Discrete Optimization

Mixed Integer Programming: Part IV

Goals of the Lecture

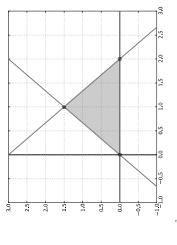
- ► Polyhedral cuts

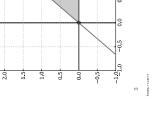
 warehouse location

 node covering

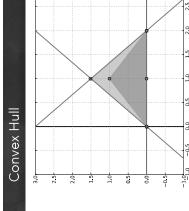


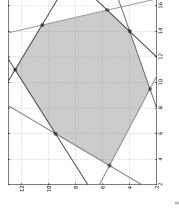
Convex Hull



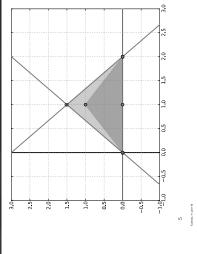


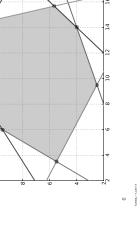


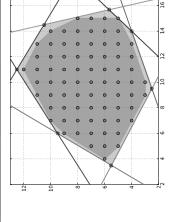


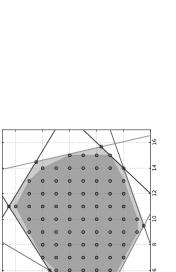


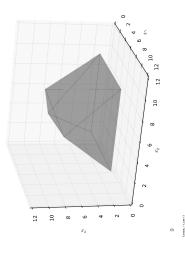
Convex Hull







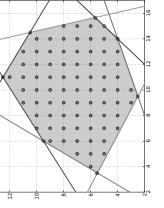


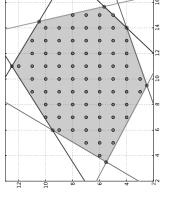


Convex Hull

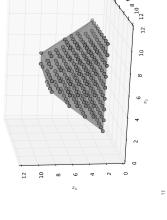
Convex Hull

Convex Hull





Convex Hull



Polyhedral Cuts

Convex Hull

▶ Polyhedral cuts

-cuts that represent the facets of the convex hull of the integer solution

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- -cuts that represent the facets of the convex hull
 - of the integer solution
- -they do not remove any solution These cuts are valid

Polyhedral Cuts ▶ Polyhedral cuts

- -cuts that represent the facets of the convex hull of the integer solution
- These cuts are valid
- -they do not remove any solution
- ▼The cuts are as strong as possible
- if we have all of them, we could use linear programming to solve the problem

Polyhedral Cuts

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- ▼They exploit the problem structure
- -they are derived from the structure of constraints -not based on information in the tableau

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Polyhedral Cuts

-they are derived from the structure of constraints

They exploit the problem structure

- not based on information in the tableau
- ▶ They share some of the spirit of syntactic cuts validity

▶ They share some of the spirit of syntactic cuts

-not based on information in the tableau

-must cut the current basic feasible solution

– validity

-do not need to generate all of them

-they are derived from the structure of constraints

They exploit the problem structure

Polyhedral Cuts

7

- -must cut the current basic feasible solution
 - -do not need to generate all of them
- ◆An application may use multiple cut types
 - -exploit different substructure

What is a Facet?

- ▼To find an facet in Rⁿ
- -find n affinely independent solutions (points)

What is a Facet?

- ▼To find an facet in Rn
- -find n affinely independent solutions (points)
- $(x_1, 1), \ldots, (x_n, 1)$ are linearly independent. x_1, \ldots, x_n are affinely independent iff ▶ Affine independence

▼ To find an facet in Rn

What is a Facet?

- find n affinely independent solutions (points)
- → Affine independence
- $(x_1, 1), \ldots, (x_n, 1)$ are linearly independent. x_1, \ldots, x_n are affinely independent iff
- Linear independence
- $x_1, \, \ldots, \, x_n$ are linearly independent iff $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$ implies that $\alpha_i = 0$ for all i.

