8. Frequency Domain Analysis – Nyquist & Bode

8.1 Introduction

- Previous work, both in this module and the first year EE114 module, concentrated on time-domain analysis, where the output of a system and its behavior were viewed and analysed from the perspective of time.
- The **frequency response** method offers a practical and important alternative approach to the design and analysis of a system.
- Here, the **response of a system is considered for sinusoidal inputs across the whole range of frequencies**.
- The resulting output waveforms for a **linear** system are sinusoidal in steady-state. They differ from the inputs only in **amplitude** and **phase angle**.
- **Frequency response is a steady-state output.** Thus, performance measures such as peak overshoot, settling time, etc. are not used here.
- One advantage of the frequency response method is the ready availability of sinusoid test signals for various ranges of frequencies and amplitudes.
- Thus the experimental determination of the frequency response of a system is easily obtained and is the most reliable and uncomplicated method for the **experimental analysis of a system**.
- A second advantage is that the transfer function describing the sinusoidal steady-state behaviour of a system can be obtained by simply replacing s with $j\omega$ in the system transfer function, i.e. $G(s) \rightarrow G(j\omega)$.
- This property exists because of the close relationship between Laplace and Fourier transforms – see text book for details.
- The magnitude and phase of the complex function $G(j\omega)$ are readily represented by graphical plots that provide important insight into the analysis and design of control systems.
- Here, we are going to consider two different graphical plots, namely the **Nyquist diagram,** and the **Bode plots**. Also, we are only considering continuous-time systems for now.
- *In the case of Bode plots, you will cover (or have already done so!) some examples of these in your circuits module EE215.*

8.2 The Nyquist Diagram

The Nyquist Diagram

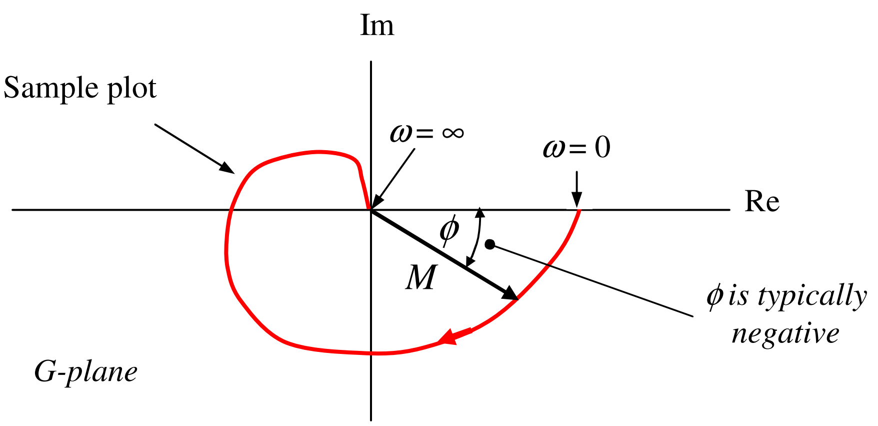
- We start with the transfer function $G(s)$ and **set $s = j\omega$** to get the frequency response function:

$$G(j\omega) = M \angle \phi \quad \text{(in polar form)}$$

- The **Nyquist diagram** gives the **polar plot** as ω varies from -∞ to ∞.

- *Note – in these lecture notes, the Nyquist plot is only shown for the range 0 to ∞. The other half (ω = -∞ to 0^-) is obtained from symmetry about the real axis. In Matlab, the complete plot is usually shown.*

- A sample Nyquist plot is shown below:



Simple Nyquist Diagram Examples

- The best way of illustrating the Nyquist diagram and Nyquist's criterion is using examples.

- **Example 8.1** – Determine the Nyquist diagram for $G(s) = \dfrac{k}{1+sT}$, where T is simply a constant value. Initially assume that k=1.

- **Solution**: Substituting $s = j\omega$ gives:

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1-j\omega T}{1+\omega^2T^2} \equiv X + jY$$

$$\text{Thus:} \quad X = \frac{1}{1+\omega^2T^2}, \quad Y = \frac{-\omega T}{1+\omega^2T^2} \quad \Rightarrow \frac{Y}{X} = -\omega T$$

1、使用X + jY的笛卡尔形式

- Substituting the last result into the equation for X gives:

$$X = \frac{1}{1 + \frac{Y^2}{X^2}} = \frac{X^2}{X^2 + Y^2}$$

- Cross-multiplying gives: $X^2 + Y^2 = X$

- By *completing the square*†, we obtain the expression: $(X - \frac{1}{2})^2 + Y^2 = (\frac{1}{2})^2$ 圆

† *Completing the square works as follows:*

> *Consider the quadratic equation $s^2 + as + b = 0$. We want to complete the square so as to express the equation in the format $(s + x)^2 = y$. How do we do this?*
>
> *Simple rule: $x = \frac{a}{2}$, i.e. half the coefficient of s, and $y = (\frac{a}{2})^2 - b$.*
> *This is easily verified as follow:*

