Lecture 10: Digital Filters

EE213 - Introduction to Signal Processing

Semester 1, 2020

Discrete-Time systems

- A discrete-time system transforms or maps an input sequence (signal) $x[n]$ into an output sequence (signal) $y[n]$ via a function or operation denoted as

$$y[n] = T\{x[n]\} \tag{1}$$

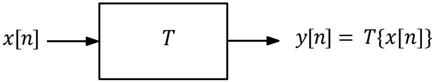


Illustration of a discrete time system

- If the system is LTI, the output is given by

$$\underbrace{y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = h[n] * x[n]}_{\textbf{time domain}} \Leftrightarrow \underbrace{Y(\omega) = H(\omega)X(\omega)}_{\textbf{frequency domain}} \tag{2}$$

- $H(\omega)$ is called the frequency response

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \tag{3}$$

- $h(n)$ is called the impulse response

$$h[n] \Leftrightarrow H(\omega)$$

A discrete-time system can be represented by

1. The function T{ }
2. The frequency response $H(\omega)$
3. The impulse response $h[n]$

Infinite Impulse Response

Example

Find the frequency response of the discrete-time LTI system described as

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{2}x[n-1] \tag{4}$$

Solution

Taking Fourier transform of both sides of the above equation results in

$$\left(1 - \frac{1}{2}e^{-j\omega}\right)Y(\omega) = \left(1 + \frac{1}{2}e^{-j\omega}\right)X(\omega) \tag{5}$$

The frequency response is

$$H(\omega) = \frac{1 + \frac{1}{2}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}} \tag{6}$$

Infinite Impulse Response

Solution (continued)

Taking the inverse Fourier transform of $H(\omega)$ we can find the system impulse response

$$h[n] = \mathcal{F}^{-1}\{H(\omega)\} = \mathcal{F}^{-1}\{\frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{\frac{1}{2}}{1-\frac{1}{2}e^{-j\omega}}e^{-j\omega}\}$$

$$= (\tfrac{1}{2})^n u[n] + \tfrac{1}{2}(\tfrac{1}{2})^{n-1}u[n-1]$$

$$= \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ (\frac{1}{2})^{n-1} & n \geq 1 \end{cases}$$

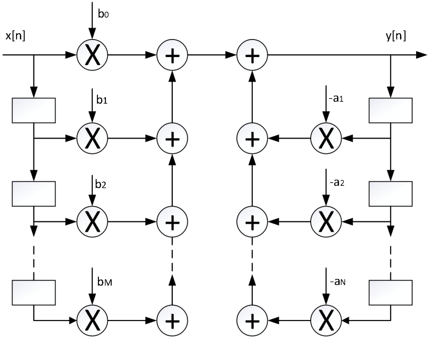
- In this example $h[n]$ is strictly positive when $n \to \infty$, i.e., the system has an **infinite impulse response** (IIR).

IIR

- An IIR system is characterised by

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k\, x]n-k[$$

- A structure of the IIR system is given in the following figure.



Example

Find the frequency response of the discrete-time LTI system described by

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]) \qquad (8)$$

Solution (method 1)

We can easily see that

$$Y(\omega) = \frac{1}{3}(X(\omega) + e^{-j\omega}X(\omega) + e^{-j2\omega}X(\omega)) \qquad (9)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{3}(1 + e^{-j\omega} + e^{-j2\omega})$$

Taking the inverse FT we obtain the impulse response as

$$h[n] = \frac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2]) \qquad (10)$$

Finite Impulse Response

Solution (method 2)

We know that the output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{11}$$

Compare (8) and (11) we obtain

$$h[k] = \begin{cases} \frac{1}{3} & n = 0,1,2 \\ 0 & otherwise \end{cases} \tag{12}$$

- In this example $h[n]$ is non-zero for some values of n, i.e., the system has a finite impulse response (FIR).

