EECS C106A: Introduction to Robotics

Fall 2018

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Midterm 1 Solutions

Name: ____________________

SID: ____________________

Lab Section/TA: ____________________

You have **90 minutes** to complete this exam.
Please do not flip or open this exam until the timer starts.

This exam consists of **9 questions**.
Tentative point values are given below for each question.

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	18	4	6	4	4	8	4	13	6	67

For full credit, please (a) show all work and (b) write your SID on every page, including the cheat sheet and Appendix.

Tear the Appendix off as soon as you begin, and turn it in separately from your test.
A cheat sheet can be found in the Appendix. No other notes or calculators are allowed.

Question 1: Inverse Kinematics .... 18 points

Consider the robotic manipulator in Figure 1 of the Appendix. You are given the desired configuration of the end effector with respect to frame W as:

$$g_{d,D} = \begin{bmatrix} & & & x_D \\ & \boldsymbol{R}_D & & y_D \\ & & & z_D \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix representing the mapping from initial to desired configuration in frame A can be written in the following form:

$$g_1 = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}$$

(a) Define the transformation g_1 using the initial configuration, $g_D(0)$, and final configuration, $g_{d,D}$, of the end effector. (2)

Solution:

$$g_{d,D} = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}e^{\hat{\xi}_4\theta_4}g_D(0) = g_1 g_D(0)$$
$$g_1 = g_{d,D}g_D^{-1}(0)$$

(b) Assume that there exist points $\boldsymbol{p}_W$, $\boldsymbol{p}_B$, $\boldsymbol{p}_C$, and $\boldsymbol{p}_D$ coinciding with the origins of frames W, B, C, and D respectively. For each of these points, determine whether or not it can be considered an invariant point. If the point can be considered invariant, list the relevant degrees of freedom. (4)

Solution:

$$\begin{aligned} p_W &: \text{invariant, } \theta_1 \\ p_B &: \text{invariant, } \theta_1, \theta_2 \\ p_C &: \text{invariant, } \theta_2, \theta_3 \\ p_D &: \text{invariant, } \theta_3 \end{aligned}$$

(c) Assume that we are given the value of θ_4. Formulate the solution to θ_1 as a Paden-Kahan subproblem. Show the steps taken to reach your solution in addition to definitions for terms like $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, and $\boldsymbol{\delta}$. (6)

Solution:

$$\text{Known} : \theta_4 \Rightarrow g_2 := g_1 e^{-\hat{\xi}_4\theta_4} = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}$$

$$\text{Choose point } p_C, \text{invariant to } \theta_2, \theta_3$$

$$\Rightarrow g_2 p_C = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p_C = e^{\hat{\xi}_1\theta_1} p_C$$

$$p = p_C,\ q = g_2 p_C,\ r = [0, 0, *]^T,\ \text{PK1}$$

Other solutions do exist! Most notably by using PK3.

(d) Formulate a solution to the remaining joints θ_2 and θ_3 using Paden-Kahan subproblems. Show the steps taken to reach your solution in addition to definitions for terms like $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, and $\boldsymbol{\delta}$. (6)

Solution:

$$\text{Known} : \theta_1, \theta_4 \Rightarrow g_3 := e^{-\hat{\xi}_1\theta_1} g_2 = e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3}$$

$$\text{Choose point } p_E,\ \text{NOT on the } \xi_2, \xi_3 \text{ axes}$$

$$\Rightarrow g_3 p_E = e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p_E$$

$$p = p_E,\ q = g_3 p_E,\ r = p_C,\ \text{PK2}$$

Again, other solutions exist.

Question 2: A Rotation Matrix .... 4 points

$$\boldsymbol{R} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\boldsymbol{x} \\ \boldsymbol{x} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

(a) Assume that $\boldsymbol{R}$ is a valid rotation matrix. Solve for $\boldsymbol{x}$: (2)

Solution: There are many ways to solve this, including by inspection. One option is to explicitly solve for solutions of x that satisfy orthonormality:

$$\begin{aligned} I &= TT^T \\ &= \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\boldsymbol{x} \\ \boldsymbol{x} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \boldsymbol{x} \\ -\boldsymbol{x} & -\frac{\sqrt{2}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{x}^2 + \frac{1}{2} & 0 \\ 0 & \boldsymbol{x}^2 + \frac{1}{2} \end{bmatrix} \\ x^2 &= \frac{1}{2} \\ x &= \pm\frac{\sqrt{2}}{2} \end{aligned}$$

Both solutions results in a +1 determinant, and are thus valid.

