EE311FZ Final Paper (21-22 version)

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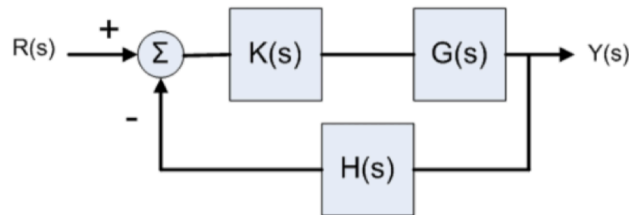
Merry Christmas, in 2022/12/25.

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1 Q1 Root Locus

Question 1 [20 marks]

Given the following system:

where

$$G(s) = \frac{1}{(s+1)(s+3)}, \qquad H(s) = \frac{1}{s+2}, \qquad K(s) = k$$

1.1 (a) Asymptotes of the root loci.

开环传递函数 $G(s)K(s)H(s) = \frac{K}{(s+1)(s+2)(s+3)}$

对应的开环极点为 $p = -1, -2, -3$, $n = 3$

没有开环零点, $m = 0$, 因此根轨迹有3个分支

根据已知, 可以求出根轨迹的渐近线为:

1. 渐近线与实轴的夹角 $\varphi_a = \frac{(2k+1)\pi}{n-m}, k = 0, 1, 2, \varphi_1 = \frac{\pi}{3}, \varphi_2 = \pi, \varphi_2 = \frac{5\pi}{3}$
2. 渐近线与实轴的交点 $\sigma_a = \frac{\sum\limits_{i=1}^{3} p_i}{3} = -2$

1.2 (b) Breakaway point & Corresponding k.

分离点, 使用试探法得到:

$\frac{1}{d+1} + \frac{1}{d+2} + \frac{1}{d+3} = 0$, $d_1 = -1.42, d_2 = -2.58$,

根轨迹活动的范围是 $[-2, -1]$ and $[-\infty, -3]$, 所以选择 $d_2 = -1.4226$

将-1.4226带回原来的式子 $(s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6$

$$k = s^3 + 6s^2 + 11s + 6|_{-1.4226} = -0.3849 \tag{1}$$

因此, $k = -0.385$

1.3 (c) Root loci & Imaginary axis.

先求出对应的闭环传递函数$G(s) = \frac{K(s+2)}{s^3+6s^2+11s+6+K}$，令$D(s) = s^3 + 6s^2 + 11s + 6 + K = 0$带入$s = j\omega$

得到 $j(-\omega^3 + 11\omega) + (-6\omega^2 + 6 + K) = 0$
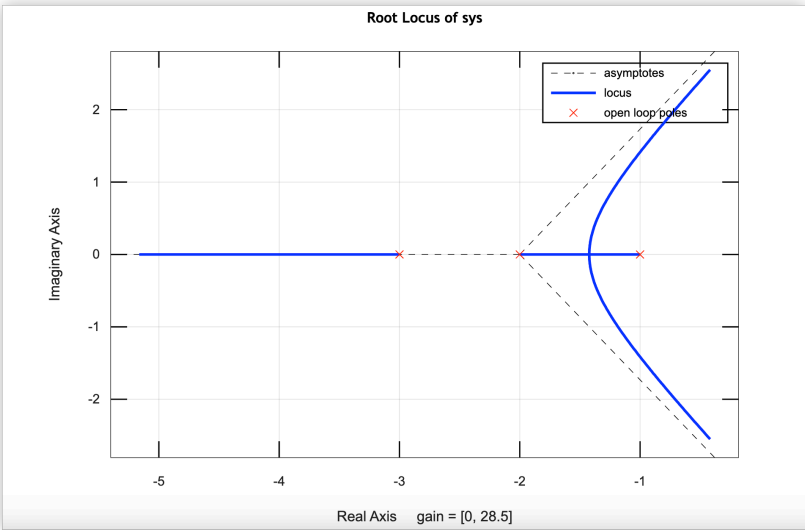
因此 $\omega = \pm\sqrt{11}, K = 60$

1.4 (d) Draw the root locus.

