3. Modelling of Static and Dynamical Systems

3.1 Introduction

- A mathematical model of a system allows us to:
  - Understand the characteristics of the system (useful for design purposes).
  - Simulate the system (useful for scenario testing, forecasting, etc.).
  - Provide a basis for control system design (for stability, optimising performance, etc.).

- Two important factors that need to be considered when modelling include:
  - Complexity/accuracy trade-off – the more accurate the model, the more complex it becomes (and hence more complex mathematical analysis required).
  - Objective of modelling – what is the purpose of the model? This dictates the level of accuracy (and hence complexity) required. Different modelling objectives include design/synthesis, analysis and control.

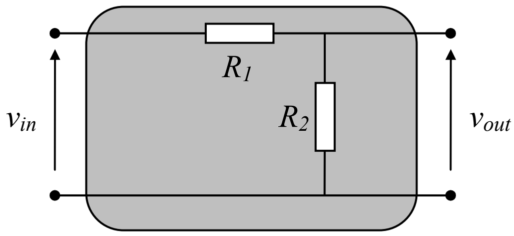
3.2 Modelling of static systems

- Before we consider dynamical systems, let us first look at some model representations of basic static systems in the areas of electrical and heat transfer.

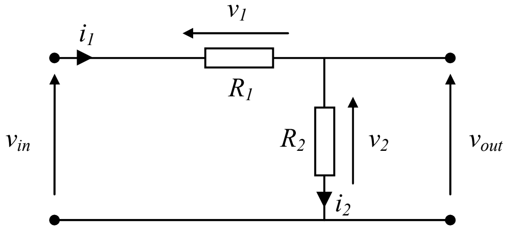
3.2.1 Electrical (covered in EE114)

- For static electrical systems, such as resistor networks, we use *Ohm's law*,

- **Example 3.1:** Derive the input-output relationship for the voltage divider circuit below:



Solution:



$v_1 = i_1R_1$ and $v_2 = i_2R_2$ - Ohm's Law

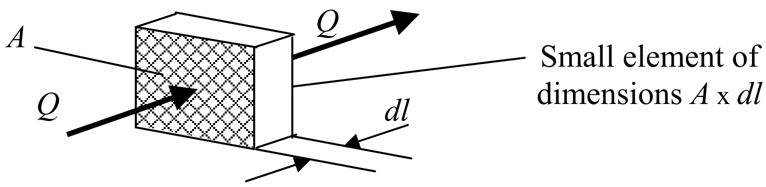
$i_1 = i_2$ - KCL

$v_{in} = v_1 + v_2$ and $v_{out} = v_2$ - KVL

Hence: $\frac{v_{out}}{v_{in}} = \frac{v_2}{v_1 + v_2} = \frac{i_2 R_2}{i_1 R_1 + i_2 R_2} = \frac{i_2 R_2}{i_2 R_1 + i_2 R_2} = \frac{R_2}{R_1 + R_2}$ (the voltage divider rule)

3.2.2 Heat transfer

- *Fourier's law of heat conduction* states that the rate of heat transfer through a material is proportional to the negative gradient in the temperature and to the area at right angles, to that gradient, through which the heat is flowing, as illustrated below:



- This leads to the basic equation for one-dimensional steady-state heat transfer:

$$Q = -kA\frac{dT}{dl}$$

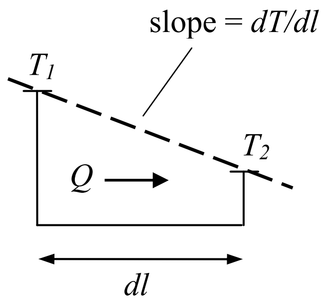
where: Q is the heat flow in Joules/sec or Watts, Like current flow

A is the cross-sectional area in m^2 (model parameter),

k is the thermal conductivity (model parameter),

dl is the length increment in metres (m), and

dT is the temperature change in degrees.

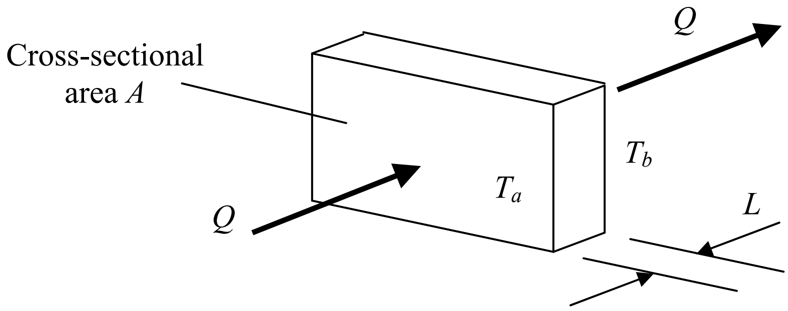


- This equation is for **steady-state** heat flow (i.e. a static scenario). 静态场景

- $\frac{dT}{dl}$ is the temperature gradient and the negative sign indicates that the heat transfer is in the direction of decreasing temperature.

为温度梯度，负号表示dl传热是在温度下降的方向。

- **Example 3.2:** Calculate the steady-state heat flow through a homogeneous (i.e. uniform in composition) rectangular block of metal when the temperatures and dimensions are as shown below:



Solution:

Assuming Q is constant and using Fourier's Law, we get:

$$Q = -kA\frac{dT}{dl} \Rightarrow Qdl = -kAdT$$

Integrating both sides and applying the limits provided gives:

$$Q\int_0^L dl = -kA\int_{T_a}^{T_b} dT$$

Solving gives: $$Q = \frac{kA}{L}(T_a - T_b) = \frac{kA}{L}\Delta T$$

- In this example, the **thermal resistance** may be defined as: $R = \dfrac{\Delta T}{Q} = \dfrac{L}{kA}$
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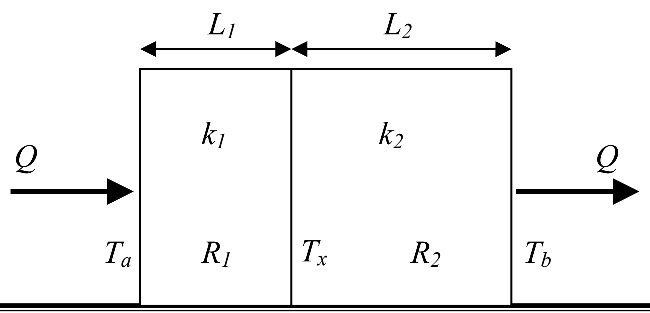
- Note that the thermal conductivity k may vary with temperature, leading to nonlinear behaviour.

- Here, the model is defined by R, which includes parameters k, A and L.

- Finally, note the similarity with the electrical system, where the resistance of a wire, is related to its length, L, cross-sectional area, A, and conductivity, σ, as follows: $R = \dfrac{L}{\sigma A}$

- Likewise, compare $R = \dfrac{V}{I}$ and $R = \dfrac{\Delta T}{Q}$. In the first case, a potential difference is required to get current flow, while in the latter case a temperature difference is required to get heat flow.

- **Example 3.3:** Calculate the total thermal resistance obtained when two blocks of metal with the same cross-sectional area A but with different values of k and L, are placed back to back, as shown below:



$R_T = ?$

Solution:

$$R = \frac{\Delta T}{Q} \Rightarrow \Delta T = QR$$

Now, for the two blocks, the total temperature difference should equate to the sum of the individual temperature difference, i.e.:

$$\begin{aligned}
&(T_a - T_x) + (T_x - T_b) = (T_a - T_b) \\
&\Rightarrow QR_1 + QR_2 = QR_T \\
&\Rightarrow Q(R_1 + R_2) = QR_T \\
&\Rightarrow R_T = R_1 + R_2
\end{aligned}$$

Note that this is the same as for resistors in series!

