Introduction to Robotics **Name:** ______________________

E106/206 Midterm 2 **SID:** ______________________

Please write your name at the top of each page

Show all working. Marks are awarded for method.

A cheat sheet is provided. No other notes are allowed.

Question	Marks
1	
2	
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7	

Introduction to Robotics **Name:** ______________________

Question 1: Rigid Body Motion .... 5 points

(a) Given rigid body transformations g_{EA}, g_{AR}, and g_{ES}, find g_{RS}. (2)

(b) Suppose that the frame S origin as seen from frame E is $[1, 1, 1]$ and that (3)

$$G_{ER} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Write down the coordinates of the frame S origin as seen from the frame R.

Question 2: Rotating Rigid Body . . . . 10 points

Referring to the figure, a rigid body, shown at the top right, rotated about the point (L,L) with angular velocity $\dot{\theta} = 1$.

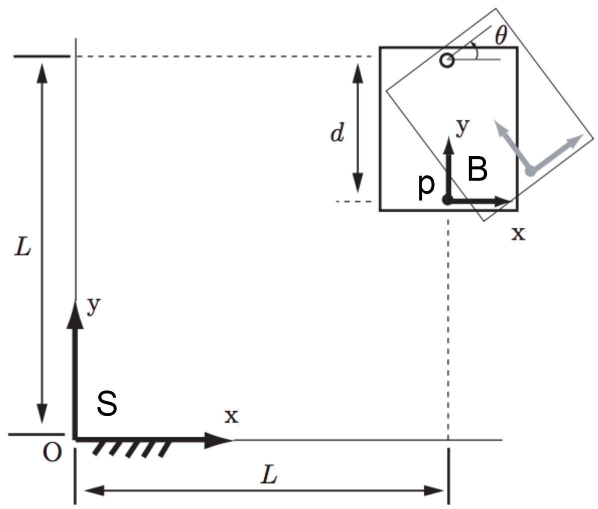


Figure 1: A rigid body rotating in the plane.

(a) Find the position of the point P on the moving body relative to the fixed reference frame $\{s\}$ in terms of θ. (2)

(b) Find the velocity of the point P in terms of the fixed reference frame $\{s\}$. (2)

(c) Find the velocity of the point P in terms of the rotating reference frame $\{s\}$. (2)

(d) Write the configuration g_{SB} in terms of θ. (2)

(e) Find the twist in body coorindates$\{s\}$. (2)

(f) Find the twist in spatial coordinates$\{s\}$. (2)

(g) What is the relationship between the twists from (d) and (e)? (2)

(h) What is the relationship between the twists from (d) and the velocity of the point from (b)? (2)

Question 3: Forward Kinematics . . . . 10 points

For this question, refer to the manipulator seen in Appendix **??**.

(a) Using matrix exponents, write an expression for the forward kinematic map $\boldsymbol{g}_{WT}(\boldsymbol{\theta})$, leaving your answer in terms of $e^{\hat{\xi}_i \theta_i}$. (2)

(b) Write the initial configuration $\boldsymbol{g}_{WT}(\mathbf{0})$ of the manipulator. (2)

(c) Write out the twists $\boldsymbol{\xi}_i$ for each of the three joints. (6)

Question 4: Inverse Kinematics . . . . 10 points

The manipulator in Appendix **??** is to be moved to some valid desired configuration $\boldsymbol{g}_{d,WT}$.

(a) The matrix $\boldsymbol{g}_1$ can be expressed as: (2)

$$\boldsymbol{g}_1 = e^{\hat{\boldsymbol{\xi}}_1 \theta_1} e^{\hat{\boldsymbol{\xi}}_2 \theta_2} e^{\hat{\boldsymbol{\xi}}_3 \theta_3}$$

Write an expression for $\boldsymbol{g}_1$ given our desired configuration $\boldsymbol{g}_{d,WT}$ and other known configurations.

