# EE114
# Intro to Systems & Control


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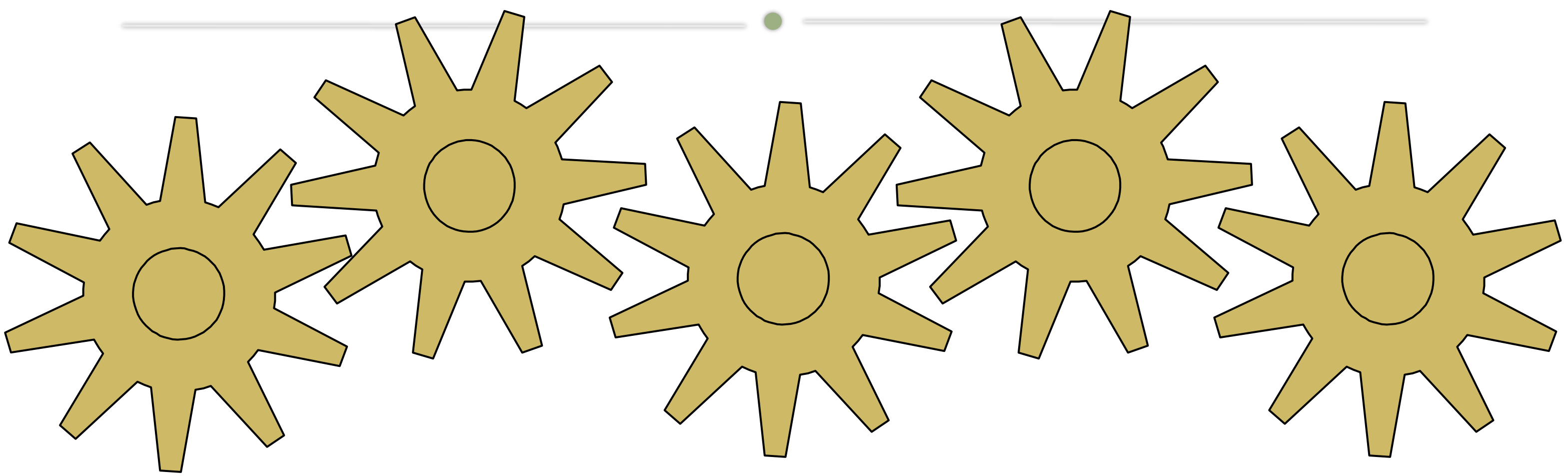

# So far …

- We’ve introduced the concept of control and, in particular, feedback control …

- We’ve illustrated the need for mathematical modelling …

- We’ve studied two simple static systems …

‘WELL, HIGGINS, I SEE EVERYTHING’S UNDER CONTROL.’

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- We’ve illustrated the need for mathematical modelling …
- We’ve studied two simple static systems …


‘WELL, HIGGINS, I SEE EVERYTHING’S UNDER CONTROL.’

- **Now, we are going to look at the mathematical modelling of a few simple dynamical systems.**

# Modelling of Dynamical Systems

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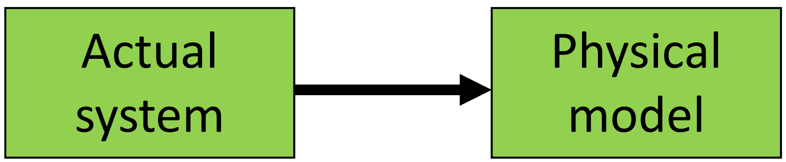

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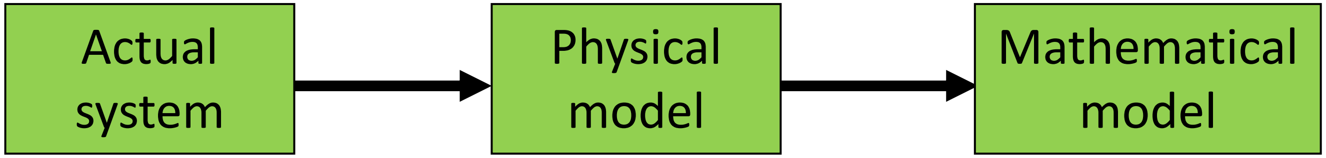

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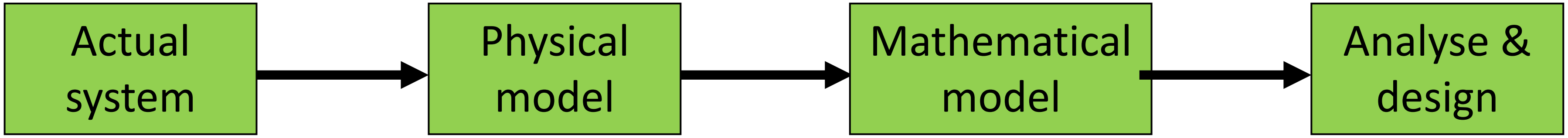


# Modelling of Dynamical Systems

- The stages of a dynamic system investigation are as follows:

Actual system → Physical model → Mathematical model → Analyse & design

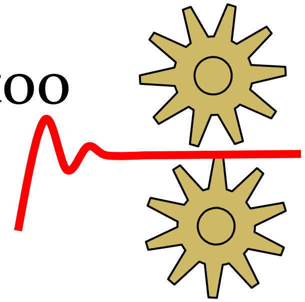
# Physical Modelling

- This involves identifying the system/sub-system to be studied and obtaining a simple physical model whose behaviour will match sufficiently closely that of the actual system.

- This typically leads to a schematic representation showing the key system components and variables and how they are physically related.

