Discrete Optimization

Linear Programming: Part III

Goals of the Lecture

- Linear programming
  - the Simplex algorithm

BFS and the Naive Algorithm

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).

- Naive algorithm
  - generate all basic feasible solutions
    - select m basic basic variables and perform Gaussian elimination
    - test whether it is feasible
  - select the BFS with the best cost
- How many basic solutions?

$$\frac{n!}{m!(n-m)!}$$

Outline of the Simplex Algorithm

Goal: You want to solve a linear program

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).
3. You can move from one BFS to a neighboring BFS
4. You can detect whether a BFS is optimal
5. From any BFS, you can move to a BFS with a better cost

Outline of the Simplex Algorithm

Goal: You want to solve a linear program

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).
3. You can move from one BFS to a neighboring BFS
4. You can detect whether a BFS is optimal
5. From any BFS, you can move to a BFS with a better cost

The Simplex Algorithm

- Local search algorithm
  - move from BFS to BFS
  - guaranteed to find the global optimum
    - because of convexity

The Simplex Algorithm

- Local search algorithm
  - move from BFS to BFS
  - guaranteed to find the global optimum
    - because of convexity
- Key idea: how to move fro BFS to BFS.

From BFS to BFS

$$\begin{array}{rcrcrcrcrcrcrcr}
3x_1 & - & 2x_2 & + & x_3 & & & & & & & = & 1 \\
2x_1 & & & & & + & x_4 & & & + & x_6 & = & 2 \\
x_1 & & & & & & & + & x_5 & + & x_6 & = & 3
\end{array}$$

$$\downarrow$$

$$\begin{array}{rcrcrcrcr}
x_3 & = & 1 & - & 3x_1 & - & 2x_2 & & \\
x_4 & = & 2 & - & 2x_1 & & & + & x_6 \\
x_5 & = & 3 & - & x_1 & & & + & x_6
\end{array}$$

From BFS to BFS

$$\begin{array}{rcrcrcrcr} x_3 & = & 1 & - & 3x_1 & - & 2x_2 & & \\ x_4 & = & 2 & - & 2x_1 & & & + & x_6 \\ x_5 & = & 3 & - & x_1 & & & + & x_6 \end{array}$$

- How to move to another BFS
  - select a non-basic variable with a negative coefficient: *entering variable*
  - introduce this variable in the basis by removing a basic variable: *leaving variable*
  - perform Gaussian elimination
- Local move: swap a basic and a non-basic variables

From BFS to BFS

$$
\begin{aligned}
x_3 &= 1 - 3x_1 - 2x_2 \\
x_4 &= 2 - 2x_1 + x_6 \\
x_5 &= 3 - x_1 + x_6
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= \tfrac{1}{2} - \tfrac{3}{2}x_1 - \tfrac{1}{2}x_3 \\
x_4 &= 2 - 2x_1 + x_6 \\
x_5 &= 3 - x_1 + x_6
\end{aligned}
$$

From BFS to BFS

$$
\begin{aligned}
x_3 &= 1 \boxed{- 3x_1} - 2x_2 \\
x_4 &= 2 \boxed{- 2x_1} + x_6 \\
x_5 &= 3 \boxed{- x_1} + x_6
\end{aligned}
$$

From BFS to BFS

$$\begin{array}{rcrcrcrcr}
x_3 & = & 1 & - & 3x_1 & - & 2x_2 & & \\
x_4 & = & 2 & - & 2x_1 & & & + & x_6 \\
x_5 & = & 3 & - & x_1 & & & + & x_6
\end{array}$$

From BFS to BFS

$$\begin{array}{rcrrrrrrr} x_3 & = & 1 & - & 3x_1 & - & 2x_2 & & \\ x_4 & = & 2 & - & 2x_1 & & & + & x_6 \\ x_5 & = & 3 & - & x_1 & & & + & x_6 \end{array}$$

$$\downarrow$$

$$\begin{array}{rcrrrrrrr} x_3 & = & -8 & - & 2x_2 & + & 3x_5 & - & 3x_6 \\ x_4 & = & -4 & & & + & 2x_5 & - & x_6 \\ x_1 & = & 3 & & & - & x_5 & + & x_6 \end{array}$$

From BFS to BFS

$$\begin{array}{rcrcrcrcrcr} x_3 & = & 1 & - & 3x_1 & - & 2x_2 & & & & \\ x_4 & = & 2 & - & 2x_1 & & & & & + & x_6 \\ x_5 & = & 3 & - & x_1 & & & & & + & x_6 \end{array}$$

$$\downarrow$$

$$\begin{array}{rcrcrcrcr} x_3 & = & -8 & - & 2x_2 & + & 3x_5 & - & 3x_6 \\ x_4 & = & -4 & & & + & 2x_5 & - & x_6 \\ x_1 & = & 3 & & & - & x_5 & + & x_6 \end{array}$$

Not a BFS: I cannot select the leaving variable arbitrarily!

