Introduction to Robotics **Name:** ______________________

E106/206 Midterm 2 **SID:** ______________________

Please write your name at the top of each page

Show all working. Marks are awarded for method.

A cheat sheet is provided. No other notes are allowed.

Question	Marks
1	
2	
3	
4	
5	
6	
7	
total	

Question 1: Exponential Maps .... 10 points

(a) Write the matrix form of the transformation $e^{\hat{\omega}\theta}$ for $\omega = [0, 0, 1]^T$, and describe what motion it represents. (2)

> **Solution:**
>
> **1** Rotation about the z axis.
>
> **1**
> $$e^{\hat{\omega}\theta} = \begin{bmatrix} cos\,(\theta) & -sin\,(\theta) & 0 \\ sin\,(\theta) & cos\,(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Write the matrix form of the transformation $e^{\hat{\omega}\theta}$ for $\omega = [0, 0, 2]^T$, and describe what motion it represents. (2)

> **Solution:**
>
> **0.5** Rotation about the z axis.
>
> **0.5** Scaled rotation or rotation by 2θ
>
> **0.5** Using z rotation matrix
>
> **0.5** Using 2θ
>
> $$e^{\hat{\omega}\theta} = \begin{bmatrix} cos\,(2\theta) & -sin\,(2\theta) & 0 \\ sin\,(2\theta) & cos\,(2\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Write the matrix form of the transformation $e^{\hat{\xi}\theta}$ for $\xi = [0, 3, 0, 0, 0, 0]^T$, and describe what motion it represents. (2)

> **Solution:**
>
> **1** Translation along the y axis (by 3θ).
>
> **0.5** Correct rotation
>
> **0.5** Correct translation
> $$e^{\hat{\xi}\theta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3\theta \\ 0 & 0 & 1 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

(d) Write the matrix form of the transformation $e^{\hat{\xi}\theta}$ for $\xi = [0,0,0,0,0,1]^T$, and describe what motion it represents.. (2)

Solution:

1 Rotation about the z axis.

0.5 Correct rotation

0.5 Correct translation

$$e^{\hat{\xi}\theta} = \begin{bmatrix} cos\,(\theta) & -sin\,(\theta) & 0 & 0 \\ sin\,(\theta) & cos\,(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

(e) Write the matrix form of the transformation $e^{\hat{\xi}\theta}$ for $\xi = [1,0,0,1,0,0]^T$, and describe what motion it represents. (2)

Solution:

1 Rotation and translation about the x axis (screw motion).

0.5 Correct rotation

0.5 Correct translation

$$e^{\hat{\xi}\theta} = \begin{bmatrix} 1 & 0 & 0 & \theta \\ 0 & cos\,(\theta) & -sin\,(\theta) & 0 \\ 0 & sin\,(\theta) & cos\,(\theta) & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

Question 2: Forward Kinematics . . . . 10 points

For this question, refer to the manipulator seen in Appendix 1.

(a) Using matrix exponents, write an expression for the forward kinematic map $\boldsymbol{g}_{WT}(\boldsymbol{\theta})$, leaving your answer in terms of $e^{\hat{\xi}_i \theta_i}$. (2)

> **Solution:** $\boldsymbol{g}_{WT}(\boldsymbol{\theta}) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} \boldsymbol{g}_{WT}(\mathbf{0})$
>
> **1** For ALL exponents in correct order.
>
> **1** For initial configuration in correct location.

