CS323 Robotics & Automation Assignment 3

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Q1   DH Parameters for the PUMA 260 (20pts).

Figure 1 shows the PUMA 260 robot and the arrangement of its joints and the schematic of the manipulator with appropriately placed coordinate frames.

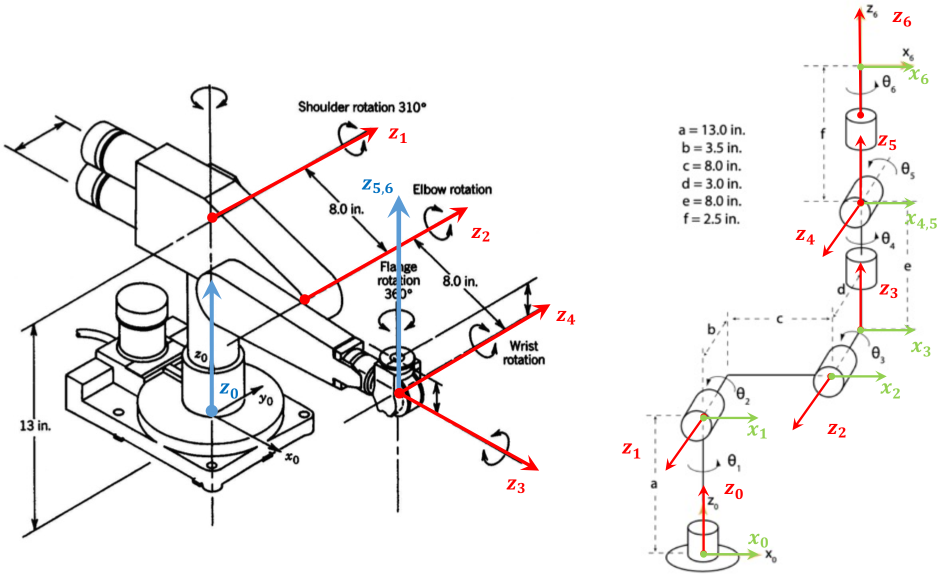

Figure 1 PUMA 260 manipulator

And the Table 1 illustrates the summary of DH parameters of this manipulator.

Table 1 DH Parameters (v1)

#	θ	d	a	α
0-1	θ_1	13.0	0	90°
1-2	θ_2	−3.5	8.0	0
2-3	θ_3	−3.0	0	−90°
3-4	θ_4	8.0	0	90°
4-5	θ_5	0	0	−90°
5-6	θ_6	2.5	0	0

The standard homogenous matrix is given by

$$T = RTTR = \begin{bmatrix} \cos\theta & -\sin\theta\cos\alpha & \sin\theta\sin\alpha & a\cos\theta \\ \sin\theta & \cos\theta\cos\alpha & -\cos\theta\sin\alpha & a\sin\theta \\ 0 & \sin\alpha & \cos\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

Then, we can get the matrix of each coordinate system, as follows

$${}^{0}T_1 = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ \sin\theta_1 & 0 & -\cos\theta_1 & 0 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$${}^{1}T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 8\cdot\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & 8\cdot\sin\theta_2 \\ 0 & 0 & 1 & -3.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$${}^{2}T_3 = \begin{bmatrix} \cos\theta_3 & 0 & -\sin\theta_3 & 0 \\ \sin\theta_3 & 0 & \cos\theta_3 & 0 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$${}^{3}T_4 = \begin{bmatrix} \cos\theta_4 & 0 & \sin\theta_4 & 0 \\ \sin\theta_4 & 0 & -\cos\theta_4 & 0 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$${}^{4}T_5 = \begin{bmatrix} \cos\theta_5 & 0 & -\sin\theta_5 & 0 \\ \sin\theta_5 & 0 & \cos\theta_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

$${}^{5}T_6 = \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 2.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

Next, we can calculate the transformation matrix by

$$[{}^{0}T_6] = [{}^{0}T_1]\cdot[{}^{1}T_2]\cdot[{}^{2}T_3]\cdot[{}^{3}T_4]\cdot[{}^{4}T_5]\cdot[{}^{5}T_6] \tag{8}$$

