Introduction to Robotics E125
Midterm 1, 2014 October 07 **SID:** ____________________

Calculators allowed, though use should be restricted to basic functions such as trigonometry and simple mathematical operations.

Please show all working. Marks are awarded for method.

A cheat sheet is provided. No other notes are allowed.

Question 1: Rotation Matrices . . . . 8 points

Consider two coordinate frames A and B. Coordinate frame B begins aligned with frame A. Frame B is then rotated by $\frac{\pi}{3}$ radians about the X axis of the A coordinate frame.

(a) Write the rotation matrix $\boldsymbol{R}_{AB}$ for these two coordinate frames. (2)

Solution:

1 Use correct Rotation matrix.

$$\boldsymbol{R}_{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$

1 Substitute in theta correctly.

$$\boldsymbol{R}_{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

(b) Write the rotation matrix $\boldsymbol{R}_{BA}$ for these two coordinate frames. (2)

Solution: Either, resolve using $\theta = -\frac{\pi}{3}$

1 Use correct Rotation matrix.

$$\boldsymbol{R}_{BA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$

1 Substitute in theta correctly.

$$\boldsymbol{R}_{BA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

or

2 Use $\boldsymbol{R}_{BA} = \boldsymbol{R}_{AB}^{-1} = \boldsymbol{R}_{AB}^{T}$ for full credit.

(c) How would a vector $\boldsymbol{p}_A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ written in the A coordinate frame be written in the B coordinate frame? (2)

Solution:

1 Use correct Rotation matrix.

$$\boldsymbol{p}_B = \boldsymbol{R}_{BA}\boldsymbol{p}_A$$

1 Perform calculation correctly.

$$\boldsymbol{p}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1+\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1.366 \\ -0.366 \end{bmatrix}$$

(d) How would a vector $\boldsymbol{p}_B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ written in the B coordinate frame be written in the A coordinate frame? (2)

Solution:

1 Use correct Rotation matrix.

$$\boldsymbol{p}_A = \boldsymbol{R}_{AB}\boldsymbol{p}_B$$

1 Perform calculation correctly.

$$\boldsymbol{p}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1-\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.366 \\ 1.366 \end{bmatrix}$$

Introduction to Robotics E125

Question 2: Rodrigues' Formula . . . . 6 points
Show that you can obtain the standard $\boldsymbol{R}_y$ rotation matrix from the Rodrigues' Formula.

Solution:

1 Write Rodrigues' formula

$$\boldsymbol{R}(\boldsymbol{\omega}, \theta) = \mathbb{I}_3 + \frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|} sin(\|\boldsymbol{\omega}\|\,\theta) + \frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2}\left(1 - cos(\|\boldsymbol{\omega}\|\,\theta)\right)$$

1 Correctly write $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$

1 Correctly writing the $\frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|}$ term:

$$\frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

1 Correctly writing the $\frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2}$ term:

$$\frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

1 Correctly writing the trig components:

$$\frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|} sin(\|\boldsymbol{\omega}\|\,\theta) = \begin{bmatrix} 0 & 0 & sin(\theta) \\ 0 & 0 & 0 \\ -sin(\theta) & 0 & 0 \end{bmatrix}$$

$$\frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2}\left(1 - cos(\|\boldsymbol{\omega}\|\,\theta)\right) = \begin{bmatrix} (cos(\theta)) - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (cos(\theta) - 1) \end{bmatrix}$$

1 Correctly arrive at $\boldsymbol{R}_y$:

$$\boldsymbol{R}(\boldsymbol{\omega}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & sin(\theta) \\ 0 & 0 & 0 \\ -sin(\theta) & 0 & 0 \end{bmatrix} + \begin{bmatrix} (cos(\theta) - 1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (cos(\theta) - 1) \end{bmatrix}$$

$$\boldsymbol{R}(\boldsymbol{\omega}, \theta) = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & (cos(\theta)) \end{bmatrix} = \boldsymbol{R}_y$$

Question 3: Multiple Rotations .... 2 points

(a) Given the rotation matrices $\boldsymbol{R}_{AB}, \boldsymbol{R}_{BC}, \boldsymbol{R}_{CD}$, write and expression for $\boldsymbol{R}_{AD}$. (1)

> **Solution:**
>
> **1** For correct expression: $\boldsymbol{R}_{AD} = \boldsymbol{R}_{AB}\boldsymbol{R}_{BC}\boldsymbol{R}_{CD}$

(b) Given the rotation matrices $\boldsymbol{R}_{BC}, \boldsymbol{R}_{CD}, \boldsymbol{R}_{AD}$, write and expression for $\boldsymbol{R}_{AB}$. (1)

