Chapter 6: Fourier Transform[*]

December 2, 2021

1 Fourier series

For a periodic function $f(t+T) = f(t)$ with period $T = 2L$, or a periodic extension of f with an interval $I = [-L, L]$, the **trigonometric Fourier series** can be expressed by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right] \tag{1}$$

where $\omega_0 = 2\pi/T = \pi/L$ is called *fundamental angular velocity*, and the coefficients a_0, a_n, b_n are determined from

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(n\omega_0 t)\ dt,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(n\omega_0 t)\ dt$$

The **complex Fourier series** can be expressed by

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{n\omega_0 t} \tag{2}$$

where the complex coefficients c_n are determined from

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-n\omega_0 t}dt$$

For real signal $f(t)$, Fourier series have the following properties

- c_n and c_{-n} are complex conjugate pairs.
- $c_0 = \frac{1}{2}a_0, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}$
- $a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad c_{-n} = i(c_n - c_{-n})$

Convergence prerequisites

- f and f' are piecewise continuous

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Convergence conditions:

If f and f' are piecewise continuous on the interval, then Fourier series of f converge to $f(t)$ at the points of continuity, and converge to

$$\frac{f(x+)+f(x-)}{2}$$

at the point of discontinuity. Here $f(x+)$ and $f(x-)$ are the limit of f at the discontinuity from the right and left, respectively.

Frequency spectrum:

if f is periodic and has fundamental period T, the plot of the points $(n\omega_0), |c_n|)$ is called **frequency spectrum.** Here ω_0, c_n are the fundamental angular frequency, and the coefficients of the complex Fourier series, respectively.

The following table shows some of the values of $|c_n|$:

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$\lvert c_n \rvert$	$\frac{1}{5\pi}$	0	$\frac{1}{3\pi}$	0	$\frac{1}{\pi}$	$\frac{1}{2}$	$\frac{1}{\pi}$	0	$\frac{1}{3\pi}$	0	$\frac{1}{5\pi}$

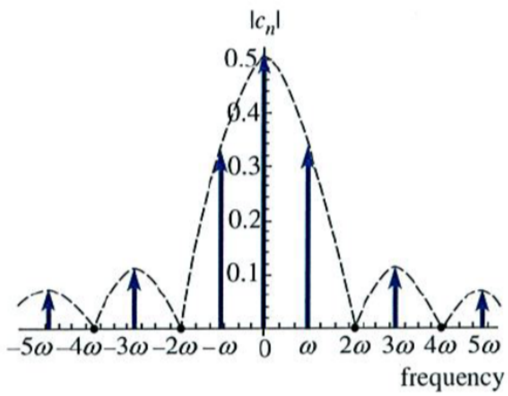

Figure 1: Fourier spectrum example (Fourier 2, page 26)

2 Fourier integral and Fourier complex integral

The **Fourier integral** of a (non-periodic) function f defined on the interval $(-\infty, \infty)$ is given by

$$f(t) = \frac{1}{\pi}\int_0^\infty [A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t)]d\omega \tag{3}$$

where

$$A(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega t)dt$$
$$B(\omega) = \int_{-\infty}^{\infty} f(t)\sin(\omega t)dt$$

the complex Fourier integral can be expressed by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \tag{4}$$

where the complex coefficients $F(\omega)$ are calculated from

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{5}$$

For real signal $f(t)$, Fourier integral and Fourier complex integral have the following properties

- $F(\omega)$ and $F(-\omega)$ are complex conjugate pairs.
- $A(\omega) = \mathrm{Re}(F(\omega)), B(\omega) = -\mathrm{Im}(F(\omega))$

Convergence prerequisites

- f and f' are piecewise continuous
- f is absolutely integrable, i.e., $\int_{-\infty}^{\infty} |f(x)|dx$ converges

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Convergence conditions:

If f and f' are piecewise continuous on the interval, and f is absolutely integrable, then Fourier series of f converge to $f(t)$ at the points of continuity, and converge to

$$\frac{f(x+) + f(x-)}{2}$$

at the point of discontinuity. Here $f(x+)$ and $f(x-)$ are the limit of f at the discontinuity from the right and left, respectively.

3 Sine and Cosine expansions

The sine/cosine expansions are introduced for

- calculation simplifications
- half-range extensions

1. The expension of even functions $f(t) = f(-t)$

   - Periodic: (trigonometric) Fourier series (Eq. (1)), for $n = 1, 2, 3, \ldots$

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \frac{2}{L}\int_{0}^{L} f(t)\,dt, \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(n\omega_0 t\right)\,dt = \frac{2}{L}\int_{0}^{L} f(t)\cos\left(n\omega_0 t\right)\,dt, \\
b_n &= 0
\end{aligned}$$

   - Non-periodic Fourier integral:

$$f(t) = \frac{1}{\pi}\int_{0}^{\infty} [A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t)]d\omega \tag{6}$$

     where

$$\begin{aligned}
A(\omega) &= \int_{-\infty}^{\infty} f(t)\cos(\omega t)dt \\
B(\omega) &= 0
\end{aligned}$$

