EE114
Intro to Systems & Control

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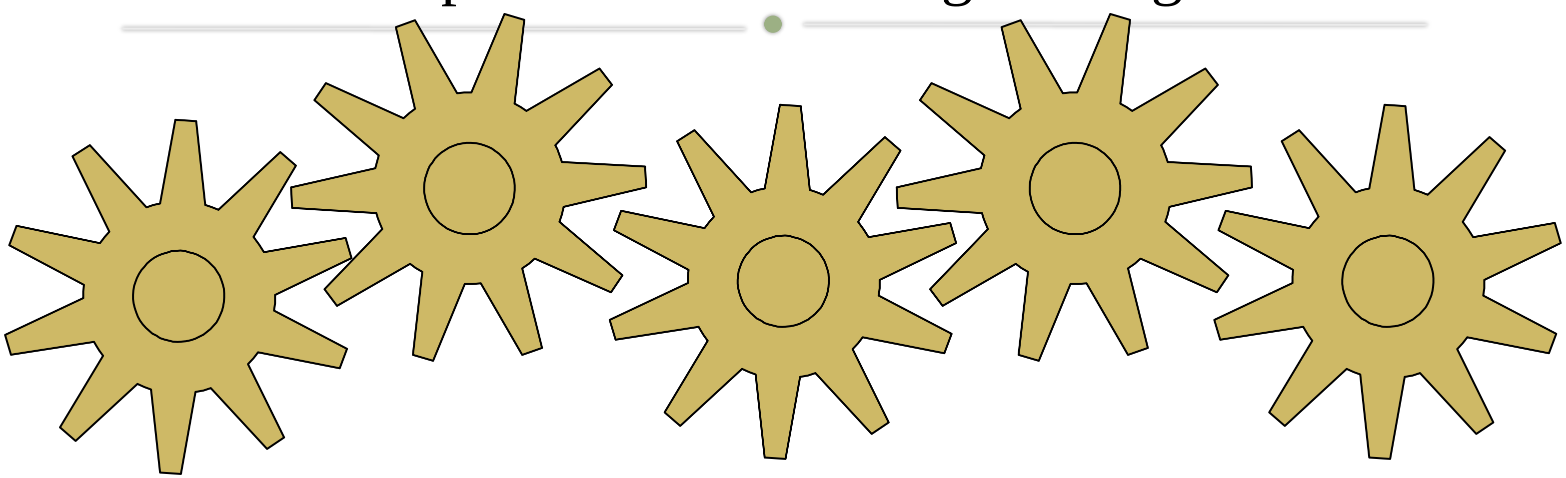
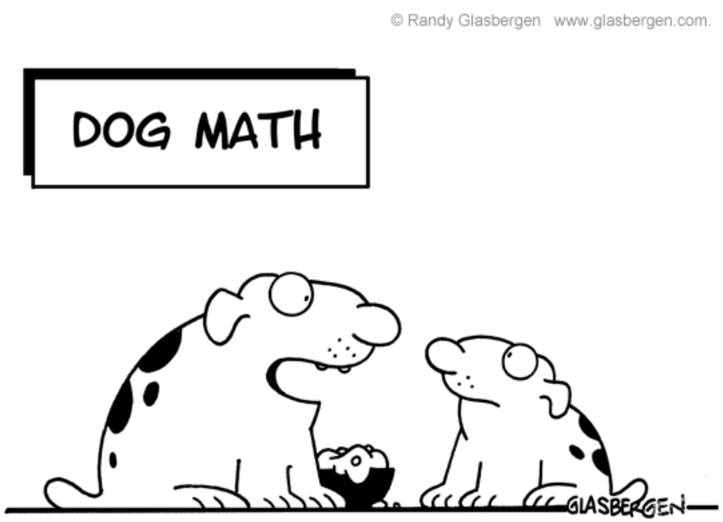


So far ...

- We've introduced the concept of control & feedback control ...
- We've started to model simple dynamical systems - an RC circuit and a mass-spring-damper (bicycle) thus far ...



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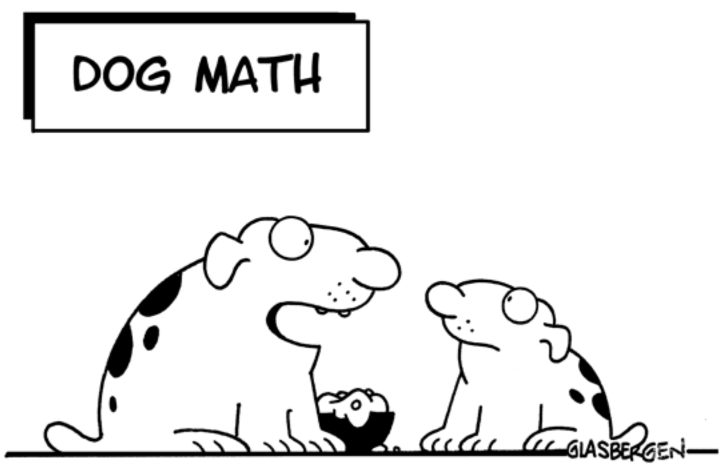


"If I have 3 bones and Mr. Jones takes away 2, how many fingers will he have left?"

