Lecture 7: Analogue to Digital Conversion

EE213 - Introduction to Signal Processing

Semester 1, 2020

Outline

- Introduce A/D conversion.
- Sampling:

  Characterize the frequency domain representation of the sampling process.

  Explain the aliasing phenomena and identify the sampling rate.
- Quantization:

  Describe uniform quantisation and determine the signal to quantisation noise ratio.
- Coding:

  Represent the quantization value using binary digits.

A/D : Analogue to Digital

- Most signal processing these days is done digitally.
- For this reason, analogue signals must first be converted into digital signals first. i.e., discrete-time & discrete-value.
- Two Step:

  Sampling: Continuous-time -> Discrete-time

  Quantization: Continuous-value -> Discrete-value

  Coding: Discrete-value -> Binary representation

Sampling: Continuous Time to Discrete Time

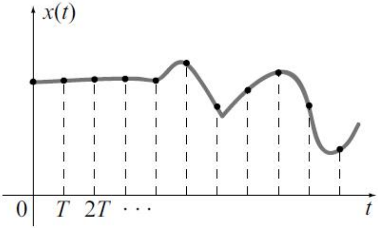
- Under certain conditions, a continuous-time signal can be completely represented by and recoverable from knowledge of its values, or samples, at points equally spaced in time.
- The first step in analogue-to-digital conversion is to sample the signal.
- We are interested in equally spaced sampling, which is also knowns as periodic sampling.

Sampling...

- **Periodic sampling**: equally spaced sampling
  Given the sampling period $T > 0$, we convert a continuous-time signal x(t) into a discrete-time signal $x[n]$, where

$$x[n] = x(nT), n = ..., -2, -1, 0, 1, 2, ... \qquad (1)$$

$fs = 1/T$, as the sampling frequency.



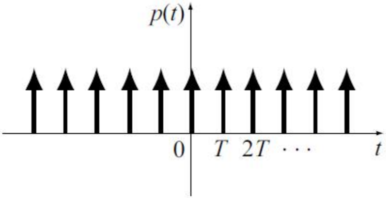
Sampling...

- **Ideal uniform sampling**: using impulse signal.

- The impulse train

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \qquad (2)$$

is referred to as the sampling function.



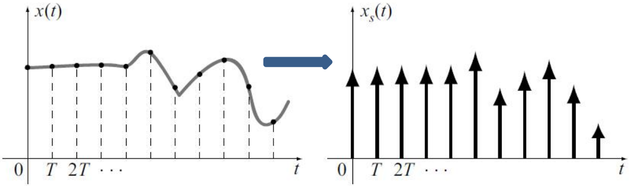
Sampling...

- We can represent the sampled signal in continuous time by

$$x_s(t) = x(t)p(t) \quad (3)$$

$$= \sum_{k=-\infty}^{\infty} x(t)\delta(t - kT)$$

In Time Domain:



- What happens in spectrum domain?

Sampling...

- The impulse train $p(t)$ is periodic signal, so we can write it as a complex exponential Fourier series

$$p(t) = \sum_{n=-\infty}^{\infty} p_n e^{jn\omega_s t} \tag{4}$$

where $\omega s = 2\pi/T$ and

$$p_n = \frac{1}{T}\int_{-T/2}^{T/2} p(t) e^{-jn\omega_s t} dt = \frac{1}{T}(!!!) \tag{5}$$

- Thus, the impulse train equals

$$p(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \tag{6}$$

Sampling...

- We have

$$x_s(t) = x(t)p(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)e^{jn\omega_s t} \qquad (7)$$

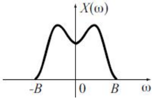
Therefore

$$\begin{aligned} X_s(\omega) &= \mathcal{F}(\frac{1}{T}\sum_{k=-\infty}^{\infty} x(t)e^{jn\omega_s t}) && (8) \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} \mathcal{F}(x(t)e^{jn\omega_s t}) && (9) \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) && (10) \end{aligned}$$

- the Fourier transform of the sampled signal $x_s(t)$ is **a sum of scaled and shifted replicas** of the Fourier transform of the original signal $x(t)$.

Sampling...

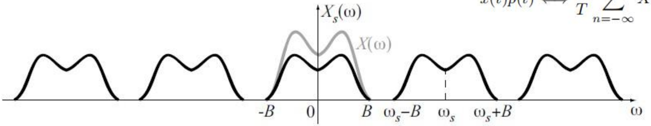
- Assume $X(\omega)$ looks like



that it is band-limited, i.e., $|X(\omega)| = 0$ for $|\omega| > B$. (refer to B as the signal bandwidth.)

- Thus, we have $\mathrm{Xs}(\omega)$, **sum of scaled and shifted replicas of $X(\omega)$**

$$x(t)p(t) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$



What happens if $w_s > 2B$ ------------ No overlapping!

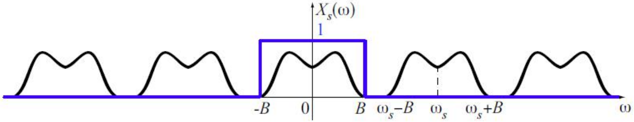
What happens if not?

