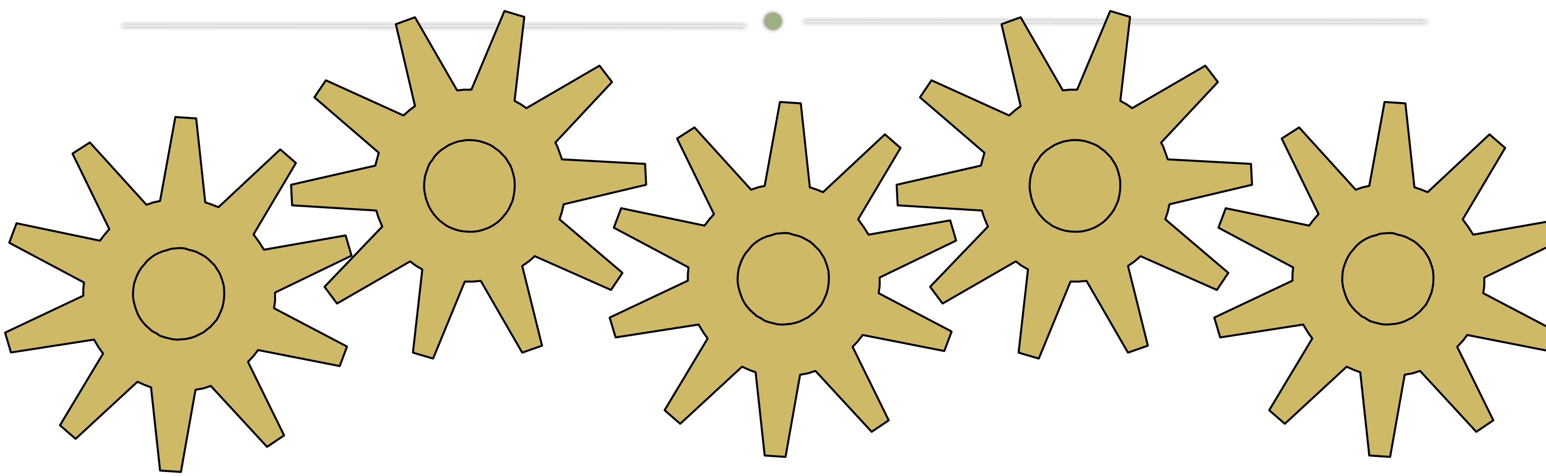
EE114
Intro to Systems & Control

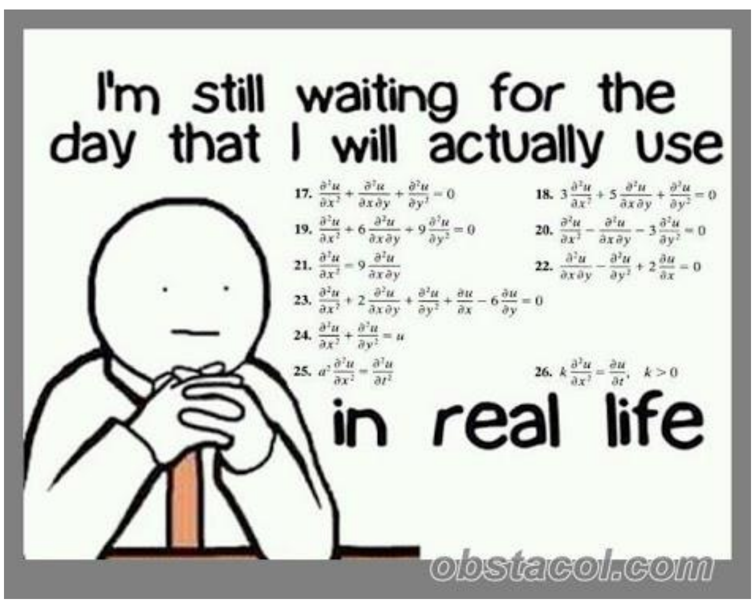
Dr. Lachman Tarachand
Dr. Chen Zhicong
Prepared by Dr. Séamus McLoone
Dept. of Electronic Engineering



