EE206 Differential Equations and Transform Methods

Tutorial 3

Problem 1b: $y'' - 10y' + 25y = 30x + 3$

First we solve the homogeneous equation:

$y'' - 10y' + 25y = 0 \qquad a = 1, b = -10, c = 25$

auxiliary equation: $am^2 + bm + c = 0$

$$\begin{aligned} m^2 - 10m + 25 &= 0 \\ (m-5)^2 &= 0 \\ m &= 5 \qquad \text{(repeated roots)} \\ y_1 &= e^{5x} \end{aligned}$$

To get y_2 we use:

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1(x)^2} dx$$

$$y'' + P(x)y' + Q(x) = 0 \quad \Rightarrow P(x) = -10$$

$$\begin{aligned} y_2(x) &= y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1(x)^2} dx \\ &= e^{5x} \int \frac{e^{10 \int dx}}{(e^{5x})^2} dx \\ &= e^{5x} \int \frac{e^{10x}}{e^{10x}} dx \\ &= e^{5x} \int dx = xe^{5x} \end{aligned}$$

This gives the complementary solution:

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$
$$y_c = c_1 e^{5x} + c_2 x e^{5x}$$

We now get the particular solution:

$$
\begin{aligned}
g(x) &= 30x + 3 \quad \Rightarrow y_p = Ax + B,\ \ y_p' = A,\ \ y_p'' = 0 \\
y'' - 10y' + 25y &= 30x + 3 \quad \Rightarrow 0 - 10A + 25Ax + 25B = 30x + 3 \\
25Ax + (-10A + 25B) &= 30x + 3 \\
\Rightarrow A &= \frac{6}{5},\ \ B = \frac{3}{5} \\
y_p &= \frac{6}{5}x + \frac{3}{5} \\
y &= y_c + y_p \\
\Rightarrow y &= c_1 e^{5x} + c_2 x e^{5x} + \frac{6}{5}x + \frac{3}{5}
\end{aligned}
$$

Problem 2c: $y'' + y = \sec(x)\tan(x)$

$y'' + y = 0; \qquad f(x) = \sec(x)\tan(x)$

Recall the solutions to this are: $\sin(x)$ and $\cos(x)$.
This gives the complementary solution: $y_c = c_1 y_1 + c_2 y_2 = c_1 \cos(x) + c_2 \sin(x)$

Now we get u_1' and u_2':

$$u_1' = -\frac{y_2(x) f(x)}{W}$$

$$u_1' = -\frac{\sin(x)\sec(x)\tan(x)}{1}$$

$$u_1 = -\int \tan^2(x) dx$$

$$u_1 = -\int (1 + \tan^2(x)) - 1 dx$$

$$u_1 = -\int \sec^2(x) - 1 dx$$

$$u_1 = -(\tan(x) - x)$$

$$u_1 = x - \tan(x)$$

$$u_2' = \frac{y_1(x)f(x)}{W}$$

$$u_2' = \frac{\cos(x)\sec(x)\tan(x)}{1}$$

$$u_2 = \int \tan(x)dx$$

$$u_2 = -\ln(\cos(x))$$

$$
\begin{aligned}
y_p &= u_1(x)y_1(x) + u_2(x)y_2(x)\\
y_p &= (x - \tan(x))\cos(x) - \ln(\cos(x))\sin(x)
\end{aligned}
$$

So our full solution is: $y = y_c + y_p$

$$y = c_1\cos(x) + c_2\sin(x) + x\cos(x) - \ln(\cos(x))\sin(x)$$

Notice that we dropped the $\tan(x)\cos(x)$ term since this is simply $\sin(x)$ which can be absorbed into the constant c_2 from y_c.