1、(a) $\ln(5x\sqrt{x+8}) = \ln 5 + \ln x + \frac{1}{2}\ln x+8$

$$\Rightarrow \frac{d\ln(5x\sqrt{x+8})}{dx} = \frac{1}{x} + \frac{1}{2(x+8)} = \frac{3x+16}{2x(x+8)}$$

(b) $$\frac{d}{dx}\frac{\sqrt{1+2x}}{e^{3x}} = -3\cdot\frac{\sqrt{1+2x}}{e^{3x}} + \frac{1}{2}\cdot 2\cdot\frac{1}{e^{3x}\cdot\sqrt{1+2x}}$$

$$= -\frac{3\sqrt{1+2x}}{e^{3x}} + \frac{1}{e^{3x}\sqrt{1+2x}} = \frac{-3(1+2x)+1}{e^{3x}\sqrt{1+2x}}$$

$$= \frac{6x-2}{e^{3x}\sqrt{1+2x}}$$

(c) $$\frac{d\ln\sin x}{dx} = \frac{\cos x}{\sin x} = \cot x$$

2. (a) $u = x^2+1 \qquad du = 2x\,dx$

$$\int x\cot(x^2+1)\,dx = \frac{1}{2}\int \cot u\,du = \frac{1}{2}\int \frac{\cos u}{\sin u}\,du$$

$$= \frac{1}{2}\ln|\sin u| + C = \frac{1}{2}\ln|\sin(x^2+1)| + C$$

(b) $\displaystyle\int \frac{\sin(3\sqrt{x})\,dx}{\sqrt{x}}$ 令 $u = 3\sqrt{x} \qquad du = \cancel{3}\,\frac{3}{2\sqrt{x}}dx$

$$= \frac{2}{3}\int \sin(3\sqrt{x})\,d\,3\sqrt{x} = \frac{2}{3}\int \sin u\,du = -\frac{2}{3}\cos u + C$$

$$= -\frac{2}{3}\cos 3\sqrt{x} + C$$

(c) $u = \ln(t) \qquad du = \frac{1}{t}dt$

$$\int \frac{1}{t\ln t}dt = \int \frac{1}{u}du = \ln|u| + C = \ln|\ln t| + C$$

(d) $$\int t\cos t\,dt = \int t\,d\sin t = t\sin t - \int \sin t\,dt$$

$$= t\sin t + \cos t$$

3. (a) linear in y : 2^{nd} order

(b) nonlinear in y : 3^{rd} order

(c) linear in v ; nonlinear in u 1^{st} order

(d) nonlinear : 2^{nd} order

4. **Verify that the indicated functions are solutions to the given differential equations and state whether they are implicit or explicit solutions. Assume an appropriate interval I of definition.**

(a) $x^2y'' + xy' + y = 0;\qquad y = \cos(\ln(x))$ **[2]**

Explicit solution

$$\begin{aligned} y &= \cos(\ln(x)) \\ y' &= -\frac{\sin(\ln(x))}{x} \\ y'' &= \frac{\sin(\ln(x))}{x^2} - \frac{\cos(\ln(x))}{x^2} \end{aligned}$$

Using these in the above equation gives:

$$x^2\frac{\sin(\ln(x))}{x^2} - x^2\frac{\cos(\ln(x))}{x^2} - x\frac{\sin(\ln(x))}{x} + \cos(\ln(x))$$

$$= \sin(\ln(x)) - \cos(\ln(x)) - \sin(\ln(x)) + \cos(\ln(x)) = 0$$

(b) $2xydx + (x^2 - y)dy = 0;\quad -2x^2y + y^2 = 1$ **[2]**

Implicit solution.

$$\begin{aligned} -2x^2y + y^2 &= 1 \\ -2(2x)y - 2x^2\frac{dy}{dx} + (2y)\frac{dy}{dx} &= 0 \\ -4xy - 2(x^2 - y)\frac{dy}{dx} &= 0 \\ 2xy + (x^2 - y)\frac{dy}{dx} &= 0 \\ 2xydx + (x^2 - y)dy &= 0 \end{aligned}$$

5. **Use the Separation of Variables technique to solve the following first order differential equations.**

(a) $(1-x^2)\frac{dy}{dx}+x(y-3)=0$ **[2]**

$$(1-x^2)\frac{dy}{dx}+x(y-3)=0$$

$$(1-x^2)\frac{dy}{dx}=x(3-y)$$

$$\int\frac{1}{3-y}dy=\int\frac{x}{1-x^2}dx$$

Let $u=1-x^2 \quad \Rightarrow du=-2xdx \quad \Rightarrow -\frac{1}{2}du=xdx$

$$\int\frac{1}{3-y}dy=-\frac{1}{2}\int\frac{1}{u}du$$

$$(-1)\ln(3-y)=-\frac{1}{2}\ln(u)+c$$

$$\ln(3-y)=\ln(u^{\frac{1}{2}})-c$$

$$\ln(3-y)=\ln\left((1-x^2)^{\frac{1}{2}}\right)-c$$

$$\ln\left(\frac{3-y}{(1-x^2)^{\frac{1}{2}}}\right)=-c$$

$$\frac{3-y}{(1-x^2)^{\frac{1}{2}}}=C$$

$$y=3-C(1-x^2)^{\frac{1}{2}}$$

(b) $e^x y\frac{dy}{dx} = e^{-y} + e^{-2x-y}$; $y(0) = 0$ **[2]**

$$e^{y-x}e^x y\ \frac{dy}{dx} = e^{y-x}\left(e^{-y} + e^{-2x-y}\right)$$
$$e^y y\ \frac{dy}{dx} = e^{-x} + e^{-3x}$$
$$e^y y\ dy = e^{-x} + e^{-3x}dx$$
$$\int e^y y\ dy = \int e^{-x} + e^{-3x}\ dx$$

$$u = y \qquad du = dy$$
$$dv = e^y\ dy \qquad v = e^y$$

$$ye^y - \int e^y dy = -e^x - \frac{1}{3}e^{-3x} + c$$
$$ye^y - e^y = -e^x - \frac{1}{3}e^{-3x} + c$$

Imposing initial conditions: $y(0) = 0$

$$-1 = -1 - \frac{1}{3} + c \implies c = \frac{1}{3}$$
$$(y-1)e^y = -e^x - \frac{1}{3}e^{-3x} + \frac{1}{3}$$