(b) Suppose $\boldsymbol{R}$ describes a rotation by an angle $\boldsymbol{\theta}$. Solve for $\boldsymbol{\theta}$: (2)

Solution:

$$\theta = \pm\frac{3\pi}{4}$$

Question 3: Transformation Ordering .... 6 points

(a) Select all operations that are always **commutative.** $AB = BA$ (3)

☐ Multiple rotation matrices, about orthogonal axes

√ **Multiple rotation matrices, about parallel axes**

√ **Multiple homogeneous transforms, where all $\boldsymbol{R} = I$**

☐ Multiple homogeneous transforms, where all $\boldsymbol{R} = \boldsymbol{R}_X(\frac{\pi}{4})$

☐ Multiple exponential mappings, with parallel revolute joints

√ **Multiple exponential mappings, with parallel prismatic joints**

(b) Select all operations that are always **associative.** $(AB)C = A(BC)$ (3)

√ **Multiple rotation matrices, about orthogonal axes**

√ **Multiple rotation matrices, about parallel axes**

√ **Multiple homogeneous transforms, where all $\boldsymbol{R} = I$**

$\surd$ **Multiple homogeneous transforms, where all** $\boldsymbol{R} = \boldsymbol{R}_X(\frac{\pi}{4})$
$\surd$ **Multiple exponential mappings, with parallel revolute joints**
$\surd$ **Multiple exponential mappings, with parallel prismatic joints**

Question 4: Another Rotation Matrix .... 4 points

$$\boldsymbol{T} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) What conditions must hold for $\boldsymbol{T}$ to be a valid rotation matrix? (2)

Solution:

$$\det(\mathbf{T}) = +1$$
$$\mathbf{T^{-1}} = \mathbf{T}^T$$

(b) Assume $\boldsymbol{T}$ is a valid rotation matrix. If we wanted to compute $\boldsymbol{T}$ using Rodrigues' Formula, what could $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$ be? (2)

Solution:

$$\omega = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \theta = -90^\circ = -\frac{\pi}{2}$$

or

$$\omega = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \theta = 90^\circ = \frac{\pi}{2}$$

Question 5: Multiple Rotation Matrices .... 4 points

Given the rotation matrices $\boldsymbol{R}_{GH}, \boldsymbol{R}_{LJ}, \boldsymbol{R}_{GJ}$, and $\boldsymbol{R}_{LK}$.

(a) Write an expression for $\boldsymbol{R}_{JH}$. (2)

Solution:

$$\boldsymbol{R}_{JH} = \boldsymbol{R}_{GJ}^{-1}\boldsymbol{R}_{GH} = \boldsymbol{R}_{GJ}^{T}\boldsymbol{R}_{GH}$$

(b) Write an expression for $\boldsymbol{R}_{KH}$. (2)

Solution:

$$\boldsymbol{R}_{KH} = \boldsymbol{R}_{LK}^{-1}\boldsymbol{R}_{LJ}\boldsymbol{R}_{GJ}^{-1}\boldsymbol{R}_{GH} = \boldsymbol{R}_{LK}^{T}\boldsymbol{R}_{LJ}\boldsymbol{R}_{GJ}^{T}\boldsymbol{R}_{GH}$$

Question 6: Homogeneous Transforms . . . . 8 points

Consider independent frames **W**, **A**, and **B**, which are initially overlapping.
The following events occur, in order:

1. Frame **A** is translated along **W**'s $+Z$ axis by 5
2. Frame **A** is rotated about its own $+X$ axis by $+\frac{\pi}{2}$
3. Frame **B** is rotated about **A**'s $+X$ axis by $-\frac{\pi}{2}$
4. Frame **B** is translated along **W**'s $+X$ axis by -1