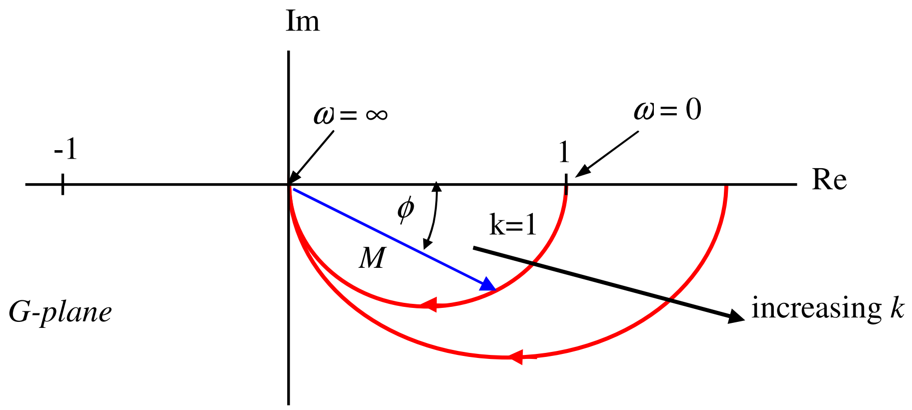
$$(s + \tfrac{a}{2})^2 = (\tfrac{a}{2})^2 - b \quad \Rightarrow s^2 + as + (\tfrac{a}{2})^2 = (\tfrac{a}{2})^2 - b \quad \Rightarrow s^2 + as + b = 0$$

- $(X - \frac{1}{2})^2 + Y^2 = (\frac{1}{2})^2$ represents a circle with centre (½, 0) and radius ½. This is the equation of **the nyquist plot for gain k = 1.**

- For general value of k, it is easily shown that the equation for the Nyquist plot is:

$$X^2 + Y^2 = kX \text{ or } (X - \tfrac{k}{2})^2 + Y^2 = (\tfrac{k}{2})^2$$

- Hence the following Nyquist diagram:



2、使用极坐标形式　　　　比较通用，但是麻烦

- **An alternative (and quicker) method** for obtaining the Nyquist plot is through the **use of polar form**, as opposed to the Cartesian form of $X + jY$. Thus:

$$\text{GH}(j\omega) \;=\; \frac{1}{1+j\omega T} = \frac{1\angle 0°}{\sqrt{1+(\omega T)^2}\,\angle\tan^{-1}(\omega T)}$$

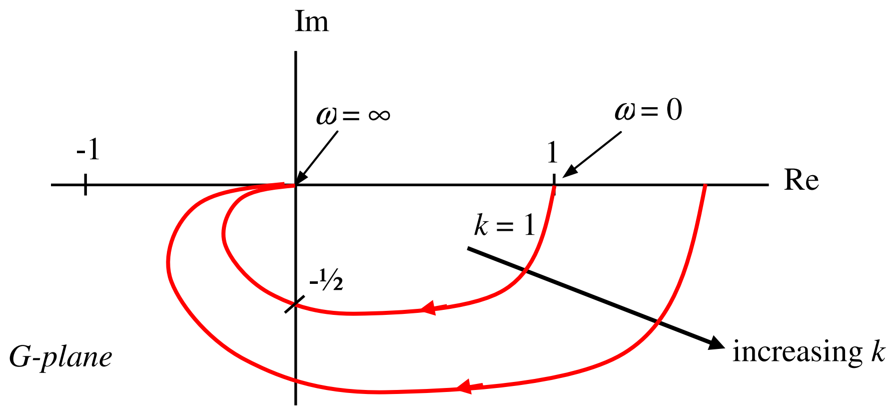
$$= \frac{1}{\sqrt{1+(\omega T)^2}}\angle -\tan^{-1}(\omega T) \;\equiv\; \text{M}\angle\phi$$

- When $\omega = 0$, $M = 1$ and $\phi = 0°$.
- When $\omega = \infty$, $M = 0$ and $\phi = -90°$.
- When $\omega = \frac{1}{T}$, $M = \frac{1}{\sqrt{2}}$ and $\phi = -45°$.
- Thus, for $k = 1$, the plot starts at a phase $\phi = 0°$ and a magnitude $M = 1$ and ends at $\phi = -90°$ and $M = 0$.
- Using a number of other values for ω, we can easily trace out the Nyquist plot, as shown earlier.
- **Example 8.2** - Use Nyquist's criterion to determine the range of values of k for which the system is stable, given that $G(s) = \dfrac{1}{(1+sT)^2}$.
- **Solution (using polar form):**

$$\text{GH}(j\omega) = \frac{1}{(1+j\omega T)^2} = \frac{1\angle 0°}{\left(\sqrt{1+(\omega T)^2}\right)^2\angle 2.\tan^{-1}(\omega T)}$$

$$\Rightarrow \text{GH}(j\omega) = \frac{1}{1+(\omega T)^2}\angle -2.\tan^{-1}(\omega T)$$

- For $\omega = 0$, $\qquad$ $\text{GH}(j0) = 1\angle\, 0°$
- For $\omega = \frac{1}{T}$, $\qquad$ $\text{GH}(j\frac{1}{T}) = \frac{1}{2}\angle\, -90°$
- For $\omega \to \infty$, $\qquad$ $\text{GH}(j\infty) \to \frac{1}{\infty}\angle\, -180° \;=\; 0\angle\, -180°$
- We can now deduce that the Nyquist plot starts at (1, j0), passes through (0, -j½) and approaches the origin along the **negative real axis**, since the angle of the end point is –180° (had it been -90°, for example, then it would have been approaching along the negative imaginary axis instead!).
- As gain k increases, the magnitude (which is the distance from the origin to the curve) will move out along a radial line as shown in the diagram below:



- For combinations of systems (e.g. first-order systems),

$$\mathrm{G}(s) = G_1(s)G_2(s)G_3(s)$$

represent each in polar form:

$$\mathrm{G}(s) = r_1 e^{j\theta_1} r_2 e^{j\theta_2} r_3 e^{j\theta_3} \quad = \quad r_1 r_2 r_3 e^{j(\theta_1+\theta_2+\theta_3)}$$

- Example

$$\mathrm{G}(s) = \frac{1}{(1+sT)^2} = \frac{1}{1+sT} \cdot \frac{1}{1+sT}$$

- Each term is represented by a semi-circle and the resultant plot is obtained by **multiplying** the magnitudes and **adding** the phase angles:

Start: Magnitude (M) = 1.1 = 1 & Phase (ϕ) = 0° + 0° = 0°
End: Magnitude (M) = 0.0 = 0 & Phase (ϕ) = -90° + -90° = -180°

- Thus the resultant plot (for k = 1) starts at (1, j0) and ends at the origin (i.e. zero magnitude) with a phase angle of -180° (as before!).

- **Example 8.3** – Plot the Nyquist diagram for $G(s) = \dfrac{k}{(1+sT)^3}$. Initially assume k=1.

- **Solution**: Here, we will deduce the plot from that of the first-order system:

$$\mathrm{G}(s) = \frac{1}{(1+sT)^3} = \frac{1}{1+sT} \cdot \frac{1}{1+sT} \cdot \frac{1}{1+sT}$$

- Magnitude is:

$$|G(j\omega)| = \frac{1}{|1+j\omega T|}\cdot\frac{1}{|1+j\omega T|}\cdot\frac{1}{|1+j\omega T|} = 1\Big/\left(\sqrt{1+\omega^2T^2}\right)^3$$

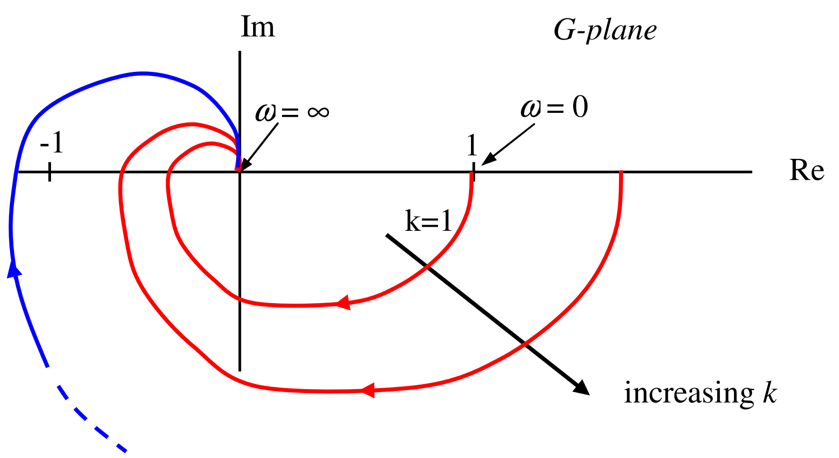
- Phase is:

$$Phase(G(j\omega)) = -\tan^{-1}(\omega T) - \tan^{-1}(\omega T) - \tan^{-1}(\omega T) = -3\tan^{-1}(\omega T)$$

- The resultant plot is obtained by multiplying the magnitudes and adding the phase angles:

  Start: Magnitude (M) = 1.1.1 = 1
  End: Phase (ϕ) = -90° + -90° + -90° = -270°

- In between the start and end points, need to put values for ω into expressions for magnitude and phase.
- Thus the resultant plot (for k = 1) starts at (1, j0) and ends at the origin (i.e. zero magnitude) with a phase angle of -270° as shown below:



- As gain *k* increases the magnitude (which is the distance from the origin to the curve) will move out along a radial line.

<u>Numerical example</u>

Plot the frequency response of the system:

$$G(s) = \frac{1+10s}{(1+s)(1+2s)(1+4s)} \quad = \quad \frac{1}{1+s}\cdot\frac{1}{1+2s}\cdot\frac{1}{1+4s}\cdot\frac{1+10s}{1}$$

Magnitude:

$$|G(j\omega)| = \frac{1}{|1+j\omega|}\cdot\frac{1}{|1+j\omega 2|}\cdot\frac{1}{|1+j\omega 4|}\cdot|1+j\omega 10| = \frac{\sqrt{1+\omega^2 100}}{\sqrt{1+\omega^2}\,\sqrt{1+\omega^2 4}\sqrt{1+\omega^2 16}}$$

Phase:

$$Phase(G(j\omega)) = -\tan^{-1}(\omega) - \tan^{-1}(2\omega) - \tan^{-1}(\omega 4) + \tan^{-1}(10\omega)$$