- Engineering judgement is needed in determining the appropriate level of detail – ***we have to decide what is important and what can be neglected.***

- Too complicated a model leads to long analysis while too simple a model is unrepresentative (i.e. not accurate enough).



# Physical Modelling

- Experience is needed – it cannot be taught! However, there are several useful guidelines (engineering approximations):
  - *neglect small effects*
    – this reduces the number and complexity of equations.
  - *assume environment is independent of the system motions*
    – this reduces the number and complexity of equations.
  - *replace distributed characteristics with appropriate lumped elements**
    – this gives ordinary differential equations rather than partial ones.

* *For example, a wire has resistance along its entire length – however we represent this by 'lumping' this distributed resistance into a single point value.*

# Physical Modelling

- Experience is needed – it cannot be taught! However, there are several useful guidelines (engineering approximations):

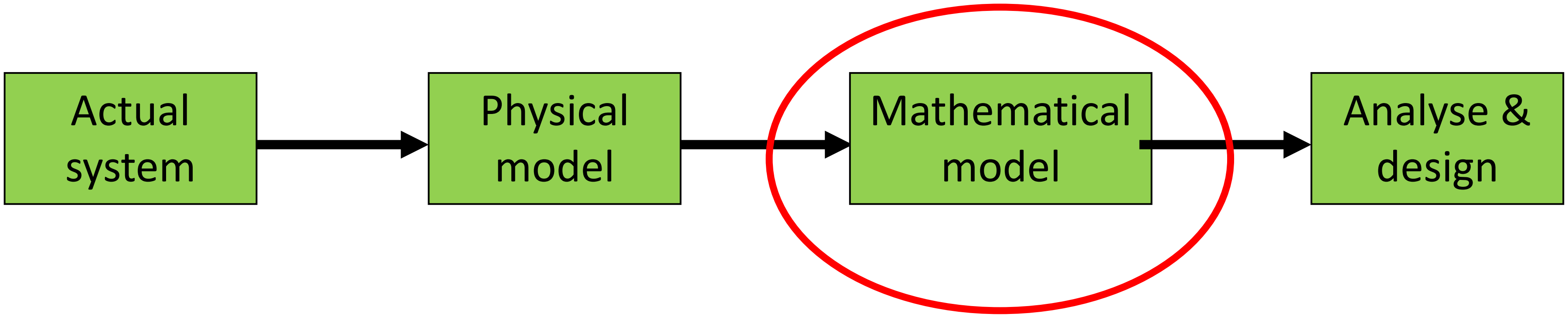
  - *assume linear relationships*
    – gives linear equations and superposition holds.

  - *assume constant parameters*
    – leads to constant coefficient in differential equations.

  - *neglect uncertainty and noise*
    – avoids statistical treatment.

# Modelling of Dynamical Systems

- The stages of a dynamic system investigation are as follows:



# Mathematical Modelling

- This involves obtaining a mathematical representation of the physical model.

- Central to this process is the writing of equations for equilibrium and/or compatibility relations.

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- This involves obtaining a mathematical representation of the physical model.
- Central to this process is the writing of equations for equilibrium and/or compatibility relations.
- **Equilibrium relations** describe the balance of forces, of flow rates, of energy, of current, etc. which must exist for the system (conservation of energy).
- **Compatibility relations** describe how motions of the system are interrelated because of the way they are connected.

# Mathematical Modelling

- Two further considerations are physical variables and physical laws.

- **Physical variables** are needed to describe the instantaneous state of the system. These can be divided into:
  - *through variables* (eg. current , flow) and
  - *across variables* (eg. voltage, pressure).

- Equilibrium relations apply to through variables while compatibility relations apply to across variables.

# Mathematical Modelling

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  - *through variables* (eg. current , flow) and
  - *across variables* (eg. voltage, pressure).

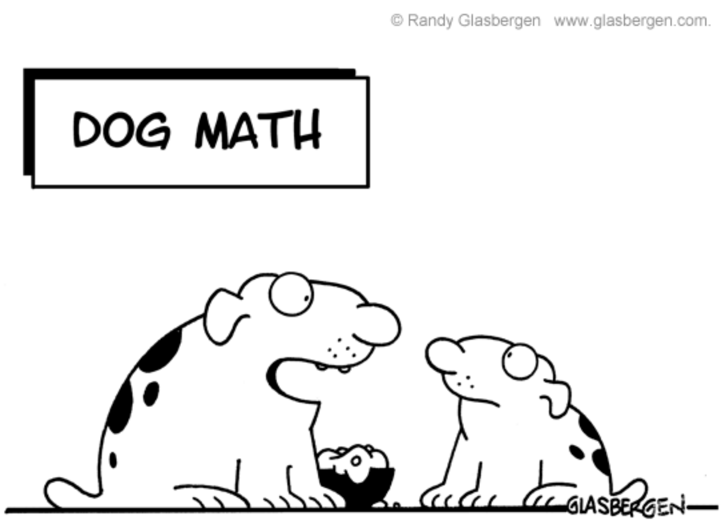
- Equilibrium relations apply to through variables while compatibility relations apply to across variables.

- **Physical laws** which individual components obey – usually between through and across variables and are generally empirical in nature.

# Mathematical Modelling

- In terms of modelling dynamical systems we are going to consider some simple electrical, mechanical and flow based ones.

- The mechanical system will be a standard simple second order mass-spring damper system while the flow-based system will be a simple first order tank system.



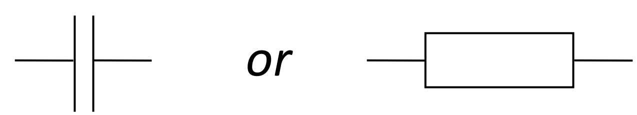

"If I have 3 bones and Mr. Jones takes away 2, how many fingers will he have left?"

