Homework 4: Velocities and Adjoints

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Feel free to use a computer to help you with this problem set. If you do write any code, to help you solve the problem, attach the code at the end of your problem set. If you use any pre-made code (such as MATLAB's pseudo-inverse function pinv()), state that you use it as a step in your solution.

Question 1: Velocities and Adjoints .... 10 points

(a) The spatial velocity $\hat{V}^s_{AB}$ and the body velocity $\hat{V}^b_{AB}$ can be written as: (3)

$$\hat{\boldsymbol{V}}^s_{AB} = \dot{\boldsymbol{g}}_{AB}\boldsymbol{g}^{-1}_{AB} \qquad \hat{\boldsymbol{V}}^b_{AB} = \boldsymbol{g}^{-1}_{AB}\dot{\boldsymbol{g}}_{AB}$$

given a rigid body transform of the form:

$$\boldsymbol{g}_{AB} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

show that the spatial and body velocities can be written as:

$$\boldsymbol{V}^s_{AB} = \begin{bmatrix} -\dot{\boldsymbol{R}}_{AB}\boldsymbol{R}^T_{AB}\boldsymbol{p}_{AB} + \dot{\boldsymbol{p}}_{AB} \\ \left(\dot{\boldsymbol{R}}_{AB}\boldsymbol{R}^T_{AB}\right)^{\vee} \end{bmatrix} \qquad \boldsymbol{V}^b_{AB} = \begin{bmatrix} \boldsymbol{R}^T_{AB}\dot{\boldsymbol{p}}_{AB} \\ \left(\boldsymbol{R}^T_{AB}\dot{\boldsymbol{R}}_{AB}\right)^{\vee} \end{bmatrix}$$

Solution:

1 Obtaining $\dot{\boldsymbol{g}}_{AB}$ and $\boldsymbol{g}^{-1}_{AB}$:

$$\dot{\boldsymbol{g}}_{AB} = \begin{bmatrix} \dot{\boldsymbol{R}}_{AB} & \dot{\boldsymbol{p}}_{AB} \\ \boldsymbol{0} & 0 \end{bmatrix} \qquad \boldsymbol{g}^{-1}_{AB} = \begin{bmatrix} \boldsymbol{R}^T_{AB} & -\boldsymbol{R}^T\boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

1 Multiplying out the matrices:

$$\begin{aligned} \hat{\boldsymbol{V}}^s_{AB} &= \dot{\boldsymbol{g}}_{AB}\boldsymbol{g}^{-1}_{AB} \\ &= \begin{bmatrix} \dot{\boldsymbol{R}}_{AB} & \dot{\boldsymbol{p}}_{AB} \\ \boldsymbol{0} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}^T_{AB} & -\boldsymbol{R}^T\boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \dot{\boldsymbol{R}}_{AB}\boldsymbol{R}^T_{AB} & -\dot{\boldsymbol{R}}_{AB}\boldsymbol{R}^T\boldsymbol{p}_{AB} + \dot{\boldsymbol{p}}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
\hat{\boldsymbol{V}}^b_{AB} &= \boldsymbol{g}^{-1}_{AB}\dot{\boldsymbol{g}}_{AB} \\
&= \begin{bmatrix} \boldsymbol{R}^T_{AB} & -\boldsymbol{R}^T\boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{R}}_{AB} & \dot{\boldsymbol{p}}_{AB} \\ \boldsymbol{0} & 0 \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{R}^T_{AB}\dot{\boldsymbol{R}}_{AB} & \boldsymbol{R}^T_{AB}\dot{\boldsymbol{p}}_{AB} \\ \boldsymbol{0} & 0 \end{bmatrix}
\end{aligned}
$$

1 performing the $\vee$ operation:

$$
\boldsymbol{V}^s_{AB} = \begin{bmatrix} -\dot{\boldsymbol{R}}_{AB}\boldsymbol{R}^T_{AB}\boldsymbol{p}_{AB} + \dot{\boldsymbol{p}}_{AB} \\ \left(\dot{\boldsymbol{R}}_{AB}\boldsymbol{R}^T_{AB}\right)^\vee \end{bmatrix} \qquad \boldsymbol{V}^b_{AB} = \begin{bmatrix} \boldsymbol{R}^T_{AB}\dot{\boldsymbol{p}}_{AB} \\ \left(\boldsymbol{R}^T_{AB}\dot{\boldsymbol{R}}_{AB}\right)^\vee \end{bmatrix}
$$

