Introduction to differential equations II: overview

- Linear first-order differential equations
- Method of variation of parameters
- Solutions by substitutions
- Bernoulli equation
- Reduction to separation of variables
- Optional material: error function, exact DE, homogeneous functions

Linear equations

A differential equation that is of the first degree in the dependent variable and all its derivatives is said to be linear.

Definition: Linear equation

A first-order differential equation of the form

$$a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x) \tag{4}$$

is said to be linear.

If $g(x) = 0$ the linear equation is said to be **homogeneous**, otherwise it is **nonhomogeneous**.

Standard form

By dividing both sides of (4) by $a_1(x)$ we get the **standard form** of a linear equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = f(x) \tag{5}$$

We seek a solution of the equation above on an interval I for which both functions P and f are continuous.

The property

The DE (5) has the property that its solution is the **sum** of two solutions, $y = y_c + y_p$, where y_c is the solution of the associated homogeneous equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = 0 \tag{6}$$

and y_p is a particular solution of the nonhomogeneous equation (5). To see this

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y_c + y_p\right] + P(x)\left[y_c + y_p\right] = \left[\frac{\mathrm{d}y_c}{\mathrm{d}x} + P(x)y_c\right] + \left[\frac{\mathrm{d}y_p}{\mathrm{d}x} + P(x)y_p\right] = 0 + f(x) = f(x)$$

The homogeneous equation (6) is also separable, so we can find y_c by integrating it

$$y_c = ce^{-\int P(x)dx} = cy_1$$

We now use the fact that $dy_1/dx + P(x)y = 0$ to determine y_p.

The procedure: Variation of parameters

Idea: to find a function u so that $y_p = u(x)y_1(x) = u(x)e^{-\int P(x)dx}$ is a solution of (5).

Substituting $y_p = uy_1$ into the equation gives

$$\begin{aligned} u\frac{\mathrm{d}y_1}{\mathrm{d}x} + y_1\frac{\mathrm{d}u}{\mathrm{d}x} + P(x)uy_1 &= f(x) \\ u\left[\frac{\mathrm{d}y_1}{\mathrm{d}x} + P(x)y_1\right] + y_1\frac{\mathrm{d}u}{\mathrm{d}x} &= f(x) \end{aligned}$$

and since y_1 is the solution of the homogeneous equation, the expression in the square bracket is zero and

$$y_1\frac{\mathrm{d}u}{\mathrm{d}x} = f(x)$$

$$y_1 \frac{\mathrm{d}u}{\mathrm{d}x} = f(x)$$

Separating variables and integrating then gives

$$du = \frac{f(x)}{y_1(x)} dx \qquad \Rightarrow \qquad u = \int \frac{f(x)}{y_1(x)} dx$$

Since $y_1(x) = e^{-\int P(x)dx}$, $1/y_1(x) = e^{\int P(x)dx}$, and therefore

$$y_p = u y_1 = \left(\int \frac{f(x)}{y_1(x)} dx \right) e^{-\int P(x)dx} = e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x) dx$$

and the solution of (5) is then of the form

$$y = y_c + y_p = ce^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)dx$$

There is an equivalent but easier way of solving (5). If the equation above is multiplied by $e^{\int P(x)dx}$ and differentiated we get

$$\begin{aligned} e^{\int P(x)dx} y &= c + \int e^{\int P(x)dx} f(x)dx \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\int P(x)dx} y\right] &= e^{\int P(x)dx} f(x) \\ e^{\int P(x)dx}\frac{\mathrm{dy}}{\mathrm{d}x} + P(x)e^{\int P(x)dx} y &= e^{\int P(x)dx} f(x) \end{aligned}$$

Dividing the result by $e^{\int P(x)dx}$ gives (5).

Method of solving a linear first-order equation

(i) Put a linear equation of form (4) into the standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = f(x)$$

and then determine $P(x)$ and the **integrating factor** $e^{\int P(x)dx}$.

(ii) Multiply the equation in its standard form by the integrating factor. The left side of the resulting equation is automatically the derivative of the integrating factor and y: write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x)$$

and then integrate both sides of this equation.

Example:

$$\frac{dy}{dx} - 3y = 6$$

Constant of integration

Considering the constant of integration in evaluation of the integrating factor $e^{\int P(x)dx}$, that is writting $e^{\int P(x)dx + c}$ is unnecessary as the integrating factor multiplies both sides of the differential equation.

Singular points

The recasting the linear equation (4) in the standard form (5) requires division by $a_1(x)$. Values of x for which the $a_1(x) = 0$ are called **singular points**. They are potentially troublesome: if $P(x)$ formed by dividing $a_0(x)$ by $a_1(x)$ is discontinuous at a point, the discontinuity may carry over to solutions of the DE.

General solution

Recall that the functions $P(x)$ and $f(x)$ in (5) are continuous on a common interval I. Also, *if* (5) has a solution on I it must be of the form

$$y = y_c + y_p = ce^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)dx$$

Conversely any function of this form is a solution of (5) on I. In other words, the solution above defines a one-parameter family of solutions of equation (5) and every solution of (5) defined on I is of this form. It is hence the **general solution**.

