EE106/206 HW3: Inverse Kinematics - Solutions

1 Paden-Kahan Subproblems (6)

1. Formulate the problem as a PK3:

$$\left\| q - e^{\hat{\xi}\theta} p \right\| = \delta$$

Define the relative coordinates u and v. Then solve for the projections, u' and v', on the plane of rotation:

$$\begin{aligned}
u &= p - r = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \\
u' &= u - \omega\omega^T u \\
&= \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
v &= q - r = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix} \\
v' &= u - \omega\omega^T u \\
&= \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}
\end{aligned}$$

Solve for the projection of δ on the plane of rotation, δ':

$$\begin{aligned}
\delta'^2 &= \delta^2 - \left|\omega^T (p - q)\right|^2 \\
&= 9 - \left| \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix} \right|^2 \\
&= 9
\end{aligned}$$

Solve for θ_0 and θ_d:

$$\begin{aligned}
\theta_0 &= atan2\left(\omega^T (u' \times v'), u'^T v'\right) \\
&= atan2\left(\begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix} - 12 \right) \\
&= \pi
\end{aligned}$$

$$
\begin{aligned}
\theta_d &= cos^{-1} \frac{\|u'\|^2 + \|v'\|^2 - \delta'^2}{2\,\|u'\|\,\|u'\|} \\
&= \frac{9 + 16 - 9}{24} \\
&= 0.8411
\end{aligned}
$$

The two solutions are:

$$\theta_3 = \theta_0 \pm \theta_d = \{-2.3005, 2.3005\}$$

2. Formulate the problem as a PK2:

$$q = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p$$

Define u and v in terms p, q, and r.

$$u = p - r = \begin{bmatrix} 0 \\ 1.999 \\ -2.235 \end{bmatrix}, \qquad v = q - r = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

Solve for α, β, and γ:

$$\begin{aligned} \alpha &= \frac{(\omega_1^T\omega_2)\omega_2^T u - \omega_1 v}{(\omega_1^T\omega_2)^2 - 1} = 1 \\ \beta &= \frac{(\omega_1^T\omega_2)\omega_1^T v - \omega_2 u}{(\omega_1^T\omega_2)^2 - 1} = 0 \\ \gamma^2 &= \frac{\|u\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\omega_1^T\omega_2}{\|\omega_1 \times \omega_2\|^2} = 7.991 \end{aligned}$$

Plug in α, β, and γ to find $z = c - r$:

$$\begin{aligned} z &= \alpha\omega_1 + \beta\omega_2 + \gamma\left(\omega_1 \times \omega_2\right) \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \pm \sqrt{7.991} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \pm 2.8268 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore, there are two values for the intersection point c:

$$c = z + r = \begin{bmatrix} 0 \\ \pm 2.8268 \\ 6 \end{bmatrix}$$

For each c, solve two PK1 Subproblems for θ_2 and θ_1:

$$c = e^{\hat{\xi}_2\theta_2} p, \quad c = e^{-\hat{\xi}_1\theta_1} q$$

- Intersection point $c_1 = \begin{bmatrix} 0 \\ 2.8268 \\ 6 \end{bmatrix}$

  PK1 for $\boldsymbol{\theta_2}$:

$$c = e^{\hat{\xi}_2\theta_2} p \quad \Longrightarrow \quad \begin{bmatrix} 0 \\ 2.8268 \\ 6 \end{bmatrix} = e^{\hat{\xi}_2\theta_2} \begin{bmatrix} 0 \\ 1.999 \\ 2.765 \end{bmatrix}$$

  Define u and v in terms p, $q = c$, and r. Then solve for the projections onto the plane of rotation u' and v':

$$\begin{aligned} u &= p - r = \begin{bmatrix} 0 \\ 1.999 \\ -2.235 \end{bmatrix} \\ u' &= u - \omega\omega^T u = \begin{bmatrix} 0 \\ 1.999 \\ -2.235 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} v &= c - r = \begin{bmatrix} 0 \\ 2.8268 \\ 1 \end{bmatrix} \\ v' &= u - \omega\omega^T u = \begin{bmatrix} 0 \\ 2.8268 \\ 1 \end{bmatrix} \end{aligned}$$

