Introduction to Robotics
E106/206 Midterm 2

Name: Brent Ji
SID: 26858652

Please write your name at the top of each page

Show all working. Marks are awarded for method.

A cheat sheet is provided. No other notes are allowed.

Introduction to Robotics **Name:** ____________________

Question 1: Rigid Body Motion . . . . 4 points

(a) Given rigid body transformations g_{EA}, g_{AR}, and g_{ES}, write an expression for g_{RS}. (1)

$$g_{RS} = g_{AR}^{-1} g_{EA}^{-1} g_{ES}$$

(b) Suppose that the origin of the S frame as seen from the E frame is $[1, 1, 1]$ and that (3)

$$g_{ER} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Write down the **coordinates** of the origin of the S frame as seen from the R frame.

$$g_{RE} = g_{ER}^{-1}$$

$$= \begin{bmatrix} R_{RE}^{-1} & -R_{RE}^{-1} p_{RE} \\ \vec{0}^T & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{RE}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad [0 \ 0 \ 0]^T$$

Question 2: Rotating Rigid Body Velocities .... 14 points

Consider the rigid body in the top right of Figure 1 right, which is rotated about the point (L,L) with angular velocity $\dot{\theta} = 1$. Use the fixed reference frame S as the spatial frame and the moving frame B as the body frame.

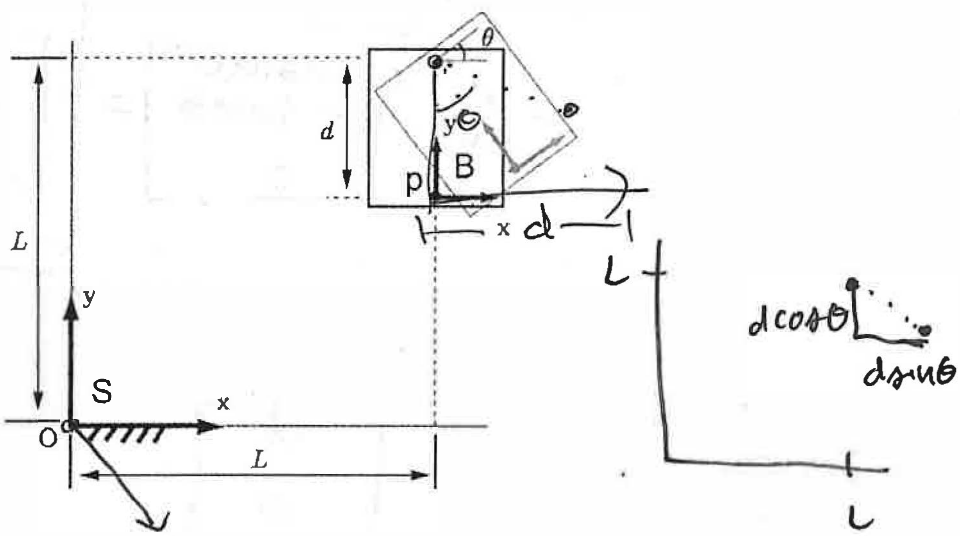


Figure 1: A rigid body rotating in the plane.

(a) The point p is at the origin of the B frame. Find p_s, the position of the point p as seen from the spatial frame S, in terms of θ. (2)

$$\cancel{\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}, \quad \omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad q = \begin{bmatrix} L \\ L \\ 0 \end{bmatrix}}$$

$$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix} \quad \omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \cancel{- \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ L & L & 0 \end{vmatrix} = iL - jL = \begin{bmatrix} L \\ -L \\ 0 \end{bmatrix}}$$

$$= \begin{bmatrix} L \\ -L \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad q = \begin{bmatrix} L \\ L \\ 0 \end{bmatrix} \quad -\omega \times q = \begin{bmatrix} L \\ -L \\ 0 \end{bmatrix}$$

$\dot{\theta}$

$$V^{S}_{AB} = \begin{bmatrix} L \\ -L \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cdot 1$$

getting ahead of myself..

$$p_s = g_{SB} \cdot p$$

$$= \boxed{\begin{bmatrix} L + d\sin\theta \\ L - d\cos\theta \\ 0 \end{bmatrix}}$$

