Introduction to Robotics **Name:** ____________________

E106/206 Midterm 2 **SID:** ____________________

Please write your name at the top of each page

Show all working. Marks are awarded for method.

A cheat sheet is provided. No other notes are allowed.

Question	Marks
1	
2	
3	
4	
5	
6	
7	
total	

Question 1: Exponential Maps . . . . 10 points

(a) Write the matrix form of the transformation $e^{\hat{\omega}\theta}$ for $\omega = [0, 0, 1]^T$, and describe what motion it represents. (2)

(b) Write the matrix form of the transformation $e^{\hat{\omega}\theta}$ for $\omega = [0, 0, 2]^T$, and describe what motion it represents. (2)

(c) Write the matrix form of the transformation $e^{\hat{\xi}\theta}$ for $\xi = [0, 3, 0, 0, 0, 0]^T$, and describe what motion it represents. (2)

(d) Write the matrix form of the transformation $e^{\hat{\xi}\theta}$ for $\xi = [0, 0, 0, 0, 0, 1]^T$, and describe what motion it represents.. (2)

(e) Write the matrix form of the transformation $e^{\hat{\xi}\theta}$ for $\xi = [1, 0, 0, 1, 0, 0]^T$, and describe what motion it represents. (2)

Question 2: Forward Kinematics . . . . 10 points

For this question, refer to the manipulator seen in Appendix 1.

(a) Using matrix exponents, write an expression for the forward kinematic map $\boldsymbol{g}_{WT}(\boldsymbol{\theta})$, leaving your answer in terms of $e^{\hat{\xi}_i \theta_i}$. (2)

(b) Write the initial configuration $\boldsymbol{g}_{WT}(\mathbf{0})$ of the manipulator. (2)

(c) Write out the twists $\boldsymbol{\xi}_i$ for each of the three joints. (6)

Question 3: Inverse Kinematics . . . . 10 points

The maniplator in Appendix 1 is to be moved to some valid desired configuration $\boldsymbol{g}_{d,WT}$.

(a) The matrix $\boldsymbol{g}_1$ can be expressed as: (2)

$$\boldsymbol{g}_1 = e^{\hat{\boldsymbol{\xi}}_1\theta_1} e^{\hat{\boldsymbol{\xi}}_2\theta_2} e^{\hat{\boldsymbol{\xi}}_3\theta_3}$$

Write an expression for $\boldsymbol{g}_1$ given our desired configuration $\boldsymbol{g}_{d,WT}$ and other known configurations.

(b) Write down the invariant points of the system and which joints they are invariant to. (2)

(c) Formulate the inverse kinematics problem to find θ_3 as in terms of Paden Kahan subproblems. Define all necessary terms to compute θ_3. (3)

(d) Given this value for θ_3, formulate the inverse kinematics problem to find θ_1 and θ_2 in terms of Paden Kahan subproblems. Define all necessary terms to compute θ_1 and θ_2. (3)

Question 4: Jacobians .... 10 points

This questions looks at the Jacobians of the manipulator shown in Appendix 1.

(a) Write the Spatial Jacobian for the manipulator in its initial configuration (Figures 2 and 3). (3)

(b) Write the Body Jacobian for the manipulator in its initial configuration (Figures 2 and 3). (3)

(c) Is this a singular configuration? If so, give an expression for the linear dependency between the columns. If not, show that the columns are linearly independent (potentially through an explicit example). (1)

(d) The spatial and body Jacobians can be written in terms of ξ' and $\xi^\dagger$. What is the relationship between ξ, ξ', and $\xi^\dagger$? What do the following matrices represent? (3)

$$\boldsymbol{M}_1 = \begin{bmatrix}\xi_1 & \xi_2 & \xi_3\end{bmatrix} \qquad \boldsymbol{M}_2 = \begin{bmatrix}\xi_1' & \xi_2' & \xi_3'\end{bmatrix} \qquad \boldsymbol{M}_3 = \begin{bmatrix}\xi_1^\dagger & \xi_2^\dagger & \xi_3^\dagger\end{bmatrix}$$

