2. The Basics – *a brief reminder*

2.1 Dynamical and static systems

- A **system** is a collection of components acting together to perform a specific task – it is anything with inputs and outputs.

- A **component** is a single functioning unit of the system.

- A **dynamical system** is where the current output (or *effect* variable) depends on past inputs (or *cause* variables). As such, a dynamical system is also referred to as a system with memory.

- On the other hand, a **static system** is where the current output depends only on the current input. It cannot depend on past values and, as such, these systems are referred to as systems with no memory.

2.2 Categorizing systems

- Systems can be categorized as:
  - linear or nonlinear
  - time-invariant or time-varying
  - continuous or discrete

- **Linear** systems possess two important properties, namely *superposition* and *homogeneity*.

- *The principle of superposition* - this states that the response produced by a combination of different inputs is the sum of the individual responses produced by each of the different inputs acting alone. In other words, the response of a multi-input linear system can be determined by dealing with each input in turn and then adding the results.

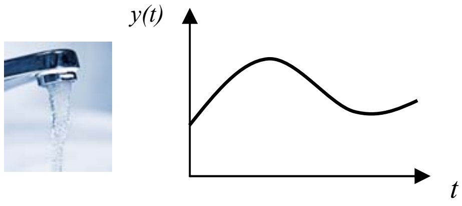
- *Homogeneity* (or *scaling*) – this states that if an input $u(t)$ produces and output $y(t)$ then, for a linear system, an input $ku(t)$ will produce an output $ky(t)$, where k is a scalar.

- If these properties are not withheld, then the system is said to be **nonlinear**. In general, analyzing such systems can be complicated, requiring difficult mathematical calculations.

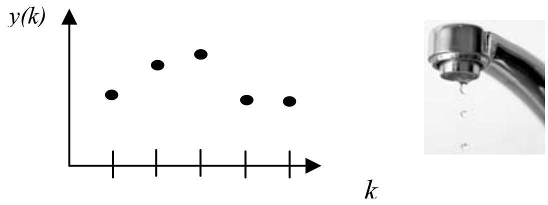
- It is, therefore, **common practice to *linearize* such systems** about specific operating points. Thus, simpler, linear mathematical techniques can now be applied to the linear approximation of the nonlinear system. *We will cover the topic of linearisation in detail later in the notes.*

- The characteristics (think defining equations, parameters) of a **time-invariant** system do not depend on time whereas the characteristics of a **time-varying** system do.
- **Continuous-time** systems involve signals that are continuous in time while **discrete-time** systems involve variables that only change at discrete instants of time.

- This concept is simply captured in the following illustration:



Continuous-time output



Discrete-time output

- *In this module, we will focus mainly on linear time-invariant (LTI) analysis. We will consider both continuous- and discrete- time systems.*

2.3 Mathematical modelling

- In order to design a system, we must first build a model of it, so that we can analyse its response and predict its performance.

- The model of the system is typically based on a mathematical description of the dynamical characteristics of the system. This mathematical description is referred to as a mathematical model.

- For many practical systems, useful mathematical models are generally described in terms of **differential equations**, for continuous-time systems, or **difference equations**, for discrete-time systems.

- While these equations can be solved in their current form, it is easier to work with them in transfer function format. *Laplace transforms* are used to convert differential equations into algebraic form, while *Z-transforms* are used to convert difference equations.

- Dynamical systems can be modelled using two related, but different-looking, methods, namely the transfer function (or input-output) method [frequency domain] and the state-space method [time domain].

- The **input-output method** captures the system behaviour by relating outputs to the inputs, in the form of a **transfer function**. *We studied this in detail in EE114. A brief summary is presented in the next section.*

- The **state-space method** also considers the inter-relationship between the outputs and the inputs, but this inter-relation involves an additional set of variables known as the **state variables**.

- *We will study state space later in the notes. We will also be looking at the relationship between state-space and transfer functions.*

2.4 Transfer Functions

- A transfer function (TF) model is the input-output relationship of a system in the Laplace Transform space.

- In this form, we no longer work with *derivatives* but rather with an *algebraic expression*.