*You will model more complicated systems in EE211 System Dynamics.*

# Modelling Electrical Systems

- Analytic procedure:
  - Physical model – circuit diagram.
  - Variables – voltages, currents.
  - Equilibrium relation – Kirchoff's Current Law (KCL).
  - Compatibility relation – Kirchoff's Voltage Law (KVL).
  - Physical relations are summarised in the following table:

# Modelling Electrical Systems

| Component | Physical Law | Symbol |
|---|---|---|
| Resistance (R) | $v = iR$ | $R$ or $R$  |
| Inductance (L) | $v = L\dfrac{di}{dt}$ | $L$ or $L$  |
| Capacitance (C) | $v = \dfrac{1}{C}\int i\,dt$ or $i = C\dfrac{dv}{dt}$ | $C$ or $C$  |

# Modelling Electrical Systems

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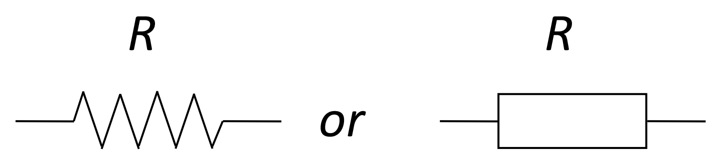
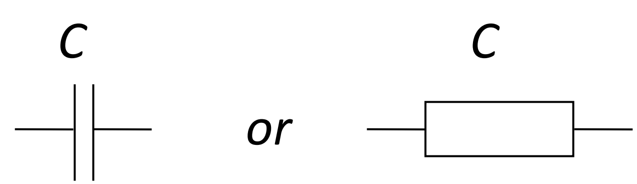
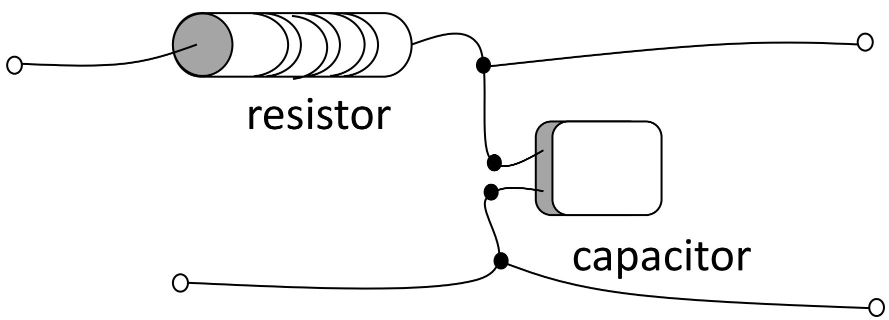
| Resistance<br>(R) | $v = iR$ |   |
| Inductance<br>(L) | $v = L\frac{di}{dt}$ |   |
| Capacitance<br>(C) | $v = \frac{1}{C}\int i dt$<br>or<br>$i = C\frac{dv}{dt}$ |   |

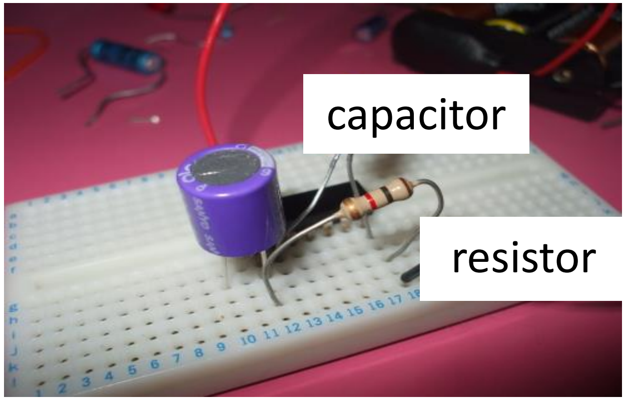
Note that we are assuming lumped parameters and ideal components.

# Modelling Electrical Systems

- ***Ex 3.3 Determine a mathematical model for the resistor/ capacitor filter circuit shown below:***

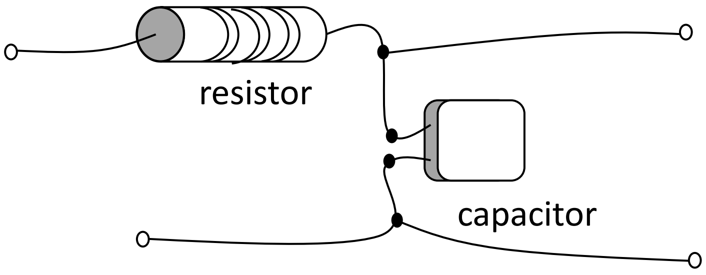



Actual system



# Modelling Electrical Systems

**Solution ...**



# Modelling Electrical Systems

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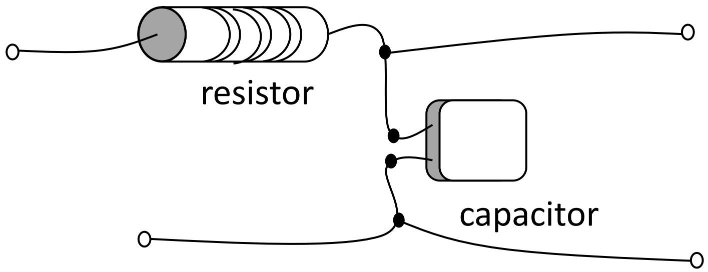
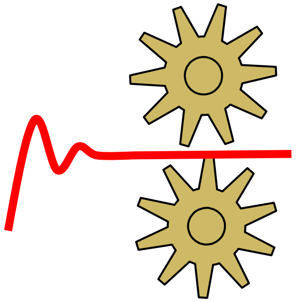
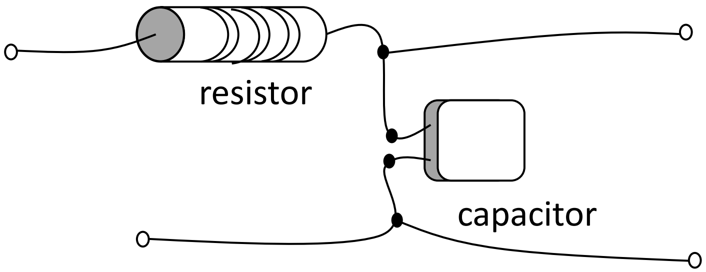



STEP 1 – Physical Model ...



