Homework 4: Velocities and Adjoints

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Homework due: Thursday 10/11/2018 @ 11:59 PM on GradeScope

Feel free to use a computer to help you with this problem set. If you do write any code, to help you solve the problem, attach the code at the end of your problem set. If you use any pre-made code (such as MATLAB's pseudo-inverse function pinv()), state that you use it as a step in your solution.

Question 1: Velocities and Adjoints .... 10 points

(a) The spatial velocity $\hat{V}^s_{AB}$ and the body velocity $\hat{V}^b_{AB}$ can be written as: (3)

$$\hat{\boldsymbol{V}}^s_{AB} = \dot{\boldsymbol{g}}_{AB}\boldsymbol{g}^{-1}_{AB} \qquad \hat{\boldsymbol{V}}^b_{AB} = \boldsymbol{g}^{-1}_{AB}\dot{\boldsymbol{g}}_{AB}$$

given a rigid body transform of the form:

$$\boldsymbol{g}_{AB} = \begin{bmatrix} \boldsymbol{R}_{AB} & \boldsymbol{p}_{AB} \\ \boldsymbol{0} & 1 \end{bmatrix}$$

show that the spatial and body velocities can be written as:

$$\boldsymbol{V}^s_{AB} = \begin{bmatrix} -\dot{\boldsymbol{R}}_{AB}\boldsymbol{R}^T_{AB}\boldsymbol{p}_{AB} + \dot{\boldsymbol{p}}_{AB} \\ \left(\dot{\boldsymbol{R}}_{AB}\boldsymbol{R}^T_{AB}\right)^\vee \end{bmatrix} \qquad \boldsymbol{V}^b_{AB} = \begin{bmatrix} \boldsymbol{R}^T_{AB}\dot{\boldsymbol{p}}_{AB} \\ \left(\boldsymbol{R}^T_{AB}\dot{\boldsymbol{R}}_{AB}\right)^\vee \end{bmatrix}$$

(b) Given two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, show: (1)

$$\hat{\boldsymbol{a}}\boldsymbol{b} = -\hat{\boldsymbol{b}}\boldsymbol{a}$$

(c) Given the following relations in $\mathcal{SO}(3)$: (6)

$$\hat{\boldsymbol{\omega}}^s_{AB} = \dot{\boldsymbol{R}}\boldsymbol{R}^{-1} \qquad \hat{\boldsymbol{\omega}}^b_{AB} = \boldsymbol{R}^{-1}\dot{\boldsymbol{R}} \qquad \boldsymbol{\omega}^s_{AB} = \boldsymbol{R}_{AB}\boldsymbol{\omega}^b_{AB}$$

show that the Adjoint relation holds:

$$\boldsymbol{V}^s_{AB} = Adj_{\boldsymbol{g}_{AB}}\boldsymbol{V}^b_{AB}$$

where $Adj_{\boldsymbol{g}_{AB}}$:

$$Adj_{\boldsymbol{g}_{AB}} = \begin{bmatrix} \boldsymbol{R}_{AB} & \hat{\boldsymbol{p}}_{AB}\boldsymbol{R}_{AB} \\ \boldsymbol{0} & \boldsymbol{R}_{AB} \end{bmatrix}$$

Question 2: Spatial and Body Velocities . . . . 20 points

For this question, consider the rotating carousel shown in Figure 1.

