Solutions for HW1: Rotations

EE106A/206A Fall 2018

Due: Thursday, September 6, 2018 at 11:59 PM on Gradescope

1 Multiple Rotation Matrices

a) You are given the rotation matrices: $\boldsymbol{R}_{AB}$, $\boldsymbol{R}_{CB}$. Write and expression for $\boldsymbol{R}_{CA}$. $\boxed{\boldsymbol{R}_{CA} = \boldsymbol{R}_{CB}\boldsymbol{R}_{AB}^T}$

b) You are given the rotation matrices: $\boldsymbol{R}_{AB}$, $\boldsymbol{R}_{CA}$. Write and expression for $\boldsymbol{R}_{BC}$. $\boxed{\boldsymbol{R}_{BC} = \boldsymbol{R}_{AB}^T\boldsymbol{R}_{CA}^T}$

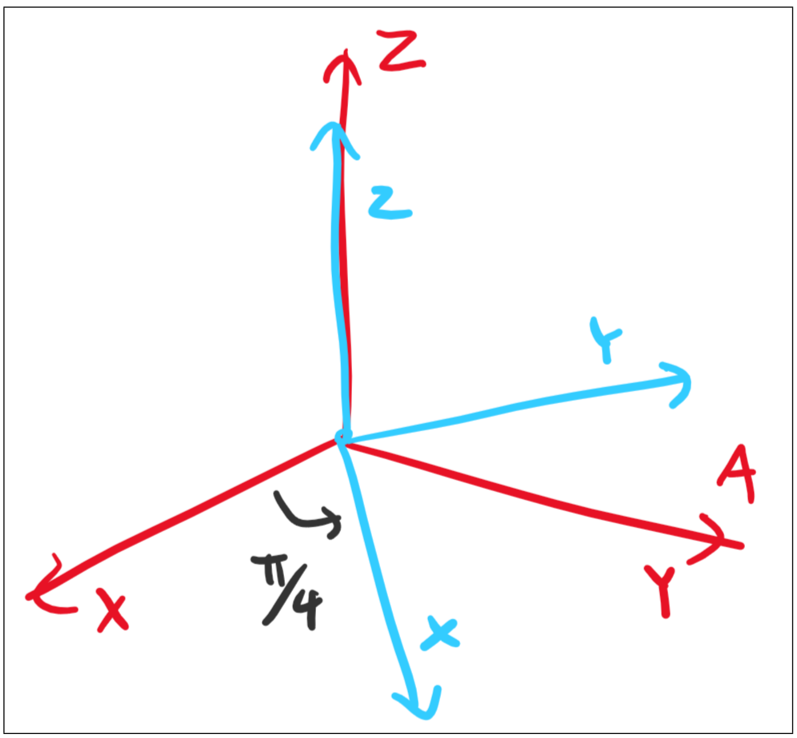
c) You are given the rotation matrices: $\boldsymbol{R}_{AB}$, $\boldsymbol{R}_{BC}$. Write and expression for $\boldsymbol{R}_{AA}$. $\boxed{\boldsymbol{R}_{AA} = \boldsymbol{R}_{AB}\boldsymbol{R}_{AB}^T}$

d) You are given the rotation matrices: $\boldsymbol{R}_{AB}^{-1}$, $\boldsymbol{R}_{BC}^T$. Write and expression for $\boldsymbol{R}_{AC}$. $\boxed{\boldsymbol{R}_{AC} = \left(\boldsymbol{R}_{AB}^{-1}\right)^T \left(\boldsymbol{R}_{BC}^T\right)^T}$

2 Euler Angles

Consider two initially overlapping frames, A and B. Frame B is then rotated about the Z axis by $\pi/4$ radians.

a) Sketch the coordinate frames A and B after the rotation.



b) Write the rotation matrix $\boldsymbol{R}_{AB}$ that will take a point from the B frame and represent it in the A frame.

$$\boxed{\boldsymbol{R}_{AB} = \begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) & 0 \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

c) Write the rotation matrix $\boldsymbol{R}_{BA}$.

$$\boxed{\boldsymbol{R}_{BA} = \boldsymbol{R}_{AB}^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

d) What are the coordinates in frame A of a point with coordinates $\boldsymbol{p}_B = [0,0,1]^T$ given with respect to frame B?

$$\boxed{\boldsymbol{p}_A = \boldsymbol{R}_{AB}\,[0,0,1]^T = [0,0,1]^T}$$

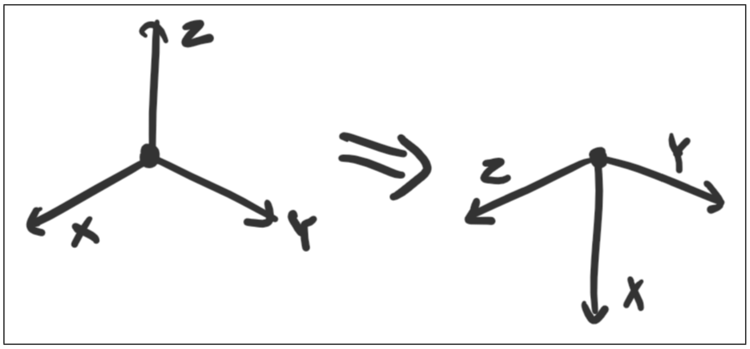
e) What are the coordinates in frame B of a point with coordinates $\boldsymbol{p}_A = [0,0,1]^T$ given with respect to frame A?

