EE206 Assignment 6

1. Use the relation between multiplication of $f(t)$ (by t^n) and differentiation of $F(s)$ to find the Laplace transforms of the following

   (a) $f(t) = te^{-t}\cos(2t)$

$$g(t) = e^{-t}\cos(2t) \quad \Rightarrow G(s) = \frac{s+1}{(s+1)^2+4}$$

$$\begin{aligned}
\mathscr{L}\{te^{-t}\cos(2t)\} &= (-1)\frac{d}{ds}(G(s)) \\
&= (-1)\frac{((s+1)^2+4)(1)-(s+1)(2(s+1))}{((s+1)^2+4)^2} \\
&= (-1)\frac{s^2+2s+5-2s^2-4s-2}{((s+1)^2+4)^2} \\
&= \frac{s^2+2s-3}{(s^2+2s+5)^2}
\end{aligned}$$

2. Use the Laplace transform to solve the given initial-value problems

(a) $y'' + y' = e^{-t}\cos t, \quad y(0) = 0, \quad y'(0) = 0.$

$$\mathscr{L}\{y'' + y'\} = \mathscr{L}\{e^{-t}\cos t\}$$

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) = \frac{(s+1)}{(s+1)^2+1}$$

$$(s^2+s)Y(s) = \frac{(s+1)}{(s+1)^2+1}$$

$$Y(s) = \frac{1}{s[(s+1)^2+1]}$$

$$\frac{A}{s} + \frac{Bs+C}{(s+1)^2+1} = \frac{1}{s[(s+1)^2+1]}$$

$$As^2 + 2As + A + A + Bs^2 + Cs = 1$$

$$2A = 1 \quad \Rightarrow A = \frac{1}{2}$$

$$2A + C = 0 \quad \Rightarrow 1 + C = 0 \quad \Rightarrow C = -1$$

$$A + B = 0 \quad \Rightarrow B = -A \quad \Rightarrow B = -\frac{1}{2}$$

$$Y(s) = \left(\frac{1}{2}\right)\frac{1}{s} - \left(\frac{1}{2}\right)\frac{s+1}{(s+1)^2+1} - \left(\frac{1}{2}\right)\frac{1}{(s+1)^2+1}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}$$

$$= \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t$$

(b) $y' + 2y = f(t), \quad y(0) = 0,$ where

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t. \end{cases}$$

We can use the formula for piecewise functions:

$$f(t) := \begin{cases} g(t) & 0 \le t \le a \\ h(t) & a \le t \end{cases}$$

Then $f(t) = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$

$$f(t) = 1 - 1\mathscr{U}(t-1) - 1\mathscr{U}(t-1) = 1 - 2\mathscr{U}(t-1)$$

We have that:

$$\mathscr{L}\{y' + 2y\} = \mathscr{L}\{1 - 2\mathscr{U}(t-1)\}$$

Then using the second shift theorem we have:

$$\begin{aligned}
sY(s) - y(0) + 2Y(s) &= \frac{1}{s} - \frac{2}{s}e^{-s} \\
(s+2)Y(s) &= \frac{1-2e^{-s}}{s} \\
Y(s) &= \frac{1-2e^{-s}}{s(s+2)} = \frac{1}{s(s+2)} - e^{-s}\frac{2}{s(s+2)} \\
\Rightarrow \frac{A}{s} + \frac{B}{s+2} &= \frac{1}{s(s+2)} \\
As + 2A + Bs &= 1 \\
A &= 1/2 \\
B &= -A \quad \Rightarrow B = -1/2 \\
\Rightarrow \frac{C}{s} + \frac{D}{s+2} &= \frac{2}{s(s+2)} \\
Cs + 2C + Ds &= 2 \\
C &= 1 \\
D &= -C \quad \Rightarrow D = -1 \\
\Rightarrow Y(s) &= \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{1}{s+2} - e^{-s}\left(\frac{1}{s} - \frac{1}{s+2}\right) \\
\Rightarrow y(t) &= \mathscr{L}^{-1}\{Y(s)\} \\
y(t) &= \frac{1}{2} - \frac{1}{2}e^{-2t} - \mathscr{U}(t-1) + e^{-2t+2}\mathscr{U}(t-1)
\end{aligned}$$

