# EE206 Assignment 4 *

Due 28th Oct.

1. Solve the given differential equations by undetermined coefficients

   (a) $y'' + 4y' + 4y = \cos(x) + 3\sin(2x)$

1. (a) $y''+4y'+4y=\cos(x)+3\sin(2x)$

$$y''+4y'+4y=0$$

auxiliary equation: $m^2+4m+4=0 \Rightarrow m=-2$

$$\therefore y_1(x)=e^{-2x},\quad y_2(x)=xe^{-2x}$$

$$y_c=C_1e^{-2x}+C_2xe^{-2x}$$

Now for $y_p$ we should consider combinations of ~~$\sin x$~~ $\cos x$ and $\sin(2x)$ and their derivatives so:

$$\cancel{y_p=A\cos x+B\sin x+}$$

$$y_p=A\sin x+B\cos x+C\sin(2x)+D\cos(2x)$$

$$y_p'=A\cos x-B\sin x+2C\cos(2x)-2D\sin(2x)$$

$$y_p''=-A\sin x-B\cos x-4C\sin(2x)-4D\cos(2x)$$

$$\therefore y_p''+4y_p'+4y_p=(4A+3B)\cos x+(3A-4B)\sin x+8C\cos(2x)-8D\sin(2x)$$

$$=\cos x+3\sin(2x)$$

$$\therefore \begin{cases}4A+3B=1\\3A-4B=0\\C=0\\-8D=3\end{cases}\quad \therefore \begin{cases}A=\frac{4}{25}\\B=\frac{3}{25}\\C=0\\D=-\frac{3}{8}\end{cases}\quad \therefore y_p=\frac{4}{25}\sin x+\frac{3}{25}\cos x-\frac{3}{8}\cos(2x)$$

$$\therefore y=y_c+y_p=C_1e^{-2x}+C_2xe^{-2x}+\frac{4}{25}\sin x+\frac{3}{25}\cos x-\frac{3}{8}\cos(2x)$$

(b) $y''-10y'+25y=40x+3$

(b) $y'' - 10y' + 25y = 40x + 3$

$\because y'' - 10y' + 25y = 0 \Rightarrow a = 1,\ b = -10,\ c = 25.$

$\therefore am^2 + bm + c = 0 \Rightarrow m = 5 \quad \therefore y_1 = e^{5x}$

$y_2 = \cancel{5x} e^{5x} \int \frac{e^{10\int dx}}{(e^{5x})^2} dx = e^{5x} \int dx = xe^{5x}$

$\therefore y_c = C_1 e^{5x} + C_2 x e^{5x}$

the particular solution:

$g(x) = 40x + 3 \Rightarrow y_p = Ax + B,\ y_p' = A,\ y_p'' = 0$

$y'' - 10y' + 25y = \cancel{B}40 + 3 \Rightarrow 0 - 10A + 25Ax + 25B = 40x + 3$

$\Rightarrow 25Ax + (-10A + 25B) = 40x + 3$

$\therefore A = \frac{8}{5},\ B = \frac{19}{25}$

$\therefore y_p = \frac{8}{5}x + \frac{19}{25}$

$\therefore y = y_c + y_p = C_1 e^{5x} + C_2 x e^{5x} + \frac{8}{5}x + \frac{19}{25}$

(c) $\dfrac{d^2x}{dt^2} + \omega^2 x = F_0 \sin(\omega t),\ x(0) = x'(0) = 0;$

►First we solve the homogeneous equation: $\frac{d^2x}{dt^2}+\omega^2 x=0$ $\quad a=1, b=0, c=\omega^2$

►auxiliary equation

$$
\begin{aligned}
am^2+bm+c&=0\\
m^2+\omega^2&=0\\
m^2&=-\omega^2\\
m&=\pm i\omega\\
x_1=e^{i\omega}&\quad x_2=e^{-i\omega}
\end{aligned}
$$

►This gives the complementary solution:

$$
\begin{aligned}
x_c&=c_1x_1(\omega)+c_2x_2(\omega)\\
x_c&=c_1\cos(\omega t)+c_2\sin(\omega t)
\end{aligned}
$$

►We now get the particular solution:

$$
\begin{aligned}
g(t)=F_0\sin(\omega t)\quad \Rightarrow x_p&=At\sin(\omega t)+Bt\cos(\omega t),\\
x_p'&=A\sin(\omega t)+B\cos(\omega t)+A\omega t\cos(\omega t)-B\omega t\sin(\omega t),\\
x_p''&=2A\omega\cos(\omega t)-2B\omega\sin(\omega t)-A\omega^2 t\sin(\omega t)-B\omega^2 t\cos(\omega t)
\end{aligned}
$$

