Tutorial 5 - Solutions

1. Consider the continuous-time signal $x(t) = \sin(20t) + \cos(40t)$.

   (a) Find the fundamental period of $x(t)$.

   (b) If $x(t)$ is sampled with a sampling period T to obtain the discrete-time signal

   $$x[n] = \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right)$$

   1. Determine a choice of T consistent with this information.
   2. Is your choice of T in the previous question (i.e., Part 1.) unique? If so, explain why. If not, specify another choice of T consistent with the information given.
      Answer: T is not unique. Another choice is $T = \frac{11\pi}{100}$.

   Solution:

   (a) We can write

   $$\begin{aligned} x(t) &= \sin(20t) + \cos(40t) \\ &= \sin(2\pi \frac{10}{\pi} t) + \cos(2\pi \frac{20}{\pi} t) \end{aligned}$$

   The period of $\sin(2\pi \frac{10}{\pi} t)$ is $T_1 = \frac{\pi}{10}$ and that of $\cos(2\pi \frac{20}{\pi} t)$ is $T_2 = \frac{\pi}{20} = \frac{T_1}{2}$. Thus, the fundamental period of $x(t)$ is $T_0 = T_1 = 2T_2 = \frac{\pi}{10}$.

   (b) If the sampling processing is carried out with a sampling period T (secs), then the value of the nth sample of the obtained sequence is given by

   $$x[n] = x(nT) = \sin(20nT) + \cos(40nT)$$

   Thus to achieve the stated discrete-time signal, we can set $20T = \frac{\pi}{5}$ and thus $T = \pi/100$.

   (c) T is not unique. Another choice is $T = \frac{11\pi}{100}$.

2. The continuous-time signal $x(t) = v_1(t) \times v_2(t)$ is sampled with an impulse train

   $$p_T(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$$

   where T is the sampling interval.

   (a) Assuming $v_1(t)$ and $v_2(t)$ are band-limited to 100 Hz and 300 Hz, respectively, compute the minimum value of the sampling rate f_s that does not introduce any aliasing.

(b) Repeat part 2a for $v_1(t) = \text{Sa}(200\pi t)$ and $v_2(t) = \text{Sa}(500\pi t)$. Assuming that a sampling interval of $T = 3$ ms is used to sample $x(t) = v_1(t) \times v_2(t)$, can $x(t)$ be accurately recovered from its samples?

(c) Repeat part 2b for a sampling interval of $T = 0.1$ ms.

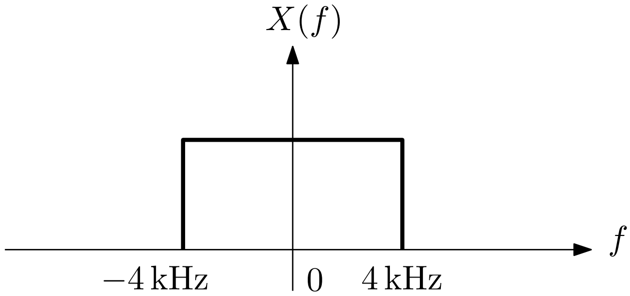
Solution:

(a) $f_s = 2 \times 400 = 800$ Hz;

(b) Aliasing occurs (you need to determine the maximum frequency of $x(t)$ in this case first, the detailed solution of this problem is on the last page);

(c) No aliasing.

3. A signal $x(t)$ has the Fourier transform $X(f)$ shown in the following figure



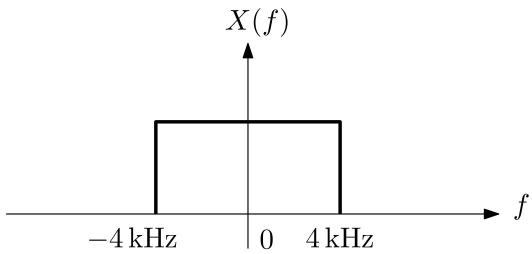
The signal $x(t)$ is sampled by an ideal uniform sampling process with a sampling rate f_s (Hz or samples/sec). The sampled signal is denoted by $x_s(t)$.

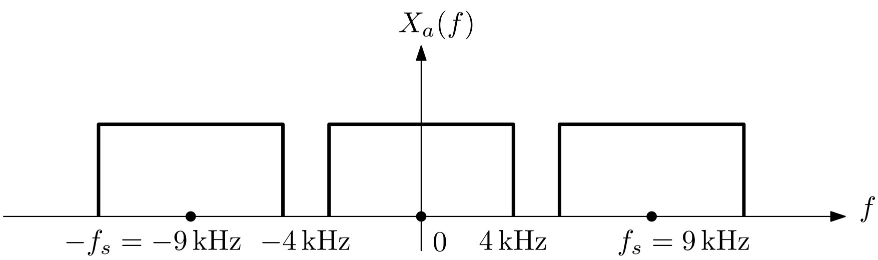
(1) Sketch the Fourier transform of $x_s(t)$ for $f_s = 9000$.

