HW2: Forward Kinematics

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Homework due: Thursday September 13, 2018 at 11:59 PM on GradeScope

Feel free to use a computer to help you with this problem set. If you do write any code, to help you solve the problem, attach the code at the end of your problem set. If you use any pre-made code (such as MATLAB's expm, state that you use it as a step in your solution.

1. **Rigid Bodies and Homogeneous Coordinates** (6)

Consider a system with three joints (one revolute joint and two prismatic joints) and five defined coordinate frames A, B, C, D, and E.

1.1. Coordinate frames A and B are on either side of a *revolute* (rotational) joint. The origin of Frame B is located at $[0, 0, 4]^T$ as given in the A frame. The joint rotates about an axis in the X direction which passes through the origin and X-axis of the Frame B. Write the rigid body transform $g_{AB}(\theta_1)$ in homogeneous coordinates.

1.2. Coordinate frame C is rigidly attached to coordinate frame B. The relative rotation R_{BC} is a rotation about the Y-axis by $\pi/2$ radians about the point with coordinates $[1, 0, 2]^T$ as given in the B frame. Write the rigid body transform g_{BC} in homogeneous coordinates.

1.3. Write an expression for the rigid body transform $g_{AC}(\theta_1)$.

1.4. *Prismatic* (translational) joints have the homogeneous form:

$$g(\theta_2) = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{T}(\theta_2) \\ \mathbb{O} & 1 \end{bmatrix} \tag{1}$$

Using Equation 1 as a reference, describe the effect the rigid body motion g_{CD} has on points given in the D frame with respect to the C frame.

$$g_{CD} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} & \begin{bmatrix} 0 \\ \theta_2 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

1.5. Coordinate frames E and D are on either side of a prismatic joint. There is no relative rotation, but there is a translation along the$X$-axis parameterized by the variable θ_3. Using Equation 1 as a reference, write the rigid body transform g_{DE} in homogeneous notation.

1.6. Show that:

$$g_{AE}(\theta_1, \theta_2, \theta_3) = \begin{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

when $[\theta_1, \theta_2, \theta_3] = [\pi/2, 4, 3]$

2. **Exponential Rigid Body Motion** (14)

General twists $\boldsymbol{\xi}$ have the form:

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} \tag{2}$$

where $\boldsymbol{v}$ and $\boldsymbol{\omega}$ are the linear and rotational velocity vectors.

The two special cases for writing twist $\boldsymbol{\xi}$ are:

$$\begin{aligned} \boldsymbol{\xi} &= \begin{bmatrix} -\boldsymbol{\omega} \times \boldsymbol{q} \\ \boldsymbol{\omega} \end{bmatrix} \quad for \quad pure\; revolute\; joints \\ \boldsymbol{\xi} &= \begin{bmatrix} \boldsymbol{v} \\ \mathbb{O} \end{bmatrix} \quad\quad for \quad pure\; prismatic\; joints \end{aligned} \tag{3}$$

The exponential map $e^{\hat{\xi}\theta}$ has the analytical forms:

$$\begin{aligned} e^{\hat{\boldsymbol{\xi}}\theta} &= \begin{bmatrix} \mathbb{I} & \boldsymbol{v}\theta \\ \mathbb{O} & 1 \end{bmatrix} \quad\quad for \quad \boldsymbol{\omega} = 0 \\ e^{\hat{\boldsymbol{\xi}}\theta} &= \begin{bmatrix} e^{\hat{\boldsymbol{\omega}}\theta} & \left(\mathbb{I} - e^{\hat{\boldsymbol{\omega}}\theta}\right)(\boldsymbol{\omega} \times \boldsymbol{v}) + \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}\theta \\ \mathbb{O} & 1 \end{bmatrix} \quad for \quad \boldsymbol{\omega} \neq 0 \end{aligned} \tag{4}$$

Revolute Joints (3)

Consider the system shown in Figure 1. In this system, a revolute joint is located at the point with coordinates $[0, 0, 5]$ defined in a coordinate frame A. The revolute joint is parametrised by angle θ_1.

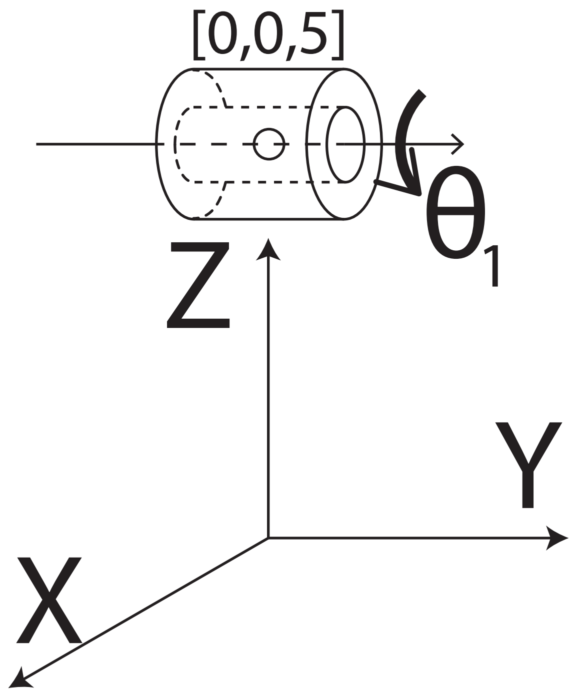

Figure 1: Revolute joint located at the point with coordinates $[0, 0, 5]$.