Now writing (5) in the normal form $y' = F(x, y)$, we see that $F(x, y) = -P(x)y + f(x)$ and $\partial F/\partial y = -P(x)$. These must be continuous on the entire interval I because of the continuity of $P(x)$ and $f(x)$.

From the uniqueness theorem we conclude that there exists one and only one solution of the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = f(x), \qquad y(x_0) = y_0$$

defined on some interval I_0 containing x_0 and that this interval of existence and uniqueness is the entire interval I.

Example: General solution

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = x^6 e^x$$

Example: General solution

$$(x^2 - 9)\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 0$$

Example: An IVP

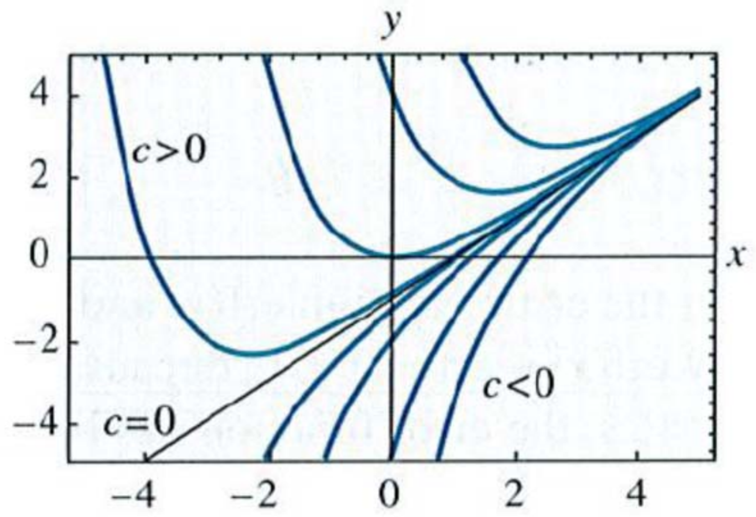
$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = x, \qquad y(0) = 4$$

The general solution of every linear first order DE is a sum, $y = y_c + y_p$, of the solution of the associated homogeneous equation (6) and a particular solution of the nonhomogeneous equation.

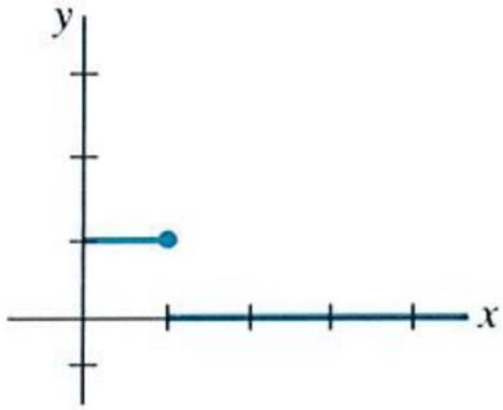
In the example above, $y_c = ce^{-x}$ and $y_p = x - 1$.

Observe that as x gets large, the graphs of all members of the family get close to the graph of y_p, as y_c becomes negligible.

We say $y_c = ce^{-x}$ is a **transient term** since $y_c \to 0$ as $x \to \infty$.



Example: A discontinuous $f(x)$



$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(x)$$

where $f(x) = 1$ for $0 \leq x \leq 1$, and $f(x) = 0$ for $x > 1$; the initial condition is $y(0) = 0$.

We solve the problem in two intervals over which f is defined. For $0 \leq x \leq 1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = 1 \qquad \Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left[e^x y\right] = e^x$$

we get $y = 1 + c_1 e^{-x}$ and since $y(0) = 0$ we have $c_1 = -1$, and so $y = 1 - e^{-x}$.

For $x > 1$ the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$

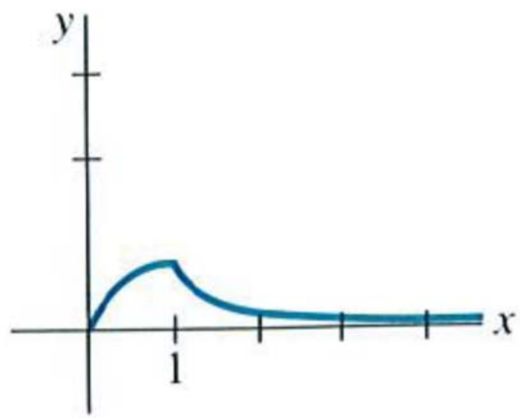
leads to the solution $y = c_2 e^{-x}$. So the solution in both intervals is

$$y = \begin{cases} 1 - e^x & \text{if } 0 \le x \le 1; \\ c_2 e^{-x} & \text{if } x > 1. \end{cases}$$

In order for y to be continuous, we want $\lim_{x \to 1^+} y(x) = y(1)$, that is, $c_2 e^{-1} = 1 - e^{-1}$ or $c_2 = e - 1$. The function

$$y = \begin{cases} 1 - e^x & \text{if } 0 \le x \le 1; \\ (e-1)e^{-x} & \text{if } x > 1. \end{cases}$$

is continuous on $(0, \infty)$.