(b) Write the initial configuration $\boldsymbol{g}_{WT}(\mathbf{0})$ of the manipulator. (2)

> **Solution:** $\begin{bmatrix} 1 & 0 & 0 & l_1 + l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
>
> **1** For the Rotational component
>
> **1** For the Translational component
>
> **-1** For incorrect form of matrix

(c) Write out the twists $\boldsymbol{\xi}_i$ for each of the three joints. (6)

Solution:

1

$$\boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

1

$$\boldsymbol{q}_2 = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \quad \boldsymbol{\omega}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

1

$$\boldsymbol{q}_3 = \begin{bmatrix} l_1 \\ 0 \\ l_0 \end{bmatrix} \quad \boldsymbol{\omega}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \boldsymbol{\xi}_2 = \begin{bmatrix} -l_0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{\xi}_3 = \begin{bmatrix} -l_0 \\ 0 \\ l_1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Question 3: Inverse Kinematics . . . . 10 points

The maniplator in Appendix 1 is to be moved to some valid desired configuration $\boldsymbol{g}_{d,WT}$.

(a) The matrix $\boldsymbol{g}_1$ can be expressed as: (2)

$$\boldsymbol{g}_1 = e^{\hat{\boldsymbol{\xi}}_1\theta_1} e^{\hat{\boldsymbol{\xi}}_2\theta_2} e^{\hat{\boldsymbol{\xi}}_3\theta_3}$$

Write an expression for $\boldsymbol{g}_1$ given our desired configuration $\boldsymbol{g}_{d,WT}$ and other known configurations.

Solution:

2

$$\boldsymbol{g}_{d,WT} = e^{\boldsymbol{\xi}_1\theta_1} e^{\boldsymbol{\xi}_2\theta_2} e^{\boldsymbol{\xi}_3\theta_3} \boldsymbol{g}_{WT}(\boldsymbol{0})$$
$$\boldsymbol{g}_1 = \boldsymbol{g}_{d,WT} \boldsymbol{g}_{WT}(\boldsymbol{0})^{-1}$$

(b) Write down the invariant points of the system and which joints they are invariant to. (2)

Solution:

1 Shoulder joint - intersection of ξ_1 and ξ_2 or $\begin{bmatrix} 0 & 0 & l_0 \end{bmatrix}^T$

1 invariant to joint 1 and joint 2

(c) Formulate the inverse kinematics problem to find θ_3 as in terms of Paden Kahan subproblems. Define all necessary terms to compute θ_3. (3)

> **Solution:** Round up:
>
> **0.5** PK 3
>
> **0.5** $\delta = \|\boldsymbol{g_1 p}_{end} - \boldsymbol{p}_{shoulder}\|$
>
> **0.5** $\boldsymbol{p} = \boldsymbol{p}_{end} = \begin{bmatrix} l_1 + l_2 & 0 & l_0 \end{bmatrix}^T$
>
> **0.5** $\boldsymbol{q} = \boldsymbol{p}_{shoulder} = \begin{bmatrix} 0 & 0 & l_0 \end{bmatrix}^T$
>
> **0.5** $\boldsymbol{\xi} = \boldsymbol{\xi}_3$
>
> **0.5** $\boldsymbol{r} = \boldsymbol{p}_{elbow} = \begin{bmatrix} l_1 & 0 & l_0 \end{bmatrix}^T$

(d) Given this value for θ_3, formulate the inverse kinematics problem to find θ_1 and θ_2 in terms of Paden Kahan subproblems. Define all necessary terms to compute θ_1 and θ_2. (3)

> **Solution:** Round up:
>
> **0.5** PK 2
>
> **0.5** $\boldsymbol{p} = \boldsymbol{p}_{end} = \begin{bmatrix} l_1 + l_2 & 0 & l_0 \end{bmatrix}^T$
>
> **0.5** $\boldsymbol{q} = \boldsymbol{g}_1 \boldsymbol{p}_{end}$
>
> **0.5** $\boldsymbol{\xi} = \boldsymbol{\xi}_1, \boldsymbol{\xi}_2$
>
> **0.5** $\boldsymbol{r} = \boldsymbol{p}_{shoulder} = \begin{bmatrix} 0 & 0 & l_0 \end{bmatrix}^T$

Question 4: Jacobians .... 10 points

This questions looks at the Jacobians of the manipulator shown in Appendix 1.

