Tsinghua University

数学物理方程

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目录

0 数学基础

0.1 二阶常系数常微分方程

针对二阶线性齐次方程 $f(x)''+a_1f(x)'+a_2f(x)=0$, 特征方程为 $\lambda^2+a_1\lambda+a_2=0$, $\Delta=a_1^2-4a_2$ λ_1,λ_2 为特征方程的两根 $(\lambda_{1,2}=\alpha\pm i\beta)$，其通解为:

$$
\begin{aligned}
&\text{两实根：}\Delta>0 && f=c_1e^{\lambda_1x}+c_2e^{\lambda_2x}\\
&\text{重根：}\Delta=0 && f=(c_1+c_2x)e^{\lambda_1x}\\
&\text{两共轭复实根：}\Delta<0 && f=(c_1\cos\beta x+c_2\sin\beta x)e^{\alpha x}
\end{aligned}
$$

0.2 傅里叶级数

0.2.1 三角函数的正交性

1. 正弦函数的半周期正交性:

   在 $[0,\pi]$ 区间内，不同频率的正弦函数 $\sin\ (nx)$ 和 $\sin\ (mx)$ 是正交的，除非 $n=m$.

   $$\int_0^\pi \sin\ (nx)\cdot\sin\ (mx)\,dx=0$$

2. 余弦函数的半周期正交性:

   在 $[0,\pi]$ 区间内，不同频率的余弦函数 $\cos\ (nx)$ 和 $\cos\ (mx)$ 是正交的，除非 $n=m$.

   $$\int_0^\pi \cos(nx)\cdot\cos(mx)\,dx=0$$

3. 正弦和余弦的正交性:

   在一个完整周期 $[0,2\pi]$ 上, 正弦函数 $\sin\ (nx)$ 和余弦函数 $\cos\ (mx)$ 是正交的, 除非 $n=m=0$.

   $$\int_0^{2\pi} \sin(nx)\cdot\cos(mx)\,dx=0$$

0.2.2 傅立叶级数

傅立叶正弦级数

$$[0,l]\quad f(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{l}$$

$$C_n=\begin{cases}\frac{1}{l}\int_0^l f(x)dx & n=0\\ \frac{2}{l}\int_0^l f(x)\cos\dfrac{n\pi x}{l}dx & n\geq 1\end{cases}$$

傅立叶余弦级数

$$[0, l] \quad f(x) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi x}{l}$$

$$C_n = \begin{cases} \frac{1}{l}\int_0^l f(x)dx & n = 0 \\ \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx & n \geq 1 \end{cases}$$

傅里叶级数

$$[-l, l] \quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx$$

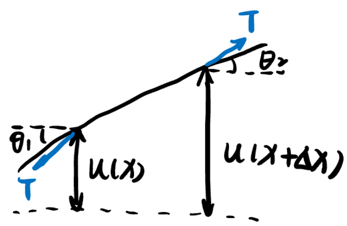
$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

1 数理方程的建立和定解条件

1.1 典型偏微分方程的建立

1.1.1 弦的横振动



小振动：$\cos\theta \approx 1$，忽略 θ^2 量级；$\sin\theta \approx \tan\theta \approx \theta = \frac{\partial u}{\partial x}$，忽略 θ^3 量级

x 方向合力：

$$F_x = T\cos\theta_2 - T\cos\theta_1 = 0$$

u 方向合力：

$$F_{合} = T\sin\theta_2 - T\sin\theta_1 \approx T\left(\frac{\partial u}{\partial x}|_{x+\Delta x} - \frac{\partial u}{\partial x}|_x\right) \approx \ T\frac{\partial^2 u}{\partial x^2}\Delta x$$

根据牛二定律：（$\dfrac{\partial^2 \bar{u}}{\partial t^2}$: 质心加速度）

$$F_{合} = \Delta m\frac{\partial^2 \bar{u}}{\partial t^2} = \rho\Delta x\frac{\partial^2 \bar{u}}{\partial t^2}$$

结合两式，得到波动方程

$$T\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

作变量代换 $a = \sqrt{\dfrac{T}{\rho}}$

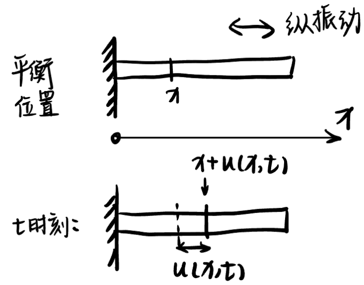
$$\boxed{\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}}$$

推广：二维均匀弹性膜横振动

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\left(\nabla^2 - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\right)u(\vec{r},t) = 0$$

1.1.2 杆的纵振动



单位面积上的拉力 $\boxed{p = E\dfrac{\partial u}{\partial x}}$，以向右为正（$E$：杨氏模量）

合力

$$
\begin{aligned}
F_{合} &= p(x+\Delta x, t)S - p(x,t)S \\
&= S\left(E|_{x+\Delta x}\frac{\partial u}{\partial x}\bigg|_{x+\Delta x} - E|_x\frac{\partial u}{\partial x}\bigg|_x\right) \\
&= S\frac{\partial}{\partial x}\left(E\frac{\partial u}{\partial x}\right) \xlongequal{E=\text{const}} ES\,\frac{\partial^2 u}{\partial x^2}\Delta x
\end{aligned}
$$

根据牛二定律:

$$F_{合} = \rho S\Delta x\,\frac{\partial^2 \bar{u}}{\partial t^2}$$

得到波动方程:

$$E\,\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}$$

作变量代换 $a = \sqrt{\dfrac{E}{\rho}}$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}}$$

2 线性偏微分方程

2.1 常微分方程的情形

2.1.1 常系数齐次 ODE

针对二阶线性齐次方程 $x'' + a_1x' + a_2x = 0, \lambda_1, \lambda_2$ 为其特征方程 $\lambda^2 + a_1\lambda + a_2 = 0$ 的两根 $(\lambda_{1,2} = \alpha \pm i\beta)$，其通解为：

$$\begin{aligned}
(i) \quad & \Delta > 0: \quad x = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} \\
(ii) \quad & \Delta = 0: \quad x = (c_1 + c_2t)e^{\lambda_1 t} \\
(iii) \quad & \Delta < 0: \quad x = (c_1\cos\beta t + c_2\sin\beta t)e^{\alpha t}
\end{aligned}$$

2.1.2 非齐次 ODE

$$\begin{cases} y''(x) + P(x)y'(x) + Q(x)y(x) = F(x) \\ y|_{x=a} = 0, y|_{x=b} = 0 \end{cases}$$

关键是先写出一个特解 $y_p(x)$ 满足 ODE，不必满足边界条件

通解 $y(x) = C_1y_1(x) + C_2y_2(x) + y_p(x)$

$y_1(x), y_2(x)$ 为齐次方程的一组线性无关解：

$$y_1'' + Py_1' + Qy_1 = 0, y_2'' + Py_2' + Qy_2 = 0$$

$y_p(x) = \int_a^x w(x;s)ds$ 是非齐次 ODE 的特解，可由齐次化原理写出

2.2 线性偏微分方程的一般理论：叠加性和解的结构

把线性偏微分方程统一写成算符形式：

$$L(u) = f$$

$$\begin{aligned}
\text{波动方程：} \quad & \frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f \quad L = \frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2} \\
\text{热传导：} \quad & \frac{\partial u}{\partial t} - D\nabla^2 u = f \quad L = \frac{\partial}{\partial t} - D\nabla^2 \\
\text{泊松方程：} \quad & \nabla^2 u = f \quad L = \nabla^2
\end{aligned}$$

其中

u：未知函数

f：已知函数，方程的非齐次项

L：线性算符

$f=0$：齐次方程

$f\neq 0$：非齐次方程

定理 2.1 (解的叠加原理).

1. $L(u_1)=0, L(u_2)=0 \Rightarrow L(c_1u_1+c_2u_2)=0$

2. $L(u_1)=0, L(u_2)=f \Rightarrow L(u_1+u_2)=f$

非齐次方程的特解 + 相应齐次方程的解仍然是非齐次方程的解

非齐次方程的通解 = 非齐次方程的任一特解 + 相应齐次方程的通解

3. $L(u_1)=f_1, L(u_2)=f_2 \Rightarrow L(c_1u_1+c_2u_2)=c_1f_1+c_2f_2$

4. $u_1,u_2,...u_n:\ L(u_1)=f_1,\ L(u_2)=f_2,...,L(u_n)=f_n$

$\Rightarrow L(c_1u_1+c_2u_2+...+c_nu_n)=c_1f_1+c_2f_2+...+c_nf_n$

2.3 齐次问题：波动方程的行波解

2.3.1 无界问题

对于无界的波动方程：

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=0,\ \ -\infty<x<\infty,\ t>0$$

通解：

$$\boxed{u(x,t)=f(x-at)+g(x+at)}$$

若给定初始条件和边界条件 $u|_{t=0}=\phi(x), \frac{\partial u}{\partial t}|_{t=0}=\psi(x)$，有

$$u(x,t)=\frac{1}{2}[\phi(x-at)+\phi(x+at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$$

其中，

$$\text{右行波：}\ f(x-at)=\frac{1}{2}\phi(x-at)-\frac{1}{2a}\int_0^{x-at}\psi(\xi)d\xi$$

$$\text{左行波：}\ g(x+at)=\frac{1}{2}\phi(x+at)+\frac{1}{2a}\int_0^{x+at}\psi(\xi)d\xi$$

2.3.2 半无界问题

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-a^2\dfrac{\partial^2 u}{\partial x^2}=0, x>0,\ t>0\\ u|_{x=0}=0,\\ u|_{t=0}=\phi(x), \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=\psi(x)\end{cases}$$

做**奇延拓**：定义

$$\Phi = \begin{cases} \phi(x), x > 0 \\ -\phi(-x), x < 0 \end{cases}, \quad \Psi = \begin{cases} \psi(x), x > 0 \\ -\psi(-x), x < 0 \end{cases}$$

$$\begin{aligned} u(x,t) =& \frac{1}{2}[\Phi(x-at)+\Phi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at}\Psi(\xi)d\xi \\ =& \begin{cases} \frac{1}{2}[\phi(x-at)+\phi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi, & t \le \frac{x}{2} \\ \frac{1}{2}[-\phi(at-x)+\phi(x+at)] + \frac{1}{2a}\int_{at-x}^{x+at}\psi(\xi)d\xi, & t > \frac{x}{2} \end{cases} \end{aligned}$$

2.4 非齐次方程的齐次化原理

例 2.1 (波动方程（对二阶的 PDE 齐次化原理）)**.**

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \; -\infty < x < \infty, \; t > 0 \\ u|_{t=0} = 0, \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \end{cases}$$

注. 若初始条件是 $u|_{t=0} = \phi(x), \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x)$，只需将问题拆解成 (1), (2) 两部分求解 $u = u_1 + u_2$

$$(1)\begin{cases} \frac{\partial^2 u_1}{\partial t^2} = a^2 \frac{\partial^2 u_1}{\partial x^2} \\ u_1|_{t=0} = \phi(x), \left.\frac{\partial u_1}{\partial t}\right|_{t=0} = \psi(x) \end{cases} \quad (2)\begin{cases} \frac{\partial^2 u_2}{\partial t^2} = a^2 \frac{\partial^2 u_2}{\partial x^2} + f \\ u_2|_{t=0} = 0, \left.\frac{\partial u_2}{\partial t}\right|_{t=0} = 0 \end{cases}$$

Step 1 分割

在 $[0, t_0]$ 区间里解方程：将 $[0, t_0]$ 分成 N 份

$$N\Delta t = t_0$$

f_n 为脉冲力，仅在小区间内非零

$$f_1(x,t) + f_2(x,t) + ... + f_N(x,t) = f(x,t)$$

$$\begin{cases} \frac{\partial^2 u_n}{\partial t^2} - a^2 \frac{\partial^2 u_n}{\partial x^2} = f(x,t), \; -\infty < x < \infty, \; t > 0 \\ u_n|_{t=0} = \left.\frac{\partial u_n}{\partial t}\right|_{t=0} = 0 \end{cases} \Rightarrow \; u(x,t) = u_1(x,t) + ... + u_N(x,t)$$

Step 2 求解脉冲力问题

$$\tau = n\Delta t$$

$$f_n|_{t<\tau-\Delta t} = 0 \; \Rightarrow \; u_n|_{t<\tau-\Delta t} = \left.\frac{\partial u_n}{\partial t}\right|_{t=0} = 0$$

在小区间内，f_n 近似为常数：$f_n(x,t) \approx f(x,\tau)\ t \in [\tau-\Delta t, \tau]$

经过区间 $[\tau-\Delta t, \tau]$，外力 f 产生速度 $\frac{\partial u_n}{\partial t}\Big|_{t=\tau} = f(x,t)\Delta t$，位移 $u_n|_{t=\tau} \sim (\Delta t)^2$ 可忽略：

$$\frac{\partial^2 u_n(x,t)}{\partial t^2} = f(x,\tau) \Rightarrow \frac{\partial u_n}{\partial t}\Big|_{t=\tau} - \frac{\partial u_n}{\partial t}\Big|_{t=\tau-\Delta t} = f(x,\tau)\Delta t$$

（$[\tau-\Delta t,\tau]$ 内，$a^2\frac{\partial^2 u_n}{\partial x^2} \sim (\Delta t)^2$，可忽略）

$$t > \tau : \begin{cases} \dfrac{\partial^2 u_n}{\partial t^2} - a^2\dfrac{\partial^2 u_n}{\partial x^2} = 0 \\ u_n|_{t=\tau} = 0, \dfrac{\partial u_n}{\partial t}\Big|_{t=\tau} = f(x,\tau)\Delta t \end{cases}$$

$$定义：w(x,t;\tau_n) = \frac{u_n(x,t)}{\Delta t} \Rightarrow \begin{cases} \dfrac{\partial^2 w}{\partial t^2} - a^2\dfrac{\partial^2 w}{\partial x^2} = 0 \\ w|_{t=\tau} = 0, \dfrac{\partial w}{\partial t}\Big|_{t=\tau} = f(x,\tau) \end{cases}$$

Step 3 $\Delta t \to 0$**，积分**

$$\begin{aligned} u(x,t) =& u_1(x,t) + u_2(x,t) + ... + u_N(x,t) \\ =& w(x,t;\tau_1)\Delta t + w(x,t;\tau_2)\Delta t + ... + w(x,t;\tau_N)\Delta t \\ \xlongequal{\Delta t \to 0}& \int_0^t w(x,t;\tau)dt \end{aligned}$$

根据行波解：

$$w(x,t;\tau) = \frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi$$

解题方法.

$$u(x,t) = \int_0^t w(x,t;\tau)d\tau$$

$$w(x,t;\tau)满足 \begin{cases} \dfrac{\partial^2 w}{\partial t^2} - a^2\dfrac{\partial^2 w}{\partial x^2} = 0, t > \tau \\ w(x,t;\tau)|_{t=\tau} = 0, \dfrac{\partial w}{\partial t}\Big|_{t=\tau} = f(x,\tau) \end{cases}$$

$$w(x,t;\tau) = \frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi$$

3 分离变量法

3.1 齐次方程

例 3.1 (热传导方程)**.** 解热传导方程:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \kappa \dfrac{\partial^2 u}{\partial t^2} \\ u|_{x=0} = u|_{x=l} = 0, u|_{t=0} = \phi(x) \end{cases}$$

1. 分离变量

假设有 $u(x,t) = X(x)T(t)$ 满足 PDE 和边界条件，暂时不考虑初始条件。

代入方程得到

$$X(x)T'(t) = \kappa X''(x)T(t) \Rightarrow \frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

（左边不依赖于 x，右边不依赖于 t，所以都不依赖，为常数）

转换为两个 ODE 问题:

$$X''(x) + \lambda X(x) = 0, T'(t) + \lambda\kappa T(t) = 0$$

边界条件:

$$u|_{x=0} = X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u|_{x=l} = X(l)T(t) = 0 \Rightarrow X(l) = 0$$

2. 本征值问题

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(l) = 0 \end{cases}$$

由 Sturm-Liouville Theory 可知: λ 为实数

i. $\lambda = 0$

$$X(x) = Ax + B$$

$X(0) = X(l) = 0 \Rightarrow \quad A = B = 0$，无解

ii. $\lambda < 0$

$$X'' - |\lambda| X = 0 \Rightarrow X = Ae^{\sqrt{|\lambda|}x} + Be^{-\sqrt{|\lambda|}x}$$

$$X(0) = 0 = A + BX(l) = 0 = Ae^{\sqrt{|\lambda|}l} + Be^{-\sqrt{|\lambda|}l}$$

无解

iii. $\lambda > 0$

$$X = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$$

$$X(0)=0 \Rightarrow B=0 \Rightarrow \quad X=A\sin\sqrt{\lambda}xX(l)=0 \Rightarrow \sqrt{\lambda}l=n,\ n=1,2,3,...$$

本征值 $\lambda_n=\left(\frac{n\pi}{l}\right)^2$

本征函数 $X_n=\sin\frac{n\pi x}{l}$

3. 乘积型解

$$T'(t)+\lambda\kappa T(t)=0$$

$$T_n'(t)+\lambda_n\kappa T_n(t) \Rightarrow T_n(t)=Be^{-\lambda_n\kappa t}=Be^{-\kappa(\frac{n}{l})^2 t}$$

$$u_n(x,t)=X_n(x)T_n(t)=B\sin\frac{nx}{l}e^{-\kappa(\frac{n}{l})^2 t}$$

此时的 $u_n|_{t=0}=B\sin\frac{nx}{l}$

4. 完整解

如果初始条件为若干 $B\sin\frac{nx}{l}$ 的和形式，可以马上得到 B_n 和 n 的值，即可通过叠加原理求得 $u(t)$

$$u|_{t=0}=\sum_{n=1}^{M}B_n\sin\frac{nx}{l} \Rightarrow u(x,t)=\sum_{n=1}^{M}B_n\sin\frac{nx}{l}e^{-\kappa(\frac{n}{l})^2 t}$$

“任意”函数 $u|_{t=0}=\phi(x)$ 可展开为

$$\phi(x)=\sum_{n=1}^{\infty}B_n\sin\frac{nx}{l}=\sum_{n=1}^{\infty}B_nX_n(x)$$

,

$$X_n=\sin\frac{n\pi}{l}x$$

求解 B_n：利用正交性

$\int_0^l X_n(x)X_m(x)dx=0, n\neq m$ 两边同乘 $X_n(x)$ 积分：

$$\int_0^l \phi(x)X_n(x)dx=B_n\int_0^l [X_n(x)]^2dx$$

$$||X_n||^2\equiv\int_0^l [X_n(x)]^2dx=\int_0^l \sin^2\frac{n\pi x}{l}dx=\frac{l}{2}$$

完整解

$$u(x,t)=\sum_{n=1}^{\infty}B_n\sin\frac{nx}{l}e^{-\kappa(\frac{n}{l})^2 t}$$

其中，

$$B_n=\frac{1}{||X_n||^2}\int_0^l \phi(x)X_n(x)dx=\frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi}{l}xdx$$

例 3.2 (波动方程：第一、二类边界条件)**.**

$$\begin{cases}\dfrac{\partial^2u}{\partial t^2}-a^2\dfrac{\partial^2u}{\partial x^2}=0, 0<x<l, t>0\\ u|_{x=0}=u|_{x=l}=0, \quad u|_{t=0}=\phi(x), \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=\psi(x)\end{cases}$$

1. 分离变量

$$u(x,t)=X(x)T(t)$$

$$XT''=a^2X''T\Rightarrow\frac{T''(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}=-\lambda$$

$$\begin{cases}X''(x)+\lambda X(x)=0\\T''(t)+\lambda a^2T(t)=0\\X(0)=X(l)=0\end{cases}$$

2. 本征值问题

本征值

$$\lambda_n=\left(\frac{n\pi}{l}\right)^2$$

本征函数

$$X_n=\sin\frac{n\pi x}{l}$$

3. 乘积型解

$$T_n''(t)+\left(\frac{n\pi a}{l}\right)^2T_n(t)=0$$

$$T_n(t)=C_n\sin\frac{n\pi a}{l}t+D_n\cos\frac{n\pi a}{l}t$$

$$u_n(x,t)=X_n(x)T_n(t)=\left(C_n\sin\frac{n\pi a}{l}t+D_n\cos\frac{n\pi a}{l}t\right)\sin\frac{n\pi x}{l}$$

4. 完整解

$$u(x,t)=\sum_{n=1}^{\infty}\left(C_n\sin\frac{n\pi a}{l}t+D_n\cos\frac{n\pi a}{l}t\right)\sin\frac{n\pi x}{l}$$

$$\begin{cases}u|_{t=0}=\phi(x)=\displaystyle\sum_{n=1}^{\infty}D_n\sin\frac{n\pi x}{l}\\\left.\dfrac{\partial u}{\partial t}\right|_{t=0}=\psi(x)=\displaystyle\sum_{n=1}^{\infty}C_n\frac{n\pi a}{l}\sin\frac{n\pi x}{l}\end{cases}$$

$$\begin{cases}D_n=\dfrac{2}{l}\displaystyle\int_0^l\phi(x)\sin\frac{n\pi x}{l}dx\\C_n=\dfrac{2}{n\pi a}\displaystyle\int_0^l\psi(x)\sin\frac{n\pi x}{l}dx\end{cases}$$

例 3.3 (波动方程：第三类边界条件)**.**

$$\begin{cases}\dfrac{\partial^2u}{\partial t^2}-a^2\dfrac{\partial^2u}{\partial x^2}=0,0<x<l,t>0\\u|_{x=0}=0,\quad\left(\dfrac{\partial u}{\partial x}+hu\right)_{x=l}=0\\u|_{t=0}=\phi(x),\quad\left.\dfrac{\partial u}{\partial t}\right|_{t=0}=\psi(x)\end{cases}$$

1. 分离变量

$$u(x,t) = X(x)T(t)$$

$$XT'' = a^2X''T \Rightarrow \frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ T''(t) + \lambda a^2 T(t) = 0 \end{cases}$$

$$u|_{x=0} = 0 \Rightarrow X(0) = 0$$

$$\left(\frac{\partial u}{\partial x} + hu\right)_{x=l} = [X'(l) + hX(l)]T(t) = 0 \Rightarrow X'(l) + hX(l) = 0$$

2. 本征值问题

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \\ X'(l) + hX(l) = 0 \end{cases}$$

Sturm-Liouville Theory: λ 为实数

i. $\lambda = 0$，无解

ii. $\lambda < 0$，无解

iii. $\lambda > 0$

$$X = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$$

$$X(0) = 0 \Rightarrow B = 0 \Rightarrow \quad X = A\sin\sqrt{\lambda}x X'(l) + hX(l) = A(\sqrt{\lambda}\cos\sqrt{\lambda}l + h\sin\sqrt{\lambda}l) = 0$$

$$\tan(\sqrt{\lambda}l) = -\frac{\sqrt{\lambda}}{h}$$

记 $\sqrt{\lambda}l = \mu$，有 $\tan\mu = -\frac{\mu}{hl}$

$$\lambda_n = \left(\frac{\mu_n}{l}\right)^2$$

$$X_n = \sin\sqrt{\lambda_n}x = \sin\frac{\mu_n x}{l}$$

3. 乘积型解

$$T_n''(t) + \lambda_n a^2 T_n(t) = 0$$

$$T_n(t) = C_n\sin\sqrt{\lambda_n}at + D_n\cos\sqrt{\lambda_n}at$$

$$u_n(x,t) = X_n(x)T_n(t) = \left(C_n\sin\sqrt{\lambda_n}at + D_n\cos\sqrt{\lambda_n}at\right)\sin\sqrt{\lambda_n}x$$

4. 完整解

仍然有正交性

$$\int_0^l X_n(x)X_m(x)dx = 0, n \neq m$$

$$||X_n||^2 \equiv \int_0^l [X_n(x)]^2 dx = \int_0^l \frac{1-\cos(2\sqrt{\lambda_n}x)}{2}dx = \frac{l}{2}\left[1 - \frac{\sin(2\sqrt{\lambda_n}l)}{2\sqrt{\lambda_n}l}\right]$$

$$u|_{t=0} = \phi(x) = \sum_{n=1}^{\infty} D_n \sin \sqrt{\lambda_n} x$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi(x) = \sum_{n=1}^{\infty} C_n \sqrt{\lambda_n} a \sin(\sqrt{\lambda_n} x)$$

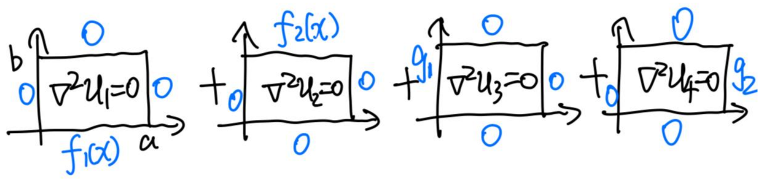
$$D_n = \frac{1}{||X_n||^2} \int_0^l \phi(x) \sin(\sqrt{\lambda_n} x) dx$$

$$C_n = \frac{1}{\sqrt{\lambda_n} a ||X_n||^2} \int_0^l \psi(x) \sin(\sqrt{\lambda_n} x) dx$$

例 3.4 (Laplace 方程)**.**

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \\ u(0, y) = u(a, y) = 0, \quad u(x, 0) = 0, \quad u(x, b) = f(x) \end{cases}$$

若边界条件为下图所示，则将以下四种边界条件的解叠加。



1. 分离变量

$$u(x, y) = X(x)Y(y)$$

$$XY'' + X''Y = 0 \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ Y''(t) - \lambda Y(y) = 0 \\ X(0) = X(a) = 0 \end{cases}$$

2. 本征值问题

本征值

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2$$

本征函数

$$X_n = \sin \frac{n\pi x}{a}$$

3. 乘积型解

$$Y_n''(y) - \left(\frac{n\pi}{l}\right)^2 Y_n(y) = 0$$

$$Y_n(y) = C_n \sinh \frac{n\pi}{a} y + D_n \cosh \frac{n\pi}{a} y$$

$$u_n(x,y) = X_n(x)Y_n(y) = \left(C_n \sinh \frac{n\pi}{a} y + D_n \cosh \frac{n\pi}{a} y\right) \sin \frac{n\pi x}{a}$$

