EE114
Intro to Systems & Control

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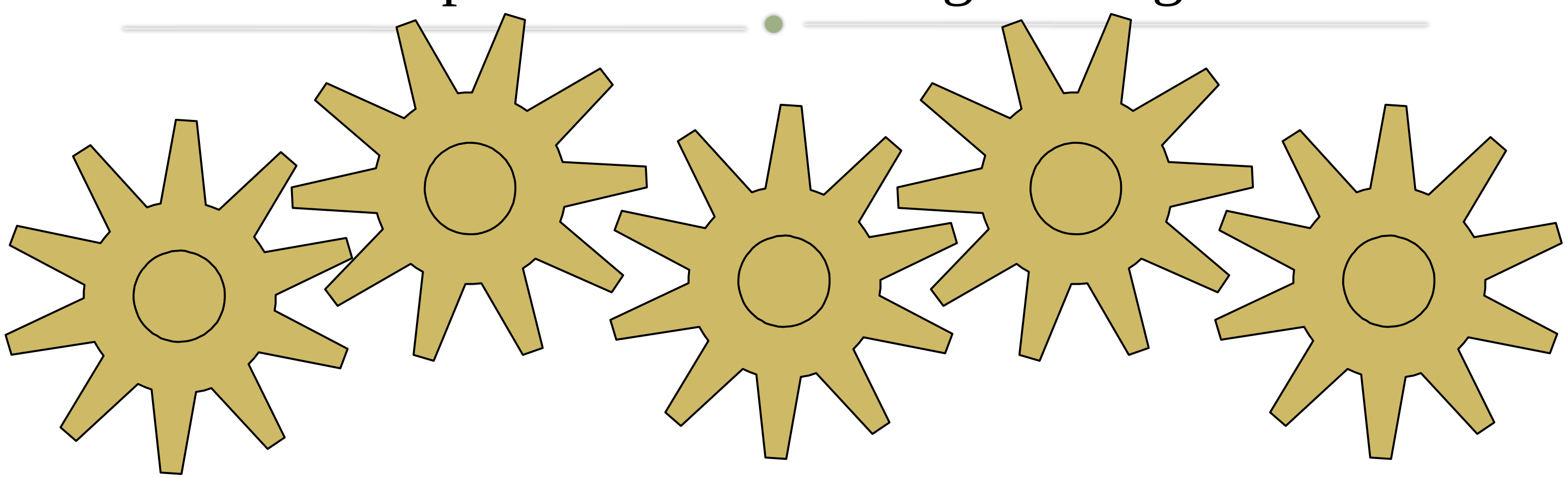
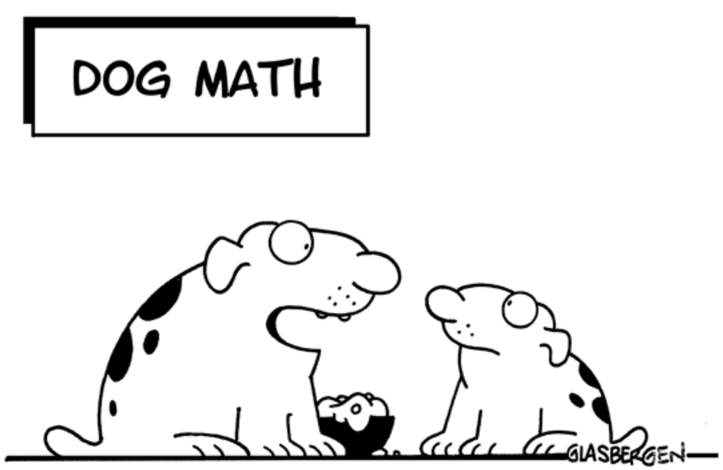


So far …

- We’ve introduced the concept of control & feedback control …
- We’ve modelled simple dynamical systems - an RC circuit, a bicycle and a water tank …
- We’ve introduced transfer functions and Laplace Transforms …



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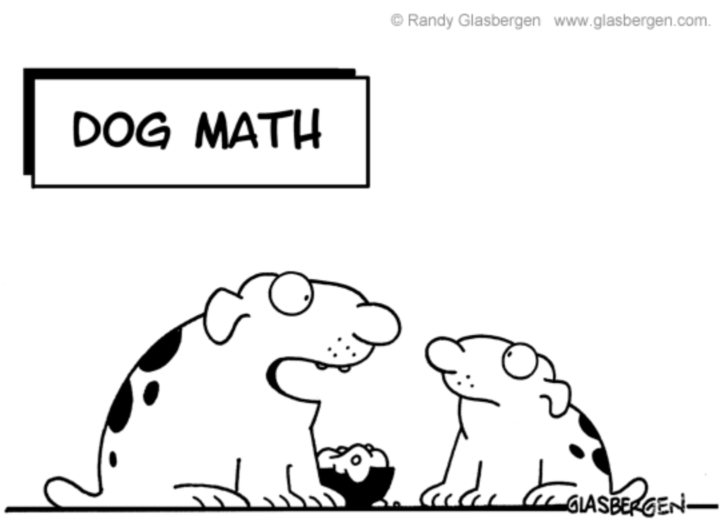


“If I have 3 bones and Mr. Jones takes away 2, how many fingers will he have left?”