To solve this equation, we need to compute some intermediate results

$$
{}^{4}T_6 = [{}^{4}T_5]\cdot[{}^{5}T_6]
$$

$$
= \begin{bmatrix} c\theta_5\cdot c\theta_6 & -c\theta_5\cdot s\theta_6 & s\theta_5 & -2.5\cdot s\theta_5 \\ s\theta_5\cdot c\theta_6 & -s\theta_5\cdot s\theta_6 & -c\theta_5 & 2.5\cdot c\theta_5 \\ s\theta_6 & c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}
$$

And then,

$$
{}^{3}T_6 = [{}^{3}T_4]\cdot[{}^{4}T_6]
$$

$$
= \begin{bmatrix} c\theta_4c\theta_5c\theta_6 - s\theta_4s\theta_6 & -c\theta_4c\theta_5s\theta_6 - s\theta_4c\theta_5 & c\theta_4s\theta_5 & 2.5\cdot c\theta_4s\theta_5 \\ s\theta_4c\theta_5c\theta_6 + c\theta_4s\theta_6 & -s\theta_4c\theta_5s\theta_6 + c\theta_4c\theta_6 & s\theta_4s\theta_5 & -2.5\cdot s\theta_4s\theta_5 \\ -s\theta_5c\theta_6 & s\theta_5s\theta_6 & c\theta_4 & 8+2.5\cdot c\theta_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}
$$

Because, Joint 2 and joint 3 are parallel (using shorthand here),

$$
{}^{1}T_3 = [{}^{1}T_2]\cdot[{}^{2}T_3]
$$

$$
= \begin{bmatrix} c_{23} - s_{23} & 0 & c_2s_3 + s_2c_3 & 8\cdot c_2 \\ s_2c_3 + c_2s_3 & 0 & s_{23} - c_{23} & 8\cdot s_2 \\ 0 & 1 & 0 & -6.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}
$$

And then,

$$
{}^{0}T_3 = [{}^{0}T_1]\cdot[{}^{1}T_3]
$$

$$
= \begin{bmatrix} c_1c_{23} - c_1s_{23} & s_1 & c_1c_2s_3 - c_1s_2c_3 & 8c_1c_3 - 6.5s_1 \\ s_1c_{23} - s_1s_{23} & -c_1 & s_1c_2s_3 - s_1s_2c_3 & s_1(a_1 + a_2c_2 + a_3c_{23}) \\ s_2c_3 + s_3c_2 & 0 & s_{23} - c_{23} & 8s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}
$$

Therefore, the transformation matrix can be simplified to

$$
[{}^{0}T_6] = [{}^{0}T_3]\cdot[{}^{3}T_6] \tag{13}
$$

Due to large number of different robot forms, it is unnecessary to expand the final calculations, so we will not provide final results here.

Supplement: *Moreover, the wrist center of the manipulator can be defined as point C. To make it easier to resolve the position of the wrist center. We can simplify the PUMA 260 as the following 6R manipulator.*

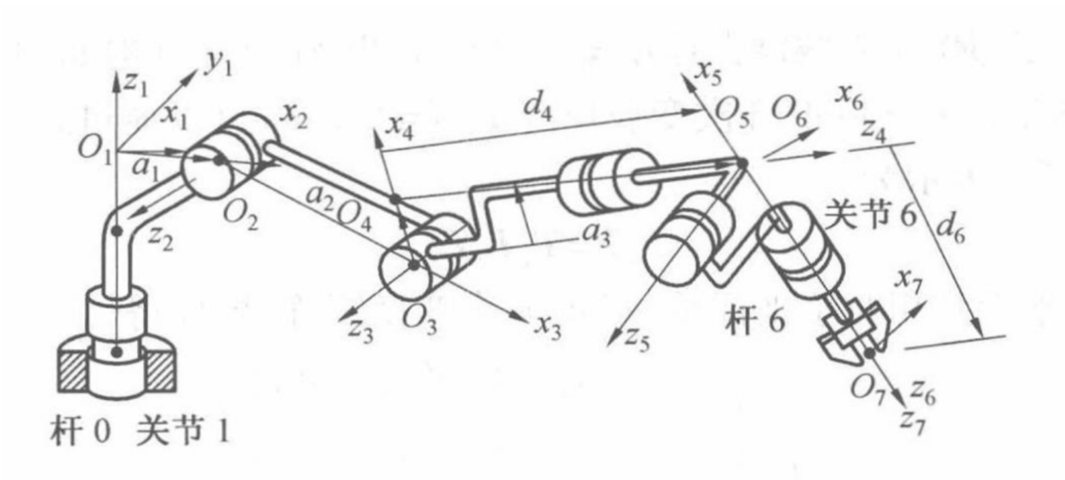

Figure 2 Simplified PUMA 260 manipulator (6R)

In this case, the DH parameters are shown in the Table 2.

