EE311FZ Final Paper (21-22 version)

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If you have any problem, please feel free to contact with us: MIEC CLUB

Merry Christmas, in 2022/12/25.

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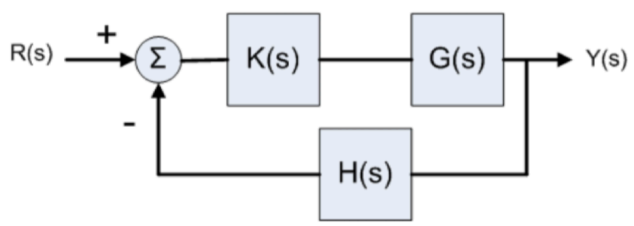
1 Q1 Root Locus

Question 1 [20 marks]

Given the following system:



where

$$G(s) = \frac{1}{(s+1)(s+3)}, \qquad H(s) = \frac{1}{s+2}, \qquad K(s) = k$$

1.1 (a) Asymptotes of the root loci.

开环传递函数$G(s)K(s)H(s) = \frac{K}{(s+1)(s+2)(s+3)}$

对应的开环极点为 $p = -1, -2, -3$，$n = 3$

没有开环零点, $m = 0$，因此根轨迹有3个分支

根据已知，可以求出根轨迹的渐近线为:

1. 渐近线与实轴的夹角 $\varphi_a = \frac{(2k+1)\pi}{n-m}, k = 0, 1, 2, \varphi_1 = \frac{\pi}{3}, \varphi_2 = \pi, \varphi_2 = \frac{5\pi}{3}$
2. 渐近线与实轴的交点 $\sigma_a = \frac{\sum_{i=1}^{3} p_i}{3} = -2$

1.2 (b) Breakaway point & Corresponding k.

分离点，使用试探法得到:

$\frac{1}{d+1} + \frac{1}{d+2} + \frac{1}{d+3} = 0$，$d_1 = -1.42, d_2 = -2.58$，

根轨迹活动的范围是 $[-2, -1]$ and $[-\infty, -3]$，所以选择 $d_2 = -1.4226$

将-1.4226带回原来的式子 $(s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6$

$$k = s^3 + 6s^2 + 11s + 6|_{-1.4226} = -0.3849 \tag{16}$$

因此，$k = -0.385$

1.3 (c) Root loci & Imaginary axis.

先求出对应的闭环传递函数$G(s)=\frac{K(s+2)}{s^3+6s^2+11s+6+K}$，令$D(s)=s^3+6s^2+11s+6+K=0$带入$s=j\omega$

得到 $j(-\omega^3+11\omega)+(-6\omega^2+6+K)=0$

因此 $\omega=\pm\sqrt{11}, K=60$

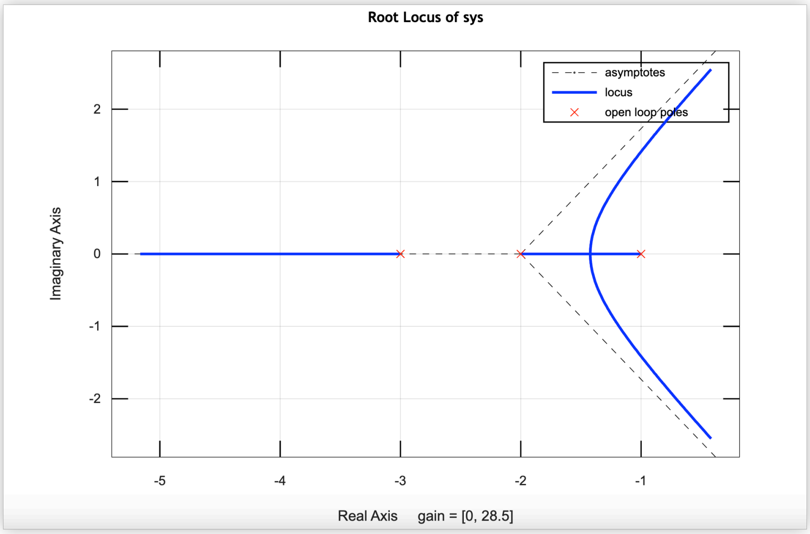
1.4 (d) Draw the root locus.