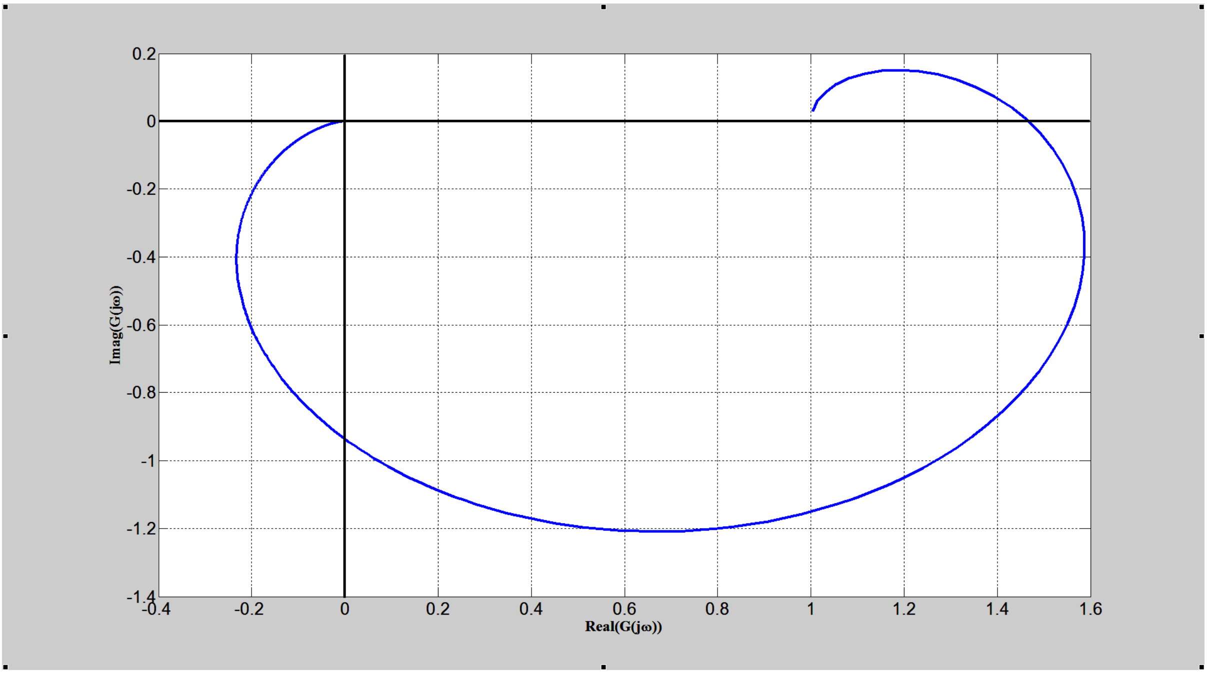


Figure generated using:

sys = zpk([-1/10],[-1 -1/2 -1/4],1.25) ----> Zero/pole/gain:

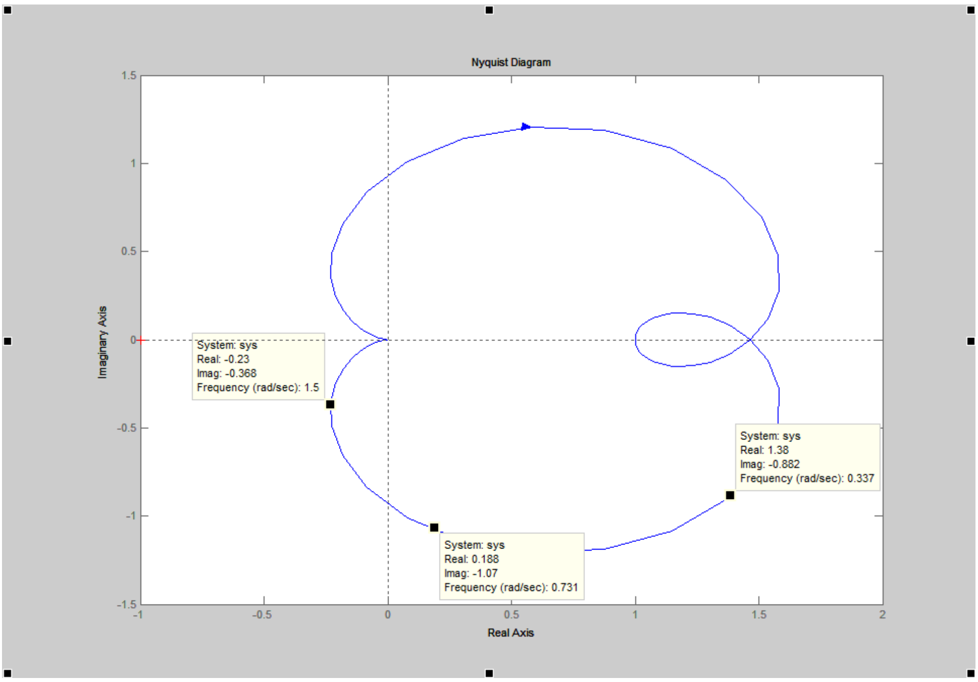
```
   1.25 (s+0.1)
----------------------
(s+1) (s+0.5) (s+0.25)
```

```
w = 0.01*(1:10000);
H = freqresp(sys,w);
NewH = squeeze(H);
Mag = abs(NewH);
Ang = angle(NewH);
polar(Ang,Mag)
```

OR

```
ReH = real(NewH);
ImH = imag(NewH);
```

nyquist(sys) ---->



A key issue is choosing the correct frequency points!