(a) Write the Spatial Jacobian for the manipulator in its initial configuration (Figures 2 and 3). (3)

Solution:

0.5 For each linear/rotational column component:

$$\boldsymbol{J}^s = \begin{bmatrix} 0 & -l_0 & -l_0 \\ 0 & 0 & 0 \\ 0 & 0 & l_1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) Write the Body Jacobian for the manipulator in its initial configuration (Figures 2 and 3). (3)

Solution:

0.5 For each linear/rotational column component:

$$\boldsymbol{J}^b = \begin{bmatrix} 0 & 0 & 0 \\ l_1 + l_2 & 0 & 0 \\ 0 & -l_1 - l_2 & -l_2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) Is this a singular configuration? If so, give an expression for the linear dependency between the columns. If not, show that the columns are linearly independent (potentially through an explicit example). (1)

Solution:

1 Not a singular configuration.

- Eg. column 1 uniquely determines v_y, columns 2, and 3 uniquely determine v_z, ω_y

(d) The spatial and body Jacobians can be written in terms of ξ' and $\xi^{\dagger}$. What is the relationship between ξ, ξ', and $\xi^{\dagger}$? What do the following matrices represent? (3)

$$\boldsymbol{M}_1 = \begin{bmatrix}\xi_1 & \xi_2 & \xi_3\end{bmatrix} \qquad \boldsymbol{M}_2 = \begin{bmatrix}\xi_1' & \xi_2' & \xi_3'\end{bmatrix} \qquad \boldsymbol{M}_3 = \begin{bmatrix}\xi_1^{\dagger} & \xi_2^{\dagger} & \xi_3^{\dagger}\end{bmatrix}$$

Solution:

1 $\boldsymbol{M}_1$ is the spatial Jacobian in the initial configuration. Its elements are the standard twists.

1 $\boldsymbol{M}_2$ is the spatial Jacobian in any given configuration. Its elements are the re-mappings of standard twists after a reconfiguration as seen in the spatial frame.

1 $\boldsymbol{M}_3$ is the body Jacobian in the initial configuration. Its elements are the re-mappings of standard twists after a reconfiguration as seen in the body frame.

Question 5: Velocities . . . . 10 points

The body and spatial Jacobians for the manipulator in configuration $\boldsymbol{g}_{WT}\left(0, 0, -\frac{\pi}{2}\right)$ (Appendix 1, Figure 4) can be written:

$$J^s = \begin{bmatrix} 0 & -l_0 & -l_0 \\ 0 & 0 & 0 \\ 0 & 0 & l_1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad J^b = \begin{bmatrix} 0 & -l_1 & 0 \\ l_1 & 0 & 0 \\ 0 & -l_2 & -l_2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) What instantaneous linear body velocities are possible in this configuration. (1)

Solution:

1 v_x, v_y, and v_z.

(b) Is this a singular configuration? Show how the joint velocities $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ relate to the instantaneous linear body velocities. (1)

Solution:

1 Not a singular configuration. $\dot{\theta}_1$ controls the v_y^b velocities. $\dot{\theta}_2$ controls the v_x^b and v_z^b velocities. $\dot{\theta}_3$ controls the v_z^b velocity.

(c) Show that the matrix that relates the velocity of a point as seen in the tool frame ($\boldsymbol{v}_{\boldsymbol{q}_T}$) to its coordinates as seen in the tool frame ($\boldsymbol{q}_T$) can be written: (3)

$$\boldsymbol{M} = \begin{bmatrix} 0 & -\dot{\theta}_1 & \dot{\theta}_2 + \dot{\theta}_3 & -l_1\dot{\theta}_2 \\ \dot{\theta}_1 & 0 & 0 & l_1\dot{\theta}_1 \\ -\dot{\theta}_2 - \dot{\theta}_3 & 0 & 0 & -l_2\left(\dot{\theta}_2 + \dot{\theta}_3\right) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

1

$$\boldsymbol{v}^b_{\boldsymbol{q_T}} = \widehat{\boldsymbol{V}}^b_{WT}\boldsymbol{q_T}$$

