LAPLACE TRANSFORM

Outline:

- Definition
- Inverse transform
- Transform of derivatives
- Translation theorems: s-axis, t-axis
- Additional properties:
  derivatives of transforms, transforms of integrals and periodic functions
- Dirac delta function

LAPLACE TRANSFORM

In linear mathematical models such as series electric circuit, the input or driving function, like the voltage impressed on a circuit, could be piecewise continuous and periodic.

The Laplace transform is an invaluable tool in simplifying the solutions of this type of problems.

Definition of the Laplace transform

We can regard the operations of differentiation, indefinite integration and definite integration as transforms which possess the linear property:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[\alpha f(x)+\beta g(x)] &= \alpha f'(x)+\beta g'(x)\\ \int[\alpha f(x)+\beta g(x)]dx &= \alpha\int f(x)dx+\beta\int g(x)dx\\ \int_a^b[\alpha f(x)+\beta g(x)]dx &= \alpha\int_a^b f(x)dx+\beta\int_a^b g(x)dx\end{aligned}$$

We will be particularly interested in **integral transforms** $\int_a^b K(s,t)f(t)dt$, which transform a function $f(t)$ into a function of a variable s, where the interval of integration is the unbounded interval $[0,\infty)$.

Definition: Laplace transform

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \tag{1}$$

is said to be the **Laplace transform** of f, provided the integral converges.

Notation:

$$\mathcal{L}\{f(t)\} = F(s), \qquad \mathcal{L}\{g(t)\} = G(s), \qquad \mathcal{L}\{y(t)\} = Y(s), \qquad \mathcal{L}\{H(t)\} = h(s)$$

Example 1: Evaluate $\mathcal{L}\{1\}$:

$$\begin{aligned}\mathcal{L}\{1\} &= \int_0^\infty e^{-st}(1)dt = \lim_{b\to\infty}\int_0^b e^{-st}dt \\ &= \lim_{b\to\infty} \left.\frac{-e^{-st}}{s}\right|_0^b = \lim_{b\to\infty} \frac{-e^{-sb}+1}{s} = \frac{1}{s}\end{aligned}$$

provided $s > 0$. In that case the exponent $-sb$ is negative and the integral converges as $e^{-sb} \to 0$ as $b \to \infty$. The integral diverges for $s < 0$.

Notation: we will write $|_0^\infty$ as a shorthand notation for $\lim_{b\to\infty}()|_0^b$.

Example 2: Evaluate $\mathcal{L}\{t\}$:

$$\mathcal{L}\{t\} \;=\; \int_0^\infty e^{-st} t\, dt = \left.\frac{-te^{-st}}{s}\right|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

where we have used integration by parts.

Example 3: Evaluate $\mathcal{L}\left\{e^{-3t}\right\}$:

$$\mathcal{L}\left\{e^{-3t}\right\} \;=\; \int_0^\infty e^{-st} e^{-3t} dt = \int_0^\infty e^{-(s+3)t} dt = \left.\frac{-e^{-(s+3)t}}{s+3}\right|_0^\infty = \frac{1}{s+3}, \qquad s > -3$$

The result follows from the fact that $\lim_{t\to\infty} e^{-(s+3)t} = 0$ for $s + 3 > 0$ or $s > -3$.

Example 4: Evaluate $\mathcal{L}\{\sin 2t\}$:

$$\begin{aligned}
\mathcal{L}\{\sin 2t\} &= \int_0^\infty e^{-st}\sin 2t\,dt = \left.\frac{-e^{-st}\sin 2t}{s}\right|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt \\
&= \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt, \qquad s>0 \\
&= \frac{2}{s}\left[\left.\frac{-e^{-st}\cos 2t}{s}\right|_0^\infty - \frac{2}{s}\int_0^\infty e^{-st}\sin 2t\,dt\right] \\
&= \frac{2}{s^2} - \frac{4}{s^2}\mathcal{L}\{\sin 2t\}
\end{aligned}$$