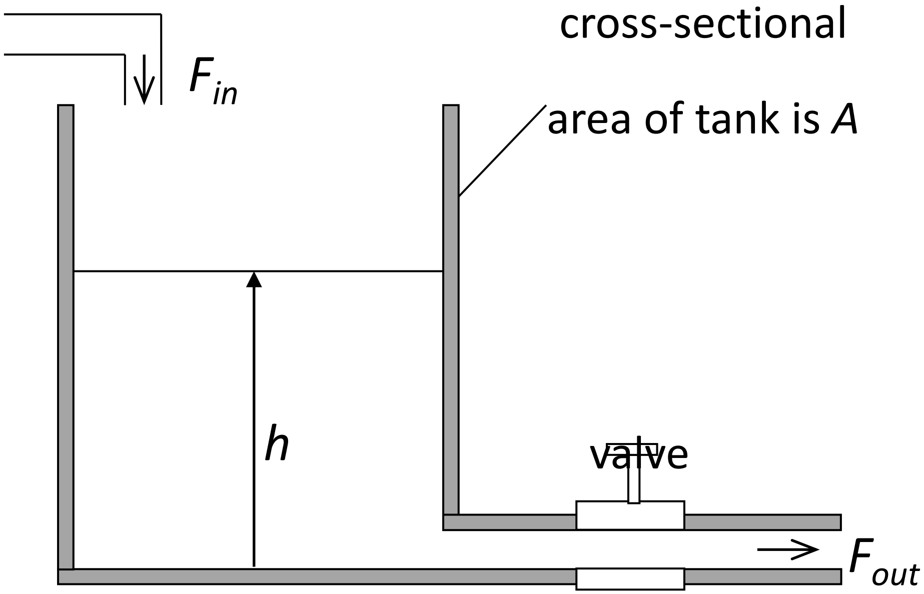
- **Today, we shall model a single tank system …**
- **… and introduce transfer function models …**

Modelling a Flow System

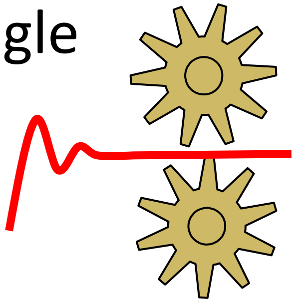
Modelling a Flow System

- Consider the following single tank flow system:

Actual system

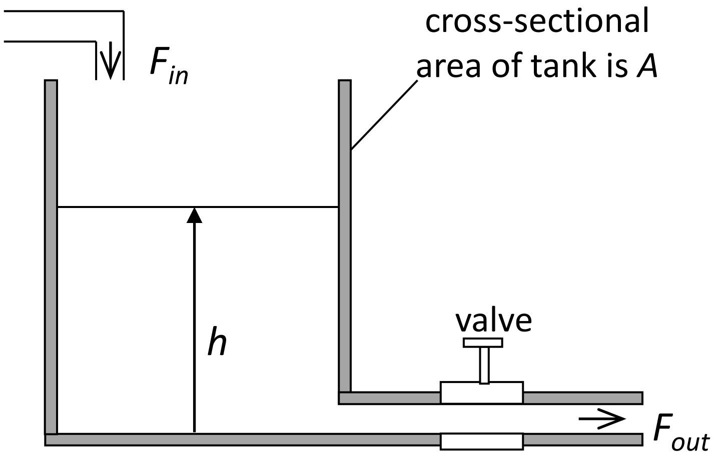


Physical model of a single tank system



Modelling a Flow System

- A tap valve controls the outlet flow f_{out}.
- The tank itself has a cross-sectional area A.



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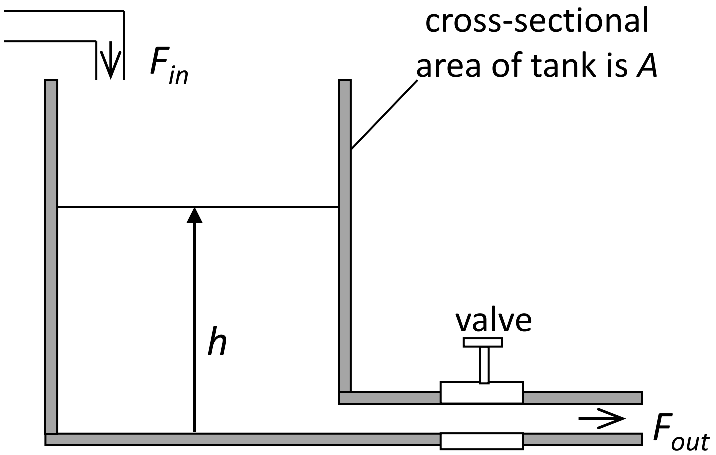

