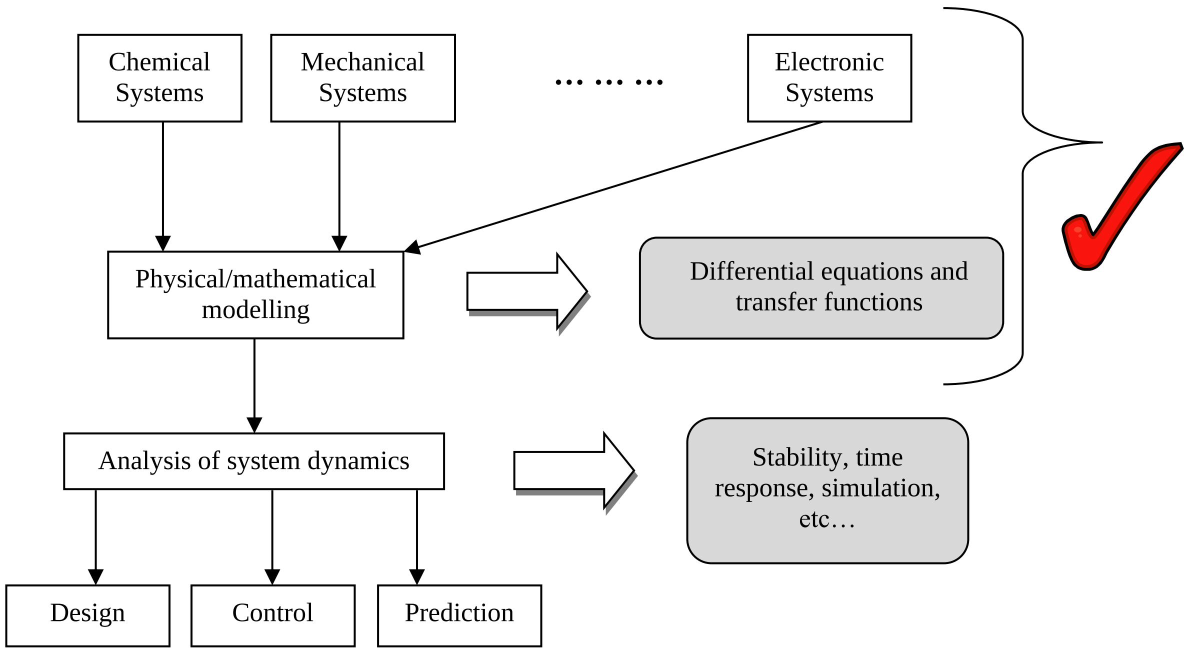
6. System Analysis (Time Response) & Simulation

6.1 Introduction

- At this stage, we have examined how to model a selection of relatively straightforward systems, which we have represented both as differential equations and, more conveniently, as transfer functions.

- This completes the first part of the 'big picture' as presented in section 2.5 of the notes and reproduced here for convenience:



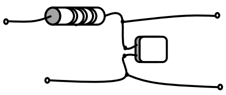
- Now, we are going to consider the analysis of such systems from both a mathematical and a simulation viewpoint.

- We will briefly look at solving the differential equation model of a system before concentrating on doing the same for the transfer function equivalent. The latter provides an easier analytical method.

- Using the Matlab and Simulink software package (to be used in the laboratories), we have (or will) simulated the systems studied thus far and can compare the responses obtained with their analytically derived counterparts.

- For the rest of this section, we are going to analyse the following 3 systems:

  - the RC electrical circuit
  - the (bicycle) mass-spring-damper and
  - the single water tank.
- The models for each of these systems are summarised in the table below:

System	Differential Equation	Transfer Function
RC Circuit 	$v_i = RC\frac{dv_C}{dt} + v_C$	$\frac{V_c(s)}{V_i(s)} = \frac{1}{1+sRC}$
Mass-spring-damper 	$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t)$	$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$
Single tank 	$A\frac{dh}{dt} = F_{in} - kh$	$\frac{H(s)}{F(s)} = \frac{1}{sA+k}$

6.2 Basic definition - the *order* of a system

- The **order of a system** is defined by the number of independent energy storage components it contains.

