Tutorial Sheet 7 – EE211

Q1 (i)

$$\frac{1+2j}{2-j} = \frac{\sqrt{5}\,\angle \tan^{-1}(2)}{\sqrt{5}\,\angle \tan^{-1}(-\frac{1}{2})} \Rightarrow 1\,\angle \tan^{-1}(2) - \tan^{-1}(-\tfrac{1}{2})$$

$$\Rightarrow \underline{\underline{1\angle 90^\circ}}$$

OR $1.\cos(90^\circ) + j.1.\sin(90^\circ) \Rightarrow 0 + j \Rightarrow \underline{\underline{j}}$

(ii)

$$\frac{j(1-3j)}{(3-2j)(5+4j)} \Rightarrow \frac{1\sqrt{10}}{\sqrt{13}\sqrt{41}}\,\angle 90^\circ + \tan^{-1}(-3) - \tan^{-1}(-\tfrac{2}{3}) - \tan^{-1}(\tfrac{4}{5})$$

$$\Rightarrow \underline{\underline{0.137\,\angle 13.47^\circ}}$$

OR $\underline{\underline{0.133 + 0.032j}}$

Q2 (i) $G(s) = \dfrac{k}{(s+1)(s+2)}$, $H(s) = 1$, $k = 1$

$$s \to j\omega \Rightarrow GH(j\omega) = \frac{1}{(1+j\omega)(2+j\omega)}$$

$$= \frac{1\,\angle 0^\circ}{\sqrt{1+\omega^2}\,\angle \tan^{-1}(\omega)\;\sqrt{4+\omega^2}\,\angle \tan^{-1}(\omega/2)}$$

$$\therefore GH(j\omega) = \frac{1}{\sqrt{1+\omega^2}\sqrt{4+\omega^2}}\,\angle -\tan^{-1}(\omega) - \tan^{-1}(\tfrac{\omega}{2})$$

- $\omega = 0 \Rightarrow |GH(j0)| = \dfrac{1}{(1)(2)} \Rightarrow \dfrac{1}{2}$

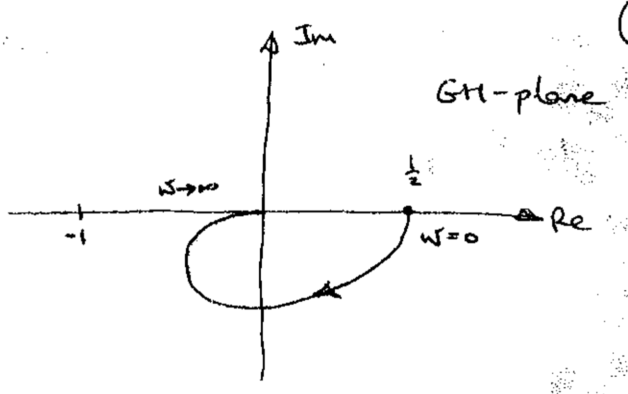
  $\angle GH(j0) = 0^\circ$

- $\omega \to \infty \Rightarrow |GH(j\infty)| = \dfrac{1}{\omega^2} = 0$

  $\angle GH(j\omega) = -2\tan^{-1}(\infty) = -180^\circ$

Q2 (i) cont.

- Hence Nyquist sketch:



- System is closed-loop stable $\forall k$

(ii)

