Engineering Mathematics 1 (Fall 2021)

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Students should be able to (after learning)

- Add, subtract and multiply complex numbers
- Convert complex numbers between Cartesian and polar forms
- Differentiate all commonly occurring functions including polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of a derivative, namely the derivative as a tangent and the derivative as a rate of change
- Integrate certain standard functions, constructed from polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of integration, namely the integral as the inverse of the derivative and the integral as the area under a curve
- Apply Taylor series to numerically approximate functions
- Apply Simpson's rule to numerically evaluate integrals
- Solve simple first and second order ordinary differential equations
- Apply and select the appropriate mathematical techniques to solve a variety of associated engineering problems

Lecture 11: Differentiation-Part 3

9. Differentiation of inverse hyperbolic functions

$\frac{d}{dx} = '$

Ex1: $y = \sinh^{-1} x$, determine $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Sol: $\because y = \sinh^{-1} x \quad \therefore \quad x = \sinh y \quad \therefore \quad \dfrac{dx}{dy} = \cosh y$

$\cosh^2 y - \sinh^2 y = 1 \quad \therefore \quad \dfrac{dx}{dy} = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}} \qquad \boxed{\therefore \left(\sinh^{-1} x\right)' = \dfrac{1}{\sqrt{1+x^2}}.}$

Ex2: $y = \cosh^{-1} x$, determine $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Sol: $\because y = \cosh^{-1} x \quad \therefore \quad x = \cosh y \quad \therefore \quad \dfrac{dx}{dy} = \sinh y = \sqrt{\cosh^2 y - 1}$

$= \sqrt{x^2 - 1}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2-1}} \qquad \boxed{\therefore \left(\cosh^{-1} x\right)' = \dfrac{1}{\sqrt{x^2-1}}.}$

Ex3: $y = \tanh^{-1} x$, determine $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Sol: $\because y = \tanh^{-1} x \quad \therefore \quad x = \tanh y \quad \therefore \quad \dfrac{dx}{dy} = \left(\dfrac{\sinh y}{\cosh y}\right)' = \dfrac{(\sinh y)' \cdot \cosh y - \sinh y \cdot (\cosh y)'}{\cosh^2 y}$

$= \dfrac{\cosh^2 y - \sinh^2 y}{\cosh^2 y} = \dfrac{1}{\cosh^2 y}$

$\therefore \dfrac{dy}{dx} = \cosh^2 y = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y}$

$\boxed{\therefore \left(\tanh^{-1} x\right)' = \dfrac{1}{1-x^2}}$

$\boxed{\text{quotient Rule}}$

$\boxed{\tanh^2 x = 1 - \operatorname{sech}^2 x}$

Ex4: $y = \sinh^{-1}(\tan x)$, determine $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Sol: $\dfrac{dy}{dx} = \left(\sinh^{-1}(\underbrace{\tan x}_{u})\right)' = \dfrac{1}{\sqrt{1 + \underbrace{\tan^2 x}_{u}}} \cdot (\tan x)' = \dfrac{1}{\cancel{\sec x}} \cdot \sec^{\cancel{2}} x$

$= \sec x$

with respect to x $\qquad \dfrac{dy}{du} \cdot \dfrac{du}{dx} \qquad$ with respect to x

chain Rule

10. Maclausrin's series and Taylor's series

$$\sin x = x^1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$+ \quad - \quad + \quad -$

Maclaurin's series for $\sin x$

x is close to 0

$\sin x \approx x \qquad$ $\boxed{\text{if } x \text{ is very small, and it is close to } 0.}$

$\sin 0.01 \approx 0.01 - \dfrac{0.01^3}{6}$ = checked by your side.

