Lecture 8: Convolution and Filtering

EE213 - Introduction to Signal Processing

Semester 1, 2019

Outline

- Explain linear time invariant (LTI) systems.
- Characterize output of discrete and continuous LTI systems.

  → Convolution sums and convolution integrals.
- Analyze LTI systems in the frequency domain.
- Identify different types of filters.

Linear Time Invariant Systems

- Consider a discrete signal processing system.
  - Both input and output signals are discrete.
  - y[n] = T{x[n]} is the output or response of the system to the input signal x[n].
  - T[ ] is called the transformation operator.
  - The output of a discrete system is determined by T[ ].
  - By designing the system properly, we can extract the required information from x[n] or change its characteristics.

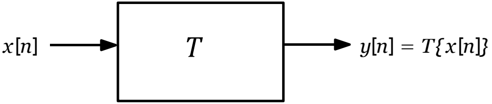


illustration of a general discrete signal processing system

Linear Time Invariant Systems...

- We consider a special type of systems, called LTI systems.
- Linearity:

$$T\{a\, x_1[n] + b\, x_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\} \tag{1}$$

where a and b are constants.

$$a\, x_1[n] \longrightarrow \boxed{T} \longrightarrow a\, T\{x_1[n]\}$$

$$b\, x_2[n] \longrightarrow \boxed{T} \longrightarrow B\, T\{x_2[n]\}$$

$$a x_1[n] + b\, x_2[n] \longrightarrow \boxed{T} \longrightarrow aT\{x_1[n]\} + bT\{x_2[n]\}$$

illustration of linearity property

Linear Time Invariant Systems...

- **Time invariance**: The principle of time invariance states that the behaviour of the system *should not change with time*. In a time invariant system if input is delayed by n_0 the output will also get delayed by n_0.

$$T\{x[n-n_0]\} = y[n-n_0]$$

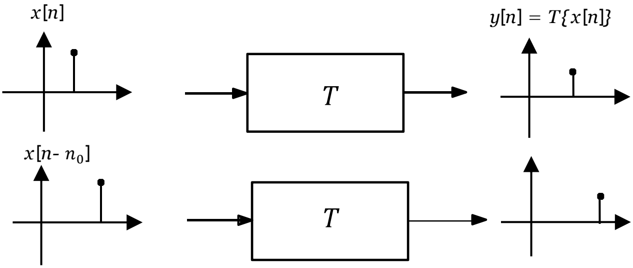


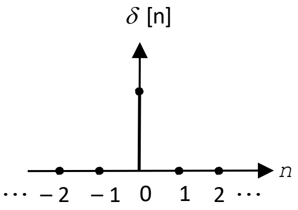
illustration of time-invariance property

- A good example of LTI systems is an **electrical circuit that is made up of resistors, capacitors, and inductors**.

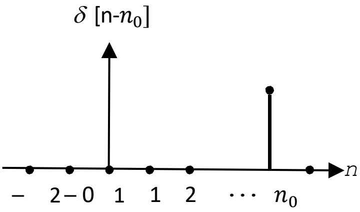
Signal Decomposition

- In order to determine the output of an LTI system, let us describe the input signal $x[n]$ in terms of impulse sequences $\delta\,[n]$.
- Impulse sequence

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



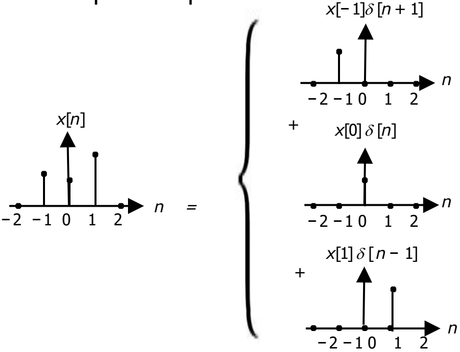
Graphical representation of $\delta\,[n]$



Graphical representation of δ [n-n_0[

Signal Decomposition...

- Graphical representation



Graphical representation of signal decomposition

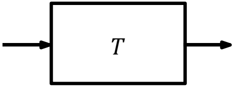
- Mathematical expression

$$X[n]=\sum_{k=-\infty}^{k=\infty} x[k]\delta[n-k]$$

Convolution Sum

- Output of LTI systems

x[n]=$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$



Output of LTI systems

y[n]=T$\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\}$

(linear property)

y[n]= $\sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\}$

(time invariance)

- Let h[n] = T$\{\delta]n]\}$.That is, h[n] is the response of the system to the impulse sequence $\delta]n]$.