Question 5: Velocities .... 10 points

The body and spatial Jacobians for the manipulator in configuration $\boldsymbol{g}_{WT}\left(0, 0, -\frac{\pi}{2}\right)$ (Appendix 1, Figure 4) can be written:

$$J^s = \begin{bmatrix} 0 & -l_0 & -l_0 \\ 0 & 0 & 0 \\ 0 & 0 & l_1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad J^b = \begin{bmatrix} 0 & -l_1 & 0 \\ l_1 & 0 & 0 \\ 0 & -l_2 & -l_2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) What instantaneous linear body velocities are possible in this configuration. (1)

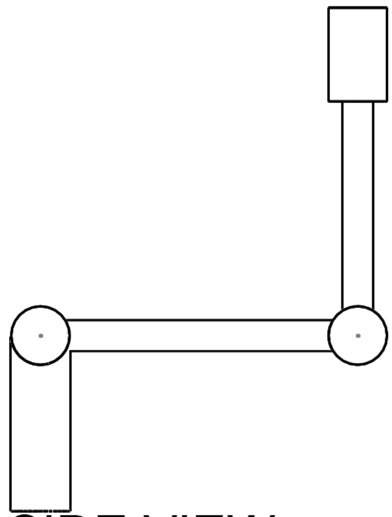
(b) Is this a singular configuration? Show how the joint velocities $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ relate to the instantaneous linear body velocities. (1)

(c) Show that the matrix that relates the velocity of a point as seen in the tool frame ($\boldsymbol{v}_{\boldsymbol{q}_T}$) to its coordinates as seen in the tool frame ($\boldsymbol{q}_T$) can be written: (3)

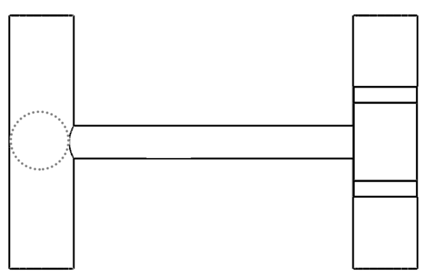
$$\boldsymbol{M} = \begin{bmatrix} 0 & -\dot{\theta}_1 & \dot{\theta}_2 + \dot{\theta}_3 & -l_1\dot{\theta}_2 \\ \dot{\theta}_1 & 0 & 0 & l_1\dot{\theta}_1 \\ -\dot{\theta}_2 - \dot{\theta}_3 & 0 & 0 & -l_2\left(\dot{\theta}_2 + \dot{\theta}_3\right) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(d) What is the instantaneous body velocity of the origin of the tool frame? (2)

(e) The figures below show the side and top views of the manipulator in configuration $\boldsymbol{g}_{WT}\left(0, 0, -\frac{\pi}{2}\right)$ (Appendix 1, Figure 4). Sketch this instantaneous body velocities from each joint on the figures below. (3)



SIDE VIEW



TOP VIEW

There are no questions on this page.

Question 6: Wrenches . . . . 10 points

The body and spatial Jacobians for the manipulator in configuration $\boldsymbol{g}_{WT}\left(0, -\frac{\pi}{2}, 0\right)$ (Appendix 1, Figure 5) can be written:

$$J^s = \begin{bmatrix} 0 & -l_0 & -l_0 - l_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad J^b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -l_1 - l_2 & -l_2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(a) What spatial forces and torques can be applied by the manipulator in this configuration? (1)

(b) How many spatial forces and torques can be controlled independently in this configuration? (1)

(c) A mass m is held by the manipulator at the point with tool frame coordinates $[0, 0, 0]$ What is the body wrench $\mathbf{\Gamma}^b$ associated with this load. Assume that the acceleration due to gravity acts in the negative Z_W direction. (2)

(d) Using the Jacobians provided, what joint torques are associated with this body wrench? (2)

(e) Based on these joint torques, explain how the mass is supported by the manipulator. (2)

(f) Without performing any computation, what is the spatial wrench Γ^s associated with this body wrench Γ^b? Explain your reasoning. (2)

Question 7: Dynamics .... 10 points

Consider the mass spring system with two masses connected by three springs shown in Figure 1.

