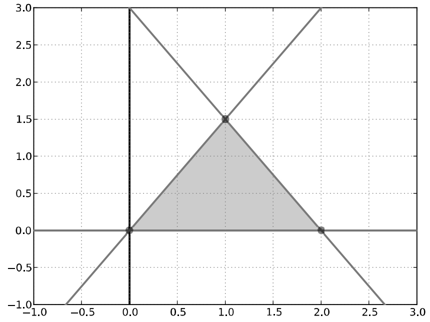
Discrete Optimization

Mixed Integer Programming: Part IV

Goals of the Lecture

- Polyhedral cuts
  - warehouse location
  - node covering

Convex Hull



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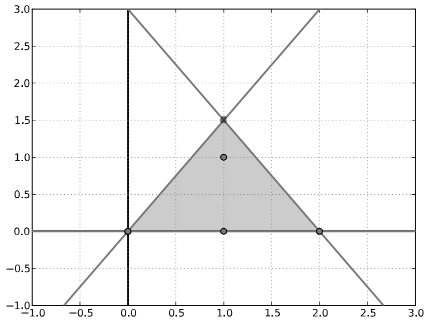

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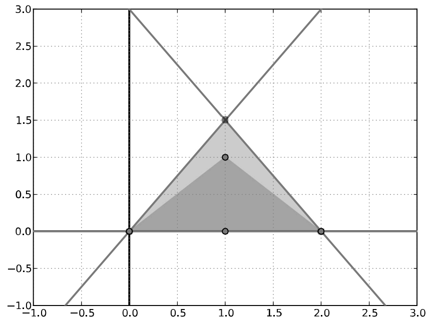

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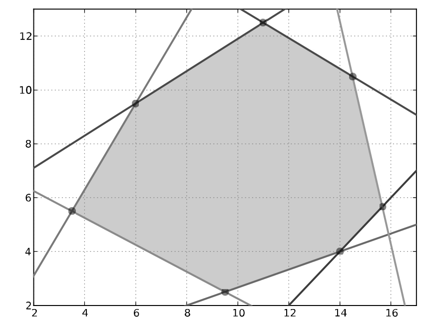

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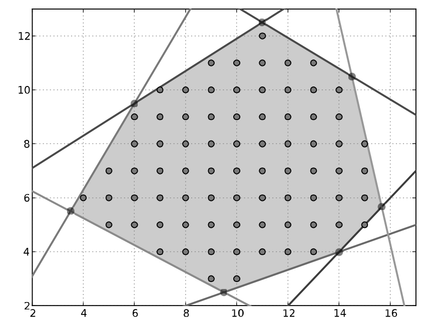

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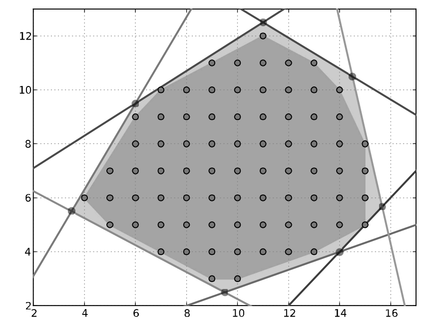

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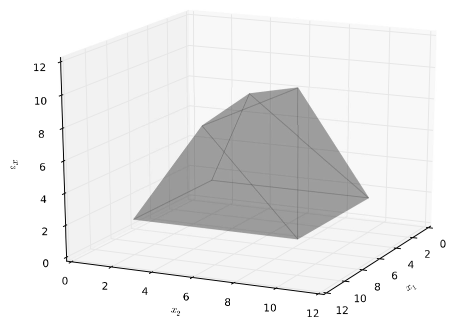

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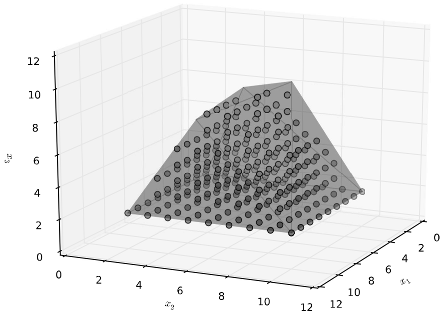

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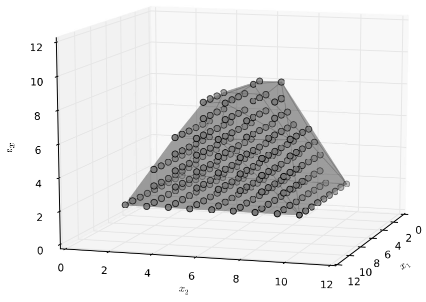

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  - cuts that represent the facets of the convex hull of the integer solution
- These cuts are valid
  - they do not remove any solution
- The cuts are as strong as possible
  - if we have all of them, we could use linear programming to solve the problem

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- They exploit the problem structure
  - they are derived from the structure of constraints
  - not based on information in the tableau

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- They share some of the spirit of syntactic cuts
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  - must cut the current basic feasible solution
  - do not need to generate all of them
- An application may use multiple cut types
  - exploit different substructure

What is a Facet?

