From Fourier Series to Fourier Integral

Fourier Series: T-periodic function $f(t) = f(t+T)$. Here $T = 2P$

Fourier series:

$$f(t) = \underbrace{\frac{a_0}{2} + \sum_{k=1}^{\infty} a_n \cos n\omega_0 t}_{\substack{\text{even part} \\ f_{even}(t)}} + \underbrace{b_n \sin n\omega_0 t}_{\substack{\text{odd part} \\ f_{odd}(t)}}$$

where $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2P} = \frac{\pi}{P}$ $\qquad f(t) = f_{even}(t) + f_{odd}(t)$

$$a_0 = \frac{1}{P}\int_{-P}^{P} f(t)\,dt$$

$$a_n = \frac{1}{P}\int_{-P}^{P} f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{1}{P}\int_{-P}^{P} f(t)\sin n\omega_0 t\,dt$$

Even part:

$$f_{even}(t) = \frac{a_0}{2} + \underbrace{a_1 \cos 1\omega_0 t}_{\substack{\text{time domain components corresponding} \\ \text{to frequency } 1\omega_0 t}} + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \cdots + \underbrace{a_n \cos n\omega_0 t}_{\substack{\text{time domain components corresponding} \\ \text{to frequency } n\omega_0 t}} + \cdots$$

$a_0 \cos 0\omega_0 t$

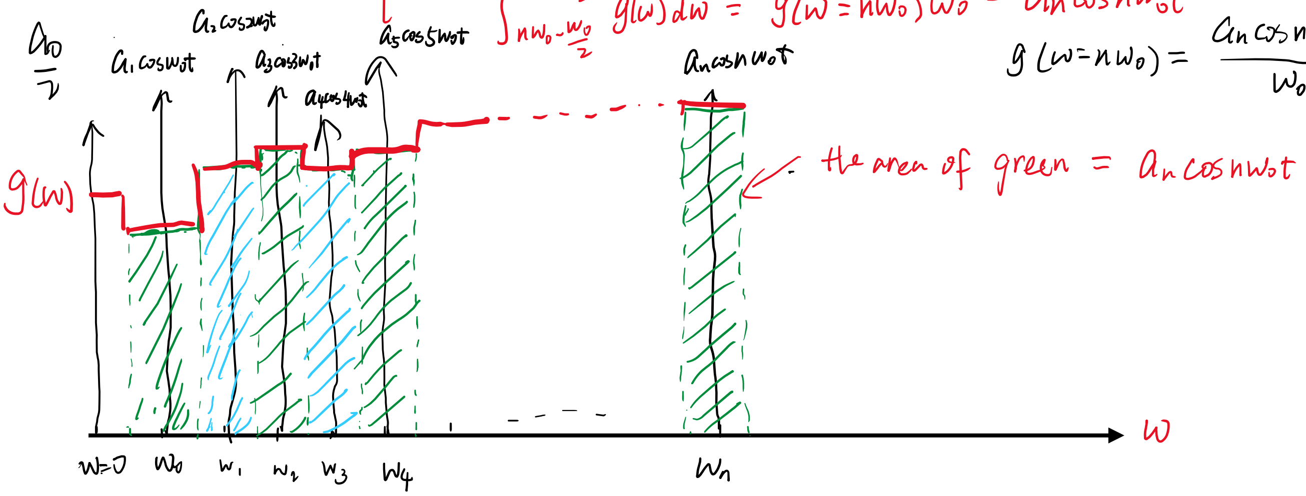
define $g(\omega)$ such that

$$\begin{cases} \int_0^{\frac{\omega_0}{2}} g(\omega)\,d\omega = g(\omega=0)\frac{\omega_0}{2} = \frac{a_0}{2} \\ \int_{\frac{\omega_0}{2}}^{\frac{3\omega_0}{2}} g(\omega)\,d\omega = g(\omega=\omega_0)\omega_0 = a_1 \cos \omega_0 t \\ \vdots \\ \int_{n\omega_0-\frac{\omega_0}{2}}^{n\omega_0+\frac{\omega_0}{2}} g(\omega)\,d\omega = g(\omega=n\omega_0)\omega_0 = a_n \cos n\omega_0 t \end{cases}$$

$\Rightarrow g(\omega=0) = \frac{a_0}{\omega_0}$

$g(\omega=\omega_0) = \frac{a_1 \cos \omega_0 t}{\omega_0}$

$g(\omega=n\omega_0) = \frac{a_n \cos n\omega_0 t}{\omega_0}$



As $w_0 \to 0$, $T \to \infty$, $p \to \infty$

For $0 < w < \frac{w_0}{2}$, $g(w) \approx g(0w_0) = \frac{a_0}{w_0} \cos 0 \cdot w_0 t \approx \frac{a_0}{w_0} \cos wt$

and $a_0 = \frac{1}{p}\int_{-p}^{p} f(t)\, dt = \frac{1}{p}\int_{-p}^{p} f(t) \cos 0 w_0 t\, dt \approx \frac{1}{p}\int_{-p}^{p} f(t) \cos wt\, dt$

$g(w) \approx \frac{a_0}{w_0} \cos wt$

$\approx \frac{1}{pw_0} \int_{-p}^{p} f(t) \cos wt\, dt \cdot \cos wt = \frac{1}{\pi}\int_{-p}^{p} f(t) \cos wt\, dt \cdot \cos wt$

Define $A(w) = \frac{1}{\pi}\int_{-p}^{p} f(t) \cos wt\, dt$, thus $g(w) = A(w) \cos wt$ ①

$w_0 - \frac{w_0}{2} < w < w_0 + \frac{w_0}{2}$, $g(w) \approx g(1 w_0) = \frac{a_1}{w_0} \cos 1 w_0 t \approx \frac{a_1}{w_0} \cos wt$

and $a_1 = \frac{1}{p}\int_{-p}^{p} f(t) \cos 1 w_0 t\, dt \approx \frac{1}{p}\int_{-p}^{p} f(t) \cos wt\, dt$

$g(w) \approx \frac{a_1}{w_0} \cos wt \approx \frac{1}{\pi}\int_{-p}^{p} f(t) \cos wt\, dt \cos wt$

$= A(w) \cos wt$ ②

Similarly

$nw_0 - \frac{w_0}{2} < w < nw_0 + \frac{w_0}{2}$, $g(w) \approx g(nw_0) = \frac{a_n}{w_0} \cos nw_0 t \approx \frac{a_n}{w_0} \cos wt$

$g(w) = A(w) \cos wt$ ③

From ① ② ③, we have that for all $w \in [0, \infty)$, and $p \to \infty$

$g(w) = A(w) \cos wt$, where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t) \cos wt\, dt$

and $f_{even}(t) = \int_0^{\infty} g(w)\, dw = \int_0^{\infty} A(w) \cos wt\, dt$

$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t) \cos wt\, dt$

Similarly $\quad f_{odd}(t) = \int_0^\infty B(w) \sin wt \, dt$

where $\quad B(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(t) \sin wt \, dt$

In summary $\quad f(t) = f_{even}(t) + f_{odd}(t)$

$$= \int_0^\infty A(w) \cos wt + B(w) \sin wt \, dt$$

where $\quad A(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(t) \cos wt \, dt$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(t) \sin wt \, dt$$