EE114
Intro to Systems & Control

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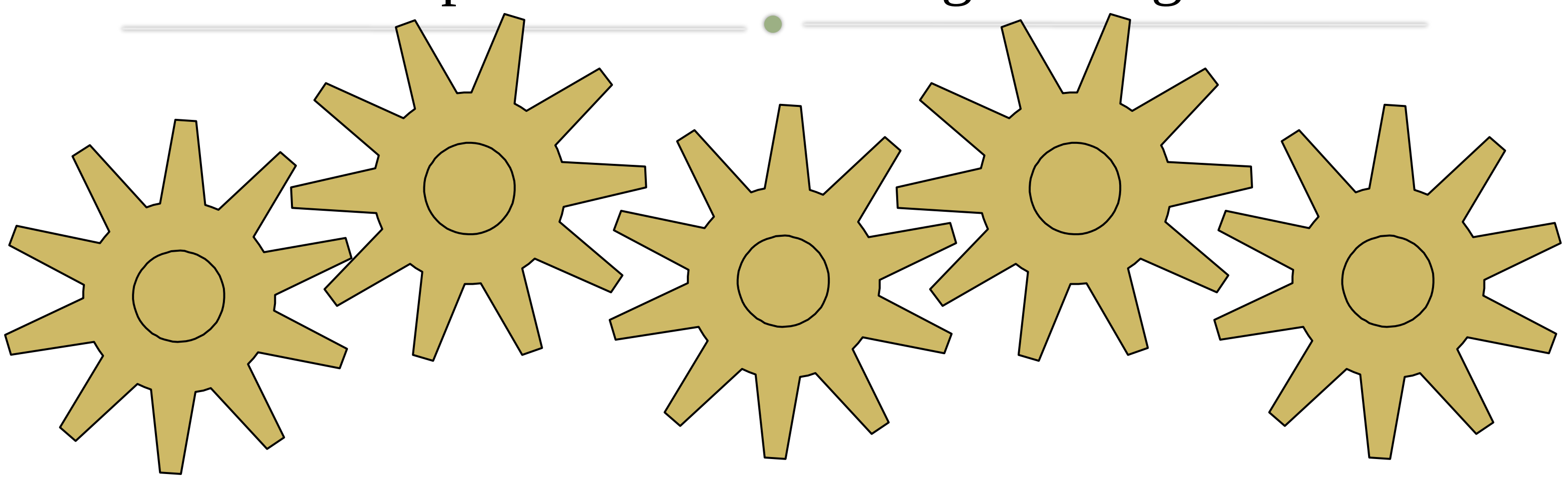


So far …

- We've modelled a range of systems – obtained differential equations & transfer function models
- Used block diagram algebra to reduce complicated systems to a single transfer function format …

"Look, Bernie, all I'm saying is I think you're riding the new guy pretty hard."

