EE206
Solutions - Assignment 2

1. Solve the following first-order equations

   (a) $y' + 3x^2 y = x^2$;   by using variation of parameters. **[2]**
   First we solve the homogeneous equation $y' + 3x^2 y = 0$, by separation of variables.

$$y' = -3x^2 y \implies \frac{1}{y} dy = -3x^2 dx$$

   Integrating we find:

$$\ln(y) = -x^3 + c \quad \text{or} \quad y_c = Ae^{-x^3}$$

   Now let $y_p = u(x) y_c$, $y_p' = u' y_c + u y_c'$. Subbing into the original equation:

$$u' y_c + u\left(y_c' + 3x^2 y_c\right) = x^2$$

   Recall that y_c satisfies $y_c' + 3x^2 y_c = 0$, so:

$$u' y_c = x^2 \quad \text{or} \quad du = \frac{1}{A} x^2 e^{x^3} dx$$

   Integrating again using a substitution we find that:

$$u = \frac{1}{3A} e^{x^3} \implies y_p = \frac{1}{3A} e^{x^3} A e^{-x^3} = \frac{1}{3}$$

   Of course we probably could've noticed at the beginning that $y = \frac{1}{3}$ is a solution since $y' = 0$ and $\frac{1}{3} 3x^2 = x^2$.
   Our general solution is then

$$y = y_c + y_p = Ae^{-x^3} + \frac{1}{3}.$$

   (b) $\cos^2(x) \sin(x) \frac{dy}{dx} + \cos^3(x) y = \sin(x)$;   solve using an integrating factor. **[2]**
   We want to put this in the standard form: $\frac{dy}{dx} + P(x) y = f(x)$, so dividing by $\cos^2(x)\sin(x)$ we have:

$$\frac{dy}{dx} + \cot(x) y = \sec^2(x)$$

   The integrating factor is given by $e^{\int p(x)dx}$. Recalling that:

$$\int p(x) dx = \int \cot(x) dx = \ln(\sin(x))$$

   which you can re-check by making the appropriate substitution, our equation becomes:

$$\frac{d}{dx}\left(e^{\ln(\sin(x))} y\right) = e^{\ln(\sin(x))} \sec^2(x)$$

   or:

$$\frac{d}{dx}(\sin(x) y) = \frac{\sin(x)}{\cos^2(x)}$$

Next integrating by substitution gives:

$$\sin(x)y = \frac{1}{\cos(x)} + c$$

Thus:

$$y = \frac{1}{\sin(x)\cos(x)} + \frac{c}{\sin(x)}$$

2. Solve the given Bernoulli equations by using an appropriate substitution.

(b) $\frac{dy}{dx} = y(xy^4 - 1)$ **[2]**

We want to get it in the form: $\frac{dy}{dx} + P(x)y = f(x)y^n$ which is:

$$\frac{dy}{dx} + y = xy^5$$

Now letting $u = y^{1-n} = y^{-4}$, we have that

$$\frac{du}{dx} = -4y^{-5}\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = -\frac{1}{4}y^5\frac{du}{dx}$$

. Subbing back in and simplifying:

$$\begin{aligned} -\frac{1}{4}y^5\frac{du}{dx} + y &= xy^5 \\ \frac{du}{dx} + -4y^{-4} &= -4x \\ \frac{du}{dx} + -4u &= -4x \end{aligned}$$

Now using the integrating factor e^{-4x} we find:

$$\frac{d}{dx}\left(e^{-4x}u\right) = -4xe^{-4x}$$
$$e^{-4x}u = xe^{-4x} - \int e^{-4x}dx$$
$$u = x + \frac{1}{4} + ce^{4x}$$

