Q1

(a) $(f_1, f_2) = \int_0^{\pi} \cos x \cdot \sin^3 x \, dx = \int_0^{\pi} \sin^3 x \, d(\sin x)$

$= \frac{1}{4} \sin^4 x \Big|_0^{\pi} = 0$

(b) $(f_1, f_2) = \int_{\pi/4}^{5\pi/4} e^x \cdot \sin x \, dx = \frac{(\sin x - \cos x) e^x}{2} \Big|_{\pi/4}^{5\pi/4} = 0$

(c) $\int_0^P \sin(\frac{n\pi x}{P}) \sin(\frac{m\pi x}{P}) dx = -\frac{P}{n\pi} \cos(\frac{n\pi x}{P}) \sin(\frac{m\pi x}{P}) \Big|_0^P + \frac{m}{n} \int_0^P \cos(\frac{n\pi x}{P}) \cos(\frac{m\pi x}{P}) dx$

$\int_0^P \cos(\frac{n\pi x}{P}) \cos(\frac{m\pi x}{P}) dx = \frac{P}{n\pi} \sin(\frac{n\pi x}{P}) \cos(\frac{m\pi x}{P}) \Big|_0^P + \frac{m}{n} \int_0^P \sin(\frac{n\pi x}{P}) \sin(\frac{m\pi x}{P}) dx$

So $\int_0^P \sin(\frac{n\pi x}{P}) \sin(\frac{m\pi x}{P}) dx = \frac{m}{n} \cdot \frac{1}{1 - \frac{m^2}{n^2}} \cdot \frac{P}{n\pi} \sin(\frac{n\pi x}{P}) \cos(\frac{m\pi x}{P}) \Big|_0^P = 0$

Q2

(a) $\int_0^{\infty} f_1(x) f_2(x) \omega(x) dx = \int_0^{\infty} (\frac{1}{2}x^2 - 2x + 1) e^{-x} dx = \frac{1}{2}(-e^{-x} \cdot x^2) \Big|_0^{\infty} - (-e^{-x} \cdot x) \Big|_0^{\infty}$

$= [-e^{-x}(\frac{1}{2}x^2 - x)] \Big|_0^{\infty} = -\lim_{x \to \infty} \frac{\frac{1}{2}x^2 - x}{e^x}$

$= -\lim_{x \to \infty} \frac{x-1}{e^x} = -\lim_{x \to \infty} \frac{1}{e^x} = 0$

(b) $\int_0^{\infty} f_1(x) f_2(x) \omega(x) dx = \int_0^{\infty} (-x+1) e^{-x} dx = -(-e^{-x} \cdot x) \Big|_0^{\infty} = \lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0$

Q3

$a_0 = \frac{1}{2} \int_{-2}^{2} f(x) dx = \frac{1}{2} \left(\int_{-2}^{0} (-1 - \frac{x}{2}) dx + \int_0^2 (1 - \frac{x}{2}) dx \right) = 0$

$a_n = \frac{1}{2} [\int_{-2}^{0} (-1 - \frac{x}{2}) \cos \frac{n\pi}{2} x \, dx + \int_0^2 (1 - \frac{x}{2}) \cos \frac{n\pi}{2} dx]$

$= \frac{1}{2} [(-1 - \frac{x}{2}) \frac{\sin \frac{n\pi}{2} x}{\frac{n\pi}{2}} \Big|_{-2}^{0} + \frac{2}{n\pi} \int_{-2}^{0} \frac{1}{2} \sin \frac{n\pi}{2} x \, dx] +$

$\frac{1}{2} [(1 - \frac{x}{2}) \frac{\sin \frac{n\pi}{2} x}{\frac{n\pi}{2}} \Big|_0^2 + \frac{2}{n\pi} \int_0^2 \frac{1}{2} \sin \frac{n\pi}{2} x \, dx]$

$= \frac{1}{2n\pi} (-\frac{2}{n\pi} \cos \frac{n\pi}{2} x) \Big|_{-2}^{0} + \frac{1}{2n\pi} (-\frac{2}{n\pi} \cos \frac{n\pi}{2} x) \Big|_0^2$

$= 0$

$b_n = \frac{1}{2} [\int_{-2}^{0} (-1 - \frac{x}{2}) \sin \frac{n\pi}{2} x \, dx + \int_0^2 (1-x) \sin \frac{n\pi}{2} x \, dx]$

$= \frac{1}{2} [(1 + \frac{x}{2}) \frac{\cos \frac{n\pi}{2} x}{\frac{n\pi}{2}} \Big|_{-2}^{0} - \frac{2}{n\pi} \int_{-2}^{0} \frac{1}{2} \cos \frac{n\pi}{2} x \, dx] + \frac{1}{2} [(-1 + \frac{x}{2}) \frac{\cos \frac{n\pi}{2} x}{\frac{n\pi}{2}} \Big|_0^2 - \frac{2}{n\pi} \int_0^2 \frac{1}{2} \cos \frac{n\pi}{2} x \, dx]$