Table 2 DH Parameters (v2)

#	θ	d	a	α
0-1	θ_1	0	a_1	$90°$
1-2	θ_2	0	a_2	0
2-3	θ_3	0	a_3	$90°$
3-4	θ_4	d_4	0	$-90°$
4-5	θ_5	0	0	$90°$
5-6	θ_6	d_6	0	0

Finally, utilizing Table 2 to calculate the location of the wrist center $\boldsymbol{p_C}$*,*

$$\boldsymbol{p_C} = \boldsymbol{p_{O_4}} + {}^{\boldsymbol{1}}\boldsymbol{Q_4 a_4}$$

$$= \begin{bmatrix} c_1(a_1 + a_2c_2 + a_3c_{23}) \\ s_1(a_1 + a_2c_2 + a_3c_{23}) \\ a_2s_2 + a_3s_{23} \end{bmatrix} + \begin{bmatrix} c_1c_{23} & s_1 & c_1s_{23} \\ s_1c_{23} & -c_1 & s_1s_{23} \\ s_{23} & 0 & -c_{23} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ d_4 \end{bmatrix} \tag{14}$$

$$= \begin{bmatrix} c_1(a_1 + a_2c_2 + a_3c_{23} + d_4s_{23}) \\ s_1(a_1 + a_2c_2 + a_3c_{23} + d_4s_{23}) \\ a_2s_2 + a_3s_{23} - d_4c_{23} \end{bmatrix}$$

Q2 Solve the Inverse Kinematics for the PUMA 260 (40 pts).

Generally speaking, it is very difficult to resolve the inverse kinematics for the space-6R manipulator. However, in this case, the PUMA 260 decouples at the Wrist Center (point C).

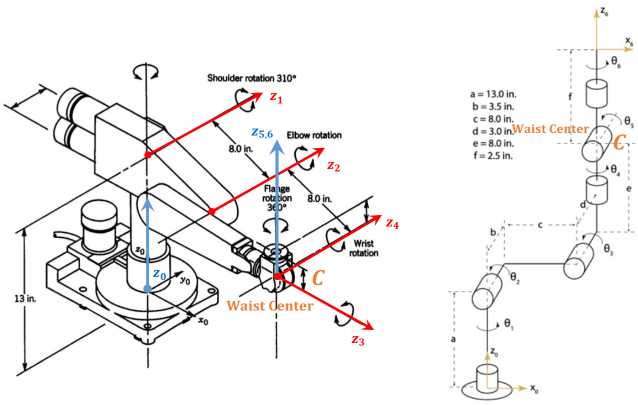

Figure 3 Waist center of PUMA 260 manipulator

Then, the inverse kinematics can be decomposed into two relatively simple problems: **(1)Position Inverse** and **(2) Orientation Inverse**. Given that:

$$\begin{cases} \boldsymbol{Q}_6^0(\theta_1, \cdots, \theta_6) = \boldsymbol{Q} \\ \boldsymbol{p}_6^0(\theta_1, \cdots, \theta_6) = \boldsymbol{p} \end{cases} \tag{15}$$

Illustrated in Equation (14) and (15), we know that

$$\boldsymbol{p} = \boldsymbol{p}_C + d_6 \boldsymbol{Q} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{16}$$

Now, we define $\boldsymbol{Q}$ and $\boldsymbol{p}$ as follows,

$$\boldsymbol{Q} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, \boldsymbol{p} = \begin{bmatrix} x_\mathrm{e} \\ y_\mathrm{e} \\ z_\mathrm{e} \end{bmatrix} \tag{17}$$

Then, $\boldsymbol{p}_C$ can be defined as

$$\boldsymbol{p}_C = \begin{bmatrix} x_C \\ y_C \\ z_C \end{bmatrix} = \begin{bmatrix} x_\mathrm{e} - d_6 r_{13} \\ y_\mathrm{e} - d_6 r_{23} \\ z_\mathrm{e} - d_6 r_{33} \end{bmatrix} \tag{18}$$

(1) Position Inverse *(Geometric method)*

Noted that, to facilitate the marking of the manipulator, the variables adopt new symbols, as shown in the following figures. Thanks a lot!

