Tutorial Sheet 4 - Solutions

Q1(i)
$$f_{k+1} = (\alpha_F + \beta_F f_k r_k) f_k,$$
$$r_{k+1} = (\alpha_R - \beta_R f_k r_k - \lambda_R r_k) r_k$$

where $\alpha_F < 1$, $\alpha_R > 1$.

For discrete systems equilibrium points given by $\boldsymbol{x}_k = \boldsymbol{x}_{k+1} = \boldsymbol{x}_e$. Therefore:

$$f_e = (\alpha_F + \beta_F f_e r_e) f_e,$$
$$r_e = (\alpha_R - \beta_R f_e r_e - \lambda_R r_e) r_e$$

Solving these equations given the equilibrium points:

$$\begin{aligned} f_e &= \alpha_F f_e + \beta_F f_e^2 r_e \\ \Rightarrow (\alpha_F - 1) f_e + \beta_F f_e^2 r_e &= 0 \\ \Rightarrow f_e((\alpha_F - 1) + \beta_F f_e r_e) &= 0 \\ \Rightarrow f_e = 0 \quad \text{or} \quad f_e &= \frac{(1-\alpha_F)}{\beta_F r_e} \end{aligned}$$

and

$$\begin{aligned} r_e &= \alpha_R r_e - \beta_R f_e r_e^2 - \lambda_R r_e^2 \\ \Rightarrow (\alpha_R - 1) r_e - (\beta_R f_e + \lambda_R) r_e^2 &= 0 \\ \Rightarrow r_e((\alpha_R - 1) - (\beta_R f_e + \lambda_R) r_e) &= 0 \\ \Rightarrow r_e = 0 \quad \text{or} \quad r_e &= \frac{\alpha_R - 1}{\beta_R f_e + \lambda_R} \end{aligned}$$

Therefore possible equilibrium points are:

(a) $f_e = 0$ we have $r_e = 0$ - extinction for both foxes and rabbits

(b) $f_e = 0$ we have $r_e = \dfrac{\alpha_R - 1}{\lambda_R}$ - extinction of foxes and large rabbit population limited only by availability of grass.

(c) $r_e = 0$ and $f_e = \dfrac{(1-\alpha_F)}{0} = \infty$ which clearly is not a valid scenario. This can be verified by setting $r_e = 0$ in equation 1 before solving for f_e. Thus, this is not an equilibrium point.

(d) $r_e = \dfrac{\alpha_R - 1}{\beta_R f_e + \lambda_R}$ and $f_e = \dfrac{(1-\alpha_F)}{\beta_F r_e}$ (both species co-exist). Solving for r_e gives:

$$r_e = \frac{\alpha_R - 1}{\beta_R\left[\dfrac{(1-\alpha_F)}{\beta_F r_e}\right] + \lambda_R} = \frac{(\alpha_R - 1)\beta_F r_e}{\beta_R(1-\alpha_F) + \lambda_R \beta_F r_e} \Rightarrow r_e = \frac{\beta_F(\alpha_R - 1) - \beta_R(1-\alpha_F)}{\lambda_R \beta_F}$$

and

$$f_e = \frac{(1-\alpha_F)}{\beta_F r_e} = \frac{(1-\alpha_F)}{\dfrac{\beta_F}{\lambda_R}\left[(\alpha_R - 1) - \dfrac{\beta_R}{\beta_F}(1-\alpha_F)\right]} = \frac{\lambda_R(1-\alpha_F)}{\beta_F(\alpha_R - 1) - \beta_R(1-\alpha_F)}$$

(b) The equations have the general form:

$$f_{k+1} = g_1(f_k, r_k) = \alpha_F f_k + \beta_F f_k^2 r_k$$

$$r_{k+1} = g_2(f_k, r_k) = \alpha_R r_k - \beta_R f_k r_k^2 - \lambda_R r_k^2$$

and the corresponding linearised model about equilibrium point (f_e, r_e) is given by:

$$\begin{cases} \Delta f_{k+1} = \left.\frac{\partial g_1}{\partial f_k}\right|_e \Delta f_k + \left.\frac{\partial g_1}{\partial r_k}\right|_e \Delta r_k \\ \Delta r_{k+1} = \left.\frac{\partial g_2}{\partial f_k}\right|_e \Delta f_k + \left.\frac{\partial g_2}{\partial r_k}\right|_e \Delta r_k \end{cases} \qquad (1)$$

$$\left.\frac{\partial g_1}{\partial f_k}\right|_e = \alpha_F + 2\beta_F f_e r_e, \quad \left.\frac{\partial g_1}{\partial r_k}\right|_e = \beta_F f_e^2$$

$$\left.\frac{\partial g_2}{\partial f_k}\right|_e = -\beta_R r_e^2, \quad \left.\frac{\partial g_2}{\partial r_k}\right|_e = \alpha_R - 2\beta_R f_e r_e - 2\lambda_R r_e$$

