the $AdS_{p+2} \times S^{p+2} \times T^{6-2p}$ vacua with self dual F_{p+2} fluxes were constructed. In here we will extend this to the dyonic brane solutions, smeared over the T^{6-2p} directions whose horizons correspond to the vacua with self dual fluxes.

The motivations behind this goal are twofold

1. Extend our understanding of p-brane bound states. Which, especially for non-SUSY bound states is essentially non-existing. So the $D2-D4$ solution in this paper is entirely new. The $D0-D6$ was implicitly known since it is the 10d lift of the dyonic Kaluza-Klein black hole. The SUSY $D-1/D7$ solution is also new and shows how our understanding of Euclidean supergravity has been superficial.

2. Holography: The bound states have AdS horizons. But for even p these AdS vacua are non-SUSY and hence at best meta-stable [11]. If so it is unclear how to define a holographic dual [12]. For $p=-1$ the dual has been conjectured in [1] to be matrix model studied in [13]. If so this constitutes a holographic pair where space is emergent from matrices alone. Note that without adding the magnetic charges the duals are non-conformal. For $D0$ branes we have the conjectured BFSS quantum mechanics [14] and for $D(-1)$ branes the IKKT matrix model [15].

2 The $D0-D6$ system

One method to construct the D0-D6 brane intersection is by lifting the so-named dyonic dilatonic black hole solution in 4d. For that, consider the following theory of gravity, a scalar field s and a one-form field A_1 with two-form field strength $F_2=dA_1$:

$$S=\int\sqrt{-g}\left(R-\tfrac{1}{2}(\partial s)^2-\tfrac{1}{4}e^{\sqrt{3}s}F_2^2\right)\,. \tag{2.1}$$

The particular coupling between the scalar s (called dilaton) and the gauge field involves $\sqrt{3}$ in the exponential coupling. For this particular coupling the system is Liouville-integrable and group theory techniques easily allow one to find the general solution, see eg [16, 17] and we review the extremal solution in the Appendix. Interestingly, exactly this $\sqrt{3}$-coupling is what one obtains by dimensionally reducing IIA supergravity on a 6-torus. The electric charge of the 4d black holes then lift to D0 charges and the magnetic charges in 4d to D6 charges. Hence uplifting the dyonic extremal solution leads to a D0-D6 bound state. All of this is reviewed in the Appendix and we simply present the solution here:

The $D0-D6$ intersection is

$$\mathrm{d}s^2_{10,s}=g_s^{1/2}\left(-H_0^{-\frac{1}{2}}H_6^{-\frac{1}{2}}\mathrm{d}t^2+H_0^{\frac{1}{2}}H_6^{\frac{1}{2}}\left(\mathrm{d}r^2+r^2\mathrm{d}\Omega_2^2\right)+H_0^{\frac{1}{2}}H_6^{-\frac{1}{2}}\delta_{ij}\mathrm{d}\theta^i\mathrm{d}\theta^j\right)\,, \tag{2.2}$$

$$e^{\phi}=g_s\left(\frac{H_0}{H_6}\right)^{\frac{3}{4}}\,, \tag{2.3}$$

$$F=Q_6\mathrm{d}\Omega_2+g_s^{-3/2}Q_0\frac{H_0^{-2}H_6}{r^2}\mathrm{d}t\wedge\mathrm{d}r\,, \tag{2.4}$$

where θ^i are coordinates on flat space, whether a 6-torus, $\mathbb{R}^6$ or something else. The functions H_0 and H_6 are

$$H_0(r)=1+g_s^{-1/2}Q_0^{2/3}F(r)\,, \tag{2.5}$$

$$H_6(r)=1+g_s^{1/2}Q_6^{2/3}F(r)\,, \tag{2.6}$$

$$F(r)\equiv\frac{1}{r}\sqrt{g_s^{-1/2}Q_0^{2/3}+g_s^{1/2}Q_6^{2/3}+\frac{Q_0^{2/3}Q_6^{2/3}}{2r^2}}\,. \tag{2.7}$$

Interestingly the effect of combining both D0 and D6 charges is to introduce subleading $1/r^2$ terms in what used to be harmonic functions on the space transversal to both branes. In what follows we will analyse one consequence of these terms and these are so-called *brane-jet instabilities* [18] which verify the Swampland conjecture that all non-SUSY AdS vacua must have some form of instability [11].

Consider for instance the action for a probe (anti-)D0 brane:

$$S_{D_0\ \text{probe}} = -\mu_0 e^{-\phi} \int \mathrm{d}\tau \sqrt{-\gamma} \pm \mu_0 \int C_1 \tag{2.8}$$

where γ is the background metric pulled back to the brane worldvolume, C_1 is the background gauge potential and μ_0 is the (absolute value of the) charge. A local expression for the t-component of the C_1-field is:

$$(C_1)_t = Q_0 \int^r \frac{H_6}{r^2 H_0^2}\,. \tag{2.9}$$

Hence the probe potential is

$$V \propto H_0^{-1} H_6^{1/2} \mp Q_0 \int^r \frac{H_6}{r^2 H_0^2}\,. \tag{2.10}$$

The integral can be computed explicitly (see appendix D) to get an analytic form of the potential:

$$V(r) \sim \frac{r^2\sqrt{1+\frac{Q_6}{r}\sqrt{1+\left(\frac{Q_0}{Q_6}\right)^{2/3}}+\frac{Q_0^{2/3}Q_6^{4/3}}{2r^2}} - r^2\sqrt{1+\left(\frac{Q_6}{Q_0}\right)^{2/3}} - rQ_0^{1/3}Q_6^{2/3}}{r^2 + rQ_0\sqrt{1+\left(\frac{Q_6}{Q_0}\right)^{2/3}}+\frac{1}{2}Q_0^{4/3}Q_6^{2/3}} \tag{2.11}$$

The potential goes to 0 as $r \to 0$, and at r going to infinity it approaches the value

$$V_\infty \sim 1 - \sqrt{1+\left(\frac{Q_6}{Q_0}\right)^{2/3}} < 0 \tag{2.12}$$

One can plot this potential for any positive value of Q_0 and Q_6 and see that it is negative everywhere and monotonically decreasing towards positive r. Therefore, the probe D_0 brane is pushed to infinity: there is indeed a brane jet instability.

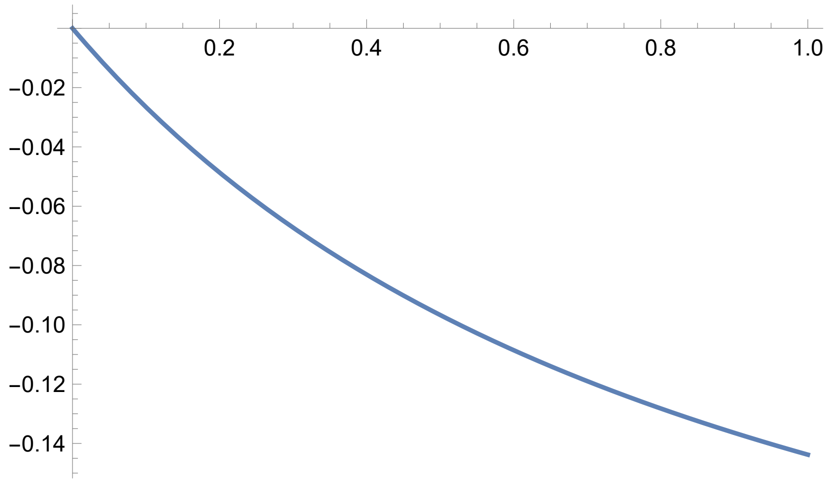

Figure 1: Potential $V(r)$ felt by the $D0$ probe for $Q_0 = 2$, $Q_6 = 1$.

[SR: This is a placeholder plot, needs to be polished.]

Another interesting feature is to consider the case where the dilaton is constant ([SR: to be revised in the future when I understand this better]), which in this case happens when $Q_0 = Q_6 = Q$:

$$H_0 = H_6 = H = 1 + \frac{\sqrt{2}Q}{r} + \frac{Q^2}{2r^2} = \left(1 + \frac{Q}{\sqrt{2}r}\right)^2 \tag{2.13}$$

Then, the formula (2.11) reduces to

$$V(r) \sim H^{-1/2}(1 - \sqrt{2}) \tag{2.14}$$

which can be understood as follows: the gravitational attraction and the electromagnetic repulsion have the same functional form as functions of r, but different overall coefficients (1 and $\sqrt{2}$) which do not cancel out.

3 The $D(-1) - D7$ solution

We consider a modified version of the Ansatz from [1]:

$$\mathrm{d}s^2 = L_x(y)^2\mathrm{d}x^2 + L_y(y)\mathrm{d}y^2 + L_1(y)^2\sum_{i=1}^{4}(\mathrm{d}\theta^i)^2 + L_5(y)^2\sum_{i=5}^{8}(\mathrm{d}\theta^i)^2 \tag{3.1}$$

$$\phi = \phi(y) \tag{3.2}$$

$$F_1 = \alpha(y)\mathrm{d}x + i\beta(y)\mathrm{d}y \tag{3.3}$$

$$F_5 = (1 - i\star)\mathcal{F}, \quad \mathcal{F} = \mathrm{d}\theta^{1234} \wedge (\gamma(y)\mathrm{d}x + \delta(y)\mathrm{d}y) \tag{3.4}$$

The metric is written in string frame. Here, the x direction is 2π periodic (corresponding to the S^1 factor in the vacuum solution from [1]), while y is a noncompact direction corresponding to the AdS_1 factor.

