School of Mathematics

Interior Point Methods for Linear Programming

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Outline

- **Part 1: IPM for LP: Motivation**
  - what is wrong with the simplex method?
  - complementarity conditions
  - first order optimality conditions
  - Newton, Lagrange and Fiacco & McCormick
  - central trajectory
  - primal-dual framework
- **Part 2: Polynomial Complexity of IPM**
  - Newton method
  - short step path-following method
  - polynomial complexity proof
- **Final Comments**

Part 1:

IPM for LP: Motivation

Simplex: What's wrong?

A **vertex** is defined by a set of n equations:

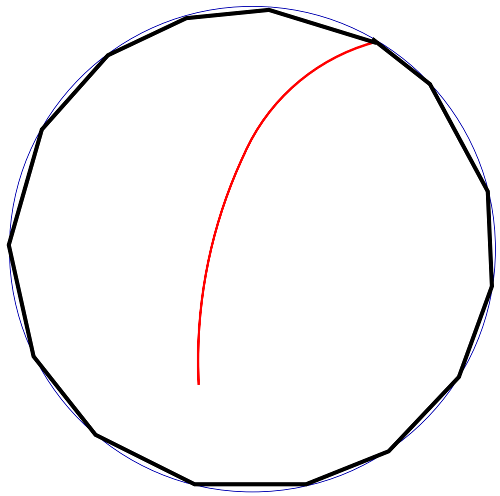
$$\begin{bmatrix} B & N \\ 0 & I_{n-m} \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

The LP program with m constraints and n variables $(n \geq m)$ may ha
as many as

$$N_V = \begin{pmatrix} \text{n} \\ \text{m} \end{pmatrix} = \frac{n!}{m!(n-m)!}$$

vertices.

Brazil 2014 view of LP



Simplex: What's wrong?

The simplex method can make a non-polynomial number of iterations to reach the optimality:

V. Klee and G. Minty constructed an example LP:
simplex method needs 2^n iterations to solve it.

Klee and Minty, How good is the simplex algorithm,
in: *Inequalities-III*, O. Shisha, ed., Academic Press, 1972, 159–175.

Narendra Karmarkar from AT&T Bell Labs:
"the simplex [method] is complex"

N. Karmarkar: A New Polynomial–time Algorithm for LP,
Combinatorica 4 (1984) 373–395.

"Elements" of the IPM

What do we need to derive the **Interior Point Method**?

- duality theory:
  Lagrangian function;
  first order optimality conditions.
- logarithmic barriers.
- Newton method.

The following 3 slides remind key facts from the duality theory applied to linear programming.

Duality in LP

Consider a **primal** program

$$\begin{array}{lrl} \min & c^T x & \\ \text{s.t.} & Ax & = b, \\ & x \geq 0, & \end{array} \qquad (1)$$

where $c, x \in \mathcal{R}^n, b \in \mathcal{R}^m$, $A \in \mathcal{R}^{m \times n}$.
With the primal we associate a *dual* program

$$\begin{array}{lrl} \max & b^T y & \\ \text{s.t.} & A^T y & \leq c, \\ & y \ \text{free}, & \end{array}$$

where $y \in \mathcal{R}^m$. We add *dual slack* $s \in \mathcal{R}^n$, $s \geq 0$, convert inequality $A^T y \leq c$ into an equation $A^T y + s = c$ and get **dual** program

$$\begin{array}{lrcl} \max & b^T y & & \\ \text{s.t.} & A^T y + & s & = c, \\ & y \ \text{free}, & s \geq 0, & \end{array} \qquad (2)$$

where $y \in \mathcal{R}^m$ and $s \in \mathcal{R}^n$.

Let $\mathcal{P}$, $\mathcal{D}$ be the feasible sets of the primal and the dual, resp.:
$\mathcal{P} = \{x \in \mathcal{R}^n \,|\, Ax = b, \ x \geq 0\}$
$\mathcal{D} = \{y \in \mathcal{R}^m, s \in \mathcal{R}^n \,|\, A^T y + s = c, \ s \geq 0\}$.

Let us introduce a convention that

$$\inf_{x \in \mathcal{P}} c^T x = +\infty, \ \text{if } \mathcal{P} = \emptyset; \qquad \sup_{y \in \mathcal{D}} b^T y = -\infty, \ \text{if } \mathcal{D} = \emptyset.$$

Weak Duality Theorem

$$\inf_{x \in \mathcal{P}} c^T x \ \geq \ \sup_{y \in \mathcal{D}} b^T y.$$

Strong Duality Theorem

If either $\mathcal{P} \neq \emptyset$ or $\mathcal{D} \neq \emptyset$ then

$$\inf_{x \in \mathcal{P}} c^T x \ = \ \sup_{y \in \mathcal{D}} b^T y.$$

If one of problems (1) and (2) is *solvable* then

$$\min_{x \in \mathcal{P}} c^T x \ = \ \max_{y \in \mathcal{D}} b^T y.$$

In **IPMs** we shall use the term **interior-point**. Let $\mathcal{P}^0$, $\mathcal{D}^0$ be t
strictly feasible sets of the primal and the dual, respectively:
$\mathcal{P}^0 = \{x \in \mathcal{R}^n \,|\, Ax = b, \ x > 0\}$
$\mathcal{D}^0 = \{y \in \mathcal{R}^m, s \in \mathcal{R}^n \,|\, A^T y + s = c, \ s > 0\}$.