4. 完整解

$$u(x,y) = \sum_{n=1}^{\infty} \left(C_n \sinh \frac{n\pi}{a} y + D_n \cosh \frac{n\pi}{a} y\right) \sin \frac{n\pi x}{a}$$

$$u|_{y=0} = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{a} = 0 \Rightarrow D_n = 0$$

$$u|_{y=b} = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a} = f(x)$$

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi}{a} y \sin \frac{n\pi x}{a}$$

$$C_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

3.2 齐次多变量问题

例 3.5 (二维扩散问题)**.**

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(D\dfrac{\partial u}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(D\dfrac{\partial u}{\partial y}\right) \\ \text{绝热边界：} \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = \left.\dfrac{\partial u}{\partial x}\right|_{x=a} = \left.\dfrac{\partial u}{\partial y}\right|_{y=0} = \left.\dfrac{\partial u}{\partial y}\right|_{y=b} = 0 \\ u|_{t=0} = \phi(x,y) \end{cases}$$

1. 分离变量

$$u(x,y,b) = X(x)Y(y)T(t)$$

$$\frac{1}{D} XYT' = X''YT + XY''T$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{D}\frac{T'}{T} = -\lambda$$

$$\therefore \frac{X''}{X} = -\mu, \frac{Y''}{Y} = -\nu, \lambda = \mu + \nu$$

2. 本征值问题

$$\begin{cases} X'' + \mu X = 0 \\ X'(0) = X'(a) = 0 \end{cases}$$

1. $\mu = 0, X_n(x) = Ax + B \Rightarrow X_0(x) = 1$
2. $\mu > 0, X_n = \cos \frac{n\pi x}{a}, \mu_n = \left(\frac{n\pi}{a}\right)^2, n = 1, 2, 3, ...$
3. $\mu < 0$，无解

合并 1, 2:

$$\mu_n = \left(\frac{n\pi}{a}\right)^2, X_n = \cos\frac{n\pi x}{a}, n = 0, 1, 2, ...$$

$$\begin{cases} Y'' + \nu X = 0 \\ Y'(0) = Y'(b) = 0 \end{cases}$$

同理可得:

$$Y_m(y) = \cos\frac{m\pi x}{b}, \nu_n = \left(\frac{m\pi}{b}\right)^2, n = 0, 1, 2, ...$$

$$\lambda_{mn} = \mu_n + \nu_m = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

3. 乘积型解

$$T'_{nm}(t) = -\lambda_{nm} DT(t)$$

$$\Rightarrow T_{nm}(t) = A_{mn} e^{-\lambda_{nm} Dt}$$

$$u_{nm}(x, y, t) = X_n(x) Y_m(y) T_{nm}(t)$$

4. 完整解

$$u(x, y, t) = \sum_{n,m} u_{nm}(x, y, t) = \sum_{n,m=0}^{\infty} \cos\frac{n\pi x}{a} \cos\frac{m\pi x}{b} e^{-[(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2]t}$$

二重傅里叶级数（1）

$$u|_{t=0} = \phi(x, y) = \sum_{n,m=0}^{\infty} A_{nm} \cos\frac{n\pi x}{a} \cos\frac{m\pi x}{b}$$

正交性:

$$\int_0^a X_n X_{n'} dx = \begin{cases} \frac{a}{2} & n = n' \neq 0 \\ a & n = n' = 0 \\ 0 & n \neq n' \end{cases} \equiv \frac{a}{2}(1 + \delta_{n0})\delta_{nn'}$$

同理:

$$\int_0^b Y_m Y_{m'} dy = \frac{b}{2}(1 + \delta_{m0})\delta_{mm'}$$

（1）式两边同乘 $X_n Y_m$ 并作积分 $\int_0^a \int_0^b dxdy$ 得:

$$\int_0^a \int_0^b \phi(x, y) \cos\frac{n\pi x}{a} \cos\frac{m\pi x}{b} dxdy = \frac{a}{2}(1 + \delta_{n0})\frac{b}{2}(1 + \delta_{m0}) A_{nm}$$

$$A_{mn} = \frac{4}{ab} \frac{1}{(1 + \delta_{n0})(1 + \delta_{m0})} \int_0^a \int_0^b \phi(x, y) \cos\frac{n\pi x}{a} \cos\frac{m\pi x}{b} dxdy$$

3.3 非齐次方程

3.3.1 齐次化原理

考虑以下非齐次方程

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = f(x,t) \quad 0 < x < l, t > 0 \\ u|_{x=0} = u|_{x=l} = 0 \\ u|_{t=0} = \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0 \end{cases}$$

引入辅助函数 $w(x,t;\tau)$，满足

$$w(x,t;\tau): \begin{cases} \dfrac{\partial^2 w}{\partial t^2} - a^2 \dfrac{\partial^2 w}{\partial x^2} = 0 \quad 0 < x < l, t > \tau \\ w|_{t=\tau} = 0, \left.\dfrac{\partial w}{\partial t}\right|_{t=\tau} = f(x,\tau) \end{cases}$$

可以得到方程的解

$$u(x,t) = \int_0^t w(x,t;\tau) d\tau$$

$$w(x,t;\tau) = \sum_{n=1}^{\infty} C_n(\tau) \sin\left[\frac{n\pi a}{l}(t-\tau)\right] \sin\frac{n\pi a}{l} dx$$

其中，

$$C_n = \frac{l}{n\pi a}\frac{2}{l}\int_0^l f(x,\tau)\sin\frac{n\pi x}{l}dx = \frac{l}{n\pi a} f_n(\tau)$$

$$\begin{aligned} u(x,t) &= \int_0^t w(x,t;\tau) d\tau \\ &= \sum_{n=1}^{\infty}\left[\int_0^t C_n(\tau)\sin\frac{n\pi a}{l}(t-\tau)d\tau\right]\sin\frac{n\pi x}{l} \\ &= \sum_{n=1}^{\infty}\left[\int_0^t \frac{l}{n\pi a} f_n(\tau)\sin\frac{n\pi a}{l}(t-\tau)d\tau\right]\sin\frac{n\pi x}{l} \\ &\equiv \sum_{n=1}^{\infty} T_n(t) X_n(x) \end{aligned}$$

3.3.2 本征函数展开法

根据齐次化原理的结果，提示解取此形式分离变量:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

展开 $f(x,t)$

$$f(x,t)=\sum_{n=1}^{\infty}f_n(t)X_n(x)$$

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=f(x,t)\Rightarrow\sum_{n=1}^{\infty}T_n''X_n-a^2T_nX_n''=\sum_{n=1}^{\infty}f_nX_n$$

又 $X_n''=-\lambda_nX_n$

$$\sum_{n=1}^{\infty}(T_n''+a^2\lambda_nT_n-f_n)X_n=0$$

$$T_n''(t)+a^2\lambda_nT_n(t)=f_n(t),\lambda_n=\left(\frac{n\pi}{l}\right)^2$$

化为解非齐次 ODE 问题:

$$T_n''(t)+\left(\frac{n\pi a}{l}\right)^2T_n(t)=f_n(t)$$

$$W(t;\tau):\begin{cases}W_n''+\left(\frac{n\pi a}{l}\right)^2W_n=0\\ W|_{t=\tau}=0,W_n'|_{t=\tau}=f_n(\tau)\end{cases}$$

$$T(x,t)=\int_0^tW(t;\tau)d\tau$$

$$W_n=(t,\tau)=f_n(\tau)\frac{l}{n\pi a}\sin\frac{n\pi a}{l}(t-\tau)$$

$$T(t)=\int_0^tW(t;\tau)d\tau=\frac{l}{n\pi a}\int_0^tf_n(\tau)\sin\frac{n\pi a}{l}(t-\tau)d\tau$$

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x)$$

3.3.3 特解法

找特解 v 满足:

$$\frac{\partial^2 v}{\partial t^2}-a^2\frac{\partial^2 v}{\partial x^2}=f(x,t),v|_{x=0}=v|_{x=l}=0$$

$u(x,t)=v(x,t)$（已取定）$+w(x,t)$（待求）

$$\frac{\partial^2(v+w)}{\partial t^2}-a^2\frac{\partial^2(v+w)}{\partial x^2}=f(x,t)$$

$$\Rightarrow\frac{\partial^2 w}{\partial t^2}-a^2\frac{\partial^2 w}{\partial x^2}=0,w|_{x=0}=w|_{x=l}=0$$

3.4 非齐次边界条件

3.4.1 边界条件齐次化

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = f(x,t) \quad 0 < x < l, t > 0 \\ u|_{x=0} = \mu(t), u|_{x=l} = \nu(t) \\ u|_{t=0} = \phi(x), \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \psi(x) \end{cases}$$

Step 1 边界条件齐次化

$$u(x,t) = p(x,t) + w(x,t)$$

$$p|_{x=0} = \mu(t), p|_{x=l} = \nu(t)$$

p 满足边界条件，但不必满足 PDE

$$\Rightarrow w|_{x=0} = u|_{x=0} - p|_{x=0} = 0, w|_{x=l} = 0$$

Step 2 求解 $w(x,t)$

$$\frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 (u-p)}{\partial t^2} - a^2 \frac{\partial^2 (u-p)}{\partial x^2} = f(x,t) - \left(\frac{\partial^2 p}{\partial t^2} - a^2 \frac{\partial^2 p}{\partial x^2} \right)$$

$$w|_{x=0} = w|_{x=l} = 0$$

$$w|_{t=0} = u|_{t=0} - p|_{t=0} = \phi(x) - p|_{t=0}$$

$$\left.\frac{\partial w}{\partial t}\right|_{t=0} = \left.\frac{\partial u}{\partial t}\right|_{t=0} - \left.\frac{\partial p}{\partial t}\right|_{t=0} = \psi(x) - \left.\frac{\partial p}{\partial t}\right|_{t=0}$$

p 的选取：自由度较大

第一类： 边界上的函数值 $u|_{x=0} = \mu(t), u|_{x=l} = \nu(t)$

$$p(x,t) = A(t)x + B(t) = \mu(t) + \frac{\nu(t) - \mu(t)}{l} x$$

$$p(x,t) = A(t)(l-x)^2 + B(t)x^2 = \frac{\mu(t)}{l^2}(l-x)^2 + \frac{\nu(t)}{l} x^2$$

第二类： 边界上的函数微商 $\frac{\partial u}{\partial x}|_{x=0} = \mu(x), \frac{\partial u}{\partial x}|_{x=l} = \nu(x)$

$$p(x,t) = A(t)x^2 + B(t)x = \left(\frac{\nu(t)}{2l} - \frac{\mu(t)}{2} \right) x^2 + \mu(t)x$$

第三类： 混合边界条件，指定边界上函数值与微商的线性关系

$$\alpha_1 u + \beta_1 \frac{\partial u}{\partial x}|_{x=0} = \mu(x), \alpha_2 u + \beta_2 \frac{\partial u}{\partial x}|_{x=l} = \nu(x)$$

$$p(x,t) = A(t)x^2 + B(t)$$

$$\begin{pmatrix} \beta_1 & \alpha_1 \\ \alpha_2 l + \beta_2 & \alpha_2 \end{pmatrix} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \begin{pmatrix} \mu(t) \\ \nu(t) \end{pmatrix}$$

$$\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \frac{1}{\beta_1\alpha_2 - \alpha_1\beta_2 - \alpha_1\alpha_2 l} \begin{pmatrix} \alpha_2 & -\alpha_1 \\ -(\alpha_2 l + \beta_2) & \beta_1 \end{pmatrix} \begin{pmatrix} \mu(t) \\ \nu(t) \end{pmatrix}$$

例 3.6 (弦受迫振动).

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \quad 0 < x < l, t > 0 \\ u|_{x=0} = 0, u|_{x=l} = A\sin\omega t \\ u|_{t=0} = 0, \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0 \end{cases}$$

$$p(x,t) = \mu(t) + \frac{\nu(t) - \mu(t)}{l} x = \frac{Ax}{l}\sin\omega t$$

$$\frac{\partial^2 w}{\partial t^2} - a^2 \frac{\partial^2 w}{\partial x^2} = -\left(\frac{\partial^2 p}{\partial t^2} - a^2 \frac{\partial^2 p}{\partial x^2}\right) = \frac{A\omega^2 x}{l}\sin\omega t$$

$$w|_{x=0} = w|_{x=l} = 0$$

$$w|_{t=0} = -p|_{t=0} = 0$$

$$\left.\frac{\partial w}{\partial t}\right|_{t=0} = -\left.\frac{\partial p}{\partial t}\right|_{t=0} = -\frac{A\omega}{l}x$$

$$w = w_1 + w_2$$

$$w_1 : \begin{cases} \dfrac{\partial^2 w_1}{\partial t^2} - a^2 \dfrac{\partial^2 w_1}{\partial x^2} = 0 \\ w_1|_{x=0} = w_1|_{x=l} = 0 \\ w_1|_{t=0} = 0, \left.\dfrac{\partial w_1}{\partial t}\right|_{t=0} = -\dfrac{A\omega}{l}x \end{cases} \qquad w_2 : \begin{cases} \dfrac{\partial^2 w_2}{\partial t^2} - a^2 \dfrac{\partial^2 w_2}{\partial x^2} = \dfrac{A\omega^2 x}{l}\sin\omega t \\ w_2|_{x=0} = w_2|_{x=l} = 0 \\ w_2|_{t=0} = \left.\dfrac{\partial w_2}{\partial t}\right|_{t=0} = 0 \end{cases}$$

解 1:

$$w_1 = \sum_{n=1}^{\infty} C_n \sin\omega_n t \sin\frac{n\pi x}{l}, \omega_n = \frac{n\pi a}{l}$$

$$\left.\frac{\partial w_1}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} C_n\omega_n \sin\frac{n\pi x}{l} - \frac{A\omega}{l}x \Rightarrow C_n = (-1)^n \frac{2A\omega}{n\pi\omega_n}$$

解 2:

$$w_2 = \sum_{n=1}^{\infty} \frac{g_n}{\omega_n} \frac{\omega\sin\omega_n t - \omega_n \sin\omega t}{\omega^2 - \omega_n^2} \sin\frac{n\pi x}{l}$$

Where

$$g_n = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}dx = \frac{2A\omega^2}{n\pi}(-1)^{n+1}$$

$$w = w_1 + w_2 = \sum_{n=1}^{\infty} (-1)^n \frac{2Aa}{l} \frac{\omega_n \sin\omega t - \omega\sin\omega_n t}{\omega^2 - \omega_n^2} \sin\frac{n\pi x}{l}$$

例 3.7 (温度周期边界条件)**.**

$$\begin{cases}\dfrac{\partial u}{\partial t}-D\dfrac{\partial^2 u}{\partial x^2}=0 \quad 0<x<l,t>0\\ u|_{x=0}=A\sin\omega t, u|_{x=l}=0\\ u|_{t=0}=0\end{cases}$$

$$p(x,t)=\mu(t)+\frac{v(t)-\mu(t)}{l}x=A(1-\frac{x}{l}\sin\omega t)$$

$$\begin{cases}\dfrac{\partial w}{\partial t}-D\dfrac{\partial^2 w}{\partial x^2}=-\left(\dfrac{\partial p}{\partial t}-D\dfrac{\partial^2 p}{\partial x^2}\right)=-A\omega(1-\dfrac{x}{l})\cos\omega t \quad 0<x<l,t>0\\ w|_{x=0}=w|_{x=l}=0\\ w|_{t=0}=-p|_{t=0}=0\end{cases}$$

齐次化原理:

$$w(x,t)=\int_0^t v(x,t;\tau)d\tau, v|_{t=\tau}=f(x,\tau)=\sum_{n=1}^{\infty}f_n(\tau)\sin\frac{n\pi x}{l}$$

$$w(x,t)=\int_0^t v(x,t;\tau)d\tau=\sum_{n=1}^{\infty}\left[f_n(\tau)e^{-D(\frac{n\pi}{l})^2(t-\tau)}d\tau\right]\sin\frac{n\pi x}{l}$$

$$f_n(\tau)=\frac{2}{l}\int_0^l f(x,\tau)\sin\frac{n\pi x}{l}dx=-\frac{2A\omega}{n\pi}\cos\omega t$$

得到

$$w(x,t)=\sum_{n=1}^{\infty}\frac{2A\omega l^2}{D^2(n\pi)^4+\omega^2l^4}\frac{1}{n\pi}\left[D(n\pi)^2e^{-D(\frac{n\pi}{l})^2t}-D(n\pi)^2\cos\omega t-\omega l^2\sin\omega t\right]\sin\frac{n\pi x}{l}$$

$$u=p+w$$

3.4.2 方程与边界条件同时齐次化

例 3.8.

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-a^2\dfrac{\partial^2 u}{\partial x^2}=0 \quad 0<x<l,t>0\\ u|_{x=0}=0, u|_{x=l}=A\sin\omega t\\ u|_{t=0}=0, \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=0\end{cases}$$

$$u=p+w$$

p 不仅满足边界条件，还满足方程:

$$\begin{cases}\dfrac{\partial^2 p}{\partial t^2}-a^2\dfrac{\partial^2 p}{\partial x^2}=0 \quad 0<x<l,t>0\\ p|_{x=0}=0, p|_{x=l}=A\sin\omega t\end{cases}$$

设 $p(x,t)=f(x)\sin\omega t, f(0)=0, f(l)=A$

$$[\omega^2 f(x)+a^2 f''(x)]\sin\omega t=0\Rightarrow f(x)=A\frac{\sin\frac{\omega x}{a}}{\sin\frac{\omega l}{a}}$$

$$p(x,t)=A\frac{\sin\frac{\omega x}{a}}{\sin\frac{\omega l}{a}}\sin\omega t$$

$$\begin{cases}\dfrac{\partial^2 w}{\partial t^2}-a^2\dfrac{\partial^2 w}{\partial x^2}=0 \quad 0<x<l, t>0\\ w|_{x=0}=w|_{x=l}=0\\ w|_{t=0}=-p|_{t=0}=0, \left.\dfrac{\partial w}{\partial t}\right|_{t=0}=-\left.\dfrac{\partial p}{\partial t}\right|_{t=0}=-A\dfrac{\sin\frac{\omega x}{a}}{\sin\frac{\omega l}{a}}\end{cases}$$

$$w(x,t)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2Aa\omega}{l(\omega^2-\omega_n^2)}\sin\omega_n t\sin\frac{n\pi x}{l}$$

$$p(x,t)=A\frac{\sin\frac{\omega x}{a}}{\sin\frac{\omega l}{a}}\sin\omega t=\sum_{n=1}^{\infty}(-1)^n\frac{2Aa\omega_n}{l}\frac{1}{\omega^2-\omega_n^2}\sin\omega t\sin\frac{n\pi x}{l}$$

$$u=p+w=\sum_{n=1}^{\infty}(-1)^n\frac{2Aa}{l}\frac{\omega_n\sin\omega t-\omega\sin\omega_n t}{\omega^2-\omega_n^2}\sin\frac{n\pi x}{l}$$

4 正交曲线坐标系

4.1 常用的正交曲线坐标系

正交曲线坐标系：空间任意一点，所有坐标线相互垂直

极坐标系：

$$x = r\cos\phi$$
$$y = r\sin\phi$$

柱坐标系：

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

球坐标系：

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

抛物坐标系：

$$x = \sqrt{\xi\eta}\cos\phi$$
$$y = \sqrt{\xi\eta}\sin\phi$$
$$z = \frac{1}{2}(\xi - \eta)$$

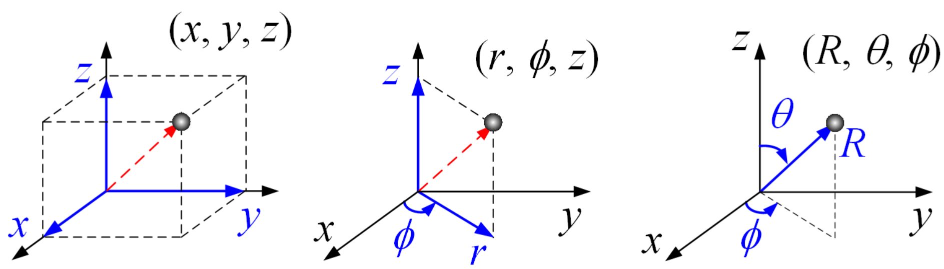


图 1: 直角坐标系，柱坐标与球坐标

4.2 正交曲线坐标的判别法：度规法

曲线坐标系 (x_1, x_2, x_3)，$x = x(x_1, x_2, x_3), y = y(y_1, y_2, y_3), z = x(x_1, x_2, x_3)$

$$\begin{aligned} dx &= \frac{\partial x}{\partial x_1}dx_1 + \frac{\partial x}{\partial x_2}dx_2 + \frac{\partial x}{\partial x_3}dx_3 \\ dy &= \frac{\partial y}{\partial x_1}dx_1 + \frac{\partial y}{\partial x_2}dx_2 + \frac{\partial y}{\partial x_3}dx_3 \\ dz &= \frac{\partial z}{\partial x_1}dx_1 + \frac{\partial z}{\partial x_2}dx_2 + \frac{\partial z}{\partial x_3}dx_3 \end{aligned}$$

距离微元

$$
\begin{aligned}
ds^2 \equiv & dx^2 + dy^2 + dz^2 \\
= & \left(\frac{\partial x}{\partial x_1}dx_1 + \frac{\partial x}{\partial x_2}dx_2 + \frac{\partial x}{\partial x_3}dx_3\right)^2 + \left(\frac{\partial y}{\partial x_1}dx_1 + \frac{\partial y}{\partial x_2}dx_2 + \frac{\partial y}{\partial x_3}dx_3\right)^2 \\
& + \left(\frac{\partial z}{\partial x_1}dx_1 + \frac{\partial z}{\partial x_2}dx_2 + \frac{\partial z}{\partial x_3}dx_3\right)^2 \\
= & \sum_{i,j=1,2,3} g_{ij} dx_i dx_j \\
\equiv & g_{11}(dx_1)^2 + g_{22}(dx_2)^2 + g_{33}(dx_3)^2 + 2g_{12}dx_1dx_2 + 2g_{13}dx_1dx_3 + 2g_{23}dx_2dx_3
\end{aligned}
$$

其中，

$$
g_{ij} = g_{ji} = \frac{\partial x}{\partial x_i}\frac{\partial x}{\partial x_j} + \frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j} + \frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}
$$

柱坐标：

$$
ds^2 = [d(r\cos\phi)]^2 + [d(r\sin\phi)]^2 + dz^2 = dr^2 + r^2 d\phi^2 + dz^2
$$

球坐标：

$$
ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2
$$

抛物坐标：

$$
ds^2 = \frac{\xi + \eta}{4\xi}(d\xi)^2 + \frac{\xi + \eta}{4\eta}(d\eta)^2 + \xi\eta(d\phi)^2
$$

定理 4.1 (度规法). **ds^2 无交叉项** ⇔ 正交曲线坐标系

$$
ds^2 = (h_1 dx_1)^2 + (h_2 dx_2)^2 + (h_3 dx_3)^2
$$

其中，

$$
g_{11} = h_1^2, g_{22} = h_2^2, g_{33} = h_3^2
$$

4.3 正交曲线坐标的微分算子

梯度 ∇u

$$
\nabla u = \frac{1}{h_1}\frac{\partial u}{\partial x_1}\vec{e}_1 + \frac{1}{h_2}\frac{\partial u}{\partial x_2}\vec{e}_2 + \frac{1}{h_3}\frac{\partial u}{\partial x_3}\vec{e}_3
$$

散度 $\nabla \cdot \vec{v}$

$$
\begin{aligned}
\nabla \cdot \vec{v} = & \frac{\text{净通量}}{h_1 h_2 h_3 dx_1 dx_2 dx_3} \\
= & \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(v_1 h_2 h_3) + \frac{\partial}{\partial x_2}(v_2 h_1 h_3) + \frac{\partial}{\partial x_3}(v_3 h_1 h_2)\right]
\end{aligned}
$$

Laplacian $\nabla^2 u = \nabla \cdot (\nabla u)$

$$\nabla^2 u = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial u}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial u}{\partial x_2}\right)+\frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right]$$

$$\text{柱坐标}\quad \nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}+\frac{\partial^2 u}{\partial z^2}$$
$$\text{球坐标}\quad \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

4.4 正交曲线坐标的分离变量

例 4.1 (极坐标下的稳态传热问题)**.**

$$\begin{cases}\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \phi^2}=0, 0<r<a, 0<\phi<2\pi\\ u|_{r=a}=f(\phi)\\ u|_{\phi=0}=u|_{\phi=2\pi}, \left.\dfrac{\partial u}{\partial \phi}\right|_{\phi=0}=\left.\dfrac{\partial u}{\partial \phi}\right|_{\phi=2\pi}\\ u|_{r=0}<\infty\end{cases}$$

1. 分离变量

设 $u = R(r)\Phi(\phi)$

$$\Rightarrow \begin{cases}\dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dR}{dr}\right)-\lambda R=0\\ R(0)<\infty\end{cases}\qquad \begin{cases}\Phi''(\phi)+\lambda\Phi(\phi)=0\\ \Phi(0)=\Phi(2\pi), \Phi'(0)=\Phi'(2\pi)\end{cases}$$

2. 本征值问题

$$\text{i.}\lambda = 0 \Rightarrow \Phi_0(0)=1$$
$$\text{ii.}\lambda < 0\text{，无解}$$
$$\text{iii.}\lambda > 0 \Rightarrow \Phi = A\sin\sqrt{\lambda}\phi + B\cos\sqrt{\lambda}\phi$$

$$\Phi(0)=\Phi(2\pi) \Rightarrow B = A\sin(2\pi\sqrt{\lambda})+B\cos(2\pi\sqrt{\lambda})$$
$$\Phi'(0)=\Phi'(2\pi) \Rightarrow A = A\cos(2\pi\sqrt{\lambda})-B\sin(2\pi\sqrt{\lambda})$$

$$\begin{vmatrix}\sin(2\pi\sqrt{\lambda}) & \cos(2\pi\sqrt{\lambda})-1\\ \cos(2\pi\sqrt{\lambda})-1 & -\sin(2\pi\sqrt{\lambda})\end{vmatrix}=0 \Rightarrow \cos(2\pi\sqrt{\lambda})=1$$

本征值：$\lambda_m = m^2, m=1,2,...,A,B$ 任意

本征函数：$\Phi_{m1}=\sin m\phi, \Phi_{m2}=\cos m\phi$

3. 乘积型解

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)-\lambda R=0$$

即欧拉方程，采用变量代换:

$$r=e^t \quad ,\frac{d}{dt}=\frac{dr}{dt}\frac{d}{dr}=r\frac{d}{dr}$$

得到关于 t 的 ODE

$$\frac{d^2R(t)}{dt^2}-\lambda R(t)=0$$

i. $\lambda=0\Rightarrow R_0(t)=C_0+D_0t=C_0+D_0\ln r$
ii. $\lambda_m=m^2,m=1,2,3,...$

$$R_m=C_me^{mt}+D_me^{-mt}=C_mr^m+D_mr^{-m}$$

$$u(r,\phi)=C_0+\sum_{m=1}^{\infty}\left(C_{m1}r^m\sin m\phi+C_{m2}r^m\cos m\phi\right)$$