3.1 An effective action for the $D - 1/D7$ system

Let us denote the metric in Einstein frame as follows

$$ds^2 = M_y^2dy^2 + M_x^2dx^2 + M_1^2[d\theta_1^2 + d\theta_2^2 + d\theta_3^2 + d\theta_4^2] + M_5^2[d\theta_5^2 + d\theta_6^2 + d\theta_7^2 + d\theta_8^2]\,. \tag{3.5}$$

So the M's are related to the L's as $M^2 = e^{-\phi/2}L^2$. The kinetic term for the M come from dimensionally reducing the EH term. After some work in which we drop multiple boundary terms we find

$$S = \int dyM_1^4M_5^4\frac{M_x}{M_y}\Big(12\frac{M_1'^2}{M_1^2} + 12\frac{M_5'^2}{M_5^2} + 32\frac{M_1'M_5'}{M_1M_5} + 8\frac{M_x'M_5'}{M_xM_5} + 8\frac{M_x'M_1'}{M_xM_1} - \frac{1}{2}\phi'^2\Big) \tag{3.6}$$

and we added the dilaton kinetic term. Let us diagonalise and normalise the kinetic term. First we will chose the gauge such that

$$M_y = M_xM_1^4M_5^4\,. \tag{3.7}$$

Then let us also write

$$M_x = \exp\left\{\frac{N_-}{\sqrt{-8\lambda_-}}\chi_- + \frac{N_+}{\sqrt{8\lambda_+}}\chi_+\right\}, \tag{3.8}$$

$$M_1 = \exp\left\{\frac{1}{4}\chi_1 - \frac{N_-}{2}\sqrt{\frac{-\lambda_-}{8}}\chi_- + \frac{N_+}{2}\sqrt{\frac{\lambda_+}{8}}\chi_+\right\}, \tag{3.9}$$

$$M_5 = \exp\left\{-\frac{1}{4}\chi_1 - \frac{N_-}{2}\sqrt{\frac{-\lambda_-}{8}}\chi_- + \frac{N_+}{2}\sqrt{\frac{\lambda_+}{8}}\chi_+\right\} \tag{3.10}$$

where we introduced the 3 scalars χ_1, χ_-, χ_+ and the constants

$$\lambda_\pm = \frac{7 \pm \sqrt{57}}{2} \tag{3.11}$$

$$N_\pm = \sqrt{\frac{2}{2 + \lambda_\pm^2}}\,. \tag{3.12}$$

The kinetic term now becomes

$$S = \int dy \left(-\tfrac{1}{2}(\phi')^2 - \tfrac{1}{2}(\chi_1')^2 - \tfrac{1}{2}(\chi_-')^2 + \tfrac{1}{2}(\chi_+')^2\right) . \tag{3.13}$$

Let us now go for the would-be potential from dimensional reduction. From reducing the magnetic contribution of F_1 we find already

$$V_{F_1} = \frac{1}{2} M_1^4 M_5^4 M_x^{-1} M_y e^{2\phi} \alpha^2 . \tag{3.14}$$

To reduce the magnetic piece of F_9 we need to be careful. Note that:

$$F_9 = \beta L_x L_y^{-1} L_1^4 L_5^4 dx \wedge d\theta_1 \dots d\theta_8 . \tag{3.15}$$

So what we hold fixed is not β but the whole coefficient

$$\tilde{\beta} = \beta L_x L_y^{-1} L_1^4 L_5^4 . \tag{3.16}$$

So we find that

$$V_{F_9} = \frac{1}{2} M_1^{-4} M_5^{-4} M_x^{-1} M_y e^{-2\phi} \tilde{\beta}^2 . \tag{3.17}$$

. We need to do something similar for F_5:

$$F_5 = (1 - i\star)\left(\gamma dx \wedge d\theta_1 \dots d\theta_4 + i\delta \frac{L_x}{L_y}\left(\frac{L_5}{L_1}\right)^4 dx \wedge d\theta_5 \dots d\theta_8\right) . \tag{3.18}$$

We keep fixed γ and

$$\tilde{\delta} = \delta \frac{L_x}{L_y}\left(\frac{L_5}{L_1}\right)^4 . \tag{3.19}$$

Such that we find:

$$V_{F_5} = \frac{1}{2} M_x^{-1} M_y \left(\frac{M_5^4}{M_1^4}\gamma^2 - \frac{M_1^4}{M_5^4}\tilde{\delta}^2\right) . \tag{3.20}$$

The total potential then is

$$V = \frac{1}{2}\frac{M_y}{M_x}\left(M_1^4 M_5^4 e^{2\phi}\alpha^2 + M_1^{-4} M_5^{-4} e^{-2\phi}\tilde{\beta}^2 + \frac{M_5^4}{M_1^4}\gamma^2 - \frac{M_1^4}{M_5^4}\tilde{\delta}^2\right) . \tag{3.21}$$

The equations of motion from the effective potential reduce to the 10d equations of motion on the conditions that

$$i\alpha\tilde{\beta} + \gamma\tilde{\delta} = 0 . \tag{3.22}$$

In other words, only 3 flux quanta in the potential are free to chose. This constraint is enforced by the 10d R_{xy} equation and needs to be imposed in the definition of the 1d effective potential.

Variation with respect to M_x implies $V = 0$ as expected. Variation with respect to ϕ puts the first term equal to the second, as expected. Variation with respect to M_1 or M_5 puts the third term equal to the fourth. This reproduces exactly our algebraic constraints! So we reproduce our vacuum. Note that with the choice (3.7) there is no M_x dependence. Instead one reproduces the condition $V = 0$ in the vacuum by keeping in mind the energy constraint enforced by M_y which sets total energy to zero.

In canonically normalised scalars we have instead:

$$V_{\text{eff}} = \frac{1}{2}\alpha^2 e^{2\phi+N_+\sqrt{8\lambda_+}\chi_+-N_-\sqrt{-8\lambda_-}\chi_-} + \frac{1}{2}\tilde{\beta}^2 e^{-2\phi} \tag{3.23}$$

$$+\frac{1}{2}\left(\gamma^2 e^{-2\chi_1} - \tilde{\delta}^2 e^{2\chi_1}\right) e^{\frac{1}{2}N_+\sqrt{8\lambda_+}\chi_+-\frac{1}{2}N_-\sqrt{-8\lambda_-}\chi_-}\,. \tag{3.24}$$

3.2 Supersymmetry and superpotential

We would like to build an effective (super)potential W for the effective action found above in terms of the canonical fields. Recall that after gauge fixing $M_y = M_x M_1^4 M_5^4$, our action read

$$S = \int \mathrm{d}y \, \left(-\frac{1}{2}\chi_1'^2 + \frac{1}{2}\chi_+'^2 - \frac{1}{2}\chi_-'^2 - \frac{1}{2}\phi'^2 - V(\chi_1, \chi_+, \chi_-, \phi)\right) \tag{3.25}$$

$$= \int \mathrm{d}y \, \left(-\frac{1}{2}\eta_{ij}\dot{\Phi}^i\dot{\Phi}^j - V(\Phi)\right), \quad \eta = \text{diag}(1,-1,1,1), \quad \Phi = \begin{pmatrix}\chi_1\\ \chi_+\\ \chi_-\\ \phi\end{pmatrix} \tag{3.26}$$

If the potential V can be expressed through a superpotential W as

$$V = \frac{1}{2}\eta^{ij}\partial_i W \partial_j W\,, \tag{3.27}$$

then the action reduces to a square up to boundary terms

$$S = -\frac{1}{2}\int \mathrm{d}y \, \left(\dot{\Phi}^i + \eta^{ij}\partial_i W\right)^2 + \int \mathrm{d}y \, \frac{\mathrm{d}W}{\mathrm{d}y} \tag{3.28}$$

The equations of motion of the effective action are then implied by the first order equations

$$\dot{\Phi}^i + \eta^{ij}\partial_i W = 0 \tag{3.29}$$

Now one can look at the 10d action and write the supersymmetry variation of the gravitino and the dilatino [1]. With our Ansatz, they become

$$\phi' = \eta_d e^{-\phi}\left(\alpha L_1^4 L_5^4 - \eta_d\tilde{\beta}\right) \tag{3.30}$$

$$\eta_d L_x' = -\frac{1}{4}L_x e^{-\phi}\left[\left(\alpha L_1^4 L_5^4 + \eta_d\tilde{\beta}\right) + i^z\eta_p\left(\gamma L_5^4 - i\eta_d\tilde{\delta}L_1^4\right)\right] \tag{3.31}$$

$$\eta_d L_1' = -\frac{1}{4}L_1 e^{-\phi}\left[-\left(\alpha L_1^4 L_5^4 - \eta_d\tilde{\beta}\right) + i^z\eta_p\left(\gamma L_5^4 + i\eta_d\tilde{\delta}L_1^4\right)\right] \tag{3.32}$$

$$\eta_d L_5' = -\frac{1}{4}L_5 e^{-\phi}\left[-\left(\alpha L_1^4 L_5^4 - \eta_d\tilde{\beta}\right) - i^z\eta_p\left(\gamma L_5^4 + i\eta_d\tilde{\delta}L_1^4\right)\right] \tag{3.33}$$

The above is expressed with the same gauge choice ($M_y = M_x M_1^4 M_5^4$) as for the effective action. The parameter $z \in \{0, 1\}$, introduced in [1], represents the supposed ambiguity in defining the supersymmetry variations in Euclidean signature. The variables $\eta_{p,d}$ verify $\eta^2 = 1$ and are simply sign choices in the definition of the projectors on the SUSY parameter:

$$(\Gamma_{\bar{x}} + i\eta_d \Gamma_{\bar{y}})\epsilon = 0 \tag{3.34}$$

$$\Gamma_{\bar{1}\bar{2}\bar{3}\bar{4}}\epsilon = \eta_p \epsilon \tag{3.35}$$

In terms of the canonically normalized fields, the first-order equations, imposed by supersymmetry, are can be derived from the following superpotential:

$$W = -\eta_d \alpha e^{\phi + 4N_+\sqrt{\frac{\lambda_+}{8}}\chi_+ - 4N_-\sqrt{\frac{-\lambda_-}{8}}\chi_-} - \tilde{\beta} e^{-\phi} \tag{3.36}$$

$$- i^z \eta_p \left(\eta_d \gamma e^{-\chi_1} - i\tilde{\delta} e^{\chi_1}\right) e^{2N_+\sqrt{\frac{\lambda_+}{8}}\chi_+ - 2N_-\sqrt{\frac{-\lambda_-}{8}}\chi_-}\,, \tag{3.37}$$

via equation (3.29).