2.1. Write the twist $\boldsymbol{\xi}_1$ associated with this rigid body motion.

2.2. Show the corresponding exponential matrix $e^{\hat{\xi}_1 \theta_1}$ has the homogeneous coordinate form:

$$e^{\hat{\boldsymbol{\xi}}_1 \theta_1} = \begin{bmatrix} \begin{bmatrix} cos(\theta_1) & 0 & sin(\theta_1) \\ 0 & 1 & 0 \\ -sin(\theta_1) & 0 & cos(\theta_1) \end{bmatrix} & \begin{bmatrix} -5sin(\theta_1) \\ 0 \\ 5 - 5cos(\theta_1) \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

2.3. Given two frames B and A on either side of the joint, with initial configuration:

$$g_{AB}(0) = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

write the homogeneous matrix expression for $g_{AB}(\theta_1)$ where:

$$g_{AB}(\theta_1) = e^{\hat{\boldsymbol{\xi}}_1 \theta_1} g_{AB}(0)$$

for $\theta_1 = \{0, \pi/4, \pi/2\}$. Describe the motion produced by $e^{\hat{\boldsymbol{\xi}}_1 \theta_1}$.

Prismatic Joints (3)
Consider the system shown in Figure 2. In this system, a prismatic joint is located at the point with coordinates $[0, 0, 5]$ defined in a coordinate frame A. The prismatic joint is parametrised by distance θ_2.

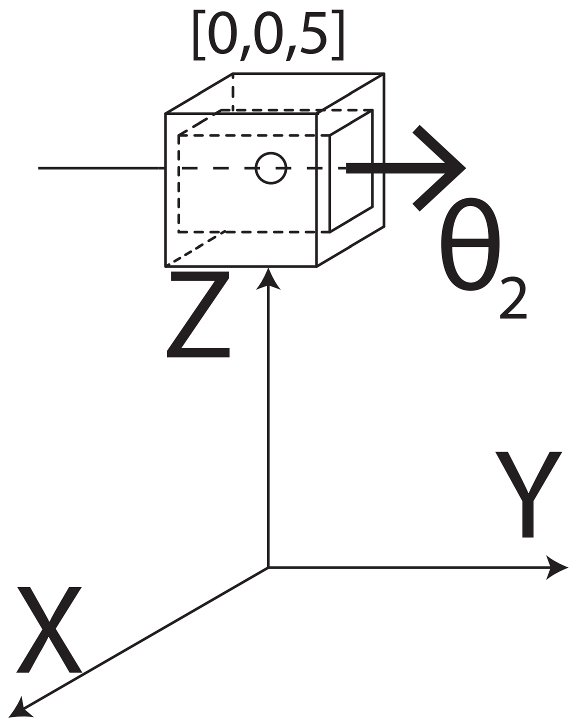

Figure 2: Prismatic joint located at the point with coordinates $[0, 0, 5]$.

2.4. Write the twist $\boldsymbol{\xi}_2$ associated with this rigid body motion.

2.5. Show the corresponding exponential matrix $e^{\hat{\xi}_2\theta_2}$ has the homogeneous coordinate form:

$$e^{\hat{\boldsymbol{\xi}}_2\theta_2} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ \theta_2 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

2.6. Given two frames B and A on either side of the joint, with initial configuration:

$$g_{AB}(0) = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

write the homogeneous matrix expression for $g_{AB}(\theta_2)$ where:

$$g_{AB}(\theta_2) = e^{\hat{\boldsymbol{\xi}}_2\theta_2} g_{AB}(0)$$

for $\theta_2 = \{0, \pi/4, \pi/2\}$. Describe the motion produced by $e^{\hat{\boldsymbol{\xi}}_2\theta_2}$.

Multiple Joints (4)
Consider the system shown in Figure 3. In this system, the revolute joint and prismatic joint have been coupled together so that the prismatic joint is rigidly attached to the revolute joint. As the revolute joint rotates, the prismatic joint rotates with it. The revolute joint is parametrised by angle θ_1, and the prismatic joint is parametrised by distance θ_2.

