Tutorial Sheet 2 - Solutions

Q1 KVL: $v_i = v_R + v_L + v_C$

We know that: $v_R = iR$ and $v_L = L\dfrac{di}{dt}$

Hence the equation becomes: $v_i = iR + L\dfrac{di}{dt} + v_C$

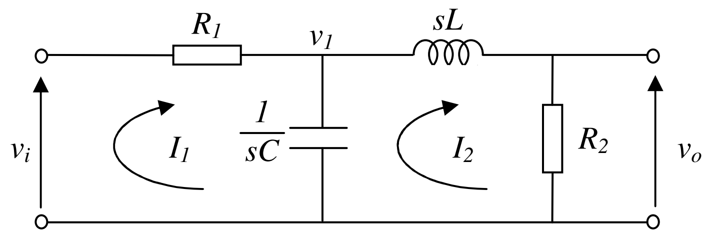
We need to eliminate *i*. We know that: $i = C\dfrac{dv_C}{dt}$

Hence: $v_i = RC\dfrac{dv_C}{dt} + L\dfrac{d}{dt}\left(C\dfrac{dv_C}{dt}\right) + v_C \quad \Rightarrow v_i = RC\dfrac{dv_C}{dt} + LC\dfrac{d^2v_C}{dt^2} + v_C$

We normally express this as:

$$LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = v_i \quad \text{or} \quad \frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_C = \frac{1}{LC}v_i$$

Q2 Circuit A:



Aside note: $L \rightarrow sL \equiv j\omega L$ and $C \rightarrow \dfrac{1}{sC} \equiv \dfrac{1}{j\omega C}$

Using mesh analysis (could also use nodal analysis) we get, in Laplace form:

$$I_1R_1 + (I_1 - I_2)\frac{1}{SC} = v_i \quad \text{and} \quad (I_2 - I_1)\frac{1}{SC} + I_2sL + I_2R_2 = 0$$

Multiplying across by sC and combining terms gives:

$$CR_1sI_1 + I_1 - I_2 = Csv_i \text{ and } I_2 - I_1 + CLs^2I_2 + CR_2sI_2 = 0$$

$$(CR_1s + 1)I_1 - I_2 = Csv_i \text{ and } (CLs^2 + CR_2s + 1)I_2 = I_1$$

Substituting the second equation into the first gives:

$$(CR_1 s+1)(CLs^2+CR_2 s+1)I_2 - I_2 = Csv_i$$

$$\Rightarrow \frac{I_2}{v_i} = \frac{Cs}{(CR_1 s+1)(CLs^2+CR_2 s+1)-1}$$

$$\Rightarrow \frac{I_2}{v_i} = \frac{Cs}{CLs^2+CR_2 s+1+CR_1 CLs^3+CR_1 CR_2 s^2+CR_1 s-1}$$

$$\Rightarrow \frac{I_2}{v_i} = \frac{Cs}{CLs^2+CR_2 s+CR_1 CLs^3+CR_1 CR_2 s^2+CR_1 s}$$

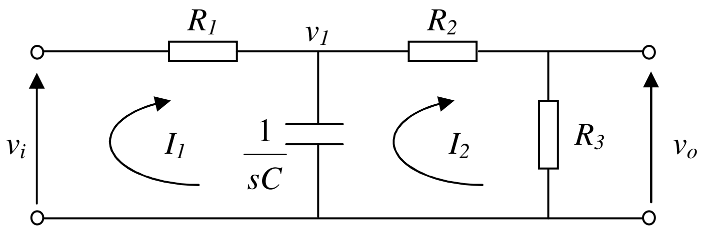
Finally dividing the numerator and denominator by Cs and collecting terms we get:

$$\frac{I_2}{v_i} = \frac{1}{R_1 CLs^2+(CR_1 R_2+L)s+(R_1+R_2)}$$

We are looking for $\frac{v_o}{v_i}$, but $v_0 = I_2 R_2$, therefore:

$$\frac{v_o}{v_i} = \frac{R_2}{R_1 CLs^2+(CR_1 R_2+L)s+(R_1+R_2)}$$

Circuit B:



Taking the mesh analysis equations in Laplace form we get:

$$I_1 R_1 + (I_1 - I_2)\frac{1}{sC} = v_i \text{ and } (I_2 - I_1)\frac{1}{sC} + I_2(R_2+R_3) = 0$$

