# Engineering Mathematics 1 (Fall 2021)

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**Students should be able to (after learning)**

- Add, subtract and multiply complex numbers
- Convert complex numbers between Cartesian and polar forms
- Differentiate all commonly occurring functions including polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of a derivative, namely the derivative as a tangent and the derivative as a rate of change
- Integrate certain standard functions, constructed from polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of integration, namely the integral as the inverse of the derivative and the integral as the area under a curve
- Apply Taylor series to numerically approximate functions
- Apply Simpson's rule to numerically evaluate integrals
- Solve simple first and second order ordinary differential equations
- Apply and select the appropriate mathematical techniques to solve a variety of associated engineering problems

## Lecture 8: Series-Part 3

### 5. Convergent/divergent series, Test for convergency:

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

If $\sum_{n=1}^{\infty} u_n$ is a definite value, then $\sum_{n=1}^{\infty} u_n$ is convergent.

If $\sum_{n=1}^{\infty} u_n$ is NOT a definite value, then $\sum_{n=1}^{\infty} u_n$ is divergent.

NOT definite numbers $\longrightarrow$ divergent.

Definite numbers $\longrightarrow$ convergent.

(i) If $\lim_{n\to\infty} u_n = 0$, then $\sum_{n=1}^{\infty} u_n$ may be convergent. General term $u_n$

If $\lim_{n\to\infty} u_n \neq 0$, then $\sum_{n=1}^{\infty} u_n$ is certainly divergent.

Ex1:

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$$

Sol: $\because u_n = \frac{1}{3^{n-1}}$ $\therefore \lim_{n\to\infty} u_n = 0$ $\therefore \sum_{n=1}^{\infty} \frac{1}{3^{n-1}}$ may be conv./div.

Let partial sum $S_n = \sum_{k=1}^{n} \frac{1}{3^{k-1}} = \frac{1\times(1-\frac{1}{3^n})}{1-\frac{1}{3}} = \frac{3}{2}(1-\frac{1}{3^n})$

$\sum_{n=1}^{\infty} \frac{1}{3^{n-1}} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{2}(1-\frac{1}{3^n}) = \frac{3}{2}$. conv.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Sol: $\because u_n = \frac{1}{n}$ $\therefore \lim_{n\to\infty} u_n = 0$ $\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ may be conv./div.

$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}} + \cdots$

$> 1 + \frac{1}{2} + 2\times\frac{1}{4} + 4\times\frac{1}{8} + 8\times\frac{1}{16} + \cdots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ is div.

$1+3+9+27+\cdots$

Sol: $\because u_n = 3^{n-1}$ $\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} 3^{n-1} = +\infty \neq 0$

$\therefore \sum_{n=1}^{\infty} 3^{n-1}$ is div.

(ii) Comparison test-Useful standard series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots+\frac{1}{n^p}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^p}$$

$p>1, \sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent;

$p\leqslant 1, \sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent.

Ex1:

$$1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\frac{1}{5^5}+\cdots$$

Sol: $\because \frac{1}{3^2} > \frac{1}{3^3}, \quad \frac{1}{4^2} > \frac{1}{4^4}, \quad \frac{1}{5^2} > \frac{1}{5^5}, \cdots$

$\therefore 1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\frac{1}{5^5}+\cdots < 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n^2}$

$\therefore$ by comparison test, $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is conv. $\therefore 1+\frac{1}{2^2}+\frac{1}{3^3}+\cdots$ is conv.

Ex2:

$$1+\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots$$

Sol: $\because \frac{1}{1\times 2} < \frac{1}{1\times 1}, \quad \frac{1}{2\times 3} < \frac{1}{2\times 2}, \quad \frac{1}{3\times 4} < \frac{1}{3\times 3}, \cdots$

$\therefore 1+\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots < 1+\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$

$= 1+\sum_{n=1}^{\infty}\frac{1}{n^2}$

$\because \sum_{n=1}^{\infty}\frac{1}{n^2}$ is conv., $\therefore 1+\sum_{n=1}^{\infty}\frac{1}{n^2}$ is conv.