We shall often refer to the **primal-dual pair**. Hence we define *prim*
dual feasible set $\mathcal{F}$ and *primal-dual strictly feasible* set $\mathcal{F}^0$:
$\mathcal{F} \ = \{(x, y, s) \,|\, Ax = b, \ A^T y + s = c, \ (x, s) \geq 0\}$
$\mathcal{F}^0 = \{(x, y, s) \,|\, Ax = b, \ A^T y + s = c, \ (x, s) > 0\}$.

Primal-Dual Pair of Linear Programs

Primal	Dual
$\begin{array}{lrl} \min & c^T x & \\ \text{s.t.} & Ax & = b, \\ & x \geq 0; & \end{array}$	$\begin{array}{lrl} \max & b^T y & \\ \text{s.t.} & A^T y + s & = c, \\ & s \geq 0. & \end{array}$

Lagrangian

$$L(x, y) \ = \ c^T x - y^T (Ax - b) - s^T x.$$

Optimality Conditions

$$\begin{array}{rcl} Ax & = & b, \\ A^T y + s & = & c, \\ XSe & = & 0, \quad (\text{ i.e., } \ x_j \cdot s_j = 0 \quad \forall j), \\ (x, s) & \geq & 0, \end{array}$$

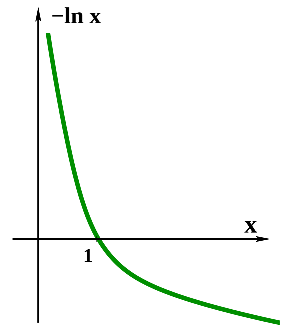
$X = diag\{x_1, \cdots, x_n\}$, $S = diag\{s_1, \cdots, s_n\}$, $e = (1, \cdots, 1) \in \mathcal{R}^n$.

Logarithmic barrier

$-\ln x_j$

"replaces" the inequality

$x_j \geq 0$.



Observe that

$$\min \mathrm{e}^{-\sum_{j=1}^{n} \ln x_j} \quad \Longleftrightarrow \quad \max \prod_{j=1}^{n} x_j$$

The minimization of $-\sum_{j=1}^{n} \ln x_j$ is equivalent to the maximization of the product of distances from all hyperplanes defining the positive orthant: it prevents all x_j from approaching zero.

Logarithmic barrier

Replace the **primal** LP

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array}$$

with the **primal barrier program**

$$\begin{array}{ll} \min & c^T x - \mu \sum_{j=1}^{n} \ln x_j \\ \text{s.t.} & Ax = b. \end{array}$$

Lagrangian: $\quad L(x, y, \mu) = c^T x - y^T (Ax - b) - \mu \sum_{j=1}^{n} \ln x_j.$

Conditions for a stationary point of the Lagrangian

$$\begin{array}{rcl} \nabla_x L(x, y, \mu) & = & c - A^T y - \mu X^{-1} e = 0 \\ \nabla_y L(x, y, \mu) & = & Ax - b = 0, \end{array}$$

where $X^{-1} = diag\{x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1}\}$.

Let us denote

$$s = \mu X^{-1} e, \quad \text{i.e.} \quad XSe = \mu e.$$

The **First Order Optimality Conditions** are:

$$\begin{array}{rcl} Ax & = & b, \\ A^T y + s & = & c, \\ XSe & = & \mu e, \\ (x, s) & > & 0. \end{array}$$

Complementarity $\quad x_j \cdot s_j = 0 \quad \forall j = 1, 2, ..., n.$

Simplex Method makes a guess of optimal partition:

For *basic* variables, $s_B = 0$ and

$$(x_B)_j \cdot (s_B)_j = 0 \quad \forall j \in \mathcal{B}.$$

For *non-basic* variables, $x_N = 0$ hence

$$(x_N)_j \cdot (s_N)_j = 0 \quad \forall j \in \mathcal{N}.$$

Interior Point Method uses ε-mathematics:

Replace $\quad x_j \cdot s_j = 0 \quad \forall j = 1, 2, ..., n$

by $\quad x_j \cdot s_j = \mu \quad \forall j = 1, 2, ..., n.$

Force convergence $\mu \to 0$.