# Modelling Electrical Systems

**Solution ...**

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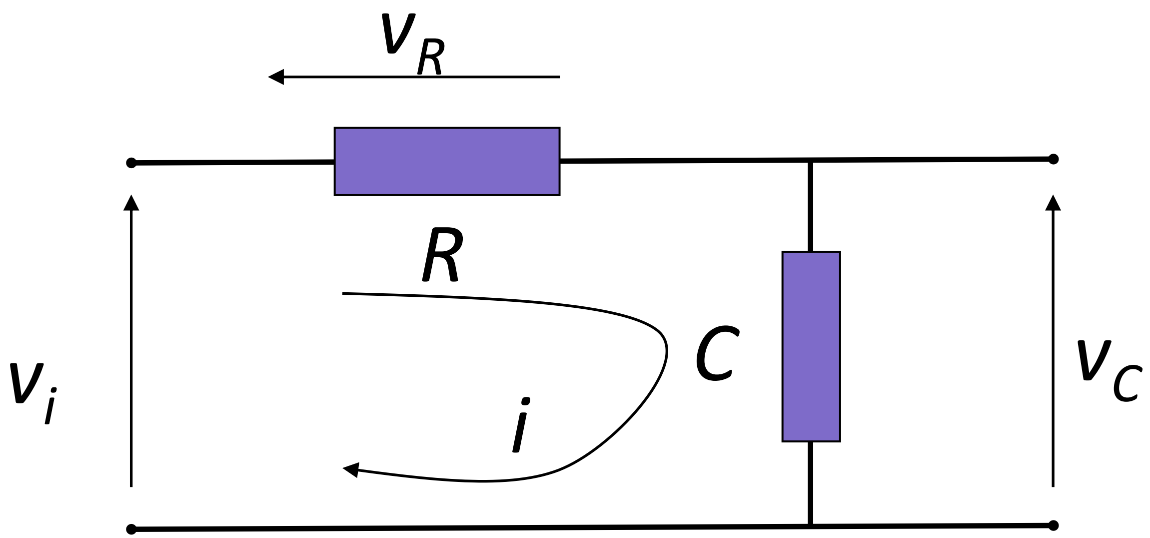
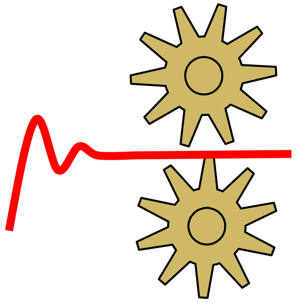
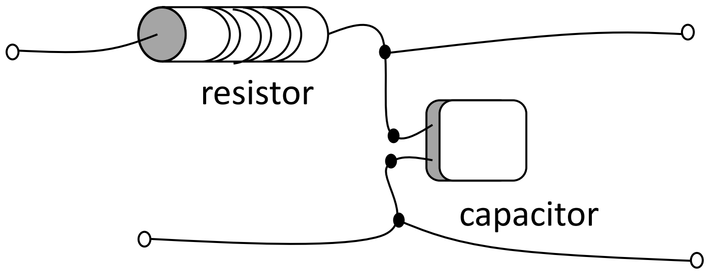


