Engineering Mathematics 1 (Fall 2021)

Prof. Fengying Wei
E-mail: weifengying@fzu.edu.cn
College of Mathematics and Statistics, Fuzhou University

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Students should be able to (after learning)

- Add, subtract and multiply complex numbers
- Convert complex numbers between Cartesian and polar forms
- Differentiate all commonly occurring functions including polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of a derivative, namely the derivative as a tangent and the derivative as a rate of change
- Integrate certain standard functions, constructed from polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of integration, namely the integral as the inverse of the derivative and the integral as the area under a curve
- Apply Taylor series to numerically approximate functions
- Apply Simpson's rule to numerically evaluate integrals
- Solve simple first and second order ordinary differential equations
- Apply and select the appropriate mathematical techniques to solve a variety of associated engineering problems

Lecture ~~12~~ 13: Differentiation-Part ~~4~~ 5

10. Maclausrin's series and Taylor's series

Maclaurin's series: $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

Taylor's series: $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$

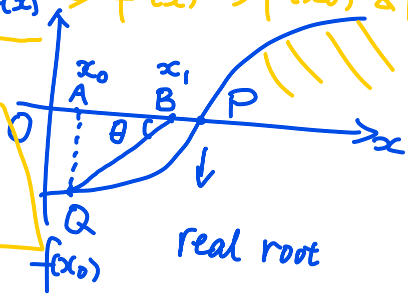
Ex1: $y = \sinh 5.02$, approximate y.

Ex2: $y = \cosh 1.01$, approximate y.

$\triangle QAB$, $\tan\theta = \dfrac{|QA|}{|AB|} = \dfrac{-f(x_0)}{x_1 - x_0}$, $\tan\theta = f'(x_0)$

$y=f(x) \Rightarrow f'(x) \Rightarrow f'(x_0)$ & $f(x_0)$

$\therefore x_0 - x_1 = \dfrac{f(x_0)}{f'(x_0)}$ $\quad \therefore \boxed{x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}}$.



11. Newton-Raphson iterative method

Aim: approximation or estimation

Curve $y = f(x)$ is given, A is the point passing through x-axis with $f(x) = 0$, P is a point on the curve near to point A, then point B (or $x = x_0$) is an approximate value of the root of $f(x) = 0$, a better approximation is given by $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$. : Curve is fixed $\Rightarrow$ first derivative $\Rightarrow$ two values

Ex1: The equation $x^3 - 3x - 4 = 0$ with properties $f(1) < 0$ and $f(3) > 0$ admits a root near 2. Find a better approximation to the root.

Ex2: The equation $\boxed{2x^3 - 7x^2 - x + 12} = 0$ has a root near to $x = 1.5$. Use the Newton-Raphson method to find the root to two decimal places.

Set $y(x) = 2x^3 - 7x^2 - x + 12$, $y'(x) = 6x^2 - 14x - 1$, $y'(x_0) = \frac{3}{2}\cdot\frac{6\times 9}{4} - 21 - 1 = 13.5 - 22 = -8.5 = -\frac{17}{2}$

See $x_0 = 1.5 = \frac{3}{2}$, $y(x_0) = 2\cdot\frac{27}{8} - 7\cdot\frac{9}{4} - \frac{3}{2} + 12 = \frac{3}{2}$

by N-R, $x_1 = x_0 - \dfrac{y(x_0)}{y'(x_0)} = \frac{3}{2} + \frac{3}{2} \times \frac{2}{17} = \frac{3}{2} \times \frac{19}{17} = \frac{57}{34} = 1.68$

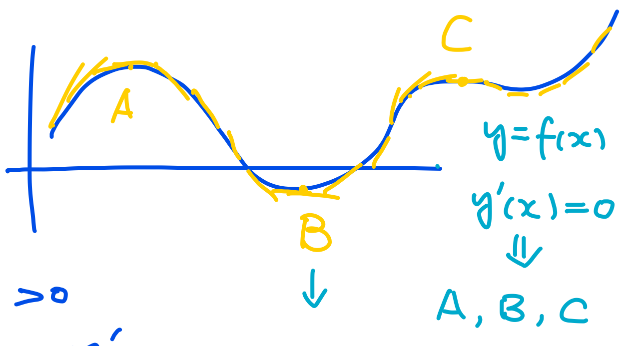
again by N-R, $x_2 = x_1 - \dfrac{y(x_1)}{y'(x_1)} = 1.68 - \dfrac{0.046}{-7.6} = 1.68 + 0.006 = 1.686 \approx \boxed{1.69}$

$\therefore$ 1.69 is a better estimation for real root. 2dp = 2 decimal point 0.01