Thus:

$$\begin{bmatrix} \Delta f_{k+1} \\ \Delta r_{k+1} \end{bmatrix} = \begin{bmatrix} \alpha_F + 2\beta_F f_e r_e & \beta_F f_e^2 \\ -\beta_R r_e^2 & \alpha_R - 2\beta_R f_e r_e - 2\lambda_R r_e \end{bmatrix} \begin{bmatrix} \Delta f_k \\ \Delta r_k \end{bmatrix}$$

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For (a) $f_e = 0$ we have $r_e = 0$ this becomes:

$$\begin{bmatrix} \Delta f_{k+1} \\ \Delta r_{k+1} \end{bmatrix} = \begin{bmatrix} \alpha_F & 0 \\ 0 & \alpha_R \end{bmatrix} \begin{bmatrix} \Delta f_k \\ \Delta r_k \end{bmatrix}$$

For (b) $f_e = 0$ we have $r_e = \frac{\alpha_R - 1}{\lambda_R}$ we get:

$$\begin{bmatrix} \Delta f_{k+1} \\ \Delta r_{k+1} \end{bmatrix} = \begin{bmatrix} \alpha_F & 0 \\ -\beta_R\left[\frac{\alpha_R - 1}{\lambda_R}\right]^2 & 2-\alpha_R \end{bmatrix} \begin{bmatrix} \Delta f_k \\ \Delta r_k \end{bmatrix}$$

and for (d) $r_e = \frac{\beta_F(\alpha_R - 1) - \beta_R(1 - \alpha_F)}{\lambda_R \beta_F}$ and $f_e = \frac{\lambda_R(1 - \alpha_F)}{\beta_F(\alpha_R - 1) - \beta_R(1 - \alpha_F)}$:

$$\begin{bmatrix} \Delta f_{k+1} \\ \Delta r_{k+1} \end{bmatrix} = \begin{bmatrix} (2 - \alpha_F) & \beta_F f_e^2 \\ -\beta_R r_e^2 & (2 - \alpha_R) \end{bmatrix} \begin{bmatrix} \Delta f_k \\ \Delta r_k \end{bmatrix}$$

NB: $f_e r_e = \frac{(1 - \alpha_F)}{\beta_F} \Rightarrow \alpha_F + 2\beta_F f_e r_e = \alpha_F + 2(1 - \alpha_F) = 2 - \alpha_F$

and $\alpha_R - 2\beta_R f_e r_e - 2\lambda_R r_e = 2 - \alpha_R$

Q2 (i) $\dot{x} = x^2 - 2x - 8$

Equilibrium points when $\dot{x} = 0 \Rightarrow x^2 - 2x - 8 = 0 \Rightarrow (x-4)(x+2) = 0 \Rightarrow x_e = -2$ or 4.

Model has a single state $\dot{x} = f(x) = x^2 - 2x - 8$.

Linearised model is:

$$\Delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_e \Delta x,\ \left.\frac{\partial f}{\partial x}\right|_e = 2x_e - 2$$

So for $x_e = -2$ model is $\Delta\dot{x} = -6\Delta x$ and for $x_e = 4$, $\Delta\dot{x} = 6\Delta x$

(ii) $\ddot{x} = \dot{x}^2 x - 2x - 8u^3$

$\ddot{x} = \dot{x} = 0$ at the equilibrium point. So... $-2x_e - 8u^3 = 0 \Rightarrow x_e = -4u_e^3$

Change to state space model format before linearising...

Let $x_1 = x$, $x_2 = \dot{x}$. This gives the state-space model:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_2^2 x_1 - 2x_1 - 8u_e^3.$$

Note that in terms of the state the equilibrium point is give by $x_{1e} = x_e = -4u_e^3$, $x_{2e} = 0$.