Now W should also reproduce our effective potential, according to equation (3.27). We find

$$\frac{1}{2}\eta^{ij}\partial_i W \partial_j W = \frac{1}{2}\alpha^2 e^{2\phi + 8N_+\sqrt{\frac{\lambda_+}{8}}\chi_+ - 8N_-\sqrt{\frac{-\lambda_-}{8}}\chi_-} + \frac{1}{2}\tilde{\beta}^2 e^{-2\phi} \tag{3.38}$$

$$+ \frac{1}{2}(-1)^z \left(\gamma^2 e^{-2\chi_1} - \tilde{\delta}^2 e^{2\chi_1}\right) e^{4N_+\sqrt{\frac{\lambda_+}{8}}\chi_+ - 4N_-\sqrt{\frac{-\lambda_-}{8}}\chi_-} \tag{3.39}$$

$$+ i\eta_d \left(i\alpha\tilde{\beta} + (-1)^z \gamma\tilde{\delta}\right) e^{4N_+\sqrt{\frac{\lambda_+}{8}}\chi_+ - 4N_-\sqrt{\frac{-\lambda_-}{8}}\chi_-} \tag{3.40}$$

Compare this with the effective potential we found earlier (3.24):

$$V_{\text{eff}} = \frac{1}{2}\alpha^2 e^{2\phi + N_+\sqrt{8\lambda_+}\chi_+ - N_-\sqrt{-8\lambda_-}\chi_-} + \frac{1}{2}\tilde{\beta}^2 e^{-2\phi} \tag{3.41}$$

$$+ \frac{1}{2}\left(\gamma^2 e^{-2\chi_1} - \tilde{\delta}^2 e^{2\chi_1}\right) e^{\frac{1}{2}N_+\sqrt{8\lambda_+}\chi_+ - \frac{1}{2}N_-\sqrt{-8\lambda_-}\chi_-} \tag{3.42}$$

They match *exactly* provided that $z = 0$ and that the constraint on flux quanta

$$i\alpha\tilde{\beta} + \gamma\tilde{\delta} = 0 \tag{3.43}$$

is satisfied.

Given that the effective action is a consistent truncation of the 10d theory, this shows that the first order Killing spinor equations square to the second order equations of motion provided that $z = 0$.

3.3 Solving the $z = 0$ flow (Thomas version)

Since the $z = 1$ flow is inconsistent we take $z = 0$ and further make the choice $\eta_d = \eta_p = 1$. To simplify the equations further we define

$$\sqrt{2}\varphi = \chi_+ - \chi_-\,, \tag{3.44}$$

$$\sqrt{2}\tilde{\varphi} = \chi_+ + \chi_-\,, \tag{3.45}$$

$$c = \frac{N_+\sqrt{\lambda_+}}{8} = \frac{N_-\sqrt{-\lambda_-}}{8}\,. \tag{3.46}$$

Then the superpotential reads

$$W = -\alpha e^{\phi + 4\sqrt{2}c\varphi} - \tilde{\beta} e^{-\phi} - \left(\gamma e^{-\chi_1} - i\tilde{\delta} e^{\chi_1}\right) e^{2\sqrt{2}c\varphi} . \tag{3.47}$$

and does not depend on $\tilde{\varphi}$. The action is given by

$$S = \int dy \Big(-\frac{1}{2} G_{ij} \dot{\Phi}^i \dot{\Phi}^j - \frac{1}{2} G^{ij} \partial_i W \partial_j W \Big) \tag{3.48}$$

with $\Phi^i = (\phi, \chi_1, \varphi, \tilde{\varphi})$ and

$$G_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = G^{ij} . \tag{3.49}$$

So the flow equations $\dot{\Phi}^i = -G^{ij}\partial_j W$ read

$$\dot{\phi} = \alpha e^{\phi + 4\sqrt{2}c\varphi} - \tilde{\beta} e^{-\phi} , \tag{3.50}$$

$$\dot{\chi_1} = -\left(\gamma e^{-\chi_1} + i\tilde{\delta} e^{\chi_1}\right) e^{2\sqrt{2}c\varphi} , \tag{3.51}$$

$$\dot{\varphi} = 0 , \tag{3.52}$$

$$\dot{\tilde{\varphi}} = -2\sqrt{2}c \Big(2\alpha e^{\phi + 4\sqrt{2}c\varphi} - \left(\gamma e^{-\chi_1} - i\tilde{\delta} e^{\chi_1}\right) e^{2\sqrt{2}c\varphi} \Big) . \tag{3.53}$$

Let us consider the solution with $\varphi = 0$. The other solutions are found by shifting and rescaling. Then

$$\dot{\phi} = \alpha e^{\phi} - \tilde{\beta} e^{-\phi} , \tag{3.54}$$

$$\dot{\chi_1} = -\left(\gamma e^{-\chi_1} + i\tilde{\delta} e^{\chi_1}\right) , \tag{3.55}$$

$$\dot{\tilde{\varphi}} = -2\sqrt{2}c \Big(2\alpha e^{\phi} - \left(\gamma e^{-\chi_1} - i\tilde{\delta} e^{\chi_1}\right) \Big) . \tag{3.56}$$

Let us take reality conditions similar to the vacuum solution, if possible, as they are physics (ie real D-1 and D7 charges). So we will insist on real $\alpha, \tilde{\beta}$. Then the reality of the scalar χ_1 forces us to take real γ and imaginary $\tilde{\delta}$. So we write $\tilde{\delta} = i\tilde{\delta}'$ and then all greek letters are real and the flow equations read:

$$\dot{\phi} = \alpha e^{\phi} - \tilde{\beta} e^{-\phi} , \tag{3.57}$$

$$\dot{\chi_1} = -\left(\gamma e^{-\chi_1} - \tilde{\delta}' e^{\chi_1}\right) , \tag{3.58}$$

$$\dot{\tilde{\varphi}} = -2\sqrt{2}c \Big(2\alpha e^{\phi} - \left(\gamma e^{-\chi_1} + \tilde{\delta}' e^{\chi_1}\right) \Big) . \tag{3.59}$$

Let us first check to what extend we can have a critical point (a vacuum). If $\tilde{\beta}$ and α have the same sign we can stabilize the dilaton. So let us for simplicity take $\alpha = \tilde{\beta}$ and fix the dilaton at zero. Similarly, if γ and $\tilde{\delta}'$ have the same sign, we can fix χ_1. But this cannot done together with stabilising the dilaton since it would violate (3.43). So the signs of γ and $\tilde{\delta}'$ has to be opposite if the signs α and β are equal. Se we reproduce that there cannot be a vacuum solution. But there is a solution with constant dilaton, which probably means self-duality of F_1 is satisfied and this might be the closest we come to an AdS_1. Let us now solve for the general solution. Using shifts and rescalings we can always take $\alpha = \tilde{\beta}$ and $\gamma = -\tilde{\delta}'$. Equation (3.43) then implies, that

$$\alpha = \tilde{\beta} = \pm\gamma = \mp\tilde{\delta}'. \tag{3.60}$$

The flow equations are then:

$$\dot{\phi} = \alpha\left(e^{\phi} - e^{-\phi}\right), \tag{3.61}$$

$$\dot{\chi_1} = \mp\alpha\left(e^{-\chi_1} + e^{\chi_1}\right), \tag{3.62}$$

$$\dot{\varphi} = -2\sqrt{2}c\alpha\Big(2e^{\phi} \mp e^{-\chi_1} \pm e^{\chi_1}\Big). \tag{3.63}$$

The solutions are

$$e^{\phi} = \frac{1 + e^{2\alpha y + c_1}}{1 - e^{2\alpha y + c_1}}, \tag{3.64}$$

$$\tag{3.65}$$

3.4 Solving the $z = 0$ flow (Seba's version)

Since the $z = 1$ flow is inconsistent we take $z = 0$. To simplify the equations further we define

$$\sqrt{2}u = \chi_+ - \chi_-, \tag{3.66}$$

$$\sqrt{2}v = \chi_+ + \chi_-, \tag{3.67}$$

$$\omega = N_+\sqrt{\lambda_+} = N_-\sqrt{-\lambda_-} = (57/4)^{-1/4}. \tag{3.68}$$

Then the superpotential reads

$$W = -\eta_d\alpha e^{\phi+2\omega u} - \tilde{\beta}e^{-\phi} - \eta_p\left(\eta_d\gamma e^{-\chi_1} - i\tilde{\delta}e^{\chi_1}\right)e^{\omega u} \tag{3.69}$$

and does not depend on v. The action is given by

$$S = \int dy\Big(-\frac{1}{2}G_{ij}\dot{\Phi}^i\dot{\Phi}^j - \frac{1}{2}G^{ij}\partial_i W\partial_j W\Big) \tag{3.70}$$

with $\Phi^i = (\chi_1, u, v, \phi)$ and

$$G_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = G^{ij}. \tag{3.71}$$

So the flow equations $\dot{\Phi}^i + G^{ij}\partial_j W = 0$ become Then the 4 equations read

$$\dot{\chi_1} = -\eta_p\left(\eta_d\gamma e^{-\chi_1} + i\tilde{\delta}e^{\chi_1}\right)e^{\omega u} \tag{3.72}$$

$$\dot{u} = 0 \tag{3.73}$$

$$\dot{v} = -2\eta_d\omega\alpha e^{\phi+2\omega u} - \eta_p\omega\left(\eta_d\gamma e^{-\chi_1} - i\tilde{\delta}e^{\chi_1}\right)e^{\omega u} \tag{3.74}$$

$$\dot{\phi} = \eta_d\alpha e^{\phi+2\omega u} - \tilde{\beta}e^{-\phi} \tag{3.75}$$

From now on we consider $u(y) = u$ constant. Let us take reality conditions similar to the vacuum solution, if possible, as they are physics (ie real D(-1) and D7 charges). So we will insist on real $\alpha, \tilde{\beta}$. Moreover, we have to impose the condition

$$i\tilde{\delta} = \frac{\alpha\tilde{\beta}}{\gamma} \tag{3.76}$$

and we take real γ, $i\tilde{\delta}$ to make sure χ is real. The flow equations can be solved as follows:

$$e^{\chi_1} = -\eta_d\eta_p\frac{\gamma}{\sqrt{\eta_d\alpha\tilde{\beta}}}\tan\left(\sqrt{\eta_d\alpha\tilde{\beta}}e^{\omega u}y + C_\chi\right) \tag{3.77}$$

$$e^{\phi} = -\eta_d\sqrt{\eta_d\frac{\tilde{\beta}}{\alpha}}e^{-\omega u}\tanh\left(\sqrt{\eta_d\alpha\tilde{\beta}}e^{\omega u}y + C_\phi\right) \tag{3.78}$$

$$e^{\frac{1}{\omega}v} = A_v\cosh^2\left(\sqrt{\eta_d\alpha\tilde{\beta}}e^{\omega u}y + C_\phi\right)\sin\left(2\sqrt{\eta_d\alpha\tilde{\beta}}e^{\omega u}y + 2C_\chi\right) \tag{3.79}$$

The signs of $\alpha, \tilde{\beta}$ and γ are arbitrary and the integration constants C_χ, C_ϕ and A_v have to be chosen accordingly so that the solution is real[2].