Draw the root locus plot on graph paper and annotate important points on the plot.

上面我们已经计算出Re轴上的分离点，以及Im轴上的交点:

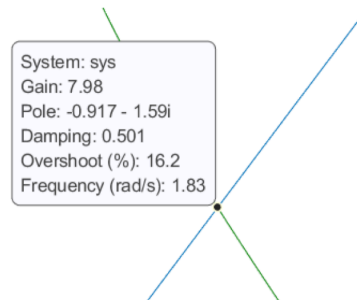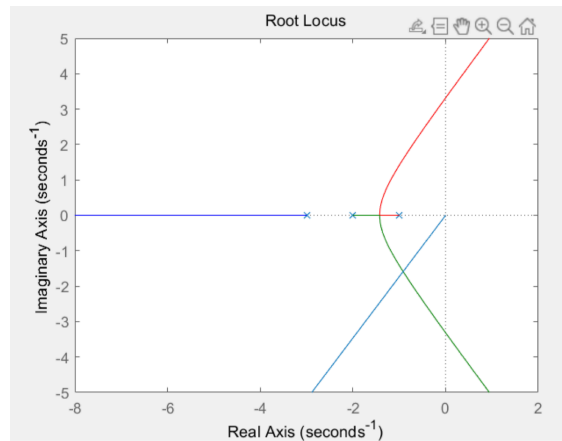| K	Re	Im
-0.385	-1.42	0
60	0	$\pm\sqrt{11}$

如下图所示:

1.5 (e) Graphical approximations

Use the root locus plot to determine the value of k such that the closed-loop system has a damping ratio ζ of approximately 0.5. Calculations based on graphical approximations are acceptable. [Hint:ζ = cosφ]
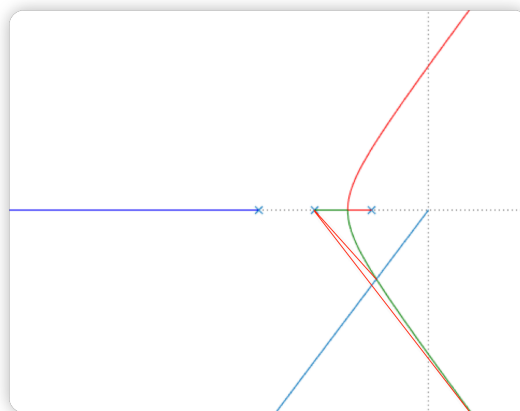
本题解仅供参考:

- 模值方程: $K^* \frac{\prod_{j=1}^{m}|s-z_j|}{\prod_{i=1}^{n}|s-p_i|} = 1$
- 相角方程: $\sum_{j=1}^{m} \angle(s - z_j) - \sum_{i=1}^{n} \angle(s - p_i) = (2k - 1)\pi$

Method 1: 使用matlab计算得出精确值解

如图 $\zeta = 0.501$, $K = 7.98$，为精确解

Method 2：使用渐近线来近似

因为 $\zeta = cos\varphi = 1/2$，则 $\varphi = 60°$

我们想要的点可以近似为 $(-1，\sqrt{3})$

代回幅值条件：$|G(s)H(s)| = 1$，可以得出 $K = \sqrt{84}$

2 Q2 State-Space Basic

A continuous-time system is described by the following state-space matrices

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 & 0 \end{bmatrix}$$

2.1 (a) Controllable

可控性的判据是$C = [B \quad AB \quad A^2B \quad \cdots \quad A^nB]$满秩，就可以确定可控

$$C = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \tag{2}$$

$Rank(C) = 3$即为满秩，那么就是可控的

2.2 (b) Observable

可观性的判断是$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix}$满秩，就可以判断可观性

$$O = \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ -6 & -3 \end{bmatrix} \tag{3}$$