First Order Optimality Conditions

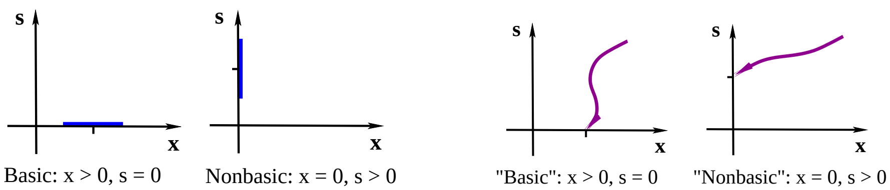
Approaching Optimality:

Simplex Method:

$$\begin{aligned} Ax &= b \\ A^T y + s &= c \\ XSe &= 0 \\ x,\, s &\geq 0. \end{aligned}$$

Interior Point Method:

$$\begin{aligned} Ax &= b \\ A^T y + s &= c \\ XSe &= \mu e \\ x,\, s &\geq 0. \end{aligned}$$



Central Trajectory

Note that the first order optimality conditions for the barrier problem

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= \mu e, \\ (x, s) &\geq 0 \end{aligned}$$

approximate the first order optimality conditions for the LP

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= 0, \\ (x, s) &\geq 0 \end{aligned}$$

more and more closely as μ goes to zero.

Central Trajectory

Parameter μ controls the distance to optimality.

$$c^T x - b^T y = c^T x - x^T A^T y = x^T (c - A^T y) = x^T s = n\mu.$$

Analytic centre (μ-centre): a (unique) point

$$(x(\mu), y(\mu), s(\mu)), \quad x(\mu) > 0,\; s(\mu) > 0$$

that satisfies FOC.

The path

$$\{(x(\mu), y(\mu), s(\mu)) : \mu > 0\}$$

is called the **primal-dual central trajectory**.

Newton Method

is used to find a stationary point of the barrier problem.

Recall how to use Newton Method to find a root of a nonlinear equati

$$f(x) = 0.$$

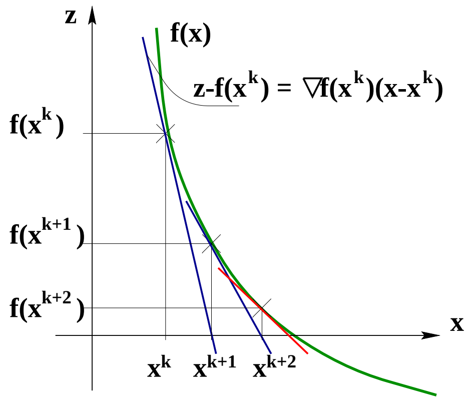
A tangent line

$$z - f(x^k) = \nabla f(x^k) \cdot (x - x^k)$$

is a local approximation of the graph of the function $f(x)$.
Substituting $z = 0$ defines a new point

$$x^{k+1} = x^k - (\nabla f(x^k))^{-1} f(x^k).$$

Newton Method



$$x^{k+1} = x^k - (\nabla f(x^k))^{-1} f(x^k).$$

Apply Newton Method to the FOC

The first order optimality conditions for the barrier problem form a large system of nonlinear equations

$$f(x,y,s) = 0,$$

where $f : \mathcal{R}^{2n+m} \mapsto \mathcal{R}^{2n+m}$ is a mapping defined as follows:

$$f(x,y,s) = \begin{bmatrix} Ax - b \\ A^T y + s - c \\ XSe - \mu e \end{bmatrix}.$$

Actually, the first two terms of it are **linear**; only the last one, corresponding to the complementarity condition, is **nonlinear**.

Newton Method (cont'd)

Note that

$$\nabla f(x,y,s) = \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix}.$$

Thus, for a given point (x,y,s) we find the Newton direction $(\Delta x, \Delta y, \Delta s)$ by solving the system of linear equations:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^T y - s \\ \mu e - XSe \end{bmatrix}.$$

Interior-Point Framework

The **logarithmic barrier**

$$-\ln x_j$$

"replaces" the inequality $x_j \geq 0.$

We derive the **first order optimality conditions** for the prim barrier problem:

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= \mu e, \end{aligned}$$

and apply **Newton method** to solve this system of (nonlinear) equ tions.

Actually, we fix the barrier parameter μ and make only **one** (dampe Newton step towards the solution of FOC. We do not solve the curre FOC exactly. Instead, we immediately reduce the barrier parameter (to ensure progress towards optimality) and repeat the process.