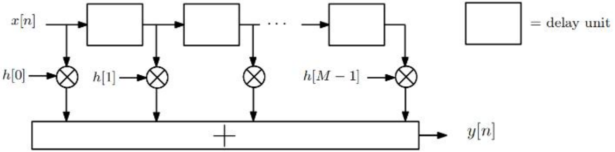
FIR

- An FIR system of length M is characterised by

$$y[n] = \sum_{k=0}^{M-1} h]k[x]n - k[\tag{14}$$

where $h[0]$, $h[1] \cdots h[M-1]$ are M filter coefficients.

- A structure of the FIR system is given in the following figure.



FIR system	IIR system
$h[n]$ has finite length. a. For causal FIR system, equation of convolution is $y[n] = \sum_{k=0}^{m} x[k]h[n-k]$ b. Need finite memory. c. These system can be designed using convolution equation. d. It is non-recursive system.	$h[n]$ has infinite length. a. For causal IIR system, equation of convolution is $y[n] = \sum_{k=0}^{\infty} x[k]h[n-k]$ b. Need infinite memory. c. These system can be designed using difference equation. d. It is recursive system.

Property of FIR – Linear Phase

- **linear phase response:**

$$H(\omega) \;=\; |H(\omega)|e^{j\varphi)w(}$$

where the phase response $\varphi(\omega)$ is given by (15)

$$\varphi(\omega) = \;\omega\;\times n_0 + \beta \qquad (16)$$

- FIR has a linear phase response when

$$h[n] = h[M-1-n] \quad ---- \;\boldsymbol{symmetric} \qquad (17)$$

or

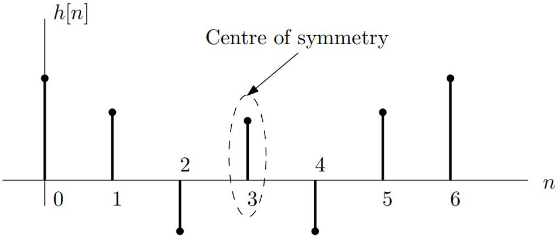
$$h[n] =- h[M-1-n] \;---- \;\boldsymbol{anti-symmetric} \qquad (18)$$

FIR Filters...

- Let us consider the symmetric case of the filter impulse response, i.e.,

$$h[n] = h[M - 1 - n] \tag{19}$$

- Suppose M is odd. An example when $M = 7$ is shown in the figure below



- We now show that the resulting FIR filter has a linear phase response.

FIR Filters...

The filter frequency response

$$\sum_{k'=0}^{\frac{M-1}{2}-1} h[k']\, e^{-j\omega(M-1-k')}$$

$$\Uparrow \quad k' = M-1-k$$

$$H(\omega) = \sum_{k=0}^{M-1} h[k]e^{-j\omega k} = \sum_{k=0}^{\frac{M-1}{2}-1} h[k]e^{-j\omega k} + h[\frac{M-1}{2}]e^{-j\omega\frac{M-1}{2}} + \sum_{k=\frac{M-1}{2}+1}^{M-1} h[k]\, e^{-j\omega k}$$

$$= \sum_{k=0}^{\frac{M-1}{2}-1} h[k](e^{-j\omega k} + e^{-j\omega(M-1-k)}) + h\left[\frac{M-1}{2}\right] e^{\,-j\omega(M-1)/2}$$

$$= e^{-j\omega(M-1)/2}\left[\sum_{k=0}^{\frac{M-1}{2}-1} h[k]\,(e^{-j\omega(k-\frac{M-1}{2})} + e^{j\omega(k-\frac{M-1}{2})})\right] + h[\frac{M-1}{2}]e^{-j\omega(M-1)/2}$$

$$= e^{-j\omega(M-1)/2}\underbrace{\left\{h\left[\frac{(M-1)}{2}\right] + 2\sum_{k=0}^{\frac{M-1}{2}-1} h[k]cos(\omega(k-\frac{M-1}{2}))\right\}}_{\text{real}}$$

FIR Filters...