- Here, we write down a ***flow balance equation***, i.e. the change of water volume in the tank is the difference between the input flow and the output flow:

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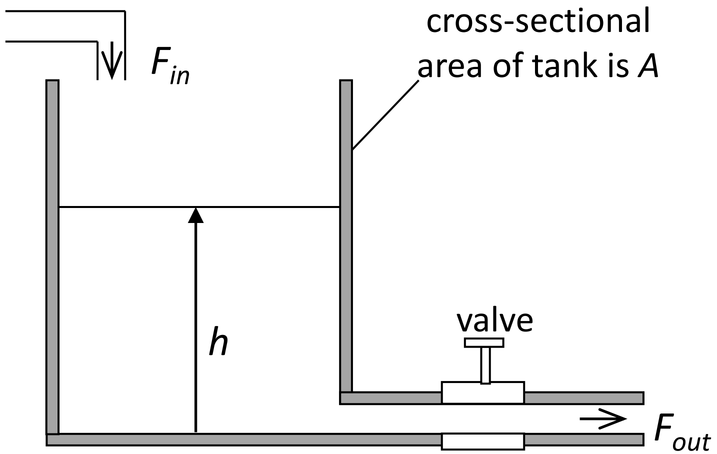

- Here, we write down a ***flow balance equation***, i.e. the change of water volume in the tank is the difference between the input flow and the output flow:

Change in Water Volume V = Flow-in - Flow-out

$$\Rightarrow \frac{dV}{dt} = F_{in} - F_{out}$$

Modelling a Flow System

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Modelling a Flow System

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- Since volume = area x height, i.e. $V = Ah$, and A is a constant value, then:

$$\frac{dV}{dt} = A\frac{dh}{dt} = F_{in} - F_{out}$$

Modelling a Flow System

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- The output flow rate F_{out} is determined by the pressure exerted by the water in the tank and, hence, we can state that the ***output flow is directly proportional to the height of the water*** (assuming laminar flow):

$$F_{out} \propto h$$

Modelling a Flow System

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$$F_{out} \propto h \quad \Rightarrow \quad F_{out} = kh$$

Modelling a Flow System

$$\frac{dV}{dt} = F_{in} - F_{out}$$

- Since volume = area x height,

Here k is a constant of proportionality that allows for several factors including the cross-sectional area of outflow pipe, frictional forces within the pipe, density of the liquid, etc. This forms part of our assumptions in the final model.

the ***output flow is directly proportional to the height of the water*** (assuming laminar flow):

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Modelling a Flow System

- Hence, our flow balance equation, and system model (relating height to input flow) is given by the first order differential equation:

$$A\frac{dh}{dt} = F_{in} - kh$$

Laminar v Turbulent Flow

Laminar v Turbulent Flow

- Water flow in a pipe can be laminar or turbulent.

- **Laminar** refers to a uniform directional flow while **turbulent**, as the name suggests, refers to a more chaotic type flow, as indicated in the following sketches:

Laminar v Turbulent Flow

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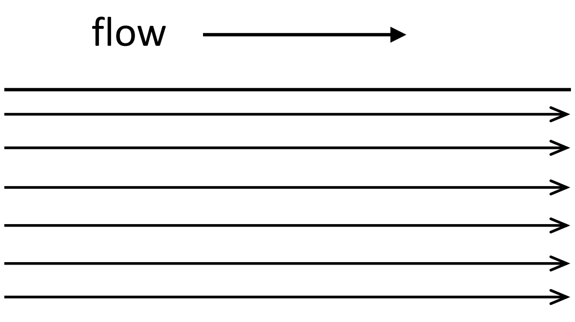
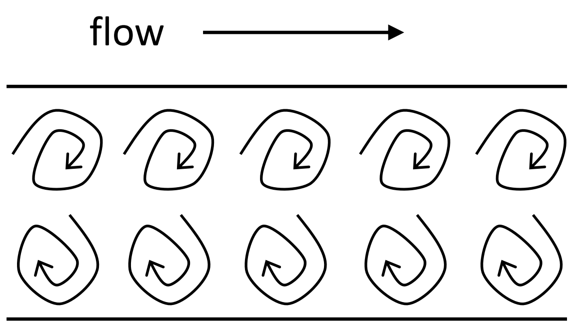
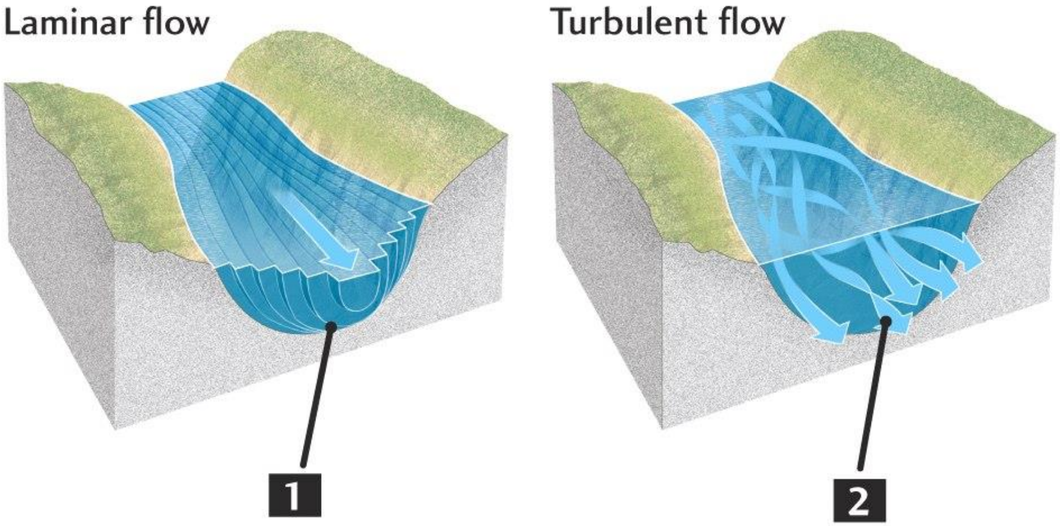


