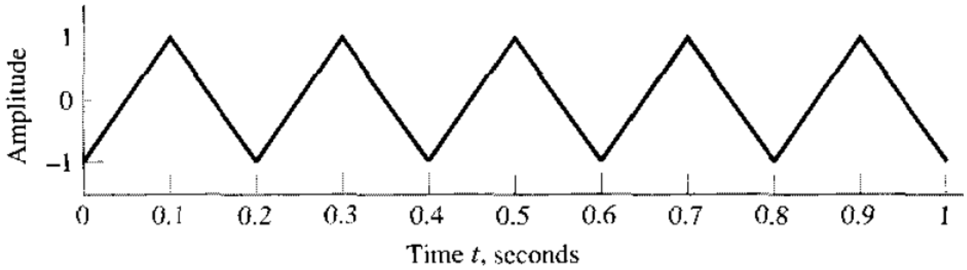
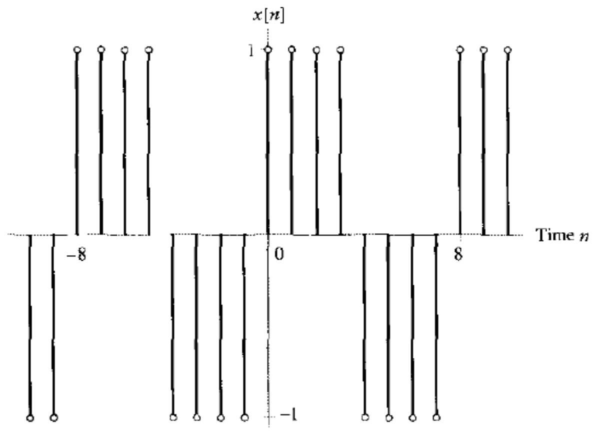
Tutorial 1 - Solutions

Q1 What is the fundamental period and frequency of the signal given in the following figure? You need to specify the unit for each parameter?



Solution: $T = 0.2$ (sec), and $f = 1/T = 5$ Hz.

Q2 What is the fundamental period and frequency of the signal given in the following figure?



Answer: The fundamental period is $N = 8$ samples, and the angular frequency is $\Omega = 2\pi/N = \pi/4$ radians, and frequency is $1/N = 1/8$.

Q3 Calculate the power of the signal in **Q1**.

Solution: The signal in **Q1** is periodic with a period of $T = 0.2$ (sec), and thus it is a power signal. To calculate its power, we only need to consider the signal over one period, i.e., $t \in [0, 0.2]$. From the figure in **Q1**, we can write the signal for the interval from $t = 0$ to $t = 0.1$ as

$$\begin{aligned} x(t) &= \frac{1-(-1)}{0.1-0}(t-0)+(-1) \\ &= 20t - 1 \end{aligned} \tag{1}$$

In the above equation, we have used the fact that the line going through (x_1, y_1) and (x_2, y_2) is given by

$$\boxed{y(t) = \frac{y_2 - y_1}{x_2 - x_1}(t - x_1) + y_1} \tag{2}$$

To arrive at (1) we substitute $(x_1, y_1) = (0, -1)$ and $(x_2, y_2) = (0.1, 1)$. Similarly, we can write the signal for the interval from $t = 0.1$ to $t = 0.2$ as

$$\begin{aligned} x(t) &= \frac{-1-(1)}{0.2-0.1}(t-0.1)+1 \\ &= -20t+3 \end{aligned}$$

In summary, we can write the signal $x(t)$ over **one period** as

$$x(t) = \begin{cases} 20t - 1 & 0 \le t < 0.1 \\ -20t + 3 & 0.1 \le t \le 0.2 \end{cases} \tag{3}$$

The signal power is given by

$$P_x = \frac{1}{T}\int_{<T>} x^2(t)dt \tag{4}$$

Since the expression of the signal is different for the two intervals as shown above, we need to compute the following two integrals

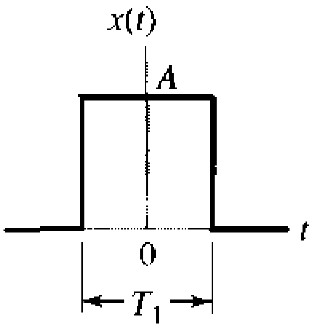
$$\begin{aligned} \frac{1}{0.2}\int_0^{0.1}(20t-1)^2 dt &= \frac{1}{0.2}\int_0^{0.1}(400t^2 - 40t + 1)dt \\ &= \frac{1}{0.2}\Big(\frac{400}{3}\times 0.1^3 - \frac{40}{2}\times 0.1^2 + 0.1\Big) = \frac{1}{6} \end{aligned} \tag{5}$$

and

$$\frac{1}{0.2}\int_{0.1}^{0.2}(-20t-3)^2 dt = \frac{1}{6} \tag{6}$$

Thus the signal power is $\frac{1}{6} + \frac{1}{6} = 1/3$. Note that the power of the signal indicates the strength of the signal. In the figure, we can see that the signal is symmetric about $t = 0.1$, meaning that the strength of the signal from $t = 0.1$ to $t = 0.2$ is equal to that from $t = 0.1$ to $t = 0.2$. This explains why the two integrals above have the same value.