3.5 Lift back to 10d

Using our solution above (3.77)-(3.79), and plugging them into the 10d fields (3.8)-(3.10), we find

$$M_x^2 = A_v e^{-\frac{7}{4}\omega u}\cosh^2\left(\sqrt{\eta_d\alpha\tilde{\beta}}e^{\omega u}y + C_\phi\right)\sin\left(2\sqrt{\eta_d\alpha\tilde{\beta}}e^{\omega u}y + 2C_\chi\right) \tag{3.80}$$

$$M_y^2 = e^{8\omega u}M_x^2 \tag{3.81}$$

$$M_1^2 = e^{\frac{1}{2}\omega u}\left(-\eta_d\eta_p\frac{\gamma}{\sqrt{\eta_d\alpha\tilde{\beta}}}\tan\left(\sqrt{\eta_d\alpha\tilde{\beta}}e^{\omega u}y + C_\chi\right)\right)^{-1/2} \tag{3.82}$$

$$M_5^2 = e^{\omega u}M_1^{-2} \tag{3.83}$$

$$e^{\phi} = -\eta_d\sqrt{\eta_d\frac{\tilde{\beta}}{\alpha}}e^{-\omega u}\tanh\left(\sqrt{\eta_d\alpha\tilde{\beta}}e^{\omega u}y + C_\phi\right) \tag{3.84}$$

[2] If considering $\eta_d\alpha\tilde{\beta} < 0$, the trig functions will become hyperbolic trig functions and a real integration constant will have to change to a purely imaginary one

4 The $D2 - D4$ system and consistent truncations

We consider the brane intersection described by the following table:

$$\begin{array}{c|ccccccccc} \mathrm{x} & \mathrm{x} & \mathrm{x} & - & - & - & - & - & - & - \\ \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & - & - & - & - & - \end{array}$$

The would-be supergravity solution could be expected to be a solution to the following truncation of type IIA supergravity:

$$S_{10} = \int \mathrm{d}^{10}x\sqrt{-g}\left(\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4!2}e^{\frac{1}{2}\phi}F_4^2\right)\,, \tag{4.1}$$

However, this cannot be correct for the following reasons. The IIA form equations of motion, in absence of F_2 and F_0 (and all fermions) are (Einstein frame):

$$d\left(e^{\phi/2}\star F_4\right) = -H_3\wedge F_4\,, \tag{4.2}$$

$$d\left(e^{-\phi}\star H_3\right) = \frac{1}{2}F_4\wedge F_4\,. \tag{4.3}$$

Since our configuration of interest would involve both electric and magnetic F_4 charges we expect $F_4\wedge F_4$ to be non-vanishing and hence the H_3 field cannot be truncated. Instead the minimal truncation of the full IIA action we need is

$$S_{10} = \int \mathrm{d}^{10}x\left(\sqrt{-g}\left(\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4!2}e^{\frac{1}{2}\phi}F_4^2 - \frac{1}{3!2}e^{-\phi}H_3^2\right) + \frac{1}{2}F_4\wedge F_4\wedge B_2\right)\,. \tag{4.4}$$

For reasons explained earlier we reduce this over $\mathbb{T}^2$ using the following truncation

$$ds_{10}^2 = e^{2\tilde{\alpha}\tilde{\varphi}}ds_8^2 + e^{2\tilde{\beta}\tilde{\varphi}}(d\theta_1^2 + d\theta_2^2) \tag{4.5}$$

$$B_2 = b\,d\theta_1\wedge d\theta_2\,, \tag{4.6}$$

$$C_3 = C_3\,. \tag{4.7}$$

The last identity means we do not consider legs of C_3 along the two torus directions $\theta_{1,2}$. The scalar $\tilde{\varphi}$ is the torus volume modulus and the numbers $\tilde{\alpha}, \tilde{\beta}$ are chosen to get 8d Einstein frame with canonically normalised volume modulus:

$$\tilde{\alpha}^2 = \frac{1}{48}\,, \qquad \tilde{\beta} = -3\tilde{\alpha}. \tag{4.8}$$

The 8d theory contains the metric, the volume scalar $\tilde{\varphi}$ and axion b and a 3-form C_3:

$$\frac{\mathcal{L}_8}{\sqrt{-g}} = \mathcal{R} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\tilde{\varphi})^2 - \frac{1}{2}e^{-\phi-4\tilde{\beta}\tilde{\varphi}}(\partial b)^2 - \frac{1}{4!2}e^{\frac{1}{2}\phi+2\tilde{\beta}\tilde{\varphi}}F_4^2 + \frac{1}{2}b\frac{F_4\wedge F_4}{\sqrt{-g}}\,. \tag{4.9}$$

We notice that the only scalar direction that couples to the forms b and C_3 is

$$\Phi \equiv \tfrac{1}{2}\phi + 2\tilde{\beta}\tilde{\varphi}\,. \tag{4.10}$$

The orthogonal field direction will be truncated to some arbitrary constant C:

$$2\tilde{\beta}\phi - \tfrac{1}{2}\tilde{\varphi} = C\,. \tag{4.11}$$

So our 8d theory becomes

$$\frac{\mathcal{L}_8}{\sqrt{-g}} = \mathcal{R} - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}e^{-2\Phi}(\partial b)^2 - \frac{1}{4!2}e^{\Phi}F_4^2 + \frac{1}{2}b\frac{F_4\wedge F_4}{\sqrt{-g}}\,. \tag{4.12}$$

A particularly powerful method to obtain stationary brane solutions relies on the hidden symmetries that become manifest once a p-brane in D dimensions is dimensionally reduced over its worldvolume to an instanton in a Euclidean theory with $D-p-1$ dimensions. This map between black holes and instantons through timelike reduction was first introduced in [16] and generalised to general p-branes in [19]. The power of this method lies in the fact that the Euclidean equations of motion are such that the Einstein equations decouple almost completely from the equations of the matter fields and the latter become the equations for a geodesic curve on some target space, often with more isometries than symmetries visible in the original theory. For instance the Einstein-Maxwell dilaton theory in $D=4$ obtained from a $\mathbb{T}^6$ reduction of IIA supergravity, enjoys a $SL(3,\mathbb{R})$ symmetry in 3D when compactified over time. The geodesic problem is then explicitly integrable and the lift of the instantons in 3d, described by the integrable geodesics, give the known black hole solutions in 4d.

Below we outline this procedure first for brane Dp-D(6-p) intersections that require no extra fields beyond F_{p+2} (D0-D6, D1-D5) and then we derive this for the case the B-field is needed as well (D2-D4). Since the method is rather general we will in the first case use the notation with arbitrary p keeping in mind we cover only the cases $p=0,1,3,5,6$.

4.1 Brane solutions from uplifting instantons: no B-field

We will now use this method to describe the Dp/D(6-p) solutions in 10d which can be seen as p-brane solutions with magnetic and electric charges in $2p+4$ dimensions with self-dual F_{p+2} field strengths at the near horizon region. The map from 10 to $2p+4$ dimensions occurs through straightforward dimensional reduction over a $\mathbb{T}^{6-2p}$ and keeping only the overall volume modulus of the torus. So we end up with the following action in $2p+4$ spacetime dimensions

$$S = \int\sqrt{|g|}\left(\mathcal{R} - \tfrac{1}{2}(\partial\Phi)^2 - \tfrac{1}{2}\tfrac{1}{(p+2)!}e^{a\Phi}F_{p+2}^2\right)\,, \tag{4.13}$$

where Φ is a particular linear combination of 10d dilaton and torus volume and a specific number whose value matters for symmetry enhancement. From 10d supergravity we can deduce

$$a^2 = 3-p\,. \tag{4.14}$$

The dimensional reduction over $\mathbb{R}^{1,p}$ proceeds as follows

$$ds^2_{2p+4} = e^{2\alpha\varphi}ds^2_{p+3} + e^{2\beta\varphi}ds^2_{p+1}\,, \tag{4.15}$$

$$C_{p+1} = \chi_E\epsilon_{p+1} \tag{4.16}$$

where χ_E will become an axion in $p+3$ dimensions whose axion charge describes electric charge in the higher dimension. We have the relations

$$\alpha^2 = \frac{1}{4(p+1)}\,, \qquad \alpha = -\beta\,. \tag{4.17}$$

The dimensional reduction and truncation then gives the following Lagrangian density in $p+3$ Euclidean dimensions

$$\frac{\mathcal{L}}{\sqrt{g}} = \mathcal{R} - \tfrac{1}{2}(\partial\Phi)^2 - \tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{2}e^{a\Phi+(p+1)(\alpha-\beta)\varphi}(\partial\chi_E)^2 - \tfrac{1}{2}\tfrac{1}{(p+2)!}e^{a\Phi+(p+1)(\beta-\alpha)\varphi}F_{p+2}^2\,, \tag{4.18}$$

The odd-sign axion kinetic term is caused by reducing over a space with 1 timelike dimension. In this $p+3$-dimensional Euclidean theory we can Hodge dualise the F_{p+2} to a 1-form axion field-strength whose axion potential we denote χ_M since it describes magnetic charges in the higher dimensions. The resulting action is (up to boundary terms)

$$\frac{\mathcal{L}}{\sqrt{g}} = \mathcal{R} - \tfrac{1}{2}(\partial\Phi)^2 - \tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{2}e^{a\Phi+(p+1)(\alpha-\beta)\varphi}(\partial\chi_E)^2 + \tfrac{1}{2}e^{-a\Phi+(p+1)(\alpha-\beta)\varphi}(\partial\chi_M)^2\,. \tag{4.19}$$

This sigmamodel is not a symmetric coset, but it can be embedded into a symmetric coset. For instance for D0-D6, it can be embedded into $SL(3,\mathbb{R}/SO(2,1)$.

If we look for null geodesics then the energy momentum of the 4-scalar fields cancels out and the $p+3$-dimensional metric is flat:

$$dr^2 + r^2 d\Omega_{p+2}^2\,. \tag{4.20}$$

If we instead work with a different radial coordinate

$$r \sim \tau^{-\frac{1}{p+2}}\,. \tag{4.21}$$

Then τ is an affine coordinate on the geodesic curve trace out in the 4d target space. In other words in that coordinate system the equations of motion are

$$\ddot{\phi}^i + \Gamma^i_{jk}\dot{\phi}^j\dot{\phi}^k = 0\,, \tag{4.22}$$

$$G_{ij}\dot{\phi}^i\dot{\phi}^j = 0,. \tag{4.23}$$

Here $i = 1,\ldots 4$ and $\phi^i = \{\Phi,\varphi,\chi_M,\chi_E\}$ and G_{ij} is the metric appearing in the kinetic term $-\frac{1}{2}G_{ij}\dot{\phi}^i\dot{\phi}^j$ and Γ its induced Christoffel symbols.