(b) Write down the invariant points of the system and which joints they are invariant to. (2)

(c) Formulate the inverse kinematics problem to find θ_3 as in terms of Paden Kahan subproblems. Define all necessary terms to compute θ_3. (3)

(d) Given this value for θ_3, formulate the inverse kinematics problem to find θ_1 and θ_2 in terms of Paden Kahan subproblems. Define all necessary terms to compute θ_1 and θ_2. (3)

Question 5: Jacobians .... 10 points

This questions looks at the Jacobians of the manipulator shown in Appendix **??**.

(a) Write the Spatial Jacobian for the manipulator in its initial configuration (Figures **??** and **??**). (3)

(b) Write the Body Jacobian for the manipulator in its initial configuration (Figures **??** and **??**). (3)

(c) Is this a singular configuration? If so, give an expression for the linear dependency between the columns. If not, show that the columns are linearly independent (potentially through an explicit example). (1)

(d) The spatial and body Jacobians can be written in terms of ξ' and $\xi^\dagger$. What is the relationship between ξ, ξ', and $\xi^\dagger$? What do the following matrices represent? (3)

$$\boldsymbol{M}_1 = \begin{bmatrix}\xi_1 & \xi_2 & \xi_3\end{bmatrix} \qquad \boldsymbol{M}_2 = \begin{bmatrix}\xi_1' & \xi_2' & \xi_3'\end{bmatrix} \qquad \boldsymbol{M}_3 = \begin{bmatrix}\xi_1^\dagger & \xi_2^\dagger & \xi_3^\dagger\end{bmatrix}$$

Question 6: Velocities .... 10 points

The body and spatial Jacobians for the manipulator in configuration $\boldsymbol{g}_{WT}\left(0, 0, -\frac{\pi}{2}\right)$ (Appendix **??**, Figure **??**) can be written:

$$J^s = \begin{bmatrix} 0 & l_0 & l_0 \\ 0 & 0 & 0 \\ 0 & 0 & l_1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad J^b = \begin{bmatrix} 0 & l_1 & 0 \\ l_1 & 0 & 0 \\ 0 & l_2 & l_1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) What instantaneous linear body velocities are possible in this configuration. (1)

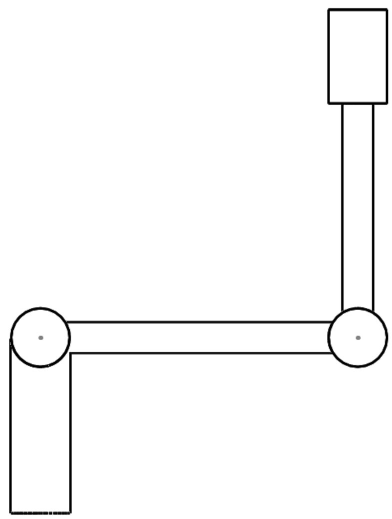
(b) Is this a singular configuration? Show how the joint velocities $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ relate to the instantaneous linear body velocities. (1)

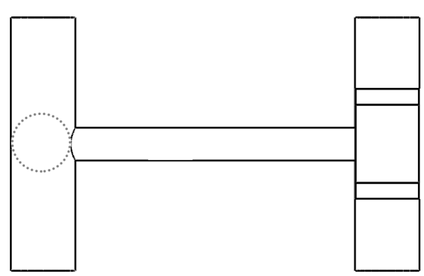
(c) Show that the matrix that relates the velocity of a point as seen in the tool frame ($\boldsymbol{v}_{\boldsymbol{q}_T}$) to its coordinates as seen in the tool frame ($\boldsymbol{q}_T$) can be written: (2)

$$\boldsymbol{M} = \begin{bmatrix} 0 & -\dot{\theta}_1 & \dot{\theta}_3 & -l_1\dot{\theta}_1 \\ \dot{\theta}_1 & 0 & -\dot{\theta}_2 & l_2\dot{\theta}_2 + l_3\dot{\theta}_1 \\ -\dot{\theta}_3 & \dot{\theta}_2 & 0 & -l_3\dot{\theta}_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(d) What is the instantaneous body velocity of the origin of the tool frame? (2)

(e) The figures below show the side and top views of the manipulator in configuration $\boldsymbol{g}_{WT}\left(0, 0, -\frac{\pi}{2}\right)$ (Appendix **??**, Figure **??**). Sketch this instantaneous body velocities from each joint on the figures below. (2)





(f) Without any further calculation, what is the spatial velocity of the same point. Explain your answer. (2)

There are no questions on this page.