(a) Write the rigid body transform $\boldsymbol{g}_{WA}$ in homogeneous form. (2)

Solution: **A** is rotated by $\frac{\pi}{2}$ about the X axis with respect to **W**, and its origin in frame **W** is $\begin{bmatrix} 0 & 0 & 5 \end{bmatrix}^T$.

$$\boldsymbol{g}_{WA} = \begin{bmatrix} & & & 0 \\ & R_X(\frac{\pi}{2}) & & 0 \\ & & & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Write the rigid body transform $\boldsymbol{g}_{WB}$ in homogeneous form. (2)

Solution: **B** is rotated by $-\frac{\pi}{2}$ about the X axis with respect to **W**, and its origin in frame **W** is $\begin{bmatrix} -1 & -5 & 5 \end{bmatrix}^T$.

$$\boldsymbol{g}_{WB} = \begin{bmatrix} & & & -1 \\ & R_X(-\frac{\pi}{2}) & & -5 \\ & & & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Write the rigid body transform $\boldsymbol{g}_{BW}$ in homogeneous form. (2)

Solution: **W** is rotated by $\frac{\pi}{2}$ about the X axis with respect to **B**, and its origin in frame **B** is $\begin{bmatrix} 1 & 5 & 5 \end{bmatrix}^T$.

$$\boldsymbol{g}_{BW} = \begin{bmatrix} & & & 1 \\ & R_X(\frac{\pi}{2}) & & 5 \\ & & & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d) What are $\boldsymbol{R}_{BA}$ and $\boldsymbol{p}_{BA}$? (2)

Solution: $\boldsymbol{g}_{BA}$ can either be computed directly by inspection, or as a product of our earlier answers:

$$\begin{aligned}
\boldsymbol{g}_{BA} &= \boldsymbol{g}_{BW}\boldsymbol{g}_{WA} \\
&= \begin{bmatrix} & & & 1 \\ & R_X(\frac{\pi}{2}) & & 5 \\ & & & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & & 0 \\ & R_X(\frac{\pi}{2}) & & 0 \\ & & & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} & & & 1 \\ & R_X(\pi) & & 0 \\ & & & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Question 7: Velocities .... 4 points

(a) Consider an axis in 3D space parallel to a vector $\boldsymbol{\omega}$. We pick a point $\boldsymbol{p}$ that lies on the axis, and an arbitrary point $\boldsymbol{q}$. (2)
If $\boldsymbol{q}$ is rotated about the axis at an angular velocity of $+\pi$ rad/ sec, what is the instantaneous linear velocity of $\boldsymbol{q}$? Express in terms of $\boldsymbol{\omega}$, $\boldsymbol{p}$, and $\boldsymbol{q}$.

Solution: Use the fact that linear velocities can be computed with a cross product, and scale it by $+\pi$ rad/ sec. Note that $\boldsymbol{\omega}$ must be normalized.

$$\boldsymbol{v} = \pi * \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} \times (\boldsymbol{q} - \boldsymbol{p})$$

(b) Consider a screw motion, which consists of the following two operations occurring simultaneously: (2)

1. A rotation about the unit axis $\boldsymbol{\omega}$ by $\boldsymbol{\theta}$
2. A translation about the same axis $\boldsymbol{\omega}$ by $\boldsymbol{\theta}$

Let $\boldsymbol{q}$ be a point on the axis $\boldsymbol{\omega}$.

What twist $\boldsymbol{\xi}$ describes this motion, in terms of $\boldsymbol{\omega}$, $\boldsymbol{\theta}$, and $\boldsymbol{q}$?

Solution: This is equivalent to the sum of the linear and angular twists.

$$\xi_{\textbf{rot}} = \begin{bmatrix} -\boldsymbol{\omega} \times \boldsymbol{q} \\ \boldsymbol{q} \end{bmatrix} \qquad \xi_{\textbf{trans}} = \begin{bmatrix} \omega \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xi = \xi_{\textbf{trans}} + \xi_{\textbf{rot}} = \begin{bmatrix} -\boldsymbol{\omega} \times \boldsymbol{q} + \boldsymbol{\omega} \\ \boldsymbol{\omega} \end{bmatrix}$$

Question 8: Forward Kinematics .... 13 points

Consider the robotics manipulator in Figure 1 of the Appendix.