So far …

- We’ve modelled a range of relatively simple dynamical systems - an RC circuit, a bicycle and a water tank … giving models in the form of ODEs

- We’ve introduced the transfer function concept, and as a result, the Laplace Transform …



So far …

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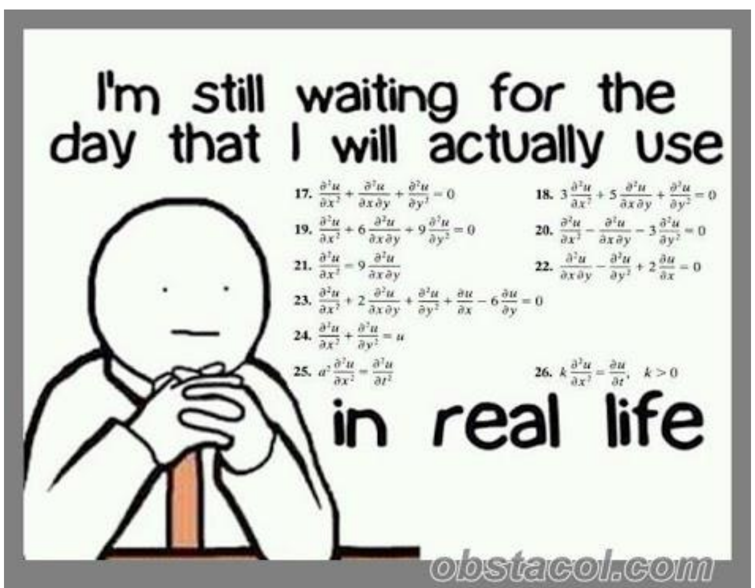
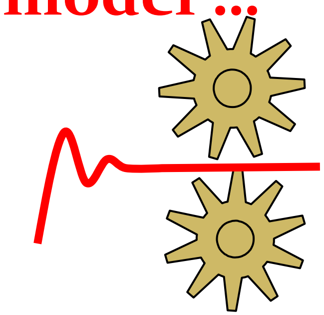
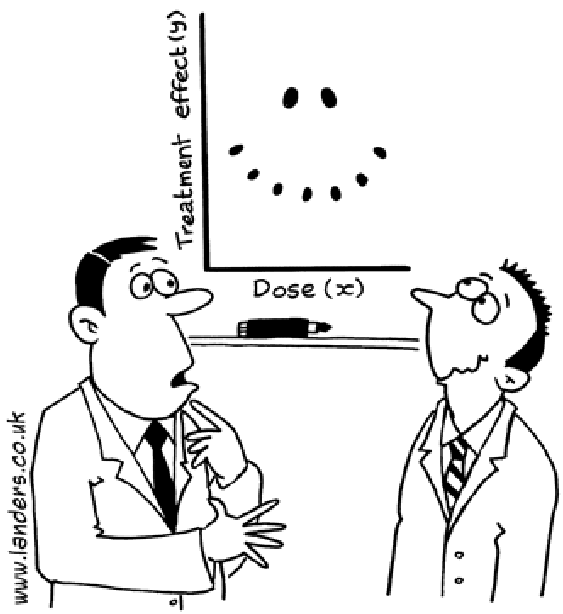


- **Today, we will continue with Laplace Transforms and formally define a transfer function model …**



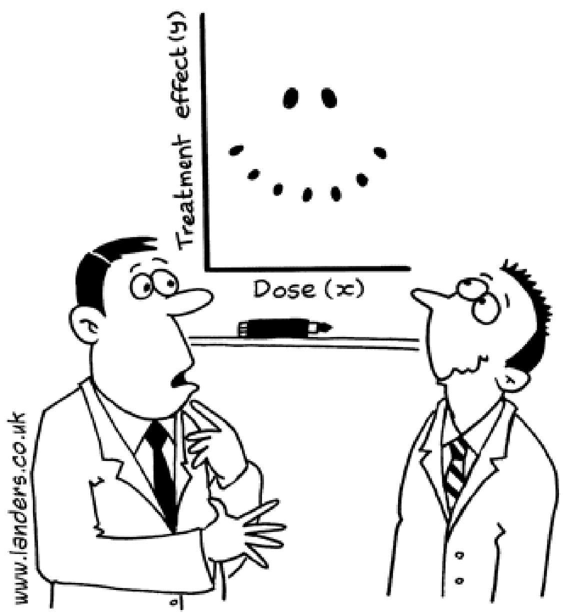
Laplace Transforms & Differential Eqn's



"It's a non-linear pattern with outliers.....but for some reason I'm very happy with the data."