12. Maximum, minimum, point of inflexion

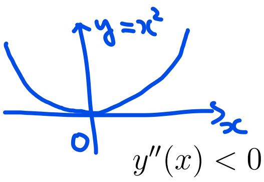
Given a function $y = f(x)$, stationary points are defined as $y'(x) = 0$. 驻点, $f'(x)=0$

$y'(x) = 0$, it may be a maximum, may be a minimum, may be a point of inflexion (i.e., S-bend form)

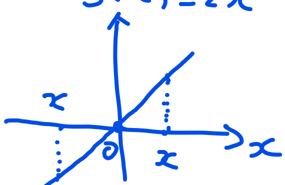


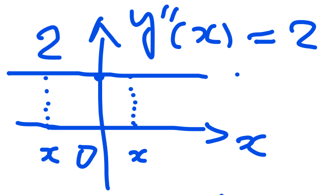
$y''(x) > 0$, ~~maximum~~ minimum;

$y(x) = x^2$, $y'(x) = 2x$, $y''(x) = 2 > 0$



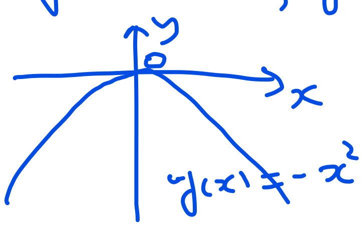
$y'(x) = 2x$ minimum x_1 x_2 x_3

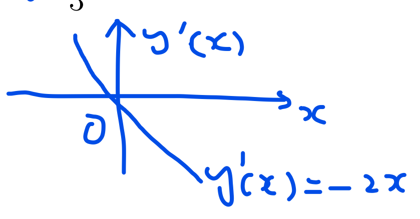


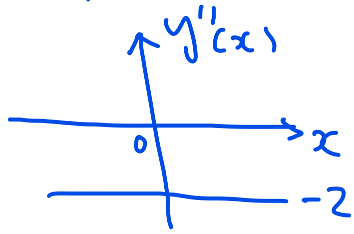


$y''(x) < 0$, ~~minimum~~ maximum;

$y(x) = -x^2$, $y'(x) = -2x$, $y''(x) = -2$







$y''(x) = 0$, may be points of inflexion (if yes, then change of sign occurs)

Ex1: $y = x^2$, to find stationary points, maximum, minimum.

Sol: $y'(x) = 2x$, Let $y'(x) = 0$ $\therefore x = 0$ is a stat. point.

$y''(x) = 2 > 0$, $\therefore x = 0$ is a minimum.

Ex2: For $y = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x - 5$, find the points of inflexion.

Sol: $y'(x) = x^2 - x - 2$, let $y'(x) = 0$ $\therefore x_1 = -1$, $x_2 = 2$

$y''(x) = 2x - 1$, $\therefore y''(x_1) = y''(-1) = -2 - 1 = -3 < 0$

$y''(x_2) = y''(2) = 4 - 1 = 3 > 0$

$\therefore x_1$ is a maximum, x_2 is a minimum

Let $y''(x) = 2x - 1 = 0$, i.e., $x = \frac{1}{2}$. So, $\frac{1}{2}$ may be a point of inflexion.

We choose a very small number $c > 0$,

For $x = \frac{1}{2}$,

At $x = \frac{1}{2} - c$, $y''(x) = 2(\frac{1}{2} - c) - 1 = -2c < 0$,

At $x = \frac{1}{2} + c$, $y''(x) = 2(\frac{1}{2} + c) - 1 = 2c > 0$,

Sign of $y''(x)$ changes, $\therefore x = \frac{1}{2}$ is a point of inflexion.

Ex3: For $y = 3x^5 - 5x^4 + x + 4$, find the points of inflexion.

Sol: $y'(x) = 15x^4 - 20x^3 + 1$, $y''(x) = 60x^3 - 60x^2 = 60x^2(x-1)$

$\therefore x_1 = x_2 = 0$, $x_3 = 1$ may be points of inflexion.

We choose a small number $c > 0$

For $x_1 = x_2 = 0$,

At $x = 0 + c$, $y''(x) = 60(+c)^2(c-1) < 0$,

At $x = 0 - c$, $y''(x) = 60(-c)^2(-c-1) < 0$,

$\therefore$ Sign of $y''(x)$ does NOT change,

$\therefore x_1 = x_2 = 0$ are not points of inflexion.

For $x_3 = 1$,

At $x = 1 + c$, $y''(x) = 60(1+c)^2(1+c-1) > 0$,

$x = 1 - c$, $y''(x) = 60(1-c)^2(1-c-1) < 0$,

$\therefore$ Sign of $y''(x)$ changes,

$\therefore x_3 = 1$ is a point of inflexion.