$$G(s) = \frac{k(s+3)}{(s+1)(s+2)}, \quad H(s) = 1, \quad k = 1$$

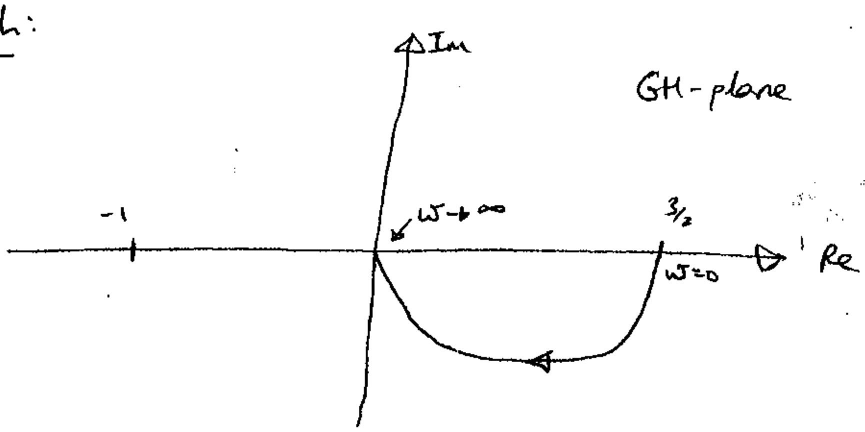
$$\therefore GH(j\omega) = \frac{1.(3+j\omega)}{(1+j\omega)(2+j\omega)} \Rightarrow \frac{1\angle 0^\circ \sqrt{9+\omega^2}\,\angle \tan^{-1}(\frac{\omega}{3})}{\sqrt{1+\omega^2}\,\angle \tan^{-1}(\omega)\,\sqrt{4+\omega^2}\,\angle \tan^{-1}(\frac{\omega}{2})}$$

$$\Rightarrow GH(j\omega) = \frac{\sqrt{9+\omega^2}}{\sqrt{1+\omega^2}\sqrt{4+\omega^2}} \angle \tan^{-1}(\tfrac{\omega}{3}) - \tan^{-1}(\omega) - \tan^{-1}(\tfrac{\omega}{2})$$

- $\omega = 0 \Rightarrow GH(j0) = \frac{3}{(1)(2)} \angle 0^\circ \Rightarrow \frac{3}{2}\angle 0^\circ$
- $\omega \to \infty \Rightarrow GH(j\infty) = \frac{\cancel{\omega}}{\cancel{\omega}\cdot\omega} \angle \cancel{90^\circ} - \cancel{90^\circ} - 90^\circ \Rightarrow \frac{1}{\omega}\angle -90^\circ$

as $\omega \to \infty \Rightarrow 0\angle -90^\circ$

$\therefore$ Nyquist sketch:



- System is closed-loop stable $\forall k$

Q3

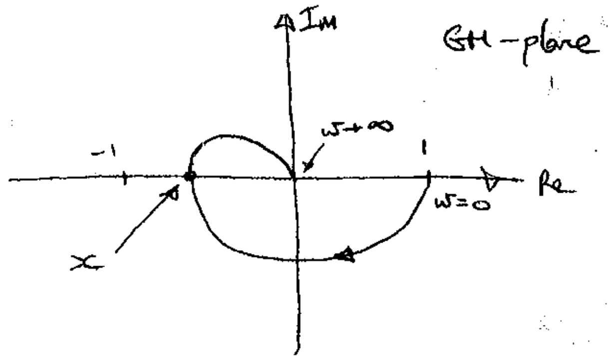
$$GH(s) = \frac{27k}{(s+3)^3} \quad , k = 1$$

$$GH(j\omega) = \frac{27}{(3+j\omega)^3} = \frac{27\angle 0^\circ}{\left(\sqrt{\omega^2+3^2}\right)^3 \angle 3\tan^{-1}\left(\frac{\omega}{3}\right)}$$

$$\Rightarrow \frac{27}{\left(\sqrt{\omega^2+3^2}\right)^3} \angle -3\tan^{-1}\left(\frac{\omega}{3}\right)$$

- $\omega = 0 \Rightarrow GH(j0) = 1 \angle 0^\circ$
- $\omega \to \infty \Rightarrow GH(j\infty) = 0 \angle -3.90^\circ \Rightarrow 0 \angle -270^\circ$

∴ Nyquist sketch:



- If $x < -1$, system is stable!