Interior Point Algorithm

Initialize

$k = 0 \qquad (x^0, y^0, s^0) \in \mathcal{F}^0$

$\mu_0 = \frac{1}{n} \cdot (x^0)^T s^0 \qquad \alpha_0 = 0.9995$

Repeat until optimality

$k = k + 1$

$\mu_k = \sigma \mu_{k-1}$, where $\sigma \in (0,1)$

$\Delta = (\Delta x, \Delta y, \Delta s) =$ Newton direction towards μ-centre

Ratio test:

$$\alpha_P := \max \{\alpha > 0 : \ x + \alpha \Delta x \geq 0\},$$
$$\alpha_D := \max \{\alpha > 0 : \ s + \alpha \Delta s \geq 0\}.$$

Make step:

$$x^{k+1} = x^k + \alpha_0 \alpha_P \Delta x,$$
$$y^{k+1} = y^k + \alpha_0 \alpha_D \Delta y,$$
$$s^{k+1} = s^k + \alpha_0 \alpha_D \Delta s.$$

Notations

$$X = diag\{x_1, x_2, \cdots, x_n\} = \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix}.$$

$e = (1, 1, \cdots, 1) \in \mathcal{R}^n, \ X^{-1} = diag\{x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1}\}.$

An equation $\qquad XSe = \mu e,$

is equivalent to $\qquad x_j s_j = \mu, \quad \forall j = 1, 2, \cdots, n.$

Notations(cont'd)

Primal feasible set $\mathcal{P} = \{x \in \mathcal{R}^n \,|\, Ax = b, \ x \geq 0\}$.
Primal strictly feasible set $\mathcal{P}^0 = \{x \in \mathcal{R}^n \,|\, Ax = b, \ x > 0\}$.

Dual feasible set $\mathcal{D} = \{y \in \mathcal{R}^m, s \in \mathcal{R}^n \,|\, A^T y + s = c, \ s \geq 0\}$.
Dual strictly feasible set $\mathcal{D}^0 = \{y \in \mathcal{R}^m, s \in \mathcal{R}^n \,|\, A^T y + s = c, \ s > 0\}$.

Primal-dual feasible set
$\mathcal{F} \ = \{(x, y, s) \,|\, Ax = b, \ A^T y + s = c, \ (x, s) \geq 0\}$.
Primal-dual strictly feasible set
$\mathcal{F}^0 = \{(x, y, s) \,|\, Ax = b, \ A^T y + s = c, \ (x, s) > 0\}$.

Interior Point Methods

Marsten, Subramanian, Saltzman, Lustig and Shanno:
"Interior point methods for linear programming:
Just call Newton, Lagrange, and Fiacco and McCormick!",
Interfaces 20 (1990) No 4, pp. 105–116.

- **Fiacco & McCormick (1968)**
  inequality constraints $\longrightarrow$ logarithmic barrier;
  a sequence of unconstrained minimizations
- **Lagrange (1788)**
  equality constraints $\longrightarrow$ multipliers;
- **Newton (1687)**
  solve unconstrained minimization problems;

Part 2:

Path-Following Method: Theory

Path-Following Algorithm

The analysis given in this lecture comes from the book of
Steve Wright: *Primal-Dual Interior-Point Methods*,
SIAM Philadelphia, 1997.

We analyze a **feasible** interior-point algorithm with the following properties:

- all its iterates are feasible and stay in a close neighbourhood of the central path;
- the iterates follow the central path towards optimality;
- systematic (though slow) reduction of duality gap is ensured.

This algorithm is called
the **short-step path-following method**.
Indeed, it makes very slow progress (short-steps) to optimality.

Central Path Neighbourhood

Assume a primal-dual strictly feasible solution $(x, y, s) \in \mathcal{F}^0$ lying in
neighbourhood of the central path is given; namely (x, y, s) satisfies:

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &\approx \mu e. \end{aligned}$$

We define a **θ-neighbourhood** of the central path $N_2(\theta)$, a set
primal-dual strictly feasible solutions $(x, y, s) \in \mathcal{F}^0$ that satisfy:

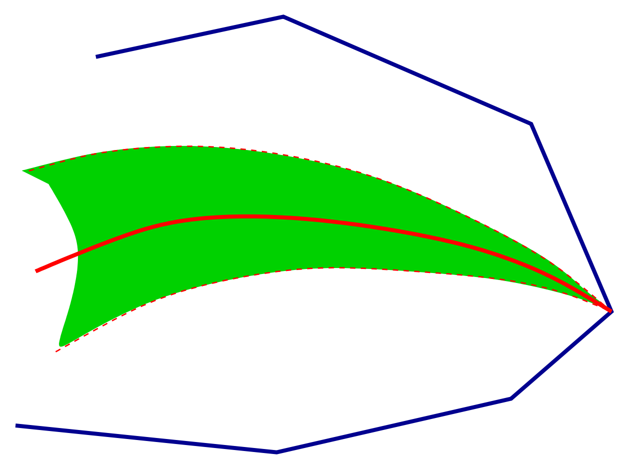
$$\|XSe - \mu e\| \leq \theta \mu,$$

where $\theta \in (0, 1)$ and the barrier μ satisfies:

$$x^T s = n\mu.$$

Hence $N_2(\theta) = \{(x, y, s) \in \mathcal{F}^0 \mid \|XSe - \mu e\| \leq \theta\mu\}$.