- A TF is defined as the ratio of the Laplace transforms of the output and input of a system **for zero initial conditions**:

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s) \qquad\qquad \text{Transfer function } = \frac{Y(s)}{U(s)} = G(s)$$

- For example, consider the system governed by the following differential equation:

$$\frac{d^2y}{dt^2} - 4y = \frac{du}{dt} - 3u$$

- Obtaining the Laplace transform of this equation gives:

$$s^2Y(s) - 4Y(s) = sU(s) - 3U(s)$$

- We can then obtain the transfer function as follows:

$$(s^2 - 4)Y(s) = (s-3)\,U(s) \qquad\qquad \Rightarrow \frac{Y(s)}{U(s)} = \frac{s-3}{s^2-4} = G(s)$$

Calculating the o/p of a system using the transfer function …

- What is the **unit-step response** of this system? The Laplace transform of a unit step input is $U(s) = \frac{1}{s}$. Hence:

$$Y(s) = G(s)\,U(s) = \frac{s-3}{(s^2-4)s}$$

- Finally, we can solve for $y(t)$ using the **partial fractions method**:

$$\frac{s-3}{(s^2-4)s} = \frac{s-3}{(s-2)(s+2)s} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s}$$

Solving for A, B and C gives: $\qquad A = -\tfrac{1}{8}, \quad B = -\tfrac{5}{8}, \quad C = \tfrac{3}{4}$

Hence: $y(t) = -\frac{1}{8}e^{2t} - \frac{5}{8}e^{-2t} + \frac{3}{4}$

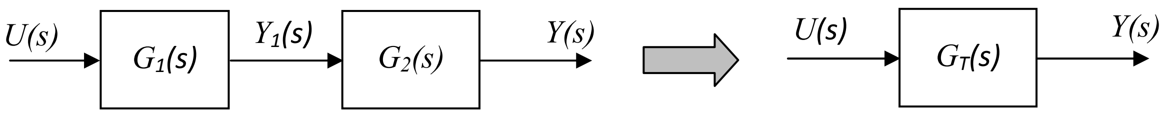
2.5 Block diagram algebra

- Large, complicated systems may consist of many components, each of which may be represented by a transfer function.

- A **block diagram** shows these transfer functions and illustrates the functional relationship between them.

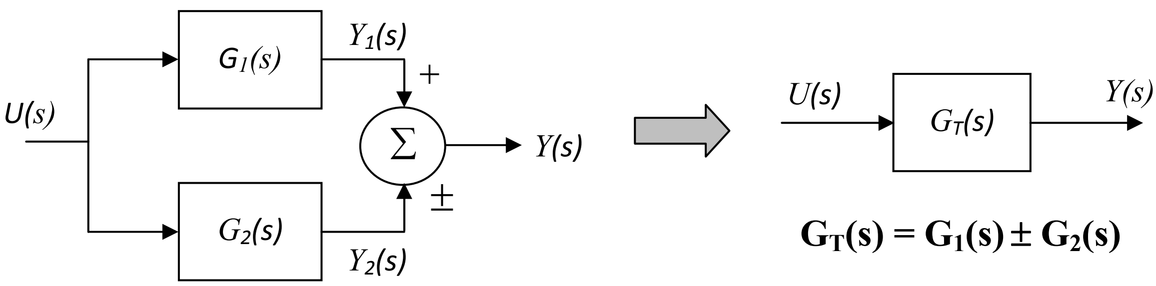
- Complicated block diagrams can be simplified by combining different blocks together using **block diagram algebra.**

- **Series (or cascade) connection – blocks in series** are combined by **multiplying** them together.



$$\mathbf{G_T(s) = G_1(s).G_2(s)}$$

- **Parallel connection – blocks in parallel** are combined by **adding** them together.