Initially, choose $\omega = 0, \infty, 0.01\ 0.1, 1, 10, 100$, then fill in gaps in plot!

8.3 Bode Diagrams

- Bode diagrams are an alternative means of representing frequency plots.

- They are much easier to draw than polar frequency plots as they are essentially asymptotic diagrams which consist of straight lines.

- In Bode diagrams, the magnitude (in decibels, dB) and the phase angle are plotted separately as functions of frequency. Thus, two diagrams are required.

- The gain in decibels is given by $\mathbf{M_{dB} = 20 \log_{10} |GH(j\omega)|}$.

- Since the logarithm of a product is the sum of the logarithms of the factors, i.e. log(ab) = log(a) + log(b), combining factors is a matter of simple addition.

- Hence the Bode Diagram is a very useful technique for complex transfer functions.

- The table below shows several values of M and their dB equivalent:

M	1	2	4	8	10	0.5	0.1
M_{dB}	0	6	12	18	20	-6	-20

Amplification (+ve dB, M >1) — Attenuation (-ve dB, M <1)

- Note that doubling M is equivalent to +6 dB and halving M gives –6 dB.

- Note also that M x 10 is equivalent to +20 dB and M / 10 gives –20 dB.

- To fit a wide range on to the frequency response graphs, we use logarithm scale graph paper.

- Hence, we plot magnitude and phase against $\log_{10}\omega$ rather than ω itself.

Simple Bode Diagram Examples

- **Example 8.7** – Determine the Bode diagram for the low pass filter (LPF) given by the transfer function:

$$G(s) = \frac{1}{1+s\tau} \quad , \quad G(j\omega) = \frac{1}{1+j\omega\tau}$$

where τ is simply a constant value (it actually represents the time constant of the filter network).

- The logarithmic gain is given by:

$$M_{dB} = 20\log|G(j\omega)| \;=\; 20\log\left(\frac{1}{1+(\omega\tau)^2}\right)^{\frac{1}{2}} \;=\; -10\log(1+(\omega\tau)^2) \qquad **$$

Recall that: $20\log_{10}(1/X)^Y = -20\,Y\log_{10}(X)$

- The phase is: $\phi = -\tan^{-1}(\omega\tau)$

- For $\omega \ll \frac{1}{\tau}$ (i.e. $\omega\tau \ll 1$): $\phi = 0°$ and $M_{dB} = -10\log(1) = 0$ dB

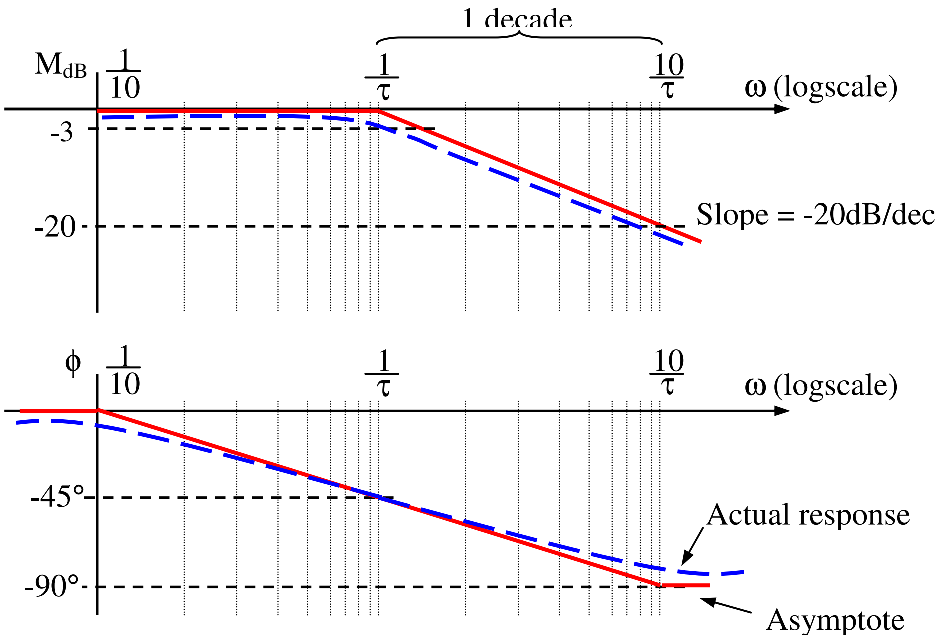
- For $\omega \gg \frac{1}{\tau}$ (i.e. $\omega\tau \gg 1$): $\phi = -90°$ and

  $M_{dB} = -10\log(\omega\tau)^2 = -20\log(\omega\tau)$ dB $\Rightarrow$ $M_{dB} = -20\log(\omega) - 20\log(\tau)$

  This is equivalent to $y = mx + c$, i.e. M_{dB} is a straight line of slope –20 dB / decade

- For $\omega = \frac{1}{\tau}$: $\phi = -45°$ and $M_{dB} = -10\log(2) \approx -3$ dB (from **)

- Hence, the **asymptotic Bode diagram** for the LPF can now be drawn as follows:



- Note that $\omega = \frac{1}{\tau}$ is often referred to as the **corner frequency** or the **break frequency**.

