Tutor 4.

Q1. sol

$$\begin{cases} f_{k+1} = (\alpha_F + \beta_F f_k r_k) f_k & ① \\ r_{k+1} = (\alpha_R - \beta_R f_k r_k - \lambda_R r_k) r_k & ② \end{cases}$$

(i) $X_k = X_{k+1} = X_e$
$$\begin{cases} f_e = (\alpha_F + \beta_F f_e r_e) f_e & ③ \\ r_e = (\alpha_R - \beta_R f_e r_e - \lambda_R r_e) r_e & ④ \end{cases}$$

$f_e = \alpha_F f_e + \beta_F f_e^2 r_e$

$\Rightarrow f_e[(\alpha_F - 1) + \beta_F f_e r_e] = 0$

$\Rightarrow f_e = 0 \;//\; f_e = \dfrac{(1-\alpha_F)}{\beta_F r_e}$

$r_e = \alpha_R r_e - \beta_R f_e r_e^2 - \lambda_R r_e^2$

$\Rightarrow r_e[(\alpha_R - 1) - (\beta_R f_e + \lambda_R) r_e] = 0$

$\Rightarrow r_e = 0 \;//\; r_e = \dfrac{(\alpha_R - 1)}{\beta_R f_e + \lambda_R}$

Δ ① If $f_e = 0$ & $r_e = 0$ extinction for both

Δ ② If $f_e = 0$ & $r_e = \dfrac{\alpha_R - 1}{\lambda_R}$ extinction for foxes and large rabbit population limited only by availability of glass

Δ ③ If $r_e = 0$ & $f_e = \dfrac{1-\alpha_F}{\beta_F r_e} (\to \infty)$ this is not an equilibrium point

Δ ④ If $f_e = \dfrac{(1-\alpha_F)}{\beta_F r_e}$ & $r_e = \dfrac{\alpha_R - 1}{\beta_R f_e + \lambda_R}$

Hence
$$\begin{cases} r_e = \dfrac{\beta_F(\alpha_R - 1) - \beta_R(1-\alpha_F)}{\lambda_R \beta_F} & ⑤ \\[2ex] f_e = \dfrac{\lambda_R(1-\alpha_F)}{\beta_F(\alpha_R - 1) - \beta_R(1-\alpha_F)} & ⑥ \end{cases}$$

(ii) $\Delta f_{k+1} = \underbrace{\dfrac{\partial g_1}{\partial f_k}\Big|_e}_{①} \cdot \Delta f_k + \underbrace{\dfrac{\partial g_1}{\partial r_k}\Big|_e}_{②} \Delta r_k$

$\Delta r_{k+1} = \underbrace{\dfrac{\partial g_2}{\partial f_k}\Big|_e}_{③} \Delta f_k + \underbrace{\dfrac{\partial g_2}{\partial r_k}\Big|_e}_{④} \Delta r_k$

$$\begin{cases} ① = \alpha_F + 2\beta_F f_e r_e & ② = \beta_F f_e \\ ③ = -\beta_R r_e^2 & ④ = \alpha_R - 2\beta_R f_e r_e - 2\lambda_R r_e \end{cases}$$

for ① $\begin{bmatrix} \Delta f_{R+1} \\ \Delta r_{R+1} \end{bmatrix} = \begin{bmatrix} \alpha_F & 0 \\ 0 & \alpha_R \end{bmatrix} \cdot \begin{bmatrix} \Delta f_R \\ \Delta r_R \end{bmatrix}$

for ② $\begin{bmatrix} \Delta f_{p+1} \\ \Delta r_{p+1} \end{bmatrix} = \begin{bmatrix} \alpha_F & 0 \\ -\beta_R \left[\frac{\alpha_{R-1}}{\alpha_R}\right]^2 & 2-\alpha_R \end{bmatrix} \begin{bmatrix} \Delta f_k \\ \Delta r_R \end{bmatrix}.$

for ③ Null

for ④ $\begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} (2-\alpha_F) & \beta_F f_e^2 \\ -\beta_R r_e^2 & (2-\alpha_F) \end{bmatrix} \begin{bmatrix} \Delta f_R \\ \Delta r_R \end{bmatrix}.$

NB: $\begin{cases} f_e r_e = \dfrac{(1-\alpha_F)}{\beta_F} \Longleftrightarrow \alpha_F + 2\beta_F f_e r_e = 2-\alpha_F \\ \alpha_R - 2\beta_R f_e r_e - 2\lambda_R r_e = 2-\alpha_R \end{cases}$

Q2 sol (i) $\dot{x} = x^2 - 2x - 8.$

when $\dot{x} = 0$ $x_e = -2$ & 4.

$\dot{x} = f(x) = x^2 - 2x - 8$

$\Delta\dot{x} = \frac{\partial f}{\partial x}\Big|_e \Delta x$ And $\frac{\partial f}{\partial x}\Big|_e = 2x_e - 2$

for $x_e = -2$ $\Delta\dot{x} = -6\Delta x$

for $x_e = 4$ $\Delta\dot{x} = 6\Delta x$

(ii) $\ddot{x} = \dot{x}^2 x - 2x - 8u^3$

① $x_e = -4u_e^3$

② let $\dot{x}_1 = x_2$; $\dot{x}_2 = x_2^2 x_1 - 2x_1 - 8u_e^3$

$x_{1e} = x_e = -4u_e^3$ $x_{2e} = 0$

③ $\dot{x}_1 = f_1(x_1, x_2, u) = x_2$

$\dot{x}_2 = f_2(x_1, x_2, u) = (x_2^2 x_1 - 2x_1 - 8u^3)$

$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -24u_e^2 \end{bmatrix} \Delta u$

✓

Q2 (iii) sol $X_{k+1} = X_k^2 - 2X_k - 8$.