2. Odd functions $-f(t) = f(-t)$

- Periodic:(trigonometric) Fourier series (Eq. (1)), for $n = 1, 2, 3, \ldots$

$$a_0 = 0, a_n = 0,$$
$$b_n = \frac{1}{L}\int\limits_{-L}^{L} f(t)\sin(n\omega_0 t)\ dt = \frac{2}{L}\int\limits_{0}^{L} f(t)\sin(n\omega_0 t)\ dt$$

- Non-periodic:

$$f(t) = \frac{1}{\pi}\int_0^\infty [A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t)]d\omega \tag{7}$$

where

$$A(\omega) = 0$$
$$B(\omega) = \int_{-\infty}^{\infty} f(t)\sin(\omega t)dt$$

Half-range expansions: Half-range expansions for a function $f(x)$ defined **only** on $I = (0, L)$:

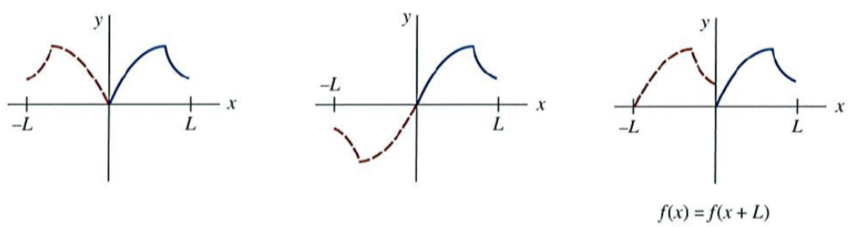

Figure 2: Left: cosine series; Mid: Sine series; Right: fourier series

Since cosine and sine expansions of a function only uses the definition of a function on $(0, p)$, we can expand the function $f(x)$ as if it is a

1. periodic even function f with a period $2L$:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t), \quad \omega_0 = \pi/L$$
$$a_0 = \frac{2}{L}\int\limits_0^L f(t)\,dt, \quad a_n = \frac{2}{L}\int\limits_0^L f(t)\cos(n\omega_0 t)\ dt$$

2. periodic odd function f with a period $2L$:

$$f(t) = \sum_{n=1}^{\infty} b_n\sin(n\omega_0 t), \quad \omega_0 = \pi/L$$
$$b_n = \frac{2}{L}\int\limits_0^L f(t)\sin(n\omega_0 t)\ dt$$

3. periodic function f with a period L:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)], \quad \omega_0 = 2\pi/L$$
$$a_0 = \frac{2}{L}\int\limits_0^L f(t)\,dt, \quad a_n = \frac{2}{L}\int\limits_0^L f(t)\cos(n\omega_0 t)\ dt, \quad b_n = \frac{2}{L}\int\limits_0^L f(t)\sin(n\omega_0 t)\ dt$$

4 Fourier Transform

Definition of Fourier Transform

$$\mathscr{F}(f(t)) = \int_{-\infty}^{\infty} f(x)e^{-i\omega t}dt = F(\omega)$$

Definition of Inverse Fourier Transform

$$\mathscr{F}^{-1}(F(w)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega = f(t)$$

Properities

Please refer to Page 20-26, Fourier 3.

5 Fourier transform of a periodic signal

The following steps extends the definition of Fourier transform to periodic signals:

1. Note that

$$e^{i\omega_0 t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{i\omega t}d\omega \tag{8}$$

   Consider the definition of inverse Fourier Transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}dt$$

   Thus, $2\pi\delta(\omega-\omega_0)$ can be viewed as the Fourier transform of $e^{i\omega_0 t}$.

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2. Periodic signal can be expanded as a (complex) Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\omega_0 t}dt, \quad \omega_0 = \pi/L \tag{9}$$

   Take the Fourier Transform of Eq. (9), we have

$$\begin{aligned} F(\omega) = \mathscr{F}\left(\sum_{n=-\infty}^{\infty} c_n e^{+in\omega_0 t}\right) &= \sum_{n=-\infty}^{\infty} c_n \mathscr{F}\left(e^{+in\omega_0 t}\right) \\ &= \sum_{n=-\infty}^{\infty} c_n \left(2\pi\delta(\omega-n\omega_0)\right) = 2\pi\sum_{n=-\infty}^{\infty} c_n\left(\delta(\omega-n\omega_0)\right) \end{aligned}$$

Examples

Question: Find the Fourier Transform of $f(t) = \sin(\omega_0(t))$ *for* $I = (-\infty, \infty)$

Solution:

$$\begin{aligned} f(t) = \sin(\omega_0 t) &= \frac{1}{2i}(e^{i\omega_0 t} - e^{-i\omega_0 t}) \text{ 展开为傅立叶级数} \\ \mathscr{F}(e^{i\omega_0 t}) &= 2\pi\delta(\omega-\omega_0) \\ \mathscr{F}(e^{-i\omega_0 t}) &= 2\pi\delta(\omega+\omega_0) \\ \mathscr{F}(f(t)) &= \frac{\pi}{i}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right) \end{aligned}$$