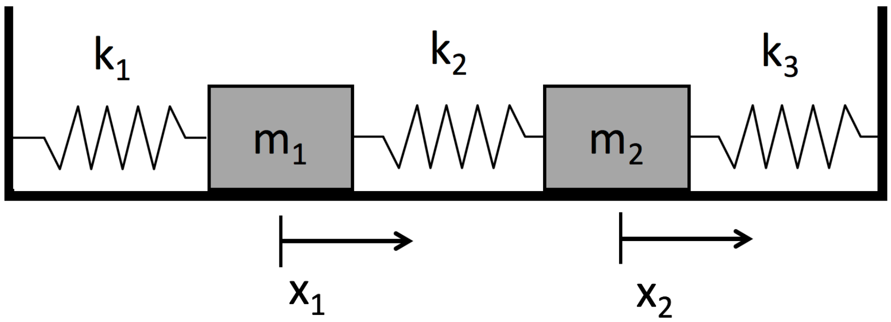


Figure 1: Two masses attached by three springs. The positions of the masses, x_1 and x_2, are defined such that equilibrium is at $x_1 = x_2 = 0$.

(a) Write the kinetic energy of this system. (2)

(b) Write an expression for the potential energy in the system. (2)

(c) Write the dynamics for this system using the Lagrange method. (6)

1 Appendix: RRR Manipulator

Robotic manipulator with three revolute joints.

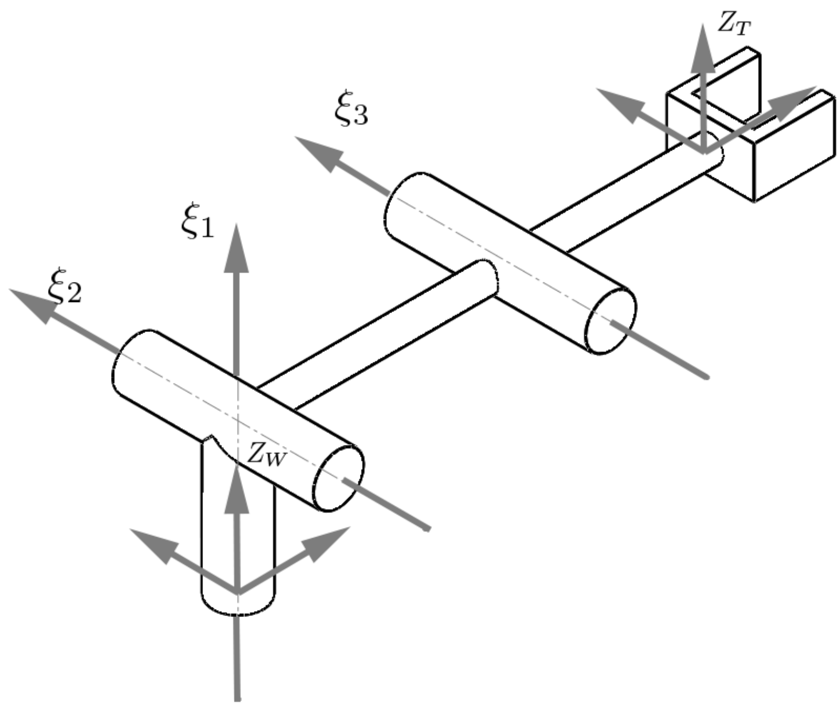


Figure 2: Schematic of a 3DoF RRR Manipulator in its initial configuration. Axes of rotation and the world and tool reference frames are shown.

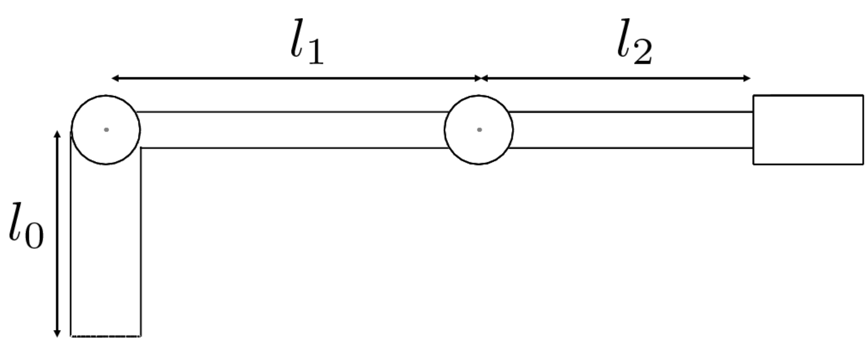


Figure 3: Side view of a 3DoF RRR Manipulator in its initial configuration. Segment lengths are shown.