2

$$\widehat{\boldsymbol{V}}^b_{WT} = \widehat{\begin{bmatrix} -l_1\dot{\theta}_2 \\ l_1\dot{\theta}_1 \\ -l_2\left(\dot{\theta}_2 + \dot{\theta}_3\right) \\ 0 \\ \dot{\theta}_2 + \dot{\theta}_3 \\ \dot{\theta}_1 \end{bmatrix}} = \boldsymbol{M}$$

(d) What is the instantaneous body velocity of the origin of the tool frame? (2)

Solution:

1

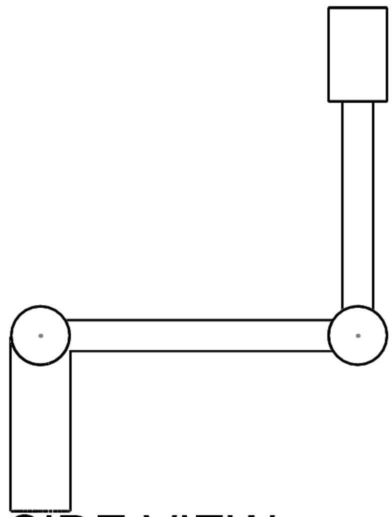
$$\boldsymbol{v}^b_{\boldsymbol{q_T}} = \widehat{\boldsymbol{V}}^b_{WT}\boldsymbol{q_T}$$

$$\boldsymbol{v}^b_{\boldsymbol{q_T}} = \begin{bmatrix} 0 & -\dot{\theta}_1 & \dot{\theta}_2 + \dot{\theta}_3 & -l_1\dot{\theta}_2 \\ \dot{\theta}_1 & 0 & 0 & l_1\dot{\theta}_1 \\ -\dot{\theta}_2 - \dot{\theta}_3 & 0 & 0 & -l_2\left(\dot{\theta}_2 + \dot{\theta}_3\right) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

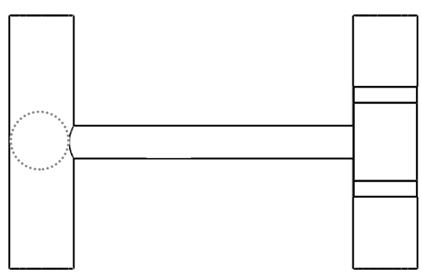
1

$$\boldsymbol{v}^b_{\boldsymbol{q_T}} = \begin{bmatrix} -l_1\dot{\theta}_2 \\ l_1\dot{\theta}_1 \\ -l_2\left(\dot{\theta}_2 + \dot{\theta}_3\right) \\ 0 \end{bmatrix}$$

(e) The figures below show the side and top views of the manipulator in configuration $\boldsymbol{g}_{WT}\left(0, 0, -\frac{\pi}{2}\right)$ (Appendix 1, Figure 4). Sketch this instantaneous body velocities from each joint on the figures below. (3)



SIDE VIEW



TOP VIEW

There are no questions on this page.

Question 6: Wrenches .... 10 points

The body and spatial Jacobians for the manipulator in configuration $\boldsymbol{g}_{WT}\left(0, -\frac{\pi}{2}, 0\right)$ (Appendix 1, Figure 5) can be written:

$$J^s = \begin{bmatrix} 0 & -l_0 & -l_0 - l_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad J^b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -l_1 - l_2 & -l_2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) What spatial forces and torques can be applied by the manipulator in this configuration? (1)

Solution:

1 F_x forces, and τ_y, and τ_z torques.

(b) How many spatial forces and torques can be controlled independently in this configuration? (1)

Solution:

1 The Jacobian is non-singular so 3.

(c) A mass m is held by the manipulator at the point with tool frame coordinates $[0, 0, 0]$ What is the body wrench $\mathbf{\Gamma}^b$ associated with this load. Assume that the acceleration due to gravity acts in the negative Z_W direction. (2)

