EE206 Differential Equations and Transform Methods

Tutorial 4

Find the Laplace Transform of the following functions.

Problem 1b: $f(t) = (1 + e^{2t})^2$

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\{(1 + e^{2t})^2\} \\
&= \mathcal{L}\{1 + 2e^{2t} + e^{4t}\} \\
&= \mathcal{L}\{1\} + 2\mathcal{L}\{e^{2t}\} + \mathcal{L}\{e^{4t}\} \\
&= \frac{1}{s} + \frac{2}{s-2} + \frac{1}{s-4}
\end{aligned}$$

Find the inverse transform of the following.

Problem 2b: $\mathcal{L}^{-1}\left\{\frac{1}{s^3+5s} - \frac{48}{s^5}\right\}$

$$\frac{1}{s^3+5s} = \frac{A}{s} + \frac{Bs+C}{s^2+5}$$

$$1 = As^2 + 5A + Bs^2 + Cs$$

$$5A = 1 \;\Rightarrow\; A = \frac{1}{5}$$

$$C = 0$$

$$A + B = 0 \;\Rightarrow\; B = -A \;\Rightarrow\; B = -\frac{1}{5}$$

$$\frac{1}{s^3+5s} = \frac{\frac{1}{5}}{s} + \frac{-\frac{1}{5}s}{s^2+5}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3+5s} - \frac{48}{s^5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3+5s}\right\} - \mathcal{L}^{-1}\left\{\frac{48}{s^5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{1}{5}}{s} + \frac{-\frac{1}{5}s}{s^2+5}\right\} - \mathcal{L}^{-1}\left\{\frac{48}{s^5}\right\}$$

Using $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$ and $\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3+5s} - \frac{48}{s^5}\right\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+5}\right\} - 2\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\}$$
$$= \frac{1}{5} - \frac{1}{5}\cos\sqrt{5}t - 2t^4$$

Use the Laplace transform to solve the given initial-value problems.

Problem 3a: $y'' + 5y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$

$$\mathcal{L}\{y'' + 5y' + 4y = 0\} = \mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 0$$

$$s^2Y(s) - sy(0) - y'(0) + 5sY(s) - 5y(0) + 4Y(s) = 0$$

$$(s^2 + 5s + 4)Y(s) - s - 5 = 0$$

$$Y(s) = \frac{s+5}{s^2 + 5s + 4}$$

$$= \frac{s+5}{(s+4)(s+1)}$$

$$\frac{A}{s+4} + \frac{B}{s+1} = \frac{A(s+1) + B(s+4)}{(s+4)(s+1)} = \frac{s+5}{(s+4)(s+1)}$$

$$s = -1 \quad A(0) + B(3) = 4 \quad \Rightarrow B = \frac{4}{3}$$

$$s = -4 \quad A(-3) + B(0) = 1 \quad \Rightarrow A = -\frac{1}{3}$$

$$Y(s) = -\left(\frac{1}{3}\right)\frac{1}{s+4} + \left(\frac{4}{3}\right)\frac{1}{s+1}$$

$$\begin{aligned} y(t) &= -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= -\frac{1}{3}e^{-4t} + \frac{4}{3}e^{-t} \end{aligned}$$