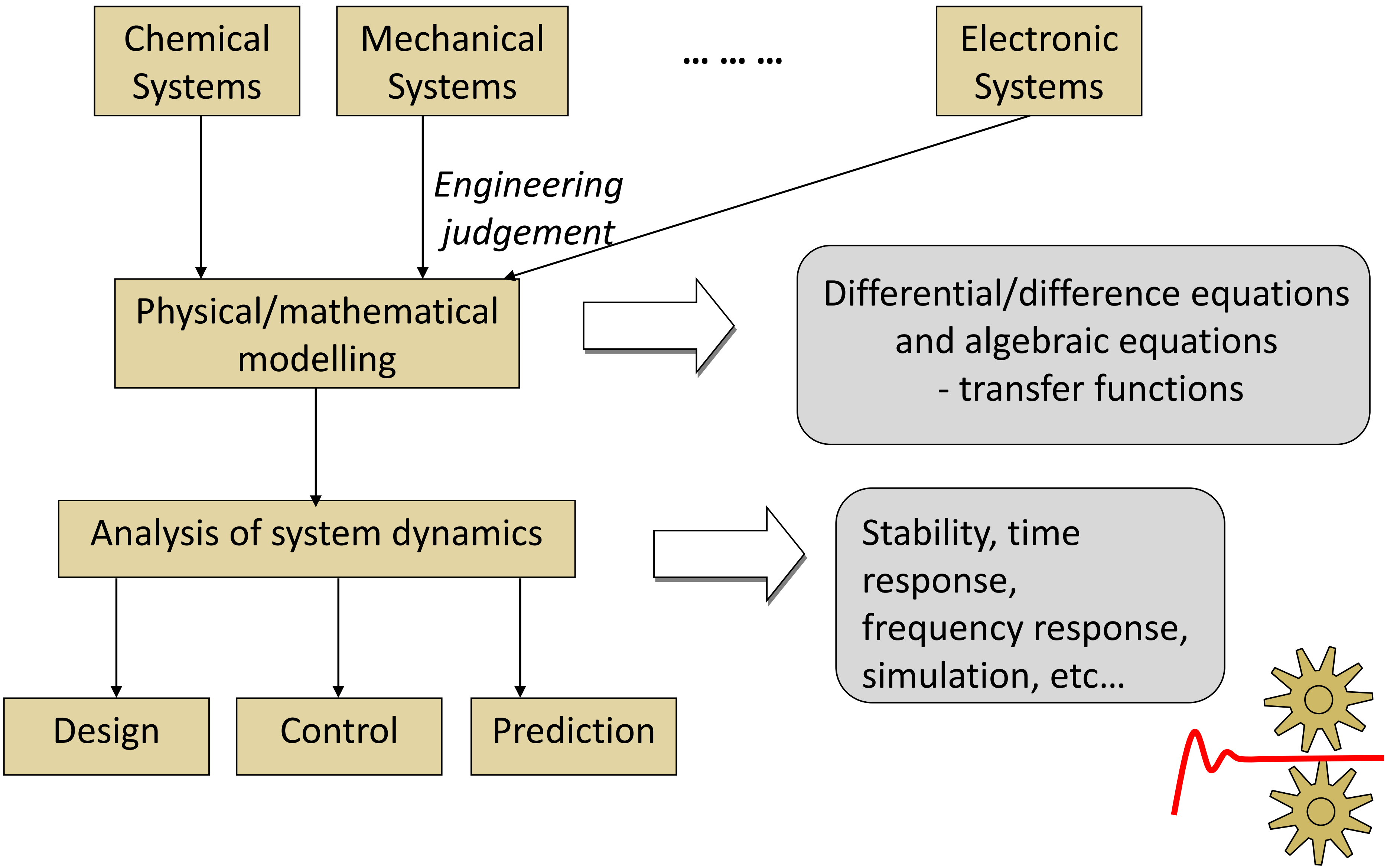
- **Today, we will start to analyse the various models obtained – first, we will obtain their time response …**

System Analysis & Simulation

- At this stage, we have examined how to model a selection of relatively straightforward systems, which we have represented both as differential equations and, more conveniently, as transfer functions.

- This completes the first part of the 'big picture' as presented in section 2.5 of the notes and reproduced here for convenience:

System Analysis & Simulation



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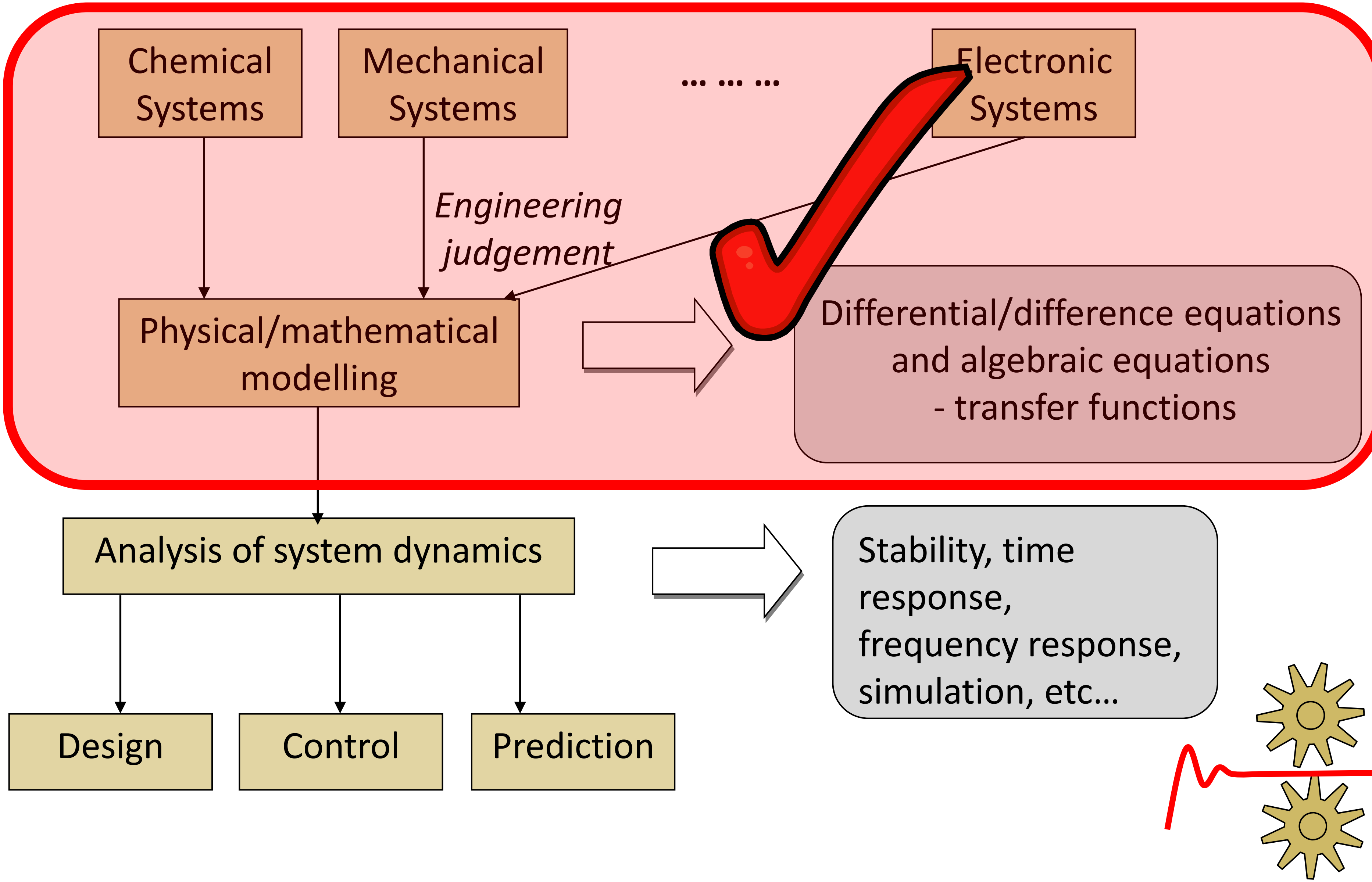

