Introduction to Robotics E106/206
Midterm1 - Fall 2016 **SID:** ____________________

Name: ____________________

Please show all working. Marks are awarded for method.

A cheat sheet is provided. No other notes or calculators are allowed.

Problem 1	/8
Problem 2	/6
Problem 3	/2
Problem 4	/4
Problem 5	/10
Problem 6	/19
Problem 7	/21
Total	/70

Question 1: Rotation Matrices .... 8 points

Consider two coordinate frames A and B. Coordinate frame B begins aligned with frame A. Frame B is then rotated by $\frac{\pi}{4}$ radians about the Y axis of the A coordinate frame.

(a) Write the rotation matrix $\boldsymbol{R}_{AB}$ for these two coordinate frames. (2)

(b) Write the rotation matrix $\boldsymbol{R}_{BA}$ for these two coordinate frames. (2)

(c) How would a vector $\boldsymbol{p}_A = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$ written in the A coordinate frame be written in the B coordinate frame? (2)

(d) How would a vector $\boldsymbol{p}_B = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$ written in the B coordinate frame be written in the A coordinate frame? (2)

Question 2: Rodrigues' Formula .... 6 points

Show that you can obtain the standard $\boldsymbol{R}_x$ rotation matrix from the Rodrigues' Formula.

Question 3: Multiple Rotations .... 2 points

(a) Given the rotation matrices $\boldsymbol{R}_{AB}, \boldsymbol{R}_{BC}, \boldsymbol{R}_{CD}$, write an expression for $\boldsymbol{R}_{AD}$. (1)

(b) Given the rotation matrices $\boldsymbol{R}_{AB}, \boldsymbol{R}_{AD}, \boldsymbol{R}_{CD}$, write an expression for $\boldsymbol{R}_{BC}$. (1)

Question 4: Valid Rotations .... 4 points

Is the transformation matrix $\boldsymbol{T}$ shown below a valid rotation matrix? If so, prove it, if not say why.

$$\boldsymbol{T} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Question 5: Rigid Body Motion .... 10 points

Consider the robotic manipulator shown in Appendix 1.

(a) Write the rigid body transform $\boldsymbol{g}_{WA}(\theta_1)$ in homogeneous form. (3)

(b) Write the rigid body transform $\boldsymbol{g}_{BT}(\theta_3, \theta_4)$ in homogeneous form. (3)

(c) Write the rigid body transform $\boldsymbol{g}_{WT}$ in terms of relative rigid body frames (such as g_{WA}) (2)

(d) How would you alter your expression for $\boldsymbol{g}_{WT}$ if we added a frame C between the B and T frames, where C is fixed to frame B. Explain your answer and any additional information required. (2)

Question 6: Forward Kinematics . . . . 19 points

Consider the robotic manipulator shown in Appendix 1.

(a) Compute $\hat{\boldsymbol{\xi}}$ for joints 1, 2, and 4: (14)

(b) Write an expression for the initial configuration $\boldsymbol{g}_{WT}\left(\mathbf{0}\right)$ of the manipulator in homogeneous coordinates. (3)

(c) Using matrix exponential terms such as $e^{\hat{\boldsymbol{\xi}}_i \theta_i}$ and $\boldsymbol{g}_{WT}\left(\mathbf{0}\right)$ write an expression for the forward kinematics map with the form $\boldsymbol{g}_{WT}\left(\boldsymbol{\theta}\right)$. (2)

Question 7: Inverse Kinematics . . . . 21 points

Consider the robotic manipulator shown in Appendix 1.