Solutions by substitutions

We first transform a given differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

by means of **substitution** $y = g(x, u)$ into another differential equation

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= g_x(x, u) + g_u(x, u)\frac{\mathrm{d}u}{\mathrm{d}x} \\ f(x, g(x, u)) &= g_x(x, u) + g_u(x, u)\frac{\mathrm{d}u}{\mathrm{d}x} \end{aligned}$$

where we assumed that $g(x, u)$ possesses the first partial derivatives, so we could apply the chain rule.

The last equation above can be reformulated as $du/dx = F(x, u)$. If we can find its solution $u = \phi(x)$, then a solution of the original equation is $y = g(x, \phi(x))$.

Bernoulli equation

is a special type of first-order ODE which can be reduced to linear form and then solved by the method for linear ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + M(x)y = N(x)y^n \tag{7}$$

where n is any real number.

It can be transformed into the linear form as follows:

$$u = y^{1-n} \quad \Rightarrow \quad u = yy^{-n} \quad \Rightarrow \quad y = y^n u$$

Differentiating this gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{y^n}{1-n}\right)\frac{\mathrm{d}u}{\mathrm{d}x}$$

Substituting this into the Bernoulli equation (7) gives

$$\left(\frac{y^n}{1-n}\right)\frac{\mathrm{d}u}{\mathrm{d}x} + M(x)\, y^n\, u = N(x)\, y^n$$

Dividing by y^n and multiplying by $(1-n)$ gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)M(x)u = (1-n)N(x)$$

which is a linear ODE with $P(x) = (1-n)M(x)$ and $f(x) = (1-n)N(x)$.

Example: A Bernoulli equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{3}y = \frac{1}{3}(1 - 2x)y^4$$

Reduction to separation of variables

A differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(Ax + By + C)$$

can always be reduced to an equation with separable variables by means of the substituition $u = Ax + By + C$.

<u>Example:</u> An IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (-2x + y)^2 - 7, \qquad y(0) = 0$$

Let $u = -2x + y$, then $du/dx = -2 + dy/dx$ and so the DE is transformed into a separable equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} + 2 = u^2 - 7 \quad \text{or} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = u^2 - 9$$

The transformed equation can be solved using the partial fractions

$$\frac{du}{(u-3)(u+3)} = dx \qquad \text{or} \qquad \frac{1}{6}\left[\frac{1}{u-3} - \frac{1}{u+3}\right]du = dx$$

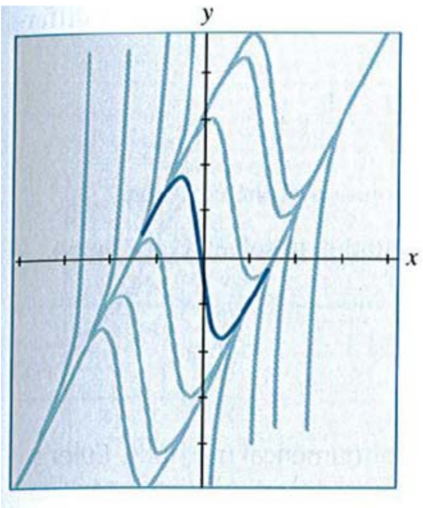
$$\Rightarrow \qquad \frac{1}{6}\ln\left|\frac{u-3}{u+3}\right| = x + c_1 \qquad \text{or} \qquad \frac{u-3}{u+3} = e^{6x+6c_1}$$

After solving the last equation for u and then resubstituting we get

$$u = \frac{3(1+ce^{6x})}{1-ce^{6x}} \qquad \text{or} \qquad y = 2x + \frac{3(1+ce^{6x})}{1-ce^{6x}}$$

and by applying the initial condition we get $c = -1$

$$y = 2x + \frac{3(1-e^{6x})}{1+e^{6x}}$$



OPTIONAL

Functions defined by integrals

Integrals of functions, which do not possess indefinite integrals that are elementary functions, are called **nonelementary**.

Error function

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Complementary error function

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

Since $2/\sqrt{\pi} \int_0^\infty e^{-t^2} dt = 1$, $erf(x) + erfc(x) = 1$. Also $erf(0) = 0$.

OPTIONAL

Example: The error function

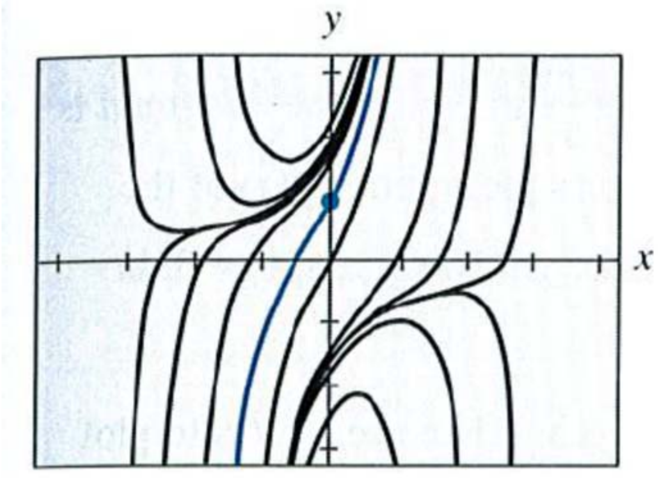
$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy = 2, \qquad y(0) = 1$$