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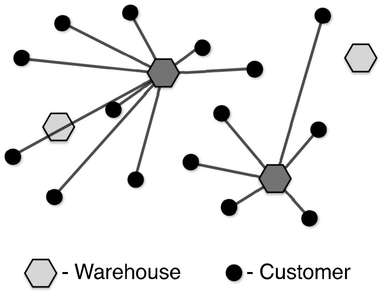
- Linear independence
  $x_1, \ldots, x_n$ are linearly independent iff $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$ implies that $\alpha_i = 0$ for all i.

Warehouse Location



Warehouse Location

$$\begin{array}{lll} \min & \sum_{w \in W} c_w \, x_w + & \sum_{w \in W, c \in C} t_{w,c} \, y_{w,c} \\ \text{subject to} & & \\ & y_{w,c} \leq x_w & (w \in W, c \in C) \\ & \sum_{w \in W} y_{w,c} = 1 & (c \in C) \\ & x_w \in \{0,1\} & (w \in W) \\ & y_{w,c} \in \{0,1\} & (w \in W, c \in C) \end{array}$$

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Facets for Warehouse Location

- Are these inequalities facets?
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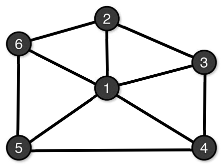
  $y_{w,c} \leq x_w$

- Consider $y_{w,1} \leq x_w$ and the following n points

w	1	2	3	$\ldots$	n	
0	0	0	0	$\ldots$	0	1
1	0	0	0	$\ldots$	0	1
1	1	0	0	$\ldots$	0	1
1	1	1	0	$\ldots$	0	1
1	1	0	1	$\ldots$	0	1
				$\ldots$		
1	1	0	0	$\ldots$	1	1

Node Packing

- Let G=(V,E) be a graph.
  - A node packing is a subset W of V such that no two nodes in W are connected by an edge. The goal is to find the node packing of maximal size.



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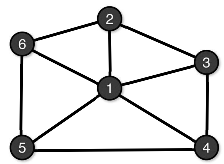

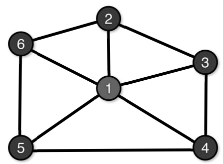

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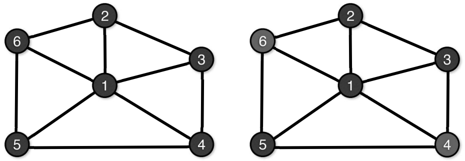

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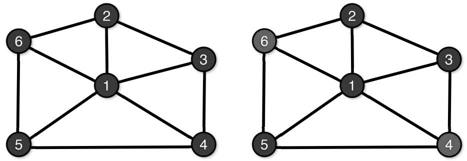

- How do we express it as a MIP?

Node Packing

$$
\begin{array}{llllll}
\max & x_1 & + & \dots & + & x_6 \\
\text{s.t.} & & & & & \\
& x_1 & + & x_2 & \leq & 1 \\
& x_1 & + & x_3 & \leq & 1 \\
& x_1 & + & x_4 & \leq & 1 \\
& x_1 & + & x_5 & \leq & 1 \\
& x_1 & + & x_6 & \leq & 1 \\
& x_2 & + & x_3 & \leq & 1 \\
& x_3 & + & x_4 & \leq & 1 \\
& x_4 & + & x_5 & \leq & 1 \\
& x_5 & + & x_6 & \leq & 1 \\
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\end{array}
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- What does the linear relaxation produce?

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\end{array}
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- What does the linear relaxation produce?

$$x_1 = \frac{1}{2}, \dots, x_6 = \frac{1}{2}$$

How do we Find Facets?

- Find a property of the solution
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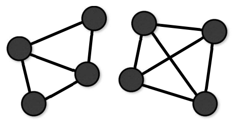
- Consider a clique



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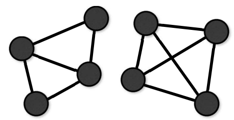
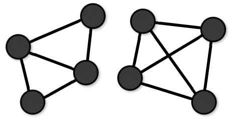


- How many nodes can be in a packing?

Clique Facets

- Consider a clique



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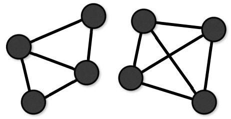

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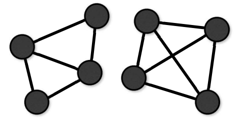

- Maximal clique
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$$x_1 + \ldots + x_5 \leq 1$$

- A maximal clique constraint is a facet!

