3. Modelling of Basic Systems

3.1 Introduction

- As mentioned already, obtaining a mathematical representation of a system allows us to:
  - Understand the characteristics of the system (useful for design purposes).
  - Simulate the system (useful for scenario testing, forecasting, etc.).
  - Provide a basis for control system design (for stability, optimising performance, etc.).

- In essence, a mathematical model of a system will increase our understanding of it.

- The basic modelling procedure is as follows:
  1 – Draw a schematic diagram of the system and define the variables.
  2 – Using physical laws, write equations for each component.
  3 – Parameterise the model (using experiment design and/or system identification techniques).
  4 – Validate the model.

- Other factors that need to be considered when modelling include:
  - Complexity/accuracy trade-off – the more accurate the model, the more complex it becomes (and hence more complex mathematical analysis required).
  - Objective of modelling – what is the purpose of the model? This dictates the level of accuracy (and hence complexity) required. Different modelling objectives include design/synthesis, analysis and control.

3.2 Modelling of static systems

- Before we consider dynamical systems, let us first look at some model representations of basic static systems. Here, we are only going to consider electrical systems.

- For static electrical systems, such as resistor networks, we use Ohm's law, which states that the voltage across a resistor is directly proportional to the current flowing through it:

$$v = iR$$

where: v is the voltage across the resistor in volts (system output),

i is the current through the resistor in amps (system input), and

R is the resistance in ohms (system model).

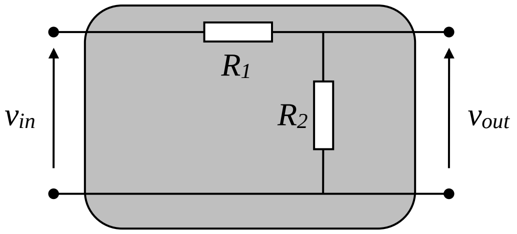
- Note that there are no dynamics in this system. Here, a step change in the input will result in a step change in the output.

- The model is given by a resistor value with a tolerance specification.

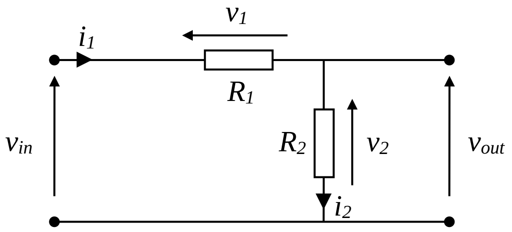
- Different resistor networks will involve parallel and series configurations.

- Furthermore, you may need to apply Kirchoff's current and voltage laws (KCL and KVL) in order to obtain a suitable model representation.

- ***Ex. 3.1 Derive, from first principles, the input-output relationship for the voltage divider circuit below:***



Solution:



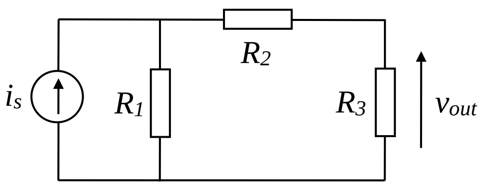
$v_1 = i_1R_1$ and $v_2 = i_2R_2$ - Ohm's Law

$i_1 = i_2$ - KCL

$v_{in} = v_1 + v_2$ and $v_{out} = v_2$ - KVL

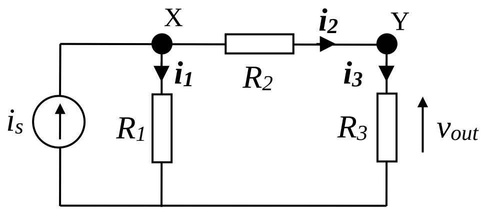
Hence: $$\frac{v_{out}}{v_{in}} = \frac{v_2}{v_1+v_2} = \frac{i_2R_2}{i_1R_1+i_2R_2} = \frac{i_2R_2}{i_2R_1+i_2R_2} = \frac{R_2}{R_1+R_2}$$ (the voltage divider rule)

- ***Ex. 3.2(a) Derive the input-output relationship for the following circuit given that the input is the current source i_s and the output is the voltage v_{out} (across resistor R_3 as shown):***



Solution:

Using nodal analysis, we apply KCL to each of the nodes X and Y, as shown.



