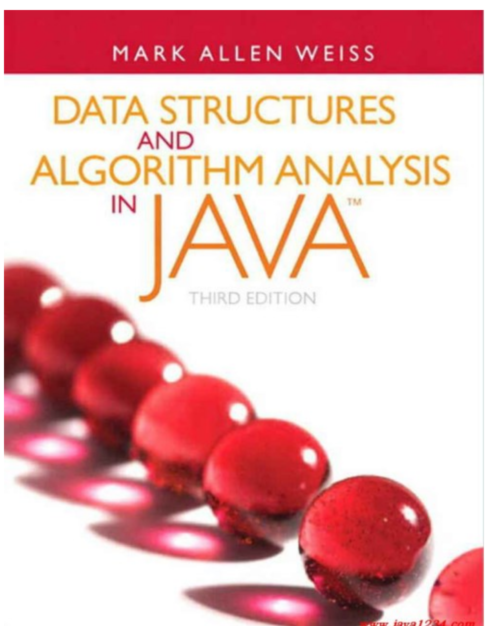
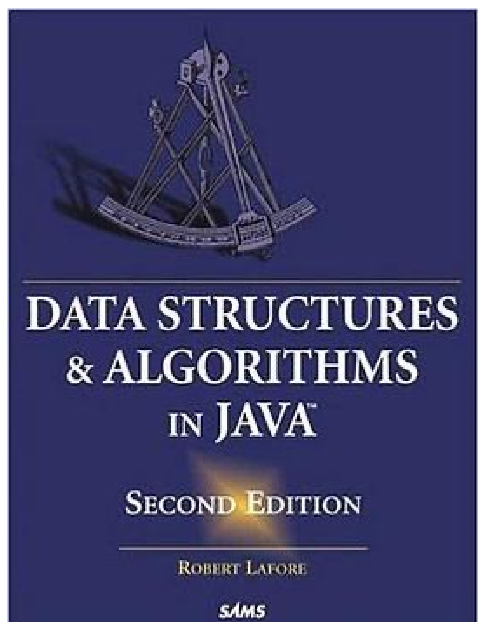
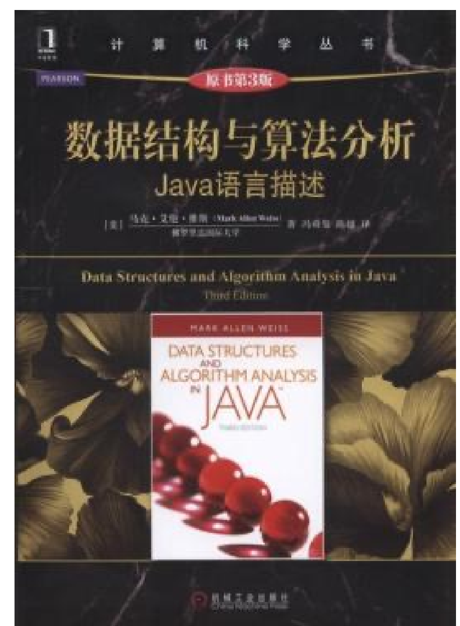
Topic 5 –
Big O Notation







Topics

- Introduction
- Programming Revision
- Methods and Objects
- Arrays and Array Algorithms
- **Big O Notation**
- Sorting Algorithms
- Stacks and Queues
- Linked Lists
- Recursion
- Bit Manipulation

Algorithm Efficiency

- Throughout the course we will seek to design efficient algorithms
- But how can we come up with a **universal standard** for algorithm efficiency?
- Imagine I compare my dice program with yours
- I run my program on my fast office computer
- You run your program on a slower lab machine

my computer

your lab machine

Running Times

- 5 Dice rolled 1 million times

	Running Time
My computer	40ms
Your computer	1000ms

- Which Dice algorithm is better?
  - Mine or yours?
- This comparison is unfair because the performances of the two computers are different

The metre

- Historically, (1889-1960) the metre was defined by the French Academy of Sciences
- It was the length between two marks on a platinum-iridium bar
- 1/10 millionth of the distance from the equator to the North Pole through Paris
- Every measurement was based on this bar

Algorithm Efficiency

- In the same way, we could try running all algorithms on the same computer
- But is this a good universal standard?
- We would need to have a **single benchmark machine** on which all of the world's programs were tested

386 25Mhz with 2MB RAM

- This machine would quickly become antiquated and need to be updated, invalidating the previous measurements

Running Times

	1 million rolls	2 million rolls	3 million rolls
My computer	40ms	160ms	360ms
Your computer	1000ms	2000ms	3000ms

- Which Dice algorithm is better?
  - Mine or yours?
- This comparison is **unfair** because the performances of the two computers are different

Standard measure

- The **relationship** between the increase in the size of a problem and the increase in the running time is platform independent (apart from a constant)
- No matter what platform you run it on, the same relationship will emerge
- We therefore use this relationship to define algorithm efficiency
- Knowing the relationship is very useful for **predicting** how long an algorithm will take to run on a particular problem

Big O Notation

- We use Big O Notation to describe this ratio
- We are not concerned with the actual time it takes to run the algorithm
  - 100 ms on a laptop
  - 10 ms on a supercomputer