One way to understand that for the correct values of the coefficient a the sigma model can be embedded as a truncation of a symmetric coset, and hence must be integrable, comes from integerating out the axion momenta. When doing so the effective geodesic action is given by the classical mechanics system:

$$L = -\tfrac{1}{2}(\dot{\Phi})^2 - \tfrac{1}{2}(\dot{\varphi})^2 - \tfrac{1}{2}e^{-a\Phi-(p+1)(\alpha-\beta)\varphi}Q_E^2 + \tfrac{1}{2}e^{a\Phi-(p+1)(\alpha-\beta)\varphi}Q_M^2 \tag{4.24}$$

This is a system of two generalised coordinates q_1, q_2 ($q_1 = \Phi, q_2 = \varphi$) in a potential that is the sum of exponentials. These systems are known to be integrable when they are of the "Toda-molecule" kind. This happens when the vectors $\vec{\alpha}_i$ of exponentials in the potential $e^{\vec{\alpha}_i\cdot\vec{q}}$ obey that the following matrix

$$A_{ij} = 2\frac{\vec{\alpha}_i\cdot\vec{\alpha}_j}{||\vec{\alpha}_i||^2}\,. \tag{4.25}$$

corresponds to the Cartan matrix of a semi-simple Lie algebra. In our case we have

$$\vec{\alpha}_1 = (-a, -2(p+1)\alpha)\,, \tag{4.26}$$

$$\vec{\alpha}_2 = (+a, -2(p+1)\alpha)\,. \tag{4.27}$$

If we combine our expressions for a and α we find:

$$\vec{\alpha}_1 \cdot \vec{\alpha}_2 = 2(p-1)\,. \tag{4.28}$$

$$\vec{\alpha}_1 \cdot \vec{\alpha}_1 = \vec{\alpha}_2 \cdot \vec{\alpha}_2 = 4\,. \tag{4.29}$$

Hence:

$$D3 - D3 : A = 2, \quad A_1 \quad \text{(just one exponential)} \tag{4.30}$$

$$D2 - D4 : A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{4.31}$$

$$D1 - D5 : A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{so} \quad A_1 \oplus A_1 \tag{4.32}$$

$$D0 - D6 : A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{so} \quad A_2 = SL(3)\,. \tag{4.33}$$

Only the $D2 - D4$ is not a Cartan matrix. Yet, the equations of motion for the D2-D4 system must be integrable since it can be obtained from a consistent truncation IIA on $\mathbb{T}^2 \times \mathbb{R}^{1,2}$ which is 5d Euclidean supergravity with coset space $E_{6(6)}/H$ with H some non-compact maximal subgroup. All such geodesics are integrable.[3] Of course this can understood since we did not consider a consistent truncation since we need to keep the B-field. We turn to this next.

4.2 With B-field: D2-D4

Our goal is to compute the extension of (4.18) when $p = 2$ that includes the B-field. We do this by reducing the Lagrangian density (4.12) over $\mathbb{R}^{1,2}$ as before but keeping now also the axion b. We find

$$\begin{aligned} \frac{\mathcal{L}_5}{\sqrt{g}} =& \mathcal{R} - \tfrac{1}{2}(\partial\Phi)^2 - \tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{2}e^{\Phi+3(\alpha-\beta)\varphi}(\partial\chi_E)^2 - \tfrac{1}{2}\tfrac{1}{4!}e^{\Phi+3(\beta-\alpha)\varphi}F_4^2 \\ & - \tfrac{1}{2}e^{-2\Phi}(\partial b)^2 + \frac{b d\chi_E \wedge F_4}{\sqrt{g}}\,. \end{aligned} \tag{4.34}$$

To do explicit computations we take $\alpha - \beta = \frac{1}{\sqrt{3}}$. We then Hodge dualise F_4 in order to display the magnetic potential χ_M through:

$$F_4 = e^{-\Phi-3(\beta-\alpha)\varphi} \star (d\chi_M + b d\chi_E)\,, \tag{4.35}$$

to find

$$\frac{\mathcal{L}_5}{\sqrt{g}} = \mathcal{R} - \tfrac{1}{2}(\partial\Phi)^2 - \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}e^{-2\Phi}(\partial b)^2 + \tfrac{1}{2}e^{\Phi+\sqrt{3}\varphi}(\partial\chi_E)^2 + \tfrac{1}{2}e^{-\Phi+\sqrt{3}\varphi}(\partial\chi_M + b\partial\chi_E)^2\,. \tag{4.36}$$

Now ... let the games begin and find the null geodesics. This seems $SL(3,\mathbb{R})/SO(2,1)$! The concrete coset representative that gives this metric can be found as follows:

$$L = \exp\{\chi_E E_{12}\}\exp\{\chi_M E_{13}\}\exp\{b E_{23}\}\exp\Big\{(-\tfrac{1}{4}\Phi - \tfrac{\sqrt{3}}{4}\varphi)H_1 + (\tfrac{\sqrt{3}}{4}\Phi - \tfrac{1}{4}\varphi)H_2\Big\} \tag{4.37}$$

[3] It has been claimed that indeed the integrability of Toda systems exactly comes from the embedding into a larger geodesic system [20]. The other way around, one can integrate out shift symmetric directions of an integrable geodesic motion to obtain an integrable system with potential.

where the Cartan generators and positive step operators are given by

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{4.38}$$

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.39}$$

The sigma model can be found from the metric

$$\mathrm{Tr}(dMdM^{-1}) \qquad \text{where} \quad M = L\eta L^T\,. \tag{4.40}$$

Where normalisations work such that the Lagrangian density (4.36) now becomes:

$$\frac{\mathcal{L}_5}{\sqrt{g}} = \mathcal{R} + \tfrac{1}{4}\mathrm{Tr}(\partial M \partial M^{-1})\,. \tag{4.41}$$

The bilinear form η which determines the $SO(2,1)$ subgroup is given by

$$\eta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \tag{4.42}$$

L is given by the rather simple expression:

$$L = \begin{pmatrix} e^{-\frac{1}{\sqrt{3}}\varphi} & e^{\frac{1}{2}\Phi+\frac{1}{2\sqrt{3}}\varphi}\chi_E & e^{-\frac{1}{2}\Phi+\frac{1}{2\sqrt{3}}\varphi}(b\chi_E+\chi_M) \\ 0 & e^{\frac{1}{2}\Phi+\frac{1}{2\sqrt{3}}\varphi} & e^{-\frac{1}{2}\Phi+\frac{1}{2\sqrt{3}}}b \\ 0 & 0 & e^{-\frac{1}{2}\Phi+\frac{1}{2\sqrt{3}}}\,. \end{pmatrix} \tag{4.43}$$

while M is more complicated:

$$M = \begin{pmatrix} e^{-\frac{2}{\sqrt{3}}\varphi}\left(-1+e^{\sqrt{3}\varphi+\Phi}\chi_E^2+e^{\sqrt{3}\varphi-\Phi}(b\chi_E+\chi_M)^2\right) & e^{\frac{1}{\sqrt{3}}\varphi-\Phi}\left((e^{2\Phi}+b)^2\chi_E+b\chi_M\right) & e^{\frac{1}{\sqrt{3}}\varphi-\Phi}(b\chi_E+\chi_M) \\ \dots & e^{\frac{1}{\sqrt{3}}\varphi-\Phi}(e^{2\Phi}+b^2) & e^{\frac{1}{\sqrt{3}}\varphi-\Phi}b \\ \dots & \dots & e^{\frac{1}{\sqrt{3}}\varphi-\Phi} \end{pmatrix} \tag{4.44}$$

4.3 The null geodesics

Once we have the null geodesics, we need to realise that the affine parameter is the radial harmonic $h(r)$ on 5d flat space

$$h(r) = A + \frac{B}{r^3}\,, \tag{4.45}$$

with A and B constants. We fix conventions such that $A = B = 1$. The geodesics through the origin are solutions given by the exponential map, at least at the level of the symmetric coset matrix M:

$$M = \eta\exp\{Qt\} \tag{4.46}$$

where Q is a matrix inside the coset algebra:

$$Q^T = \eta Q \eta \,. \tag{4.47}$$

These are all geodesics through the origin. The expression can trivially be generalised. The general Q element that vanishes at order 3 is given by:

$$Q = \begin{pmatrix} -\frac{\alpha+\beta}{2} & -\frac{\beta}{2}\sqrt{\frac{\beta}{\beta-\alpha}} & \frac{\alpha}{2}\sqrt{\frac{\alpha}{\alpha-\beta}} \\ \frac{\beta}{2}\sqrt{\frac{\beta}{\beta-\alpha}} & \frac{\beta}{2} & 0 \\ -\frac{\alpha}{2}\sqrt{\frac{\alpha}{\alpha-\beta}} & 0 & \frac{\alpha}{2} \end{pmatrix} . \tag{4.48}$$

This almost the same situation as the solutions discussed in [17] but with a slightly different Q matrix (columns and rows 2 and 3 interchanged) and a slightly different η-matrix (columns and rows 2 and 3 interchanged). But so it should be easy to find the solutions using the technology of [21, 17].

By comparing $M = \eta \exp(Qh)$ and $M = L\eta L^T$, one can extract the fields $b(h)$, $\varphi(h)$, $\Phi(h)$, and $\chi_{E/M}(h)$:

$$b(h) = \frac{(\alpha\beta)^{3/2} h^2}{h^2\alpha^2\beta - 4h\alpha(\alpha-\beta) - 8(\alpha-\beta)} \tag{4.49}$$

$$\chi_E(h) = -\frac{(-\beta)^{3/2}}{\sqrt{\alpha-\beta}} \frac{h^2\alpha + 4h}{h^2\alpha\beta + 4h(\alpha+\beta) + 8} \tag{4.50}$$

$$\chi_M(h) = -\frac{\alpha^{3/2}}{\sqrt{\alpha-\beta}} \frac{h^2\beta + 4h}{h^2\alpha\beta + 4h(\alpha+\beta) + 8} \tag{4.51}$$

$$e^{\Phi(h)} = 2\sqrt{2}(\alpha-\beta)\sqrt{\frac{h^2\alpha\beta + 4h(\alpha+\beta) + 8}{(h^2\alpha^2\beta - 4h\alpha(\alpha-\beta) - 8(\alpha-\beta))^2}} \tag{4.52}$$

$$e^{\frac{1}{\sqrt{3}}\varphi(h)} = \frac{1}{2}\left(\frac{1}{2}h^2\alpha\beta + 2h(\alpha+\beta) + 4\right)^{1/2} \tag{4.53}$$

[SR: see mathematica for details on this]. The constants α and β can be fixed in terms of the charges Q_E, Q_M from the axion shift symmetries (see below).