Hence:

$$R_T = \frac{L_1}{Ak_1} + \frac{L_2}{Ak_2} = \frac{1}{A}\left(\frac{L_1}{k_1} + \frac{L_2}{k_2}\right)$$

3.2.3 Review of modeling static systems

- Overall, in the case of modelling of static systems, we can say:

  - Models are based on the principle of conversion of energy.
  - A static relationship exists between input and output.
  - Static = memory-less = instantaneous. 瞬时的
  - We start with some basic laws for given system type, for example: Ohm's law, Fourier's law, etc.
  - Model parameters may be subject to error (e.g. resistance tolerance values).
  - We should state any assumptions made in deriving model.
  - In general, all models are subject to assumptions and all models have errors.

- Complete model description gives:

  - model structure,
  - model parameters,
  - modelling assumptions, including range of model validity, and
  - some measure of the error in the model.

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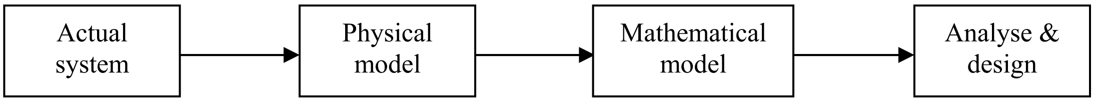
- In practice, most real systems have dynamics associated with them – even resistors! Sometimes these dynamics have a negligible effect on the system behaviour that we are interested in studying allowing us to assume steady state (or static) conditions.

3.3 Modelling of dynamical systems

- The stages of a dynamic system investigation are as follows:



- **Physical modelling** involves identifying the system/sub-system to be studied and obtaining a simple physical model whose behaviour will match sufficiently closely that of the actual system.

- Engineering judgement is needed in determining the appropriate level of detail. Too complicated a model leads to long analysis while too simple a model is unrepresentative (i.e. not accurate enough).

- Experience is needed – it cannot be taught! However, there are several useful guidelines (engineering approximations):

  - *neglect small effects*
    – this reduces the number and complexity of equations.
  - *assume environment is independent of the system motions*
    – this reduces the number and complexity of equations.
  - *replace distributed characteristics with appropriate lumped elements**
    – this gives ordinary differential equations rather than partial ones.
  - *assume linear relationships*
    – gives linear equations and superposition holds.
  - *assume constant parameters*
    – leads to constant coefficient in differential equations.
  - *neglect uncertainty and noise*
    – avoids statistical treatment.

* For example, a wire has resistance along its entire length – however we represent this by 'lumping' this distributed resistance into a single point value.

- **Mathematical modelling** involves obtaining a mathematical representation of the physical model.

- Central to this process is the writing of equations for equilibrium and/or compatibility relations.

- *Equilibrium relations* describe the balance of forces, of flow rates, of energy, of current, etc. which must exist for the system (conservation of energy).

- *Compatibility relations* describe how motions of the system are interrelated because of the way they are connected.

- The procedure for obtaining a mathematical model can be summarised as follows:

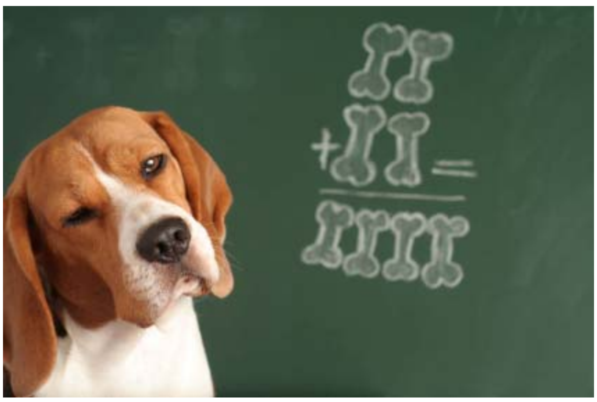
> - Develop a physical model for the system.
> - Define the system variables
> - Write equations for the equilibrium and/or compatibility relations in the system.
> - Use physical relations/laws to relate the variables for each component in the system.

- In the next section we will summarise, for convenience, the relatively simple continuous-time systems that were covered in the EE114 module.

- We will provide the models in both differential equations and transfer functions format.

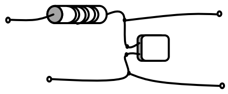
- For the remainder of this section of the notes, we will consider more complex continuous-time dynamical systems.

- We will also consider models of some discrete-time systems.



3.3.1 Basic systems covered in module EE114 (revision)

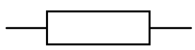
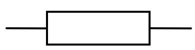
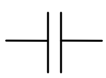
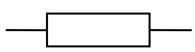
- The RC circuit, the mass-spring-damper (the bicycle) and the single water tank were the main systems studied in EE114. The models for each of these systems are as follows:

System	Differential Equation	Transfer Function
RC Circuit 	$v_i = RC\dfrac{dv_C}{dt} + v_C$	$\dfrac{V_c(s)}{V_i(s)} = \dfrac{1}{1+sRC}$
Mass-spring-damper 	$M\dfrac{d^2x}{dt^2} + B\dfrac{dx}{dt} + Kx = f(t)$	$\dfrac{X(s)}{F(s)} = \dfrac{1}{Ms^2 + Bs + K}$
Single tank 	$A\dfrac{dh}{dt} = F_{in} - kh$	$\dfrac{H(s)}{F(s)} = \dfrac{1}{sA+k}$

- We will remind ourselves of the modelling procedure by working through each of these examples in turn.

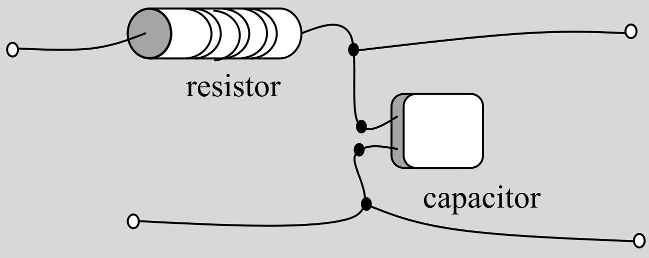
Modelling the RC circuit (first-order electrical system) …

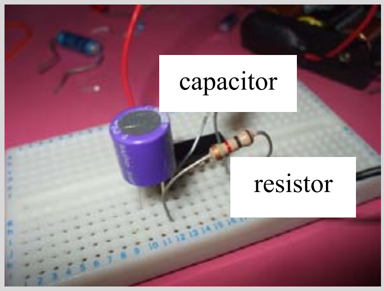
- The key physical relations for electrical systems are:

Component	Physical Law	Symbol		
Resistance (R)	$v = iR$		*or*	
Inductance (L)	$v = L\dfrac{di}{dt}$		*or*	
Capacitance (C)	$v = \dfrac{1}{C}\int idt$ $\left(i = C\dfrac{dv}{dt}\right)$		*or*	

- **Example 3.4:** Determine a mathematical model for a resistor/capacitor filter circuit as shown below:

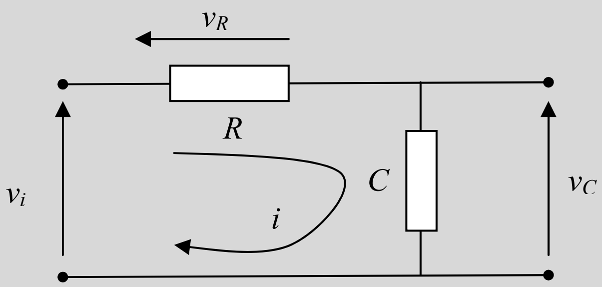
In such systems, we generally assume lumped parameters and ideal components.





Solution:

Step 1: Physical model assuming ideal components:



Step 2: Model variables defined on the physical model – current i and voltages v_i, v_R, v_C.

Step 3: Compatibility relation required – KVL: $v_i = v_R + v_C$.

