EE206 Assignment 5

1. Find the Laplace Transform of the following functions, **using the definition, NOT the tables.**

   (a) $f(t) = t^n$, where n is a natural number, i.e. $n = 0, 1, 2, \ldots$
   (Hint: write $\mathcal{L}\{t^n\}$) in terms of $\mathcal{L}\{t^{n-1}\}$ using integration by parts. Then use the result to write $\mathcal{L}\{t^n\}$ in terms of $\mathcal{L}\{1\}$

Solution:

First following the hint we use the definition to write $\mathscr{L}\{t^n\}$ in terms of $\mathscr{L}\{t^{n-1}\}$ using integration by parts, and for now assume $n > 1$ since we know $\mathscr{L}\{1\} = \frac{1}{s}$:

$$\mathscr{L}\{t^n\} = \int_0^\infty t^n e^{-st} dt$$

$$\int u dv = uv - \int v du$$

$$u = t^n \qquad du = nt^{n-1}\, dt$$

$$dv = e^{-st} dt \qquad v = -\frac{e^{-st}}{s}$$

So we have

$$\mathscr{L}\{t^n\} = -t^n \frac{e^{-st}}{s}\bigg|_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}$$

Or since the first term is zero:

$$\mathscr{L}\{t^n\} = \frac{n}{s}\mathscr{L}\{t^{n-1}\}$$

Now we can continue using this to find $\mathscr{L}\{t^n\}$ in terms of $\mathscr{L}\{1\}$, for example replace n by $n-1$ in the formula above to get $\mathscr{L}\{t^{n-1}\} = \frac{n-1}{s}\mathscr{L}\{t^{n-2}\}$, and sub it back in.

$$\mathscr{L}\{t^n\} = \frac{n}{s}\cdot\frac{n-1}{s}\mathscr{L}\{t^{n-2}\}$$

$$\mathscr{L}\{t^n\} = \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\mathscr{L}\{t^{n-3}\}$$

Notice the pattern: the last fraction has $n-2$ on top and the power in the laplace transform has $n-3$, 3 is one greater than two. So if we keep going down to the power $n - n = 0$ then in the fraction we should have $n - (n-1) = 1$:

$$\mathscr{L}\{t^n\} = \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\frac{2}{s}\frac{1}{s}\mathscr{L}\{t^0 = 1\}$$

Now remember that the laplace transform of 1, $\mathscr{L}\{1\} = \frac{1}{s}$ so:

$$\mathscr{L}\{t^n\} = \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\frac{2}{s}\cdot\frac{1}{s}\cdot\frac{1}{s}$$

Tidying this up a bit remember that $n! := n\cdot n-1\cdots 2\cdot 1$, and counting that there's $n+1$ s's we have that:

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

(b) $f(t) = 2\sinh 3t + \cos 2t$

Solution:

We have two separate functions in the one we'd like to take the laplace transform of so we'll do them separately and add the result together.

$$\mathscr{L}\{\sinh(3t)\} = \int_0^\infty \sinh(3t)e^{-st}dt = \frac{1}{2}\int_0^\infty \left(e^{3t} - e^{-3t}\right)e^{-st}dt$$

We know the Laplace transform of e^{at} is $\frac{1}{s-a}$ but lets check anyway:

$$\begin{aligned}
&= \frac{1}{2}\int_0^\infty e^{(3-s)t}dt - \frac{1}{2}\int_0^\infty e^{-(s+3)t}dt \\
&= \frac{1}{2}\frac{e^{(3-s)t}}{3-s}\bigg|_0^\infty + \frac{1}{2}\frac{e^{-(3+s)t}}{3+s}\bigg|_0^\infty \\
&= \frac{1}{2}\left(0 - \frac{1}{3-s}\right) + \frac{1}{2}\left(0 - \frac{1}{3+s}\right) \quad \text{for } s > 3 \\
&= \frac{1}{2}\left(\frac{1}{s-3} - \frac{1}{s+3}\right) \\
&= \frac{3}{s^2-9}
\end{aligned}$$