4.4 The uplift

The uplift to 10d *string frame* can be shown to be

$$\begin{aligned} ds^2_{10} =& e^{-\frac{1}{\sqrt{3}}\varphi - \frac{2C}{\sqrt{3}}}\left(-dt^2 + dx^2 + dy^2\right) + e^{\frac{1}{\sqrt{3}}\varphi - \frac{2C}{\sqrt{3}}}\left(dr^2 + r^2 d\Omega_4^2\right) + e^{\Phi}\left(d\theta_1^2 + d\theta_2^2\right) , && (4.54) \\ H_3 =& b'\, dr \wedge d\theta_1 \wedge d\theta_2 \,, && (4.55) \\ F_4 =& \chi_E' dr \wedge dt \wedge dx \wedge dy + e^{-\Phi + \sqrt{3}\varphi}\left(\chi_M' + b\chi_E'\right) r^4 d\Omega_4 \,. && (4.56) \end{aligned}$$

Primes denote derivatives wrt to r and the r-dependence of the quantities $\Phi, \varphi, b, \chi_M, \chi_E$ all comes from them being functions of the affine parameter $h(r)$. [4] This is how harmonics are introduced.

[4] In particular $f' = (df/dh)(dh/dr) = -3\dot{f}r^{-4}$.

Using the axion shift symmetries we deduce the two Noether charges, equal to the RR charges under F_4 (Q_E) and F_6 (Q_M):

$$Q_M = e^{-\Phi+\sqrt{3}\varphi}\left(\chi_M' + b\chi_E'\right) r^4 \tag{4.57}$$

$$Q_E = e^{\Phi+\sqrt{3}\varphi}\chi_E' r^4 + bQ_M\,. \tag{4.58}$$

This allows us to rewrite the F_4 as

$$F_4 = \frac{(Q_E - bQ_M)}{r^4} e^{-\Phi-\sqrt{3}\varphi}(dr\wedge dt\wedge dx\wedge dy) + Q_M d\Omega_4\,. \tag{4.59}$$

Using our explicit expressions for the fields, we find

$$Q_M = \frac{3B}{2}\frac{\alpha^{3/2}}{\sqrt{\alpha-\beta}}\,, \tag{4.60}$$

$$Q_E = \frac{3B}{2}\frac{(-\beta)^{3/2}}{\sqrt{\alpha-\beta}}\,, \tag{4.61}$$

where B is the constant appearing in (4.45). Note that this implies that the sign of Q_E is equal to the sign of Q_M

4.5 Consistency check: reducing to pure $D2$ and pure $D4$

The $D2$ brane solution can be obtained by setting $Q_M = 0$. In that case, we find the string frame metric

$$\mathrm{d}s_{10}^2 = H_2^{-1/2} e^{-\frac{2C}{\sqrt{3}}}\left(-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2\right) + H_2^{1/2} e^{-\frac{2C}{\sqrt{3}}}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_4^2\right) + H_2^{1/2}\left(d\theta_1^2 + d\theta_2^2\right) \tag{4.62}$$

where we set $A = 0$ in the harmonic function h and we used

$$e^{\Phi(h)} = e^{\frac{1}{\sqrt{3}}\varphi(h)} = \left(1 - \frac{Q_E}{3r^3}\right)^{1/2} \equiv H_2^{1/2}\,. \tag{4.63}$$

In this case, F_4 can be expressed as

$$F_4 = -\mathrm{d}H_2^{-1}\wedge \underbrace{\mathrm{d}t\wedge\mathrm{d}x\wedge\mathrm{d}y}_{D2\text{ worldvolume}} \tag{4.64}$$

Note the minus sign in the harmonic function: this describes an orientifold for positive Q_E and a D-brane for negative Q_M. Doing the same reasoning for the special case of $D4$, one ends up with $H_4 = 1 + \frac{Q_M}{3r^3}$, and since the sign of Q_E has to be equal to the sign of Q_M (see comment at the end of the previous section), this means that whenever we reduce to an orientifold for one, we have to get a brane for the other ($D2 - O4$ or $O4 - D2$). Below we generalize this argument slightly.

4.6 Trying to get branes and not orientifolds: disconnected components of $O(2,1)$.

Here are the conditions that the Q matrix needs to satisfy:

$$Q^3 = 0, \quad Q^T = \eta Q \eta, \quad \mathrm{Tr}\big(Q^2\big) = 0\,. \tag{4.65}$$

where

$$\eta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.66}$$

Note then that given a Q matrix, we can generate others by defining

$$Q' = \Lambda^{-1} Q \Lambda \tag{4.67}$$

provided that

$$\Lambda \eta \Lambda^T = \eta \tag{4.68}$$

This is reminiscent of the disconnected components of the Lorentz group. The analogy of the parity and time reversal transformations here are the matrices

$$P = \mathrm{diag}(+1, -1, +1) \tag{4.69}$$

$$T = \mathrm{diag}(-1, +1, +1). \tag{4.70}$$

Note that these matrices verify

$$\Lambda \eta \Lambda^T = \eta, \qquad \Lambda \in \{P, T, PT\} \tag{4.71}$$

Therefore, we can define the "disconnected" Q's as

$$Q_P = PQP \tag{4.72}$$

$$Q_T = TQT \tag{4.73}$$

$$Q_{PT} = TPQPT \tag{4.74}$$

and hopefully they give us the 4 different possibilities of brane-brane, brane-orientifold, orientifold-brane and orientifold-orientifold. We investigate this below, but long story short: it does not work.

When implementing this in Mathematica, we find all the same expressions for the fields as before (in terms of α, β and h), but the rules for inverting α and β in terms of Q_E and Q_M are different.

We find

$$Q_M = \frac{3B}{2}\frac{\alpha^{3/2}}{\sqrt{\alpha-\beta}}, \qquad Q_E = \frac{3B}{2}\frac{(-\beta)^{3/2}}{\sqrt{\alpha-\beta}} \tag{4.75}$$

$$Q_M^{(P)} = \frac{3B}{2}\frac{\alpha^{3/2}}{\sqrt{\alpha-\beta}}, \qquad Q_E^{(P)} = -\frac{3B}{2}\frac{(-\beta)^{3/2}}{\sqrt{\alpha-\beta}} \tag{4.76}$$

$$Q_M^{(T)} = -\frac{3B}{2}\frac{\alpha^{3/2}}{\sqrt{\alpha-\beta}}, \qquad Q_E^{(T)} = -\frac{3B}{2}\frac{(-\beta)^{3/2}}{\sqrt{\alpha-\beta}} \tag{4.77}$$

$$Q_M^{(PT)} = -\frac{3B}{2}\frac{\alpha^{3/2}}{\sqrt{\alpha-\beta}}, \qquad Q_E^{(PT)} = \frac{3B}{2}\frac{(-\beta)^{3/2}}{\sqrt{\alpha-\beta}} \tag{4.78}$$

Note that for the normal convention and T, the signs of Q_E and Q_M have to be equal, whereas for P and PT, they have to be opposite. The sign of B is equal to the sign of Q_M for the normal and the P case, whereas it is opposite for the T and PT cases.

Let's see how the single brane limits emerge out of this. Recall that the formulas we found for the fields ϕ and Φ are the same as in the original Q choice when expressed in terms of α and β, i.e.

$$e^{\Phi(h)} = 2\sqrt{2}(\alpha-\beta)\sqrt{\frac{8+h^2\alpha\beta+4h(\alpha+\beta)}{(h^2\alpha^2\beta-4h\alpha(\alpha-\beta)-8(\alpha-\beta))^2}} \tag{4.79}$$

$$e^{\frac{1}{\sqrt{3}}\varphi(h)} = \frac{1}{2}\left(\frac{1}{2}h^2\alpha\beta+2h(\alpha+\beta)+4\right)^{1/2} \tag{4.80}$$

Clearly, if $Q_M=0$ (i.e. the $D2$ case), we have $\alpha=0$ and

$$Q_E = Q_E^{(PT)} = \frac{3B}{2}(-\beta) \tag{4.81}$$

$$Q_E^{(P)} = Q_E^{(T)} = -\frac{3B}{2}(-\beta) \tag{4.82}$$

$$\tag{4.83}$$

Therefore,

$$e^{\Phi(h)} = \sqrt{1+\frac{1}{2}h\beta} \equiv H_2^{1/2} \tag{4.84}$$

$$e^{\frac{1}{\sqrt{3}}\varphi(h)} = \left(1+\frac{1}{2}h\beta\right)^{1/2} \equiv H_2^{1/2} \tag{4.85}$$

which means that we find (if we set $A=0$ in the h function)

$$H_2^{(0,PT)} = 1-\frac{Q_E^{(0,PT)}}{3r^3} \tag{4.86}$$

$$H_2^{(P,T)} = 1+\frac{Q_E^{(P,T)}}{3r^3} \tag{4.87}$$

(here the superscript $^{(0)}$ denotes the base case of our original Q matrix.)

Similarly, for the case of $Q_E=0$ (i.e. the $D4$ case), we have $\beta=0$ and

$$Q_M = Q_M^{(P)} = \frac{3B}{2}\alpha \tag{4.88}$$

$$Q_M^{(T)} = Q_M^{(PT)} = -\frac{3B}{2}\alpha \tag{4.89}$$

which leads to

$$e^{\Phi(h)} = \frac{1}{\sqrt{1+\frac{1}{2}h\alpha}} \equiv H_4^{-1/2} \tag{4.90}$$

$$e^{\frac{1}{\sqrt{3}}\varphi(h)} = \sqrt{1+\frac{1}{2}h\alpha} \equiv H_4^{1/2} \tag{4.91}$$

where

$$H_4^{(0,P)} = 1 + \frac{Q_M^{(0,P)}}{3r^3} \tag{4.92}$$

$$H_4^{(T,PT)} = 1 - \frac{Q_M^{(T,PT)}}{3r^3} \tag{4.93}$$

To summarize, we have

$$H_2^{(0)} = 1 - \frac{Q_E^{(0)}}{3r^3} \qquad H_4^{(0)} = 1 + \frac{Q_M^{(0)}}{3r^3} \tag{4.94}$$

$$H_2^{(P)} = 1 + \frac{Q_E^{(P)}}{3r^3} \qquad H_4^{(P)} = 1 + \frac{Q_M^{(P)}}{3r^3} \tag{4.95}$$

$$H_2^{(T)} = 1 + \frac{Q_E^{(T)}}{3r^3} \qquad H_4^{(T)} = 1 - \frac{Q_M^{(T)}}{3r^3} \tag{4.96}$$

$$H_2^{(PT)} = 1 - \frac{Q_E^{(PT)}}{3r^3} \qquad H_4^{(PT)} = 1 - \frac{Q_M^{(PT)}}{3r^3} \tag{4.97}$$

but recall that for $(0, T)$, Q_E and Q_M have the same sign whereas for (P, PT), Q_E and Q_M have opposite sign. This means that it is **never** possible to have positive (or negative) tension for both branes.