Subbing this into $\frac{d^2x}{dt^2}+\omega^2x=F_0\sin(\omega t)$ gives:

$$A=0,\;\; B=-\frac{F_0}{2\omega},\quad \Rightarrow x_p=-\frac{F_0t}{2\omega}\cos(\omega t)$$

►Which gives our full solution:

$$x=x_c+x_p\quad \Rightarrow x=c_1\cos(\omega t)+c_2\sin(\omega t)-\frac{F_0t}{2\omega}\cos(\omega t)$$

►Imposing initial conditions: $x(0)=0,\; x'(0)=0$

$$
\begin{aligned}
x'(t)&=-c_1\omega\sin(\omega t)+c_2\omega\cos(\omega t)-\frac{F_0}{2\omega}\cos(\omega t)+\frac{F_0t}{2}\sin(\omega t)\\
x(0)&=c_1=0\quad \Rightarrow c_1=0\\
x'(0)&=c_2\omega-\frac{F_0}{2\omega}=0\quad \Rightarrow c_2=\frac{F_0}{2\omega^2}\\
x(t)&=\frac{F_0}{2\omega^2}\sin(\omega t)-\frac{F_0t}{2\omega}\cos(\omega t)
\end{aligned}
$$

2. Solve the given differential equations by variation of parameters

(a) $4y''-y=xe^{\frac{x}{2}}$, $y(0)=0$, $y'(0)=1$

$$y'' - \frac{1}{4}y = \frac{1}{4}xe^{\frac{x}{2}}; \qquad a = 1, b = 0, c = -\frac{1}{4}, f(x) = \frac{1}{4}xe^{\frac{x}{2}}$$

►auxiliary equation: $am^2 + bm + c = 0$

$$m^2 - \frac{1}{4} = 0$$
$$m = \pm\frac{1}{2}$$

►This gives the complementary solution: $y_c = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x}$

►Now we get $u_1'$ and $u_2'$:

$$\begin{array}{ll} u_1' = -\frac{y_2(x)f(x)}{W}, & u_2' = \frac{y_1(x)f(x)}{W} \\ u_1' = -e^{-\frac{1}{2}x}\frac{1}{4}xe^{\frac{x}{2}}(-1) & u_2' = e^{\frac{1}{2}x}\frac{1}{4}xe^{\frac{x}{2}}(-1) \\ \Rightarrow \frac{du_1}{dx} = \frac{x}{4} & \Rightarrow \frac{du_2}{dx} = -\frac{1}{4}xe^x \\ \int du_1 = \frac{1}{4}\int xdx & \int du_2 = -\frac{1}{4}\int xe^x dx \\ u_1 = \frac{1}{8}x^2, & u_2 = -\frac{1}{4}e^x(x-1) \end{array}$$

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$
$$y_p = \frac{1}{8}x^2 e^{\frac{1}{2}x} - \frac{1}{4}xe^{\frac{1}{2}x} + \frac{1}{4}e^{\frac{1}{2}x}$$

►So our full solution is: $y = y_c + y_p$

$$y(x) = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x} + \frac{1}{8}x^2 e^{\frac{1}{2}x} - \frac{1}{4}xe^{\frac{1}{2}x}$$

►Imposing initial conditions: $y(0) = 0, \quad y'(0) = 1$

$$y'(x) = \frac{c_1}{2}e^{\frac{1}{2}x} - \frac{c_2}{2}e^{-\frac{1}{2}x} + \frac{1}{4}xe^{\frac{1}{2}x} + \frac{1}{16}x^2 e^{\frac{1}{2}x} - \frac{1}{4}e^{\frac{1}{2}x} - \frac{1}{8}xe^{\frac{1}{2}x}$$
$$y(0) = c_1 + c_2 = 0 \qquad \Rightarrow c_1 = -c_2$$
$$y'(0) = \frac{c_1}{2} - \frac{c_2}{2} - \frac{1}{4} = 1 \qquad \Rightarrow c_2 = -\frac{5}{4} \qquad \Rightarrow c_1 = \frac{5}{4}$$
$$y(x) = \frac{5}{4}e^{\frac{1}{2}x} - \frac{5}{4}e^{-\frac{1}{2}x} + \frac{1}{8}x^2 e^{\frac{1}{2}x} - \frac{1}{4}xe^{\frac{1}{2}x}$$