Warehouse Location



O- Warehouse

Customer

4

$$\min \sum_{w \in W} c_w \; x_w + \sum_{w \in W, c \in C} t_{w,c} \; y_{w,c}$$
 subject to

$$\sum_{\substack{w \in W \\ x_w \in \{0,1\}}} y_{w,c} \leq x_w \qquad (w \in W, c \in C)$$

$$\sum_{\substack{w \in W \\ x_w \in \{0,1\}}} (w \in W)$$

$$\{0,1\} \qquad (w \in W)$$

$$x_w \in \{0, 1\}$$
 $(w \in W)$
 $y_{w,c} \in \{0, 1\}$ $(w \in W, c \in C)$

$$\min \sum_{w \in W} c_w \ x_w + \sum_{w \in W_c \in \mathcal{C}} t_{w,c} \ y_{w,c}$$
subject to

Warehouse Location

$$\sum_{w \in W} y_{w,c} = x_w \qquad (w \in W, c \in C)$$
$$\sum_{w \in W} y_{w,c} = 1 \quad (c \in C)$$
$$x_w \in \{0,1\} \qquad (w \in W)$$

 $(w \in W, c \in C)$

 $y_{w,c} \in \{0,1\}$

▶ Are these inequalities facets? $y_{w,c} \le x_w$

Facets for Warehouse Location

Warehouse Location

$$\leq x_w \qquad (w \in W, c \in C)$$

 $y_{w,c} = 1 \quad (c \in C)$

$$\{0,1\} \qquad (w \in W)$$

► Let G=(V,E) be a graph.

Node Packing

-A node packing is a subset W of V such that no goal is to find the node packing of maximal size, two nodes in W are connected by an edge. The



▶ How do we express it as a MIP?

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0

0

goal is to find the node packing of maximal size, - A node packing is a subset W of V such that no two nodes in W are connected by an edge. The

▶ Consider $y_{w,1} \le x_w$ and the following n points

► Let G=(V,E) be a graph.

Node Packing

Facets for Warehouse Location

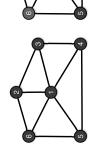
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► Are these inequalities facets?

 $y_{w,c} \le x_w$

Node Packing

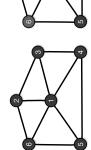
- ► Let G=(V,E) be a graph.
- goal is to find the node packing of maximal size. -A node packing is a subset W of V such that no two nodes in W are connected by an edge. The



► Let G=(V,E) be a graph.

Node Packing

goal is to find the node packing of maximal size. -A node packing is a subset W of V such that no two nodes in W are connected by an edge. The



► How do we express it as a MIP?

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F		VI	VI	VI	VI	VI	VI	VI	VI	VI		
:		x_2	x_3	x_4	x_5	x^{e}	x_3	x_4	x_5	x^{e}	$\{0, 1\}$	
H		+	+	+	+	+	+	+	+	+	Ψ	
7		x_1	x_1	x_1	x_1	x_1	x_2	x_3	x_4	x_5	x_i	
IIIGY	s.t.											

Node Packing

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MIP?
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Node Packing	$\max_{s.t.} x_1 + \dots + x_6$	$+ x_2$			$x_1 + x_5 \le 1$	+ x ₆		+ x4	+ x5	$+$ x^{e}	$0 \leq x_1 \dots x_6 \leq 1$
Node Packing	$\max_{s,t} x_1 + \dots + x_6$	$+ x_2$	+	+ x4	$x_1 + x_5 \le 1$	+ x ₆	+ x3	+ x4	+ x5	$+ x_6$	$0 \le x_1 \dots x_6 \le 1$

▶ What does the linear relaxation produce?

20

20

 $\max_{\text{s.t.}} \quad x_1 \quad + \quad$ $\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}$ 0

► What does the linear relaxation produce?

$$x_1 = \frac{1}{2}, \dots, x_6 = \frac{1}{2}$$

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How do we Find Facets?

- ▶ Find a property of the solution
- -constraints satisfied by all solutions

-constraints satisfied by all solutions How do we Find Facets? ▶ Find a property of the solution ▼Consider a clique

How do we Find Facets?

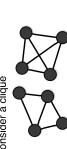
- -constraints satisfied by all solutions ▶ Find a property of the solution
- ▼Consider a clique



▶ How many nodes can be in a packing?