The voltage at node X is given by v_X while the voltage at node Y is v_{out} in this case.

KCL @ node X: $$i_s = i_1 + i_2 \Rightarrow i_s = \frac{v_X}{R_1} + \frac{v_X - v_{out}}{R_2}$$

KCL @ node Y: $$i_2 = i_3 \Rightarrow \frac{v_X - v_{out}}{R_2} = \frac{v_{out}}{R_3}$$

Rearranging the last equation gives: $$v_X - v_{out} = \frac{R_2}{R_3} v_{out} \Rightarrow v_X = v_{out}\left(1 + \frac{R_2}{R_3}\right)$$

Subbing this expression in the first equation we obtain:

$$i_s = \frac{v_{out}\left(1 + \frac{R_2}{R_3}\right)}{R_1} + \frac{v_{out}}{R_3} \qquad \text{... note that: } \frac{v_X - v_{out}}{R_2} = \frac{v_{out}}{R_3}$$

$$\Rightarrow i_s = \left(\frac{\left(1 + \frac{R_2}{R_3}\right)}{R_1} + \frac{1}{R_3}\right) v_{out} \qquad \Rightarrow \frac{i_s}{v_{out}} = \left(\frac{(R_3 + R_2) + R_1}{R_1 R_3}\right)$$

$$\Rightarrow \frac{v_{out}}{i_s} = \frac{R_1 R_3}{R_1 + R_2 + R_3}$$

- ***Ex. 3.2(b) Given that $R_1 = R_3 = 0.5\Omega$, $R_2 = 1\Omega$ and $i_s = 8A$, determine the voltage v_{out} for the above circuit.***

Solution:

$$v_{out} = \frac{R_1 R_3}{R_1 + R_2 + R_3} i_s = \frac{(0.5)(0.5)}{0.5 + 1 + 0.5}(8) = \frac{0.25}{2}(8) = 1V$$

- Overall, in the case of modelling of static systems, we can say:

- A static relationship exists between input and output.
- Static = memoryless = instantaneous.
- We start with some basic laws for given system type - for example, Ohm's law for an electrical circuit.
- Model parameters may be subject to error (e.g. resistance tolerance values).
- We should also state any assumptions made in deriving model.

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- In general, all models are subject to assumptions and all models have errors.
- In the case of the previous examples, the wire in the circuit has resistance along its entire length – we can argue that this is negligible in relation to the value of the resistor itself.
- We also assume that the components are ideal.
- Ideally, the complete model description should give:
  - the model structure,
  - the parameters of the model,
  - modelling assumptions, including range of model validity, and
  - some measure of the error in the model.

- In practice, most real systems have dynamics associated with them – even resistors!
- Sometimes these dynamics have a negligible effect on the system behaviour that we are interested in studying allowing us to assume steady state (or static) conditions.

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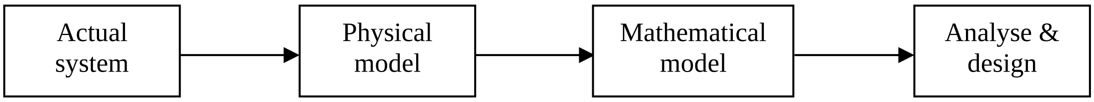
3.3 Modelling of dy

- The stages of a dynamic system investigation are as follows:



3.3.1 Physical modelling

- This involves identifying the system/sub-system to be studied and obtaining a simple physical model whose behaviour will match sufficiently closely that of the actual system.

- This typically leads to a schematic representation showing the key system components and variables and how they are physically related.

- Engineering judgement is needed in determining the appropriate level of detail – we have to decide what is important and what can be neglected.

- Too complicated a model leads to long analysis while too simple a model is unrepresentative (i.e. not accurate enough).

- Experience is needed – it cannot be taught! However, there are several useful guidelines (engineering approximations):

  - *neglect small effects*
    – this reduces the number and complexity of equations.

  - *assume environment is independent of the system motions*
    – this reduces the number and complexity of equations.

  - *replace distributed characteristics with appropriate lumped elements*[*]
    – this gives ordinary differential equations rather than partial ones.