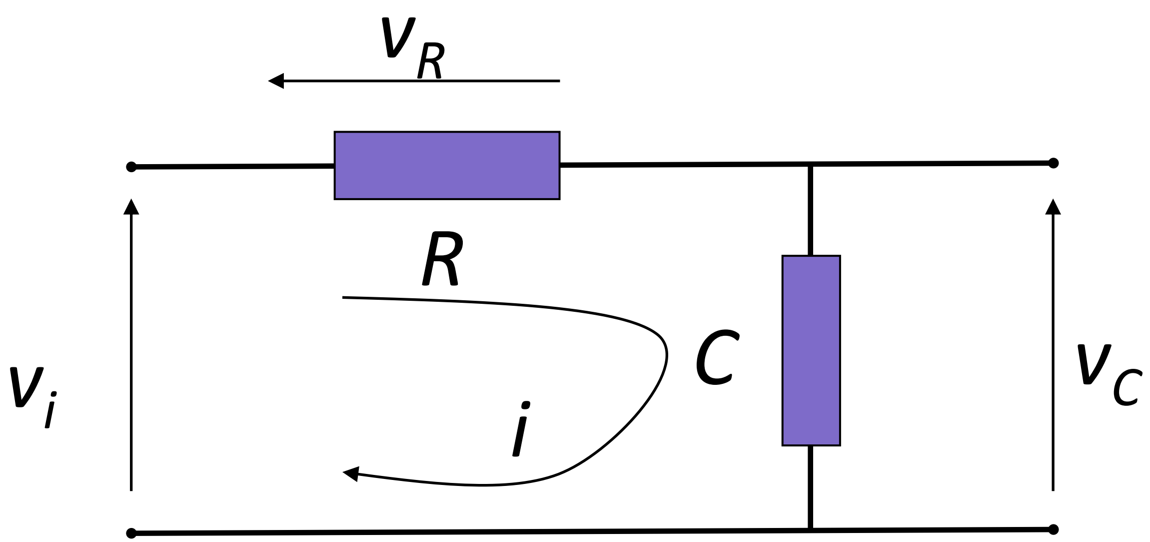



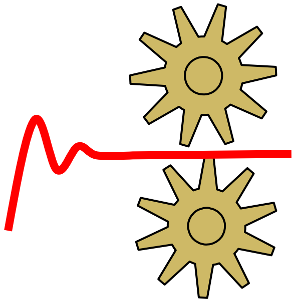
# Modelling Electrical Systems

**Solution ...**

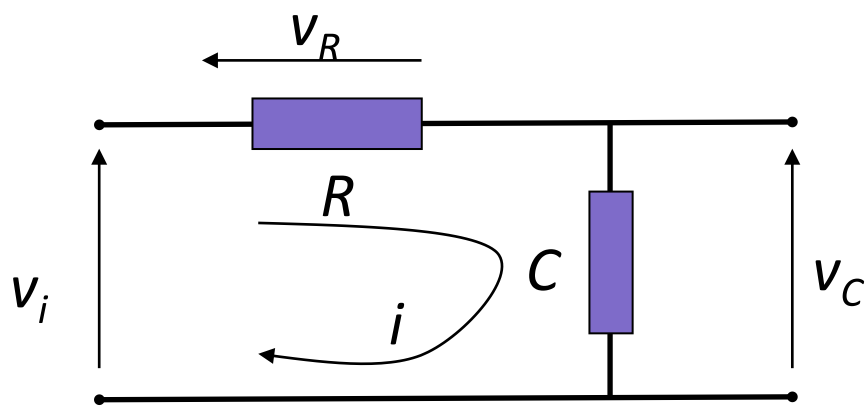



STEP 2 – Model variables as defined ...



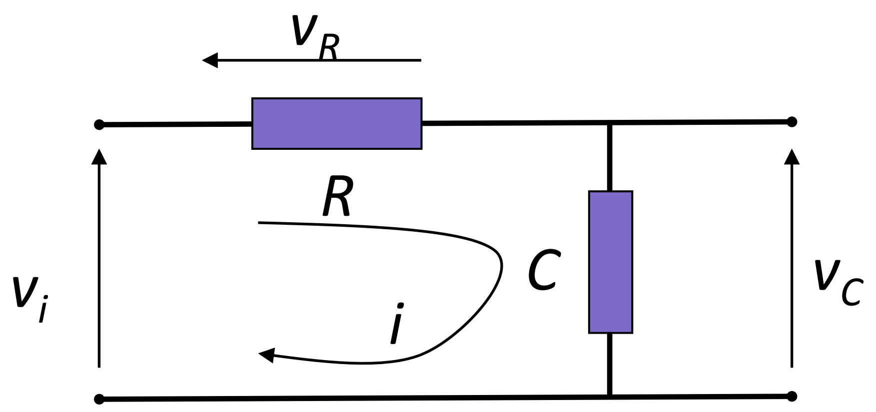



# Modelling Electrical Systems




STEP 3 – Compatibility relation – KVL ...

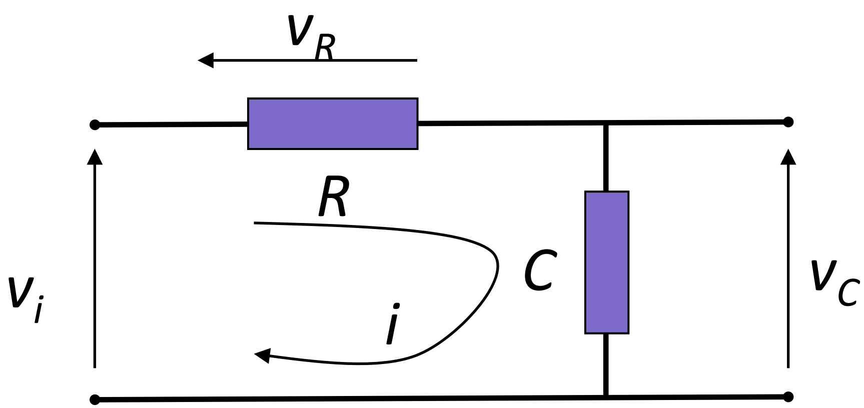
# Modelling Electrical Systems




STEP 3 – Compatibility relation – KVL ...

$$v_i = v_R + v_C$$

# Modelling Electrical Systems

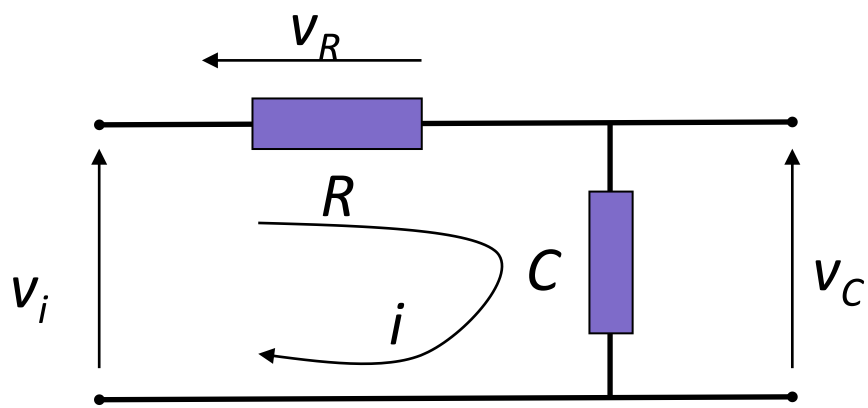



STEP 3 – Compatibility relation – KVL ...

$$v_i = v_R + v_C$$

Equilibrium relation implied by choice of current variable in this example ... i.e.