(b) Given two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, show: (1)

$$
\hat{\boldsymbol{a}}\boldsymbol{b} = -\hat{\boldsymbol{b}}\boldsymbol{a}
$$

Solution:

1 Expanding out both sides:

$$
\hat{\boldsymbol{a}}\boldsymbol{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -a_3b_2 + a_2b_3 \\ a_3b_1 - a_1b_3 \\ -a_2b_1 + a_1b_2 \end{bmatrix}
$$

$$
-\boldsymbol{b}\hat{\boldsymbol{a}} = -\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -a_3b_2 + a_2b_3 \\ a_3b_1 - a_1b_3 \\ -a_2b_1 + a_1b_2 \end{bmatrix}
$$

(c) Given the following relations in $\mathcal{SO}(3)$: (6)

$$
\hat{\boldsymbol{\omega}}^s_{AB} = \dot{\boldsymbol{R}}\boldsymbol{R}^{-1} \qquad \hat{\boldsymbol{\omega}}^b_{AB} = \boldsymbol{R}^{-1}\dot{\boldsymbol{R}} \qquad \boldsymbol{\omega}^s_{AB} = \boldsymbol{R}_{AB}\boldsymbol{\omega}^b_{AB}
$$

show that the Adjoint relation holds:

$$
\boldsymbol{V}^s_{AB} = Adj_{\boldsymbol{g}_{AB}}\boldsymbol{V}^b_{AB}
$$

where $Adj_{\boldsymbol{g}_{AB}}$:

$$
Adj_{\boldsymbol{g}_{AB}} = \begin{bmatrix} \boldsymbol{R}_{AB} & \hat{\boldsymbol{p}}_{AB}\boldsymbol{R}_{AB} \\ \boldsymbol{0} & \boldsymbol{R}_{AB} \end{bmatrix}
$$

Solution:

1 Form the Adjoint correctly:

$$Adj_{\boldsymbol{g}_{AB}} = \begin{bmatrix} \boldsymbol{R}_{AB} & \hat{\boldsymbol{p}}_{AB}\boldsymbol{R}_{AB} \\ \boldsymbol{0} & \boldsymbol{R}_{AB} \end{bmatrix}$$

2 Finding $Adj_{\boldsymbol{g}_{AB}}\boldsymbol{V}_{AB}^{b}$ and substituting in $\boldsymbol{\omega}_{AB}^{b}$

$$Adj_{\boldsymbol{g}_{AB}}\boldsymbol{V}_{AB}^{b} = \begin{bmatrix} \boldsymbol{R}_{AB} & \hat{\boldsymbol{p}}_{AB}\boldsymbol{R}_{AB} \\ \boldsymbol{0} & \boldsymbol{R}_{AB} \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_{AB}^{T}\dot{\boldsymbol{p}}_{AB} \\ \left(\boldsymbol{R}_{AB}^{T}\dot{\boldsymbol{R}}_{AB}\right)^{\vee} \end{bmatrix}$$

$$Adj_{\boldsymbol{g}_{AB}}\boldsymbol{V}_{AB}^{b} = \begin{bmatrix} \boldsymbol{R}_{AB} & \hat{\boldsymbol{p}}_{AB}\boldsymbol{R}_{AB} \\ \boldsymbol{0} & \boldsymbol{R}_{AB} \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_{AB}^{T}\dot{\boldsymbol{p}}_{AB} \\ \boldsymbol{\omega}_{AB}^{b} \end{bmatrix}$$

$$Adj_{\boldsymbol{g}_{AB}}\boldsymbol{V}_{AB}^{b} = \begin{bmatrix} \boldsymbol{R}_{AB}\boldsymbol{R}_{AB}^{T}\dot{\boldsymbol{p}}_{AB} + \hat{\boldsymbol{p}}\boldsymbol{R}_{AB}\boldsymbol{\omega}_{AB}^{b} \\ \boldsymbol{R}_{AB}\boldsymbol{\omega}_{AB}^{b} \end{bmatrix}$$

1 Applying $\mathcal{SO}(3)$ Adjoint:

$$Adj_{\boldsymbol{g}_{AB}}\boldsymbol{V}_{AB}^{b} = \begin{bmatrix} \dot{\boldsymbol{p}}_{AB} + \hat{\boldsymbol{p}}\boldsymbol{\omega}_{AB}^{s} \\ \boldsymbol{\omega}_{AB}^{s} \end{bmatrix}$$