- We want a way to describe the rate with which the running time of the algorithm increases compared to the rate at which the size of the problem (**n**) increases
- Big O is always concerned with **worst** case time requirement

Examples

- $O(n)$ – The rate at which the running time increases is proportional to the rate at which the size increases
  - Example. If the running time is $100n + 53$ or $2n - 1000$, then we say it is $O(n)$.
  - Note. We don't care about the constant.
- $O(n^2)$ – Running time increases proportional to the square of the size of the problem
  - Example. If the running time is $5n^2 - 99n + 1$, then we say it is $O(n^2)$
- $O(1)$ – Running time is not related to the size of the problem
  - The running time is a constant
- $O(\log n)$ – Time increases slowly at log the rate of the size
  - Example. If the running time is $10 \log n + 100$, then we say it is $O(\log n)$

Question

- We already know that
  - $O(100n + 53 \text{ or } 2n - 1000)$ ➔ $O(n)$
  - $O(5n^2 - 99n + 1)$ ➔ $O(n^2)$
  - $O(99)$ ➔ $O(1)$
  - $O(10 \log n + 100)$ ➔ $O(\log n)$

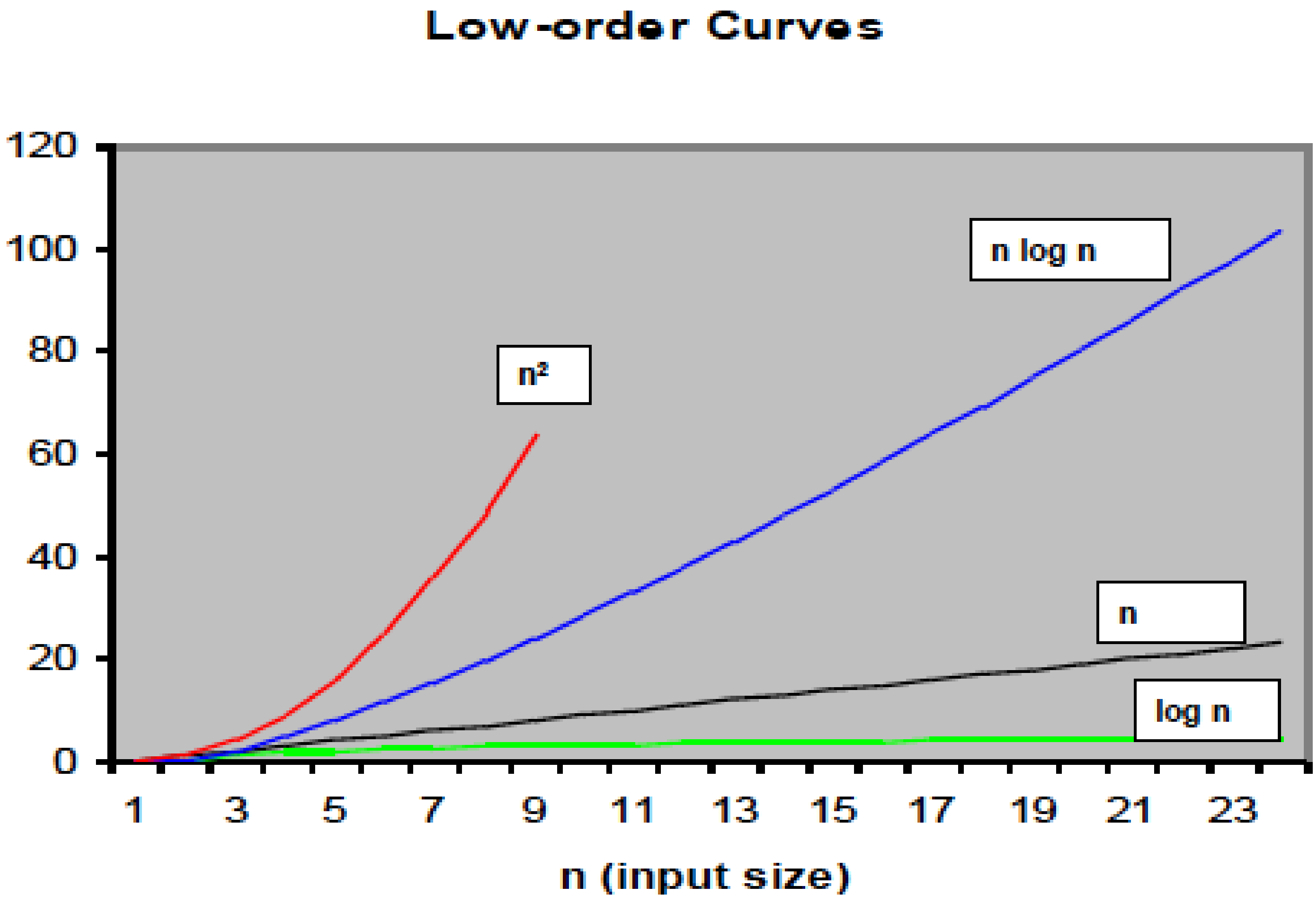
- So,
  - $O(1000 \log n + 3n^2 + 3n + 1)$ ➔ **?**