Linearised state model is given by:

$$\Delta\dot{x}_1 = \left.\frac{\partial f_1}{\partial x_1}\right|_e \Delta x_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_e \Delta x_2 + \left.\frac{\partial f_1}{\partial u}\right|_e \Delta u \text{ and } \Delta\dot{x}_2 = \left.\frac{\partial f_2}{\partial x_1}\right|_e \Delta x_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_e \Delta x_2 + \left.\frac{\partial f_2}{\partial u}\right|_e \Delta u$$

where $\dot{x}_1 = f_1(x_1, x_2, u) = x_2 \Rightarrow \left.\frac{\partial f_1}{\partial x_1}\right|_e = 0,\ \left.\frac{\partial f_1}{\partial x_2}\right|_e = 1,\ \left.\frac{\partial f_1}{\partial u}\right|_e = 0$

and $\dot{x}_2 = f_2(x_1, x_2, u) = x_2^2 x_1 - 2x_1 - 8u^3$

$$\left.\frac{\partial f_2}{\partial x_1}\right|_e = x_{2e}^2 - 2 = -2,\ \left.\frac{\partial f_2}{\partial x_2}\right|_e = 2x_{2e}x_{1e} = 0 \text{ and } \left.\frac{\partial f_2}{\partial u}\right|_e = -24u_e^2$$

Therefore the linear state space model is:

$$\begin{bmatrix}\Delta\dot{x}_1 \\ \Delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -2 & 0\end{bmatrix}\begin{bmatrix}\Delta x_1 \\ \Delta x_2\end{bmatrix} + \begin{bmatrix}0 \\ -24u_e^2\end{bmatrix}\Delta u \quad (2)$$

(iii) $x_{k+1} = x_k^2 - 2x_k - 8$

Discrete system, therefore $x_{k+1} = x_k = x_e$ at the equilibrium points:

$$x_e = x_e^2 - 2x_e - 8 \Rightarrow x_e^2 - 3x_e - 8 = 0 \Rightarrow x_e = \frac{3 \pm \sqrt{41}}{2}$$

Linear model given by:

$$\Delta x_{k+1} = \left.\frac{\partial f}{\partial x_k}\right|_e \Delta x_k = (2x_e - 2)\Delta x_k$$

For $x_e = \frac{3+\sqrt{41}}{2}$ this gives $\Delta x_{k+1} = [1+\sqrt{41}]\Delta x_k$

and for $x_e = \frac{3-\sqrt{41}}{2}$, $\Delta x_{k+1} = [1-\sqrt{41}]\Delta x_k$

(iv) $x_{k+2} = x_{k+1}^2 x_k - 2x_k - 2u_k$

$$x_{k+2} = x_{k+1} = x_k = x_e$$

$$x_e = x_e^2 x_e - 2x_e - 2u_e \quad \Rightarrow x_e^3 - 3x_e - 2u_e$$

This cannot be solved explicitly without knowing u_e. e.g. if $u_e = 0$ we get:

$$x_e^3 - 3x_e = 0 \Leftrightarrow x_e = 0, \text{ or } x_e = \pm\sqrt{3}$$

i.e there are 3 equilibrium points in this case. For other values of u_e we may only have 1 or 2 equilibrium points.

State space model obtained by choosing states $x_1(k) = x_k$, $x_2(k) = x_{k+1}$:

$$x_1(k+1) = x_2(k)$$

$$x_2(k+1) = [x_2(k)]^2 x_1(k) - 2x_1(k) - 2u_k$$

and the equilibrium state is $x_{1e} = x_{2e} = x_e$

The linear model is $\begin{bmatrix}\Delta x_1(k+1)\\ \Delta x_2(k+1)\end{bmatrix} = \begin{bmatrix}0 & 1\\ x_{2e}^2 - 2 & 2x_{2e}x_{1e}\end{bmatrix}\begin{bmatrix}\Delta x_1(k)\\ \Delta x_2(k)\end{bmatrix} + \begin{bmatrix}0\\ -2\end{bmatrix}$

which becomes $\begin{bmatrix}\Delta x_1(k+1)\\ \Delta x_2(k+1)\end{bmatrix} = \begin{bmatrix}0 & 1\\ x_e^2 - 2 & 2x_e^2\end{bmatrix}\begin{bmatrix}\Delta x_1(k)\\ \Delta x_2(k)\end{bmatrix} + \begin{bmatrix}0\\ -2\end{bmatrix}$

