Question 1: Rotation Matrices .... 9 points

Consider two coordinate frames A and B. Coordinate frame B begins aligned with frame A. Frame B is then rotated by $\frac{\pi}{2}$ radians about the Y axis of the A coordinate frame.

(a) Write the rotation matrix $\boldsymbol{R}_{AB}$ for these two coordinate frames. (2)

$$R_{AB} = R_Y\left(\frac{\pi}{2}\right) = \begin{bmatrix} \cos \pi/2 & 0 & \sin \pi/2 \\ 0 & 1 & 0 \\ -\sin \pi/2 & 0 & \cos \pi/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

(b) Write the rotation matrix $\boldsymbol{R}_{BA}$ for these two coordinate frames. (2)

$$R_{BA} = R_{AB}^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

OR

$$R_{BA} = R_Y\left(-\frac{\pi}{2}\right) = \begin{bmatrix} \cos -\pi/2 & 0 & \sin -\pi/2 \\ 0 & 1 & 0 \\ -\sin -\pi/2 & 0 & \cos -\pi/2 \end{bmatrix}$$

(c) How would a vector $\boldsymbol{p}_A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ written in the A coordinate frame be written in the B coordinate frame? (2)

$$P_B = R_{BA} P_A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

(d) Consider two coordinate frames A and B. Coordinate frame B begins aligned with frame A. Frame B undergoes the following sequence of rotations: (3)

- Rotation by ϕ about the Z axis
- Rotation by θ about the current X axis
- Rotation by γ about the current Z axis

Write the resulting rotation matrix R_{AB} (You do not need to compute the product).

$$R_{AB} = R_Z(\phi)\, R_X(\theta)\, R_Z(\gamma)$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 2: Axis Angle Rotations .... 6 points

Write the rotation matrix R for a rotation of π radians about the axis $\omega = [1, -1, 1]^T$

(1) $\frac{\omega}{\|\omega\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(1) $\hat{\omega} = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

(1) $\hat{\omega}^2 = \frac{1}{3}\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$

(1)

$R = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$

$\sin(\pi) = 0 \qquad \cos(\pi) = -1$

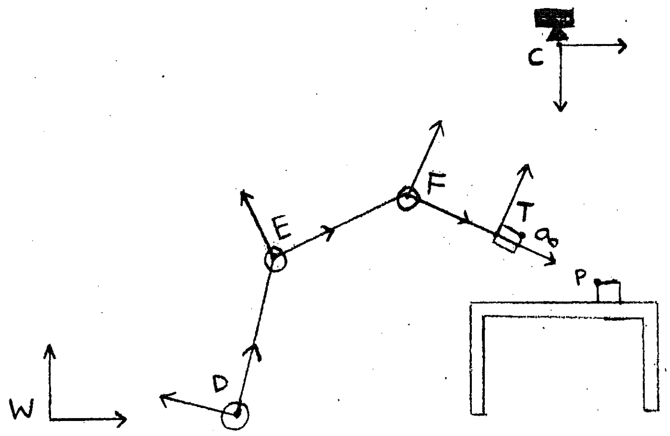
$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -2/3 & -1/3 & 1/3 \\ -1/3 & -2/3 & -1/3 \\ 1/3 & -1/3 & -2/3 \end{bmatrix} (2)$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -4/3 & -2/3 & 2/3 \\ -2/3 & -4/3 & -2/3 \\ 2/3 & -2/3 & -4/3 \end{bmatrix}$

(2) $= \begin{bmatrix} -1/3 & -2/3 & 2/3 \\ -2/3 & -1/3 & -2/3 \\ 2/3 & -2/3 & -1/3 \end{bmatrix}$

Question 3: Homogeneous Transformations .... 11 points

Consider the diagram below to answer the following questions. Pictured is a robot adjacent to a table with a camera mounted overhead.



(a) You are given the homogeneous transforms g_{WT} and g_{WC}. Write g_{CT} in terms of $R_{WT}, p_{WT}, R_{WC}, p_{WC}$. In words, describe what g_{CT} does. (4)

$$g_{CT} = g_{WC}^{-1} g_{WT}$$

$$= \begin{bmatrix} R_{WC}^{-1} & -R_{WC}^{-1} p_{WC} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{WT} & p_{WT} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_{WC}^{-1} R_{WT} & R_{WC}^{-1} p_{WT} - R_{WC}^{-1} p_{WC} \\ 0 & 1 \end{bmatrix}$$

Tool coordinate frame relative to camera frame

(b) Given point q_T, defined in the T frame and point p_C, defined in the C frame, write an expression for the distance between q and p. (2)

$\| g_{CT} q_T - p_C \|$ or $\| g_{WT} q_T - g_{WC} p_C \|$

(c) Write the rigid body transform g_{WT} in terms of relative rigid body transformations (use frames defined in the figure). (2)