Multiplying across by sC and combining terms gives:

$$CR_1 sI_1 + I_1 - I_2 = Csv_i \text{ and } I_2 - I_1 + C(R_2+R_3)sI_2 = 0$$

$$(CR_1 s+1)I_1 - I_2 = Csv_i \text{ and } (C(R_2+R_3)s+1)I_2 = I_1$$

Substituting the second equation into the first gives:

$$(CR_1 s+1)(C(R_2+R_3)s+1)I_2 - I_2 = Csv_i$$

$$\Rightarrow \frac{I_2}{v_i} = \frac{Cs}{(CR_1 s+1)(C(R_2+R_3)s+1)-1}$$

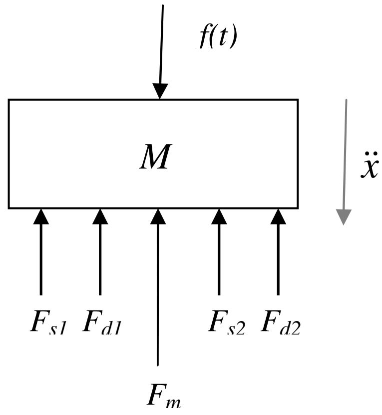
$$\Rightarrow \frac{I_2}{v_i} = \frac{Cs}{CR_1 s+1+C(R_2+R_3)s+C^2 R_1(R_2+R_3)s^2-1}$$

$$\Rightarrow \frac{I_2}{v_i} = \frac{1}{R_1+R_2+R_3+sCR_1(R_2+R_3)}$$

We are looking for $\frac{v_o}{v_i}$, but $v_0 = I_2 R_3$, therefore:

$$\frac{v_o}{v_i} = \frac{R_3}{(R_1 + R_2 + R_3) + sCR_1(R_2 + R_3)}$$

Q3 The free body diagram for this system is as shown.



Free body diagram

From the free body diagram showing only the forces acting on M we obtain the Force equilibrium equation:

$$F_m + F_{d1} + F_{s1} + F_{d2} + F_{s2} = f(t)$$

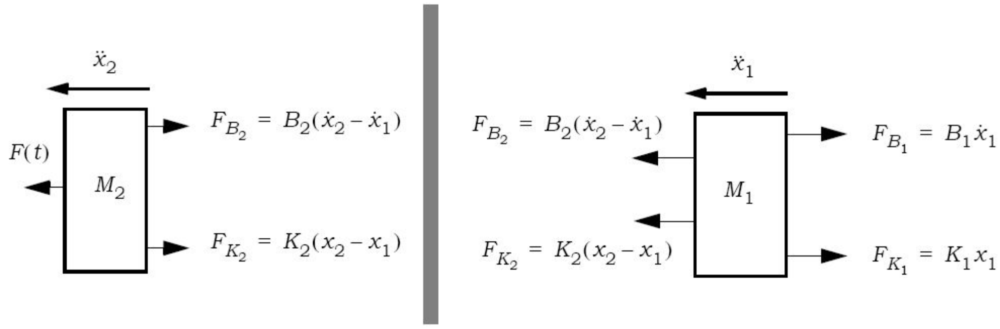
Using the physical force-geometry relations this becomes:

$$M\frac{d^2x}{dt^2} + B_1\frac{dx}{dt} + K_1 x + B_2\frac{dx}{dt} + K_2 x = f(t)$$

or:

$$M\frac{d^2x}{dt^2} + (B_1 + B_2)\frac{dx}{dt} + (K_1 + K_2)x = f(t)$$

Q4 The free body diagram for the masses in the system are:



Therefore for M_2, Newton's 2nd law gives:

$$F(t) - B_2(\dot{x}_2 - \dot{x}_1) - K_2(x_2 - x_1) = M_2\ddot{x}_2$$

$$\Rightarrow \ddot{x}_2 = -\frac{B_2}{M_2}\dot{x}_2 - \frac{K_2}{M_2}x_2 + \frac{B_2}{M_2}\dot{x}_1 + \frac{K_2}{M_2}x_1 + \frac{F(t)}{M_2}$$

and for M_1:

$$B_2(\dot{x}_2 - \dot{x}_1) + K_2(x_2 - x_1) - B_1\dot{x}_1 - K_1x_1 = M_1\ddot{x}_1$$

$$\Rightarrow \ddot{x}_1 = -\frac{B_1 + B_2}{M_1}\dot{x}_1 - \frac{K_1 + K_2}{M_1}x_1 + \frac{B_2}{M_1}\dot{x}_2 + \frac{K_2}{M_1}x_2$$