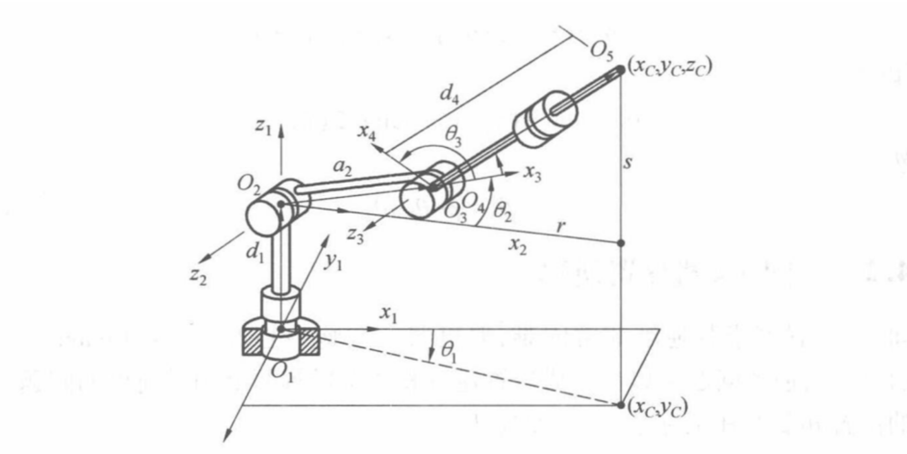

Figure 4 First three joint (as 3R manipulator)

First and foremost, we resolve the first three joints. Figure 4 illustrates the general organization. Then, we can easily define the θ_1 as:

$$\theta_1 = \begin{cases} \arctan 2(y_C, x_C) \\ \arctan 2(y_C, x_C) + \pi \end{cases} \tag{19}$$

So, there are two possible singularity posture in this case, as shown in Figure 5.

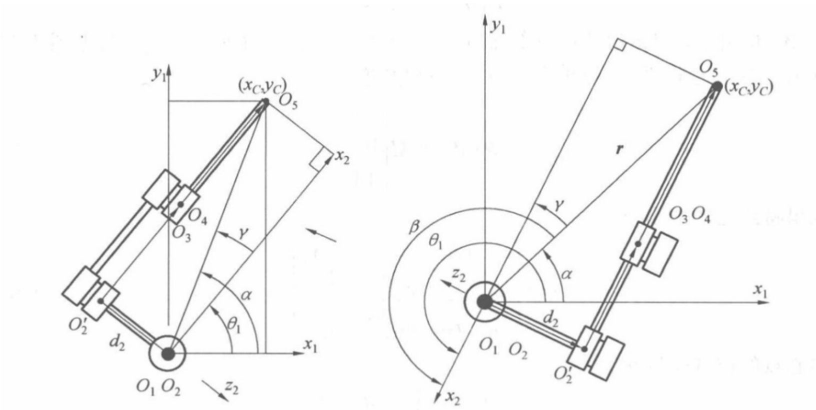

Figure 5 Shoulder Right & Shoulder Left

To simplify the calculation, as for the specific PUMA 260 shown in Figure 1, we will only consider the manipulator follows the condition of **Shoulder Left**. So, as illustrated in Figure 5

$$\theta_1 = \alpha - \beta \tag{20}$$

In the Equation (20),

$$\begin{cases} \alpha = \text{atan } 2(y_C, x_C) \\ \beta = \pi + \gamma \\ \gamma = \text{atan } 2\left(d_2, \sqrt{x_C^2 + y_C^2 - d_2^2}\right) \end{cases} \tag{21}$$

Then, we get

$$\theta_1 = \text{atan}\,2(y_C, x_C) - \text{atan}\,2\left(d_2, \sqrt{x_C^2 + y_C^2 - d_2^2}\right) + \pi \tag{22}$$

After calculating θ_1, now we consider θ_2 and θ_3, the following Figure 6 illustrates the projection of the Joint 2 and Joint 3.