Solve for θ_2:

$$\begin{aligned} \theta_2 &= atan2\left(\omega^T\left(\omega \times v'\right), u'^T v'\right) \\ &= atan2\left(-8.3169, 3.4158\right) \\ &= -1.18109 \end{aligned}$$

PK1 for θ_1 :

$$c = e^{-\hat{\xi}_1\theta_1} q \quad \Longrightarrow \quad \begin{bmatrix} 0 \\ 2.8268 \\ 6 \end{bmatrix} = e^{-\hat{\xi}_1\theta_1} \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix}$$

Define u and v in terms p, $q = c$, and r. Then solve for the projections onto the plane of rotation u' and v':

$$\begin{aligned} u &= p - r = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \\ u' &= u - \omega\omega^T u = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} v &= c - r = \begin{bmatrix} 0 \\ 2.8268 \\ 1 \end{bmatrix} \\ v' &= u - \omega\omega^T u = \begin{bmatrix} 0 \\ 2.8268 \\ 0 \end{bmatrix} \end{aligned}$$

Solve for θ_1:

$$\begin{aligned} \theta_2 &= atan2\left(\omega^T\left(\omega \times v'\right), u'^T v'\right) \\ &= atan2\left(5.6569, 5.6536\right) \\ &= 0.7857 \end{aligned}$$

- Intersection point $c_2 = \begin{bmatrix} 0 \\ -2.8268 \\ 6 \end{bmatrix}$

  PK1 for θ_2 :

$$c = e^{\hat{\xi}_2\theta_2} p \quad \Longrightarrow \quad \begin{bmatrix} 0 \\ -2.8268 \\ 6 \end{bmatrix} = e^{\hat{\xi}_2\theta_2} \begin{bmatrix} 0 \\ 1.999 \\ 2.765 \end{bmatrix}$$

Define u and v in terms p, $q = c$, and r. Then solve for the projections onto the plane of rotation u' and v':

$$\begin{aligned} u &= p - r = \begin{bmatrix} 0 \\ 1.999 \\ -2.235 \end{bmatrix} \\ u' &= u - \omega\omega^T u = \begin{bmatrix} 0 \\ 1.999 \\ -2.235 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} v &= c - r = \begin{bmatrix} 0 \\ -2.8268 \\ 1 \end{bmatrix} \\ v' &= u - \omega\omega^T u = \begin{bmatrix} 0 \\ -2.8268 \\ 1 \end{bmatrix} \end{aligned}$$

Solve for θ_2:

$$\begin{aligned} \theta_2 &= atan2\left(\omega^T\left(\omega \times v'\right), u'^T v'\right) \\ &= atan2\left(4.3189, 7.8858\right) \\ &= 2.6405 \end{aligned}$$

PK1 for $\boldsymbol{\theta_1}$:

$$c = e^{-\hat{\xi}_1 \theta_1} q \quad \Longrightarrow \quad \begin{bmatrix} 0 \\ -2.8268 \\ 6 \end{bmatrix} = e^{-\hat{\xi}_1 \theta_1} \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix}$$

Define u and v in terms p, $q = c$, and r. Then solve for the projections onto the plane of rotation u' and v':

$$\begin{aligned} u &= p - r = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \\ u' &= u - \omega\omega^T u = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} v &= c - r = \begin{bmatrix} 0 \\ -2.8268 \\ 1 \end{bmatrix} \\ v' &= u - \omega\omega^T u = \begin{bmatrix} 0 \\ -2.8268 \\ 0 \end{bmatrix} \end{aligned}$$

Solve for θ_1:

$$\begin{aligned} \theta_2 &= atan2\left(\omega^T\left(\omega \times v'\right), u'^T v'\right) \\ &= atan2\left(-5.6536, -5.6536\right) \\ &= -2.3562 \end{aligned}$$

