Tutorial 3 - Solutions

1. For sinusoids the best way to find their Fourier series is to use Euler's formula which is written as
$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta} \tag{1}$$
As a result we have
$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \tag{2}$$
$$\sin\theta = \frac{1}{j2}\left(e^{j\theta} - e^{-j\theta}\right) = \frac{-j}{2}\left(e^{j\theta} - e^{-j\theta}\right) = \frac{-j}{2}e^{j\theta} + \frac{j}{2}e^{-j\theta} \tag{3}$$
Now recall that for a signal $x(t)$ with frequency f_0 (Hz), then its Fourier series has the general form given by
$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi n f_0 t} = x_0 + \underbrace{x_1 e^{j2\pi f_0 t}}_{n=1} + \underbrace{x_{-1} e^{-j2\pi f_0 t}}_{n=-1} + + \underbrace{x_2 e^{j4\pi f_0 t}}_{n=2} + \underbrace{x_{-2} e^{-j4\pi f_0 t}}_{n=-2} + \underbrace{x_3 e^{j6\pi f_0 t}}_{n=3} + \underbrace{x_{-3} e^{-j6\pi f_0 t}}_{n=-3} + \cdots \tag{4}$$
If we denote $\omega_0 = 2\pi f_0$ to be the angular frequency (rads), the Fourier series can be equivalently written as
$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t} = \underbrace{x_0}_{n=0} + \underbrace{x_1 e^{j\omega_0 t}}_{n=1} + \underbrace{x_{-1} e^{-j\omega_0 t}}_{n=-1} + + \underbrace{x_2 e^{j2\omega_0 t}}_{n=2} + \underbrace{x_{-2} e^{-j2\omega_0 t}}_{n=-2} + \underbrace{x_3 e^{j3\omega_0 t}}_{n=3} + \underbrace{x_{-3} e^{-j3\omega_0 t}}_{n=-3} + \cdots \tag{5}$$
Depending on the given information (i.e., the normal frequency or the angular frequency), we can use either (4) or (5) to find the Fourier series of sinusoids.

   (a) For $x(t) = \cos(\omega_0 t)$, we can see that its **angular frequency** is ω_0. Using equation (2) we can rewrite $x(t)$ as
$$x(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} \tag{6}$$
   Equation (6) implies that $x(t)$ contains only two complex sinusoids having the same angular frequency ω_0. Thus, the Fourier series of $x(t)$ contains only two non-zero coefficients corresponding to $n = 1$ (associated with the term $e^{j\omega_0 t}$) and $n = -1$ (associated with the term $e^{-j\omega_0 t}$). Specifically, matching (6) and (5), we can conclude that the Fourier series of $x(t)$ is given by
$$x_n = \begin{cases} 0 & n = 0, \pm 2, \pm 3 \\ \frac{1}{2} & n = \pm 1 \end{cases} \tag{7}$$
   .

(b) For $x(t) = \cos(2t + \frac{\pi}{4})$ we follow the same steps above. First note that the angular frequency of $x(t)$ is 2 (rads). Then rewrite $x(t)$ as

$$\begin{aligned} \cos(2t + \frac{\pi}{4}) &= \frac{1}{2}e^{j\left(2t+\frac{\pi}{4}\right)} + \frac{1}{2}e^{-j\left(2t+\frac{\pi}{4}\right)} && (8) \\ &= \frac{1}{2}e^{j\frac{\pi}{4}}e^{j2t} + \frac{1}{2}e^{-j\frac{\pi}{4}}e^{-j2t} && (9) \end{aligned}$$

Now we can conclude that the Fourier series of $x(t)$ is given by

$$x_n = \begin{cases} 0 & n = 0, \pm 2, \pm 3 \\ \frac{1}{2}e^{j\frac{\pi}{4}} = \frac{1}{2\sqrt{2}}(1+j) & n = 1 \\ \frac{1}{2}e^{-j\frac{\pi}{4}} = \frac{1}{2\sqrt{2}}(1-j) & n = -1 \end{cases} \quad (10)$$

Note that we have $x_{-1} = x_1^*$. That is, the conjugate of x_1 is equal to x_{-1}.

