Introduction to Robotics E106/206
Midterm1 - Fall 2016 - SOLUTIONS **SID:** ______________________

Name: ______________________

Please show all working. Marks are awarded for method.

A cheat sheet is provided. No other notes or calculators are allowed.

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Introduction to Robotics 106/206

Question 1: Rotation Matrices .... 8 points

Consider two coordinate frames A and B. Coordinate frame B begins aligned with frame A. Frame B is then rotated by $\frac{\pi}{4}$ radians about the Y axis of the A coordinate frame.

(a) Write the rotation matrix $\boldsymbol{R}_{AB}$ for these two coordinate frames. (2)

Solution:

1 Use correct Rotation matrix.

$$\boldsymbol{R}_{AB} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

1 Substitute in theta correctly.

$$\boldsymbol{R}_{AB} = \begin{bmatrix} \cos(\frac{\pi}{4}) & 0 & \sin(\frac{\pi}{4}) \\ 0 & 1 & 0 \\ -\sin(\frac{\pi}{4}) & 0 & \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

(b) Write the rotation matrix $\boldsymbol{R}_{BA}$ for these two coordinate frames. (2)

Solution: Either, resolve using $\theta = -\frac{\pi}{4}$

1 Use correct Rotation matrix.

$$\boldsymbol{R}_{BA} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

1 Substitute in theta correctly.

$$\boldsymbol{R}_{BA} = \begin{bmatrix} \cos(-\frac{\pi}{4}) & 0 & \sin(-\frac{\pi}{4}) \\ 0 & 1 & 0 \\ -\sin(-\frac{\pi}{4}) & 0 & \cos(-\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

or

2 Use $\boldsymbol{R}_{BA} = \boldsymbol{R}_{AB}^{-1} = \boldsymbol{R}_{AB}^{T}$ for full credit.

(c) How would a vector $\boldsymbol{p}_A = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$ written in the A coordinate frame be written in the B coordinate frame? (2)

Solution:

1 Use correct Rotation matrix.

$$\boldsymbol{p}_B = \boldsymbol{R}_{BA}\boldsymbol{p}_A$$

1 Perform calculation correctly.

$$\boldsymbol{p}_B = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \sqrt{2} \end{bmatrix}$$

(d) How would a vector $\boldsymbol{p}_B = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$ written in the B coordinate frame be written in the A coordinate frame? (2)

Solution:

1 Use correct Rotation matrix.

$$\boldsymbol{p}_A = \boldsymbol{R}_{AB}\boldsymbol{p}_B$$

1 Perform calculation correctly.

$$\boldsymbol{p}_A = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \end{bmatrix}$$

Question 2: Rodrigues' Formula .... 6 points

Show that you can obtain the standard $\boldsymbol{R}_x$ rotation matrix from the Rodrigues' Formula.

Solution:

1 Write Rodrigues' formula

$$\boldsymbol{R}(\boldsymbol{\omega}, \theta) = \mathbb{I}_3 + \frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|}\sin(\theta) + \frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2}(1-\cos(\theta))$$

1 Correctly write $\boldsymbol{\omega}$:

$$\boldsymbol{\omega} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$$

1 Correctly writing the $\frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|}$ term:

$$\frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|} = \begin{bmatrix}0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix}$$

1 Correctly writing the $\frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2}$ term:

$$\frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}^2\|} = \begin{bmatrix}0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0\end{bmatrix}^2 = \begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{bmatrix}$$

1 Correctly writing the trig components:

$$\frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|}\sin(\theta) = \begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & -\sin(\theta) \\ 0 & \sin(\theta) & 0\end{bmatrix}$$

$$\frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2}(1-\cos(\theta)) = \begin{bmatrix}0 & 0 & 0 \\ 0 & (\cos(\theta)-1) & 0 \\ 0 & 0 & (\cos(\theta))-1\end{bmatrix}$$

1 Correctly arrive at $\boldsymbol{R}_y$:

$$\boldsymbol{R}(\boldsymbol{\omega}, \theta) = \begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix} + \begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & -\sin(\theta) \\ 0 & \sin(\theta) & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0 \\ 0 & (\cos(\theta)-1) & 0 \\ 0 & 0 & (\cos(\theta))-1\end{bmatrix}$$

$$\boldsymbol{R}(\boldsymbol{\omega}, \theta) = \begin{bmatrix}1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & (\cos(\theta))\end{bmatrix} = \boldsymbol{R}_x$$

