Lecture 09: Laplace Transform and Circuit Analysis

Semester 1, 2021

Outline

Laplace Transform

Transfer Function

The one-sided Laplace Transform Property

Laplace transform method in circuit analysis

Laplace Transform

From Fourier transform to Laplace transform

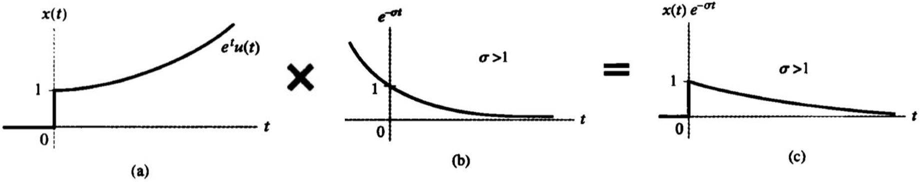
Some functions do not satisfy the condition of absolute convergence and it is difficult to solve the Fourier transform.

For this reason, the signal $x(t)$ can be multiplied by an attenuation factor $e^{-\sigma t}$.

Choose the value of σ appropriately so that the signal amplitude of the product signal $x(t)e^{-\sigma t}$ approaches zero when the time t goes to infinity.

So that the Fourier transform of the signal $x(t)e^{-\sigma t}$ exists.

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty$$



Laplace Transform

Definition

two-sided Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

one-sided Laplace transform of $x(t)$ is defined as

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

In this module, we only work with the one-sided version.

$$s = \sigma + j\omega$$

When $s = j\omega$, the Laplace transform is equal to the Fourier transform.

$$X(\omega) = X(s)|_{\sigma=0}$$

Laplace Transform...

Example

Find the Laplace transform of $x(t) = e^{-at}$.

$$X(s) = \int_0^{\infty} e^{-at} e^{-st} dt$$

$$= (-\frac{1}{s+a}) e^{-(a+s)t} |_0^{+\infty}$$

$$= \frac{1}{s+a}$$

Laplace Transform...

Example

Find the Laplace transform of $x(t) = te^{-\ at}$.

$$X(s) = \int_0^\infty te^{-at}e^{-st}dt \quad = \int_0^\infty te^{-(a+s)t}dt$$

$$= \frac{-1}{a+s}\int_0^\infty td(e^{-(a+s)t})$$

$$= \frac{-1}{a+s}(te^{-(a+s)t}|_0^\infty - \int_0^\infty e^{-(a+s)t}dt)$$

$$= \frac{-1}{a+s}(0 + \frac{1}{a+s}e^{-(a+s)t}|_0^\infty)$$

$$= \frac{1}{(s+a)^2}$$

Laplace transform of common functions

	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	e^{-at}	$\frac{1}{s+a}$
5	te^{-at}	$\frac{1}{(s+a)^2}$
6	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
8	$t^n (n=1,2,3,\cdots)$	$\frac{n!}{s^{n+1}}$

	$x(t)$	$X(s)$
9	$t^n e^{-at} (n=1,2,3\cdots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
11	$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
12	$\frac{1}{ab}[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})]$	$\frac{1}{s(s+a)(s+b)}$
13	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
14	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

The one-sided Laplace Transform Property...

Laplace transform has many properties similar to those of Fourier transform.

Linearity

$$\boxed{ax(t) + by(t) \xleftrightarrow{\mathcal{L}_u} aX(s) + bY(s).}$$

Scaling

$$\boxed{x(at) \xleftrightarrow{\mathcal{L}_u} \frac{1}{a}X\left(\frac{s}{a}\right) \quad \text{for } a > 0.}$$

Time shift

$$\boxed{x(t-\tau) \xleftrightarrow{\mathcal{L}_u} e^{-s\tau}X(s)}$$

Example

If $f_1(t) \longleftrightarrow F_1(s)$, determine $f_2(t) \longleftrightarrow F_2(s)$