Central Path Neighbourhood



$N_2(\theta)$ neighbourhood of the central path

Progress towards optimality

Assume a primal-dual strictly feasible solution $(x, y, s) \in N_2(\theta)$ for some $\theta \in (0, 1)$ is given.

Interior point algorithm tries to move from this point to another one that also belongs to a θ-neighbourhood of the central path but corresponds to a smaller μ. The required reduction of μ is small:

$$\mu^{k+1} = \sigma \mu^k, \qquad \text{where} \qquad \sigma = 1 - \beta/\sqrt{n},$$

for some $\beta \in (0, 1)$.

This is a **short-step** method:
It makes short steps to optimality.

Progress towards optimality

Given a new μ-centre, interior point algorithm computes Newton direction:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma \mu e - XSe \end{bmatrix},$$

and makes step in this direction.

Magic numbers (will be explained later):

$$\theta = 0.1 \quad \text{and} \quad \beta = 0.1.$$

θ controls the proximity to the central path;
β controls the progress to optimality.

How to prove the $\mathcal{O}(\sqrt{n})$ complexity result

We will prove the following:

- full step in Newton direction is feasible;
- the new iterate $(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k, y^k, s^k) + (\Delta x^k, \Delta y^k, \Delta s^k)$ belongs to the θ-neighbourhood of the new μ-centre (with $\mu^{k+1} = \sigma \mu^k$);
- duality gap is reduced $1 - \beta/\sqrt{n}$ times.

$\mathcal{O}(\sqrt{n})$ complexity result

Note that since at one iteration duality gap is reduced $1 - \beta/\sqrt{n}$ tim after $\sqrt{n}$ iterations the reduction achieves:

$$(1 - \beta/\sqrt{n})^{\sqrt{n}} \approx e^{-\beta}.$$

After $C \cdot \sqrt{n}$ iterations, the reduction is $e^{-C\beta}$. For sufficiently lar constant C the reduction can thus be arbitrarily large (i.e. the duali gap can become arbitrarily small).

Hence this algorithm has complexity $\mathcal{O}(\sqrt{n})$.

This should be understood as follows:

> *"after the number of iterations proportional to* $\sqrt{n}$ *the algorithm solves the problem".*

Technical Results

Lemma 1

Newton direction $(\Delta x, \Delta y, \Delta s)$ defined by the equation system

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma\mu e - XSe \end{bmatrix}, \qquad (3)$$

satisfies:

$$\Delta x^T \Delta s = 0.$$

Proof:

From the first two equations in (3) we get

$$A\Delta x = 0 \quad \text{and} \quad \Delta s = -A^T \Delta y.$$

Hence

$$\Delta x^T \Delta s = \Delta x^T \cdot (-A^T \Delta y) = -\Delta y^T \cdot (A \Delta x) = 0.$$

Technical Results (cont'd)

Lemma 2

Let $(\Delta x, \Delta y, \Delta s)$ be the Newton direction that solves the system (3). The new iterate

$$(\bar{x}, \bar{y}, \bar{s}) = (x, y, s) + (\Delta x, \Delta y, \Delta s)$$

satisfies

$$\bar{x}^T \bar{s} = n\bar{\mu},$$

where

$$\bar{\mu} = \sigma\mu.$$

Proof: From the third equation in (3) we get

$$S\Delta x + X\Delta s = -XSe + \sigma\mu e.$$

By summing the n components of this equation we obtain

$$\begin{aligned} e^T(S\Delta x + X\Delta s) = s^T\Delta x + x^T\Delta s &= -e^T XSe + \sigma\mu e^T e \\ &= -x^T s + n\sigma\mu = -x^T s \cdot (1-\sigma). \end{aligned}$$

Thus

$$\begin{aligned} \bar{x}^T\bar{s} &= (x+\Delta x)^T(s+\Delta s) \\ &= x^T s + (s^T\Delta x + x^T\Delta s) + (\Delta x)^T\Delta s \\ &= x^T s + (\sigma - 1)x^T s + 0 = \sigma x^T s, \end{aligned}$$

which is equivalent to:

$$n\bar{\mu} = \sigma n\mu.$$

Reminder: Norms of the vector $x \in \mathcal{R}^n$.

$$\begin{aligned} \|x\| &= (\sum_{j=1}^{n} x_j^2)^{1/2} \\ \|x\|_\infty &= \max_{j\in\{1..n\}} |x_j| \\ \|x\|_1 &= \sum_{j=1}^{n} |x_j| \end{aligned}$$

For any $x \in \mathcal{R}^n$:

$$\begin{aligned} \|x\|_\infty &\le \|x\|_1 \\ \|x\|_1 &\le n\cdot\|x\|_\infty \\ \|x\|_\infty &\le \|x\| \\ \|x\| &\le \sqrt{n}\cdot\|x\|_\infty \\ \|x\| &\le \|x\|_1 \\ \|x\|_1 &\le \sqrt{n}\cdot\|x\| \end{aligned}$$