Question 3: Multiple Rotations . . . . 2 points

(a) Given the rotation matrices $\boldsymbol{R}_{AB}, \boldsymbol{R}_{BC}, \boldsymbol{R}_{CD}$, write and expression for $\boldsymbol{R}_{AD}$. (1)

> **Solution:**
>
> **1** For correct expression: $\boldsymbol{R}_{AD} = \boldsymbol{R}_{AB}\boldsymbol{R}_{BC}\boldsymbol{R}_{CD}$

(b) Given the rotation matrices $\boldsymbol{R}_{AB}, \boldsymbol{R}_{AD}, \boldsymbol{R}_{CD}$, write and expression for $\boldsymbol{R}_{BC}$. (1)

> **Solution:**
>
> **1** For correct expression: $\boldsymbol{R}_{BC} = \boldsymbol{R}_{AB}^{-1}\boldsymbol{R}_{AD}\boldsymbol{R}_{CD}^{-1}$

Question 4: Valid Rotations . . . . 4 points

Is the transformation matrix $\boldsymbol{T}$ shown below a valid rotation matrix? If so, prove it, if not say why.

$$\boldsymbol{T} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

> **Solution:**
>
> **1** Checking for $det\,(T) = 1$.
>
> **1** For correct calculation $det\,(T) == 0 * 0 - (-1) * 1 = 1$.
>
> **1** Checking for $TT^T = \mathbb{I}$.
>
> **1** For correct calculation $TT^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \mathbb{I}$.
>
> or
>
> **2** By inspection, this is an element of $\mathcal{SO}\,(2)$.
>
> or
>
> **4** By inspection, this is the $2D$ rotation matrix about $-\frac{\pi}{2}$.

Question 5: Rigid Body Motion . . . . 10 points

Consider the robotic manipulator shown in Appendix 1.

(a) Write the rigid body transform $\boldsymbol{g}_{WA}(\theta_1)$ in homogeneous form. (3)

Solution:

1 For writing in homogeneous coordinate form:

$$\boldsymbol{g}_{WA} = \begin{bmatrix} \boldsymbol{R}_{WA} & \boldsymbol{p}_{WA} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

1 For correctly using the $\boldsymbol{R}_Z$ rotation matrix:

$$\boldsymbol{R}_{WA} = \boldsymbol{R}_Z(\theta) = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1 For correctly finding $\boldsymbol{p}_{WA}$:

$$\boldsymbol{p}_{WA} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) Write the rigid body transform $\boldsymbol{g}_{BT}(\theta_3, \theta_4)$ in homogeneous form. (3)

Solution:

1 For writing in homogeneous coordinate form:

$$\boldsymbol{g}_{BT} = \begin{bmatrix} \boldsymbol{R}_{BT} & \boldsymbol{p}_{BT} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

1 For correctly using the $\boldsymbol{R}_Y$ rotation matrix:

$$\boldsymbol{R}_{BT} = \boldsymbol{R}_Y = \begin{bmatrix} \cos(\theta_3) & 0 & \sin(\theta_3) \\ 0 & 1 & 0 \\ -\sin(\theta_3) & 0 & \cos(\theta_3) \end{bmatrix}$$

1 For correctly finding $\boldsymbol{p}_{BT}$:

$$\boldsymbol{p}_{BT} = \begin{bmatrix} 0 \\ l_1 + l_2 + \theta_4 \\ 0 \end{bmatrix}$$

(c) Write the rigid body transform $\boldsymbol{g}_{WT}$ in terms of relative rigid body frames (such as g_{WA}) (2)

Solution:

2 For writing in homogeneous coordinate form:

$$\boldsymbol{g}_{WT} = \boldsymbol{g}_{WA}\boldsymbol{g}_{AB}\boldsymbol{g}_{BT}$$

(d) How would you alter your expression for $\boldsymbol{g}_{WT}$ if we added a frame C between the B and T frames, where C is fixed to frame B. Explain your answer and any additional information required. (2)

Solution:

1 No change

1 C is a reference frame and does not actually induce any motion to the subsequent frames. As we already know $\boldsymbol{g}_{BT}$ we do not need to include the C frame.

