# EE206
# Assignment 7

*Due by next Tutorial, November $17^{th}$. Starred questions will be done out in tutorials and do NOT need to be handed in.*

1. Show the given functions are orthogonal on the indicated interval.

   (a) $f_1(x) = \cos x, \quad f_2(x) = \sin^3 x; \qquad [0, \pi]$ **[2]**

$$\int_0^{\pi} (\cos x)(\sin^3 x)dx$$

$$u = \sin x \qquad du = \cos x dx$$

$$\int u^3 du = \frac{u^4}{4} = \frac{1}{4}[\sin^4 x]_0^{\pi} = \frac{1}{4}\sin^4(\pi) - \frac{1}{4}\sin^4(0) = 0 - 0 = 0.$$

   *(b) $f_1(x) = e^x, \quad f_2(x) = \sin x; \qquad [\frac{\pi}{4}, \frac{5\pi}{4}]$

$$\int e^x \sin x dx$$

$$u = e^x \qquad dv = \sin x dx$$

$$du = e^x dx \quad v = -\cos x$$

$$\Rightarrow \int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$$

$$\int e^x \cos x dx$$

$$u = e^x \qquad dv = \cos x dx$$

$$du = e^x dx \quad v = \sin x$$

$$\Rightarrow \int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

$$\Rightarrow \int e^x \sin x dx = -e^x \cos x + e^x \sin x - \int e^x \sin x dx$$

$$2\int e^x \sin x dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x dx = \frac{1}{2}(-e^x \cos x + e^x \sin x)$$

$$\begin{aligned}
\Rightarrow \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} e^x \sin x dx &= \frac{1}{2}[(-e^x \cos x + e^x \sin x)]\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\
&= \frac{1}{2}\left(-e^{\frac{5\pi}{4}}\left(-\frac{1}{\sqrt{2}}\right) + e^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}}\right) + e^{\frac{5\pi}{4}}\left(-\frac{1}{\sqrt{2}}\right) - e^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}}\right)\right) \\
&= \frac{1}{2}(0) \\
&= 0
\end{aligned}$$

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(c) $\{\sin\left(\frac{n\pi}{p}x\right)\}$ $\quad n = 1, 2, 3...;$ $\quad [0, p]$ **[2]**

$$\int_0^p \left(\sin\left(\frac{m\pi}{p}x\right)\right)\left(\sin\left(\frac{n\pi}{p}x\right)\right)dx$$

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

$$\begin{aligned}
\int_0^p \left(\sin\left(\frac{m\pi}{p}x\right)\right)\left(\sin\left(\frac{n\pi}{p}x\right)\right)dx &= \frac{1}{2}\int_0^p \left(\cos\left(\frac{(m-n)\pi}{p}x\right) - \cos\left(\frac{(m+n)\pi}{p}x\right)\right) \\
&= \frac{1}{2}\left[\frac{\sin\left(\frac{(m-n)\pi}{p}x\right)}{\frac{(m-n)\pi}{p}} + \frac{\sin\left(\frac{(m+n)\pi}{p}x\right)}{\frac{(m+n)\pi}{p}}\right]_0^p \\
&= \frac{p}{2}\left[\frac{\sin(\frac{(m-n)\pi}{p}x)}{(m-n)\pi} + \frac{\sin(\frac{(m+n)\pi}{p}x)}{(m+n)\pi}\right]_0^p \\
&= \frac{p}{2}\left(\frac{\sin((m-n)\pi) - \sin(0)}{(m-n)\pi} + \frac{\sin((m+n)\pi) - \sin(0)}{(m+n)\pi}\right) \\
&= 0
\end{aligned}$$

For $m \neq n$.

2. Verify by direct integration that the functions are orthogonal with respect to the indicated weight functions on the given interval.