  From the principle of time invariance, we know that T$\{\delta]n-k]\}$ = h[n − k]. Thus, we have

$$y]n] = \sum_{k=-\infty}^{\infty} x[k]h]n-k[$$
$$= \mathrm{x}]n] * h]n]$$

which is called *convolution sum*

Properties

Commutativity

$$f * g = g * f$$

Proof: By definition

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) \, d\tau$$

Changing the variable of integration to $u = t - \tau$ the result follows.

Associativity

$$f * (g * h) = (f * g) * h$$

Proof: This follows from using Fubini's theorem (i.e., double integrals can be evaluated as iterated integrals in either order).

Distributivity

$$f * (g + h) = (f * g) + (f * h)$$

Proof: This follows from linearity of the integral.

Associativity with scalar multiplication

$$a(f * g) = (af) * g$$

for any real (or complex) number a.

Convolution Sum...

Compute the convolution sum of $x_1[n] = \{\underset{\uparrow}{1}, 2\}$ and $x_2[n] = \{1, \underset{\uparrow}{-1}, 2\}$.

Solution:

$$x[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

or

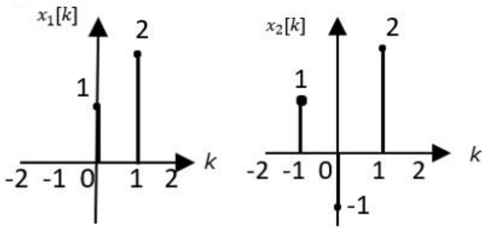
$$x[n] = x_2[n] * x_1[n] = \sum_{k=-\infty}^{\infty} x_2[k]x_1[n-k]$$

Convolution sum includes four steps:

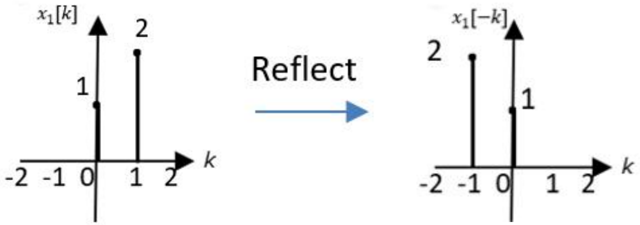
1. Replace n with k
2. Reflect: $x_1[k] \rightarrow x_1[-k]$
3. Shift: $x_1[-k] \rightarrow x_1[-(k-n)]$
4. Multiply & Sum

Convolution Sum...

Step 1:

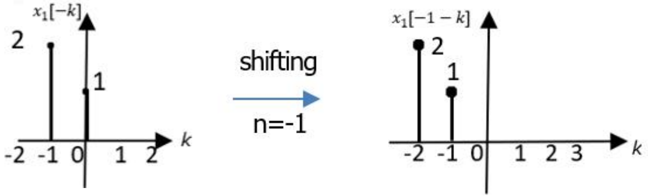


Step 2:

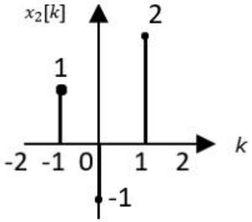


Convolution Sum...

Step 3:



Step 4:



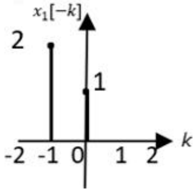
Multiple & Sum

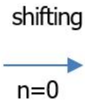
$2 \times 0 + 1 \times 1 + 0 \times (-1) + 0 \times 2 = 1$

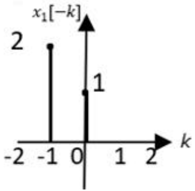
$x[-1] = 1$

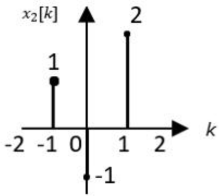
Convolution Sum...

Step 3:







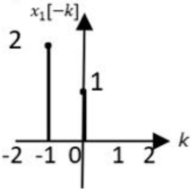


Step 4:

Multiple & Sum: $x[0] = 1$

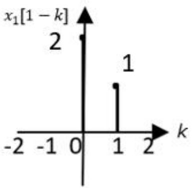
Convolution Sum...