- Here, the bandwidth is $\frac{1}{\tau}$. In general, it is defined as the frequency range over which $M_{dB} > -3$ dB.

- A piecewise linear phase approximation is not as easy because the high and low frequency asymptotes don't intersect. Instead we use a rule that follows the exact function fairly closely, but is also arbitrary. Its main advantage is that it is easy to remember. The rule can be stated as: ***Follow the low frequency asymptote until one tenth the break frequency (0.1 $\frac{1}{\tau}$) then decrease linearly to meet the high frequency asymptote at ten times the break frequency (10 $\frac{1}{\tau}$).*** Note that there is no error at the break frequency and about 5.7° of error at one tenth and ten times the break frequency.

- ***Exercise:*** *Sketch the Bode diagram for a system with transfer function ($1+5s$)$^{-1}$. Note that analytical detail is not required!*

- **Example 8.8** – Determine the Bode diagram for the transfer function of a pure integrator, $G(j\omega) = \dfrac{1}{j\omega\tau}$.

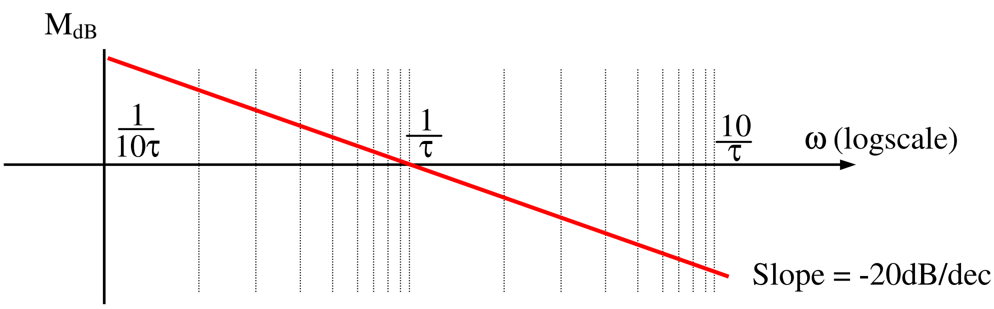
- **Solution**:

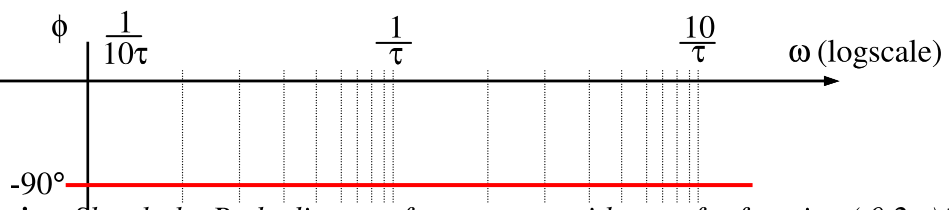
  $\phi = -90°$ and

  $M_{dB} = -20 \log(\omega\tau)$ dB $= -20 \log(\omega) - 20 \log(\tau)$

  i.e. M_{dB} is a straight line of slope –20 dB / decade

- For $\omega = \frac{1}{\tau}$, the logarithmic gain is 0 dB.
- Hence, the asymptotic Bode diagrams for the pure integrator can now be drawn as follows:

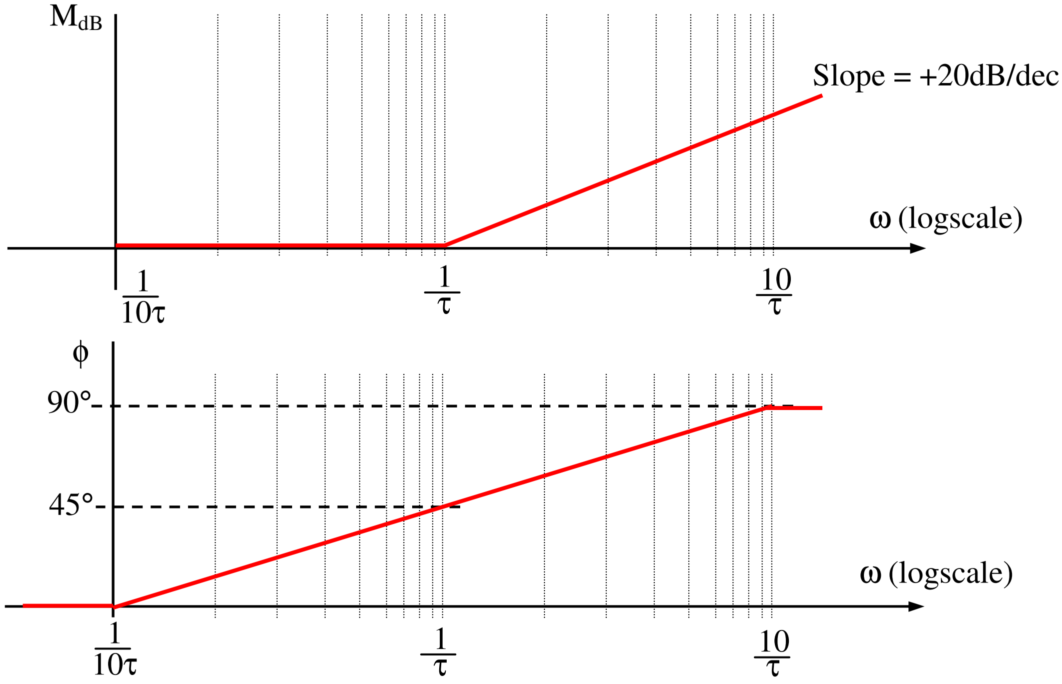




- ***Exercise:*** *Sketch the Bode diagram for a system with transfer function ($0.2s$)$^{-1}$.*