*(b) $L_0(x) = 1, \quad L_1(x) = -x + 1; \quad w(x) = e^{-x}, \quad [0, \infty)$

$$(L_0(x), L_1(x)) = \int_0^\infty e^{-x}(1)(-x+1)dx = -\int_0^\infty xe^{-x}dx + \int_0^\infty e^{-x}dx$$

$$\int xe^{-x}dx$$

$$u = xdv = e^{-x}dx$$

$$du = dx \qquad v = -e^{-x}$$

$$\Rightarrow \int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} + -e^{-x}$$

$$L_0(x)L_1(x) = \int_0^\infty e^{-x}(1)(-x+1)dx = [xe^{-x} + e^{-x} - e^{-x}]_0^\infty = 0$$

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(a) $L_0(x) = 1, \quad L_2(x) = \frac{1}{2}x^2 - 2x + 1; \quad w(x) = e^{-x}, \quad [0, \infty)$ **[2]**

$$(L_0(x), L_2(x)) = \int_0^\infty e^{-x}(1)\left(\frac{1}{2}x^2 - 2x + 1\right)dx = \int_0^\infty e^{-x}\left(\frac{1}{2}x^2 - 2x + 1\right)dx$$

$$u = \frac{1}{2}x^2 - 2x + 1 \quad dv = e^{-x}dx$$

$$du = x - 2\,dx \quad v = -e^{-x}$$

$$= -e^{-x}\left(\frac{1}{2}x^2 - 2x + 1\right)\Big|_0^\infty + \int_0^\infty e^{-x}\,(x-2)\,dx$$

$$= 1 + \int_0^\infty e^{-x}\,(x-2)\,dx$$

$$u = x - 2 \quad dv = e^{-x}dx$$

$$du = dx \quad v = -e^{-x}$$

$$= 1 - e^{-x}(x-2)\Big|_0^\infty + \int_0^\infty e^{-x}dx$$

$$= 1 - 2 + \int_0^\infty e^{-x}dx$$

$$= -1 - e^{-x}\Big|_0^\infty = -1 + 1 = 0$$

3. Find the Fourier series of $f$ on the given interval.

(a) **[2]**

$$f_1(x) = \begin{cases} -1 - x/2, & -2 \le x < 0 \\ 1 - x/2, & 0 < x \le 2 \end{cases}$$

$a_0 = 0$ since the function looking at the graph is odd. For the same reason $a_n = 0$. Using the fact it is an odd function we can multiply by 2 and only integrate from 0 to 2.

$$\Rightarrow b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi}{p}x\right)dx$$

$$= \frac{2}{2}\int_0^2 (1 - x/2)\sin(n\pi x/2)dx$$

integrationg by parts:

$$= \frac{-2}{n\pi}[(1 - x/2)\cos(n\pi x/2)]_2^0 - \frac{2}{n\pi}\int_0^2 \cos(n\pi x/2)dx$$

$$= \frac{2}{n\pi} - \frac{4}{n^2\pi^2}[\sin(n\pi x/2)]_0^2$$

$$= \frac{2}{n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{nx\pi}{p}\right) + b_n\sin\left(\frac{nx\pi}{p}\right)\right)$$

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x/2)$$

(b) **[2]**

$$f_2(x) = \begin{cases} 0, & -1 < x < 0 \\ \dfrac{e^{-10x} - e^{-10}}{1 - e^{-10}}, & 0 \le x < 1 \end{cases}$$

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$$
\begin{aligned}
a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)dx \\
&= \int_0^1 \frac{e^{-10x}-e^{-10}}{1-e^{-10}}dx \\
&= \frac{1}{1-e^{-10}}[-0.1e^{-10x}-e^{-10}x]_0^1 \\
&= \frac{1}{1-e^{-10}}[-1.1e^{-10}+0.1]
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi}{p}x\right)dx \\
&= \frac{1}{1-e^{-10}}\int_0^1 \left(e^{-10x}-e^{-10}\right)\cos(n\pi x)dx
\end{aligned}
$$

integration by parts

$$
\begin{aligned}
a_n &= \frac{1}{1-e^{-10}}\left[\frac{1}{n\pi}\left(e^{-10x}-e^{-10}\right)\sin(n\pi x)\Big|_0^1 + \frac{10}{n\pi}\int_0^1 e^{-10x}\sin(n\pi x)dx\right] \\
&= \frac{1}{1-e^{-10}}\cdot\frac{10}{n\pi}\int_0^1 e^{-10x}\sin(n\pi x)dx
\end{aligned}
$$