(c) The signal $x(t) = \cos 4t + \sin 6t$ is a sum of two sinusoids of different frequencies. Thus, $x(t)$ is not a sinusoid but still a periodic signal. To find the Fourier series of $x(t)$ we need to two steps:

Step 1: Find the fundamental frequency of $x(t)$

Step 2: Express $x(t)$ as sum of complex exponentials and conclude its Fourier series. Now to find the frequency of $x(t)$, we note the following

- The sinusoid $\cos 4t$ has an angular frequency of $\omega_1 = 4$ (rads) and thus its period is $T_1 = \frac{1}{f_1} = \frac{2\pi}{\omega_1} = \pi/2$ (secs).
- The sinusoid $\sin 6t$ has an angular frequency of $\omega_2 = 6$ (rads) and thus its period is $T_2 = \frac{1}{f_2} = \frac{2\pi}{\omega_2} = \pi/3$ (secs).

To find the period of $x(t)$, we consider the ratio between T_1 and T_2

$$\frac{T_1}{T_2} = \frac{\pi/2}{\pi/3} = \frac{3}{2} \quad (11)$$

and thus we have

$$2T_1 = 3T_2 \quad (12)$$

From the above equation, we can conclude that the period of $x(t)$ is

$$\boxed{T = 2T_1 = 3T_2 = \pi \text{ (secs)}} \quad (13)$$

and thus is angular frequency is given by

$$\boxed{\omega = 2\pi f = 2\pi \times \frac{1}{T} = 2} \quad (14)$$

The second part is to express $x(t)$ as sum of complex exponentials. It is clear that we can write $x(t)$ as

$$x(t) = \underbrace{\frac{1}{2}e^{j4t} + \frac{1}{2}e^{-j4t}}_{\cos(4t)} + \underbrace{\frac{-j}{2}e^{j6t} + \frac{j}{2}e^{-j6t}}_{\sin(6t)} \quad (15)$$

To conclude the Fourier series of $x(t)$ we need to rewrite the exponentials in term of the basic exponential which is e^{j2t} in this case because the angular frequency of $x(t)$ is 2. Now we have

$$x(t) = \frac{1}{2}e^{j2\times 2t} + \frac{1}{2}e^{-j2\times 2t} + \frac{-j}{2}e^{j3\times 2t} + \frac{j}{2}e^{-j3\times 2t} \tag{16}$$

The above expression means that the term $\frac{1}{2}e^{j2\times 2t}$ corresponds to the Fourier series coefficient x_2 and $x_2 = \frac{1}{2}$. Similarly, the term $\frac{j}{2}e^{-j3\times 2t}$ corresponds to the Fourier series coefficient x_{-3} and $x_{-3} = \frac{j}{2}$. In summary, the Fourier series of $x(t)$ has only 4 non-zero coefficients given by

$$x_n = \begin{cases} \frac{1}{2} & n = 2 \\ \frac{1}{2} & n = -2 \\ \frac{-j}{2} & n = 3 \\ \frac{j}{2} & n = -3 \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

(d) We simply rewrite $x(t)$ as $x(t) = \sin^2(t) = \frac{1}{2}(1 - \cos(2t))$. Thus $x(t)$ is periodic with an angular frequency of 2 rads. We can further express $x(t)$ as sum of exponentials as

$$x(t) = \frac{1}{2} - \frac{1}{4}e^{j2t} - \frac{1}{4}e^{-j2t} \tag{18}$$

Thus we can conclude that the Fourier series of $x(t)$ has 3 non-zero coefficients

$$x_n = \begin{cases} \frac{1}{2} & n = 0 \\ -\frac{1}{4} & n = -1 \\ -\frac{1}{4} & n = 1 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

2. (a) The fundamental frequency is $f = 100$ Hz and period is $T = 1/f = 10$ (ms).

(b) Power of $x(t)$

$$P_x = \frac{1}{T}\int_0^T x(t)^2 dt = \frac{4}{T}\int_0^T \cos^2(2\pi 100t)dt$$
$$= \frac{2}{T}\int_0^T (1 + \cos(4\pi 100t))\, dt = 2$$

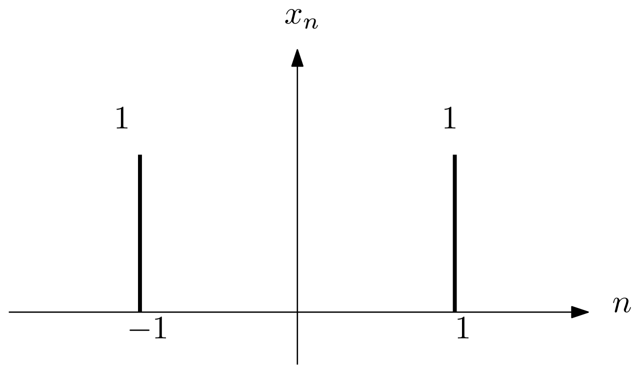
(c) We can write $x(t) = \quad (e^{-j2\pi 100t} + e^{j2\pi 100t})$.