At this point we have an equation for $\mathcal{L}\{\sin 2t\}$ whose solution is

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}, \qquad s>0.$$

The Laplace transform is linear:

$$\int_0^\infty e^{-st}[\alpha f(x) + \beta g(x)]\, dt = \alpha \int_0^\infty e^{-st} f(x)\, dt + \beta \int_0^\infty e^{-st} g(x)\, dt$$

whenever both integrals converge for $s > c$. Hence it follows that

$$\mathcal{L}\{\alpha f(x) + \beta g(x)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s)$$

<u>Example:</u>

$$\mathcal{L}\{1 + 5t\} = \mathcal{L}\{1\} + 5\mathcal{L}\{t\} = \frac{1}{s} + \frac{5}{s^2}$$

$$\mathcal{L}\left\{4e^{-3t} - 10\sin 2t\right\} = 4\mathcal{L}\left\{e^{-3t}\right\} + 10\mathcal{L}\{\sin 2t\} = \frac{4}{s+3} - \frac{20}{s^2+4}$$

Transforms of some basic functions

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad n = 1,\ 2,\ 3, \ldots$$

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$

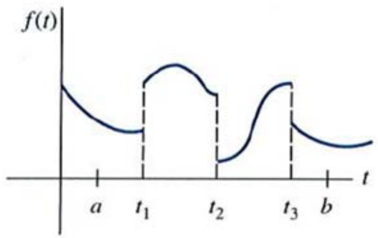
$$\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$$

$$\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$$

Sufficient conditions for existence of $\mathcal{L}\{f(t)\}$:

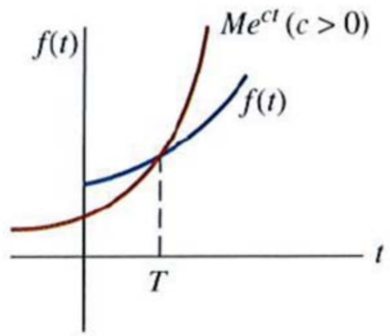
- f has to be piecewise continuous on $[0, \infty)$;

  A function is piecewise continuous on $[0, \infty)$ if, in any interval defined by $0 \le a \le t \le b$, there are at most a finite number of points t_k, $k = 1, 2, 3, ..., n$ $(t_{k-1} < t_k)$, at which f has finite discontinuities and is continuous on each open interval defined by $t_{k-1} < t < t_k$.



- f has to be of exponential order for $t > T$.

  A function f is said to be of **exponential order c** if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \le Me^{ct}$ for all $t > T$.

