4. Laplace Transforms & Transfer Functions

4.1 Introduction

- In the previous section, we looked at the mathematical modelling of a range of both static and dynamic systems.

- In each case, the system model was represented by a linear ordinary differential equation (ODE), which were either first or second order in the examples provided. Here, the order is the highest degree of the differential equation.

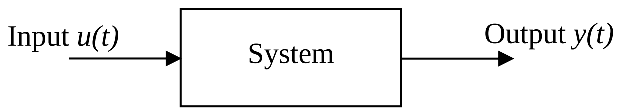
- While ODEs describe the input-output relation of the system, they are not a satisfactory representation from a system's perspective.

- For example, consider the following mass-spring-damper system equation:

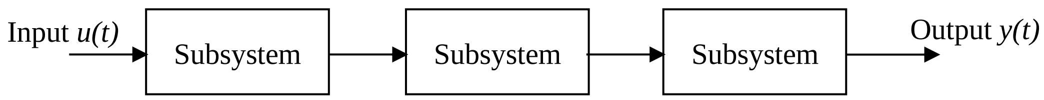
$$M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Kx(t) = u(t)$$

- The system parameters (M, B and K) and the output $x(t)$, for this particular system, appear throughout the equation.

- Ideally, it is more desirable to represent the system so that the input, output and system are distinct parts, as illustrated in the block diagram below:



- Furthermore, it would also be desirable to conveniently represent the interconnection of several subsystems, where applicable, such as the cascaded system illustrated below:



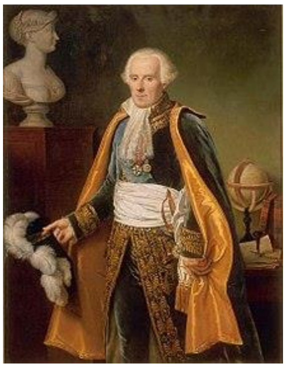
- The differential equation does not provide such convenience. However, an alternative model representation, known as the transfer function, does!

- The **transfer function** is a compact representation of the relationship between the input and an output for a linear system. *In this form, we no longer work with differentials but rather with an algebraic expression.*

- Combined with block diagrams, transfer functions offer a powerful means for dealing with complex linear systems, as we will see in section 5.

- Here, we will use the Laplace Transform to convert a differential equation to transform function form.

4.2 Laplace Transforms – a brief overview



Pierre-Simon Laplace

- Laplace Transforms will be covered in significant detail in the mathematics modules (EE112 and EE206).

- *Additional detail is available at the end of these notes for the EngSc students.*

- Here, we will give a very brief overview of the key features of Laplace Transforms before we examine how they can be applied to differential equations (in section 4.3).

- It is the latter part that is essential to obtaining the transfer function representation.

- The **Laplace Transform is defined as**:

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

where s is a *complex operator* given by $s = \sigma + j\omega$.

- This is a special transformation that transforms a differential equation into an expression with no derivatives.

- Note, the transformed function is a function of s only. The time variable t does not exist in this form (see examples below).

- *Also note the convention of using **a capital letter for the Laplace transform**!*

- Example **Laplace transforms of common functions** include:

  - $\boldsymbol{f(t)} = \boldsymbol{1}, \quad t \geq 0$ (i.e. the unit step)

$$\boldsymbol{F(s)} = L[f(t)] = \int_0^\infty 1.e^{-st}dt$$

$$= \left(-\frac{1}{s}e^{-st}\right)_0^\infty \qquad = \left(-\frac{1}{s}(e^{-\infty} - e^0)\right) \qquad = \left(-\frac{1}{s}(0-1)\right)$$

$$= \frac{\boldsymbol{1}}{\boldsymbol{s}}$$

  - $\boldsymbol{f(t)} = k,$ where k is a constant value

$$\boldsymbol{F(s)} = L[f(t)] = \int_0^\infty k.e^{-st}dt$$

$$= \left(-\frac{k}{s}e^{-st}\right)_0^\infty \qquad = \frac{\boldsymbol{k}}{\boldsymbol{s}}$$