(2) Repeat (1) for $f_s = 6000$ and give your comments on this case.

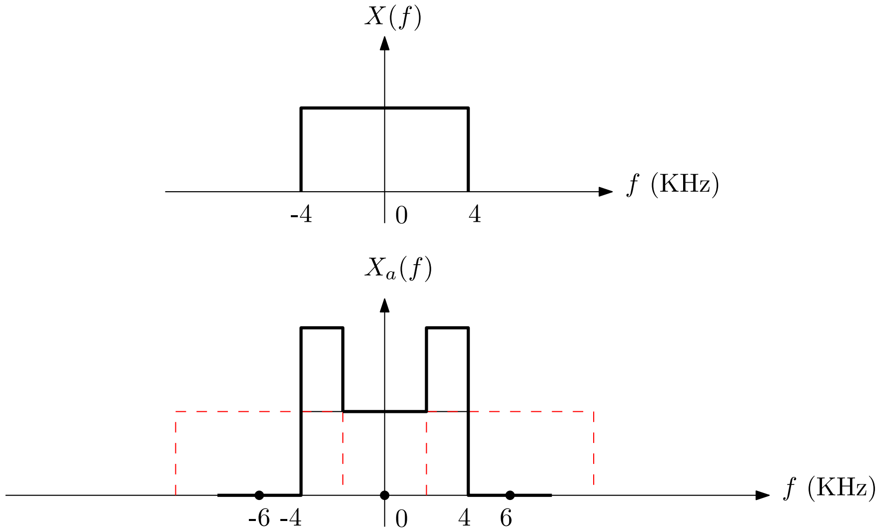
Solution:

(1) The spectrum of the sampled signal is **a sum of scaled and shifted replicas** of the Fourier transform of the original signal $x(t)$. When $f_s = 9$ KHz, there is no overlapping between these replicas, and thus the spectrum of the sampled signal is given below





(2) When $f_s = 6$ KHz, these replicas will overlap, and thus the spectrum of the sampled signal is given below

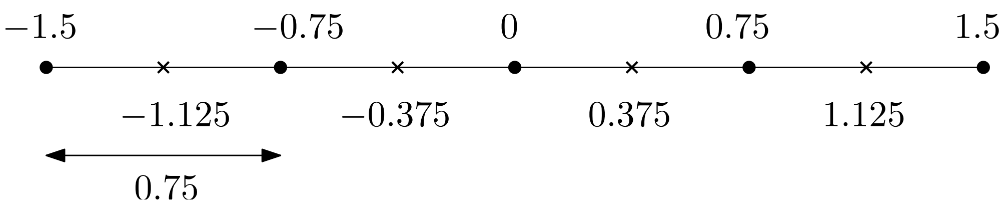


In this case aliasing occurs.

4. Consider a uniform quantiser in the range of $(-1.5, 1.5)$ with 4 levels.
   (a) Sketch the 4 quantisation levels of the quantiser.
   (b) Compute the step size and the maximum quantisation error of the quantiser.
   (c) Compute the output of the quantiser for the following input sequence $\{1.2, -0.2, 0.4, -0.89\}$
   (d) Assume the input signal of the quantiser is uniformly distributed, compute the signal-to-quantisation noise.

   Solution:

   (a) The quantiser has 4 levels, and thus there are 4 quantised levels as shown in the following figure



   (b) The step size is $(1.5 - (-1.5))/4 = 0.75$ and the maximum quantisation error is half of the step size, i.e. $0.75/2 = 0.375$.
   (c) The quantised sequence is $\{1.125, -0.375, 0.375, -1.125\}$
   (d) The quantiser has 4 levels, so it needs 2 bits to represent all levels. Thus the SQNR $= 6.02 \times 2 = 12.04$ dB.

5. Explain how you would (approximately) measure the system impulse response of an LTI system, without knowing its components.

   Solution: We know that for an LTI system, if the input is an impulse signal then, the output will be the system impulse response. In fact, impulse signal can be viewed as a rectangular signal with a very small width and very large height. Thus, in practice we can estimate the system impulse response by inputting a pulse signal of *very small width* and measuring the output.