Let:

$$
\begin{aligned}
I &= \int_0^1 e^{-10x}\sin(n\pi x)dx \\
&= -\frac{1}{n\pi}e^{-10x}\cos(n\pi x)\Big|_0^1 - \frac{10}{n\pi}\int_0^1 e^{-10x}\cos(n\pi x)dx \\
&= \frac{1+(-1)^{n+1}e^{-10}}{n\pi} - \frac{100}{n^2\pi^2}I \\
I &= \frac{n\pi}{n^2\pi^2+100}\left(1+(-1)^{n+1}e^{-10}\right) \\
a_n &= \frac{10\left(1+(-1)^{n+1}e^{-10}\right)}{(1-e^{-10})(n^2\pi^2+100)}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi}{p}x\right)dx \\
&= \frac{1}{1-e^{-10}}\int_0^1 \left(e^{-10x}-e^{-10}\right)\sin(n\pi x)dx
\end{aligned}
$$

Using I from before

$$
\begin{aligned}
&= \frac{1}{1-e^{-10}}\left(I - e^{-10}\int_0^1 \sin(n\pi x)dx\right) \\
&= \frac{1}{1-e^{-10}}\left[\frac{n\pi(1+(-1)^{n+1}e^{-10})}{n^2\pi^2+100} + \frac{e^{-10}}{n\pi}\cos(n\pi x)\Big|_0^1\right] \\
&= \frac{1}{1-e^{-10}}\left[\frac{n\pi(1+(-1)^{n+1}e^{-10})}{n^2\pi^2+100} + \frac{e^{-10}((-1)^n-1)}{n\pi}\right]
\end{aligned}
$$

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{nx\pi}{p}\right) + b_n \sin\left(\frac{nx\pi}{p}\right)\right)$$

$$f(x) = \frac{1}{1-e^{-10}}[-1.1e^{-10} + 0.1] + \sum_{n=1}^{\infty} \frac{10\left(1+(-1)^{n+1}e^{-10}\right)}{(1-e^{-10})(n^2\pi^2+100)} \cos(n\pi x) +$$
$$\frac{1}{1-e^{-10}}\left[\frac{n\pi(1+(-1)^{n+1}e^{-10})}{n^2\pi^2+100} + \frac{e^{-10}((-1)^n - 1)}{n\pi}\right] \sin(n\pi x)$$

*(d)

$$f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$$

$$\begin{aligned}
\Rightarrow a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)dx \\
&= \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{0}^{1} xdx + \frac{1}{2}\int_{1}^{2} dx \\
&= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}[x]_1^2 \\
&= \frac{1}{2}\left[\frac{1}{2} - 0\right] + \frac{1}{2}[2-1] \\
&= \frac{1}{4} + \frac{1}{2} \\
&= \frac{3}{4}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi}{p}x\right)dx \\
&= \frac{1}{2}\int_{0}^{1} x\cos(\frac{n\pi}{2}x)dx + \frac{1}{2}\int_{1}^{2} \cos(\frac{n\pi}{2}x)dx
\end{aligned}$$

$$\int x\cos(\frac{n\pi}{2}x)dx$$

$$u = x \qquad dv = \cos\left(\frac{n\pi}{2}x\right)dx$$
$$du = dx \quad v = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}x\right)$$
$$\int udv = uv - \int vdu$$
$$\int x\cos(\frac{n\pi}{2}x)dx = \frac{2x}{n\pi}\sin(\frac{n\pi}{2}x) - \frac{2}{n\pi}\int \sin(\frac{n\pi}{2}x)dx$$

$$\begin{aligned}
\Rightarrow a_n &= \frac{1}{2}\left[\frac{2x}{n\pi}\sin(\frac{n\pi}{2}x) + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi}{2}x\right)\right]_0^1 + \frac{1}{2}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}x\right)\right]_1^2 \\
&= \frac{1}{2}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) - 0 + \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\cos(0)\right] \\
&+ \frac{1}{2}\left[\frac{2}{n\pi}\sin(n\pi) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right] \\
&= \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) - 0 + \frac{2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n^2\pi^2} - \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) \\
&= \frac{2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n^2\pi^2}
\end{aligned}$$