(a) What are $\hat{\boldsymbol{\xi}}_2$ and $\hat{\boldsymbol{\xi}}_4$? (2)

Solution:

$$
\begin{aligned}
\boldsymbol{\omega}_2 &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \\
\boldsymbol{q}_2 &= \begin{bmatrix} 0 & * & 10 \end{bmatrix} \\
\boldsymbol{\xi}_2 &= \begin{bmatrix} -\boldsymbol{\omega}_2 \times \boldsymbol{q}_2 \\ \boldsymbol{\omega}_2 \end{bmatrix} \\
&= \begin{bmatrix} -10 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \\
\hat{\boldsymbol{\xi}}_2 &= \begin{bmatrix} 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\xi}_4 &= \begin{bmatrix} \boldsymbol{v} \\ \vec{0} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \\
\hat{\boldsymbol{\xi}}_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

(b) What is the robot's initial configuration, $\boldsymbol{g}_{WD}(0)$? (2)

Solution:

$$
\boldsymbol{g}_{WD}(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

(c) What is the robot's final configuration after its joints are moved to $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, $\boldsymbol{\theta}_3$, and $\boldsymbol{\theta}_4$? Leave unsimplified, in terms of the angles, twists, and $\boldsymbol{g}_{WD}(0)$. (1)

Solution:

$$\boldsymbol{g}_{WD}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} e^{\hat{\xi}_4 \theta_4} \boldsymbol{g}_{WD}(0)$$

(d) The robot was programmed with ROS. You want to run a single command in your terminal to print out the current joint positions, which are being published to the **`robot/joint_states`** topic. What command should you run? (2)

Solution: `rostopic echo /joint_states`

(e) You run the command, and see that: (6)

$$\boldsymbol{\theta}_1 = \frac{\pi}{2}$$
$$\boldsymbol{\theta}_2 = 0$$
$$\boldsymbol{\theta}_3 = 0$$
$$\boldsymbol{\theta}_4 = 2$$

What is $\boldsymbol{g}_{WD}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4)$? Solve **using exponential mappings**, and provide a numerical answer. Show all work for full credit.

Solution:

$$\begin{aligned}
\boldsymbol{g}_{WD}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3, \boldsymbol{\theta}_4) &= e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} e^{\hat{\xi}_4 \theta_4} \boldsymbol{g}_{WD}(0) \\
&= e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_4 \theta_4} \boldsymbol{g}_{WD}(0) \\
&= \begin{bmatrix} & R_Z(\frac{\pi}{2}) & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & I_3 & & 0 \\ & & & 0 \\ & & & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} & R_Z(\frac{\pi}{2}) & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} & R_Z(\frac{\pi}{2}) & & -12 \\ & & & 0 \\ & & & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Question 9: Programming with ROS .... 6 points

(a) Name two mechanics for communicating between ROS nodes. (3)
When should we use each one?

Solution:

1. Topics: continuous streams of data, many-to-one
2. Services: request-response communication, one-to-one

or

1. Publishers: for broadcasting data to ROS topics
2. Subscribers/Listeners: for receiving data from ROS topics

(b) Describe at least one function of the `package.xml` file. (1)

Solution: Possible answers include:

- Declaring package dependencies
- Declaring build dependencies
- Declaring runtime dependencies
- Declaring package metadata (name, description, author, etc)

(c) Consider the following Python script. It tries to print Baxter's joint states, which are published to the **`robot/joint_states`** topic: (1)

```python
#!/usr/bin/env python

import rospy
from std_msgs.msg import String

def callback(data):
    print data

if __name__ == "__main__":
    rospy.init_node("fk_map", anonymous=True)

    rospy.Subscriber("robot/joint_states", String, callback)
    rospy.spin()
```

Assume the code is syntactically correct. Will it print the published data? Explain.

Solution: No. `String` is the wrong message type for the `robot/joint_states` topic.