- **Example 8.9** – Determine the Bode diagram for the transfer function $G(j\omega) = 1 + j\omega\tau$.

- **Solution**: Note that $\log\left(\dfrac{1}{1+j\omega\tau}\right) = -\log(1+j\omega\tau)$ **.**
- Hence the bode diagram for $1+j\omega\tau$ is simply the negative of the plot for $1/(1+j\omega\tau)$.

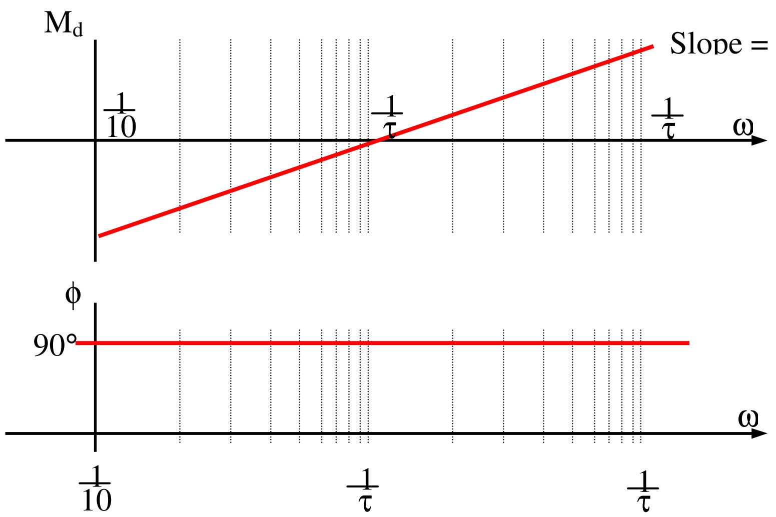


- ***Exercise:*** *Sketch the Bode diagram for a system with transfer function (1+5s).*

- **Example 8.10** – Determine the Bode diagram for the transfer function $G(j\omega) = j\omega\tau$.

- **Solution**: Again, note that $\log\left(\dfrac{1}{j\omega\tau}\right) = -\log(j\omega\tau)$ **=>** negative of the plot for

$1/j\omega\tau$.



- ***Exercise:*** *Sketch the Bode diagram for a system with transfer function (0.2s).*

- One of the main advantages of using Bode plots is that the frequency response of more complicated networks can easily be constructed by superposition (see example below).

- The Bode diagrams in the last four examples provide the *basic building blocks* for more complicated transfer functions.

- ***Note – you should be able to easily and quickly sketch the bode diagrams of the basic building blocks.***

- ***The shape of each one will always remain the same - only the break frequency value will differ for different examples.***

- **Example 8.11** – Determine the asymptotic Bode diagram for the following transfer function:

$$GH(j\omega) = \frac{50}{j\omega(5 + j\omega)^2}$$

- **Solution**: In order to facilitate the sketching of Bode diagrams it is often desirable to express the transfer function in terms such as:

$$\frac{j\omega}{\omega_1} \text{ or } 1 + \frac{j\omega}{\omega_1}$$

where ω_1 is then the break frequency, i.e. the frequency at which the gain is 0 dB.

- Thus, we rewrite the transfer function as follows:

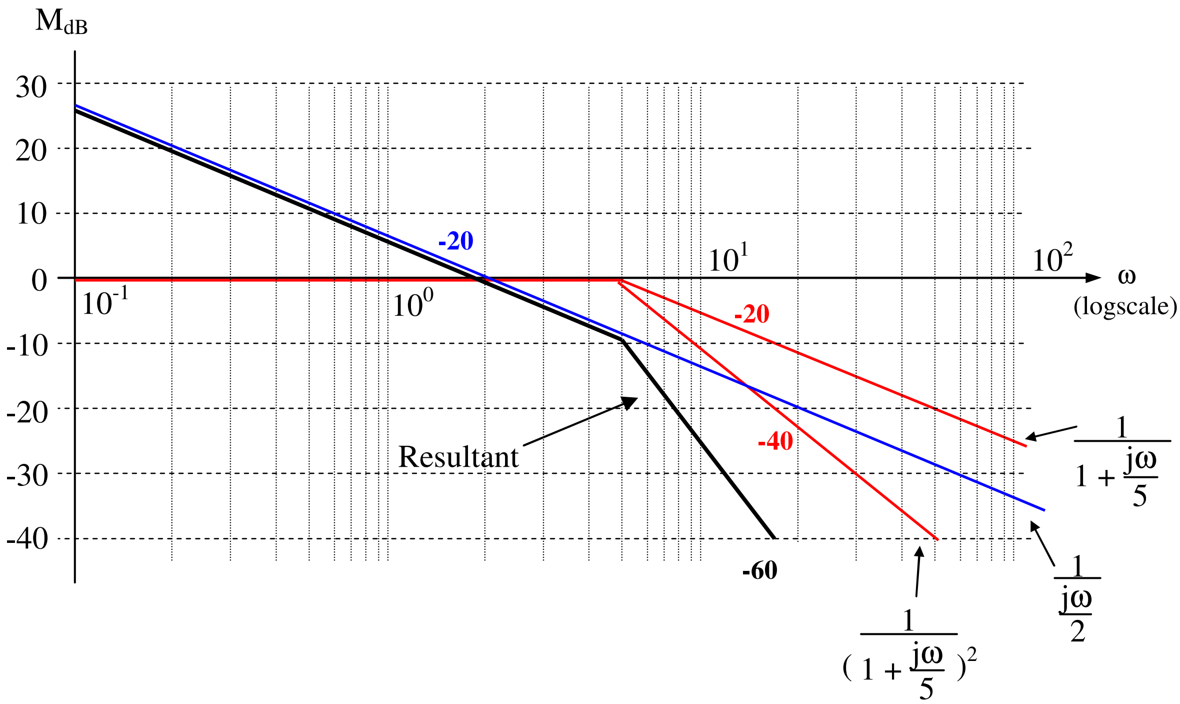
$$GH(j\omega) = \frac{1}{\frac{j\omega}{2}(1 + \frac{j\omega}{5})^2}$$

- The factors of this transfer function are an integrator and two identical first-order LPFs.

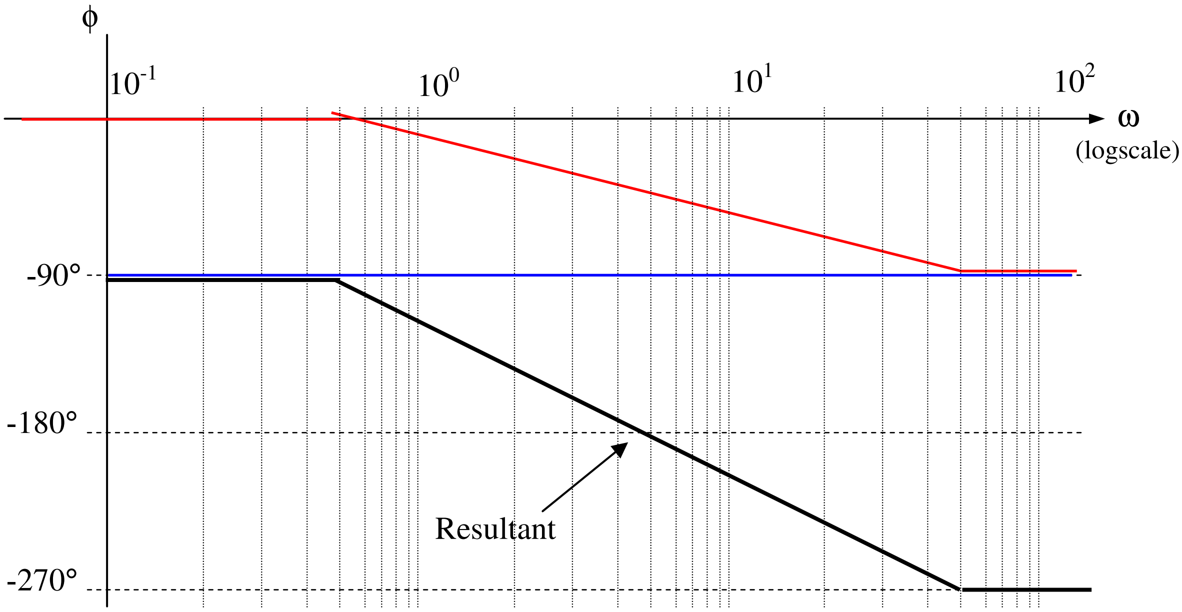
- Thus, we can construct the Bode diagram of the transfer function from the Bode diagrams of the individual factors as shown in the next diagram.

- *Alternatively:* $$\frac{50}{j\omega(5 + j\omega)^2} = \frac{2}{j\omega(1 + \frac{j\omega}{5})^2}.$$

This consists of one integrator (breakpoint of 1 rad/s), two LPFs (as before) and a gain of 2. The gain needs to be converted to dB, i.e. $20\log_{10}(2) = 6dB$. This corresponds to a straight line on the gain plot only at 6dB. Adding all the terms together will result in the same final bode diagrams as with the previous breakdown.



M_{dB}
30
20
10
0
-10
-20
-30
-40
10^{-1}
10^{0}
10^{1}
10^{2}
ω
(logscale)
-20
-20
-40
-60
Resultant
$\frac{1}{1+\frac{j\omega}{5}}$
$\frac{1}{\frac{j\omega}{2}}$
$\frac{1}{(1+\frac{j\omega}{5})^2}$



φ
10^{-1}
10^{0}
10^{1}
10^{2}
ω
(logscale)
-90°
-180°
-270°
Resultant