Solution:

$$\mathbf{\Gamma}^b = \begin{bmatrix} -mg \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1 For the linear components

1 For the rotational components

(d) Using the Jacobians provided, what joint torques are associated with this body wrench? (2)

Solution:

1 $\boldsymbol{\tau} = \left(J^b_{ST}\right)^T \boldsymbol{\Gamma}^b$

1

$$\boldsymbol{\tau} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -l_1 - l_2 & 0 & 1 & 0 \\ 0 & 0 & -l_2 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -mg \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(e) Based on these joint torques, explain how the mass is supported by the manipulator. (2)

Solution:

2 The mass is supported by the structure of the robot or the mass is not supported by any actuator torques.

(f) Without performing any computation, what is the spatial wrench Γ^s associated with this body wrench Γ^b? Explain your reasoning. (2)

Solution:

1

$$\boldsymbol{\Gamma}^s = \begin{bmatrix} 0 \\ 0 \\ -mg \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1 Redrawing the same force on the manipulator in the spatial frame.
OR

1 For writing adjoint relationship.

Question 7: Dynamics .... 10 points

Consider the mass spring system with two masses connected by three springs shown in Figure 1.

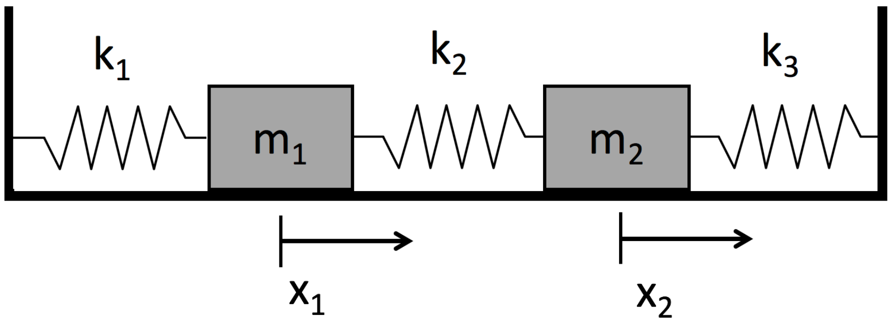


Figure 1: Two masses attached by three springs. The positions of the masses, x_1 and x_2, are defined such that equilibrium is at $x_1 = x_2 = 0$.

(a) Write the kinetic energy of this system. (2)

Solution:

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

1 For any $\frac{1}{2}mv^2$ expression

1 For correct terms.

(b) Write an expression for the potential energy in the system. (2)

Solution:

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3x_2^2$$

1 For any $\frac{1}{2}kx^2$ expression

1 For correct terms.

(c) Write the dynamics for this system using the Lagrange method. (6)

Solution:

1

$$L = T - V$$

1

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2(x_1 - x_2)^2 - \frac{1}{2}k_3x_2^2$$

0.5

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1$$

0.5

$$\frac{\partial L}{\partial x_1} = -k_1x_1 - k_2\left(x_1 - x_2\right)$$

0.5

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2$$

0.5

$$\frac{\partial L}{\partial x_2} = k_2\left(x_1 - x_2\right) - k_3x_2$$

1

$$m_1\ddot{x}_1 = -k_1x_1 - k_2\left(x_1 - x_2\right)$$

1

$$m_2\ddot{x}_2 = k_2\left(x_1 - x_2\right) - k_3x_2$$

1 Appendix: RRR Manipulator

Robotic manipulator with three revolute joints.