From BFS to BFS

$$
\begin{array}{rclrrrrrr}
x_3 & = & 1 & \boxed{-} & \boxed{3x_1} & - & 2x_2 & & \\
x_4 & = & 2 & \boxed{-} & \boxed{2x_1} & & & + & x_6 \\
x_5 & = & 3 & \boxed{-} & \boxed{x_1} & & & + & x_6
\end{array}
\qquad
\boxed{\begin{array}{c} \frac{1}{3} \\ 1 \\ 3 \end{array}}
$$

- How to choose the leaving variable?
  – we must maintain feasibility

$$l = \underset{i:a_{ie}<0}{\text{arg-min}} \frac{b_i}{-a_{ie}}$$

guarantee the feasibility because a_ie < 0

From BFS to BFS

$$
\begin{array}{lclllllll}
x_3 & = & 1 & \boxed{-} & \boxed{3x_1} & - & 2x_2 & & \\
x_4 & = & 2 & \boxed{-} & \boxed{2x_1} & & & + & x_6 \\
x_5 & = & 3 & \boxed{-} & \boxed{x_1} & & & + & x_6
\end{array}
\qquad
\boxed{\begin{array}{c} \frac{1}{3} \\ 1 \\ 3 \end{array}}
$$

$$
\begin{array}{lcllllllll}
\boxed{x_1 = \frac{1}{3} - \frac{2}{3}x_2 - \frac{1}{3}x_3} & & & & & & & & & \\
x_4 & = & \frac{4}{3} & + & \frac{4}{3}x_2 & + & \frac{2}{3}x_3 & + & x_6 \\
x_5 & = & \frac{8}{3} & + & \frac{2}{3}x_2 & + & \frac{1}{3}x_3 & + & x_6
\end{array}
$$

The Local Move

- Moving from BFS to BFS
  - select the entering variable x_e
    - non-basic variable with negative coefficients in the right-hand side
  - select the leaving variable x_l to maintain feasibility

$$l = \operatorname*{arg\text{-}min}_{i:a_{ie}<0} \frac{b_i}{-a_{ie}}$$

  - apply Gaussian elimination
    - eliminate x_e from the right-hand side
- This operation is called pivoting in linear programming
  - pivot(e,l)

Outline of the Simplex Algorithm

Goal: You want to solve a linear program

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).
3. You can move from one BFS to a neighboring BFS
4. You can detect whether a BFS is optimal
5. From any BFS, you can move to a BFS with a better cost

Outline of the Simplex Algorithm

Goal: You want to solve a linear program

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).
3. You can move from one BFS to a neighboring BFS
4. You can detect whether a BFS is optimal
5. From any BFS, you can move to a BFS with a better cost

The Simplex Algorithm

A BFS is optimal if its objective function, after having eliminated all the basic variables, is of the form

$$c_0 + c_1 x_1 + \ldots + c_n x_n$$

with

$$c_i \geq 0 \ \ (1 \leq i \leq n).$$

From BFS to BFS

$$
\begin{array}{lrcrcrcrcrcl}
\min & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & & \\
\text{subject to} & & & & & & & & & & & \\
 & 3x_1 & + & 2x_2 & + & x_3 & & & & & = & 1 \\
 & 5x_1 & + & x_2 & + & x_3 & + & x_4 & & & = & 3 \\
 & 2x_1 & + & 5x_2 & + & x_3 & & & + & x_5 & = & 4
\end{array}
$$

- First, find a BFS and eliminate the basic variable from the objective function

From BFS to BFS

$$
\begin{array}{lrcrcrcrcrcl}
\min & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & & \\
\text{subject to} & & & & & & & & & & & \\
 & 3x_1 & + & 2x_2 & + & x_3 & & & & & = & 1 \\
 & 5x_1 & + & x_2 & + & x_3 & + & x_4 & & & = & 3 \\
 & 2x_1 & + & 5x_2 & + & x_3 & & & + & x_5 & = & 4
\end{array}
$$

$$
\begin{array}{llcrcrcr}
\min & & & \boxed{6 - 3x_1 - 3x_2} \\
\text{subject to} & & & & & & & \\
 & x_3 & = & 1 & - & 3x_1 & - & 2x_2 \\
 & x_4 & = & 2 & - & 2x_1 & + & x_2 \\
 & x_5 & = & 3 & + & x_1 & - & 3x_2
\end{array}
$$