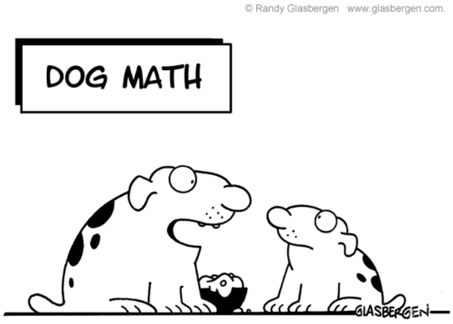
  - *assume linear relationships*
    – gives linear equations and superposition holds.

  - *assume constant parameters*
    – leads to constant coefficient in differential equations.

  - *neglect uncertainty and noise*
    – avoids statistical treatment.

[*] For example, a wire has resistance along its entire length – however we represent this by 'lumping' this distributed resistance into a single point value.

3.3.2 Mathematical modelling



"If I have 3 bones and Mr. Jones takes away 2, how many fingers will he have left?"

- This involves obtaining a mathematical representation of the physical model.
- Central to this process is the writing of equations for equilibrium and/or compatibility relations.
- **Equilibrium relations** describe the balance of forces, of flow rates, of energy, of current, etc. which must exist for the system (conservation of energy).
- **Compatibility relations** describe how motions of the system are interrelated because of the way they are connected.
- Two further considerations are physical variables and physical laws.
- **Physical variables** are needed to describe the instantaneous state of the system. These can be divided into:
    - *through variables* (eg. current , flow) and
    - *across variables* (eg. voltage, pressure).
- Equilibrium relations apply to through variables while compatibility relations apply to across variables.
- **Physical laws** which individual components obey – usually between through and across variables and are generally empirical in nature.
- The procedure for obtaining a mathematical model can be summarised as follows:
    - Develop a physical model for the system.
    - Define the system variables (through and across).
    - Write equations for the equilibrium (through variables) and/or compatibility (across variables) relations in the system.
    - Use physical relations/laws to relate the through and across variables for each component in the system.
- In terms of modelling dynamical systems we are going to consider some simple electrical, mechanical and flow based ones.
- The mechanical system will be a standard simple second order mass-spring damper system while the flow-based system will be a simple first order tank system.
- You will model more complicated systems in EE211 System Dynamics.

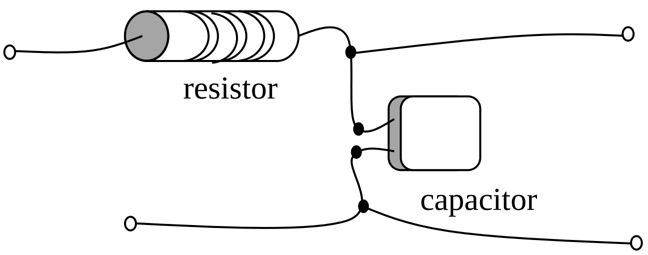
3.3.3 Modelling electrical systems

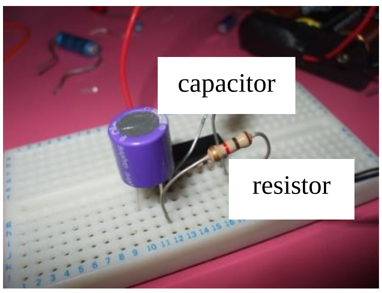
- Analytic procedure:
    - Physical model – circuit diagram.
    - Variables – voltages, currents.
    - Equilibrium relation – Kirchoff's Current Law (KCL).
    - Compatibility relation – Kirchoff's Voltage Law (KVL).
    - Physical relations are summarised in the following table:

Component	**Physical Law**	**Symbol**
Resistance (R)	$v = iR$	R *or* R
Inductance (L)	$v = L\dfrac{di}{dt}$	L *or* L
Capacitance (C)	$v = \dfrac{1}{C}\int idt$ or $i = C\dfrac{dv}{dt}$	C *or* C

- Note that we are assuming lumped parameters and ideal components.