For cosine as in the notes we use integration by parts twice: and recalling that cosine and sine are bounded the limits of integration that we use are:

$$e^{-st}\cos(2t)\bigg|_0^\infty = -1 \quad e^{-st}\sin(2t)\bigg|_0^\infty = 0$$

So

$$\begin{aligned}
\mathscr{L}\{\cos(2t)\} &= \int_0^\infty \cos(2t)e^{-st}dt \\
&= -\frac{1}{s}e^{-st}\cos(2t)\bigg|_0^\infty - \frac{2}{s}\int_0^\infty \sin(2t)e^{-st}dt \\
&= \frac{1}{s} - \frac{2}{s}\left(-\frac{1}{s}e^{-st}\sin(2t)\bigg|_0^\infty + \frac{2}{s}\int_0^\infty \cos(2t)e^{-st}dt\right) \\
&= \frac{1}{s} - \frac{4}{s^2}\mathscr{L}\{\cos(2t)\} \\
\frac{s^2+4}{s^2}\mathscr{L}\{\cos(2t)\} &= \frac{1}{s} \\
\mathscr{L}\{\cos(2t)\} &= \frac{1}{s}\frac{s^2}{s^2+4} = \frac{s}{s^2+4}
\end{aligned}$$

Finally we can add these together:

$$\mathscr{L}\{2\sinh(3t) + 3\cos 2t\} = \frac{6}{s^2-9} + \frac{3s}{s^2+4}$$

2. Find the inverse Laplace transform of the following

(a) $\mathcal{L}^{-1}\left\{\frac{6}{s^2+36s}\right\}$

$$
\begin{aligned}
\frac{6}{s^2+36s} &= \frac{6}{s(s+36)} \\
&= \frac{A}{s} + \frac{B}{s+36} \\
6 &= A(s+36) + B(s) \\
s = -36 \Rightarrow B &= -\frac{1}{6} \\
s = 0 \Rightarrow A &= \frac{1}{6} \\
\frac{6}{s^2+36s} &= \frac{1}{6}\cdot\frac{1}{s} - \frac{1}{6}\cdot\frac{1}{s+36} \\
\mathscr{L}^{-1}\left\{\frac{6}{s^2+36s}\right\} &= \frac{1}{6} - \frac{1}{6}e^{-36t}
\end{aligned}
$$

(b) $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)(s-5)(s-7)}\right\}$

$$
\begin{aligned}
\frac{s}{(s-2)(s-5)(s-7)} &= \frac{A}{s-2} + \frac{B}{s-5} + \frac{C}{s-7} \\
s &= A(s-5)(s-7) + B(s-2)(s-7) + C(s-2)(s-5) \\
s &= 2 \;\Rightarrow\; 2 = A(-3)(-5) \;\Rightarrow\; A = \frac{2}{15} \\
s &= 5 \;\Rightarrow\; 5 = B(3)(-2) \;\Rightarrow\; B = -\frac{5}{6} \\
s &= 7 \;\Rightarrow\; 7 = C(5)(2) \;\Rightarrow\; C = \frac{7}{10}
\end{aligned}
$$

(c) $\mathcal{L}^{-1}\left\{\frac{(s-1)^3}{s^4}\right\}$

$$
\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{(s-1)^3}{s^4}\right\} &= \mathscr{L}^{-1}\left\{\frac{s^3-3s^2+3s-1}{s^4}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s^3}{s^4}\right\} - 3\mathscr{L}^{-1}\left\{\frac{s^2}{s^4}\right\} + 3\mathscr{L}^{-1}\left\{\frac{s}{s^4}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 3\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\}
\end{aligned}
$$

Using $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$

$$
\mathscr{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\} = 1 - 3t + \frac{3}{2}t^2 - \frac{1}{6}t^3
$$