From BFS to BFS

$$
\begin{array}{lrcrcrcr}
\text{min} & & & 6 & - & 3x_1 & - & 3x_2 \\
\text{subject to} & & & & & & & \\
& x_3 & = & 1 & - & 3x_1 & - & 2x_2 \\
& x_4 & = & 2 & - & 2x_1 & + & x_2 \\
& x_5 & = & 3 & + & x_1 & - & 3x_2
\end{array}
$$

From BFS to BFS

$$
\begin{array}{lrcrcrcr}
\min & & & 6 & - & 3x_1 & - & 3x_2 \\
\text{subject to} & & & & & & & \\
& x_3 & = & 1 & - & 3x_1 & - & 2x_2 \\
& x_4 & = & 2 & - & 2x_1 & + & x_2 \\
& x_5 & = & 3 & + & x_1 & - & 3x_2
\end{array}
$$

From BFS to BFS

$$
\begin{array}{lrcrcrcr}
\text{min} & & & 6 & - & 3x_1 & - & 3x_2 \\
\text{subject to} & & & & & & & \\
& x_3 & = & 1 & - & 3x_1 & - & 2x_2 \\
& x_4 & = & 2 & - & 2x_1 & + & x_2 \\
& x_5 & = & 3 & + & x_1 & - & 3x_2
\end{array}
$$

From BFS to BFS

$$
\begin{array}{lrcrcrcr}
\text{min} & & & 6 & - & 3x_1 & - & 3x_2 \\
\text{subject to} & & & & & & & \\
 & x_3 & = & 1 & - & 3x_1 & - & 2x_2 \\
 & x_4 & = & 2 & - & 2x_1 & + & x_2 \\
 & x_5 & = & 3 & + & x_1 & - & 3x_2
\end{array}
$$

$$\downarrow$$

$$
\begin{array}{lrcrcrcr}
\text{min} & & & \frac{9}{2} & + & \frac{3}{2}x_1 & + & \frac{3}{2}x_3 \\
\text{subject to} & & & & & & & \\
 & x_2 & = & \frac{1}{2} & - & \frac{3}{2}x_1 & - & \frac{1}{2}x_3 \\
 & x_4 & = & \frac{5}{2} & - & \frac{7}{2}x_1 & - & \frac{1}{2}x_3 \\
 & x_5 & = & \frac{3}{2} & + & \frac{11}{2}x_1 & + & \frac{3}{2}x_3
\end{array}
$$

From BFS to BFS

$$
\begin{array}{llcrcrcr}
\min & & & 6 & - & 3x_1 & - & 3x_2 \\
\text{subject to} & & & & & & & \\
& x_3 & = & 1 & - & 3x_1 & - & 2x_2 \\
& x_4 & = & 2 & - & 2x_1 & + & x_2 \\
& x_5 & = & 3 & + & x_1 & - & 3x_2
\end{array}
$$

$$
\begin{array}{llcrcrcr}
\min & & & \frac{9}{2} & + & \frac{3}{2}x_1 & + & \frac{3}{2}x_3 \\
\text{subject to} & & & & & & & \\
& x_2 & = & \frac{1}{2} & - & \frac{3}{2}x_1 & - & \frac{1}{2}x_3 \\
& x_4 & = & \frac{5}{2} & - & \frac{7}{2}x_1 & - & \frac{1}{2}x_3 \\
& x_5 & = & \frac{3}{2} & + & \frac{11}{2}x_1 & + & \frac{3}{2}x_3
\end{array}
$$

Outline of the Simplex Algorithm

Goal: You want to solve a linear program

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).
3. You can move from one BFS to a neighboring BFS
4. You can detect whether a BFS is optimal
5. From any BFS, you can move to a BFS with a better cost

Outline of the Simplex Algorithm

Goal: You want to solve a linear program

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).
3. You can move from one BFS to a neighboring BFS
4. You can detect whether a BFS is optimal
5. From any BFS, you can move to a BFS with a better cost

Look Ma, My Local Move is Improving

- Assume that, in the BFS,
  - $b_1>0$, ... , $b_m>0$
  - there exists an entering variable e with $c_e < 0$
  - there exists a leaving variable l
- then the move pivot(e,l) is improving

The Simplex Algorithm

while $\exists 1 \leq i \leq n : c_i < 0$ do

choose e such that $c_e < 0$;

$$l = \operatorname*{arg\text{-}min}_{i : a_{ie} < 0} \frac{b_i}{-a_{ie}};$$

pivot(e,l);

The Simplex Algorithm

$$
\begin{aligned}
&\text{while } \exists 1 \leq i \leq n : c_i < 0 \text{ do}\\
&\quad \text{choose } e \text{ such that } c_e < 0;\\
&\quad l = \operatorname*{arg-min}_{i : a_{ie} < 0} \frac{b_i}{-a_{ie}};\\
&\quad \text{pivot}(e, l);
\end{aligned}
$$

- Assume that during the execution
  - b_1 , ... , b_m are always strictly positive
  - the objective function is bounded by below
- then the simplex algorithm terminates with an optimal solution

Where is My Leaving Variable?