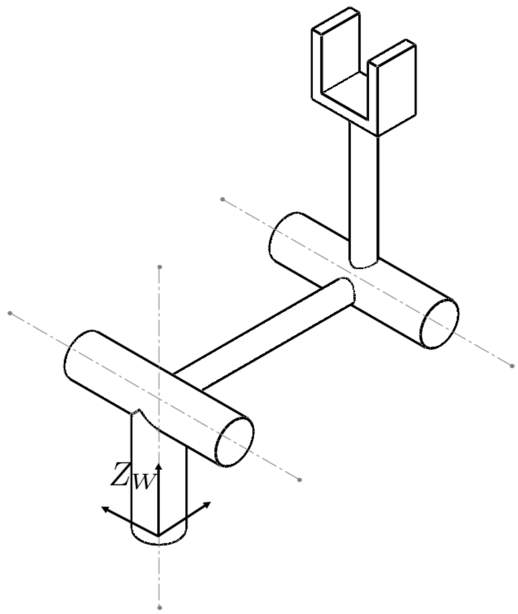


Figure 4: Manipulator in the configuration $\boldsymbol{\theta} = \left[0, 0, -\frac{\pi}{2}\right]$

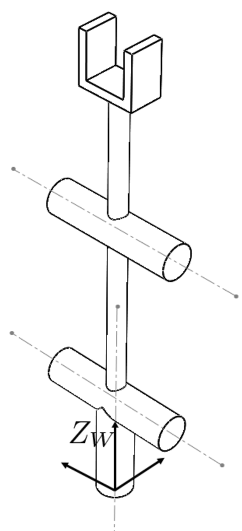


Figure 5: Manipulator in the configuration $\boldsymbol{\theta} = \left[0, -\frac{\pi}{2}, 0\right]$.

2 Appendix: Cheat Sheet

This is the cheat sheet that will be provided for every midterm.

2.1 Trigonometry

Pythagoras's theorem $h^2 = x^2 + y^2$ for a right angled triangle where h is the hypotenuse and x and y are the lengths of the two remaining sides.

Sine, Cosine Relation $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$

Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos(\theta_C)$ where a, b, c are the lengths of the triangle and θ_A, θ_B and θ_C are the angles of their opposing corner.

2.2 Linear Algebra

For orthogonal matrices $A^{-1} = A^T$

Orthogonality A matrix $\begin{bmatrix} \boldsymbol{v}_1, ..., \boldsymbol{v}_n \end{bmatrix}$ is said to be orthogonal if:

$$\boldsymbol{v}_i^T \boldsymbol{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

2.3 Special Operators

Hat

$$\hat{\boldsymbol{\omega}} = \widehat{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Wedge

$$\hat{\boldsymbol{\xi}} = \widehat{\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \boldsymbol{0} & 0 \end{bmatrix}$$

2.4 Rotations

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{x}}\theta}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{y}}\theta}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{\hat{\boldsymbol{z}}\theta}$$

2.5 Rodrigues' Formula

$$R(\boldsymbol{\omega}, \theta) = e^{\hat{\boldsymbol{\omega}}\theta} = \mathbb{I}_3 + \frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|}\sin(\theta) + \frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2}(1 - \cos(\theta))$$

2.6 Rigid Body Motion

$$\boldsymbol{g}_{AB} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \mathbf{0} & 1 \end{bmatrix} \qquad \boldsymbol{g}_{AB}^{-1} = \begin{bmatrix} \boldsymbol{R}_{AB}^{-1} & -\boldsymbol{R}_{AB}^{-1}\boldsymbol{p}_{AB} \\ \mathbf{0} & 1 \end{bmatrix}$$

2.7 Exponential Notation

$$\boldsymbol{R}_{AB}(\theta_1) = e^{\hat{\omega}_1\theta_1}$$

$$\boldsymbol{g}_{AB}(\theta_1) = e^{\hat{\xi}_1\theta_1}\boldsymbol{g}_{AB}(0)$$

$$\boldsymbol{g}_{ST}(\theta_1, \ldots, \theta_n) = e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_n\theta_n}\boldsymbol{g}_{ST}(0)$$

2.7.1 Special Cases

Pure Rotation

$$\boldsymbol{\xi} = \begin{bmatrix} -\boldsymbol{\omega} \times \boldsymbol{q} \\ \boldsymbol{\omega} \end{bmatrix}$$

Pure Translation

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} \\ \mathbf{0} \end{bmatrix}$$

Pure Rotations, Screws (Rotation and Translation)