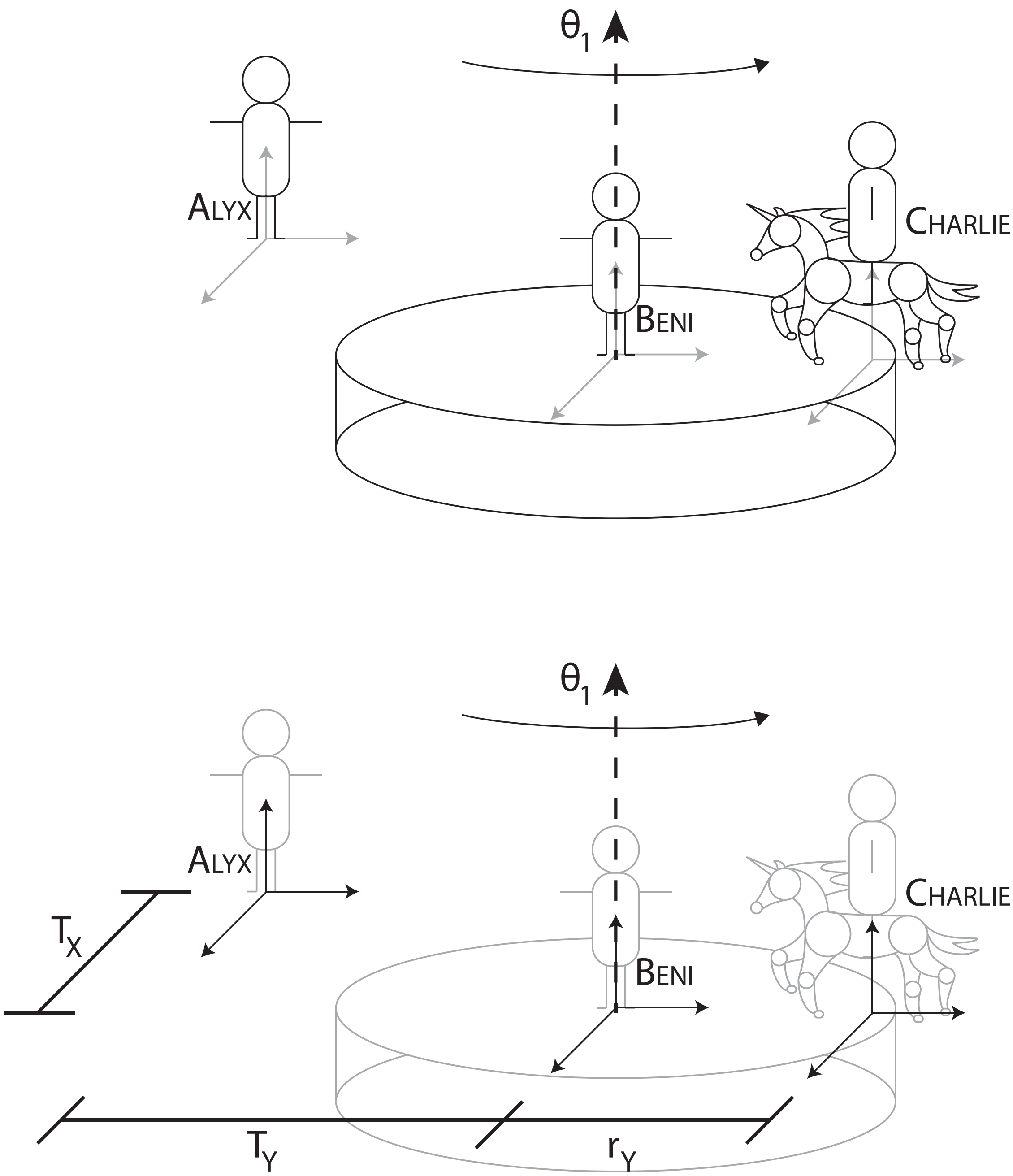

Figure 1: Rotating Carousel. Alyx is standing fixed on the ground. The carousel is located at the the point with coordinates $[T_x, T_y, 0]$ as seen in Alyx's frame. Beni is standing at the centre of the carousel. Charlie is located at the point with the coordinates $[0, r_y, 0]$ as seen in Beni's frame. All coordinate axes start aligned, with their Z axis being vertical. The carousel rotates about the Z axis.

