CS422FZ (CS323FZ) Final Paper 21-22

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If you have any problem, please feel free to contact with us: MIEC CLUB

Best Wishes! Last update in 2022/12/29.

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1 Q1 Comprehensive Questions

1.1 (a) Homogenous Transformation

(a) Find the homogeneous transformation H matrix corresponding to the following transformation:

- First rotate by $\pi/2$ about the fixed y-axis. Call the new frame 1
- Then translate 7 units along the current y-axis. Call the new frame 2
- Then rotate by $\pi/4$ about the current z-axis. Call the new frame 3
- Finally translate 5 units along the fixed z-axis. Call this new frame 4

Please first evaluate the expression symbolically (i.e., in terms of variables) and then evaluate the resulting expression by substituting the values of the variables.

[5 marks]

$$H_1 = Trans_{z,l_2} Rot_{y,\theta} Trans_{y,l_1} Rot_{z,\alpha}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

$$H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 7 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

1.2 (b) Standard DH

If joint J_{i+1} is positioned at joint angle $\pi/4$, and link L_i has link length 110 units, and link twist $\pi/2$, what is A_i (Standard DH).

α	a	d	θ
$\pi/2$	110	0	$\pi/4$

$$A_i = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 55\sqrt{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 55\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{15}$$

首先必须要明白他说的这几个参数到底是什么，相关知识点：

a_i **Link Length**	the distance perpendicular to z_i and z_{i-1}, measured along x_i
α_i **Link Twist**	the angle between z_{i-1} and z_i, measured in the plane normal to x_i (right-hand rule around x_i)
d_i **Link Offset**	the distance along z_{i-1} from o_{i-1} to the intersection with x_i
θ_i **Joint Angle**	the angle between x_{i-1} and x_i, measured in the plane normal to z_{i-1} (right-hand rule around z_{i-1})

1.3 (c) Rotation Matrix

(c) Suppose that three coordinate frames $o_1x_1y_1z_1$, $o_2x_2y_2z_2$ and $o_3x_3y_3z_3$ are given, and suppose

$$R_2^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \qquad R_3^1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Find the rotation matrix R_3^2.

$$R_3^2 = R_1^2 R_3^1 = \begin{bmatrix} 0 & 0 & -1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{bmatrix} \tag{16}$$

相关知识点：旋转矩阵为正交矩阵，因此转制==取逆

- $R_1^0 = (R_0^1)^{-1} = (R_0^1)^T$.

1.4 (d) Multiplication of Homogeneous

(d) In general, multiplication of homogeneous transformation matrices is not commutative. Consider the matrix product

$$\begin{aligned} H &= \mathrm{Rot}_{z,\theta}\mathrm{Trans}_{z,d}\mathrm{Trans}_{x,a}\mathrm{Rot}_{x,\alpha} \\ &= \begin{bmatrix} c_\theta & -s_\theta c_\alpha & s_\theta s_\alpha & a c_\theta \\ s_\theta & c_\theta c_\alpha & -c_\theta s_\alpha & a s_\theta \\ 0 & s_\alpha & c_\alpha & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Determine which pairs of the four matrices on the right-hand side commute. Explain why these pairs commute. Find all permutations of these four matrices that yield the same homogeneous transformation matrix H.

本题需要找到另外4种齐次变换的表示方法（算上given，一共5种）：

$$
\begin{aligned}
H &= Rot_{z,\theta} Trans_{z,d} Trans_{x,a} Rot_{x,\alpha} \\
&= Trans_{z,d} Rot_{z,\theta} Trans_{x,a} Rot_{x,\alpha} \\
&= Rot_{z,\theta} Trans_{z,d} Rot_{x,\alpha} Trans_{x,a} \\
&= Rot_{z,\theta} Trans_{x,a} Trans_{z,d} Rot_{x,\alpha} \\
&= Trans_{z,d} Rot_{z,\theta} Rot_{x,\alpha} Trans_{x,a}
\end{aligned} \tag{17}
$$

Explain why:

Dr. Zhan: 说明$Trans_{z,d}$和$Rot_{z,\theta}$可以互换位置，结果不变即可。

代数证明方法，分别列写出 $Rot_{z,\theta} Trans_{z,d} Trans_{x,a} Rot_{x,\alpha}$ 四项的表达式，然后证明两两交换的结果不变，可以参考本文与具体运算过程；还有另一种解法可参考这里。

1.5 (e) Artificial potential field (Section 7)

(e) Given the artificial potential field

$$U(q) = U_{att}(q) + U_{rep}(q)$$

where

$$
\begin{aligned}
U_{att}(q) &= \frac{1}{2}\zeta\|q - q_{final}\|^2 \\
U_{rep}(q) &= \begin{cases} \frac{1}{2}\eta\left(\frac{1}{\rho(q)} - \frac{1}{\rho_0}\right)^2, & \text{if } \rho(q) \le \rho_0 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

(1) calculate the virtual force from the artificial potential field defined above; (2) explain the principle of constructing artificial potential field and comment on the following terms:

- U_{att}:
- U_{rep}:
- $\rho(q)$:
- ρ_0:

1.5.1 (e-1) Calculation

$$
U_{att}(q) = \frac{1}{2}\zeta\|q - q_{\text{final}}\|^2
$$

$$
U_{rep}(q) = \begin{cases} \frac{1}{2}\eta\left(\frac{1}{\rho(q)} - \frac{1}{\rho_0}\right)^2, & \text{if } \rho(q) \le \rho_0 \\ 0, & \text{otherwise} \end{cases}
$$

$$
\begin{gathered}
F_{\text{att}}(q) = -\nabla U_{\text{att}}(q) = -\zeta\left(q - q_{\text{final}}\right) \\
F_{\text{rep}}(q) = -\nabla U_{\text{rep}}(q) = \begin{cases} \eta\left(\frac{1}{\rho(q)} - \frac{1}{\rho_0}\right)\frac{1}{\rho^2(q)}\nabla\rho(q) & : \quad \rho(q) \le \rho_0 \\ 0 & : \quad \rho(q) > \rho_0 \end{cases}
\end{gathered} \tag{18}
$$

$$
\text{Hence,} \quad F(q) = -\nabla U_{\text{att}}(q) - \nabla U_{\text{rep}}(q)
$$

$$
F(q) = \begin{cases} -\zeta\left(q - q_{\text{final}}\right) - \eta\left(\frac{1}{\rho(q)} - \frac{1}{\rho_0}\right)\frac{1}{\rho^2(q)}\nabla\rho(q) & : \quad \rho(q) \le \rho_0 \\ -\zeta\left(q - q_{\text{final}}\right) - 0 & : \quad \rho(q) > \rho_0 \end{cases}
$$

相关知识点:

$$U_{\text{att}}(q) = \begin{cases} \frac{1}{2}\zeta\rho_f^2(q) & : \rho_f(q) \le d \\ d\zeta\rho_f(q) - \frac{1}{2}\zeta d^2 & : \rho_f(q) > d \end{cases}$$

$\rho_f(q) = ||q - q_{\text{final}}||$ $\quad$ d is a distance to be designed (to scale down $U_{att}(q)$)

ζ is a parameter used to scale the effects of the attractive potential.

$$F_{\text{att}}(q) = -\nabla U_{\text{att}}(q) = \begin{cases} -\zeta(q - q_{\text{final}}) & : \rho_f(q) \le d \\ -\dfrac{d\zeta(q - q_{\text{final}})}{\rho_f(q)} & : \rho_f(q) > d \end{cases} \qquad (5.6)$$

$$U_{\text{rep}}(q) = \begin{cases} \frac{1}{2}\eta\left(\frac{1}{\rho(q)} - \frac{1}{\rho_0}\right)^2 & : \rho(q) \le \rho_0 \\ 0 & : \rho(q) > \rho_0 \end{cases}$$

$\rho(q)$ is the shortest distance from q to a configuration space obstacle boundary,
η is a scalar gain coefficient that determines the influence of the repulsive field
ρ_0 is a distance of influence of a particle

$$F_{\text{rep}}(q) = \begin{cases} \eta\left(\frac{1}{\rho(q)} - \frac{1}{\rho_0}\right)\frac{1}{\rho^2(q)}\nabla\rho(q) & : \rho(q) \le \rho_0 \\ 0 & : \rho(q) > \rho_0 \end{cases}$$

1.5.2 (e-2) Explanation

Here we introduce one search method: artificial potential field.

- Robot is treated as a point particle in the configuration space, under the influence of an artificial potential field U
- The field U is constructed so that the robot is attracted to the final configuration, $\boldsymbol{q}_{final}$, while being repelled from the boundaries of $\boldsymbol{QO}$

$$U(q) = U_{att}(q) + U_{rep}(q)$$

$U_{att}(q)$ is the attractive field, $U_{rep}(q)$ is the repulsive field

$\rho(q)$ is the shortest distance from q to a configuration space obstacle boundary,
η is a scalar gain coefficient that determines the influence of the repulsive field
ρ_0 is a distance of influence of a particle

1.6 (f) Velocity Kinematics - Jacobian (Section 6)

(f) Find the Jacobian of

$$X = \begin{bmatrix} x_1(\theta_1^*, \theta_2^*, d_3^*) \\ x_2(\theta_1^*, \theta_2^*, d_3^*) \\ x_3(\theta_1^*, \theta_2^*, d_3^*) \end{bmatrix} = \begin{bmatrix} d_3^* \cos\theta_1^* \sin\theta_2^* - d_2 \sin\theta_1^* \\ d_3^* \sin\theta_1^* \sin\theta_2^* + d_2 \cos\theta_1^* \\ d_3^* \cos\theta_2^* \end{bmatrix}$$

and identify the singular positions. Here θ_1^*, θ_2^* and d_3^* are joint variables.