A Chirality and sign issues

The SUSY spinor chirality condition is present because we are considering type IIB supergravity. We follow here the notation from the appendix of [22]. We will compare their notation to our notation (all done in Lorentzian signature). In [22], they explicitly write two real spinors $\epsilon = (\epsilon_1\ \epsilon_2)^T$, which matches our notation by setting our $\epsilon = \epsilon_2 + i\epsilon_1$. The chirality of the SUSY parameter is set to be positive by the constraint

$$\epsilon_i = \Gamma^{01\bar{\ }...9}\epsilon_i \tag{A.1}$$

Given this, they write explicit KS equations which exactly reduce to the Lorentzian KS equations from [1]. Note that their notation for the actions and fields is the same as [1] with $\Phi = \phi$. This means that positive chirality of ϵ is set *from the beginning* and is not a freedom we have. If we were to consider negative chirality, we would have to consider different KS equations to begin with.

An interesting hypothesis by Thomas is that this could also have consequences on the i^z story: maybe this sign issue also switches F_5 from self-dual to anti-self-dual, which could give an extra factor of i in some cases. [SR: Worth investigating further, but it would be nice to find the KS equations for IIB with negative chirality spinors to compare... Toine did not write these in his appendix.]

B Details about D(-1)/D7: squaring to the EOM

We summarize below the equations of motion and killing spinor equations for our Ansatz ((3.1)-(3.4)). Note that throughout this section we have gauge fixed $L_y = 1$. [SR: to do: write everything in the other gauge choice.]

Equations of motion

Bianchi and EOM for the gauge fields:

$$\delta_1 L_x L_5^4 L_1^{-4} = \tilde{\delta}, \quad \beta L_x L_1^4 L_5^4 = \tilde{\beta}, \quad \alpha(y) = \alpha, \quad \gamma(y) = \gamma \tag{B.1}$$

Dilaton EOM:

$$\phi'' - 2\phi'^2 = \phi' \frac{\beta'}{\beta} + e^{2\phi}\left(\frac{\alpha^2}{L_x^2} - \beta^2\right) \tag{B.2}$$

Einstein equations:

$$(\mathbf{xx}): \quad \frac{L_x''}{L_x} + 4\frac{L_x'}{L_x}\left(\frac{L_1'}{L_1} + \frac{L_5'}{L_5}\right) = 2\frac{L_x'}{L_x}\phi' - \frac{1}{4}e^{2\phi}\left(\frac{\alpha^2}{L_x^2} + \beta^2\right) - \frac{1}{4}e^{2\phi}\frac{1}{L_1^8}\left(\frac{\gamma^2}{L_x^2} - \delta_1^2\right) \tag{B.3}$$

$$(\mathbf{yy}): \quad \frac{L_x''}{L_x} + 4\left(\frac{L_1''}{L_1} + \frac{L_5''}{L_5}\right) = 2\phi'' + \frac{1}{4}e^{2\phi}\left(\frac{\alpha^2}{L_x^2} + \beta^2\right) + \frac{1}{4}e^{2\phi}\frac{1}{L_1^8}\left(\frac{\gamma^2}{L_x^2} - \delta_1^2\right) \tag{B.4}$$

$$(\mathbf{xy}): \quad i\alpha\beta + \frac{\gamma\delta_1}{L_1^8} = 0 \tag{B.5}$$

$$(\mathbb{T}_1): \quad \frac{L_1''}{L_1} + 3\frac{L_1'^2}{L_1^2} + \frac{L_1'}{L_1}\left(\frac{4L_5'}{L_5} + \frac{L_x'}{L_x}\right) = 2\frac{L_1'}{L_1}\phi' + \frac{1}{4}e^{2\phi}\left(\frac{\alpha^2}{L_x^2} - \beta^2\right) - \frac{1}{4}e^{2\phi}\frac{1}{L_1^8}\left(\frac{\gamma^2}{L_x^2} + \delta_1^2\right) \tag{B.6}$$

$$(\mathbb{T}_5): \quad \frac{L_5''}{L_5} + 3\frac{L_5'^2}{L_5^2} + \frac{L_5'}{L_5}\left(\frac{4L_1'}{L_1} + \frac{L_x'}{L_x}\right) = 2\frac{L_5'}{L_5}\phi' + \frac{1}{4}e^{2\phi}\left(\frac{\alpha^2}{L_x^2} - \beta^2\right) + \frac{1}{4}e^{2\phi}\frac{1}{L_1^8}\left(\frac{\gamma^2}{L_x^2} + \delta_1^2\right) \tag{B.7}$$

Killing spinor equations

Dilatino:

$$\phi' = \eta_d e^{\phi}\left(\frac{\alpha}{L_x} - \eta_d \beta\right) \tag{B.8}$$

$$(\Gamma_{\bar{x}} + i\eta_d \Gamma_{\bar{y}})\epsilon = 0 \tag{B.9}$$

Recall that $\eta_d = \pm 1$ comes from the choice of the projector for the SUSY parameter. [SR: As far as I understand this is undetermined.]

Gravitino - x and y directions

A combination of the two equations allow us to solve for $\epsilon(y)$ in terms of L_x:

$$\epsilon(y) = \frac{\epsilon_0}{\sqrt{L_x(y)}} \tag{B.10}$$

The remaining equation can be written as

$$\eta_d L_x' + \frac{1}{4} L_x e^{\phi} \left[\left(\frac{\alpha}{L_x} + \eta_d \beta \right) + \frac{i^z}{L_1^4} \eta_p \left(\frac{\gamma}{L_x} - i \eta_d \delta_1 \right) \right] = 0 \tag{B.11}$$

Gravitino - torus directions

$$\eta_d L_1' + \frac{1}{4} L_1 e^{\phi} \left[- \left(\frac{\alpha}{L_x} - \eta_d \beta \right) + \frac{i^z}{L_1^4} \eta_p \left(\frac{\gamma}{L_x} + i \eta_d \delta_1 \right) \right] = 0 \tag{B.12}$$

$$\eta_d L_5' + \frac{1}{4} L_5 e^{\phi} \left[- \left(\frac{\alpha}{L_x} - \eta_d \beta \right) - \frac{i^z}{L_1^4} \eta_p \left(\frac{\gamma}{L_x} + i \eta_d \delta_1 \right) \right] = 0 \tag{B.13}$$

C Details about dilatonic BH uplift

[SR: I don't know yet how much of this is necessary but I put it here for completeness.]

C.1 The 4-dimensional dilatonic black hole

We consider the 4d action

$$S = \int \sqrt{|g|} \left\{ R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{a\phi}F^2 \right\} . \tag{C.1}$$

The following solution describes the well-known dilatonic black hole [SR: refs?]

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left(dr^2 + r^2 \mathrm{d}\Omega_2^2\right) , \qquad e^{2U} = (BC)^{-1/2} , \tag{C.2}$$

$$e^{\phi} = (B/C)^{\frac{\sqrt{3}}{2}} , \tag{C.3}$$

$$F = Q_M \mathrm{d}\Omega_2 + Q_E \frac{e^{2U - a\phi}}{r^2} \mathrm{d}t \wedge \mathrm{d}r\, . \tag{C.4}$$

where

$$B = 1 - \beta r^{-1} + \frac{\alpha\beta^2}{2(\alpha-\beta)} r^{-2} , \tag{C.5}$$

$$C = 1 + \alpha r^{-1} - \frac{\alpha^2\beta}{2(\alpha-\beta)} r^{-2} . \tag{C.6}$$

The constants α and β can be written in terms of mass and charges as

$$M = \frac{\sqrt{2}}{4}(\alpha - \beta) , \quad Q_E = \sqrt{\frac{\beta^3}{\beta - \alpha}} , \quad Q_M = \sqrt{\frac{\alpha^3}{\alpha - \beta}} \tag{C.7}$$

C.2 The uplift to 10d

Consider IIA string theory and truncate down to the bosonic action with RR 2-form field strength. This gives

$$S = \int \sqrt{-g} \left(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}e^{3\phi/2}F_2^2\right) . \tag{C.8}$$

Now reducing on a 6-torus keeping only the volume modulus and performing a field rotation should give us the 4d action we started from in the previous section.

Indeed, let us take

$$ds_{10}^2 = e^{2\alpha\varphi} ds_4^2 + e^{2\beta\varphi} ds_6^2 \tag{C.9}$$

where 4d Einstein frame requires $3\beta = -\alpha$, and canonical normalisation of φ requires $\alpha^2 = 3/16$. The reduced action in 4d then is

$$S = \int \sqrt{-g} \left(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}(\partial\varphi)^2 - \frac{1}{4} e^{3\phi/2 - 2\alpha\varphi} F_2^2 \right) . \tag{C.10}$$

Now we make a rotation in field space

$$s = \frac{1}{\sqrt{3}}\left(\frac{3}{2}\phi - 2\alpha\varphi\right), \tag{C.11}$$

$$t = \frac{1}{\sqrt{3}}\left(2\alpha\phi + \frac{3}{2}\varphi\right). \tag{C.12}$$

we then get

$$S = \int \sqrt{-g}\left(R - \tfrac{1}{2}(\partial s)^2 - \tfrac{1}{2}(\partial t)^2 - \tfrac{1}{4}e^{\sqrt{3}s}F_2^2\right). \tag{C.13}$$

So the scalar t decouples and we can put it to a constant and then we recovered the dilatonic black-hole Lagrangian. This allows us to lift dilatonic black holes.

