Tutorial Sheet 4 - Solutions

Q1 (i) 2nd order as there are 2 poles (or highest power of *s* in denominator is 2)

(ii) 3rd order as there are 3 poles (or highest power of *s* in denominator is 3)

(iii) 1st order as the degree of the highest (output) derivative is 1

(iv) 2nd order as the degree of the highest (output) derivative is 2

Q2 (i) $L\dfrac{di}{dt} + Ri = v_i$

Solution for $i(t)$ involves two components, i.e.: $i(t) = i_n(t) + i_f(t)$

Finding $i_n(t)$, the zero-input response:

Set $v_i = 0$ $\Rightarrow L\dfrac{di_n}{dt} + Ri_n = 0$

Use *separation of variables* method:

$$L\frac{di_n}{dt} = -Ri_n \Rightarrow \frac{di_n}{i_n} = -\frac{R}{L}dt$$

$$\therefore \int \frac{di_n}{i_n} = -\frac{R}{L}\int dt \qquad \Rightarrow \ln(i_n) = -\frac{R}{L}t + K$$

$$\Rightarrow i_n = e^{\frac{-R}{L}t+K} = e^K e^{\frac{-R}{L}t} = Ae^{\frac{-R}{L}t}$$

Finding $i_f(t)$, the steady-state response:

Set all derivatives to zero: $\dfrac{di}{dt} = 0 \Rightarrow L(0) + Ri_f = v_i \Rightarrow i_f = \dfrac{v_i}{R}$

Complete solution:

$$i(t) = i_n(t) + i_f(t) = Ae^{\frac{-R}{L}t} + \frac{v_i}{R}$$

Since $i(0) = 0$: $0 = Ae^{\frac{-R}{L}(0)} + \dfrac{v_i}{R} \Rightarrow 0 = A(1) + \dfrac{v_i}{R} \Rightarrow A = -\dfrac{v_i}{R}$

Therefore: $i(t) = -\dfrac{v_i}{R}e^{\frac{-R}{L}t} + \dfrac{v_i}{R}$ or $i(t) = \dfrac{v_i}{R}\left(1 - e^{-\frac{R}{L}t}\right)$

Q2 (ii) $\dfrac{I(s)}{V_i(s)} = \dfrac{1}{sL+R}$

The output is given by: $I(s) = \dfrac{1}{sL+R}V_i(s)$

The input is a constant value, v_i, hence: $V_i(s) = \dfrac{v_i}{s}$

Hence: $I(s) = v_i\left(\dfrac{1}{s(sL+R)}\right)$

Using the partial fraction method: $\dfrac{1}{s(sL+R)} \equiv \dfrac{A}{s} + \dfrac{B}{sL+R} = \dfrac{A(sL+R)+Bs}{s(sL+R)}$

Equating the coefficients of s gives: $A = \frac{1}{R}$ and $AL + B = 0 \Rightarrow B = -AL = -\frac{L}{R}$

Hence: $I(s) = v_i\left(\dfrac{1}{s(sL+R)}\right) = v_i\left(\dfrac{\frac{1}{R}}{s} - \dfrac{\frac{L}{R}}{sL+R}\right) = \dfrac{v_i}{R}\left(\dfrac{1}{s} - \dfrac{1}{s+\frac{R}{L}}\right)$

Obtaining the Inverse Laplace Transforms: $i(t) = \dfrac{v_i}{R}\left(1 - e^{-\frac{R}{L}t}\right)$

This, as expected, is the same as the solution in part (i).

Q3 $\dfrac{Y(s)}{U(s)} = \dfrac{1}{s^2+6s+8} \quad \Rightarrow \quad Y(s) = \dfrac{1}{s^2+6s+8}U(s)$

$u(t) = 1 \;\Rightarrow\; U(s) = \dfrac{1}{s}$

Hence: $Y(s) = \dfrac{1}{s(s^2+6s+8)} = \dfrac{1}{s(s+2)(s+4)} \equiv \dfrac{A}{s} + \dfrac{B}{s+2} + \dfrac{C}{s+4}$

$$= \frac{A(s+2)(s+4) + Bs(s+4) + Cs(s+2)}{s(s+2)(s+4)}$$

Setting $s = 0$: $1 = A(2)(4) \;\Rightarrow A = \dfrac{1}{8}$

Setting $s = -2$: $$1 = B(-2)(2) \Rightarrow B = -\frac{1}{4} = -\frac{2}{8}$$

Setting $s = -4$: $$1 = C(-4)(-2) \Rightarrow C = \frac{1}{8}$$

Hence: $$Y(s) = \frac{1}{8}\left(\frac{1}{s} - \frac{2}{s+2} + \frac{1}{s+4}\right)$$