Where we have used integration by parts, $u = -4x$, $dv = e^{-4x}dx$, $du = -4$, $v = (-1/4)e^{-4x}$
Finally then:

$$y^{-4} = x + \frac{1}{4} + ce^{4x}$$

or

$$y = \frac{1}{\sqrt[4]{x + 1/4 + ce^{4x}}}$$

(c) $y' + \frac{y}{x} - \sqrt{y} = 0; \quad y(4) = 1/9$ **[2]**
We can rewrite it in the usual form:

$$y' + \frac{y}{x} = y^{1/2} \quad n = \frac{1}{2}$$

We again make the substitution $u = y^{1-n} = y^{1/2}$ so:

$$\frac{du}{dx} = \frac{1}{2}y^{-1/2}\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = 2y^{1/2}\frac{du}{dx}$$

Subbing back in:

$$2y^{1/2}\frac{du}{dx} + \frac{y}{x} = y^{1/2}$$
$$\frac{du}{dx} + \frac{y^{1/2}}{2x} = \frac{1}{2}$$
$$\frac{du}{dx} + \frac{u}{2x} = \frac{1}{2}$$

Once again the integrating factor is $e^{\frac{1}{2}\ln(x)} = x^{1/2}$, so the equation becomes:

$$\frac{d}{dx}\left(ux^{1/2}\right) = \frac{1}{2}x^{1/2}$$
$$ux^{1/2} = \frac{1}{3}x^{3/2} + c$$
$$u = \frac{1}{3}x + cx^{-1/2}$$
$$y^{1/2} = \frac{1}{3}x + cx^{-1/2}$$

Finally $y(4) = \frac{1}{9}$ so

$$\frac{1}{3} = \frac{4}{3} + c\frac{1}{2} \implies c = -2.$$

Thus:

$$y^{1/2} = \frac{1}{3}x - 2x^{-1/2}$$
$$y = \left(\frac{1}{3}x - 2x^{-1/2}\right)^2$$
$$y = \frac{1}{9}x^2 - \frac{4}{9}x^{1/2} + 4x^{-1}$$

3. Solve the following differential equations.

(a) $\frac{dy}{dx} = \tan^2(x+y)$ **[2]**

$$u = x + y, \quad \frac{du}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\frac{du}{dx} - 1 = \tan^2 u$$

$$\frac{du}{dx} = \tan^2 u + 1 = \frac{1}{\cos^2 u} \quad (\text{recalling } \sin^2(u) + \cos^2(u) = 1)$$

$$\cos^2 u \ du = dx$$

$$\frac{1+\cos(2u)}{2} du = dx$$

$$(1+\cos(2u))du = 2dx$$

$$\int (1+\cos(2u))du = 2\int dx$$

$$u + \frac{\sin(2u)}{2} = 2x + c; \quad u = x + y$$

$$x + y + \frac{\sin(2(x+y))}{2} = 2x + c$$

$$y - x + \frac{\sin(2(x+y))}{2} = c$$

(b) $\frac{dy}{dx} = (x+y+1)^2$ **[2]**

$$u = x + y + 1 \quad \frac{du}{dx} = 1 + \frac{dy}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\frac{du}{dx} - 1 = u^2$$

$$\frac{du}{dx} = u^2 + 1$$

$$\frac{1}{u^2+1} du = dx$$

$$\int \frac{1}{u^2+1} du = \int 1dx$$

$$\tan^{-1}(u) = x + c$$

$$u = \tan(x+c); \quad u = x + y + 1$$

$$x + y + 1 = \tan(x+c)$$

$$y(x) = \tan(x+c) - x - 1$$

4. State the type of differential equation, or type of technique required to solve the following differential equations.
eg: separation of variables, linear first-order, substitution (Bernoulli, reduction to separation of variables)

(a) $y^2\frac{dy}{dx} = x$ - Separation of Variables **[2]**

(b) $\frac{dy}{dx} = \sin(x + y)$ - Reduction to Separation of Variables **[2]**

(c) $x\frac{dy}{dx} - y = x^2 \sin x$ - Linear First Order **[2]**

(d) $x\frac{dy}{dx} - (1 + x)y = xy^2$ - Bernoulli **[2]**