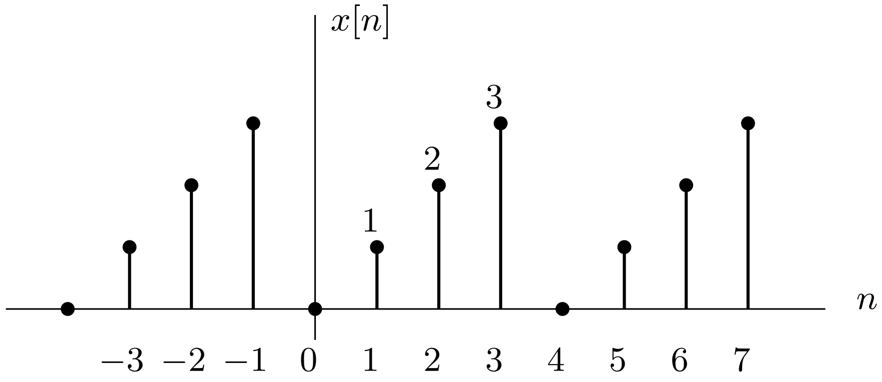
Tutorial 4 - Solutions

1. Determine the Fourier coefficients for the periodic sequence $x[n]$ shown in the figure below.



<u>**Solution**</u>: From the plot of $x[n]$ we know that $x[n]$ is a periodic signal with a period $N = 4$. Thus its discrete time Fourier series (DTFS) has $\boldsymbol{N} = 4$ **coefficients:** $X[0]$, $X[1]$, $X[2]$, $X[3]$. Recall that the definition of DTFS is

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \tag{1}$$

To compute $X[0]$, we substitute $k = 0$ to 1 which produces

$$X[0] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] = \frac{1}{4}(0 + 1 + 2 + 3) = \frac{3}{2} \tag{2}$$

For $X[1]$ we have

$$\begin{aligned} X[1] &= \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi n/4} = \frac{1}{4} \left(0 + \underbrace{e^{-j\pi/2}}_{-j} + 2 \underbrace{e^{-j\pi}}_{-1} + 3 \underbrace{e^{-j3\pi/2}}_{j} \right) \\ &= \frac{1}{4}(-j - 2 + 3j) = -1/2 + j/2, \end{aligned} \tag{3}$$

Continuing with $k = 2$ and $k = 3$ we can check that $X[2] = -1/2$ and $X[3] = -1/2 - j/2$.

2. Consider the discrete sinusoid $x[n] = 2\cos\left(\frac{8\pi n}{31}\right)$.

   (a) Find the fundamental period and fundamental frequency of $x[n]$.

   (b) Express $x[n]$ in terms of complex exponential functions.

(c) Find the discrete-time Fourier series (DTFS) coefficients of $x[n]$.

Solution:

(a) We write $x[n] = 2\cos\left(\frac{8\pi n}{31}\right) = 2\cos\left(2\pi\frac{4}{31}n\right)$. fundamental period N=31; fundamental frequency 1/N=1/31.

(b) Using Euler's formula, we can write

$$x[n] = e^{j\frac{8\pi n}{31}} + e^{-j\frac{8\pi n}{31}} \tag{4}$$

(c) The DTFS of $x[n]$ is defined as

$$x[n] \;=\; \textstyle\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N} = \displaystyle\sum_{k=0}^{N-1} X[k]e^{jn\Omega_0 k} \tag{5}$$

where $N = 31$ and $\Omega_0 = 2\pi/N$. In the above equation, $X[k]$, $k = 0, 1, \ldots 30$ are 31 Fourier coefficients (that we need to compute). Note that we can rewrite (4) as

$$\begin{aligned} x[n] &= e^{j4n\Omega_0} + e^{-j4n\Omega_0} \\ &= e^{j4n\Omega_0} + e^{j27n\Omega_0} \end{aligned} \tag{6}$$

Here we use that equality

$$e^{-j4n\Omega_0} = e^{-j4n2\pi/31}\underbrace{e^{j2\pi n}}_{1} = e^{j27n2\pi/31} = e^{j27n\Omega_0} \tag{7}$$

By matching (5) with (6), we can conclude that

$$\begin{aligned} X_4 &= 1 \\ X_{27} &= 1 \end{aligned}$$

3. Determine the discrete Fourier series representation for each of the following sequences.

(a) $x[n] = \cos(\frac{\pi}{3}n) + \sin(\frac{\pi}{4}n)$

(b) $x[n] = \cos^2(\frac{\pi}{8}n)$

Hint: Use Euler's formula.

Solution: We use the same steps as shown in the previous question, which can be summarised as

- Find the fundamental frequency of a signal.
- Use Euler's formula to express the signal as sum of complex exponential functions.
- Conclude the Fourier series.