Draw the root locus plot on graph paper and annotate important points on the plot.

上面我们已经计算出Re轴上的分离点，以及Im轴上的交点:

K	Re	Im
-0.385	-1.42	0
60	0	$\pm\sqrt{11}$

如下图所示:

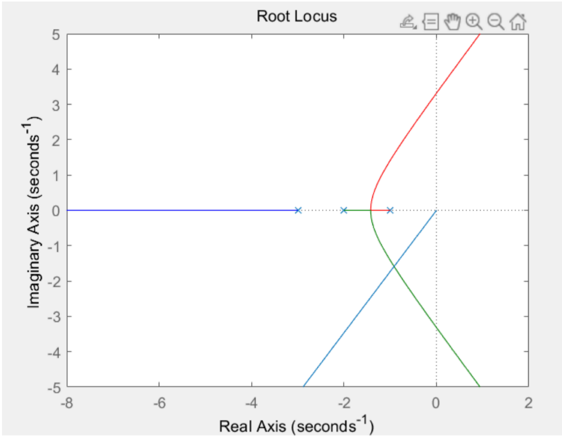


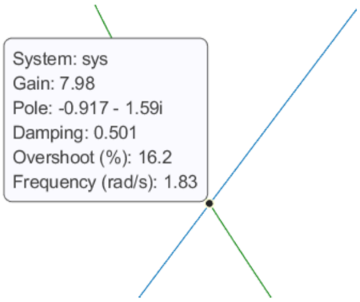
1.5 (e) Graphical approximations

Use the root locus plot to determine the value of k such that the closed-loop system has a damping ratio ζ of approximately 0.5. Calculations based on graphical approximations are acceptable. [Hint:$\zeta=cos\varphi$]

本题解仅供参考:

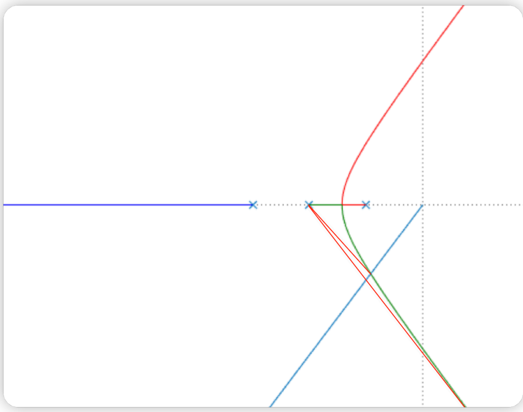
Method 1: 使用matlab计算得出精确值解





如图$\zeta = 0.501$, $K = 7.98$，为精确解

Method 2：使用渐近线来近似



因为 $\zeta = cos\varphi = 1/2$，则 $\varphi = 60^\circ$

我们想要的点可以近似为$(-1，\sqrt{3})$

代回幅值条件：$|G(s)H(s)| = 1$，可以得出$K = \sqrt{84}$

2 Q2 State-Space Basic

A continuous-time system is described by the following state-space matrices

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 & 0 \end{bmatrix}$$

2.1 (a) Controllable

可控性的判据是$C = [B \quad AB \quad A^2B \quad \cdots \quad A^nB]$满秩，就可以确定可控

$$C = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \tag{17}$$

$Rank(C) = 2$，满秩，是可控的

2.2 (b) Observable

可观性的判断是$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix}$满秩，就可以判断可观性

$$O = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \tag{18}$$

$Rank(O) = 2$，满秩，是可观的

2.3 (c) Transfer function & Stability

转换公式为 $G(s) = C(sI - A)^{-1}B$

首先要知道矩阵求逆的步骤，假设我们已知矩阵F，要求逆矩阵，首先要确定该矩阵不是奇异矩阵，即$det(F)! = 0$

$$F^{-1} = \frac{adj(F)}{\det(F)} \tag{19}$$

其中分子是F的伴随矩阵，伴随矩阵的求法是，主对角线对调，其余的元素改变符号

因此，对于本题目：

$$
\begin{aligned}
G(s) &= C(sI-A)^{-1}B \\
G(s) &= \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 0 \end{bmatrix} \frac{\begin{bmatrix} s+1 & 1 \\ -2 & s \end{bmatrix}}{s^2+s+2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{3}{s^2+s+2}
\end{aligned}
\tag{20}
$$