(v)
$$\dot{x}_1 = -2x_1x_2 + x_2^2 u$$
$$\dot{x}_2 = -4x_1^2 x_2 + 2u$$

Equilibrium when $\dot{x}_1 = \dot{x}_2 = 0$, i.e.
$$\begin{cases} 0 = -2x_{1e}x_{2e} + x_2^2 u_e \\ 0 = -4x_{1e}^2 x_{2e} + 2u_e \end{cases}$$

First equation gives:

$x_{2e}(x_{2e}u_e - 2x_{1e}) = 0 \quad \Rightarrow x_{2e} = 0$ or $x_{2e} = 2\dfrac{x_{1e}}{u_e}$ at an equilibrium point

while the second equation gives $x_{1e}^2 x_{2e} = \frac{1}{2}u_e$

If $x_{2e} = 0$ then $x_{1e} \to \infty$, hence not an equilibrium point. Therefore only equilibrium point is when $x_{2e} = 2\dfrac{x_{1e}}{u_e}$:

$$x_{1e}^2\left[2\frac{x_{1e}}{u_e}\right] = \frac{1}{2}u_e \Rightarrow x_{1e}^3 = \frac{u_e^2}{4} \Rightarrow x_{1e} = \sqrt[3]{\frac{u_e^2}{4}} \text{ and } x_{2e} = \frac{2}{u_e}x_{1e} = \sqrt[3]{\frac{2}{u_e}}$$

i.e equilibrium state is $\underline{x}_e = \begin{bmatrix} \sqrt[3]{\dfrac{u_e^2}{4}} & \sqrt[3]{\dfrac{2}{u_e}} \end{bmatrix}^T$

Linear model given by:
$$\begin{cases} \Delta\dot{x}_1 = [-2x_{2e}]\Delta x_1 + [2x_2 u_e - 2x_{1e}]\Delta x_2 + [x_{2e}^2]\Delta u \\ \Delta\dot{x}_2 = [-8x_{1e}x_{2e}]\Delta x_1 + [-4x_{1e}^2]\Delta x_2 + 2\Delta u \end{cases}$$

Writing this in matrix from gives:
$$\underline{\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_{2e} & 2x_2u_e - 2x_{1e} \\ -8x_{1e}x_{2e} & -4x_{1e}^2 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} x_{2e}^2 \\ 2 \end{bmatrix}\Delta u}$$

(vi)
$$\dot{x}_1 = -2x_1x_2 + x_2u^2$$
$$\dot{x}_2 = -4x_1^2 + x_2 + 2u$$

Equilibrium when:

$$0 = -2x_{1e}x_{2e} + x_{2e}u_e^2 \qquad \Rightarrow x_{2e}(-2x_{1e} + u_e^2) = 0 \Rightarrow x_{2e} = 0 \quad \text{or} \quad x_{1e} = \frac{u_e^2}{2}$$
$$0 = -4x_{1e}^2 + x_{2e} + 2u_e \qquad \text{and} \qquad x_{2e} = 4x_{1e}^2 - 2u_e$$

For $x_{2e} = 0$ we get $4x_{1e}^2 - 2u_e = 0 \Rightarrow x_{1e} = \sqrt{\dfrac{u_e}{2}}$

For $x_{1e} = \dfrac{u_e^2}{2}$ we get $x_{2e} = u_e^4 - 2u_e$

The linearised state space model is obtained as before, i.e.:

$$\Delta\dot{x}_1 = \left.\frac{\partial f_1}{\partial x_1}\right|_e \Delta x_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_e \Delta x_2 + \left.\frac{\partial f_1}{\partial u}\right|_e \Delta u \text{ and } \Delta\dot{x}_2 = \left.\frac{\partial f_2}{\partial x_1}\right|_e \Delta x_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_e \Delta x_2 + \left.\frac{\partial f_2}{\partial u}\right|_e \Delta u$$

or in matrix form as:

$$\begin{bmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}\left.\frac{\partial f_1}{\partial x_1}\right|_e & \left.\frac{\partial f_1}{\partial x_2}\right|_e\\ \left.\frac{\partial f_2}{\partial x_1}\right|_e & \left.\frac{\partial f_2}{\partial x_2}\right|_e\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix} + \begin{bmatrix}\left.\frac{\partial f_1}{\partial u}\right|_e\\ \left.\frac{\partial f_2}{\partial u}\right|_e\end{bmatrix}\Delta u \qquad (3)$$

where $f_1 = -2x_1x_2 + x_2u^2$ and $f_2 = -4x_1^2 + x_2 + 2u$.