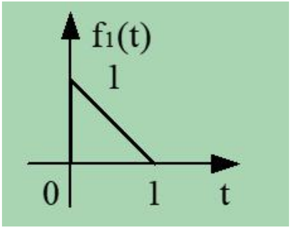
Solution:

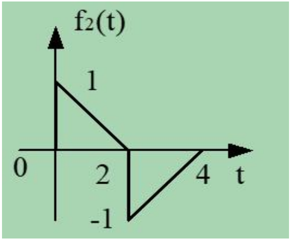
$$f_2(t) = f_1(0.5t) - f_1[0.5(t-2)]$$

$$f_1(0.5t) \longleftrightarrow 2F_1(2s)$$

$$f_1[0.5(t-2)] \longleftrightarrow 2F_1(2s)e^{-2s}$$

$$f_2(t) \longleftrightarrow 2F_1(2s)(1 - e^{-2s})$$





The one-sided Laplace Transform Property...

s-domain shift

$$\boxed{e^{s_o t}x(t) \xleftrightarrow{\mathcal{L}_u} X(s - s_o).}$$

Example: **If** $f(t) \leftrightarrow \dfrac{s}{s^2+1}$ **Determine the Laplace transform of function** $e^{-t}f(3t-2)$

$$f(t-2) \leftrightarrow F(s)e^{-2s} = \frac{s}{s^2+1}e^{-2s}$$

$$f(3t-2) \longleftrightarrow \frac{1}{3}\frac{\frac{s}{3}}{\left(\frac{s}{2}\right)^2+1}e^{-\frac{2}{3}s}$$

$$e^{-t}f(3t-2) \longleftrightarrow \frac{s+1}{(s+1)^2+9}\mathrm{e}^{-\frac{2}{3}(s+1)}$$

The one-sided Laplace Transform Property...

We will pay particular attention to the property of time-differentiation which is the key in analysing linear circuits.

$$\boxed{\frac{d}{dt}x(t) \xleftrightarrow{\mathcal{L}_u} sX(s) - x(0^-).}$$

Prove: $\int_{0-}^{\infty} \frac{d}{dt} x(t) e^{-st} dt\ ] = \int_{0-}^{\infty} e^{-st}\, d(x(t))$