（由于 $u|_{r=0}<\infty$, 没有 D_mr^{-m} 和 $D_0\ln r$ 项）

$$u|_{r=a}=f(\phi)=C_0+\sum_{m=1}^{\infty}a^m\left(C_{m1}\sin m\phi+C_{m2}\cos m\phi\right)$$

4. 完整解

$$C_0=\frac{1}{2\pi}\int_0^{2\pi}f(\phi)d\phi$$
$$C_{m1}=\frac{1}{\pi a^m}\int_0^{2\pi}f(\phi)\sin(m\phi)d\phi$$
$$C_{m2}=\frac{1}{\pi a^m}\int_0^{2\pi}f(\phi)\cos(m\phi)d\phi$$

$$u(r,\phi)=C_0+\sum_{m=1}^{\infty}r^m\left(C_{m1}\sin m\phi+C_{m2}\cos m\phi\right)$$

5 Sturm-Liouville 理论

5.1 S-L 问题

应对本征值问题的复杂性

1. S-L 理论：直接导出解的性质
2. 级数解法

Def (S-L 型方程的一般形式)**.**

$$\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right]-q(x)y(x)+\lambda\rho(x)y(x)=0 \quad ,0<x<b$$

引进算符

$$L=-\frac{d}{dx}(p(x)\frac{d}{dx})+q(x)$$

方程可以写成：

$$Ly(x)=\lambda\rho(x)y(x)$$

加适当的齐次边界条件

$p(x),q(x),\rho(x)$ 非常数：非均匀性的体现（材料不均匀/曲线坐标系）
考虑 $p(x),q(x),\rho(x)$ 为实值函数，连续函数在开区间 (a,b) 上

$$\begin{cases} p(x)>0 \\ q(x)\geq 0 \\ \rho(x)>0 \end{cases}$$

5.2 自伴算符的本征值问题

5.2.1 自伴算符

Def (自伴性)**.**

$$\langle Lf,g\rangle=\langle f,Lg\rangle$$

则称 L 具有自伴性 L 的伴算子 $L^\dagger$ 定义为：

$$\langle L^\dagger f,g\rangle=\langle f,Lg\rangle$$

自伴：

$$L^\dagger=L$$

注. 函数内积定义

$$\langle f,g\rangle=\int_a^b f^*(x)g(x)dx$$

5.2.2 L 算符的自伴性

$$(Lf)^*g-f^*(Lg)=\frac{d}{dx}\left[p(x)\left(f^*\frac{dg}{dx}-g\frac{df^*}{dx}\right)\right]$$

两边积分:

$$\int_a^b[(Lf)^*g-f^*(Lg)]dx=p(x)\left[f^*\frac{dg}{dx}-g\frac{df^*}{dx}\right]\Bigg|_a^b$$

$$\langle Lf,g\rangle-\langle f,Lg\rangle=\text{边界项}$$

若边界项 $=0\Rightarrow\langle Lf,g\rangle=\langle f,Lg\rangle$，L 具有自伴性

自伴性强烈依赖边界条件，只有当边界条件使得边界项 $=0$ 时，L 具有自伴性

边界项 $=0$ 的几种典型实现方式

$$p(x)\left[f^*\frac{dg}{dx}-g\frac{df^*}{dx}\right]\Bigg|_a^b=0$$

1. 每个端点 (a,b) 处，加第一、二、三类边界条件，使两边界各自为零

第一类:

$$\begin{cases}f(a)=g(a)=0\\f(b)=g(b)=0\end{cases}$$

第二类:

$$\begin{cases}f'(a)=g'(a)=0\\f'(b)=g'(b)=0\end{cases}$$

第三类:

$$\begin{cases}f'(a)=\alpha f(a),g'(a)=\alpha g(a)\\f'(b)=\beta f(b),g'(b)=\beta g(b)\end{cases}$$

2. 两端点边界项抵消

$$p(a)=p(b)$$

周期边界条件:

$$\begin{cases}f(a)=f(b),f'(a)=f'(b)\\g(a)=g(b),g'(a)=g'(b)\end{cases}$$

3. 端点 $p(x)=0$，加有界条件

$$p(a)=p(b)=0,y(a),y(b),y'(a),y'(b)<\infty$$

5.2.3 自伴算符的基本性质

1. 本征值的可数性：自伴算符的本征值必然存在。本征值有无穷多个，构成可数集

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...$$

$$\lim_{n\to\infty} \lambda_n = +\infty$$

2. 本征值的实数性 $\lambda = \lambda^*$

3. 本征函数的正交性：对应不同本征值的本征函数一定正交

$$\int_a^b \rho(x) f^*(x) g(x) dx = 0$$

4. 本征值的非负性 $\lambda \geq 0$

若 $\rho(a)f^*(a)f'(a) - \rho(b)f^*(b)f'(b) \geq 0$，则 $\lambda \geq 0$

第一、二类边界条件显然满足

第三类边界条件

$$\begin{cases} \alpha_1 f(a) + \beta_1 f'(a) = 0 & \Rightarrow f'(a) = -\frac{\alpha_1}{\beta_1} f(a) \\ \alpha_2 f(b) + \beta_2 f'(b) = 0 & \Rightarrow f'(b) = -\frac{\alpha_2}{\beta_2} f(b) \end{cases}$$

$$\alpha_1\beta_1 < 0, \alpha_2\beta_2 > 0 \Rightarrow \lambda \geq 0$$

排除非物理情形，第三类边界条件也有 $\lambda \geq 0$

5. 完备性

自伴算符的本征函数（的全体）构成一个完备的函数组，即任意一个在区间 $[a,b]$ 中有连续二阶导数、且满足和自伴算符 L 相同的边界条件的函数 $f(x)$，均可按本征函数 $\{y_n(x)\}$ 展开为绝对而且一致收敛的级数。

$$f(x) = \sum_{n=1}^{\infty} C_n X_n(x)$$

$$C_n = \frac{\int_a^b \rho(x) X_n^* f(x)}{||X_n||^2} = \frac{\int_a^b \rho(x) X_n^* f(x)}{\int_a^b \rho(x)|X_n(x)|^2 dx}$$

$$||X_n||^2 = \int_a^b \rho(x)|X_n(x)|^2 dx$$

5.3 S-L 型方程的本征值问题

将方程化为 S-L 方程的标准形式

$$\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] - q(x)y(x) + \lambda\rho(x)y(x) = 0$$

一般方程：

$$y''(x) + a(x)y'(x) + b(x)y(x) + \lambda c(x)y(x) = 0$$

$$e^{\int^x a(x')dx'}[y'' + ay' + by + \lambda cy] = 0$$

$$\frac{d}{dx}[e^{\int^x a(x')dx'}\frac{dy}{dx}] + b(x)e^{\int^x a(x')dx'}y + \lambda c(x)e^{\int^x a(x')dx'}y = 0$$

$$p(x) = e^{\int^x a(x')dx'}$$

$$q(x) = -b(x)e^{\int^x a(x')dx'}$$

$$\rho(x) = c(x)e^{\int^x a(x')dx'}$$

6 二阶线性常微分方程的幂级数解法

6.1 幂级数

Def (幂级数是通项为幂函数的函数项级数). 幂级数是通项为幂函数的函数项级数

$$\sum_{n=0}^{\infty} c_n(z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + ... + c_n(z-n)^a + ...$$

6.1.1 函数的幂级数展开

设 $f(x)$ 在 x_0 的邻域内任意阶可导，则

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

其中，

$$a_n = \frac{f^{(n)}(x_0)}{n!} \quad (n = 0, 1, 2, ...)$$

常见函数在 $x=0$ 处的幂级数

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \quad (-1 < x < 1) \\
\frac{1}{1+x} &= \sum_{n=0}^{\infty} (-1)^n x^n \quad (-1 < x < 1) \\
\ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad (-1 < x \leq 1) \\
-\ln(1-x) &= \sum_{n=1}^{\infty} \frac{x^n}{n} \quad (-1 \leq x < 1)
\end{aligned}$$

6.1.2 幂级数的收敛半径

在幂级数的收敛点与发散点之间存在一个分界线，而且这个分界线一定是圆周. 圆内区域称为幂级数的收敛圆. 收敛圆的半径称为收敛半径. 作为特殊情况，收敛半径可以是 0(收敛圆退化为一

个点. 除 $z=a$ 点外，幂级数在全平面处处发散)，也可以是 ∞(收敛圆就是全平面. 幂级数在全平面收敛，但在 ∞ 点肯定发散，除非此幂级数只有常数项一项).

定理 6.1 (达朗贝尔判别法)**.** 对于级数 $u_0+u_1+u_2+\dots$,

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \begin{cases} >1\text{发散} \\ <1\text{收敛} \\ =1\text{无法判别} \end{cases}$$

幂级数 $\sum_{n=0}^{\infty} c_n(z-a)^n$ 的收敛半径为

$$R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

定理 6.2 (高斯判别法)**.** 对于正项级数

$$\sum_{k=0}^{\infty} u_k \quad ,(u_k>0)$$

若 $\lim_{k\to\infty}\frac{u_k}{u_{k+1}}$ 可以写成

$$\lim_{k\to\infty}\frac{u_k}{u_{k+1}} = 1+\frac{\mu}{k}+\frac{\theta_k}{k^2} \quad (k\to\infty, |\theta_k|<\infty)$$

可以根据 μ 的大小判断收敛性

$$\mu\begin{cases} >1\text{收敛} \\ \le 1\text{发散} \end{cases}$$

（$\mu=1$ 的情况：调和级数 $\sum_{k=1}^{\infty}\frac{1}{k}$，发散）

6.2 二阶线性常微分方程的常点和奇点

二阶线性齐次常微分方程的标准形式:

$$y''(x)+p(x)y'(x)+q(x)y(x)=0$$

根据 $p(x), q(x)$ 在 x_0 附近的行为分类:

1. x_0 为常点：$p(x), q(x)$ 在 x_0 点解析（$p(x_0), q(x_0)<\infty$）

$$p(x)=\sum_{k=0}^{\infty} A_k(x-x_0)^k$$

$$q(x)=\sum_{k=0}^{\infty} B_k(x-x_0)^k$$

2. x_0 为奇点：$p(x)$ 或 $q(x)$ 在 x_0 点不解析

- 正则奇点：$x \to x_0 : (x-x_0)p(x) < \infty, (x-x_0)^2 q(x) < \infty$

$$(x-x_0)p(x) = \sum_{k=0}^{\infty} A_k (x-x_0)^k$$

$$(x-x_0)^2 q(x) = \sum_{k=0}^{\infty} B_k (x-x_0)^k$$

- 非正则奇点 $x \to x_0 : (x-x_0)p(x) \to \infty$ 或 $(x-x_0)^2 q(x) \to \infty$

6.3　方程常点邻域内的解

定理 6.3. 若 $p(x), q(x)$ 在 x_0 的邻域 $|x-x_0| < R$ 内解析，则 $y''(x) + p(x)y'(x) + q(x)y(x) = 0$，$y(x_0) = C_0, y'(x_0) = C_1$ 在 $|x-x_0| < R$ 内有唯一解，且解具有解析性

$$y(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k$$

例 6.1 (Legendre 方程的本征值问题)**.** Legendre 方程：$(1-x^2)y'' - 2xy' + \lambda y = 0$

$$y'' - \frac{2x}{1-x^2}y' + \frac{\lambda}{1-x^2}y = 0$$

$x = 0$ 为常点，$x = \pm 1$ 是奇点

在 $x = 0$ 附近求解级数

$$\begin{aligned}
y(x) &= \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + ... \\
y'(x) &= \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k = a_1 + 2a_2 x + 3a_3 x^2 + ... \\
y''(x) &= \sum_{k=0}^{\infty} (k+1)(k+2) a_{k+2} x^k = 2\cdot 1 a_2 + 3\cdot 2 a_3 x + 4\cdot 3 a_4 x^2 + ... \\
-x^2 y'' &= -\sum_{k=2}^{\infty} k(k-1) a_k x^k = -2\cdot 1 a_2 x^2 - 3\cdot 2 a_3 x^3 - 4\cdot 3 a_4 x^4 - ... \\
-2xy' &= -2\sum_{k=1}^{\infty} k a_k x^k = -2a_1 x - 2\cdot 2a_2 x^2 - 2\cdot 3a_3 x^3 + ... \\
\lambda y &= \lambda \sum_{k=0}^{\infty} a_k x^k = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + ...
\end{aligned}$$

代入原方程：

$$\begin{aligned}
&(1-x^2)y'' - 2xy' + \lambda y \\
=&(2a_2 + \lambda a_0) + [6a_3 + (\lambda - 2)a_1]x + \\
&\sum_{k=2}^{\infty} \{(k+2)(k+1)a_{k+2} + [\lambda - k(k+1)]a_k\} x^k = 0
\end{aligned}$$

递推关系：$(k+2)(k+1)a_{k+2}+[\lambda-k(k+1)]a_k=0 \quad k\geq 0$

$$a_{k+2}=\frac{k(k+1)-\lambda}{(k+1)(k+2)}a_k$$
$$a_{2n}=\prod_{j=1}^{n}\frac{(2j-1)(2j-2)-\lambda}{2j(2j-1)}a_0$$
$$a_{2n+1}=\prod_{j=1}^{n}\frac{2j(2j-1)-\lambda}{2j(2j+1)}a_1$$

得到方程的级数解：

$$y(x)=(a_0+a_2x^2+...)+(a_1x+a_3x^3+...)=a_0y_0(x)+a_1y_1(x)$$
$$y_0(x)=1+\sum_{n=1}^{\infty}\prod_{j=1}^{n}\frac{(2j-1)(2j-2)-\lambda}{2j(2j-1)}x^{2n}$$
$$y_1(x)=x+\sum_{n=1}^{\infty}\prod_{j=1}^{n}\frac{2j(2j-1)-\lambda}{2j(2j+1)}x^{2n+1}$$

计算 $a_0y_0(x)=a_0+a_2x^2+a_4x^4+...$ 的收敛区间：
利用达朗贝尔判别法 6.1，相邻项比为

$$\lim_{k\to\infty}\left|\frac{a_{2n+2}x^{2n+2}}{a_{2n}x^{2n}}\right|=\lim_{k\to\infty}|x^2|\frac{2n(2n+1)-\lambda}{(2n+2)(2n+1)}=\lim_{k\to\infty}|x^2|$$

$$\begin{cases}|x^2|>1\text{发散}\\|x^2|<1\text{收敛}\\|x^2|=1\text{无法判别}\end{cases}$$

对于 $|x|=1$ 的情况使用高斯判别法 6.2

$x=1$ 时：

$$\begin{aligned}\frac{a_{2n}x^{2n}}{a_{2n+2}x^{2n+2}}&=\frac{a_{2n}}{a_{2n+2}}=\frac{(2n+2)(2n+1)}{2n(2n+1)-\lambda}\\ &\xrightarrow{n\to\infty}\frac{(2n+2)(2n+1)}{2n(2n+1)}\frac{1}{1-\frac{\lambda}{2n(2n+1)}}\\ &=(1+\frac{1}{n})\left[1+\frac{\lambda}{2n(2n+1)}+\left(\frac{\lambda}{2n(2n+1)}\right)^2+...\right]\\ &=1+\frac{1}{n}+\frac{\theta_n}{n^2}\quad(|\theta_n|<\infty)\\ &>1\quad\text{发散}\end{aligned}$$

和 $y(\pm1)<\infty$ 矛盾。出路：级数退化为多项式

$$a_{k+2}=\frac{k(k+1)-\lambda}{(k+1)(k+2)}a_k,\quad \lambda=l(l+1),l=0,1,2,...$$
$$a_{l+2}=\frac{l(l+1)-l(l+1)}{(l+1)(l+2)}a_l=0\Rightarrow a_{l+4}=a_{l+6}=0\text{从}l+4\text{项开始退化}$$

$$\begin{cases} l\text{为偶：}\ y_0(x) = a_0 + a_2x^2 + ... + a_lx^l & \text{退化} \\ l\text{为奇：}\ y_1(x) = a_1x + a_3x^3 + ... + a_lx^l & \text{退化} \end{cases}$$

Legendre 方程在 $y(\pm 1)$ 有界这一边界条件下：
本征值 $\lambda_l = l(l+1)$
本征函数为 $P_l(x), l = 0, 1, 2, ...$

$$P_l(x) = \begin{cases} y_0(x)/y_0(1) & l\text{为偶，偶函数} \\ y_1(x)/y_1(1) & l\text{为奇，奇函数} \end{cases}$$

$$\begin{aligned} P_0(x) =& 1 \\ P_1(x) =& x \\ P_2(x) =& \frac{1}{2}(3x^2 - 1) \\ P_3(x) =& \frac{1}{2}(5x^3 - 3x) \\ P_4(x) =& \frac{1}{8}(35x^4 - 30x^2 + 3) \\ P_5(x) =& \frac{1}{8}(63x^5 - 70x^3 + 15x) \end{aligned}$$

6.4 方程正则奇点领域内的解

定理 6.4 (Fuchs 定理)**.** 广义幂级数解 (Frobenius 方法)

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

在正则奇点 x_0 的去心邻域 $0 < |x - x_0| < R$ 内，有两个线性无关解：

$$\begin{aligned} y_1(x) &= (x - x_0)^{\rho_1} \sum_{k=0}^{\infty} c_k(x - x_0)^k \quad c_0 \neq 0 \\ y_2(x) &= gy_1(x)ln(x - x_0) + (x - x_0)^{\rho_2} \sum_{k=0}^{\infty} d_k(x - x_0)^k \quad ,g\text{或}d_0 \neq 0 \end{aligned}$$

解 6.1 (一般正则奇点解法：对系数进行幂级数展开)**.**

$$\begin{aligned} & y'' + p(x)y' + q(x)y = 0 \Rightarrow x^2y'' + x[xp(x)]y' + x^2q(x)y = 0 \\ & xp(x) = A_0 + A_1x + A_2x^2 + ... \\ & x^2q(x) = B_0 + B_1x + B_2x^2 + ... \end{aligned}$$

设解取以下形式（由欧拉方程启发）

$$y = \sum_{k=0}^{\infty} a_kx^{k+f}(a_0 \neq 0)$$

得到一阶导和二阶导

$$y' = \sum_{k=0}^{\infty} a_k (k+f) x^{k+f-1}$$

$$y'' = \sum_{k=0}^{\infty} a_k (k+f)(k+f-1) x^{k+f-2}$$

定义 $L[y] = x^2 y'' + x(A_0 + A_1 x + ...)y' + (B_0 + B_1 x + ...)y = 0$ ，代入级数解：

$$L[y] = L\left[\sum_{k=0}^{\infty} a_k x^{k+f} (a_0 \neq 0)\right]$$

$$= x^\rho \left[\sum_{k=0}^{\infty} (k+\rho)(k+\rho+1) a_k x^k + \left(\sum_{k=0}^{\infty} A_m x^m\right)\left(\sum_{k=0}^{\infty} (k+\rho) a_k x^k\right) + \left(\sum_{k=0}^{\infty} B_m x^m\right)\left(\sum_{k=0}^{\infty} a_k x^k\right)\right]$$

$$\equiv a_0 f_0(\rho) x^\rho + \sum_{k=1}^{\infty} F_k x^{k+\rho}$$

其中，

$$f_0(\rho) \equiv \rho(\rho-1) + A_0 \rho + B_0$$

$$F_k(\rho) \equiv [(k+\rho)(k+\rho-1) + A_0(k+\rho) + B_0] a_k + \sum_{k=1}^{\infty} [(m+\rho) A_{k-m} + B_{k-m}] a_m$$

$$= f_0(\rho+k) a_k + \sum_{k=1}^{\infty} [(m+\rho) A_{k-m} + B_{k-m}] a_m$$

上式每项为零，需要

1. ρ 满足指标方程 $f_0(\rho) = 0$
2. $\{a_k\}$ 满足递推关系：$F_k = 0 (k \geq 1)$

可以得到广义幂级数解（系数由递推关系确定）

$$y = \sum_{k=0}^{\infty} a_k x^{k+\rho}$$

解题方法 (第一解的求法).

1. 将解写成广义幂级数形式 $y(x) = \sum_{k=0}^{\infty} a_k x^{k+\rho}$
2. 计算常微分方程的每一项，相同次数对齐书写
3. 代入常微分方程相加，每项系数都为零
4. x^ρ 项对应系数为指标方程，得到两个指标根 ρ_1 和 ρ_2，根据 $\rho_1 - \rho_2$ 情况分类讨论第二解解法
5. 令求和项系数 $F_k = 0$，得到递推关系
6. 如果求和项前面有两项，第二项可以判断奇/偶数项首项是否为零；如果只有一项，可以任取 a_0。一般取 $a_0 = 1$，Bessel 函数取 $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$

6.4.1 $\rho_1 - \rho_2 \neq$ 整数

方程的两个线性无关解为

$$y_1 = \sum_{k=0}^{\infty} a_k(\rho_1)x^{k+\rho_1}, y_2 = \sum_{k=0}^{\infty} a_k(\rho_2)x^{k+\rho_2}$$

6.4.2 $\rho_1 = \rho_2$

第一解:

$$y_1 = \sum_{k=0}^{\infty} a_k x^{k+\rho_1}$$

第二解:

$$y_2 = \left.\frac{\partial y(x;p)}{\partial \rho}\right|_{\rho=\rho_1} = gy_1(x)\ln x + \sum_{k=0}^{\infty}\left(\frac{\partial a_k}{\partial \rho}\right)_{\rho=\rho_1} x^{k+\rho_1}$$

取 $y(x;p) = \sum_{k=0}^{\infty} a_k x^{k+\rho}$ 满足递推关系 $F_k = 0$

$$L[y(x;p)] = a_0 f_0(p)x^\rho = a_0(\rho-\rho_1)(\rho-\rho_2)x^\rho = a_0(\rho-\rho_1)^2 x^\rho$$

对 ρ 求导得

$$\begin{aligned}\frac{\partial}{\partial \rho}L[y(x;p)] = L\left[\frac{\partial y}{\partial \rho}\right] = \frac{\partial}{\partial \rho}[a_0(\rho-\rho_1)^2 x^\rho]\\ =2a_0(\rho-\rho_1)x^\rho + a_0(\rho-\rho_1)^2 x^\rho \ln x = 0\end{aligned}$$

$$\Rightarrow L\left[\left.\frac{\partial y}{\partial \rho}\right|_{\rho=\rho_1}\right] = 0$$

，即第二解为

$$\boxed{\left.\frac{\partial y(x;p)}{\partial \rho}\right|_{\rho=\rho_1}}$$

$$\begin{aligned}y(x;p) &= \sum_{k=0}^{\infty} a_k x^{k+\rho}\\ \Rightarrow y_2(x) &= \left.\frac{\partial y(x;p)}{\partial \rho}\right|_{\rho=\rho_1}\\ &= \sum_{k=0}^{\infty}\left(a_k x^{k+\rho}\ln x + \frac{\partial a_k}{\partial \rho}x^{k+\rho}\right)_{\rho=\rho_1}\\ &= y_1(x)\ln x + \sum_{k=0}^{\infty}\left(\frac{\partial a_k}{\partial \rho}\right)_{\rho=\rho_1} x^{k+\rho_1}\end{aligned}$$

$g \neq 0$，为常数

例 6.2. 在 $x=0$ 附近求解 $xy''+y'-4y=0$

$$y(x)=\sum_{k=0}^{\infty}a_k x^{\rho+k}$$

代入 $xy''+y'-4y=0$:

$$\begin{aligned}x^2y''&=\rho(\rho-1)a_0x^{\rho}+(\rho+1)\rho a_1x^{\rho+1}+...+(\rho+k)(\rho+k-1)a_kx^{\rho+k}+...\\xy'&=\rho a_0x^{\rho}+(\rho+1)a_1x^{\rho+1}+...+(\rho+k)a_kx^{\rho+k}+...\\-4xy&=-4a_0x^{\rho+1}-...-4a_{k-1}x^{\rho+k}-...\end{aligned}$$

指标方程 $f_0(\rho)=\rho(\rho-1)+\rho=\rho^2\Rightarrow\rho_1=\rho_2=0$

$$F_k=0\Rightarrow a_k=\frac{4}{(\rho+k)^2}a_{k-1}=\frac{4^k}{(\rho+1)^2...(\rho+k)^2}a_0$$

取 $\rho=\rho_1=0$，得

$$\begin{aligned}y_1(x)&=1+\sum_{k=1}^{\infty}\frac{4^k}{(k!)^2}x^k\quad(a_0=1)\\y_2(x)&=\left.\frac{\partial y(x;p)}{\partial\rho}\right|_{\rho=\rho_1}=\frac{\partial}{\partial\rho}\left(\sum_{k=0}^{\infty}a_kx^{\rho+k}\right)_{\rho=0}\\&=\sum_{k=0}^{\infty}\left(a_kx^{k+\rho}\ln x+\frac{\partial a_k}{\partial\rho}x^{k+\rho}\right)_{\rho=0}\\&=y_1(x)\ln x-2\sum_{k=1}^{\infty}\frac{4^k}{(\rho+1)^2...(\rho+k)^2}\left(\frac{1}{\rho+1}+...+\frac{1}{\rho+k}\right)x^{\rho+k}\Bigg|_{\rho=0}\\&=y_1(x)\ln x-2\sum_{k=1}^{\infty}\frac{4^kH_k}{(k!)^2}x^k\quad\left(H_k=1+\frac{1}{2}+...+\frac{1}{k}\right)\end{aligned}$$

6.4.3 $\rho_1-\rho_2=$ 整数

表面看来和 $\rho_1-\rho_2\neq$ 整数时并无不同，实则不然

第一解:

$$y_1=\sum_{k=0}^{\infty}a_kx^{k+\rho_1}$$

第二解: $g=0$ 或 $g\neq0$ 当求 $y_2=\sum_{k=0}^{\infty}a_k(\rho_2)x^{k+\rho_2}$ 时，递推关系 $F_n=0$ 出现意外:

$$f_0(\rho_2+n)a_n=(...)a_{n-1}+(...)a_{n-2}+...+(...)a_0=(......)a_0$$

而 $\rho_1=rho_2+n$ 是指标方程之根，即 $f_0(\rho_2+n)=0$

1. 若 a_0 系数 $=0$，则 a_n 任意，$y_2(x)$ 含 a_0,a_n 两个任意常数，是方程的通解
2. 若 a_0 系数 $\neq0$，则递推关系无法满足，无 $y_2=\sum_{k=0}^{\infty}a_k(\rho_2)x^{k+\rho_2}$ 形式的解。第二解需要另行求出，含对数项。

6.5 贝塞尔方程的解

Def (Gamma 函数)**.**

$$\boxed{\Gamma(n+1) \equiv n! \quad , n = 0, 1, 2, ...}$$

将定义域拓宽到实数域:

保持递推关系: $\boxed{\Gamma(x+1) = x\Gamma(x)}$

$$\begin{cases} x > 0, & \Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt \\ x < 0, & \Gamma(x)\text{由}\Gamma(x+1) = x\Gamma(x)\text{定义} \end{cases}$$

性质:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

Def (Bessel 方程)**.** $x^2y'' + xy' + (x^2 - \nu^2)y = 0$

$$y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y = 0$$

$xp(x) < \infty, x^2g(x) < \infty \Rightarrow x = 0$ 为正则奇点

取

$$y(x) = \sum_{k=0}^{\infty} a_k x^{\rho+k} = a_0 x^{\rho} + a_1 x^{\rho+1} + ...(a_0 \neq 0)$$

代入方程得

$$\begin{aligned} & x^2y'' + xy' + (x^2 - \nu^2)y \\ =& a_0(\rho^2 - \nu^2)x^{\rho} + [(\rho+1)^2 - \nu^2]a_1 x^{\rho+1} + \sum_{k=2}^{\infty}\{[(\rho+k)^2 - \nu^2]a_k + a_{k-2}\} \\ \equiv& a_0 f_0(\rho)x^{\rho} + \sum_{k=1}^{\infty} F_k(\rho)x^{\rho+k} \end{aligned}$$

指标方程

$$f_0(\rho) = \rho^2 - \nu^2 = 0, \quad \rho_1 = \nu, \rho_2 = -\nu$$

$$a_1[(\rho+1)^2 - \nu^2] = (\rho+1+\nu)(\rho+1-\nu)a_1 = 0 \Rightarrow a_1 = 0$$

先考虑 $\rho = \rho_1 = \nu$

$$\begin{aligned} a_k &= -\frac{1}{(\rho+k)^2 - \nu^2} a_{k-2} (k \geq 2) \\ &= -\frac{1}{k(2\nu+k)} a_{k-2} \end{aligned}$$

由于 $a_1=0$，所有奇数项为零; 偶数项:

$$\begin{aligned}a_{2n}=&-\frac{1}{2n(2\nu+2n)}a_{2n-2}=-\frac{1}{4}\frac{1}{n(\nu+n)}a_{2n-2}\\=&(-1)^n\frac{1}{2^{2n}}\frac{1}{n!(\nu+n)(\nu+n-1)...(\nu+1)}a_0\end{aligned}$$

得到第一个级数解:

$$\begin{aligned}y_1(x)=&a_0x^\nu[1-\frac{1}{\nu+1}\left(\frac{x}{2}\right)^2+\frac{1}{2!}\frac{1}{(\nu+1)(\nu+2)}\left(\frac{x}{2}\right)^4+...\\&+(-1)^n\frac{1}{n!}\frac{1}{(\nu+1)...(\nu+n)}\left(\frac{x}{2}\right)^{2n}+...]\end{aligned}$$

根据达朗贝尔判别法，

$$\lim\frac{a_kx^k}{a_{k-2}x^{k-2}}=x^2\lim_{k\to\infty}\left[-\frac{1}{k(2\nu+k)}\right]=0$$

得收敛半径 $R=\infty$

习惯取

$$a_0=\frac{1}{2^\nu\Gamma(\nu+1)}$$

可以化简 a_{2n} 的形式为:

$$\begin{aligned}\Rightarrow a_{2n}=&(-1)^n\frac{1}{2^{2n}}\frac{1}{n!(\nu+n)(\nu+n-1)...(\nu+1)}\frac{1}{2^\nu\Gamma(\nu+1)}\\=&(-1)^n\frac{1}{2^{2n+\nu}}\frac{1}{n!(\nu+n)(\nu+n-1)...(\nu+2)\Gamma(\nu+2)}\\=&...\\=&(-1)^n\frac{1}{2^{2n+\nu}}\frac{1}{n!\Gamma(\nu+n+1)}\end{aligned}$$

$y_1(x)$ 为 ν 阶 Bessel 函数.

$$y_1(x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!\Gamma(\nu+n+1)}\left(\frac{x}{2}\right)^{2n+\nu}\equiv J_\nu(x)$$

6.5.1 $\nu\neq$ 整数：第一类贝塞尔函数

$$k(2\nu-k)a_k=a_{k-2}$$

可以看到此时当 k 为偶数时，$k(2\nu-k)\neq0$

同理，取 $\rho=\rho_2=-\nu$

$$\begin{aligned}y_2(x)=&b_0x^{-\nu}[1-\frac{1}{-\nu+1}\left(\frac{x}{2}\right)^2+\frac{1}{2!}\frac{1}{(-\nu+1)(-\nu+2)}\left(\frac{x}{2}\right)^4+...\\&+(-1)^n\frac{1}{n!}\frac{1}{(-\nu+1)...(-\nu+n)}\left(\frac{x}{2}\right)^{2n}+...]\end{aligned}$$

取

$$b_0 = \frac{1}{2^{-\nu}\Gamma(-\nu+1)} = \frac{2^{\nu}}{\Gamma(-\nu+1)}$$

$y_2(x)$ 为 $-\nu$ 阶 Bessel 函数.

$$y_2(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n!\Gamma(-\nu+n+1)}\left(\frac{x}{2}\right)^{2n-\nu} \equiv J_{-\nu}(x)$$

通解:

$$y(x) = C_1 J_{\nu}(x) + C_2 J_{-\nu}(x)$$

6.5.2 $\nu = 0$

见柱函数一章

7 球函数

7.1 Legendre 方程

将 Laplace 方程 $\nabla^2 u(r,\theta,\phi)=0$ 在球坐标系下分离变量（见球谐函数），将会得到以下连带 Legendre 方程:

$$\boxed{\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[\lambda-\frac{m^2}{\sin^2\theta}\right]\Theta=0}$$

其中，$m=0,1,2,\ldots$

变量代换 $x=\cos\theta(-1\le x\le 1)$; $\quad y=\Theta$

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+\left(\lambda-\frac{m^2}{1-x^2}\right)y=0$$

$$\boxed{(1-x^2)y''-2xy'+\left(\lambda-\frac{m^2}{1-x^2}\right)y=0}$$

$m=0$：Legendre 方程

$$\boxed{(1-x^2)y''-2xy'+\lambda y=0}$$

本征值：$\lambda_l=l(l+1)\quad(l=0,1,2,\ldots)$（推导见Legendre 方程的本征值问题）;

本征函数：$y_l(x)=\mathrm{P}_l(x)$ 称为 **Legendre 多项式**

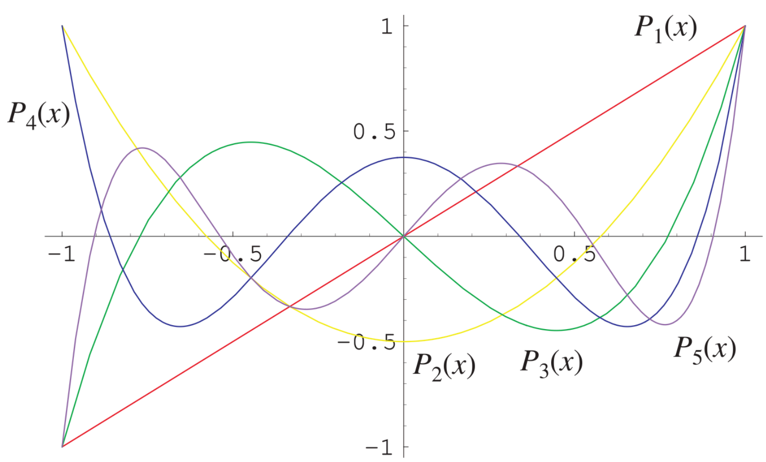


图 2: Legendre 多项式（注意到奇偶性与 l 一致）

利用 $(1-x^2)\mathrm{P}_l''-2x\mathrm{P}_l'+l(l+1)\mathrm{P}_l=0$ 可以证明

$$\int_x^1 \mathrm{P}_k(x)\mathrm{P}_l(x)\mathrm{d}x=\left(1-x^2\right)\frac{\mathrm{P}_k'(x)\mathrm{P}_l(x)-\mathrm{P}_l'(x)\mathrm{P}_k(x)}{k(k+1)-l(l+1)},\quad k\neq l.$$

7.1.1 微分表示

定理 7.1 (Leibniz rule).

$$(uv)^{(n)} = \sum_{k=0}^{n} \mathrm{C}_n^k u^{(n-k)} n^{(k)}$$

$$\boxed{\mathrm{P}_l(x) = \frac{1}{2^l l!}[(x^2-1)^l]^{(l)}}$$

$$\begin{aligned}
\text{例：}\ \mathrm{P}_1(x) =& \frac{1}{2}\frac{d}{dx}(x^2-1) = x \\
\mathrm{P}_2(x) =& \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) = \frac{1}{2}(3x^2-1)
\end{aligned}$$

证明 7.1 ($\mathrm{P}_l(x)$ 满足 Legendre 方程)**.**

设 $y = (x^2-1)^l$

$$y' = 2lx(x^2-1)^{l-1} \Rightarrow (x^2-1)y' = 2lx(x^2-1)^l = 2lxy$$

对上式求 $l+1$ 阶导数

$$(x^2-1)y^{(l+2)} + (l+1)y^{(l+1)}2x + \frac{l(l+1)}{2}y^{(l)} \cdot 2 = 2lxy^{(l+1)} + 2l(l+1)y^{(l)}$$

$$(1-x^2)y^{(l+2)} - 2xy^{(l+1)} + l(l+1)y^{(l)} = 0$$

因此 $y^{(l)}$ 满足 Lengendre 方程

由于我们通常约定 $\mathrm{P}_l(1) = 1$, 因此 $\mathrm{P}_l(x) = \frac{1}{y^{(l)}(1)}y^{(l)}(x)$，其中

$$\begin{aligned}
&y^{(l)}(1) = [(x^2-1)^l]^{(l)}|_{x=1} \\
=&[(x-1)^l(x+1)^l]^{(l)}|_{x=1} \\
=&[(x-1)^l]^{(l)}(x+1)^l|_{x=1} + l[(x-1)^l]^{(l-1)}[(x+1)^l]'|_{x=1} + \ldots \\
=&l!2^l \\
\Rightarrow&\mathrm{P}_l(x) = \frac{1}{2^l l!}[(x-1)^l]^{(l)}
\end{aligned}$$

例 7.1. 计算：

$$\int_{-1}^{1} x^k \mathrm{P}_l(x) dx$$

由 $\mathrm{P}_l(x)=(-1)^l\mathrm{P}_l(-x)$ 可知，l 为奇数时为奇函数，为偶数时为偶函数；若 $k\pm l$ 为奇数，积分为零，则当 $k\pm l$ 为偶数时

$$\begin{aligned}
&\int_{-1}^{1}x^k\mathrm{P}_l(x)dx\\
=&\frac{1}{2^l l!}\int_{-1}^{1}x^k[(x^2-1)^l]^{(l)}dx\\
=&\frac{1}{2^l l!}x^k[(x^2-1)^l]^{(l-1)}|_{-1}^{1}-\frac{1}{2^l l!}\int_{-1}^{1}(x^k)'[(x^2-1)^l]^{(l-1)}dx\\
=&\dots\\
=&\frac{1}{2^l l!}\int_{-1}^{1}(x^k)^{(l)}(1-x)^2dx
\end{aligned}$$

$k<l$：$(x^k)^{(l)}=0\Rightarrow 0$

$k\geq l$：$k=l+2n(n\geq 0)\Rightarrow$

$$\begin{aligned}
&\int_{-1}^{1}x^{l+2n}\mathrm{P}_l(x)dx\\
=&\frac{1}{2^l l!}\int_{-1}^{1}(x^{l+2n})^{(l)}(1-x^2)^l dx\\
=&\frac{1}{2^l l!}\frac{(l+2n)!}{(2n)!}\int_{-1}^{1}x^{2n}(1-x^2)^l dx\\
\xlongequal{x^2=t}&\frac{1}{2^l l!}\frac{(l+2n)!}{(2n)!}\int_{0}^{1}t^{n-\frac{1}{2}}(1-t)^l dx\\
=&\frac{1}{2^l l!}\frac{(l+2n)!}{(2n)!}\frac{\Gamma(n+\frac{1}{2})\Gamma(l+1)}{\Gamma(n+l+\frac{3}{2})}\\
=&2^{l+1}\frac{(l+2n)!}{n!}\frac{(l+n)!}{(2l+2n+1)!}
\end{aligned}$$

注. 多次利用 $\Gamma(x+1)=x\Gamma(x)$

$$\begin{aligned}
\Gamma(n+l+\tfrac{3}{2})=&(n+l+\tfrac{1}{2})(n-1+l+\tfrac{1}{2})\dots(1+\tfrac{1}{2})\frac{1}{2}\Gamma(\tfrac{1}{2})=\frac{1}{2^{2(n+l)+1}}\frac{(2n+2l+1)!}{(n+l)!}\Gamma(\tfrac{1}{2})\\
\Gamma(n+\tfrac{1}{2})=&\frac{1}{2^{2n}n!}\Gamma(\tfrac{1}{2})
\end{aligned}$$

Def (Beta 函数)**.**

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

例 7.2.

$$I_l=\int_0^1\mathrm{P}_l(x)dx$$

l 为偶数时：利用对称性

$$I_l=I_{2n}=\int_0^1\mathrm{P}_l(x)dx=\frac{1}{2}\int_{-1}^{1}\mathrm{P}_{2n}(x)dx=0\quad l=2n(n=1,2,3,\dots):\quad \mathrm{P}_{2n}(x)\mathrm{P}_0(x)\text{正交}$$

l 为奇数时:

$$\begin{aligned}
I_{2n+1} =& \frac{1}{2^{2n+1}(2n+1)!}\int_0^1 [(x^2-1)^{2n+1}]^{(2n+1)}dx \\
=& \frac{1}{2^{2n+1}(2n+1)!}[(x^2-1)^{2n+1}]^{(2n)}\Big|_0^1 \\
=& -\frac{1}{2^{2n+1}(2n+1)!}[(x^2-1)^{2n+1}]^{(2n)}\Big|_{x=0} \\
=& -\frac{1}{2^{2n+1}(2n+1)!}[\mathrm{C}_{2n+1}^{n}(x^2)^{(n)}(-1)^{n+1}]^{(2n)}\Big|_{x=0} \\
=& (-1)^n\frac{1}{2^{2n+1}(2n+1)!}\frac{(2n+1)!}{(n+1)!n!}(2n)! \\
=& \boxed{(-1)^n\frac{(2n-1)!!}{(2n+2)!!}}
\end{aligned}$$

7.1.2 广义傅里叶级数

正交性:

$$\int_{-1}^{1}\mathrm{P}_k(x)\mathrm{P}_l(x)dx = 0 \quad (k\neq l)$$

$x=\cos\theta$

$$\int_0^\pi \mathrm{P}_k(\cos\theta)\mathrm{P}_l(\cos\theta)\sin\theta d\theta = 0 \quad (k\neq l)$$

$$\boxed{||\mathrm{P}_l||^2 = \int_{-1}^{1}[\mathrm{P}_l(x)]^2dx = \frac{2}{2l+1}}$$

利用正交性确定系数:

$$[-1,1]: \quad f(x) = \sum_{l=0}^{\infty}C_l\mathrm{P}_l(x) \qquad C_l = \frac{1}{||\mathrm{P}_l||^2}\int_{-1}^{1}f(x)\mathrm{P}_l(x)dx = \frac{2l+1}{2}\int_{-1}^{1}f(x)\mathrm{P}_l(x)dx$$

$$[0,\pi]: \quad f(\cos\theta) = \sum_{l=0}^{\infty}C_l\mathrm{P}_l(\cos\theta) \qquad C_l = \frac{2l+1}{2}\int_{-1}^{1}f(\cos\theta)\mathrm{P}_l(\cos\theta)\sin\theta d\theta$$

7.1.3 生成函数（母函数, Generating Function）

生成函数：$\{a_n\}$ 的信息翻译为 $f(t)$ 的信息，后者容易处理，可以导出更多信息，然后再翻译为 $\{a_n\}$ 的信息。

例 7.3 (斐波那契数列的通项公式)**.** 递推关系：$a_n = a_{n-1} + a_{n-2}(n\geq 2)$

生成函数：$f(t) = a_0 + a_1t + a_2t^2 + \dots$

$$\begin{aligned}
\sum_{n=2}^{\infty}a_nt^n =& \sum_{n=2}^{\infty}(a_{n-1}t^n + a_{n-2}t^n) \\
f(t) - a_0 - a_1t =& t[f(t)-a_0] + t^2f(t)
\end{aligned}$$

$$\Rightarrow (1-t-t^2)f(t) = a_0 + a_1 t - a_0 t = 1$$
$$f(t) = \frac{1}{1-t-t^2}$$

对 $f(t)$ 泰勒展开可得 a_n

解 7.1 (以物理图像证明：北极放置点电荷)**.**

$$g(x,t) = \sum_{l=0}^{\infty} \mathrm{P}_l(x) t^l = \frac{1}{\sqrt{1-2xt+t^2}}$$

设在距原点 r 处放有一个单位点电荷, 取点电荷所在点的方向为 z 轴方向.

这时点电荷在 (r', θ, ϕ) 点的电势 (显然与 ϕ 无关)

$$d = \sqrt{1-2r\cos\theta + r^2}$$
$$u = \frac{1}{d} = \frac{1}{\sqrt{1-2r\cos\theta+r^2}}$$

球内无电荷，电势满足

$$\nabla^2 u = \nabla^2\left(\frac{1}{d}\right) = 0$$

分离变量法得到 u 的级数解正是 $\sum_{l=0}^{\infty} \mathrm{P}_l(x)t^l$，计算如下：

分离变量 $u = R(r)\Theta(\theta)$，得

$$\begin{cases} \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\right) + \lambda\Theta = 0 \\ \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \lambda R = 0 \end{cases}$$

θ 方向：Legendre 方程，本征值和本征函数分别为 $\lambda_l = l(l+1), \mathrm{P}_l(\cos\theta)$

r 方向：欧拉方程，变量代换 $r = e^t$，

$$\frac{d^2R(t)}{dt^2} + \frac{dR(t)}{dt} - l(l+1)R(t) = 0$$

$$\begin{aligned} R =& A_l e^{lt} + B_l e^{(l+1)t} = A_l r^l + B_l \frac{1}{r^{l+1}} \\ \Rightarrow u =& \frac{1}{d} = \sum_{l=0}^{\infty}\left(A_l r^l + B_l\frac{1}{r^{l+1}}\right)\mathrm{P}_l(\cos\theta) \quad ,r<1 \\ =& \sum_{l=0}^{\infty} A_l r^l \mathrm{P}_l(\cos\theta) \end{aligned}$$

$\theta = 0$ 时，上式可得

$$u = \sum_{l=0}^{\infty} A_l r^l$$

另一方面，

$$u = \frac{1}{d} = \frac{1}{1-r} = 1 + r + r^2 + r^3 + \dots$$

因此 $A_l = 1$

$$\frac{1}{d} = \frac{1}{\sqrt{1-2r\cos\theta + r^2}} = \sum_{l=0}^{\infty} r^l \mathrm{P}_l(\cos\theta)$$

令 $r = t, \cos\theta = x$，得到生成函数

$$\boxed{g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} t^l \mathrm{P}_l(x) \quad |t| < 1}$$

$t > 1$:

$$\begin{aligned} g(x,t) =& \frac{1}{\sqrt{1-2xt+t^2}} = \frac{1}{t}\frac{1}{\sqrt{1-2x\frac{1}{t}+\left(\frac{1}{t}\right)^2}} \\ =& \frac{1}{t}\sum_{l=0}^{\infty}\left(\frac{1}{l}\right)^l \mathrm{P}_l(x) = \frac{1}{t}\sum_{l=0}^{\infty}\frac{1}{t^{l+1}}\mathrm{P}_l(\cos\theta) \end{aligned}$$

应用 7.1 (计算 $||\mathrm{P}_l||^2$)**.**

$$[g(x,t)]^2 = \frac{1}{1-2xt+t^2} = \left[\sum_{l=0}^{\infty} t^l \mathrm{P}_l(x)\right]^2$$

两边积分，利用正交性 $\int_{-1}^{1} \mathrm{P}_l \mathrm{P}_k dx = 0, l \neq k$

$$\int_{-1}^{1} \frac{dx}{1-2xt+t^2} = \sum_{l=0}^{\infty} t^{2l} \int_{-1}^{1} [\mathrm{P}_l(x)]^2 dx$$

对左边进行泰勒展开

$$\frac{1}{2t}\int_{(1-t)^2}^{(1+t)^2} \frac{dy}{y} = \frac{1}{t}\ln\frac{1+t}{1-t} = 2\sum_{l=0}^{\infty}\frac{t^{2l}}{2l+1}$$

得到

$$\int_{-1}^{1} [\mathrm{P}_l(x)]^2 dx = \frac{2}{2l+1}$$

应用 7.2. 递推关系

1. $x\mathrm{P}_l(x) = \frac{l+1}{2l+1}\mathrm{P}_{l+1}(x) + \frac{l}{2l+1}\mathrm{P}_{l-1}(x) \leftarrow g(x,t)$ 对 t 求导
2. $\mathrm{P}'_{l+1}(x) + \mathrm{P}'_{l-1}(x) = 2xP'_l(x) + \mathrm{P}_l(x) \leftarrow g(x,t)$ 对 x 求导
3. $\mathrm{P}'_{l+1}(x) = xP'_l(x) + (l+1)\mathrm{P}_l(x)$
4. $\mathrm{P}'_{l-1}(x) = xP'_l(x) - l\mathrm{P}_l(x)$

例 7.4 (地球外部引力场)**.** 球外 Laplace 方程

$$U(r,\theta) = \frac{GM}{R}\left[\frac{R}{r} - \sum_{l=2}^{\infty} a_l \left(\frac{R}{r}\right)^{l+1} \mathrm{P}_l(\cos\theta)\right]$$

7.2 连带 Legendre 方程

$$(1-x^2)y''-2xy'+\left(\lambda-\frac{m^2}{1-x^2}\right)y=0 \quad (m=1,2,3,...) \quad y(\pm 1)<\infty$$

7.2.1 微分表示

1. 将 $y(x)$ 写成 $y(x)=(1-x^2)^{\frac{m}{2}}v(x)$ 形式

   在 $x=1$ 附近求解 $y, t=x-1$

$$-t(t+2)y''-2(t+1)y'+\left[\lambda+\frac{m^2}{t(t+2)}\right]y=0$$

   指标方程：$\rho^2=\frac{m^2}{4}$，得 $\rho=\pm\frac{m}{2}$，由于 y 不发散，$\rho=\frac{m}{2}$

$$y=t^{\frac{m}{2}}(a_0+a_1t+\dots)$$

$$y(x)=(x-1)^{\frac{m}{2}}[a_0+a_1(x-1)+\dots]$$

   $x=-1$ 附近求解：$y(x)=(x+1)^{\frac{m}{2}}[b_0+b_1(x-1)+\dots]$

$$y(x)=(1-x)^{\frac{m}{2}}(1+x)^{\frac{m}{2}}v(x)=(1-x^2)^{\frac{m}{2}}v(x)$$

   代入原方程：

$$(1-x^2)^{\frac{m}{2}+1}v''-2(m+1)x(1-x^2)^{\frac{m}{2}}v'+(1-x^2)^{\frac{m}{2}}[\lambda-m(m+1)]v=0$$

   $v(x)$ 满足以下方程：

$$(1-x^2)v''-2(m+1)xv'+[\lambda-m(m+1)]v=0$$

2. $v(x)$ 方程 $\leftrightarrow$：Legendre 方程

   $\mathrm{P}_l(x)$ 满足 Legendre 方程：$[(1-x^2)\mathrm{P}_l']'+\lambda\mathrm{P}_l=0, \mathrm{P}_l(\pm 1)<\infty, \lambda=l(l+1)$

   求 m 次导数：

$$[(1-x^2)\mathrm{P}_l']^{(m+1)}+\lambda\mathrm{P}_l^{(m)}=0$$

$$(1-x^2)(\mathrm{P}_l')^{(m+1)}+(m+1)(\mathrm{P}_l')^{(m)}(-2x)+\frac{m}{2}(m+1)(\mathrm{P}_l')^{(m-1)}(-2)+\lambda\mathrm{P}_l^{(m)}=0$$

$$(1-x^2)\mathrm{P}_l^{(m+2)}-2(m+1)x\mathrm{P}_l^{(m+1)}+[\lambda-m(m+1)]\mathrm{P}_l^{(m)}=0$$

   $\mathrm{P}_l^{(m)}$ 与 v 满足同一方程，因此 $v(x)\propto[\mathrm{P}_l(x)]^{(m)}$。习惯性取比例为 $(-1)^m$

$$v(x)=(-1)^m[\mathrm{P}_l(x)]^{(m)}$$

   本征函数：m 阶 l 次连带 Legendre 函数

$$\boxed{y_l(x)=\mathrm{P}_l^m(x)=(1-x^2)^{\frac{m}{2}}v=(-1)^m(1-x^2)^{\frac{m}{2}}[\mathrm{P}_l(x)]^{(m)}}$$

其中，

$$\mathrm{P}_l^m(x)=\frac{1}{2^l l!^l}(-1)^m(1-x^2)^{\frac{m}{2}}[(x^2-1)^l]^{(l+m)}$$

本征值： $\lambda_l=l(l+1), l=m,m+1,m+2,\ldots$（当 $m>l$ 时，$\mathrm{P}_l^m(x)=0$）

$$\begin{array}{lll}\mathrm{P}_0^0=1 & \mathrm{P}_1^0=x & \mathrm{P}_1^1=-(1-x^2)^{\frac{1}{2}}\\ \mathrm{P}_2^0=\frac{1}{2}(3x^2-1) & \mathrm{P}_2^1=-3x(1-x^2)^{\frac{1}{2}} & \mathrm{P}_2^2=3(1-x^2)\end{array}$$

在 θ 变量下：

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(\lambda-\frac{m^2}{\sin^2\theta}\right)\Theta=0,\quad \Theta(0),\Theta(\pi)<\infty$$

本征值： $\lambda_l=l(l+1), l=m,m+1,m+2,\ldots$

本征函数： $\mathrm{P}_l^m(\cos\theta)$

7.2.2 广义傅立叶级数

正交性：

$$\int_{-1}^{1}\mathrm{P}_k^m(x)\mathrm{P}_l^m(x)dx=0\quad(k\neq l)$$

$$||\mathrm{P}_l^m||^2=\int_{-1}^{1}[\mathrm{P}_l^m(x)]^2dx=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}$$

$$[-1,1]:\quad f(x)=\sum_{l=m}^{\infty}C_l\mathrm{P}_l^m(x)\qquad C_l=\frac{1}{||\mathrm{P}_l^m||^2}\int_{-1}^{1}f(x)\mathrm{P}_l^m(x)dx$$

$$[0,\pi]:\quad f(\cos\theta)=\sum_{l=m}^{\infty}C_l\mathrm{P}_l^m(\cos\theta)\qquad C_l=\frac{1}{||\mathrm{P}_l^m||^2}\int_{-1}^{1}f(\cos\theta)\mathrm{P}_l(\cos\theta)\sin\theta d\theta$$