Question

- We already know that
  - $O(100n + 53 \text{ or } 2n - 1000)$ ➔ $O(n)$
  - $O(5n^2 - 99n + 1)$ ➔ $O(n^2)$
  - $O(99)$ ➔ $O(1)$
  - $O(10 \log n + 100)$ ➔ $O(\log n)$

- So,
  - $O(1000 \log n + 3n^2 + 3n + 1)$ ➔ $\boldsymbol{O(n^2)}$
  - Because $n^2 > \log n$ when n is **large enough**

Big O graph



Low-order Curves
120
100
80
60
40
20
0
1
3
5
7
9
11
13
15
17
19
21
23
n (input size)
n log n
n²
n
log n

Insertion in an Unordered Array

- For insertion into an unordered array running time doesn't depend on the size of the array – we just stick the element on at the end
- We can say that running time (**T**) = some constant time (**K**) which won't change ➔ **T = K**
- K can depend on factors such as the speed of the computer, the amount of RAM etc.
- We don't care what **K** actually is – Big O Notation is only concerned with describing the relationship between the running time and the size of the problem
- We just say the algorithm is $O(1)$ – running time is unaffected by ***n***

Linear Search

- We have to search through all the elements in an array
- On average, we'll have to check half of them
- So $T = K \times \frac{n}{2}$
- Because K is a constant, $\frac{K}{2}$ will still be a constant (value doesn't depend on n)
- So $T = K \times n$
- This algorithm is $O(n)$

Binary Search

- We have already shown that $\mathbf{iterations} = \boldsymbol{log_2}(\mathbf{size})$
- Therefore $T = K \times log_2(n)$
- As it happens $log_2(n) = \frac{log_{10}(n)}{log_{10}(2)}$
- Incorporating the above equation to *T*, we get
$$T = (1/\,log_{10}(2)) \times K \times log_{10}(n)$$
- $(1/\,log_{10}(2)) \times K$ is just a constant which is irrelevant to Big O Notation
- This algorithm is
$$\boldsymbol{O(log\,n)}$$

Operations in an Ordered Array

- Ordered arrays are handy because we can use binary search on them and this is $O(\log n)$
- However, if we want to insert or delete we have to make space / remove a space
- On average, we will have to move half of the items up or down ➔ $K \times \frac{n}{2}$
- Therefore, these operations are $\boldsymbol{O(n)}$

Running times in Big O Notation

Algorithm	**Running Time**
Linear Search	$O(n)$
Binary Search	$O(\log n)$
Insertion in unordered array	$O(1)$
Insertion in ordered array	$O(n)$
Deletion in unordered array	$O(n)$
Deletion in ordered array	$O(n)$

Expressing iterations in terms of $\boldsymbol{n}$

- Usually we can look at a piece of code and derive a function $f(n)$ which describes the number of loop steps in it
  - How many loop iterations in this code?
  - In other words, how many time will counter++ be run?

```
for (int i = 10; i < n; i++){
    for (int j = 10; j > 0; j--) {
        counter++;
    }
}
```

Expressing iterations in terms of n

```
for (int i = 10; i < n; i++){          // Run (n-10) times
    for (int j = 10; j > 0; j--) {     // Run 10 times
        counter++;
    }
}
```

- Analysis:

  i = 10: "for (int j = 10; j > 0; j--){counter++;}" runs 10 times

  i = 11: "for (int j = 10; j > 0; j--){counter++;}" runs 10 times

  …

  i = n-1: "for (int j = 10; j > 0; j--){counter++;}" runs 10 times

- There are (n - 10) * 10 iterations = 10n - 100
- Thus, its running times in Big O notation is $\boldsymbol{O(n)}$

Formalities

- Formal mathematical definition of Big O
- A function $f(n) = O(g(n))$ if
  - a positive real number c and positive integer n_0 **exist** such that

$$f(n) \leq c \times g(n) \text{ for all } n \geq n_0$$

- Example. $10n - 100 = O(n)$
  - We set $c = 20$ and $n_0 = 1$
  - We have $20n - (10n - 100) = 10n + 100 \geq 0$ for all $n \geq 1$
  - Thus, $10n - 100 \leq 20n$ for all $n \geq 1$