① $X_e = \frac{3 \pm \sqrt{41}}{2}$.

② $\Delta X_{k+1} = \frac{\partial f}{\partial X_k}\Big|_e \Delta X_k = (2X_e - 2)\Delta X_k$

For $X_e = \frac{3+\sqrt{41}}{2}$ $\Delta X_{k+1} = [1+\sqrt{41}]\,\Delta X_k$

For $X_e = \frac{3-\sqrt{41}}{2}$ $\Delta X_{k+1} = [1-\sqrt{41}]\,\Delta X_k$.

(iv)

$$\begin{bmatrix} \Delta X_1(k+1) \\ \Delta X_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ X_e^2 - 2 & 2X_e^2 \end{bmatrix} \begin{bmatrix} \Delta X_1(k) \\ \Delta X_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

(v)

$$\begin{bmatrix} \Delta \dot{X}_1 \\ \Delta \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -2X_{2e} & 2X_2 u_e - 2X_{1e} \\ -8X_{1e}X_{2e} & -4X_{1e}^2 \end{bmatrix} \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \end{bmatrix} + \begin{bmatrix} X_{2e}^2 \\ 2 \end{bmatrix} \Delta u$$

(vi)

$$\begin{bmatrix} \Delta \dot{X}_1 \\ \Delta \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 4u_e - 2u_e^2 & 0 \\ -4u_e^2 & 1 \end{bmatrix} \begin{bmatrix} \Delta X_1 \\ \Delta X_2 \end{bmatrix} + \begin{bmatrix} 2u_e^5 - 4u_e^2 \\ 2 \end{bmatrix} \Delta u$$

(vii) Equilibrum point $\Leftrightarrow X_e = \frac{-1 \pm \sqrt{-3}}{2} \notin R$

So there are not ~~real~~ real equilibrum point.

(iv) sol.

$$\begin{bmatrix} \Delta X_1(k+1) \\ \Delta X_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\sqrt{1-u_e^2} & 1 \end{bmatrix} \begin{bmatrix} \Delta X_1(k) \\ \Delta X_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u$$

(Q3) (i) sol.

$$\frac{dh}{dt} = -\frac{k}{\pi \tan^2\theta}\cdot\frac{1}{h} + \frac{1}{\pi\tan^2\theta}\cdot\frac{f_{in}}{h^2}.$$

Q point

$$f_{in} = k h_e \Rightarrow h_e = \frac{f_{in}}{k}$$

because flow rate out of the tank depend on pressure (which is directly proportional to height, h)

Since θ only affects the volume but not the equilibrium liquid level.

(ii) For $k=\frac{1}{2}$ & $\theta = \frac{\pi}{4}$. $\Rightarrow \frac{dh}{dt} = -\frac{1}{2\pi}\cdot\frac{1}{h} + \frac{1}{\pi}\cdot\frac{f_{in}}{h^2}$

$$\begin{cases} \dot{h} = g(h, f_{in}) = -\frac{1}{2\pi}\cdot h^{-1} + \frac{1}{\pi} f_{in} h^{-2} \\ h_e = 2 f_{in} \end{cases}$$

$$\Delta\dot{h} = \frac{\partial g}{\partial h}\Big|_e \Delta h + \frac{\partial g}{\partial f_{in}}\Big|_e \Delta f_{in}$$

$$\Downarrow \qquad \Downarrow$$

$$\left(-\frac{1}{8\pi f_{in}^2}\right) \qquad \left(\frac{1}{4\pi f_{in}^2}\right)$$

Hence $\Delta\dot{h} = \left[-\frac{1}{8\pi f_{in}^2}\right]\Delta h + \left[\frac{1}{4\pi f_{in}^2}\right]\Delta f_{in}$

When $\begin{cases} f_{in} = 1; & \Delta\dot{h} = -\frac{1}{8\pi}\Delta h + \frac{1}{4\pi}\Delta f_{in} \\ f_{in} = 2; & \Delta\dot{h} = -\frac{1}{32\pi}\Delta h + \frac{1}{16\pi}\Delta f_{in} \end{cases}$

Q4.

(i) sol

$$\begin{cases} \dot{x}_1 = x_1 \sin(x_1) + x_2 \\ \dot{x}_2 = x_2 + u \end{cases}$$

$$\begin{cases} x_{1e}\sin(x_{1e}) + x_{2e} = 0 \\ x_{2e} + u_e = 0 \end{cases} \Longrightarrow \begin{cases} x_{2e} = -u_e \\ x_{1e}\sin(x_{1e}) = u_e \end{cases}$$

① when $u_e = 0$ $\quad x_{1e}\sin(x_{1e}) = 0$

$\Longrightarrow x_{1e} = 0 \,;\, -\pi \,;\, \pi$,

② when $u_e = 2$ $\quad x_{1e}\sin(x_{1e}) = 2$

$\Longrightarrow$ In $|x_1| < 2\pi$, there are no equilibrium point.

(ii)
$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_{1e}\cos(x_{1e}) + \sin(x_{1e}) & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\Delta u$$

① for $x_{1e} = 0 \Longrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\Delta u$

② for $x_{1e} = (-\pi) \Longrightarrow \begin{bmatrix} \pi & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\Delta u$

③ for $x_{1e} = (\pi) \Longrightarrow \begin{bmatrix} -\pi & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\Delta u$