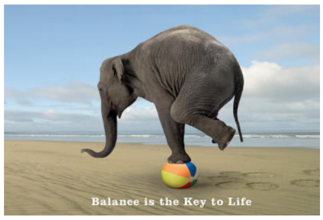
7. Stability of dynamical systems

7.1 Definitions of stability

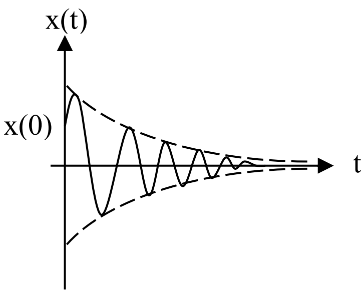
- A linear system is said to be **asymptotically stable** if ***all*** the states of the system decay to zero asymptotically with time, for all initial states and **independently of system inputs** (i.e. we are only interested in the natural response of the system).

- A system is said to be **stable** (or **marginally stable**) if one or more states achieve some non-zero final value, or oscillate with consistent amplitude, while all other states are asymptotically stable.

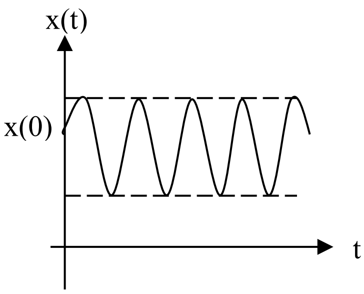


- A system is said to be **unstable** if one or more states becomes infinite with increasing time.

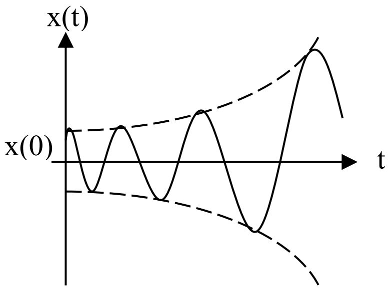
- Each of these concepts are illustrated below:



Asymptotically stable
$\Rightarrow$ ***all*** states $\rightarrow 0$ as $t \rightarrow \infty$



Marginally stable - note, cannot have any unstable states



Unstable if **any** state is not bounded as $t \rightarrow \infty$ (even if all other states $\rightarrow 0$)

7.2 Stability of discrete-time systems

- Recall that there are many possible state representations of the same system.

- As we have seen previously, by choosing an appropriate transformation matrix, we can convert to a state representation with a diagonal ***A*** matrix, i.e.:

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}u(k)$$

where: $$\boldsymbol{A} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

- Since **stability is independent of the system input**, we only need to consider the zero-input system response, i.e.:

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k)$$

- This has a solution given by:

$$\boldsymbol{x}(k) = \boldsymbol{A}^k \boldsymbol{x}(0) = \begin{bmatrix} \lambda_1^{\,k} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{\,k} \end{bmatrix} \boldsymbol{x}(0)$$

- Therefore, we have n independent difference equations, one for each state, that is:

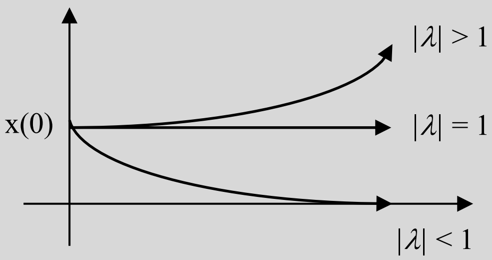
$$x_i(k) = \lambda_i^{\,k} x_i(0)$$

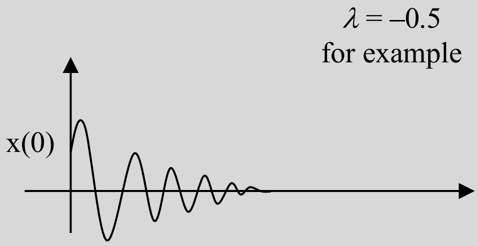
- Hence, the stability of the system depends on the eigenvalues as follows.