(a) You are given a desired configuration of the tool (T) frame: $\boldsymbol{g}_{d,WT}$. The matrix $\boldsymbol{g}_1$ can be written as: (2)

$$\boldsymbol{g}_1 = e^{\hat{\boldsymbol{\xi}}_1\theta_1} e^{\hat{\boldsymbol{\xi}}_2\theta_2} e^{\hat{\boldsymbol{\xi}}_3\theta_3} e^{\hat{\boldsymbol{\xi}}_4\theta_4}$$

Write $\boldsymbol{g}_1$ in terms of known configurations

(b) In words, describe how each joint affects the configuration of the manipulator. (4)

(c) What are the invariant point(s) in this system. Give their coordinates in the initial configuration, and state what joints they are invariant to. (2)

(d) Let $q_{d,t}$ be the W frame coordinates of the origin of the T frame in the desired configuration. Write $q_{d,t}$ in terms of known values. (1)

(e) Suppose $q_{d,t} = \begin{bmatrix}1 & 1 & 2\end{bmatrix}^T$. Formulate an inverse kinematics problem and solve for θ_4 in terms of known lengths (ie. l_0, l_1, l_2). (3)

(f) Assume θ_4 is known. Formulate the Inverse Kinematics for θ_1 and θ_2 as a Paden-Kahan problem. List which sub-problem and define necessary terms (ie. p, q, r, δ). (4)

(g) The matrix $\boldsymbol{g_2}$ can be written as: (1)

$$\boldsymbol{g}_2 = e^{\hat{\boldsymbol{\xi}}_3 \theta_3}$$

Given θ_1, θ_2, and θ_4 and $\boldsymbol{g_1}$ write an expression for $\boldsymbol{g}_2$:

(h) Assume θ_1, θ_2, and θ_4 are known. Formulate the Inverse Kinematics for θ_3 as a Paden-Kahan problem. List which sub-subproblem and define necessary terms (ie. p, q, r, δ). (4)

Appendix 1: 4 DoF Manipulator

Consider the four degree of freedom robotic manipulator shown in Figures 1, 2, and 3. The manipulator has three revolute joints and one prismatic joint.

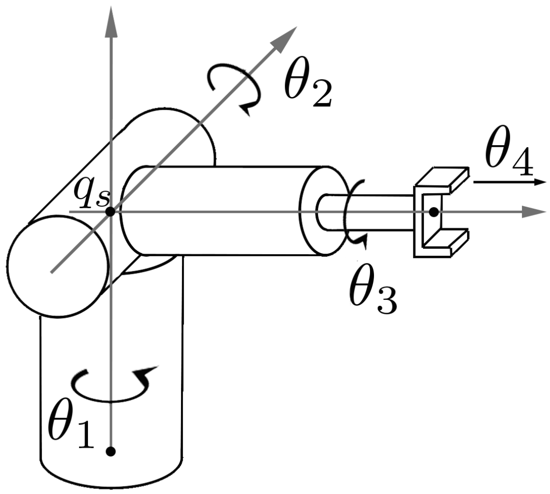

Figure 1: Joints 1, 2, and 3 are revolute joints. Joint 4 is a prismatic joint.

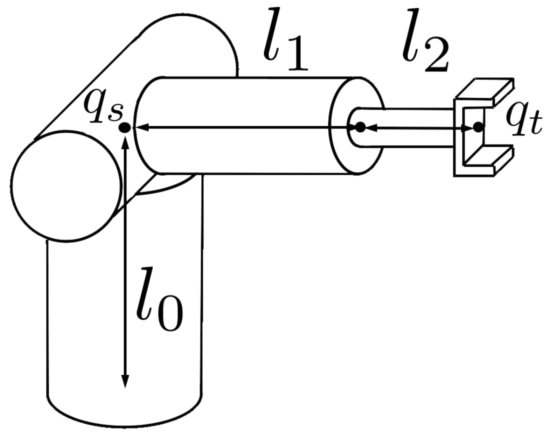

Figure 2: The zero configuration with initial manipulator lengths.