Hence linear state-space model is $\begin{bmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}-2x_{2e} & u_e^2 - 2x_{1e}\\ -8x_{1e} & 1\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix} + \begin{bmatrix}2x_{2e}u_e\\ 2\end{bmatrix}\Delta u$

For $x_{1e} = \sqrt{\frac{u_e}{2}}$, $x_{2e} = 0$ this gives $\begin{bmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & u_e^2 - 2\sqrt{\frac{u_e}{2}}\\ -8\sqrt{\frac{u_e}{2}} & 1\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix} + \begin{bmatrix}0\\ 2\end{bmatrix}\Delta u$

For $x_{1e} = \frac{u_e^2}{2}$, $x_{2e} = u_e^4 - 2u_e$ this gives $\begin{bmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}4u_e - 2u_e^2 & 0\\ -4u_e^2 & 1\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix} + \begin{bmatrix}2u_e^5 - 4u_e^2\\ 2\end{bmatrix}\Delta u$

(vii) $\dot{x} = x^2 + x + 1$

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Equilibrium points when $0 = x_e^2 + x_e + 1$:

$$\Rightarrow x_e = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2},$$

i.e there are no real valued solutions, hence this system does not have equilibrium points.

(viii) $x_{k+1} = x_k - sin(x_{k-1}) + u_k$

Equilibrium points $x_e = x_e - sin(x_e) + u_e \Rightarrow sin(x_e) = u_e$, i.e. $x_e = sin^{-1}(u_e)$.

This means that for a particular value of u_e there are an infinite number of equilibrium points provided, $|u_e| \le 1$.

e.g for $u_e = 0$, $x_e = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.

Defining the states as $x_1(k) = x_{k-1}$ and $x_2(k) = x_k$ then the state space model is:

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = x_2(k) - sin(x_1(k)) + u_k$$

and the linearised state space model is:

$$\begin{bmatrix} \Delta x_1(k+1) \\ \Delta x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -cos(x_e) & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1(k) \\ \Delta x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u .$$

From trigonometry if $sin(x_e) = u_e$ then $cos(x_e) = \sqrt{1 - u_e^2}$, hence the linear model can be written:

$$\begin{bmatrix} \Delta x_1(k+1) \\ \Delta x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\sqrt{1 - u_e^2} & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1(k) \\ \Delta x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u$$

This means that even though there are an infinite number of equilibrium points for a given input, they all have the same linearised state space model.

Q3 $\quad \dfrac{dh}{dt} = -\dfrac{k}{\pi tan^2\theta} \cdot \dfrac{1}{h} + \dfrac{1}{\pi tan^2\theta} \cdot \dfrac{f_{in}}{h^2}$

(i) $\quad \dfrac{dh}{dt} = 0$ at equilibrium point,

$$\Rightarrow \frac{1}{\pi tan^2\theta} \cdot \frac{f_{in}}{h_e^2} = \frac{k}{\pi tan^2\theta} \cdot \frac{1}{h_e} \Rightarrow f_{in} = kh_e \Rightarrow h_e = \frac{f_{in}}{k}$$

Not a function of θ because flow rate out of the tank depends on pressure (which is directly proportional to height, h), i.e. flow out if tank is NOT dependent on tank volume. Since θ affects only the volume of the tank it does not influence the equilibrium liquid level.

(ii) For $k = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$ equation becomes $\dfrac{dh}{dt} = -\dfrac{1}{2\pi} \cdot \dfrac{1}{h} + \dfrac{1}{\pi} \cdot \dfrac{f_{in}}{h^2}$

i.e. $\dot{h} = g(h, f_{in}) = -\frac{1}{2\pi} \cdot h^{-1} + \frac{1}{\pi} \cdot f_{in} h^{-2}$ and also, $\quad h_e = 2f_{in}$

and the linearised state model is given by $\Delta \dot{h} = \left.\dfrac{\partial g}{\partial h}\right|_e \Delta h + \left.\dfrac{\partial g}{\partial f_{in}}\right|_e \Delta f_{in}$