# Modelling Electrical Systems

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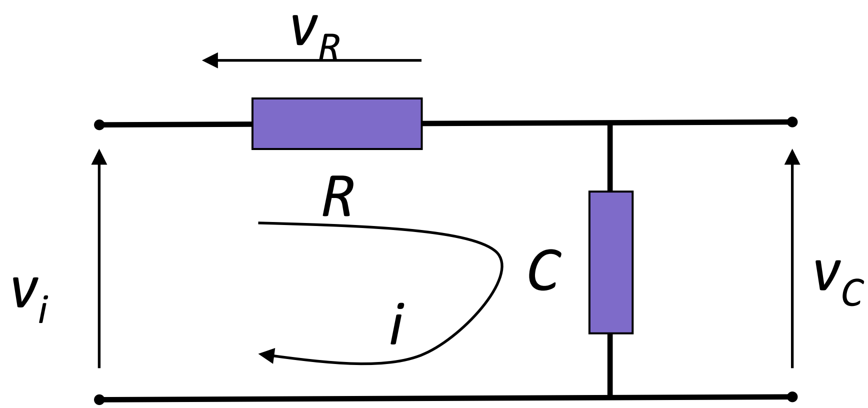
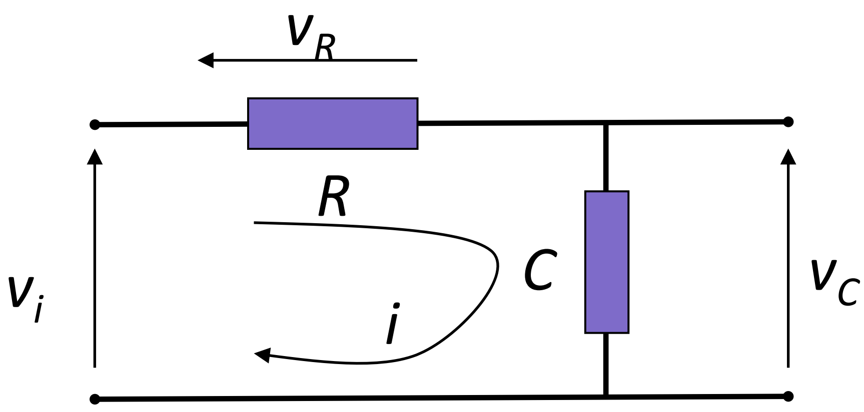
$$i = i_R = i_C$$

# Modelling Electrical Systems




STEP 4 – Applying physical relations to compatibility equation ...

# Modelling Electrical Systems




STEP 4 – Applying physical relations to compatibility equation ...

$$v_i = v_R + v_C \quad \rightarrow \quad v_i = iR + \frac{1}{C}\int i\,dt$$

# Modelling Electrical Systems

Differentiate once ...

$$v_i = iR + \frac{1}{C}\int i\,dt \quad \rightarrow \quad \frac{dv_i}{dt} = R\frac{di}{dt} + \frac{1}{C}i$$

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Here, we want relationship between $v_i$ and $v_C$ ...

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Recall ...

$$i = C\frac{dv_C}{dt}$$

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Recall ...

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# Modelling Electrical Systems

Differentiate once ...

$$v_i = iR + \frac{1}{C}\int i\,dt \quad \rightarrow \quad \frac{dv_i}{dt} = R\frac{di}{dt} + \frac{1}{C}i$$

Here, we want relationship between $v_i$ and $v_C$ ...

Recall ...

$$i = C\frac{dv_C}{dt}$$

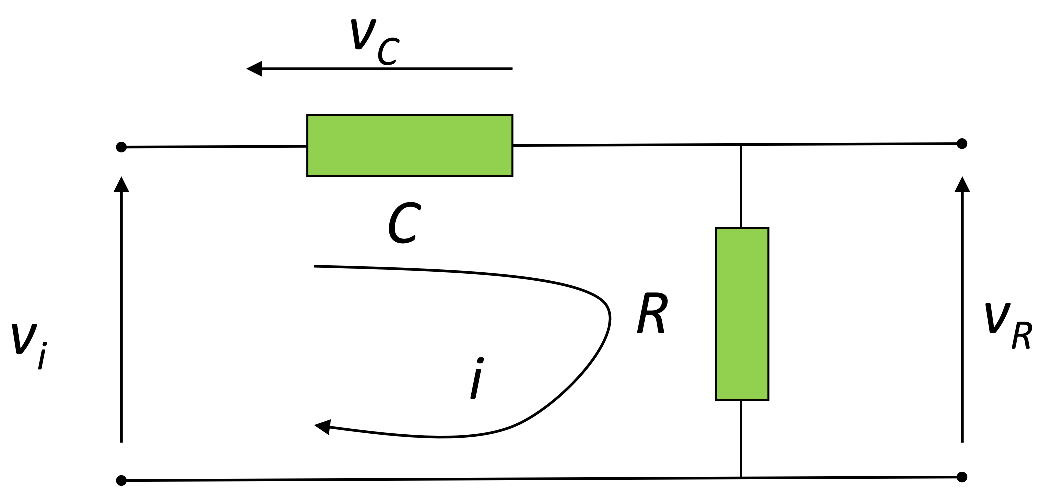
Hence ...