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$$
\begin{aligned}
\Rightarrow b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi}{p}x\right)dx \\
&= \frac{1}{2}\int_{0}^{1} x\sin\left(\frac{n\pi}{2}x\right)dx + \frac{1}{2}\int_{1}^{2}\sin\left(\frac{n\pi}{2}x\right)dx
\end{aligned}
$$

$$\int x\sin\left(\frac{n\pi}{2}x\right)dx$$

$$u = x \qquad dv = \sin\left(\frac{n\pi}{2}x\right)dx$$

$$du = dx \quad v = -\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$$

$$\int udv = uv - \int vdu$$

$$\int x\sin\left(\frac{n\pi}{2}x\right)dx = -\frac{2x}{n\pi}\cos\left(\frac{n\pi}{2}x\right) + \frac{2}{n\pi}\int\cos\left(\frac{n\pi}{2}x\right)dx$$

$$
\begin{aligned}
\Rightarrow b_n &= \frac{1}{2}\left[-\frac{2x}{n\pi}\cos\left(\frac{n\pi}{2}x\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}x\right)\right]_0^1 - \frac{1}{2}\left[\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)\right]_1^2 \\
&= \frac{1}{2}\left[-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - 0 + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\sin(0)\right] \\
&- \frac{1}{2}\left[\frac{2}{n\pi}\cos(n\pi) - \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right] \\
&= -\frac{1}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - 0 - \frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}\cos\left(\frac{n\pi}{2}\right) \\
&= \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{(-1)^n}{n\pi}
\end{aligned}
$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{nx\pi}{p}\right) + b_n\sin\left(\frac{nx\pi}{p}\right)\right)$$

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n^2\pi^2}\right)\cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{(-1)^n}{n\pi}\right)\sin\left(\frac{n\pi x}{2}\right)$$

(c) **[2]**

$$f_4(x) = \begin{cases} (x+1)^2, & -1 \le x \le 0 \\ (x-1)^2, & 0 \le x \le 1 \end{cases}$$

$$
\begin{aligned}
\Rightarrow a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)dx \\
&= 2\int_{0}^{1}(x-1)^2dx = 2\int_{-1}^{0}u^2du \\
&= 2[u^3/3]_{-1}^{0} \\
&= 2[1/3] = 2/3
\end{aligned}
$$



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$$
\begin{aligned}
\Rightarrow a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi}{p}x\right)dx \\
&= 2\int_0^1 (x-1)^2\cos(n\pi x)dx \\
u &= (x-1)^2 \qquad dv = \cos(n\pi x)dx \\
du &= 2(x-1)dx \quad v = \frac{1}{n\pi}\sin(n\pi x) \\
\int udv &= uv - \int vdu \\
\Rightarrow &= \frac{2(x-1)^2}{n\pi}\sin(n\pi x)\Big|_0^1 - \frac{4}{n\pi}\int_0^1 (x-1)\sin(n\pi x)dx \\
&= -\frac{4}{n\pi}\int_0^1 (x-1)\sin(n\pi x)dx \\
u &= x-1 \qquad dv = \sin(n\pi x)dx \\
du &= dx \quad v = -\frac{1}{n\pi}\cos(n\pi x) \\
\int udv &= uv - \int vdu \\
\Rightarrow &= \frac{4}{n^2\pi^2}\left[(x-1)\cos(n\pi x)\Big|_0^1 - \int_0^1 \cos(n\pi x)dx\right] \\
&= \frac{4}{n^2\pi^2}
\end{aligned}
$$

$b_n = 0$ since the function is even.

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{nx\pi}{p}\right) + b_n\sin\left(\frac{nx\pi}{p}\right)\right) \\
f(x) &= \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\cos(n\pi x)
\end{aligned}
$$

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