$g_{WT} = g_{WD}\, g_{DE}\, g_{EF}\, g_{FT}$

(d) For a general case, show that a homogeneous transformation preserves the distance between two points. (3)

Consider two points q_1, q_2

$$\| g q_1 - g q_2 \| = \left\| \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ 1 \end{bmatrix} - \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_2 \\ 1 \end{bmatrix} \right\|$$

$$= \| R q_1 - p - R q_2 + p \|$$

$$= \| R q_1 - R q_2 \|$$

$$= \| R(q_1 - q_2) \| = \| q_1 - q_2 \|$$

Rotations preserve distance:

$$\| R q \|^2 = (Rq)^T (Rq) = q^T R^T R q = q^T q = \| q \|^2$$

Question 4: Forward Kinematics .... 20 points

For this problem, consider the robotic manipulator shown in Appendix 1.

(a) Define forward kinematics and inverse kinematics. (2)

Forward kinematics: Given joint angles, find robot configuration

Inverse: Given end-effector configuration, find joint angles

(b) Compute the twists $\boldsymbol{\xi}$ for joints 1, 3, and 5. (8)

(1) $\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (1) $\quad q_1 = \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix}$ (1) $\quad -\omega \times q = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (1)

(3) $\omega_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (1) $\quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (1)

$\xi_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(5) $\omega_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (1) $\quad q_5 = \begin{bmatrix} 0 \\ l_2 \\ l_1 - l_3 \end{bmatrix}$ (1) $\quad -\omega \times q = -\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ l_2 \\ l_1 - l_3 \end{bmatrix}$

$= -\begin{bmatrix} l_1 - l_3 \\ 0 \\ 0 \end{bmatrix}$

$\xi_5 = \begin{bmatrix} l_3 - l_1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (1)

(c) Write an expression for rigid body transform $g_{WB}(\theta_1, \theta_2)$ in homogeneous form (by inspection). (3)

$$g_{WB} = \begin{bmatrix} R_Z(\theta_1) & \begin{bmatrix} 0 \\ 0 \\ l_1 + \theta_2 \end{bmatrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

(d) Write an expression for the initial configuration $g_{WT}(\mathbf{0})$ of the manipulator in homogeneous form (by inspection). (3)

$$g_{WT}(0) = \begin{bmatrix} R_Y\left(\frac{\pi}{2}\right) & \begin{bmatrix} 0 \\ l_2 \\ l_1 - l_3 \end{bmatrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

(e) Using matrix exponential terms such as $e^{\hat{\xi}_i \theta_i}$, write an expression for the forward kinematics map with the form $g_{WB}(\boldsymbol{\theta})$. (2)

$$g_{WB}(\theta_1, \theta_2) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} g_{WB}(0)$$

(f) Using matrix exponential terms such as $e^{\hat{\xi}_i \theta_i}$, write an expression for the forward kinematics map with the form $g_{WT}(\boldsymbol{\theta})$. (2)

$$g_{WT}(\theta_1, \theta_2, \theta_3, \theta_4) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} e^{\hat{\xi}_4 \theta_4} e^{\hat{\xi}_5 \theta_5} g_{WT}(0)$$

Question 5: Inverse Kinematics . . . . 25 points

Consider the robotic manipulator shown in Appendix 1. You are given a desired configuration of the tool (T) frame:

$$\boldsymbol{g}_{d,WT} = \begin{bmatrix} & R_d & & d_x \\ & & & d_y \\ & & & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) The matrix $\boldsymbol{g}_1$ can be written as: (2)

$$\boldsymbol{g}_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5}$$

Write $\boldsymbol{g}_1$ in terms of known configurations

$g_{d,WT} = g_1 \, g_{WT}(0)$

$\rightarrow \quad g_1 = g_{d,WT} \, g_{WT}^{-1}(0)$

(b) In words, describe how each joint affects the configuration of the manipulator. (5)

θ_1 : polar position of T frame in x,y plane

θ_2 : vertical position of T frame (z coord)

θ_3 : radial position (distance) of T frame in the x,y plane

θ_4, θ_5 : orientation of T frame

(c) Let p_1 be a point invariant to joint 1, and p_2 be a point invariant to joints 4 and 5. Give potential coordinates for p_1 and p_2. (2)

$$P_1 = \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 \\ \ell_2 \\ \ell_1 - \ell_3 \end{bmatrix}$$

(d) Using the initial and desired configurations, find θ_2. Hint: Leave in terms of d_x, d_y, and/or d_z. You do not need to use a Paden-Kahan sub-problem. (3)

$$d_z = (\ell_1 - \ell_3) + \theta_2$$

$$\Rightarrow \theta_2 = d_z - (\ell_1 - \ell_3)$$

$$= d_z - \ell_1 + \ell_3$$

(e) Using the initial and desired configurations, find θ_3. Hint: Leave in terms of d_x, d_y, and/or d_z. You do not need to use a Paden-Kahan sub-problem. (3)

$$\sqrt{d_x^2 + d_y^2} = \ell_2 + \theta_3$$

$$\Rightarrow \theta_3 = \sqrt{d_x^2 + d_y^2} - \ell_2$$

(f) Given values of θ_2 and θ_3, formulate the inverse kinematics for θ_1 as a Paden-Kahan sub-problem. List the sub-problem and define necessary parameters (ie. p, q, r, δ). (4)