7.3 球谐函数

球坐标系下的定解问题：

$$\nabla^2u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\phi^2}=0$$

$$u|_{\theta=0}=u|_{\theta=2\pi}<\infty$$

$$u|_{\phi=0}=u|_{\phi=\pi}<\infty\qquad \left.\frac{\partial u}{\partial\phi}\right|_{\phi=0}=\left.\frac{\partial u}{\partial\phi}\right|_{\phi=2\pi}$$

$$u|_{r=0}<\infty\ u|_{r=a}=f(\theta,\phi)$$

分离变量：$u(r,\theta,\phi)=R(r)S(\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right]-\lambda R(r)=0\\ \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(\lambda-\frac{\mu}{\sin^2\theta}\right)\Theta=0\\ \Phi''+\mu\Phi=0\end{cases}$$

ϕ 方向：$\Phi'' + \mu\Phi = 0$

本征值 $\mu_m = m^2, m = 0, 1, 2, \ldots$

本征函数 $\Phi_0 = 1 \quad \Phi_{m1} = \sin m\phi, \quad \Phi_{m2} = \cos m\phi$

θ 方向：

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[\lambda - \frac{\mu}{\sin^2\theta}\right]\Theta = 0$$

本征值：$\lambda_l = l(l+1) \quad (l = 0, 1, 2, ...)$

本征函数：$y_l(x) = \mathrm{P}_l^m(x)$

$$S_{lm}(\theta,\phi) = \begin{cases} \mathrm{P}_l^m(\cos\theta)\cos m\phi & m = 0, 1, \cdots, l \\ \mathrm{P}_l^m(\cos\theta)\sin m\phi & m = 1, 2, \cdots, l \end{cases}$$

r 方向：

$$\frac{d}{dr}(r^2\frac{dR}{dr}) - l(l+1)R = 0$$

欧拉方程，变量代换 $r = e^t$ 可得

$$R = C_l r^l + D_l r^{-(l+1)} = C_l r^l$$

最后得到：

$$u(r,\theta,\phi) = \boxed{\sum_{l=0}^{\infty}\sum_{m=0}^{l} r^l \mathrm{P}_l^m(\cos\theta)(A_{lm}\cos m\phi + B_{lm}\sin m\phi)}$$
$$= \sum_{l=0}^{\infty}\sum_{m=0}^{l} r^l (A_{lm}S_{lm1} + B_{lm}S_{lm2})$$

正交性：

当且仅当 $l = l', m = m', \alpha = \alpha'$ 时

$$\int_0^{\pi}\int_0^{2\pi} S_{lm\alpha}(\theta,\phi)S_{l'm'\alpha'}(\theta,\phi)\sin\theta d\theta d\phi \neq 0$$

$$||S_{lm1}||^2 = \int_0^{\pi}\int_0^{2\pi}[\mathrm{P}_l^m(\cos\theta)]^2\cos^2 m\phi\sin\theta d\theta d\phi = \frac{(l+m)!}{(l-m)!}\frac{2\pi}{2l+1}(1+\delta_{m0})$$
$$||S_{lm2}||^2 = \int_0^{\pi}\int_0^{2\pi}[\mathrm{P}_l^m(\cos\theta)]^2\sin^2 m\phi\sin\theta d\theta d\phi = \frac{(l+m)!}{(l-m)!}\frac{2\pi}{2l+1}$$

边界条件：

$$u|_{r=a} = f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} a^l (A_{lm}S_{lm1} + B_{lm}S_{lm2})$$

$$a^l A_{lm} = \frac{1}{||S_{lm1}||^2}\int_0^{\pi}\int_0^{2\pi} f(\theta,\phi)S_{lm1}(\theta,\phi)\sin\theta d\theta d\phi$$

$$a^l B_{lm} = \frac{1}{||S_{lm2}||^2}\int_0^{\pi}\int_0^{2\pi} f(\theta,\phi)S_{lm2}(\theta,\phi)\sin\theta d\theta d\phi$$

$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}(\ldots)S_{lm1} + (\ldots)S_{lm2}$ 称为 Laplace 级数

8 柱函数

8.1 Bessel 函数和 Neumann 函数

在贝塞尔方程的解中，已经求得 Bessel 方程 $x^2y''+xy'+(x^2-\nu^2)y=0$ 的基本解：
当 $\nu\neq$ 整数时，Bessel 方程的两个线性无关正则解为

$$\boxed{\mathrm{J}_{\pm\nu}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma\left(k\pm\nu+1\right)}\left(\frac{x}{2}\right)^{2k\pm\nu}}$$

其中 $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt \quad ,x>0$
当 $\nu=n,n=0,1,2,3,...$ 时，J_n,J_{-n} 不独立：$\mathrm{J}_{-n}(x)=(-1)^n\mathrm{J}_n(x)$
Bessel 方程的第一解仍是 $J_\nu(x)$，第二解则可取为

Def (Neumann 函数)**.**

$$\boxed{\mathrm{N}_\nu(x)=\frac{\cos\nu\pi\mathrm{J}_\nu(x)-\mathrm{J}_{-\nu}(x)}{\sin\nu\pi}}$$

$$\mathrm{N}_n(x)=\lim_{\nu\to n}\mathrm{N}_\nu(x)=\lim_{\nu\to n}\frac{\cos\nu\pi\mathrm{J}_\nu(x)-\mathrm{J}_{-\nu}(x)}{\sin\nu\pi}$$

应用 L'Hospital 法则, 可得

$$\begin{aligned}\mathrm{N}_n(x)=&\frac{2}{\pi}\mathrm{J}_n(x)\ln\frac{x}{2}-\frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n}\\&-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\Big[\psi(n+k+1)+\psi(k+1)\Big]\left(\frac{x}{2}\right)^{2k+n}\end{aligned}$$

$$\mathrm{N}_0(x)=\frac{2}{\pi}\mathrm{J}_0(x)\ln\frac{x}{2}-\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}2\psi(k+1)\left(\frac{x}{2}\right)^{2k}$$

例 8.1.

$$\int_0^\infty \mathrm{e}^{-ax}\mathrm{J}_0(bx)\,\mathrm{d}x\quad a>0$$

代入 Bessel 函数的级数表示, 并逐项积分

$$
\begin{aligned}
\int_0^\infty \mathrm{e}^{-ax}\mathrm{J}_0(bx)\mathrm{d}x &= \int_0^\infty \mathrm{e}^{-ax}\sum_{k=0}^\infty \frac{(-1)^k}{(k!)^2}\left(\frac{bx}{2}\right)^{2k}\mathrm{d}x \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{(k!)^2}\left(\frac{b}{2}\right)^{2k}\int_0^\infty \mathrm{e}^{-ax}x^{2k}\mathrm{d}x \\
&= \sum_{k=0}^\infty \frac{(-1)^k}{(k!)^2}\left(\frac{b}{2}\right)^{2k}\frac{(2k)!}{a^{2k+1}} \\
&= \frac{1}{a}\sum_{k=0}^\infty \frac{1}{k!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\times\left(-\frac{5}{2}\right)\cdots\left(-\frac{2k-1}{2}\right)\left(\frac{b}{a}\right)^{2k} \\
&= \frac{1}{a}\left[1+\left(\frac{b}{a}\right)^2\right]^{-1/2} = \frac{1}{\sqrt{a^2+b^2}}
\end{aligned}
$$

8.1.1 递推关系

$$
\boxed{\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^\nu \mathrm{J}_\nu(x)\right] = x^\nu \mathrm{J}_{\nu-1}(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\nu}\mathrm{J}_\nu(x)\right] = -x^{-\nu}\mathrm{J}_{\nu+1}(x)\end{gathered}}
\qquad
\boxed{\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^\nu \mathrm{N}_\nu(x)\right] = x^\nu \mathrm{N}_{\nu-1}(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\nu}\mathrm{N}_\nu(x)\right] = -x^{-\nu}\mathrm{N}_{\nu+1}(x)\end{gathered}}
$$

证明 8.1 (递推关系 1)**.**

$$
\begin{aligned}
\frac{d}{dx}(x^\nu J_\nu) =& \frac{d}{dx}\left[\sum_{k=0}^\infty (-1)^k \frac{1}{k!\Gamma(k+\nu+1)}\frac{1}{2^{2k+\nu}}x^{2k+2\nu}\right] \\
\text{逐项对}x\text{求导} \quad =& \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(\nu+k+1)}\frac{2(k+\nu)}{2^{2k+\nu}}x^{2k+2\nu-1} \\
=& \sum_{k=0}^\infty (t)^k \frac{1}{k!\Gamma(\nu+k)}\left(\frac{x}{2}\right)^{2k+\nu-1}x^\nu \\
=& x^\nu \mathrm{J}_{\nu-1}(x)
\end{aligned}
$$

注. 由 $\Gamma(x+1) = x\Gamma(x)$ 可得

$$
\frac{k+\nu}{\Gamma(\nu+k+1)} = \frac{1}{\Gamma(\nu+k)}
$$

证明 8.2 (递推关系 2)**.**

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-\nu}\mathrm{J}_\nu(x)\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma\left(k+\nu+1\right)}\frac{x^{2k}}{2^{2k+\nu}} \\
&= \sum_{k=0}^\infty \frac{(-1)^{k+1}}{k!\Gamma\left(k+\nu+2\right)}\frac{x^{2k+1}}{2^{2k+\nu+1}} = -x^{-\nu}\mathrm{J}_{\nu+1}(x)
\end{aligned}
$$

将此二递推关系写成

$$
\nu x^{\nu-1}\mathrm{J}_\nu(x) + x^\nu \mathrm{J}_\nu'(x) = x^\nu \mathrm{J}_{\nu-1}(x)
$$

$$-\nu x^{-\nu-1}\mathrm{J}_\nu(x)+x^{-\nu}\mathrm{J}'_\nu(x)=-x^{-\nu}\mathrm{J}_{\nu+1}(x)$$

消去 $J_\nu(x)$ 或 $J'\nu(x)$, 又可以得到两个新的递推关系

$$\boxed{\begin{array}{c}\mathrm{J}_{\nu-1}(x)-\mathrm{J}_{\nu+1}(x)=2\mathrm{J}'_\nu(x)\\ \mathrm{J}_{\nu-1}(x)+\mathrm{J}_{\nu+1}(x)=\frac{2\nu}{x}\mathrm{J}_\nu(x)\end{array}}$$

例 8.2.

$$\int_0^1\left(1-x^2\right)\mathrm{J}_0(\mu x)x\,\mathrm{d}x$$

其中 μ 是 $J_0(x)$ 的零点，$J_0(\mu)=0$

由递推关系 1，

$$\frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}x}\left[x\mathrm{J}_1(\mu x)\right]=x\mathrm{J}_0(\mu x)$$

积分得

$$\begin{aligned}
&\int_0^1\left(1-x^2\right)\mathrm{J}_0(\mu x)x\,\mathrm{d}x\\
&=\frac{1}{\mu}\int_0^1(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\left[x\mathrm{J}_1(\mu x)\right]\,\mathrm{d}x\\
\text{分部积分}\quad&=(1-x^2)\frac{1}{\mu}xJ_1(\mu x)\Big|_0^1+\frac{2}{\mu}\int_0^1x^2J_1(\mu x)dx\\
\text{第一项为零，第二项继续使用递推关系 1}\quad&=\frac{2}{\mu^2}\int_0^1\frac{d}{dx}[x^2J_2(\mu x)]dx\\
&=\frac{2}{\mu^2}x^2J_2(\mu x)\Big|_0^1\\
&=\frac{2}{\mu^2}J_2(\mu)=\frac{4}{\mu^3}J_1(\mu)
\end{aligned}$$

应用 8.1 (幂函数乘 Bessel 函数积分)**.**

$$\begin{aligned}
&\int_a^b x^\mu J_\nu(x)dx\\
&=\int_a^b x^{\mu-\nu-1}x^{\nu+1}J_\nu dx=\int_a^b x^{\mu-\nu-1}\frac{d}{dx}[x^{\nu+1}J_{\nu+1}]dx\\
&=x^{\mu-\nu-1}x^{\nu+1}J_{\nu+1}\Big|_a^b-(\mu-\nu-1)\int_a^b x^{\mu-\nu-2}x^{\nu+1}J_{\nu+1}dx\\
&=\dots\\
&=
\end{aligned}$$

两类易算情况

1. $(u-n)-(\nu+n)=1$，即 $\mu-\nu=2n+1$

$$\int_a^b x^{\nu+n+1}J_{\nu+n}dx=\int_a^b \frac{d}{dx}[x^{\nu+n+1}J_{\nu+n+1}]dx$$

2. $(u-n)+(\nu+n)=1$，即 $\mu+\nu=1$

$$\int_a^b x^{-(\nu+n-1)}J_{\nu+n}dx=\int_a^b \frac{d}{dx}[x^{-(\nu+n-1)}J_{\nu+n-1}]dx$$

8.2 渐进行为

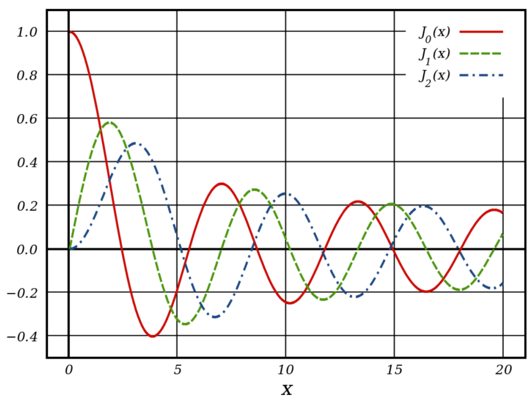


图 3: Bessel 函数

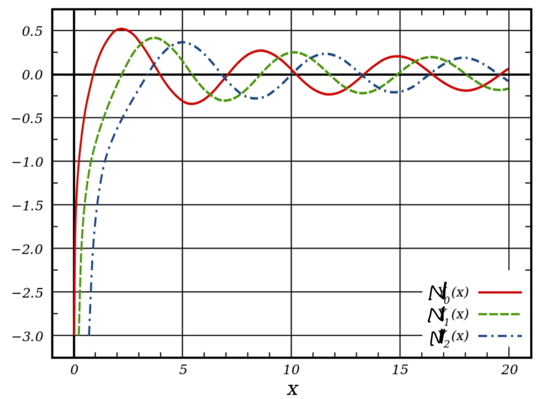


图 4: Neumann 函数

8.2.1 $x\to 0$

1. Bessel 函数

$$J_\nu(x)=\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(u+k+1)}\left(\frac{x}{2}\right)^{2k+\nu}\quad (k=0\text{主导})\sim\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}$$

$$\begin{cases}\nu=0 & J_0(x)\to 1\\ \nu>0 & J_\nu(x)\sim\frac{1}{\Gamma(\nu+1)}(\frac{x}{2})^2\to 0\end{cases}$$

2. Neumann 函数

$$\begin{cases}\nu=0 & N_0(x)\to \infty\\ \nu>0 & J_\nu(x)\sim\frac{1}{\Gamma(\nu+1)}(\frac{x}{2})^2\to \infty\end{cases}$$

$\nu > 0$ 时:

$$
\begin{aligned}
\mathrm{N}_\nu(x) =& \frac{\cos\nu\pi \mathrm{J}_\nu(x) - \mathrm{J}_{-\nu}(x)}{\sin\nu\pi} \\
(\mathrm{J}_{-\nu}\text{主导}) \sim& -\frac{J_{-\nu}(x)}{\sin\nu\pi} \\
(k=0\text{主导}) \sim& -\frac{1}{\sin\nu\pi}\frac{1}{\Gamma(1-\nu)}(\frac{x}{2})^{-u} \\
\left[\text{由}\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}\right] \sim& -\frac{T(\nu)T(1-\nu)}{\pi}\frac{1}{\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu} \\
\sim& -\frac{\Gamma(\nu)}{\pi}\left(\frac{x}{2}\right)^{-\nu} \to \infty
\end{aligned}
$$

$\nu = 0$ 时:

$$
N_0(x) \sim \frac{2}{\pi}J_0(x)\ln\frac{x}{2} + \text{幂级数} \sim \frac{2}{\pi}\ln\frac{x}{2} \to \infty
$$

8.2.2 $x \to \infty$

$$
\left.\begin{aligned}
J_\nu(x) &\sim \sqrt{\frac{2}{\pi x}}\cos(x - \frac{\nu\pi}{2} - \frac{\pi}{4}) \\
N_\nu(x) &\sim \sqrt{\frac{2}{\pi x}}\sin(x - \frac{\nu\pi}{2} - \frac{\pi}{4})
\end{aligned}\right. \quad \text{振荡衰减}
$$

0 级近似:

$$
y'' + \frac{1}{x}y' + (1 - \frac{\nu^2 f}{x^2})y = 0 \Rightarrow y'' + y \approx 0 \Rightarrow y(x) \sim \cos(x - x_0)
$$

1 级近似：待定 $y(x) = f(x)\cos(x - x_0)$

$$
y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y = (f'\cos - 2f'\sin - f\cos) + \frac{1}{x}(f'\cos - f\sin) + (1 - \frac{\nu^2}{x})f\cos = 0
$$

$$
(f''\cos - 2f'\sin) + \frac{1}{x}(f'\cos - f\sin) - \frac{\nu^2}{x^2}f\cos = 0
$$

$$
-(2f' + \frac{f}{x})\sin x + (f'' + \frac{1}{x}f' - \frac{\nu^2}{x^2}f)\cos x = 0
$$

$$
\begin{aligned}
&\sin\text{系数} = 0 \Rightarrow 2f' + \frac{f}{x} = 0 \Rightarrow f(x) \sim x^{-\frac{1}{2}} \\
&\cos\text{系数} f'' + \frac{1}{x}f' - \frac{\nu^2}{x^2}f \sim x^{-\frac{5}{2}}, \text{是更高阶小量}
\end{aligned}
$$

因此 $y(x) \sim f(x)\cos(x - x_0) \sim \sqrt{\frac{1}{x}}\cos(x - x_0)$

8.3 Bessel 函数的应用

例 8.3 (扩散问题).

$$\frac{\partial u}{\partial t} = D\nabla^2 u, r < a$$

$$u|_{\phi=0} = u|_{\phi=2\pi}, \quad \left.\frac{\partial u}{\partial \phi}\right|_{\phi=0} = \left.\frac{\partial u}{\partial \phi}\right|_{\phi=2\pi}$$

$$u|_{t=a} = 0, \qquad u|_{t=0} = f(r,\phi)$$

分离变量 $u = V(r,\phi)T(t)$ 得

$$\frac{\nabla^2 V}{V} = \frac{T'}{DT} = -E$$

时间上：$T' = -DET$

空间上：

$$\begin{cases} \nabla^2 V + EV = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + EV = 0, \\ V|_{r=a} = 0, V|_{r=0} < \infty \\ V|_{\phi=0} = V|_{\phi=2\pi}, \frac{\partial V}{\partial \phi}|_{\phi=0} = \frac{\partial V}{\partial \phi}|_{\phi=2\pi} \end{cases}$$

空间上，进一步分离变量：$V(r,\phi) = R(r)\Phi(\phi)$

角向

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(0) = \Phi(2\pi), \Phi'(0) = \Phi'(2\pi) \end{cases} \Rightarrow \lambda = m^2, \quad \Phi_m = \begin{cases} \cos m\phi & m = 0,1,2\ldots \\ \sin m\phi & m = 1,2\ldots \end{cases}$$

径向

$$\begin{cases} \frac{1}{r}\frac{d}{dr}(r\frac{dR}{dr}) + (E - \frac{\lambda}{r^2})R = 0 \\ R(0) < \infty, R(a) = 0 \end{cases}$$

$$(E = k^2) \quad r\frac{d}{dr}(r\frac{dR}{dr}) + (k^2r^2 - m^2)R = 0$$

$$(x = kr) \quad x\frac{d}{dx}(x\frac{dR}{dx}) + (x^2 - m^2)R = 0$$

$$x^2R'' + xR' + (x^2 - m^2)R = 0$$

$$\Rightarrow R = CJ_m(x) + DN_m(x)$$

由于 $R(0) < \infty, N_m(x)$ 项系数为零，

$$R = CJ_m(x) = CJ_m(kr)$$

$$R(a) = 0 \Rightarrow J_m(ka) = 0 \Rightarrow ka = \mu_i^{(m)}$$

其中 μ_i 为 $J_m(x)$ 的第 i 个正零点 $(m = 0,1,2,\ldots; \quad i = 1,2,3,\ldots)$

$$k_{mi} = \frac{\mu_i^{(m)}}{a}$$

本征值

$$E_{mi} = \left(\frac{\mu_i^{(m)}}{a}\right)^2$$

径向本征函数

$$R_{mi}(r) = J_m(k_{mi}r) = J_m\left(\frac{\mu_i^{(m)}}{a}r\right)$$

V 的本征函数 $(i = 1, 2, 3, \ldots)$

$$\begin{cases} V_{mi1} = R_{mi}(r)\cos m\phi = J_m\left(\frac{\mu_i^{(m)}}{a}r\right)\cos m\phi & m = 0, 1, 2, \ldots \\ V_{mi2} = R_{mi(r)}\sin m\phi = J_m\left(\frac{\mu_i^{(m)}}{a}r\right)\sin m\phi & m = 1, 2, 3, \ldots \end{cases}$$

得到一般解形式:

$$\begin{aligned} u(r,\phi,t) &= \left(\sum_m\sum_i A_{mi}V_{mi1} + \sum_m\sum_i B_{mi}V_{mi2}\right)e^{-DE_{mi}t} \\ &= \left[\sum_{m=0}^{\infty}\sum_{i=1}^{\infty} A_{mi}R_{mi}(r)\cos m\phi + \sum_{m=1}^{\infty}\sum_{i=1}^{\infty} B_{mi}R_{mi}(r)\sin m\phi\right]e^{-DE_{mi}t} \end{aligned}$$

利用初始条件确定 A_{mi}, B_{mi}:

$$\begin{aligned} u|_{t=0} = f(r,\phi) &= \sum_{m=0}^{\infty}\sum_{i=1}^{\infty} A_{mi}R_{mi}(r)\cos m\phi & &+ \sum_{m=1}^{\infty} B_{mi}R_{mi}(r)\sin m\phi \\ f(r,\phi) &= \sum_{m=0}^{\infty} f_{m1}(r)\cos m\phi & &+ \sum_{m=1}^{\infty} f_{m2}(r)\sin m\phi \end{aligned}$$

比较 cos 和 sin 的系数

$$f_{m1}(r) = \sum_{i=1}^{\infty} A_{m_i}R_{m_i}(r) = \sum_{i=1}^{\infty} A_{mi} - J_m\left(\frac{\mu_i^{(m)}}{a}r\right)$$

$$f_{m2}(r) = \sum_{i=1}^{\infty} B_{m_i}R_{m_i}(r) = \sum_{i=1}^{\infty} B_{mi} - J_m\left(\frac{\mu_i^{(m)}}{a}r\right)$$

即 $f_{m1}(r), f_{m2}(r)$ 按 $J_m\left(\frac{\mu_i^{(m)}}{a}r\right)$ 展开.