> **Solution:**
>
> **1** For correct expression: $\boldsymbol{R}_{AB} = \boldsymbol{R}_{AD}\left[\boldsymbol{R}_{BC}\boldsymbol{R}_{CD}\right]^{-1}$

Question 4: Valid Rotations .... 4 points

Is the transformation matrix $\boldsymbol{T}$ shown below a valid rotation matrix? If so, prove it, if not say why.

$$\boldsymbol{T} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}$$

> **Solution:**
>
> **1** Checking for orthonormality, orthogonality.
>
> **1** For showing $\boldsymbol{t}_1^T\boldsymbol{t}_1 = cos(\theta)^2 + sin(\theta)^2 = 1$, $\boldsymbol{t}_2^T\boldsymbol{t}_2 = cos(\theta)^2 + sin(\theta)^2 = 1$.
>
> **1** For showing $\boldsymbol{t}_1^T\boldsymbol{t}_2 = cos(\theta)sin(\theta) - cos(\theta)sin(\theta) = 0$.
>
> **1** For saying T is a valid rotation matrix.

Question 5: Rigid Body Motion . . . . 20 points

Consider a robotics manipulator with three joints. We represent the rotation of these joints as three rigid body motions:

$\boldsymbol{g}_{AB}$ Frame B starts aligned with frame A. Frame B is then translated so that its origin is at $\begin{bmatrix} l_0 & 0 & 0 \end{bmatrix}^T$ as seen in the A frame. Frame B is then rotated by $\frac{\pi}{2}$ radians about the X axis of the A frame.

$\boldsymbol{g}_{BC}$ Frame C starts aligned with frame B. Frame C is then translated so that its origin is at $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ as seen in the B frame. Frame C is then rotated by 0 radians about the Y axis of the B frame.

$\boldsymbol{g}_{CD}$ Frame D starts aligned with frame C. Frame D is then translated so that its origin is at $\begin{bmatrix} 0 & -l_1 & 0 \end{bmatrix}^T$ as seen in the C frame. Frame D is then rotated by $\frac{\pi}{2}$ radians about the Y axis of the C frame.

(a) Write the rigid body transform $\boldsymbol{g}_{AB}$ in homogeneous form. (4)

Solution:

1 For writing in homogeneous coordinate form:

$$\boldsymbol{g}_{AB} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

1 For correctly using the $\boldsymbol{R}_x$ rotation matrix:

$$\boldsymbol{R}_{AB} = \boldsymbol{R}_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$

1 For correctly finding $\boldsymbol{R}_{AB}$:

$$\boldsymbol{R}_{AB} = \boldsymbol{R}_X\left(\frac{\pi}{2}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

1 For correctly substituting in translation:

$$\boldsymbol{g}_{AB} = \begin{bmatrix} 1 & 0 & 0 & l_0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Write the rigid body transform $\boldsymbol{g}_{BC}$ in homogeneous form. (4)

Solution:

1 For writing in homogeneous coordinate form:

$$\boldsymbol{g}_{BC} = \begin{bmatrix} \boldsymbol{R}_{BC} & \boldsymbol{p}_{BC} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

1 For correctly using the $\boldsymbol{R}_Y$ rotation matrix:

$$\boldsymbol{R}_{BC} = \boldsymbol{R}_Y\left(\theta\right) = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & & cos(\theta) \end{bmatrix}$$

1 For correctly finding $\boldsymbol{R}_{BC}$:

$$\boldsymbol{R}_{BC} = \boldsymbol{R}_Y\left(0\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1 For correctly substituting in translation:

$$\boldsymbol{g}_{BC} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Write the rigid body transform $\boldsymbol{g}_{CD}$ in homogeneous form. (4)

Solution:

1 For writing in homogeneous coordinate form:

$$\boldsymbol{g}_{CD} = \begin{bmatrix} \boldsymbol{R}_{CD} & \boldsymbol{p}_{CD} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

1 For correctly using the $\boldsymbol{R}_Y$ rotation matrix:

$$\boldsymbol{R}_{CD} = \boldsymbol{R}_Y\left(\theta\right) = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & & cos(\theta) \end{bmatrix}$$

1 For correctly finding $\boldsymbol{R}_{CD}$:

$$\boldsymbol{R}_{CD} = \boldsymbol{R}_Y\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

1 For correctly substituting in translation:

$$\boldsymbol{g}_{CD} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d) Write the rigid body transform $\boldsymbol{g}_{AD}$ in homogeneous form. (2)

Solution:

1 For the order of multiplication correctly:

$$\boldsymbol{g}_{AD} = \boldsymbol{g}_{AB}\boldsymbol{g}_{BC}\boldsymbol{g}_{CD}$$