Sampling...

- When the condition $w_s > 2B$ is satisfied, we can recover the original signal $x(t)$ from $x_s(t)$ by lowpass filtering and scaling

$$X(\omega) = T \underbrace{H_{LP}(\omega)}_{\text{lowpass filter}} X_s(\omega) \tag{12}$$

- Illustration of lowpass filtering of $x_s(t)$



Sampling...

- We have the following important result, discovered in various forms by Shannon, Nyquist, Whittaker and Kotelnikov:

The Sampling Theorem.

Consider a bandlimited signal x(t) with bandwidth B. Then, provided the sampling frequency $w_s > 2B$, the signal $x(t)$ can be recovered exactly from its sampled version $x_s(t)$ by scaling and lowpass filtering.

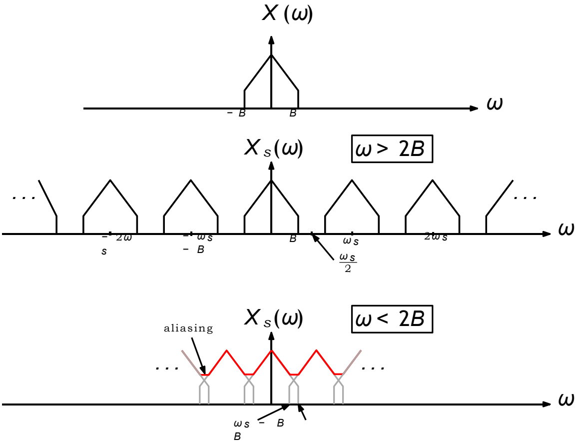
- The minimum required sampling frequency $w_s \triangleq 2\mathrm{B}$ is called the Nyquist frequency or the Nyquist sampling rate.

Aliasing

- If $w_s < 2B, Xs(\omega)$ will overlap, called aliasing.
- In this case the original signal cannot be recovered.

Aliasing...

Illustration of aliasing



Preventing Aliasing

In practice, two approaches to prevent aliasing:

- Apply an analog (continuous-time) lowpass filter before sampling to ensure the signal is bandlimited
  - These lowpass filters are often called anti-aliasing filters
- Use a higher sampling rate than required by the sampling theorem (e.g. $w_s = 2.2B$)

Quantisation

- Note that the sampling process produces discrete time, but **continuous-amplitude** signals.
- Converting the continuous-amplitude into discrete-amplitude is called **quantisation.**

- The error between the sampled continuous-valued amplitude and the approximated discrete amplitude is called **quantisation error** or **quantisation noise**.
- The output of quantizer is

$$x_q[n] = Q(x_a[n]) \tag{14}$$

- The quantizer error is

$$e_q[n] = x_a[n] - x_q[n] \tag{15}$$

Quantization...

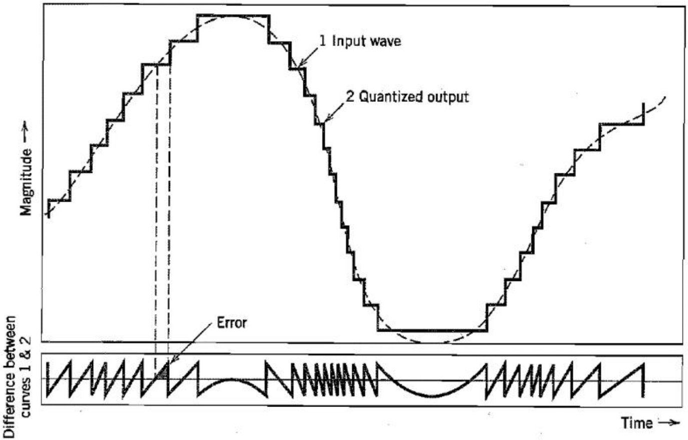


FIGURE 3.11 Illustration of the quantization process. (Adapted from Bennett, 1948, with permission of AT&T.)

Uniform Quantisation

Given the signal is in a finite range ($x_{\min}$, $x_{\max}$)

- The entire data range is divided into L equal intervals

quantisation Interval: $\Delta = \dfrac{x_{\max} - x_{\min}}{L}$ (16)

If L increases, Δ decreases. Hence, the quantisation error decreases and the accuracy of the quantizer increases.