- The order is given by the **highest derivative involved in the linear differential equation** describing the system.

- Alternatively, the order is generally given by the **highest power of *s* in the denominator of the transfer function**.

- So, for the above systems we can state that the RC circuit and the single tank are both first order systems while the mass-spring-damper is a second-order system.

6.3 Solving the differential equation model (first order)

- Consider the first order RC system whose model is given by: $v_i = RC\frac{dv_C}{dt} + v_C$

- For convenience and clarity, we will rewrite this as:

$$RC\frac{dy}{dt} + y = u$$

where the output v_C is replaced with y and the input v_i is replaced with u.

- The solution of this first order ordinary differential equation (ODE) involves the superposition (or sum) of two components, i.e.:

$$y(t) = y_n(t) + y_f(t)$$

- Here, $y_n(t)$ is **the zero-input response or natural response** – this is found by solving the ODE with **all inputs set to zero**.

- Also, $y_f(t)$ is the **steady-state response or forced response** – this is found by solving the ODE with **all derivatives set to zero** (*note, in steady-state derivatives are zero).*

- The final solution is simply the sum of these two responses.

Finding $y_n(t)$, the zero-input response:

- Set all inputs to zero. Hence: $u = 0 \Rightarrow RC\dfrac{dy_n}{dt} + y_n = 0$

- Use the *separation of variables* method to solve this problem. Hence:

$$RC\frac{dy_n}{dt} = -y_n \Rightarrow \frac{dy_n}{y_n} = -\frac{1}{RC}dt$$

- Integrating both sides gives:

$$\int \frac{dy_n}{y_n} = -\frac{1}{RC}\int dt$$

$$\Rightarrow \ln(y_n) = -\frac{1}{RC}t + K$$

$$\Rightarrow y_n = e^{\frac{-t}{RC}+K} = e^K e^{\frac{-t}{RC}} = Ae^{\frac{-t}{RC}}$$

- Here, K (and effectively A) is a *constant of integration*. We will obtain a value for this constant once we have a complete solution.

Finding $y_f(t)$, the steady-state response:

- Set all derivatives to zero. Here:

$$\frac{dy}{dt} = 0 \Rightarrow RC(0) + y_f = u \Rightarrow y_f = u$$

Complete solution:

- Adding both responses gives: $y(t) = y_n(t) + y_f(t) = Ae^{\frac{-t}{RC}} + u$

- To find the value of *A*, we apply any initial conditions.

- Here, we will take zero initial conditions, i.e. at time *t* = 0, the output *y* = 0. In other words, there is no voltage across the capacitor at time *t* = 0.

- Hence:

$$y(0) = 0 \quad \Rightarrow 0 = Ae^0 + u$$
$$\Rightarrow 0 = A(1) + u$$
$$\Rightarrow A = -u$$

- Therefore the complete solution for the RC circuit is:

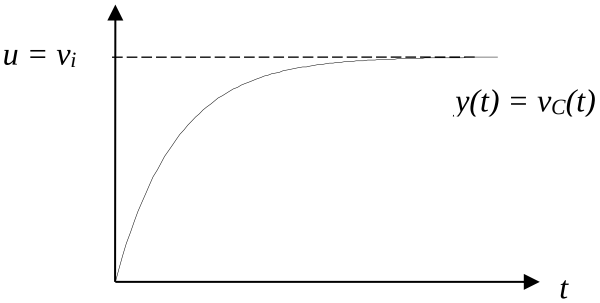
$$y(t) = -ue^{\frac{-t}{RC}} + u$$

$$\Rightarrow y(t) = u\left(1 - e^{\frac{-t}{RC}}\right)$$

- *A quick sketch of output y over time gives:*

- *Intuitively, this is the expected response of a charging capacitor.*

- *At the start time, the capacitor has not been charged and the voltage across it is 0.*



- *As time passes, the capacitor charges up and the voltage across it increases.*

- *Once the capacitor is fully charged, it acts as an open circuit and here the voltage across it is equal to the input voltage.*

- Here, we are only going to consider the solution to a first order differential equation (mainly for the purpose of illustration).