由正交性确定系数:

将方程改写为 S-L 标准形式，可见 r 为权函数

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (k^2r - \frac{m^2}{r})R = 0$$

$$R(0) < \infty, R(a) = 0$$

可得正交性

$$\int_a^b J_m\left(\frac{\mu_i^{(m)}}{a}r\right) J_m\left(\frac{\mu_j^{(m)}}{a}r\right) rdr = 0(i \neq j)$$

广义傅里叶级数

$$[0,a] = f(r) = \sum_{i=1}^{\infty} b_i J_m\left(\frac{\mu_i^{(m)}}{a}r\right)$$

$$b_i = \frac{1}{||J_m(\frac{\mu_i^{(m)}}{a}r)||^2} \int_0^a f(r) J_m\left(\frac{\mu_i^{(m)}}{a}r\right) rdr$$

模方

$$\boxed{\left|\left|J_m\left(\frac{\mu_i^{(m)}}{a}r\right)\right|\right|^2 \equiv \int_0^a \left[J_m\left(\frac{\mu_i^{(m)}}{a}r\right)\right]^2 rdr = \frac{a^2}{2}\left[J_{m+1}\left(\mu_i^{(m)}\right)\right]^2}$$

计算过程：

$$\int_0^a [J_m(\frac{\mu_i^{(m)}}{a}r)]^2 rdr \xlongequal{x=\frac{\mu_i^{(m)}}{a}r} \left(\frac{a}{\mu_i^{(m)}}\right)^2 \int_0^{\mu_i^{(m)}} [J_m(x)]^2 xdx$$

其中，

$$\begin{aligned}\int_0^a [J_m(x)]^2 xdx =& \frac{a^2}{2}[J_m'(a)]^2 + \frac{1}{2}(a^2 - m^2)[J_m(a)]^2, \quad m \geq 0 \\ \xlongequal{\text{if } J_m(a)=0} & \frac{a^2}{2}[J_{m+1}(a)]^2\end{aligned}$$

例 8.4 (Laplace 方程：圆柱体内稳定温度分布)**.**

$$\nabla^2 u = 0 \quad r < a, \quad 0 < z < h$$

$$u|_{r=a} = 0, \quad u|_{z=0} = 0, u|_{z=h} = u_0$$

分离变量 $u(r,z) = R(r)Z(z)$（对称性）

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{\partial^2 u}{\partial z^2} = 0$$

r 方向解本征值问题

$$\begin{cases}(rR')' + \lambda rR = 0 \\ R(0) < \infty, R(a) = 0\end{cases}$$

$$r^2 R'' + rR' + \lambda r^2 R = 0$$

变量代换 $x = \sqrt{\lambda}r$，得到 0 阶 Bessel 方程形式

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - 0^2)R = 0$$

解得 r 方向本征函数的形式（由于 $N_0(x)$ 在 $x = 0$ 发散，需要舍弃）

$$R = J_0(x) = J_0(\sqrt{\lambda}r)$$

利用边界条件求本征值，其中 μ_i 为 0 阶 Bessel 方程的第 i 个正零点

$$R(a)=0\Rightarrow J_0(\sqrt{\lambda}a)=0\Rightarrow \lambda_i=(\frac{\mu_i}{a})^2$$

最终得到 r 方向本征函数

$$R_i(r)=J_0\left(\frac{\mu_i}{a}r\right)\quad i=1,2,3\dots$$

z 方向

$$z''-\lambda z=0$$

$$z_i(z)=A_ie^{\frac{\mu_i}{\alpha}z}+B_ie^{-\frac{\mu_i}{\alpha}z}$$

一般解形式

$$u(r,z)=\sum_{i=1}^{\infty}R_i(r)Z_i(z)$$

$z=0$ 处边界条件：$u|_{z=0}=0\Rightarrow A_i+B_i=0$

$$u(r,z)=\sum_{i=1}^{\infty}C_i\sinh(\frac{\mu_i}{a}z)J_0(\frac{\mu_i}{a}r)$$

$z=h$ 处边界条件：

$$u|_{z=h}=u_0=\sum_{i=1}^{\infty}C_i\sinh\frac{\mu_ih}{a}J_0(\frac{\mu_i}{a}r)$$

利用正交关系确定 C_i

$$\begin{aligned}
C_i\sinh\frac{\mu_ih}{a}=&\frac{1}{||J_0(\frac{\mu_i}{a}r)||^2}\int_0^a u_0J_0(\frac{\mu_i}{a}r)rdr\\
\xlongequal{x=\frac{\mu_i}{a}r}&\frac{1}{\frac{a^2}{2}[J_1(\mu_i)]^2}u_0(\frac{a}{\mu_i})^2\int_0^{\mu_i}J_0(x)xdx\\
=&\frac{1}{\frac{a^2}{2}[J_1(\mu_i)]^2}u_0(\frac{a}{\mu_i})^2\int_0^{\mu_i}\frac{d}{dx}(xJ_1)dx\\
=&\frac{1}{\frac{a^2}{2}[J_1(\mu_i)]^2}u_0(\frac{a}{\mu_i})^2\mu_iJ_1(\mu_i)=\frac{2\mu_0}{\mu_iJ_1(\mu_i)}\\
\Rightarrow C_i=&\frac{2u_0}{\mu_iJ_1(\mu_i)}\frac{1}{\sinh(\frac{\mu_ih}{a})}
\end{aligned}$$

最终得到

$$u=\sum_{i=1}^{\infty}\frac{2u_0}{\mu_i}\frac{\sinh(\frac{\mu_i}{a}z)}{\sinh(\frac{\mu_i}{a}h)}\frac{J_0(\frac{\mu_i}{a}r)}{J_1(\mu_i)}$$

8.4 虚宗量 Bessel 函数

$J_\nu(ix)$ 仍然满足 Bessel 方程

$$(ix)^2\frac{d^2J_\nu(ix)}{d(ix)^2}+ix\frac{dJ_\nu(ix)}{d(ix)}+\left[(ix)^2-\nu^2\right]J_\nu(ix)=0$$

$$\Rightarrow x^2\frac{d^2J_\nu(ix)}{dx^2}+x\frac{dJ_\nu(ix)}{dx}+(-x^2-\nu^2)J_\nu(ix)=0$$

解得

$$\begin{aligned}J_\nu(ix)=&J_\nu(e^{\frac{\pi}{2}i}x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}e^{\frac{\pi}{2}i}\right)^{2k+\nu}\\=&e^{\frac{\pi u}{2}i}\sum_{k=0}^{\infty}\frac{(-1)^ke^{k\pi i}}{k!\Gamma(k+\nu+1)}(\frac{x}{2})^{2k+\nu}\\=&e^{\frac{\pi\nu}{2}i}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k+\nu+1)}(\frac{x}{2})^{2k+\nu}\end{aligned}$$

Def (第一类虚宗量 Bessel 函数)**.**

$$\begin{aligned}\mathrm{I}_\nu(x)&=\mathrm{e}^{-\mathrm{i}\pi\nu/2}\mathrm{J}_\nu(x\mathrm{e}^{\mathrm{i}\pi/2})\\&=\sum_{k=0}^{\infty}\frac{1}{k!\Gamma\left(k+\nu+1\right)}\left(\frac{x}{2}\right)^{2k+\nu}\end{aligned}$$

I_ν 满足方程:

$$\boxed{x^2I_\nu''+xI_\nu'(x)+(-x^2-\nu^2)I_\nu(x)=0}$$

$I_{-n}=I_n$, $\quad n=0,1,2,\ldots$, 不独立

Def (第二类虚宗量 Bessel 函数，Mc-Donald 函数)**.**

$$\mathrm{K}_\nu(x)=\frac{\pi}{2\sin\nu\pi}\left[\mathrm{I}_{-\nu}(x)-\mathrm{I}_\nu(x)\right]$$

8.4.1 渐进行为

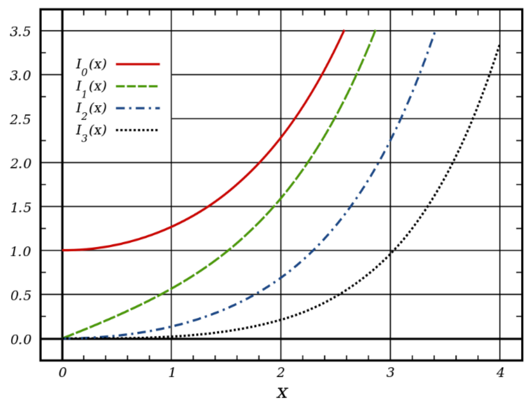


图 5: 第一类虚宗量 Bessel 函数

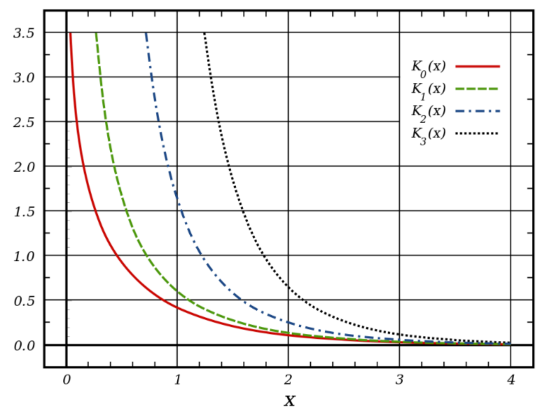


图 6: 第二类虚宗量 Bessel 函数

1. 当 $x \to \infty$ 时

$$\mathrm{I}_\nu(x) \sim \sqrt{\frac{1}{2\pi x}}\mathrm{e}^x \quad \mathrm{K}_\nu(x) \sim \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}$$

2. 当 $x \to 0$ 时（约定 $\nu \geq 0$），$\mathrm{I}_\nu(x)$ 有界， $\mathrm{K}_\nu(x)$ 无界

8.4.2 应用

例 8.5 (圆柱体 Laplace 方程).

$$\begin{aligned}
&\nabla^2 u = 0 \quad r < a, \quad 0 < z < h \\
&u|_{\phi=0} = u|_{\phi=2\pi}, \quad \frac{\partial u}{\partial \phi}\Big|_{\phi=0} = \frac{\partial u}{\partial \phi}\Big|_{\phi=2\pi} \\
&u|_{r=0} < \infty, \quad u|_{r=a} = f(\phi, z) \\
&u|_{z=0} = u|_{z=h} = 0
\end{aligned}$$

分离变量 $u(r,\phi,z) = R(r)\Phi(\phi)Z(z)$

ϕ 方向:

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(0) = \Phi(2\pi), \Phi'(0) = \Phi'(2\pi) \end{cases} \Rightarrow \lambda = m^2, \quad \Phi_m = \begin{cases} \cos m\phi & m = 0, 1, 2 \ldots \\ \sin m\phi & m = 1, 2 \ldots \end{cases}$$

z 方向:

$$\begin{cases} Z'' + \lambda Z = 0 \\ Z(0) = Z(h) = 0 \end{cases} \Rightarrow \lambda_n = (\frac{n\pi}{h})^2 \quad n = 1, 2 \ldots \quad Z_n = \sin\frac{n\pi}{h}z$$

r 方向:

$$\frac{1}{r}\frac{d}{dr}(r\frac{dR}{dr}) + (-\lambda - \frac{\mu}{r^2})R = 0$$

$$\Rightarrow r\frac{d}{dr}(r\frac{dR}{dr})+[-(\frac{n\pi}{h}r)^2-m^2]R=0$$

变量代换：$x=\frac{n\pi}{h}r$

$$x\frac{d}{dx}(x\frac{dR}{dx})+[-x^2-m^2]R=0$$

得到 r 方向本征函数形式

$$\begin{aligned}R=&CI_m(x)+DK_m(x)\\=&I_m(\frac{n\pi}{h}r)\end{aligned}$$

一般解形式

$$u=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(A_{mn}\cos m\phi+B_{mn}\sin m\phi)I_m(\frac{n\pi}{h}r)\sin\frac{n\pi}{h}z$$

边界条件

$$u|_{r=a}=f(\phi,z)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(A_{mn}\cos m\phi+B_{mn}\sin m\phi)I_m(\frac{n\pi}{h}a)\sin\frac{n\pi}{h}z$$

利用正交性得

$$A_{mn}I_m(\frac{n\pi}{h}a)=\frac{1}{\pi}\frac{1}{1+\delta_{m0}}\frac{2}{h}\int_0^{2\pi}d\phi\int_0^h dzf(\phi,z)\cos m\phi\sin\frac{n\pi}{h}z$$

$$B_{mn}I_m(\frac{n\pi}{h}a)=\frac{1}{\pi}\frac{2}{h}\int_0^{2\pi}d\phi\int_0^h dzf(\phi,z)\sin m\phi\sin\frac{n\pi}{h}z$$

8.5 球 Bessel 函数

球坐标 Helmholtz 方程分离变量时，得到

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)+\left(k^2-\frac{\lambda}{r^2}\right)R=0$$

在一般情况下，

$$\lambda_l=l(l+1),l=0,1,2,\cdots$$

当 $k=0$ 时，方程的解是 r^l 和 r^{-l-1}

当 $k\neq0$ 时：作变量代换 $x=kr$，得

Def (球 Bessel 方程).

$$\frac{d}{dx}(x^2\frac{dR}{dx})+[x^2-l(l+1)]R=0$$

可以将球 Bessel 方程化为 $l+1$ 阶 Bessel 方程：

令 $R(x)=\sqrt{\frac{\pi}{2x}}\nu(x)$

$$x\frac{d}{dx}(x\frac{dv}{dx})+[x^2-(l+\frac{1}{2})^2]\nu=0$$

Def (l 阶球 Bessel 函数)**.**

$$\mathrm{j}_l(x) \equiv \sqrt{\frac{\pi}{2x}}\mathrm{J}_{l+1/2}(x)$$

Def (l 阶球 Neumann 函数)**.**

$$\mathrm{n}_l(x) \equiv \sqrt{\frac{\pi}{2x}}\mathrm{N}_{l+1/2}(x) = (-1)^{l+1}\sqrt{\frac{\pi}{2x}}\mathrm{J}_{-(l+1/2)}(x)$$

8.5.1 $j_l(x)$，$n_l(x)$ 的初等表达式

0 阶球 Bessel 和球 Neumann 函数

$$\begin{aligned} J_{\frac{1}{2}}(x) &= \sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(k+\frac{3}{2})}\left(\frac{x}{2}\right)^{2k+\frac{1}{2}} \\ &= \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)!}x^{2k+\frac{1}{2}}\sqrt{\frac{2}{\pi}} \end{aligned}$$

又有

$$\sin x = \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)!}x^{2k+1}$$

因此

$$\begin{aligned} J_{\frac{1}{2}}(x) &= \sqrt{\frac{z}{\pi x}}\sin x \\ j_0(x) &= \sqrt{\frac{\pi}{2x}}J_{\frac{1}{2}}(x) = \frac{1}{x}\sin x \\ J_{-\frac{1}{2}}(x) &= \sqrt{\frac{2}{\pi x}}\cos x = -N_{\frac{1}{2}}(x) \\ n_0(x) &= -\frac{1}{x}\cos x \end{aligned}$$

l 阶球 Neumann 函数

利用递推关系，

$$\frac{d}{dx}(x^\nu J_\nu) = x^\nu J_{\nu-1} \quad \Rightarrow \quad \frac{1}{x}\frac{d}{dx}(x^\nu J_\nu) = x^{\nu-1}J_{\nu-1}$$

$$(\frac{1}{x}\frac{d}{dx})^l(x^\nu J_\nu) = x^{\nu-l}J_{\nu-l}$$

$$(\nu = -\frac{1}{2}) \qquad (\frac{1}{x}\frac{d}{dx})^l(x^{-\frac{1}{2}}J_{-\frac{1}{2}}) = x^{-\frac{1}{2}-l}J_{-(l+\frac{1}{2})}$$

$$J_{-(l+\frac{1}{2})}(x) = \sqrt{\frac{2}{\pi}}x^{l+\frac{1}{2}}(\frac{1}{x}\frac{d}{dx})^l\left(\frac{\cos x}{x}\right)$$

$$\boxed{n_l(x) = (-1)^{l+1}x^l(\frac{l}{x}\frac{d}{dx})^l(\frac{\cos x}{x})}$$

例如

$$\mathrm{n}_0(x) = -\frac{\cos x}{x} \qquad \mathrm{n}_1(x) = -\frac{1}{x^2}(\cos x + x\sin x)$$

$$\mathrm{n}_2(x) = -\frac{1}{x^3}\Big[\,\big(3 - x^2\big)\cos x + 3x\sin x\Big]$$

l 阶球 Bessel 函数

利用递推关系，

$$\frac{d}{dx}(x^{-\nu}J_\nu) = -x^{-\nu}J_{\nu+\frac{1}{2}} \quad \Rightarrow \quad \frac{1}{x}\frac{d}{dx}(x^{\nu}J_\nu) = -x^{-(\nu+1)}J_{\nu+1}$$

$$(\frac{1}{x}\frac{d}{dx})^l(x^{-\nu}J_\nu) = (-1)^l x^{-(\nu+l)}J_{\nu+l}$$

$$(\nu = \frac{1}{2}) \qquad (\frac{1}{x}\frac{d}{dx})^l(x^{-\frac{1}{2}}J_{\frac{1}{2}}) = (-1)^l x^{-(l+\frac{1}{2})}J_{l+\frac{1}{2}}$$

$$J_{l+\frac{1}{2}}(x) = (-1)^l\sqrt{\frac{2}{\pi}}x^{l+\frac{1}{2}}(\frac{1}{x}\frac{d}{dx})^l(\frac{\sin x}{x})$$

$$\boxed{j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+\frac{1}{2}} = (-1)^l x^l(\frac{1}{x}\frac{d}{dx})^l(\frac{\sin x}{x})}$$

例如

$$\mathrm{j}_0(x) = \frac{\sin x}{x} \quad \mathrm{j}_1(x) = \frac{1}{x^2}\big(\sin x - x\cos x\big)$$

$$\mathrm{j}_2(x) = \frac{1}{x^3}\Big[\big(3 - x^2\big)\sin x - 3x\cos x\Big]$$

8.5.2 渐进行为

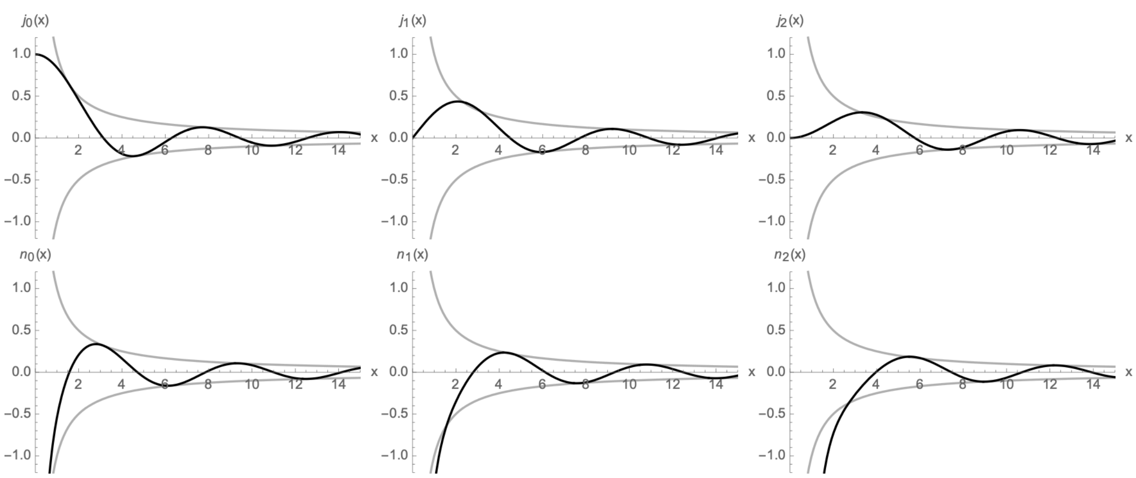


图 7: 球 Bessel 函数 $j_l(x)$ 和球 Neumann 函数 $n_l(x)$，其中细灰线为它们的渐进线 $y = \pm\frac{1}{x}$

$$x \to 0: \quad j_0(x) \to 1, \quad j_1(x), j_2(x)\cdots \to 0$$

$$n_0(x), n_1(x), n_2(x)\cdots \to \infty$$

$$x \to \infty: \quad j_0, j_1, j_2, n_0, n_1, n_2 \to 0$$

8.5.3 应用

例 8.6 (球内热传导问题)**.**

$$\frac{\partial u}{\partial t} = k\nabla^2 u, \quad u|_{r=a} = 0, \quad u|_{t=0} = f(r, \theta, \phi)$$

r 方向本征值问题:

$$\begin{cases} \frac{1}{r^2}\frac{d}{dr}(r^2\frac{dR}{dr}) + [\lambda - \frac{l(l+1)}{r^2}]R = 0 \\ R(0) < \infty, R(a) = 0 \end{cases}$$

写成 S-L 标准形式，可见权函数 $\rho(r) = r^2$

$$\frac{d}{dr}(r^2\frac{dR}{dr}) + [\lambda r^2 - l(l+1)]R = 0$$

变量代换 $x = \sqrt{\lambda}r$

$$\frac{d}{dx}(x^2\frac{dR}{dx}) + [x^2 - l(l+1)]R = 0$$

$$R = j_l(x) = j_l(\sqrt{\lambda}r)$$

边界条件:

$$R(a) = 0 \Rightarrow j_l(\sqrt{\lambda}a) = 0 \Rightarrow \sqrt{\lambda_n}a = \mu_n^{(l)}$$

$$l = 0: \quad j_0(x) = \frac{1}{x}\sin x \Rightarrow \quad \mu_n^{(0)} = n\pi, \quad n = 1, 2, 3\ldots$$

广义傅里叶级数:

$$[0, a] = f(r) = \sum_{n=1}^{\infty} c_n j_l(\frac{\mu_n^{(l)}}{a}r)$$

$$C_n = \frac{1}{||j_l(\frac{\mu_n^{(l)}}{a}r)||^2}\int_0^a f(r)j_l(\frac{\mu_n^{(l)}}{a}r)r^2 dr$$

9 二阶偏微分方程的分类和通解

9.1 二阶偏微分方程的分类

	典型的二阶 PDE	二次曲线
波动方程	$\frac{\partial u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
热传导	$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0$	$y = x^2$
Laplace	$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

二次曲线的分类：$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$

$$\Rightarrow a(x + \frac{b}{a}y)^2 + (c - \frac{b^2}{a})y^2 + dx + ey + f = 0$$

记 $\xi = x + \frac{b}{a}, \eta = y, \Delta = b^2 - ac$

$$\Rightarrow a^2\xi^2 - \Delta\eta^2 + \cdots = 0$$

$$\Delta \begin{cases} > 0, & \text{双曲线} \\ = 0, & \text{抛物线} \\ < 0, & \text{椭圆} \end{cases}$$

二阶 PDE 的分类:

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu + g = 0$$

9.1.1 常系数

$$au_{xx} + 2bu_{xy} + cu_{yy} = \Phi(u, u_x, u_y)$$

其中 a, b, c 为常数

设 $a \neq 0$，配方

$$a\left(\frac{\partial}{\partial x} + \frac{b}{a}\frac{\partial}{\partial y}\right)^2 u + \left(c - \frac{b^2}{a}\right)\frac{\partial^2 u}{\partial y^2} = \Phi$$

换元 (ξ, η)

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} + \frac{b}{a}\frac{\partial}{\partial y} = x_\xi\frac{\partial}{\partial x} + y_\xi\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial y} = x_\eta\frac{\partial}{\partial x} + y_\eta\frac{\partial}{\partial y}$$

即

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

Jacobian 矩阵:

$$\begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{pmatrix}$$

方程变为

$$au_{\xi\xi} - \frac{\Delta}{a} u_{\eta\eta} = \Phi' \quad \Delta = b^2 - ac$$

$$\Delta \begin{cases} > 0, & \text{双曲型} \\ = 0, & \text{抛物型} \\ < 0, & \text{椭圆型} \end{cases}$$

9.1.2 一般的情况

系数 $a, b, ..., g$ 皆为 (x, y) 的函数

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu + g = 0$$

换元 $\xi(x, y), \eta(x, y)$，系数 $A, B, ...G$ 为 (ξ, η) 的函数

$$Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta} + Du_\xi + Eu_\eta + fu + g = 0$$

寻找变换下的**不变量**：一个 PDE 的内在属性不应依赖于变量选取，所以尝试根据变量代换下的不变量对 PDE 分类。

$$\left(\begin{array}{c} u_x \\ u_y \end{array} \right) = \left(\begin{array}{cc} \xi_x & \eta_x \\ \xi_y & \eta_y \end{array} \right) \left(\begin{array}{c} u_y \\ u_\eta \end{array} \right) \equiv J \left(\begin{array}{c} u_y \\ u_\eta \end{array} \right)$$

$$\begin{aligned} u_x &= \xi_{xx} u_\xi + \xi_x (u_\xi)_x + \eta_{xx} u_\eta + \eta_x (u_\eta)_x \\ &= \xi_{xx} u_\xi + \xi_x (\xi_x u_{\xi\xi} + \eta_x u_{\xi\eta}) + \eta_{xx} u_y + \eta_x (\xi_x u_{\eta\xi} + \eta_x u_{\eta\eta}) \\ &= (\xi_x)^2 u_{\xi\xi} + 2\xi_x \eta_x u_{\xi\eta} + (\eta_x)^2 u_{\eta\eta} + \xi_{xx} u_\xi + \eta_{xx} u_y \end{aligned}$$

u_{xy}, u_{yy}, u_x, u_y 同理

$$\begin{aligned} A &= a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2 \\ B &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y \\ C &= a(\eta_x)^2 + 2b\eta_x\eta_y + c(\eta_y)^2 \\ D &= a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y \\ E &= a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\xi_y \\ F &= f \\ G &= g \end{aligned}$$

可计算得到

$$\bar{\Delta} = B^2 - AC = (\xi_x\eta_y - \xi_y\eta_x)^2\Delta$$

$$\bar{M} = \begin{pmatrix} A & B \\ B & C \end{pmatrix} = J^T M J$$

$$\det \overline{M} = (\det J)^2 \det M \Rightarrow \overline{\Delta} = (\det J)^2\Delta$$

因此，$\boxed{\text{sgn}(\Delta)\text{是不变量}}$

$$\text{sgn}(\Delta) = \begin{cases} 1 & \Delta > 0 \quad \text{双曲型} \\ 0 & \Delta = 0 \quad \text{抛物型} \\ -1 & \Delta < 0 \quad \text{椭圆型} \end{cases}$$

9.1.3 特征方程

希望二阶偏导项系数 A, B, C 中有些为 0，以简化方程。

求 ξ 满足

$$A = a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2 = 0$$

定理 9.1. 若 $\phi(x, y) = k$（k 为常数）是以下一阶常微分方程（称其为特征方程）的一个通解（即由 k 标记的一族积分曲线）

$$a(\frac{dy}{dx})^2 - 2b\frac{dy}{dx} + c = 0$$

则 $\xi = \phi(x, y)$ 是以下一阶偏微分方程的一个特解

$$a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2 = 0$$

证明 9.1.

$$\phi(x, y) = k \Rightarrow \varphi_x dx + \phi_y dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\phi_x}{\phi_y}$$

代入常微分方程：

$$\begin{aligned} & a(-\frac{\phi_x}{\phi_y})^2 - 2b(-\frac{\phi_x}{\phi_y}) + c = 0 \\ & a(\phi_x)^2 + 2b\phi_x\phi_y + c(\phi_y)^2 = 0 \\ \xi = \phi(x, y) \Rightarrow \quad & a(\xi_x)^2 + 2b\xi_x\xi_y + c(\xi_y)^2 = 0 \end{aligned}$$

Def (特征方程).

$$\boxed{a(\frac{dy}{dx})^2 - 2b\frac{dy}{dx} + c = 0}$$

1. $\Delta = b^2 - ac > 0$

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} = \frac{b \pm \sqrt{\Delta}}{a}$$

$$\frac{dy}{dx}=\frac{b+\sqrt{\Delta}}{a} \quad \Rightarrow \phi(x,y)=k_1 \quad \Rightarrow \xi=\phi(x,y)\Rightarrow A=0$$
$$\frac{dy}{dx}=\frac{b-\sqrt{\Delta}}{a} \quad \Rightarrow \psi(x,y)=k_2 \quad \Rightarrow \eta=\psi(x,y)\Rightarrow C=0$$

其中，$\phi(x,y)=k_1$ 和 $\psi(x,y)=k_2$ 称为**特征线**

$A=C=0, B\neq 0$，方程只剩下交叉项.