6. Suppose that the input is a continuous non-periodic signal $x(t)$ and its continuous-time Fourier transform $X(\omega)$ is given. The system impulse response is denoted by $h(t)$ and its frequency response is $H(\omega)$. Describe two methods to find the output of the system.

   Solution: The output of an LTI system can be computed by two methods. In the time domain we have

   $$y(t) = x(t) * h(t)$$

   where $*$ denotes the convolution integral. In the frequency domain, we have

   $$Y(\omega) = H(\omega)X(\omega)$$

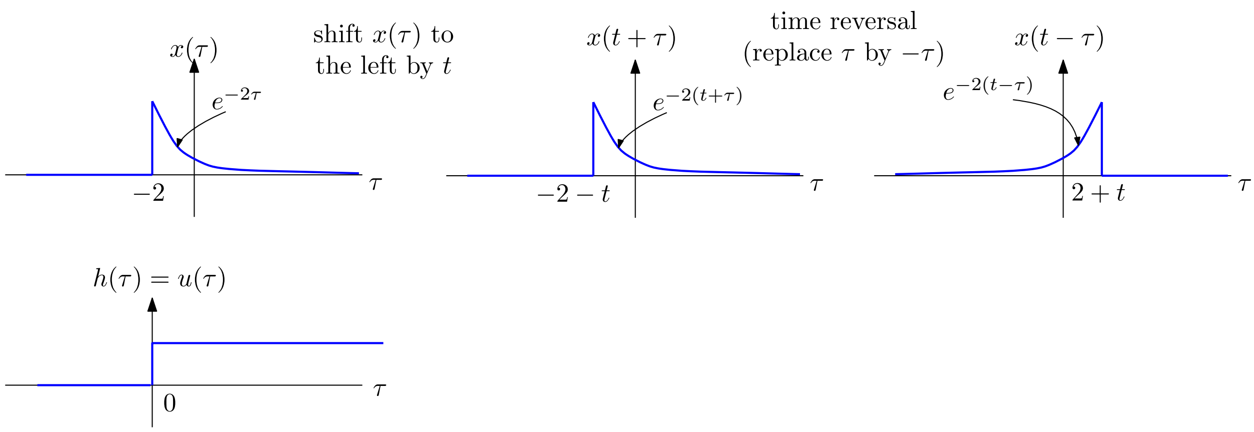
   After $Y(\omega)$ is computed, inverse FT is used to find the signal in the time domain.

7. Consider an LTI system having an impulse response $h(t) = u(t)$. Find the output signal of the system if the input signal is $x(t) = e^{-2t}u(t+2)$.

   Solution: The output of the system is the convolution integral of the input and the system impulse response:

   $$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \tag{1}$$

   To evaluate the above integral we first need to sketch $h(\tau)$ and $x(t-\tau)$, which are shown below.



   Then we consider two cases:

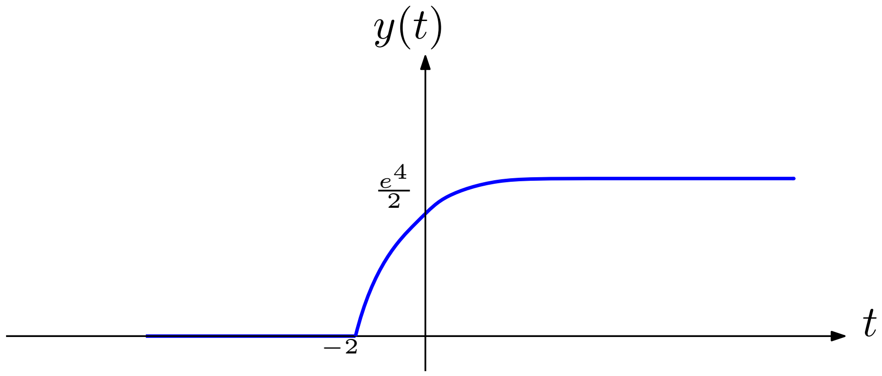
   - $t+2 < 0$, i.e. $t < -2$. In this case the product $h(\tau)x(t-\tau)$ is equal to 0 (use the above figure to check that !!!), and thus $y(t) = 0$.
   - $t+2 \geq 0$, i.e. $t \geq -2$. In this case we can see that

   $$\begin{aligned} y(t) &= \int_0^{t+2} e^{-2(t-\tau)}d\tau = e^{-2t}\int_0^{t+2} e^{2\tau}d\tau \\ &= \frac{e^{-2t}}{2}(e^{2(t+2)} - 1) = \frac{1}{2}(e^4 - e^{-2t}) \\ &= \frac{e^4}{2}(1 - e^{-2(t+2)}) \end{aligned}$$

In summary, the output of the system is given by

$$y(t) = \begin{cases} 0 & t < -2 \\ \frac{e^4}{2}(1 - e^{-2(t+2)}) & t \geq -2 \end{cases} \tag{2}$$

In a compact form we can write $y(t) = \frac{e^4}{2}(1 - e^{-2(t+2)})u(t+2)$. The output of the system is plotted in the figure below.