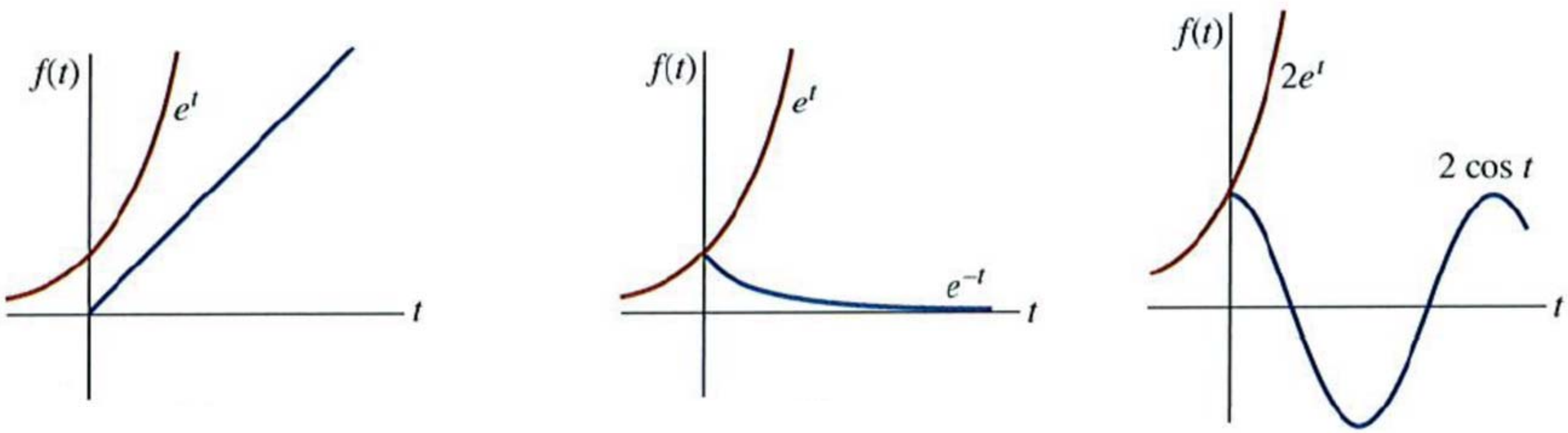


If f is an *increasing* function, then the condition $|f(t)| \leq Me^{ct}$, $t > T$ simply states that the graph of f on the interval (T, ∞) does not grow faster than the graph of the exponential function Me^{ct}.

<u>Examples:</u>

$f(t) = t$, $f(t) = e^{-t}$, and $f(t) = 2\cos t$ are all of exponential order $c = 1$ for $t > 0$:

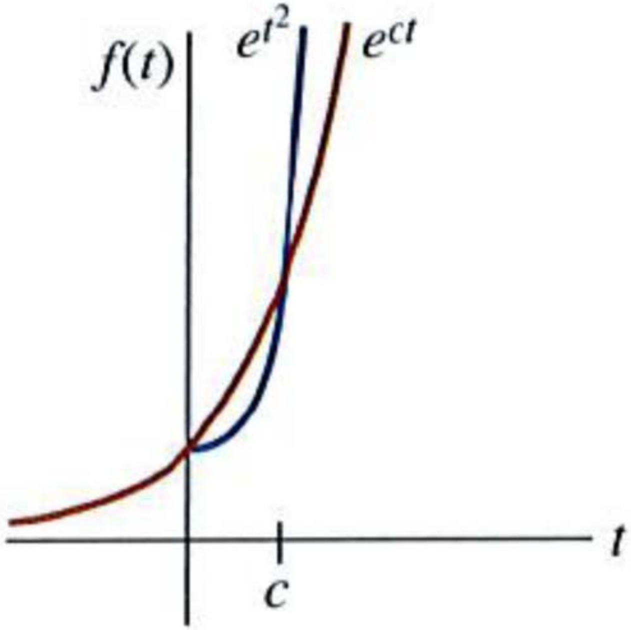
$$|t| \leq e^t, \qquad \left|e^{-t}\right| \leq e^t, \qquad |2\cos t| \leq 2e^t$$



A positive integral power of t, $f(t) = t^n$ is always of exponential order since, for $c > 0$

$$\left|t^n\right| \leq Me^{ct}, \qquad t > 0$$

A function $f = e^{t^2}$ is not of exponential order as its graph grows faster than any positive linear power of e for $t > 0$ and $c > 0$.



Theorem: Sufficient conditions for existence

If $f(t)$ is a piecewise continuous on the interval $[0,\infty)$ and of exponential order c, then $\mathcal{L}\{f(t)\}$ exists for $s > c$.

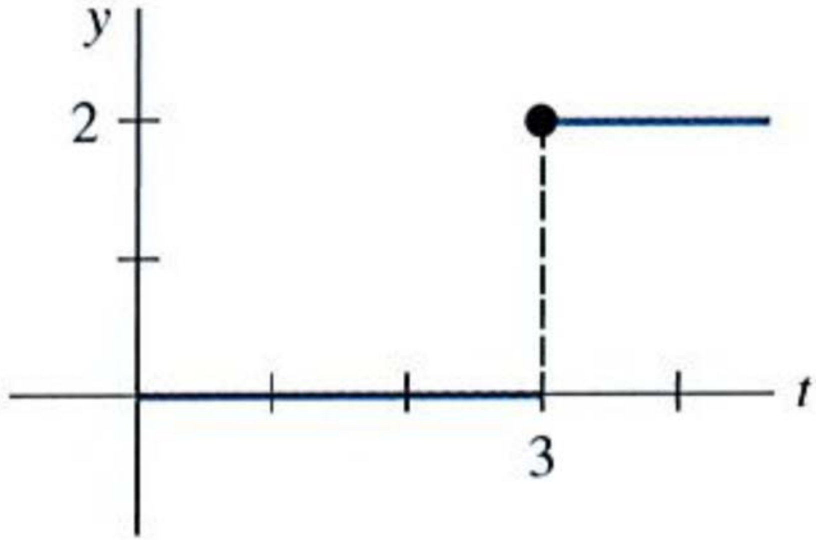
The Proof: See D.G. Zill et al., Advanced Engineering Mathematics, 4th Ed., p. 200.