Question 6: Forward Kinematics . . . . 19 points

Consider the robotic manipulator shown in Appendix 1.

(a) Compute $\hat{\boldsymbol{\xi}}$ for joints 1, 2, and 4: (14)

Solution:

3 One point for each correct $\boldsymbol{\omega}$:

$$\boldsymbol{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{\omega}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{\omega}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2 For each correct $\boldsymbol{q}$. '$\cdot$' represents any number.

$$\boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 0 \\ \cdot \end{bmatrix} \qquad \boldsymbol{q}_2 = \begin{bmatrix} \cdot \\ 0 \\ l_0 \end{bmatrix}$$

3 Calculating the $\hat{\boldsymbol{\omega}}$ terms:

$$\hat{\boldsymbol{\omega}}_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \hat{\boldsymbol{\omega}}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad \hat{\boldsymbol{\omega}}_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3 Calculating $\boldsymbol{v}$:

$$\boldsymbol{v}_1 = -\boldsymbol{\omega}_1 \times \boldsymbol{q}_1 \qquad \boldsymbol{v}_2 = -\boldsymbol{\omega}_2 \times \boldsymbol{q}_2$$

$$\boldsymbol{v}_1 = -\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \cdot \end{bmatrix} \qquad \boldsymbol{v}_2 = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \cdot \\ 0 \\ l_0 \end{bmatrix}$$

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ -l_0 \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

3 Calculating the twists $\hat{\boldsymbol{\xi}}$ terms:

$$\hat{\boldsymbol{\xi}}_1 = \begin{bmatrix} \hat{\boldsymbol{\omega}}_1 & \boldsymbol{v}_1 \\ \mathbf{0} & 0 \end{bmatrix} \qquad \hat{\boldsymbol{\xi}}_2 = \begin{bmatrix} \hat{\boldsymbol{\omega}}_2 & \boldsymbol{v}_2 \\ \mathbf{0} & 0 \end{bmatrix} \qquad \hat{\boldsymbol{\xi}}_4 = \begin{bmatrix} \hat{\boldsymbol{\omega}}_4 & \boldsymbol{v}_4 \\ \mathbf{0} & 0 \end{bmatrix}$$

(b) Write an expression for the initial configuration $\boldsymbol{g}_{WT}(\mathbf{0})$ of the manipulator in homogeneous coordinates (3)

Solution:

1 For writing in homogeneous coordinate form:

$$\boldsymbol{g}_{WT} = \begin{bmatrix} \boldsymbol{R}_{WT} & \boldsymbol{p}_{WT} \\ \mathbf{0} & 1 \end{bmatrix}$$

1 For correctly setting $\boldsymbol{R}_{WT}$ as the identity matrix.

1 For correctly finding $\boldsymbol{p}_{WT}$:

$$\boldsymbol{p}_{WT} = \begin{bmatrix} 0 \\ l_1 + l_2 \\ l_0 \end{bmatrix}$$

(c) Using matrix exponential terms such as $e^{\hat{\boldsymbol{\xi}}_i \theta_i}$ and $\boldsymbol{g}_{WT}(\mathbf{0})$ write an expression for the forward kinematics map with the form $\boldsymbol{g}_{WT}(\boldsymbol{\theta})$. (2)

Solution:

$$\boldsymbol{g}_{WT}(\boldsymbol{\theta}) = e^{\hat{\boldsymbol{\xi}}_1 \theta_1} e^{\hat{\boldsymbol{\xi}}_2 \theta_2} e^{\hat{\boldsymbol{\xi}}_3 \theta_3} e^{\hat{\boldsymbol{\xi}}_4 \theta_4} \boldsymbol{g}_{WT}(\mathbf{0})$$

1 For the correct exponents in the correct order.

1 For correctly putting the initial configuration term at the end.

Question 7: Inverse Kinematics .... 21 points

Consider the robotic manipulator shown in Appendix 1.