Tidying up and taking the Laplace transform gives:

$$M_2\ddot{x}_2 + B_2(\dot{x}_2 - \dot{x}_1) + K_2(x_2 - x_1) = F(t)$$

$$B_2(\dot{x}_2 - \dot{x}_1) + K_2(x_2 - x_1) = M_1\ddot{x}_1 + B_1\dot{x}_1 + K_1 x_1$$

$$\Downarrow$$

$$M_2 s^2 x_2(s) + B_2 s x_2(s) - B_2 s x_1(s) + K_2 x_2(s) - K_2 x_1(s) = F(s)$$

$$\Rightarrow (M_2 s^2 + B_2 s + K_2)x_2(s) - (B_2 s + K_2)x_1(s) = F(s)$$

and

$$B_2 s x_2(s) - B_2 s x_1(s) + K_2 x_2(s) - K_2 x_1(s) = M_1 s^2 x_1(s) + B_1 s x_1(s) + K_1 x_1(s)$$

$$\Rightarrow (B_2 s + K_2)x_2(s) = (M_1 s^2 + (B_1 + B_2)s + K_1 + K_2)x_1(s)$$

Therefore:

$$x_2(s) = \frac{M_1 s^2 + (B_1 + B_2)s + K_1 + K_2}{B_2 s + K_2} x_1(s)$$

Subbing for x_2 in the first equation gives:

$$\frac{(M_2 s^2 + B_2 s + K_2)(M_1 s^2 + (B_1 + B_2)s + K_1 + K_2)}{B_2 s + K_2} x_1(s) - (B_2 s + K_2)x_1(s) = F(s)$$

$$\Rightarrow \frac{(M_2 s^2 + B_2 s + K_2)(M_1 s^2 + (B_1 + B_2)s + K_1 + K_2) - (B_2 s + K_2)^2}{B_2 s + K_2} x_1(s) = F(s)$$

$$\Rightarrow \frac{x_1(s)}{F(s)} = \frac{B_2 s + K_2}{(M_2 s^2 + B_2 s + K_2)(M_1 s^2 + (B_1 + B_2)s + K_1 + K_2) - (B_2 s + K_2)^2}$$

$$\Rightarrow \frac{x_1(s)}{F(s)} = \frac{B_2 s + K_2}{M_1 M_2 s^4 + \alpha s^3 + \beta s^2 + [B_2 K_1 + B_1 K_2]s + K_1 K_2}$$

where:

$$\alpha = M_1 B_2 + M_2(B_1 + B_2) \quad \text{and} \quad \beta = M_1 K_2 + M_2(K_1 + K_2) + B_1 B_2$$

Note that x_1 represents the position of mass M_1 as required.

Q5(i) **1st order model** is:

$$y_k = a_1 y_{k-1} + b_1 u_{k-1}$$

Using the data in the table we can write:

$$\begin{aligned} 0.109 &= 0.098a_1 + 0.065b_1 \\ 0.117 &= 0.109a_1 + 0.065b_1 \\ -0.019 &= 0.117a_1 - 0.15b_1 \\ -0.128 &= -0.019a_1 - 0.15b_1 \\ -0.1 &= -0.128a_1 - 0.1b_1 \end{aligned}$$

We can write these equations in matrix form as:

$$\begin{bmatrix} 0.109 \\ 0.117 \\ -0.019 \\ -0.128 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 0.098 & 0.065 \\ 0.109 & 0.065 \\ 0.117 & -0.15 \\ -0.019 & -0.15 \\ -0.128 & -0.1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

This has the form $y = X\theta$, where X is not square.The best solution for θ is given by the least squares solution (students are NOT expected to remember this formula):

$$\theta = (X^T X)^{-1} X^T y$$

Using Matlab to perform these calculation gives:

$$\theta = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0.5607 \\ 0.6345 \end{bmatrix} \approx \begin{bmatrix} 0.56 \\ 0.63 \end{bmatrix}$$

Hence the 1st order model is:

$$y_k = 0.56y_{k-1} + 0.63u_{k-1}$$

The **2nd order model** is given by:

$$y_k = a_1 y_{k-1} + a_2 y_{k-2} + b_1 u_{k-1} + b_2 u_{k-2}$$

Using the data in the table we obtain the following matrix equation for the unknown parameters:

$$\begin{bmatrix} 0.117 \\ -0.019 \\ -0.128 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 0.109 & 0.098 & 0.065 & 0.065 \\ 0.117 & 0.109 & -0.15 & 0.065 \\ -0.019 & 0.117 & -0.15 & -0.15 \\ -0.128 & -0.019 & -0.1 & -0.15 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}$$