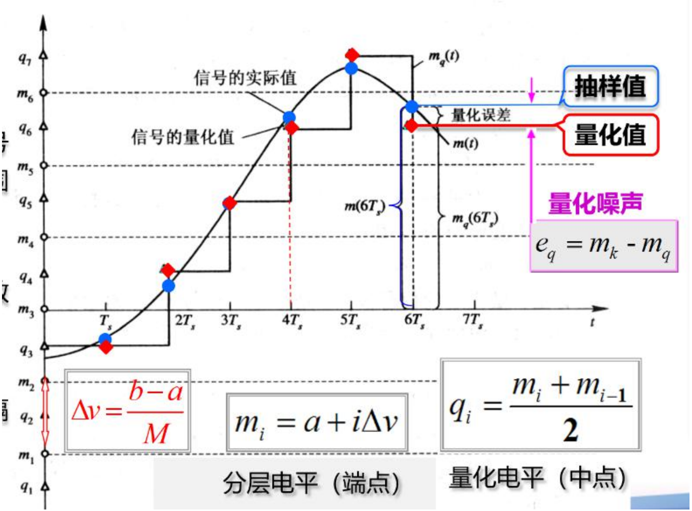
- The values allowed in the digital signal are called **quantisation levels**.

$$\text{Value\_k} = x_{min} + k^{*}\Delta - \frac{\Delta}{2}$$

- Distance between two quantization levels is called **quantisation step size** or **resolution**, for uniform quantisation, also Δ .

Uniform Quantisation

Sampled value is mapped to the **middle value** of its interval.



Uniform Quantisation...

- The quantisation error is the range of

$$\frac{-\Delta}{2} \leq e_q(n) \leq \frac{\Delta}{2} \Longleftrightarrow |e_q(n)| \leq \frac{\Delta}{2} \tag{17}$$

and thus the maximum error is

$$e_q^{\max}(n) = \frac{\Delta}{2} \tag{18}$$

Example

For the following sequence {1.2,-0.2,-0.5,0.4,0.89,1.3. . . }, quantise it using a uniform quantiser in the range of (-1.5,1.5) with 4 levels, and write the quantised sequence.
(Δ=0.75, {1.125,-0.375,-0.375,0.375,1.125,1.125...})

Uniform Quantisation

- An important indicator for the performance of a quantiser is the ratio of the signal average power to the noise power is the signal-quantization noise ratio (SQNR) defined as

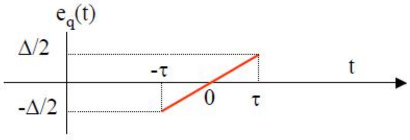
$$\mathrm{SQNR} = \frac{P_x}{P_q} \tag{19}$$

where

- P_q is the quantisation noise power.
- P_x is the average power of the signal.

Uniform Quantisation - Quantization Error...

- Example: case when the analogue signal $x_a(t)$ is (almost) linear between quantization levels.
- Consider one interval



- Quantization error:

$$\begin{aligned} e_q(t) &= x_a(t) - x_q(t) && (20) \\ &= \frac{\Delta}{2\tau} t && (21) \end{aligned}$$

for $-\tau \leq t \leq \tau$.

The power of quantisation error

$$P_q = \frac{1}{\tau} \int_0^\tau e_q^2(t)dt = \frac{1}{\tau} \int_0^\tau \left(\frac{\Delta}{2\tau} t \right)^2 dt = \frac{\Delta^2}{12} \qquad (22)$$

Uniform Quantisation - Uniformly Distributed Signals

- Consider the case where signals are uniformly distributed over an interval $[-A, A]$
  - Examples include full amplitude triangle waves and sawtooth waves.
- The quantisation error is uniformly distributed betwee $\frac{\Delta}{2}$ and $\frac{\Delta}{2}$ and the power of the quantisation error is $P_q = \Delta^2/12$.
- Let b be the number of bits to represent all quantisation levels between $-A$ and A. The quantisation interval is

$$\Delta = \frac{2A}{2^b} \tag{27}$$

- The power of the signal (triangle waves) in this case is given by

$$P_x = 2 \times \frac{1}{T} \int_{-\frac{T}{2}}^{0} \left(\frac{4A}{T} t + A\right)^2 dt = \frac{A^2}{3} \tag{28}$$

Uniform Quantisation - Quantization Error...

- The SQNR in this case is given by

$$\text{SQNR} = \frac{A^2/3}{\Delta^2/12} = 2^{2b} \tag{22}$$

- In dB scale

$$\text{SQNR}_{\text{dB}} = 10\log_{10} 2^{2b} = 6.02 \times b \tag{29}$$

Uniform Quantisation - Sinusoidal Signals

- Example: a sinusoid signal. Its quantisation error $e_q(t)$ is NOT uniformly distributed between $\frac{\Delta}{2}$ and $\frac{\Delta}{2}$.

- The quantisation error is said to be almost linear between $\frac{\Delta}{2}$ and $\frac{\Delta}{2}$ and the power of quantisation error is approximated as

$$P_q \approx \frac{\Delta^2}{12} \tag{23}$$

- Assume the signal is a sine wave between $-A$ and A. The power of the signal is

$$P_x = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} (A\sin t)^2 \, dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2} A^2 (1 - \cos 2t) dt = \frac{A^2}{2}. \tag{24}$$

Uniform Quantisation - Sinusoidal Signals...

- The SQNR in this case is given by

$$\mathrm{SQNR} = \frac{A^2/2}{\Delta^2/12} = \frac{3}{2}2^{2b} \qquad (25)$$

In dB scale

$$\mathrm{SQNR_{dB}} = 10\log_{10}\frac{3}{2}2^{2b} = 1.76 + 6.02b \qquad (26)$$