PK1:

$$\underbrace{g_1 P_2}_{q} = e^{\hat{\xi}_1\theta_1} \underbrace{e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} P_2}_{p} \qquad r = \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix}$$

where $P_2 = \begin{bmatrix} 0 \\ \ell_2 \\ \ell_1 - \ell_3 \end{bmatrix}$ so that $P_2 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} P_2$

(g) The matrix g_2 can be written as: (2)

$$g_2 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5}$$

Given solutions for θ_1, θ_2, and θ_3 and g_1, write an expression for g_2 in terms of known matrices:

$$g_2 = e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_1$$

(h) Given values of θ_1, θ_2 and θ_3, formulate the inverse kinematics for θ_4 and θ_5 as a Paden-Kahan sub-problem. List the sub-problem and define necessary parameters (ie. p, q, r, δ). (4)

PK 2 :

Let p_f be a point not on ξ_4, ξ_5

$$g_2 \, p_f = e^{\hat{\xi}_4 \theta_4} e^{\hat{\xi}_5 \theta_5} p_f \quad , \quad r = \begin{bmatrix} 0 \\ l_2 \\ l_1 - l_3 \end{bmatrix}$$

ex:

$$p_f = \begin{bmatrix} 1 \\ l_2 \\ l_1 - l_3 \end{bmatrix}$$

1 Appendix: 5 DoF Manipulator

Consider the robotic manipulator shown in Figure 1. The manipulator is shown in its initial configuration.

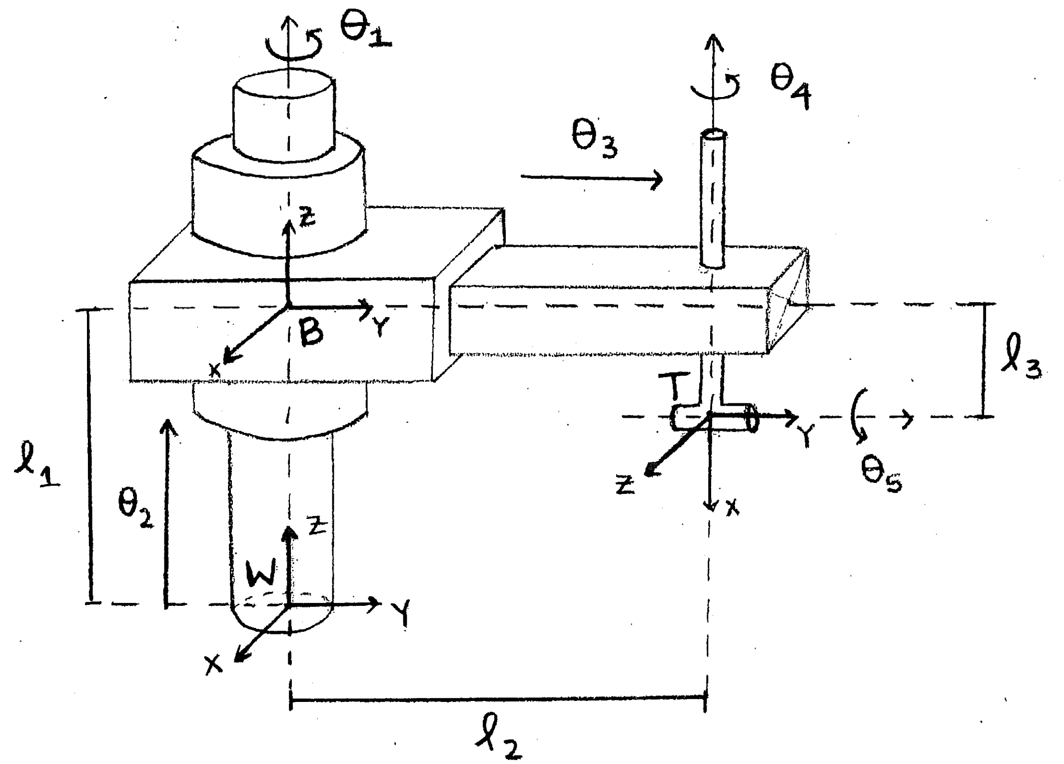


Figure 1: Joints 1, 4, and 5 are revolute joints. Joint 2 and 3 are prismatic joints. The world and tool frame labelled as W and T respectively. Frame B rotates with joint 1 and translates with joint 2.