Solving using Matlab gives:

$$\theta = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -3.087 \\ 2.5057 \\ 0.6459 \\ 2.5529 \end{bmatrix} \approx \begin{bmatrix} -3.1 \\ 2.51 \\ 0.65 \\ 2.55 \end{bmatrix}$$

Hence the 2nd order model is:

$$y_k = -3.1y_{k-1} + 2.51y_{k-2} + 0.65u_{k-1} + 2.55u_{k-2}$$

Q5(ii) Taking the Z-transform of:

$$y_k = 0.56y_{k-1} + 0.63u_{k-1}$$

gives:

$$y(z) = 0.56z^{-1}y(z) + 0.63z^{-1}u(z)$$

$$\Rightarrow \frac{y(z)}{u(z)} = \frac{0.63z^{-1}}{1-0.56z^{-1}}$$

We can also write this as:

$$\Rightarrow \frac{y(z)}{u(z)} = \frac{0.63}{z-0.56}$$

Similarly for the second order model:

$$y(z) = -3.1z^{-1}y(z) + 2.51z^{-2}y(z) + 0.65z^{-1}u(z) + 2.55z^{-2}u(z)$$

$$\Rightarrow \frac{y(z)}{u(z)} = \frac{0.65z^{-1} + 2.55z^{-2}}{1+3.1z^{-1} - 2.51z^{-2}}$$

or

$$\Rightarrow \frac{y(z)}{u(z)} = \frac{0.65z + 2.55}{z^2 + 3.1z - 2.51}$$

Q6(i) Writing flow balance equation for the tank gives: $\frac{dV}{dt} = F_{in} - F_{out}$

Since volume $V = Ah$: $\frac{dV}{dt} = A\frac{dh}{dt} = F_{in} - F_{out}$

Given that $F_{out} = \frac{h}{R}$ (for laminar flow) then the flow balance equation is:

$$A\frac{dh}{dt} = F_{in} - \frac{h}{R} \quad \text{or} \quad A\frac{dh}{dt} + \frac{h}{R} = F_{in}$$

Transfer function:

$$AsH(s) = F_{in}(s) - \frac{1}{R}H(s) \qquad \Rightarrow H(s)\left(sA + \frac{1}{R}\right) = F_{in}(s)$$

$$\Rightarrow \frac{H(s)}{F_{in}(s)} = \frac{1}{sA + \frac{1}{R}} = \frac{R}{1+sAR}$$

Q6(ii) Turbulent $\Rightarrow F_{out} = \frac{\sqrt{h}}{R}$ and hence model becomes: $A\frac{dh}{dt} + \frac{\sqrt{h}}{R} = F_{in}$

Q7 From Q6(i), we can easily write out the system model for each tank as follows:

Tank 1: $$A_1 \frac{dh_1}{dt} + \frac{h_1}{R_1} = F_{in}$$

Tank 2: $$A_2 \frac{dh_2}{dt} + \frac{h_2}{R_2} = F_{12}$$

Since F_{12} is the output flow of Tank 1, then we know that $F_{12} = \frac{h_1}{R_1}$ and hence the model for Tank 2 becomes:

$$A_2 \frac{dh_2}{dt} + \frac{h_2}{R_2} = \frac{h_1}{R_1}$$

Now, in order to obtain the desired model, i.e. relating F_{in} to h_2, we need to eliminate h_1 as follows:

From the Tank 2 equation:

$$\frac{h_1}{R_1} = A_2 \frac{dh_2}{dt} + \frac{h_2}{R_2} \quad \Rightarrow h_1 = A_2 R_1 \frac{dh_2}{dt} + \frac{R_1 h_2}{R_2} \quad \Rightarrow \frac{dh_1}{dt} = A_2 R_1 \frac{d^2 h_2}{dt^2} + \frac{R_1}{R_2}\frac{dh_2}{dt}$$