Laplace Transforms & Differential Eqn's

- One of the key advantages (particularly from a system analysis viewpoint) of Laplace transforms is the **transformation of linear differential equations into algebraic equations**.



"It's a non-linear pattern with outliers.....but for some reason I'm very happy with the data."

Laplace Transforms & Differential Eqn's

- It can be shown that:

$$L\left[f^{'}(t)\right] = sF(s) - f(0)$$

$$L\left[f^{''}(t)\right] = s^{2}F(s) - sf(0) - f^{'}(0)$$

$$L\left[f^{'''}(t)\right] = s^{3}F(s) - s^{2}f(0) - sf^{'}(0) - f^{''}(0)$$

$$\ldots$$

$$\ldots$$

$$L\left[f^{n}(t)\right] = s^{n}F(s) - s^{n-1}f(0) - s^{n-2}f^{'}(0) - \ \ldots \ - f^{n-1}(0)$$

Laplace Transforms & Differential Eqn's

- It can be shown that:

$$L\left[f^{'}(t)\right] = sF(s) - f(0)$$

$$L\left[f^{''}(t)\right] = s^{2}F(s)$$

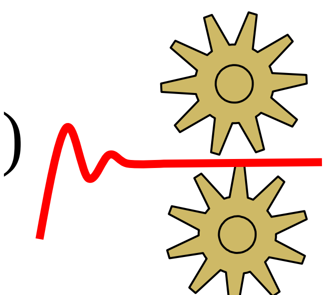
First Order
Differential Equation

$$L\left[f^{'''}(t)\right] = s^{3}F(s) - s^{2}f(0) - sf^{'}(0) - f^{''}(0)$$

$$\ldots$$

$$\ldots$$

$$L\left[f^{n}(t)\right] = s^{n}F(s) - s^{n-1}f(0) - s^{n-2}f^{'}(0) - \ldots - f^{n-1}(0)$$



Laplace Transforms & Differential Eqn's

- It can be shown that:

$$L\left[f^{'}(t)\right] = sF(s) - f(0)$$

$$L\left[f^{''}(t)\right] = s^2F(s) - sf(0) - f^{'}(0)$$

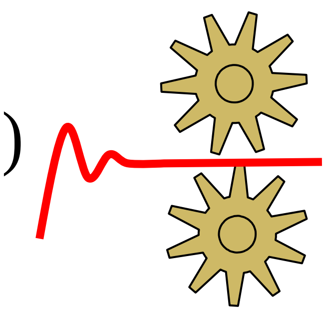
$$L\left[f^{'''}(t)\right] = s^3F(s) - s^2 f$$

Second Order Differential Equation

...

...

$$L\left[f^{n}(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f^{'}(0) - \ \dots\ - f^{n-1}(0)$$



Laplace Transforms & Differential Eqn's

- It can be shown that:

$$L\left[f'(t)\right] = sF(s) - f(0)$$

$$L\left[f''(t)\right] = s^2F(s) - sf(0) - f'(0)$$

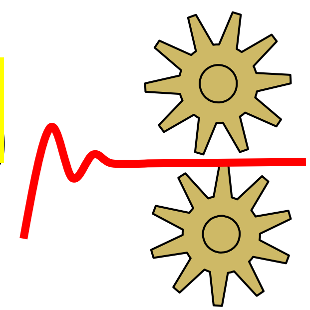
$$L\left[f'''(t)\right] = s^3F(s) - s^2 f(0) - sf'(0) - f''(0)$$

$$\ldots$$

$$\ldots$$

$$L\left[f^n(t)\right] = s^nF(s) - s^{n-1}f(\ \ldots$$

Initial Conditions



Laplace Transforms & Differential Eqn's

- It can be shown that:

$$L\left[f^{'}(t)\right] = sF(s) - f(0)$$

$$d/dt \rightarrow s$$

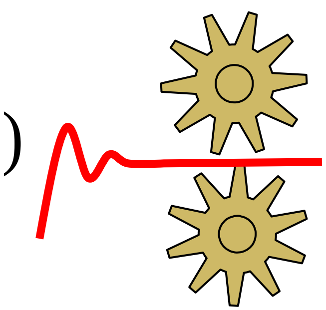
$$L\left[f^{''}(t)\right] = s^2F(s) - \ldots$$

$$L\left[f^{'''}(t)\right] = s^3F(s) - s^2f(0) - sf^{'}(0) - f^{''}(0)$$

$$\ldots$$

$$\ldots$$

$$L\left[f^{n}(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f^{'}(0) - \ \ldots \ - f^{n-1}(0)$$



Laplace Transforms & Differential Eqn's

- ***Ex 4.2 Express the following differential equation in terms of Laplace transforms given that at time t = 0, x = 1:***

$$\frac{dx(t)}{dt} - 2x(t) = 4$$

Laplace Transforms & Differential Eqn's

Solution:

$$L\left(\frac{dx}{dt} - 2x = 4\right)$$

Laplace Transforms & Differential Eqn's

Solution:

$$L\left(\frac{dx}{dt} - 2x = 4\right) \qquad \Rightarrow L\left(\frac{dx}{dt}\right) - 2L\left(x\right) = L(4)$$

Laplace Transforms & Differential Eqn's

Solution:

$$L\left(\frac{dx}{dt} - 2x = 4\right) \qquad \Rightarrow L\left(\frac{dx}{dt}\right) - 2L\left(x\right) = L(4)$$

$$\Rightarrow sX(s) - x(0) - 2X(s) = \frac{4}{s}$$

Laplace Transforms & Differential Eqn's

Solution:

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We know that $x(0) = 1$, hence:

Laplace Transforms & Differential Eqn's

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We know that $x(0) = 1$, hence:

$$sX(s) - 1 - 2X(s) = \frac{4}{s}$$

Laplace Transforms & Differential Eqn's

Solution:

$$sX(s) - x(0) - 2X(s) = \frac{4}{s}$$

We know that $x(0) = 1$, hence:

$$sX(s) - 1 - 2X(s) = \frac{4}{s} \qquad \Rightarrow X(s)(s-2) = \frac{4}{s} + 1 = \frac{4+s}{s}$$

Laplace Transforms & Differential Eqn's

Solution:

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$$\Rightarrow X(s) = \frac{s+4}{s(s-2)}$$

Laplace Transforms & Differential Eqn's

Solution:

$$sX(s) - x(0) - 2X(s) = \frac{4}{s}$$

Note that, if needed, we can now solve the differential equation by obtaining the partial factions for X(s) and finding the inverse Laplace transform. This gives:

$$\Rightarrow X(s) = \frac{s+4}{s(s-2)}$$

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$$x(t) = 3e^{2t} - 2$$

Transfer Function Representation

Transfer Function Representation

- The **transfer function model** is the input-output relationship of a system in the Laplace Transform space.

- It is defined as **the ratio of the Laplace transforms of the output and input of a system for zero initial conditions.**

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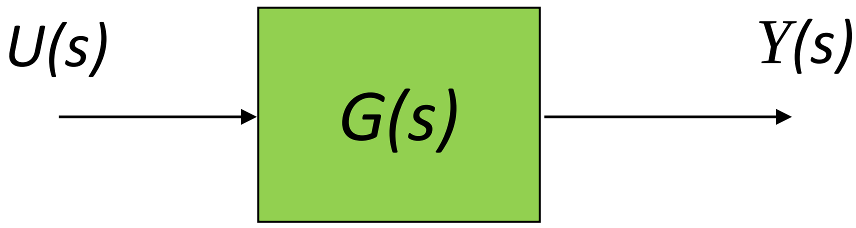

$$\text{Transfer function } = \frac{Y(s)}{U(s)} = G(s)$$

Transfer Function Representation

- Since there are zero initial conditions, then the Laplace transform of differential expressions, in this context, is simply reduced to:

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$$\dots$$

$$\dots$$

$$L\left[f^{n}(t)\right] = s^{n}F(s)$$

Transfer Function Representation

- So, for example, consider the system governed by the following differential equation:

$$\frac{d^2y}{dt^2} - 4y = \frac{du}{dt} - 3u$$

Transfer Function Representation

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$$\frac{d^2 y}{dt^2} - 4y = \frac{du}{dt} - 3u$$

- Obtaining the Laplace transform of this equation gives:

$$s^2 Y(s) - 4Y(s) = sU(s) - 3U(s)$$

Transfer Function Representation

- So, for example, consider the system governed by the following differential equation:

$$\frac{d^2y}{dt^2} - 4y = \frac{du}{dt} - 3u$$

- Obtaining … gives:

No need to worry about initial conditions in this case!

$$s^2Y(s) - 4Y(s) = sU(s) - 3U(s)$$

Transfer Function Representation

$$s^2Y(s) - 4Y(s) = sU(s) - 3U(s)$$

- We can then obtain the transfer function as follows:

Transfer Function Representation

$$s^2Y(s) - 4Y(s) = sU(s) - 3U(s)$$

- We can then obtain the transfer function as follows:

$$(s^2 - 4)Y(s) = (s - 3)U(s)$$

Transfer Function Representation

$$s^2Y(s) - 4Y(s) = sU(s) - 3U(s)$$

- We can then obtain the transfer function as follows:

$$(s^2 - 4)Y(s) = (s - 3)U(s)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{s-3}{s^2-4} = G(s)$$

Transfer Function Representation

- ***Ex 4.3 Obtain the transfer function for the system given by the following ordinary differential equation:***

$$\frac{dx(t)}{dt} - 2x(t) = 4u(t)$$

Transfer Function Representation

Solution:

$$L\left(\frac{dx(t)}{dt} - 2x(t) = 4u(t)\right)$$

Transfer Function Representation

Solution:

$$L\left(\frac{dx(t)}{dt}-2x(t)=4u(t)\right)$$

$$\Rightarrow sX(s)-2X(s)=4U(s)$$

Transfer Function Representation

Solution:

$$L\left(\frac{dx(t)}{dt} - 2x(t) = 4u(t)\right)$$

$$\Rightarrow sX(s) - 2X(s) = 4U(s)$$

$$\Rightarrow X(s)(s-2) = 4U(s)$$

Transfer Function Representation

Solution:

$$L\left(\frac{dx(t)}{dt} - 2x(t) = 4u(t)\right)$$

$$\Rightarrow sX(s) - 2X(s) = 4U(s)$$

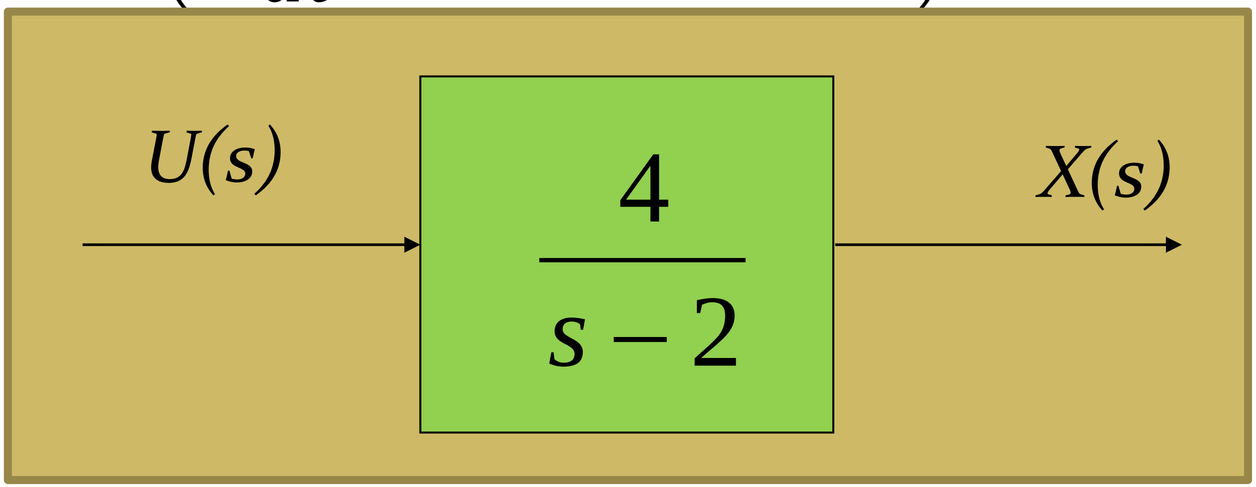
$$\Rightarrow X(s)(s-2) = 4U(s)$$

$$\Rightarrow G(s) = \frac{X(s)}{U(s)} = \frac{4}{s-2}$$

Transfer Function Representation

Solution:

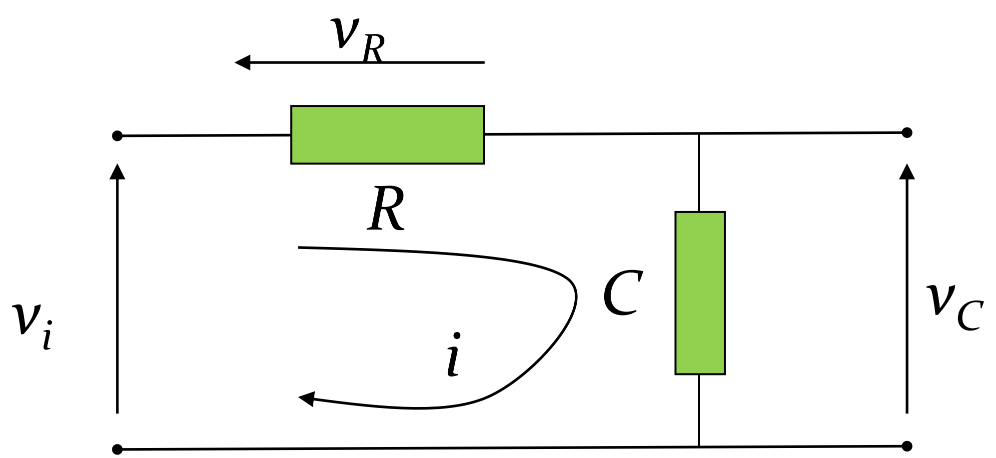
$$L\left(\frac{dx(t)}{dt} - 2x(t) = 4u(t)\right)$$

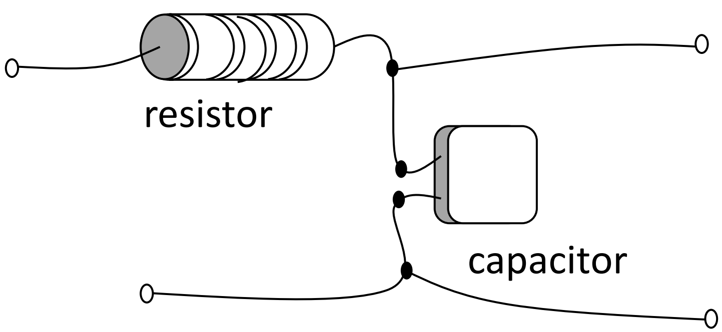


$$\Rightarrow G(s) = \frac{X(s)}{U(s)} = \frac{4}{s-2}$$

Transfer Function Representation

- ***Ex 4.4 Obtain the transfer function for the following circuit-based system (Ex3.3):***





Transfer Function Representation

- **Solution:**

Recall from Ex 3.3: $$v_i = RC\frac{dv_C}{dt} + v_C$$

Transfer Function Representation

- **Solution:**

Recall from Ex 3.3:
$$v_i = RC\frac{dv_C}{dt} + v_C$$

Taking the Laplace transform gives:
$$V_i(s) = RCsV_c(s) + V_c(s)$$

Transfer Function Representation

- **Solution:**

Recall from Ex 3.3: $$v_i = RC\frac{dv_C}{dt} + v_C$$

Taking the Laplace transform gives: $$V_i(s) = RCsV_c(s) + V_c(s)$$

$$\Rightarrow V_i(s) = V_c(s)(1 + sRC)$$

Transfer Function Representation

- **Solution:**

Recall from Ex 3.3:

$$v_i = RC\frac{dv_C}{dt} + v_C$$

Taking the Laplace transform gives:

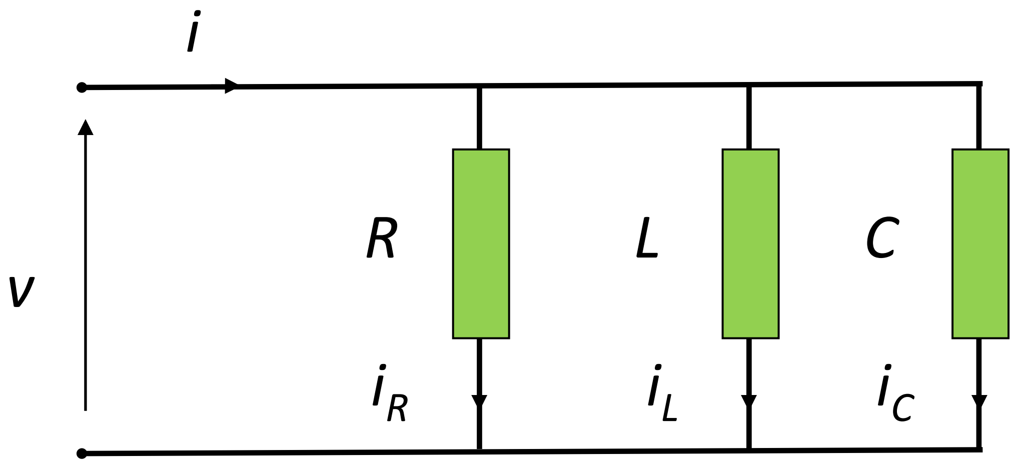
$$V_i(s) = RCsV_c(s) + V_c(s)$$

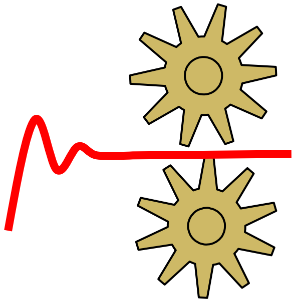
$$\Rightarrow V_i(s) = V_c(s)(1 + sRC)$$

$$\Rightarrow \frac{V_c(s)}{V_i(s)} = \frac{1}{1 + sRC}$$

Transfer Function Representation

- ***Ex 4.5 Obtain the transfer function representation for the following circuit-based system, relating voltage to current (Ex3.5):***





Transfer Function Representation

- **Solution:**

Recall from Ex 3.5:

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{C}\frac{di}{dt}$$

Transfer Function Representation

- **Solution:**

Recall from Ex 3.5:

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{C}\frac{di}{dt}$$

Taking the Laplace transform gives:

$$s^2V(s) + \frac{1}{RC}sV(s) + \frac{1}{LC}V(s) = \frac{1}{C}I(s)$$

Transfer Function Representation

- **Solution:**

Recall from Ex 3.5:
$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=\frac{1}{C}\frac{di}{dt}$$

Taking the Laplace transform gives:

$$s^2V(s)+\frac{1}{RC}sV(s)+\frac{1}{LC}V(s)=\frac{1}{C}I(s)$$

$$\Rightarrow\left(s^2+\frac{s}{RC}+\frac{1}{LC}\right)V(s)=\frac{1}{C}I(s)$$

Transfer Function Representation

- **Solution:**

Hence the required transfer function is given by :

$$\frac{V(s)}{I(s)} = \frac{\dfrac{1}{C}}{s^2 + \dfrac{s}{RC} + \dfrac{1}{LC}}$$

Transfer Function Representation

- **Solution:**

Hence the required transfer function is given by :

$$\frac{V(s)}{I(s)} = \frac{\frac{1}{C}}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

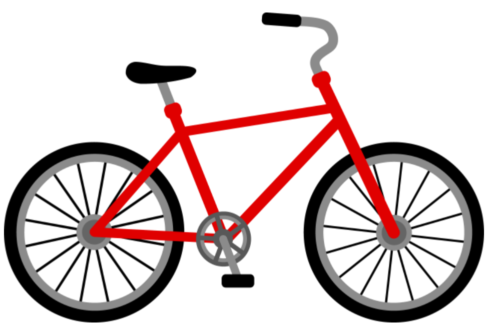
or:

$$\frac{V(s)}{I(s)} = \frac{RL}{RLCs^2 + Ls + R}$$

Transfer Function Representation

- ***Ex 4.6 Determine the transfer function model for the spring-mass-damper system given in Ex 3.6(a):***

