EE206
Assignment 9

Due by next Tutorial, December 4^{th}. Starred questions will be done out in tutorials and do NOT need to be handed in.

1. Find the Fourier integral representation of the given function.

*(a) $f(x) = \begin{cases} 0, & x < 0 \\ \sin x, & 0 \leq x \leq \pi \\ 0, & x > \pi \end{cases}$

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)]d\alpha$$

$$\begin{aligned}
A(\alpha) &= \int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx \\
&= \int_0^\pi \sin(x)\cos(\alpha x)dx \\
&= \frac{1}{2}\int_0^\pi (\sin((1+\alpha)x) + \sin((1-\alpha)x))\,dx \\
&= \frac{1}{2}\left[-\frac{1}{1+\alpha}\cos((1+\alpha)x) - \frac{1}{1-\alpha}\cos((1-\alpha)x)\right]_0^\pi \\
&= \frac{1}{2}\left[-\frac{1}{1+\alpha}\cos((1+\alpha)\pi) - \frac{1}{1-\alpha}\cos((1-\alpha)\pi) + \frac{1}{1+\alpha}\cos(0) + \frac{1}{1-\alpha}\cos(0)\right] \\
&= \frac{1}{2}\left[\frac{-(1-\alpha)\cos((1+\alpha)\pi) - (1+\alpha)\cos((1-\alpha)\pi) + 1 + \alpha + 1 - \alpha}{1-\alpha^2}\right] \\
&= \frac{1}{2}\left[\frac{-(\cos((1+\alpha)\pi) + \cos((1-\alpha)\pi)) + \alpha(\cos((1+\alpha)\pi) - \cos((1-\alpha)\pi)) + 2}{1-\alpha^2}\right] \\
&= \frac{1}{2}\left[\frac{-2\cos(\pi)\cos(\alpha\pi) - 2\alpha\sin(\pi)\sin(\alpha\pi) + 2}{1-\alpha^2}\right] \\
&= \frac{\cos(\alpha\pi) + 1}{1-\alpha^2}
\end{aligned}$$

$$\begin{aligned}
B(\alpha) &= \int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx \\
&= \int_0^\pi \sin(x)\sin(\alpha x)dx \\
&= \frac{1}{2}\int_0^\pi (\cos((1-\alpha)x) - \cos((1+\alpha)x))\,dx \\
&= \frac{1}{2}\left[\frac{1}{1-\alpha}\sin((1-\alpha)x) - \frac{1}{1+\alpha}\sin((1+\alpha)x)\right]_0^\pi \\
&= \frac{1}{2}\left[\frac{1}{1-\alpha}\sin((1-\alpha)\pi) - \frac{1}{1+\alpha}\sin((1+\alpha)\pi) - \frac{1}{1-\alpha}\sin(0) + \frac{1}{1+\alpha}\sin(0)\right] \\
&= \frac{1}{2}\left[\frac{(\sin((1-\alpha)\pi) - \sin((1+\alpha)\pi)) + \alpha(\sin((1-\alpha)\pi) + \sin((1+\alpha)\pi))}{1-\alpha^2}\right]
\end{aligned}$$

$$= \frac{1}{2}\left[\frac{-2\cos(\pi)\sin(\alpha\pi) + 2\alpha\sin(\pi)\cos(\alpha\pi)}{1-\alpha^2}\right]$$

$$= \frac{\sin(\alpha\pi)}{1-\alpha^2}$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \left(\frac{\cos(\alpha\pi)+1}{1-\alpha^2}\cos(\alpha x) + \frac{\sin(\alpha\pi)}{1-\alpha^2}\sin(\alpha x)\right)d\alpha$$

(b) $f(x) = \begin{cases} 0, & x < \pi \\ 3, & \pi < x < 2\pi \\ 0, & x > 2\pi \end{cases}$ **[5]**

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)]d\alpha$$

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx$$

$$= 3\int_\pi^{2\pi} \cos(\alpha x)dx$$

$$= \frac{3}{\alpha}[\sin(\alpha x)]_\pi^{2\pi}$$

$$= \frac{3}{\alpha}[\sin(2\pi\alpha) - \sin(\pi\alpha)]$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx$$

$$= 3\int_\pi^{2\pi} \sin(\alpha x)dx$$

$$= -\frac{3}{\alpha}[\cos(\alpha x)]_\pi^{2\pi}$$

$$= -\frac{3}{\alpha}[\cos(2\pi\alpha) - \cos(\pi\alpha)]$$

$$f(x) = \frac{3}{\pi\alpha}\int_0^\infty \Big([\sin(2\pi\alpha) - \sin(\pi\alpha)]\cos(\alpha x) + [-\cos(2\pi\alpha) + \cos(\pi\alpha)]\sin(\alpha x)\Big)d\alpha$$