- ***Ex 3.3 Determine a mathematical model for the resistor/capacitor filter circuit shown below:***

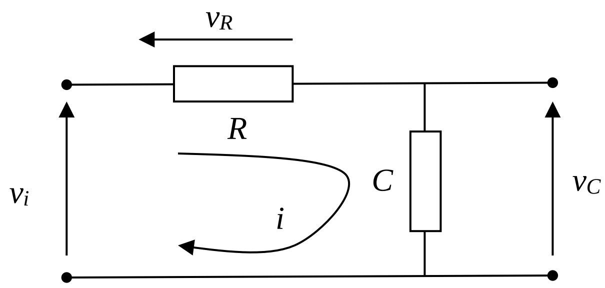




Actual system

Solution:

Step 1: Physical model assuming ideal components:



Step 2: Model variables defined on the physical model – current i and voltages v_i, v_R, v_C.

Step 3: Compatibility relation required – KVL:

$$v_i = v_R + v_C$$

Note that the equilibrium condition is implied in our choice of current variable and therefore, does not need to be stated explicitly (i.e. $i = i_C = i_R$)

Step 4: Combining physical relations with the compatibility equation gives:

$$v_i = iR + \frac{1}{C}\int i dt$$

Differentiating once gives a differential equation in terms of v_i and i:

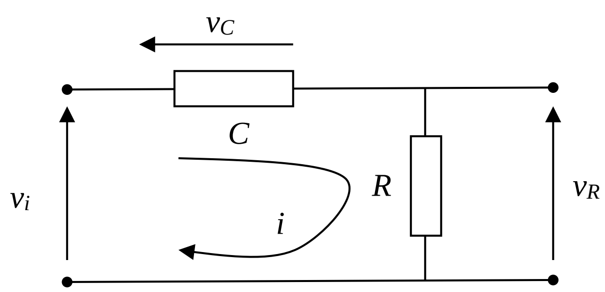
$$\frac{dv_i}{dt} = R\frac{di}{dt} + \frac{1}{C}i$$

Here we are interested in finding out the relationship between v_C and v_i.

This can be obtained by noting that: $$i = C\frac{dv_C}{dt}$$

Substituting into equation for v_i gives: $$v_i = RC\frac{dv_C}{dt} + v_C$$

- ***Ex 3.4 Determine a mathematical model for the capacitor/resistor filter circuit given by the following physical model:***



Solution:

Note, this is almost identical to the previous example, but we are now interested in finding out the relationship between v_R and v_i.

From the previous example, we know that: $$v_i = RC\frac{dv_C}{dt} + v_C$$

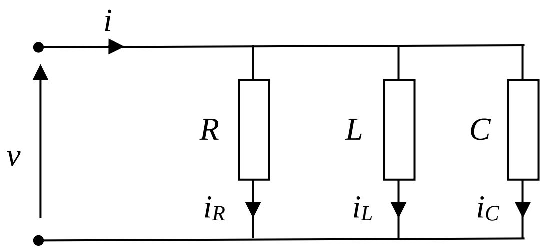
We also know that: $v_i = v_R + v_C$ Hence: $v_C = v_i - v_R$

Substituting for v_C into the first equation: $$v_i = RC\frac{d(v_i - v_R)}{dt} + v_i - v_R$$

$$\Rightarrow 0 = RC\frac{dv_i}{dt} - RC\frac{dv_R}{dt} - v_R \qquad \Rightarrow 0 = \frac{dv_i}{dt} - \frac{dv_R}{dt} - \frac{v_R}{RC}$$

$$\Rightarrow \frac{dv_i}{dt} = \frac{dv_R}{dt} + \frac{v_R}{RC}$$

- ***Ex 3.5 Develop the mathematical model for the LRC circuit that is described by the following physical model (step 1):***



Solution:

Step2: Variables are as defined in physical model.

Step 3: Compatibility relationship implied (i.e. $v = v_R = v_L = v_C$). By KCL the equilibrium relationship for the currents is:

$$i = i_R + i_C + i_L$$

Step 4: Combining the physical relationships between current and voltage R, C and L with the above equation gives

$$i = \frac{v}{R} + \frac{1}{L}\int v dt + C\frac{dv}{dt}$$

which can be rewritten as:

$$\frac{di}{dt} = \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v + C\frac{d^2v}{dt^2}$$

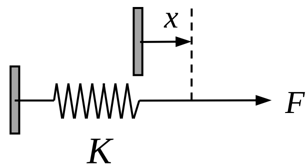
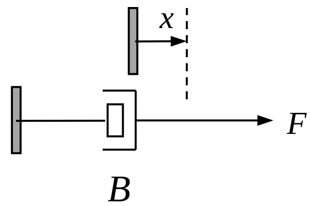
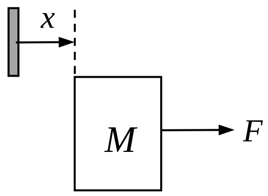
or

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{C}\frac{di}{dt}$$

- For more complex circuits this first principles approach is tedious and we generally use more efficient circuit analysis techniques (i.e. nodal, mesh) to obtain the model equations.