~~shifted~~ L, L

(b) Find v_{ps}, the velocity of the point p as seen from the spatial frame. (2)

$$V_{ps} = \frac{d}{dt}\left[p_s \right]$$

$$= \frac{d}{dt}\begin{bmatrix} L + d\sin\theta \\ L - d\cos\theta \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} d\cos\theta \\ d\sin\theta \\ 0 \end{bmatrix}}$$

(c) Find v_{pB}, the velocity of the point p as seen from the body frame. (2)

$$\begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix}$$

(d) Write the configuration g_{SB} in terms of θ. (2)

$$g_{SB} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix},$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & L + d\sin\theta \\ \sin\theta & \cos\theta & 0 & L - d\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) Find the twist in body frame coordinates. (2)

$$\xi^{b} = \begin{bmatrix} d \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(f) Find the twist in spatial frame coordinates. (2)

$$\xi^{s} = \begin{bmatrix} L \\ -L \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(g) What is the relationship between the twists from (e) and (f)? (1)

Multiplied by Adjoint

$\xi^{b} = Ad_{g^{-1}(0)} \xi^{s}$

(h) What is the relationship between the spatial twist and the spatial velocity of the point? (1)

$V^{s}_{p} = \hat{V}^{s}_{AB}\bar{p}$

$V^{s}_{AB} = \xi\dot{\theta} \rightarrow \boxed{V^{s}_{p} = (\hat{\xi}\dot{\theta})\bar{p}}$

spatial velocity = spatial twist · joint velocity

Question 3: Forward Kinematics .... 6 points

For this question, refer to the manipulator seen in Appendix 2.

(a) Write an expression for $g_{WT}(\theta)$, using the forwards kinematics map (leaving your answer in terms of $e^{\hat{\xi}_i \theta_i}$). (2)

$$g_{WT}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} g_{WT}(0)$$

(b) Write an expression for $g_{WT}(\theta)$, using relative coordinate frame transformations. Include θ_1, θ_2, θ_3 in your expression. (2)

$$g_{WT}(\theta) = g_{WA}(\theta_1)\, g_{AB}(\theta_2)\, g_{BC}(\theta_3)\, g_{CT}$$

(c) Write the initial configuration $g_{WT}(0)$ of the manipulator. (2)

$$g_{WT}(0) = \begin{bmatrix} 1 & 0 & 0 & L1+L3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L0-L2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Introduction to Robotics Name: Bret Yi

Question 4: Inverse Kinematics .... 14 points

The manipulator in Appendix 2 is to be moved to a valid desired end-effector configuration $g_{d,WT}$.

(a) Given $g_{d,WT}$, write an expression for $g_{d,WC}$, the configuration of the C frame in the desired configuration. If you use additional configurations in your expression, explicitly define the configurations in homogeneous matrix form. (2)

$$g_{d,WT} = g_{d,WC}\, g_{CT}$$

$$g_{d,WC} = g_{d,WT}\, g_{CT}^{-1}$$

$$g_{CT} = \begin{bmatrix} 1 & 0 & 0 & L3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{CT}^{-1} = \begin{bmatrix} 1 & 0 & 0 & -L3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Why is $g_{d,WC}$ useful in solving the inverse kinematics problem? (1)

There is only one valid C frame for each desired T frame. We can use $g_{d,WC}$ to solve a simpler IK problem for the first two joint angles, then solve for the last with $g_{d,WT}$.

(c) The matrix g_1 can be expressed as: (1)

$$g_1 = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3}$$

Write an expression for g_1 in terms of known configurations.

$$g_{d,WT} = g_1\, g_{WT}(0)$$

$$g_1 = g_{d,WT}\, g_{WT}(0)^{-1}$$

(d) Decompose the manipulator inverse kinematics into Paden-Kahan subproblems: State 1) the order in which you would solve the joint angles, and 2) which PK subproblem you would use to solve for each joint angle. (4)

a) PK 3 to solve for θ_2
b) PK 1 to solve for θ_1
c) PK 1 to solve for θ_3, with an off-axis point