- ***f(t)* = e^{at}**

$$\boldsymbol{F(s)} = L\left[f(t)\right]$$

$$= \int_0^\infty e^{at}.e^{-st}\,dt \qquad = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left(-\frac{1}{s-a}e^{-(s-a)t}\right)_0^\infty \qquad = \left(-\frac{1}{s-a}(e^{-\infty}-e^{0})\right)$$

$$= \boldsymbol{\frac{1}{s-a}}$$

- ***f(t)* = *sin(kt)***

To make the transformation easier to compute, we note that:

$$e^{j\theta} = \cos\theta + j\sin\theta \quad \Rightarrow \sin\theta \equiv \operatorname{Im}\left\{e^{j\theta}\right\}$$

Hence, we can state that: $\sin(kt) \equiv \operatorname{Im}\left\{e^{jkt}\right\}$

Thus:

$$\boldsymbol{F(s)} = L\left[f(t)\right] = L\left[\operatorname{Im}\left\{e^{jkt}\right\}\right]$$

$$= \operatorname{Im}\left\{\int_0^\infty e^{jkt}.e^{-st}\,dt\right\} \qquad = \operatorname{Im}\left\{\int_0^\infty e^{-(s-jk)t}\,dt\right\}$$

$$= \operatorname{Im}\left\{\left(-\frac{1}{s-jk}e^{-(s-jk)t}\right)_0^\infty\right\} \qquad = \operatorname{Im}\left\{\left(-\frac{1}{s-jk}(e^{-\infty}-e^{0})\right)\right\}$$

$$= \operatorname{Im}\left\{\frac{1}{s-jk}\right\} \qquad = \operatorname{Im}\left\{\frac{1}{s-jk}.\frac{s+jk}{s+jk}\right\} \qquad = \operatorname{Im}\left\{\frac{s+jk}{s^2+k^2}\right\}$$

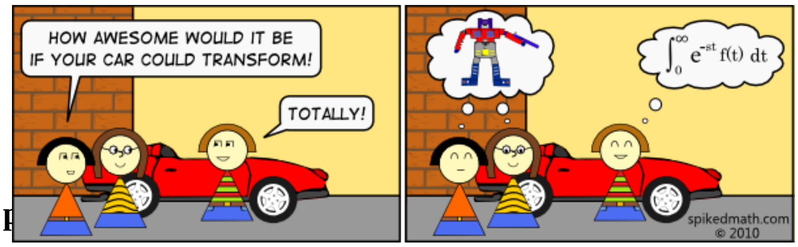
$$= \operatorname{Im}\left\{\frac{s}{s^2+k^2} + j\frac{k}{s^2+k^2}\right\}$$

$$= \boldsymbol{\frac{k}{s^2+k^2}}$$

- *The good news! We don't need to remember the above (and other) examples, as the more common functions and their transforms are typically available in look up tables, similar to the one on the next page.*
- *Such a table will be available in exam situations if needed.*

Function name	**Time domain function** $f(t)$	**Laplace transform** $F(s) = L\{f(t)\}$
Constant	a	$\dfrac{a}{s}$
Linear	t	$\dfrac{1}{s^2}$
Power	t^n	$\dfrac{n!}{s^{n+1}}$
Exponent	e^{at}	$\dfrac{1}{s-a}$
Sine	$\sin at$	$\dfrac{a}{s^2+a^2}$
Cosine	$\cos at$	$\dfrac{s}{s^2+a^2}$
Hyperbolic sine	$\sinh at$	$\dfrac{a}{s^2-a^2}$
Hyperbolic cosine	$\cosh at$	$\dfrac{s}{s^2-a^2}$
Delta function	$\delta(t)$	1
Delayed delta	$\delta(t-a)$	e^{-as}
Derivative	$\dfrac{df}{dt}$	$sF(s)-f(0)$
Derivative (k^{th})	$\dfrac{d^k f}{dt^k}$	$s^k F(s)-s^{k-1}f(0)-s^{k-2}f^1(0)\ldots-f^{k-1}(0)$
Integral	$\int_0^t f(\tau)d\tau$	$\dfrac{F(s)}{s}$

Table of common Laplace Transforms – all functions are defined for $t \geq 0$



- Some **useful p**

- Linearity: if $F(s) = L[f(t)]$ and $G(s) = L[g(t)]$

  then : $L[af(t) + bg(t)] = aF(s) + bG(s)$

- (Shift theorem) Multiplying by e^{at}: $L[e^{at} f(t)] = F(s-a)$

- Final value theorem (end point): $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

- Initial value theorem (start point): $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Inverse Laplace Transform

- The **Inverse Laplace Transform** relates to the process of finding $f(t)$ from the corresponding Laplace transform $F(s)$, i.e.:

$$f(t) = L^{-1}[F(s)]$$

- This is normally carried out using the ***partial fraction method***. This method is best illustrated using an example.