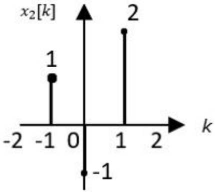
Step 3:



shifting

n=1

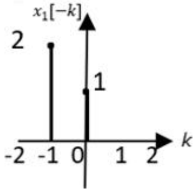




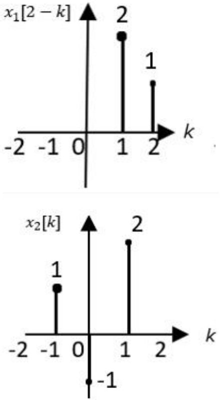
Step 4:

Multiple & Sum: $x[1] = 0$

Step 3:



$\xrightarrow[\text{n=2}]{\text{shifting}}$

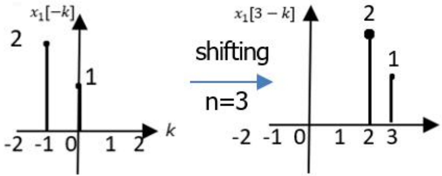


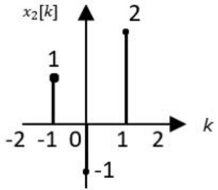
Step 4:

Multiple & Sum: $x[2] = 4$

Convolution Sum...

Step 3:



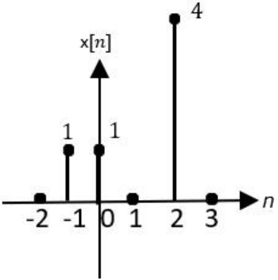


Step 4:

Multiple & Sum: $x[3] = 0$

Convolution Sum...

$$x[n] = \{\dots 1, \underset{\uparrow}{1}, 0, 4 \dots\}$$



Convolution Integral

Extend Discrete Time to Continuous Time

Convolutional Sum:

$$\mathrm{y}]n] = \sum_{k=-\infty}^{\infty} x[k]h]n-k[$$
$$= \mathrm{x}]n] * h]n]$$

Convolution integral:

$$\mathrm{y(t)} = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution Integral

- Define the impulse response of a continuous time system :

$$T\{\delta(t)\}=h(t)$$

- The response of the input signal $x(t)$ entering the system T is

$$\begin{aligned} y(t) &= T\{x(t)\} \\ &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \end{aligned}$$

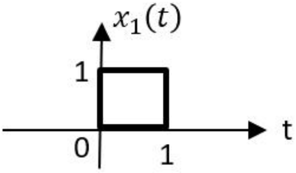
Convolution Integral...

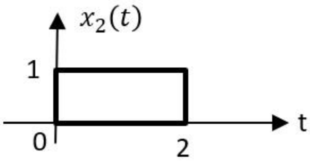
Example

Compute the convolution of $x_1(t) = u(t) - u(t-1)$ and $x_2(t) = u(t) - u(t-2)$.

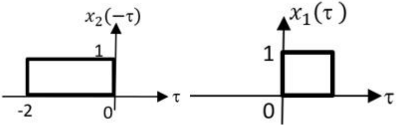
$$y(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) * x_2(t-\tau)d\tau$$

$$y(t) = x_1(t) * x_2(t) = x_2(t) * x_2(t)$$

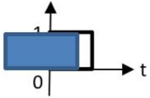




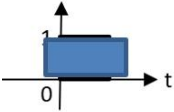
Convolution Integral



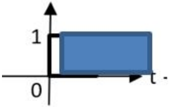
(1). $t < 0$: $y(t) = 0$



(2). $0 < t < 1$: $y(t) = \int_0^t 1 d\tau = t$

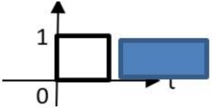


(3). $1 < t < 2$: $y(t) = \int_0^1 1 d\tau = 1$



(4). $2 < t < 3$: $y(t) = \int_{t-2}^1 1 d\tau = 3 - t$

(5). $3 < t$: $y(t) = 0$



Convolution Integral

$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 < t < 1 \\ 1 & 1 < t < 2 \\ 3 - t & 2 < t < 3 \\ 0 & t > 3 \end{cases}$$

Convolution integral includes four steps also:

1. Replace t with τ

2. Reflect : $x_1[\tau] \rightarrow x_1[-\tau]$

3. Shift: $x_1[-\tau]$ --> $x_1[t-\tau]$

4. Multiply overlapping parts of two signals and integrate overlapping parts

Important Property of Convolution

- The following table shows two important properties of convolution integral

Time domain	Frequency domain
$y(t) = x_1(t) * x_2(t)$	$Y(\omega) = X_1(\omega)X_2(\omega)$
$y(t) = x_1(t)x_2(t)$	$Y(\omega) = \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$