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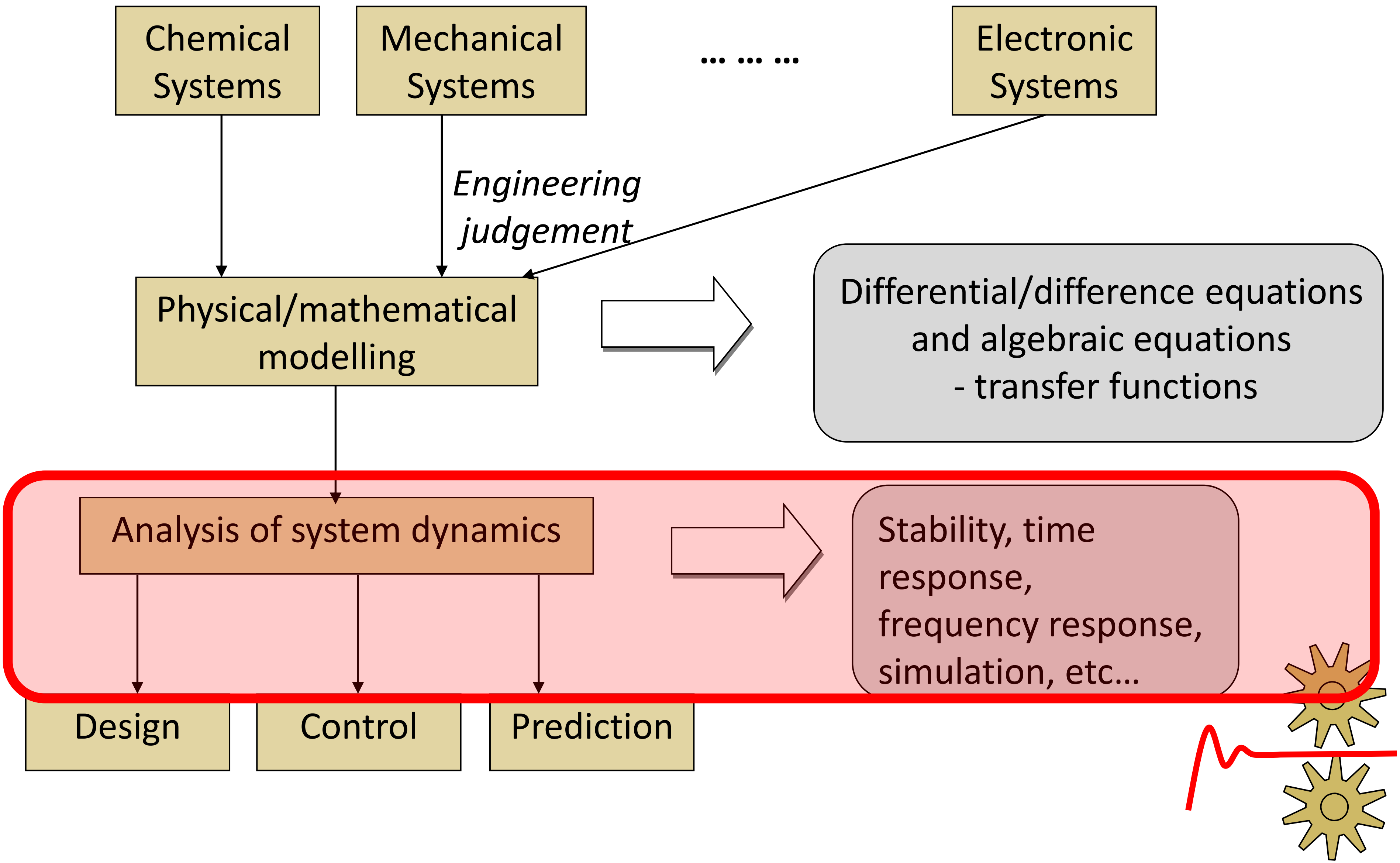


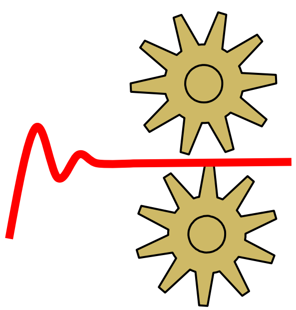
System Analysis & Simulation

- Now, we are going to consider the analysis of such systems from both a mathematical and a simulation viewpoint.

- We will briefly look at solving the differential equation model of a system before concentrating on doing the same for the transfer function equivalent.

- The latter provides an easier analytical method.

- Using the Matlab and Simulink software package (in the laboratories), we have (or will) simulated the systems studied thus far and can compare the responses obtained with their analytically derived counterparts.



System Analysis & Simulation

- For the rest of this section, we are going to analyse the following 3 systems:

  - the RC electrical circuit

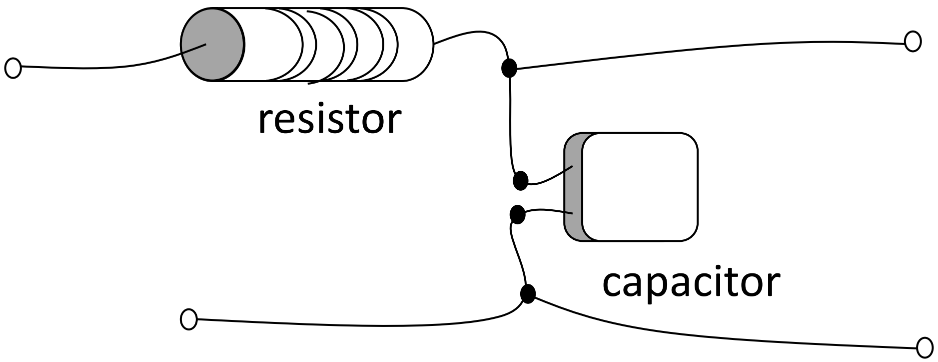
  - the (bicycle) mass-spring-damper and

  - the single water tank.