(c) $y' + 3y = f(t), \quad y(0) = 0, \quad$ where

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & 1 \le t. \end{cases}$$

$$f(t) = 1 - \mathscr{U}(t-1)$$

$$\mathscr{L}\{y' + 3y\} = \mathscr{L}\{1 - \mathscr{U}(t-1)\}$$

$$sY(s) - y(0) + 3Y(s) = \frac{1}{s} - \frac{1}{s}e^{-s}$$

$$(s+3)Y(s) = \frac{1-e^{-s}}{s}$$

$$Y(s) = \frac{1-e^{-s}}{s(s+3)} = \frac{1}{s(s+3)} - e^{-s}\frac{1}{s(s+3)}$$

$$\Rightarrow \frac{A}{s} + \frac{B}{s+3} = \frac{1}{s(s+3)}$$

$$As + 3A + Bs = 1$$

$$A = \frac{1}{3}$$

$$B = -A \quad \Rightarrow B = -\frac{1}{3}$$

$$Y(s) = \frac{1}{3}\left(\frac{1}{s} - \frac{e^{-s}}{s} - \frac{1}{s+3} + \frac{e^{-s}}{s+3}\right)$$

$$\Rightarrow y(t) = \mathscr{L}^{-1}\{Y(s)\}$$

$$y(t) = \frac{1}{3}[1 - \mathscr{U}(t-1) - e^{-3t} + e^{-3t+3}\mathscr{U}(t-1)]$$

$$= \frac{1}{3} - \frac{1}{3}e^{-3t} - \frac{1}{3}\mathscr{U}(t-1) - e^{-3t+3}\mathscr{U}(t-1)$$

3. Use the Convolution Theorem to find the Laplace Transform of the following functions (* stands for convolution)

(a) $f(t) = t^3 * te^{-t}$

$$
\begin{aligned}
\mathcal{L}\{t^3 * te^{-t}\} &= \mathcal{L}\{t^2\}\mathcal{L}\{te^t\} \\
&= \left(\frac{3!}{s^{3+1}}\right)\left(\left[\frac{1}{s^2}\right]_{s\to s+1}\right) \\
&= \left(\frac{6}{s^4}\right)\left(\frac{1}{(s+1)^2}\right) \\
&= \frac{6}{s^4(s+1)^2}
\end{aligned}
$$

(b) $f(t) = e^{2t} * \sin 3t$

$$
\begin{aligned}
\mathcal{L}\{e^{2t} * \sin 3t\} &= \mathcal{L}\{e^{2t}\}\mathcal{L}\{\sin 3t\} \\
&= \frac{1}{s-2}.\frac{3}{s^2+9} \\
&= \frac{3}{(s-2)(s^2+9)}
\end{aligned}
$$

4. Evaluate the given Laplace transforms without evaluating the integrals (Convolution theorem)

(a) $\mathcal{L}\left\{\int_0^t \tau \sin\tau d\tau\right\}$

$$\begin{aligned}
\mathcal{L}\left\{\int_0^t \tau \sin\tau d\tau\right\} &= \frac{\mathcal{L}\{t\sin t\}}{s} \\
&= \frac{(-1)}{s}\frac{d}{ds}(\frac{1}{s^2+1}) \\
&= \frac{(-1)(-1)(2s)}{s(s^2+1)^2} \\
&= \frac{2}{(s^2+1)^2}
\end{aligned}$$