Table: Properties of convolution integral

- Roughly we can say that
  - **Convolution in time is equivalent to multiplication in frequency**
  - **Multiplication in time is equivalent to convolution in frequency,** subject to a scaling factor of $\frac{1}{2\pi}$

Frequency Domain Representation of LTI Systems

Convolution in time domain equivalent to multiplication in frequency domain:

$$\text{If}\quad f_1(t) \longleftrightarrow F_1(j\omega),\quad f_2(t) \longleftrightarrow F_2(j\omega)$$

$$\text{Then}\quad f_1(t) * f_2(t) \longleftrightarrow F_1(j\omega)F_2(j\omega)$$

Proof:

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau$$

Interchanging the order of integration

$$F[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau\right] \mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t$$

Using time shifting

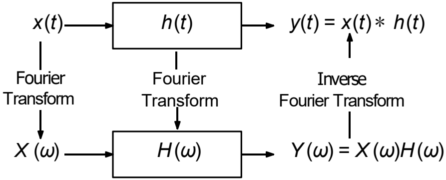
$$= \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_2(t-\tau)\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t\right]\mathrm{d}\tau$$

$$= \int_{-\infty}^{\infty} f_1(\tau) F_2(\mathrm{j}\omega)\mathrm{e}^{-\mathrm{j}\omega\tau}\,\mathrm{d}\tau$$

$$= F_1(j\omega)F_2(j\omega)$$

Frequency Domain Representation of LTI Systems

- If the input signal $x(t)$, the following figure describes the relationship between input and output of continuous LTI systems.



1. Calculate $X(\omega)$, the FT of $x(t)$.
2. Calculate $H(\omega)$, the FT of the impulse response $h(t)$.
3. Find the FT of the output of the system as $Y(\omega) = X(\omega)H(\omega)$.
4. Calculate the inverse FT of $Y(\omega)$ to find the time domain representation of the output $\boldsymbol{y}(t)$.

Frequency Response $H(\omega)$

$$h(t) \leftrightarrow H(\omega)$$

$$H(\omega)=|H(\omega)|e^{j\theta(\omega)}=\frac{|Y(\omega)|}{|F(\omega)|}\, e^{j[\varphi_y(\omega)\,-\,\varphi_f(\omega)]}$$

$|H(\omega)|$ is known as amplitude response, $\theta(\omega)$ is phase response.

Given a system with frequency response $H(\omega)$, the system response $y(t)$ to a sinusoid $\cos(\omega t + \theta)$ is given by

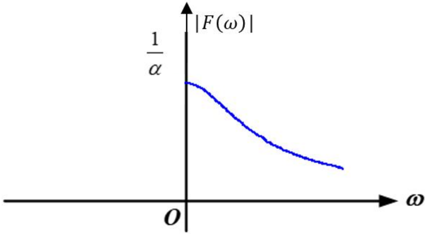
$$y(t) = |H(\omega)|\cos(\omega t + \theta + \theta(\omega))$$

Frequency Response

Given $H(\omega) = \frac{1}{\alpha + j\omega}$

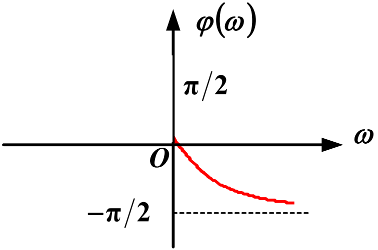
Amplitude response $|\mathbf{H}(\boldsymbol{\omega})| = \frac{\mathbf{1}}{\sqrt{\boldsymbol{\alpha}^2 + \boldsymbol{\omega}^2}}$

$$\begin{cases} \omega \to 0, & |H(\omega)| = \dfrac{1}{\alpha} \\ \omega \to +\infty, & |H(\omega)| \to 0 \end{cases}$$



Phase response $\Phi(\boldsymbol{\omega}) = \mathbf{arctan}\,(-\frac{\boldsymbol{\omega}}{\boldsymbol{\alpha}})$

$$\begin{cases} \omega \to 0, & \phi(\omega) = 0 \\ \omega \to +\infty, & \phi(\omega) \to -\dfrac{\pi}{2} \end{cases}$$



Filtering

- An LTI system yield an output signal with the spectrum $Y(\omega) = X(\omega)H(\omega)$. In a sense, $H(\omega)$ acts as a spectral shaping filter.
- The term **filter** is commonly used to describe a device that discriminates, according to some attribute of the objects applied at its input, what passes through it.
- We can see that an LTI sytem also performs a type of discrimination or filtering among various frequency components at its input.
- Filtering is one of the most basic operations in signal processing.