(d) Consider the following code snippet from Lab 3. The script is given a list of desired joint angles and commands Baxter to move to them. (1)

```
Done = False
while not Done:
    # Send a list of target joint angles to the right arm
    set_joint_angles(right, desired_angles)
    # Get the current joint angles from the right arm
    measured_angles = get_joint_angles(right)

    Done = True
    for i in range(0,len(desired_angles)):
        if desired_angles[i] != measured_angles[i]:
            Done = False
            break
```

Assume that the code has no syntactical errors and runs without throwing any exceptions. All called functions operate correctly.
Do you expect the code to terminate in normal operation? Explain.

Solution: No, it will not. Due to imperfect physical systems, `measured_angles` will never exactly equal the desired angles.

Note: this isn't a floating point precision problem. It refers to the fact that real-world systems will have noise and error, and thus the controller will never converge exactly.

1 Appendix: 4 DoF Manipulator

Consider the four degree of freedom robotic manipulator shown in Figure 1. The manipulator has three revolute joints and one prismatic joint.

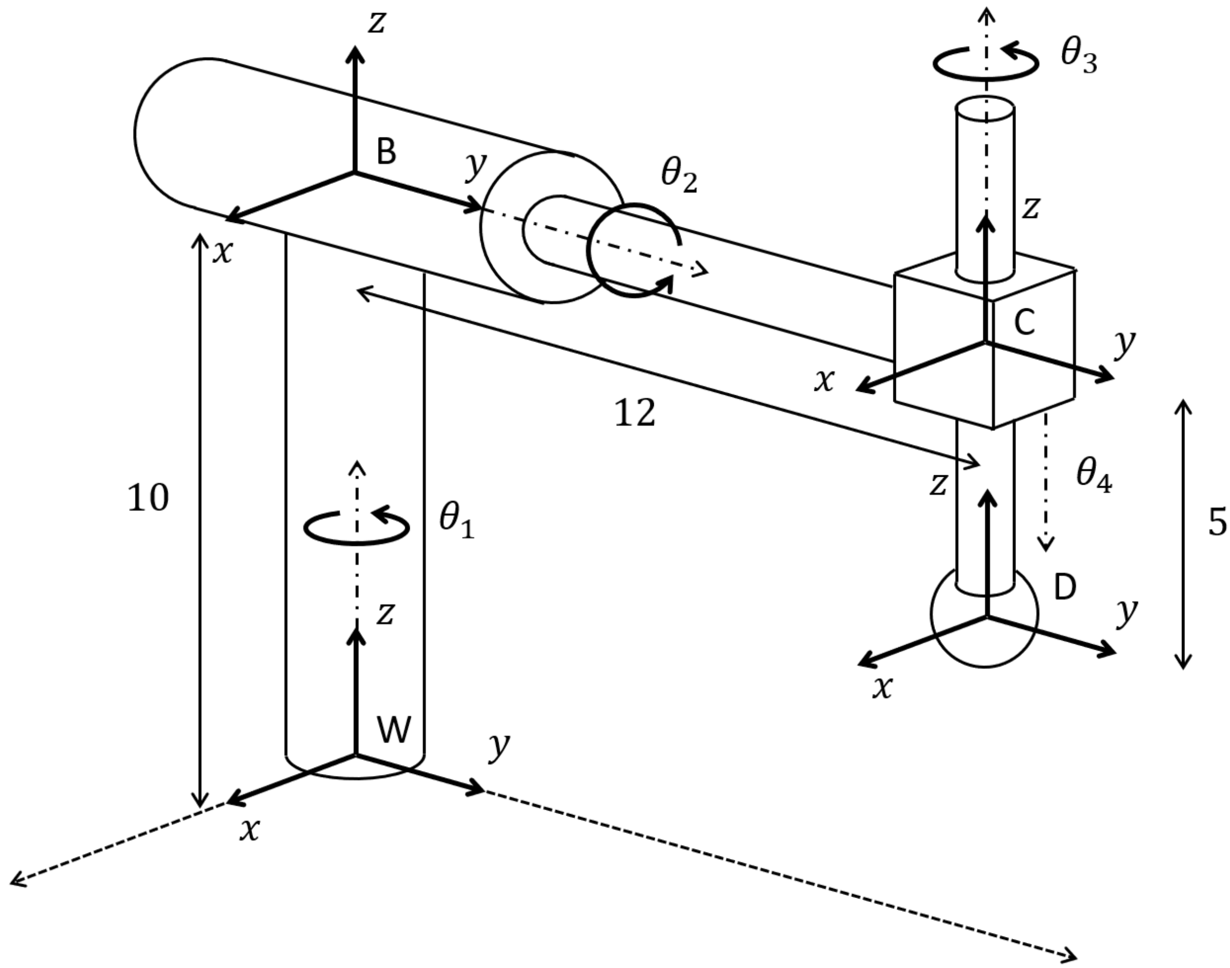

Figure 1: 4-DOF Robotic Manipulator

2 Appendix: Cheat Sheet

2.1 Trigonometry

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Pythagoras's theorem $h^2 = x^2 + y^2$ for a right angled triangle where h is the hypotenuse and x and y are the lengths of the two remaining sides.

Sine, Cosine Relation $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$

Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos(\theta_C)$ where a, b, c are the lengths of the triangle and θ_A, θ_B and θ_C are the angles of their opposing corner.

2.2 Linear Algebra

For orthogonal matrices $A^{-1} = A^T$

Orthogonality A matrix $\begin{bmatrix}\boldsymbol{v}_1, ..., \boldsymbol{v}_n\end{bmatrix}$ is said to be orthogonal if:

$$\boldsymbol{v}_i^T \boldsymbol{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

2.3 Special Operators

Hat

$$\hat{\boldsymbol{\omega}} = \widehat{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Wedge

$$\hat{\boldsymbol{\xi}} = \widehat{\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix}$$

