Engineering Mathematics 1 (Fall 2021)

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Students should be able to (after learning)

- Add, subtract and multiply complex numbers
- Convert complex numbers between Cartesian and polar forms
- Differentiate all commonly occurring functions including polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of a derivative, namely the derivative as a tangent and the derivative as a rate of change
- Integrate certain standard functions, constructed from polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of integration, namely the integral as the inverse of the derivative and the integral as the area under a curve
- Apply Taylor series to numerically approximate functions
- Apply Simpson's rule to numerically evaluate integrals
- Solve simple first and second order ordinary differential equations
- Apply and select the appropriate mathematical techniques to solve a variety of associated engineering problems

Lecture 7: Series-Part 2

3. Series of power of natural number:

Sum of natural numbers $1+2+3+\cdots+n=\sum_{r=1}^{n} r=\frac{n(n+1)}{2}$

Proof: $1+2+3+\cdots+n=\sum_{k=1}^{n}k$ ①

$n+n-1+\cdots+2+1=\sum_{k=1}^{n}k$ ②

①+②: $n(n+1)=2\sum_{k=1}^{n}k \quad \therefore \sum_{k=1}^{n}k=\frac{1}{2}n(n+1)$

Sum of squares $1^2+2^2+3^2+\cdots+n^2=\sum_{r=1}^{n} r^2=\frac{n(n+1)(2n+1)}{6}$

Proof: $\because (n+1)^3=n^3+3n^2+3n+1$

$\therefore (n+1)^3-n^3=3n^2+3n+1$

$n^3-(n-1)^3=3(n-1)^2+3(n-1)+1$

$(n-1)^3-(n-2)^3=3(n-2)^2+3(n-2)+1$

$\vdots$

$2^3-1^3=3\cdot1^2+3\cdot1+1$

Taking sum: $(n+1)^3-1^3=3[n^2+(n-1)^2+\cdots+1^2]+3[n+(n-1)+\cdots+1]+n$

$\therefore n^3+3n^2+3n=3\sum_{r=1}^{n}r^2+3\cdot\frac{n(n+1)}{2}+n$

$\therefore n^3+3n^2+2n-\frac{3}{2}n(n+1)=3\sum_{r=1}^{n}r^2$

$\therefore \sum_{r=1}^{n}r^2=\frac{1}{3}\left[n^3+\frac{3}{2}n^2+\frac{n}{2}\right]$

$=\frac{n(n+1)(2n+1)}{6}$

Sum of cubes $1^3+2^3+3^3+\cdots+n^3=\sum_{r=1}^{n} r^3=\left(\frac{n(n+1)}{2}\right)^2$

Ex1:

$$\sum_{n=1}^{20}(3n-2)$$

Sol: $\sum_{n=1}^{20}(3n-2)=\sum_{n=1}^{20}3n-\sum_{n=1}^{20}2$

$=3\sum_{n=1}^{20}n-40=3\times\frac{1}{2}\times20(20+1)-40$

$=630-40=590.$

$$\sum_{n=1}^{20}(5n^2-1)$$

Sol: $\sum_{n=1}^{20}(5n^2-1)=\sum_{n=1}^{20}5n^2-\sum_{n=1}^{20}1=5\sum_{n=1}^{20}n^2-20=5\times\frac{1}{6}\times20\times21\times41-20$

$=14350-20=14330.$

4. Infinite series and limiting values:

$$\lim_{n\to\infty}\sum_{r=0}^{n-1}(a+rd) = \lim_{n\to\infty}\left[na+\frac{n(n-1)d}{2}\right] = \infty \text{ or } -\infty, \quad \text{depends on } a \text{ and } d.$$

partial sum: $S_n = \sum_{r=0}^{n-1}(a+rd) = na + \frac{n(n-1)d}{2}$ is a quadratic form of n, the highest order is 2, d is the coefficient of n, so the sign of d is important. $\lim_{n\to\infty} S_n = \begin{cases} +\infty, & d>0. \\ -\infty, & d<0. \end{cases}$

$$\lim_{n\to\infty}\sum_{k=0}^{n-1} ar^k = \lim_{n\to\infty}\frac{a(1-r^n)}{1-r}$$