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t)$$



Transfer Function Representation

Solution:

$$M\frac{d^2x}{dt^2}+B\frac{dx}{dt}+Kx=f(t)$$

Transfer Function Representation

Solution:

$$M\frac{d^2x}{dt^2}+B\frac{dx}{dt}+Kx=f(t)$$

Taking the Laplace transform gives:

$$Ms^2X(s)+BsX(s)+KX(s)=F(s)$$

Transfer Function Representation

Solution:

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t)$$

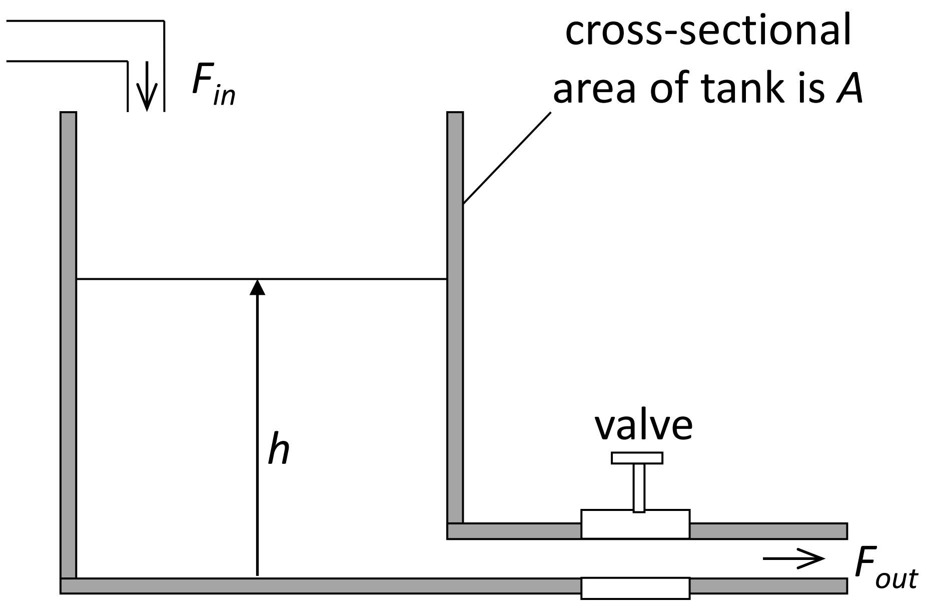
Taking the Laplace transform gives:

$$Ms^2X(s) + BsX(s) + KX(s) = F(s)$$

$$\Rightarrow \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

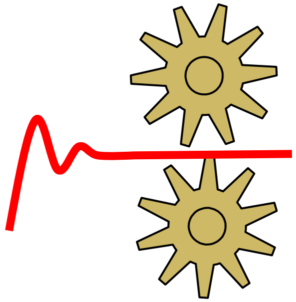
Transfer Function Representation

- ***Ex 4.7 Determine the transfer function model for the single water tank system represented by the differential equation:***



Physical model of a single
tank system

$$A\frac{dh}{dt} = F_{in} - kh$$



Transfer Function Representation

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Transfer Function Representation

Solution:

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Taking the Laplace transform gives:

$$AsH(s) = F(s) - kH(s)$$

Transfer Function Representation

Solution:

$$A\frac{dh}{dt} = F_{in} - kh$$

Taking the Laplace transform gives:

$$AsH(s) = F(s) - kH(s) \quad \Rightarrow H(s)(sA + k) = F(s)$$

Transfer Function Representation

Solution:

$$A\frac{dh}{dt} = F_{in} - kh$$

Taking the Laplace transform gives:

$$AsH(s) = F(s) - kH(s) \quad \Rightarrow H(s)(sA + k) = F(s)$$

$$\Rightarrow \frac{H(s)}{F(s)} = \frac{1}{sA + k}$$