Tutorial Sheet 5 - Solutions

Q1

The eigenvalue-eigenvector method for the discrete case is

$$A^k = M\Lambda^k M^{-1}$$

and the continuous case

$$e^{At} = Me^{\Lambda t}M^{-1}$$

where Λ is the diagonal matrix of eigenvalues and M is the matrix of eigenvectors (Modal matrix)

(a) $A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Eigenvectors given by cofactors of a row of $|A - \lambda I| = \begin{bmatrix} 1-\lambda & 0 \\ 2 & 2-\lambda \end{bmatrix}$

$$\boldsymbol{m}(\lambda) = \begin{bmatrix} 2-\lambda \\ -2 \end{bmatrix},\ \boldsymbol{m}(1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ \boldsymbol{m}(2) = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \Rightarrow M = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}$$

$$M^{-1} = \frac{1}{-2}\begin{bmatrix} -2 & 0 \\ 2 & 1 \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}^k = \frac{1}{-2}\begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^k\begin{bmatrix} -2 & 0 \\ 2 & 1 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2^k \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 2 & 1 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 2(2^k) & 2^k \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} -2 & 0 \\ 4-4(2^k) & -2(2^k) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2(2^k)-2 & (2^k) \end{bmatrix}$$

$$e^{At} = \frac{1}{-2}\begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 2 & 1 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} -2e^t & 0 \\ 2e^{2t} & e^{2t} \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} -2e^t & 0 \\ 4e^t - 4e^{2t} & -2e^{2t} \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 2e^{2t} - 2e^t & e^{2t} \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8} \end{bmatrix}$, eigenvalues = $\lambda = 0, \frac{9}{8}$, $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \frac{9}{8} \end{bmatrix}$.

Eigenvectors given by cofactors of a row of $|A-\lambda I| = \begin{bmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8}-\lambda \end{bmatrix}$.

$$\boldsymbol{m}(\lambda) = \begin{bmatrix} \frac{1}{8}-\lambda \\ -\frac{1}{4} \end{bmatrix},\ \boldsymbol{m}(0) = \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ \boldsymbol{m}\left(\frac{9}{8}\right) = \begin{bmatrix} -1 \\ -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\Rightarrow M = \begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix} \text{ and } M^{-1} = \frac{1}{9}\begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8} \end{bmatrix}^k = \frac{1}{9}\begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \frac{9}{8} \end{bmatrix}^k\begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{9}{8}\right)^k \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2\left(\frac{9}{8}\right)^k & \left(\frac{9}{8}\right)^k \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 8\left(\frac{9}{8}\right)^k & 4\left(\frac{9}{8}\right)^k \\ 2\left(\frac{9}{8}\right)^k & \left(\frac{9}{8}\right)^k \end{bmatrix} = \begin{bmatrix} \frac{8}{9}\left(\frac{9}{8}\right)^k & \frac{4}{9}\left(\frac{9}{8}\right)^k \\ \frac{2}{9}\left(\frac{9}{8}\right)^k & \frac{1}{9}\left(\frac{9}{8}\right)^k \end{bmatrix}$$

$$e^{At} = \frac{1}{9}\begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{9}{8}t} \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 2e^{\frac{9}{8}t} & e^{\frac{9}{8}t} \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 8e^{\frac{9}{8}t}+1 & 4e^{\frac{9}{8}t}-4 \\ 2e^{\frac{9}{8}t}-2 & e^{\frac{9}{8}t}+8 \end{bmatrix} = \begin{bmatrix} \frac{8}{9}e^{\frac{9}{8}t}+\frac{1}{9} & \frac{4}{9}e^{\frac{9}{8}t}-\frac{4}{9} \\ \frac{2}{9}e^{\frac{9}{8}t}-\frac{2}{9} & \frac{1}{9}e^{\frac{9}{8}t}+\frac{8}{9} \end{bmatrix}$$

(c) $\quad A = \begin{bmatrix} -2 & -1 \\ -4 & -5 \end{bmatrix},\ \lambda = -1, -6,\ \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}$

Eigenvectors given by cofactors of a row of $|A-\lambda I| = \begin{bmatrix} -2-\lambda & -1 \\ -4 & -5-\lambda \end{bmatrix}$.