(e) Formulate the inverse kinematics problem to find θ_2 using a PK subproblem. Based on your answer to part (d), you may assume knowledge of previously solved joint angles. State the subproblem and define all necessary terms to compute θ_2 (ie. p, q, r, δ). (4)

$$\xi = \xi_2 = \begin{bmatrix} -L_0 \\ 0 \\ L_1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad q = [L_1 \;\; 0 \;\; L_0]$$

$$-\omega \times q = -\begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ L_1 & 0 & L_0 \end{vmatrix} = -L_0 i + L_1 k = [-L_0 \;\; 0 \;\; L_1]$$

$$\delta = \|p_A - g_1 p_c\|_2 = \|[0 \;\; 0 \;\; L_0]^T - g_1 [L_1 \;\; 0 \;\; L_0 - L_2]^T\|_2$$

$$q = p_A = [0 \;\; 0 \;\; L_0]^T$$

$$p = [L_1 \;\; 0 \;\; L_0 - L_2]^T$$

$$r = [L_1 \;\; 0 \;\; L_0]^T$$

(f) For a valid desired configuration of this manipulator, how many valid inverse kinematics solutions exist? (1)

1

(g) For this manipulator, give an example of a desired configuration with no inverse kinematics solution. Explain why there is no solution. (1)

~~$L_1 + L_3 + 1$~~

$$g_{d,wt} = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_3 + 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_0 - L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

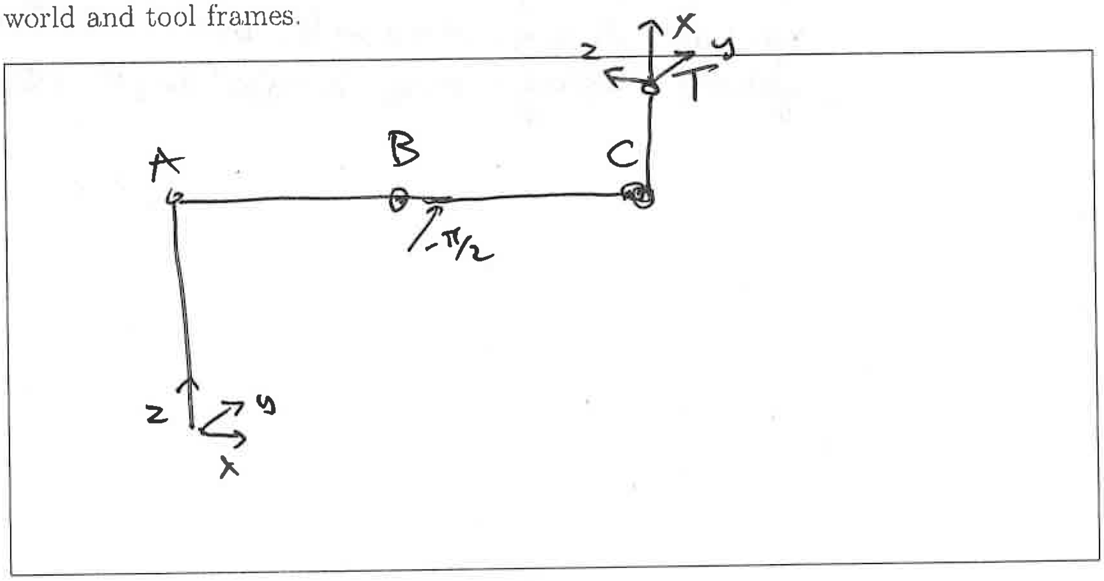
this is impossible because the joint is not long enough. We'd need a prismatic joint along the x-axis to reach it.

Question 5: Velocities .... 12 points

The body and spatial Jacobians for the manipulator in configuration $g_{WT}\left(0, -\frac{\pi}{2}, 0\right)$ (Appendix 2, Figure 3) can be written:

$$J^s = \begin{bmatrix} 0 & -l_0 & 0 \\ 0 & 0 & -l_1 - l_2 \\ 0 & l_1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad J^b = \begin{bmatrix} 0 & -l_2 & 0 \\ l_1 + l_2 & 0 & 0 \\ 0 & -l_3 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Sketch the manipulator in the configuration $g_{WT}\left(0, -\frac{\pi}{2}, 0\right)$. Label the axes of the world and tool frames. (2)