$s^2+s+2=0$, 那么$s_1=-0.5+1.32i, s_2=-0.5-1.32i$

两者都在左半平面，所以是稳定的

2.4 (d) Performance Specs (unit step input)

2.4.1 (i) the steady-state system response

Hint: Laplace final value Theorem 终值定理

终值定理 $\lim_{s\to\infty} G(s) = \lim_{s\to 0} sG(s)$

带入之后 $\lim_{s\to 0} \frac{1}{s} sG(s) = \lim_{s\to 0} \frac{3}{s^2+s+2} = \frac{3}{2}$

2.4.2 (ii) the settling time for the system

已知 $T_s = \frac{4}{\xi\omega_n}$

我们通过传递函数可以知道两个参数 $\xi = \sqrt{2}/4, \omega_n = \sqrt{2}$

那么 $T_s = \frac{4}{\xi\omega_n} = 8s$

2.4.3 (iii) the PO% overshoot for the system

系统的超调计算公式为 $PO = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$

带入数据可以得到PO=30.5%

2.4.4 (iv) the frequency of oscillation of the response

知道 $\omega_d = \sqrt{1-\xi^2}\omega_n = 1.3229$

2.5 (e) State feedback Controller

(e) It is desired that the closed-loop system will have poles of $[-2,-3]$. Design a state feedback controller $u = \begin{bmatrix} k_1 & k_2 \end{bmatrix} x$ to achieve these specifications (determine the value of k_1 and k_2).

$$
\begin{aligned}
&|sI - A + BK| = (s+2)(s+3) \\
&\begin{vmatrix} s & -1 \\ 2+k_1 & s+1+k_2 \end{vmatrix} = s^2 + 5s + 6 \\
&s(s+1+k_2) + 2 + k_1 = s^2 + 5s + 6 \\
&s^2 + s + k_2 s + 2 + k_1 = s^2 + 5s + 6 \\
&k_1 = 4, k_2 = 4
\end{aligned} \tag{21}
$$

Hence, the state feedback controller is:

$$
u = [4, 4]x \tag{22}
$$

2.6 (f) State Observer

(f) Due to an inability to measure the states, a state estimator is required. Design a state observer to place the poles of the observer at $[-6, -6]$. (You should write down the observer dynamics and then calculate the observer gain $K_e = \begin{bmatrix} k_{e,1} & k_{e,2} \end{bmatrix}^T$)

$$
\begin{aligned}
|sI - A + KC| &= det \begin{bmatrix} s+3k_1 & -1 \\ 2+3k_2 & s+1 \end{bmatrix} \\
&= s^2 + (3k_1 + 1)s + 3k_1 + 3k_2 + 2
\end{aligned} \tag{23}
$$

Because pole is [-6, -6] , the expected characteristic equation is:

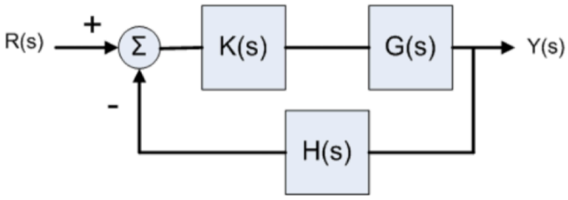
$$
\begin{gathered}
s^2 + 12s + 36 \\
k_1 = \frac{11}{3}, k_2 = \frac{23}{3}
\end{gathered} \tag{24}
$$