3.3.4 Modelling mechanical systems (one dimensional)

- Analytic procedure:
  - Physical model – schematic showing geometry of the system with respect to an arbitrary configuration and reference co-ordinate frame.
  - Variables – force and position.
  - Equilibrium/ Compatibility relations – Energy conservation or force equilibrium.
  - Physical relations (assuming ideal components):

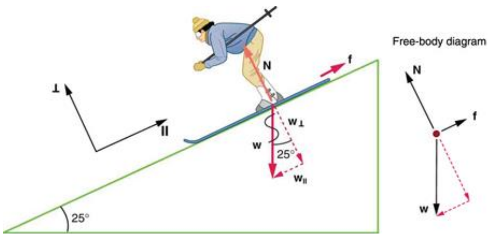
Component	Physical Law	Symbol
Spring (K)	$F = Kx$	
Damper (or Dashpot) (B)	$F = B\dot{x}$	
Mass (M)	$F = M\ddot{x}$	

Note – x represents position (displacement) or distance, $\dot{x}$ represents velocity (v) and $\ddot{x}$ represents acceleration (a).

- **Free body diagram** – In mechanics the concept of a free body diagram (FBD) (or force diagram) is used to analyse mechanical systems.

- Each mass is viewed as a free body isolated from the rest of the system with only the forces acting on it shown.

- Force balance equations are then written for each mass.

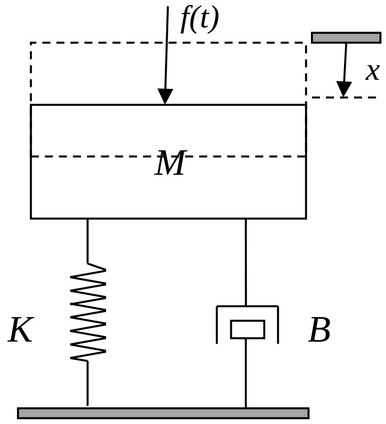


- ***Ex 3.6(a) Determine a mathematical model for the spring-mass-damper system, whose physical model is shown in step 1 below:***

  Solution:

  Step 1: Physical model assuming ideal components:

Actual system – ‘Suspension’ within a shoe or a bicycle for example

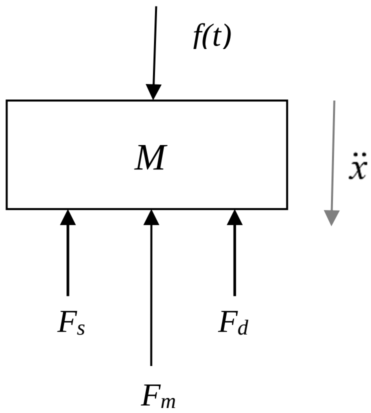


Physical model

  Step 2: Model variables defined on the physical model

  Step 3: From the free body diagram showing only the Force acting on *M* we obtain the Force equilibrium equation:

$$F_m + F_d + F_s = f(t)$$



Free body diagram

  Step 4: Using the physical force-geometry relations this becomes:

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t)$$

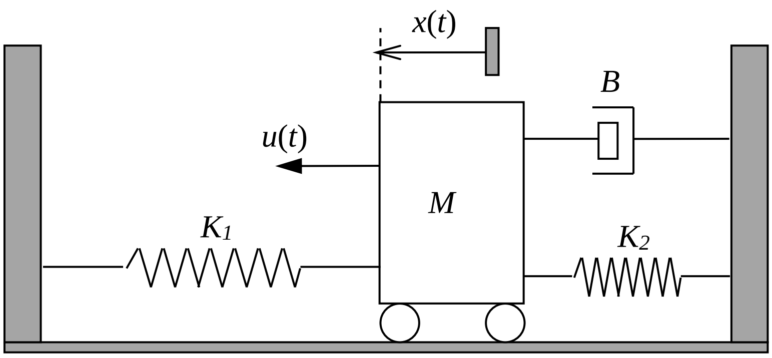
  or simply:

$$M\ddot{x} + B\dot{x} + Kx = f(t)$$

- ***Ex 3.6(b) Determine a mathematical model for a pogo stick (when in contact with the ground).*** *Carry out this exercise yourself.*

Hint: The pogo stick is effectively a mass-spring system!

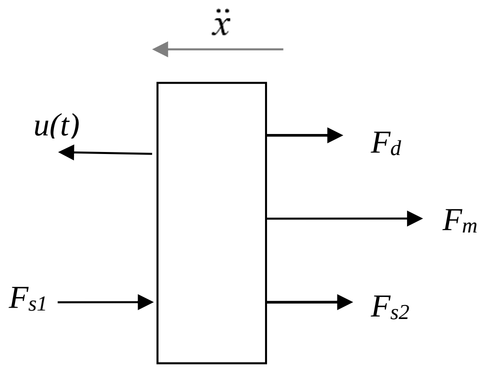
- ***Ex 3.7 Determine a mathematical model for the system, whose physical model is shown below:***



Physical Model

Solution

Free Body Diagram for Mass M is given by:



Free body diagram

Thus, the Force equilibrium equation is:

$$F_m + F_d + F_{s1} + F_{s2} = u(t)$$

Using the physical force-geometry relations this becomes:

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + K_1x + K_2x = u(t)$$

or simply:

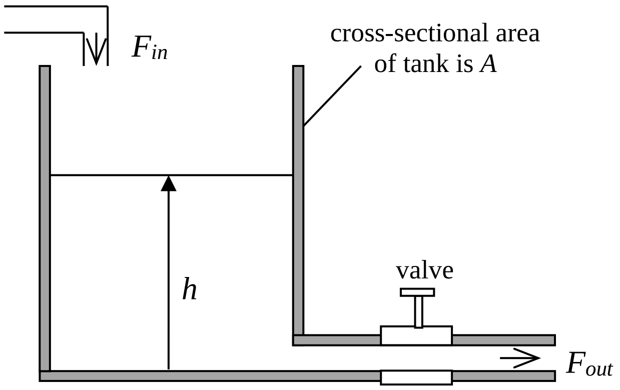
$$M\ddot{x} + B\dot{x} + (K_1 + K_2)x = u(t)$$

- Note, by way of assumptions in the previous examples, we assume that we are using ideal components (springs, dampers) and that there is no friction between the mass and the ground.

3.3.5 Modelling a flow system

- Consider the following single tank flow system:

Actual system



Physical model of a single tank

- A tap valve controls the outlet flow f_{out}.
- The tank itself has a cross-sectional area A.
- Here, we write down a ***flow balance equation***, i.e. the change of water volume in the tank is the difference between the input flow and the output flow:

Change in Water Volume V = Flow-in - Flow-out

$$\Rightarrow \frac{dV}{dt} = F_{in} - F_{out}$$

- Since volume = area x height, i.e. $V = Ah$, and A is a constant value, then:

$$\frac{dV}{dt} = A\frac{dh}{dt} = F_{in} - F_{out}$$

- The output flow rate F_{out} is determined by the pressure exerted by the water in the tank and, hence, we can state that the ***output flow is directly proportional to the height of the water*** (assuming laminar flow):

$$F_{out} \propto h \qquad \text{or} \qquad F_{out} = kh$$

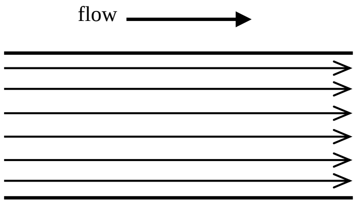
- Here k is a constant of proportionality that allows for several factors including the cross-sectional area of outflow pipe, frictional forces within the pipe, density of the liquid, etc. This forms part of our assumptions in the final model.
- Hence, our flow balance equation, and system model (relating height to input flow) is given by the first order differential equation:

$$A\frac{dh}{dt} = F_{in} - kh$$

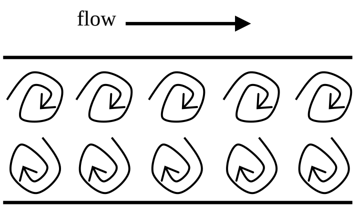
Aside note – laminar v turbulent flow:

- Water flow in a pipe can be laminar or turbulent.