More examples

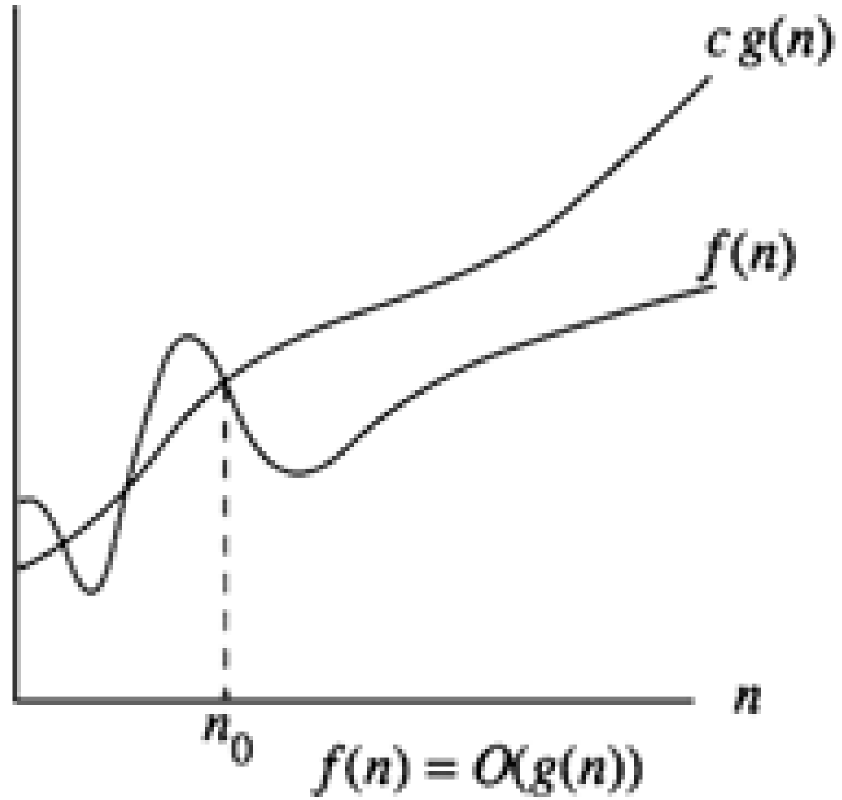
- Example. $9n^2 + 100n = O(n^2)$
  - We set $c = 10$ and $n_0 = 100$
  - We have $10n^2 - (9n^2 + 100n) = n^2 - 100n \geq 0$ for all $n \geq 100$
    - *Note.* $100^2 = 10000 \geq 100n = 10000$
  - Thus, $9n^2 + 1000n \leq 10n^2$ for all $n \geq 100$

- Example. $20n^3 - 100n^2 + 50n - 75 = O(n^3)$
  - We set $c = 30$ and $n_0 = 10$
  - We have $30n^3 - (20n^3 + 50n) = 10n^3 - 50n \geq 0$ for all $n \geq 10$, and $20n^3 + 50n \geq 20n^3 - 100n^2 + 50n - 75$
  - Thus, $20n^3 - 100n^2 + 50n - 75 \leq 30n^3$ for all $n \geq 10$

Graph

$$f(n) \leq c \times g(n) \text{ for all } n \geq n_0$$

- $c \times g(n)$ is the upper bound on $f(n)$ when n is **sufficiently large**



Interpretation

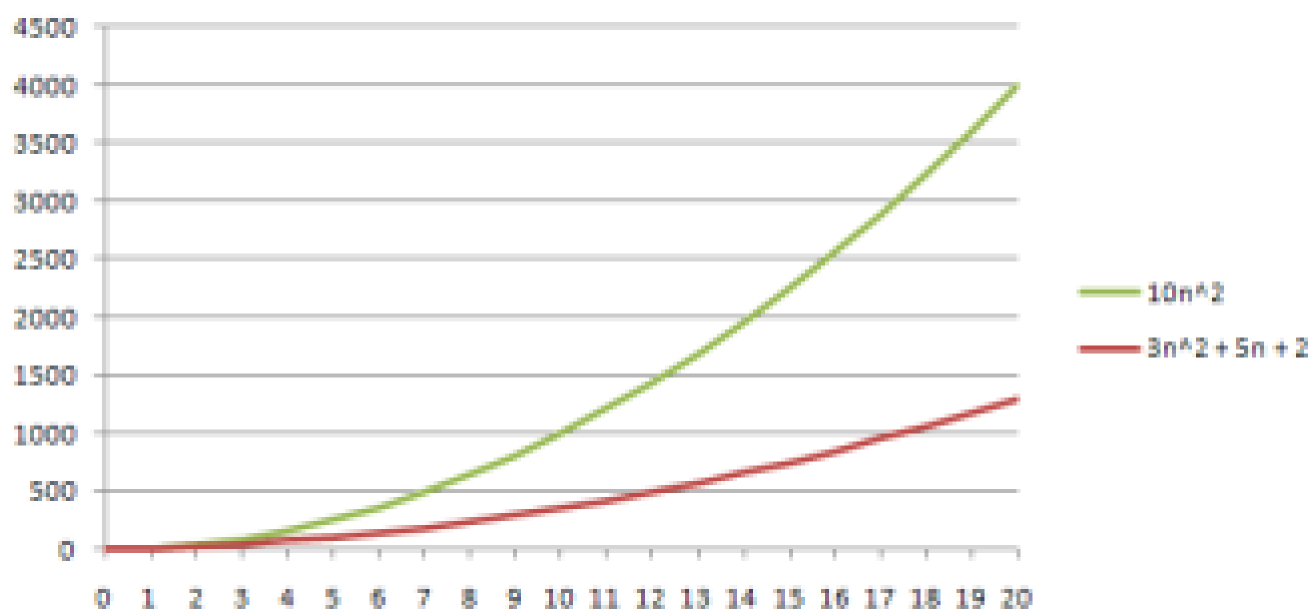
- We want to describe how the size of a function $f(n)$ (which describes the running time of a program) increases as n gets really huge
- The biggest power of n will always dominate
- Accordingly, we pick this as the Big O complexity $\boldsymbol{g(n)}$
- **We don’t care about constants**
- To justify that this pick is a good description of $f(n)$, we show that $f(n)$ is always bounded by the Big O complexity $g(n)$ (multiplied by some constant, which doesn’t matter as we don’t care about constants!) as long as n is bigger than some value n_0
- In other words, to show $g(n)$ provides a good description of $f(n)$ we show that $f(n) \leq c \times g(n)$ for all $n \geq n_0$