The integrating factor is e^{-x^2}, and so from

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-x^2} y\right] = 2e^{-x^2} \qquad \Rightarrow \qquad y = 2e^{x^2}\int_0^x e^{-t^2}dt + ce^{x^2}$$

From the initial value we get $c = 1$ and thus the solution of the IVP is

$$y = 2e^{x^2}\int_0^x e^{-t^2}dt + e^{x^2} = e^{x^2}\left[1 + \sqrt{\pi}\, erf(x)\right]$$



OPTIONAL

Exact equations

A differential expression $M(x, y)dx + N(x, y)dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$, i.e.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

A first order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be an **exact equation** if the expression on the l. h. s. is an exact differential.

Example: $x^2y^3dx + x^3y^2dy = 0$ is exact as $d(x^3y^3/3) = x^2y^3dx + x^3y^2dy$.

OPTIONAL

Theorem: Criterion for an exact differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in the region R defined by $a < x < b$ and $c < y < d$. Then a necessary and sufficient condition that $M(x, y)dx + N(x, y)dy$ be an exact differential is

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

Proof:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial x}$$

OPTIONAL

Example: Solution of an exact equation

$$2xydx + (x^2 - 1)dy = 0$$

$M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$, we get $\partial M/\partial y = 2x = \partial N/\partial x$, so the equation is exact and there exist a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 2xy \quad \text{or} \quad \frac{\partial f}{\partial y} = x^2 - 1$$

Integrating the first equation gives

$$f(x, y) = x^2 y + g(y)$$

OPTIONAL

By taking now the partial derivative w.r.t. y we obtain

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1$$

from which it follows that $g'(y) = -1$ and $g(y) = -y$.

Hence $f(x, y) = x^2y - y$, and so the solution of the equation in implicit form is

$$x^2y - y = c$$

The explicit solution is $y = c/(x^2 - 1)$ and is defined on any interval not containing $x = \pm 1$.

OPTIONAL

Homogeneous equations

A first-order DE in differential form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be **homogeneous** if both coefficients M and N are **homogeneous functions** of the same degree α, i.e.

$$M(tx, ty) = t^{\alpha} M(x, y) \qquad N(tx, ty) = t^{\alpha} N(x, y)$$

Introducing $u = y/x$ and $v = x/y$, we can rewrite the coefficients as

$$M(x, y) = x^{\alpha} M(1, u) \qquad N(x, y) = x^{\alpha} N(1, u)$$
$$M(x, y) = y^{\alpha} M(v, 1) \qquad N(x, y) = y^{\alpha} N(v, 1)$$

OPTIONAL

Either of the substitutions above, $y = ux$ or $x = vy$, will reduce a homogeneous equation to a *separable* first order ODE:

$$\begin{aligned} M(x,y)dx + N(x,y)dy &= 0 \\ \Rightarrow \quad x^{\alpha}M(1,u)dx + x^{\alpha}N(1,u)dy &= 0 \\ \Rightarrow \quad M(1,u)dx + N(1,u)dy &= 0 \end{aligned}$$

By substituting the differential $dy = udx + xdu$, we get a separable DE in the variables u and x:

$$\begin{aligned} M(1,u)dx + N(1,u)\,[udx + xdu] &= 0 \\ [M(1,u) + uN(1,u)]\,dx + xN(1,u)du &= 0 \\ \Rightarrow \quad \frac{dx}{x} + \frac{N(1,u)\,du}{M(1,u) + u\,N(1,u)} &= 0 \end{aligned}$$

OPTIONAL

Example: Solving a homogeneous DE

$$\left(x^2+y^2\right)dx+\left(x^2-xy\right)dy=0$$

The coefficients $M(x,y)=x^2+y^2$ and $N(x,y)=x^2-xy$ are homogeneous functions of the degree 2. Let $y=ux$, then $dy=udx+xdu$, and the given DE becomes

$$\begin{aligned}\left(x^2+u^2x^2\right)dx+\left(x^2-ux^2\right)[udx+xdu] &= 0\\ x^2(1+u)\,dx+x^3(1-u)\,du &= 0\\ \frac{1-u}{1+u}du+\frac{dx}{x}=0&\\ \left[-1+\frac{2}{1+u}\right]du+\frac{dx}{x}=0&\end{aligned}$$

OPTIONAL

$$\left[-1 + \frac{2}{1+u}\right] du + \frac{dx}{x} = 0$$

After integration, and transformation back to the original variables, we get

$$-u + 2\ln|1+u| + \ln|x| = \ln|c| \qquad \Rightarrow \qquad -\frac{y}{x} + 2\ln\left|1 + \frac{y}{x}\right| + \ln|x| = \ln|c|$$

Using the properties of logarithms, the solution can be written as $(x+y)^2 = cxe^{y/x}$.

Intuitive interpretation of a linear ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x) = f(x)$$

The function $f(x)$ often represents some controllable quantity, such as a force or an applied voltage, which can be interpreted as the **input** to the system. Within this interpretation, we can view the dependent variable $y(x)$ as an **output** or as an effect which is produced **in response** to the **input(s)**.