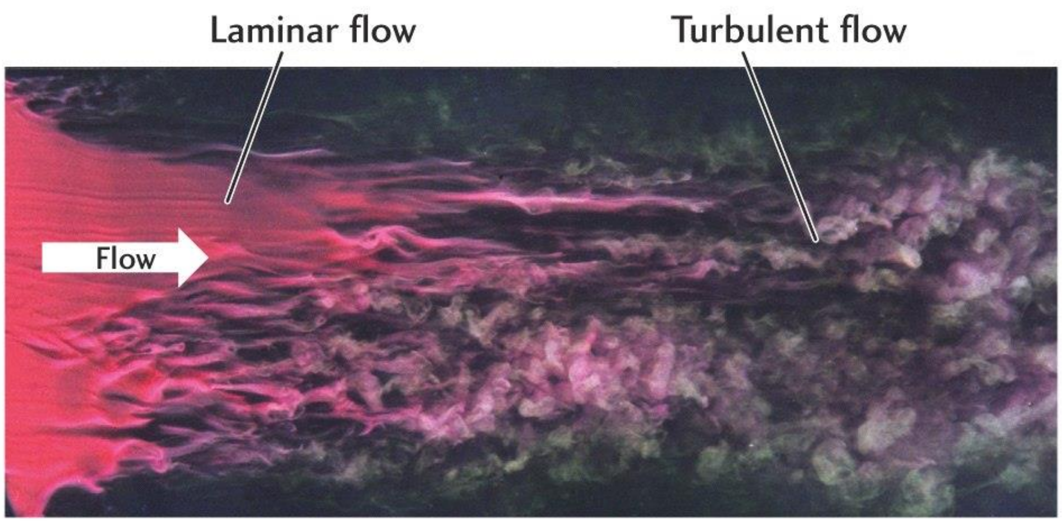
laminar flow



turbulent flow

Laminar v Turbulent Flow





Laminar v Turbulent Flow

- The relationship between the output flow and the height of water in the tank is also dependent on the type of flow involved, as follows:

Laminar v Turbulent Flow

- The relationship between the output flow and the height of water in the tank is also dependent on the type of flow involved, as follows:

$$F_{out} = kh$$

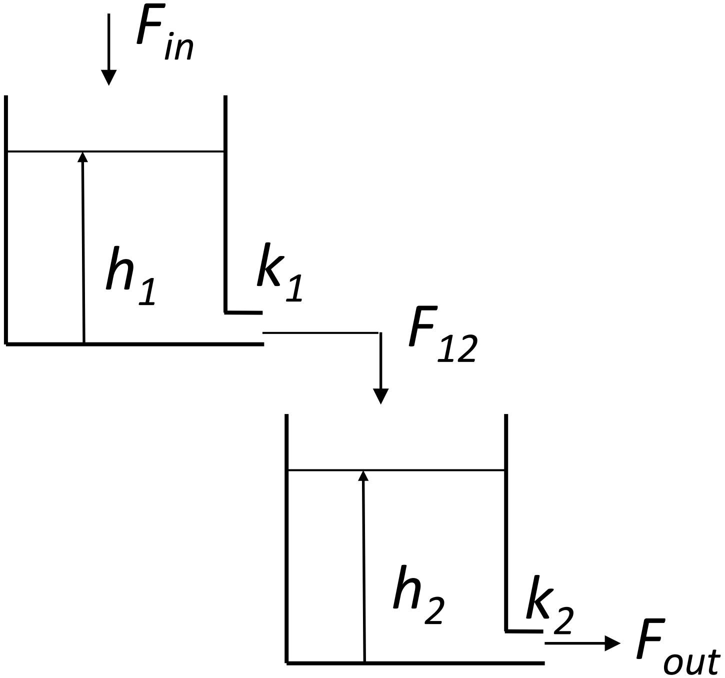
for **laminar** flow (*this is a* ***linear*** *relationship*)