3. Use the Laplace transform to solve the given initial-value problems

(a) $y'' + 5y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$

$$\mathscr{L}\{y'' + 5y' + 4y = 0\} = \mathscr{L}\{y''\} + 5\mathscr{L}\{y'\} + 4\mathscr{L}\{y\} = 0$$

$$s^2Y(s) - sy(0) - y'(0) + 5sY(s) - 5y(0) + 4Y(s) = 0$$

$$(s^2 + 5s + 4)Y(s) - s - 5 = 0$$

$$\begin{aligned} Y(s) &= \frac{s+5}{s^2+5s+4} \\ &= \frac{s+5}{(s+4)(s+1)} \end{aligned}$$

$$\frac{A}{s+4} + \frac{B}{s+1} = \frac{A(s+1) + B(s+4)}{(s+4)(s+1)} = \frac{s+5}{(s+4)(s+1)}$$

$$s = -1 \quad A(0) + B(3) = 4 \quad \Rightarrow B = \frac{4}{3}$$

$$s = -4 \quad A(-3) + B(0) = 1 \quad \Rightarrow A = -\frac{1}{3}$$

$$Y(s) = -\left(\frac{1}{3}\right)\frac{1}{s+4} + \left(\frac{4}{3}\right)\frac{1}{s+1}$$

$$\begin{aligned} y(t) &= -\frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\} + \frac{4}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= -\frac{1}{3}e^{-4t} + \frac{4}{3}e^{-t} \end{aligned}$$

(b) $2\frac{dy}{dt} - y = 0, \quad y(0) = 5$ **[7]**

$$\begin{aligned}
\mathscr{L}\left\{2\frac{dy}{dt} - y\right\} &= \mathscr{L}\{0\} \\
2\mathscr{L}\left\{\frac{dy}{dt}\right\} - \mathscr{L}\{y\} &= 0 \\
2sY(s) - 2y(0) - Y(s) &= 0 \\
(2s-1)Y(s) - 10 &= 0 \\
Y(s) &= \frac{10}{2s-1} \\
y(t) &= \mathscr{L}^{-1}\{Y(s)\} \\
&= 10\mathscr{L}^{-1}\left\{\frac{1}{2s-1}\right\} \\
&= 10\mathscr{L}^{-1}\left\{\frac{\frac{1}{2}}{s-\frac{1}{2}}\right\} \\
&= 5\mathscr{L}^{-1}\left\{\frac{1}{s-\frac{1}{2}}\right\} \\
y(t) &= 5e^{\frac{1}{2}t}
\end{aligned}$$

(c) $y' - y = 2\cos 6t, \quad y(0) = 0$ **[7]**

$$\begin{aligned}
\mathscr{L}\{y' - y\} &= \mathscr{L}\{2\cos 6t\} \\
\mathscr{L}\{y'\} - \mathscr{L}\{y\} &= 2\mathscr{L}\{\cos 6t\} \\
sY(s) - y(0) - Y(s) &= 2 \cdot \frac{s}{s^2+36} \\
(s-1)Y(s) &= 2\frac{s}{s^2+36} \\
Y(s) &= \frac{2s}{(s^2+36)(s-1)} \\
\frac{2s}{(s^2+36)(s-1)} &= \frac{A}{s-1} + \frac{Bs+C}{s^2+36} \\
2s &= As^2 + 36A + Bs^2 - Bs + Cs - C \\
36A - C = 0 &\;\Rightarrow\; C = 36A \\
-B + C = 2 &\;\Rightarrow\; B = -2 + C = -2 + 36A \\
A + B = 0 &\;\Rightarrow\; B = -A = -2 + 36A \;\Rightarrow\; A = \frac{2}{37} \\
B = -A &\;\Rightarrow\; B = -\frac{2}{37} \\
C = 36A &\;\Rightarrow\; C = \frac{72}{37} \\
Y(s) &= \frac{2s}{(s^2+36)(s-1)} = \frac{2}{37}\cdot\frac{1}{s-1} - \frac{2}{37}\cdot\frac{s}{s^2+36} + \frac{72}{37}\cdot\frac{1}{s^2+36} \\
y(t) &= \mathscr{L}^{-1}\{Y(s)\} \\
&= \frac{2}{37}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{2}{37}\mathscr{L}^{-1}\left\{\frac{s}{s^2+36}\right\} + \frac{72}{6\cdot 37}\mathscr{L}^{-1}\left\{\frac{6}{s^2+36}\right\} \\
&= \frac{2}{37}e^t - \frac{2}{37}\cos(6t) + \frac{12}{37}\sin(6t)
\end{aligned}$$