- Let's remind ourselves of each of these models again ....

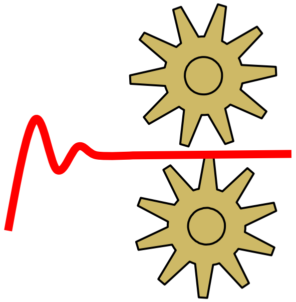
System Analysis & Simulation

- **The RC electrical circuit …**



$$v_i = RC\frac{dv_C}{dt} + v_C$$

$$\frac{V_c(s)}{V_i(s)} = \frac{1}{1+sRC}$$

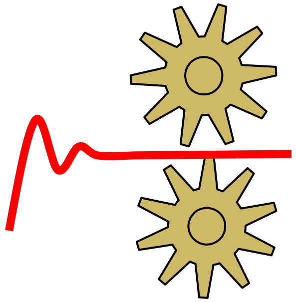


System Analysis & Simulation

- **The (bicycle) mass-spring-damper…**

$$M\frac{d^2x}{dt^2}+B\frac{dx}{dt}+Kx=f(t)$$

$$\frac{X(s)}{F(s)}=\frac{1}{Ms^2+Bs+K}$$

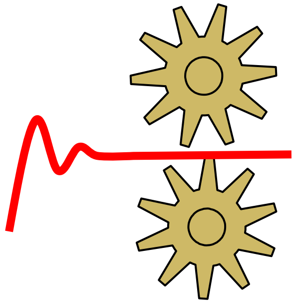


System Analysis & Simulation

- **The single water tank…**

$$A\frac{dh}{dt} = F_{in} - kh$$

$$\frac{H(s)}{F(s)} = \frac{1}{sA+k}$$

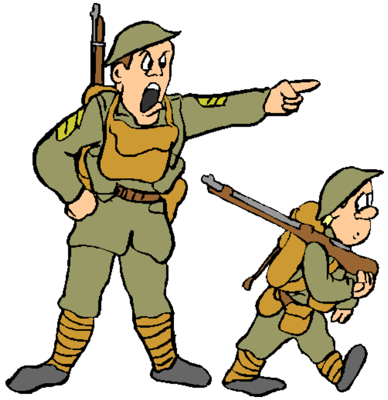


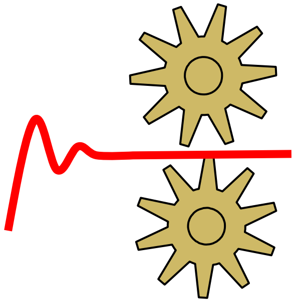
The *Order* of a System

- The **order of a system** is defined by the number of independent energy storage components it contains.

- The order is given by the **highest derivative involved in the linear differential equation** describing the system

- Alternatively, the order is generally given by the **highest power of *s* in the denominator of the transfer function**.





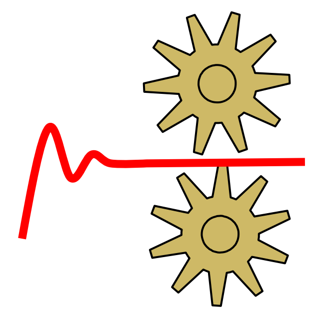
The *Order* of a System

- So, we can state that the RC circuit and the single tank are both first order systems while the mass-spring-damper is a second-order system.

$$\frac{V_c(s)}{V_i(s)} = \frac{1}{1+sRC}$$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2+Bs+K}$$

$$\frac{H(s)}{F(s)} = \frac{1}{sA+k}$$



Solving the differential equation model (1st order)

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- For convenience and clarity, we will rewrite this as:

$$RC\frac{dy}{dt} + y = u$$

where the output v_C is replaced with y and the input v_i is replaced with u.

Solving the differential equation model (1st order)

- The solution of this first order ordinary differential equation (ODE) involves the superposition (or sum) of two components:

$$y(t) = y_n(t) + y_f(t)$$

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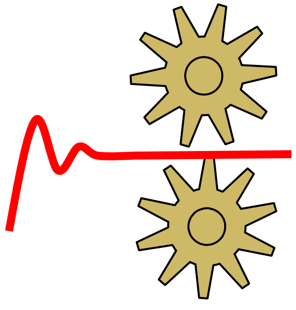
- The final solution is simply the sum of these two responses.



