Tutorial 6 - Solutions

1. Given an LTI system with frequency response

$$H(\omega) = \frac{1}{1 + j\omega} \tag{1}$$

Find the response to the input signal $x(t) = |\sin(t)|$.

Solution: First note that $x(t)$ is a periodic signal with period $T_0 = \pi$. The fundamental frequency is thus $f_0 = 1/\pi$. The important step to find the output of the system is compute the Fourier series of $x(t)$. The nth coefficient of the FS of $x(t)$ is given by

$$\begin{aligned}
x_n &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi n f_0 t}dt = \frac{1}{T_0}\int_0^{T_0} \sin(t)e^{-j2\pi n f_0 t}dt \\
&= \frac{1}{2jT_0}\int_0^{T_0} \left(e^{jt} - e^{-jt}\right) e^{-j2\pi n f_0 t}dt = \frac{1}{2jT_0}\int_0^{T_0} \left(e^{j(1-2\pi n f_0)t} - e^{-j(1+2\pi n f_0)t}\right) dt \\
&= \frac{1}{2jT_0}\left[\frac{1}{j(1-2\pi n f_0)}\, e^{j(1-2\pi n f_0)t}\Big|_{t=0}^{t=T_0} - \frac{1}{-j(1+2\pi n f_0)}\, e^{-j(1+2\pi n f_0)t}\Big|_{t=0}^{t=T_0}\right] \\
&= \frac{1}{2jT_0}\left[\frac{1}{j(1-2\pi n f_0)}\left(e^{j(1-2\pi n f_0)T_0} - 1\right) + \frac{1}{j(1+2\pi n f_0)}\left(e^{-j(1+2\pi n f_0)T_0} - 1\right)\right] \\
&= -\frac{1}{jT_0}\left[\frac{1}{j(1-2\pi n f_0)} + \frac{1}{j(1+2\pi n f_0)}\right] = \frac{2}{\pi(1-4n^2)}
\end{aligned} \tag{2}$$

The system output is a period signal with the FS representation given by

$$y(t) = \sum_{n=-\infty}^{\infty} y_n e^{j2\pi n f_0 t} \tag{3}$$

where

$$y_n = x_n H(2\pi n f_o) = \frac{2}{\pi(1+j2n)(1-4n^2)} \tag{4}$$

2. Consider an LTI system having an impulse response $h(t) = u(t)$. Find the output signal of the system if the input signal is $x(t) = e^{-2t}u(t+2)$. Compare with the result of Q7 in Tutorial 5.

Solution: Let $Y(\omega)$ be the Fourier transform of the system output. By the convolution theorem we have

$$Y(\omega) = X(\omega)H(\omega) \tag{5}$$

where $X(\omega)$ and $H(\omega)$ are the Fourier transform of $x(t)$ and $h(t)$ respectively. Using the FT table we have

$$H(\omega) = \left(\pi\delta(\omega) + \frac{1}{j\omega}\right) \tag{6}$$

To find the FT of $x(t)$ we write $x(t) = e^4 e^{-2(t+2)} u(t+2)$. Using the FT table we know that the FT of $e^{-2t}u(t)$ is $\frac{1}{2+j\omega}$. By the time-shifting property of FT, the FT of $e^{-2(t+2)}u(t+2)$ is given by $\frac{1}{2+j\omega}e^{j2\omega}$. Thus, FT of $x(t)$ is given by

$$X(\omega) = e^4 e^{j2\omega} \frac{1}{2+j\omega} \tag{7}$$

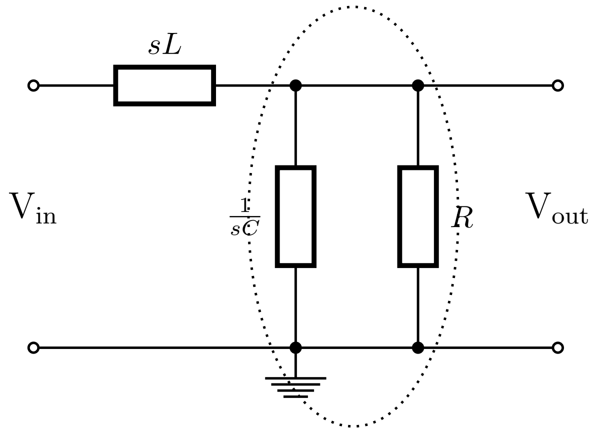
Substitute (6) and (7), we have

$$\begin{aligned} Y(\omega) &= X(\omega)H(\omega) = e^4(\pi\delta(\omega) + \frac{1}{j\omega})\frac{1}{2+j\omega}e^{j2\omega} \\ &= e^4\left(\frac{\pi}{2}\delta(\omega) + \frac{1}{j\omega(2+j\omega)}e^{j2\omega}\right) = \frac{e^4}{2}\left(\pi\delta(\omega) + \frac{1}{j\omega} - \frac{1}{(2+j\omega)}\right)e^{j2\omega} \end{aligned}$$

Taking the inverse FT of $Y(\omega)$, we have $y(t) = \frac{e^4}{2}(1-e^{-2(t+2)})u(t+2)$. We can see that this result is the same as what we derived in Q7 in Tutorial 5 using the convolution integral (i.e., in time domain).

3. (a) For low frequency, the inductor will act as a short circuit, meaning V_{in} is passed through the system.The opposite occurs for high frequencies. Thus the circuit is indeed a low-pass filter.

   (b) Equivalent circuit of the filter on the s-domain.



   (c) Equivalent impedance

$$Z_{RC} = \frac{R \times \frac{1}{sC}}{R + \frac{1}{sC}} = \frac{R}{sRC+1}$$

   (d) The transfer function is

$$\begin{aligned} H(s) &= \frac{Z_{RC}}{Z_{RC}+Z_L} = \frac{\frac{R}{sRC+1}}{\frac{R}{sRC+1}+sL} = \frac{R}{sL + s^2RLC + R} \\ &= \frac{1}{s\frac{L}{R} + s^2LC + 1} \end{aligned}$$

Replace $s = j\omega$, we obtained the frequency response as

$$H(\omega) = H(s)|_{s=j\omega} = \frac{1}{1 - \omega^2 LC + \frac{j\omega L}{R}}$$

(e) The magnitude frequency response of the circuit $|H(\omega)|$ is given by

$$\begin{aligned} |H(\omega)| &= \frac{1}{\sqrt{(1-\omega^2/\omega_0^2)^2 + \left(\frac{\omega L}{R}\right)^2}} \\ &= \frac{1}{\sqrt{1+(\omega/\omega_0)^4}} \end{aligned}$$

This is a second order low-pass filter.

4. Consider a discrete-time LTI system with the impulse response $h[n]$ given by

$$h[n] = \begin{cases} 1 & n=-1 \\ -1 & n=0 \\ 2 & n=1 \end{cases}$$

Determine its response $y[n]$ to the input

$$x[n] = \cos\left(\frac{\pi}{2}n\right)$$

Solution: $H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = e^{j\omega} - 1 + 2e^{-j\omega}$. The output of the system is given by

$$y[n] = |H(\pi/2)| \cos\left(\frac{\pi}{2}n + \angle H(\pi/2)\right) \tag{8}$$

$H(\pi/2) = 2e^{-j\pi/2}$. Thus, $y[n] = \sqrt{2}\cos\left(\frac{\pi}{2}n + \frac{5\pi}{4}\right)$