EE206 Differential Equations and Transform Methods

Tutorial 2

Problem 1c: $\frac{dy}{dx} + 2\sin(2x)y = 2e^{\cos(2x)}$; $\quad y(0) = 0$

We can see that this is in the standard form: $\frac{dy}{dx} + P(x)y = f(x)$:

$$P(x) = 2\sin(2x) \quad f(x) = 2e^{\cos(2x)}$$

Using the integrating factor $e^{\int P(x)dx}$ and

$$\int P(x)dx = 2\int \sin(2x)dx = \frac{-2}{2}\cos(2x) = -\cos(2x)$$

The equation becomes after multiplying across by $e^{-\cos(2x)}$:

$$e^{-\cos(2x)}\frac{dy}{dx} + y\left[2\sin(2x)e^{-\cos(2x)}\right] = 2e^{\cos(2x)}e^{-\cos(2x)}$$

$$e^{-\cos(2x)}\frac{dy}{dx} + y\frac{d}{dx}\left[e^{-\cos(2x)}\right] = 2$$

$$\frac{d}{dx}(ye^{-\cos(2x)}) = 2$$

So:

$$ye^{-\cos(2x)} = 2x + c \implies y(x) = e^{\cos(2x)}(2x + c)$$

Imposing initial conditions $y(0) = 0$,

$$y(0) = e^{1}(0 + c) = 0 \quad \Rightarrow c = 0$$

Thus:

$$y(x) = 2xe^{\cos(2x)}$$

Problem 2a: $x\frac{dy}{dx} + y = \frac{1}{y^2}$

We want to get it in the form: $\frac{dy}{dx} + P(x)y = f(x)y^n$

$$x\frac{dy}{dx} + y = \frac{1}{y^2}$$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x}y^{-2}, \quad n = -2$$

We then make the substitution $u = y^{1-n}; \quad u = y^3 \quad \Rightarrow y = uy^{-2}$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{3}y^{-2}\frac{du}{dx}$$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x}y^{-2}$$
$$\frac{1}{3}y^{-2}\frac{du}{dx} + \frac{1}{x}y = \frac{1}{x}y^{-2}$$
$$\frac{1}{3}\frac{du}{dx} + \frac{y^3}{x} = \frac{1}{x}; \;\; u = y^3$$
$$\frac{du}{dx} + \frac{3}{x}u = \frac{3}{x}; \quad P(x) = \frac{3}{x}, \;\; f(x) = \frac{3}{x}$$

$$\int P(x)dx = 3\int \frac{1}{x}dx = 3\ln(x) = \ln(x^3)$$

$$e^{\int P(x)dx} = e^{\ln(x^3)} = x^3$$

$$x^3 \frac{du}{dx} + 3x^2 u = 3x^2$$

$$\frac{d}{dx}(x^3 u) = 3x^2$$

$$x^3 u = 3 \int x^2 dx = \frac{3x^3}{3} + c = x^3 + c$$

$$x^3 u = x^3 + c; \quad u = y^3$$

$$x^3 y^3 = x^3 + c$$

$$y^3 = 1 + cx^{-3}$$

$$y(x) = \sqrt[3]{1 + cx^{-3}}$$

Problem 3c: $\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$

$$u = y - 2x + 3 \quad \frac{du}{dx} = \frac{dy}{dx} - 2 \quad \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + 2$$

$$\frac{du}{dx} + 2 = 2 + \sqrt{u}$$

$$\frac{1}{\sqrt{u}}du = dx$$

$$\int u^{-\frac{1}{2}}du = \int 1dx$$

$$2u^{\frac{1}{2}} = x + c$$

$$u = \left(\frac{x + c}{2}\right)^2; \quad u = y - 2x + 3$$

$$y - 2x + 3 = \left(\frac{x + c}{2}\right)^2$$

$$y(x) = \left(\frac{x + c}{2}\right)^2 + 2x - 3$$