- GAIN MARGIN $= \frac{1}{x}$ where $x = \left|GH(j\omega)\right|_{\angle GH(j\omega) = -180^\circ}$

- $\angle GH(j\omega) = -180^\circ = -3\tan^{-1}\left(\frac{\omega}{3}\right)$

$$\Rightarrow 60^\circ = \tan^{-1}\left(\frac{\omega}{3}\right) \Rightarrow \omega = 3 \cdot \tan 60^\circ$$
$$\Rightarrow \omega = 5.196$$

- $$\left|GH(j\omega)\right|_{\omega = 5.196} = \left.\frac{27}{\left(\sqrt{\omega^2+3^2}\right)^3}\right|_{\omega=5.196} \Rightarrow \frac{1}{8} \left(= 0.125\right)$$

∴ GAIN MARGIN $= \frac{1}{\frac{1}{8}} = \underline{8}$ $\left(\simeq 18_{dB}\right)$

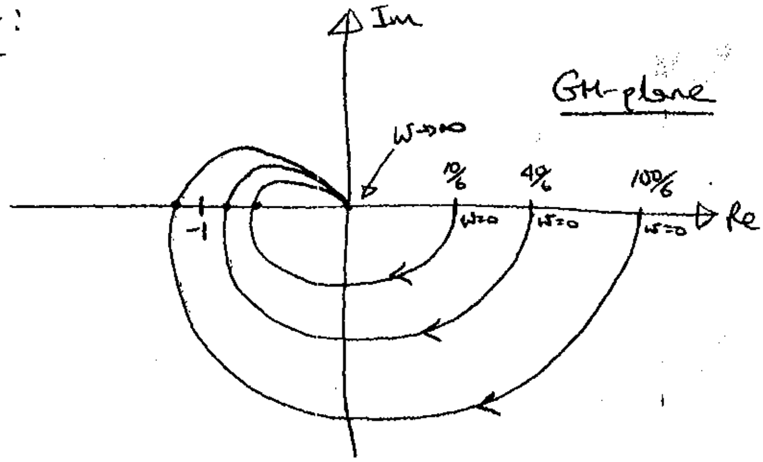
Q4 (i) $G(s) = \dfrac{k}{(s+1)(s+2)}$, $H(s) = \dfrac{1}{s+3}$ $\underline{k = 10, 40, 100}$

$$GH(j\omega) = \frac{k}{(1+j\omega)(2+j\omega)(3+j\omega)} = \frac{k\angle 0^\circ}{\sqrt{1+\omega^2}\angle \tan^{-1}(\omega)\sqrt{4+\omega^2}\angle \tan^{-1}(\omega/2)\sqrt{9+\omega^2}\angle \tan^{-1}(\omega/3)}$$

$$\Rightarrow \frac{k}{\sqrt{1+\omega^2}\sqrt{4+\omega^2}\sqrt{9+\omega^2}} \angle\left(-\tan^{-1}(\omega) - \tan^{-1}(\omega/2) - \tan^{-1}(\omega/3)\right)$$

- $\omega = 0 \Rightarrow \dfrac{k}{1.2.3} \angle 0^\circ \Rightarrow \dfrac{k}{6}\angle 0^\circ$
- $\omega \to \infty \Rightarrow \dfrac{k}{\infty} \angle -3.90^\circ \Rightarrow 0 \angle -270^\circ$

$\therefore$ Sketch of Nyquist Plot:



(ii) Crosses negative real axis when $\angle GH(j\omega) = -180^\circ$

i.e. $-180^\circ = \underbrace{-\tan^{-1}(\omega) - \tan^{-1}(\omega/2) - \tan^{-1}(\omega/3)}$

@ $\omega = \sqrt{11}$ gives $-179.99\ldots \simeq -180^\circ$

$\therefore$ Nyquist plot crosses negative real axis @ $\omega = \sqrt{11}$ $\underline{\text{QED}}$

$\rightarrow$

Q4 (ii) cont.

$$GM = \frac{1}{|GH(j\omega)|}\Bigg|_{\omega=\sqrt{11}}$$

$$\Rightarrow \frac{1}{\frac{k}{\sqrt{1+\omega^2}\sqrt{4+\omega^2}\sqrt{9+\omega^2}}}\Bigg|_{\omega=\sqrt{11}} \Rightarrow \frac{\sqrt{1+\omega^2}\sqrt{4+\omega^2}\sqrt{9+\omega^2}}{k}\Bigg|_{\sqrt{11}}$$

$$\Rightarrow \frac{\sqrt{12}\sqrt{15}\sqrt{20}}{k} \Rightarrow \frac{60}{k}$$

- For $k = 10, 40, 100$

$$GM = \underbrace{6, 1.5}_{\substack{\text{Stable as} \\ GM>1}} \text{ and } \underbrace{0.6}_{\text{Unstable as } GM<1}$$

