# Introduction to Robotics E125
# Midterm 1, 2014 October 07 SID: ____________

Calculators allowed, though use should be restricted to basic functions such as trigonometry and simple mathematical operations.

Please show all working. Marks are awarded for method.

A cheat sheet is provided. No other notes are allowed.

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Question 1: Rotation Matrices . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8 points

Consider two coordinate frames $A$ and $B$. Coordinate frame $B$ begins aligned with frame $A$. Frame $B$ is then rotated by $\frac{\pi}{3}$ radians about the $X$ axis of the $A$ coordinate frame.

(a) Write the rotation matrix $\boldsymbol{R}_{AB}$ for these two coordinate frames. (2)

(b) Write the rotation matrix $\boldsymbol{R}_{BA}$ for these two coordinate frames. (2)

(c) How would a vector $\boldsymbol{p}_A = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$ written in the $A$ coordinate frame be written in the $B$ coordinate frame? (2)

(d) How would a vector $\boldsymbol{p}_B = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$ written in the $B$ coordinate frame be written in the $A$ coordinate frame? (2)

Question 2: Rodrigues' Formula . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6 points

Show that you can obtain the standard $\boldsymbol{R}_y$ rotation matrix from the Rodrigues' Formula.

Question 3: Multiple Rotations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 2 points

(a) Given the rotation matrices $\boldsymbol{R}_{AB}, \boldsymbol{R}_{BC}, \boldsymbol{R}_{CD}$, write and expression for $\boldsymbol{R}_{AD}$. (1)

(b) Given the rotation matrices $\boldsymbol{R}_{BC}, \boldsymbol{R}_{CD}, \boldsymbol{R}_{AD}$, write and expression for $\boldsymbol{R}_{AB}$. (1)

Question 4: Valid Rotations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 points

Is the transformation matrix $\boldsymbol{T}$ shown below a valid rotation matrix? If so, prove it, if not say why.

$$\boldsymbol{T} = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}$$

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Question 5: Rigid Body Motion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20 points

Consider a robotics manipulator with three joints. We represent the rotation of these joints as three rigid body motions:

$\boldsymbol{g}_{AB}$ Frame $B$ starts aligned with frame $A$. Frame $B$ is then translated so that its origin is at $\begin{bmatrix} l_0 & 0 & 0 \end{bmatrix}^T$ as seen in the $A$ frame. Frame $B$ is then rotated by $\frac{\pi}{2}$ radians about the $X$ axis of the $A$ frame.

$\boldsymbol{g}_{BC}$ Frame $C$ starts aligned with frame $B$. Frame $C$ is then translated so that its origin is at $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ as seen in the $B$ frame. Frame $C$ is then rotated by 0 radians about the $Y$ axis of the $B$ frame.

$\boldsymbol{g}_{CD}$ Frame $D$ starts aligned with frame $C$. Frame $D$ is then translated so that its origin is at $\begin{bmatrix} 0 & -l_1 & 0 \end{bmatrix}^T$ as seen in the $C$ frame. Frame $D$ is then rotated by $\frac{\pi}{2}$ radians about the $Y$ axis of the $C$ frame.

(a) Write the rigid body transform $\boldsymbol{g}_{AB}$ in homogeneous form. (4)

(b) Write the rigid body transform $\boldsymbol{g}_{BC}$ in homogeneous form. (4)

(c) Write the rigid body transform $\boldsymbol{g}_{CD}$ in homogeneous form. (4)

(d) Write the rigid body transform $\boldsymbol{g}_{AD}$ in homogeneous form. (2)

(e) How would a point $\boldsymbol{p}$ with the coordinates $\boldsymbol{p}_D = \begin{bmatrix} 0 & 0 & -l_2 \end{bmatrix}^T$ in the $D$ frame be represented in the $A$ frame? (2)

(f) How would you alter your expression for $\boldsymbol{g}_{AD}$ if we added another joint $F$ between the $C$ and $D$ frames? Explain your answer and any additional information required. (2)

(g) How would you alter your expression for $\boldsymbol{g}_{AD}$ if we added a frame $F$ between the $C$ and $D$ frames, where $F$ is a frame that is fixed to frame $C$? Explain your answer and any additional information required. (2)

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Question 6: Forward Kinematics . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 30 points

In this question we will be computing the Forward Kinematics of the 3 Degree of Freedom manipulator shown in Figure 1.