Example: Evaluate $\mathcal{L}\{f(t)\}$ of the piecewise continuous function defined as

$$f(t) = \begin{cases} 0, & 0 \leq t < 3 \\ 2, & t \geq 3 \end{cases}$$

Since f is defined in two pieces, $\mathcal{L}\{f(t)\}$is expressed as the sum of two integrals:

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\, dt = \int_0^3 e^{-st}(0)\, dt + \int_3^\infty e^{-st}(2)\, dt \\ &= -\frac{2e^{-st}}{s}\bigg|_3^\infty \\ &= \frac{2e^{-3s}}{s}, \quad s > 0 \end{aligned}$$



Inverse transform and transforms of derivatives

Inverse transforms

If $F(s)$ is the Laplace transform of a function $f(t)$, i.e. $\mathcal{L}\{f(t)\} = F(s)$, we then say $f(t)$ is the **inverse Laplace transform** of $F(s)$ and write $\mathcal{L}^{-1}\{F(s)\}$.

Examples:

$$1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}, \qquad t = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}, \qquad e^{-3t} = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

Some inverse transforms

$$
\begin{aligned}
1 &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} \\
t^n &= \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} \\
e^{at} &= \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} \\
\sin kt &= \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} \\
\cos kt &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} \\
\sinh kt &= \mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} \\
\cosh kt &= \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}
\end{aligned}
$$

Example 1:

Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}\,t^4$$

Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin\,\sqrt{7}t$$

The inverse Laplace transform is linear:

$$\mathcal{L}^{-1}\left\{\alpha F(s)+\beta G(s)\right\}=\alpha\mathcal{L}^{-1}\left\{F(s)\right\}+\beta\mathcal{L}^{-1}\left\{G(s)\right\}$$

where α and β are constants. This extends to any linear combination.

<u>Example 2:</u> Termwise division and linearity

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} &= \mathcal{L}^{-1}\left\{\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right\}\\
&= -2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{6}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\\
&= -2\cos 2t+3\sin 2t
\end{aligned}$$

Partial fractions

Partial fractions play an important role in finding the inverse Laplace transforms.

Example 3: Partial fractions and linearity

Evaluate $\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$:

there exist unique constants A, B, C such that

$$\begin{aligned}\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} &= \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}\\ &= \frac{A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)}{(s-1)(s-2)(s+4)}\end{aligned}$$

Comparing the coefficients at different powers in the numerators

$$s^2 + 6s + 9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)$$

yields the system of three equations for A, B and C. Alternatively, we can insert the zeros of the common denominator $s = 1$, $s = 2$ and $s = -4$ and we get $A = -16/5$, $B = 25/6$ and $1/30$. The final result is then

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} &= -\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} \\
&= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}
\end{aligned}$$

Transform of derivatives

Our ultimate goal is to use the Laplace transform to solve differential equations. But how do we evaluate derivatives?

$$\begin{aligned}
\mathcal{L}\left\{f'(t)\right\} &= \int_0^\infty e^{-st} f'(t)\, dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt \\
&= -f(0) + s\mathcal{L}\{f(t)\}
\end{aligned}$$

$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0) \tag{2}$$

Here we have assumed that $e^{-st} f(t) \to 0$ as $t \to \infty$.