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} e^{\hat{\boldsymbol{\omega}}\theta} & \left(\mathbb{I}_3 - e^{\hat{\boldsymbol{\omega}}\theta}\right)(\boldsymbol{\omega} \times \boldsymbol{v}) + \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}$$

Pure Translation

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} \mathbb{I}_3 & \boldsymbol{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}$$

2.8 Paden-Kahan

Subproblem 1: Rotation about a single axis

$$e^{\widehat{\xi}\theta}p = q$$

Subproblem 2: Rotation about two subsequent axes

$$e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}p = q$$

Subproblem 3: Rotation to a distance

$$\left\| e^{\widehat{\xi}\theta}p - q \right\| = \delta$$

2.9 Velocities

Spatial Velocities

$$\widehat{V}^s_{AB} = \dot{g}_{AB} g^{-1}_{AB} \qquad V^s_{AB} = \begin{bmatrix} -\dot{R}R^T p + \dot{p} \\ \left(\dot{R}R^T\right)^{\vee} \end{bmatrix} = \xi \dot{\theta}$$

Body Velocities

$$\widehat{V}^b_{AB} = g^{-1}_{AB} \dot{g}_{AB} \qquad V^b_{AB} = \begin{bmatrix} R^T \dot{p} \\ \left(R^T \dot{R}\right)^{\vee} \end{bmatrix} = \left(Ad_{g^{-1}_{AB}(0)} \xi\right) \dot{\theta}$$

Adjoint

$$Ad_g = \begin{bmatrix} R & \widehat{p}R \\ 0 & R \end{bmatrix} \qquad V^s = Ad_g V^b$$

$$V^s_{AC} = V^s_{AB} + Ad_{g_{AB}} V^s_{BC} \qquad V^b_{AC} = Ad_{g^{-1}_{BC}} V^b_{AB} + V^b_{BC}$$

2.10 Jacobians

Spatial Jacobian

$$V^s_{ST} = J^s_{ST} \dot{\theta}$$
$$J^s_{ST} = \begin{bmatrix} \xi_1 & \xi'_2 & \dots & \xi'_n \end{bmatrix}$$
$$\xi'_i = Ad_{\left(e^{\widehat{\xi}_1 \theta_1} \dots e^{\widehat{\xi}_{i-1} \theta_{i-1}}\right)} \xi_i$$

Body Jacobian

$$V^b_{ST} = J^b_{ST} \dot{\theta}$$
$$J^b_{ST} = \begin{bmatrix} \xi^{\dagger}_1 & \xi^{\dagger}_2 & \dots & \xi^{\dagger}_n \end{bmatrix}$$
$$\xi^{\dagger}_i = Ad^{-1}_{\left(e^{\widehat{\xi}_i \theta_i} \dots e^{\widehat{\xi}_n \theta_n} g_{ST}(0)\right)} \xi_i$$

2.11 Wrenches

$$\mathbf{\Gamma} = \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{\tau} \end{bmatrix}$$

$$\mathbf{\Gamma}^b = \left(Ad_{g_{ST}}\right)^T \mathbf{\Gamma}^s$$

Spatial Wrench

$$\boldsymbol{\tau} = \left(J^s_{ST}\right)^T \mathbf{\Gamma}^s$$

Body Wrench

$$\boldsymbol{\tau} = \left(J^b_{ST}\right)^T \mathbf{\Gamma}^b$$

2.12 Euler Lagrange

$$\mathcal{L}(q, \dot{q}) = T(q, \dot{q}) - V(q)$$
$$\Gamma_i = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i}$$

2.13 Forces

Gravitational Force

$$W = -mg$$

where m is the mass, g is the acceleration due to gravity.

Elastic Force

$$F_k = -k\delta$$

where k is the spring constant and δ is the extension of the spring.

2.14 Energies

Kinetic

$$KE = \frac{1}{2}mv^2$$

where m is the mass and v is the velocity of the object.

Gravitational Potential

$$GPE = mgh$$

where m is the mass, g is the acceleration due to gravity and h is the distance along the gravitational axis.

Elastic Potential

$$EPE = \frac{1}{2}k\delta^2$$

where k is the spring constant and δ is the extension of the spring.

2.15 Moments of Inertia

Mass at a Radius

$$I = mr^2$$

where m is the mass, and r is the radius of the mass.