1 For correctly performing the matrix multiplication:

$$\boldsymbol{g}_{AD} = \begin{bmatrix} 0 & 0 & 1 & l_0 \\ 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) How would a point $\boldsymbol{p}$ with the coordinates $\boldsymbol{p}_D = \begin{bmatrix} 0 & 0 & -l_2 \end{bmatrix}^T$ in the D frame be represented in the A frame? (2)

Solution:

1 post multiplying $\boldsymbol{g}_{AD}$ by $\boldsymbol{p}_D$:

$$\boldsymbol{p}_A = \boldsymbol{g}_{AD}\boldsymbol{p}_D$$

1 Multiplying correctly the homogenous coordinate of $\boldsymbol{p}_D$:

$$\boldsymbol{p}_A = \begin{bmatrix} 0 & 0 & 1 & l_0 \\ 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -l_2 \\ 1 \end{bmatrix} = \begin{bmatrix} l_0 - l_2 \\ l_1 \\ 0 \\ 1 \end{bmatrix}$$

(f) How would you alter your expression for $\boldsymbol{g}_{AD}$ if we added another joint F between the C and D frames? Explain your answer and any additional information required. (2)

Solution:

1 Need expressions for $\boldsymbol{g}_{CF}$ and $\boldsymbol{g}_{FD}$

1 For adding it after the $\boldsymbol{g}_{BC}$ term:

$$\boldsymbol{g}_{AE} = \boldsymbol{g}_{AB}\boldsymbol{g}_{BC}\boldsymbol{g}_{CF}\boldsymbol{g}_{FD}$$

(g) How would you alter your expression for $\boldsymbol{g}_{AD}$ if we added a frame F between the C and D frames, where F is a frame that is fixed to frame C? Explain your answer (2) and any additional information required.

Solution:

1 No change.

1 F is a reference frame and does not actually induce any motion to the subsequent frames. As we already know $\boldsymbol{g}_{CD}$ we do not need to include the F frame.

Question 6: Forward Kinematics . . . . 30 points
In this question we will be computing the Forward Kinematics of the 3 Degree of Freedom manipulator shown in Figure 1.

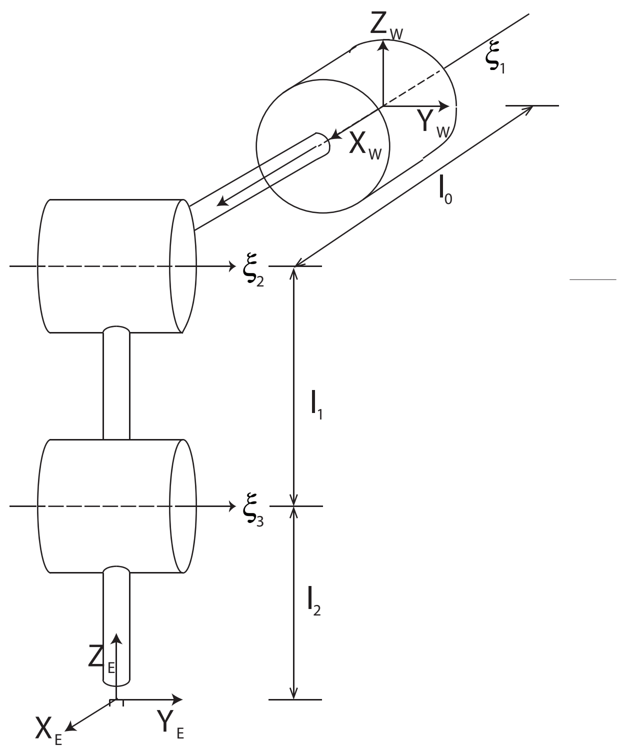

Figure 1: Schematic of a 3DoF Manipulator with axes of rotation shown in its initial configuration.

(a) Write the twists that represent each joint of the manipulator. (8)

Solution:

3 One Mark for each correct $\boldsymbol{\omega}$:

$$\boldsymbol{\omega}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{\omega}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{\omega}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3 One Mark for each correct $\boldsymbol{q}$. '$\cdot$' represents any number.

$$\boldsymbol{q}_1 = \begin{bmatrix} \cdot \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{q}_2 = \begin{bmatrix} l_0 \\ \cdot \\ 0 \end{bmatrix} \quad \boldsymbol{q}_3 = \begin{bmatrix} l_0 \\ \cdot \\ -l_1 \end{bmatrix}$$

1 Calculating $-\hat{\boldsymbol{\omega}} \times \boldsymbol{q}$:

$$-\hat{\boldsymbol{\omega}}_1 \times \boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad -\hat{\boldsymbol{\omega}}_2 \times \boldsymbol{q}_2 = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \quad -\hat{\boldsymbol{\omega}}_3 \times \boldsymbol{q}_3 = \begin{bmatrix} l_1 \\ 0 \\ l_0 \end{bmatrix}$$