The phase response

$$\angle H(\omega) = -\frac{\omega(M-1)}{2} + \beta$$

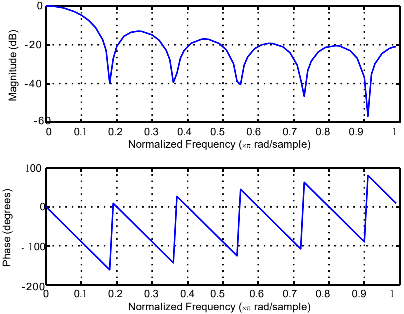
where β is 0, if $h[n] = h[M-1-n]$. If $h[n] = -h[M-1-n]$, β is $\frac{\pi}{2}$, and hence it is a linear function of ω in the generalized sense.

Example

The following figure shows the frequency response of a moving average filter of order 10.

Matlab code

```
N=11;
h=ones(1,N)/N;
freqz(h, 1,100);
```



FIR Filters...

- If the phase response $\angle H(\omega($ is linear with ω ,
  then $\frac{\angle H(\omega(}{\omega} = n_0$ for all frequencies. (20)
  That is, the filter delays all the frequency components of a signal by **the same amount** .

- A filter with linear phase response has no phase distortion. Thus linear phase response filters are important for applications that are sensitive to phase distortion such as image processing.

FIR Filters Design Methods...

- By designing the system impulse response properly, we can obtain *a frequency response satisfying desired filtering effects*.

  This process is commonly known as digital filter design.
- There are three basic design methods for FIR filters
  - windows
  - frequency sampling
  - equiripple desgin

FIR Filters Design Methods

Example

Consider the desired low-pass filter with frequency response

$$H_d(\Omega)\begin{cases} e^{-\frac{jM\Omega}{2}} & |\Omega| \leq \Omega_c \\ 0 & \Omega_c < |\Omega| \leq \pi \end{cases} \tag{21}$$

i.e., a digital ideal low-pass filter with a linear phase.

$\Omega_c = 0.2\pi$ radians, $M = 12$.

Solution

Basic idea: using $h[n]$ to design an FIR filter, i.e., Eq. (14).

Key: Find out $h[n]$, a finite length $h[n]$.

FIR Filters Design Methods...

Solution(continued)

The desired response is

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\Omega)e^{jn\Omega}d\Omega = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega(n-M/2)}\,\mathrm{d}\Omega \qquad (23)$$

Invoking the definition of the $Sa()$ function, $h_d[n]$ is

$$h_d[n] = \frac{\Omega_c}{\pi} Sa\left[\Omega_c\left(n - \frac{M}{2}\right)\right], \quad -\infty < n < \infty \qquad (24)$$

However, $h_d[n]$ is infinite length! We have to cut-off it.

For example: use rectangular window $w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & otherwise \end{cases}$ (25)

to cut $h_d[n]$ to $h[n]$: $h[n] = \begin{cases} \frac{\Omega_c}{\pi} Sa\left[\Omega_c\left(n - \frac{M}{2}\right)\right] & , \; 0 \le n \le M \\ 0, & otherwise \end{cases}$

In math: $h[n]$= $h_d[n]w[n]$.

$$(26)$$

FIR Filters Design Methods - Window Method

- window method:

$$h[n] = w]n[h_d]n[$$

where $w[n]$ is a finite-length window; equal to 0 outside $0 \leq n \leq M$

- *Multiplex in time domain = convolution in frequency domain.*

The frequency response of the FIR filter is

$$H(\omega) = \frac{1}{2\pi} H_d(\omega) * W(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\theta) W(\omega - \theta) d\theta$$

What does $H(\omega)$ look like? Still an ideal low-pass filter?

FIR Filters Design Methods - Window Method

- $H(\omega)$ is a "smeared" version of the desired response $H_d(\omega)$.