$$\left.\frac{\partial g}{\partial h}\right|_e = \frac{1}{2\pi} h_e^{-2} - 2\frac{f_{in}}{\pi} h_e^{-3} = \frac{1}{2\pi}(2f_{in})^{-2} - 2\frac{f_{in}}{\pi}(2f_{in})^{-3} = \frac{1}{8\pi f_{in}^2} - \frac{2}{8\pi f_{in}^2} = -\frac{1}{8\pi f_{in}^2}$$

$$\left.\frac{\partial g}{\partial f_{in}}\right|_e = \frac{h_e^{-2}}{\pi} = \frac{1}{4\pi f_{in}^2}$$

Thus the linearised model in terms of f_{in} is: $\quad \Delta \dot{h} = \left[-\dfrac{1}{8\pi f_{in}^2}\right] \Delta h + \left[\dfrac{1}{4\pi f_{in}^2}\right] \Delta f_{in}$

When $f_{in} = 1$ this gives $\Delta \dot{h} = -\frac{1}{8\pi}\Delta h + \frac{1}{4\pi}\Delta f_{in}$ and $f_{in} = 2$ gives $\Delta \dot{h} = -\frac{1}{32\pi}\Delta h + \frac{1}{16\pi}\Delta f_{in}$.

Q4. $\dot{x}_1 = x_1 \sin(x_1) + x_2$

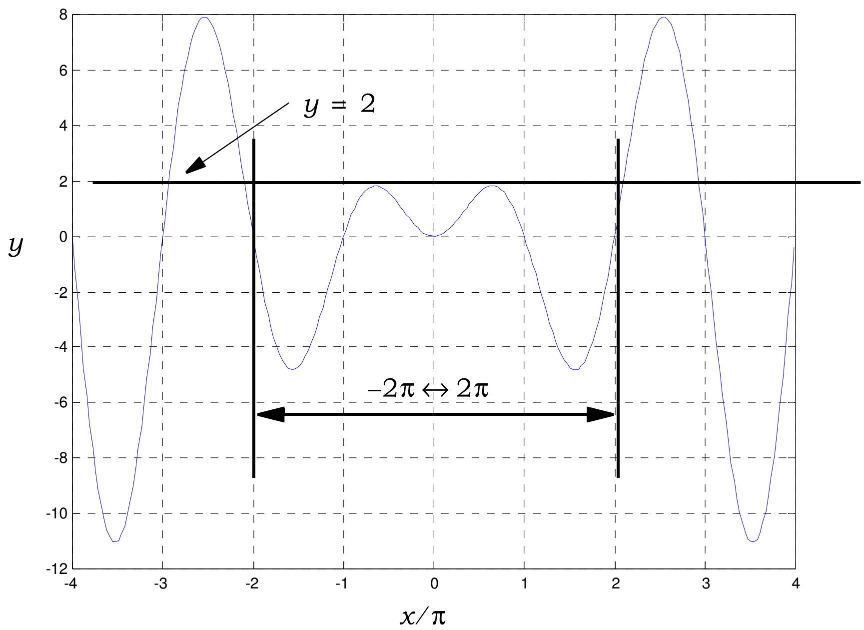
$$\dot{x}_2 = x_2 + u$$

(i) Equilibrium when $\begin{array}{l} 0 = x_{1e}\sin(x_{1e}) + x_{2e} \\ 0 = x_{2e} + u_e \end{array}$, hence:

$$x_{2e} = -u_e \text{ and } x_{1e}\sin(x_{1e}) = u_e$$

When $u_e = 0$, $x_{1e}\sin(x_{1e}) = 0$, $\Rightarrow x_{1e} = 0, -\pi, \pi$

When $u_e = 2$, $x_{1e}\sin(x_{1e}) = 2$. This can only be solved numerically. Easiest way is to plot the graph of $y = x\sin(x)$ as shown below. As can be seen in the interval $|x_1| < 2\pi$ there are no equilibrium points.



(ii)The linearised state-space model is given by:

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_{1e}\cos(x_{1e}) + \sin(x_{1e}) & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u$$

$x_{1e} = 0$ gives:

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u$$

$x_{1e} = -\pi$ gives:

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} \pi & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u$$

$x_{1e} = \pi$ gives:

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\pi & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u$$