Denote partial sum $S_n = \sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r}$

$= \frac{a}{1-r}$, as $|r| < 1$;

$\therefore \sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{a(1-r^n)}{1-r} = \begin{cases} \frac{a}{1-r}, & |r|<1 \\ \pm\infty, & r\geq 1 \\ \text{None}, & r\leq -1 \end{cases}$

$= \pm\infty$ as $r = 1$;

$= \pm\infty$ as $r > 1$;

NOT confirmed, as $r < -1$;

Ex1:

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$$

Sol: The first term is $a=1$, the common ratio is $r=\frac{1}{3}$,

$\therefore S_n = \sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r} = \frac{1\cdot(1-\frac{1}{3^n})}{1-\frac{1}{3}} = \frac{3}{2}(1-\frac{1}{3^n})$

$\because \lim_{n\to\infty}\frac{1}{3^n} = 0 \quad \therefore \lim_{n\to\infty} S_n = \frac{3}{2} \quad \therefore 1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots = \frac{3}{2}.$

Ex2:

$$1 + 3 + 9 + 27 + \cdots$$

Sol: The first term is $a=1$, the common ratio $r=3$.

$\therefore S_n = \frac{1(1-3^n)}{1-3} = -\frac{1}{2}(1-3^n)$

$\because \lim_{n\to\infty} 3^n = +\infty \quad \therefore \lim_{n\to\infty} S_n = +\infty \quad \therefore 1+3+9+27+\cdots = +\infty.$

Here $n = 1, 2, 3, \ldots$ Choose the highest order to divide $= HO$

Ex3:

$$\lim_{n\to\infty} \frac{5n+3}{2n-7} \quad \begin{matrix} 5n+3 \to \infty \\ 2n-7 \to \infty \end{matrix}$$

Sol: $\lim\limits_{n\to\infty} \frac{5n+3}{2n-7} = \lim\limits_{n\to\infty} \frac{5+3/n}{2-7/n} = \frac{\lim\limits_{n\to\infty}(5+3/n)}{\lim\limits_{n\to\infty}(2-7/n)} = \frac{5}{2}.$

$HO = 1$

$$\lim_{n\to\infty} \frac{2n^2+4n+3}{5n^2-6n+1} \qquad HO = 2$$

Sol: $\lim\limits_{n\to\infty} \frac{(2n^2+4n+3)/n^2}{(5n^2-6n+1)/n^2} = \frac{\lim\limits_{n\to\infty}(2+4/n+3/n^2)}{\lim\limits_{n\to\infty}(5-6/n+1/n^2)} = \frac{2}{5},$

due to $\lim\limits_{n\to\infty} 1/n = \lim\limits_{n\to\infty} 1/n^2 = 0.$

$$\lim_{n\to\infty} \frac{n+3}{2n^3+3n-4} \qquad HO = 3$$

Sol: $\lim\limits_{n\to\infty} \frac{(n+3)/n^3}{(2n^3+3n-4)/n^3} = \frac{\lim\limits_{n\to\infty}(1/n^2+3/n^3)}{\lim\limits_{n\to\infty}(2+3/n^2-4/n^3)} = \frac{0}{2} = 0,$

due to $\lim\limits_{n\to\infty} 1/n^2 = \lim\limits_{n\to\infty} 1/n^3 = 0.$

Wrong sol: $\lim\limits_{n\to\infty} \frac{(n+3)/n}{(2n^3+3n-4)/n^3} = \frac{\lim\limits_{n\to\infty}(1+3/n)}{\lim\limits_{n\to\infty}(2+3/n^2-4/n^3)} = \frac{1}{2}.$

HO must be the same scale for the whole fraction!