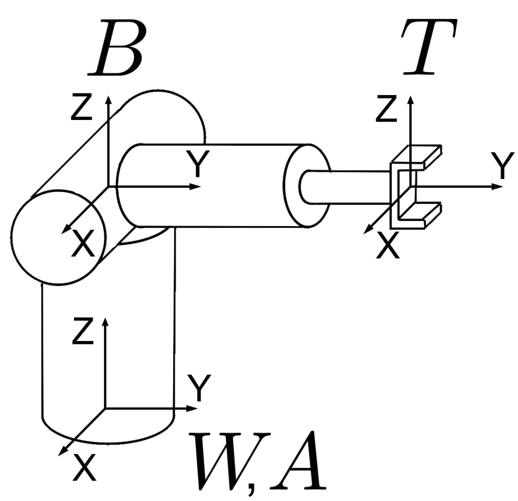

Figure 3: Schematic of a 4DoF Manipulator in its zero configuration. Frame naming conventions are shown, with the world and tool frame labelled as W and T respectively. Frames A and B refer to the local frame for joints 1 and 2 respectively. In the initial configuration (shown) the W and A frames are aligned.

Appendix: Cheat Sheet

0.1 Trigonometry

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Pythagoras's theorem $h^2 = x^2 + y^2$ for a right angled triangle where h is the hypotenuse and x and y are the lengths of the two remaining sides.

Sine, Cosine Relation $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$

Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos(\theta_C)$ where a, b, c are the lengths of the triangle and θ_A, θ_B and θ_C are the angles of their opposing corner.

0.2 Linear Algebra

For orthogonal matrices $A^{-1} = A^T$

Orthogonality A matrix $\begin{bmatrix}\boldsymbol{v}_1, ..., \boldsymbol{v}_n\end{bmatrix}$ is said to be orthogonal if:

$$\boldsymbol{v}_i^T\boldsymbol{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

0.3 Special Operators

Hat

$$\hat{\boldsymbol{\omega}} = \widehat{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Wedge

$$\hat{\boldsymbol{\xi}} = \widehat{\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \boldsymbol{0} & 0 \end{bmatrix}$$

0.4 Rotations

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{x}}\theta}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{y}}\theta}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{\hat{\boldsymbol{z}}\theta}$$

0.5 Rodrigues' Formula

$$R(\boldsymbol{\omega}, \theta) = e^{\hat{\boldsymbol{\omega}}\theta} = \mathbb{I}_3 + \frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|} \sin(\theta) + \frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2} (1 - \cos(\theta))$$

0.6 Rigid Body Motion

$$\boldsymbol{g}_{AB} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \mathbf{0} & 1 \end{bmatrix} \qquad \boldsymbol{g}_{AB}^{-1} = \begin{bmatrix} \boldsymbol{R}_{AB}^{-1} & -\boldsymbol{R}_{AB}^{-1}\boldsymbol{p}_{AB} \\ \mathbf{0} & 1 \end{bmatrix}$$

0.7 Exponential Notation

$$\boldsymbol{R}_{AB}(\theta_1) = e^{\hat{\omega}_1\theta_1}$$
$$\boldsymbol{g}_{AB}(\theta_1) = e^{\hat{\xi}_1\theta_1}\boldsymbol{g}_{AB}(0)$$
$$\boldsymbol{g}_{ST}(\theta_1, \ldots, \theta_n) = e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_n\theta_n}\boldsymbol{g}_{ST}(0)$$

0.7.1 Special Cases

Pure Rotation

$$\boldsymbol{\xi} = \begin{bmatrix} -\boldsymbol{\omega} \times \boldsymbol{q} \\ \boldsymbol{\omega} \end{bmatrix}$$

Pure Translation

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} \\ \mathbf{0} \end{bmatrix}$$

Pure Rotations, Screws (Rotation and Translation)

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} e^{\hat{\boldsymbol{\omega}}\theta} & \left(\mathbb{I}_3 - e^{\hat{\boldsymbol{\omega}}\theta}\right)(\boldsymbol{\omega} \times \boldsymbol{v}) + \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}$$

Pure Translation

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} \mathbb{I}_3 & \boldsymbol{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}$$

0.8 Paden-Kahan

Subproblem 1: Rotation about a single axis

$$e^{\widehat{\xi}\theta}p = q$$

Subproblem 2: Rotation about two subsequent axes

$$e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}p = q$$

Subproblem 3: Rotation to a distance

$$\left\| e^{\widehat{\xi}\theta}p - q \right\| = \delta$$