Introduction to Robotics **Name:** ____________________

Midterm 1 - F2017 **SID:** ____________________

Please write your name at the top of each page

Show all working. Marks are awarded for method.

A cheat sheet is provided. No other notes are allowed.

Problem 1	/9
Problem 2	/6
Problem 3	/10
Problem 4	/20
Problem 5	/25
Total	/70

Question 1: Rotation Matrices . . . . 9 points

Consider two coordinate frames A and B. Coordinate frame B begins aligned with frame A. Frame B is then rotated by $\frac{\pi}{2}$ radians about the Y axis of the A coordinate frame.

(a) Write the rotation matrix $\boldsymbol{R}_{AB}$ for these two coordinate frames. (2)

(b) Write the rotation matrix $\boldsymbol{R}_{BA}$ for these two coordinate frames. (2)

(c) How would a vector $\boldsymbol{p}_A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ written in the A coordinate frame be written in the B coordinate frame? (2)

(d) Consider two coordinate frames A and B. Coordinate frame B begins aligned with frame A. Frame B undergoes the following sequence of rotations: (3)

- Rotation by ϕ about the Z axis
- Rotation by θ about the current X axis
- Rotation by γ about the current Z axis

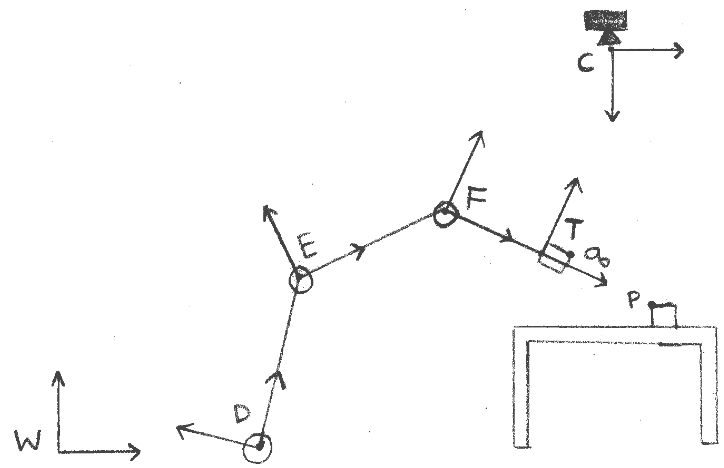
Write the resulting rotation matrix R_{AB} (You do not need to compute the product).

Question 2: Axis Angle Rotations . . . . 6 points

Write the rotation matrix R for a rotation of π radians about the axis $\omega = [1, -1, 1]^T$

Question 3: Homogeneous Transformations .... 11 points

Consider the diagram below to answer the following questions. Pictured is a robot adjacent to a table with a camera mounted overhead.



(a) You are given the homogeneous transforms g_{WT} and g_{WC}. Write g_{CT} in terms of $R_{WT}, p_{WT}, R_{WC}, p_{WC}$. In words, describe what g_{CT} does. (4)

(b) Given point q_T, defined in the T frame and point p_C, defined in the C frame, write an expression for the distance between q and p. (2)

(c) Write the rigid body transform g_{WT} in terms of relative rigid body transformations (use frames defined in the figure). (2)

(d) For a general case, show that a homogeneous transformation preserves the distance between two points. (3)

Question 4: Forward Kinematics . . . . 20 points

For this problem, consider the robotic manipulator shown in Appendix 1.

(a) Define forward kinematics and inverse kinematics. (2)

(b) Compute the twists $\boldsymbol{\xi}$ for joints 1, 3, and 5. (8)

(c) Write an expression for rigid body transform $\boldsymbol{g}_{WB}(\theta_1, \theta_2)$ in homogeneous form (by inspection). (3)

(d) Write an expression for the initial configuration $\boldsymbol{g}_{WT}(\mathbf{0})$ of the manipulator in homogeneous form (by inspection). (3)

(e) Using matrix exponential terms such as $e^{\hat{\boldsymbol{\xi}}_i \theta_i}$, write an expression for the forward kinematics map with the form $\boldsymbol{g}_{WB}(\boldsymbol{\theta})$. (2)

(f) Using matrix exponential terms such as $e^{\hat{\boldsymbol{\xi}}_i \theta_i}$, write an expression for the forward kinematics map with the form $\boldsymbol{g}_{WT}(\boldsymbol{\theta})$. (2)