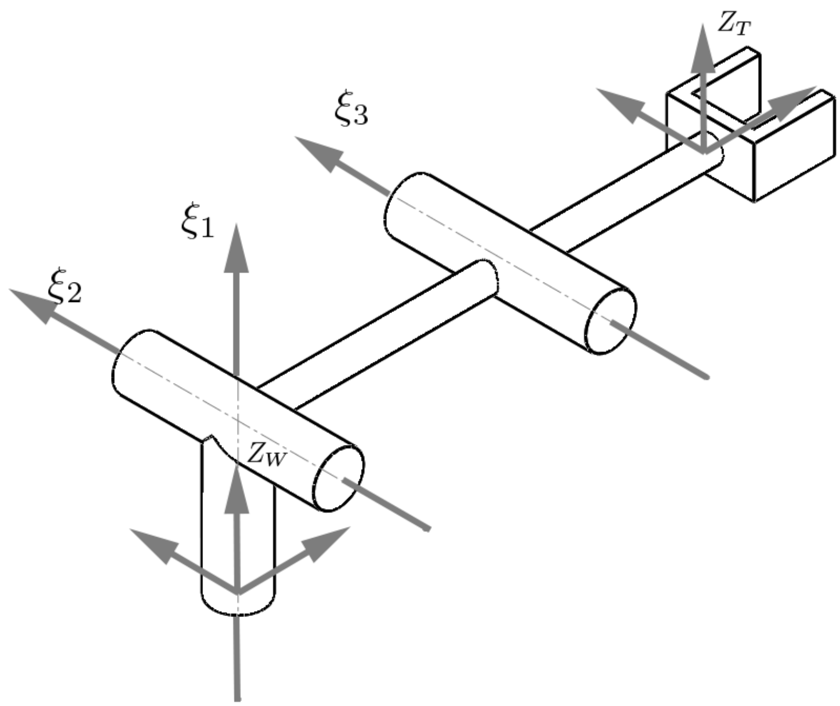


Figure 2: Schematic of a 3DoF RRR Manipulator in its initial configuration. Axes of rotation and the world and tool reference frames are shown.

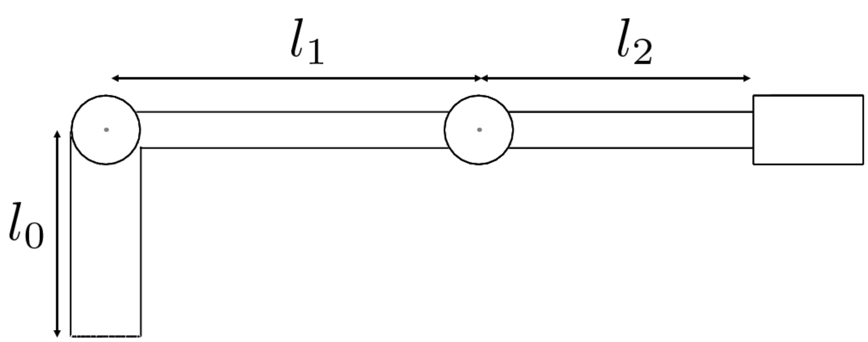


Figure 3: Side view of a 3DoF RRR Manipulator in its initial configuration. Segment lengths are shown.

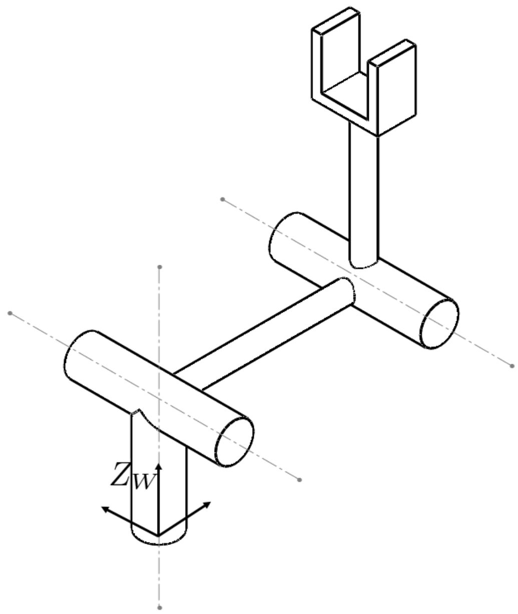


Figure 4: Manipulator in the configuration $\boldsymbol{\theta} = \left[0, 0, -\frac{\pi}{2}\right]$