$$= e^{-st}x(t)|_{0-}^{\infty} - \int_{0-}^{\infty} x(t)\, d(e^{-st})$$

$$= -x(0\text{-}) + s\int_{0-}^{\infty} e^{-st}x(t)\, dt$$

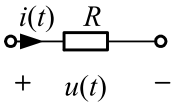
$$= sX(s) - x(0\text{-})$$

If the initial condition is zero, then $\mathcal{L}\{\frac{dx(t)}{dt}\} = sX(s)$

Laplace transform method in circuit analysis

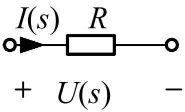
s-domain model of resistor

$\boldsymbol{u}(\mathbf{t}) = \boldsymbol{R}\ \mathbf{i}(\mathbf{t})$



LT

$U(s) = R\ I(s)$

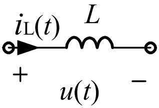


s domain

Laplace transform method in circuit analysis

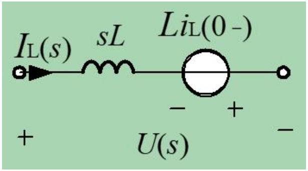
s-domain model of inductor

$$u(t) = L\frac{\mathrm{d}\,i_L(t)}{\mathrm{d}\,t}$$



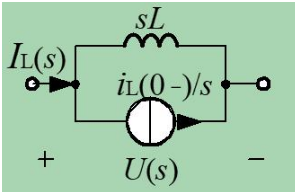
Laplace transform

$$U(s) = sLI_L(s) - Li_L(0-)$$



S domain

$$I_L(s) = \frac{1}{sL}U(s) + \frac{i_L(0\_)}{s}$$



s-domain model

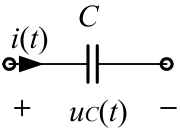
s-domain model of capacitor

$$i(t) = C\frac{\mathrm{d}u_C(t)}{\mathrm{d}t}$$

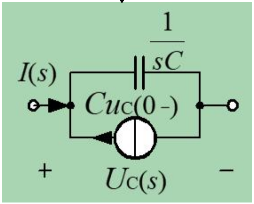
Laplace transform

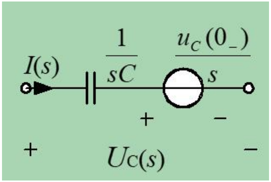
$$I(s) = sCU_S(s) - Cu_c(0\text{-})$$

$$U_C(s) = \frac{1}{sC}I(s) + \frac{u_C(0_-)}{s}$$



S domain





Laplace transform method in circuit analysis

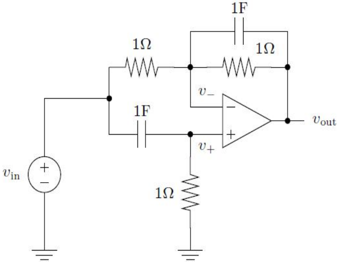
If the initial condition is zero, s-domain impedance of resistors, capacitors, and inductors are listed as follows:

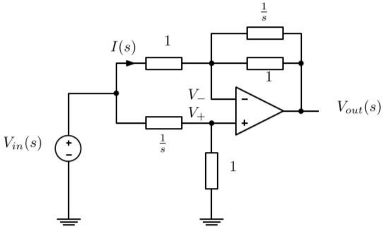
	Time domain	s-domain
Resistors	$v(t) = Ri(t)$	$V(s) = RI(s)$ $Z_R = R$
Capacitors	$i(t) = C\frac{dv(t)}{dt}$	$V(s) = \frac{1}{sC}I(s)$ $Z_C = \frac{1}{sC}$
Inductors	$v(t) = L\frac{di(t)}{dt}$	$V(s) = sLI(s)$ $Z_L = sL$

s-domain impedance makes it easier to analyse linear circuits.

Example

Determine the transfer function and the impulse response of the following circuit, assuming zero initial voltages for capacitors.





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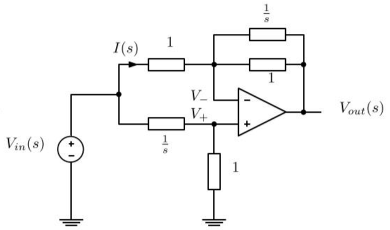
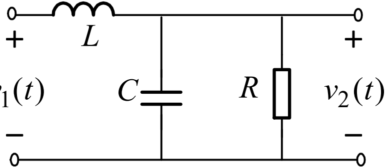


- $V_+(s) = \frac{1}{1+1/s} V_{in}(s) = \frac{s}{s+1} V_{in}(s)$
- $I(s) = \frac{V_{in}(s) - V_-}{1} = V_{in}(s) - V_+ = \frac{1}{s+1} V_{in}(s)$
- $V_{out}(s) = V_- - I(s)\frac{1}{s+1} = V_{in}(s)\frac{s^2+s-1}{(s+1)^2}$
- The transfer function: $H(s) = \frac{s^2+s-1}{(s+1)^2} = 1 - \frac{1}{s+1} - \frac{1}{(s+1)^2}$
- The impulse response is $\boldsymbol{h(t) = \delta(t) - e^{-t}u(t) - te^{-t}u(t)}$

Analogue Filters

An achievable low pass filter



$$V_2(s) = V_1(s)\frac{\frac{1}{sC}||R}{sL+\frac{1}{sC}||R}$$

$$= V_1(s)\frac{\frac{1}{sC+\frac{1}{R}}}{sL+\frac{1}{sC+\frac{1}{R}}}$$

$$= V_1(s)\frac{1}{1+sL(sC+\frac{1}{R})}$$

$$= V_1(s)\frac{1}{1+s^2LC+s\frac{L}{R}}$$

$$H(\omega)=H(s)|_{s=j\omega}=\frac{1}{1-\omega^2LC+j\omega\frac{L}{R}}$$