In the general solution of the linear ODE

$$y = e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)dx + ce^{-\int P(x)dx}$$

the first term can be viewed as the system response to the input $f(x)$ and the second term as the influence of the initial state of the system.

Modelling an RC-circuit

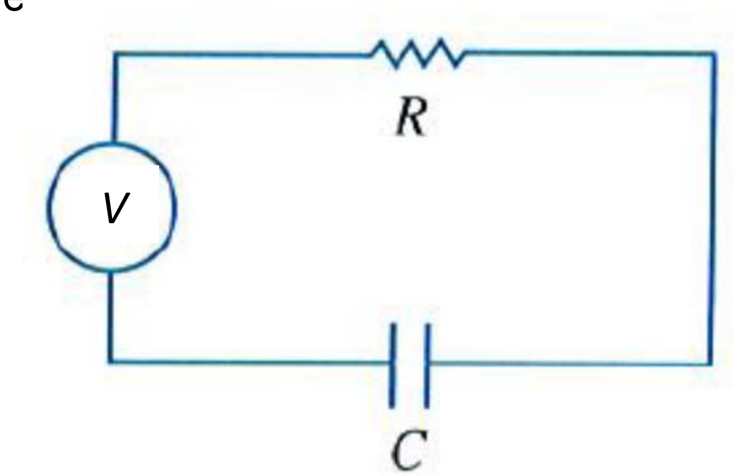
A resistor of resistance R is connected in series with a capacitor of capacitance C and a source of electromotive force in the form of an applied voltage, $V(t)$. When the circuit is closed, a current $i(t)$ will flow through it.

According to the Kirchhoff second law with this circuit, the voltage drops at the capacitor and resistor equal the applied voltage:

$$V_R + V_C = V(t)$$

where $V_R = Ri$ and $V_C = q/C = \int idt/C$. Thus we get

$$Ri + \frac{1}{C}\int idt = V(t)$$



Let us differentiate w.r.t. t and divide by R, to get

$$\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{RC}i = \frac{1}{R}\frac{\mathrm{d}V(t)}{\mathrm{d}t}$$

This equation has the form which is the standard form of the linear equation where $P(t) = 1/RC$ and $f(t) = (1/R)dV(t)/dt$. The integrating factor is then

$$e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$$

so the general solution becomes

$$i(t) = e^{-\frac{t}{RC}}\left(\frac{1}{R}\int e^{\frac{t}{RC}}\frac{\mathrm{d}V(t)}{\mathrm{d}t}dt + c\right)$$

Case 1: $V(t) =$ **constant**

In this case we get $\frac{\mathrm{d}V(t)}{\mathrm{d}t} = 0$ and so

$$i(t) = ce^{-\frac{t}{RC}}$$

The current in this case decays with time eventually approaching zero

Case 2: $V(t) = V_0 \sin(\omega t)$
Substituting this into the general form of the solution we get

$$i(t) = e^{-\frac{t}{RC}} \left(\frac{1}{R} \int e^{\frac{t}{RC}} V_0 \omega \cos(\omega t) dt + c \right)$$

Integrating by parts and using trigonometric relations gives

$$\begin{aligned} i(t) &= c e^{-\frac{t}{RC}} + \frac{\omega V_0 C}{1 + (\omega RC)^2} [\cos(\omega t) + \omega RC \sin(\omega t)] \\ &= c e^{-\frac{t}{RC}} - \frac{\omega V_0 C}{\sqrt{1 + (\omega RC)^2}} \sin(\omega t - \phi) \end{aligned}$$

where $\tan(\phi) = -1/\omega RC$.
The response involves two terms: an exponential decay and steady state response to oscillating external voltage, oscillating with ω and the amplitude $\frac{\omega V_0 C}{\sqrt{1+(\omega RC)^2}}$.

HIGHER ORDER DIFFERENTIAL EQUATIONS

Theory of linear equations

Initial-value and boundary-value problem

nth-order initial value problem is

Solve: $$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ \ldots \ + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

Subject to: $$y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots, y^{(n-1)} = y_{n-1} \qquad (1)$$

we seek a function defined on an interval I, containing x_0, that satisfies the DE and the n initial conditions above.

Higher-order ODEs: overview

- General aspects
- Reduction of order
- Homogeneous linear equation with constant coefficients
- Method of undetermined coefficients
- Method of variation of parameters
- Linear models
- Examples

Existence and uniqueness

Theorem: Existence of a unique solution

Let $a_n(x)$, $a_{n-1}(x)$, ... , $a_1(x)$, $a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ in any point in this interval, then a solution $y(x)$ of the initial value problem (1) exists on the interval and is unique.