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Note that the equilibrium condition is implied in our choice of current variable and therefore, does not need to be stated explicitly (i.e. $i = i_C = i_R$)

Step 4: Combining physical relations with the compatibility equation, and noting that we are interested in the relationship between v_C and v_i, gives:

$$v_i = iR + v_C$$

Substituting $i = C\dfrac{dv_C}{dt}$ gives the **differential equation**: $$v_i = RC\frac{dv_C}{dt} + v_C$$

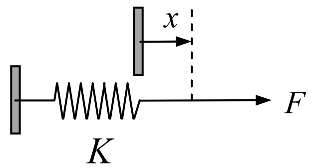
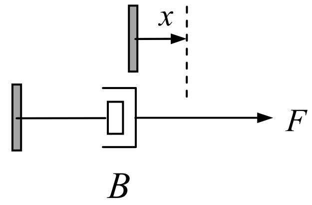
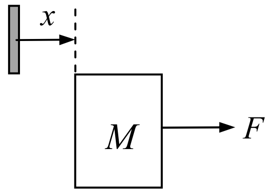
Taking the Laplace transform gives: $$V_i(s) = RCsV_c(s) + V_c(s)$$

$$\Rightarrow V_i(s) = V_c(s)(1 + sRC)$$

Hence the **transfer function** model is given by: $$\frac{V_c(s)}{V_i(s)} = \frac{1}{1+sRC}$$

Modelling the bicycle (second-order mechanical system) …

- The key physical relations for (one-dimensional) mechanical systems (assuming ideal components) are:

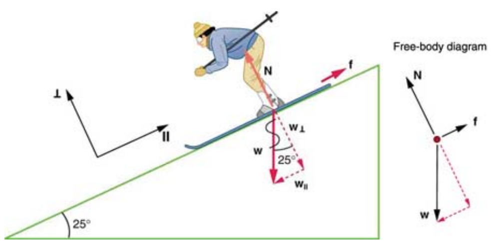
Component	Physical Law	Symbol
Spring (K)	$F = Kx$	
Damper (or Dashpot) (B)	$F = B\dot{x}$	
Mass (M)	$F = M\ddot{x}$	

Note – x represents position (displacement) or distance, $\dot{x}$ represents velocity (v) and $\ddot{x}$ represents acceleration (a)

- **Free body diagram** – In mechanics the concept of a free body diagram (FBD) (or force diagram) is used to analyse mechanical systems.

- Each mass is viewed as a free body isolated from the rest of the system with only the forces acting on it shown.

- Force balance equations are then written for each mass.

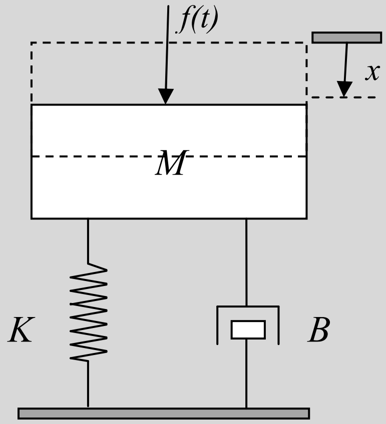


- **Example 3.5:** Determine a mathematical model for the spring-mass-damper system, whose physical model is shown in step 1 below:

Solution:

Step 1: Physical model assuming ideal components:

Actual system – 'Suspension' within a shoe or a bicycle for example

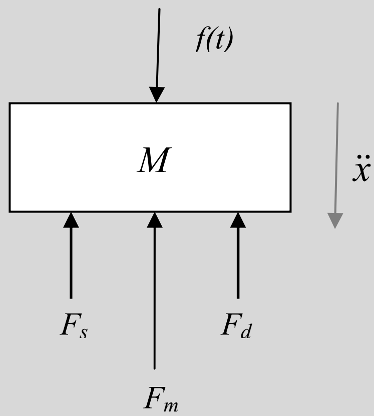


Physical model

Step 2: Model variables defined on the physical model

Step 3: From the free body diagram showing only the Force acting on *M* we obtain the Force equilibrium equation:

$$F_m + F_d + F_s = f(t)$$



Free body diagram

Step 4: Using the physical force-geometry relations gives the **differential equation**:

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t)$$

or simply:

$$M\ddot{x} + B\dot{x} + Kx = f(t)$$

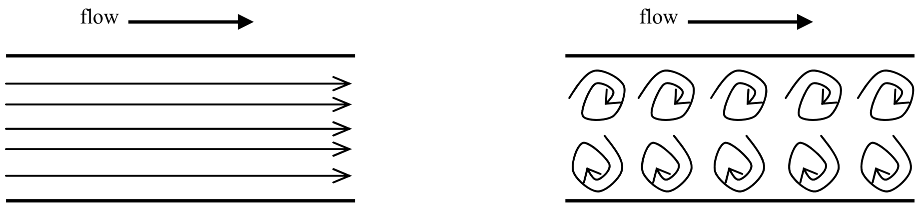
Taking the Laplace transform gives: $$Ms^2X(s) + BsX(s) + KX(s) = F(s)$$

Hence the **transfer function** model is: $$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

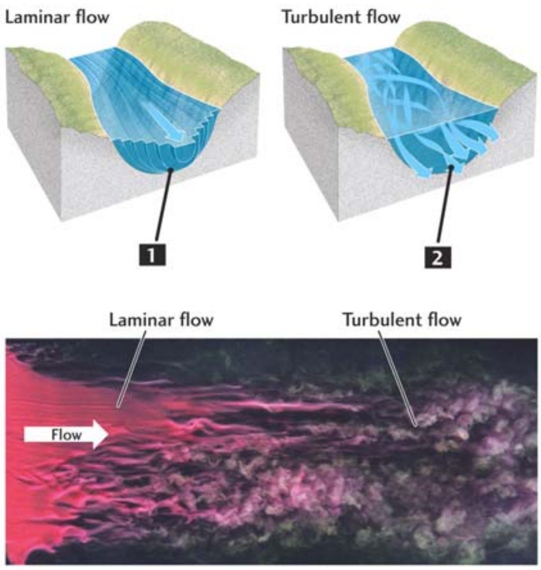
Modelling the single tank (first-order flow system) …

- Water flow in a pipe can be laminar or turbulent.

- Laminar refers to a uniform directional flow while turbulent, as the name suggests, refers to a more chaotic type flow, as indicated in the following sketches:



laminar flow　　　　　　　　turbulent flow



- The relationship between the output flow F_{out} and the height h of water in the tank is also dependent on the type of flow involved, as follows:

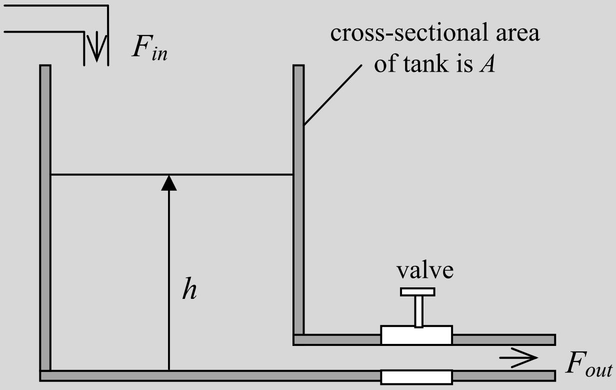
$F_{out} = kh$ for **laminar** flow (*this is a **linear** relationship*)

$F_{out} = k\sqrt{h}$ for **turbulent** flow (*this is a **nonlinear** relationship*)

- Here k is a constant of proportionality that allows for several factors including the cross-sectional area of outflow pipe, frictional forces within the pipe, density of the liquid, etc.