Subbing into the Tank 1 equation gives:

$$A_1\left(A_2 R_1 \frac{d^2 h_2}{dt^2} + \frac{R_1}{R_2}\frac{dh_2}{dt}\right) + \left(A_2 \frac{dh_2}{dt} + \frac{h_2}{R_2}\right) = F_{in}$$

$$\Rightarrow A_1\left(A_2 R_1 R_2 \frac{d^2 h_2}{dt^2} + R_1 \frac{dh_2}{dt}\right) + \left(A_2 R_2 \frac{dh_2}{dt} + h_2\right) = R_2 F_{in}$$

$$\Rightarrow \frac{d^2 h_2}{dt^2}\left(A_1 A_2 R_1 R_2\right) + \frac{dh_2}{dt}\left(A_1 R_1 + A_2 R_2\right) + h_2 = R_2 F_{in}$$

Transfer function:

$$s^2 H_2(s)\left(A_1 A_2 R_1 R_2\right) + sH_2(s)\left(A_1 R_1 + A_2 R_2\right) + H_2(s) = R_2 F_{in}(s)$$

$$\Rightarrow \frac{H_2(s)}{F_{in}(s)} = \frac{R_2}{s^2\left(A_1 A_2 R_1 R_2\right) + s\left(A_1 R_1 + A_2 R_2\right) + 1}$$

Alternatively:

The transfer function model of both tanks is of the same form as that obtained for Q6, i.e.:

$$\frac{H_1(s)}{F_{in}(s)} = \frac{R_1}{1+sA_1R_1} \quad \text{for the upper tank}$$

and

$$\frac{H_2(s)}{F_{12}(s)} = \frac{R_2}{1+sA_2R_2} \quad \text{for the lower tank}$$

In addition, we need to note that:

$$F_{12} = \frac{h_1}{R_1} \Rightarrow \frac{F_{12}(s)}{H_1(s)} = \frac{1}{R_1} \quad \text{for the upper tank}$$

and

$$F_{out} = \frac{h_2}{R_2} \Rightarrow \frac{F_{out}(s)}{H_2(s)} = \frac{1}{R_2} \quad \text{for the lower tank}$$

Here, we want a model for $\frac{H_2(s)}{F_{in}(s)}$. We obtain this as follows:

$$\frac{H_2(s)}{F_{in}(s)} = \frac{H_1(s)}{F_{in}(s)} \text{x} \frac{F_{12}(s)}{H_1(s)} \text{x} \frac{H_2(s)}{F_{12}(s)}$$

$$= \left(\frac{R_1}{1+sA_1R_1}\right)\left(\frac{1}{R_1}\right)\left(\frac{R_2}{1+sA_2R_2}\right)$$

$$= \frac{R_2}{(1+sA_1R_1)(1+sA_2R_2)}$$

$$\equiv \frac{R_2}{s^2(A_1A_2R_1R_2)+s(A_1R_1+A_2R_2)+1}$$

Aside ... *Model of output flow of lower tank to input flow of upper tank:*

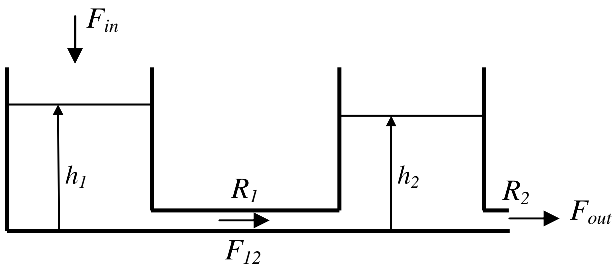
$$\frac{F_{out}(s)}{F_{in}(s)} = \frac{H_1(s)}{F_{in}(s)} \text{x} \frac{F_{12}(s)}{H_1(s)} \text{x} \frac{H_2(s)}{F_{12}(s)} \text{x} \frac{F_{out}(s)}{H_2(s)}$$

$$= \left(\frac{R_1}{1+sA_1R_1}\right)\left(\frac{1}{R_1}\right)\left(\frac{R_2}{1+sA_2R_2}\right)\left(\frac{1}{R_2}\right)$$

$$= \frac{1}{(1+sA_1R_1)(1+sA_2R_2)}$$

Q8 Based, on the solution to Q6(i), we know that the flow out of the tank is proportional to the height of the water in the tank and inversely proportional to the resistance to flow.

In the case of the flow between connected tanks, as in this question, we have to allow for both heights in our model of flow. In this case, the flow between tanks is proportional to the difference in heights on either side of the connection.



Note, if $h_1 > h_2$ then the liquid flows from tank 1 to tank 2, if $h_2 > h_1$ then the liquid flows from tank 2 to tank 1 and if $h_1 = h_2$ then the liquid does not flow!