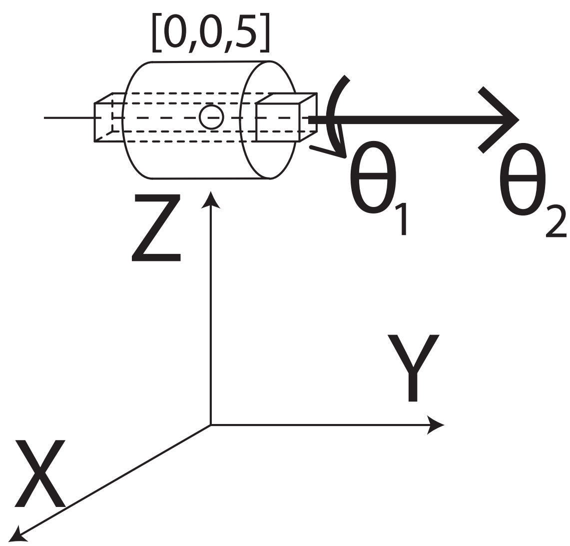

Figure 3: System of a revolute and prismatic joint located at the point with coordinates $[0, 0, 5]$.

2.7. Show the corresponding exponential matrix $e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}$ has the homogeneous coordinates form:

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2} = \begin{bmatrix} \begin{bmatrix} cos\left(\theta_1\right) & 0 & sin\left(\theta_1\right) \\ 0 & 1 & 0 \\ -sin\left(\theta_1\right) & 0 & cos\left(\theta_1\right) \end{bmatrix} & \begin{bmatrix} -5sin\left(\theta_1\right) \\ \theta_2 \\ 5 - 5cos\left(\theta_1\right) \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

2.8. Given two frames B and A on either side of the joints, with initial configuration:

$$g_{AB}\left(0\right) = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

write the homogeneous matrix expression for $g_{AB}\left(\theta_1, \theta_2\right)$ where:

$$g_{AB}\left(\theta_1, \theta_2\right) = e^{\hat{\boldsymbol{\xi}}_1\theta_1}e^{\hat{\boldsymbol{\xi}}_2\theta_2}g_{AB}\left(0\right)$$

for $\theta_1 = \left[0, \pi/4, \pi/2\right], \theta_2 = 0$. Describe the motion produced by $e^{\hat{\boldsymbol{\xi}}_1\theta_1}e^{\hat{\boldsymbol{\xi}}_2\theta_2}$.

2.9. Write the homogeneous matrix expression for $g_{AB}\left(\theta_1, \theta_2\right)$ for $\theta_1 = 0, \theta_2 = \{0, \pi/4, \pi/2\}$. Describe the motion produced by $e^{\hat{\boldsymbol{\xi}}_2 0}e^{\hat{\boldsymbol{\xi}}_2\theta_2}$.

2.10. Consider the special case when $\theta_1 = \theta_2$. Write the homogeneous matrix expression for $g_{AB}\left(\theta_1, \theta_2\right)$ for $\theta_1 = \theta_2 = \{0, \pi/4, \pi/2\}$. Describe the motion produced by $e^{\hat{\boldsymbol{\xi}}_2 0}e^{\hat{\boldsymbol{\xi}}_2\theta_1}$.

Compound Joints (4)
Consider the joint with twist $\boldsymbol{\xi}_3 = \begin{bmatrix} -5 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}^T$.

2.11. Show the corresponding exponential matrix $e^{\hat{\xi}_3 \theta_3}$ has the homogeneous coordinates form:

$$e^{\hat{\xi}_3 \theta_3} = \begin{bmatrix} \begin{bmatrix} cos\left(\theta_3\right) & 0 & sin\left(\theta_3\right) \\ 0 & 1 & 0 \\ -sin\left(\theta_3\right) & 0 & cos\left(\theta_3\right) \end{bmatrix} & \begin{bmatrix} -5sin\left(\theta_3\right) \\ \theta_3 \\ 5 - 5cos\left(\theta_3\right) \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

2.12. Given two frames B and A, with initial configuration:

$$g_{AB}\left(0\right) = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

show the corresponding rigid body map $g_{AB}\left(\theta_3\right)$ has the form:

$$g_{AB}\left(\theta_3\right) = \begin{bmatrix} \begin{bmatrix} cos\left(\theta_3\right) & 0 & sin\left(\theta_3\right) \\ 0 & 1 & 0 \\ -sin\left(\theta_3\right) & 0 & cos\left(\theta_3\right) \end{bmatrix} & \begin{bmatrix} 5sin\left(\theta_3\right) \\ \theta_3 \\ 5 + 5cos\left(\theta_3\right) \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

2.13. Write the homogeneous matrix expression for $g_{AB}\left(\theta_3\right)$ for $\theta_3 = \{0, \pi/4, \pi/2\}$. Describe the motion produced by $e^{\hat{\boldsymbol{\xi}}_3 \theta_3}$. How does this motion relate to the motion produced by $e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2}$.

2.14. Comparing $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_3$, what do you notice? How does this relate to the observed motion of $e^{\hat{\boldsymbol{\xi}}_3 \theta_3}$