(iii)

$$|GH(j\omega)| = \frac{k}{\sqrt{1+\omega^2}\sqrt{4+\omega^2}\sqrt{9+\omega^2}}$$

- $k = 10, \omega = 1 \Rightarrow \frac{10}{\sqrt{2}\sqrt{5}\sqrt{10}} = \underline{\underline{1}}$ QED
- $k = 40, \omega \simeq 2.73 \Rightarrow \frac{40}{\sqrt{8.4529}\sqrt{11.4529}\sqrt{16.4529}} = \underline{\underline{1}}$ QED
- $k = 100, \omega \simeq 4.14 \Rightarrow 0.9988 \simeq \underline{\underline{1}}$ QED

$$PM = 180° + \angle GH(j\omega)\Big|_{\omega=1,2.73,4.14}$$

$$\Rightarrow 180° - \tan^{-1}(\omega) - \tan^{-1}(\omega/2) - \tan^{-1}(\omega/3)\Bigg|_{\omega=1,2.73,4.14}$$

This gives $PM = \underbrace{90°, 14°}_{\text{Stable as } PM>0°}$ and $\underbrace{-14.7°}_{\text{Unstable as } PM<0°}$ respectively.

Q5(i)

$$GH(s) = \frac{1}{(s+1)(s+2)}$$

$$\therefore GH(j\omega) = \frac{1}{(j\omega+1)(j\omega+2)} = \frac{1}{(1+j\omega)\cdot 2(1+j\omega/2)}$$

$$\Rightarrow \frac{1}{1+j\omega} \cdot \frac{1}{1+j\omega/2} \cdot \frac{1}{2} \qquad \underline{20\log_{10}\left(\tfrac{1}{2}\right) = -6\,dB}$$

– See Bode Plot GH(jω) – Q5(i) on Page 7

- System is <u>closed-loop stable</u> as neither the 0dB line nor the −180° line is crossed (ie. <u>GM = PM = Infinite!</u>)

(Note: an infinite PM effectively means PM = 180°!)

Q5(ii)

$$GH(s) = \frac{k(s+3)}{(s+1)(s+2)}, \quad k = 1$$

$$\therefore GH(j\omega) = \frac{1}{1+j\omega} \cdot \frac{1}{1+j\omega/2} \cdot \frac{1}{2} \cdot \underbrace{3\cdot(1+j\omega/3)}$$

Can add this to part (i) to get <u>new plot</u>

– See Bode Plot GH(jω) – Q5(ii) on Page 7

- System is <u>closed-loop stable</u> as GM = Inf and PM ≃ 120°

[<u>Note</u>: MATLAB gives PM = 127° @ ω = 1 rad/sec]