(d) The Fourier series representation of $x(t)$ has only two non-zero coefficients

$$x_{-1} = x_1 = 1 \;\; -$$

That is $x_n = 0$ if $n \neq \pm 1$.

(e) The phase is always zero and the amplitude is plotted in the figure below.



3. It is easy to check that the period of signal is T. The DC component of $x(t)$ is given by

$$x_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\int_0^{T/4} dt = \frac{1}{4} \tag{20}$$

Now we compute the FS coefficient x_n for $n \neq 0$. By the definition of the FS we have

$$\begin{aligned} x_n &= \frac{1}{T}\int_0^T x(t)e^{-j2\pi n f_0 t}dt = \frac{1}{T}\int_0^{T/4} e^{-j2\pi n f_0 t}dt \\ &= \frac{1}{T(-j2\pi n f_0)}\left(e^{-j2\pi n f_0 t}\Big|_0^{T/4}\right) \\ &= -\frac{1}{j2\pi n}\left(e^{-j\pi n/2} - 1\right) \end{aligned}$$

We can rewrite the term $e^{-j\pi n/2}$ as

$$e^{-j\pi n/2} = \cos(\pi n/2) - j\sin(\pi n/2) \tag{21}$$

Now consider the following special cases for n.

- If $n = \pm 4, \pm 8, \pm 12, \ldots$, then $\cos(\pi n/2) = 1$ and $\sin(\pi n/2) = 0$, i.e, $e^{-j\pi n/2} = 1$, and thus $x_n = 0$.
- If $n = \pm 2, \pm 6, \pm 10, \ldots$, then $\cos(\pi n/2) = -1$ and $\sin(\pi n/2) = 0$,i.e, $e^{-j\pi n/2} = -1$, and thus $x_n = \frac{1}{j\pi n}$.
- If $n = 1, 5, 9, \ldots$, then $\cos(\pi n/2) = 0$ and $\sin(\pi n/2) = 1$,i.e, $e^{-j\pi n/2} = -j$, and thus $x_n = \frac{1}{j2\pi n}(1 + j)$.
- If $n = -1, -5, -9, \ldots$, then $\cos(\pi n/2) = 0$ and $\sin(\pi n/2) = -1$,i.e, $e^{-j\pi n/2} = j$, and thus $x_n = \frac{1}{j2\pi n}(1 - j)$.

In summary, the Fourier series of $x(t)$ is given by

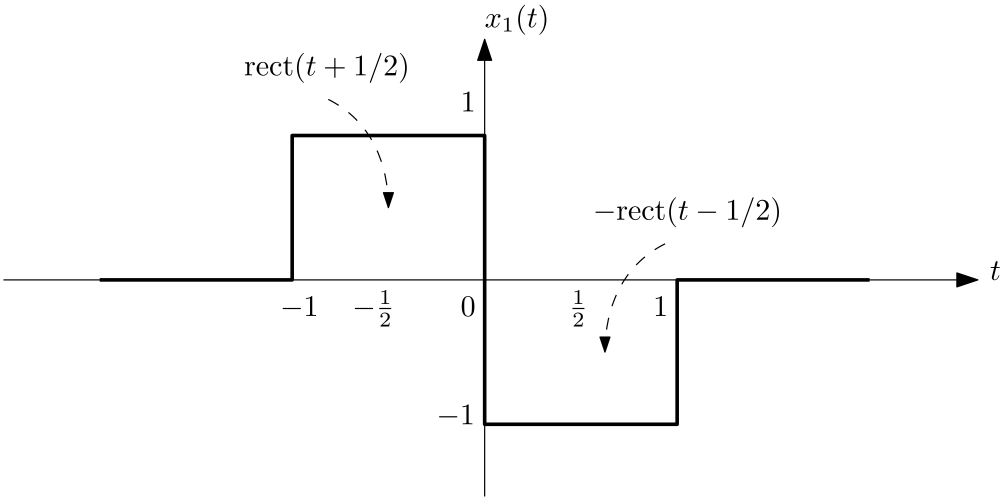
$$x_n = \begin{cases} 0 & n = \pm 4, \pm 8, \pm 12, \ldots \\ \frac{1}{j\pi n} & n = \pm 2, \pm 6, \pm 10, \ldots \\ \frac{1}{j2\pi n}(1 + j) & n = 1, 5, 9, \ldots, \\ \frac{1}{j2\pi n}(1 - j) & n = -1, -5, -9, \ldots, \end{cases} \tag{22}$$