1 Calculating the twists $\boldsymbol{\xi}$:

$$\boldsymbol{\xi}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{\xi}_2 = \begin{bmatrix} 0 \\ 0 \\ l_0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \boldsymbol{\xi}_3 = \begin{bmatrix} l_1 \\ 0 \\ l_0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(b) Write an expression for the homogeneous transformation $\boldsymbol{g}_{WE}$ when $\theta_1 = \frac{\pi}{2}$, $\theta_2 = 0$ and $\theta_3 = \frac{\pi}{2}$. (15)

Solution:

1 Writing the expression:

$$\boldsymbol{g}_{WE}\left(\theta_1, \theta_2, \theta_3\right) = e^{\hat{\boldsymbol{\xi}}_1 \theta_1} e^{\hat{\boldsymbol{\xi}}_2 \theta_2} e^{\hat{\boldsymbol{\xi}}_3 \theta_3} \boldsymbol{g}_{WE}\left(0\right)$$

2 Writing the expression (1 point for rotation, 1 point for translation):

$$\boldsymbol{g}_{WE}\left(0\right) = \begin{bmatrix} 1 & 0 & 0 & l_0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -l_1 - l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1 Writing the expression:

$$\hat{\boldsymbol{\xi}}_i = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix}$$

3 Writing the rotations:

$$e^{\hat{\boldsymbol{\omega}}_1\theta_1} = \boldsymbol{R}_X\left(\frac{\pi}{2}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$e^{\hat{\boldsymbol{\omega}}_2\theta_2} = \boldsymbol{R}_Y\left(0\right) = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{\hat{\boldsymbol{\omega}}_3\theta_3} = \boldsymbol{R}_Y\left(\frac{\pi}{2}\right) = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

3 Computing the cross products:

$$\boldsymbol{\omega}_1 \times \boldsymbol{v}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{\omega}_2 \times \boldsymbol{v}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} = \begin{bmatrix} l_0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{\omega}_3 \times \boldsymbol{v}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 \\ 0 \\ l_0 \end{bmatrix} = \begin{bmatrix} l_0 \\ 0 \\ -l_1 \end{bmatrix}$$

3 Computing $e^{\hat{\boldsymbol{\xi}}_i\theta_i}$:

$$e^{\hat{\boldsymbol{\xi}}_1\theta_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{\hat{\boldsymbol{\xi}}_2\theta_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{\hat{\boldsymbol{\xi}}_3\theta_3} = \begin{bmatrix} 0 & 0 & 1 & l_0 + l_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & l_0 - l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2 Combining to form $\boldsymbol{g}_{WE}$:

$$\boldsymbol{g}_{WE}\left(\frac{\pi}{2}, 0, \frac{\pi}{2}\right) = \begin{bmatrix} 0 & 0 & 1 & l_0 - l_2 \\ 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) How would your expression change if we added another joint (joint 4) between joint 3 and the E frame? (2)

Solution:

1 Addition of a new $e^{\hat{\boldsymbol{\xi}}_1\theta_1}$ ONLY.

1 Correct location:

$$\boldsymbol{g}_{WE}\left(\theta_1, \theta_2, \theta_3\right) = e^{\hat{\boldsymbol{\xi}}_1\theta_1} e^{\hat{\boldsymbol{\xi}}_2\theta_2} e^{\hat{\boldsymbol{\xi}}_3\theta_3} e^{\hat{\boldsymbol{\xi}}_4\theta_4} \boldsymbol{g}_{WE}\left(0\right)$$

(d) How would your expression change if we wanted the transformation $\boldsymbol{g}_{WF}$ where the frame F is attached to the limb segment connecting joint 2 and joint 3? (2)

Solution:

1 Removal of the $e^{\hat{\boldsymbol{\xi}}_3\theta_3}$ term

1 Change $\boldsymbol{g}_{WE}\left(0\right)$ to $\boldsymbol{g}_{WF}\left(0\right)$:

$$\boldsymbol{g}_{WF}\left(\theta_1, \theta_2\right) = e^{\hat{\boldsymbol{\xi}}_1\theta_1} e^{\hat{\boldsymbol{\xi}}_2\theta_2} \boldsymbol{g}_{WF}\left(0\right)$$

(e) Write the twist $\boldsymbol{\xi}_T$ that describes a prismatic joint moving along the positive Y (1) axis.

Solution:

1

$$\boldsymbol{\xi}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

(f) ξ_3 is now modified so that it now takes the form: (2)

$$\boldsymbol{\xi}_3^{'} = \boldsymbol{\xi}_3 + \boldsymbol{\xi}_T$$

Describe the motion of a point that is fixed in the E frame as seen in W frame as θ_3 varies about this new $\boldsymbol{\xi}_3^{'}$.

Solution:

1 Addition of a translational velocity.

1 The point will move in a helical path about a screw with a non-zero pitch.