Tutorial Sheet 4 - Linearisation

Q1 Let f_k denote the population of foxes and r_k denote the population of rabbits on an island in year k. The rabbit food source is grass while the fox food source is rabbits. The fox-rabbit predator-prey interaction is proportional to the product of both populations, i.e. $f_k r_k$. The interaction has a positive effect on the fox population and a negative effect on the rabbit population. A model which describes the growth of the two species is as follows:

$$f_{k+1} = (\alpha_F + \beta_F f_k r_k) f_k$$
$$r_{k+1} = (\alpha_R - \beta_R f_k r_k - \lambda_R r_k) r_k$$

where $\alpha_F < 1, \alpha_R > 1$. Parameters α_F and α_R represent growth factors, β_F and β_R represent predator-prey impact factors and λ_R is a limited food supply (i.e. grass) factor.

(i) Determine the equilibrium points of this system.

(ii) For each equilibrium point, find the equivalent linear state-space model.

Q2 Determine the equilibrium points and corresponding linear state-space model for each of the following dynamical systems:

(i) $\dot{x} = x^2 - 2x - 8$

(ii) $\ddot{x} = \dot{x}^2 x - 2x - 8u^3$

(iii) $x_{k+1} = x_k^{\ 2} - 2x_k - 8$

(iv) $x_{k+2} = x_{k+1}^2 x_k - 2x_k - 2u_k$

(v) $\dot{x}_1 = -2x_1 x_2 + x_2^{\ 2} u$, $\dot{x}_2 = -4x_1^{\ 2} x_2 + 2u$

(vi) $\dot{x}_1 = -2x_1 x_2 + x_2 u^2$, $\dot{x}_2 = -4x_1^{\ 2} + x_2 + 2u$

(vii) $\dot{x} = x^2 + x + 1$

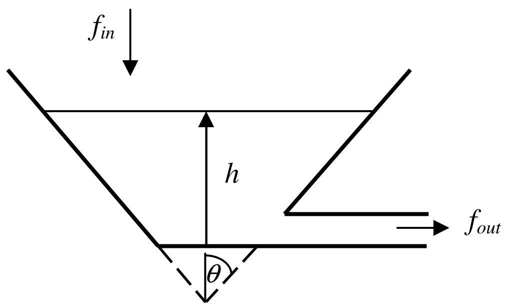
(viii) $x_{k+1} = x_k - \sin(x_{k-1}) + u_k$

Q3 The dynamical equation for a conical tank system, as shown below, has the form:

$$\frac{dh}{dt} = -\frac{k}{\pi \tan^2\theta} \cdot \frac{1}{h} + \frac{1}{\pi \tan^2\theta} \cdot \frac{f_{in}}{h^2}$$

(i) Determine the equilibrium point for this system as a function of the inflow f_{in}. Explain why this is not a function of θ.

(ii) Determine the linearised model about the operating points $f_{in} = 1$ and $f_{in} = 2$ when the tank parameters are $k = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$.



Conical tank system

Q4 A coupled dynamical system is defined by the equations:

$$\dot{x}_1 = x_1 \sin(x_1) + x_2$$
$$\dot{x}_2 = x_2 + u$$

(i) Determine the equilibrium points for the operating points $u = 0$ and $u = 2$ in the interval $|x_1| < 2\pi$.

(ii) Calculate the state-matrix for the linearised model about each of these operating points.