Reminder: Triangle Inequality

For any vectors p, q and r and for any norm $\|.\|$

$$\|p - q\| \leq \|p - r\| + \|r - q\|.$$

The relation between *algebraic* and *geometric* means.
For any scalars a and b such that $ab \geq 0$:

$$\sqrt{|ab|} \leq \frac{1}{2} \cdot |a + b|.$$

Technical Result (algebra)

Lemma 3 Let u and v be any two vectors in $\mathcal{R}^n$ such that $u^T v \geq 0$. Then

$$\|UVe\| \leq 2^{-3/2} \|u + v\|^2,$$

where $U = diag\{u_1, \cdots, u_n\}$, $V = diag\{v_1, \cdots, v_n\}$.

Proof: Let us partition all products $u_j v_j$ into positive and negative ones:

$$\mathcal{P} = \{j \,|\, u_j v_j \geq 0\} \quad \text{and} \quad \mathcal{M} = \{j \,|\, u_j v_j < 0\} :$$

$$0 \leq u^T v = \sum_{j \in \mathcal{P}} u_j v_j + \sum_{j \in \mathcal{M}} u_j v_j = \sum_{j \in \mathcal{P}} |u_j v_j| - \sum_{j \in \mathcal{M}} |u_j v_j|.$$

Proof (cont'd)

We can now write

$$\begin{aligned}
\|UVe\| &= (\|[u_j v_j]_{j \in \mathcal{P}}\|^2 + \|[u_j v_j]_{j \in \mathcal{M}}\|^2)^{1/2} \\
&\leq (\|[u_j v_j]_{j \in \mathcal{P}}\|_1^2 + \|[u_j v_j]_{j \in \mathcal{M}}\|_1^2)^{1/2} \\
&\leq (2\|[u_j v_j]_{j \in \mathcal{P}}\|_1^2)^{1/2} \\
&\leq \sqrt{2} \|[\tfrac{1}{4}(u_j + v_j)^2]_{j \in \mathcal{P}}\|_1 \\
&= 2^{-3/2} \sum_{j \in \mathcal{P}} (u_j + v_j)^2 \\
&\leq 2^{-3/2} \sum_{j=1}^{n} (u_j + v_j)^2 \\
&= 2^{-3/2} \|u + v\|^2, \qquad \text{as requested.}
\end{aligned}$$

IPM Technical Results (cont'd)

Lemma 4
If $(x, y, s) \in N_2(\theta)$ for some $\theta \in (0, 1)$, then

$$(1 - \theta)\mu \leq x_j s_j \leq (1 + \theta)\mu \quad \forall j.$$

In other words,

$$\min_{j \in \{1..n\}} x_j s_j \geq (1 - \theta)\mu,$$

$$\max_{j \in \{1..n\}} x_j s_j \leq (1 + \theta)\mu.$$

Proof:

Since $\|x\|_\infty \le \|x\|$, from the definition of $N_2(\theta)$,

$$N_2(\theta) = \{(x,y,s) \in \mathcal{F}^0 \,|\, \|XSe - \mu e\| \le \theta\mu\},$$

we conclude

$$\|XSe - \mu e\|_\infty \le \|XSe - \mu e\| \le \theta\mu.$$

Hence

$$|x_j s_j - \mu| \le \theta\mu \quad \forall j,$$

which is equivalent to

$$-\theta\mu \le x_j s_j - \mu \le \theta\mu \quad \forall j.$$

Thus

$$(1-\theta)\mu \le x_j s_j \le (1+\theta)\mu \quad \forall j.$$

IPM Technical Results (cont'd)

Lemma 5
If $(x,y,s) \in N_2(\theta)$ for some $\theta \in (0,1)$, then

$$\|XSe - \sigma\mu e\|^2 \le \theta^2\mu^2 + (1-\sigma)^2\mu^2 n.$$

Proof:
Note first that

$$e^T(XSe - \mu e) = x^T s - \mu e^T e = n\mu - n\mu = 0.$$

Therefore

$$\begin{aligned} &\|XSe - \sigma\mu e\|^2 \\ &= \|(XSe-\mu e) + (1-\sigma)\mu e\|^2 \\ &= \|XSe-\mu e\|^2 + 2(1-\sigma)\mu e^T(XSe-\mu e) + (1-\sigma)^2\mu^2 e^T e \\ &\le \theta^2\mu^2 + (1-\sigma)^2\mu^2 n. \end{aligned}$$

IPM Technical Results (cont'd)

Lemma 6
If $(x,y,s) \in N_2(\theta)$ for some $\theta \in (0,1)$, then

$$\|\Delta X \Delta S e\| \le \frac{\theta^2 + n(1-\sigma)^2}{2^{3/2}(1-\theta)}\mu.$$