Hence, the state observer is

$$
K_e = \begin{bmatrix} \frac{11}{3} \\ \frac{23}{3} \end{bmatrix} \tag{25}
$$

3 Q3 Nyquist Stability Criterion

3.1 (a) Nyquist Plot



where

$$G(s) = \frac{200}{(s+5)(s+10)}, \qquad H(s) = \frac{1}{s+1}, \qquad K(s) = 1$$

$$\begin{aligned} G_{op}(s) &= \frac{200}{(s+1)(s+5)(s+10)} = \frac{200(50-16\omega^2)+200j(-65\omega-\omega^3)}{(1+\omega^2)(25+\omega^2)(100+\omega^2)} \\ |G_{op}(j\omega)| &= \frac{200}{\sqrt{1+\omega^2}\sqrt{25+\omega^2}\sqrt{100+\omega^2}} \\ \angle G_{op}(j\omega) &= 0 - \arctan\omega - \arctan\frac{\omega}{5} - \arctan\frac{\omega}{10} \end{aligned} \tag{26}$$

$$G_{(S)}H_{(S)} = \frac{200}{(s+1)(s+5)(s+10)}$$

$$s = j\omega$$

$$\frac{200}{(j\omega+1)(j\omega+5)(j\omega+10)}$$

$$\frac{200(1-j\omega)(5-j\omega)(10-j\omega)}{(j\omega+1)(j\omega+5)(j\omega+10)(1-j\omega)(5-j\omega)(10-j\omega)}$$

$$\frac{200(j\omega^3-16\omega^2-65j\omega+50)}{(1+\omega^2)(25+\omega^2)(100+\omega^2)}$$

$$Im[G_{(j\omega)}] = 0, \omega = \pm 8.06226$$

$$\omega = \pm 8.06226:$$

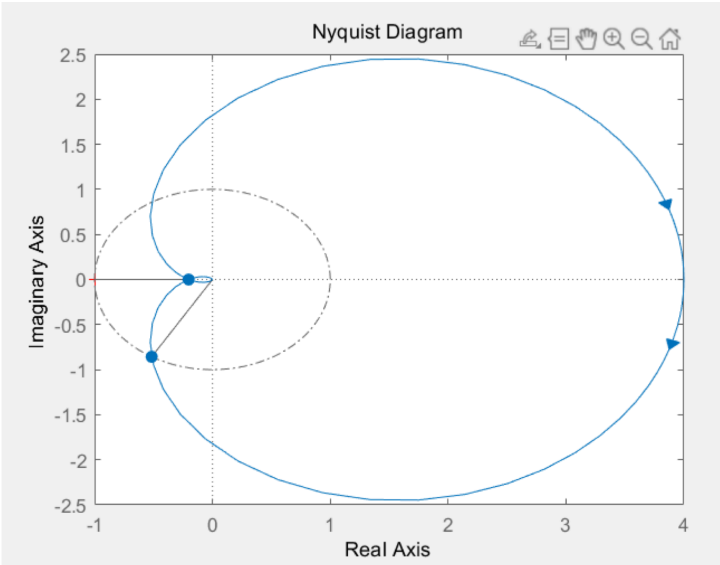
$$Re[G_{(j8.06226)}] = \frac{-198000}{980100} = -0.20202$$

$$Re[G_{(j\omega)}] = 0, \omega = \pm 1.76777$$

$$Im[G_{(j\pm 1.76777)}] = \frac{-21,876}{11964} = -1.82848$$

画奈氏图步骤:

Frequency	Amplitude	Phase
0	4	0
∞	0	-270°
± 8.06226		
± 1.76777		



3.2 (b) Gain Margin

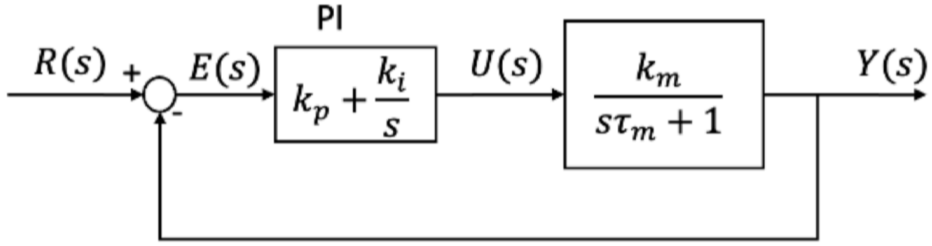
由上可知 cross phase frequency为:

$$\omega_g = \pm 8.06226$$

$$\frac{1}{|G_{(j\omega_g)}|} = \frac{(\sqrt{\omega_g^2 + 1})(\sqrt{\omega_g^2 + 25})(\sqrt{\omega_g^2 + 100})}{200} = 4.95 \tag{27}$$

$$GM = 20 \log 4.95 = 13.892dB$$

4 Q4 PID Controller



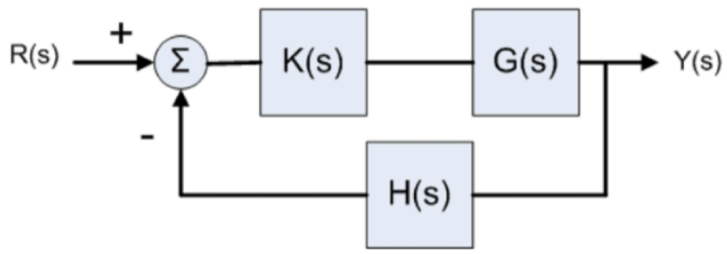
4.1 (a) PI-controlled system & Steady-state error

$$
\begin{aligned}
\frac{Y(s)}{R(s)} &= \frac{(sKp + Ki)(Km)}{s(s\tau_m + 1) + (sKp + Ki)Km} \\
\frac{E(s)}{R(s)} &= \frac{R(s) - Y(s)}{R(s)} = \frac{s(s\tau_m + 1)}{s(s\tau_m + 1) + (sKp + Ki)Km} \\
\lim_{t\to\infty} E(s) &= \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{s(s\tau_m + 1)}{s(s\tau_m + 1) + (sKp + Ki)Km}\frac{1}{s} = 0
\end{aligned}
\tag{28}
$$

4.2 (b) only P-controlled & Steady-state error

$$
\begin{aligned}
\frac{Y(s)}{R(s)} &= \frac{KpKm}{s\tau_m + 1 + KpKm} \\
\frac{E(s)}{R(s)} &= \frac{R(s) - Y(s)}{R(s)} = \frac{s\tau_m + 1}{s\tau_m + 1 + KpKm} \\
\lim_{t\to\infty} E(s) &= \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{s\tau_m + 1}{s\tau_m + 1 + KpKm}\frac{1}{s} = \frac{1}{1 + KpKm}
\end{aligned}
\tag{29}
$$

4.3 (c) PID controller using closed-loop ZN method



where

$$G(s) = \frac{100}{(s+1)(s+2)(s+3)}, \qquad K(s) = k_p + k_d s + \frac{k_i}{s}, \qquad H(s) = 1$$

[10 marks]

For your convenience, the closed-loop Ziegler-Nichols rules are given in the following table ($k_d = k_p t_d$, $k_i = k_p/t_i$)

	k_p	t_i	t_d
P	$0.5k_c$		
PI	$0.45k_c$	$t_c/1.2$	
PID	$0.6k_c$	$t_c/2$	$t_c/8$

$$
\begin{gathered}
1+G_{(S)}k_p=0\\
1+\frac{100k_p}{s^3+6s^2+11s+6}=0\\
s^3+6s^2+11s+6+100k_p=0\\
-j\omega^3-6\omega^2+11j\omega+6+100k_p=0\\
\begin{cases}-j\omega^3+11j\omega=0\\-6\omega^2+6+100k_p=0\end{cases}\\
k_c=0.6,\omega_c=\sqrt{11}rad/s\\
t_c=2\pi/\omega_c=1.8945\\
k_P=0.36,t_i=0.9472,t_d=0.2368\\
K_P=0.36,K_i=0.38,K_d=0.08525
\end{gathered}
\tag{30}
$$