$$\boldsymbol{m}(\lambda) = \begin{bmatrix} -5-\lambda \\ 4 \end{bmatrix},\ \boldsymbol{m}(-1) = \begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\qquad \boldsymbol{m}(-6) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\Rightarrow M = \begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix} \text{ and } M^{-1} = \frac{1}{5}\begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} -2 & -1 \\ -4 & -5 \end{bmatrix}^k = \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} (-1)^k & 0 \\ 0 & (-6)^k \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -4(-1)^k & (-1)^k \\ (-6)^k & (-6)^k \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 4(-1)^k + (-6)^k & -(-1)^k + (-6)^k \\ -4(-1)^k + 4(-6)^k & (-1)^k + 4(-6)^k \end{bmatrix} = \begin{bmatrix} \frac{4}{5}(-1)^k + \frac{1}{5}(-6)^k & -\frac{1}{5}(-1)^k + \frac{1}{5}(-6)^k \\ -\frac{4}{5}(-1)^k + \frac{4}{5}(-6)^k & \frac{1}{5}(-1)^k + \frac{4}{5}(-6)^k \end{bmatrix}$$

$$e^{At} = \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -4e^{-t} & e^{-t} \\ e^{-6t} & e^{-6t} \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 4e^{-t} + e^{-6t} & -e^{-t} + e^{-6t} \\ -4e^{-t} + 4e^{-6t} & e^{-t} + 4e^{-6t} \end{bmatrix} = \begin{bmatrix} \frac{4}{5}e^{-t} + \frac{1}{5}e^{-6t} & -\frac{1}{5}e^{-t} + \frac{1}{5}e^{-6t} \\ -\frac{4}{5}e^{-t} + \frac{4}{5}e^{-6t} & \frac{1}{5}e^{-t} + \frac{4}{5}e^{-6t} \end{bmatrix}$$

Q2

$$\underline{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\underline{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k),\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\underline{x}(k) \qquad (1)$$

(i) Zero-input state and output response when initial state is $\underline{x}(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$:

Need to determine $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^k$.

Eigenvalues $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0, \quad \Rightarrow \lambda = -1, -2$

Eigenvectors given by cofactors of a row of $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix}$.

$$\boldsymbol{m}(\lambda) = \begin{bmatrix} -1 \\ -\lambda \end{bmatrix}, \quad \boldsymbol{m}(-1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{m}(-2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\Rightarrow M = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \text{ and } M^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^k = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} (-1)^k & 0 \\ 0 & (-2)^k \end{bmatrix}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2(-1)^k & -(-1)^k \\ (-2)^k & (-2)^k \end{bmatrix}$$

$$= \begin{bmatrix} 2(-1)^k - (-2)^k & (-1)^k - (-2)^k \\ -2(-1)^k + 2(-2)^k & -(-1)^k + 2(-2)^k \end{bmatrix}$$

$$\underline{x}(k) = A^k \underline{x}(0) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(-1)^k - (-2)^k & (-1)^k - (-2)^k \\ -2(-1)^k + 2(-2)^k & -(-1)^k + 2(-2)^k \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3(-1)^k - 2(-2)^k \\ -3(-1)^k + 4(-2)^k \end{bmatrix}$$

The output is given by $y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\underline{x}(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 3(-1)^k - 2(-2)^k \\ -3(-1)^k + 4(-2)^k \end{bmatrix} = 3(-1)^k - 2(-2)^k$

(ii) Calculate the output of the system when $u(k) = (-1)^k$:

$$y(k) = \sum_{i=1}^{k} CA^{k-i}Bu(i-1) = \sum_{i=1}^{k} \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 2(-1)^{k-i} - (-2)^{k-i} & (-1)^{k-i} - (-2)^{k-i} \\ -2(-1)^{k-i} + 2(-2)^{k-i} & -(-1)^{k-i} + 2(-2)^{k-i} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}(-1)^{i-1}$$

$$= \sum_{i=1}^{k} \left[(-1)^{k-i} - (-2)^{k-i}\right](-1)^{i-1} = \sum_{i=1}^{k} \left[(-1)^{k-1} + (-1)^i(-2)^{k-i}\right] \qquad (2)$$

$$= k(-1)^{k-1} + (-2)^k \sum_{i=1}^{k} \left[(-1)^i(-2)^{-i}\right]$$

$$\sum_{i=1}^{k} \left[(-1)^i(-2)^{-i}\right] = \sum_{i=1}^{k} \left[(-1)^i\left(-\frac{1}{2}\right)^i\right] = \sum_{i=1}^{k} \left(\frac{1}{2}\right)^i = \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^k\right)}{1 - \frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^k$$