Def (双曲型 PDE 的标准形式).

$$u_{\xi\eta}=\Phi_1(\xi,\eta,u,u_\xi,u_\eta)$$

变量代换：$\psi=\xi+\eta,\sigma=\xi-\eta$

$$\psi_{\rho\rho}-u_{\sigma\sigma}=\Phi_2(\rho,\sigma,u,u_\rho,u_\sigma)$$

2. $\Delta=b^2-ac<0$

$$\frac{dy}{dx}=\frac{b\pm i\sqrt{|\Delta|}}{a}$$

$$\frac{dy}{dx}=\frac{b+i\sqrt{|\Delta|}}{a} \quad \Rightarrow \phi(x,y)=k_1 \quad \Rightarrow \xi=\phi(x,y)\Rightarrow A=0$$
$$\frac{dy}{dx}=\frac{b-i\sqrt{|\Delta|}}{a} \quad \Rightarrow \psi(x,y)=k_2 \quad \Rightarrow \eta=\psi(x,y)\Rightarrow C=0$$

两方程互为复共轭，可取 $\psi(x,y)=\phi^*(x,y)$

Def (椭圆型 PDE 的标准形式).

$$\begin{cases}\xi=\phi(x,y)\\ \eta=\phi^*(x,y)\end{cases} \Rightarrow u_{\xi\eta}=\Phi_1(\xi,\eta,u,u_\xi,u_\eta)$$

变量代换：$\rho=\xi+\eta,\sigma=i(\xi-\eta)\Rightarrow \dfrac{\partial}{\partial\xi}=\dfrac{\partial}{\partial\rho}+i\dfrac{\partial}{\partial\sigma},\dfrac{\partial}{\partial\eta}=\dfrac{\partial}{\partial\rho}-i\dfrac{\partial}{\partial\sigma}$

$$u_{\rho\rho}+u_{\sigma\sigma}=\Phi_2(\rho,\sigma,u,u_\rho,u_\sigma)$$

3. $\Delta=b^2-ac=0$

$$\frac{dy}{dx}=\frac{b}{a}\Rightarrow\varphi(x,y)=k\Rightarrow\xi=\phi(x,y)\Rightarrow A=0$$

$$\Delta=B^2-AC=0,A=0\Rightarrow B=0,C\neq 0$$

$$u_{\eta\eta}=\Phi(\xi,\eta,u,u_\xi,u_\eta)$$

η 可任选，需和 ξ 独立，即

$$\det\begin{pmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{pmatrix}\neq 0$$

例 9.1. $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0$

由 $\Delta = (-xy)^2 - x^2y^2 = 0$ 可知，方程为抛物型。由特征方程可得

$$a(\frac{dy}{dx})^2 - 2b\frac{dy}{dx} + c = 0 \Rightarrow \frac{dy}{dx} = \frac{b}{a} = \frac{-xy}{x^2} = -\frac{y}{x}$$
$$\Rightarrow \frac{dy}{y} + \frac{dx}{x} = 0 \Rightarrow \ln y + \ln x = k' \Rightarrow xy = k$$

取 $\xi = xy, \eta$ 可任取，取 $\eta = y$

$$C = y^2 = \eta^2$$
$$E = y = \eta \quad D = F = G = 0$$

简化后的方程为:

$$\eta^2 u_{\eta\eta} + \eta u_\eta = 0 \Rightarrow \eta u_{\eta\eta} + u_\eta = 0$$

令 $\nu_\eta + v = 0$，可得 $\eta\nu_\eta + \nu = 0$，为欧拉型的方程。再令 $\eta = e^t$，有

$$v = \phi(\xi)e^{-t} = \frac{\phi(\xi)}{\eta}$$

因此

$$\begin{aligned} u &= \varphi(\xi)\ln|\eta| + \phi(\xi) \\ &= \varphi(xy)\ln|y| + \psi(x,y) \end{aligned}$$

例 9.2 (Tricomi 方程)**.**

$$yu_{xx} + u_{yy} = 0$$

可以计算得 $\Delta = -y$

1. $y > 0$：椭圆型

$$\text{特征方程：} y(\frac{dy}{dx})^2 + 1 = 0 \ \Rightarrow \frac{dy}{dx} = \pm\frac{i}{\sqrt{y}} \quad \text{特征曲线：} x \pm i\frac{2}{3}y^{\frac{3}{2}} = k$$

做变量代换:

$$\begin{cases} \xi = x \\ \eta = \frac{2}{3}y^{\frac{3}{2}} \end{cases}$$

计算可得在这组变量下方程的标准型为:

$$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\eta}u_\eta = 0$$

2. $y < 0$

$$\text{特征方程：} y(\frac{dy}{dx})^2 + 1 = 0 \ \Rightarrow \frac{dy}{dx} = \pm\frac{1}{\sqrt{-y}} \quad \text{特征曲线：} x \pm \frac{2}{3}(-y)^{\frac{3}{2}} = k$$

做变量代换:

$$\begin{cases} \rho = x \\ \sigma = \frac{2}{3}(-y)^{\frac{3}{2}} \end{cases}$$

计算可得在这组变量下方程的标准型为:

$$u_{\rho\rho}-u_{\sigma\sigma}-\frac{1}{3\sigma}u_{\sigma}=0.$$

3. $y=0\Rightarrow u_{yy}|_{y=0}=0$

例 9.3 (常系数二阶 PDE 可消去一阶偏导项)**.**

$$au_{xx}+2bu_{xy}+cu_{yy}+du_x+eu_y+fu+g=0$$

$$u=e^{\lambda x+\mu y}v(x,y)$$

$$\begin{aligned}&a(v_{xx}+2v_x\lambda+\lambda^2v)e^{\lambda x+\mu y}+2b(v_{xy}+v_x\mu+v_y\lambda+\mu\lambda)e^{\lambda x+\mu y}\\&+c(v_{yy}+2v_y\mu+\mu^2v)e^{\lambda x+\mu y}\\&+d(v_x+\lambda v)e^{\lambda x+\mu y}+e(v_y+\mu v)e^{\lambda x+\mu y}+fve^{\lambda x+\mu y}+g=0\end{aligned}$$

$$\begin{aligned}&av_{xx}+2bv_{xy}+Cv_{yy}+(2\lambda a+2\mu b+d)v_x+(2\lambda b+2\mu c+e)v_y\\&+(a\lambda^2+2\lambda\mu b+c\mu^2+\lambda d+\mu e+f)v+ge^{-\lambda x-\mu y}=0\end{aligned}$$

$$\text{一阶偏导项}=0\Leftarrow\begin{cases}v_x\text{系数}\quad 2\lambda a+2\mu b+d=0\\v_y\text{系数}\quad 2\lambda b+2\mu c+e=0\end{cases}$$

$$\begin{pmatrix}a\\b\end{pmatrix}\begin{pmatrix}2\lambda\\2\mu\end{pmatrix}=\begin{pmatrix}-d\\-e\end{pmatrix}$$

$$\begin{pmatrix}\lambda\\\mu\end{pmatrix}=\frac{1}{2}\begin{pmatrix}a&b\\b&c\end{pmatrix}^{-1}\begin{pmatrix}-d\\-e\end{pmatrix}=\frac{1}{2\Delta}\begin{pmatrix}c&-b\\-b&a\end{pmatrix}\begin{pmatrix}d\\e\end{pmatrix}$$

$$\lambda=\frac{cd-be}{2\Delta},\mu=\frac{ae-bd}{2\Delta}$$

10 积分变换

10.1 Fourier 变换

Fourier 级数:

$$[-L/2, L/2] \quad f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{2\pi n}{L}x + b_n \sin\frac{2\pi n}{L}x)$$

$$\text{正交性} \quad a_0 = \frac{1}{L}\int_{-L/2}^{L/2} f(x)dx$$

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\frac{2\pi n}{L}x, \quad b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\frac{2\pi n}{L}xdx$$

Fourier 级数就是按照如下本征值问题的解所构成的完备集展开

$$\begin{cases} y''(x) + \lambda y(x) = 0 \\ y(-L/2) = y(L/2), \quad y'(-L/2) = y'(L/2) \end{cases}$$

将 sin, cos 写成复数形式

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\begin{aligned} f(x) =& a_0 + \sum_{n=1}^{\infty} a_n \frac{e^{i\frac{2\pi n}{L}x} + e^{-i\frac{2\pi n}{L}x}}{2} + \sum_{n=1}^{\infty} b_n \frac{e^{i\frac{2\pi n}{L}x} - e^{-i\frac{2\pi n}{L}x}}{2i} \\ =& a_0 + \sum_{n=1}^{\infty}\frac{1}{2}(a_n - ib_n)e^{i\frac{2\pi n}{L}x} + \sum_{n=1}^{\infty}\frac{1}{2}(a_n + ib_n)e^{-i\frac{2\pi n}{L}x} \\ =& \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{L}x} \end{aligned}$$

其中，系数 c_n 满足

$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-\mathrm{i}\frac{2n\pi}{L}x}\mathrm{d}x$$

或者将 Fourier 级数写成复数形式:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2n\pi\mathrm{i}}{L}x}$$

正交性:

$$\int_{-L/2}^{L/2}(e^{i\frac{2\pi m}{L}x})^* e^{i\frac{2\pi n}{L}x}dx = \begin{cases} L, & m = n \\ 0, & m \neq n \end{cases}$$

令区间长度 $L\to\infty$, 将级数变成积分

$$\begin{aligned}f(x)=&\sum_{k=\frac{2\pi}{L}n}\frac{1}{L}F(k)e^{ikx}\quad n=0,\pm1,\pm2,...\\=&\frac{1}{2\pi}\sum_{k=\frac{2\pi}{L}n}\frac{2\pi}{L}F(k)e^{ikx}\\\xRightarrow{L\to\infty}&\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(k)e^{ikx}dk\end{aligned}$$

Def (Fourier 变换)**.**

$$\boxed{F(k)=\mathscr{F}[f(x)]\equiv\int_{-\infty}^{\infty}f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x}$$

Def (Fourier 逆变换)**.**

$$\boxed{f(x)=\mathscr{F}^{-1}[F(k)]\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k}$$

为使变换**存在且可逆**，要求：

1. 任意有限区间上，$f(x)$ 分段光滑
2. $f(x)$ 在 $(-\infty,\infty)$ 上绝对可积，即 $\int_{-\infty}^{\infty}|f(x)|dx<\infty\Rightarrow f(\pm\infty)\to0$

10.1.1 Fourier 变换的性质

1. 线性：$c_1f_1(x)+c_2f_2(x)\longleftrightarrow c_1F_1(k)+c_2F_2(k)$
2. 微分公式：$f'(x)\longleftrightarrow ikF(k)\Rightarrow f^{(n)}(x)\longleftrightarrow(ik)^nF(k)$
3. 卷积公式：$f*g(x)\longleftrightarrow F(k)G(k)$

证明 10.1 (微分公式)**.**

$$\mathscr{F}[f(k)]=\int_{-\infty}^{+\infty}f(x)e^{-ikx}dx=e^{-ikx}f(x)\Big|_{-\infty}^{+\infty}+ik\int_{-\infty}^{+\infty}f(x)e^{-ikx}dx=ik\int_{-\infty}^{+\infty}f(x)e^{-ikx}dx$$

Def (卷积)**.**

$$a=a_0,a_1,...,a_n,b=b_0,b_1,...,b_n$$

$$a\longleftrightarrow f_a(t)=a_0+a_1t+...+a_nt_n$$

$$b\longleftrightarrow f_b(t)=b_0+b_1t+...+b_nt_n$$

$$\begin{aligned}f_a(t)f_b(t)=&a_0b_0+(a_0b_1+a_1b_0)t+...\\\equiv&c_0+c_1t+...+c_nt^n\\\equiv&f_{a*b}\end{aligned}$$

$$\boxed{f*g=\int_{-\infty}^{\infty}f(\xi)g(x-\xi)d\xi}$$

例 10.1 (卷积的实际应用)**.** 两个独立连续随机变量 X, Y，概率密度为 $f_X(x), f_Y(y), Z = X + Y$ 的概率密度为

$$f(z) = \int_{-\infty}^{\infty} f_X(t) f_Y(z-t) dt$$

10.1.2 高维 Fourier 变换

三维 $\vec{r} = (x, y, z), \vec{k} = (k_x, k_y, k_z)$

10.1.3 Fourier 变换的应用

例 10.2 (一维无限热传导)**.**

$$\begin{cases} \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad t > 0 \\ u|_{t=0} = \phi(x) \end{cases}$$

$$\int_{-\infty}^{+\infty} (\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2}) e^{-ikx} dk = 0$$

$$\Rightarrow \begin{cases} \frac{\mathrm{d}U(k,t)}{\mathrm{d}t} + \kappa k^2 U(k,t) = 0 \\ U|_{t=0} = \mathscr{F}[\phi(x)] \equiv \Phi(k) \end{cases}$$

$$U(k,t) = \Phi(k) e^{-\kappa k^2 t}$$

对两项分别做傅里叶逆变换:

$$\begin{aligned} &\mathscr{F}^{-1}(\Phi(k)) = \phi(x) \\ &\mathscr{F}^{-1}(e^{-Kk^2 t}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\kappa k^2 t} e^{ikx} dk = \sqrt{\frac{\pi}{\kappa t}} e^{-\frac{x^2}{4\kappa t}} \end{aligned}$$

u 为两项逆变换的卷积

$$\begin{aligned} u(x,t) &= \mathscr{F}^{-1}[\Phi(k) e^{-\kappa k^2 t}] \\ &= \int_{-\infty}^{+\infty} \phi(\xi) \frac{1}{2\sqrt{\pi\kappa t}} e^{-\frac{(x-\xi)^2}{4kt}} d\xi \end{aligned}$$

类比 Fourier 变换法和分离变量法

$$\begin{array}{ccc} \frac{2\pi n}{L} & T_n'(t) + k(\frac{2\pi n}{L})^2 T_n(t) = 0 & u = \sum_n X_n(x) T_n(t) \\ k & \frac{dU(t,t)}{dt} + kR^2 U(t,t) = 0 & u = \frac{1}{2\pi}\int \Phi(t) e^{-\kappa R^2 t} dR \end{array}$$

例 10.3 (非齐次方程)**.**

$$\begin{cases} \frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} = f(x,t) \quad -\infty < x < \infty, t > 0 \\ u|_{t=0} = 0 \end{cases}$$

做 Fourier 变换 $u(x,t)\longleftrightarrow U(k,t)\quad f(x,t)\longleftrightarrow F(k,t)$

$$\begin{cases}\frac{dU(k,t)}{dt}+\kappa k^2U(k,t)=F(k,t)\\ U|_{t=0}\end{cases}$$

根据齐次化原理:

$$U(k,t)=\int_0^t e^{-\kappa k^2(t-\tau)}F(k,\tau)d\tau$$

$$\begin{aligned}u(x,t)&=\mathscr{F}^{-1}\left[\int_0^t e^{-k^2(t-\tau)}F(k,\tau)d\tau\right]\\&=\int_0^t d\tau\mathscr{F}^{-1}\left[e^{-\kappa^2(t-\tau)}F(k,\tau)\right]\end{aligned}$$

分别对 $e^{-\kappa^2(t-\tau)},F(k,\tau)$ 求逆变换，再求卷积

$$\mathscr{F}(e^{-\kappa^2(t-\tau)})=\frac{1}{2\sqrt{\pi\kappa(t-\tau)}}e^{-\frac{x^2}{4\kappa(t-\tau)}}$$

$$u(x,t)=\frac{1}{2\sqrt{\pi\kappa}}\int_0^t d\tau\int_{-\infty}^{+\infty}d\xi f(\xi,\tau)\frac{1}{\sqrt{t-\tau}}e^{-\frac{(x-\xi)^2}{4\kappa(t-\tau)}}$$

例 10.4 (三维齐次热传导问题)**.**

$$\begin{cases}\frac{\partial u}{\partial t}-k\nabla^2u=0 & -\infty<x,y,z<\infty,t>0\\ u|_{t=0}=\phi(\vec{r}) & \vec{r}=(x,y,z)\end{cases}$$

$$U(\vec{k},t)=\Phi(\vec{k})e^{-\kappa\vec{k}^2t}$$

分别对 $e^{-\kappa\vec{k}^2t},\Phi(\vec{k})$ 求逆变换，再求卷积

$$\mathscr{F}(e^{-\kappa\vec{k}^2t})=\frac{1}{(4\pi\kappa t)^{3/2}}e^{-\frac{|\vec{x}-\vec{\xi}|^2}{4\kappa t}}$$

$$u(\vec{r},t)=\frac{1}{(4\pi\kappa t)^{3/2}}\iiint_{-\infty}^{\infty}\phi(\vec{\xi})e^{-\frac{|\vec{x}-\vec{\xi}|^2}{4\kappa t}}d\xi^3$$

10.2 Laplace 变换

Def (Laplace 变换).

$$L[x(t)] = X(p) = \int_0^\infty x(t)e^{-pt}dt$$

Def (Laplace 逆变换).

$$x(t) = L^{-1}[X(p)] = \frac{1}{2j}\int_{\sigma+j\infty}^{\sigma-j\infty} X(p)e^{pt}ds$$

只关心 $t>0$：$f(t)$ 理解为 $f(t)H(t)$，其中

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

收敛性优于 Fourier 变换

10.2.1 Laplace 变换的性质

性质	表达式	性质	表达式
线性	$L[ax_1(t)+bx_2(t)] = aX_1(s) \pm X_2(s)$	时标变换	$L\left[x\left(\frac{t}{a}\right)\right] = aX(as)$
实微分	$L\left[\frac{dx(t)}{dt}\right] = sX(s) - x(0)$	复微分	$L[tx(t)] = -\frac{dX(s)}{ds}$
	$L\left[\frac{d^2x(t)}{dt^2}\right] = s^2X(s) - sx(0) - x'(0)$		$L[t^nx(t)] = (-1)^n\frac{d^nX(s)}{ds^n}$
	$L\left[\frac{d^nx(t)}{dt^n}\right] = s^nX(s) - [s^{n-1}x(0) + s^{n-2}x'(0) + ... + x^{(n-1)}(0)]$		
实积分	$L\left[\int x(t)dt\right] = \frac{1}{s}X(s)$	复积分	$L\left[\frac{1}{t}x(t)\right] = \int_s^\infty X(s)ds$
实平移	$L[x(t-\tau)u(t-\tau)] = e^{-\tau s}X(s)$	复平移	$L[e^{-at}x(t)] = X(s+a)$
实卷积	$L[x_1(t) * x_2(t)] \equiv L\left[\int_0^t x_1(t-\tau)x_2(\tau)d\tau\right] = X_1(s)X_2(s)$	复卷积	$L[x_1(t)x_2(t)] = \frac{1}{2\pi j}\int_{\sigma+j\infty}^{\sigma-j\infty} X_2(p)X_1(s-p)dp$
初值	$\lim_{t\to 0^+} x(t) = \lim_{s\to\infty} sX(s)$	终值	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

图 8: Laplace 变换的性质

10.2.2 常用函数的 Laplace 变换

$f(t)$	$\mathcal{L}[f(t)] = F(s)$
1	$\dfrac{1}{s}$
$e^{-at}f(t)$	$F(s+a)$
$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$
$\delta(t)$	1
$\delta(t-t_0)$	e^{-st_0}
$t^n f(t)$	$(-1)^n \dfrac{d^n F(s)}{ds^n}$
$f'(t)$	$sF(s) - f(0)$
$f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) -$ $\dots - f^{(n-1)}(0)$
$\displaystyle\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$
$t^n\ (n = 0, 1, 2, \dots)$	$\dfrac{n!}{s^{n+1}}$
$t^x\ (x \geq -1 \in \mathbb{R})$	$\dfrac{\Gamma(x+1)}{s^{x+1}}$
$\sin kt$	$\dfrac{k}{s^2+k^2}$
$\cos kt$	$\dfrac{s}{s^2+k^2}$
e^{at}	$\dfrac{1}{s-a}$
$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(s-a)(s-b)}$

$f(t)$	$\mathcal{L}[f(t)] = F(s)$
$\dfrac{ae^{at}-be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$
te^{at}	$\dfrac{1}{(s-a)^2}$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$
$e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$
$t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$
$t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$
$t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$
$t\cosh kt$	$\dfrac{s^2-k^2}{(s^2-k^2)^2}$
$\dfrac{\sin at}{t}$	$\arctan\dfrac{a}{s}$
$\dfrac{1}{\sqrt{\pi t}}e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$\dfrac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
$\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$

10.2.3 Laplace 变换的应用

例 10.5 (ODE：受迫振动)**.**

$$\begin{cases} y''(t) + w^2 y(t) = f(t) \\ y(0) = y'(0) = 0 \end{cases}$$

$$p^2Y(p) - Py(0) - y'(0) + w^2Y(p) = F(p)$$

$$p^2Y(p) + w^2Y(p) = F(p) \Rightarrow Y(p) = \frac{1}{p^2+w^2}F(p)$$

对 $\frac{1}{p^2+w^2}, F(p)$ 两项分别做 Laplace 变换，然后做卷积。

$$L\left(\frac{1}{p^2+w^2}\right) = \frac{1}{\omega}\sin\omega t$$

$$y(t) = \int_0^t \frac{1}{w}\sin(t-\tau)f(\tau)d\tau$$

例 10.6 (半无界端点外力下的波动方程)**.**

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \quad x > 0, \quad t > 0 \\ \frac{\partial u}{\partial x}|_{x=0} = f(t) \\ u|_{t=0} = \frac{\partial y}{\partial t}|_{t=0} = 0 \end{cases}$$

$$\frac{\partial u}{\partial t^2} \longleftrightarrow p^2U - pu|_{t=0} - \left.\frac{\partial u}{\partial t}\right|_{t=0} = p^2U$$

$$\begin{cases} p^2U(x,p) = a^2\frac{d^2}{dx^2}U(x,p) \\ \frac{dU}{dx}|_{x=0} = F(p), \quad U|_{x=+\infty} = 0 \end{cases}$$

$$\begin{aligned} U(x,p) &= ce^{\frac{p}{a}x} + De^{-\frac{p}{a}x} \\ &= -\frac{a}{p}e^{-\frac{p}{a}x}F(p) \end{aligned}$$

法 1: 组合 $-\frac{a}{p}e^{-\frac{x}{a}p}$; 查表得 $L^{-1}[\frac{1}{p}e^{-\lambda p}] = H(t-\lambda)$，令 $\lambda = \frac{x}{a}$

$$L^{-1}[-\frac{a}{p}e^{-\frac{x}{a}p}] = -aH(t-\frac{x}{a})$$

$$\begin{aligned} u(x,t) &= L^{-1}[-\frac{a}{p}e^{-\frac{x}{a}p}F(p)] \\ &= -a\int_0^t H(t-\tau-\frac{x}{a})f(\tau)d\tau \\ &= \begin{cases} 0, t < \frac{x}{a} \\ -a\int_0^{t-\frac{x}{a}} f(\tau)d\tau, t \geq \frac{x}{a} \end{cases} \\ &= -aH(t-\frac{x}{a})\int_0^{t-\frac{x}{a}} f(\tau)d\tau \end{aligned}$$

法 2: 组合 $-\frac{a}{p}F(p)$

$$L^{-1}[-\frac{a}{p}F(p)] = -a\int_0^t f(\tau)d\tau$$

$$L^{-1}[e^{-\lambda p}G(p)] = g(t-\lambda)H(t-\lambda)$$

取 $\lambda = \frac{x}{a}$, $\quad G(p) = -\frac{a}{p}F(p)$

$$\begin{aligned} L^{-1}[-\frac{a}{p}e^{-\frac{x}{a}p}F(p)] &= H(t-\frac{x}{a})g(t-\frac{x}{a}) \\ &= H(t-\frac{x}{a})(-a\int_0^{t-\frac{x}{a}} f(\tau)d\tau) \end{aligned}$$

11 格林函数

11.1 数学基础：δ 函数

Def (Dirac δ 函数). 一维：

$$p(x)=\delta_l(x)=\begin{cases}0 & |x|>\frac{l}{2}\\ \frac{1}{l} & |x|\le\frac{l}{2}\end{cases},\int_{-\infty}^{\infty}\delta_l(x)dx=1$$

$$\delta(x)=\lim_{l\to 0}\delta_l(x),\int_{-\infty}^{+\infty}\delta(x)dx=1$$

$\delta(x)$ 由积分性质定义

$$\int_a^b\delta(x)dx=\begin{cases}0 & a<b<0\text{或}0<a<b,0\notin(a,b)\\ 1 & a<0<b\end{cases}$$

利用 $\delta(x)=\lim_{n\to\infty}\delta_n(x)$ 定义,$\delta_n(x)$ 可以有以下定义方式

$$1.\delta_n(x)=\frac{n}{\sqrt{\pi}}e^{-n^2x^2}$$
$$2.\delta_n(x)=\frac{n}{\pi}\frac{1}{n^2x^2+1}$$
$$3.\delta_n(x)=\frac{1}{\pi}\frac{\sin x}{x}$$

11.1.1 Dirac 函数的性质

1. 筛选性

$$\int_{-\infty}^{+\infty}f(x)\delta(x)dx=f(0)$$

2. $\delta(x)$ 为偶函数

$$\int_{-\infty}^{+\infty}\delta(-x)f(x)dx\overset{x'=-x}{=\!=\!=}\int_{-\infty}^{+\infty}\delta(x')f(-x')dx'=f(0)$$

3. 导数运算

$$\int_{-\infty}^{+\infty}\delta'(x)f(x)dx=\delta(x)f(x)\Big|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}f(x)\delta(x)dx=-f'(0)$$

$$\int_{-\infty}^{\infty}\delta^{(n)}(x)f(x)dx=(-1)^nf^{(n)}(0)$$

2. $\delta(x)$ 与普通函数的复合

$$\delta[\phi(x)]=\sum_k\frac{\delta(x-x_k)}{|\phi'(x_k)|}$$

其中 x_k 是 $\phi(x)=0$ 的根

例：$\delta(ax)=\frac{1}{|a|}\delta(x)$

11.1.2 高维 Dirac 函数

二维：$\vec{r}=(x,y)\quad \delta(\vec{r})=\delta(x)\delta(y)$

$$\begin{aligned}\int_{-\infty}^{+\infty}\int f(\vec{r})\delta(\vec{r})d\vec{r} &\equiv \int_{-\infty}^{+\infty} f(x,y)\delta(x)\delta(y)dxdy \\ &= \int_{-\infty}^{+\infty} f(0,y)\delta(y)dy = f(0,0)\end{aligned}$$

三维：$\vec{r}=(x,y,z),\quad \partial(\vec{r})=\delta(x)\delta(y)\delta(z)$

$$\iiint_{-\infty}^{+\infty} f(\vec{r})\delta(\vec{r})d\vec{r}=f(0,0,0)$$
$$\iiint_{-\infty}^{+\infty} f(\vec{r})\delta(\vec{r}-\vec{r}_0)d\vec{r}=f(\vec{r}_0)$$

11.1.3 Dirac 函数的傅里叶变换

$$\boxed{\delta(x)\longleftrightarrow 1}$$

取 $\delta_n(x)=\frac{n}{\pi}\frac{1}{n^2x^2+1}$

$$\begin{aligned}\Rightarrow \int_{-\infty}^{+\infty}\delta_n\alpha e^{-ikx}dx &= \int_{-\infty}^{+\infty}\frac{n}{\pi}\frac{1}{n^2x^2+1}e^{-ikx}dx \\ &= e^{-\frac{|k|}{n}}\end{aligned}$$

$$\begin{aligned}\delta_n(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ikx}e^{-\frac{|k|}{n}}dk &= \frac{1}{2\pi}\int_0^{\infty}e^{-\frac{k}{n}}(e^{ikx}+e^{-ikx})dx \\ &= \frac{1}{2\pi}(\frac{1}{\frac{1}{n}-ix}+\frac{1}{\frac{1}{n}+ix}) \\ &= \frac{n}{\pi}\frac{1}{n^2x^2+1}\end{aligned}$$

11.2 格林函数的定义与性质

例 11.1 (三维无界空间静电场)**.** 电荷密度 $\rho(\vec{r})$，电势 $u(\vec{r})$.