$Rank(O) = 3$也是满秩，那么就是可观的

2.3 (c) Transfer function & Stability

转换公式为 $G(s) = C(sI - A)^{-1}B$

首先要知道矩阵求逆的步骤，假设我们已知矩阵F，要求逆矩阵，首先要确定该矩阵不是奇异矩阵，即$det(F)! = 0$

$$F^{-1} = \frac{adj(F)}{\det(F)} \tag{4}$$

其中分子是F的伴随矩阵，伴随矩阵的求法是，主对角线对调，其余的元素改变符号

因此，对于本题目：

$$G(s) = C(sI - A)^{-1}B$$

$$G(s) = \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \end{bmatrix} \frac{\begin{bmatrix} s+1 & 1 \\ -2 & s \end{bmatrix}}{s^2 + s + 2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{3}{s^2 + s + 2}$$

(5)

$s^2 + s + 2 = 0$, 那么 $s_1 = -0.5 + 1.32i, s_2 = -0.5 - 1.32i$

两者都在左半平面，所以是稳定的

2.4 (d) Performance Specs (unit step input)

2.4.1 (i) the steady-state system response

Hint: Laplace final value Theorem 终值定理

终值定理 $\lim_{s \to \infty} G(s) = \lim_{s \to 0} sG(s)$

带入之后 $\lim_{s \to 0} \frac{1}{s} sG(s) = \lim_{s \to 0} \frac{3}{s^2 + s + 2} = \frac{3}{2}$

2.4.2 (ii) the settling time for the system

已知 $T_s = \frac{4}{\xi \omega_n}$

我们通过传递函数可以知道两个参数 $\xi = \sqrt{2}/4, \omega_n = \sqrt{2}$

那么 $T_s = \frac{4}{\xi \omega_n} = 8s$

2.4.3 (iii) the PO% overshoot for the system

系统的超调计算公式为 $PO = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$

带入数据可以得到PO=30.5%

2.4.4 (iv) the frequency of oscillation of the response

知道 $\omega_d = \sqrt{1 - \xi^2} \omega_n = 1.3229$

2.5 (e) State feedback Controller

(e) It is desired that the closed-loop system will have poles of $[-2, -3]$. Design a state feedback controller $u = \begin{bmatrix} k_1 & k_2 \end{bmatrix} x$ to achieve these specifications (determine the value of k_1 and k_2).

$$|sI - A + BK| = (s+2)(s+3)$$

$$\begin{vmatrix} s & -1 \\ 2+k_1 & s+1+k_2 \end{vmatrix} = s^2 + 5s + 6$$

$$s(s+1+k_2) + 2 + k_1 = s^2 + 5s + 6$$

$$s^2 + s + k_2 s + 2 + k_1 = s^2 + 5s + 6$$

$$k_1 = 4, k_2 = 4$$

(6)

Hence, the state feedback controller is:

$$u = [4, 4]x \tag{7}$$

2.6 (f) State Observer

(f) Due to an inability to measure the states, a state estimator is required. Design a state observer to place the poles of the observer at $[-6, -6]$. (You should write down the observer dynamics and then calculate the observer gain $K_e = \begin{bmatrix} k_{e,1} & k_{e,2} \end{bmatrix}^T$)

$$|sI - A + KC| = \det \begin{bmatrix} s + 3k_1 & -1 \\ 2 + 3k_2 & s+1 \end{bmatrix}$$

$$= s^2 + (3k_1 + 1)s + 3k_1 + 3k_2 + 2 \tag{8}$$

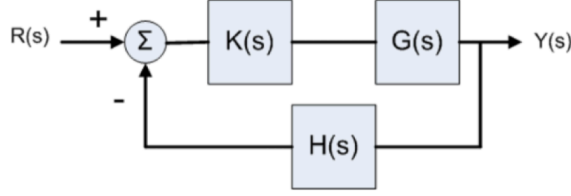Because pole is [-6, -6] , the expected characteristic equation is:

$$s^2 + 12s + 36$$

$$k_1 = \frac{11}{3}, k_2 = \frac{23}{3} \tag{9}$$

Hence, the state observer is

$$K_e = \begin{bmatrix} \frac{11}{3} \\ \frac{23}{3} \end{bmatrix} \tag{10}$$

3 Q3 Nyquist Stability Criterion

3.1 (a) Nyquist Plot

where
$$G(s) = \frac{200}{(s+5)(s+10)}, \qquad H(s) = \frac{1}{s+1}, \qquad K(s) = 1$$

$$G_{op}(s) = \frac{200}{(s+1)(s+5)(s+10)} = \frac{200(50 - 16\omega^2) + 200j(-65\omega - \omega^3)}{(1+\omega^2)(25+\omega^2)(100+\omega^2)}$$

$$|G_{op}(j\omega)| = \frac{200}{\sqrt{1+\omega^2}\sqrt{25+\omega^2}\sqrt{100+\omega^2}}$$ (11)

$$\angle G_{op}(j\omega) = 0 - \arctan\omega - \arctan\frac{\omega}{5} - \arctan\frac{\omega}{10}$$

$$G_{(S)}H_{(S)} = \frac{200}{(s+1)(s+5)(s+10)}$$

$$s = j\omega$$

$$\frac{200}{(j\omega+1)(j\omega+5)(j\omega+10)}$$

$$\frac{200(1-j\omega)(5-j\omega)(10-j\omega)}{(j\omega+1)(j\omega+5)(j\omega+10)(1-j\omega)(5-j\omega)(10-j\omega)}$$

$$\frac{200(j\omega^3 - 16\omega^2 - 65j\omega + 50)}{(1+\omega^2)(25+\omega^2)(100+\omega^2)}$$

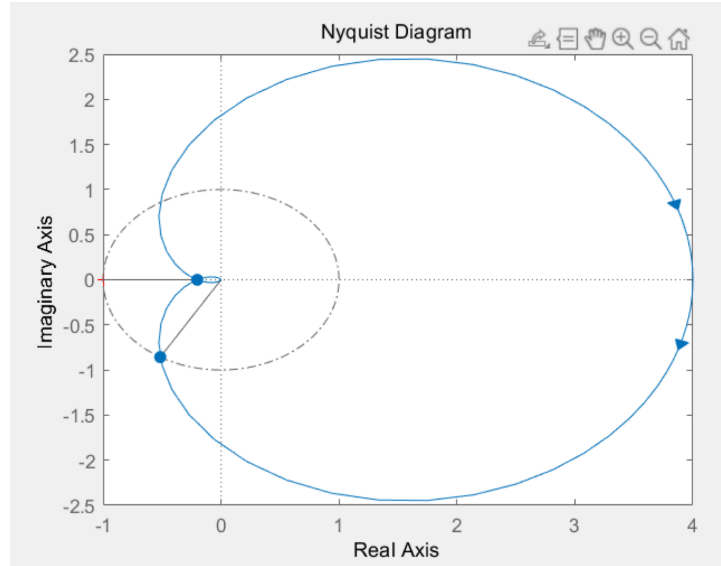$$Im[G_{(j\omega)}] = 0, \omega = \pm 8.06226$$

$$\omega = \pm 8.06226 :$$

$$Re[G_{(j8.06226)}] = \frac{-198000}{980100} = -0.20202$$

$$Re[G_{(j\omega)}] = 0, \omega = \pm 1.76777$$

$$Im[G_{(j\pm1.76777)}] = \frac{-21,876}{11964} = -1.82848$$

画奈氏图步骤:

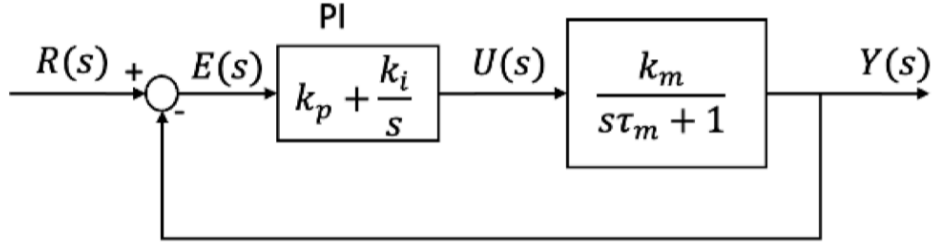| Frequency	Amplitude	Phase
0	4	0
∞	0	-270°
±8.06226		
±1.76777		