- Selecting the leaving variable

$$l = \underset{i:a_{ie}<0}{\text{arg-min}} \frac{b_i}{-a_{ie}}$$

$$\begin{array}{lccccccc}
\text{min} & & & 6 & - & 3x_1 & - & 3x_2 \\
\text{subject to} & & & & & & & \\
& x_3 & = & 1 & + & 3x_1 & - & 2x_2 \\
& x_4 & = & 2 & + & 2x_1 & + & x_2 \\
& x_5 & = & 3 & + & x_1 & - & 3x_2
\end{array}$$

- There are no leaving variables

Where is My Leaving Variable?

- Selecting the leaving variable

$$l = \operatorname*{arg\text{-}min}_{i : a_{ie} < 0} \frac{b_i}{-a_{ie}}$$

$$\begin{array}{lrcrcrcr}
\text{min} & & & 6 & - & 3x_1 & - & 3x_2 \\
\text{subject to} & & & & & & & \\
& x_3 & = & 1 & + & 3x_1 & - & 2x_2 \\
& x_4 & = & 2 & + & 2x_1 & + & x_2 \\
& x_5 & = & 3 & + & x_1 & - & 3x_2
\end{array}$$

- There are no leaving variables

Where is My Leaving Variable?

$$
\begin{array}{lrcrcrcr}
\min & & & 6 & - & 3x_1 & - & 3x_2 \\
\text{subject to} & & & & & & & \\
& x_3 & = & 1 & + & 3x_1 & - & 2x_2 \\
& x_4 & = & 2 & + & 2x_1 & + & x_2 \\
& x_5 & = & 3 & + & x_1 & - & 3x_2
\end{array}
$$

- What is the basic solution?

$$\{x_1 = 0; x_2 = 0; x_3 = 1; x_4 = 2; x_5 = 3\}$$

- What happens if I increase the value of x_1
  - the solution remains feasible
  - the value of the objective can decrease arbitrarily

Where is My Leaving Variable?

- What if some b_i becomes zero?

$$
\begin{array}{llcrcrcr}
\min & & & 5 & - & 3x_1 & - & 3x_2 \\
\text{subject to} & & & & & & & \\
& x_3 & = & 0 & - & 3x_1 & - & 2x_2 \\
& x_4 & = & 2 & - & 2x_1 & + & x_2 \\
& x_5 & = & 3 & + & x_1 & - & 3x_2
\end{array}
$$

- What is the leaving variable?

$$
\begin{array}{llcrcrcr}
\min & & & \boxed{5} & + & \frac{3}{2}x_1 & + & \frac{3}{2}x_3 \\
\text{subject to} & & & & & & & \\
& x_2 & = & 0 & - & \frac{3}{2}x_1 & - & \frac{1}{2}x_3 \\
& x_4 & = & 2 & - & \frac{7}{2}x_1 & - & \frac{1}{2}x_3 \\
& x_5 & = & 3 & + & \frac{11}{2}x_1 & + & \frac{3}{2}x_3
\end{array}
$$

Outline of the Simplex Algorithm

1. An optimal solution is located at a vertex.
2. A vertex is a Basic Feasible Solution (BFS).
3. You can move from one BFS to a neighboring BFS
4. You can detect whether a BFS is optimal
5. From any BFS, you can move to a BFS with a better cost

We need to find a new way to guarantee termination

Termination of the Simplex Algorithm

- Many approaches
  - Bland rule
    - select always the first entering variable lexicographically
  - Lexicographic pivoting rule
    - break ties when selecting the leaving variable by using a lexicographic rule

$$l = \operatorname*{arg\text{-}lex\text{-}min}_{i:a_{ie}<0} \frac{b_i}{-a_{ie}}$$

  - Perturbation methods

Where is My BFS?

$$
\begin{array}{lrcrcrcrcrcl}
\min & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & & \\
\text{subject to} & & & & & & & & & & & \\
 & 3x_1 & + & 2x_2 & + & x_3 & & & & & = & 1 \\
 & 5x_1 & + & x_2 & + & x_3 & + & x_4 & & & = & 3 \\
 & 2x_1 & + & 5x_2 & + & x_3 & & & + & x_5 & = & 4
\end{array}
$$

- How do I find my first BFS?