- The discrete-time system is:
  - asymptotically stable if $|\lambda_i| < 1$ for ***all*** eigenvalues
  - is marginally stable if $|\lambda_i| = 1$ for some distinct, non-overlapping eigenvalues (different position in the complex plane) and $|\lambda_i| < 1$ for the rest.
  - is unstable if $|\lambda_i| > 1$ for ***any*** of the eigenvalues or if $|\lambda_i| = 1$ for some overlapping eigenvalues (same position in the complex plane)

- Note that we are working with the **magnitude of the eigenvalues**, $|\lambda|$, so these definitions also apply for system with complex eigenvalues!

- **Also, if λ is negative, then the response will be oscillatory in nature.**
- In short, for discrete-time systems:





$\lambda < 0$, response is oscillatory

7.3 Stability of continuous-time systems

- The rules for the stability of continuous-time systems can be obtained in a similar fashion, as follows.

- Any continuous-time state space system can be transformed into one with a diagonal ***A*** matrix:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

- This has a solution $\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$ and for a diagonal matrix this becomes:

$$\boldsymbol{x}(t) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \boldsymbol{x}(0)$$

- Therefore, we have n independent equations, one for each state, that is:

$$x_i(t) = e^{\lambda_i t} x_i(0)$$

- Once again, λ can be real or complex, but stability only depends on the real parts, as follows.

- The continuous-time system is:

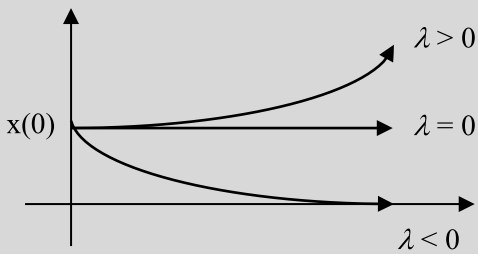
  - asymptotically stable if $Re\ (\lambda_i) < 0$ for ***all*** eigenvalues

  - stable if $Re\ (\lambda_i) = 0$ for some distinct, non-overlapping eigenvalues (different positions in the complex plane) and $Re\ (\lambda_i) < 0$ for the rest

  - unstable if $Re\ (\lambda_i) > 0$ for ***any*** of the eigenvalues or if $Re\ (\lambda_i) = 0$ for some overlapping eigenvalues (same position in the complex plane).

- **Note, if λ is complex, then the response will be oscillatory in nature.**

- In short, for continuous-time systems:



General notes:

- **Stability of dynamical systems depends only on the eigenvalues of the *A* matrix**.

- If the system is not in diagonal form, then to determine stability we first calculate the eigenvalues of the characteristic equations:

$$|\lambda \mathbf{I} - \mathbf{A}| = 0$$

- The rules for continuous-time and discrete-time system stability are different! *Don't confuse them!*

有界的稳定

- Not all states may be connected to the system output. Hence, it is possible for the output of a system to be stable but still have unstable internal dynamics!

- **Example 7.1:** Determine the stability of the following discrete-time system:

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

Solution:

$$\begin{vmatrix} \lambda - 1 & -2 \\ -1 & \lambda - 3 \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4\lambda + 1 = 0 \Rightarrow \lambda_1 = 0.27, \lambda_2 = 3.73$$

$$|\lambda_2| > 1 \Rightarrow \text{unstable}$$

- **Example 7.2:** Determine the stability of the following discrete-time system:

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.4 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

Solution:

Clearly: $\lambda_1 = 0.4, \lambda_2 = 1$

$|\lambda_2| = 1, |\lambda_1| < 1 \Rightarrow$ marginally stable

- **Example 7.3:** Determine the stability of the following discrete-time system:

$$\mathbf{x}(k+1) = \begin{bmatrix} -0.5 & 1.5 \\ 0 & -0.8 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

Solution:

Clearly: $\lambda_1 = -0.5, \lambda_2 = -0.8$

All $|\lambda| < 1 \Rightarrow$ asymptotically stable

- **Example 7.4:** Determine the stability of the following discrete-time system:

$$\boldsymbol{x}(k+1)=\begin{bmatrix}0 & 8 & 0\\ 0 & 0 & 8\\ -1 & -7 & -14\end{bmatrix}\boldsymbol{x}(k)+\begin{bmatrix}1\\1\\0\end{bmatrix}u(k)$$