Proof: 3rd equation in the Newton system gives

$$S\Delta x + X\Delta s = -XSe + \sigma\mu e.$$

Having multiplied it with $(XS)^{-1/2}$, we obtain

$$X^{-1/2}S^{1/2}\Delta x + X^{1/2}S^{-1/2}\Delta s = (XS)^{-1/2}(-XSe + \sigma\mu e).$$

Proof (cont'd)
Define $u = X^{-1/2}S^{1/2}\Delta x$ and $v = X^{1/2}S^{-1/2}\Delta s$ and observe that (by Lemma 1) $u^T v = \Delta x^T \Delta s = 0$. Now apply Lemma 3:

$$\begin{aligned} \|\Delta X \Delta S e\| &= \|(X^{-1/2}S^{1/2}\Delta X)(X^{1/2}S^{-1/2}\Delta S)e\| \\ &\le 2^{-3/2}\|X^{-1/2}S^{1/2}\Delta x + X^{1/2}S^{-1/2}\Delta s\|^2 \\ &= 2^{-3/2}\|X^{-1/2}S^{-1/2}(-XSe + \sigma\mu e)\|^2 \\ &= 2^{-3/2}\sum_{j=1}^{n}\frac{(-x_j s_j + \sigma\mu)^2}{x_j s_j} \\ &\le 2^{-3/2}\frac{\|XSe - \sigma\mu e\|^2}{\min_j x_j s_j} \\ &\le \frac{\theta^2 + n(1-\sigma)^2}{2^{3/2}(1-\theta)}\mu \qquad \text{(by Lemmas 4 and 5).} \end{aligned}$$

Magic Numbers

We have previously set two parameters for the short-step path-following method:

$$\theta = 0.1 \quad \text{and} \quad \beta = 0.1.$$

Now it's time to justify this particular choice.

Both θ and β have to be small to make sure that a full step in the Newton direction does not take the new iterate outside the neighbourhood $N_2(\theta)$.

θ controls the proximity to the central path;
β controls the progress to optimality.

Magic numbers choice lemma

Lemma 7 If $\theta = 0.1$ and $\beta = 0.1$, then

$$\frac{\theta^2 + n(1-\sigma)^2}{2^{3/2}(1-\theta)} \leq \sigma\theta.$$

Proof:
Recall that

$$\sigma = 1 - \beta/\sqrt{n}.$$

Hence

$$n(1-\sigma)^2 = \beta^2$$

and for $\beta = 0.1$ (for any $n \geq 1$)

$$\sigma \geq 0.9.$$

Substituting $\theta = 0.1$ and $\beta = 0.1$, we obtain

$$\frac{\theta^2 + n(1-\sigma)^2}{2^{3/2}(1-\theta)} = \frac{0.1^2 + 0.1^2}{2^{3/2} \cdot 0.9} \leq 0.02 \leq 0.9 \cdot 0.1 \leq \sigma\theta.$$

Full Newton step in $N_2(\theta)$

Lemma 8 Suppose $(x, y, s) \in N_2(\theta)$ and $(\Delta x, \Delta y, \Delta s)$ is the Newt direction computed from the system (3). Then the new iterate

$$(\bar{x}, \bar{y}, \bar{s}) = (x, y, s) + (\Delta x, \Delta y, \Delta s)$$

satisfies $(\bar{x}, \bar{y}, \bar{s}) \in N_2(\theta)$, i.e. $\|\bar{X}\bar{S}e - \bar{\mu}e\| \leq \theta\bar{\mu}$.

Proof: From Lemma 2, the new iterate $(\bar{x}, \bar{y}, \bar{s})$ satisfies

$$\bar{x}^T\bar{s} = n\bar{\mu} = n\sigma\mu,$$

so we have to prove that $\|\bar{X}\bar{S}e - \bar{\mu}e\| \leq \theta\bar{\mu}$.
For a given component $j \in \{1..n\}$, we have

$$\begin{aligned} \bar{x}_j\bar{s}_j - \bar{\mu} &= (x_j + \Delta x_j)(s_j + \Delta s_j) - \bar{\mu} \\ &= x_js_j + (s_j\Delta x_j + x_j\Delta s_j) + \Delta x_j\Delta s_j - \bar{\mu} \\ &= x_js_j + (-x_js_j + \sigma\mu) + \Delta x_j\Delta s_j - \sigma\mu \\ &= \Delta x_j\Delta s_j. \end{aligned}$$

Proof (cont'd)

Thus, from Lemmas 6 and 7, we get

$$\begin{aligned} \|\bar{X}\bar{S}e - \bar{\mu}e\| &= \|\Delta X\Delta Se\| \\ &\leq \frac{\theta^2 + n(1-\sigma)^2}{2^{3/2}(1-\theta)}\mu \\ &\leq \sigma\theta\mu \\ &= \theta\bar{\mu}. \end{aligned}$$