Hence we can state the following: $F_{12} = \dfrac{h_1 - h_2}{R_1}$ and $F_{out} = \dfrac{h_2}{R_2}$

Our flow balance equations are:

$$A_1 \frac{dh_1}{dt} = F_{in} - F_{12} \qquad\qquad A_2 \frac{dh_2}{dt} = F_{12} - F_{out}$$

$$\Rightarrow A_1 \frac{dh_1}{dt} = F_{in} - \left(\frac{h_1 - h_2}{R_1}\right) \quad \text{and} \quad \Rightarrow A_2 \frac{dh_2}{dt} = \left(\frac{h_1 - h_2}{R_1}\right) - \left(\frac{h_2}{R_2}\right)$$

$$\Rightarrow A_1 R_1 \frac{dh_1}{dt} = R_1 F_{in} - h_1 + h_2 \qquad\qquad \Rightarrow A_2 R_1 R_2 \frac{dh_2}{dt} = R_2 h_1 - h_2 (R_1 + R_2)$$

Obtaining the Laplace transforms:

$$sA_1R_1H_1(s) = R_1F_{in}(s) - H_1(s) + H_2(s)$$
$$\Rightarrow H_1(s)(1 + sA_1R_1) = R_1F_{in}(s) + H_2(s)$$

and
$$sA_2R_1R_2H_2(s) = R_2H_1(s) - H_2(s)(R_1 + R_2)$$
$$\Rightarrow H_2(s)(R_1 + R_2 + sA_2R_1R_2) = R_2H_1(s)$$

We want a model relating output height $H_2(s)$ to input $F_{in}(s)$.

Hence, we need to eliminate $H_1(s)$ from the first equation above using the second equation, as follows:

$$H_2(s)(R_1 + R_2 + sA_2R_1R_2) = R_2H_1(s)$$
$$\Rightarrow H_1(s) = \frac{H_2(s)(R_1 + R_2 + sA_2R_1R_2)}{R_2}$$

Substituting into left hand equation gives:

$$\frac{H_2(s)(R_1 + R_2 + sA_2R_1R_2)}{R_2}(1 + sA_1R_1) = R_1F_{in}(s) + H_2(s)$$
$$\Rightarrow H_2(s)\left(\frac{(R_1 + R_2 + sA_2R_1R_2)(1 + sA_1R_1)}{R_2} - 1\right) = R_1F_{in}(s)$$
$$\Rightarrow \frac{H_2(s)}{F_{in}(s)} = \frac{R_1}{\frac{(R_1 + R_2 + sA_2R_1R_2)(1 + sA_1R_1)}{R_2} - 1}$$
$$\Rightarrow \frac{H_2(s)}{F_{in}(s)} = \frac{R_2R_1}{(R_1 + R_2 + sA_2R_1R_2)(1 + sA_1R_1) - R_2}$$
$$\Rightarrow \frac{H_2(s)}{F_{in}(s)} = \frac{R_2R_1}{s^2(A_1A_2R_1{}^2R_2) + s(A_2R_1R_2 + A_1R_1{}^2 + A_1R_1R_2) + R_1 + R_2 - R_2}$$
$$\Rightarrow \frac{H_2(s)}{F_{in}(s)} = \frac{R_2R_1}{s^2(A_1A_2R_1{}^2R_2) + s(A_2R_1R_2 + A_1R_1{}^2 + A_1R_1R_2) + R_1 + R_2 - R_2}$$
$$\Rightarrow \frac{H_2(s)}{F_{in}(s)} = \frac{R_2R_1}{s^2(A_1A_2R_1{}^2R_2) + s(A_2R_1R_2 + A_1R_1{}^2 + A_1R_1R_2) + R_1}$$
$$\Rightarrow \frac{H_2(s)}{F_{in}(s)} = \frac{R_2}{s^2(A_1A_2R_1R_2) + s(A_2R_2 + A_1R_1 + A_1R_2) + 1}$$

Aside ... *Model of output flow to input flow:*

$$\frac{F_{out}(s)}{F_{in}(s)} = \frac{H_2(s)}{F_{in}(s)} \text{x} \frac{F_{out}(s)}{H_2(s)}$$
$$= \left(\frac{R_2}{s^2(A_1A_2R_1R_2) + s(A_2R_2 + A_1R_1 + A_1R_2) + 1}\right)\left(\frac{1}{\mathrm{R}_2}\right)$$
$$= \frac{1}{s^2(A_1A_2R_1R_2) + s(A_2R_2 + A_1R_1 + A_1R_2) + 1}$$