$\nabla^2u(\vec{r})=-\frac{\rho(\vec{r})}{\epsilon_r}$，但为书写方便，取 $\epsilon_r=1$ ，或者说，将 $\frac{\rho(\vec{r})}{\epsilon_r}$ 记为 $\rho(\vec{r})$

$$\nabla u(\vec{r})=-\rho(\vec{r})\quad -\infty<x,y,z<\infty$$

根据静电场的线性叠加性

$$\begin{aligned}u(\vec{r}) &= \iiint_{-\infty}^{\infty}\frac{1}{4\pi(\vec{r}-\vec{r'})}\rho(\vec{r'})d\vec{r'} \\ &\equiv \iiint_{-\infty}^{\infty}(\vec{G}(\vec{r},\vec{r'}))\rho(\vec{r'})d\vec{r'}\end{aligned}$$

$$\boxed{G(\vec{r},\vec{r}') = \frac{1}{4\pi|\vec{r}-\vec{r}'|}}$$

代表 $\vec{r}'$ 处都单位点电荷在 $\vec{r}$ 处产生的电势

无界空间相对简单，半无界、有界空间呢？例如, 给定长方体、球体内部电荷分布边界上的电势, 如何用类似思路得出其中的 $u(\vec{r})$

Green 函数的意义

1. **物理上**：点源产生的场（函数）在时空中的分布。在空间是**源函数**；在时空是**传播函数**。
2. **数学上**：具有点源的偏微分方程在齐次边界条件或者无界区域、初值条件下的解。

解决 PDE 非齐次方项或非齐次边界条件。非齐次项 = 源，将源的效果分解为点源的叠加。

Green 函数的定义

Def (Green 函数)**.** Green 函数由线性算子 $\hat{L}$ 和边界条件和初始条件决定:

$$\hat{L}G(\vec{r},t;\vec{r}',t') = \delta(\vec{r}-\vec{r}')\delta(t-t')$$

加上齐次边界条件和初始条件。

对于泊松方程:

用定解问题

$$\boxed{\begin{array}{l}\nabla^2 G(\boldsymbol{r};\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}'),\quad \boldsymbol{r},\boldsymbol{r}'\in V\\ \text{齐次边界条件}\end{array}}$$

的解 $G(\boldsymbol{r};\boldsymbol{r}')$ 叠加出

$$\begin{cases}\nabla^2 u(\boldsymbol{r}) = -\rho(\boldsymbol{r}),\quad \boldsymbol{r}\in V\\ u|_\Sigma = f(\Sigma)\end{cases}$$

的解 $u(\boldsymbol{r})$, 即把 $u(\boldsymbol{r})$ 用 $\rho(\boldsymbol{r}), f(\Sigma)$ 和 $G(\boldsymbol{r};\boldsymbol{r}')$ 表示出来。而 $G(\boldsymbol{r};\boldsymbol{r}')$ 即称为原定解问题的 Green 函数。

Green 函数的性质

Def (Green 公式一维形式)**.**

$$\boxed{\int_a^b [u(x)v''(x)-v(x)u''(x)]\mathrm{d}x = [u(x)v'(x)-v(x)u'(x)]|_a^b}$$

Def (Green 公式三维形式)**.**

$$\boxed{\iiint_V \left[u(\boldsymbol{r})\nabla^2 v(\boldsymbol{r}) - v(\boldsymbol{r})\nabla^2 u(\boldsymbol{r})\right]\mathrm{d}^3\boldsymbol{r} = \iint_\Sigma [u\nabla v - v\nabla u]\cdot\,\mathrm{d}\boldsymbol{\Sigma}}$$

定理 11.1 (Green 函数的对称性).

若算子 $\hat{L}$ 是厄米的，则由 $\hat{L}$ 产生的 G 有 $G^*(\vec{r};\vec{r}') = G(\vec{r}';\vec{r})$；特别地，对于实变 Green 函数，$G(\vec{r};\vec{r}') = G(\vec{r}';\vec{r})$.

时间传播函数没有对称性：$G(\vec{r},t;\vec{r}',t') \neq G(\vec{r}',t';\vec{r},t)$.（因果律引起）

证明 11.1 (对于泊松方程：$\hat{L} = \nabla^2$). 格林公式中取 $u = G(x;x'),\quad v(x) = G(x;x'')$

$$\int_0^L [G(x;x')\frac{d^2G(x;x'')}{dx^2} - G(x;x'')\frac{d^2G(x-x')}{dx^2}]dx = (G(x;x')\frac{dG(x;x'')}{dx} - G(x;x'')\frac{dG(x;x'')}{dx})\Big|_0^L$$

$$\Rightarrow G(x'';x') - G(x';x'') = 0$$

11.3 格林函数解偏微分方程

解题方法 (偏微分方程的积分解).

1. 求格林函数 $G(\vec{r};\vec{r}')$
2. 利用迭加原理给出物理问题 $u(\vec{r})$ 的积分形式解

11.3.1 含时格林函数解决非齐次方程问题

回顾非齐次方程的齐次化原理：考虑一下非齐次方程

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2\dfrac{\partial^2 u}{\partial x^2} = f(x,t) \quad 0<x<l, t>0 \\ u|_{x=0} = u|_{x=l} = 0 \\ u|_{t=0} = \dfrac{\partial u}{\partial t}\Big|_{t=0} = 0 \end{cases}$$

引入辅助函数 $w(x,t;\tau)$，满足

$$w(x,t;\tau): \begin{cases} \dfrac{\partial^2 w}{\partial t^2} - a^2\dfrac{\partial^2 w}{\partial x^2} = 0 \quad 0<x<l, t>\tau \\ w|_{t=\tau} = 0, \dfrac{\partial w}{\partial t}\Big|_{t=\tau} = f(x,\tau) \end{cases}$$

可以得到方程的解

$$u(x,t) = \int_0^t w(x,t;\tau)d\tau$$

而格林函数 $G(x,t;x',t')$ 满足如下方程

$$\begin{cases} \frac{\partial^2 G}{\partial t^2} - a^2\frac{\partial^2 G}{\partial x^2} = \delta(x-x')\delta(t-t'), \quad 0<x,x'<l, t,t'>0 \\ G(x,t;x',t')|_{x=0} = G(x,t;x',t')|_{x=l} = 0 \\ G(x,t;x',t') = 0, t<t' \end{cases}$$

在这个情况下，辅助函数 $w(x,t;\tau)$ 实际上可以看作是特定初始条件下的格林函数的卷积。具体来说，$w(x,t;\tau)$ 可以用格林函数表示为

$$w(x,t;\tau)=\int_0^l G(x,t;x',\tau)f(x',\tau)dx'$$

这就意味着 $u(x,t)$ 可以表示为:

$$u(x,t)=\int_0^t w(x,t;\tau)d\tau=\int_0^t\int_0^l G(x,t;x',\tau)f(x',\tau)dx'd\tau$$

11.3.2 格林函数解决泊松方程

例 11.2 (一维电势分布：齐次边界条件)**.**

$$\begin{cases}\frac{d^2u(x)}{dx^2}=-f(x), \quad 0<x<L\\ u(0)=u(L)=0\end{cases}$$

如果将 $f(x')$ 视为脉冲的连续分布，则总体响应 $u(x)$ 可视为对这些脉冲的响应的叠加. 和三维电势问题类比，我们想求如下形式的解:

$$u(x)=\int_0^L G(x;x')f(x')dx'$$

为使其满足 ODE,

$$-f(x)=\frac{d^2u(x)}{dx^2}=\int_0^L\frac{d^2G(x;x')}{dx^2}f(x')dx'$$

x' 处点源的效果 $G(x,x')$ 为以下定解问题的解:

$$\begin{cases}\frac{d^2G(x;x')}{dx^2}=-\delta(x-x'), \quad 0<x,x'<L\\ u(0)=u(L)=0\Leftarrow G(0;x')=G(L;x')=0\end{cases}$$

$$\Rightarrow\left.\frac{d^2G(x;x')}{dx^2}\right|_{x\neq x'}=-\delta(x-x')|_{x\neq x'}=0$$

x' 点两侧导数跳跃为 -1:

$$\left.\frac{dG}{dx}\right|_{x=x'+0}-\left.\frac{dG}{dx}\right|_{x=x'-0}=\int_{x'-0}^{x'+0}\frac{d^2G}{dx^2}dx=\int_{x'-0}^{x'+0}[-\delta(x'-x)]dx=-1$$

$$G(x;x')=\begin{cases}a+bx\\ c+dx\end{cases}\xlongequal{G|_{x=0}=G|_{x=L}=0}\begin{cases}bx & x<x'\\ c+(b-1)x & x>x'\end{cases}\xlongequal{G(x;x')\text{连续}}G(x,x')=\begin{cases}\frac{x}{L}(L-x'), & x<x'\\ \frac{x'}{L}(L-x), & x>x'\end{cases}$$

$$\Rightarrow u(x)=\int_0^L G(x;x)f(x')dx'=\int_0^x\frac{x'}{L}(L-x)f(x')dx'+\int_x^L\frac{x}{L}(L-x')f(x)dx$$

例 11.3 (一维电势分布：非齐次边界条件).

$$\begin{cases}\frac{d^2u}{dx^2}=-f(x),0<x<L,\\ u|_{x=0}=\alpha,u|_{x=L}=\beta\end{cases}$$

代入格林函数 $G(x;x')$

$$\Rightarrow\int_0^L\left[u(x)\frac{d^2G(x;x')}{dx^2}-G(x;x')\frac{d^2u}{dx^2}\right]=\left[u(x)\frac{dG}{dx}-G(x;x')\frac{du}{dx}\right]\Big|_0^L$$

$\frac{d^2G(x;x')}{dx^2}=-\delta(x-x'),G|_{x=0},G|_{x=L}=0$

得到

$$u(x')=\int_0^L G(x;x')f(x')dx'-\beta\frac{dG(x;x')}{dx}\Big|_{x=L}+\alpha\frac{dG(x;x')}{dx}|_{x=0}$$

交换 (x,x')

$$u(x)=\int_0^L G(x;x')f(x')dx'-\beta\frac{dG(x;x')}{dx}\Big|_{x'=L}+\alpha\frac{dG(x;x')}{dx}|_{x'=0}$$

例 11.4 (三维电势分布).

$$\begin{cases}\nabla^2u(\vec{r})=-\rho(\vec{r})\quad \vec{r}\in V\\ u|_\Sigma=f(\Sigma)\end{cases}$$

$$\iiint_v[u(\vec{r})v^2G(\vec{r};\vec{r}')-G(\vec{r};\vec{r}')v^2u(\vec{r})]dr$$

$$=\iint_\Sigma[u(\vec{r})\frac{\partial G(\vec{r},\vec{r})}{\partial n}-G(\vec{r},\vec{r}')\frac{\partial u(r)}{\partial n}]d\Sigma$$

$$\Rightarrow u(\vec{r}')=\iiint_v G(\vec{r};\vec{r}')\rho(r)d\vec{k}\iint_\Sigma u(\vec{r})\frac{\partial G(\vec{r};\vec{r}')}{\partial n}d\Sigma$$

同一维，有交换律 $G(x'';x')=G(x';x'')$

$$u(\vec{r})=\iiint_V G(\vec{r},\vec{r}')\rho(\vec{r}')d\vec{r}-\iint_\Sigma f(\vec{r}')\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}d\Sigma'$$

11.4 求格林函数的典型方法

11.4.1 特殊方法：电像法

大家知道, 一旦在接地圆中放上点电荷后, 在圆周上必然出现感生电荷. 圆内任意一点的电势, 就是点电荷的电势和感生电荷的电势的叠加. 前者在点电荷所在点是对数发散的, 而后者在圆内是处处连续的. 如果我们能够方便地求出感生电荷在圆内所产生的电势, 当然也就求出了整个圆内 Poisson 方程第一边值问题的 Green 函数.

电像法的基本思想

- 将边界上的感生电荷用一个 (或者几个, 尽可能少) 等价的点电荷代替. 换句话说, 就是把接地圆内的点电荷的问题等价地转化为无界空间中的两个点电荷 (一个是真实的点电荷, 另一个是等价的"虚"电荷) 的问题.
- 圆内的电荷分布不能变
- 边界条件不变

解题方法 (电像法解题步骤).

1. 写出格林函数定义与齐次边界条件
2. 在 r' 处放置一个 +1 点电荷
3. 寻找边界以外的点电荷，使边界上电势为 0（对于平面，镜像地放置一个等量异号点电荷；对于球体，在半径延长线上 $r'r''=a^2$ 处放置等量异号点电荷）
4. 将 $G(r;r')$ 写成这些点电荷（包括在 r' 处的）在 r 处产生的电势之和

例 11.5 ($V=$ 上半空间 $z>0$).

$$\begin{cases}\nabla^2 u=0 \quad (z>0)\\ u|_{z=0}=f(x,y)\end{cases}$$

$$\Rightarrow \begin{cases}\nabla^2 G(\vec{r};\vec{r}')=-\delta(\vec{r}-\vec{r}'), \quad \vec{r},\vec{r}'\in V\\ G(\vec{r};\vec{r}')|_{z=0}=0\end{cases}$$

取 $\vec{r}'=(x',y',z')$ 的像点 $\vec{r}''=(x',y',-z')$. 在 $\vec{r}'$ 和 $\vec{r}'$ 分别放置电量 $+1$ 和 -1，则可验证这两个点电荷产生的电势可作为 $G(\vec{r};\vec{r}')$

证明 11.2.

$$G(\vec{r};\vec{r}')=\frac{1}{4\pi}(\frac{1}{|\vec{r}-\vec{r}'|}-\frac{1}{|\vec{r}-\vec{r}''|})$$

首先，$G(\vec{r};\vec{r}')$ 满足方程

$$\nabla^2 G(\vec{r};\vec{r}')=-\delta(\vec{r}-\vec{r}')+\delta(\vec{r}-\vec{r}'')$$

当 $\vec{r}$ 限制在上半空间时，第二项 $\delta(\vec{r}-\vec{r}'')$ 可扔掉

其次，边界条件 $G|_{z=0}=0$ 也满足

$$\begin{aligned}u(\vec{r})=&-\int_{z'=0} f(x',y')(-\frac{\partial G(\vec{r};\vec{r}')}{\partial z'})dx'dy'\\ =&\frac{z}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x',y')\frac{1}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}dx'dy'\end{aligned}$$

例 11.6 (三维泊松方程球内问题)**.** 若 V 为半径为 a 的球体,

$$\begin{cases}\nabla^2 u = -\rho(\vec{r}), \quad r < a \\ u|_{r=a} = f(\theta, \phi).\end{cases}$$

其格林函数满足

$$\begin{cases}\nabla^2 G(\vec{r}, \vec{r}') = -\delta(\vec{r} - \vec{r}'), \quad r < a, r' < a \\ G|_{r=a} = 0\end{cases}$$

设 $\vec{r}'$ 为球内一点 $(r' < a)$

利用平面几何知识(相似三角形),易证明:

当 $\vec{r}''$ 取在 $\vec{r}'$ 的延长线上,且且 $r'r'' = a^2$ 时,$\dfrac{1}{|\vec{r} - \vec{r}'|} : \dfrac{1}{|\vec{r} - \vec{r}''|} = \dfrac{1}{r'} : \dfrac{1}{a}$ 对球面上的任意点成立。

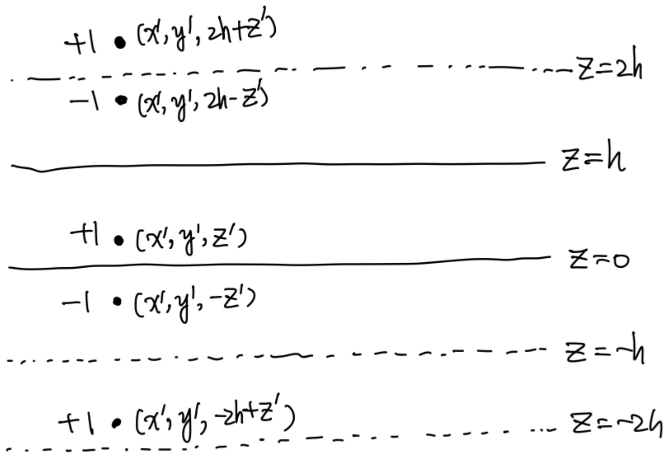
$$G(\vec{r}; \vec{r}') = \frac{1}{4\pi} \frac{1}{(\vec{r} - \vec{r}')} - \frac{1}{4\pi} \frac{a}{r'} \frac{1}{(\vec{r} - \vec{r}'')}$$

$$u(\vec{r}) = \iiint_V G(\vec{r}; \vec{r}') P(\vec{r}') d\vec{r}' - \iint_\Sigma f(\vec{r}') \frac{\partial G(\vec{r}; \vec{r}')}{\partial n'} d\Sigma'$$

例 11.7 (层状空间 $0 < z < h$)**.**

$$\begin{cases}v^2 G(\vec{r}; \vec{r}') = -\delta(\vec{r} - \vec{r}') \\ G|_{z=0} = G|_{z=h} = 0\end{cases}$$

记 $\vec{r} = (x, y, z), \vec{r'} = (x', y', z')$, 按如下方式选取无限个像点.(像是电梯里相对两面各放一个镜子,产生无限个像)



在 $\vec{r_n'} = (x', y', z' + znh)$ 放单位正电荷,

在 $\vec{r_n''} = (x', y', -z' + znh)$ 放单位负电荷,

则可使 $z = 0$ 和 $z = h$ 平面的电势为 0,满足 $G|_{z=0} = G|_{z=h} = 0$ 的要求

$$G(\vec{r}; \vec{r}') = \frac{1}{4\pi} \sum_{n=-\infty}^{+\infty} \left(\frac{1}{|\vec{r} - \vec{r_n'}|} - \frac{1}{|\vec{r} - \vec{r_n''}|}\right)$$

关于此级数的收敛性：

$$n \to \pm\infty : |\vec{r} - \vec{r}'_n| \approx |2nh + z' - z|$$
$$|\vec{r} - \vec{r}''_n| \approx |2nh - z' - z|$$

$$|\frac{1}{|\vec{r} - \vec{r}_n|} - \frac{1}{|\vec{r} - r_n|}| = |\frac{|\vec{r} - r''_n| - |\vec{r} - r'_n|}{|\vec{r} - r'_n||\vec{r} - \vec{r}''_n|}|$$
$$\approx \frac{2|z'|}{(2nh)^2} \approx \frac{|z'|}{2h^2}\frac{1}{n^2}$$

$\Rightarrow G(\vec{r};\vec{r}')$ 的级数收敛

11.4.2 本征函数展开法

例 11.8.

$$\begin{cases} \nabla^2 u(\vec{r}) + k^2 u(\vec{r}) = -\rho(\vec{r}), \quad \vec{r} \in V \\ u|_\Sigma = f(\Sigma) \end{cases}$$

取 V 为长方体：$0 < x < a, 0 < y < b, 0 < z < c$

先求 $G(\vec{r};\vec{r}')$ 满足

$$\begin{cases} (\nabla^2 + k^2)G(\vec{r};\vec{r}') = -\delta(\vec{r} - \vec{r}'), \vec{r}, \vec{r}' \in V \\ G(\vec{r};\vec{r}')|_z = 0 \end{cases}$$

$$\Rightarrow u(\vec{r}) = \iiint_V G(\vec{r},\vec{r}')\rho(\vec{r}')d\vec{r}' - \iint_\Sigma f(\vec{r}')\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}d\Sigma'$$

为求 $G(\vec{r};\vec{r}')$ 可将其按齐次方程问题的本征函数展开，并求出系数

设 $G(\vec{r},\vec{r}) = \sum_n C_n(\vec{r}')u_n(\vec{r})$. 先求本征函数问题：

$$\begin{cases} (\nabla^2 + k^2 + \lambda_n)u_n = 0 \\ u_n|_\Sigma = 0 \end{cases}$$

$$\Rightarrow \lambda_{n_x n_y n_z} = \pi^2(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}) - k^2, \quad n_x, n_y, n_z = 1, 2, 3 \ldots$$

$$u_{n_x n_y n_z}(x, y, z) = \sqrt{\frac{2}{a}}\sqrt{\frac{2}{b}}\sqrt{\frac{2}{c}} \sin\frac{n\pi}{a}x \sin\frac{n\pi}{b}y \sin\frac{n\pi}{c}z$$

$$||u_{n_x n_y n_z}||^2 = \iiint_V |u_{n_x n_y n_z}(x, y, z)|^2 dxdydz = 1$$

将 $G(\vec{r},\vec{r}) = \sum_n C_n(\vec{r}')u_n(\vec{r})$，代入方程

$$(\nabla^2 + k^2)G(\vec{r};\vec{r}) = -\delta(\vec{r} - \vec{r})$$
$$\Rightarrow -\sum_n C_n \lambda_n u_n(\vec{r}) = -\delta(\vec{r} - \vec{r})$$

两边同乘 $u_m^*(\vec{r})$，然后积分，利用正交性：

$$\begin{aligned}-c_m\lambda_m||u_m||^2 &= -\int\int u_m^*(\vec{r})\delta(\vec{r}-\vec{r}')d\vec{r} \\ &= -u_m^*(\vec{r}')\end{aligned}$$

$$\begin{aligned}\Rightarrow c_m &= \frac{1}{\lambda_m}\frac{u_m^*(\vec{r}')}{||u_m||^2} \\ \Rightarrow G(\vec{r};\vec{r}) &= \sum_m \frac{1}{\lambda_m}\frac{u_m^*(\vec{r}')}{||u_m||^2}u_m(\vec{r}) \\ &= \sum_n \frac{1}{||u_n||^2}\frac{u_n(\vec{r})u_n^*(\vec{r}')}{\lambda_n}\end{aligned}$$

11.4.3 积分变换法

例 11.9 (无穷长弦横振动)**.** 一个无限长弦，$t=t_0$ 时在 $x=x_0$ 处受到瞬时打击，冲量为 I. 求解弦的横振动. 设初位移和初速度均为 0.

记此解为 $G(x,t;x_0,t_0)$，则其满足

$$\begin{cases}\frac{\partial^2 G}{\partial t^2}-\alpha^2\frac{\partial^2 G}{\partial x^2}=\frac{I}{\rho}\delta(x-x_0)\delta(t-t_0) \\ G|_{t=0}=0, \quad \frac{\partial G}{\partial t}|_{t=0}=0\end{cases}$$

空间上：作傅里叶变换

$$g(k,t;x_0,t_0)=\int_{-\infty}^{+\infty}G(x,t;x_0,t_0)e^{-ikx}dx$$

$$\Rightarrow \frac{d^2g}{dt^2}+a^2k^2g=\frac{I}{\rho}e^{-ikx_0}\delta(t-t_0)$$

时间上：再作 Laplace 变换

$$\bar{g}(k,p;x_0,t_0)=\int_0^{\infty}g(k,t;x_0,t_0)e^{-pt}dt$$

$$\Rightarrow p^2\bar{g}+k^2a^2\bar{g}=\frac{I}{p}e^{-ikx_0}e^{-pt_0}$$

得到

$$\bar{g}=\frac{I}{p}\frac{1}{p^2+k^2a^2}e^{-ikx_0}e^{-pt_0}$$

作 Laplace 反演：

$$g(k,t;x_0,t_0)=\frac{I}{\rho}e^{-ikx_0}\frac{1}{ka}\sin ka(t-t_0)H(t-t_0)$$

作傅里叶反演:

$$
\begin{aligned}
G(x,t;x_0,t_0) &= \frac{I}{\rho}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}e^{ikx}e^{-ikx_0}\frac{1}{ka}\sin[ka(t-t_0)]H(t-t_0) \\
&= \frac{I}{\rho}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\int_0^{t-t_0}d\tau\cos(kat)e^{ik(x-x_0)}H(t-t_0) \\
&= \frac{I}{\rho}\int_0^{t-t_0}d\tau\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\frac{e^{-ikat}+e^{-ikat}}{2}e^{ik(x-x_0)}H(t-t_0) \\
&= \frac{I}{2\rho}H(t-t_0)\int_0^{t-t_0}d\tau[\delta(x-x_0+a\tau)+\delta(x-x_0-a\tau)] \\
&= \frac{I}{2\rho}H(t-t_0)\int_0^{t-t_0}d\tau\frac{1}{a}[\delta(t+\frac{x-x_0}{a})+\delta(\tau-\frac{x-x_0}{a})] \\
&= \begin{cases} \frac{I}{2a\rho}H(t-t_0), & |\frac{x-x_0}{a}|<t-t_0 \\ 0, & |\frac{x-x_0}{a}|>t-t_0 \end{cases}
\end{aligned}
$$

附录：积分公式

$$\int \sin^2(ax)dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$
$$\int x\sin(ax)dx = \frac{1}{a^2}[\sin(ax) - ax\cos(ax)]$$
$$\int x\cos(ax)dx = \frac{1}{a^2}[\cos(ax) + ax\sin(ax)]$$
$$\int x^2\sin(ax) = \frac{1}{a^3}[-(a^2x-2)\cos(ax) + 2ax\sin(ax)]$$
$$\int x^2\cos(ax) = \frac{1}{a^3}[(a^2x-2)\sin(ax) + 2ax\cos(ax)]$$
$$\int_{-\infty}^{+\infty} e^{-Ak^2-Bk}dk = \sqrt{\frac{\pi}{A}}e^{\frac{B^2}{4A}}$$

$\cos(n\pi) = (-1)^n$

$$\int_0^l x(l-x)\sin(\frac{n\pi x}{l})dx = \begin{cases} 4\left(\frac{l}{n\pi}\right)^3 & n \in \text{odd} \\ 0 & n \in \text{even} \end{cases}$$

$$\int_0^l x^2(l-x)^2\sin(\frac{n\pi x}{l})dx = \begin{cases} \frac{-4l^5(n\pi^2-12)}{n^5\pi^5} & n \in \text{odd} \\ 0 & n \in \text{even} \end{cases}$$

（广义）傅里叶级数模方计算：

$$\int_0^l \sin^2\frac{n\pi x}{l}dx = \int_0^l \cos^2\frac{n\pi x}{l}dx = \frac{l}{2}$$
$$||P_l^m||^2 = \int_{-1}^1 [P_l^m(x)]^2dx = \frac{2}{2l+1}$$
$$^*||P_l^m||^2 = \int_{-1}^1 [P_l^m(x)]^2dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}$$
$$\left|\left|J_m\left(\frac{\mu_i^{(m)}}{a}r\right)\right|\right|^2 \equiv \int_0^a \left[J_m\left(\frac{\mu_i^{(m)}}{a}r\right)\right]^2 rdr = \frac{a^2}{2}\left[J_{m+1}\left(\mu_i^{(m)}\right)\right]^2$$
$$||j_0(x)||^2$$