4. Note that $x_a(t) = e^{-at}$ if $t \geq 0$ and $x_a(t) = e^{at}$ if $t < 0$. Thus

$$
\begin{aligned}
X_a(\omega) &= \int_{-\infty}^{\infty} x_a(t)e^{-j\omega t}dt = \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \\
&= \int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt \\
&= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}
\end{aligned}
$$

5. The first step is to express the given signals in terms of the unit rectangle signal. To do this it is better to plot the $x_1(t)$ first.

   (a) The plot of $x(t)$ is given in the following figure.



The left part of $x_1(t)$ is obtained by shifting rect(t) to the left by $\frac{1}{2}$ time unit. This results in rect$(t+\frac{1}{2})$. Similarly the right part of $x_1(t)$ is written as $-$rect$(t-1/2)$. Thus we can write

$$x_1(t) = \text{rect}(t+\frac{1}{2}) - \text{rect}(t-\frac{1}{2}) \tag{23}$$

From the above equation we can write

$$
\begin{aligned}
\text{Fourier tranform of } x_1(t) &= \text{Fourier tranform of rect}(t+\frac{1}{2}) \\
&\quad - \text{Fourier tranform of rect}(t-\frac{1}{2})
\end{aligned} \tag{24}
$$

Since the Fourier transform of rect(t) is $\frac{\sin(\omega/2)}{\omega/2}$, the Fourier transform of rect$(t+\frac{1}{2})$ is $e^{j\omega\frac{1}{2}}\frac{\sin(\omega/2)}{\omega/2}$ (due to the time-shifting property of Fourier transform). In the same way, the Fourier transform of rect$(t-\frac{1}{2})$ is $e^{-j\omega\frac{1}{2}}\frac{\sin(\omega/2)}{\omega/2}$. Thus we have

$$\begin{aligned} X_1(\omega) = \text{Fourier tranform of } x_1(t) &= e^{j\omega\frac{1}{2}}\frac{\sin(\omega/2)}{\omega/2} - e^{-j\omega\frac{1}{2}}\frac{\sin(\omega/2)}{\omega/2} \\ &= \left(e^{j\omega\frac{1}{2}} - e^{-j\omega\frac{1}{2}}\right)\frac{\sin(\omega/2)}{\omega/2} \\ &= 2j\sin(\omega/2)\frac{\sin(\omega/2)}{\omega/2} \\ &= 2j\frac{\sin^2(\omega/2)}{\omega/2} \qquad (25) \end{aligned}$$

(b) We can equivalently rewrite $x_2(t) = \text{rect}(t-\frac{1}{2})$. Thus the FT of $x_2(t)$ is given by

$$X_2(\omega) = e^{-j\omega\frac{1}{2}}\frac{\sin(\omega/2)}{\omega/2} \qquad (26)$$

(c) We can equivalently rewrite $x_3(t) = \text{rect}(\frac{t}{2}-\frac{1}{2})$. In words, $x_3(t)$ is obtained by two steps:

- shift rect(t) to the right by $1/2$, which yields the signal rect$(t-\frac{1}{2})$. We know that the FT of rect$(t-\frac{1}{2})$ is given by $e^{-j\omega\frac{1}{2}}\frac{\sin(\omega/2)}{\omega/2}$.
- Scale (i.e. expand) rect$(t-\frac{1}{2})$ by a factor of 2, i.e., $x_3(t) = \text{rect}(\frac{1}{2}t-\frac{1}{2})$. By the time and frequency scaling property of the Fourier transform (see Lecture 6) we have

$$\begin{aligned} X_3(\omega) = \text{Fourier tranform of } x_3(t) &= 2\times\left(\text{Fourier tranform of rect}(t-\frac{1}{2})\Big|_{\text{replace } \omega \text{ by } 2\omega}\right) \\ &= 2\times\left(e^{-j\omega\frac{1}{2}}\frac{\sin(\omega/2)}{\omega/2}\Big|_{\text{replace } \omega \text{ by } 2\omega}\right) \\ &= 2\times e^{-j\omega}\frac{\sin(\omega)}{\omega} \end{aligned}$$