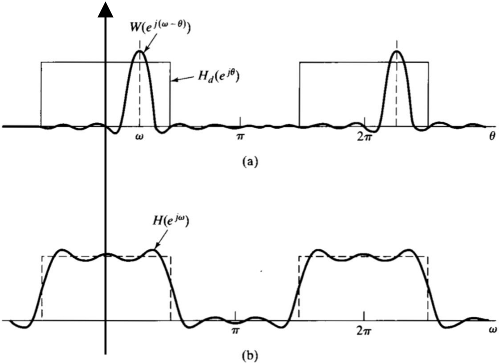


Figure 7.19 (a) Convolution process implied by truncation of the ideal impulse response. (b) Typical approximation resulting from windowing the ideal impulse response.

- Popular windows:
  - Rectangular

$$w[n] = \begin{cases} 1 & 0 \leq n \leq M \\ 0 & else \end{cases} \tag{34}$$

  - Hanning

$$w[n] = \begin{cases} 0.5 - 0.5\cos(\frac{2\pi n}{M}) & 0 \leq n \leq M \\ 0 & else \end{cases} \tag{35}$$

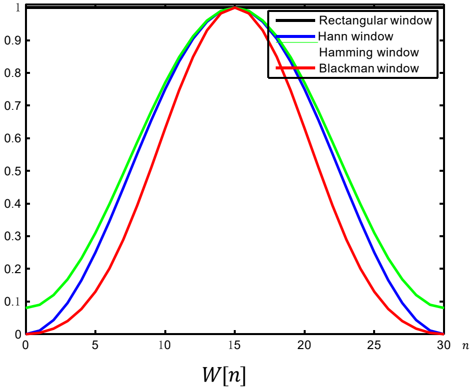
  - Hamming

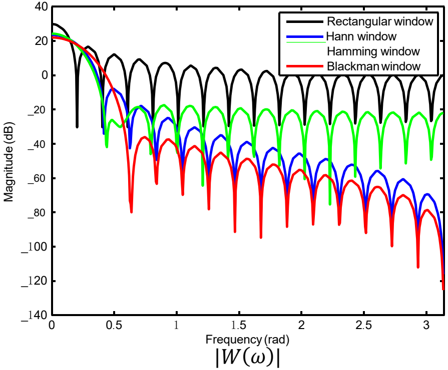
$$w[n] = \begin{cases} 0.54 - 0.46\cos(\frac{2\pi n}{M}) & 0 \leq n \leq M \\ 0 & else \end{cases} \tag{36}$$

  - Blackman

$$w[n] = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{M}\right) - 0.8\cos(\frac{4\pi n}{M}) & 0 \leq n \leq M \\ 0 & else \end{cases} \tag{37}$$

FIR Filters Design Methods - Window Method...





- The rectangular window clearly has the narrowest main lobe, and thus, for a given length, it should yield the sharpest transitions of $H(\omega)$ at a discontinuity of $H_d(\omega)$. However, the first side lobe is only about 13 dB below the main peak.

- The Hamming, Hanning, and Blackman windows, the side lobes (second column) are greatly reduced in amplitude; but a much wider main lobe and thus wider transition.

Example

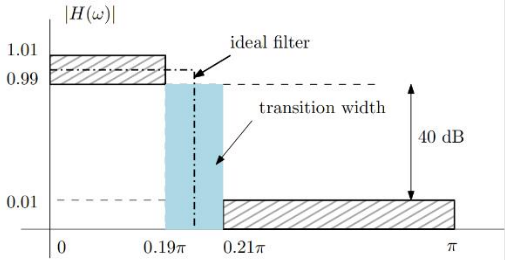
Suppose that we would like to design an FIR linear phase lowpass filter according to the following specifications:

$$0.99 \le |H(\omega)| \le 1.01 \qquad 0 \le |\omega| \le 0.19\pi$$

$$|H(\omega)| \le 0.01 \qquad 0.21\pi \le |\omega| \le \pi$$

Solution

- For a stopband attenuation of -40 dB, we may use a Hanning window.
- A transition width of $\Delta\omega = 0.02\pi$. To achieve this requirement, we can choose $M = 310$.