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Clique Facets Consider a clique ▼Consider a clique Clique Facets



► Maximal clique

-a clique that cannot be extended further

Maximal clique

-a clique that cannot be extended further

► Clique constraints for x₁,...,x₅

$$x_1 + \ldots + x_5 \le 1$$

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▼Consider a clique

Clique Facets



▼Maximal clique

-a clique that cannot be extended further

 $x_1 + \ldots + x_5 \le 1$ ▶ Clique constraints for x₁,...,x₅

▶ A maximal clique constraint is a facet!

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MIP with Clique Facets

MIP with Clique Facets

 x_1

max

$\bigvee |\bigvee |\bigvee |\bigvee |\bigvee |\bigvee |$ x_5 x_4 x_4 MIP with Clique Facets $\max x_1$ x_1 x_1

 x_5 x^{e} x_1

 x^6

▼What is the linear relaxation?

 y_x^0 + + x_{2} x_5 x_3 VI y x_2 x_1 x_1 x_1 x_1 max

▶ What does the linear relaxation produce?

23

23

▶ What does the linear relaxation produce? x^{e} x_2 x_1

VI VI VI VI VI x_6 + +

 x_5

 x_4

 $\begin{matrix} x_1 \\ x_1 \end{matrix}$

 x_3

23

 $x_1 = 0, x_2 = \frac{1}{2}, \dots, x_6 = \frac{1}{2}$

Coloring a Map with 0/1 Variables

LP: $\begin{array}{ll} obj & = 0.5 \\ color_{c,0} & = 0.5 \\ color_{c,1} & = 0.5 \end{array}$

≥çç g mir objColoring a Map

Variables	= 0.5	= 0.5 = 0.5	0 = 0	Need at least 2 colors!	$(c \in C)$	$(c \in C)$
coloring a Map with U/I Variables	LP: obj	$color_{c,0}$ $color_{c,1}$		Need at lea	$\sum_{v=0}^{3} v \times \operatorname{color}_{c,v}$	$\sum_{i=0}^{3} \operatorname{color}_{c,v} = 1$
oloring a M	IIP:	optimal - 9 nodes Proof - 41 nodes	$j \in \{0, 1, 2, 3\}$ $\text{lor}_{c,v} \in \{0, 1\}$	in obj	bject to $obj \ge \sum_{v=0}^{3} v \times \operatorname{color}_{c,v}$	\sum_{col}^{3}

subject to
$$obj \geq \sum_{v=0}^{3} v \times \operatorname{color}_{c,v}$$
 $(c \in C)$

$$\sum_{v=0}^{3} \operatorname{color}_{c_1,v} = 1 \qquad (c \in C)$$

$$\operatorname{color}_{c_1,v} + \operatorname{color}_{c_2,v} \leq 1 \quad (c_1,c_2 \in C \text{ and adjacent}, v \in 0..3)$$

v=0 $\operatorname{color}_{c_1,v}+\operatorname{color}_{c_2,v}\leq 1\quad (c_1,c_2\in C \text{ and adjacent}, v\in 0..3)$

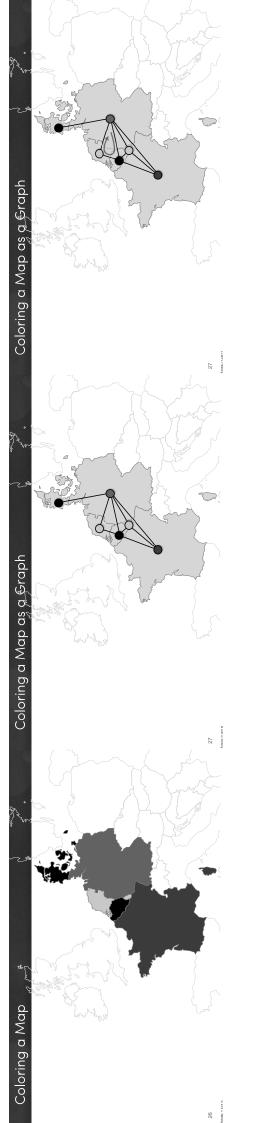
Need at least 2 colors!