(b) For this configuration, write $\hat{V}^S_{WT}$. (3)

$$\hat{V}^s_{WT} = \widehat{J^s_{st}\dot{\theta}} = \begin{bmatrix} -l_0\dot{\theta}_2 \\ \dot{\theta}_3(-l_1-l_2) \\ l_1\dot{\theta}_2 \\ 0 \\ \dot{\theta}_2 \\ \dot{\theta}_1+\dot{\theta}_3 \end{bmatrix}$$

hat

$$= \begin{bmatrix} 0 & -l_1\dot{\theta}_2 & \dot{\theta}_3(-l_1l_2) & 0 \\ l_1\dot{\theta}_2 & 0 & l_0\dot{\theta}_2 & \dot{\theta}_2 \\ -\dot{\theta}_3(l_1-l_2) & -l_0\dot{\theta}_2 & 0 & \dot{\theta}_1+\dot{\theta}_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) Given a point q_T defined in the T frame, write an expression for v_{qw} (the velocity of the point with reference to the world frame. (2)

$V_{qW} = \hat{V}^{s}_{WT}\, g_{WT}\, q_T$

(d) For this configuration, how does the body velocity of the same point, v_{qT}, relate to v_{qw}? Explain your answer. (1)

Multiplied by inverse of adjoint, because frame needs to be ~~trans~~ changed

OR: same, but ~~flipped~~ swapped axes ($x = z$, $z = -x$) because of relative frame positions

(e) What instantaneous linear and rotational body velocities are possible in this configuration? (2)

~~Any ω_x and V_y, and (V_x or V_z) ω_x, V~~

Any linear combination of the twists.

V_y & ω_x, V_x & V_z & ω_y, ω_x

(f) Is it possible to independently control the v_x, v_y, and ω_y body velocities? Explain. (1)

No — no joint has control of V_y

(V_y columns are all zero)

(g) Is this a singular configuration? Explain. (1)

No — rank is 3 (full col. rank)

Question 6: Wrenches . . . . 9 points

Again consider the configuration $g_{WT}\left(0, -\frac{\pi}{2}, 0\right)$ whose body and spatial Jacobians can be written:

$$J^s = \begin{bmatrix} 0 & -l_0 & 0 \\ 0 & 0 & -l_1 - l_2 \\ 0 & l_1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad J^b = \begin{bmatrix} 0 & -l_2 & 0 \\ l_1 + l_2 & 0 & 0 \\ 0 & -l_3 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) What spatial forces and torques can be applied by the manipulator end-effector in this configuration? (2)

~~Arbitrary~~ ~~ω_z, ω~~

ω_z, V_y, and $(\underbrace{V_x, V_z, \text{or } \omega_y}_{\text{coupled}})$

(b) Suppose that a force $f = [1, 2, 3]$ and a torque $\tau = [4, 5, 6]$ are applied at the origin of the T frame. What are the resulting torques experienced at each joint? (3)

$$T = \left(J^b_{st}\right)^T \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2(l_1 + l_2) + 4 \\ -l_2 - 3l_3 + 5 \\ 4 \end{bmatrix}$$

(c) A mass m is held by the manipulator at the origin of the T frame. What is the body wrench $\mathbf{\Gamma}^b$ associated with this load in the initial configuration (Fig. 3)? Is this the same as the body wrench in the configuration $g_{WT}\left(0,-\frac{\pi}{2},0\right)$? (2)

$$\Gamma^b = \begin{bmatrix} F \\ \tau \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -mg \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Not same — different ~~axes~~ axis facing down in world frame.

(d) Given a wrench Γ_1 applied at the T frame origin and a wrench Γ_2 applied at the B frame origin, write an expression for the total wrench experienced at the world frame. (2)

$$\Gamma^b = (Ad_{g_{WT}})^T \Gamma^s$$

$$\Gamma^s = (Ad_{g_{WT}}^T)^{-1} \Gamma^b$$

use the adjoint

$$\Gamma_W = (Ad_{g_{WT}}^T)^{-1} \Gamma_1 + (Ad_{g_{WB}}^T)^{-1} \Gamma_2$$

Question 7: Dynamics .... 11 points

A torsion spring is a spring that works by torsion or twisting; that is, a flexible elastic object that stores mechanical energy when it is twisted. When it is twisted, it exerts a torque in the opposite direction, proportional to the amount (angle) it is twisted.

Torsion springs obey an angular form of Hooke's law:

$$\tau = -\kappa\theta$$

where τ is the torque exerted by the spring, and θ is the angle of twist from its equilibrium position. κ is the spring's torsion coefficient, analogous to the spring constant of a linear spring. The negative sign indicates that the direction of the torque is opposite to the direction of twist.