(c) $f(x) = \begin{cases} \frac{2}{3}x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$ **[5]**

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)]d\alpha$$

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx$$

$$= \frac{2}{3}\int_{-\pi}^{\pi} x\cos(\alpha x)dx \quad [\text{odd times even function } = 0]$$

$$= \frac{2}{3}\left(\left[\frac{x}{\alpha}\sin(\alpha x)\right]_{-\pi}^{\pi} - \frac{1}{\alpha}\int_{-\pi}^{\pi}\sin(\alpha x)dx\right)$$

$$
\begin{aligned}
&= \frac{2\pi}{3\alpha}\sin(\alpha\pi) + \frac{2\pi}{3\alpha}\sin(-\alpha\pi) + \frac{2}{3\alpha^2}[\cos(\alpha x)]_{-\pi}^{\pi} \\
&= \frac{2\pi}{3\alpha}\sin(\alpha\pi) - \frac{2\pi}{3\alpha}\sin(\alpha\pi) + \frac{2}{3\alpha^2}(\cos(\alpha\pi) - \cos(-\alpha\pi)) \\
&= 0 + \frac{2}{3\alpha^2}(\cos(\alpha\pi) - \cos(\alpha\pi)) \\
&= 0 \\
B(\alpha) &= \int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx \\
&= \frac{2}{3}\int_{-\pi}^{\pi} x\sin(\alpha x)dx \\
&= \frac{2}{3}[-\frac{x}{\alpha}\cos(\alpha x)]_{-\pi}^{\pi} + \frac{2}{3\alpha}\int_{-\pi}^{\pi}\cos(\alpha x)dx \\
&= -\frac{2\pi}{3\alpha}\cos(\alpha\pi) - \frac{2\pi}{3\alpha}\cos(-\alpha\pi) + \frac{2}{3\alpha^2}[\sin(\alpha x)]_{-\pi}^{\pi} \\
&= -\frac{2\pi}{3\alpha}\cos(\alpha\pi) - \frac{2\pi}{3\alpha}\cos(\alpha\pi) + \frac{2}{3\alpha^2}\left(\sin(\alpha\pi) - \sin(-\alpha\pi)\right) \\
&= -\frac{4\pi}{3\alpha}\cos(\alpha\pi) + \frac{2}{3\alpha^2}\left(\sin(\alpha\pi) + \sin(\alpha\pi)\right) \\
&= -\frac{4\pi}{3\alpha}\cos(\alpha\pi) + \frac{4}{3\alpha^2}\sin(\alpha\pi) \\
&= \frac{4}{3\alpha^2}\sin(\alpha\pi) - \frac{4\pi}{3\alpha}\cos(\alpha\pi) \\
f(x) &= \int_0^{\infty}\left([0]\cos(\alpha x) + \left[\frac{2}{\alpha^2}\sin(\alpha\pi) - \frac{2\pi}{\alpha}\cos(\alpha\pi)\right]\sin(\alpha x)\right)d\alpha \\
&= \int_0^{\infty}\left[\frac{2}{\alpha^2}\sin(\alpha\pi) - \frac{2\pi}{\alpha}\cos(\alpha\pi)\right]\sin(\alpha x)d\alpha
\end{aligned}
$$

2. Find the cosine and sine integral representations of the given function.
   $f(x) = e^{-x/2}, \qquad x > 0$ [**5**]
   cosine

$$
\begin{aligned}
f(x) &= \frac{2}{\pi}\int_0^{\infty} A(\alpha)\cos(\alpha x)d\alpha \\
A(\alpha) &= \int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx \\
&= \int_0^{\infty} e^{-x/2}\cos(\alpha x)dx
\end{aligned}
$$