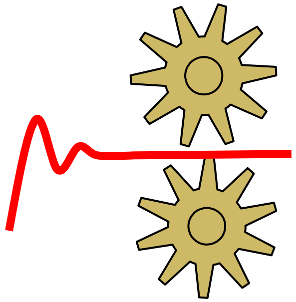
$$F_{out} = k\sqrt{h}$$

for **turbulent** flow (*this is a* ***nonlinear*** *relationship*)

Modelling a Flow System

- ***Ex 3.8 Determine a mathematical model for the dual non-interacting water tank system, whose physical model is shown.***





Modelling a Flow System

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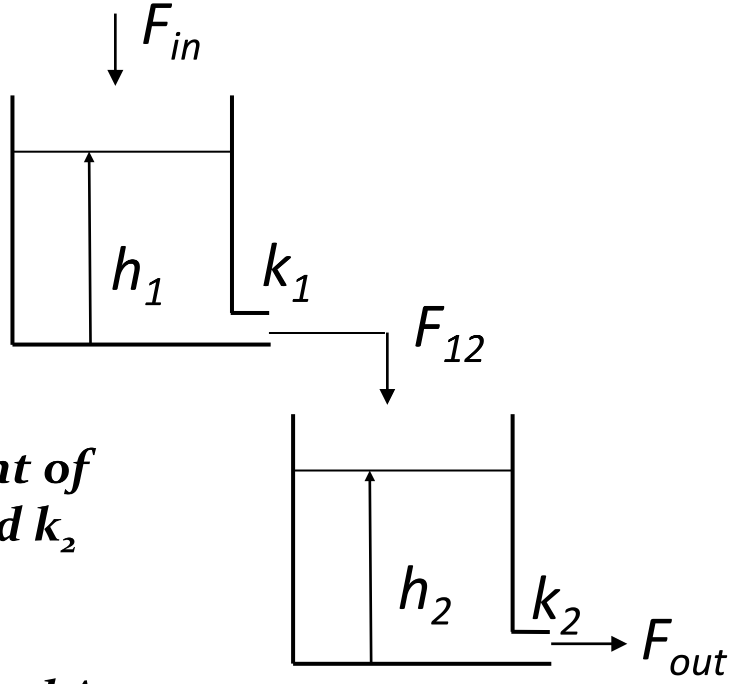


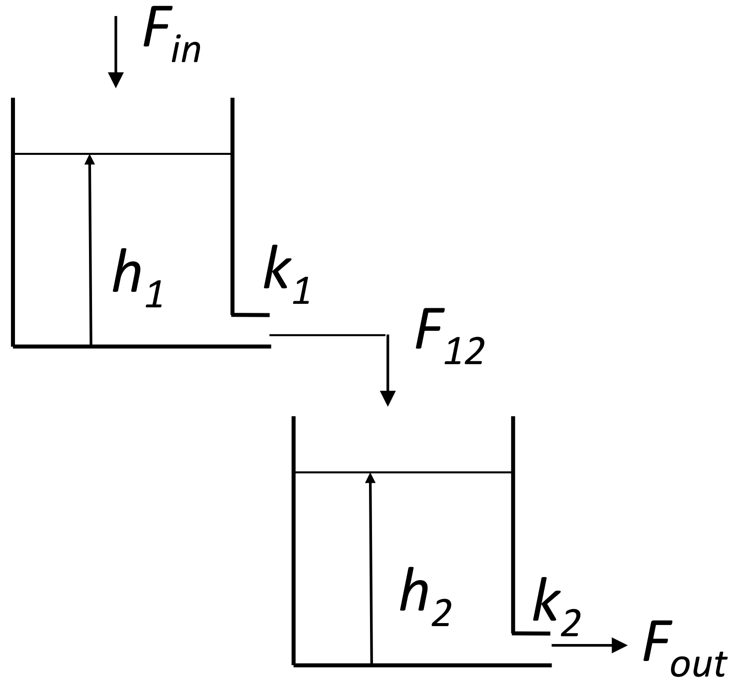
Assume laminar flow conditions and the constant of proportionality between flow and height is k_1 and k_2 for the respective tanks, as indicated.

Also, the cross-sectional area of each tank is A_1 and A_2 respectively.

Modelling a Flow System

Solution …

From the previous section, we can easily write out the system model for each tank as follows:



Modelling a Flow System

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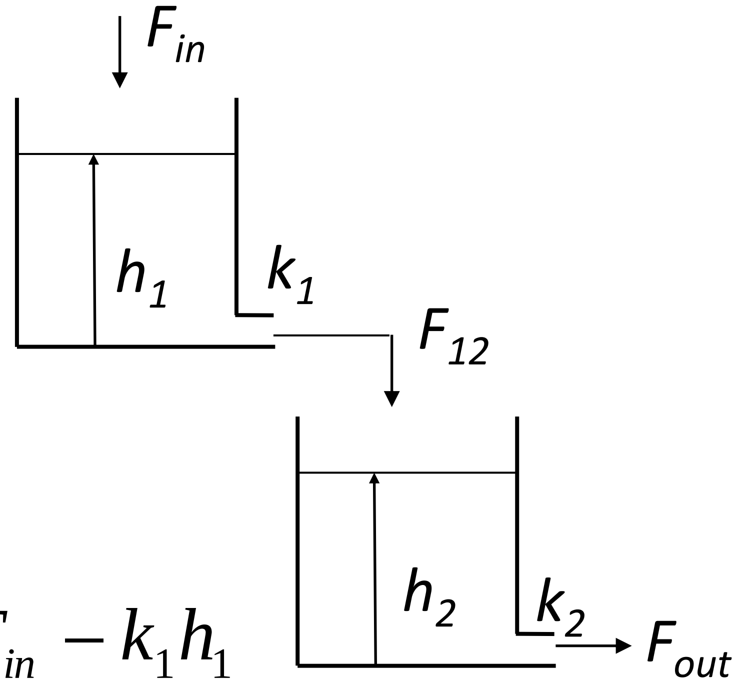

$$A_1 \frac{dh_1}{dt} = F_{in} - F_{12} \qquad \Rightarrow A_1 \frac{dh_1}{dt} = F_{in} - k_1 h_1$$

Modelling a Flow System

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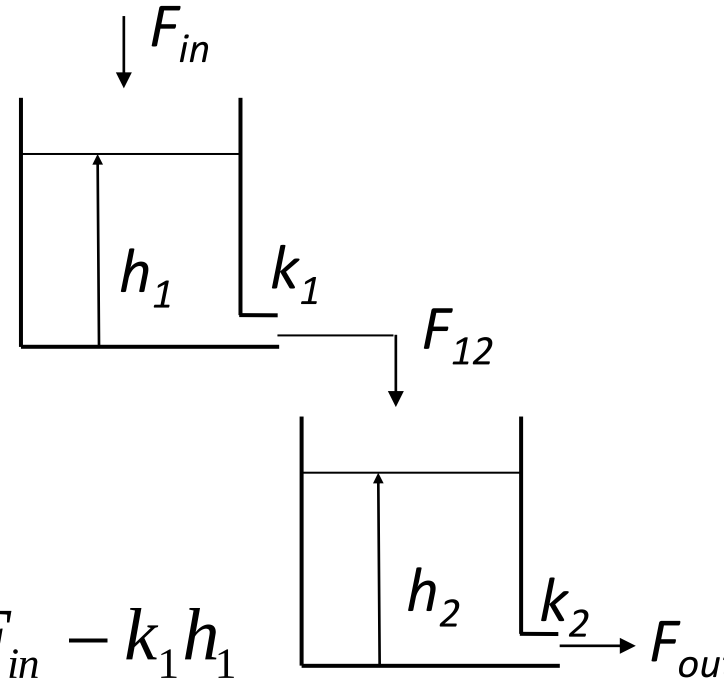
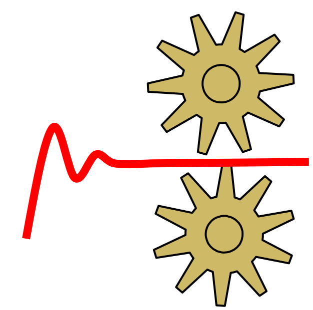


$$A_1 \frac{dh_1}{dt} = F_{in} - F_{12} \qquad \Rightarrow A_1 \frac{dh_1}{dt} = F_{in} - k_1 h_1$$

$$A_2 \frac{dh_2}{dt} = F_{12} - F_{out} \qquad \Rightarrow A_2 \frac{dh_2}{dt} = k_1 h_1 - k_2 h_2$$



Modelling a Flow System

- Note – we will revisit these systems in the next section of the notes when we examine transfer function representation.

- We will also simulate, and hence analyse, these systems when we use Simulink and Matlab.

Review - Modelling Physical Systems

Review - Modelling Physical Systems

- There is a common generalized approach to the modelling of physical systems.
- The first step is to select the fundamental variables whose values at any instance in time contain all the information about the system (we refer to these as state variables).
- Three such state variables include **mass**, **energy** and **momentum**.
- Most models are based on the **conservation** of these quantities.
- Often, the fundamental variables are not conveniently measured.

Review - Modelling Physical Systems

- Hence, we use characterising dependent variables instead, such as density, temperature, pressure, flow rate, etc.
- The values of all the characterising variables at any instance in time define the state of the system.

Laplace Transforms & Transfer Functions

Laplace Transforms & Transfer Functions

- In the previous section, we looked at the mathematical modelling of a range of both static and dynamic systems.

- In each case, the system model was represented by a linear ordinary differential equation (ODE), which was either first or second order in the examples provided.

- Here, the order is the highest degree of the differential equation.