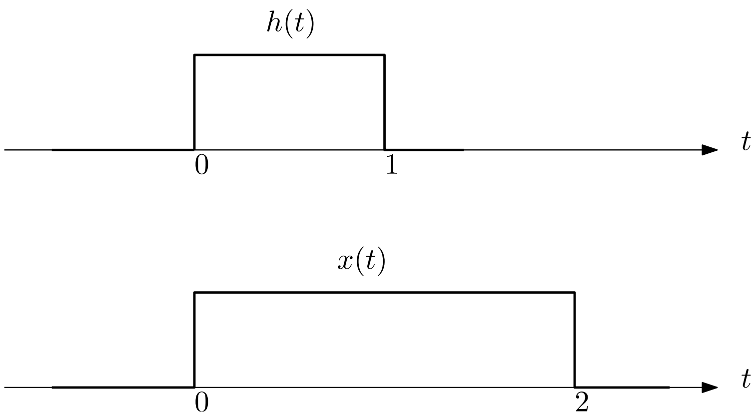
8. Consider a continuous-time LTI system with impulse response $h(t) = u(t) - u(t-1)$ and input $x(t) = u(t) - u(t-2)$. Compute the output signal of the system.

Solution:

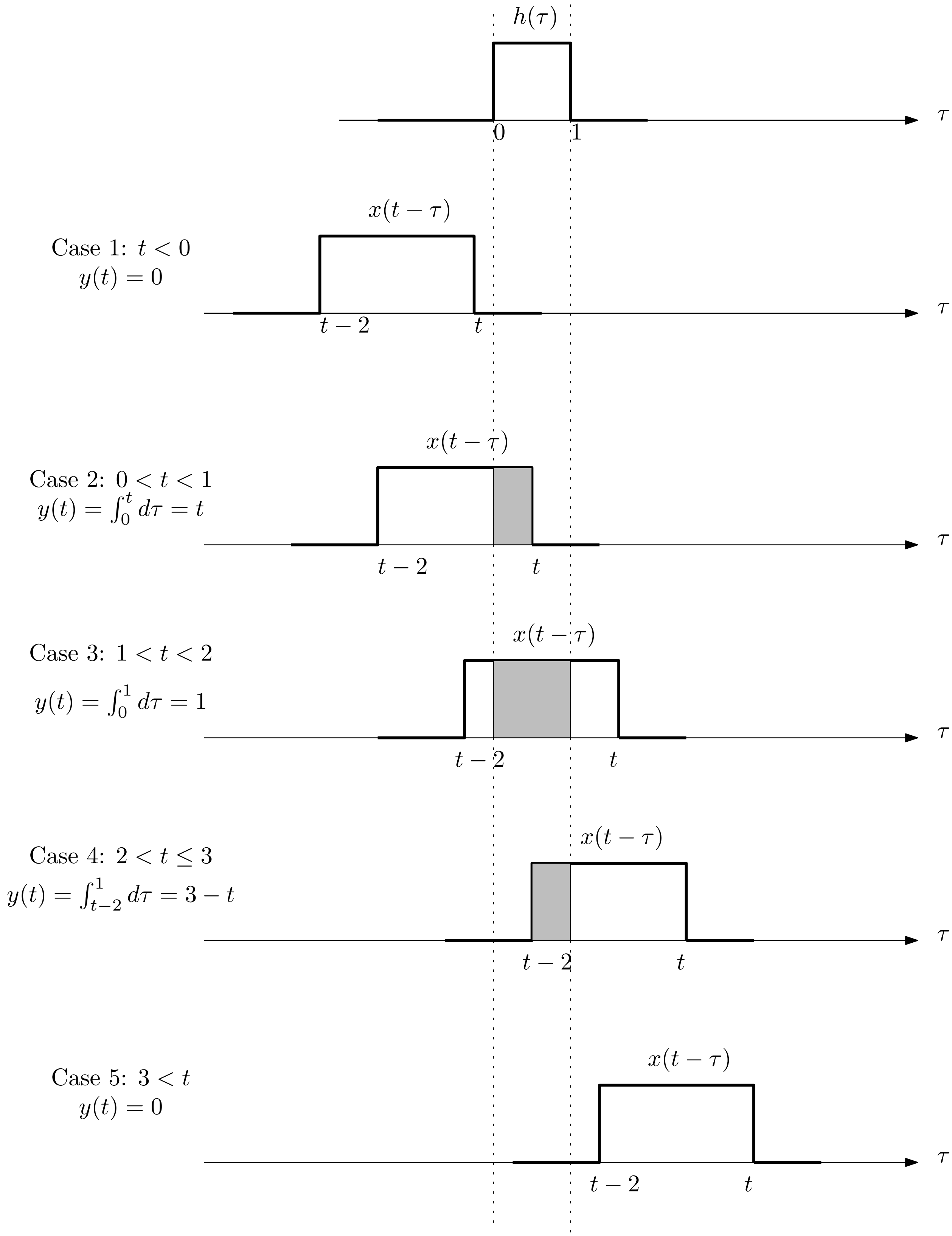
The output of the system is the convolution integral of the input and the system impulse response:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \tag{3}$$