Similar to Q2 (d) Problem set 1 we have:

$$\begin{aligned}
\int e^{ax}\cos(bx)dx &= e^{ax}\frac{a\cos(bx)+b\sin(bx)}{a^2+b^2} \\
&= \left[\frac{e^{-x/2}(-0.5\cos(\alpha x)+\alpha\sin(\alpha x))}{0.25+\alpha^2}\right]_0^\infty \\
&= 0-\frac{e^0(-0.5\cos(0)+\alpha\sin(0))}{0.25+\alpha^2} \\
&= \frac{1/2}{1/4+\alpha^2}=\frac{2}{1+4\alpha^2} \\
f(x) &= \frac{2}{\pi}\int_0^\infty \frac{2}{1+4\alpha^2}\cos(\alpha x)d\alpha
\end{aligned}$$

sine

$$\begin{aligned}
f(x) &= \frac{2}{\pi}\int_0^\infty B(\alpha)\cos(\alpha x)d\alpha \\
B(\alpha) &= \int_{-\infty}^\infty f(x)\sin(\alpha x)dx \\
&= \int_0^\infty e^{-x/2}\cos(\alpha x)dx
\end{aligned}$$

This was actually Q2(d) PS1:

$$\begin{aligned}
\int e^{ax}\sin(bx)dx &= e^{ax}\frac{a\sin(bx)-b\cos(bx)}{a^2+b^2} \\
&= \left[\frac{e^{-x/2}(-0.5\sin(\alpha x)-\alpha\cos(\alpha x))}{0.25+\alpha^2}\right]_0^\infty \\
&= 0-\frac{e^0(-0.25\sin(0)-\alpha\cos(0))}{0.25+\alpha^2} \\
&= \frac{\alpha}{1/4+\alpha^2} \\
f(x) &= \frac{2}{\pi}\int_0^\infty \frac{4\alpha}{1+4\alpha^2}\sin(\alpha x)d\alpha
\end{aligned}$$

3. Find the Fourier transforms of the following function.

(a) $f(x) = \begin{cases} 0, & x < -1 \\ e^{2ix}, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$ **[5]**

$$\begin{aligned} F(\alpha) &= \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx \\ &= \int_{-1}^{1} e^{2ix}e^{-i\alpha x}dx \\ &= \int_{-1}^{1} e^{i(2-\alpha)x}dx \\ &= \frac{1}{i(2-\alpha)}[e^{i(2-\alpha)x}]_{-1}^{1} \\ &= \frac{1}{i(2-\alpha)}[e^{i(2-\alpha)} - e^{-i(2-\alpha)}] \\ &= \frac{2}{2-\alpha}\sin(2-\alpha) \\ &= \frac{2\sin(\alpha-2)}{\alpha-2} \end{aligned}$$

(b) $f(x) = e^{-|x|}$ **[5]**

$$\begin{aligned} F(\alpha) &= \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx \\ &= \int_{-\infty}^{0} e^{x}e^{-i\alpha x}dx + \int_{0}^{\infty} e^{-x}e^{-i\alpha x}dx \\ &= \int_{-\infty}^{0} e^{(1-i\alpha)x}dx + \int_{0}^{\infty} e^{-(1+i\alpha)x}dx \\ &= \frac{1}{1-i\alpha}[e^{(1-i\alpha)x}]_{-\infty}^{0} - \frac{1}{1+i\alpha}[e^{-(1+i\alpha)x}]_{0}^{\infty} \\ &= \frac{1}{1-i\alpha}[e^{0}-0] - \frac{1}{1+i\alpha}[0-e^{0}] \\ &= \frac{1}{1-i\alpha}[1] + \frac{1}{1+i\alpha}[1] \\ &= \frac{1}{1-i\alpha} + \frac{1}{1+i\alpha} \\ &= \frac{1+i\alpha+1-i\alpha}{1+\alpha^2} \\ &= \frac{2}{1+\alpha^2} \end{aligned}$$

*(c) $f(x) = \begin{cases} 0, & x < -1 \\ x, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$

$$
\begin{aligned}
F(\alpha) &= \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx \\
&= \int_{-1}^{1} x e^{-i\alpha x} dx \\
&= -\left[\frac{x e^{-i\alpha x}}{i\alpha}\right]_{-1}^{1} + \frac{1}{i\alpha}\int_{-1}^{1} e^{-i\alpha x} dx \\
&= -\frac{e^{-i\alpha}}{i\alpha} - \frac{e^{i\alpha}}{i\alpha} + \frac{1}{\alpha^2}[e^{-i\alpha x}]_{-1}^{1} \\
&= -\frac{e^{-i\alpha} + e^{i\alpha}}{i\alpha} + \frac{e^{-i\alpha} - e^{i\alpha}}{\alpha^2} \\
&= \frac{2i\cos\alpha}{\alpha} - \frac{2i\sin\alpha}{\alpha^2}
\end{aligned}
$$