Solution (continued)

- We can set the cut-off frequency of the ideal lowpass filter to be $\omega_c = 0.2\pi$.

$$H_d(\omega) = \begin{cases} 1 & 0 \le |\omega| \le 0.2\pi \\ 0 & 0.2\pi \le |\omega| \le 2\pi \end{cases}$$

- The unit impulse response is:

$$h_d[n] = \frac{\omega_\pi}{\pi} sinc(\omega_c) = \frac{\sin(0.2\pi n)}{\pi n}$$

- shift $h_d[n]$ to the midpoint of the window

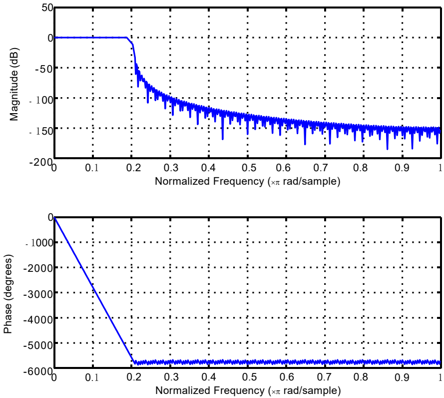
$$h_d\ [n] = \frac{\sin(0.2\pi(n-155))}{\pi(n-155)}$$

- The designed filter is given by

$$h[n] = \frac{\sin\left(0.2\pi(n-155)\right)}{\pi(n-155)}\left(0.5 - 0.5\cos\left(\frac{2\pi n}{M}\right)\right) \quad 0 \le n \le M$$

FIR Filters Design Methods - Window Method...

- The frequency response of the designed filter is shown in the figure below. We can check that it satisfies all the requirements.



FIR Filters Design Methods - Frequency sampling

- The desired frequency response $H_d)\omega($ is first uniformly sampled at N equally spaced points between 0 and 2π or between $-\pi$ and π.

$$H(k) = H_d(\omega)\Big|_{\omega=\frac{2\pi k}{M}} = H_d\left(\frac{2\pi k}{M}\right) = \,, \; k = 0,1,\ldots,M-1$$

  These frequency samples constitute an M-point DFT.

- The impulse response of FIR filter is computed by inverse DFT:

$$h[n] = \frac{1}{M}\sum_{k=0}^{M-1} H(k)e^{\frac{j2\pi nk}{M}}$$

- For linear phase filters, with positive symmetrical impulse response, we can write

$$h[n] = \frac{1}{M}\left[\sum_{k=0}^{\frac{M}{2}-1} 2|H(k)|cos\left[\frac{2\pi k(n-\alpha)}{M}\right] + H(0)\right]$$

where $\alpha = (N-1)/2$

FIR Filters Design Methods - Frequency sampling

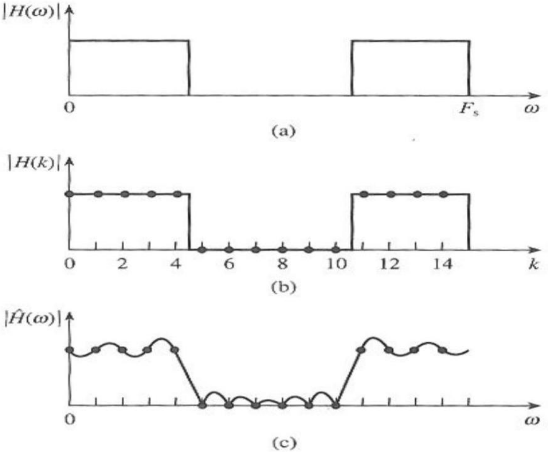


Figure 3 (a) Frequency response of an ideal lowpass filter. (b) Samples of the ideal lowpass filter. (c) Frequency response of lowpass filter derived from the frequency samples of (b).