Similarly

$$\begin{aligned}
\mathcal{L}\left\{f''(t)\right\} &= \int_0^\infty e^{-st} f''(t)\, dt = e^{-st} f'(t)\Big|_0^\infty + s \int_0^\infty e^{-st} f'(t)\, dt \\
&= -f'(0) + s\mathcal{L}\left\{f'(t)\right\} \\
&= s\left[sF(s) - f(0)\right] - f'(0)
\end{aligned}$$

$$\mathcal{L}\left\{f''(t)\right\} = s^2 F(s) - sf(0) - f'(0) \tag{3}$$

And in a similar manner

$$\mathcal{L}\left\{f'''(t)\right\} = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0) \tag{4}$$

Theorem: Transform of a derivative

If f, f', ..., $f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$ then

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - ... - f^{(n-1)}(0)$$

where $F(s) = \mathcal{L}\{f(t)\}$.

Solving linear ODEs

The Laplace transform is ideally suited for solving linear initial-value problems in which the differential equation

$$\begin{aligned} a_n \frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + ... + a_0 y &= g(t) \\ y(0) = y_0, \quad y'(0) = y_1, ..., y^{(n-1)}(0) &= y_{n-1} \end{aligned}$$

has *constant coefficients*, a_i, $i = 0, 1, ..., n$. Also, y_0, y_1, ... ,y_{n-1} are constants.

By linearity, the Laplace transform of this linear combination is a linear combination of Laplace transforms

$$a_n \mathcal{L}\left\{\frac{\mathrm{d}^n y}{\mathrm{d}t^n}\right\} + a_{n-1} \mathcal{L}\left\{\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}}\right\} + ... + a_0 \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

From the Theorem above, this becomes

$$a_n \left[s^n Y(s) - s^{n-1} y(0) - ... - y^{(n-1)}(0) \right]$$
$$+ a_{n-1} \left[s^{n-1} Y(s) - s^{n-2} y(0) - ... - y^{(n-2)}(0) \right] + ... + a_0 Y(s) = G(s) \qquad (5)$$

where $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{g(t)\} = G(s)$.

The Laplace transform of a linear differential equation with constant coefficients becomes an algebraic equation in $Y(s)$.

Solving the equation

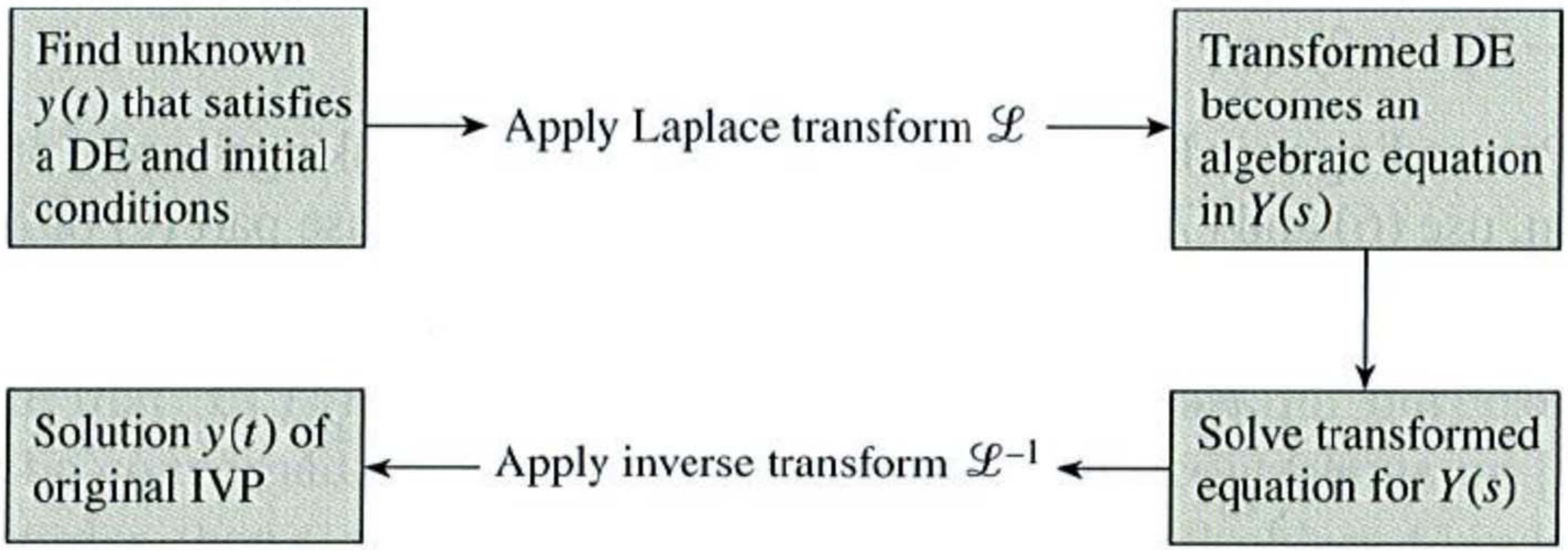
$$\begin{aligned}
&a_n\left[s^nY(s) - s^{n-1}y(0) - ... - y^{(n-1)}(0)\right] \\
&+a_{n-1}\left[s^{n-1}Y(s) - s^{n-2}y(0) - ... - y^{(n-2)}(0)\right] + ... + a_0Y(s) = G(s)
\end{aligned}$$