$$v_i = RC\frac{dv_C}{dt} + v_C$$

# Continuing …

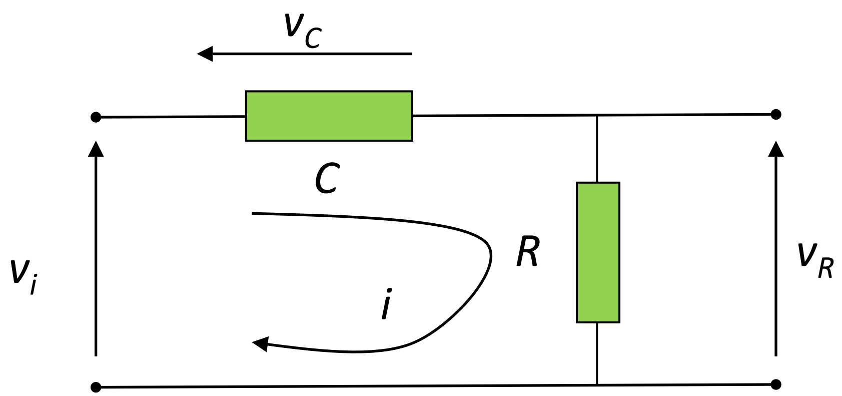
# Modelling Electrical Systems

- ***Ex 3.4 Determine a mathematical model for the capacitor/ resistor filter circuit given by the following physical model:***



# Modelling Electrical Systems

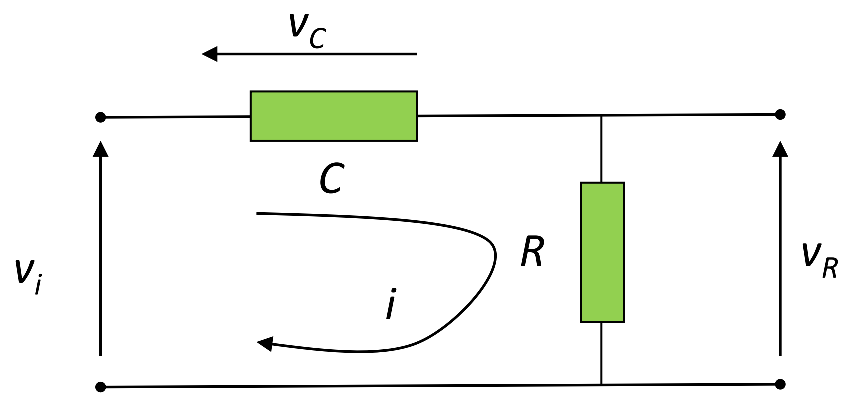
**Solution …**




Note, this is almost identical to the previous example, but we are now interested in finding out the relationship between $v_R$ and $v_i$.

# Modelling Electrical Systems

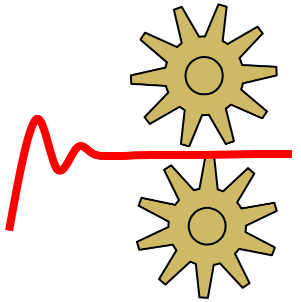
**Solution …**




Note, this is almost identical to the previous example, but we are now interested in finding out the relationship between $v_R$ and $v_i$.

From the previous example, we know that:

$$v_i = RC\frac{dv_C}{dt} + v_C$$



# Modelling Electrical Systems

**Solution …**

$$v_i = RC\frac{dv_C}{dt} + v_C$$

We also know that: $v_i = v_R + v_C$

# Modelling Electrical Systems

**Solution …**

$$v_i = RC\frac{dv_C}{dt} + v_C$$

We also know that: $v_i = v_R + v_C$

Hence: $v_C = v_i - v_R$

# Modelling Electrical Systems

**Solution …**

$$v_i = RC\frac{dv_C}{dt} + v_C$$

We also know that: $v_i = v_R + v_C$

Hence: $v_C = v_i - v_R$

Substituting for $v_C$ into the first equation:

$$v_i = RC\frac{d(v_i - v_R)}{dt} + v_i - v_R$$

# Modelling Electrical Systems

**Solution …**

$$v_i = RC\frac{d(v_i - v_R)}{dt} + v_i - v_R$$

# Modelling Electrical Systems

**Solution …**

$$v_i = RC\frac{d(v_i - v_R)}{dt} + v_i - v_R$$

$$\Rightarrow 0 = RC\frac{dv_i}{dt} - RC\frac{dv_R}{dt} - v_R$$

# Modelling Electrical Systems

**Solution …**

$$v_i = RC\frac{d(v_i - v_R)}{dt} + v_i - v_R$$

$$\Rightarrow 0 = RC\frac{dv_i}{dt} - RC\frac{dv_R}{dt} - v_R$$

$$\Rightarrow 0 = \frac{dv_i}{dt} - \frac{dv_R}{dt} - \frac{v_R}{RC}$$

# Modelling Electrical Systems

**Solution ...**

$$v_i = RC\frac{d(v_i - v_R)}{dt} + v_i - v_R$$

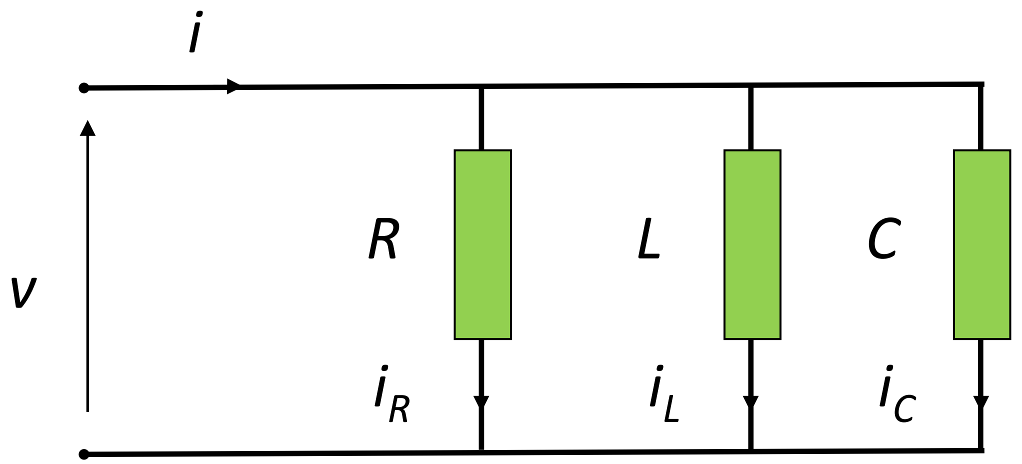
$$\Rightarrow 0 = RC\frac{dv_i}{dt} - RC\frac{dv_R}{dt} - v_R$$