1 Using $\hat{\boldsymbol{a}}\boldsymbol{b} = -\hat{\boldsymbol{b}}\boldsymbol{a}$.

$$Adj_{\boldsymbol{g}_{AB}}\boldsymbol{V}_{AB}^{b} = \begin{bmatrix} \dot{\boldsymbol{p}}_{AB} - \hat{\boldsymbol{\omega}}_{AB}^{s}\boldsymbol{p} \\ \boldsymbol{\omega}_{AB}^{s} \end{bmatrix}$$

1 Using $\hat{\boldsymbol{a}}\boldsymbol{b} = -\hat{\boldsymbol{b}}\boldsymbol{a}$.

$$Adj_{\boldsymbol{g}_{AB}}\boldsymbol{V}_{AB}^{b} = \begin{bmatrix} \dot{\boldsymbol{p}}_{AB} - \dot{\boldsymbol{R}}_{AB}\boldsymbol{R}_{AB}^{T}\boldsymbol{p}_{AB} \\ \left(\dot{\boldsymbol{R}}_{AB}\boldsymbol{R}_{AB}^{T}\right)^{\vee} \end{bmatrix}$$

Question 2: Spatial and Body Velocities .... 20 points

For this question, consider the rotating carousel shown in Figure 1.

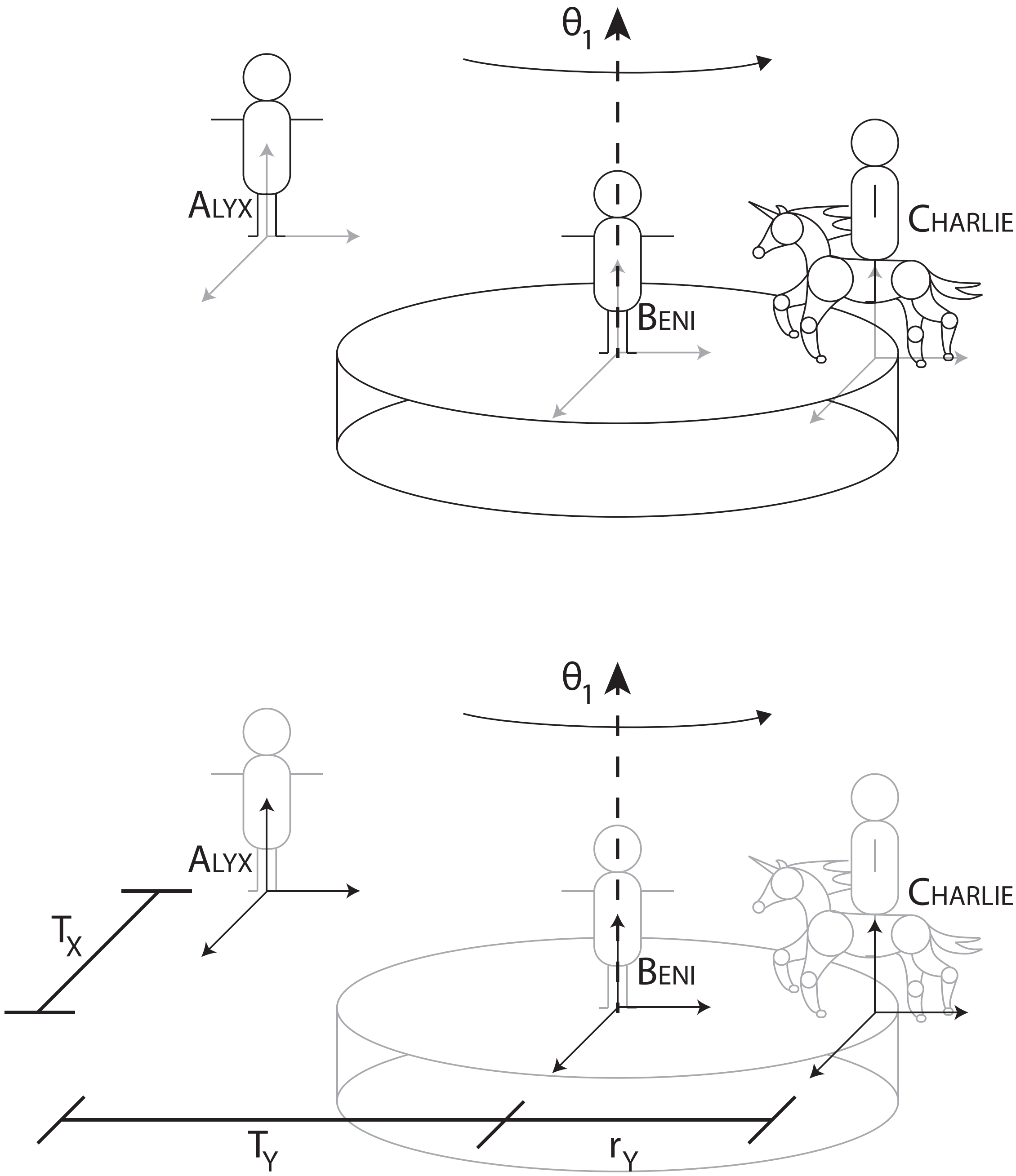

Figure 1: Rotating Carousel. Alyx is standing fixed on the ground. The carousel is located at the the point with coordinates $[T_x, T_y, 0]$ as seen in Alyx's frame. Beni is standing at the centre of the carousel. Charlie is located at the point with the coordinates $[0, r_y, 0]$ as seen in Beni's frame. All coordinate axes start aligned, with their Z axis being vertical. The carousel rotates about the Z axis.

(a) Write the rigid body transform $\boldsymbol{g}_{AB}(\theta_1)$ that relates Alyx's and Beni's frames. (1)

Solution:

1

$$\boldsymbol{g}_{AB} = \begin{bmatrix} \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} T_x \\ T_y \\ 0 \end{bmatrix} \\ \mathbf{0} & 1 \end{bmatrix}$$

(b) Show that $\boldsymbol{V}_{AB}^s$ and $\boldsymbol{V}_{AB}^b$ can be written as: (5)

$$\boldsymbol{V}_{AB}^s = \begin{bmatrix} T_y \\ -T_x \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1 \qquad \boldsymbol{V}_{AB}^b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1$$

Solution:

1 Computing $\dot{\boldsymbol{g}}_{AB}$:

$$\dot{\boldsymbol{g}}_{AB} = \begin{bmatrix} \begin{bmatrix} -s_1 & -c_1 & 0 \\ c_1 & -s_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{0} & 0 \end{bmatrix} \dot{\theta}_1$$

1 Computing $\boldsymbol{g}_{AB}^{-1}$:

$$\boldsymbol{g}_{AB}^{-1} = \begin{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} -T_x c_1 - T_y s_1 \\ T_x s_1 - T_y c_1 \\ 0 \end{bmatrix} \\ \mathbf{0} & 1 \end{bmatrix}$$

1 Computing $\hat{\boldsymbol{V}}_{AB}^s$

$$\hat{\boldsymbol{V}}_{AB}^s = \dot{\boldsymbol{g}}_{AB} \boldsymbol{g}_{AB}^{-1}$$

$$\hat{\boldsymbol{V}}_{AB}^s = \begin{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} T_y \\ -T_x \\ 0 \end{bmatrix} \\ \mathbf{0} & 0 \end{bmatrix} \dot{\theta}_1$$

1 Computing $\hat{\boldsymbol{V}}_{AB}^b$

$$\hat{\boldsymbol{V}}_{AB}^b = \boldsymbol{g}_{AB}^{-1} \dot{\boldsymbol{g}}_{AB}$$

$$\hat{\boldsymbol{V}}_{AB}^{b} = \begin{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{0} & 0 \end{bmatrix} \dot{\theta}_1$$

1 Using the $\vee$ operator to un-wedge the $\mathbb{R}^{4\times 4}$ matrices into $\mathbb{R}^{6\times 1}$ vectors.

(c) Charlie sits on the Carousel at a point with coordinates $\boldsymbol{q}_B = [0, r_y, 0]^T$ as seen from Beni's frame (frame B). Using your results for $\hat{\boldsymbol{V}}_{AB}^{s}$ and $\hat{\boldsymbol{V}}_{AB}^{b}$, show that the representations for Charlie's velocity as seen in Alyx's frame ($\boldsymbol{v}_{\boldsymbol{q}_B}^{s}$) and in Beni's frame ($\boldsymbol{v}_{\boldsymbol{q}_B}^{b}$) are: (2)

$$\boldsymbol{v}_{\boldsymbol{q}_B}^{s} = \begin{bmatrix} -r_y c_1 \\ -r_y s_1 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}_1 \qquad \boldsymbol{v}_{\boldsymbol{q}_B}^{b} = \begin{bmatrix} -r_y \\ 0 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}_1$$