- This generally forms part of our assumptions in the final model.
- **Example 3.6:** Assuming laminar flow, determine a mathematical model for the following single tank flow system, relating the height of the water to the input flow F_{in}:

Actual system



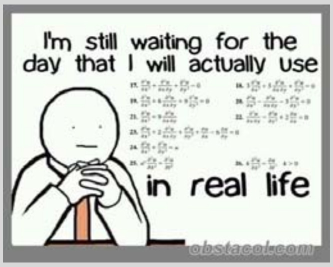
Physical model of a single tank

Solution:

The ***flow balance equation*** is given by:

Change in Water Volume V = Flow-in - Flow-out

$$\Rightarrow \frac{dV}{dt} = F_{in} - F_{out}$$



Since $V = Ah$, and A is a constant value, then:

$$\frac{dV}{dt} = A\frac{dh}{dt} = F_{in} - F_{out}$$

Since $F_{out} = kh$, we obtain the first order **differential equation**:

$$A\frac{dh}{dt} = F_{in} - kh$$

Taking the Laplace transform gives:

$$AsH(s) = F(s) - kH(s)$$

$$\Rightarrow H(s)(sA + k) = F(s)$$

Hence the **transfer function** model is: $\quad \dfrac{H(s)}{F(s)} = \dfrac{1}{sA + k}$

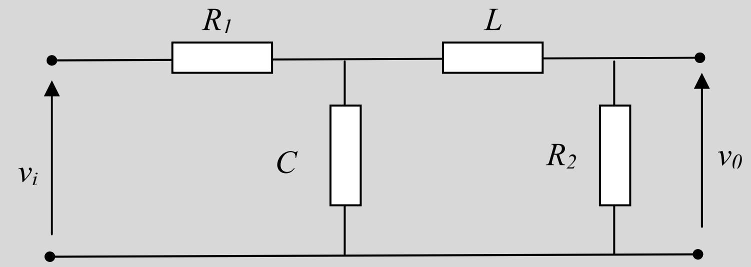
3.3.2 Higher order continuous-time systems

Higher order electrical circuits …

- For more complex circuits the first principles approach, as used previously, is tedious.

- Instead, we generally use more efficient circuit analysis techniques to obtain the model equations – typically nodal and mesh analysis.

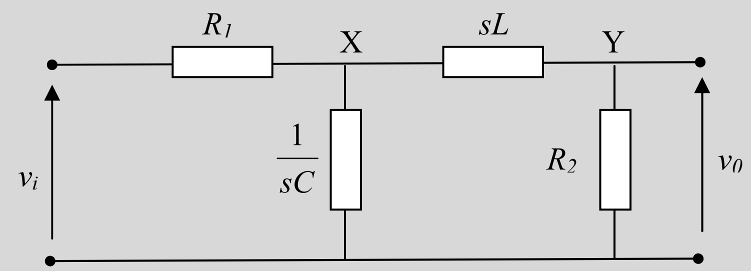
- **Example 3.7:** Develop a mathematical model for the following circuit, using nodal analysis:



Solution:

We begin by replacing C and L with their impedances (*note that* $s \equiv j\omega$).

Hence: $C \rightarrow \dfrac{1}{j\omega C} \equiv \dfrac{1}{sC}$ and $L \rightarrow j\omega L \equiv sL$



Using nodal analysis – *recall that the sum of the currents flowing in to a node equals the sum of the currents flowing out* – if we assume that all currents flow out of the node, then the sum of these currents must be 0:

At node X (with voltage v_x): $$\frac{v_X - v_i}{R_1} + \frac{v_X - v_0}{sL} + \frac{v_X - 0}{\frac{1}{sC}} = 0$$

At node Y (with voltage v_0): $$\frac{v_0 - 0}{R_2} + \frac{v_0 - v_X}{sL} = 0$$

Rearranging gives:

$$\left[\frac{L}{R_1}s + 1 + LCs^2\right]v_X - v_0 - \frac{L}{R_1}sv_i = 0 \qquad \text{(A)}$$

$$v_X = \left[\frac{L}{R_2}s + 1\right]v_0 \qquad \text{(B)}$$

Substituting equation B into equation A gives:

$$\left[\frac{L}{R_1}s + 1 + LCs^2\right]\left[\frac{L}{R_2}s + 1\right]v_0 - v_0 - \frac{L}{R_1}sv_i = 0$$

This simplifies to:

$$\left[(R_1 LC)s^2 + (R_1 R_2 C + L)s + (R_1 + R_2)\right] v_0 = R_2 v_i$$

We can rewrite this as a **differential equation** by converting back to the time domain:

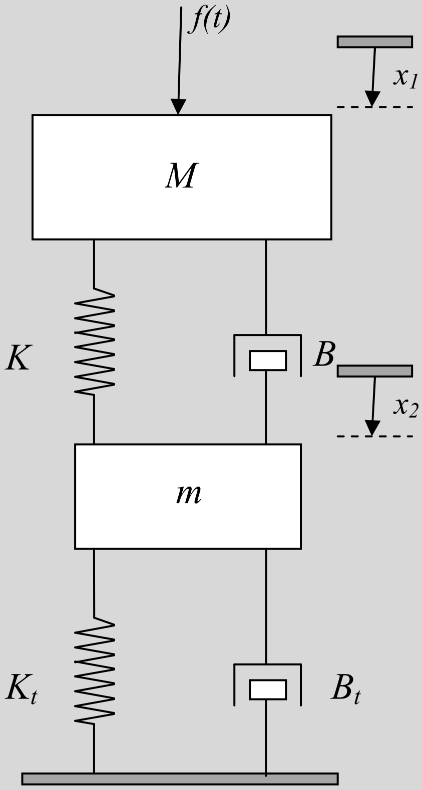
$$(R_1 LC)\ddot{v}_0 + (R_1 R_2 C + L)\dot{v}_0 + (R_1 + R_2)v_0 = R_2 v_i$$

Or we can stay in the Laplace domain and express the answer in **transfer function** form:

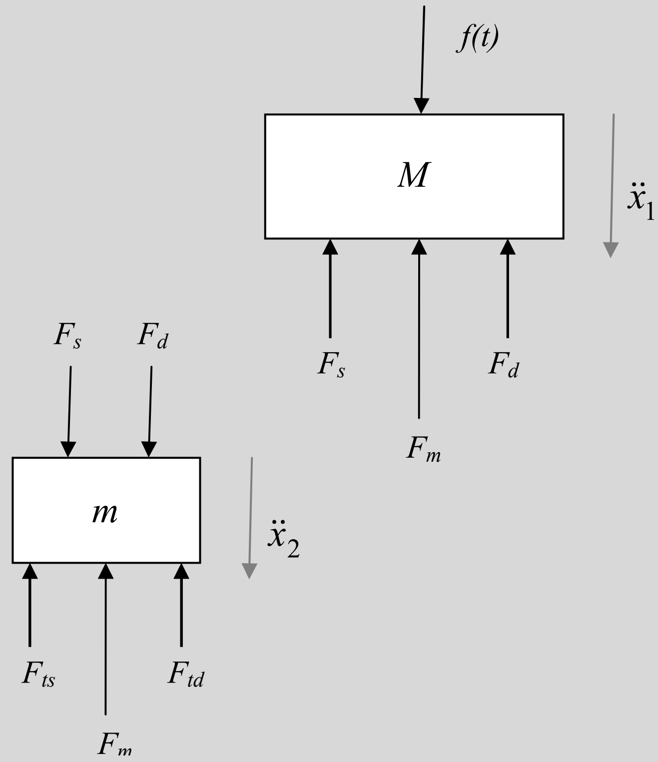
$$\frac{V_0(s)}{V_i(s)} = \frac{R_2}{(R_1 LC)s^2 + (R_1 R_2 C + L)s + (R_1 + R_2)}$$

- ***Exercise** – Determine a mathematical model for the circuit using mesh analysis.*
- Note – nodal analysis effectively combines steps 3 and 4 of the modelling procedure shown previously.
- The equations obtained from nodal analysis are the equilibrium relationships for the currents at each node, but here the currents are expressed in terms of voltages using the physical laws governing each component.
- Similarly, mesh analysis also combines steps 3 and 4, expressing the compatibility relationships for the voltages across each component in terms of '*loop*' currents.