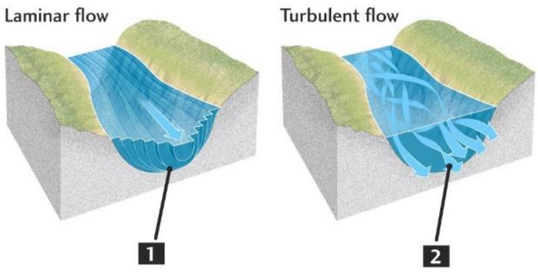
- Laminar refers to a uniform directional flow while turbulent, as the name suggests, refers to a more chaotic type flow, as indicated in the following sketches:

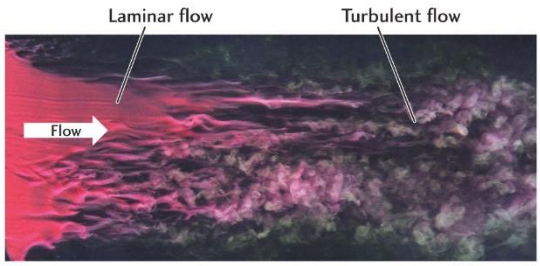


laminar flow



turbulent flow





- The relationship between the output flow and the height of water in the tank is also dependent on the type of flow involved, as follows:

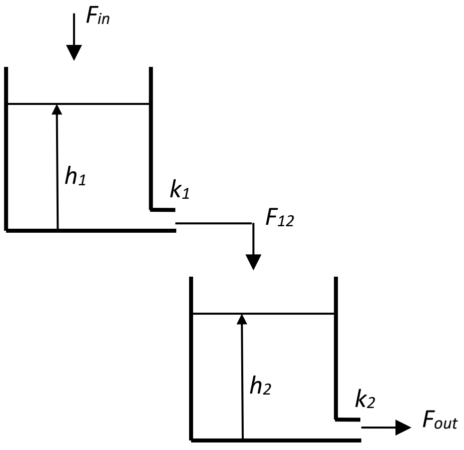
$F_{out} = kh$ for **laminar** flow (*this is a **linear** relationship*)

$F_{out} = k\sqrt{h}$ for **turbulent** flow (*this is a **nonlinear** relationship*)

- ***Ex 3.8 Determine a mathematical model for the dual non-interacting water tank system, whose physical model is shown:***

  Assume laminar flow conditions and the constant of proportionality between flow and height is k_1 and k_2 for the respective tanks, as indicated.

  Also, the cross-sectional area of each tank is A_1 and A_2 respectively.



  Solution

  From the previous section, we can easily write out the system model for each tank as follows:

  Tank 1: $$A_1 \frac{dh_1}{dt} = F_{in} - F_{12} \qquad \Rightarrow A_1 \frac{dh_1}{dt} = F_{in} - k_1 h_1$$

  Tank 2: $$A_2 \frac{dh_2}{dt} = F_{12} - F_{out} \qquad \Rightarrow A_2 \frac{dh_2}{dt} = k_1 h_1 - k_2 h_2$$

- Note – we will revisit these systems in the next section of the notes when we examine transfer function representation.

- We will also simulate, and hence analyse, these systems when we use Simulink and Matlab.

3.3.6 Review of modelling physical (dynamical) systems

- There is a common generalized approach to the modelling of physical systems.

- The first step is to select the fundamental variables whose values at any instance in time contain all the information about the system (we refer to these as state variables). Three such state variables include **mass**, **energy** and **momentum**.

- Most models are based on the **conservation** of these quantities.

- Often, the fundamental variables are not conveniently measured. Hence, we use characterising dependent variables instead, such as density, temperature, pressure, flow rate, etc.

- The values of all the characterising variables at any instance in time define the state of the system.