$\therefore 1+\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots$ is conv.

(iii) D'Alembert's ratio test for positive terms

If $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$, then $\sum\limits_{n=1}^{\infty} u_n$ is convergent.

If $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$, then $\sum\limits_{n=1}^{\infty} u_n$ is divergent.

If $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, then $\sum\limits_{n=1}^{\infty} u_n$ is NOT confirmed.

Ex1:

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$$

Sol: $\because u_n = \frac{n}{n+1}$ $\therefore u_{n+1} = \frac{n+1}{n+2}$ $\therefore \frac{u_{n+1}}{u_n} = \frac{n+1}{n+2} \times \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n}$

$\therefore \lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = \lim\limits_{n\to\infty} \frac{1+2/n+1/n^2}{1+2/n} = 1$ $\therefore \sum\limits_{n=1}^{\infty} \frac{n}{n+1}$ is NOT conf.

by general term, $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$ $\therefore \sum\limits_{n=1}^{\infty} \frac{n}{n+1}$ is div.

Ex2:

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{4+n}$$

Sol: $\because u_n = \frac{2^{n-1}}{4+n}$, $\therefore u_{n+1} = \frac{2^n}{5+n}$ $\therefore \frac{u_{n+1}}{u_n} = \frac{2^n}{5+n} \times \frac{4+n}{2^{n-1}} = \frac{2(4+n)}{5+n}$

$\therefore \lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{8+2n}{5+n} = \lim\limits_{n\to\infty} \frac{8/n+2}{5/n+1} = 2 > 1$

$\therefore \sum\limits_{n=1}^{\infty} \frac{2^{n-1}}{4+n}$ is div.

Ex3:

$$\sum_{n=1}^{\infty} \frac{x^n}{(n+1)7^n}$$ find a positive $x$ such that the series is conv.

Sol: $\because u_n = \frac{x^n}{(n+1)7^n}$ $\therefore u_{n+1} = \frac{x^{n+1}}{(n+2)7^{n+1}}$ $\therefore \frac{u_{n+1}}{u_n} = \frac{x(n+1)}{7(n+2)}$

$\therefore \lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{x(n+1)}{7(n+2)} = \lim\limits_{n\to\infty} \frac{x+x/n}{7+14/n} = \frac{x}{7} < 1$



$\therefore 0 < x < 7$ is the required.

For general series

If $\sum_{n=1}^{\infty} |u_n|$ is convergent, then $\sum_{n=1}^{\infty} u_n$ is absolutely convergent.

If $\sum_{n=1}^{\infty} |u_n|$ is divergent, but $\sum_{n=1}^{\infty} u_n$ is convergent, $\sum_{n=1}^{\infty} u_n$ is conditionally convergent.

Ex1: Find the range of values for $x$ such that

$$\frac{x}{2\times 5} - \frac{x^2}{3\times 5^2} + \frac{x^3}{4\times 5^3} - \frac{x^4}{5\times 5^4} + \frac{x^5}{6\times 5^5} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{(n+1)5^n}$$

is absolutely convergent.

Sol: $\because u_n = (-1)^{n+1} \frac{x^n}{(n+1)5^n}$ $\therefore u_{n+1} = (-1)^{n+2} \frac{x^{n+1}}{(n+2)5^{n+1}}$

$$\therefore \left|\frac{u_{n+1}}{u_n}\right| = \frac{|x|(n+1)}{5(n+2)} \quad \therefore \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{|x|(n+1)}{5(n+2)}$$

$$= \lim_{n\to\infty} \frac{|x|(1+\frac{1}{n})}{5(1+\frac{2}{n})} = \frac{|x|}{5} < 1$$

$\therefore -5 < x < 5$ is the solution.

Analysis

$$\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{x^n}{(n+1)5^n}\right| = \sum_{n=1}^{\infty} \frac{|x|^n}{(n+1)5^n} \text{ is conv.}$$