- **Example 3.8:** Determine a mathematical model for a car suspension that takes into account the effect of the tyres and wheel mass, as shown below:



Physical model



Free body diagrams

Solution:

Force balance equation for mass *M*: $F_m + F_d + F_s = f(t)$

Force balance equation for mass *m*: $F_m + F_{td} + F_{ts} = F_s + F_d$

The physical force-geometry relations for *m* are:

$$F_m = m\ddot{x}_2,\ F_{td} = B_t\dot{x}_2 \text{ and } F_{ts} = K_t x_2$$

The physical force-geometry relations for *M* are:

$$F_m = M\ddot{x}_1,\ F_d = B(\dot{x}_1 - \dot{x}_2) \text{ and } F_s = K(x_1 - x_2)$$

Note that the spring and damper extensions for mass *M* are denoted by the difference between x_1 and x_2.

Substituting these expressions into the force balance equations gives the following **differential equation model** for the system:

$$M\ddot{x}_1 + B(\dot{x}_1 - \dot{x}_2) + K(x_1 - x_2) = f(t)$$

$$m\ddot{x}_2 + B_t\dot{x}_2 + K_t x_2 = B(\dot{x}_1 - \dot{x}_2) + K(x_1 - x_2)$$

This is a pair of coupled 2nd order differential equations. The variables x_1 and x_2 define the position of the car and wheel respectively w.r.t. an equilibrium position.

If we define x_1 as the output variable then we can obtain a **transfer function model** relating x_1 to the input $f(t)$ by taking the Laplace transform of both equations and eliminating x_2 from the resulting simultaneous equations.

This gives:

$$\frac{X_1(s)}{F(s)} = \frac{ms^2 + (B_t + B)s + (K_t + K)}{mMs^4 + \alpha s^3 + \beta s^2 + (BK_t + KB_t)s + KK_t}$$

where: $\alpha = M(B_t + B) + mB$ and $\beta = \mathrm{M}(\mathrm{K}_t + K) + B_t B + Km$

Alternatively, by eliminating x_1 we can get a **transfer function model for x_2** as follows:

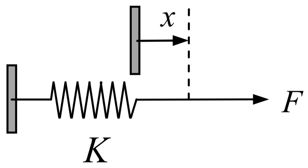
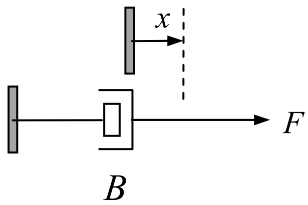
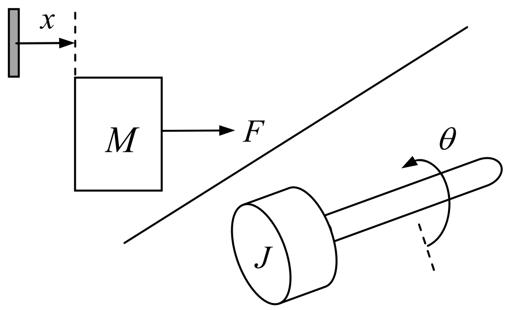
$$\frac{X_2(s)}{F(s)} = \frac{Bs + K}{mMs^4 + \alpha s^3 + \beta s^2 + (BK_t + KB_t)s + KK_t}$$

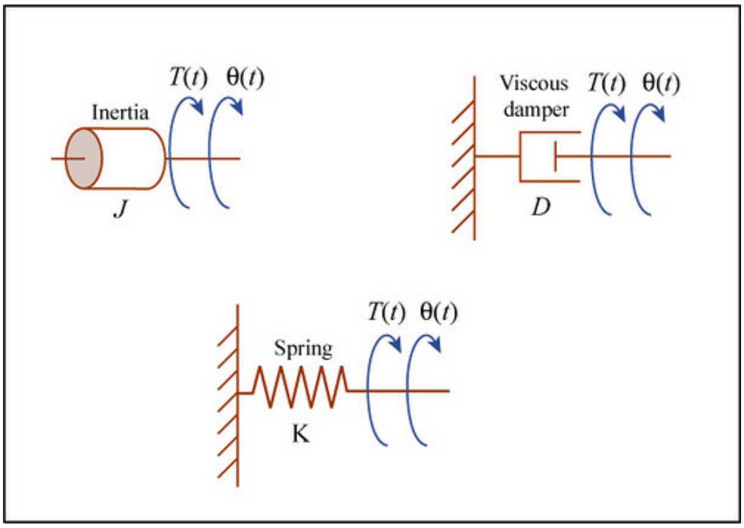
- *Exercise – derive these transfer functions.*
- Note – the denominator of the transfer function is 4th order and hence the dynamics of the system are 4th order.
- This is as expected since each free body mass in the system contributed second order dynamics.
- **A transfer function model can only describe the relationship between one input and one output at a time**.

- The previous examples relating to mechanical systems involve translational movement.

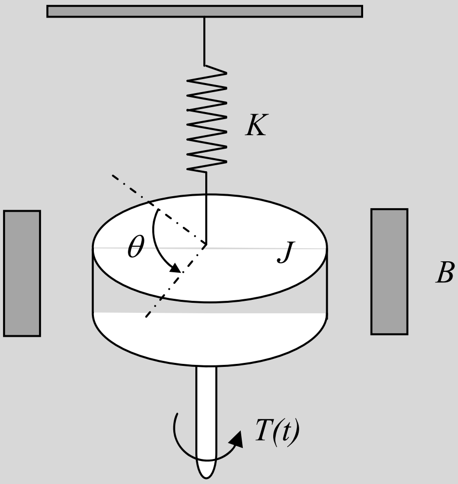
- A similar procedure exists for systems involving rotational movement. In this case, the equations are in terms of Torque, T and the rotational displacement is given by θ.

- Hence, the key physical relations for mechanical systems (assuming ideal components) can be updated as follows:

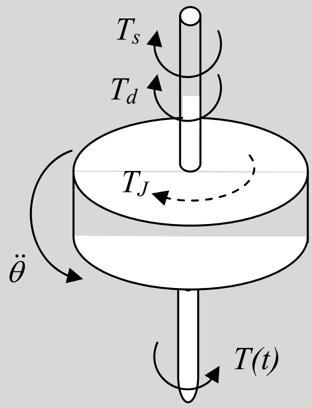
Component	**Physical Law**	**Symbol**
Spring (K)	$F = Kx$ $T = K\theta$	
Damper (B)	$F = B\dot{x}$ $T = B\dot{\theta}$	
惯性 Mass/Inertia (M/J)	$F = M\ddot{x}$ $T = J\ddot{\theta}$	



- **Example 3.9:** Develop a mathematical model for a torsional pendulum. The physical model, assuming ideal components, and the free body diagram are as follows:



Physical model



Free body diagram

Solution:

扭矩方程

From the free body diagram, the torque equilibrium equation is:

$$T_J + T_d + T_s = T(t)$$

T(t) represents externally applied torque

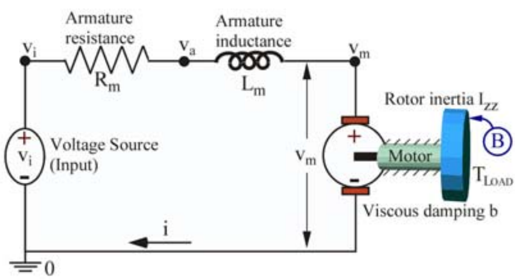
Substituting the physical-force-geometry relations gives the **differential equation**:

$$J\ddot{\theta} + B\dot{\theta} + K\theta = T(t)$$

The corresponding **transfer function model** is:

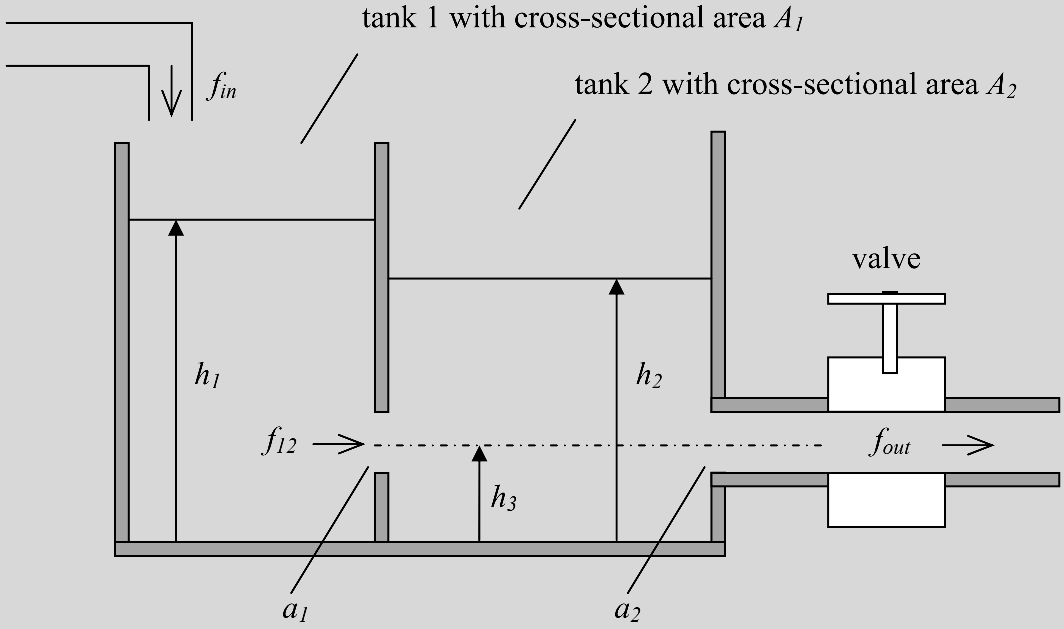
$$\frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}$$

- *Mechanical and electrical systems are often combined in practice (known as electromechanical systems) – you will consider the modelling of such systems in EE312 (Electromagnetics, Fields and Waves).*



Coupled-tank flow systems …

- Consider the following physical model of a coupled-tank flow system:



physical model of two-tank system

- This represents a classic reservoir height control problem. The tanks are connected by a small valve of cross-sectional area a_1. A tap valve controls the outlet flow f_{out} and the cross-sectional area of this outlet is a_2. Inter-tank flow is given by f_{12}.

- Recall – two possible equations relating output flow F_{out} to height h in a single tank are:

$$F_{out} = kh \quad \textbf{(laminar)} \qquad \text{and} \qquad F_{out} = k\sqrt{h} \quad \textbf{(turbulent)}$$

- Hence, we can readily obtain the following equations relating output flow f_{out} to the different in heights (i.e. $h_2 - h_3$) as follows:

$$f_{out} = k\,(h_2 - h_3) \quad \textbf{(laminar)} \qquad \text{and} \qquad f_{out} = k\sqrt{(h_2 - h_3)} \quad \textbf{(turbulent)}$$

- Likewise, the equations relating inter-tank flow f_{12} to the different in heights $h_1 - h_2$ are:

$$f_{12} = k\,(h_1 - h_2) \quad \textbf{(laminar)} \qquad \text{and} \qquad f_{12} = k\sqrt{(h_1 - h_2)} \quad \textbf{(turbulent)}$$

- Note, k in the above equations represents a general constant of proportionality – however, it does not necessarily have the same value in each case.

- **Example 3.10:** Develop a mathematical model for the coupled-tank system, relating both tank height dynamics to the input flow rate, under *turbulent flow* conditions.

  Solution:

  Writing flow balance equations for each tank (i.e. flow-in = flow-out) gives:

  $$\frac{dV_1}{dt} = f_{in} - f_{12} \qquad \text{and} \qquad \frac{dV_2}{dt} = f_{12} - f_{out}$$

  Here, V_1 and V_2 are the volumes of liquid in tanks 1 and 2 respectively.

  Since $V_1 = A_1 h_1$ and $V_2 = A_2 h_2$ and the cross-sectional areas are constant:

  $$\frac{dV_1}{dt} = A_1 \frac{dh_1}{dt} \qquad \text{and} \qquad \frac{dV_2}{dt} = A_2 \frac{dh_2}{dt}$$

  Hence:

  $$\frac{dh_1}{dt} = \frac{1}{A_1}(f_{in} - f_{12}) \qquad \text{and} \qquad \frac{dh_2}{dt} = \frac{1}{A_2}(f_{12} - f_{out})$$

  As notes previously, under turbulent conditions we can state that:

  $$f_{12} = k_1\sqrt{(h_1 - h_2)} \qquad \text{and} \qquad f_{out} = k_2\sqrt{(h_2 - h_3)}$$

  Hence, we obtain our final dynamic model of the two tank system as follows:

  $$\dot{h}_1 = \frac{1}{A_1} f_{in} - \frac{k_1}{A_1}\sqrt{h_1 - h_2} \qquad \text{and} \qquad \dot{h}_2 = \frac{k_1}{A_2}\sqrt{h_1 - h_2} - \frac{k_2}{A_2}\sqrt{h_2 - h_3}$$

  These **differential equations** have the general form:

  $$\dot{h}_1 = f_1(h_1, h_2, f_{in}) \quad \text{and} \quad \dot{h}_2 = f_2(h_1, h_2)$$

- This system is not linear. *Why not?* Note – we can't obtain a transfer function model for this particular system, as it is nonlinear and transfer functions generally only represent linear systems.

- The order of a system is given by the highest derivative – however, in this case we have first order equations which are coupled. Thus, this leads to **a second order system**!

- *How would you increase the steady-state height difference between the tanks?*

  Increase f_{in} *(set $\dot{h}_1 = 0$ and see what happens!)*

3.3.3 Equivalences between system types

- The equivalences between the various systems types are indicated below:

Electrical	Mechanical (Translational)	Mechanical (Rotational)	Thermal	Fluid	Chemical
charge q	position x	position θ	- -	volume V	mass m
current i	speed $v = \dot{x}$	speed $\omega = \dot{\theta}$	flow rate q	flow rate q (or f)	flow rate -
- di/dt	acceleration $a = \ddot{x}$	acceleration $\alpha = \ddot{\theta}$	- -	- -	- -
voltage v	force F	torque T	temperature T	pressure p	concentration -
inductance L	mass M	inertia J	- -	- -	- -
resistance R	damping B	damping B	resistance R_t	resistance R_f	- -
Capacitance $1/C$	spring K	spring K	- $1/(Mc_p)$	- -	- -

- For example, consider the electrical RLC circuit:

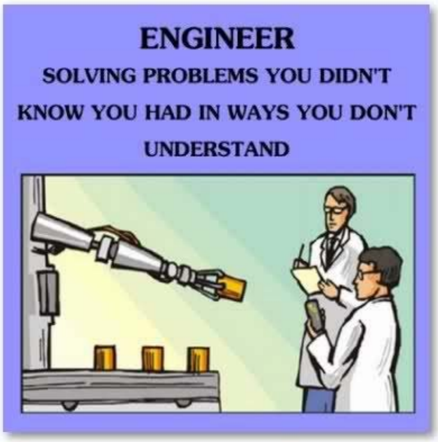
$$v = L\frac{di}{dt} + Ri + \frac{1}{C}\int idt$$

- Since $i = \dfrac{dq}{dt}$:

$$v = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q$$

or

$$v = L\ddot{q} + R\dot{q} + \frac{1}{C}q$$



- The equivalent mechanical translational spring-mass-damper system is:

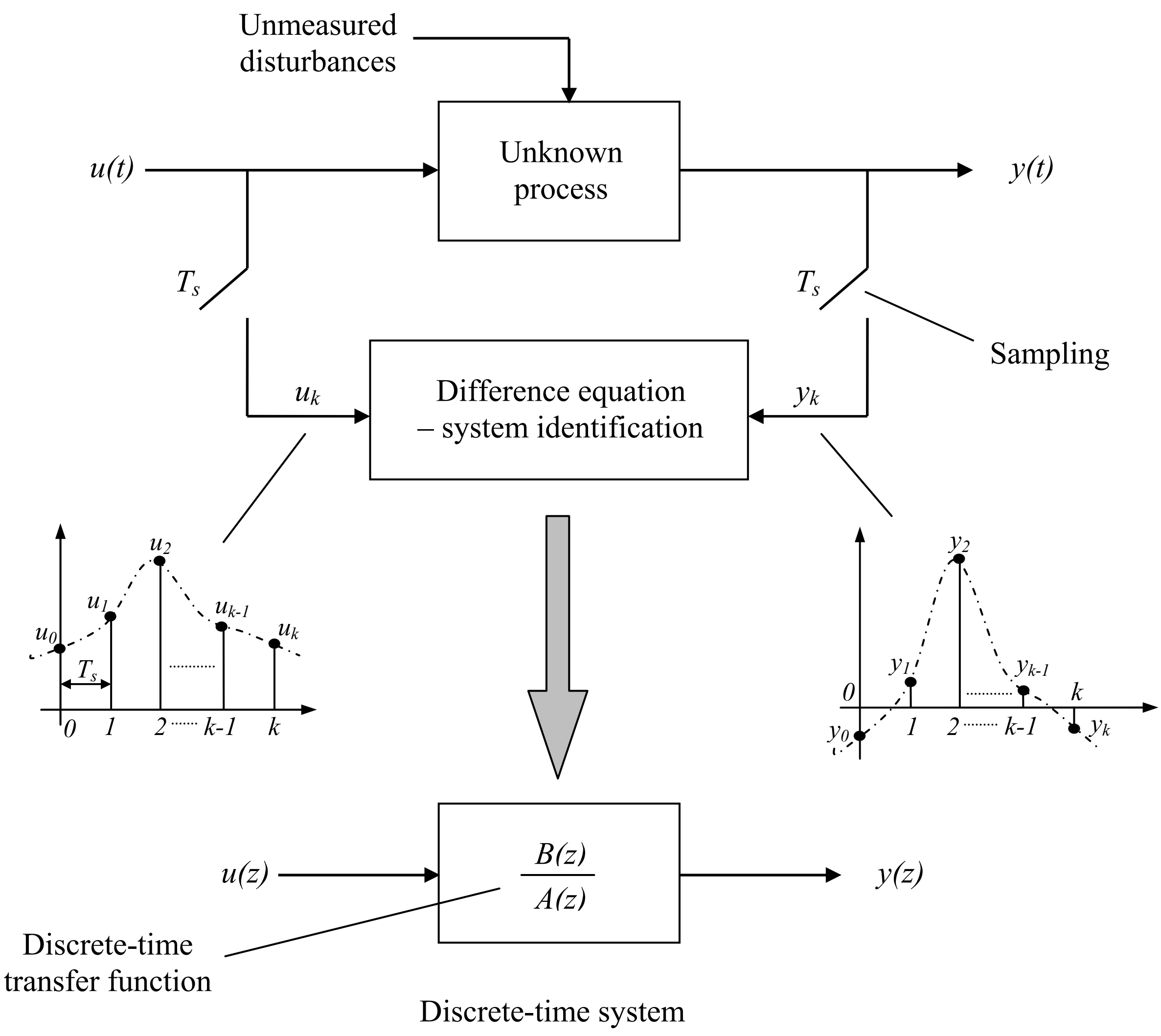
$$F = M\ddot{x} + B\dot{x} + Kx$$

3.4 Discrete-time modelling

- In the previous section, the examples all involved continuous-time systems.
- Discrete-time systems are those where one or more variables change only at discrete instants of time (typically equally spaced).
- **Difference equation models** are used to describe the relationship between variables that change at discrete times.
- Discrete-time systems often result from system identification and computer controlled systems. However, they can also occur naturally.

3.4.1 System Identification

- In system identification, discrete-time models often arise when modelling real systems from experimental data.



- Experimental data is generally only available at discrete times (referred to as sample instants). For example, a data logger might record the temperature in a furnace once every 5 seconds, etc.

- We can use this data to identify a difference equation model of the system.

- This model relates the output of the system at the current sample instant $t = kT_s$ to output and input values from previous sample instants.

- The general form of this model is:

$$y(kT_s) = f(y((k-1)T_s), y((k-2)T_s),..., u((k-1)T_s), u((k-2)T_s),...)$$

- For linear models we get:

$$y(kT_s) = a_1 y((k-1)T_s) + a_2 y((k-2)T_s) + ... + b_1 u((k-1)T_s) + b_2 u((k-2)T_s) + ...$$

- We generally write this as:

$$y(k) = a_1 y(k-1) + a_2 y(k-2) + ... + b_1 u(k-1) + b_2 u(k-2) + ...$$

or

$$y_k = a_1 y_{k-1} + a_2 y_{k-2} + ... + b_1 u_{k-1} + b_2 u_{k-2} + ...$$

- In general, an n^{th} order linear system can be modeled at discrete sample instants by an n^{th} order difference equation model.

- A first order difference equation model is given by:

$$y_k = a y_{k-1} + b u_{k-1}$$

- A second order difference equation model is given by:

$$y_k = a_1 y_{k-1} + a_2 y_{k-2} + b_1 u_{k-1} + b_2 u_{k-2}$$

- **Example 3.11:** Determine the parameters of a first order difference equation model for a system given the following experimental data:

sample time (k)	input (u)	output (y)
1	0.2	0.200
2	0.3	0.240
3	0.4	0.330
4	0.3	0.445

Solution:

$$y_k = ay_{k-1} + bu_{k-1}$$

$$y_2 = ay_1 + bu_1 \rightarrow 0.24 = 0.2a + 0.2b$$
$$y_3 = ay_2 + bu_2 \rightarrow 0.33 = 0.24a + 0.3b$$
$$y_4 = ay_3 + bu_3 \rightarrow 0.445 = 0.33a + 0.4b$$

Now, we have two unknown and 3 equations. Taking the first two equations gives:

$$\begin{bmatrix} 0.24 \\ 0.33 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.2 \\ 0.24 & 0.3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \equiv \mathbf{y} = \mathbf{X}\boldsymbol{\theta}\ , \quad \boldsymbol{\theta} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Solving for $\boldsymbol{\theta}$ gives: $\boldsymbol{\theta} = \mathbf{X}^{-1}\mathbf{y} = \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix}$

Can solve this using maths, or using the command `inv(X)*y` in Matlab!

Thus, $a = 0.5$ and $b = 0.7$

- Note – can use any two of the equations to give the same result.

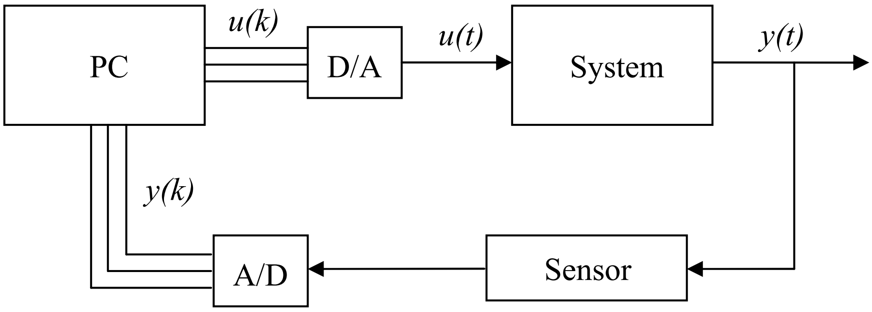
- However, if input and/or output measurement are corrupted with noise, the parameter estimates will be biased!