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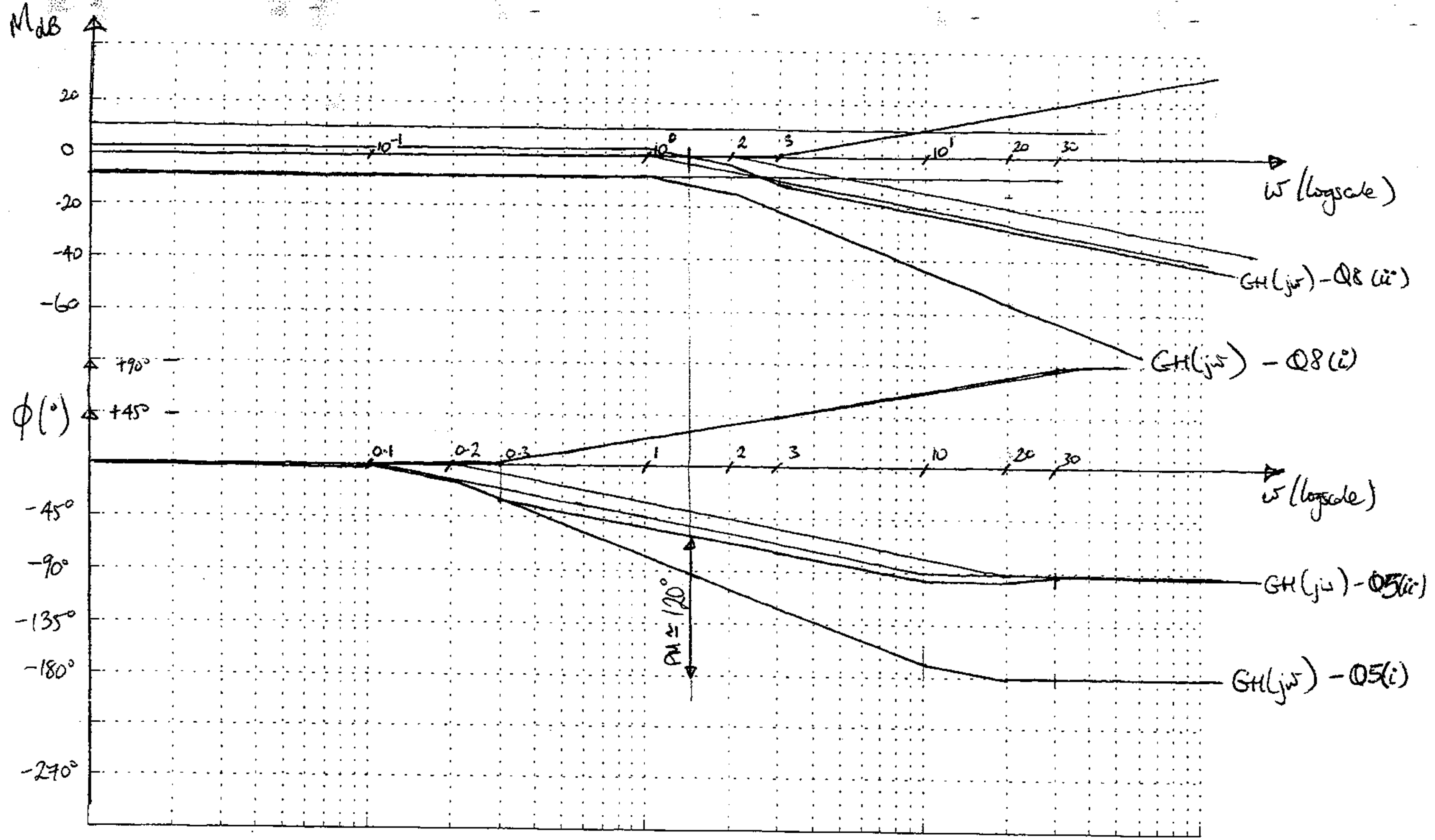
Q5(i), (ii)

(i) $GH(j\omega) = \frac{1}{1+j\omega} \cdot \frac{1}{1+j\omega/2} \cdot \frac{1}{2}$