Interpretation

- For example $O(n^2)$ is a good description of $3n^2 + 5n + 2$ since n^2 multiplied by the arbitrary constant $\mathbf{10}$ will always be bigger than $3n^2 + 5n + 2$ for every value of n greater than 1
- $O(n^2)$ manages to capture the behavior of this function as n becomes bigger (with only a constant amount of inaccuracy)
- We don't care that $3n^2 + 5n + 2$ could be up to $\mathbf{10}$ times bigger than $O(n^2)$
- $\mathbf{10}$ is only a constant and in the long run as n gets huge, constants will become insignificant

Example

- The function $10n^2$ will always exceed $3n^2 + 5n + 2$, so long as n is 2 or greater
- Therefore $3n^2 + 5n + 2$ is $O(n^2)$ because...
  - $10n^2 = 3n^2 + 5n^2 + 2n^2$ which is $>$ than $3n^2 + 5n + 2$ when $n \geq 2$
  - $3n^2 = 3n^2$
  - $5n^2 > 5n$
  - $2n^2 > 2$



Explain?

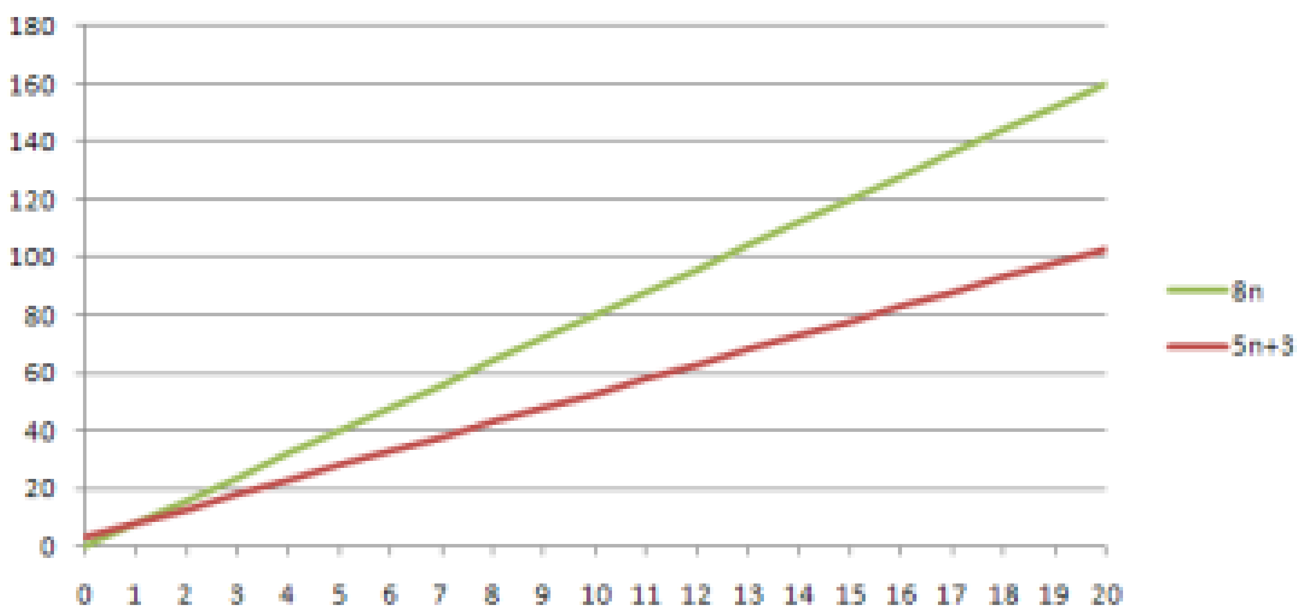
- I'm looking for the Big O function which is the closest description of the performance of my function (i.e. computer program)
- My function must be bounded by the Big O function beyond a certain problem size $\boldsymbol{n_0}$
- The Big O function can be multiplied by any constant in order to meet this requirement
- For example, if I describe my function as being $O(n)$, what I mean is that my function always has a running of less than $\boldsymbol{k \times n}$ when n is bigger than $\boldsymbol{n_0}$
    - k can be a million, a billion, a trillion, it doesn't matter
    - n_0 can be any value too, but it is usually more sensible to keep it low
    - Even though it is huge, 2100 is actually a constant because it has no $\boldsymbol{n}$ term