(d) $y'' - 10y' + 25y = 3e^{3t}$, $\quad y(0) = 0$, $\quad y'(0) = -1$ **[7]**

$$\mathscr{L}\{y''\} - 10\mathscr{L}\{y'\} + 25\mathscr{L}\{y\} = 3\mathscr{L}\{e^{3t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 10sY(s) + 10y(0) + 25Y(s) - 25y(0) = \frac{3}{s-3}$$

$$(s^2 - 10s + 25)Y(s) + 1 = \frac{3}{(s-3)}$$

$$(s-5)^2Y(s) = \frac{3}{(s-3)} - 1$$

$$Y(s) = \frac{6-s}{(s-3)(s-5)^2}$$

$$\frac{6-s}{(s-3)(s-5)^2} = \frac{A}{s-3} + \frac{B}{s-5} + \frac{C}{(s-5)^2}$$

$$6 - s = A(s-5)^2 + B(s-3)(s-5) + C(s-3)$$

For $s = 3$ we find $3 = 4A$ so $A = \frac{3}{4}$. For $s = 5$ we find that $1 = 2C$ so $C = \frac{1}{2}$. Finally comparing the s^2 terms we find that $A + B = 0$ or $B = -\frac{3}{4}$
Lastly then we have that:

$$Y(s) = \frac{3}{4}\frac{1}{s-3} - \frac{3}{4}\frac{1}{s-5} + \frac{1}{2}\frac{1}{(s-5)^2}$$

So that taking the inverse laplace transform and using that $\mathscr{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = te^{at}$ which was given:

$$y(t) = \frac{3}{4}e^{3t} - \frac{3}{4}e^{5t} + \frac{1}{2}te^{5t}$$

4. Use the First Translation (Shift) Theorem to find either $F(s)$ or $f(t)$, as indicated. State in each case how the translation theorem applies.

   (a) $\mathcal{L}\{\cosh(t)\cos(t)\}$

We have that:

$$\mathscr{L}\{\cosh(t)\cos(t)\} = \frac{1}{2}\left(\mathscr{L}\{e^t\cos(t)\} + \mathscr{L}\{e^{-t}\cos(t)\}\right)$$

We can apply the shift theorem that $\mathscr{L}\{e^{at}\cos(t)\} = \frac{s}{s^2+1}_{s\to s-a}$ so the above becomes:

$$\frac{1}{2}\left(\frac{s-1}{(s-1)^2+1} + \frac{s+1}{(s+1)^2+1}\right)$$

Simplifying:

$$\begin{aligned}
\mathscr{L}\{\cosh(t)\cos(t)\} &= \frac{1}{2}\frac{[s+1+s-1](s+1)(s-1)+s+1+s-1}{[(s+1)(s-1)]^2+(s+1)^2+(s-1)^2+1} \\
&= \frac{1}{2}\frac{2s(s+1)(s-1)+2s}{[s^2-1]^2+s^2+2s+1+s^2-2s+1+1} \\
&= \frac{1}{2}\frac{2s(s^2-1)+2s}{[s^2-1]^2+2s^2+3} \\
&= \frac{1}{2}\frac{2s^3}{s^4-2s^2+1+2s^2+3} \\
&= \frac{s^3}{s^4+4}
\end{aligned}$$

(b) $\mathcal{L}^{-1}\left\{\frac{(s-1)^2}{(s+2)^4}\right\}$

Again if we has simply an s^4 on the denominator we could just expand out the numerator and cancel a few powers of s and take ordinary inverse Laplace transforms of the inverse powers of s.