(b) $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}; \quad y(0) = 1, \quad y'(0) = 0$ **[2]**

(b). $y''+2y'-8y=2e^{-3x}-e^{-x}$, $y(0)=1$, $y'(0)=0$

$y''+2y'-8y=0 \Rightarrow a=1, b=2, c=-8, f(x)=2e^{-3x}-e^{-x}$

auxiliary equa: $am^2+bm+c=0 \Rightarrow m^2+2m-8=0$

$\therefore m_1=-4, m_2=2$

$\therefore y_c=C_1e^{-4x}+C_2e^{2x}$

Now we get $u_1'$ and $u_2'$:

$$u_1'=-\frac{e^{2x}[2e^{-3x}-e^{-x}]}{6e^{-2x}}=-\frac{1}{6}e^{4x}[2e^{-3x}-e^{-x}]$$

$\therefore u_1=-\frac{1}{3}e^{x}+\frac{1}{18}e^{3x}$

$$u_2'=\frac{e^{-4x}[2e^{-3x}-e^{-x}]}{6e^{-2x}}=\frac{1}{3}e^{-5x}-\frac{1}{6}e^{-3x}$$

$\therefore u_2'=-\frac{1}{15}e^{-5x}+\frac{1}{18}e^{-3x}$

$\cancel{\therefore y=y_c+y_p=C_1e^{-4x}+C_2e^{2x}-\frac{1}{3}e^{x}+\frac{1}{18}e^{3x}-\frac{1}{15}e^{-5x}+\frac{1}{18}e^{-3x}}$

$\therefore y_p=u_1(x)y_1(x)+u_2(x)y_2(x)=\frac{1}{9}e^{-x}-\frac{2}{5}e^{-3x}$

$y=y_c+y_p=C_1e^{-4x}+C_2e^{2x}+\frac{1}{9}e^{-x}-\frac{2}{5}e^{-3x}$

$\therefore y'(x)=-4C_1e^{-4x}+2C_2e^{2x}-\frac{1}{9}e^{-x}+\frac{6}{5}e^{-3x}$

$\therefore y(0)=C_1+C_2+\frac{1}{9}-\frac{2}{5}=1$

$y'(0)=-4C_1+2C_2-\frac{1}{9}+\frac{6}{5}=0$ $\therefore C_1=\frac{11}{18}$, $C_2=\frac{61}{90}$

$\therefore y=\frac{11}{18}e^{-4x}+\frac{61}{90}e^{2x}+\frac{1}{9}e^{-x}-\frac{2}{5}e^{-3x}$

*(c) $y'' + y = \sec(x)\tan(x)$

$$y'' + y = 0; \qquad f(x) = \sec(x)\tan(x)$$

Recall the solutions to this are

$$\sin(x) \text{ and } \cos(x).$$

►This gives the complementary solution: $y_c = c_1 y_1 + c_2 y_2 = c_1 \cos(x) + c_2 \sin(x)$

►Now we get $u_1'$ and $u_2'$:

$$\begin{array}{ll} u_1' = -\frac{y_2(x)f(x)}{W}, & u_2' = \frac{y_1(x)f(x)}{W} \\ u_1' = -\frac{\sin(x)\sec(x)\tan(x)}{1}, & u_2' = \frac{\cos(x)\sec(x)\tan(x)}{1} \\ u_1 = -\int \tan^2(x)dx, & u_2' = \int \tan(x)dx \\ u_1 = -\int 1 + \tan^2(x) - 1dx, & u_2' = -\ln(\cos(x)) \\ u_1 = -\int \sec^2(x) - 1dx, & u_2' = -\ln(\cos(x)) \\ u_1 = -(\tan(x) - x), & u_2' = -\ln(\cos(x)) \\ u_1 = x - \tan(x), & \underline{u_2' = -\ln(\cos(x))} \end{array}$$

?( ) How

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$
$$y_p = (x - \tan(x))\cos(x) - \ln(\cos(x))\sin(x)$$

►So our full solution is: $y = y_c + y_p$

$$y = c_1\cos(x) + c_2\sin(x) + x\cos(x) - \ln(\cos(x))\sin(x)$$

Notice that we dropped the $\tan(x)\cos(x)$ term since this is simply $\sin(x)$ which can be absorbed into the constant $c_2$ from $y_c$.