(a) Write the rigid body transform $\boldsymbol{g}_{AB}(\theta_1)$ that relates Alyx's and Beni's frames. (1)

(b) Show that $\boldsymbol{V}^s_{AB}$ and $\boldsymbol{V}^b_{AB}$ can be written as: (5)

$$\boldsymbol{V}^s_{AB} = \begin{bmatrix} T_y \\ -T_x \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1 \qquad \boldsymbol{V}^b_{AB} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1$$

(c) Charlie sits on the Carousel at a point with coordinates $\boldsymbol{q}_B = [0, r_y, 0]^T$ as seen from Beni's frame (frame B). Using your results for $\hat{\boldsymbol{V}}^s_{AB}$ and $\hat{\boldsymbol{V}}^b_{AB}$, show that the representations for Charlie's velocity as seen in Alyx's frame ($\boldsymbol{v}^s_{\boldsymbol{q}_B}$) and in Beni's frame ($\boldsymbol{v}^b_{\boldsymbol{q}_B}$) are: (2)

$$\boldsymbol{v}^s_{\boldsymbol{q}_B} = \begin{bmatrix} -r_y c_1 \\ -r_y s_1 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}_1 \qquad \boldsymbol{v}^b_{\boldsymbol{q}_B} = \begin{bmatrix} -r_y \\ 0 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}_1$$

(d) Describe with words and sketches the geometric intuition behind $\boldsymbol{V}^s_{AB}$, $\boldsymbol{V}^b_{AB}$, $\boldsymbol{v}^s_{\boldsymbol{q}_B}$, and $\boldsymbol{v}^b_{\boldsymbol{q}_B}$. (2)

We now want to find the velocity representations with respect to Charlie's frame ($\boldsymbol{V}^s_{AC}$, $\boldsymbol{V}^b_{AC}$).

(e) Show that the rigid-body transform $\boldsymbol{g}_{AC}$ can be written as: (1)

$$\boldsymbol{g}_{AC} = \begin{bmatrix} \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} T_x - r_y s_1 \\ T_y + r_y c_1 \\ 0 \end{bmatrix} \\ \mathbf{0} & 1 \end{bmatrix}$$

(f) Using this expression for $\boldsymbol{g}_{AC}$ show that the spatial velocity $\boldsymbol{V}^s_{AC}$ can be written as: (1)

$$\boldsymbol{V}^s_{AC} = \begin{bmatrix} T_y \\ -T_x \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1$$

(g) Show that the Adjoint to find $\boldsymbol{V}_{AC}^b$ from $\boldsymbol{V}_{AC}^s$ can be written as: (1)

$$\begin{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & T_x s_1 - T_y c_1 - r_y \\ 0 & 0 & T_x c_1 + T_y s_1 \\ T_y + r_y c_1 & -T_x + r_y s_1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{bmatrix}$$

(h) Using this Adjoint, show that $\boldsymbol{V}_{AC}^b$ can be written as: (1)

$$\boldsymbol{V}_{AC}^b = \begin{bmatrix} -r_y \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1$$

(i) Using your results for $\hat{\boldsymbol{V}}_{AC}^s$ and $\hat{\boldsymbol{V}}_{AC}^b$, show that the representations for Charlie's velocity as seen in Alyx's frame ($\boldsymbol{v}_{\boldsymbol{q}_C}^s$) and in Charlie's frame ($\boldsymbol{v}_{\boldsymbol{q}_C}^b$) are: (2)

$$\boldsymbol{v}_{\boldsymbol{q}_C}^s = \begin{bmatrix} -r_y c_1 \\ -r_y s_1 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}_1 \qquad \boldsymbol{v}_{\boldsymbol{q}_C}^b = \begin{bmatrix} -r_y \\ 0 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}_1$$

(j) Describe with words and sketches the geometric intuition behind $\boldsymbol{V}_{AC}^s$, $\boldsymbol{V}_{AC}^b$, $\boldsymbol{v}_{\boldsymbol{q}_C}^s$, and $\boldsymbol{v}_{\boldsymbol{q}_C}^b$. (2)

(k) Compare your results you obtained for $\boldsymbol{V}_{AB}^s$ and $\boldsymbol{V}_{AC}^s$. What did you notice, and how does it affect your results for $\boldsymbol{v}_{\boldsymbol{q}_B}^s$ and $\boldsymbol{v}_{\boldsymbol{q}_C}^s$. (1)

(l) Compare your results you obtained for $\boldsymbol{V}_{AB}^b$ and $\boldsymbol{V}_{AC}^b$. What did you notice, and how does it affect your results for $\boldsymbol{v}_{\boldsymbol{q}_B}^b$ and $\boldsymbol{v}_{\boldsymbol{q}_C}^b$. (1)