Solution:

$$\begin{vmatrix}\lambda & -8 & 0\\ 0 & \lambda & -8\\ 1 & 7 & \lambda+14\end{vmatrix}=0$$

$$\Rightarrow \lambda(\lambda(\lambda+14)+56))+8(0+8)=0$$
$$\Rightarrow \lambda^3+14\lambda^2+56\lambda+64=0$$
$$\Rightarrow (\lambda+2)(\lambda+4)(\lambda+8)=0$$
$$\Rightarrow \lambda_1=-2, \lambda_2=-4, \lambda_3=-8$$

$|\lambda_1|>1 \Rightarrow$ unstable *(response will diverge to infinity in an oscillatory fashion)*

- **Example 7.5:** Determine the stability of the following continuous-time system:

$$\dot{\boldsymbol{x}}(t)=\begin{bmatrix}1 & 2\\ 5 & 3\end{bmatrix}\boldsymbol{x}(t)+\begin{bmatrix}0\\1\end{bmatrix}u(t)$$

Solution:

$$\begin{vmatrix}\lambda-1 & -2\\ -5 & \lambda-3\end{vmatrix}=0 \Rightarrow \lambda^2-4\lambda-7=0 \Rightarrow \lambda_1=-1.316, \lambda_2=5.3166$$

$Re(\lambda_2)>0 \Rightarrow$ unstable

- **Example 7.6:** Determine the stability of the following continuous-time system:

$$\dot{\boldsymbol{x}}(t)=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\boldsymbol{x}(t)+\begin{bmatrix}1\\1\end{bmatrix}u(t)$$

Solution:

$$\begin{vmatrix}\lambda & -1\\ 2 & \lambda+3\end{vmatrix}=0 \Rightarrow \lambda^2+3\lambda+2=0 \Rightarrow \lambda_1=-2, \lambda_2=-1$$

All $Re(\lambda)<0 \Rightarrow$ asymptotically stable

- **Example 7.7:** Determine the stability of the following continuous-time system:

$$\dot{\boldsymbol{x}}(t)=\begin{bmatrix}-0.5 & 1.5\\ 0 & -0.8\end{bmatrix}\boldsymbol{x}(t)+\begin{bmatrix}1\\1\end{bmatrix}u(t)$$

Solution:

Clearly: $\lambda_1=-0.5, \lambda_2=-0.8$

All $Re(\lambda)<0 \Rightarrow$ asymptotically stable

- **Example 7.8:** Determine the stability of the following continuous-time system:

$$\dot{\boldsymbol{x}}(t)=\begin{bmatrix}-6 & -24 & 12\\ -2 & 1 & -2\\ -4 & 29 & -22\end{bmatrix}\boldsymbol{x}(t)+\begin{bmatrix}1\\1\\0\end{bmatrix}u(t)$$

Solution:

$$\begin{vmatrix}\lambda+6 & 24 & -12\\ 2 & \lambda-1 & 2\\ 4 & -29 & \lambda+22\end{vmatrix}=0$$

$$\Rightarrow (\lambda+6)((\lambda-1)(\lambda+22)+58))-24(2\lambda+44-8)-12(-58-4\lambda+4)=0$$

$$\Rightarrow (\lambda+6)(\lambda^2+21\lambda+36)-48\lambda-864=0$$

$$\Rightarrow \lambda^3+27\lambda^2+162\lambda+216-216=0$$

$$\Rightarrow \lambda^3+27\lambda^2+162\lambda=0$$

$$\Rightarrow \lambda(\lambda+9)(\lambda+18)=0$$

$$\Rightarrow \lambda_1=0, \lambda_2=-9, \lambda_3=-18$$

$Re(\lambda_1)=0, Re(\lambda_2,\lambda_3)<0 \Rightarrow$ marginally stable

"I'm disappointed; if anyone should have seen the red flags, it's you."