Example

- Show that $f(n) = 5n + 3 = O(n)$
  - **Find a $\boldsymbol{g(n)}$, $\boldsymbol{c}$ and $\boldsymbol{n_0}$ such that $\boldsymbol{f(n) \leq c \times g(n)}$ for all $\boldsymbol{n \geq n_0}$**
  - How about $g(n) = n, c = 8, n_0 = 1$?
  - $f(n) \leq 8n$ for every value of n greater than 1
  - $\mathbf{5}\boldsymbol{n} + \mathbf{3}$ is always less than $\mathbf{5}\boldsymbol{n} + \mathbf{3}\boldsymbol{n}$ when n is at least 1
  - Therefore, we can say $f(n)$ is $O(n)$
- Why don't we let $g(n) = n^2$?
- Although the conclusion is correct since $f(n)$ will always be less than $O(n^2)$ as well, this is not the closest description of the algorithm

Example

- $8n$ will always bound $5n + 3$ when n is bigger than 1
- Therefore, we can say that a program with $5n + 3$ steps is $O(n)$
- Of course, it would also be bounded by $6n$ but so long as we show it for any constant then that's sufficient



Usage

- Always use the **most parsimonious formula** for the O-notation.
- We write

$$3n^2 + 2n + 5 = O(n^2)$$

- The followings are all correct but we want the most concise
  - $3n^2 + 2n + 5 = O(3n^2 + 2n + 5)$
  - $3n^2 + 2n + 5 = O(n^2 + n)$
  - $3n^2 + 2n + 5 = O(100n^2)$

- Note. $3n^2 + 2n + 5 \neq O(1000n)$

Tip

- In order to figure out what the order of a function is, just look at the highest order of n
- If there's an $\boldsymbol{n^2}$ term, then the formula is $O(n^2)$
- Always put $g(n)$ equal to this power
  - $\boldsymbol{g(n) = n^2}$
- Now choose $\boldsymbol{c}$ so that it equals the sum of all the variables in the function
  - If $f(n) = 3n^2 + 2n + 5$, then choose $\boldsymbol{c}$ to be $10 = 3 + 2 + 5$
- This makes it easy to show that $3n^2 + 2n + 5 < 3n^2 + 2n^2 + 5n^2$
- Finally, figure out what value $\boldsymbol{n_0}$ needs to have in order to make the above statement $\boldsymbol{f(n) \leq c \times g(n)}$ true

Example of O-notation

- Show that $3n^2 + 2n + 5 = O(n^2)$
  - $\boldsymbol{g(n)} = n^2$, $\boldsymbol{c} = 10$, $\boldsymbol{n_0} = 1$
  - Pick $\boldsymbol{c} = 10$ because it's easy to show $10n^2 \geq 3n^2 + 2n + 5$
  - $10n^2 = 3n^2 + 2n^2 + 5n^2$
  - $10n^2 = 3n^2 + 2n^2 + 5n^2 \geq 3n^2 + 2n + 5$ for all $\boldsymbol{n_0} = 1$
    - Note.
      - $3n^2 = 3n^2$
      - $2n^2 \geq 2n$
      - $5n^2 \geq 5$

Formalities

- The following identities hold for Big O notation:

$$\boldsymbol{O\big(k \times f(n)\big) = O\big(f(n)\big)}$$

  - If an algorithm is doubled in complexity, it still has the same Big O Notation

$$\boldsymbol{O(f(n) + g(n)) = O\big(f(n)\big) + O\big(g(n)\big)}$$

  - If we run one algorithm after the other, the complexity is added
  - However, if algorithm 1 is $O(n^2)$ and algorithm 2 is $O(n)$ then $O(n^2 + n)$ can be more parsimoniously described as $O(n^2)$

$$\boldsymbol{O(f(n) \times g(n)) = O\big(f(n)\big) \times O\big(g(n)\big)}$$

  - If algorithm 1 is $O(n^2)$ and algorithm 2 is $O(n)$ and one algorithm is run inside the other as a loop then the Big O Notation is $O(n^3)$

Big-O Examples

$\mathbf{7n-2}$ is $O(n)$

- need $c > 0$ and $n_0 \geq 1$
- such that $7n - 2 \leq cn$ for $n \geq n_0$
- this is true for $c = 9$ and $n_0 = 1$

$\mathbf{3n^3 + 20n^2 + 5}$ is $O(n^3)$

- need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq cn^3$ for $n \geq n_0$
- this is true for $c = 28$ and $n_0 = 1$

$\mathbf{3\log n + 5}$ is $O(\log n)$

- need $c > 0$ and $n_0 \geq 1$ such that $3\log n + 5 \leq c\,log\,n$ for $n \geq n_0$
- this is true for $c = 8$ and $n_0 = 2$
    - Note that $\log 2 = log_2 2$ = 1 so n_0 has to be 2 before $\log n$ exceeds 1

Keeping it simple

- $f(n) = 10n + 25n^2$ is $O(n^2)$
- $f(n) = 20n\log n + 5n$ is $O(n\log n)$
- $f(n) = 12n\log n + 0.05n^2$ is $O(n^2)$
- $f(n) = n^{\frac{1}{2}} + 3n\log n$ is $O(n\log n)$