Ideal Filter Characteristics

- An *ideal* frequency-selective filter is one that exactly passes signals at one set of frequencies and completely rejects the rest.
- Ideal filters are classified as

→ **Lowpass filters.**

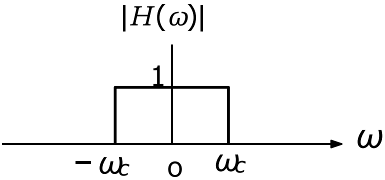
→ **Highpass filters.**

→ **Bandpass filters.**

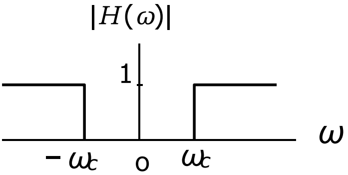
→ **Bandstop filters.**

Ideal Filter Characteristics...

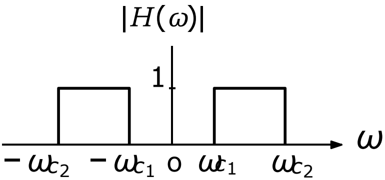
- Lowpass filters



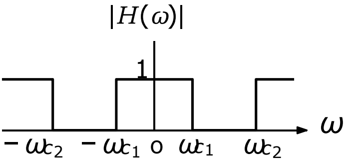
- Highpass filters



- Bandpass filters



- Bandstop filters



Ideal Filter Characteristics...

- Ideal filters are characterised by a linear phase response:

$$\angle H(\omega) = -\omega t_d$$

  where t_d is a constant.
- An ideal bandpass filter

$$H(\omega) = \begin{cases} Ce^{-j\omega t_d} & \omega_1 \leq \omega \leq \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

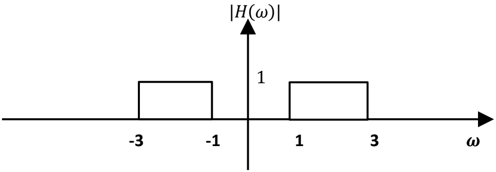
  where C and t_d are constants.
- The signal at the output of the filter has a spectrum

$$Y(\omega) = H(\omega)X(\omega) = CX(\omega)e^{-j\omega t_d} \qquad \omega_1 \leq \omega \leq \omega_2$$

- The time-domain output is a delayed and amplified-scaled version of the input.

$$y(t) = C\hat{x}(t - t_d)$$

The frequency response of the band-pass filter is shown in below, where the phase response $\varphi(\omega) = 0$.



Given the input signal

$$x(t) = \frac{sin3t}{t},$$

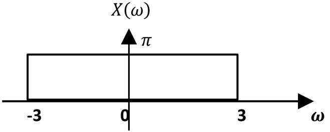
determine the spectrum of out signal $y(t)$.

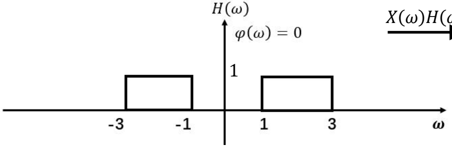
Solution:

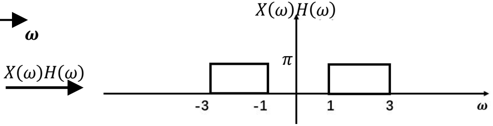
$$\text{rect}\left(\frac{t}{\tau}\right) \longleftrightarrow \tau \cdot Sa\left(\frac{\omega\tau}{2}\right) \qquad \text{with } \tau = 6\text{: } \text{rect}\left(\frac{t}{6}\right) \longleftrightarrow 6 \cdot Sa(3\omega)$$

$$\textit{Duality:}\ \ Sa(3t) \longleftrightarrow 2\pi\,\frac{1}{6}\text{rect}\left(-\frac{\omega}{6}\right)$$

$$x(t) = \frac{sin3t}{t} = 3Sa(3t) \longleftrightarrow \pi \cdot rect\left(\frac{\omega}{6}\right) = \pi g_6(\omega)$$







$\mathbf{Y}(\boldsymbol{\omega}) = \boldsymbol{X}(\boldsymbol{\omega})\boldsymbol{H}(\boldsymbol{\omega})$
$= \boldsymbol{\pi}[\boldsymbol{g}_2(\boldsymbol{\omega}+\mathbf{2}) + \boldsymbol{g}_2(\boldsymbol{\omega}-\mathbf{2})]$