Example: Unique solution of an IVP

$$3y''' + 5y'' - y' + 7y = 0, \qquad y(1) = 0, y'(1) = 0, y''(1) = 0$$

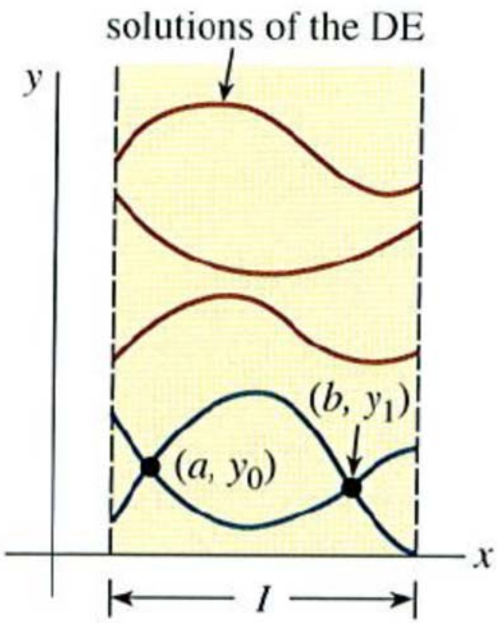
has the trivial solution $y = 0$. Since the DE is linear with constant coefficients, all the conditions of the theorem are fulfilled, and thus $y = 0$ is the *only* solution on any interval containing $x = 1$.

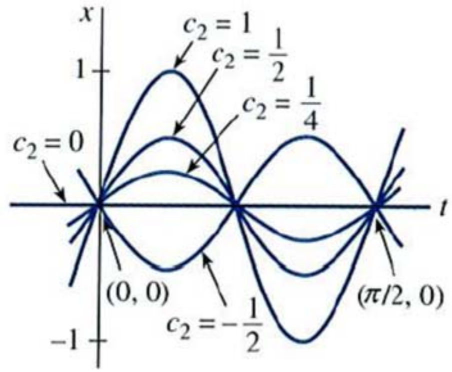
Boundary-value problem

consists of solving a linear DE of order two or greater in which the dependent variable y or its derivatives are specified at *different points*. Example: a two-point BVP

Solve: $$a_2(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x)$$

Subject to boundary conditions: $$y(x_0) = y_0,\ y(b) = y_1 \qquad (2)$$





A BVP can have many, one or no solutions:

The DE $x''+16x = 0$ has the two-parameter family of solutions $x = c_1 \cos 4t+c_2 \sin 4t$. Consider the BVPs:

(1) $x(0) = 0$, and $x(\pi/2) = 0 \Rightarrow c_1 = 0$ and the solution satisfies the DE for any value of c_2, thus the solution of this BVP is the one-parameter family $x = c_2 \sin 4t$.

(2) $x(0) = 0$, and $x(\pi/8) = 0 \Rightarrow c_1 = 0$ and $c_2 = 0$, so the only solution to this BVP is $x = 0$.

(3) $x(0) = 0 \Rightarrow c_1 = 0$ again but the second condition $x(\pi/2) = 1$ leads to the contradiction: $1 = c_2 \sin 2\pi = c_2.0 = 0$.

Homogeneous equations

nth-order **homogeneous** differential equation

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ \ldots \ + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0 \tag{3}$$

nth-order **nonhomogeneous** differential equation ($g(x) \neq 0$)

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ \ldots \ + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x) \tag{4}$$

Examples:

(1) Homogeneous DE: $2y'' + 3y' - 5y = 0$

(2) Nonhomogeneous DE: $x^2 y''' + 6y' + 10y = e^x$.

To solve a nonhomogeneous DE, we must first be able to solve the **associated homogeneous equation**.

We will soon proceed to the general theory of nth-order linear equations which we will present through a number of definitions and theorems. To avoid needless repetition, we make (and remeber) the following assumptions:

on some common interval I

- the coefficients $a_i(x)$, $i = 0, 1, 2, ..., n$ are continuous;
- the function $g(x)$ on r. h. s. is continuous; and
- $a_n(x) \neq 0$ for every x in the interval.

Differential operators

Examples:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}y = Dy \quad \text{or} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = D\,(Dy) = D^2y \quad \text{and in general} \quad \frac{\mathrm{d}^ny}{\mathrm{d}x^n} = D^ny$$

***n*th-order differential operator:**

polynomial expressions involving D are also differential operators

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \ \ldots \ + a_1(x)D + a_0(x)$$

An nth-order differential operator is a **linear operator**, that is, it satisfies

$$L\left[\alpha f(x) + \beta g(x)\right] = \alpha L\left(f(x)\right) + \beta L\left(g(x)\right) \tag{5}$$

Differential equations

Any linear differential equation can be expressed in terms of the D notation.

Example

$$\begin{aligned} y'' + 5y' + 6y &= 5x - 3 \\ D^2y + 5Dy + 6y &= 5x - 3 \\ (D^2 + 5D + 6)y &= 5x - 3 \end{aligned}$$

The nth-order linear differential equations can be written compactly as

Homogeneous: $L(y) = 0$

Non-homogeneous: $L(y) = g(x)$

Superposition principle

Theorem: Superposition principle - homogeneous equations

Let y_1, y_2, ... , y_k be solutions of the homogeneous nth-order DE (3) on an interval I, then the linear combination

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_k y_k(x) = \sum_{i=1}^{k} c_i y_i(x),$$

where the c_i, $i = 1, 2, ..., k$ are arbitrary constants, is also a trivial solution.