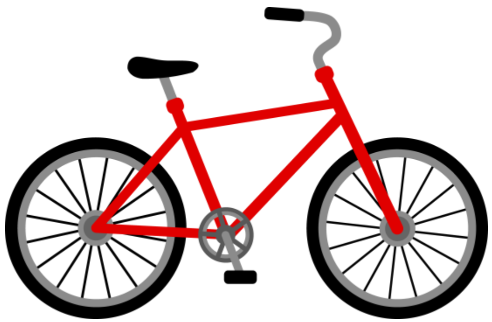
- While ODEs describe the input-output relation of the system, they are not a satisfactory representation from a system's perspective.

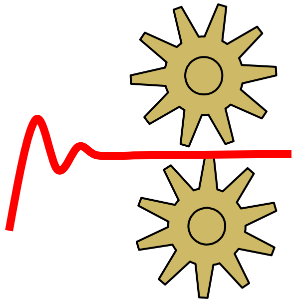
Laplace Transforms & Transfer Functions

- For example, consider the following mass-spring-damper system equation:

$$M\frac{d^2x(t)}{dt^2}+B\frac{dx(t)}{dt}+Kx(t)=u(t)$$

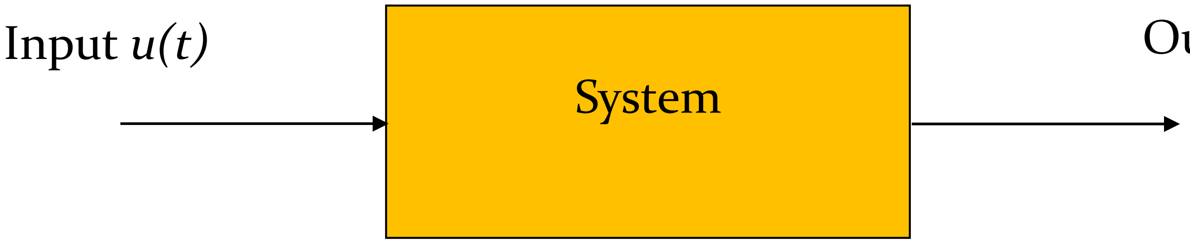
- The system parameters (M, B and K) and the output $x(t)$, for this particular system, appear throughout the equation.

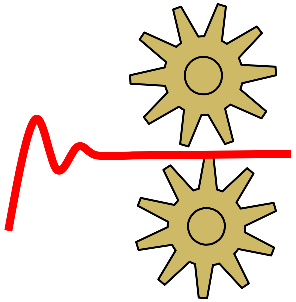




Laplace Transforms & Transfer Functions

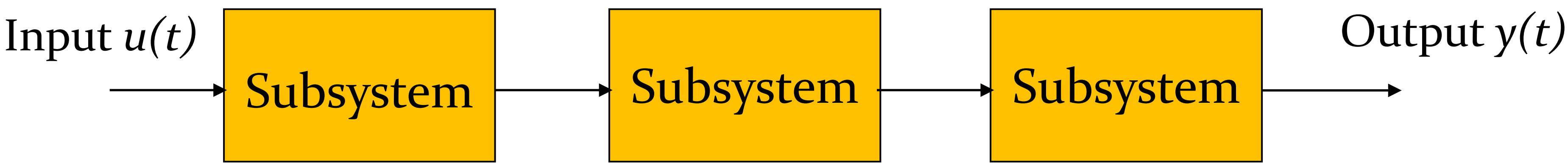
- Ideally, it is more desirable to represent the system so that the input, output and system are distinct parts, as illustrated in the block diagram below:

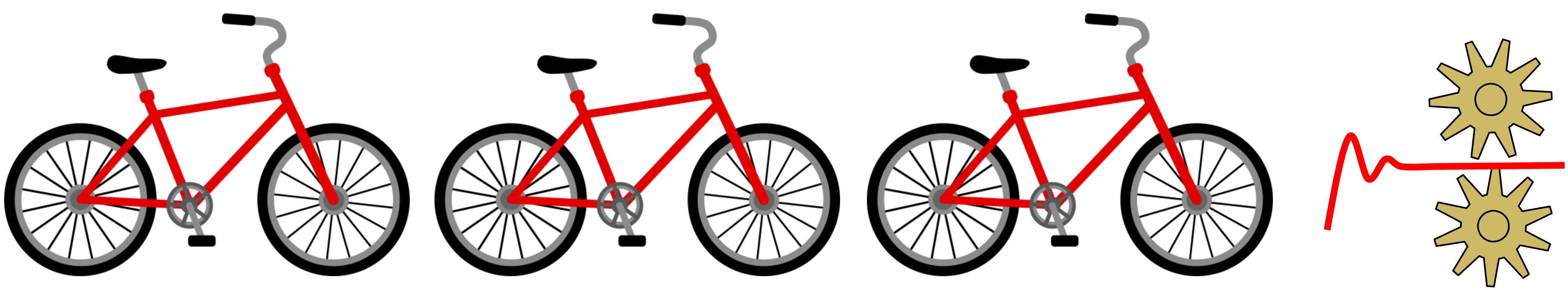




Laplace Transforms & Transfer Functions

- Furthermore, it would also be desirable to conveniently represent the interconnection of several subsystems, where applicable, such as the cascaded system illustrated below:





Laplace Transforms & Transfer Functions

- The differential equation does not provide such convenience. However, an alternative model representation, known as the transfer function, does!