3.2 (b) Gain Margin

由上可知 cross phase frequency为:

$$\omega_g = \pm 8.06226$$

$$\frac{1}{|G_{(j\omega_g)}|} = \frac{(\sqrt{\omega_g^2 + 1})(\sqrt{\omega_g^2 + 25})(\sqrt{\omega_g^2 + 100})}{200} = 4.95 \tag{12}$$
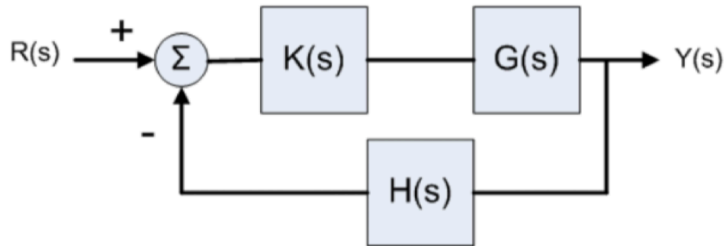
$$GM = 20 \log 4.95 = 13.892 dB$$

4 Q4 PID Controller

4.1 (a) PI-controlled system & Steady-state error

$$\frac{Y(s)}{R(s)} = \frac{(sKp + Ki)(Km)}{s(s\tau_m + 1) + (sKp + Ki)Km}$$

$$\frac{E(s)}{R(s)} = \frac{R(s) - Y(s)}{R(s)} = \frac{s(s\tau_m + 1)}{s(s\tau_m + 1) + (sKp + Ki)Km} \tag{13}$$

$$\lim_{t\to\infty} E(s) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{s(s\tau_m + 1)}{s(s\tau_m + 1) + (sKp + Ki)Km}\frac{1}{s} = 0$$

4.2 (b) only P-controlled & Steady-state error

$$\frac{Y(s)}{R(s)} = \frac{KpKm}{s\tau_m + 1 + KpKm}$$

$$\frac{E(s)}{R(s)} = \frac{R(s) - Y(s)}{R(s)} = \frac{s\tau_m + 1}{s\tau_m + 1 + KpKm} \tag{14}$$

$$\lim_{t\to\infty} E(s) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{s\tau_m + 1}{s\tau_m + 1 + KpKm}\frac{1}{s} = \frac{1}{1 + KpKm}$$

4.3 (c) PID controller using closed-loop ZN method

where

$$G(s) = \frac{100}{(s + 1)(s + 2)(s + 3)}, \qquad K(s) = k_p + k_d s + \frac{k_i}{s}, \qquad H(s) = 1$$

[10 marks]

For your convenience, the closed-loop Ziegler-Nichols rules are given in the following table ($k_d = k_p t_d$, $k_i = k_p/t_i$)

	k_p	t_i	t_d
P	$0.5k_c$		
PI	$0.45k_c$	$t_c/1.2$	
PID	$0.6k_c$	$t_c/2$	$t_c/8$

$$1 + G_{(S)}k_p = 0$$

$$1 + \frac{100k_p}{s^3 + 6s^2 + 11s + 6} = 0$$

$$s^3 + 6s^2 + 11s + 6 + 100k_p = 0$$

$$-j\omega^3 - 6\omega^2 + 11j\omega + 6 + 100k_p = 0$$

$$\begin{cases} -j\omega^3 + 11j\omega = 0 \\ -6\omega^2 + 6 + 100k_p = 0 \end{cases} \tag{15}$$

$$k_c = 0.6, \omega_c = \sqrt{11}rad/s$$

$$t_c = 2\pi/\omega_c = 1.8945$$

$$k_P = 0.36, t_i = 0.9472, t_d = 0.2368$$

$$K_P = 0.36, K_i = 0.38, K_d = 0.08525$$