Question 7: Wrenches .... 10 points

The body and spatial Jacobians for the manipulator in configuration $\boldsymbol{g}_{WT}\left(-\frac{\pi}{2}, 0, 0\right)$ (Appendix **??**, Figure **??**) can be written:

$$J^s = \begin{bmatrix} 0 & l_0 & l_0 \\ 0 & 0 & 0 \\ l_0 & 0 & -l_1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad J^b = \begin{bmatrix} 0 & l_0 & l_0 + l_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) What spatial forces and torques can be applied by the manipulator in this configuration? (1)

(b) How many spatial forces and torques can be controlled independently in this configuration? (1)

(c) A mass m is held by the manipulator at the point with tool frame coordinates $[0, 0, 0]$ What is the body wrench $\boldsymbol{\Gamma}^b$ associated with this load. Assume that the acceleration due to gravity acts in the negative Z_W direction. (2)

(d) Using the Jacobians provided, what joint torques are associated with this body wrench? (2)

(e) Based on these joint torques, explain how the mass is supported by the manipulator. (2)

(f) Without performing any computation, what is the spatial wrench Γ^s associated with this body wrench Γ^b? Explain your reasoning. sdads z (2)

Question 8: Dynamics . . . . 10 points

A torsion spring is a spring that works by torsion or twisting; that is, a flexible elastic object that stores mechanical energy when it is twisted. When it is twisted, it exerts a torque in the opposite direction, proportional to the amount (angle) it is twisted.

Torsion springs obey an angular form of Hooke's law:

$$\tau = -\kappa\theta$$

where τ is the torque exerted by the spring, and θ is the angle of twist from its equilibrium position. κ is the spring's torsion coefficient, analogous to the spring constant of a linear spring. The negative sign indicates that the direction of the torque is opposite to the direction of twist.

The energy U, in joules, stored in a torsion spring is:

$$U = \frac{1}{2}\kappa\theta^2$$

.

Consider the pendulum with a point mass and a torsion spring. The angle θ is the displacement from the horizontal. Denote the equilibrium position of the mass θ_0.

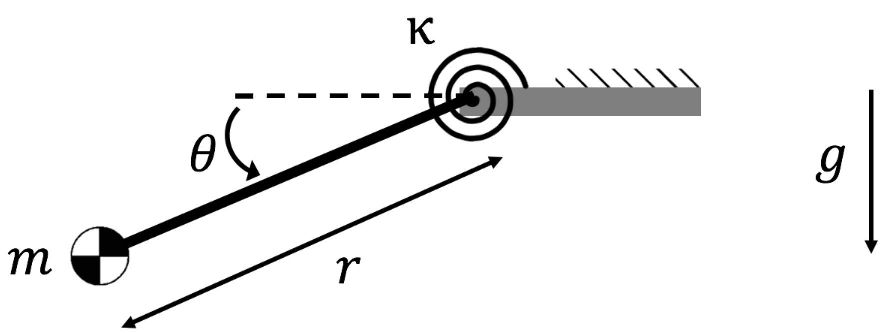


Figure 2: Pendulum with torsion spring.

(a) Write an expression for the kinetic energy in this system. (1)

(b) Write an expression for the potential energy in the system. (1)

(c) Derive the equations of motion using Lagrangian mechanics (use energy). (4)

(d) How would you determine the equilibrium position of the mass? (Note: there is no closed form solution). (1)

(e) Now consider a mass m with inertia I attached a a radius r. Write the equation of motion for this system. (1)

1 Appendix: RRR Manipulator

Robotic manipulator with three revolute joints.