2 Forward Kinematics (4)

1. For each joint, find the axis of rotation ω and choose a point q along the axis. For a revolute joint, the twist is:

$$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$$

2.

$$\begin{aligned} g_{WB}(\theta_1, \theta_2) &= e^{\hat{\xi_1}\theta_1} e^{\hat{\xi_2}\theta_2} g_{WB}(0) \\ &= e^{\hat{\xi_1}\theta_1} e^{\hat{\xi_2}\theta_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

3.

$$\begin{aligned} g_{WD}(\theta_1, \theta_2, \theta_3, \theta_4) &= e^{\hat{\xi_1}\theta_1} e^{\hat{\xi_2}\theta_2} e^{\hat{\xi_3}\theta_3} e^{\hat{\xi_4}\theta_4} g_{WD}(0) \\ &= e^{\hat{\xi_1}\theta_1} e^{\hat{\xi_2}\theta_2} e^{\hat{\xi_3}\theta_3} e^{\hat{\xi_4}\theta_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

4.

$$\begin{aligned} g_{WG}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_5) &= e^{\hat{\xi_1}\theta_1} e^{\hat{\xi_2}\theta_2} e^{\hat{\xi_3}\theta_3} e^{\hat{\xi_4}\theta_4} e^{\hat{\xi_5}\theta_5} e^{\hat{\xi_6}\theta_6} g_{WG}(0) \\ &= e^{\hat{\xi_1}\theta_1} e^{\hat{\xi_2}\theta_2} e^{\hat{\xi_3}\theta_3} e^{\hat{\xi_4}\theta_4} e^{\hat{\xi_5}\theta_5} e^{\hat{\xi_6}\theta_6} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

3 Inverse Kinematics (10)

1. Use the known initial configuration to solve for the fixed transform g_{FG}:

$$
\begin{aligned}
g_{FG} &= g_{WF}^{-1}(0) g_{WG}(0) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

   Then, use the desired end effector configuration $g_{d,WG}$ to find the desired $g_{d,WF}$

$$
\begin{aligned}
g_{d,WF} &= g_{d,WG} g_{FG}^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

2. Define g_1 in terms of known values:

$$
g_1 \doteq g_{d,WF} g_{WF}^{-1} = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6}
$$

   We have the invariant points q_{wrist} and q_{sho} with the following relations:

$$
q_{wrist} = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_{wrist}
$$

$$
q_{sho} = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} q_{sho}
$$

   Use the invariant points to obtain a PK3 Subproblem:

$$
\begin{aligned}
\|g_1 q_{wrist} - q_{sho}\|_2 &= \left\| e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} q_{wrist} - q_{sho} \right\|_2 \\
&= \left\| e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_{wrist} - e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} q_{sho} \right\|_2 \\
&= \left\| e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \left(e^{\hat{\xi}_3\theta_3} q_{wrist} - q_{sho} \right) \right\|_2 \\
&= \left\| e^{\hat{\xi}_3\theta_3} q_{wrist} - q_{sho} \right\|_2
\end{aligned}
$$

   Solve for δ:

$$
\delta = \|g_1 q_{wrist} - q_{sho}\|_2 = \left\| \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 7 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\|_2 = 3
$$

3. Solved in problem 1.1
4. Apply g_1 to q_{wrist}:

$$g_1 q_{wrist} = e^{\hat{\xi_1}\theta_1} e^{\hat{\xi_2}\theta_2} e^{\hat{\xi_3}\theta_3} q_{wrist}$$

   To formulate the PK2 Subproblem, define q and p:

$$q = g_1 q_{wrist} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 7 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 6 \\ 0 \end{bmatrix}$$

$$p = e^{\hat{\xi_3}\theta_3} q_{wrist} = \begin{bmatrix} 0 \\ 2 \\ 2.7689 \\ 0 \end{bmatrix}$$