- Note. for $k \geq 2$,

$$O(n^{k+1}) \geq O(n^k) \geq O(n\log n) \geq O(n) \geq O(n^{\frac{1}{2}}) \geq O(\log n) \geq O(1)$$

Getting Big O of a program

- When trying to determine the Big O Notation of a computer program, look at the loop structure
- Statements that are run the same number of times regardless of the size of the problem are just **constants**
- All you're interested in is how increasing the size of $\boldsymbol{n}$ increases the number of iterations of the loops
- Increasing the size of $\boldsymbol{n}$ will only have an effect is there a loop structure which depends on $\boldsymbol{n}$
  - A single loop running n times indicates $O(n)$
  - A nested loop each running n times indicates $O(n^2)$

Picturing Efficiency

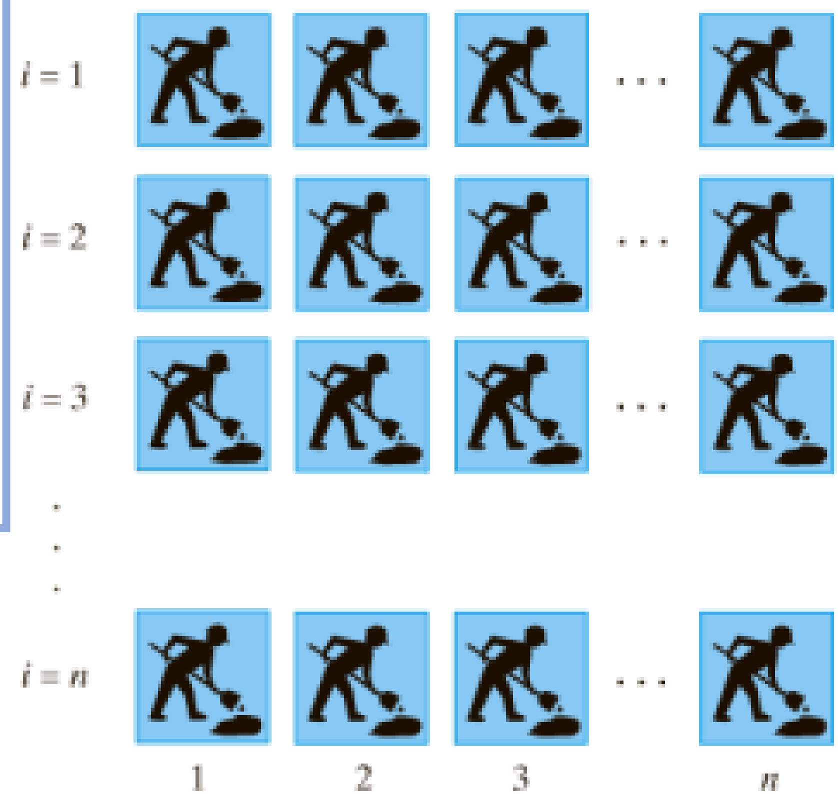
- Consider this algorithm:

```
for( int i = 1; i <= n; i++) {
    sum = sum + i;
}
```

- The work done by the body of the loop (i.e. sum = sum + i) requires a constant amount of time $O(1)$
- This body is executed n times
- Therefore, the algorithm is $O(n)$

Picturing Efficiency

```
for( int i = 1; i <= n; i++) {
    for( int j = 1; j <= n; j++) {
        sum sum + i;
    }
}
```



- n steps of work are repeated n times
- An $O(n^2)$ algorithm

Shaking hands at a party

- If there are n people at the party, we will need to shake $n-1$ hands
- The next person will have to shake $n-2$ hands (they don't have to shake your hand again)
- The last person has to shake 0 hands because everybody has already shaken his hand
- Total number of handshakes is

$$(n-1)+(n-2)+\cdots+0=\frac{n\times(n-1)}{2}=\frac{1}{2}n^2-\frac{1}{2}n$$

- Using our usual methodology we can show that this is $O(n^2)$

Compute average of array

```
double average(int[] array) {
    double sum = 0;
    int n = array.length;
    for (int i = 0; i < n; i++) {
        sum += array[i];
    }
    return sum / n;
}
```

- One loop running n times $= O(n)$

Nested Loops

```
double sum = 0;
for(int i = 0; i < n; i++) {
    for(int j = 0; j < n; j++) {
        sum += 5;
    }
}
```

- Nested for loops each running n times $= O(n^2)$ steps

Loop running constant number of times

- Suppose that your implementation of a particular algorithm appears in Java as follows:

```
for (int pass = 1; pass <= n; pass++) {
    for (int index = 0; index < n; index++) {
        for (int count = 1; count <= 10; count++) {
            . . .
        } // end for
    } // end for
} // end for
```

- Two loops running to n, third loop runs a **constant** number of times is $O(n^2)$
  - It is $O(n \times n \times 10) = O(10n^2) = O(n^2)$

How about this loop?

```
for(int i = 0; i < 10; i++) {
    for(int j = 0; j < 20; j++) {
        counter++;
    }
}
```

- Analyze the complexity of the above algorithm
- Notice that the loops do not depend on the size of n
- No matter what size n is, the loops will run the same number of times
- Therefore, the running time will always be the same
- The order of the above algorithm is $O(1)$

Exam Question

- A function involves the following number of steps where n is the size of the problem:

$$f(n) = \log n + \frac{n}{2} + 5$$

- State the Big-O complexity of the function and prove that this is the case using the mathematical definition.