(b) $\mathcal{L}\left\{\int_0^t 2\sin\tau\cos(t-\tau)d\tau\right\}$

$$\begin{aligned}
2\mathcal{L}\left\{\int_0^t \sin\tau\cos(t-\tau)d\tau\right\} &= 2\mathcal{L}\{\sin t\}\mathcal{L}\{\cos t\} \\
&= 2(\frac{1}{s^2+1})(\frac{s}{s^2+1}) \\
&= \frac{2s}{(s^2+1)^2}
\end{aligned}$$

5. Use the Laplace transform to solve the following problems

(a) $f(t)+\int_0^t f(\tau)d\tau = 1$

$$\begin{aligned} F(s)+\frac{F(s)}{s} &= \frac{1}{s} \\ F(s)(\frac{s+1}{s}) &= \frac{1}{s} \\ F(s) &= \frac{1}{s+1} \\ f(t) &= e^{-t} \end{aligned}$$

(b) $y''+9y=\cos 3t, \quad y(0)=1, \quad y'(0)=4$

$$\begin{aligned} s^2Y(s)-sy(0)-y'(0)+9Y(s) &= \frac{s}{s^2+9} \\ (s^2+9)Y(s)-1s-4 &= \frac{s}{s^2+9} \\ Y(s) &= \frac{s}{(s^2+9)^2}+1\frac{s}{s^2+9}+4\frac{1}{s^2+9} \\ y(t) &= \frac{1}{6}t\sin 3t+\cos 3t+\frac{4}{3}\sin 3t \end{aligned}$$

(c) $y''+4y'+13y=\delta(t-\pi)+\delta(t-3\pi), \quad y(0)=1, \quad y'(0)=0$

$$\begin{aligned} s^2Y(s)-sy(0)-y'(0)+4sY(s)-4y(0)+13Y(s) &= e^{-\pi s}+e^{-3\pi s} \\ (s^2+4s+13)Y(s)-s-4 &= e^{-\pi s}+e^{-3\pi s} \\ [(s^2+4s+4)+9]Y(s) &= e^{-\pi s}+e^{-3\pi s}+s+4 \\ [(s+2)^2+9]Y(s) &= e^{-\pi s}+e^{-3\pi s}+s+4 \end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{e^{-\pi s}}{(s+2)^2+9}+\frac{e^{-3\pi s}}{(s+2)^2+9}+\frac{s}{(s+2)^2+9}+\frac{4}{(s+2)^2+9} \\ Y(s) &= \frac{e^{-\pi s}}{(s+2)^2+9}+\frac{e^{-3\pi s}}{(s+2)^2+9}+\frac{s+2}{(s+2)^2+9}+\frac{2}{(s+2)^2+9} \\ y(t) &= \frac{1}{3}e^{-2(t-\pi)}\sin 3(t-\pi)\mathcal{U}(t-\pi)+\frac{1}{3}e^{-2(t-3\pi)}\sin 3(t-3\pi)\mathcal{U}(t-3\pi) \\ &+\frac{2}{3}e^{-2t}\sin 3t+e^{-2t}\cos 3t \end{aligned}$$

(d) $y'' + 2y' = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 1$

$$\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) &= e^{-s} \\
(s^2 + 2s)Y(s) &= e^{-s} + 1 \\
Y(s) &= \frac{e^{-s}}{s(s+2)} + \frac{1}{s(s+2)} \\
\frac{A}{s} + \frac{B}{s+2} &= \frac{1}{s(s+2)} \\
As + 2A + Bs &= 1 \\
A + B &= 0 \quad \Rightarrow A = -B \\
2A &= 1 \quad A = \frac{1}{2} \quad B = -\frac{1}{2} \\
Y(s) &= \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2} + \frac{e^{-s}}{s} - \frac{e^{-s}}{s+2}\right) \\
y(t) &= \frac{1}{2} - \frac{1}{2}e^{-2t} + \frac{1}{2}\mathcal{U}(t-1)\left(1 - e^{-2t+2}\right)
\end{aligned}$$