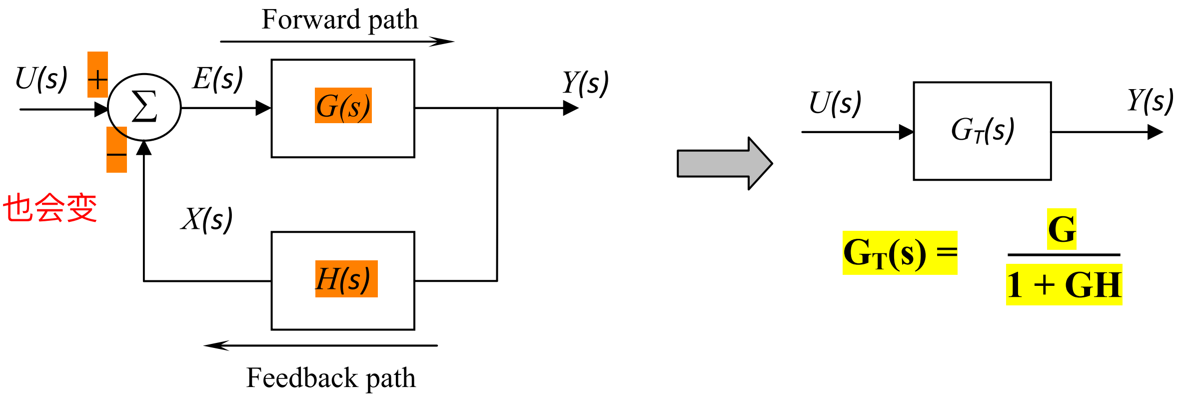
$$\mathbf{G_T(s) = G_1(s) \pm G_2(s)}$$

- **Feedback connection** – when the output of a block is fed back to the input of an earlier block in the block diagram, a feedback mechanism is introduced.

- This is usually referred to as ***closing the loop***. 闭环

- The resulting transfer function is called the **closed loop transfer function**.

- The fundamental closed loop diagram for a feedback system is as follows:



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$$G_T(s) = \frac{G}{1 + GH}$$

- This is known as the **canonical block diagram** and is characterised by:

典型框图

  - *G(s)* is the direct or forward path transfer function.
  - *H(s)* is the feedback transfer function (a unity feedback system has $H(s) = 1$).
  - $\frac{X(s)}{U(s)} = G(s)H(s)$, **with feedback disconnected**, is the **open-loop transfer function (OLTF).**
  - $\frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)}$ is the **closed-loop transfer function (CLTF).**

2.6 Matrix algebra and z-transforms

- These are key concepts that *you should have learned* in previous modules.
- You will *need to know these fundamental concepts* in order to successfully work through the material covered in this module.

Matrix algebra (covered in EE112)

矩阵复习

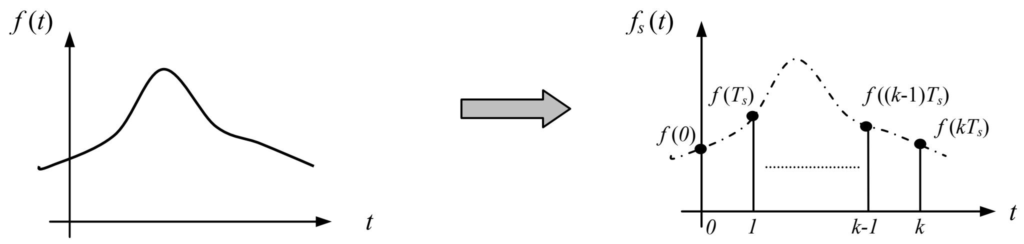
- In terms of matrices, you should be able to carry out the following: multiplication; addition; transpose; inverse; rank; calculate the eigenvalues and eigenvectors.
- *We will revise how to calculate the eigenvalues and eigenvectors later in the notes, when required.*

Z-transforms (covered in EE206)

- The z-transform can be viewed as the discrete-time equivalent to the Laplace transform (difference equations → algebraic equations).

- Consider a continuous function $f(t)$ sampled at time intervals of T_s, giving the discrete sequence:

$$f(0), f(T_s), f(2T_s), \ldots, f(kT_s), \ldots$$



- We can write down the equation for the sampled version of $f(t)$, denoted $f_s(t)$, by using the impulse function $\delta(t-T)$, i.e.:

$$f_s(t) = f(0)\delta(t) + f(T_s)\delta(t-T_s) + f(2T_s)\delta(t-2T_s) + \ldots$$

$$= \sum_{k=0}^{\infty} f(kT_s)\delta(t-kT_s)$$

- The Laplace transform of this function is:

$$F(s) = \sum_{k=0}^{\infty} f(kT_s)\, e^{-ksT_s}$$

- If we define a new complex variable $z = e^{sT_s}$, then we can write:

$$F(z) = Z[f(kT_s)] = \sum_{k=0}^{\infty} f(kT_s)\, z^{-k}$$

- This is called the z-transform of a discrete-time signal and can be viewed as the discrete-time equivalent of the Laplace transform.