Finding $y_n(t)$, the zero-input response

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- Use the *separation of variables* method to solve this problem.

- Hence:

$$RC\frac{dy_n}{dt} = -y_n \Rightarrow \frac{dy_n}{y_n} = -\frac{1}{RC}dt$$

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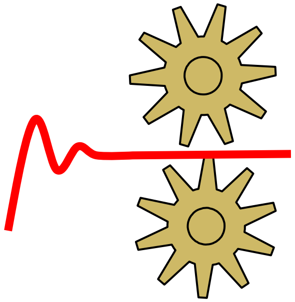
- Integrating both sides gives:

$$\int \frac{dy_n}{y_n} = -\frac{1}{RC}\int dt$$



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- Here, K (and effectively A) is a *constant of integration*.
- We will obtain a value for this constant once we have a complete solution.

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- Adding both responses gives:

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- Hence: $y(0) = 0 \quad \Rightarrow 0 = Ae^0 + u$

$$\Rightarrow 0 = A(1) + u$$

$$\Rightarrow A = -u$$

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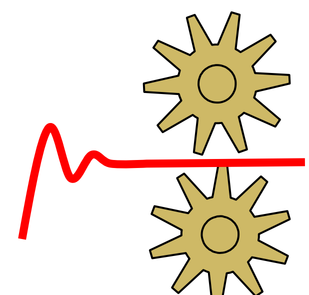

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- Therefore the complete solution for the RC circuit is:

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Complete solution

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Complete solution

1

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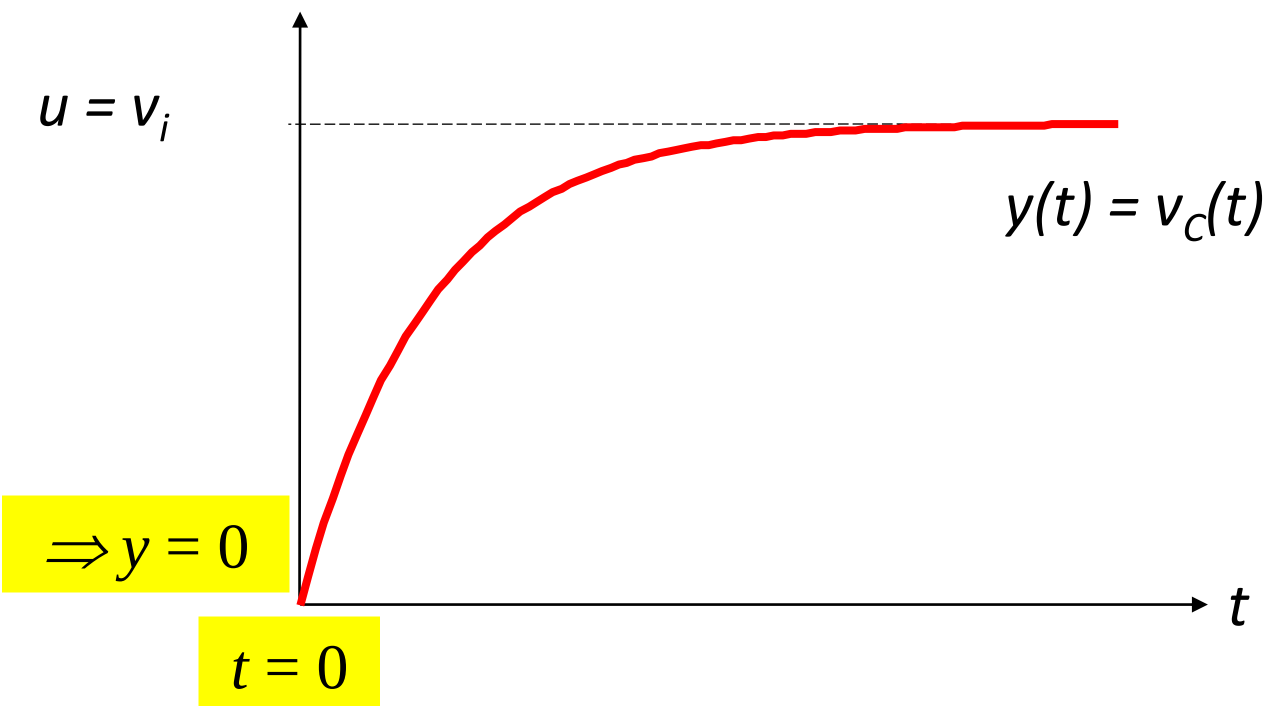