- **Today, we will continue with Laplace Transforms and also consider the Inverse Laplace Transform …**

Laplace Transforms of Common Functions

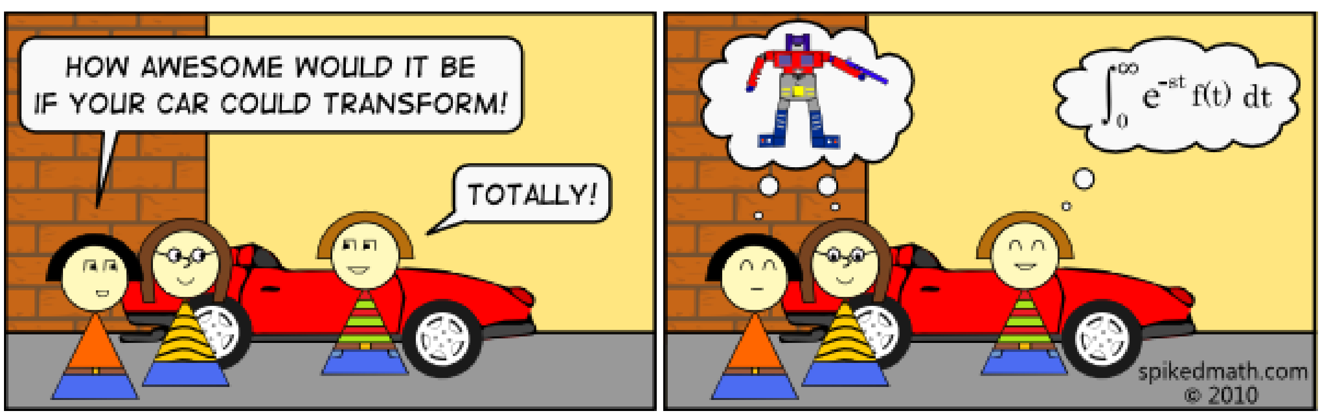
The good news! We don't need to remember the above (and other) examples, as the more common functions and their transforms are typically available in look up tables.

Such a table will be available in exam situations if needed.

Laplace Transforms of Common Functions

Function name	Time domain function $f(t)$	Laplace transform $F(s) = L\{f(t)\}$
Constant	a	$\dfrac{a}{s}$
Linear	t	$\dfrac{1}{s^2}$
Power	t^n	$\dfrac{n!}{s^{n+1}}$
Exponent	e^{at}	$\dfrac{1}{s-a}$
Sine	$\sin at$	$\dfrac{a}{s^2+a^2}$
Cosine	$\cos at$	$\dfrac{s}{s^2+a^2}$
Hyperbolic sine	$\sinh at$	$\dfrac{a}{s^2-a^2}$

Useful Properties of Laplace Transforms

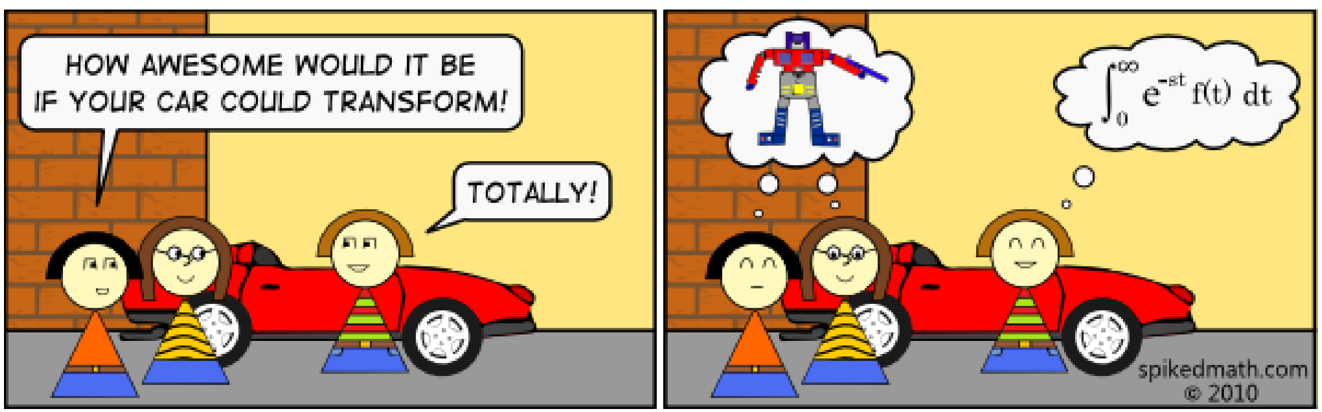


Useful Properties of Laplace Transforms

- Linearity:

$$\text{if} \quad F(s) = L\left[f(t)\right] \ \text{and} \ G(s) = L\left[g(t)\right]$$

$$\text{then:} \quad L\left[af(t) + bg(t)\right] = aF(s) + bG(s)$$



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- (Shift theorem) Multiplying by e^{at}:

$$L[e^{at} f(t)] = F(s-a)$$

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- Final value theorem (end point):