for $Y(s)$ we first get $P(s)Y(s) = Q(s) + G(s)$, and then we write

$$Y(s) \quad = \quad \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where $P(s) = a_ns^n + a_{n-1}s^{n-1} + ... + a_0$, $Q(s)$ is a polynomial in s of degree less than or equal to $n - 1$ consisting of the products of the coefficients a_i, $i = 1, ..., n$, and the prescribed initial conditions y_0, y_1, ... , y_{n-1}, and $G(s)$ is the Laplace transform of $g(t)$.

Typically, we put both terms over the least common denominator and then decompose the expression into the partial fractions. Finally the solution $y(t)$ of the original initial value problem is $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.


Find unknown $y(t)$ that satisfies a DE and initial conditions
Apply Laplace transform $\mathcal{L}$
Transformed DE becomes an algebraic equation in $Y(s)$
Solve transformed equation for $Y(s)$
Apply inverse transform $\mathcal{L}^{-1}$
Solution $y(t)$ of original IVP

Example 1:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + 3y = 13\sin 2t, \qquad y(0) = 6$$

Solution:

$$\mathcal{L}\left\{\frac{\mathrm{d}y}{\mathrm{d}t}\right\} + 3\mathcal{L}\{y\} = 13\mathcal{L}\{\sin 2t\}$$

where $\mathcal{L}\left\{\frac{\mathrm{d}y}{\mathrm{d}t}\right\} = sY(s) - y(0) = sY(s) - 6$ and $\mathcal{L}\{\sin 2t\} = 2/(s^2 + 4)$, so we get the algebraic equation

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2 + 4} \quad \text{or} \quad (s + 3)Y(s) = 6 + \frac{26}{s^2 + 4}$$

Solving the equation for $Y(s)$ we get

$$Y(s) = \frac{6}{s + 3} + \frac{26}{(s + 3)(s^2 + 4)} = \frac{6s^2 + 50}{(s + 3)(s^2 + 4)}$$

Since the quadratic polynomial $s^2 + 4$ does not factor using real numbers, the partial fraction decomposition reads as

$$\frac{6s^2 + 50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs + C}{s^2 + 4}$$

Setting $s = -3$, we get $A = 8$, and by equating the coefficients of s^2 and s we get $6 = A + B$ and $0 = 3B + C$. From these we calculate $B = -2$ and $C = 6$. Thus

$$Y(s) = \frac{6s^2 + 50}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s + 6}{s^2 + 4}$$

We use now the termwise division to rewrite the last expression on r.h.s.

$$y(t) = 8\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

The final solution is then $y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$.