Note using $S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} = a\sum_{i=0}^{n-1} r^i = \frac{a(1 - r^n)}{1 - r}$ to get the sum of the geometric series.

i.e. We have $a = r$ giving $S_n = r + r^2 + r^3 + \ldots + r^n = \sum_{i=0}^{n-1} r^{i+1} = \sum_{i=1}^{n} r^i = \frac{r(1 - r^n)}{1 - r}$

Therefore:

$$y(k) = k(-1)^{k-1} + (-2)^k\left[1 - \left(\frac{1}{2}\right)^k\right] = k(-1)^{k-1} + (-2)^k - (-1)^k = (-2)^k - (1 + k)(-1)^k$$

(iii) State transformation matrix T is the modal matrix M, i.e:

$$T = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = M$$

$$M^{-1} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\Lambda = M^{-1}AM = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},\ B = M^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} M = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \end{bmatrix}$$

Therefore diagonalised state-space model:

$$\underline{z}(k+1) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\underline{z}(k) + \begin{bmatrix} -1 \\ 1 \end{bmatrix}u(k),\ y(k) = \begin{bmatrix} -1 & -1 \end{bmatrix}\underline{z}(k) \tag{3}$$

(iv) - (i) Since $\underline{x} = M\underline{z} \Rightarrow \underline{z} = M^{-1}\underline{x},\ \underline{z}(0) = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

$$\underline{z}(k) = A^k\underline{z}(0) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}^k\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} (-1)^k & 0 \\ 0 & (-2)^k \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3(-1)^k \\ 2(-2)^k \end{bmatrix}$$

$$y(k) = \begin{bmatrix} -1 & -1 \end{bmatrix}\underline{z}(k) = \begin{bmatrix} -1 & -1 \end{bmatrix}\begin{bmatrix} -3(-1)^k \\ 2(-2)^k \end{bmatrix} = 3(-1)^k - 2(-2)^k$$

(iv) - (ii)

$$y(k) = \sum_{i=1}^{k} CA^{k-i}Bu(i-1) = \sum_{i=1}^{k}\begin{bmatrix} -1 & -1 \end{bmatrix}\begin{bmatrix} (-1)^{k-i} & 0 \\ 0 & (-2)^{k-i} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}(-1)^{i-1}$$

$$= \sum_{i=1}^{k}\begin{bmatrix} -1 & -1 \end{bmatrix}\begin{bmatrix} -(-1)^{k-i} \\ (-2)^{k-i} \end{bmatrix}(-1)^{i-1} = \sum_{i=1}^{k}\left[(-1)^{k-i} - (-2)^{k-i}\right](-1)^{i-1} \tag{4}$$

This is identical to equation (2) from earlier, hence $y(k) = (-2)^k - (1+k)(-1)^k$.

Q3 $\quad \dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t),\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\underline{x}$

(i) zero-input response for $\underline{x}(0) = [0 \quad 1]^T$.

$$\underline{x}(t) = e^{At}x(0) \text{ and } y(t) = Ce^{At}x(0)$$

Eigenvalues of $\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{vmatrix} -\lambda & 1 \\ 0 & -2-\lambda \end{vmatrix} = (2+\lambda)\lambda = 0. \quad \Rightarrow \lambda = 0, -2$

The modal matrix M (taking row 2 and allowing for the correct sign):

$$\boldsymbol{m}(\lambda) = \begin{bmatrix} -1 \\ -\lambda \end{bmatrix}, \quad \boldsymbol{m}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{m}(-2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow M = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix},\ M^{-1} = \frac{1}{-2}\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