   Then, we have the PK2 Subproblem:

$$q = e^{\hat{\xi_1}\theta_1} e^{\hat{\xi_2}\theta_2} q$$

   Compute c_1 and c_2 as in problem 1.2
5. Solved in problem 1.2
6. Define g_2:

$$g_2 \doteq e^{-\hat{\xi_3}\theta_3} e^{-\hat{\xi_2}\theta_2} e^{-\hat{\xi_1}\theta_1} g_1 = e^{\hat{\xi_4}\theta_4} e^{\hat{\xi_5}\theta_5} e^{\hat{\xi_6}\theta_6}$$

   Choose a point on the ξ_6 axis so that it is invariant to rotations of joint 6:

$$p_6 = \begin{bmatrix} 0 \\ 9 \\ 5 \\ 1 \end{bmatrix}$$

   Apply g_2 to p_6:

$$g_2 p_6 = e^{\hat{\xi_4}\theta_4} e^{\hat{\xi_5}\theta_5} p_6$$

   To formulate the PK2 Subproblem, define q, p, r:

$$\begin{aligned} q &= g_2 p_6 = \begin{bmatrix} 1.4142 \\ 7.6166 \\ 6.2727 \\ 1 \end{bmatrix} \\ p &= p_6 = \begin{bmatrix} 0 \\ 9 \\ 5 \\ 1 \end{bmatrix} \\ r &= \text{wrist center} = \begin{bmatrix} 0 \\ 7 \\ 5 \\ 1 \end{bmatrix} \end{aligned}$$

Solving PK2, we obtain:

$$\alpha = 1.2727, \quad \beta = 0, \quad \gamma^2 = 2.3802 \quad \Longrightarrow c_{1,2} = \begin{bmatrix} 0 \\ 7 \pm 1.5428 \\ 6.2727 \end{bmatrix}$$

7. For both intersection points, c_1 and c_2, we solve for the solutions of the PK1 problem:

$$e^{-\hat{\xi}_4 \theta_4} q = c - e^{-\hat{\xi}_4 \theta_4 q} e^{\hat{\xi}_5 \theta_5 q} p$$

- $c_1 = \begin{bmatrix} 0 \\ 5.4572 \\ 6.2727 \end{bmatrix} \quad \Longrightarrow \quad \theta_4 = 1.9820, \theta_5 = -2.4518$
- $c_2 = \begin{bmatrix} 0 \\ 8.5428 \\ 6.2727 \end{bmatrix} \quad \Longrightarrow \quad \theta_4 = -1.1596, \theta_5 = -0.6898$

8. Choose a point p_6' not along the ξ_6 axis:

$$p_6' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + p_6 = \begin{bmatrix} 0 \\ 9 \\ 6 \end{bmatrix}$$

Define g_3:

$$g_3 \doteq e^{-\hat{\xi}_5 \theta_5} e^{-\hat{\xi}_4 \theta_4} g_2 = e^{\hat{\xi}_6 \theta_6}$$

$$g_2 p_6' = e^{\hat{\xi}_5 \theta_6} p_6'$$

Solve PK1 Subproblem $q = e^{\hat{\xi}_6 \theta_6} p$ for q, p, and r:

$$\begin{aligned} p &= p_6' = \begin{bmatrix} 0 \\ 9 \\ 6 \end{bmatrix} \\ q &= g_3 p_6^{-1} = \begin{bmatrix} 0.8249 \\ 9 \\ 5.5652 \end{bmatrix} \\ r &= \text{wrist center} = \begin{bmatrix} 0 \\ 7 \\ 5 \\ 1 \end{bmatrix} \end{aligned}$$

9. There are 8 sets of unique inverse kinematics solutions:

$$(2 \;\; \theta_3) \times (2 \;\; \theta_1, \theta_2) \times (2 \;\; \theta_4, \theta_5) \times (1 \;\; \theta_6) = 8$$

10.
- θ_3 will change to set the radial distance of the wrist δ
- θ_1 and θ_2 will set the polar coordinates of the wrist. θ_2 will change. The unique values of θ_2 will remain the same.
- θ_4, θ_5, and θ_6 may very to set the orientation of the wrist based on the new θ_1 and θ_2