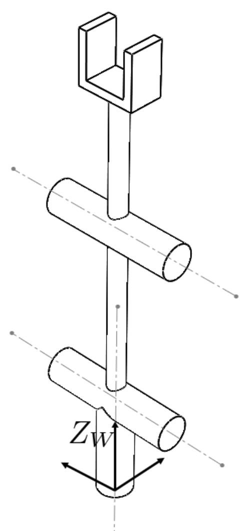


Figure 5: Manipulator in the configuration $\boldsymbol{\theta} = \left[0, -\frac{\pi}{2}, 0\right]$.

2 Appendix: Cheat Sheet

This is the cheat sheet that will be provided for every midterm.

2.1 Trigonometry

Pythagoras's theorem $h^2 = x^2 + y^2$ for a right angled triangle where h is the hypotenuse and x and y are the lengths of the two remaining sides.

Sine, Cosine Relation $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$

Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos(\theta_C)$ where a, b, c are the lengths of the triangle and θ_A, θ_B and θ_C are the angles of their opposing corner.

2.2 Linear Algebra

For orthogonal matrices $A^{-1} = A^T$

Orthogonality A matrix $\begin{bmatrix}\boldsymbol{v}_1, ..., \boldsymbol{v}_n\end{bmatrix}$ is said to be orthogonal if:

$$\boldsymbol{v}_i^T \boldsymbol{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

2.3 Special Operators

Hat

$$\hat{\boldsymbol{\omega}} = \widehat{\begin{bmatrix}\omega_1 \\ \omega_2 \\ \omega_3\end{bmatrix}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Wedge

$$\hat{\boldsymbol{\xi}} = \widehat{\begin{bmatrix}\boldsymbol{v} \\ \boldsymbol{\omega}\end{bmatrix}} = \begin{bmatrix}\hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \boldsymbol{0} & 0\end{bmatrix}$$

2.4 Rotations

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{x}}\theta}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{y}}\theta}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{\hat{\boldsymbol{z}}\theta}$$

2.5 Rodrigues' Formula

$$R(\boldsymbol{\omega}, \theta) = e^{\hat{\boldsymbol{\omega}}\theta} = \mathbb{I}_3 + \frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|} \sin(\theta) + \frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2} (1 - \cos(\theta))$$

2.6 Rigid Body Motion

$$\boldsymbol{g}_{AB} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix} \qquad \boldsymbol{g}_{AB}^{-1} = \begin{bmatrix} \boldsymbol{R}_{AB}^{-1} & -\boldsymbol{R}_{AB}^{-1}\boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

2.7 Exponential Notation

$$\boldsymbol{R}_{AB}(\theta_1) = e^{\hat{\omega}_1\theta_1}$$

$$\boldsymbol{g}_{AB}(\theta_1) = e^{\hat{\xi}_1\theta_1}\boldsymbol{g}_{AB}(0)$$

$$\boldsymbol{g}_{ST}(\theta_1, \ldots, \theta_n) = e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_n\theta_n}\boldsymbol{g}_{ST}(0)$$

2.7.1 Special Cases

Pure Rotation

$$\boldsymbol{\xi} = \begin{bmatrix} -\boldsymbol{\omega} \times \boldsymbol{q} \\ \boldsymbol{\omega} \end{bmatrix}$$

Pure Translation

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{0} \end{bmatrix}$$

Pure Rotations, Screws (Rotation and Translation)

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} e^{\hat{\boldsymbol{\omega}}\theta} & \left(\mathbb{I}_3 - e^{\hat{\boldsymbol{\omega}}\theta}\right)(\boldsymbol{\omega} \times \boldsymbol{v}) + \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}\theta \\ \boldsymbol{0} & 1 \end{bmatrix}$$

Pure Translation

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} \mathbb{I}_3 & \boldsymbol{v}\theta \\ \boldsymbol{0} & 1 \end{bmatrix}$$