(a) We rewrite $x[n]$ as $x[n] = 1/2(\cos(2\pi\frac{1}{8}n) + 1)$ to conclude that the fundamental period of $x[n]$ is $N = 8$ and thus the fundamental frequency is $\Omega_0 = 2\pi/N = \pi/4$. Using Euler's formula, we can write

$$x[n] = \frac{1}{2}\left(\frac{1}{2}e^{j\frac{\pi}{4}n} + \frac{1}{2}e^{-j\frac{\pi}{4}n} + 1\right) = \frac{1}{2}\left(\frac{1}{2}e^{j\Omega_0 n} + \frac{1}{2}e^{-j\Omega_0 n} + 1\right) \tag{8}$$

According to the DTFS representation we can write

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N} = \sum_{k=0}^{N-1} X[k]e^{jkn\Omega_0} \tag{9}$$

Since $e^{-j\Omega_0 n} = e^{(8-1)j\Omega_0 n} = e^{j\Omega_0 7n}$, we can write

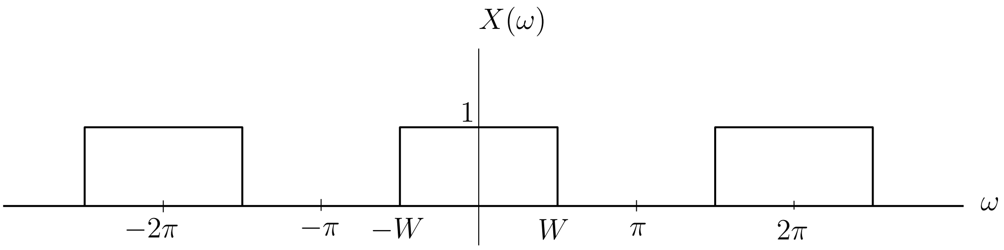
$$x[n] = \frac{1}{2}\left(\frac{1}{2}e^{jn\Omega_0} + \frac{1}{2}e^{j7n\Omega_0} + 1\right) \tag{10}$$

Comparing (9) and (10), we can see that $X[0] = 1/2\ X[1] = X[7] = 1/4$ and $X[k] = 0$ for $k = 2, 3, \ldots, 6$.

4. (a) Find the inverse Fourier transform $x[n]$ of the rectangular pulse spectrum $X(\omega)$ defined by

$$X(\omega) = \begin{cases} 1 & |\omega| \leq W \\ 0 & W < |\omega| \leq \pi \end{cases} \tag{11}$$

which is shown in the following figure



(b) Plot $x[n]$ for $W = \pi/4$.

Solution:

(a) The inverse Fourier transform of $X(\omega)$ will produce the signal $x[n]$. By the definition of inverse Fourier transform we have

$$x[n] = \frac{1}{2\pi}\int_{<2\pi>} X(\omega)e^{j\omega n}d\omega \tag{12}$$

In the above equation the integral is compute over a period of 2π. (Recall that Fourier transform of discrete signals is periodic with a period of 2π). Thus equation (12) becomes

$$x[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}d\omega = \frac{1}{j2\pi n}\left(e^{jWn} - e^{-jWn}\right) = \frac{\sin(Wn)}{\pi n} \tag{13}$$

5. Find the DTFT of each of the following sequences:

(a) $x[n] = \left(\frac{1}{2}\right)^n u[n+3]$

(b) $x[n] = \alpha^n \sin(n\omega_0)u[n]$

Solution:

(a) By definition, the DTFT of $x[n]$ is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} = \sum_{n=-3}^{\infty} \left(\frac{1}{2}\right)^n e^{-jn\omega}$$
$$= 8e^{j3\omega} \sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\omega}\right)^n \tag{14}$$

To evaluate the finite series in the above equation, we use the inequality

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

for $|a| < 1$. Applying the above equality with $a = \frac{1}{2}e^{-j\omega}$we have

$$X(\omega) = \frac{8e^{j3\omega}}{1 - \frac{1}{2}e^{-j\omega}} \tag{15}$$

(b) First we write $x[n]$ as

$$x[n] \;=\; \alpha^n \sin(n\omega_0)u[n] = \frac{1}{2j}\left[\alpha^n e^{jn\omega_0} - \alpha^n e^{-jn\omega_0}\right] u[n]$$
$$= \frac{1}{2j}\left[\left(\alpha e^{j\omega_0}\right)^n - \left(\alpha e^{-j\omega_0}\right)^n\right] u[n] \tag{16}$$

In the above equation we have used the fact that

$$\sin(n\omega_0) = \frac{1}{2j}\left(e^{jn\omega_0} - e^{-jn\omega_0}\right) \tag{17}$$

Using the same steps as done for (a), we can see that the DTFT of $x[n]$ is given by

$$X(\omega) = \frac{1}{2j}\left[\frac{1}{1 - \alpha e^{-j(\omega-\omega_0)}} - \frac{1}{1 - \alpha e^{-j(\omega+\omega_0)}}\right] = \frac{\alpha \sin(\omega_0)e^{-j\omega}}{1 - 2\alpha\cos(\omega_0)e^{-j\omega} + \alpha^2 e^{-j2\omega}}$$

.

Determine the magnitude frequency response.