- You will study how to solve second order differential equations in your mathematics modules.

- You will also analyse various circuits in more detail in your circuit module in Year 2.

6.4 Solving the transfer function model

- Once again, let us consider the first order RC system but this time we will solve its transfer function representation instead.

$$\frac{V_c(s)}{V_i(s)} = \frac{1}{1+sRC} \equiv \frac{Y(s)}{U(s)}$$

- Remember also, that this function is based on zero initial conditions by default.

- Okay, so we need to use the inverse Laplace transform in order to obtain the solution as a function of time.

- The output is given by:

$$Y(s) = \frac{U(s)}{1+sRC}$$

- The input is a constant value, say *u*, hence:

$$U(s) = \frac{u}{s} \qquad \textit{(from the table of Laplace transforms)}$$

- Hence:

$$Y(s) = u\left(\frac{1}{s(1+sRC)}\right)$$

- We now apply the partial fraction method as follows:

$$\frac{1}{s(1+sRC)} \equiv \frac{A}{s} + \frac{B}{1+sRC} = \frac{A(1+sRC)+Bs}{s(1+sRC)}$$

- Equating the coefficients of *s* gives:

$$A = 1 \quad \text{and} \qquad B + A(RC) = 0 \quad \Rightarrow B = -RC$$

- Hence:

$$Y(s) = u\left(\frac{1}{s} - \frac{RC}{1+sRC}\right) = u\left(\frac{1}{s} - \frac{1}{s+\frac{1}{RC}}\right)$$

- Referring to the table of commonly used Laplace transforms:

$$y(t) = L^{-1}\left(u\left(\frac{1}{s} - \frac{1}{s+\frac{1}{RC}}\right)\right) \quad = u\left(1 - e^{\frac{-t}{RC}}\right)$$

- Hence, as expected, we have obtained the exact same solution as in the previous section.

- However, this method is arguably a lot easier to perform and avoids the need for integration.

- As such, we will use this method to solve the remaining systems.

- ***Ex. 6.1 Obtain a solution for h(t) for the single tank system whose model is given by:***

$$\frac{H(s)}{F(s)} = \frac{1}{sA + k}$$

Solution:

Output height is given by: $$H(s) = \frac{F(s)}{sA + k}$$

Here, the input is a constant flow rate, say *fin*, hence: $F(s) = \frac{fin}{s}$

Thus: $$H(s) = fin\left(\frac{1}{s\,(sA + k)}\right)$$

Using the method of partial fractions:

$$\frac{1}{s\,(sA + k)} \equiv \frac{X}{s} + \frac{Y}{sA + k} = \frac{X\,(sA + k) + Ys}{s\,(sA + k)} = \frac{s(XA + Y) + Xk}{s\,(sA + k)}$$

Equating the coefficients of *s* gives:

$$Xk = 1 \quad \Rightarrow X = \frac{1}{k} \qquad \text{and} \qquad XA + Y = 0 \quad \Rightarrow Y = -\frac{A}{k}$$