The energy U, in joules, stored in a torsion spring is:

$$U = \frac{1}{2}\kappa\theta^2$$

Consider the pendulum with a point mass and a torsion spring. The angle θ is the displacement from the horizontal. Denote the equilibrium position of the mass θ_0.

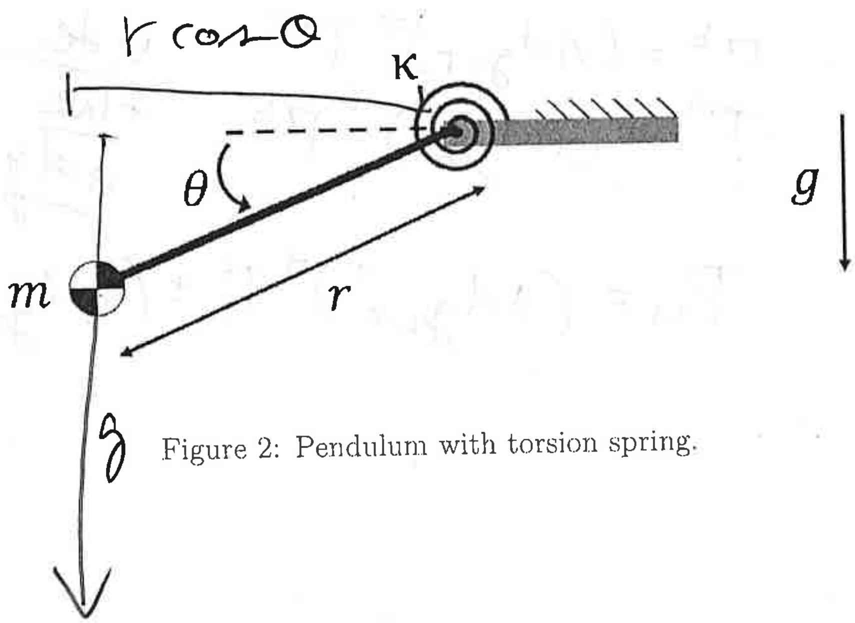


Figure 2: Pendulum with torsion spring.

 Name: Brent J

(a) Write an expression for the kinetic energy in this system. (1)

$$T = KE = \frac{1}{2} m r^2 \dot{\theta}^2$$

(b) Write an expression for the potential energy in the system. (1)

$$V = PE = PE_{gravity} + PE_{spring}$$

$$\cancel{= -mg\sin}$$

$$= -mgr\sin\theta + \frac{1}{2} k \theta^2$$

(c) Derive the equations of motion using Lagrangian mechanics (use energy). (4)

$$L = \frac{1}{2} m r^2 \dot{\theta}^2 + mgr\sin\theta - \frac{1}{2} k \theta^2$$

$$\Gamma = \frac{d}{dt} \frac{dL}{\partial \dot{\theta}} - \frac{dL}{d\theta}$$

$$= \frac{d}{dt}\left(mr^2\dot{\theta}\right) - \left(mgr\cos\theta - k\theta\right)$$

$$mr^2\ddot{\theta} = mgr\cos\theta - k\theta$$

(d) Derive the equations of motion using Newtonian mechanics. (3)

$\tau_{spring} = -k\theta$

$\tau_{gravity} = mgr\cos\theta$

$\Sigma\tau = I\ddot{\theta}$

$\Sigma F = ma$

$\Sigma\tau = I\ddot{\theta}$

$\Sigma\tau = mr^2\ddot{\theta}$

$mr^2\ddot{\theta} = mgr\cos\theta - k\theta$

(e) How would you determine the equilibrium position of the mass? (Note: there is no closed form solution). (1)

Set the sum of the spring and gravity torques ~~equal~~ to ~~each other~~ zero:

$\tau_{spring} + \tau_{gravity} = 0$

$-k\theta + mgr\cos\theta = 0$

and solve for θ

(f) Now consider the same system but with a mass m with inertia I attached at a radius r. Write the equation of motion for this system. (1)

$I' = I + mr^2$

$\Sigma\tau = I'\ddot{\theta}$

$(I+mr^2)\ddot{\theta} = mgr\cos\theta - k\theta$