Q4 Calculate the energy of the following signal.



Solution: The signal is only defined from $t = -T_1/2$ to $t = T_1/2$, and thus it is a time-limited signal. The energy of $x(t)$ is given by

$$E_x = \int_{-T_1/2}^{T_1/2} x(t)^2 dt = \int_{-T_1/2}^{T_1/2} A^2 dt = A^2\int_{-T_1/2}^{T_1/2} dt = A^2 T_1 \tag{7}$$

The signal $x(t)$ is called a rectangle signal and it has many applications in signal processing. Suppose we want to create a rectangle signal of unit energy, i.e, $E_x = 1$ with a very small width ($T_1 \to 0$), then the height of the signal will be immensely large $A = \sqrt{\frac{E_x}{T_1}} = \sqrt{\frac{1}{T_1}} \to \infty$. The resulting signal is called *an impulsive signal* which is one of the fundamental signals in signal processing.

$\boxed{\textbf{Q5}}$ Determine whether or not the signals below are periodic and, for each signal that is periodic, determine the fundamental period.

(a) $x[n] = \cos(0.125\pi n)$
(b) $x[n] = \text{Re}\{e^{j\pi n/12}\} + \text{Im}\{e^{j\pi n/18}\}$
(c) $x[n] = \sin(\pi + 0.2n)$

Solution: (a) $N = 16$. (b) $N = 72$, which is the least common multiple of 24 and 36. (c) $x[n]$ is not periodic.

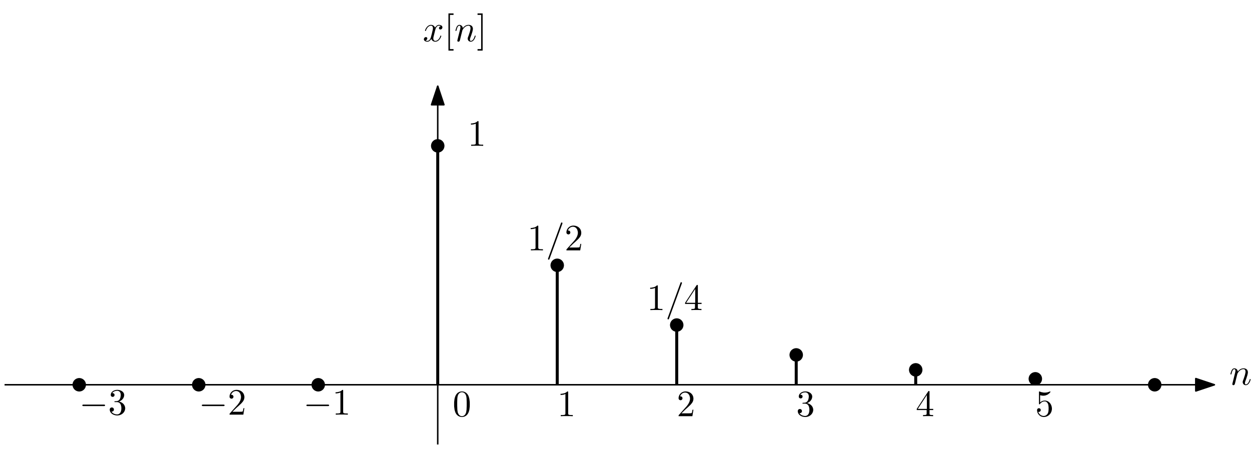
$\boxed{\textbf{Q6}}$ Given the following signal $x[n] = \alpha^n u[n]$.

(a) Plot the signal for $\alpha = 1/2$ and $\alpha = 2$.
(b) Determine if this signal is an energy or power signal.