2.8 Paden-Kahan

Subproblem 1: Rotation about a single axis

$$e^{\widehat{\xi}\theta}p = q$$

Subproblem 2: Rotation about two subsequent axes

$$e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}p = q$$

Subproblem 3: Rotation to a distance

$$\left\| e^{\widehat{\xi}\theta}p - q \right\| = \delta$$

2.9 Velocities

Spatial Velocities

$$\widehat{V}^s_{AB} = \dot{g}_{AB} g^{-1}_{AB} \qquad V^s_{AB} = \begin{bmatrix} -\dot{R}R^T p + \dot{p} \\ \left(\dot{R}R^T\right)^\vee \end{bmatrix} = \xi \dot{\theta}$$

Body Velocities

$$\widehat{V}^b_{AB} = g^{-1}_{AB} \dot{g}_{AB} \qquad V^b_{AB} = \begin{bmatrix} R^T \dot{p} \\ \left(R^T \dot{R}\right)^\vee \end{bmatrix} = \left(Ad_{g^{-1}_{AB}(0)} \xi\right) \dot{\theta}$$

Adjoint

$$Ad_g = \begin{bmatrix} R & \widehat{p}R \\ 0 & R \end{bmatrix} \qquad V^s = Ad_g V^b$$

$$V^s_{AC} = V^s_{AB} + Ad_{g_{AB}} V^s_{BC} \qquad V^b_{AC} = Ad_{g^{-1}_{BC}} V^b_{AB} + V^b_{BC}$$

2.10 Jacobians

Spatial Jacobian

$$V^s_{ST} = J^s_{ST} \dot{\theta}$$
$$J^s_{ST} = \begin{bmatrix} \xi_1 & \xi'_2 & \dots & \xi'_n \end{bmatrix}$$
$$\xi'_i = Ad_{\left(e^{\widehat{\xi}_1 \theta_1} \dots e^{\widehat{\xi}_{i-1} \theta_{i-1}}\right)} \xi_i$$

Body Jacobian

$$V^b_{ST} = J^b_{ST} \dot{\theta}$$
$$J^b_{ST} = \begin{bmatrix} \xi^\dagger_1 & \xi^\dagger_2 & \dots & \xi^\dagger_n \end{bmatrix}$$
$$\xi^\dagger_i = Ad^{-1}_{\left(e^{\widehat{\xi}_i \theta_i} \dots e^{\widehat{\xi}_n \theta_n} g_{ST}(0)\right)} \xi_i$$

2.11 Wrenches

$$\mathbf{\Gamma} = \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{\tau} \end{bmatrix}$$

$$\mathbf{\Gamma}^b = \left(Ad_{g_{ST}}\right)^T \mathbf{\Gamma}^s$$

Spatial Wrench

$$\boldsymbol{\tau} = \left(J^s_{ST}\right)^T \mathbf{\Gamma}^s$$

Body Wrench

$$\boldsymbol{\tau} = \left(J^b_{ST}\right)^T \mathbf{\Gamma}^b$$

2.12 Euler Lagrange

$$\mathcal{L}(q, \dot{q}) = T(q, \dot{q}) - V(q)$$
$$\Gamma_i = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i}$$

2.13 Forces

Gravitational Force

$$W = -mg$$

where m is the mass, g is the acceleration due to gravity.

Elastic Force

$$F_k = -k\delta$$

where k is the spring constant and δ is the extension of the spring.

2.14 Energies

Kinetic

$$KE = \frac{1}{2}mv^2$$

where m is the mass and v is the velocity of the object.

Gravitational Potential

$$GPE = mgh$$

where m is the mass, g is the acceleration due to gravity and h is the distance along the gravitational axis.

Elastic Potential

$$EPE = \frac{1}{2}k\delta^2$$

where k is the spring constant and δ is the extension of the spring.

2.15 Moments of Inertia

Mass at a Radius

$$I = mr^2$$

where m is the mass, and r is the radius of the mass.