$= \frac{1}{2} \cdot \frac{2}{\frac{n\pi}{2}} = \frac{2}{n\pi}$

(b) $a_0 = \int_{-1}^{0} 0\,dx + \int_0^1 \frac{e^{-10x}-e^{-10}}{1-e^{-10}}\,dx = \frac{1}{1-e^{-10}}\left(-\frac{1}{10}e^{-10x} - e^{-10}x\right)\Big|_0^1 = \frac{1-11e^{-10}}{10-10e^{-10}}$

$$a_n = \int_0^1 \frac{e^{-10x}-e^{-10}}{1-e^{-10}}\cos n\lambda x\,dx = \frac{e^{-10x}-e^{-10}}{1-e^{-10}}\frac{\sin(n\lambda x)}{n\lambda}\Big|_0^1 + \frac{1}{n\lambda}\int_0^1 \frac{-10e^{-10x}}{1-e^{-10}}\sin(n\lambda x)\,dx$$

$$= \frac{10}{(e^{-10}-1)n\lambda}\int_0^1 e^{-10x}\sin(n\lambda x)\,dx$$

$$= \frac{n\lambda}{100+n^2\lambda^2}(1-e^{-10}\cos n\lambda)$$

$$b_n = \int_0^1 \frac{e^{-10x}-e^{-10}}{1-e^{-10}}\sin n\lambda x\,dx = -\frac{e^{-10x}-e^{-10}}{1-e^{-10}}\cdot\frac{\cos(n\lambda x)}{n\lambda}\Big|_0^1 - \frac{1}{n\lambda}\int_0^1 \frac{-10e^{-10x}}{1-e^{-10}}\cos n\lambda x\,dx$$

$$= \frac{1}{n\lambda} + \frac{10}{(1-e^{-10})n\lambda}\int_0^1 e^{-10x}\cos n\lambda x\,dx$$

$$= \frac{1}{n\lambda} + \frac{10-10e^{-10}\cos n\lambda}{100+n^2\lambda^2}$$

(c) $a_0 = \int_{-1}^{0}(x+1)^2\,dx + \int_0^1 (x-1)^2\,dx = \frac{(x+1)^3}{3}\Big|_{-1}^{0} + \frac{(x-1)^3}{3}\Big|_0^1 = \frac{1}{3} - 0 + (0 - -\frac{1}{3}) = \frac{2}{3}$

$$a_n = \int_{-1}^{0}(x+1)^2\cos n\lambda x\,dx = (x+1)^2\frac{\sin n\lambda x}{n\lambda}\Big|_{-1}^{0} - \frac{2}{n\lambda}\int_{-1}^{0}(x+1)\sin n\lambda x\,dx$$

$$= -\frac{2}{n\lambda}\left(-(x+1)\frac{\cos n\lambda x}{n\lambda}\Big|_{-1}^{0} + \frac{1}{n\lambda}\int_{-1}^{0}\cos n\lambda x\,dx\right)$$

$$= -\frac{2}{n\lambda}\left(-\frac{1}{n\lambda} + 0\right) = \frac{2}{n^2\lambda^2}$$

$$b_n = \int_0^1 (x-1)^2\sin n\lambda x\,dx = -(x-1)^2\frac{\cos n\lambda x}{n\lambda}\Big|_0^1 + \frac{2}{n\lambda}\int_0^1 (x-1)\cos n\lambda x\,dx$$

$$= \frac{1}{n\lambda} + \frac{2}{n\lambda}\left((x-1)\frac{\sin n\lambda x}{n\lambda}\Big|_0^1 - \frac{1}{n\lambda}\int_0^1 \sin n\lambda x\,dx\right)$$

$$= \frac{1}{n\lambda} + \frac{2}{n\lambda}\left(0 + \frac{1}{n\lambda}\frac{\cos n\lambda x}{n\lambda}\Big|_0^1\right)$$

$$= \frac{1}{n\lambda} + \frac{2}{n\lambda}\left(\frac{\cos n\lambda}{n^2\lambda^2} - \frac{1}{n^2\lambda^2}\right) = \frac{1}{n\lambda} + \frac{2\cos n\lambda - 2}{n^3\lambda^3}$$

(d) $a_0 = \frac{1}{2}\left(\int_{-2}^{0} 0\,dx + \int_0^1 x\,dx + \int_1^2 1\,dx\right) = \frac{1}{2}\left(\frac{1}{2}x^2\Big|_0^1 + x\Big|_1^2\right) = \frac{1}{2}\times\left(\frac{1}{2}+1\right) = \frac{3}{4}$

$a_n = \frac{1}{2}\left(\int_0^1 x\cdot\cos\frac{n\pi}{2}x\,dx + \int_1^2 \cos\frac{n\pi}{2}x\,dx\right) = \frac{1}{2}\left(x\cdot\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}}\Big|_0^1 - \frac{2}{n\pi}\int_0^1 \sin\frac{n\pi}{2}x\,dx + \frac{2}{n\pi}\sin\frac{n\pi}{2}x\Big|_1^2\right)$

$= \frac{1}{2}\left(\frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}x\Big|_0^1 - \frac{2}{n\pi}\sin\frac{n\pi}{2}\right) = \frac{2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{1}{n\pi}\sin\frac{n\pi}{2} - \frac{2}{n^2\pi^2}$

$b_n = \frac{1}{2}\left(\int_0^1 x\cdot\sin\frac{n\pi}{2}x\,dx + \int_1^2 \sin\frac{n\pi}{2}x\,dx\right) = \frac{1}{2}\left(-x\frac{\cos\frac{n\pi}{2}x}{\frac{n\pi}{2}}\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos\frac{n\pi}{2}x\,dx - \frac{2}{n\pi}\cos\frac{n\pi}{2}x\Big|_1^2\right)$

$= \frac{1}{2}\left(-\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{2}{n\pi}\cos n\pi + \frac{2}{n\pi}\cos\frac{n\pi}{2}\right)$

$= \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{1}{n\pi}\cos n\pi$