Solution: Recall that $u[n] = 0$ for $n < 0$ and thus $x[n] = 0$ for $n < 0$. If $n \geq 1$ then $u[n] = 1$ and $x[n] = \alpha^n$. In order to plot $x[n]$ we can compute the value of $x[n]$ for some first n, i.,e., $n = 0, 1, 2, \ldots$... For $\alpha = 1/2$ we have

$$x[n] = \begin{cases} \frac{1}{2^n} & n = 0, 1, 2, ... \\ 0 & n = \ldots, -2, -1 \end{cases} \tag{8}$$

In this case $x[n]$ is a decreasing signal as plotted below



Note that $x[n]$ is a discrete signal and it is **only defined for integers** $n = -\infty, \ldots, -2, 1, 0, 1, 2, 3, \ldots, \infty$. For example, the value of $x[1.2]$ is not defined (this does NOT mean $x[1.2] = 0$). From the figure we can see that $x[n] \to 0$ as $n \to \infty$. This is a good indicator suggesting $x[n]$ is an energy signal. To confirm this hypothesis we follow the definition of the energy of discrete signals. In particular, if $\alpha = 1/2$, then

$$E_x = \sum_{n=-\infty}^{\infty} x[n]^2 = \sum_{n=0}^{\infty} \alpha^{2n} = \sum_{n=0}^{\infty} (\alpha^2)^n = \frac{1}{1-\alpha^2} = 4/3 \tag{9}$$

In the above equation we have used the equality

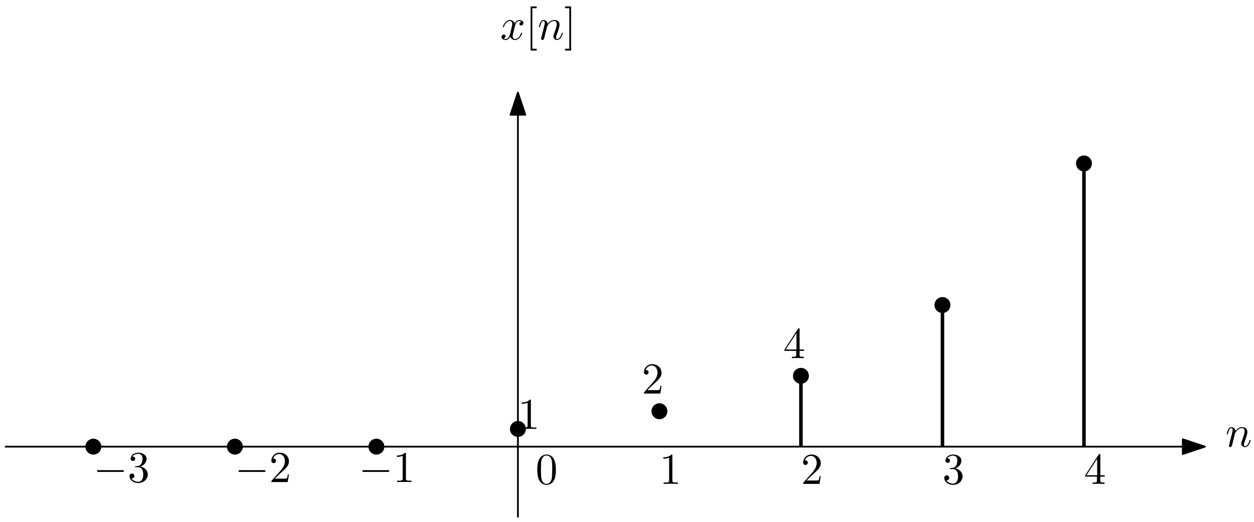
$$\sum_{n=0}^{\infty} q^n = \lim_{N\to\infty} \sum_{n=0}^{N-1} q^n = \lim_{N\to\infty} \frac{1-q^N}{1-q} = \frac{1}{1-q} \tag{10}$$

for $q < 1$ since $\lim_{N\to\infty} q^N = 0$.

For $\alpha = 2$ we have

$$x[n] = \begin{cases} 2^n & n = 0, 1, 2, ... \\ 0 & n = \ldots, -2, -1 \end{cases} \tag{11}$$

In this case $x[n]$ is increasing with n as shown below.



Since $x[n] \to \infty$ as $n \to \infty$ in this case, $x[n]$ is not an energy signal. We can easily check that for $\alpha = 2$

$$E_x = \sum_{n=-\infty}^{\infty} x[n]^2 == \sum_{n=0}^{\infty} 4^n \to \infty \tag{12}$$

Now we need to see if the signal is a power one or not. To do so, consider the following definition

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 4^n = \lim_{N\to\infty} \frac{1}{2N+1} \times \frac{4^{N+1}-1}{4-1} \to \infty \tag{13}$$

Since E_x and P_x don't take on finite value, $x(t)$ is neither an energy nor a power signal.