Solution:

1 Computing $\boldsymbol{v}_{\boldsymbol{q}_B}^{s}$:

$$\boldsymbol{v}_{\boldsymbol{q}_B}^{s} = \hat{\boldsymbol{V}}_{AB}^{s} \boldsymbol{q}_A = \hat{\boldsymbol{V}}_{AB}^{s} \boldsymbol{g}_{AB} \boldsymbol{q}_B$$

$$\boldsymbol{v}_{\boldsymbol{q}_B}^{s} = \begin{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} T_y \\ -T_x \\ 0 \end{bmatrix} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} -T_x c_1 - T_y s_1 \\ T_x s_1 - T_y c_1 \\ 0 \end{bmatrix} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ r_y \\ 0 \\ 1 \end{bmatrix}$$

1 Computing $\boldsymbol{v}_{\boldsymbol{q}_B}^{b}$:

$$\boldsymbol{v}_{\boldsymbol{q}_B}^{b} = \hat{\boldsymbol{V}}_{AB}^{b} \boldsymbol{q}_B$$

$$\boldsymbol{v}_{\boldsymbol{q}_B}^{b} = \begin{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ r_y \\ 0 \\ 1 \end{bmatrix}$$

(d) Describe with words and sketches the geometric intuition behind $\boldsymbol{V}_{AB}^{s}$, $\boldsymbol{V}_{AB}^{b}$, $\boldsymbol{v}_{\boldsymbol{q}_B}^{s}$, and $\boldsymbol{v}_{\boldsymbol{q}_B}^{b}$. (2)

We now want to find the velocity representations with respect to Charlie's frame ($\boldsymbol{V}_{AC}^{s}$, $\boldsymbol{V}_{AC}^{b}$).

(e) Show that the rigid-body transform $\boldsymbol{g}_{AC}$ can be written as: (1)

$$\boldsymbol{g}_{AC} = \begin{bmatrix} \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} T_x - r_y s_1 \\ T_y + r_y c_1 \\ 0 \end{bmatrix} \\ \mathbf{0} & 1 \end{bmatrix}$$

Solution:

$$\boldsymbol{g}_{AC} = e^{\hat{\xi}_1 \theta_1} \boldsymbol{g}_{AC}$$

1

$$\xi_1 = \begin{bmatrix} T_y \\ -T_x \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

1

$$\boldsymbol{g}_{AC} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} T_x \\ T_y + r_y \\ 0 \end{bmatrix} \\ \mathbf{0} & 1 \end{bmatrix}$$

(f) Using this expression for $\boldsymbol{g}_{AC}$ show that the spatial velocity $\boldsymbol{V}_{AC}^s$ can be written as: (1)

$$\boldsymbol{V}_{AC}^s = \begin{bmatrix} T_y \\ -T_x \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1$$

Solution:

$$\boldsymbol{V}_{AC}^s = \dot{\boldsymbol{g}}_{AC} \boldsymbol{g}_{AC}^{-1}$$

$$\dot{\boldsymbol{g}}_{AC} = \begin{bmatrix} \begin{bmatrix} -s_1 & -c_1 & 0 \\ c_1 & -s_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} -r_y c_1 \\ -r_y s_1 \\ 0 \end{bmatrix} \\ \mathbf{0} & 0 \end{bmatrix} \dot{\theta}_1$$

$$g_{AC}^{-1} = \begin{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} -T_x c_1 - T_y s_1 \\ T_x s_1 - T_y c_1 - r_y \\ 0 \end{bmatrix} \\ \mathbf{0} & 1 \end{bmatrix}$$

(g) Show that the Adjoint to find $\boldsymbol{V}_{AC}^b$ from $\boldsymbol{V}_{AC}^s$ can be written as: (1)

$$\begin{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & T_x s_1 - T_y c_1 - r_y \\ 0 & 0 & T_x c_1 + T_y s_1 \\ T_y + r_y c_1 & -T_x + r_y s_1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{bmatrix}$$

Solution:

$$\boldsymbol{V}_{AC}^b = Ad_{\boldsymbol{g}_{AC}^{-1}} \boldsymbol{V}_{AC}^s$$

(h) Using this Adjoint, show that $\boldsymbol{V}_{AC}^b$ can be written as: (1)

$$\boldsymbol{V}_{AC}^b = \begin{bmatrix} -r_y \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1$$

