Engineering Mathematics 1 (Fall 2021)

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Students should be able to (after learning)

- Add, subtract and multiply complex numbers
- Convert complex numbers between Cartesian and polar forms
- Differentiate all commonly occurring functions including polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of a derivative, namely the derivative as a tangent and the derivative as a rate of change
- Integrate certain standard functions, constructed from polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of integration, namely the integral as the inverse of the derivative and the integral as the area under a curve
- Apply Taylor series to numerically approximate functions
- Apply Simpson's rule to numerically evaluate integrals
- Solve simple first and second order ordinary differential equations
- Apply and select the appropriate mathematical techniques to solve a variety of associated engineering problems

Lecture 12: Differentiation-Part 4

10. Maclausrin's series and Taylor's series

Maclaurin's series: $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$

x: small

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

x is small

$\because \sinh x = \frac{1}{2}(e^x - e^{-x})$, $\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots,}$

$\therefore \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ even order terms are cancelled

x: small

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

Similarly, $\cosh x = \frac{1}{2}(e^x + e^{-x})$ odd order terms disappeared

x: small

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$y(0) = 0$ Denote $\boxed{y(x) = \ln(1+x).}$

Let $u = 1+x$, $y = \ln(1+x)$, then $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{u}\cdot 1 = \frac{1}{1+x}$, $y'(0) = 1$,

$\frac{d^2y}{dx^2}\ \frac{d}{dx}\left(\frac{1}{1+x}\right) = \left[(1+x)^{-1}\right]' = (-1)(1+x)^{-2}\cdot(1+x)' = -(1+x)^{-2}$, $y''(0) = -1$,

$\frac{d^3y}{dx^3} = \left[-(1+x)^{-2}\right]' = (-1)(-2)(1+x)^{-3}\cdot(1+x)' = 2(1+x)^{-3}$, $y'''(0) = 2$,

$y = y(x) = y(0) + xy'(0) + x^2\frac{y''(0)}{2!} + x^3\frac{y'''(0)}{3!} + \cdots$

$= 0 + x\cdot 1 + x^2\cdot\left(-\frac{1}{2}\right) + x^3\cdot\frac{2}{3!} + \cdots$

$= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$\boxed{x : \text{small}}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Denote $f(x) = (1+x)^n$, $f(0) = 1$, $f'(x) = n(1+x)^{n-1} \cdot (1+x)' = n(1+x)^{n-1}$, $f'(0) = n$,

$f''(x) = n(n-1)(1+x)^{n-2}$, $f''(0) = n(n-1)$,

$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$, $f'''(0) = n(n-1)(n-2)$.

Maclaurin's series

h : small

$x + h$ is increatment of x

Taylor's series: $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$

OR $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$

where $x = \underbrace{x - x_0}_{h} + \underbrace{x_0}_{x}$, $f(x) = f(x - x_0 + x_0)$

$f(h + x)$

Ex1: $y = \sinh^{\cancel{-1}} 5.02$, approximate y.

Sol: $y = \sinh(5.02) = \sinh(5 + 0.02)$, by Taylor's series,

$\sinh(5 + 0.02) = \sinh(5) + 0.02\cosh(5) + \frac{0.02^2}{2!}\sinh(5) + \cdots$

$\sinh(5 + 0.02) \approx \sinh 5 + 0.0002\sinh 5 + 0.02\cosh 5$

Ex2: $y = \cosh^{\cancel{-1}} 1.01$, approximate y.

check Table for $\sinh x$ and $\cosh x$

Sol: $y = \cosh(1.01) = \cosh(1 + 0.01)$, by Taylor's series,

$y = \cosh(1 + 0.01) = \cosh 1 + 0.01\sinh 1 + \frac{0.01^2}{2!}\cosh 1$

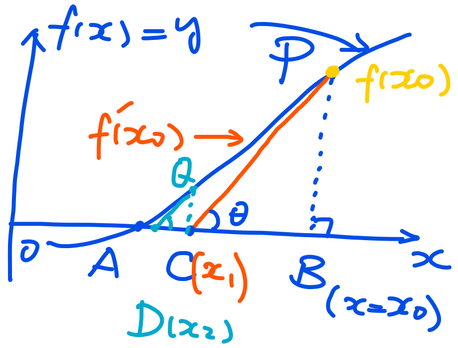
check Table for $\sinh x$ & $\cosh x$

11. Newton-Raphson iterative method

N-R

Aim: for approximation or estimation

Curve $y = f(x)$ is given, A is the point passing through x-axis with $f(x) = 0$, P is a point on the curve near to point A, then point B (or $x = x_0$) is an approximate value of the root of $f(x) = 0$, a better approximation is given by $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.



$\because \triangle PBC$, $\tan\theta = \frac{PB}{CB} = \frac{f(x_0)}{x_0 - x_1} = f'(x_0)$

$\therefore x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$

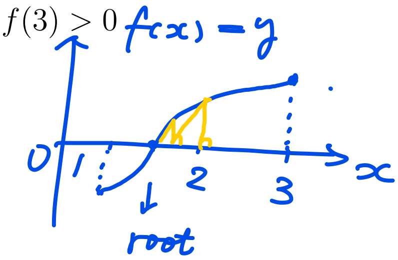
$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ First estimation

Again, $\triangle QCD$, $\tan\alpha = \dfrac{QC}{DC} = \dfrac{f(x_1)}{x_1 - x_2} = f'(x_1)$

$\therefore x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ Second estimation, a better estimation

Starting from $\underset{(x_0)}{B} \Rightarrow \underset{(x_1)}{C} \Rightarrow \underset{(x_2)}{D} \Rightarrow \underset{(\text{real value})}{A}$

Ex1: The equation $x^3 - 3x - 4 = 0$ with properties $f(1) < 0$ and $f(3) > 0$ admits a root near 2. Find a better approximation to the root.



Sol: Let $f(x) = x^3 - 3x - 4$, $f'(x) = 3x^2 - 3$

Let $x_0 = 2$, $f(x_0) = 8 - 6 - 4 = -2$, $f'(x_0) = 12 - 3 = 9$,

by N-R method, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 2 - \dfrac{-2}{9} = 2.22$

by N-R again, $x_2 = x_1 - f(x_1)/f'(x_1) = 2.22 - [2.22^3 - 6.66 - 4]/[3 \times 2.22^2 - 3] = 2.19$

$\therefore$ 2.19 is a better approximation.

Ex2: The equation $2x^3 - 7x^2 - x + 12 = 0$ has a root near to $x = 1.5$. Use the Newton-Raphson method to find the root to two decimal places.

12. Maximum, minimum, point of inflexion

Given a function $y = f(x)$, stationary points are defined as $y'(x) = 0$.

$y'(x) = 0$, it may be a maximum, may be a minimum, may be a point of inflexion (i.e., S-bend form)

$y''(x) > 0$, maximum

$y''(x) < 0$, minimum

$y''(x) = 0$, may be points of inflexion (if yes, then change of sign occurs)

Ex1: $y = x^2$, to find stationary points, maximum, minimum.

Ex2: For $y = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x - 5$, find the points of inflexion.

Ex3: For $y = 3x^5 - 5x^4 + x + 4$, find the points of inflexion.