$$\Rightarrow 0 = \frac{dv_i}{dt} - \frac{dv_R}{dt} - \frac{v_R}{RC}$$

$$\Rightarrow \frac{dv_i}{dt} = \frac{dv_R}{dt} + \frac{v_R}{RC}$$

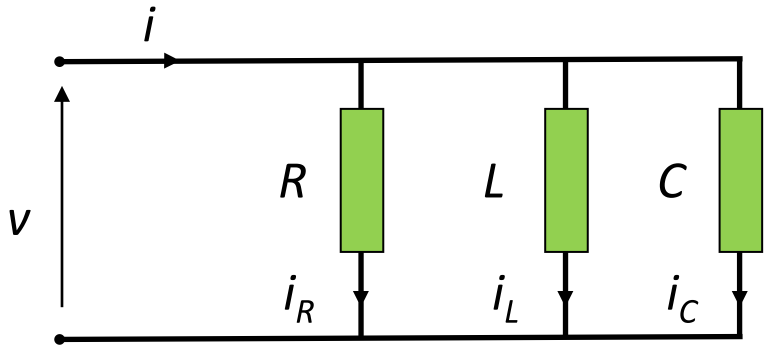
# Modelling Electrical Systems

- ***Ex 3.5 Develop the mathematical model for the LRC circuit that is described by the following physical model (step 1):***



# Modelling Electrical Systems

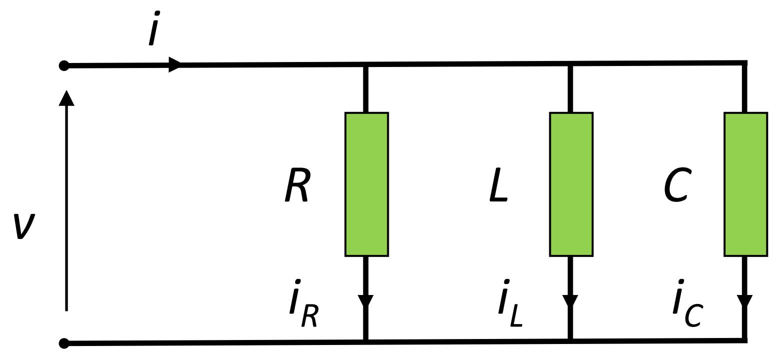
**Solution …**




Step2: Variables are as defined in physical model.

# Modelling Electrical Systems

**Solution …**




Step2: Variables are as defined in physical model.

Step 3: Compatibility relationship implied, i.e.:

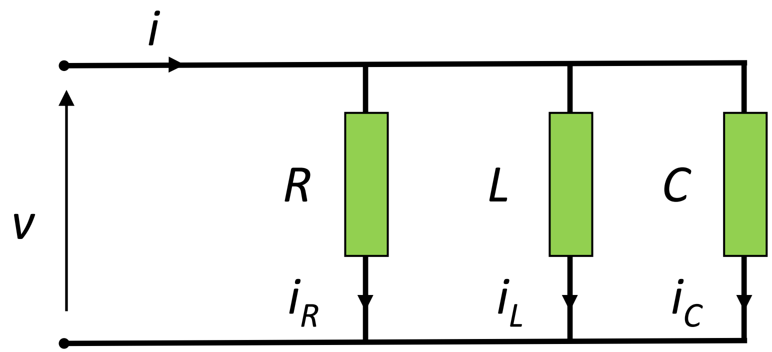
$$v = v_R = v_L = v_C$$

By KCL the equilibrium relationship for the currents is:

$$i = i_R + i_C + i_L$$

# Modelling Electrical Systems

**Solution …**




Step 4: Combining the physical relationships between current and voltage *R*, *C* and L with the previous equation gives:

$$i = \frac{v}{R} + \frac{1}{L}\int v dt + C\frac{dv}{dt}$$

# Modelling Electrical Systems

**Solution …**

$$i = \frac{v}{R} + \frac{1}{L}\int v\,dt + C\frac{dv}{dt}$$

# Modelling Electrical Systems

**Solution ...**

$$i = \frac{v}{R} + \frac{1}{L}\int v\,dt + C\frac{dv}{dt}$$

$$\frac{di}{dt} = \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v + C\frac{d^2v}{dt^2}$$

# Modelling Electrical Systems

**Solution ...**

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$$\frac{di}{dt} = \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v + C\frac{d^2v}{dt^2}$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{C}\frac{di}{dt}$$

# Modelling Electrical Systems

**Solution …**

$$i = \frac{v}{R} + \frac{1}{L}\int v\,dt + C\frac{dv}{dt}$$

$$\frac{di}{dt} = \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v + C\frac{d^2v}{dt^2} \quad \rightarrow \quad \frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{C}\frac{di}{dt}$$

- For more complex circuits this first principles approach is tedious and we generally use more efficient circuit analysis techniques (i.e. nodal, mesh) to obtain the model equations.