Example 2: Solving second-order IVP

$$y'' - 3y' + 2y = e^{-4t}, \qquad y(0) = 1, \quad y'(0) \;=\; 5.$$

Little bit more of theory

Not every arbitrary function of s is a Laplace transform of a piecewise continuous function of exponential order

Theorem: Behavior of $F(s)$ as $s \to \infty$

If f is piecewise continuous on $[0, \infty)$ and of exponential order, then $\lim_{s\to\infty} \mathcal{L}\{f(t)\} = 0$.

<u>Example:</u> $F_1(s) = 1$ and $F_2(s) = s/(s+1)$ are not the Laplace transforms of piecewise continuous functions of exponential order. You should not conclude that they are not Laplace transform. There are other types of functions.

Remarks:

(i) The inverse Laplace transform may not be unique, i.e. $\mathcal{L}\{f_1(t)\} = \mathcal{L}\{f_2(t)\}$ and yet $f_1(t) \neq f_2(t)$. However

- if f_1 and f_2 are piecewise continuous on $[0, \infty)$ and of exponential order, then f_1 and f_2 are *essentially* the same;

- if f_1 and f_2 are continuous on $[0, \infty)$ and $\mathcal{L}\{f_1(t)\} = \mathcal{L}\{f_2(t)\}$, then $f_1 = f_2$ on the interval.

(ii) **Cover-up method**

for determining the coefficients in a partial fraction decomposition

<u>Example:</u>

$$\frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4} \tag{6}$$

First multiply by $s - 1$, simplify and then set $s = 1$ (the zero of $s - 1$):

$$\left.\frac{s^2 + 6s + 9}{(s-2)(s+4)}\right|_{s=1} = A \qquad \Rightarrow \qquad A = -\frac{16}{5}$$

Now to obtain B, multiply the expression (6) by $s - 2$ and then set $s = 2$, and similarly for C

$$\left.\frac{s^2 + 6s + 9}{s-1)(s+4)}\right|_{s=2} = B \qquad \Rightarrow \qquad B = \frac{25}{6}, \quad \left.\frac{s^2 + 6s + 9}{(s-1)(s-2)}\right|_{s=-4} = C \qquad \Rightarrow \qquad C = \frac{1}{30},$$

(iii) Dynamical systems

The Laplace transform is well adapted to *linear* dynamical systems. In $Y(s) = Q(s)/P(s) + G(s)/P(s)$, the polynomial $P(s) = a_n s^n + a_{n-1}s^{n-1} + ... + a_0$ is the total coefficient of $Y(s)$ in

$$\begin{aligned} &a_n\left[s^nY(s) - s^{n-1}y(0) - ... - y^{(n-1)}(0)\right] \\ &+a_{n-1}\left[s^{n-1}Y(s) - s^{n-2}y(0) - ... - y^{(n-2)}(0)\right] + ... + a_0Y(s) = G(s) \end{aligned}$$

and is simply the l.h.s. of the DE with the derivatives d^ky/dt^k replaced by powers s^k, $k = 0, 1, ..., n$.

Introducing the **transfer function** $W(s) = 1/P(s)$ allows us to rewrite the expression for $Y(s)$ as

$$Y(s) = W(s)Q(s) + W(s)G(s)$$

where the first term on r.h.s. represents the effects on the response that are due to the initial conditions, and the second term represents the effects on the response that are due to the input $g(t)$.

Hence the response $y(t)$ of the system is a superposition of two responses

$$y(t) = \mathcal{L}^{-1}\{W(s)Q(s)\} + \mathcal{L}^{-1}\{W(s)G(s)\} = y_0(t) + y_1(t)$$

If $g(t) = 0$, then the solution of the problem is $y_0(t) = \mathcal{L}^{-1}\{W(s)Q(s)\}$ and it is called the **zero-input response** of the systems. If the initial state of the system is the zero state (all the initial conditions are zero), we get for nonzero $g(t)$ the **zero-state response** of the system. Both $y_0(t)$ and $y_1(t)$ are particular solutions.