# EE206 Differential Equations and Transform Methods

Tutorial 1

**Problem 1a:** $\dfrac{d}{dx} \quad \ln(4x\sqrt{x+7})$

$$\ln(4x\sqrt{x+7}) = \ln(4) + \ln(x) + \tfrac{1}{2}\ln(x+7)$$

$$\implies \tfrac{d}{dx}\ln(4x\sqrt{x+7}) = \tfrac{1}{x} + \tfrac{1}{2(x+7)} = \tfrac{3x+14}{2x(x+7)}$$

**Problem 2a:** $\int x\cot(x^2+1)dx$

$$u = x^2 + 1 \qquad du = 2x\,dx$$

$$\begin{aligned}\int x\cot(x^2+1)dx &= \frac{1}{2}\int \cot(u)du\\ &= \frac{1}{2}\int \frac{\cos(u)}{\sin(u)}du\end{aligned}$$

$$v = \sin(u) \qquad dv = \cos(u)\,du$$

$$\begin{aligned}&= \frac{1}{2}\int \frac{1}{v}\,dv\\ &= \frac{1}{2}\ln|v| + c = \frac{1}{2}\ln|\sin(u)| + c\\ &= \frac{1}{2}\ln|\sin(x^2+1)| + c\end{aligned}$$

**Problem 2d:** $\int e^{ax}\sin(bx)dx$

$$\int udv = uv - \int vdu$$

$$u = \sin(bx) \qquad du = b\cos(bx)\,dx$$

$$dv = e^{ax}\,dx \qquad v = \frac{1}{a}e^{ax}$$

$$\Longrightarrow \int e^{ax}\sin(bx)dx = \frac{1}{a}\sin(bx)e^{ax} - \frac{b}{a}\int e^{ax}\cos(bx)\,dx$$

$$u = \cos(bx) \qquad du = -b\sin(bx)\,dx$$

$$dv = e^{ax}\,dx \qquad v = \frac{1}{a}e^{ax}$$

$$\Longrightarrow \underline{\int e^{ax}\sin(bx)dx} = \frac{1}{a}\sin(bx)e^{ax} - \frac{b}{a}\left[\frac{1}{a}\cos(bx)e^{ax} + \frac{b}{a}\underline{\int e^{ax}\sin(bx)\,dx}\right]$$

$$\left(1+\frac{b^2}{a^2}\right)\int e^{ax}\sin(bx)dx = \frac{1}{a}\sin(bx)e^{ax} - \frac{b}{a^2}\cos(bx)e^{ax}$$

$$\frac{a^2+b^2}{a^2}\int e^{ax}\sin(bx)dx = \frac{1}{a}\sin(bx)e^{ax} - \frac{b}{a^2}\cos(bx)e^{ax}$$

$$\begin{aligned}\implies \int e^{ax}\sin(bx)dx &= \frac{a}{a^2+b^2}\sin(bx)e^{ax} - \frac{b}{a^2+b^2}\cos(bx)e^{ax}\\ &= e^{ax}\frac{a\sin(bx)-b\cos(bx)}{a^2+b^2}+c\end{aligned}$$

**Problem 3: State whether the following differential equations are linear or non-linear. Give the order of each equation.**

(a) $(1 - x)y'' - 4xy' + 5y = \cos x$
linear (in y): $2^{nd}$ order

(b) $(y^2 - 1)dx + xdy = 0$
non linear in y: $1^{st}$ order
linear in x: $1^{st}$ order

(c) $t^5y^{(4)} - t^3y'' + 6y = 0$

**Problem 4c :Verify that the indicated functions are solutions to the given differential equations and state whether they are implicit or explicit solutions. Assume an appropriate interval $I$ of definition**

$xy' + xy^2 - y = 0; \qquad y = \frac{2x}{x^2+c}$

Explicit solution

$$\begin{aligned} y &= \frac{2x}{x^2 + c} \\ y^2 &= \frac{4x^2}{(x^2 + c)^2} \\ y' &= \frac{2(x^2 + c) - 4x^2}{(x^2 + c)^2} \\ &= \frac{-2x^2 + 2c}{(x^2 + c)^2} \end{aligned}$$

Using these in the above equation gives:

$$\frac{-2x^3 + 2cx}{(x^2 + c)^2} + \frac{4x^3}{(x^2 + c)^2} - \frac{2x}{x^2 + c}$$

$$= \frac{-2x^3 + 2cx + 4x^3 - 2x(x^2 + c)}{(x^2 + c)^2}$$

$$= \frac{-2x^3 + 2cx + 4x^3 - 2x^3 - 2cx}{(x^2 + c)^2} = 0$$

**Problem 5c: Use the Separation of Variables technique to solve the following first order differential equations.**

$(1+x^2)\frac{dy}{dx}+y^2=0; \quad y(0)=\frac{2}{\pi}$

$$
\begin{aligned}
(1+x^2)\frac{dy}{dx}+y^2 &= 0\\
\frac{1}{y^2}\frac{dy}{dx} &= -\frac{1}{1+x^2}\\
\int \frac{1}{y^2}dy &= -\int \frac{1}{1+x^2}dx
\end{aligned}
$$

Let $x = \tan(u)$; $\quad dx = \sec^2(u)du$

$$\begin{aligned}
-\frac{1}{y} &= -\int \frac{\sec^2(u)}{1+\tan^2(u)}\,du + c \\
\frac{1}{y} &= \int \frac{\sec^2(u)}{\sec^2(u)}\,du - c \\
\frac{1}{y} &= \int 1\,du - c \\
\frac{1}{y} &= u - c \\
\frac{1}{y} &= \arctan(x) - c \\
y(x) &= \frac{1}{\arctan(x) - c}
\end{aligned}$$

Imposing initial conditions: $y(0) = \frac{2}{\pi}$

$$y(0) = \frac{1}{-c} = \frac{2}{\pi} \implies c = -\frac{\pi}{2}$$

$$y(x) = \frac{1}{\arctan(x) + \pi/2}$$