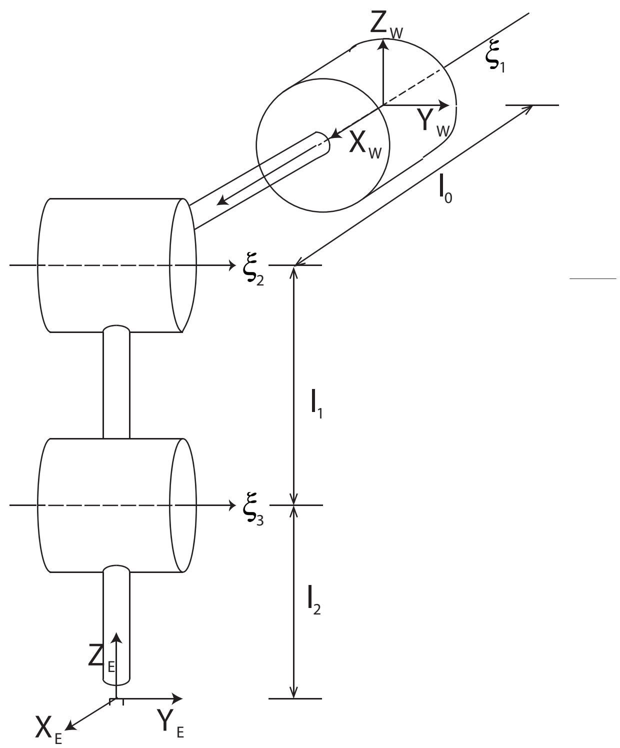


Figure 1: Schematic of a 3DoF Manipulator with axes of rotation shown in its initial configuration.

(a) Write the twists that represent each joint of the manipulator. (8)

(b) Write an expression for the homogeneous transformation $\boldsymbol{g}_{WE}$ when $\theta_1 = \frac{\pi}{2}$, $\theta_2 = 0$ and $\theta_3 = \frac{\pi}{2}$. (15)

(c) How would your expression change if we added another joint (joint 4) between joint 3 and the $E$ frame? (2)

(d) How would your expression change if we wanted the transformation $\boldsymbol{g}_{WF}$ where the frame $F$ is attached to the limb segment connecting joint 2 and joint 3? (2)

(e) Write the twist $\boldsymbol{\xi}_T$ that describes a prismatic joint moving along the positive Y axis. (1)

(f) $\xi_3$ is now modified so that it now takes the form: (2)

$$\boldsymbol{\xi}_3^{'} = \boldsymbol{\xi}_3 + \boldsymbol{\xi}_T$$

Describe the motion of a point that is fixed in the $E$ frame as seen in $W$ frame as $\theta_3$ varies about this new $\boldsymbol{\xi}_3^{'}$.

Question 3: Inverse Kinematics . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20 points

Consider the six DoF manipulator shown in figure 4. Joints 1, 2, 4, 5, 6 are revolute, and joint 3 is prismatic.

We wish to compute the Inverse Kinematics for this manipulator.

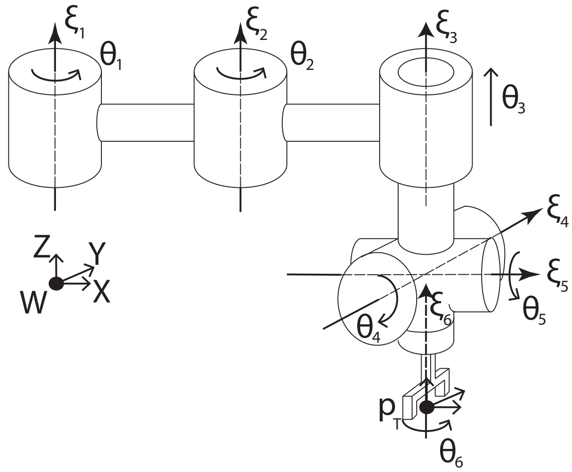


Figure 4: Six DoF manipulator with joints shown

(a) In words, describe how joints 1-6 affect the configuration of the tool frame. Break the manipulator down into a translational and a rotational components. (2)

(b) Write an expression for the forward Kinematics of the manipulator in the product of exponentials form. You expression should relate an initial configuration $g_{WT}(0)$, with the configuration after applying a set of twists: $g_{WT}(\theta_1, \ldots, \theta_6)$. (1)

(c) Use this product of exponentials form to find a matrix $g_1$ where: (1)

$$g_1 = e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}e^{\widehat{\xi}_3\theta_3}e^{\widehat{\xi}_4\theta_4}e^{\widehat{\xi}_5\theta_5}e^{\widehat{\xi}_6\theta_6}$$

and $g_1$ relates some desired configuration $g_{DT}$ to the initial configuration.