- The **transfer function** is a compact representation of the relationship between the input and an output for a linear system. **In this form, we no longer work with differentials but rather with an algebraic expression.**

Laplace Transforms & Transfer Functions

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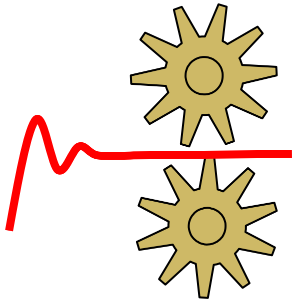
- The **transfer function** is a compact representation of the relationship between the input and an output for a linear system. **In this form, we no longer work with differentials but rather with an algebraic expression.**

- Combined with block diagrams, transfer functions offer a powerful means for dealing with complex linear systems, as we will see in section 5.

- Here, we will use the **Laplace Transform to convert a differential equation to transform function form**.



Laplace Transforms (overview)

- Laplace Transforms will be covered in significant detail in the mathematics modules (EE112 and EE206).
- Here, we will give a very brief overview of the key features of Laplace Transforms before we examine how they can be applied to differential equations (in section 4.3).
- It is the latter part that is essential to obtaining the transfer function representation.

Pierre-Simon Laplace

Laplace Transforms (overview)

- The **Laplace Transform is defined as**:

$$L\left[f(t)\right] = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

where s is a *complex operator* given by $s = \sigma + j\omega$.

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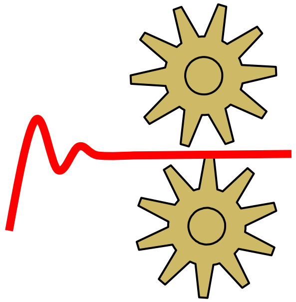
where s is a *complex operator* given by $s = \sigma + j\omega$.

- This is a special transformation that **transforms a differential equation into an expression with no derivatives.**

- Note, the transformed function is a **function of *s* only**. The time variable t does not exist in this form.

- *Also note the convention of using* ***a capital letter for the Laplace transform!***



Laplace Transforms of Common Functions

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$$= \left(-\frac{1}{s}e^{-st}\right)_0^\infty$$

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$$= \frac{1}{s-a}$$

Laplace Transforms of Common Functions

- $f(t) = \sin(kt)$

Laplace Transforms of Common Functions

- ***f(t) = sin(kt)***

To make the transformation easier to compute, we note that:

$$e^{j\theta} = \cos\theta + j\sin\theta \quad \Rightarrow \sin\theta \equiv \mathrm{Im}\left\{e^{j\theta}\right\}$$

Laplace Transforms of Common Functions

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To make the transformation easier to compute, we note that:

$$e^{j\theta} = \cos\theta + j\sin\theta \quad \Rightarrow \sin\theta \equiv \operatorname{Im}\left\{e^{j\theta}\right\}$$

Hence, we can state that:

$$\sin(kt) \equiv \operatorname{Im}\left\{e^{jkt}\right\}$$

Laplace Transforms of Common Functions

Thus:

$$\boldsymbol{F(s)} = L\left[f(t)\right] = L\left[\mathrm{Im}\left\{e^{jkt}\right\}\right]$$

Laplace Transforms of Common Functions

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Laplace Transforms of Common Functions

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Laplace Transforms of Common Functions

$$= \mathrm{Im}\left\{\frac{1}{s - jk}\right\} \qquad = \mathrm{Im}\left\{\frac{1}{s - jk} \cdot \frac{s + jk}{s + jk}\right\} \qquad = \mathrm{Im}\left\{\frac{s + jk}{s^2 + k^2}\right\}$$

Laplace Transforms of Common Functions

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$$= \mathrm{Im}\left\{\frac{s}{s^2+k^2}+j\frac{k}{s^2+k^2}\right\}$$

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$$= \operatorname{Im}\left\{\frac{s}{s^2 + k^2} + j\frac{k}{s^2 + k^2}\right\}$$

$$= \boldsymbol{\frac{k}{s^2 + k^2}}$$

Laplace Transforms of Common Functions

The good news! We don't need to remember the above (and other) examples, as the more common functions and their transforms are typically available in look up tables.

Such a table will be available in exam situations if needed.

Laplace Transforms of Common Functions

Function name	Time domain function $f(t)$	Laplace transform $F(s) = L\{f(t)\}$
Constant	a	$\frac{a}{s}$
Linear	t	$\frac{1}{s^2}$
Power	t^n	$\frac{n!}{s^{n+1}}$
Exponent	e^{at}	$\frac{1}{s-a}$
Sine	$\sin at$	$\frac{a}{s^2+a^2}$
Cosine	$\cos at$	$\frac{s}{s^2+a^2}$
Hyperbolic sine	$\sinh at$	$\frac{a}{s^2-a^2}$