(a) You are given a desired configuration of the tool (T) frame: $\boldsymbol{g}_{d,WT}$. The matrix $\boldsymbol{g}_1$ can be written as: (2)

$$\boldsymbol{g}_1 = e^{\hat{\boldsymbol{\xi}}_1\theta_1} e^{\hat{\boldsymbol{\xi}}_2\theta_2} e^{\hat{\boldsymbol{\xi}}_3\theta_3} e^{\hat{\boldsymbol{\xi}}_4\theta_4}$$

Write $\boldsymbol{g}_1$ in terms of known configurations

Solution:

1 For writing the expression

$$\boldsymbol{g}_{d,WT} = e^{\hat{\boldsymbol{\xi}}_1\theta_1} e^{\hat{\boldsymbol{\xi}}_2\theta_2} e^{\hat{\boldsymbol{\xi}}_3\theta_3} e^{\hat{\boldsymbol{\xi}}_4\theta_4} \boldsymbol{g}_{WT}(\mathbf{0})$$

1 For finding $\boldsymbol{g}_1$:

$$\boldsymbol{g}_1 = \boldsymbol{g}_{d,WT} \boldsymbol{g}_{WT}^{-1}(\mathbf{0})$$

or

2 For writing $\boldsymbol{g}_1$:

$$\boldsymbol{g}_1 = \boldsymbol{g}_{d,WT} \boldsymbol{g}_{WT}^{-1}(\mathbf{0})$$

(b) In words, describe how each joint affects the configuration of the manipulator. (4)

Solution:

2 θ_1, θ_2 determine the angular position of the frame T (end effector).

1 θ_3 determines the orientation of the frame T (end effector).

1 θ_4 determines the radial position of the frame T (distance from the shoulder to the end effector).

(c) What are the invariant point(s) in this system. Give their coordinates in the initial configuration, and state what joints they are invariant to. (2)

Solution:

1 $q_s = \begin{bmatrix} 0 & 0 & l_0 \end{bmatrix}^T$ and is invariant to rotations about ξ_1, ξ_2, (ξ_3, ξ_4) .

1 $q_t = \begin{bmatrix} 0 & l_1 + l_2 & l_0 \end{bmatrix}^T$ is invariant to rotations about ξ_3

(d) Let $q_{d,t}$ be the W frame coordinates of the origin of the T frame in the desired configuration. Write $q_{d,t}$ in terms of known values. (1)

Solution:

1 $\boldsymbol{q}_{d,t} = g_1 q_t$

or

1 Note that $q_{d,t}$ is the translation from the origin of the W frame to the origin of the T frame in the desired configuration matrix:

$$g_{d,WT} = \begin{bmatrix} R_d & q_{d,t} \\ 0 & 1 \end{bmatrix}$$

(e) Suppose $q_{d,t} = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}^T$. Formulate an inverse kinematics problem and solve for θ_4 in terms of known lengths (ie. l_0, l_1, l_2). (3)

Solution:

2 $\|\boldsymbol{q_s} - \boldsymbol{q_{d,t}}\|_2 = l_1 + l_2 + \theta_4$

1 $\theta_4 = \left\| \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\|_2 - (l_1 + l_2)$

or

1 $\theta_4 = \sqrt{2 + (l_0 - 2)^2}_2 - (l_1 + l_2)$

(f) Assume θ_4 is known. Formulate the Inverse Kinematics for θ_1 and θ_2 as a Paden-Kahan problem. List which sub-problem and define necessary terms (ie. p, q, r, δ). (4)

Solution:

1 PKII

1 $\boldsymbol{p} = e^{\hat{\boldsymbol{\xi}}_4 \theta_4} \boldsymbol{q}_t = \begin{bmatrix} 0 & l_1 + l_2 & l_0 \end{bmatrix}^T$

1 $\boldsymbol{q} = \boldsymbol{g}_1 \boldsymbol{q}_t = \boldsymbol{q_{d,t}}$

1 $\boldsymbol{r} = \boldsymbol{q}_s = \begin{bmatrix} 0 & 0 & l_0 \end{bmatrix}^T$

(g) The matrix $\boldsymbol{g}_2$ can be written as: (2)

$$\boldsymbol{g}_2 = e^{\hat{\boldsymbol{\xi}}_3 \theta_3}$$

Given θ_1, θ_2, and θ_4 and $\boldsymbol{g_1}$ write an expression for $\boldsymbol{g}_2$:

Solution:

2 $\boldsymbol{g}_2 = e^{-\hat{\boldsymbol{\xi}}_2 \theta_2} e^{-\hat{\boldsymbol{\xi}}_1 \theta_1} \boldsymbol{g}_1 e^{-\hat{\boldsymbol{\xi}}_4 \theta_4}$

(h) Assume θ_1, θ_2, and θ_4 are known. Formulate the Inverse Kinematics for θ_3 as a Paden-Kahan problem. List which sub-subproblem and define necessary terms (ie. p, q, r, δ). (4)

Solution:

1 PKI

1 $\boldsymbol{p} = \boldsymbol{q}_{END} = \begin{bmatrix} l_1 + l_2 & 0 & l_0 + 1 \end{bmatrix}^T$

1 $\boldsymbol{q} = \boldsymbol{g}_2 \boldsymbol{q}_{END}$

1 $\boldsymbol{r} = \boldsymbol{q}_t$

where $\boldsymbol{q}_{END}$ is any point not on the $\boldsymbol{\xi}_3$ axis.

1 Appendix: 4 DoF Manipulator

Consider the four degree of freedom robotic manipulator shown in Figures 1, 2, and 3. The manipulator has three revolute joints and one prismatic joint.

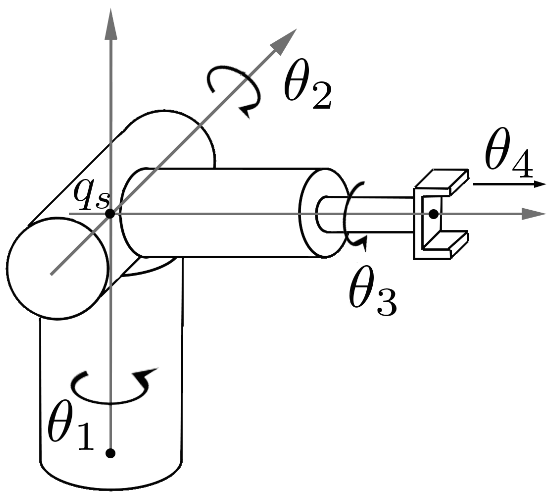

Figure 1: Joints 1, 2, and 3 are revolute joints. Joint 4 is a prismatic joint.

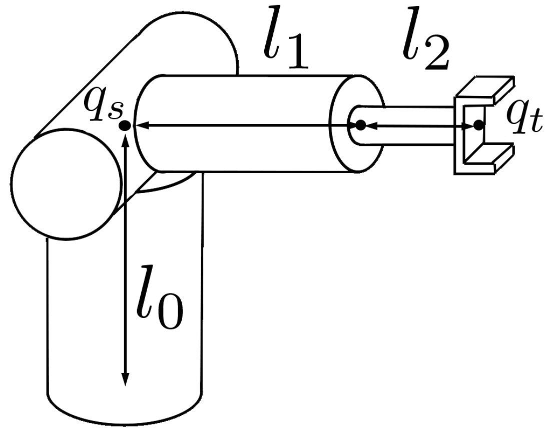

Figure 2: The zero configuration with initial manipulator lengths.

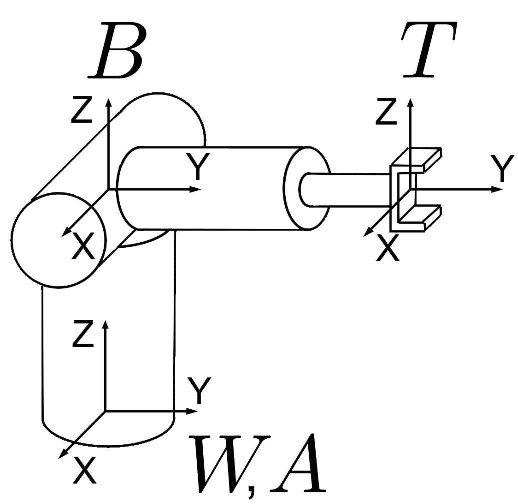

Figure 3: Schematic of a 4DoF Manipulator in its zero configuration. Frame naming conventions are shown, with the world and tool frame labelled as W and T respectively. Frames A and B refer to the local frame for joints 1 and 2 respectively. In the initial configuration (shown) the W and A frames are aligned.