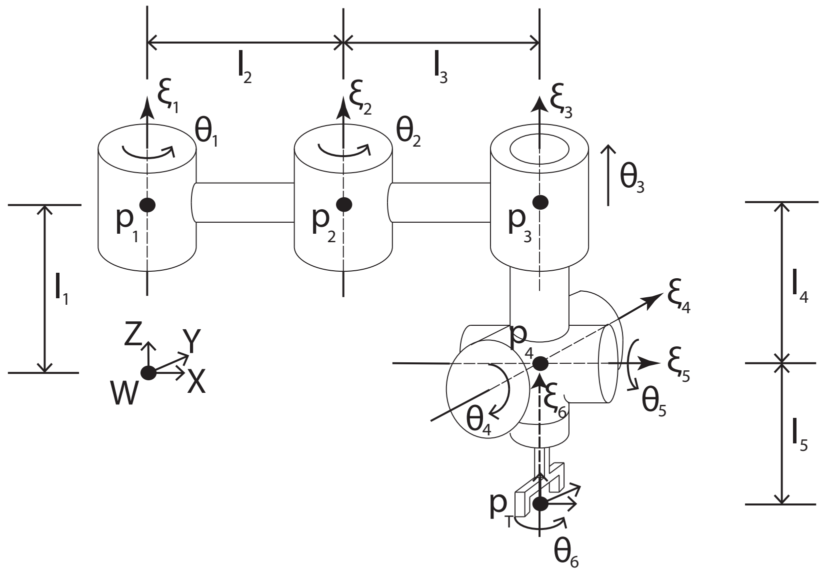


Figure 5: Six DoF manipulator with joints and key points shown. Point 3 is rigidly attached to joint 2 at a distance $l_3$ and does not move with $\theta_3$.

(d) Using the joint labelling and points shown in Figure 5, we wish to compute the Inverse Kinematics of this manipulator. Assume we are given a desired tool configuration of: (2)

$$g_{DT} = \begin{bmatrix} R_{DT} & p_{DT} \\ 0 & 1 \end{bmatrix}$$

Using the fact that joints 4,5,6 rotate the tool frame about point $p_4$ from an initial position $\begin{bmatrix} 0 & 0 & -l_5 \end{bmatrix}^T$ to our desired point $p_{DT}$, show that the desired position of point 4 can be written:

$$p_{D4} = p_{DT} - R_{DT} \begin{bmatrix} 0 \\ 0 \\ -l_5 \end{bmatrix}$$

(e) Given that we now know the desired position $p_{D4}$, find the desired position of a point $p_3$. Point $p_3$ is rigidly attached a distance of $l_3$ from joint 2 and does not move with prismatic joint 3. Write this desired point $p_{D3}$ in terms of $p_{D4}$ entries and manipulator lengths. (1)

(f) We now have expressions for the desired points $p_{D3}$ and $p_{D4}$. How can we solve for $\theta_3$ given these two points. (Note, you do not have to use a Paden-Kahan subproblem to solve for this joint.) (1)

(g) Looking at the first two joints of the manipulator we can write the expression: (1)

$$e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}p_3 = p_{D3}$$

where $p_3$ is the initial position of point 3 and $p_{D3}$ is the desired end point. Why can we say that the position of the point $p_3$ only varies with joints 1,2?

(h) Use this expression for $e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}$ to find $\theta_1$ and $\theta_2$ using Paden-Kahan subproblems. State clearly what Paden Kahan subproblems you are using and why you are using particular points on the manipulator. You do not have to solve these expressions to find the $\theta$ terms explicitly, however you do need to get write what the Paden-Kahan vectors $p$ and $q$ would be. (5)

(i) We now have values for $\theta_1$, $\theta_2$ and $\theta_3$. Consider the matrix $g_3$ of form: (1)

$$g_3 = e^{\widehat{\xi}_4\theta_4}e^{\widehat{\xi}_5\theta_5}e^{\widehat{\xi}_6\theta_6}$$

Write an expression for matrix $g_3$ in terms of $g_1$, and the exponential representations of joints 1-3.

(j) Use this matrix $g_3$ to find $\theta_4$, $\theta_5$ and $\theta_6$ using Paden-Kahan subproblems. State clearly what Paden Kahan subproblems you are using and why you are using particular points on the manipulator. You do not have to solve these expressions to find the $\theta$ terms explicitly, however you do need to get write what the Paden-Kahan vectors $p$ and $q$ would be. (5)