$20\log_{10}(0.5) = \underline{-6\,dB}$

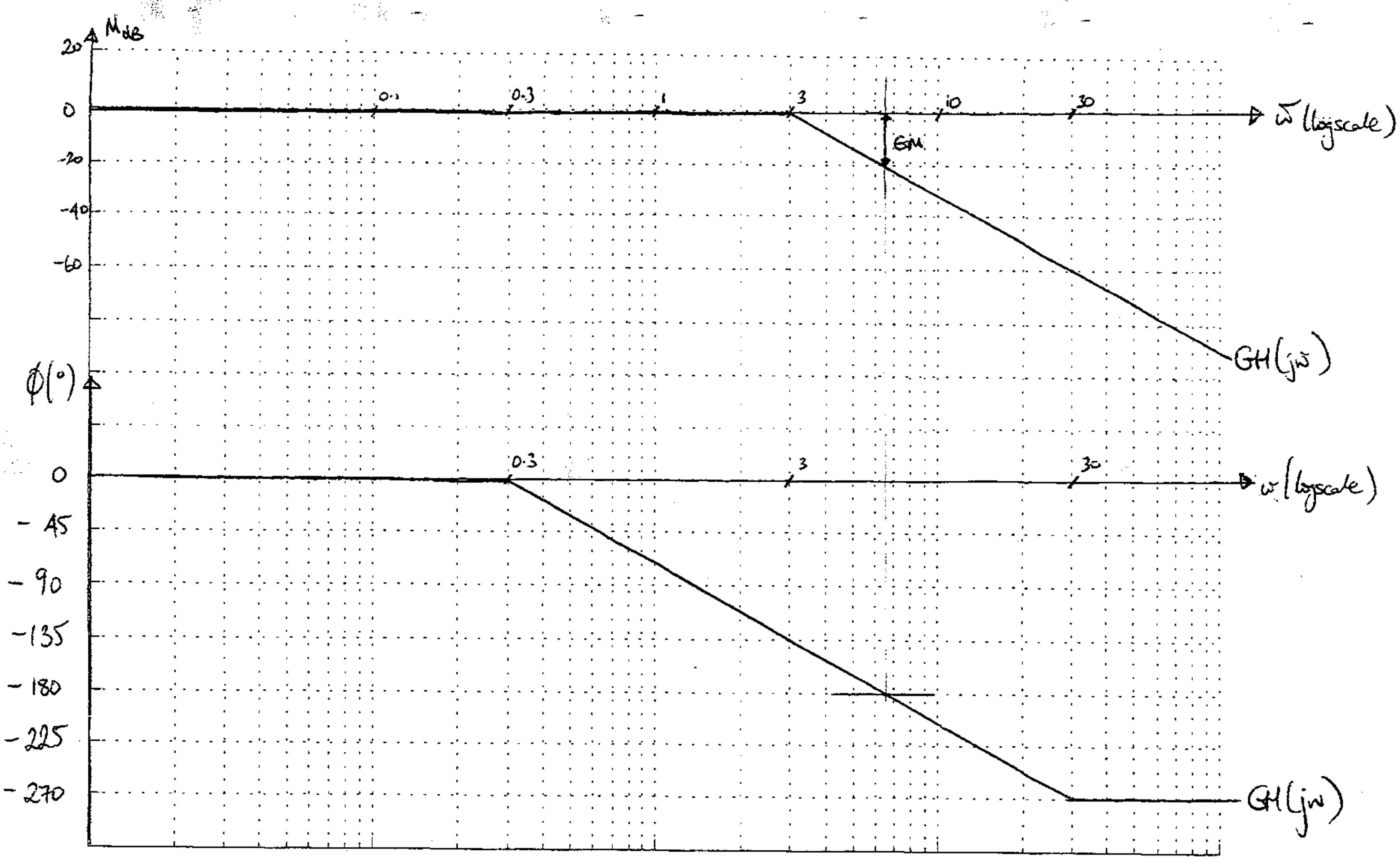


(ii) $\frac{1}{1+j\omega} \cdot \frac{1}{1+j\omega/2} \cdot \frac{1}{2} \cdot 3 \cdot (1+j\omega/3)$

$20\log_{10}(3) = 9.54\,dB$

(8)

Q6 $GH(j\omega) = \frac{27}{(j\omega+3)^3} = \frac{27}{3^3(1+j\omega/3)^3} \Rightarrow \frac{1}{(1+j\omega/3)^3}$



PM = 180°, GM = 20dB [Analytically result is 18dB]