Question 5: Inverse Kinematics .... 25 points

Consider the robotic manipulator shown in Appendix 1. You are given a desired configuration of the tool (T) frame:

$$\boldsymbol{g}_{d,WT} = \begin{bmatrix} & & & d_x \\ & R_d & & d_y \\ & & & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) The matrix $\boldsymbol{g}_1$ can be written as: (2)

$$\boldsymbol{g}_1 = e^{\hat{\boldsymbol{\xi}}_1\theta_1} e^{\hat{\boldsymbol{\xi}}_2\theta_2} e^{\hat{\boldsymbol{\xi}}_3\theta_3} e^{\hat{\boldsymbol{\xi}}_4\theta_4} e^{\hat{\boldsymbol{\xi}}_5\theta_5}$$

Write $\boldsymbol{g}_1$ in terms of known configurations

(b) In words, describe how each joint affects the configuration of the manipulator. (5)

(c) Let p_1 be a point invariant to joint 1, and p_2 be a point invariant to joints 4 and 5. Give potential coordinates for p_1 and p_2. (2)

(d) Using the initial and desired configurations, find θ_2. Hint: Leave in terms of d_x, d_y, and/or d_z. You do not need to use a Paden-Kahan sub-problem. (3)

(e) Using the initial and desired configurations, find θ_3. Hint: Leave in terms of d_x, d_y, and/or d_z. You do not need to use a Paden-Kahan sub-problem. (3)

(f) Given values of θ_2 and θ_3, formulate the inverse kinematics for θ_1 as a Paden-Kahan sub-problem. List the sub-problem and define necessary parameters (ie. p, q, r, δ). (4)

(g) The matrix $\boldsymbol{g}_2$ can be written as: (2)

$$\boldsymbol{g}_2 = e^{\hat{\boldsymbol{\xi}}_4 \theta_4} e^{\hat{\boldsymbol{\xi}}_5 \theta_5}$$

Given solutions for θ_1, θ_2, and θ_3 and $\boldsymbol{g_1}$, write an expression for $\boldsymbol{g}_2$ in terms of known matrices:

(h) Given values of θ_1, θ_2 and θ_3, formulate the inverse kinematics for θ_4 and θ_5 as a Paden-Kahan sub-problem. List the sub-problem and define necessary parameters (ie. p, q, r, δ). (4)

1 Appendix: 5 DoF Manipulator

Consider the robotic manipulator shown in Figure 1. The manipulator is shown in its initial configuration.