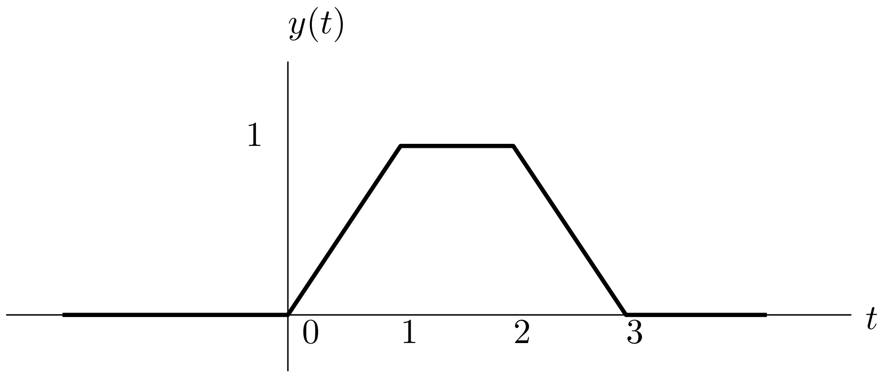
To evaluate the above integral we first need to sketch $h(t)$ and $x(t)$, which is shown below.



To compute $y(t)$ for at time t we need to create $x(t-\tau)$. We consider several cases for t as illustrated in the following figure



Thus $y(t)$ is plotted in the following figure



9. A discrete-time LTI system has the impulse response given below

$$h[n] = \begin{cases} 1 & n = -1 \\ 3 & n = 0 \\ 2 & n = 1 \\ -1 & n = 2 \end{cases} \quad (4)$$

Given the input $x[n] = u[n] - u[n-3]$, determine the system output $y[n]$.

Solution:

We can write $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$, and thus $y[n] = x[n] * h[n] = (\delta[n] + \delta[n-1] + \delta[n-2]) * h[n] = h[n] + h[n-1] + h[n-2]$. Recall that $\delta[n - n_0] * h[n] = h[n - n_0]$.

2.b,c

$$v_1(t) = Sa(200\pi t) = \frac{sin200\pi t}{200\pi t} \leftrightarrow \frac{1}{200} rect\left(\frac{\omega}{400\pi}\right), \tau = 400\pi$$

$$v_2(t) = Sa(500\pi t) = \frac{sin500\pi t}{500\pi t} \leftrightarrow \frac{1}{500} rect\left(\frac{\omega}{1000\pi}\right), \tau = 1000\pi$$

$We\ can\ find\ that\ \omega_{1,max} = 200\pi\ rad/s; \omega_{2,max} = 500\pi\ rad/s.$

When two signals in the time domain are multiplied, they are convolved in the frequency domain, so the maximum frequency of *x(t)* is the sum of the maximum frequencies of the two signals.

$$\omega_{max} = \omega_{1,max} + \omega_{2,max} = 700\pi \approx 2198\ rad/s$$

(b) When T=3ms, Sampling angle frequency

$$\omega_s = \frac{2\pi}{T} = \frac{2\pi}{3 \times 10^{-3}} \approx 2093\ rad/s$$

$\omega_s < \omega_{max}$, so it will appear aliasing.

(c) When T=0.1ms, Sampling angle frequency

$$\omega_s = \frac{2\pi}{T} = \frac{2\pi}{0.1 \times 10^{-3}} \approx 62800\ rad/s$$

So there is no aliasing.