$$\boxed{\boldsymbol{p}_B = \boldsymbol{R}_{BA}\,[0,0,1]^T = [0,0,1]^T}$$

3 Multiple Euler Angles

A frame is rotated first about the Z axis by angle $\frac{\pi}{2}$, then about the mobile Y axis by an angle of $\frac{\pi}{2}$, then about the mobile X axis an angle of $\frac{\pi}{2}$.

a) Draw the frame before and after the rotation. Label all axes.

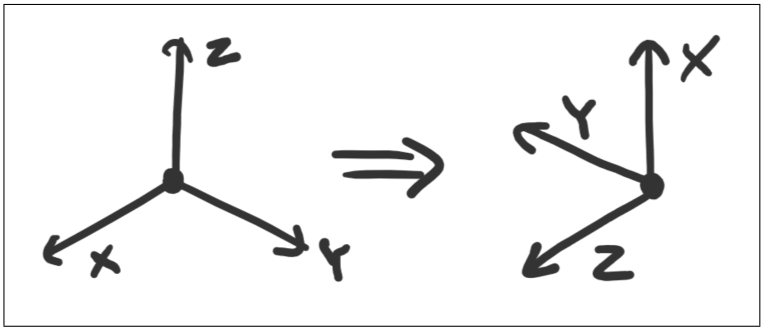


b) Write the net rotation matrix.

$$\begin{aligned}
\boldsymbol{R} &= \boldsymbol{R}_X \boldsymbol{R}_Y \boldsymbol{R}_Z \\
&= \begin{bmatrix} \cos\theta_Z & -\sin\theta_Z & 0 \\ \sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_Y & 0 & \sin\theta_Y \\ 0 & 1 & 0 \\ -\sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & -\sin\theta_X \\ 0 & \sin\theta_X & \cos\theta_X \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}
\end{aligned}$$

A frame is rotated first about the Z axis by angle $\frac{\pi}{2}$, then about the original Y axis by an angle of $\frac{\pi}{2}$, then about the original X axis an angle of $\frac{\pi}{2}$.

c) Draw the frame before and after the rotation. Label all axes.



d) Write the net rotation matrix.

$$
\begin{aligned}
\boldsymbol{R} &= \boldsymbol{R}_X \boldsymbol{R}_Y \boldsymbol{R}_Z \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & -\sin\theta_X \\ 0 & \sin\theta_X & \cos\theta_X \end{bmatrix} \begin{bmatrix} \cos\theta_Y & 0 & \sin\theta_Y \\ 0 & 1 & 0 \\ -\sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix} \begin{bmatrix} \cos\theta_Z & -\sin\theta_Z & 0 \\ \sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

4 Properties of Rotations

State whether each transformation matrix below is a valid rotation. Justify.

a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\det \boldsymbol{R} = ad - bc = -1$
Not a valid rotation matrix.

b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\det \boldsymbol{R} = ad - bc = 1$
$\boldsymbol{R}^T \boldsymbol{R} = I$

Valid rotation matrix.

c) $\begin{bmatrix} \frac{1}{2} & \sqrt{2} \\ -\sqrt{2} & 0 \end{bmatrix}$

$\det \boldsymbol{R} = ad - bc = 2$

Not a valid rotation matrix.

5 Axis Angle Notation

a) Use the Rodrigues formula to show that a rotation of θ radians about the Y axis results in the same rotation matrix as the Euler Y equation.

$$\boldsymbol{R_Y} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\begin{aligned}
\omega &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
\boldsymbol{R_{\omega,\theta}} &= I + \hat{w}\sin\theta + \hat{w}^2(1-\cos\theta) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + (1-\cos\theta) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & 0 & \sin\theta \\ 0 & 0 & 0 \\ -\sin\theta & 0 & 0 \end{bmatrix} + (1-\cos\theta) \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \\
&= \boldsymbol{R}_Y
\end{aligned}$$

b) Use the Rodrigues formula to find the rotation matrix for a rotation of $\frac{\pi}{4}$ about the axis given by the vector $[1, 2, 3]$.

Using definition from class (only theta inside cosine and sine):

$$R = \begin{bmatrix} 0.72802773 & -0.52510482 & 0.44072731 \\ 0.6087886 & 0.79079056 & -0.06345657 \\ -0.31520164 & 0.3145079 & 0.89539528 \end{bmatrix}$$

Using definition from textbook (omega inside cosine and sine):

$$R = \begin{bmatrix} 0.8380941 & 0.1212143 & 0.5318885 \\ 0.44435312 & -0.41391854 & 0.79449465 \\ 0.31646262 & 0.90220759 & 0.29304073 \end{bmatrix}$$