Complete solution

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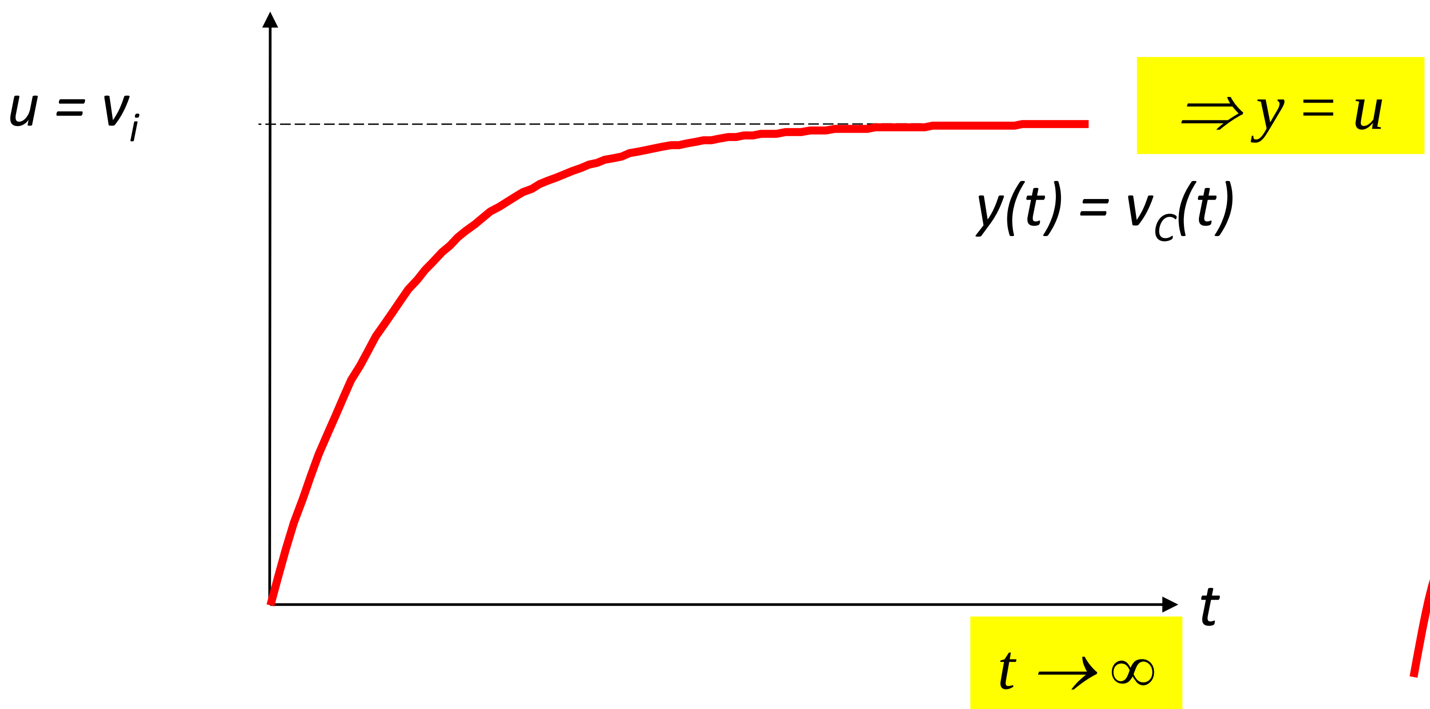
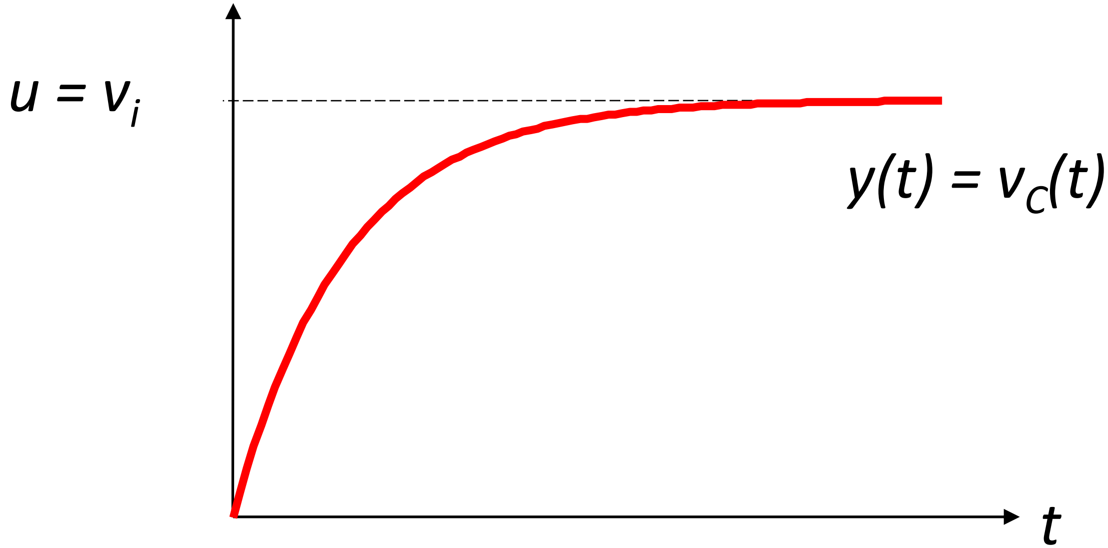