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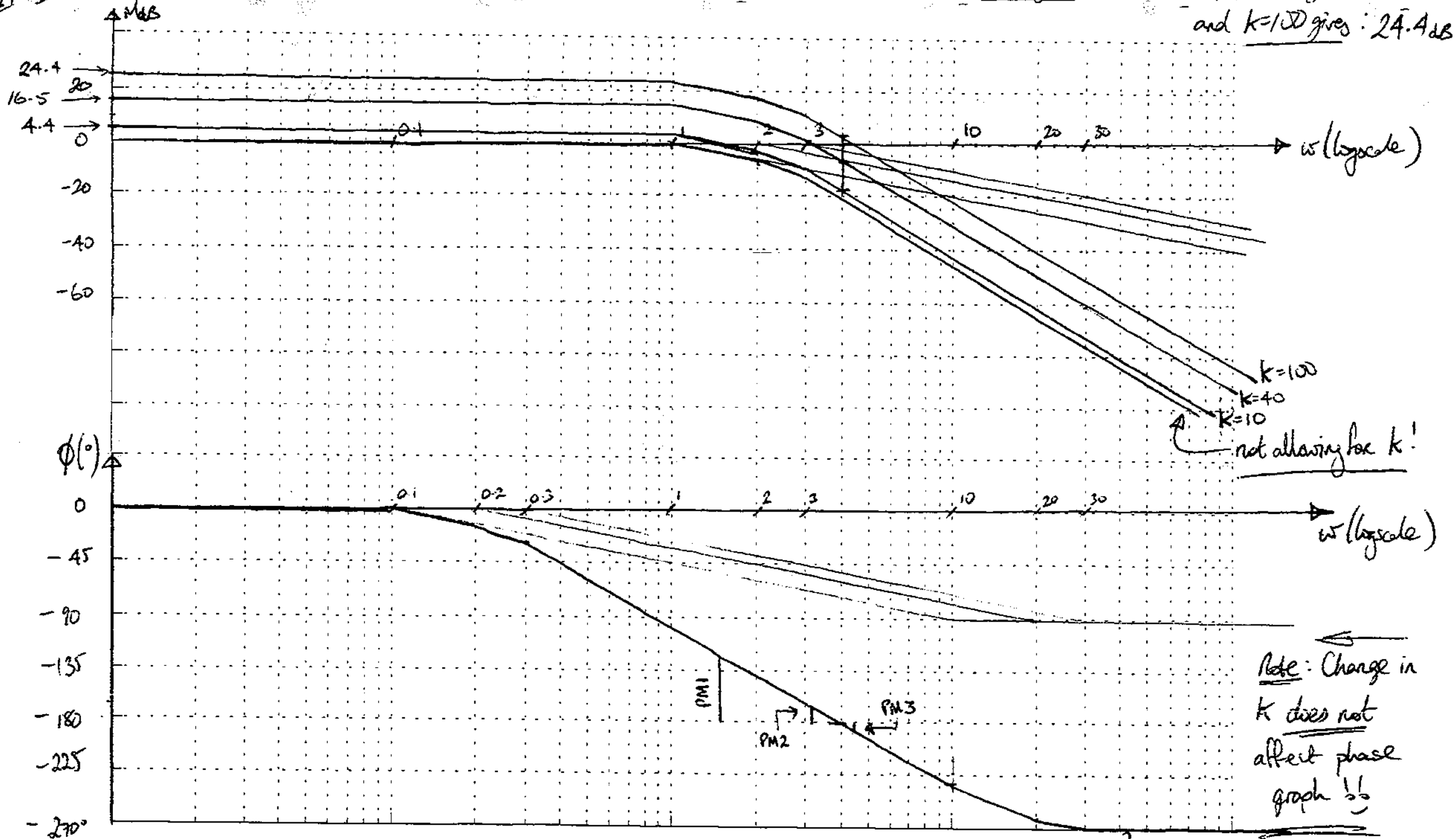
Q7

(i), (ii)

$$GH(j\omega) = \frac{K}{(1+j\omega)(2+j\omega)(3+j\omega)} \Rightarrow \frac{K}{1+j\omega} \cdot \frac{K}{1+j\omega/2} \cdot \frac{K}{1+j\omega/3} \cdot \frac{K}{6}$$

$20\log_{10}\left(\frac{1}{6}\right) = \underline{-15.56\,dB}$ $(K=1)$

$\underline{K=10 \text{ gives}}$: $4.4\,dB$, $\underline{K=40 \text{ gives}}$: $16.5\,dB$ and $\underline{K=100 \text{ gives}}$: $24.4\,dB$



not allowing for K!

Note: Change in K does not affect phase graph!!

$\angle GH(j\omega)$ $(\forall K)$

(ii) K=10 : GM = 18 dB, PM = 55° } STABLE
K=40 : GM = 5 dB, PM = 12° } STABLE
K=100 : GM = −3 dB PM = −8° } UNSTABLE

NOTE: ① Asymptotic Bode Plots
② Inaccurate Sketching
∴ Gross Approximations