Tutorial Sheet 5 - Solving the state equations

Q1 Using the eigenvalue-eigenvector method (i.e. the modal matrix method), determine the state transition matrix for each of the following state matrices given that, in each case, the system is (i) discrete-time and (ii) continuous-time:

(a) $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ (b) $\boldsymbol{A} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8} \end{bmatrix}$ (c) $\boldsymbol{A} = \begin{bmatrix} -2 & -1 \\ -4 & -5 \end{bmatrix}$

Q2 Consider the following system:

$$\boldsymbol{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x}(k)$$

where $\boldsymbol{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$.

(i) Determine the zero-input state and output response of this system when the initial state is $\boldsymbol{x}(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

(ii) Calculate the output of the system for zero initial conditions when $u(k) = (-1)^k$.

(iii) Determine the state transformation matrix T which converts the above state-space model into one with a diagonalised state-space matrix. Hence calculate the diagonalised state space model:

$$\boldsymbol{z}(k+1) = \boldsymbol{\Lambda}\boldsymbol{z}(k) + \boldsymbol{B}u(k), \quad y(k) = \boldsymbol{C}\boldsymbol{z}(k)$$

(iv) Repeat parts (i) and (ii) for the diagonalised state-space model from part (iii). Note that the initial state $\boldsymbol{x}(0)$ has also to be transformed to $\boldsymbol{z}(0)$.

Q3 A continuous-time dynamical system is defined by the following equations:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}$$

(i) Determine the zero-input response of this system when the initial state is $\boldsymbol{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.

(ii) Calculate the output of the system for a unit step input $u(t)$ and zero initial conditions i.e. $\boldsymbol{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

Q4 Using the appropriate state transformation matrix, convert the following state-space model into one with a diagonalised state matrix:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -4 & -1 \\ -3 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & -1 \end{bmatrix} \boldsymbol{x}$$

Hence, determine the system output when the input is a unit step and the initial condition is $\boldsymbol{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$.