8.4 Determining stability from the transfer function

- In section 4.6, we saw how to convert a state-space representation to transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \boldsymbol{C}(sI - \boldsymbol{A})^{-1}\boldsymbol{B} \qquad \text{for continuous-time}$$

and

$$G(z) = \frac{Y(z)}{U(z)} = \boldsymbol{C}(zI - \boldsymbol{A})^{-1}\boldsymbol{B} \qquad \text{for discrete-time}$$

- Note, that for any given matrix $\boldsymbol{A}$, we calculate the inverse as:

$$\boldsymbol{A}^{-1} = \frac{adj(\boldsymbol{A})}{|\boldsymbol{A}|}$$

- Hence we can write:

$$G(s) = \frac{Y(s)}{U(s)} = \boldsymbol{C}(sI - \boldsymbol{A})^{-1}\boldsymbol{B} = \frac{\boldsymbol{C}\, adj(sI - \boldsymbol{A})\boldsymbol{B}}{|sI - \boldsymbol{A}|}$$

and

$$G(z) = \frac{Y(z)}{U(z)} = \boldsymbol{C}(zI - \boldsymbol{A})^{-1}\boldsymbol{B} = \frac{\boldsymbol{C}\, adj(zI - \boldsymbol{A})\boldsymbol{B}}{|zI - \boldsymbol{A}|}$$

- Thus, the denominator of the transfer function is given by the characteristic equation of $\boldsymbol{A}$ (i.e. the equation used to calculate the eigenvalues), but with λ replaced by s for continuous time (or z for discrete-time).

- We know that the stability of a system is determined by the eigenvalues of $\boldsymbol{A}$, i.e. the roots of the characteristic equation.

- Therefore, this implies that the **stability depends on the roots of the denominator of *G*(*s*) (or *G*(*z*)) only**.

- These roots are referred to as the **poles of the system**.

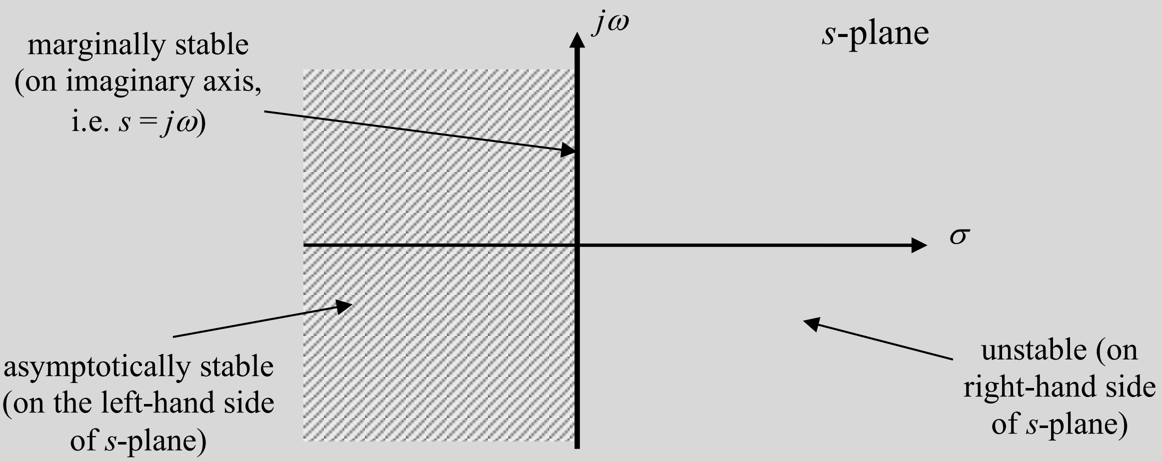
- *We covered how to determine stability from the transfer function in EE114, but will do so again now.*

8.4.1 Stability in the *s*-plane

- When we solve the poles (or roots) of $G(s)$, we obtain values for the complex variable $s = \sigma + j\omega$. These can be plotted on the *s*-plane.
- As $s \equiv \lambda$ (continuous-time), we know that the same stability conditions also apply to s as follows:

  - asymptotically stable if $Re\,(s_i) < 0$ for ***all*** poles
  - stable if $Re\,(s_i) = 0$ for some distinct, non-overlapping poles (different positions on imaginary axis) and $Re\,(s_i) < 0$ for the other poles
  - unstable if $Re\,(s_i) > 0$ for ***any*** of the poles or $Re\,(s_i) = 0$ for some overlapping poles (same position on imaginary axis)

- We can easily visualise the criteria for stability of continuous-time systems by viewing these conditions on the *s*-plane as follows:



- Simply put, if any poles of the continuous system lie on the right-hand side of the *s*-plane, the system is unstable.
- If all poles lie on the left-hand side of the *s*-plane, the system is stable.