- Example z-transforms of common functions include:

  - $f(k) = 1$, $k \geq 0$ (i.e. the unit step)

$$Z\left[f(k)\right] = \sum_{k=0}^{\infty} 1.z^{-k} = 1 + z^{-1} + z^{-2} + \ldots = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

- $f(k) = e^{ak}$

$$Z\left[f(k)\right] = \sum_{k=0}^{\infty} e^{ak} z^{-k} = \sum_{k=0}^{\infty} \left(e^{a} z^{-1}\right)^{k} = \frac{1}{1 - e^{a} z^{-1}} = \frac{z}{z - e^{a}}$$

- $f(k) = \delta(k-p)$

$$Z\left[f(k)\right] = \sum_{k=0}^{\infty} \delta(k-p) z^{-k} = z^{-p}$$

- Note, in the last example, when $p = 0$ we get: $Z\left[\delta(k)\right] = z^{0} = 1$

Inverse *z*-transform

- The inverse *z*-transform is determined in the same way as the inverse Laplace transform, i.e. using the partial fraction method and a look-up table of standard z-transforms.

- For example, consider the following z-transform:

$$F(z) = \frac{4z}{3z^2 - 2z - 1}$$

- We can express this in partial fraction form as follows:

$$\frac{F(z)}{z} = \frac{4}{3z^2 - 2z - 1} = \frac{\frac{4}{3}}{z^2 - \frac{2}{3}z - \frac{1}{3}} = \frac{\frac{4}{3}}{(z-1)(z+\frac{1}{3})} = \frac{A}{z-1} + \frac{B}{z+\frac{1}{3}} = \frac{A(z+\frac{1}{3}) + B(z-1)}{(z-1)(z+\frac{1}{3})}$$

- Comparing terms: $A + B = 0$ and $\frac{1}{3}A - B = \frac{4}{3}$

- Solving for A and B gives: $A = 1, \quad B = -1$

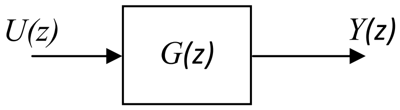
- Hence: $\frac{F(z)}{z} = \frac{1}{z-1} - \frac{1}{z+\frac{1}{3}} \Rightarrow F(z) = \frac{z}{z-1} - \frac{z}{z+\frac{1}{3}}$

- This can now be easily converted back to the time domain to give:

$$F(z) = \frac{z}{z-1} - \frac{z}{z+\frac{1}{3}} \quad \rightarrow \quad f(k) = 1 - \left(-\tfrac{1}{3}\right)^{k} = 1 - (-3)^{-k}$$

Discrete-time transfer function from difference equation

- z-transforms are a convenient way of representing discrete-time systems as they allow them to be described as transfer functions, i.e.:



$$\text{Transfer function } = \frac{Y(z)}{U(z)} = G(z)$$

- For example, consider the system governed by the following difference equation:

$$y_k = 0.5y_{k-1} + 0.8y_{k-2} - 3u_{k-1} + u_{k-3}$$

- Taking the z-transform gives:

$$Y(z) = 0.5z^{-1}Y(z) + 0.6z^{-2}Y(z) - 3z^{-1}U(z) + z^{-3}U(z)$$

- Rearranging gives:

$$\frac{Y(z)}{U(z)} = \frac{-3z^{-1}+z^{-3}}{1-0.5z^{-1}-0.6z^{-2}} = \frac{1-3z^2}{z^3-0.5z^2-0.6z}$$

- Note that discrete-time models are not as intuitive as continuous-time models since model parameters do not, in general, represent physical quantities such as time constants, damping ratio, etc.

Note – other material from the EE114 module, including modelling and stability, will be revised at appropriate points, later in these notes.