Instead to apply the same procedure we should try to write $(s-1)^2$ in terms of $s+2$ and it's powers. This way factors of $s+2$ will cancel off and we apply the shift theorem to inverse powers of $s+2$. So:

$$\frac{(s-1)^2}{(s+2)^4} = \frac{s^2-2s+1}{(s+2)^4} = \frac{(s+2)^2-6s-3}{(s+2)^4}$$

Remember we want powers of $s+2$ so we used the identity

$$(s+2)^2 = s^2+4s+4$$

to replace s^2 by $(s+2)^2$. Just take

$$s^2 = (s+2)^2-4s-4$$

and sub it into the numerator and simplify. We'll do a similar thing again to replace the $-6s$ by $-6(s+2)$:

$$-6(s+2) = -6s-12 \implies -6s = -6(s+2)+12$$

This gives in total:

$$\frac{(s-1)^2}{(s+2)^4} = \frac{s^2-2s+1}{(s+2)^4} = \frac{(s+2)^2-6s-3}{(s+2)^4} = \frac{(s+2)^2-6(s+2)+9}{(s+2)^4}$$

Now separating the terms, we can then take an inverse Laplace transform using the shift theorem:

$$\frac{(s-1)^2}{(s+2)^4} = \frac{1}{(s+2)^2} - \frac{6}{(s+2)^3} + \frac{9}{(s+2)^4}$$

$$\mathscr{L}^{-1}\left\{\frac{(s-1)^2}{(s+2)^4}\right\} = e^{-2t}\left(t-3t^2+\frac{3}{2}t^3\right)$$

5. Use the Second Translation (Shift) Theorem to find either $F(s)$ or $f(t)$, as indicated. State in each case how the translation theorem applies.

(a) $\mathcal{L}\{(3t+1)\mathcal{U}(t-1)\}$

For this we'll use the alternative formulation of the second shift theorem namely:

$$\mathscr{L}\{f(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{f(t+a)\}$$

So then:

$$\mathscr{L}\{(3t+1)\mathscr{U}(t-1)\} = e^{-s}\mathscr{L}\{3t+4\}$$

Then:

$$\begin{aligned}\mathscr{L}\{(3t+1)\mathscr{U}(t-1)\} &= 3e^{-s}\mathscr{L}\{t\} + 4e^{-s}\mathscr{L}\{1\} \\ &= (3e^{-s})\frac{1}{s^2} + (4e^{-s})\frac{1}{s} \\ &= \frac{3e^{-s}}{s^2} + \frac{4e^{-s}}{s}\end{aligned}$$

(b) $\mathcal{L}\{\cos(4t-8)\mathcal{U}(t-2)\}$

This time we'll use the first version of the second shift theorem:

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s)$$

If $f(t) = \cos(4t)$ then $f(t-2) = \cos(4t-8)$, $F(s) = \mathscr{L}\{f(t)\}$,

$$\begin{aligned}\mathscr{L}\{\cos(4t-8)\mathscr{U}(t-2)\} &= e^{-2s}F(s) \\ &= e^{-2s}\frac{s}{s^2+16}\end{aligned}$$

(c) $\mathcal{L}^{-1}\left\{\dfrac{(1+e^{-s})^2}{s+3}\right\}$

$$\begin{aligned}\frac{(1+e^{-s})^2}{s+3} &= \frac{1}{s+3} + 2\frac{e^{-s}}{s+3} + \frac{e^{-2s}}{s+3} \\ \mathscr{L}^{-1}\left\{\frac{(1+e^{-s})^2}{s+3}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\} + 2\mathscr{L}^{-1}\left\{\frac{e^{-s}}{s+3}\right\} + \mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s+3}\right\} \\ &= e^{-3t} + 2(e^{-3t})_{t\to t-1}\mathscr{U}(t-1) + (e^{-3t})_{t\to t-2}\mathscr{U}(t-2) \\ &= e^{-3t} + 2e^{-3t+3}\mathscr{U}(t-1) + e^{-3t+6}\mathscr{U}(t-2)\end{aligned}$$