Engineering Mathematics 1 (Fall 2021)

Prof. Fengying Wei
E-mail: weifengying@fzu.edu.cn
School of Mathematics and Statistics, Fuzhou University

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Students should be able to (after learning)

- Add, subtract and multiply complex numbers
- Convert complex numbers between Cartesian and polar forms
- Differentiate all commonly occurring functions including polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of a derivative, namely the derivative as a tangent and the derivative as a rate of change
- Integrate certain standard functions, constructed from polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of integration, namely the integral as the inverse of the derivative and the integral as the area under a curve
- Apply Taylor series to numerically approximate functions
- Apply Simpson's rule to numerically evaluate integrals
- Solve simple first and second order ordinary differential equations
- Apply and select the appropriate mathematical techniques to solve a variety of associated engineering problems

Lecture 19: Integration-Part 5

Integration applications 1

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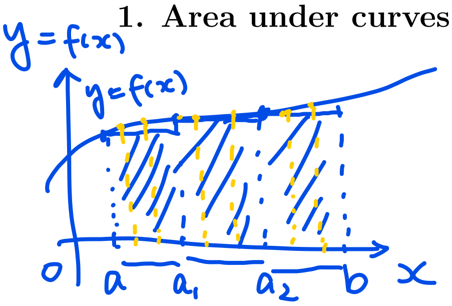
* max $\{a_1-a,\ a_2-a_1,\ b-a_2\} \triangleq \delta x$

1. Area under curves $A = \int_a^b y\,\mathrm{d}x,\ y = f(x)$

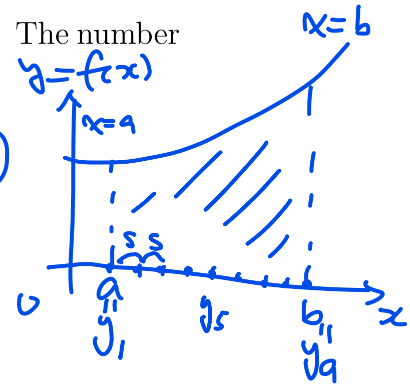


$A = \int_a^b y\,dx = \lim_{\delta x \to 0} \sum_{\delta x} y\,\delta x$ Definite Integral

2. Simpson's rule

(a) Divide the figure into any even number (n) of equal-width strips (width s).

(b) Number and measure each ordinate: $y_1, y_2, y_3, \cdots, ..., y_{n+1}$. The number of ordinates will be one more than the number of strips.



(c) The area A of the figure is then given by Simpson's rule:

$A \approx \frac{s}{3}[F + L + 4E + 2R]$ $\quad s = \frac{b-a}{8}$

where s width of each strip,

$F + L$ sum of the first and last ordinates, $y_1 + y_9$

$4E$ 4 × the sum of the even-numbered ordinates $4(y_2 + y_4 + y_6 + y_8)$

$2R$ 2 × the sum of the remaining odd-numbered ordinates. $2 \times (y_3 + y_5 + y_7)$

Note: Each ordinate is used once C and only once.

Ex Evaluate the integral $\int_2^6 y\mathrm{d}x, y = f(x)$.

$a = 2,\ b = 6,\ s = \frac{b-a}{8} = \frac{6-2}{8} = \frac{1}{2}$

Sol: Simpson's rule is $A \approx \frac{s}{3}[F+L+4E+2R]$

$\because s = \frac{b-a}{8} = \frac{6-2}{8} = \frac{1}{2}$ $\qquad = \frac{1}{6}[19+176.8+70]$

$\therefore F+L = 7.5+11.5 = 19$ $\qquad = 44.3.$

$\therefore 4E = 4[8.2+11.5+12.8+11.7] = 4 \times 44.2 = 176.8$

$2R = 2[10.3+12.4+12.3] = 2\times 35 = 70$

Ordinate no.	1	2	3	4	5	6	7	8	9 −1
Length	7.5	8.2	10.3	11.5	12.4	12.8	12.3	11.7	11.5

3. Definite integrals

$\int_a^b [f_1(x)+f_2(x)]dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx$

$\int_a^b k f(x)dx = k\int_a^b f(x)dx, \quad k \in \mathbb{R}$

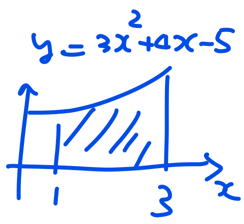
$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad c \in (a,b)$

Regular equation $y = f(x)$:

$$\int_a^b y\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a), \text{ where } y = f(x) = F'(x).$$

Parametric equation $x = f(t)$, $y = F(t)$: $\qquad \frac{dx}{dt} = \frac{d}{dt}x = x' = f'(t)$

$$\int_{x_1}^{x_2} y\,\mathrm{d}x = \int_{x_1}^{x_2} F(t)\,\mathrm{d}x = \int_{t_1}^{t_2} F(t)\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = \int_{t_1}^{t_2} F(t) f'(t)\,dt$$



Ex1: Find the area under curve $y = 3x^2 + 4x - 5$ between $x = 1$ and $x = 3$.

Sol: $\int_1^3 y\,dx = \int_1^3 (3x^2+4x-5)dx = [x^3 + 2x^2 - 5x]\Big|_1^3$

$= 27+18-15-(1+2-5) = 32.$

Ex2: Find the area under curve $y = \dfrac{1}{x+5}$ between $x = 0$ and $x = 5$.