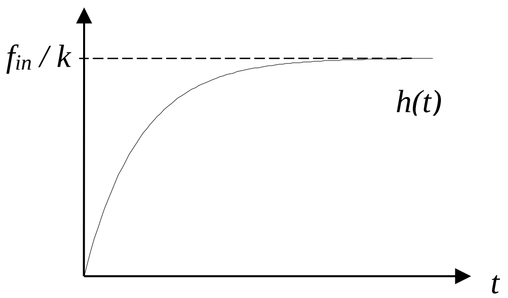
Hence:

$$H(s) = fin\left(\frac{\frac{1}{k}}{s} - \frac{\frac{A}{k}}{sA + k}\right) = \frac{fin}{k}\left(\frac{1}{s} - \frac{A}{sA + k}\right) = \frac{fin}{k}\left(\frac{1}{s} - \frac{1}{s + \frac{k}{A}}\right)$$

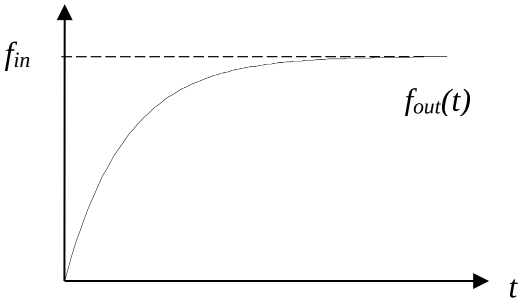
Using the Inverse Laplace transform, we obtain:

$$h(t) = L^{-1}\left(\frac{fin}{k}\left(\frac{1}{s} - \frac{1}{s + \frac{k}{A}}\right)\right) = \frac{fin}{k}\left(1 - e^{\frac{-kt}{A}}\right)$$

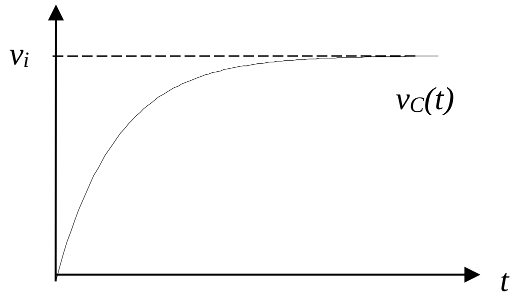
- A few noteworthy points about this solution:



  - A sketch of the change in height with time is shown to the right:
  - Recall that $h = k\, f_{out}$, hence, if we let $k = 1$, then $h = f_{out}$, the output flow and vice versa, $f_{out} = h$.
  - Hence we can sketch the relationship between the output flow and the input flow as shown on the left below.



Single tank system



RC circuit system

  - This is identical to the relationship between the output voltage and input voltage of the RC circuit, as shown above to the right.
  - In addition, the charge $q(t)$ in a capacitor is given by: $q(t) = Cv_C(t)$

    where C is the capacitance (farads).
  - This is similar to the tank system where: $h(t) = k\, f_{out}(t)$
  - In the tank system, the output flow is initially zero and increases from time t = 0 as the height of the water in the tank increases. At some point the pressure due to the volume of water (directly related to the height) causes the output flow to match the input flow (for $k = 1$) and the system settles at this point, i.e. the height of the water no longer changes.
  - Similarly, in the RC circuit, the voltage across the capacitor is initially zero and from time t = 0 the capacitor charges. At some point the capacitor becomes fully charged, the current stops flowing and the system settles at this point, i.e. the charge no longer changes and the output voltage matches the input.
  - ***So, in brief, although we have two very different physical systems (one a single tank and one an RC circuit) they are both first order systems and exhibit identical dynamical behaviour and thus can effectively be represented by the same mathematical model!***
- ***Ex. 6.2 Obtain a solution for x(t) for the mass-spring-damper system whose model is:***

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

given that the input force is 1N, M = 1kg, B = 5Ns/m and K = 6N/m.