$\sin 0 = a + b\cdot 0 + c\cdot 0^2 + d\cdot 0^3 + e\cdot 0^4 + \cdots$ $\therefore a = 0$ ✓ $(\sin x)' = \cos x$

$\to \cos x = 0 + b + 2cx + 3dx^2 + 4ex^3 + \cdots$

$\cos 0 = b + 2c\cdot 0 + 3d\cdot 0^2 + 4e\cdot 0^3 + \cdots$ $\therefore 1 = b$ ✓

$\to -\sin x = 2c + 6dx + 12ex^2 + \cdots$ $(\cos x)' = -\sin x$

$-\sin 0 = 2c + 6d\cdot 0 + 12e0^2 + \cdots$ $\therefore 0 = c$ ✓

$\to -\cos x = 0 + 6d + 24ex + \cdots$

Proof. Let $\sin x = a + bx + cx^2 + dx^3 + ex^4 + \cdots$, $\boxed{\text{put } x = 0}$

$-\cos 0 = 6d + 24e\cdot 0 + \cdots$ $\therefore -1 = 6d$ $\therefore d = -\frac{1}{6}$ ✓

$\to \sin x = 24e + f(x) + \cdots$

$0 = 24e + 0 + \cdots$ $\therefore e = 0$ ✓ $6 = 1\times 2\times 3 = 3!$

$\sin x = a + bx + cx^2 + dx^3 + ex^4 + \cdots$

$= 0 + 1\cdot x + 0\cdot x^2 - \frac{1}{6}x^3 + 0\cdot x^4 + \cdots$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

x is very small, close to 0

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$ ✓ $\leftarrow$ taking prime on both sides of $\sin x$

approximation (estimation)

$\cos 0.05 \approx 1 - \frac{0.05^2}{2} = 0.99875$

$$\text{Maclaurin's series: } f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$ ☆

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$ ✓

Ex: $f(x) = \tan x$, $f(0) = 0$

$f'(x) = \sec^2 x = 1 + \tan^2 x$, $f'(0) = 1 + 0^2 = 1$

$f''(x) = (1 + \tan^2 x)' = 2\tan x\cdot \sec^2 x$

$= 2\tan x(1 + \tan^2 x)$

$f''(0) = 2\cdot 0(1 + 0^2) = 0$ $\boxed{\sec^2 x = 1 + \tan^2 x}$

$f'''(x) = (2\tan x + 2\tan x\tan^2 x)'$

$= 2\sec^2 x + 2(\tan x)'\tan^2 x + 2\tan x(\tan^2 x)'$

$= 2\sec^2 x + 2\sec^2 x\cdot\tan^2 x + 2\tan x\cdot 2\tan x(\tan x)'$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

$$\text{Taylor's series: } f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

Ex1: $y = \sinh^{-1} 5.02$, approximate y.

$f'''(0) = 2\cdot 1 + 2\cdot 0 + 2\cdot 0 = 2$

$\vdots$

$\therefore \tan x = x + \frac{x^3}{3} + \cdots$

$\tan 0.02 \approx x + \frac{x^3}{3} = 0.02 + \frac{0.02^3}{3}$.

Note: $y = \sin^{-1} x = \arcsin x$, single function $x \to \sin^{-1} x$

$y = (\sin x)^{-1} = \frac{1}{\sin x}$ compound function $x \to \sin x \to \frac{1}{\sin x}$

$\sin^{-1} x \neq (\sin x)^{-1} = \frac{1}{\sin x}$.

Ex2: $y = \cosh^{-1} 1.01$, approximate y.

11. Newton-Raphson iterative method

Curve $y = f(x)$ is given, A is the point passing through x-axis with $f(x) = 0$, P is a point on the curve near to point A, then point B (or $x = x_0$) is an approximate value of the root of $f(x) = 0$, a better approximation is given by $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.

Ex1: The equation $x^3 - 3x - 4 = 0$ with properties $f(1) < 0$ and $f(3) > 0$ admits a root near 2. Find a better approximation to the root.

Ex2: The equation $2x^3 - 7x^2 - x + 12 = 0$ has a root near to $x = 1.5$. Use the Newton-Raphson method to find the root to two decimal places.

12. Maximum, minimum, point of inflexion

Given a function $y = f(x)$, stationary points are defined as $y'(x) = 0$.

$y'(x) = 0$, it may be a maximum, may be a minimum, may be a point of inflexion (i.e., S-bend form)

$y''(x) > 0$, maximum

$y''(x) < 0$, minimum

$y''(x) = 0$, may be points of inflexion (if yes, then change of sign occurs)

Ex1: $y = x^2$, to find stationary points, maximum, minimum.

Ex2: For $y = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x - 5$, find the points of inflexion.

Ex3: For $y = 3x^5 - 5x^4 + x + 4$, find the points of inflexion.