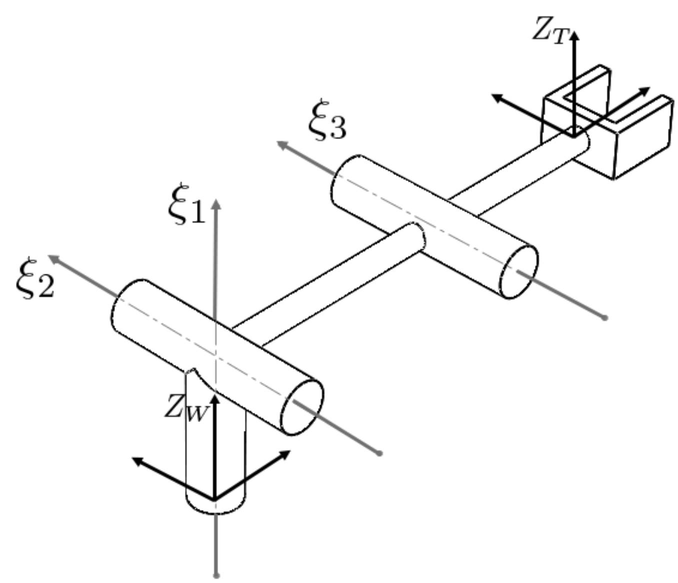


Figure 3: Schematic of a 3DoF RRR Manipulator in its initial configuration. Axes of rotation and the world and tool reference frames are shown.

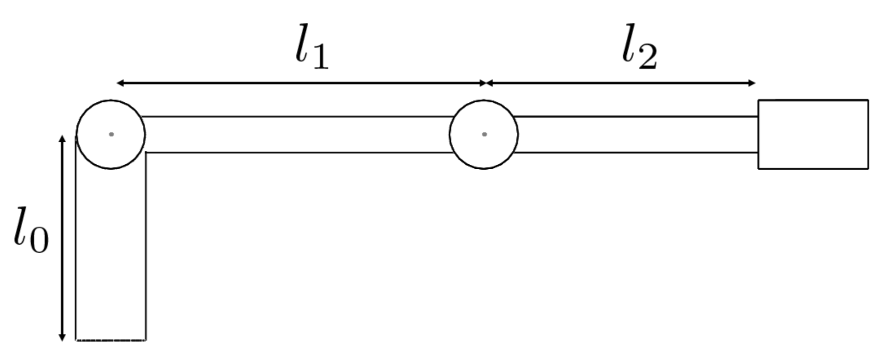


Figure 4: Side view of a 3DoF RRR Manipulator in its initial configuration. Segment lengths are shown.

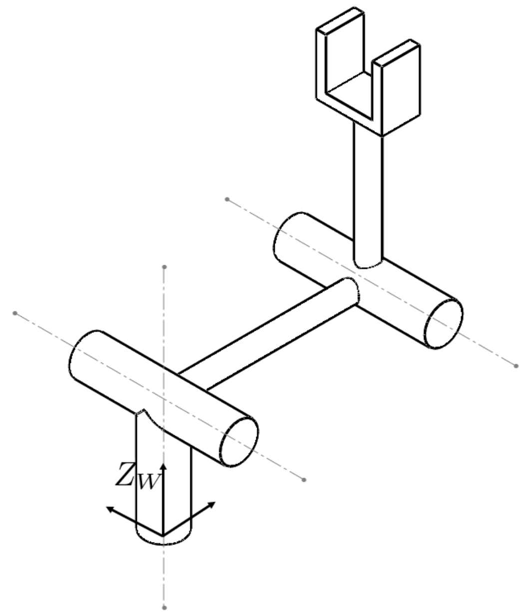


Figure 5: Manipulator in the configuration $\boldsymbol{\theta} = \left[0, 0, -\frac{\pi}{2}\right]$