Finally:
$$y(t) = \frac{1}{8}\left(1 - 2e^{-2t} + e^{-4t}\right)$$

Q4 $$\frac{d^2x(t)}{dt} - 4x(t) = 4 \to s^2X(s) - 4X(s) = \frac{4}{s} \Rightarrow X(s) = \frac{4}{s(s^2-4)}$$

$$\frac{4}{s(s^2-4)} = \frac{4}{s(s-2)(s+2)} \equiv \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+2} = \frac{A(s-2)(s+2) + Bs(s+2) + Cs(s-2)}{s(s-2)(s+2)}$$

Setting $s = 0$: $$4 = A(-2)(2) \Rightarrow A = -1$$

Setting $s = 2$: $$4 = B(2)(4) \Rightarrow B = \frac{1}{2}$$

Setting $s = -2$: $$4 = C(-2)(-4) \Rightarrow C = \frac{1}{2}$$

Hence: $$X(s) = \frac{1}{2}\left(-\frac{2}{s} + \frac{1}{s-2} + \frac{1}{s+2}\right)$$

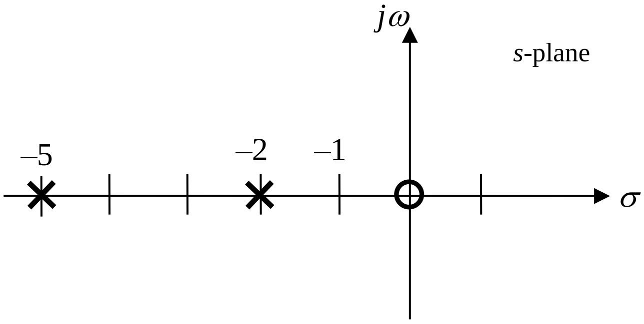
Finally:
$$x(t) = \frac{1}{2}\left(-2 + e^{2t} + e^{-2t}\right)$$

Q5 Refer to Notes

Q6 (i) $$\frac{s}{(s+2)(s+5)}$$

Zero: $s = 0$ **Poles:** $(s + 2)(s + 5) = 0 \Rightarrow s = -2, s = -5$

Hence:

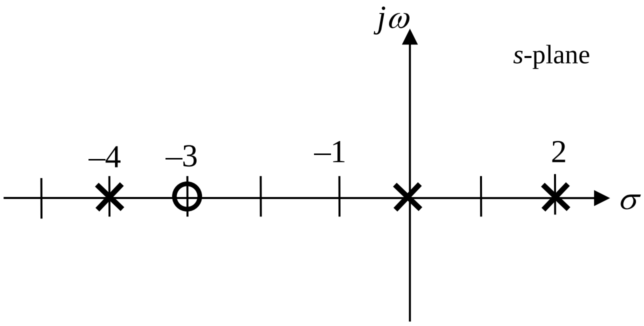


System is **stable** as both poles are on the LHS of the imaginary axis.

(ii) $\dfrac{s+3}{s(s^2+2s-8)}$

Zero: $s + 3 = 0 \Rightarrow s = -3$ **Poles:** $s\,(s-2)\,(s+4) = 0 \Rightarrow s = 0,\ s = 2,\ s = -4$

Hence:

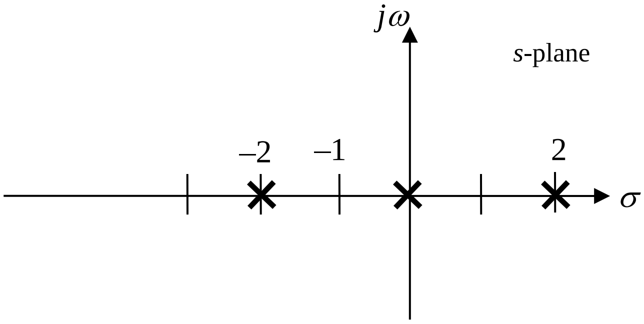


System is **unstable** as one of the poles is on the RHS of the imaginary axis.