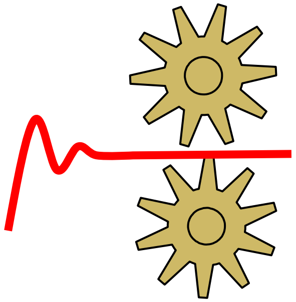
Complete solution

- *Intuitively, this is the expected response of a charging capacitor.*

- *At the start time, the capacitor has not been charged and the voltage across it is 0. As time passes, the capacitor charges up and the voltage across it increases.*

- *Once the capacitor is fully charged, it acts as an open circuit and here the voltage across it is equal to the input voltage.*





Solving the differential equation model (1st order)

- Here, we are only going to consider the solution to a first order differential equation (***mainly for the purpose of illustration***).

- You will study how to solve second order differential equations in your mathematics modules.

- You will also analyse various circuits in more detail in your circuit module in Year 2.

Who's been wearing my 'cool' t-shirt?

Solving the transfer function model

- Once again, let us consider the first order RC system but this time we will solve its transfer function representation instead.

$$\frac{V_c(s)}{V_i(s)} = \frac{1}{1+sRC} \equiv \frac{Y(s)}{U(s)}$$

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- Remember also, that this function is based on **zero initial conditions** by default.
- Okay, so we need to use the inverse Laplace transform in order to obtain the solution as a function of time.

Solving the transfer function model

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$$U(s) = \frac{u}{s}$$

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- Hence:

$$Y(s) = u\left(\frac{1}{s\,(1 + sRC)}\right)$$

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- We now apply the partial fraction method as follows:

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- Equating the coefficients of *s* gives:

$$A = 1, \qquad B + A(RC) = 0 \quad \Rightarrow B = -RC$$

Solving the transfer function model

- Hence:

$$Y(s) = u\left(\frac{1}{s} - \frac{RC}{1+sRC}\right) = u\left(\frac{1}{s} - \frac{1}{s+\frac{1}{RC}}\right)$$

Solving the transfer function model

- Hence:

$$Y(s) = u\left(\frac{1}{s} - \frac{RC}{1+sRC}\right) = u\left(\frac{1}{s} - \frac{1}{s+\frac{1}{RC}}\right)$$

- Referring to the table of commonly used Laplace transforms:

$$y(t) = L^{-1}\left(u\left(\frac{1}{s} - \frac{1}{s+\frac{1}{RC}}\right)\right) \quad = u\left(1 - e^{\frac{-t}{RC}}\right)$$

Solving the transfer function model

- Hence, as expected, we have obtained the exact same solution as in the previous section.
- However, this method is arguably a lot easier to perform and avoids the need for integration.
- As such, we will use this method to solve the remaining systems.