 $\begin{array}{ll}
\operatorname{color}_{c,2} &= 0\\
\operatorname{color}_{c,3} &= 0
\end{array}$

 $\begin{array}{ll} obj & \in \{0,1,2,3\} \\ \operatorname{color}_{c,v} & \in \{0,1\} \end{array}$

Proof - 41 nodes

Optimal - 9 nodes



obj = 0.5 Need at least 2 colors. Improving the Coloring Relaxation ▶ With the best model from before Coloring a Map as a Graph

obj = 0.5 Need at least 2 colors.

 $\sum_{c \in \{0,3,4\}}^{3} \sum_{v=0}^{3} v \times \operatorname{color}_{c,v} \ge 3$

◆ Add the 3-Clique

Improving the Coloring Relaxation

▶ With the best model from before

Improving the Coloring Relaxation

Improving the Coloring Relaxation

- With the best model from before
- obj = 0.5 Need at least 2 colors.
 - ► Add the 3-Clique

obj = 0.5 Need at least 2 colors.

obj = 1.0 Still at least 2 colors.

 $\sum_{c \in \{0,3,4\}} \sum_{v=0}^{\sim} v \times \operatorname{color}_{c,v} \ge 3$

▶ Add the 3-Clique

Improving the Coloring Relaxation

▼With the best model from before

$$\sum_{c \in \{0.3.4\}}^{3} \sum_{v=0}^{3} v \times \operatorname{color}_{c,v} \ge c$$

$$\sum_{c\in\{0.3,4\}}^3\sum_{v=0}^3v\times\operatorname{color}_{c,v}\geq 3$$

$$\circ\mathsf{Add} \text{ the } 4\text{-Clique}$$

$$\sum_{c\in\{0.2,3,5\}}^3\sum_{v=0}^3v\times\operatorname{color}_{c,v}\geq 6$$

obj = 0.5 Need at least 2 colors. $\sum_{c\in\{0,3,4\}}^3\sum_{v=0}^3v\times {\rm color}_{c,v}\geq 3$ $obj=1.0\quad {\rm Still}\ {\rm at\ least\ 2\ colors}.$ > Add the 4-Clique ► With the best model from before $\sum_{c \in \{0,2,3,5\}}^{3} \sum_{v=0}^{3} v \times \operatorname{color}_{c,v} \ge 6$ Add the 3-Clique

obj = 1.5 Need at least 3 colors!!!

 $\begin{array}{ll} \textit{obj} = 0.5 \text{ Need at least 2 colors.} \\ \textbf{Add the 3-Clique} \\ \sum\limits_{c \in \{0.3.4\}}^{3} \sum\limits_{v = 0}^{y} v \times \operatorname{color}_{c,v} \geq 3 & \operatorname{Proof} - 41 \operatorname{nodes} \\ obj = 1.0 \\ \textbf{Add the 4-Clique} \\ \sum\limits_{c \in \{0.2.3.5\}}^{3} \sum\limits_{v = 0}^{y} v \times \operatorname{color}_{c,v} \geq 6 \\ oe \{0.2.3.5\}_{v = 0}^{v = 0} & obj = 1.5 \text{ Need at least 3 colors!!!} \end{array}$

▼With the best model from before

 $\begin{array}{lll} & \textit{obj} = 0.5 \text{ Need at least 2 colors.} \\ & \text{Add the 3-Clique} & & \text{MIP:} \\ & \sum\limits_{c \in \{0.3,4\}} \sum\limits_{v = 0}^{s} v \times \text{color}_{c,v} \geq 3 & \text{Proof} & -41 \text{ nodes} \\ & c \in \{0.3,4\} & v \geq 0 \\ & obj = 1.0 & \text{MIP:} \\ & \sum\limits_{c \in \{0.2,3.5\}} \sum\limits_{v = 0}^{s} v \times \text{color}_{c,v} \geq 6 & \text{Optimal -5 nodes} \\ & \sum\limits_{c \in \{0.2,3.5\}} \sum\limits_{v = 0}^{s} v \times \text{color}_{c,v} \geq 6 & \text{Optimal -5 nodes} \\ & obj = 1.5 & \text{Need at least 3 colors!!!} \\ \end{array}$

29 Tuestey, 11 June 13

28 Toeste, 11 Jans 13

28 Tamelloy, 11 Aces 13