- Consider the following Laplace transform:

$$F(s) = \frac{s+2}{(s+3)(s+4)}$$

- We express this in partial fraction form as:

$$\frac{s+2}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

- In other words, we separate the denominator into its individual factors.

- We now determine the value of the unknown variables A and B, as follows:

$$\frac{s+2}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4} = \frac{A(s+4) + B(s+3)}{(s+3)(s+4)}$$

$$= \frac{s(A+B) + (4A+3B)}{(s+3)(s+4)}$$

- Now we compare terms on the numerator and match the coefficients as follows:

  $s = (A+B)s \;\Rightarrow\; 1 = A+B$ and $2 = 4A+3B$

- Solving for A and B, we get:

From the first equation: $1 = A + B \Rightarrow B = 1 - A$

Substituting into the second equation gives:

$$2 = 4A + 3(1 - A) \qquad \Rightarrow 2 = 4A + 3 - 3A \qquad \Rightarrow A = -1$$

Hence: $B = 1 - (-1) = 2$

- An **alternative** and quicker method, known as the **cover-up method**, is as follows:

  Compare the numerators: $s + 2 = A(s + 4) + B(s + 3)$

  Now cover-up the (s + 4) factor by setting s = –4, i.e. (s + 4) = 0:

  $$-4 + 2 = 0 + B(-4 + 3) \;\Rightarrow \mathrm{B} = 2$$

  Now cover-up the (s + 3) factor by setting s = –3, i.e. (s + 3) = 0:

  $$-3 + 2 = A(-3 + 4) + 0 \;\Rightarrow \mathrm{A} = -1$$

  We obtain the same result as above!

- Therefore, we can express $F(s)$ as: $F(s) = -\dfrac{1}{s+3} + \dfrac{2}{s+4}$

- We now use the table of Laplace transforms to convert this expression to the time domain.

- From the table we see that: $L\left(e^{at}\right) = \dfrac{1}{s-a} \quad \Rightarrow L^{-1}\left(\dfrac{1}{s-a}\right) = e^{at}$

- Hence: $L^{-1}\left(\dfrac{-1}{s+3}\right) = -L^{-1}\left(\dfrac{1}{s+3}\right) = -e^{-3t}$

  $$L^{-1}\left(\dfrac{2}{s+4}\right) = 2L^{-1}\left(\dfrac{1}{s+4}\right) = 2e^{-4t}$$

- Finally: $F(s) = -\dfrac{1}{s+3} + \dfrac{2}{s+4} \quad \rightarrow \quad f(t) = -e^{-3t} + 2e^{-4t}$

- ***Note – there are additional rules the need to be taken into consideration when working with partial fractions. These will be covered in your mathematics modules (or in the last section of these notes for the EngSc students). Here, we are simply illustrating the key concept.***
- ***Ex 4.1 Find the function f(t) given that its Laplace transform is:***

$$F(s) = \frac{3s+1}{s^2 - s - 6}$$

Solution:

$$\frac{3s+1}{s^2 - s - 6} = \frac{3s+1}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

$$\frac{A}{s-3} + \frac{B}{s+2} \quad = \frac{A(s+2) + B(s-3)}{(s-3)(s+2)}$$

Compare the numerators: $\quad 3s+1 = A(s+2) + B(s-3)$

Now cover-up the (s + 2) factor by setting s = –2:

$$-6+1 = 0 + B(-2-3) \quad \Rightarrow \mathrm{B} = 1$$

Now cover-up the (s – 3) factor by setting s = 3:

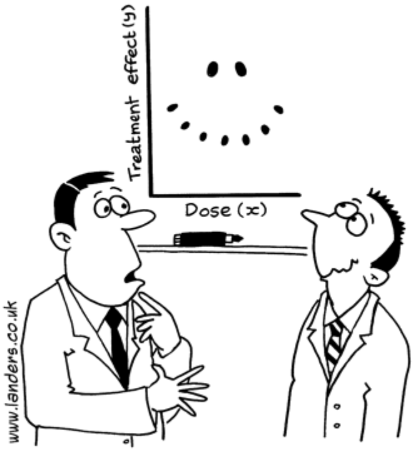
$$9+1 = A(3+2) + 0 \;\Rightarrow \mathrm{A} = 2$$

Thus: $F(s) = \dfrac{2}{s-3} + \dfrac{1}{s+2}$

Getting the inverse Laplace transform gives:

$$f(t) = L^{-1}\left(\frac{2}{s-3}\right) + L^{-1}\left(\frac{1}{s+2}\right)$$

$$\Rightarrow f(t) = 2e^{3t} + e^{-2t}$$



"It's a non-linear pattern with outliers.....but for some reason I'm very happy with the data."