Where is My BFS?

- How do I find my first BFS?

$$
\begin{array}{lrcccccl}
\min & c_1 x_1 & + & \ldots & + & c_n x_n & & \\
\text{subject to} & & & & & & & \\
 & a_{11} x_1 & + & \ldots & + & a_{1n} x_n & = & b_1 \\
 & \ldots & & & & & & \\
 & a_{m1} x_1 & + & \ldots & + & a_{mn} x_n & = & b_m
\end{array}
$$

- Introduce artificial variables

Where is My BFS?

- Introduce artificial variables

$$
\begin{array}{llllllllllll}
\min & c_1x_1 & + & \dots & + & c_nx_n & & & & & & \\
\text{subject to} & & & & & & & & & & & \\
& a_{11}x_1 & + & \dots & + & a_{1n}x_n & + & y_1 & & & = & b_1 \\
& \dots & & & & & & & & & & \\
& a_{i1}x_1 & + & \dots & + & a_{in}x_n & & & + \quad y_i & & = & b_i \\
& \dots & & & & & & & & & & \\
& a_{m1}x_1 & + & \dots & + & a_{mn}x_n & & & & + \quad y_m & = & b_m
\end{array}
$$

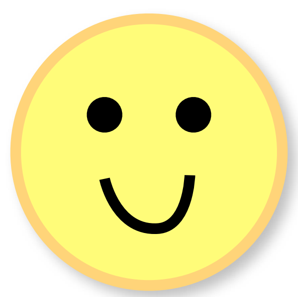
- I have an easy BFS

- But to another problem

Where is My BFS?

- Introduce artificial variables

$$
\begin{array}{lrcccccccccc}
\min & c_1x_1 & + & \ldots & + & c_nx_n & & & & & & \\
\text{subject to} & & & & & & & & & & & \\
 & a_{11}x_1 & + & \ldots & + & a_{1n}x_n & + & y_1 & & & = & b_1 \\
 & \ldots & & & & & & & & & & \\
 & a_{i1}x_1 & + & \ldots & + & a_{in}x_n & & & + \quad y_i & & = & b_i \\
 & \ldots & & & & & & & & & & \\
 & a_{m1}x_1 & + & \ldots & + & a_{mn}x_n & & & & + \quad y_m & = & b_m
\end{array}
$$

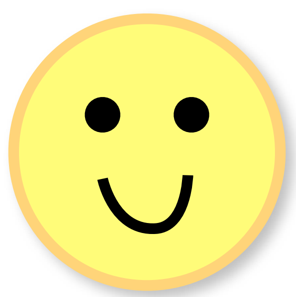
- I have an easy BFS 

- But to another problem

Where is My BFS?

- Introduce artificial variables

$$
\begin{array}{llllllllll}
\min & c_1x_1 & + & \ldots & + & c_nx_n & & & & \\
\text{subject to} & & & & & & & & & \\
& a_{11}x_1 & + & \ldots & + & a_{1n}x_n & + \; y_1 & & & = \; b_1 \\
& \ldots & & & & & & & & \\
& a_{i1}x_1 & + & \ldots & + & a_{in}x_n & & + \; y_i & & = \; b_i \\
& \ldots & & & & & & & & \\
& a_{m1}x_1 & + & \ldots & + & a_{mn}x_n & & & + \; y_m & = \; b_m
\end{array}
$$

- I have an easy BFS 
- But to another problem 

Two-Phase Method

- First find a BFS
  - if there is one
- Then find an optimal BFS

Two-Phase Method

- First find a BFS

$$
\begin{array}{lcccccccccccc}
\boxed{\min} & & & & & & & y_1 & + & \dots & + & y_m & & \\
\text{subject to} & & & & & & & & & & & & & \\
& a_{11}x_1 & + & \dots & + & a_{1n}x_n & + & y_1 & & & & & = & b_1 \\
& \dots & & & & & & & & & & & & \\
& a_{i1}x_1 & + & \dots & + & a_{in}x_n & & & + & y_i & & & = & b_i \\
& \dots & & & & & & & & & & & & \\
& a_{m1}x_1 & + & \dots & + & a_{mn}x_n & & & & & + & y_m & = & b_m
\end{array}
$$

- Feasible if the objective value reaches 0
  - all the y_i are thus zeros
    - I have a BFS without the y_i
    - really? always?

Until Next Time