- **Example 7.9:** Determine the stability of the system described by the following transfer function:

$$G(s) = \frac{s+1}{(s+2)(s+3)^2}$$

Solution:

Poles (or roots) of $G(s)$ are: $s = -2, \; -3 \;$ and $\; -3$

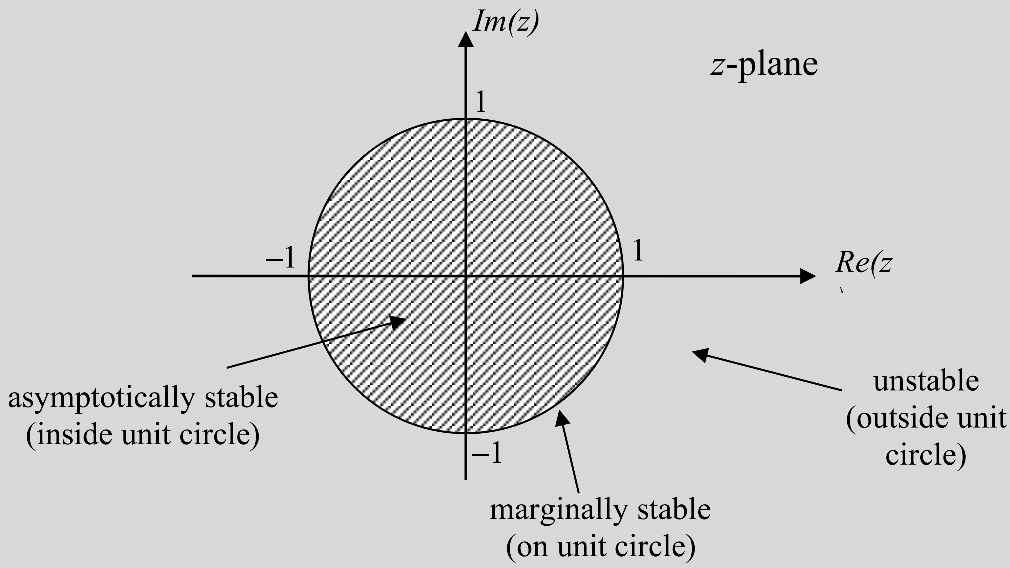
These poles all lie on the left-hand side of the *s*-plane, hence the system is asymptotically stable.

8.4.2 Stability in the *z*-plane

- When we solve the poles (or roots) of $G(z)$, we obtain values for the complex variable z and these poles can be plotted on the z-plane.

- As $z \equiv \lambda$ (discrete-time), we know that the same stability conditions also apply to z as follows:
  - asymptotically stable if $|z_i| < 1$ for ***all*** poles
  - is marginally stable if $|z_i| = 1$ for some distinct, non-overlapping poles (different position on unit circle) and $|z_i| < 1$ for the rest
  - is unstable if $|z_i| > 1$ for ***any*** of the poles or if $|z_i| = 1$ for some overlapping poles (same position on unit circle)

- We can easily visualise the criteria for stability of discrete-time systems by viewing these conditions on the z-plane as follows:



- Simply put, if any poles of the discrete system lie outside the z-plane unit circle, the system is unstable. If all poles lie inside the z-plane unit circle, the system is stable.

- **Example 7.10:** Determine the stability of the system described by the following transfer function:

$$G(z) = \frac{z+1}{(z-0.5)(z+0.8)(z-1)}$$

Solution:
Poles (or roots) of G(z) are: $z = 0.5, \ -0.8 \ \text{ and } \ 1$

One of these poles lies on the unit circle, while the others are inside it. Hence the system is marginally stable.

