(b) $\int_0^t 2\sin\tau \cos(t-\tau)d\tau = 2\sin t * \cos t$

$\therefore F(s) = \mathcal{L}\{2\sin t\}\cdot\mathcal{L}\{\cos t\}$

$= \frac{2}{s^2+1}\cdot\frac{s}{s^2+1} = \frac{2s}{(s^2+1)^2}$

5.

(a) $F(s) + \mathcal{L}\{f(t)*1\} = \frac{1}{s}$ $\quad F(s) + F(s)\cdot\frac{1}{s} = \frac{1}{s}$

$F(s) = \frac{1}{s+1}$ $\quad f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-t}$

(b) $s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{s}{s^2+9}$

$(s^2+9)Y(s) - s - 4 = \frac{s}{s^2+9}$ $\quad Y(s) = \frac{s}{(s^2+9)^2} + \frac{s+4}{s^2+9}$

$y = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{6}t\sin 3t + \cos 3t + \frac{4}{3}\sin 3t$

(c) $s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 13Y(s) = e^{-\pi s} + e^{-3\pi s}$

$(s^2+4s+13)Y(s) - s - 4 = e^{-\pi s} + e^{-3\pi s}$

$Y(s) = \frac{s+4}{(s+2)^2+9} + \frac{e^{-\pi s}+e^{-3\pi s}}{(s+2)^2+9}$

$y = \mathcal{L}^{-1}\{Y(s)\} = e^{-2t}\cos 3t + \frac{2}{3}e^{-2t}\sin 3t + \frac{1}{3}e^{-2(t-\pi)}\sin 3(t-\pi)\mathcal{U}(t-\pi) + \frac{1}{3}e^{-2(t-3\pi)}\sin 3(t-3\pi)\mathcal{U}(t-3\pi)$

(d) $s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) = e^{-s}$

$(s^2+2s)Y(s) = 1 + e^{-s}$ $\quad Y(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right) + \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)e^{-s}$

$y = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}(1 - e^{-2t}) + \frac{1}{2}(1 - e^{-2(t-1)})\mathcal{U}(t-1)$

1. $\mathcal{L}\{t\cdot(e^{-t}\cdot\cos 2t)\} = -\frac{d}{ds}F(s)$

$= -\frac{d}{ds}\mathcal{L}\{e^{-t}\cdot\cos 2t\} = -\frac{d}{ds}\left(\frac{s+1}{(s+1)^2+4}\right) = \underline{\frac{(s+1)^2-4}{((s+1)^2+4)^2}}$

2. (a) $s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) = \frac{s+1}{(s+1)^2+1}$

$Y(s) = \frac{1}{s}\cdot\frac{1}{(s+1)^2+1}$ $\quad y = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2} - \frac{1}{2}e^{-t}(\cos t + \sin t)$

$= \underline{\frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}s+1}{(s+1)^2+1}}$

(b) $sY(s) - y(0) + 2Y(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{1-2\mathcal{U}(t-1)\}$

$(s+2)Y(s) = \frac{1}{s} - 2\frac{e^{-s}}{s}$ $\quad Y(s) = \frac{1}{s(s+2)} - \frac{2}{s(s+2)}e^{-s}$

$y = \mathcal{L}^{-1}\{Y(s)\} = \underline{\frac{1}{2}(1-e^{-2t}) - (1-e^{-2(t-1)})}$

(c) $sY(s) - y(0) + 3Y(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{1-\mathcal{U}(t-1)\}$

$(s+3)Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$ $\quad Y(s) = \frac{1}{s(s+3)} - \frac{1}{s(s+3)}e^{-s}$

$y = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}(1-e^{-3t}) - \frac{1}{3}(1-e^{-3(t-1)})$

3. (a) $F(s) = \mathcal{L}\{t^3 * t\cdot e^{-t}\} = \int_0^\infty\int_0^t \tau^3\cdot(t-\tau)e^{-(t-\tau)}d\tau\, e^{-st}dt$

$= \int_0^\infty\int_\tau^\infty \tau^3\cdot(t-\tau)e^{-(t-\tau)}e^{-st}d\tau\, dt$

we let $t-\tau = u$, $t = u+\tau$

$dt = du + d\tau = du$

$= \int_0^\infty\int_0^\infty e^{-s\tau}\cdot\tau^3 d\tau \int_0^\infty u\cdot e^{-u}\cdot e^{-su}du$

$= \mathcal{L}\{t^3\}\cdot\mathcal{L}\{t\cdot e^{-t}\} = \frac{6}{s^4}\cdot\frac{1}{(s+1)^2} = \underline{\frac{6}{s^4(s+1)^2}}$

(b) $F(s) = \mathcal{L}\{e^{2t} * \sin 3t\} = \mathcal{L}\{e^{2t}\}\cdot\mathcal{L}\{\sin 3t\}$

$= \frac{1}{s-2}\cdot\frac{3}{s^2+9} = \underline{\frac{3}{(s-2)(s^2+9)}}$

4. (a) $\int_0^t \tau\cdot\sin\tau\, d\tau = t\cdot\sin t * 1$

$\therefore F(s) = \mathcal{L}\{t\cdot\sin t\}\cdot\mathcal{L}\{1\}$

$\mathcal{L}\{t\cdot\sin t\} = -\frac{d}{ds}\mathcal{L}\{\sin t\} = \frac{2s}{(s^2+1)^2}$

$\therefore \underline{F(s) = \frac{2}{(s^2+1)^2}}$