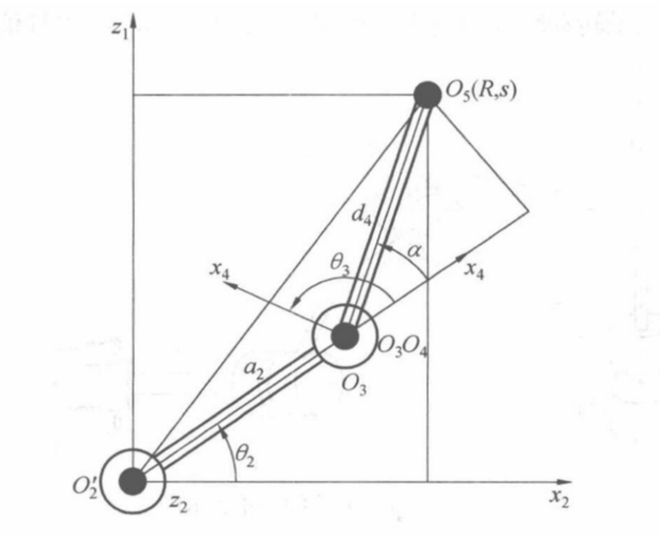

Figure 6 Projection of the Joint 2 and Joint 3

$$\begin{cases} \cos\,\alpha = \dfrac{R^2 + s^2 - a_2^2 - d_4^2}{2a_2 d_4} \equiv D \\ R = \sqrt{x_C^2 + y_C^2 - d_2^2} \\ s = z_C - d_1 \\ \alpha = \text{atan } 2\left(\pm\sqrt{1 - D^2}, D\right) \end{cases} \tag{23}$$

So, we can get

$$\begin{cases} \theta_2 = \text{atan}\,2(s, R) - \text{atan}\,2(d_4 s\alpha, a_2 + d_4 c\alpha) \\ \theta_3 = \dfrac{\pi}{2} + \alpha \end{cases} \tag{24}$$

(2) Orientation Inverse *(Algebraic + Geometric)*

After resolving the $\boldsymbol{\theta}_{1,2,3}$, we can get the corresponding rotation matrix, defined as $\boldsymbol{Q}_{1,2,3}$. So, the rotation matrix from the coordinate system 4 to coordinate system 6 can be defined as

$$\boldsymbol{M} = (\boldsymbol{Q}_3)^{\mathrm{T}}(\boldsymbol{Q}_2)^{\mathrm{T}}(\boldsymbol{Q}_1)^{\mathrm{T}}\boldsymbol{Q} = ({}^1\boldsymbol{Q}_4)^{\mathrm{T}}\boldsymbol{Q} \tag{25}$$

Similar as Equation (17), we define $\boldsymbol{M}$ as follows

$$\boldsymbol{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{26}$$

Now, considering the last three Joint 4, 5 and 6. Figure 7 illustrates the general organization, where the three axes of rotation intersect at one point.

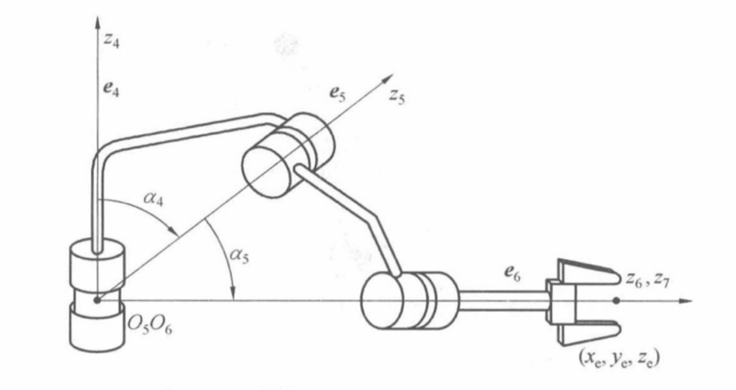

Figure 4 Last three joint

To begin with, we can define the Joint i possess the axis unit vector $\boldsymbol{e}_i$.

$${}^4\boldsymbol{e}_6 = \begin{bmatrix} m_{13} \\ m_{23} \\ m_{33} \end{bmatrix}, {}^4\boldsymbol{e}_5 = \begin{bmatrix} \mu_4 s_4 \\ -\mu_4 c_4 \\ \lambda_4 \end{bmatrix} \tag{27}$$