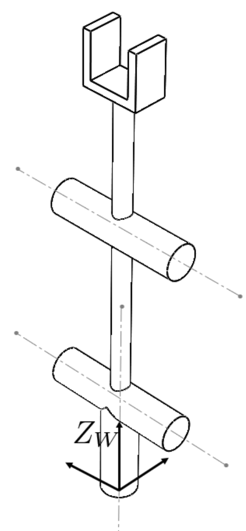


Figure 6: Manipulator in the configuration $\boldsymbol{\theta} = \left[-\frac{\pi}{2}, 0, 0\right]$.

2 Appendix: Cheat Sheet

This is the cheat sheet that will be provided for every midterm.

2.1 Trigonometry

Pythagoras's theorem $h^2 = x^2 + y^2$ for a right angled triangle where h is the hypotenuse and x and y are the lengths of the two remaining sides.

Sine, Cosine Relation $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$

Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos(\theta_C)$ where a, b, c are the lengths of the triangle and θ_A, θ_B and θ_C are the angles of their opposing corner.

2.2 Linear Algebra

For orthogonal matrices $A^{-1} = A^T$

Orthogonality A matrix $\begin{bmatrix}\boldsymbol{v}_1, ..., \boldsymbol{v}_n\end{bmatrix}$ is said to be orthogonal if:

$$\boldsymbol{v}_i^T \boldsymbol{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

2.3 Special Operators

Hat

$$\hat{\boldsymbol{\omega}} = \widehat{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Wedge

$$\hat{\boldsymbol{\xi}} = \widehat{\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \boldsymbol{0} & 0 \end{bmatrix}$$

2.4 Rotations

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{x}}\theta}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{y}}\theta}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{\hat{\boldsymbol{z}}\theta}$$

2.5 Rodrigues' Formula

$$R(\boldsymbol{\omega}, \theta) = e^{\hat{\boldsymbol{\omega}}\theta} = \mathbb{I}_3 + \frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|}\sin(\theta) + \frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2}(1 - \cos(\theta))$$

2.6 Rigid Body Motion

$$\boldsymbol{g}_{AB} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix} \qquad \boldsymbol{g}_{AB}^{-1} = \begin{bmatrix} \boldsymbol{R}_{AB}^{-1} & -\boldsymbol{R}_{AB}^{-1}\boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

2.7 Exponential Notation

$$\boldsymbol{R}_{AB}(\theta_1) = e^{\hat{\omega}_1\theta_1}$$

$$\boldsymbol{g}_{AB}(\theta_1) = e^{\hat{\xi}_1\theta_1}\boldsymbol{g}_{AB}(0)$$

$$\boldsymbol{g}_{ST}(\theta_1, \ldots, \theta_n) = e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_n\theta_n}\boldsymbol{g}_{ST}(0)$$

2.7.1 Special Cases

Pure Rotation

$$\boldsymbol{\xi} = \begin{bmatrix} -\boldsymbol{\omega} \times \boldsymbol{q} \\ \boldsymbol{\omega} \end{bmatrix}$$

Pure Translation

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{0} \end{bmatrix}$$

Pure Rotations, Screws (Rotation and Translation)

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} e^{\hat{\boldsymbol{\omega}}\theta} & \left(\mathbb{I}_3 - e^{\hat{\boldsymbol{\omega}}\theta}\right)(\boldsymbol{\omega} \times \boldsymbol{v}) + \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}\theta \\ \boldsymbol{0} & 1 \end{bmatrix}$$

Pure Translation

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} \mathbb{I}_3 & \boldsymbol{v}\theta \\ \boldsymbol{0} & 1 \end{bmatrix}$$