Now in this top down language, the dilatonic black hole reads

$$ds_4^2 = -e^{2U}dt^2 + e^{-2U}\left(dr^2 + r^2\mathrm{d}\Omega_2^2\right), \qquad e^{2U} = (BC)^{-1/2}, \tag{C.14}$$

$$e^s = (B/C)^{\frac{\sqrt{3}}{2}}, \tag{C.15}$$

$$F = Q_M \mathrm{d}\Omega_2 + Q_E \frac{e^{2U-\sqrt{3}s}}{r^2}\mathrm{d}t\wedge \mathrm{d}r\,. \tag{C.16}$$

The uplift of the metric is fairly easy: we have the 10d metric in Einstein frame

$$ds_{10,e}^2 = -e^{\sqrt{3}\alpha t}B^{-\frac{7}{8}}C^{-\frac{1}{8}}\mathrm{d}t^2 + e^{\sqrt{3}\alpha t}B^{\frac{1}{8}}C^{\frac{7}{8}}\left(dr^2 + r^2\mathrm{d}\Omega_2^2\right) + e^{-\frac{1}{\sqrt{3}}\alpha t}B^{\frac{1}{8}}C^{-\frac{1}{8}}\mathrm{d}s_6^2 \tag{C.17}$$

In string frame, this becomes

$$ds_{10,s}^2 = -B^{-\frac{1}{2}}C^{-\frac{1}{2}}\mathrm{d}t^2 + B^{\frac{1}{2}}C^{\frac{1}{2}}\left(dr^2 + r^2\mathrm{d}\Omega_2^2\right) + B^{\frac{1}{2}}C^{-\frac{1}{2}}\mathrm{d}s_6^2\,, \tag{C.18}$$

where we set $t = 0$ for simplicity [SR: I am slightly confused: when $t \neq 0$, we do not get asymptotically flat space?]. Note that the translation between einstein and string frame uses the 10d dilaton ϕ and not the 4d dilaton s.

This is very reminiscent of the usual harmonic ansatz for brane intersections. Recall that the functions B and C are given by

$$B = 1 - \beta r^{-1} + \frac{\alpha\beta^2}{2(\alpha-\beta)}r^{-2}, \tag{C.19}$$

$$C = 1 + \alpha r^{-1} - \frac{\alpha^2\beta}{2(\alpha-\beta)}r^{-2}, \tag{C.20}$$

where the constants α and β are written in terms of mass and charges as

$$M = \frac{\sqrt{2}}{4}(\alpha-\beta)\,, \quad Q_E = \sqrt{\frac{\beta^3}{\beta-\alpha}}\,, \quad Q_M = \sqrt{\frac{\alpha^3}{\alpha-\beta}}\,. \tag{C.21}$$

We can invert the above relations to get the functions B and C only in terms of the charges: we use

$$\begin{cases} \alpha = Q_M\sqrt{1+\left(\frac{Q_E}{Q_M}\right)^{2/3}} \\ \beta = -Q_E\sqrt{1+\left(\frac{Q_M}{Q_E}\right)^{2/3}} \\ M = \frac{\sqrt{2}}{4}\left(Q_E^{2/3}+Q_M^{2/3}\right)^{3/2} \end{cases} \tag{C.22}$$

to obtain

$$B = 1 + \frac{Q_E\sqrt{1+\left(\frac{Q_M}{Q_E}\right)^{2/3}}}{r} + \frac{Q_E Q_M}{2r^2}\left(\frac{Q_E}{Q_M}\right)^{1/3}, \tag{C.23}$$

$$C = 1 + \frac{Q_M\sqrt{1+\left(\frac{Q_E}{Q_M}\right)^{2/3}}}{r} + \frac{Q_E Q_M}{2r^2}\left(\frac{Q_M}{Q_E}\right)^{1/3}. \tag{C.24}$$

Our dilaton and field strength are:

$$e^{\phi} = \left(\frac{B}{C}\right)^{\frac{3}{4}} \tag{C.25}$$

$$F = Q_M \mathrm{d}\Omega_2 + Q_E\frac{B^{-2}C}{r^2}\mathrm{d}t\wedge\mathrm{d}r \tag{C.26}$$

This is the result in the main text, with $B = H_0$, $C = H_6$, $Q_E = Q_0$ and $Q_M = Q_6$.

D Probe brane and jet instabilities

D.1 D0-D6

In the main text we argued that a probe (anti-) $D0$ brane feels a potential

$$V(r) \sim H_0^{-1}(r)H_6^{1/2}(r) \mp Q_0\int^r \mathrm{d}\hat{r}\frac{H_6(\hat{r})}{\hat{r}^2 H_0^2(\hat{r})}. \tag{D.1}$$

The sign can be fixed by looking at a probe D_0 on a pure D_0 background and checking that it does not feel a force (when the probe charge and background charge have same sign). If we set $Q_6 = 0$, we should find $V(r) =$ constant. Indeed,

$$V(r; Q_6 = 0) \sim H_0^{-1}(r) \mp Q_0\int^r \mathrm{d}\hat{r}\frac{1}{\hat{r}^2 H_0^2(\hat{r})} \tag{D.2}$$

$$\sim \frac{1\pm\frac{Q_0}{r}}{1+\frac{Q_0}{r}} \tag{D.3}$$

Clearly this gives $V(r) = 1$ if we choose the $-$ sign in equation (D.1), or in other words if we use a probe brane and not a probe antibrane.

For generic charges Q_0, Q_6, we can compute the integral explicitly in Mathematica, and we find the potential

$$V(r) \sim \frac{r^2\sqrt{1+\frac{Q_6}{r}\sqrt{1+\left(\frac{Q_0}{Q_6}\right)^{2/3}}+\frac{Q_0^{2/3}Q_6^{4/3}}{2r^2}} - r^2\sqrt{1+\left(\frac{Q_6}{Q_0}\right)^{2/3}} - rQ_0^{1/3}Q_6^{2/3}}{r^2 + rQ_0\sqrt{1+\left(\frac{Q_6}{Q_0}\right)^{2/3}} + \frac{1}{2}Q_0^{4/3}Q_6^{2/3}} \tag{D.4}$$

References

[1] S. E. Aguilar-Gutierrez, K. Parmentier, and T. Van Riet, "Towards an "$\mathrm{AdS}_1/\mathrm{CFT}_0$" correspondence from the D(−1)/D7 system?," *JHEP* **09** (2022) 249, arXiv:2207.13692 [hep-th].

[2] A. A. Tseytlin, "Harmonic superpositions of M-branes," *Nucl. Phys. B* **475** (1996) 149–163, arXiv:hep-th/9604035.

[3] E. Bergshoeff, M. de Roo, E. Eyras, B. Janssen, and J. P. van der Schaar, "Multiple intersections of D-branes and M-branes," *Nucl. Phys. B* **494** (1997) 119–143, arXiv:hep-th/9612095.

[4] D. J. Smith, "Intersecting brane solutions in string and M theory," *Class. Quant. Grav.* **20** (2003) R233, arXiv:hep-th/0210157.

[5] A. Hanany and E. Witten, "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics," *Nucl. Phys. B* **492** (1997) 152–190, arXiv:hep-th/9611230.

[6] U. Danielsson, G. Ferretti, and I. R. Klebanov, "Creation of fundamental strings by crossing D-branes," *Phys. Rev. Lett.* **79** (1997) 1984–1987, arXiv:hep-th/9705084.

[7] M. Massar and J. Troost, "D0 - D8 - F1 in massive IIA SUGRA," *Phys. Lett. B* **458** (1999) 283–287, arXiv:hep-th/9901136.

[8] Y. Imamura, "1/4 BPS solutions in massive IIA supergravity," *Prog. Theor. Phys.* **106** (2001) 653–670, arXiv:hep-th/0105263.

[9] B. Janssen, P. Meessen, and T. Ortin, "The D8-brane tied up: String and brane solutions in massive type IIA supergravity," *Phys. Lett. B* **453** (1999) 229–236, arXiv:hep-th/9901078.

[10] E. Bergshoeff, U. Gran, R. Linares, M. Nielsen, and D. Roest, "Domain walls and the creation of strings," *Class. Quant. Grav.* **20** (2003) 3465–3482, arXiv:hep-th/0303253.

[11] H. Ooguri and C. Vafa, "Non-supersymmetric AdS and the Swampland," *Adv. Theor. Math. Phys.* **21** (2017) 1787–1801, arXiv:1610.01533 [hep-th].

[12] J. M. Maldacena, J. Michelson, and A. Strominger, "Anti-de Sitter fragmentation," *JHEP* **02** (1999) 011, arXiv:hep-th/9812073.

[13] M. Billò, M. Frau, F. Fucito, L. Gallot, A. Lerda, and J. F. Morales, "On the D(–1)/D7-brane systems," *JHEP* **04** (2021) 096, arXiv:2101.01732 [hep-th].

[14] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, "M theory as a matrix model: A Conjecture," *Phys. Rev. D* **55** (1997) 5112–5128, arXiv:hep-th/9610043.

[15] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, "A Large N reduced model as superstring," *Nucl. Phys. B* **498** (1997) 467–491, arXiv:hep-th/9612115.

[16] P. Breitenlohner, D. Maison, and G. W. Gibbons, "Four-Dimensional Black Holes from Kaluza-Klein Theories," *Commun. Math. Phys.* **120** (1988) 295.

[17] W. Chemissany, P. Fre, J. Rosseel, A. S. Sorin, M. Trigiante, and T. Van Riet, "Black holes in supergravity and integrability," *JHEP* **09** (2010) 080, arXiv:1007.3209 [hep-th].

[18] I. Bena, K. Pilch, and N. P. Warner, "Brane-Jet Instabilities," *JHEP* **10** (2020) 091, arXiv:2003.02851 [hep-th].

[19] E. Bergshoeff, W. Chemissany, A. Ploegh, M. Trigiante, and T. Van Riet, "Generating Geodesic Flows and Supergravity Solutions," *Nucl. Phys. B* **812** (2009) 343–401, arXiv:0806.2310 [hep-th].

[20] L. A. Ferreira and D. I. Olive, "Noncompact Symmetric Spaces and the Toda Molecule Equations," *Commun. Math. Phys.* **99** (1985) 365.

[21] W. Chemissany, J. Rosseel, M. Trigiante, and T. Van Riet, "The Full integration of black hole solutions to symmetric supergravity theories," *Nucl. Phys. B* **830** (2010) 391–413, arXiv:0903.2777 [hep-th].

[22] L. Martucci, J. Rosseel, D. Van den Bleeken, and A. Van Proeyen, "Dirac actions for D-branes on backgrounds with fluxes," *Class. Quant. Grav.* **22** (2005) 2745–2764, arXiv:hep-th/0504041.