<u>Proof:</u> The case $k = 2$. Let $y_1(x)$ and $y_2(x)$ be solutions of $L(y) = 0$, then also

$$L(y) = L\left[c_1 y_1(x) + c_2 y_2(x)\right] = c_1 L(y_1) + c_2 L(y_2) = 0$$

Corollaries

(a) A constant multiple $y = c_1 y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear DE is also a solution.

(b) A homogeneous linear DE always possesses the trivial solution $y = 0$.

Example: Superposition - homogeneous DE
Let $y_1 = x^2$ and $y_2 = x^2 \ln x$ be both solutions of the homogeneous linear DE $x^3 y''' - 2xy' + 4y = 0$ on the interval $I = (0, \infty)$.

Show that by superposition principle, the linear combination

$$y = c_1 x^2 + c_2 x^2 \ln x$$

is also a solution of the equation on the interval.

Linear dependence and linear independence

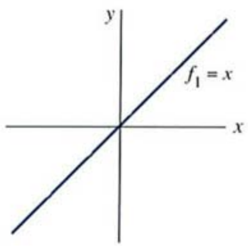
Definition:
A set of functions $f_1(x), f_2(x), ..., f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, ..., c_n$, not all zero, s.t.

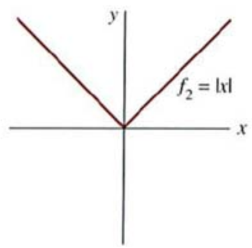
$$c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x) = 0 \tag{6}$$

for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent**.

Example: If two functions are linearly dependent, then one is simply a constant multiple of the other: assuming $c_1 \neq 0$, $c_1 f_1(x) + c_2 f_2(x) = 0 \Rightarrow f_1(x) = -(c_2/c_1) f_2(x)$. For example $f_1(x) = \sin(x)\cos(x)$ and $f_2(x) = \sin(2x) = 2f_1(x)$.

Two functions are linearly independent when neither is a constant multiple of the other on an interval. For example $f_1(x) = x$ and $f_2(x) = |x|$ on $I = (-\infty, \infty)$.





Solutions of differential equations

We are primarily interested in linearly independent solutions of linear DEs.

How to decide whether n solutions $y_1, y_2, ..., y_n$ of a homogeneous linear nth-order DE (3) are linearly independent?

Definition: Wronskian

Suppose each of the functions $f_1(x), f_2(x), ..., f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, ..., f_n) = \begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ f_1^{(n-1)} & f_2^{(n-1)} & ... & f_n^{(n-1)} \end{vmatrix}$$

is called the **Wronskian** of the functions.

Theorem: Criterion for linearly independent solutions

Let $y_1, y_2, ..., y_n$ be n solutions of the homogeneous linear nth-order DE (3) on an interval I. Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, ..., y_n) \neq 0$ for every x in the interval.

Definition: Fundamental set of solutions

Any set $y_1, y_2, ..., y_n$ of n linearly independent solutions of the homogeneous linear nth-order DE (3) on an interval I is said to be a **fundamental set of solutions** on the interval.

Theorem: Existence of a fundamental set

There exists a fundamental set of solutions for the homogeneous linear nth-order DE (3) on an interval I.

Theorem: General solution - homogeneous equations

Let $y_1, y_2, ..., y_n$ be a fundamental set of solutions of the homogeneous linear nth-order DE (3) on an interval I. Then the **general** solution of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x)$$

where c_i, $i = 1, 2, ..., n$ are arbitrary constants.
For proof for the case $n = 2$ see D.G. Zill et al., Advanced Engineering Mathematics, 4th Edition, p. 104.

Example 1:

The functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear DE $y'' - 9y = 0$ on $(-\infty, \infty)$.

Calculate the Wronskian and determine whether the functions form a fundamental set of solutions. If yes, determine a general solution.

Example 2:

The function $y = 4\sinh 3x - 5e^{3x}$ is a solution of the DE in Example 1 above. Verify this.

We must be able to obtain this solution from the general solution $y = c_1 e^{3x} + c_2 e^{-3x}$. What values the constants c_1 and c_2 have to have to get the solution above.

Example 3:

The functions $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ satisfy the third order DE $y''' - 6y'' + 11y' - 6y = 0$. Determine whether these functions form the fundamental set of solutions on $(-\infty, \infty)$, and write down the general solution.

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ \ldots\ + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = g(x) \tag{4}$$

Nonhomogeneous equations

Any function y_p free of any arbitrary parameters that satisfies (4) is said to be a **particular solution** of the equation.

For example, $y_p = 3$ is a particular solution of $y'' + 9y = 27$.

Theorem: General solution - nonhomogeneous equations

Let y_p be any particular solution of the nonhomogeneous linear nth-order DE (4) on an interval I, and let $y_1, y_2, ..., y_n$ be a fundamental set of solutions of the associated homogeneous DE (3) on I. Then the **general solution** of the equation on I is

$$y = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) + y_p \tag{7}$$

where the $c_i, i = 1, 2, ..., n$ are arbitrary constants.