2.8 Paden-Kahan

Subproblem 1: Rotation about a single axis

$$e^{\widehat{\xi}\theta}p = q$$

Subproblem 2: Rotation about two subsequent axes

$$e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}p = q$$

Subproblem 3: Rotation to a distance

$$\left\| e^{\widehat{\xi}\theta}p - q \right\| = \delta$$

2.9 Velocities

Spatial Velocities

$$\widehat{V}^s_{AB} = \dot{g}_{AB} g^{-1}_{AB} \qquad V^s_{AB} = \begin{bmatrix} -\dot{R}R^T p + \dot{p} \\ \left(\dot{R}R^T\right)^\vee \end{bmatrix} = \xi\dot{\theta}$$

Body Velocities

$$\widehat{V}^b_{AB} = g^{-1}_{AB}\dot{g}_{AB} \qquad V^b_{AB} = \begin{bmatrix} R^T\dot{p} \\ \left(R^T\dot{R}\right)^\vee \end{bmatrix} = \left(Ad_{g^{-1}_{AB}(0)}\xi\right)\dot{\theta}$$

Adjoint

$$Ad_g = \begin{bmatrix} R & \widehat{p}R \\ 0 & R \end{bmatrix} \qquad V^s = Ad_g V^b$$

$$V^s_{AC} = V^s_{AB} + Ad_{g_{AB}} V^s_{BC} \qquad V^b_{AC} = Ad_{g^{-1}_{BC}} V^b_{AB} + V^b_{BC}$$

2.10 Jacobians

Spatial Jacobian

$$\begin{aligned} V^s_{ST} &= J^s_{ST}\dot{\theta} \\ J^s_{ST} &= \begin{bmatrix} \xi_1 & \xi'_2 & \dots & \xi'_n \end{bmatrix} \\ \xi'_i &= Ad_{\left(e^{\widehat{\xi}_1\theta_1}\dots e^{\widehat{\xi}_{i-1}\theta_{i-1}}\right)}\xi_i \end{aligned}$$

Body Jacobian

$$\begin{aligned} V^b_{ST} &= J^b_{ST}\dot{\theta} \\ J^b_{ST} &= \begin{bmatrix} \xi^\dagger_1 & \xi^\dagger_2 & \dots & \xi^\dagger_n \end{bmatrix} \\ \xi^\dagger_i &= Ad^{-1}_{\left(e^{\widehat{\xi}_i\theta_i}\dots e^{\widehat{\xi}_n\theta_n}g_{ST}(0)\right)}\xi_i \end{aligned}$$

2.11 Wrenches

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{\tau} \end{bmatrix}$$

$$\boldsymbol{\Gamma}^b = \left(Ad_{g_{ST}}\right)^T \boldsymbol{\Gamma}^s$$

Spatial Wrench

$$\boldsymbol{\tau} = \left(J^s_{ST}\right)^T \boldsymbol{\Gamma}^s$$

Body Wrench

$$\boldsymbol{\tau} = \left(J^b_{ST}\right)^T \boldsymbol{\Gamma}^b$$

2.12 Euler Lagrange

$$\begin{aligned} \mathcal{L}(q,\dot{q}) &= T(q,\dot{q}) - V(q) \\ \Gamma_i &= \tfrac{d}{dt}\tfrac{\partial\mathcal{L}}{\partial\dot{q}_i} - \tfrac{\partial\mathcal{L}}{\partial q_i} \end{aligned}$$

2.13 Forces

Gravitational Force

$$W = -mg$$

where m is the mass, g is the acceleration due to gravity.

Elastic Force

$$F_k = -k\delta$$

where k is the spring constant and δ is the extension of the spring.

2.14 Energies

Kinetic

$$KE = \frac{1}{2}mv^2$$

where m is the mass and v is the velocity of the object.

Gravitational Potential

$$GPE = mgh$$

where m is the mass, g is the acceleration due to gravity and h is the distance along the gravitational axis.

Elastic Potential

$$EPE = \frac{1}{2}k\delta^2$$

where k is the spring constant and δ is the extension of the spring.

2.15 Moments of Inertia

Mass at a Radius

$$I = mr^2$$

where m is the mass, and r is the radius of the mass.