Hence:

$$e^{At} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & -e^{-2t} \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} -2 & -1+e^{-2t} \\ 0 & -2e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}-\frac{1}{2}e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

$$y(t) = Ce^{At}x(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2}-\frac{1}{2}e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2}-\frac{1}{2}e^{-2t} \\ e^{-2t} \end{bmatrix} = \frac{1}{2}-\frac{1}{2}e^{-2t}$$

(ii) output for a unit step input $u(t)$, $\underline{x}(0) = [0 \quad 0]^T$.

$$y(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2}-\frac{1}{2}e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot 1\,d\tau = \int_0^t \left[\frac{1}{2}-\frac{1}{2}e^{-2(t-\tau)}\right]d\tau$$

$$\Rightarrow y(t) = \int_0^t \frac{1}{2}d\tau - \frac{1}{2}e^{-2t}\int_0^t e^{2\tau}d\tau = \frac{1}{2}t - \frac{1}{2}e^{-2t}\left[\frac{1}{2}e^{2t}-\frac{1}{2}e^{0}\right] = \frac{1}{2}t + \frac{1}{4}[e^{-2t}-1]$$

Q4 $\quad \dot{\underline{x}} = \begin{bmatrix} -4 & -1 \\ -3 & -2 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)\,,\ y(t) = \begin{bmatrix} 1 & -1 \end{bmatrix}\underline{x}$

The required state transformation matrix is the modal matrix, M, (i.e the matrix of eigenvectors of A):

$$|A - \lambda I| = |\lambda I - A| = \begin{vmatrix} \lambda+4 & 1 \\ 3 & \lambda+2 \end{vmatrix} = (\lambda+4)(\lambda+2) - 3 = \lambda^2 + 6\lambda + 5 = 0$$

$$\Rightarrow \lambda = -1, -5$$

$$\boldsymbol{m}(\lambda) = \begin{bmatrix} \lambda+2 \\ -3 \end{bmatrix},\ \boldsymbol{m}(-1) = \begin{bmatrix} 1 \\ -3 \end{bmatrix},\ \boldsymbol{m}(-6) = \begin{bmatrix} -3 \\ -3 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow M = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix},\ M^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$$

Transformation is given by $\begin{aligned} \dot{\underline{z}} &= M^{-1}AM\underline{z} + M^{-1}Bu \\ y &= CM\underline{z} \end{aligned}$

$$A_z = M^{-1}AM = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -4 & -1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}$$

$$B_z = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 \\ 1 \end{bmatrix},\ C_z = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \end{bmatrix}$$

Therefore diagonalised system is:

$$\dot{\underline{z}} = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}\underline{z} + \begin{bmatrix} -0.25 \\ 0.25 \end{bmatrix}u(t)\,,\ y(t) = \begin{bmatrix} 4 & 0 \end{bmatrix}\underline{z}$$

System output when the input is a unit step and the initial condition is $\underline{x}(0) = [1 \quad 0]^T$.

First step is to convert the initial condition to the new states $\underline{z}$, i.e.

$$\underline{z}(0) = M^{-1}\underline{x}(0) = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$$

The output is given by:

$$y(t) = Ce^{At}\underline{z}(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau$$

where:

$$Ce^{At}\underline{z}(0) = \begin{bmatrix} 4 & 0 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-5t} \end{bmatrix}\begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 4 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{4}e^{-t} \\ \frac{3}{4}e^{-5t} \end{bmatrix} = e^{-t}$$

and:

$$\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} e^{-(t-\tau)} & 0 \\ 0 & e^{-5(t-\tau)} \end{bmatrix} \begin{bmatrix} -0.25 \\ 0.25 \end{bmatrix} \cdot 1\,d\tau = \int_0^t \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} -0.25e^{-(t-\tau)} \\ 0.25e^{-5(t-\tau)} \end{bmatrix} d\tau$$

$$= \int_0^t -e^{-(t-\tau)}d\tau = -e^{-t}\int_0^t e^{\tau}d\tau = -e^{-t}[e^t - 1] = e^{-t} - 1$$

$$\Rightarrow y(t) = e^{-t} + e^{-t} - 1 = 2e^{-t} - 1$$