Exam Question

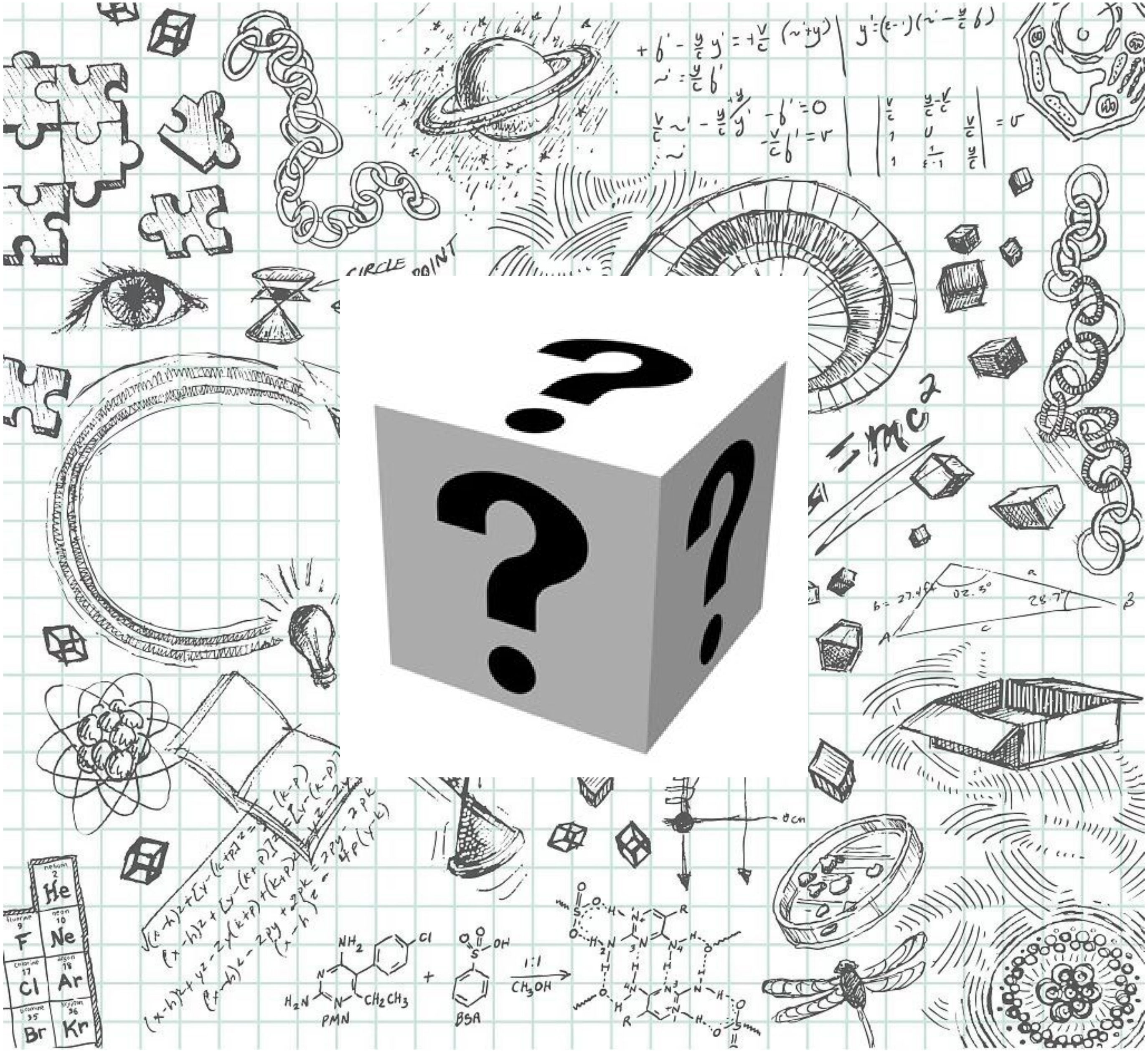
$$f(n) = \log n + \frac{n}{2} + 5$$

- We set $g(n) = n$ since n is the biggest term
- Let $c = 7$ since there are 7 units in the function
- We must show $c \times g(n) \geq f(n)$ above some threshold n_0
- Thus, $7n = n + n + 5n \geq \log n + \frac{n}{2} + 5$ as long as $n \geq 1$
- That is $f(n)$ is $O(n)$

Timing Programs

- You can check how long your program has been running
- There is a System method that allows us to store the current value of the system clock
- By comparing two different system clock values we can figure out how long the program has been running

```
long start = System.currentTimeMillis();
long elapsed = System.currentTimeMillis() - start;
```


CIRCLE
POINT
He
F
Ne
Cl
Ar
Br
Kr
PMN
BSA
1:1
CH3OH