Sol: $\int_0^5 y\,dx = \int_0^5 \frac{1}{x+5}dx = \int_0^5 \frac{1}{x+5}\,d\underbrace{(x+5)}_{u} = \ln(x+5)\Big|_0^5$

$= \ln 10 - \ln 5 = \ln 2.$

Ex3: Evaluate $\displaystyle\int_1^e x^2 \ln x\,\mathrm{d}x$. $\qquad u = \ln x \qquad x^2dx = dV \Rightarrow V = \frac{x^3}{3}$

Sol: $\int_1^e x^2\ln x\,dx = \ln x \cdot \frac{x^3}{3} - \int_1^e \frac{x^3}{3}\cdot d(\ln x) = \frac{x^3}{3}\ln x - \int_1^e \frac{x^3}{3}\cdot\frac{1}{x}dx$

$= \frac{x^3}{3}\ln x - \int_1^e \frac{x^2}{3}dx = \left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]\Big|_1^e$

$= \frac{e^3}{3}\cdot 1 - \frac{e^3}{9} - \left[\frac{1}{3}\cdot 0 - \frac{1}{9}\right] = \frac{2}{9}e^3 + \frac{1}{9} = \frac{1}{9}(2e^3+1)$

$f'(t) = x' = (at^2)' = 2at$

$x = f(t) \quad y = F(t)$

Ex4: A curve has parametric equations $x = at^2, y = 2at$, find the area bounded by the curve, the x-axis and the ordinates $t = 1$ and $t = 2$.

Sol: $A = \int_{t_1}^{t_2} F(t) f'(t)\,dt = \int_1^2 2at \cdot 2at\,dt$

$= \int_1^2 4a^2t^2\,dt = 4a^2 \cdot \int_1^2 t^2\,dt = 4a^2 \cdot \frac{t^3}{3}\Big|_1^2$

$= 4a^2\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{7}{3}4a^2 = \frac{28a^2}{3}$.

Ex5: Let $x = \theta - \sin\theta, y = 1 - \cos\theta$, find the area between $\theta = 0$ and $\theta = \pi$.

$x = f(t) \quad y = F(t)$

Sol: $A = \int_{t_1}^{t_2} F(t) f'(t)\,dt = \int_0^\pi (1-\cos\theta)(\theta - \sin\theta)'\,d\theta$

$= \int_0^\pi (1-\cos\theta)(1-\cos\theta)\,d\theta = \int_0^\pi (1 + \cos^2\theta - 2\cos\theta)\,d\theta$

$= \int_0^\pi (1 + \frac{1}{2}(1+\cos 2\theta) - 2\cos\theta)\,d\theta = \int_0^\pi \frac{3}{2}\,d\theta + \frac{1}{2}\int_0^\pi \cos 2\theta\,d\theta - 2\int_0^\pi \cos\theta\,d\theta$

$= \frac{3}{2}\theta\Big|_0^\pi + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\Big|_0^\pi - 2\sin\theta\Big|_0^\pi$

$= \frac{3}{2}\pi + \frac{1}{4}\cdot 0 - 2\cdot 0 = \frac{3}{2}\pi$.

4. Means and root mean square (RMS) values

$$\text{Mean} = \frac{\text{Area}}{\text{Length of the base line}}$$

Ex1: To find the mean value of $y = x^2 + 4x + 1$ between $x = -1$ and $x = 2$.

Sol: $A = \int_{-1}^2 y\,dx = \int_{-1}^2 (x^2 + 4x + 1)\,dx = \left(\frac{x^3}{3} + 2x^2 + x\right)\Big|_{-1}^2$

$= \frac{8}{3} + 8 + 2 - \left(\frac{-1}{3} + 2 - 1\right) = 3 + 10 - 1 = 12$.

Length of base line $= 2 - (-1) = 3 \quad \therefore M = \frac{12}{3} = 4$.

Ex2: Find mean value of $y = 3\sin 5t + 2\cos 3t$ between $t = 0$ and $t = \pi$.

Sol: $A = \int_0^\pi y\,dx = \int_0^\pi (3\sin 5t + 2\cos 3t)\,dt = \left[3\cdot\frac{1}{5}(-\cos 5t) + 2\cdot\frac{1}{3}\cdot\sin 3t\right]_0^\pi$

$= -\frac{3}{5}(\cos 5\pi - \cos 0) + \frac{2}{3}(\sin 3\pi - \sin 0) = \frac{6}{5}$

Length of base line $= \pi - 0 = \pi \quad \therefore M = \frac{6}{5}\cdot\frac{1}{\pi} = \frac{6}{5\pi}$.

$$\text{RMS} = \sqrt{\text{Mean value of } y^2 \text{ between } x = a \text{ and } x = b}$$

$\Updownarrow$

$\bigstar \ (\text{RMS})^2 = \frac{1}{b-a}\underbrace{\int_a^b y^2\,dx}_{A}$

$b-a \downarrow$ Length

Ex1: Find RMS value of $y = 400 \sin 200\pi t$ between $\underbrace{t = 0}_{}$ (a) and $t = \frac{1}{100}$ (b).

Sol: $\text{Length} = b - a = \frac{1}{100} - 0 = \frac{1}{100}$

$$A = \int_0^{\frac{1}{100}} y^2 dx = \int_0^{\frac{1}{100}} 160000 \sin^2 200\pi t \, dt = \int_0^{\frac{1}{100}} 160000 \cdot \frac{1}{2}(1 - \cos 400\pi t) dt$$

$$= 80000 \int_0^{\frac{1}{100}} (1 - \cos 400\pi t) dt = 80000 \cdot t \Big|_0^{\frac{1}{100}} - 800\cancel{00} \sin 400\pi t \cdot \frac{1}{\cancel{400}\pi} \Big|_0^{\frac{1}{100}}$$

$$= 800 - \frac{200}{\pi}\left[\sin 4\pi - 0\right] = 800$$

$$(\text{RMS})^2 = \frac{A}{\text{Length}} = \frac{800}{\frac{1}{100}} = 80000$$

$\therefore \text{RMS} \approx 280.$