MIP with Clique Facets

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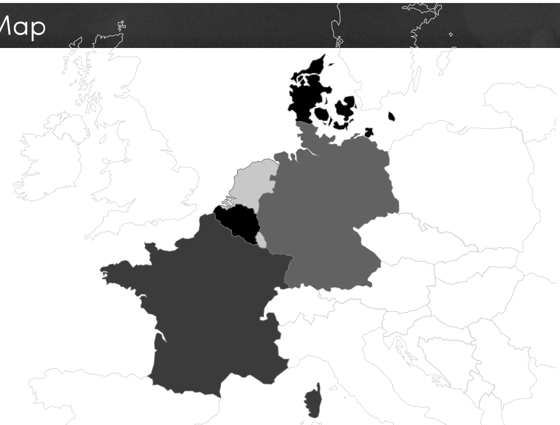
Coloring a Map



Coloring a Map with 0/1 Variables

MIP:
Optimal - 9 nodes
Proof - 41 nodes

LP:
$obj = 0.5$
$\text{color}_{c,0} = 0.5$
$\text{color}_{c,1} = 0.5$
$\text{color}_{c,2} = 0$
$\text{color}_{c,3} = 0$

$$\begin{array}{ll}
obj & \in \{0,1,2,3\} \\
\text{color}_{c,v} & \in \{0,1\}
\end{array}$$

Need at least 2 colors!

$$\begin{array}{lll}
\text{min} & obj & \\
\text{subject to} & obj \geq \sum_{v=0}^{3} v \times \text{color}_{c,v} & (c \in C) \\
 & \sum_{v=0}^{3} \text{color}_{c,v} = 1 & (c \in C) \\
 & \text{color}_{c_1,v} + \text{color}_{c_2,v} \leq 1 & (c_1, c_2 \in C \text{ and adjacent}, v \in 0..3)
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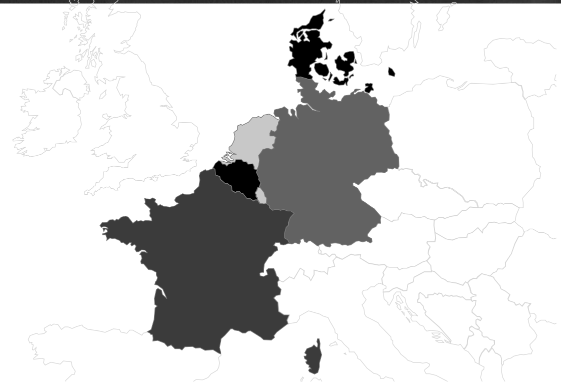
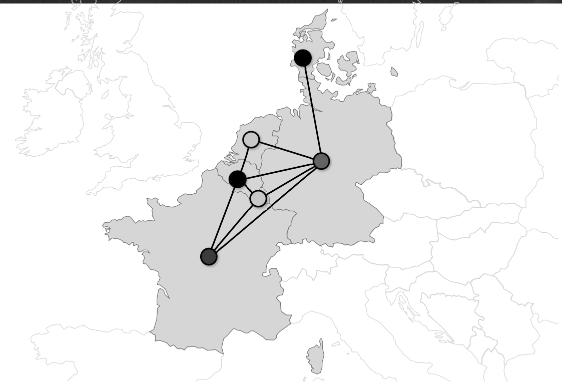


Coloring a Map as a Graph



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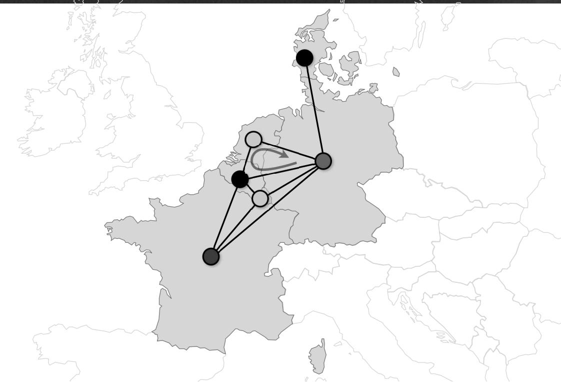

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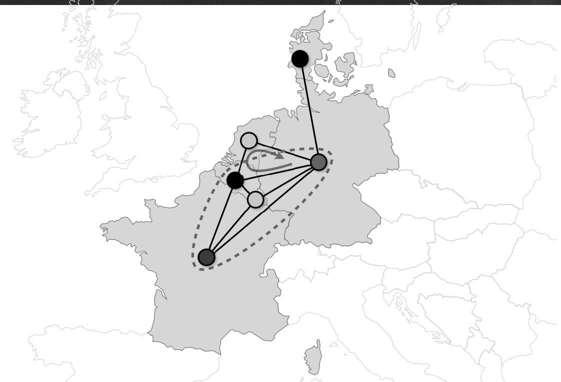

Improving the Coloring Relaxation

- With the best model from before

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$$\sum_{c\in\{0,3,4\}}^{3} \sum_{v=0}^{3} v \times \mathrm{color}_{c,v} \geq 3$$

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Proof - 41 nodes

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MIP:
Optimal - 5 nodes
Proof - 9 nodes

$obj = 1.5$ Need at least 3 colors!!!

Until Next Time