Solution:

1 Multiplication

(i) Using your results for $\hat{\boldsymbol{V}}_{AC}^s$ and $\hat{\boldsymbol{V}}_{AC}^b$, show that the representations for Charlie's velocity as seen in Alyx's frame ($\boldsymbol{v}_{\boldsymbol{q}_C}^s$) and in Charlie's frame ($\boldsymbol{v}_{\boldsymbol{q}_C}^b$) are: (2)

$$\boldsymbol{v}_{\boldsymbol{q}_C}^s = \begin{bmatrix} -r_y c_1 \\ -r_y s_1 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}_1 \qquad \boldsymbol{v}_{\boldsymbol{q}_C}^b = \begin{bmatrix} -r_y \\ 0 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}_1$$

Solution:

1

$$\boldsymbol{v}^s_{\boldsymbol{q}_C} = \hat{\boldsymbol{V}}^s_{AC}\boldsymbol{g}_{AC}\boldsymbol{q}_C$$

1

$$\boldsymbol{v}^b_{\boldsymbol{q}_C} = \hat{\boldsymbol{V}}^b_{AC}\boldsymbol{q}_C$$

where $\boldsymbol{q}_C = [0,0,0,0,1]^T$

(j) Describe with words and sketches the geometric intuition behind $\boldsymbol{V}^s_{AC}$, $\boldsymbol{V}^b_{AC}$, $\boldsymbol{v}^s_{\boldsymbol{q}_C}$, and $\boldsymbol{v}^b_{\boldsymbol{q}_C}$. (2)

Solution:

1 $\boldsymbol{V}^s_{AC}$ is the velocity of a point on a disc centered at the joint, instantaneously at the origin of the spatial frame as seen in the spatial frame. $\boldsymbol{V}^b_{AC}$ is the velocity of a point on a disc centered at the joint, instantaneously at the end effector as seen in the body frame.

1 $\boldsymbol{v}^s_{\boldsymbol{q}_C}$ is the velocity of the point $\boldsymbol{q}$ between the A frame and the C frame as seen in the A frame. $\boldsymbol{v}^b_{\boldsymbol{q}_C}$ is the velocity of the point $\boldsymbol{q}$ between the A frame and the C frame as seen in the C frame.

(k) Compare your results you obtained for $\boldsymbol{V}^s_{AB}$ and $\boldsymbol{V}^s_{AC}$. What did you notice, and how does it affect your results for $\boldsymbol{v}^s_{\boldsymbol{q}_B}$ and $\boldsymbol{v}^s_{\boldsymbol{q}_C}$. (1)

Solution:

1 $\boldsymbol{V}^s_{AB}$ and $\boldsymbol{V}^s_{AC}$ are identical. While the body frame is different, these are spatial velocities, and are therefore identical.

$\boldsymbol{v}^s_{\boldsymbol{q}_B}$ and $\boldsymbol{v}^s_{\boldsymbol{q}_C}$ are also identical. The spatial velocties $\boldsymbol{V}^s$ always operate on points in the spatial frame. While $\boldsymbol{V}^s_{AB}$ and $\boldsymbol{V}^s_{AC}$ are defined with respect to frames B and C respectivley, they operate in the A frame. As the points $\boldsymbol{q}_B$ and $\boldsymbol{q}_C$ are written in the A frame (via $\boldsymbol{g}_{AB}$ and $\boldsymbol{g}_{AC}$), the spatial velocities $\boldsymbol{v}^s_{\boldsymbol{q}_B}$ and $\boldsymbol{v}^s_{\boldsymbol{q}_C}$ will be the same.

(l) Compare your results you obtained for $\boldsymbol{V}^b_{AB}$ and $\boldsymbol{V}^b_{AC}$. What did you notice, and how does it affect your results for $\boldsymbol{v}^b_{\boldsymbol{q}_B}$ and $\boldsymbol{v}^b_{\boldsymbol{q}_C}$. (1)

Solution:

1 $\boldsymbol{V}^b_{AB}$ and $\boldsymbol{V}^b_{AC}$ are not the same as they refer to two different body frames.

$\boldsymbol{v}^b_{\boldsymbol{q}_B}$ and $\boldsymbol{v}^b_{\boldsymbol{q}_C}$ are also identical. $\boldsymbol{q}$ is the same point, just written in the two different frames. When the velocity is evaluated using the correct $\boldsymbol{V}^b$, the body velocity of that point should be the same.