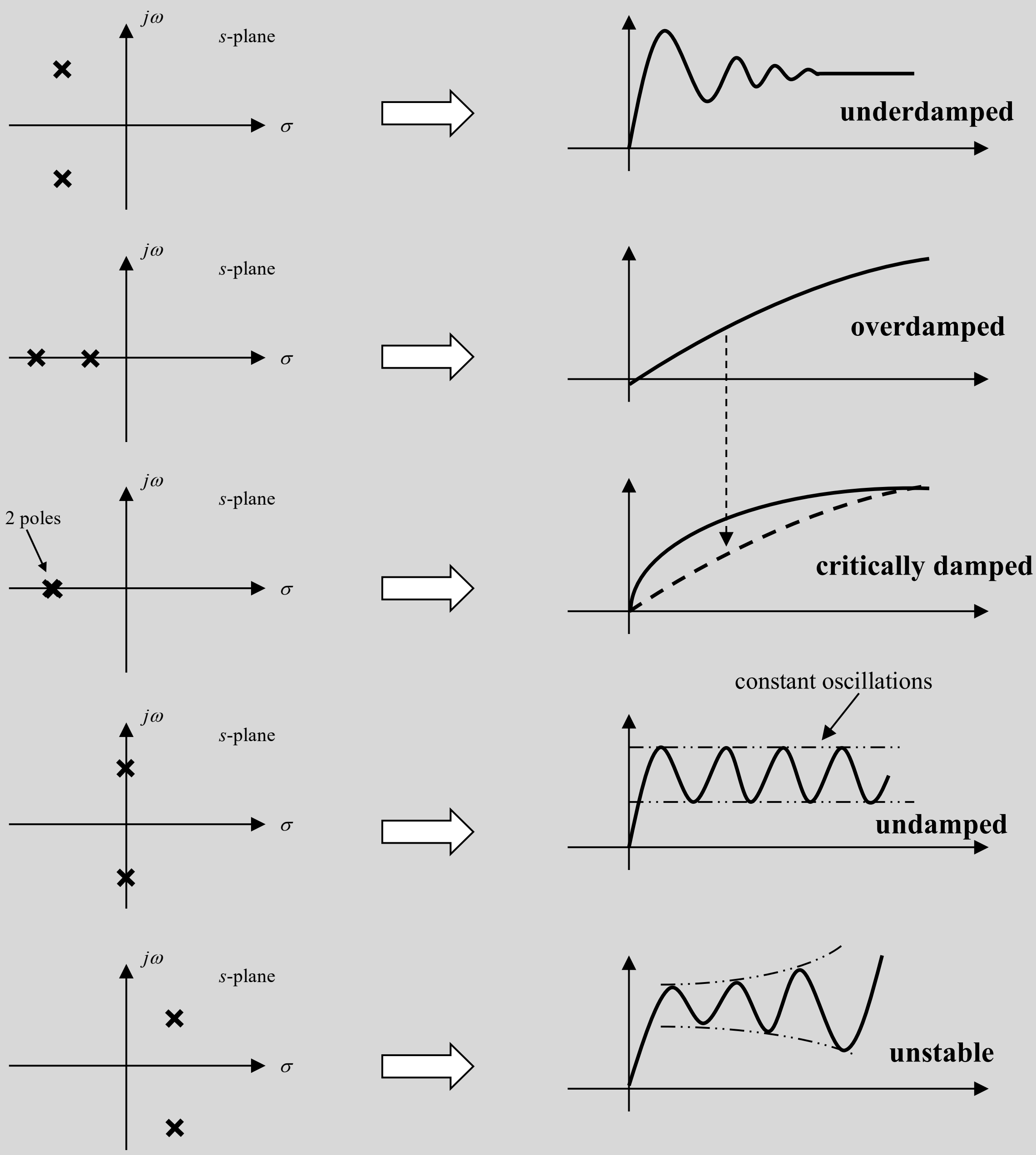
8.5 Summary of Pole locations & Transient response & Stability

- Recall that the transient response refers to the initial part of the output response of the system, i.e. that part of the response before it settles to a steady condition. When the response settles, it is said to be in steady-state.