Solution:

Output position is given by: $$X(s) = \frac{F(s)}{Ms^2 + Bs + K}$$

Here, the input is a constant force of value 1N, hence: $$F(s) = \frac{1}{s}$$

Thus, substituting for the parameters, we obtain: $$X(s) = \frac{1}{s\left(s^2 + 5s + 6\right)}$$

Using the method of partial fractions:

$$\frac{1}{s\left(s^2 + 5s + 6\right)} = \frac{1}{s\,(s+2)(s+3)} \equiv \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$\equiv \frac{A(s+2)(s+3) + Bs(s+3) + Cs(s+2)}{s\,(s+2)(s+3)}$$

Compare the numerators: $$1 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2)$$

Setting $s = 0$: $$1 = A(2)(3) \Rightarrow \boldsymbol{A} = \frac{\boldsymbol{1}}{\boldsymbol{6}}$$

Setting $s = -2$: $$1 = B(-2)(1) \Rightarrow \boldsymbol{B} = -\frac{\boldsymbol{1}}{\boldsymbol{2}} = -\frac{\boldsymbol{3}}{\boldsymbol{6}}$$

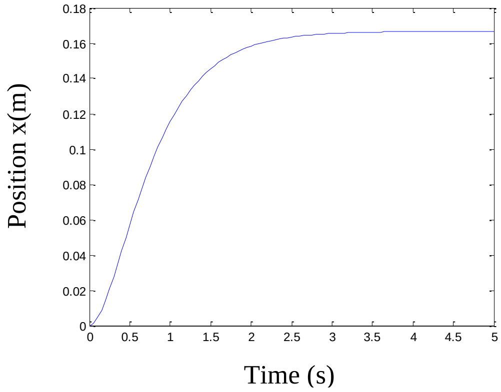
Setting $s = -3$: $$1 = C(-3)(-1) \Rightarrow \boldsymbol{C} = \frac{\boldsymbol{1}}{\boldsymbol{3}} = \frac{\boldsymbol{2}}{\boldsymbol{6}}$$

Hence: $$X(s) = \frac{1}{6}\left(\frac{1}{s} - \frac{3}{s+2} + \frac{2}{s+3}\right)$$

Finally:

$$x(t) = L^{-1}\left(X(s)\right) = \frac{1}{6}\left(1 - 3e^{-2t} + 2e^{-3t}\right)$$

- A plot of $x(t)$ looks like:

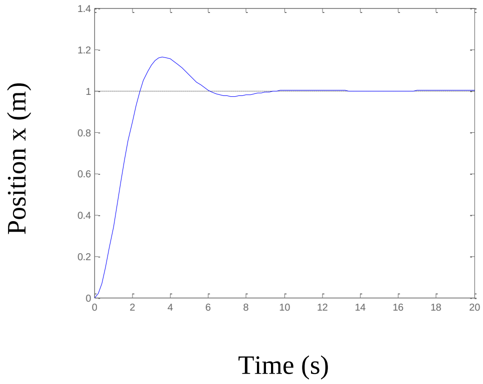


- While this response is similar in nature to those obtained from the previous systems, this is simply due to the parameters chosen for this example.

- Consider a mass-spring-damper system given by the transfer function:

$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + s + 1}$$

- This produces the response to the right:



- Clearly, we can see that is very different to our previous systems, with more complicated dynamics.

- This is to be expected, however, as we are now dealing with a 2nd order system as opposed to a 1st order system, as in the previous examples.

- Try solving this latter transfer function using inverse Laplace transforms ... you will find that it is not quite as straightforward as the previous example!

- Thankfully, we don't need to worry about solving more complicated systems.

- Instead we can use suitable software to simulate the systems for us and we can use such simulations to analyse the given system.

6.5 System simulation

MATLAB SIMULINK

- In this module, we use **Simulink**, a graphical modeling and simulation environment that can be used to design, simulate, implement and test mathematical models of real systems.

- Simulink is integrated with **Matlab**, the latter being a high-level language and interactive environment for numerical computation, data analysis and visualisation, etc.

- **You will be introduced to, and learn to use both of, these environments in our labs.**

- *It is important to learn to use these environments well, as you will use them (Matlab in particular) for a range of activities, across numerous different modules in the BE programme, and possibly even ...*