2 Appendix: Cheat Sheet

2.1 Trigonometry

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Pythagoras's theorem $h^2 = x^2 + y^2$ for a right angled triangle where h is the hypotenuse and x and y are the lengths of the two remaining sides.

Sine, Cosine Relation $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$

Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos(\theta_C)$ where a, b, c are the lengths of the triangle and θ_A, θ_B and θ_C are the angles of their opposing corner.

2.2 Linear Algebra

For orthogonal matrices $A^{-1} = A^T$

Orthogonality A matrix $\left[\boldsymbol{v}_1, ..., \boldsymbol{v}_n\right]$ is said to be orthogonal if:

$$\boldsymbol{v}_i^T \boldsymbol{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

2.3 Special Operators

Hat

$$\hat{\boldsymbol{\omega}} = \widehat{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Wedge

$$\hat{\boldsymbol{\xi}} = \widehat{\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix}$$

2.4 Rotations

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{x}}\theta}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} = e^{\hat{\boldsymbol{y}}\theta}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{\hat{\boldsymbol{z}}\theta}$$

2.5 Rodrigues' Formula

$$R(\boldsymbol{\omega}, \theta) = e^{\hat{\boldsymbol{\omega}}\theta} = \mathbb{I}_3 + \frac{\hat{\boldsymbol{\omega}}}{\|\boldsymbol{\omega}\|} \sin(\theta) + \frac{\hat{\boldsymbol{\omega}}^2}{\|\boldsymbol{\omega}\|^2} (1 - \cos(\theta))$$

2.6 Rigid Body Motion

$$\boldsymbol{g}_{AB} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \mathbf{0} & 1 \end{bmatrix} \qquad \boldsymbol{g}_{AB}^{-1} = \begin{bmatrix} \boldsymbol{R}_{AB}^{-1} & -\boldsymbol{R}_{AB}^{-1}\boldsymbol{p}_{AB} \\ \mathbf{0} & 1 \end{bmatrix}$$

2.7 Exponential Notation

$$\boldsymbol{R}_{AB}(\theta_1) = e^{\hat{\omega}_1\theta_1}$$
$$\boldsymbol{g}_{AB}(\theta_1) = e^{\hat{\xi}_1\theta_1}\boldsymbol{g}_{AB}(0)$$
$$\boldsymbol{g}_{ST}(\theta_1, \ldots, \theta_n) = e^{\hat{\xi}_1\theta_1} \ldots e^{\hat{\xi}_n\theta_n}\boldsymbol{g}_{ST}(0)$$

2.7.1 Special Cases

Pure Rotation

$$\boldsymbol{\xi} = \begin{bmatrix} -\boldsymbol{\omega} \times \boldsymbol{q} \\ \boldsymbol{\omega} \end{bmatrix}$$

Pure Translation

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} \\ \mathbf{0} \end{bmatrix}$$

Pure Rotations, Screws (Rotation and Translation)

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} e^{\hat{\boldsymbol{\omega}}\theta} & \left(\mathbb{I}_3 - e^{\hat{\boldsymbol{\omega}}\theta}\right)(\boldsymbol{\omega} \times \boldsymbol{v}) + \boldsymbol{\omega}\boldsymbol{\omega}^T\boldsymbol{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}$$

Pure Translation

$$e^{\hat{\boldsymbol{\xi}}\theta} = \begin{bmatrix} \mathbb{I}_3 & \boldsymbol{v}\theta \\ \mathbf{0} & 1 \end{bmatrix}$$

2.8 Paden-Kahan

Subproblem 1: Rotation about a single axis

$$e^{\widehat{\xi}\theta}p = q$$

Subproblem 2: Rotation about two subsequent axes

$$e^{\widehat{\xi}_1\theta_1}e^{\widehat{\xi}_2\theta_2}p = q$$

Subproblem 3: Rotation to a distance

$$\left\| e^{\widehat{\xi}\theta}p - q \right\| = \delta$$