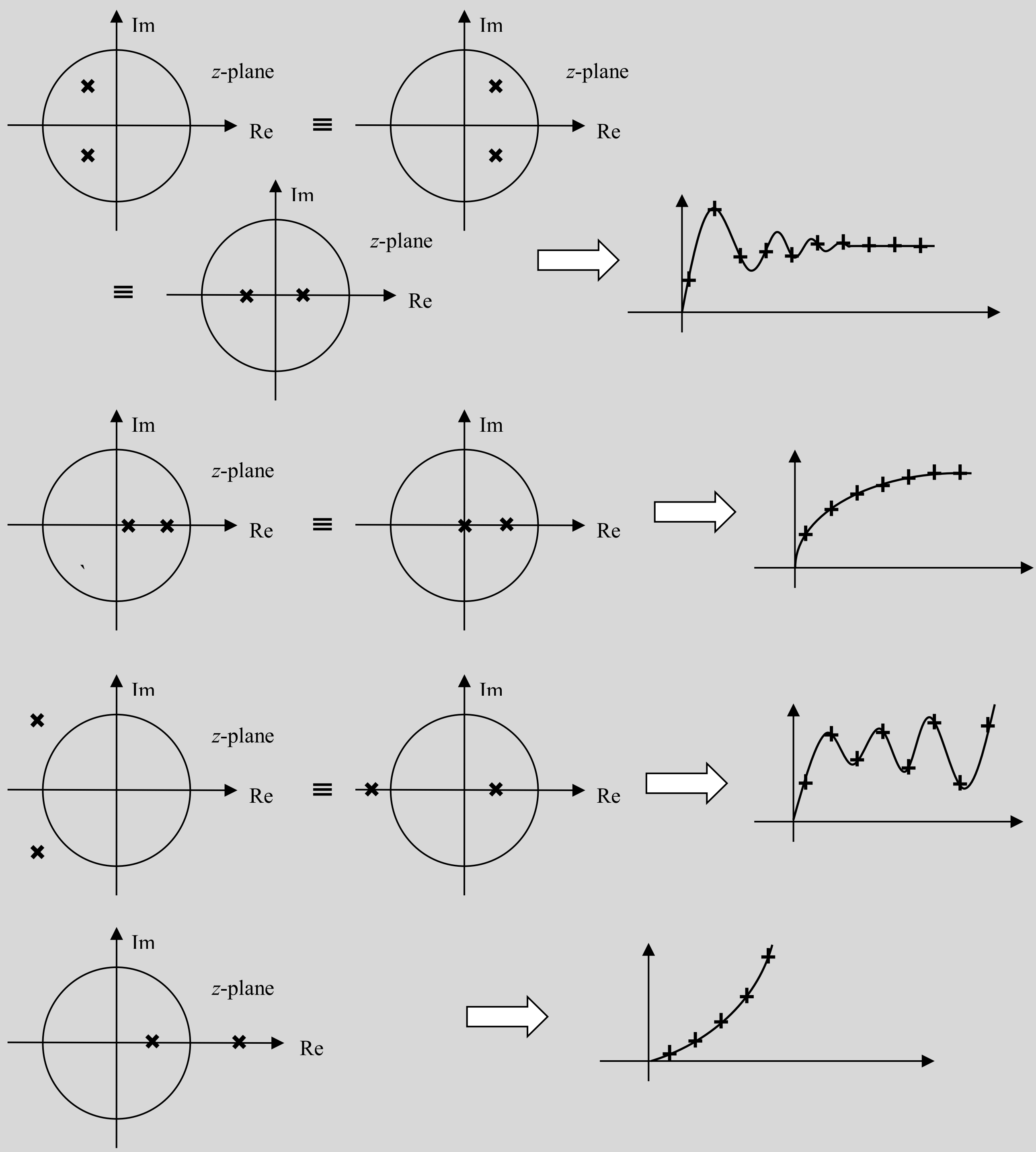
- The location of the poles affects both the transient performance and the stability of a system (both discrete-time and continuous-time).

- For the sake of completeness and convenience, the following two diagrams summarize the key response of a standard two-pole (i.e. second order) system for various pole locations.

- For the continuous-time (s-domain) we get:



- For the discrete-time (z-domain) we get:



- These general trends still apply for more general transfer functions, with multiple poles and zeros, but the interactions are much more complex.

- Transient performance specifications are often expressed in terms of desired pole positions. The objective of controllers is to move poles to these desired positions. *This will be studied in much more detail in the Control module next year.*