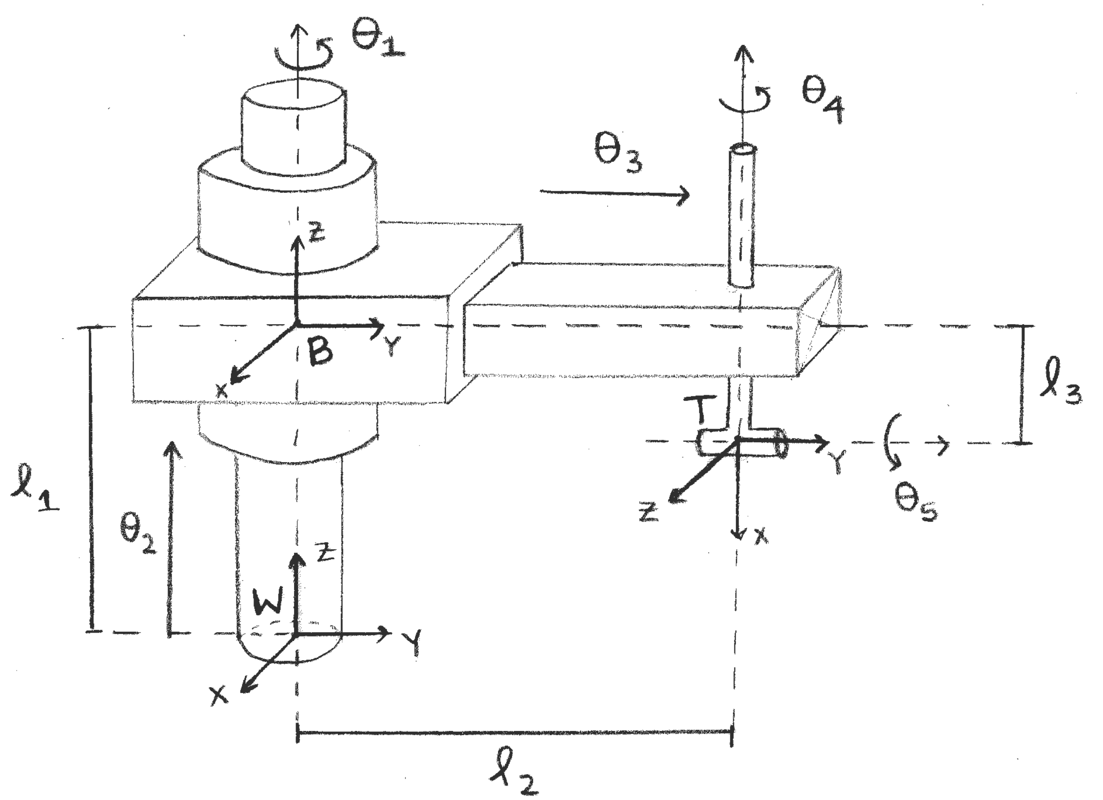


Figure 1: Joints 1, 4, and 5 are revolute joints. Joint 2 and 3 are prismatic joints. The world and tool frame labelled as W and T respectively. Frame B rotates with joint 1 and translates with joint 2.

2 Appendix: Cheat Sheet

2.1 Trigonometry

Pythagoras's theorem $h^2 = x^2 + y^2$ for a right angled triangle where h is the hypotenuse and x and y are the lengths of the two remaining sides.

Sine, Cosine Relation $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$

Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos(\theta_C)$ where a, b, c are the lengths of the triangle and θ_A, θ_B and θ_C are the angles of their opposing corner.

2.2 Linear Algebra

For orthogonal matrices $A^{-1} = A^T$

Orthogonality A matrix $\begin{bmatrix}\boldsymbol{v}_1, ..., \boldsymbol{v}_n\end{bmatrix}$ is said to be orthogonal if:

$$\boldsymbol{v}_i^T \boldsymbol{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

2.3 Special Operators

Hat

$$\hat{\boldsymbol{\omega}} = \widehat{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Wedge

$$\hat{\boldsymbol{\xi}} = \widehat{\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \boldsymbol{0} & 0 \end{bmatrix}$$

2.4 Rotations

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{x}}\theta}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{y}}\theta}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{\hat{\boldsymbol{z}}\theta}$$

2.5 Rodrigues' Formula

For $\|\omega\| = 1$:

$$R(\boldsymbol{\omega}, \theta) = e^{\hat{\boldsymbol{w}}\theta} = \mathbb{I}_3 + \hat{\boldsymbol{\omega}} \sin(\theta) + \hat{\boldsymbol{\omega}}^2 (1 - \cos(\theta))$$

2.6 Rigid Body Motion

$$\boldsymbol{g}_{AB} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix} \qquad \boldsymbol{g}_{AB}^{-1} = \begin{bmatrix} \boldsymbol{R}_{AB}^{-1} & -\boldsymbol{R}_{AB}^{-1}\boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

2.7 Exponential Notation

$$\boldsymbol{R}_{AB}(\theta_1) = e^{\hat{\omega}_1\theta_1}$$
$$\boldsymbol{g}_{AB}(\theta_1) = e^{\hat{\xi}_1\theta_1}\boldsymbol{g}_{AB}(0)$$
$$\boldsymbol{g}_{ST}(\theta_1, \ldots, \theta_n) = e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_n\theta_n}\boldsymbol{g}_{ST}(0)$$

2.7.1 Special Cases

Pure Rotation

$$\boldsymbol{\xi} = \begin{bmatrix} -\boldsymbol{\omega} \times \boldsymbol{q} \\ \boldsymbol{\omega} \end{bmatrix}$$

Pure Translation

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{0} \end{bmatrix}$$

Pure Rotations, Screws (Rotation and Translation)

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} e^{\hat{\boldsymbol{\omega}}\theta} & \left(\mathbb{I}_3 - e^{\hat{\boldsymbol{\omega}}\theta}\right)(\boldsymbol{\omega} \times \boldsymbol{v}) + \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}\theta \\ \boldsymbol{0} & 1 \end{bmatrix}$$

Pure Translation

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} \mathbb{I}_3 & \boldsymbol{v}\theta \\ \boldsymbol{0} & 1 \end{bmatrix}$$

2.8 Paden-Kahan

Subproblem 1: Rotation about a single axis

$$e^{\widehat{\xi}\theta}p = q$$

Subproblem 2: Rotation about two subsequent axes

$$e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}p = q$$

Subproblem 3: Rotation to a distance

$$\left\| e^{\widehat{\xi}\theta}p - q \right\| = \delta$$