A property of log function

Lemma 9 For all $\delta > -1$:

$$\ln(1+\delta) \leq \delta.$$

Proof:
Consider the function

$$f(\delta) = \delta - \ln(1+\delta)$$

and its derivative

$$f^{'}(\delta) = 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}.$$

Obviously $f^{'}(\delta) < 0$ for $\delta \in (-1, 0)$ and $f^{'}(\delta) > 0$ for $\delta \in (0, \infty)$. Hence $f(.)$ has a *minimum* at $\delta = 0$. We find that $f(\delta = 0) = 0$. Consequently, for any $\delta \in (-1, \infty)$, $f(\delta) \geq 0$, i.e.

$$\delta - \ln(1+\delta) \geq 0.$$

$\mathcal{O}(\sqrt{n})$ Complexity Result

Theorem 10

Given $\epsilon > 0$, suppose that a feasible starting point $(x^0, y^0, s^0) \in N_2(0.1)$ satisfies

$$(x^0)^T s^0 = n\mu^0, \text{ where } \mu^0 \leq 1/\epsilon^{\kappa},$$

for some positive constant κ. Then there exists an index K with $K = \mathcal{O}(\sqrt{n}\ \ln(1/\epsilon))$ such that

$$\mu^k \leq \epsilon, \quad \forall k \geq K.$$

$\mathcal{O}(\sqrt{n})$ Complexity Result

Proof: From Lemma 2, $\mu^{k+1} = \sigma\mu^k$. Having taken logarithms of bo sides of this equality we obtain

$$\ln\mu^{k+1} = \ln\sigma + \ln\mu^k.$$

By repeatedly applying this formula and using $\mu^0 \leq 1/\epsilon^{\kappa}$, we get

$$\ln\mu^k = k\ln\sigma + \ln\mu^0 \leq k\ln(1-\beta/\sqrt{n}) + \kappa\ln(1/\epsilon).$$

From Lemma 9 we have $\ln(1-\beta/\sqrt{n}) \leq -\beta/\sqrt{n}$. Thus

$$\ln\mu^k \leq k(-\beta/\sqrt{n}) + \kappa\ln(1/\epsilon).$$

To satisfy $\mu^k \leq \epsilon$, we need:

$$k(-\beta/\sqrt{n}) + \kappa\ln(1/\epsilon) \leq \ln\epsilon.$$

This inequality holds for any $k \geq K$, where

$$K = \frac{\kappa+1}{\beta} \cdot \sqrt{n} \cdot \ln(1/\epsilon).$$

Polynomial Complexity Result

Main ingredients of the polynomial complexity result for the short-st path-following algorithm:

Stay close to the central path:
all iterates stay in the $N_2(\theta)$ neighbourhood of the central path.

Make (slow) progress towards optimality:
reduce systematically duality gap

$$\mu^{k+1} = \sigma\mu^k,$$

where

$$\sigma = 1 - \beta/\sqrt{n},$$

for some $\beta \in (0, 1)$.

Interior Point Methods

Theory: convergence in $\mathcal{O}(\sqrt{n})$ or $\mathcal{O}(n)$ iterations
Practice: convergence in $\mathcal{O}(\log n)$ iterations

Expected number of IPM iterations:

Problem Dimension	LP	QP
1,000	5 - 10	5 - 10
10,000	10 - 15	10 - 15
100,000	15 - 20	10 - 15
1,000,000	20 - 25	15 - 20
10,000,000	25 - 30	15 - 20
100,000,000	30 - 35	20 - 25
1000,000,000	35 - 40	20 - 25

... but one iteration may be expensive!

Reading about IPMs

S. Wright
Primal-Dual Interior-Point Methods, SIAM Philadelphia, 1997.

Gondzio
Interior point methods 25 years later,
European J. of Operational Research 218 (2012) 587–601.
`http://www.maths.ed.ac.uk/~gondzio/reports/ipmXXV.html`

Gondzio and Grothey
Direct solution of linear systems of size 10^9 arising in optimization with interior point methods, in: *Parallel Processing and Applied Mathematics PPAM 2005*, R. Wyrzykowski, J. Dongarra, N. Meyer and J. Wasniewski (eds.), *Lecture Notes in Computer Science*, 3911, Springer-Verlag, Berlin, 2006, pp 513–525.

OOPS: Object-Oriented Parallel Solver
`http://www.maths.ed.ac.uk/~gondzio/parallel/solver.html`

IPMs: Open Questions/Current Research

- iterative methods for indefinite linear systems (very close relation to PDEs)
- preconditioners for iterative methods
- extension to non-convex nonlinear problems

Interior Point Methods:

- Unified view of optimization
  $\rightarrow$ from LP via QP to NLP
- Predictable behaviour
  $\rightarrow$ small number of iterations
- Unequalled efficiency
  - competitive for small problems ($n \leq 10^6$)
  - beyond competition for large problems ($n \geq 10^6$)

Use IPMs in your research!