4.3 Laplace Transforms and differential equations

- One of the key advantages (particularly from a system analysis viewpoint) of Laplace transforms is the transformation of linear differential equations into algebraic equations.

- It can be shown that:

$$L\left[f^{'}(t)\right] = sF(s) - f(0)$$

$$L\left[f^{''}(t)\right] = s^2F(s) - sf(0) - f^{'}(0)$$

$$L\left[f^{'''}(t)\right] = s^3F(s) - s^2f(0) - sf^{'}(0) - f^{''}(0)$$

$$\ldots$$

$$\ldots$$

$$L\left[f^{n}(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f^{'}(0) - \ \ldots \ - f^{n-1}(0)$$

- Note how the initial conditions are introduced in the above equations. These are important when solving differential equations.

- ***Ex 4.2 Express the following differential equation in terms of Laplace transforms given that at time t = 0, x = 1:***

$$\frac{dx(t)}{dt} - 2x(t) = 4$$

Solution:

$$L\left(\frac{dx}{dt} - 2x = 4\right) \qquad \Rightarrow L\left(\frac{dx}{dt}\right) - 2L(x) = L(4)$$

$$\Rightarrow sX(s) - x(0) - 2X(s) = \frac{4}{s}$$

We know that $x(0) = 1$, hence:

$$sX(s) - 1 - 2X(s) = \frac{4}{s} \qquad \Rightarrow X(s)(s-2) = \frac{4}{s} + 1 = \frac{4+s}{s}$$

$$\Rightarrow X(s) = \frac{s+4}{s(s-2)}$$

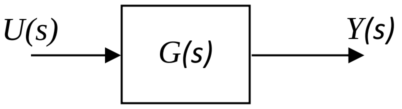
- *Note that, if needed, we can now solve the differential equation by obtaining the partial factions for X(s) and finding the inverse Laplace transform. This gives:*

$$x(t) = 3e^{2t} - 2 \qquad \textit{(Verify this for yourself)}$$

4.4 Transfer function representation

- The **transfer function model** is the input-output relationship of a system in the Laplace Transform space.

- It is defined as **the ratio of the Laplace transforms of the output and input of a system for zero initial conditions**.



$$\text{Transfer function } = \frac{Y(s)}{U(s)} = G(s)$$

- Since there are zero initial conditions, then the Laplace transform of differential expressions, in this context, is simply reduced to:

$$L\left[f^{'}(t)\right] = sF(s)$$

$$L\left[f^{''}(t)\right] = s^2F(s)$$

$$\dots$$

$$\dots$$

$$L\left[f^{n}(t)\right] = s^nF(s)$$

- So, for example, consider the system governed by the following differential equation:

$$\frac{d^2y}{dt^2} - 4y = \frac{du}{dt} - 3u$$

- Obtaining the Laplace transform of this equation gives:

$$s^2Y(s) - 4Y(s) = sU(s) - 3U(s)$$

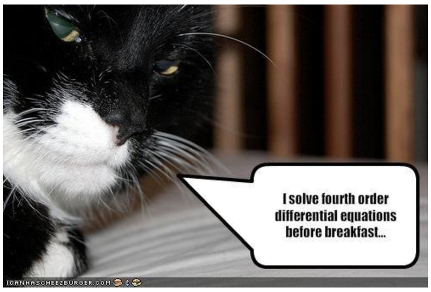
- We can then obtain the transfer function as follows:

$$(s^2 - 4)Y(s) = (s - 3)U(s)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{s-3}{s^2-4} = G(s)$$

- ***Ex 4.3 Obtain the transfer function for the system given by the following ordinary differential equation:***

$$\frac{dx(t)}{dt} - 2x(t) = 4u(t)$$



Solution:

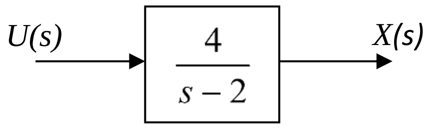
$$L\left(\frac{dx(t)}{dt} - 2x(t) = 4u(t)\right)$$

$$\Rightarrow sX(s) - 2X(s) = 4U(s)$$

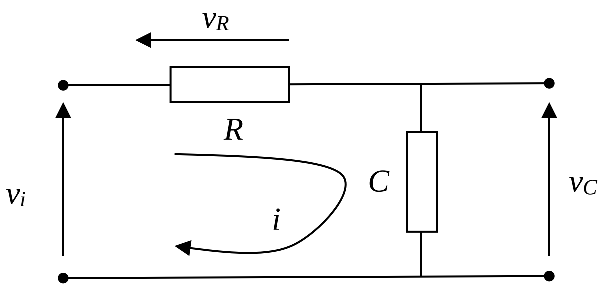
The transfer function is thus:

$$X(s)(s-2) = 4U(s)$$

$$\Rightarrow G(s) = \frac{X(s)}{U(s)} = \frac{4}{s-2}$$



- ***Ex 4.4 Obtain the transfer function for the following circuit-based system (Ex2.3):***



Solution:

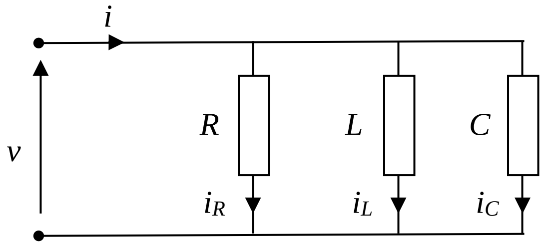
Recall from Ex 2.3: $v_i = RC\dfrac{dv_C}{dt} + v_C$

Taking the Laplace transform gives: $V_i(s) = RCsV_c(s) + V_c(s)$

$$\Rightarrow V_i(s) = V_c(s)(1 + sRC)$$

Hence the required transfer function is given by: $\dfrac{V_c(s)}{V_i(s)} = \dfrac{1}{1+sRC}$

- ***Ex 4.5 Obtain the transfer function representation for the following circuit-based system, relating voltage to current (Ex2.5):***



Solution:

Recall from Ex 2.3: $$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=\frac{1}{C}\frac{di}{dt}$$

Taking the Laplace transform gives:

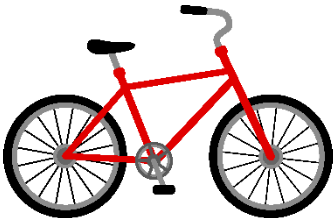
$$s^2V(s)+\frac{1}{RC}sV(s)+\frac{1}{LC}V(s)=\frac{1}{C}sI(s)$$

$$\Rightarrow\left(s^2+\frac{s}{RC}+\frac{1}{LC}\right)V(s)=\frac{s}{C}I(s)$$

Hence the required transfer function is: $$\frac{V(s)}{I(s)}=\frac{\frac{s}{C}}{s^2+\frac{s}{RC}+\frac{1}{LC}}=\frac{RLs}{RLCs^2+Ls+R}$$

- ***Ex 4.6 Determine the transfer function model for the spring-mass-damper system given in Ex 2.6(a):***

Solution:



Recall from Ex 2.6(a): $$M\frac{d^2x}{dt^2}+B\frac{dx}{dt}+Kx=f(t)$$

Taking the Laplace transform gives:

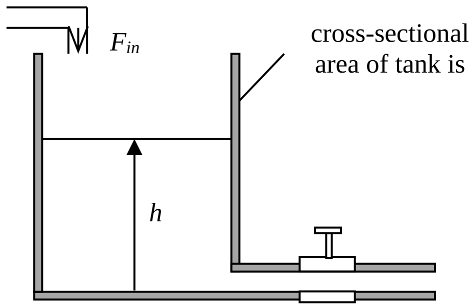
$$Ms^2X(s)+BsX(s)+KX(s)=F(s)$$

Hence the required transfer function is:

$$\frac{X(s)}{F(s)}=\frac{1}{Ms^2+Bs+K}$$

- ***Ex 4.7 Determine the transfer function model for the single water tank system represented by the differential equation:***

$$A\frac{dh}{dt} = F_{in} - kh$$



Solution:

Taking the Laplace transform gives:

$$AsH(s) = F(s) - kH(s)$$

$$\Rightarrow H(s)(sA + k) = F(s)$$

Hence the required transfer function is: $$\frac{H(s)}{F(s)} = \frac{1}{sA + k}$$