Then, we get

$$\begin{cases} {}^4\boldsymbol{e}_6 \cdot {}^4\boldsymbol{e}_5 = \lambda_5 \\ m_{13}\mu_4 s_4 - m_{23}\mu_4 c_4 + m_{33}\lambda_4 = \lambda_5 \end{cases} \tag{28}$$

Now, we assume that

$$\begin{cases} A = m_{13}\mu_4 \\ B = m_{23}\mu_4 \\ C = \lambda_5 - m_{33}\lambda_4 \end{cases} \tag{29}$$

Thus we have $As_4 - Bc_4 = C$, then utilizing the **Transformation of Triangle,**

$$\begin{cases} A = \rho\cos\varphi \\ B = \rho\sin\varphi \\ \rho = \sqrt{A^2 + B^2} \\ \varphi = \text{atan } 2(B, A) \end{cases} \tag{30}$$

We can get the $\theta_{4,5,6}$, as shown in following procedures:

a. For θ_4

$$\left.\begin{aligned} &\sin(\theta_4 - \varphi) = C/\rho, \cos(\theta_4 - \varphi) = \pm\sqrt{1 - (C/\rho)^2} \\ &\theta_4 - \varphi = \text{atan } 2\left(C/\rho, \pm\sqrt{1 - (C/\rho)^2}\right) \\ &\boldsymbol{\theta_4} = \textbf{atan } \mathbf{2}(\boldsymbol{B}, \boldsymbol{A}) + \textbf{atan } \mathbf{2}\left(\boldsymbol{C}/\boldsymbol{\rho}, \pm\sqrt{\mathbf{1} - (\boldsymbol{C}/\boldsymbol{\rho})^2}\right) \end{aligned}\right\} \tag{31}$$

b. For θ_5

$$\boldsymbol{Q}_5 = (\boldsymbol{Q}_4)^{\mathrm{T}} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} (\boldsymbol{Q}_6)^{\mathrm{T}} \tag{32}$$

So, we can resolve the $\boldsymbol{\theta_5}$ through the following equations (33),

$$\begin{cases} \mu_5 s_5 = (m_{12}\mu_6 + m_{13}\lambda_6)c_4 + (m_{22}\mu_6 + m_{23}\lambda_6)s_4 \\ \mu_5 c_5 = \lambda_4(m_{12}\mu_6 + m_{13}\lambda_6)s_4 - \lambda_4(m_{22}\mu_6 + m_{23}\lambda_6)c_4 - \mu_4(m_{32}\mu_6 + m_{33}\lambda_6) \end{cases} \tag{33}$$

c. For θ_6

$$\boldsymbol{Q}_6 = \boldsymbol{Q}_5^{\mathrm{T}} \boldsymbol{Q}_4^{\mathrm{T}} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{34}$$

Also, we can get the result of $\boldsymbol{\theta_6}$ through the Equation (34). As for the different $(\boldsymbol{\theta_4}, \boldsymbol{\theta_5})$ correspond to different $\boldsymbol{\theta_6}$, thus the Orientation Inverse possesses two sets of solution, which will be suggested in the following summary.

(3) Summary

Noted that, to facilitate the marking of the manipulator, the variables adopt new symbols, as shown in the above figures. Thanks a lot!

Refer to the above Equation (17, 26, 23, 30, 33), the summary of the Inverse Kinematics for the PUMA 260 is shown as follows,

$$\begin{cases} \theta_1 = \operatorname{atan} 2(y_C, x_C) - \operatorname{atan} 2\left(d_2, \sqrt{x_C^2 + y_C^2 - d_2^2}\right) + \pi \\ \theta_2 = \operatorname{atan} 2(s, R) - \operatorname{atan} 2(d_4 s\alpha, a_2 + d_4 c\alpha) \\ \theta_3 = \dfrac{\pi}{2} + \alpha \\ \theta_4 = \operatorname{atan}\ 2(B, A) + \operatorname{atan}\ 2\left(C/\rho, \pm\sqrt{1 - (C/\rho)^2}\right) \\ \theta_5 = \operatorname{atan}\ 2(c_4 m_{13} + s_4 m_{23}, m_{33}) \\ \theta_6 = \operatorname{atan}\ 2(-s_4 m_{11} + c_4 m_{21}, -s_4 m_{12} + c_4 m_{22}) \end{cases} \tag{35}$$

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Hanlin CAI

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