Complementary function

The general solution of a homogeneous linear equation consists of the sum of two functions

$$y = c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x) + y_p(x) = y_c(x) + y_p(x)$$

The linear combination $y = c_1y_1(x)+c_2y_2(x)+...+c_ny_n(x)$ which is the general solution of the homogeneous DE (3), is called the **complementary solution** for equation (4).

Thus to solve the nonhomogeneous linear DE, we first solve the associated homogeneous equation and then find any particular solution of the nonhomogeneous equation. The general solution is then

y = complementary function + any particular solution.

Another superposition principle

Theorem: Superposition principle - nonhomogeneous equations

Let $y_{p_1}, y_{p_2}, ..., y_{p_k}$ be k particular solutions of the nonhomogeneous linear nth-order DE (4) on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, ..., g_k$. That is, suppose y_{p_i} denotes a particular solution of the corresponding DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ \ldots \ + a_1(x)y' + a_0(x)y = g_i(x) \tag{8}$$

where $i = 1, 2, ..., k$. Then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + ... + y_{p_k}(x) \tag{9}$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ \ldots \ + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + ... + g_k(x) \tag{10}$$

For proof for the case $k = 2$ see D.G. Zill et al., Advanced Engineering Mathematics, 4th Edition, p. 104.

<u>Example:</u>

Verify that

$$\begin{aligned} y_{p_1} &= -4x^2 && \text{is a particular solution of} \quad y'' - 3y' + 4y = -16x^2 + 24x - 8 \\ y_{p_2} &= e^{2x} && \text{is a particular solution of} \quad y'' - 3y' + 4y = 2e^{2x} \\ y_{p_3} &= xe^x && \text{is a particular solution of} \quad y'' - 3y' + 4y = 2xe^x - e^x \end{aligned}$$

and that $y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^x$ is a solution of

$$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^x - e^x$$

Remarks:

A dynamical system whose mathematical model is a linear nth-order DE

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ . \ . \ . \ + a_1(t)y' + a_0(t)y = g(t)$$

is said to be a **linear system**. The set of n time dependent functions $y(t), y'(t)$, ... , $y^{(n-1)}(t)$ are the **state variables** of the system. Their values at some time t give the **state of the system**. The function g is called the **input function**, **forcing function**, or **excitation function**. A solution $y(t)$ of the DE is said to be the **output** or **response of the system**. The output or response $y(t)$ is uniquely determined by the input and the state of the system prescribed at a time t_0; that is, by the initial conditions $y(t_0), y'(t_0), ..., y^{(n-1)}(t_0)$.

Reduction of order

Suppose $y(x)$ denotes a known solution of a homogeneous linear second-order equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{11}$$

we seek the second solution $y_2(x)$ so that y_1 and y_2 are linearly independent on some interval I. That is we are looking for y_2 s. t. $y_2/y_1 = u(x)$, or $y_2(x) = u(x)y_1(x)$.

The idea is to find $u(x)$ by substituting $y_2(x) = u(x)y_1(x)$ into the DE. This method is called **reduction of order** since we must solve a first-order equation to find u.

Example:

Given $y_1 = e^x$ is a solution of $y'' - y = 0$ on $(-\infty, \infty)$, use the reductions of order to find a second solution y_2.

General case

We put the equation (11) into the standard form by dividing by $a_2(x)$:

$$y'' + P(x)y' + Q(x)y = 0 \tag{12}$$

where $P(x)$ and $Q(x)$ are continuous on some interval I. Assume that $y_1(x)$ is a known solution of (12) on I and that $y_1(x) \neq 0$ for every $x \in I$. We define $y = u(x)y_1(x)$

$$y' = uy_1' + y_1u', \qquad y'' = uy_1'' + 2y_1'u' + y_1u''$$

$$y'' + Py' + Qy = u\left[y_1'' + Py_1' + Qy_1\right] + y_1u'' + \left(2y_1' + Py_1\right)u' = 0$$

where the term in the square bracket equals to zero.

This implies

$$y_1 u'' + \left(2y_1' + Py_1\right) u' = 0 \quad \text{or} \quad y_1 w' + \left(2y_1' + Py_1\right) w = 0$$

where we used $w = u'$. The last equation can be solved by separating variables and integrating

$$\frac{dw}{w} + 2\frac{y_1'}{y_1}dx + Pdx = 0$$

$$\ln\left|wy_1^2\right| = -\int Pdx + c$$

or

$$wy_1^2 = c_1 e^{-\int Pdx}$$

Solving the last equation for w, and using $w = u'$ and integrating again gives

$$u = c_1 \int \frac{e^{-\int P dx}}{y_1^2} dx + c_2$$

By choosing $c_1 = 1$ and $c_2 = 0$ and by using $y = u(x)y_1(x)$ we find the second solution of the equation (12):

$$y_2 = y_1(x) \int \frac{e^{-\int P dx}}{y_1(x)^2} dx \tag{13}$$

Example:

$y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Find the generals solution on $(0, \infty)$.
From the standard form of the equation

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

we find using the formula above

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4} dx = x^2 \int \frac{dx}{x} = x^2 \ln x$$

The general solution on $(0, \infty)$ is given by

$$y = c_1y_1 + c_2y_2 = c_1x^2 + c_2x^2 \ln x$$