(iii) $\dfrac{1}{s(s+2)(s-2)}$

Zero: None **Poles:** $s\,(s+2)\,(s-2) = 0 \Rightarrow s = 0,\ s = -2,\ s = 2$

Hence:



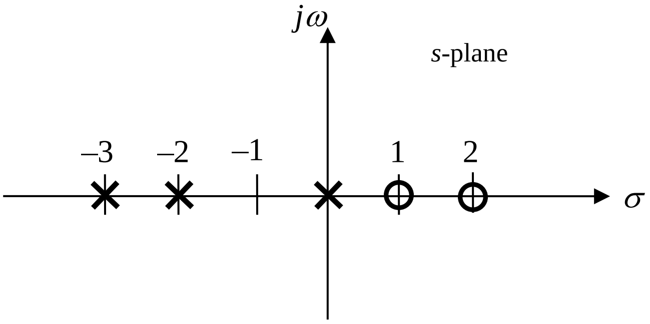
System is **unstable** as one of the poles is on the RHS of the imaginary axis.

(iv) $$\frac{s^2 - 3s + 2}{(s^2 + 2s)(s+3)}$$

Zero: $(s-1)(s-2) = 0 \;\Rightarrow s = 1, s = 2$

Poles: $s(s+2)(s+3) = 0 \;\Rightarrow s = 0, s = -2, s = -3$

Hence:



System is **marginally stable** as one of the poles is on the imaginary axis, while the other poles are all on the LHS.

Q7 (i) $$\frac{1}{(s+2)(s+\alpha)}$$ $\Rightarrow$ poles at -2 and $-\alpha$ $\quad\therefore$ for stability, $\alpha > 0$

(ii) $$\frac{s+\alpha}{(s^2 + 4s + 4)}$$

Here, α does not affect the location of the poles (only the zero) and hence it does not affect stability. Since both system poles are at -2 we can state that the system is **always stable** irrespective of α.

(iii) $$\frac{s}{(s-2)(s+\alpha)}$$ $\Rightarrow$ poles at 2 and $-\alpha$

Since one pole is always on the RHS of the imaginary axis then this system is **always unstable** irrespective of α.

Q8 (i) Combine forward path blocks to give: $\dfrac{2(s+2)}{8(s+2.5)} = \dfrac{s+2}{4(s+2.5)}$

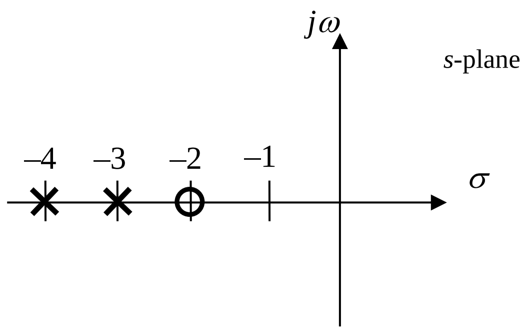
Now, consider the feedback connection. Hence:

$$\frac{G}{1+GH} \to \frac{\dfrac{s+2}{4(s+2.5)}}{1+\dfrac{s+2}{4(s+2.5)}(s+1)} = \frac{s+2}{4(s+2.5)+(s+2)(s+1)} = \frac{s+2}{s^2+7s+12}$$

(ii) Order = 2

(iii) **Zero:** $s+2=0 \ \Rightarrow s=-2$

Poles: $s^2 + 7s + 12 = 0$
$\Rightarrow (s+3)(s+4) = 0$
$\Rightarrow s=-3,\ s=-4$



(iv) System is **stable** as both poles are on the LHS of the imaginary axis

(v) $$\frac{Y(s)}{U(s)} = \frac{s+2}{(s+3)(s+4)} \quad \Rightarrow \quad Y(s) = \frac{s+2}{(s+3)(s+4)}U(s)$$

$$u(t)=1 \ \Rightarrow \ U(s)=\frac{1}{s}$$

Hence: $$Y(s)=\frac{s+2}{s(s+3)(s+4)} \equiv \frac{A}{s}+\frac{B}{s+3}+\frac{C}{s+4}$$

$$=\frac{A(s+3)(s+4)+Bs(s+4)+Cs(s+3)}{s(s+3)(s+4)}$$

Setting $s = 0$: $$2=A(3)(4) \ \Rightarrow A=\frac{1}{6}$$

Setting $s = -3$: $$-1=B(-3)(1) \ \Rightarrow B=\frac{1}{3}=\frac{2}{6}$$

Setting $s = -4$: $$-2=C(-4)(-1) \ \Rightarrow C=-\frac{1}{2}=-\frac{3}{6}$$

Hence: $$Y(s)=\frac{1}{6}\left(\frac{1}{s}+\frac{2}{s+3}-\frac{3}{s+4}\right)$$

Finally: $$y(t)=\frac{1}{6}\left(1+2e^{-3t}-3e^{-4t}\right)$$