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

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- Initial value theorem (start point):

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

Inverse Laplace Transform

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- The **Inverse Laplace Transform** relates to the process of finding *f(t)* from the corresponding Laplace transform *F*(*s*), i.e.:

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- This is normally carried out using the ***partial fraction method***.

- This method is best illustrated using an example.

Inverse Laplace Transform

- Consider the following Laplace transform:

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- In other words, we separate the denominator into its individual factors.

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- Now we compare terms on the numerator and match the coefficients.

$$= \frac{s(A+B)+(4A+3B)}{(s+3)(s+4)}$$

Inverse Laplace Transform

- Hence: $s = (A + B)s \;\Rightarrow\; 1 = A + B$

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- Solving: $1 \;=\; A+B \Rightarrow B = 1-A$

$$2 = 4A+3(1-A) \;\Rightarrow 2 = 4A+3-3A \;\Rightarrow A = -1$$

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- Solving: $$1 = A+B \Rightarrow B = 1 - A$$

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$$B = 1-(-1) = 2$$

Inverse Laplace Transform

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Compare the numerators:

$$\frac{s+2}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$= \frac{A(s+4) + B(s+3)}{(s+3)(s+4)}$$

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$$s+2=A(s+4)+B(s+3)$$

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Now cover-up the (s + 4) factor by setting s = −4,
i.e. (s + 4) = 0:

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$$-3 + 2 = A(-3+4) + 0 \Rightarrow A = -1$$

Same values obtained for A and B !!

Inverse Laplace Transform

- Therefore, we can express $F(s)$ as:

$$\frac{s+2}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

$$F(s) = -\frac{1}{s+3} + \frac{2}{s+4}$$

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- From the table we see that:

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- From the table we see that:

$$L\left(e^{at}\right) = \frac{1}{s-a} \quad \Rightarrow L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

Inverse Laplace Transform

$$F(s) = -\frac{1}{s+3} + \frac{2}{s+4}$$

- We now use the table of Laplace transforms to convert this expression to the time domain.
- From the table we see that:

$$L^{-1}\left(\frac{-1}{s+3}\right) = -L^{-1}\left(\frac{1}{s+3}\right) = -e^{-3t}$$

Inverse Laplace Transform

$$F(s) = -\frac{1}{s+3} + \frac{2}{s+4}$$

- We now use the table of Laplace transforms to convert this expression to the time domain.

- From the table we see that:

$$L^{-1}\left(\frac{2}{s+4}\right) = 2L^{-1}\left(\frac{1}{s+4}\right) = 2e^{-4t}$$

Inverse Laplace Transform

- Finally:

$$F(s) = -\frac{1}{s+3} + \frac{2}{s+4} \quad \rightarrow \quad f(t) = -e^{-3t} + 2e^{-4t}$$

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Note – there are additional rules the need to be taken into consideration when working with partial fractions. These will be covered in your mathematics modules. Here, we are simply illustrating the key concept.

Inverse Laplace Transform

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$$F(s) = \frac{3s+1}{s^2 - s - 6}$$

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$$\frac{3s+1}{s^2 - s - 6} = \frac{3s+1}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

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Solution:

$$3s + 1 = A(s+2) + B(s-3)$$

Now cover-up the (s + 2) factor by setting s = −2:

$$-6 + 1 = 0 + B(-2-3) \quad \Rightarrow \mathrm{B} = 1$$

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Solution:

$$3s+1=A(s+2)+B(s-3)$$

Now cover-up the (s – 3) factor by setting s = 3:

$$9+1=A(3+2)+0 \Rightarrow \mathrm{A}=2$$

Inverse Laplace Transform

- ***Ex 4.1 Find the function f(t) given that its Laplace transform is*:**

$$F(s)=\frac{3s+1}{s^2-s-6}$$

Solution:

$$F(s)=\frac{2}{s-3}+\frac{1}{s+2}$$

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Solution:

$$F(s)=\frac{2}{s-3}+\frac{1}{s+2}$$

$$f(t)=L^{-1}\left(\frac{2}{s-3}\right)+L^{-1}\left(\frac{1}{s+2}\right)$$

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Solution:

$$F(s) = \frac{2}{s-3} + \frac{1}{s+2}$$

$$f(t) = L^{-1}\left(\frac{2}{s-3}\right) + L^{-1}\left(\frac{1}{s+2}\right) \quad \Rightarrow f(t) = 2e^{3t} + e^{-2t}$$