2.4 Rotations

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{x}}\theta}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{y}}\theta}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{\hat{\boldsymbol{z}}\theta}$$

2.5 Rodrigues' Formula

$$R(\boldsymbol{\omega}, \theta) = e^{\hat{\boldsymbol{\omega}}\theta} = \mathbb{I}_3 + \frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|} \sin\left(\|\boldsymbol{\omega}\| \theta\right) + \frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2} \left(1 - \cos\left(\|\boldsymbol{\omega}\| \theta\right)\right)$$

2.6 Rigid Body Motion

$$\boldsymbol{g}_{AB} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \mathbf{0} & 1 \end{bmatrix} \qquad \boldsymbol{g}_{AB}^{-1} = \begin{bmatrix} \boldsymbol{R}_{AB}^{-1} & -\boldsymbol{R}_{AB}^{-1}\boldsymbol{p}_{AB} \\ \mathbf{0} & 1 \end{bmatrix}$$

2.7 Exponential Notation

$$\boldsymbol{g}_{AB}\left(\theta_1\right) = e^{\hat{\xi}_1\theta_1}\boldsymbol{g}_{AB}\left(0\right)$$
$$\boldsymbol{g}_{ST}\left(\theta_1, \ldots, \theta_n\right) = e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_n\theta_n}\boldsymbol{g}_{ST}\left(0\right)$$

2.7.1 Special Cases

Pure Rotation

$$\boldsymbol{\xi} = \begin{bmatrix} -\boldsymbol{\omega} \times \boldsymbol{q} \\ \boldsymbol{\omega} \end{bmatrix}$$

Pure Translation

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} \\ \mathbf{0} \end{bmatrix}$$

Pure Rotations, Screws (Rotation and Translation)

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} e^{\hat{\boldsymbol{\omega}}\theta} & \left(\mathbb{I}_3 - e^{\hat{\boldsymbol{\omega}}\theta}\right)\left(\boldsymbol{\omega} \times \boldsymbol{v}\right) + \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}$$

Pure Translation

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} \mathbb{I}_3 & \boldsymbol{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}$$

2.8 Paden-Kahan

Subproblem 1: Rotation about a single axis

$$e^{\widehat{\xi}\theta}p = q$$

Subproblem 2: Rotation about two subsequent axes

$$e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}p = q$$

Subproblem 3: Rotation to a distance

$$\left\| e^{\widehat{\xi}\theta}p - q \right\| = \delta$$