- For data which is subject to measurement noise (nearly all data!), we can use *System Identification* to get reliable parameter estimates

- Note – we must sample our system using at least twice the highest frequency in the system (Shannon's sampling theorem).

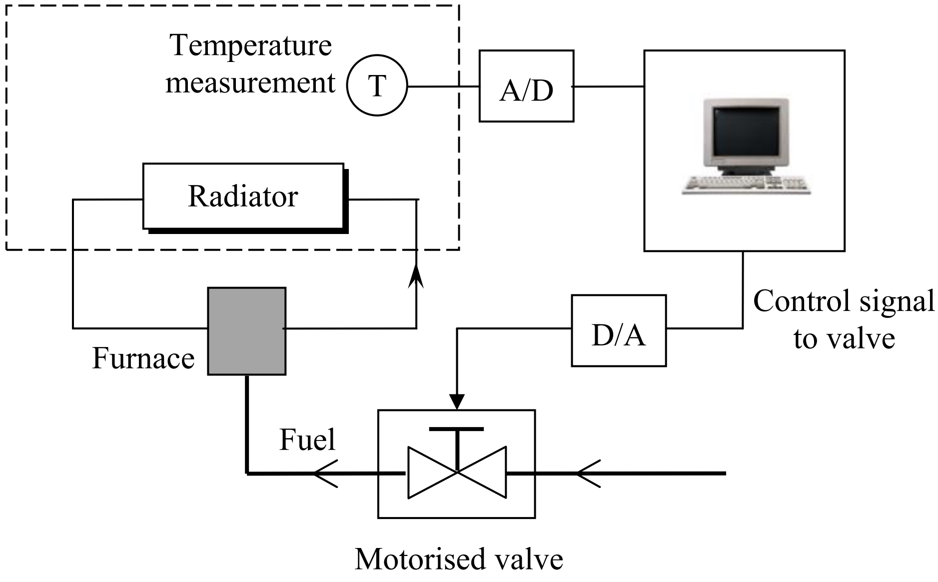
3.4.2 Computer control

- Computers work in discrete time (clock speed). When they interface with real systems, they collect or transmit data at discrete sample instants.

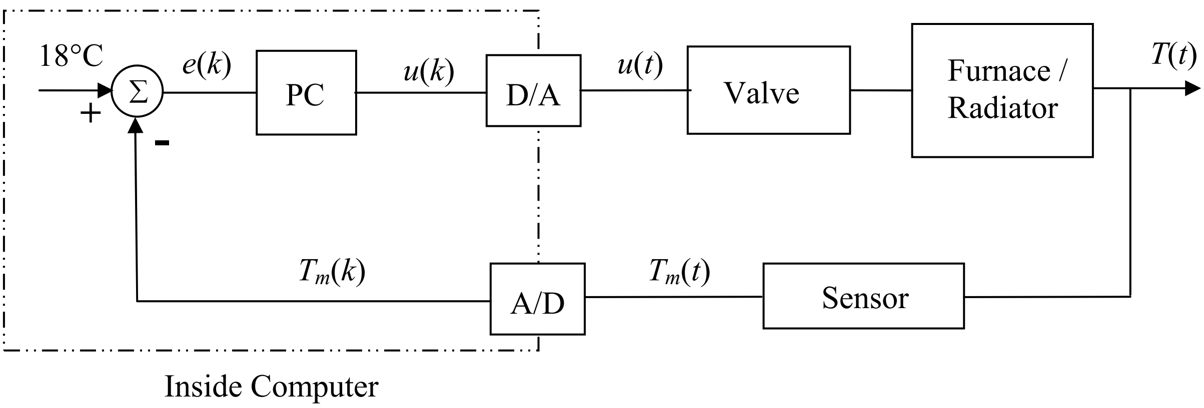


- D/A are digital-to-analogue converters while A/D are analogue-to-digital converters.

- For example, consider the computer control system for regulating room temperature, as shown below:



- The objective is to control the temperature of the room to 18 degrees. A schematic of the control systems looks like:



- The control algorithm in the computer is of the form:

$$u(k) = K\, e(k), \qquad \text{where } \; e(k) = 18 - T_m(k)$$

- The error $e(k)$ represents the deviation of the room temperature from the desired value (referred to as the setpoint).

- Is the controller static or dynamic? *Static (only current values used)*

- Is the controller linear? *Yes (we're using proportional control)*

- Is the controlled system (valve, furnace, radiator and sensor) static or dynamic? *Dynamic (system described by a differential equation – has memory)*

- Is the controlled system linear? *In reality, no! Over a specific range, perhaps!*

- The computer calculates the control signal at discrete (regular) points in time. What determines the maximum rate? *Speed of processor (and calculations required)*

- Interface between the computer and the rest of the system is an important consideration – needs to be part of the system model.

- This control scheme is called proportional control.

- Control scheme uses negative feedback and works as follows:
  - Large error $\rightarrow$ large control signal
  - Negative error $\Rightarrow T_{room} > 18°C$ gives decrease in fuel flow
  - Positive error $\Rightarrow T_{room} < 18°C$ gives an increase in fuel flow
  - No error $\Rightarrow T_{room} = 18°C$ gives no change in fuel flow.

- A PC is a somewhat overkill for this application – why? Alternative? *We're only carrying out a basic gain change as a control mechanism – a simple op-amp circuit would suffice. Even if we wanted something more complicated, a microprocessor would be sufficient.*

- The control scheme is typical of industrial boilers, but a much simpler form is used for domestic heating!

- Control schemes can be much more complex:
  - Could be based on past errors (e.g. PI, PID) if desired.
  - Choice of controller depends on the dynamics of the system.
  - Wide range of alternative strategies.
  - Some controllers have a plant model inbuilt into the controller.

3.4.3 Naturally discrete systems

- Many systems and relationships exist that are naturally in discrete form. Examples include:
  - Average daily rainfall, temperature, number of sunlight hours, etc.
  - Weekly cars sales in Ireland.
  - Annual monthly earnings.
  - Annual employment figures.
  - Number of solar flares each month.

- Note that most of these discrete systems are governed by underlying continuous systems.

- For example, a possible world *population model* might be:

$$p_k = p_{k-1} + b\,p_{k-1} - d\,p_{k-1} + a_k - s_k$$

where:

- p represents the population in billions
- k represents the year number
- b is the mean birth rate ($0 < b < 1$)
- d is the mean death rate ($0 < d < 1$)
- a is ??? (*for aliens*!)
- s is ??? (*for space tourists!*)

- Combining common terms in this equation, we get: $p_k = (1 + b - d)\,p_{k-1} + a_k - s_k$

- A more realistic population model is achieved by realizing that, as the population gets large, people compete with each other for living space, food, etc. (Velhurst model):

$$b - d = r = \alpha - \beta\,p_{k-1} \qquad (\alpha \text{ and } \beta \text{ are constant values})$$

- **Example 3.12:** Population statistics tell us that the population doubles every 35 years. Calculate the relationship between b and d, using the first population model, assuming a_k and s_k are negligible.

Solution:

$$p_k = (1 + b - d)\,p_{k-1}$$

Taking $k = 2001$ for example, gives:

$$p_{2001} = (1 + b - d)\,p_{2000}$$
$$p_{2002} = (1 + b - d)\,p_{2001} = (1 + b - d)^2\,p_{2000}$$
$$p_{2003} = (1 + b - d)\,p_{2002} = (1 + b - d)^3\,p_{2000} \qquad etc...$$

Hence: $p_{2035} = (1 + b - d)^{35}\,p_{2000}$, but $p_{2035} = 2 \times p_{2000} \Rightarrow (1 + b - d)^{35} = 2$

Therefore: $b = \left(\sqrt[35]{2}\right) + d - 1$