[15 marks]

$$J = \begin{bmatrix} -d3s1s2 - d2c1 & d3c1c2 & c1s2 \\ d3c1s2 - d2s1 & d3s1c2 & s1s2 \\ 0 & -d3s2 & c2 \end{bmatrix} \qquad (19)$$

$$\det(J) = 0 \Rightarrow {d_3}^2 \sin\theta_2 = 0$$

$$d_3 = 0 \quad or \quad \sin\theta_2 = 0 \Rightarrow \theta_2 = 0, \theta_2 = \pi \tag{20}$$

相关知识:

- Define generalised position kinematics $X := [x_1, x_2, \ldots, x_m]^T$ and generalised joint variable $q := [q_1, q_2, \ldots, q_n]^T$, where $q_i = \theta_i$ for revolute joints and $q_i = d_i$ for prismatic joints. The forward kinematics is described by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} f_1(q_1, q_2, \ldots, q_n) \\ f_2(q_1, q_2, \ldots, q_n) \\ \vdots \\ f_m(q_1, q_2, \ldots, q_n) \end{bmatrix} \tag{15}$$

The expression of Jacobian is obtained by taking the derivative of both sides of Eq. (15)

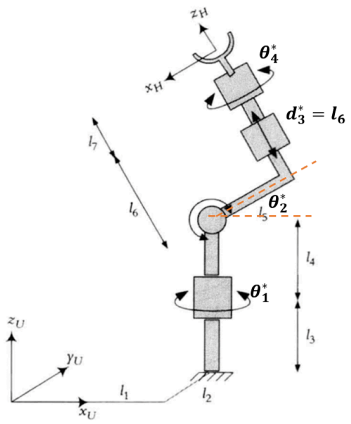
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_m \end{bmatrix} = J(q) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}, \qquad J(q) = \begin{bmatrix} \frac{f_1}{q_1} & \cdots & \frac{f_1}{q_n} \\ \vdots & \ddots & \vdots \\ \frac{f_m}{q_1} & \cdots & \frac{f_m}{q_n} \end{bmatrix}$$

Here $J(q)$ is called the **Jacobian matrix**.

- Singularities exist when Jacobian matrix loses rank. For square Jacobian matrix, singularities can be found by setting $\det(J) = 0$

2 Q2 Forward Kinematics

For the given specialty-designed 4-DOF robot

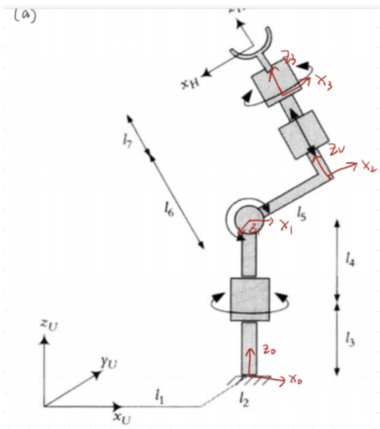


2.1 (a) Frames

(a) Assign appropriate frames (Frame 0, Frame 1, Frame 2, Frame 3) on the above diagram for the Denavit-Hartenberg representation. Here Frame H denotes the ground frame and Frame U denotes the end-effector frame. (You may annotate the positive directions of variables θ_1^*, θ_2^*, d_3^*, θ_4^*)

[8 marks]

首先我们需要建立一个标注DH坐标系，这里采用经典建系方法：



2.2 (b) HT Matrix

(b) Find the homogeneous transformation matrix from Frame U to Frame 0.

$$T_U^0 =$$

这道题是存在争议的，from Frame U to Frame 0 应该指的是T_0^U

正确的做法是：

$$T_0^U = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$T_U^0 = (T_0^U)^{-1} = \begin{bmatrix} 1 & 0 & 0 & -l_1 \\ 0 & 1 & 0 & -l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{21}$$

- Inverse Homogeneous Transformation

$$H_0^1 = H_1^{0^{-1}} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix} \tag{7}$$

2021年改卷时，写T_U^0或T_0^U，都给分。

2.3 (c) DH Table

对应的DH Table参数是:

#	θ	**d**	**a**	α
0-1	θ_1^*	$l_3 + l_4$	0	90
1-2	θ_2^*	0	l_5	-90
2-3	0	d_3^*	0	0
3-H(4)	(θ_4^*+180)	l_7	0	0

2.4 (d) Homogeneous transformation A

(d) Calculate each of the A matrices symbolically:
$A_1^0 =$
$A_2^1 =$
$A_3^2 =$
$A_H^3 =$

$$A_1^0 = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & 0 \\ s\theta_1 & 0 & -c\theta_1 & 0 \\ 0 & 1 & 0 & l_3 + l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$A_2^1 = \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 & l_5 c\theta_2 \\ s\theta_2 & 0 & c\theta_2 & l_5 s\theta_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$A_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$
$$A_H^3 = \begin{bmatrix} -c\theta_4 & s\theta_4 & 0 & 0 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 1 & l_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.5 (e) T & A

(e) Write an equation in terms of T_0^U and A-matrices that shows how T_H^U can be calculated (You do not need to perform the matrix multiplications)

[3 marks]

$$
\begin{aligned}
&T_H^0 = A_1^0 A_2^1 A_3^2 A_H^3 \\
&T_H^U = T_0^U T_H^0 = T_0^U A_1^0 A_2^1 A_3^2 A_H^3
\end{aligned}
\tag{23}
$$

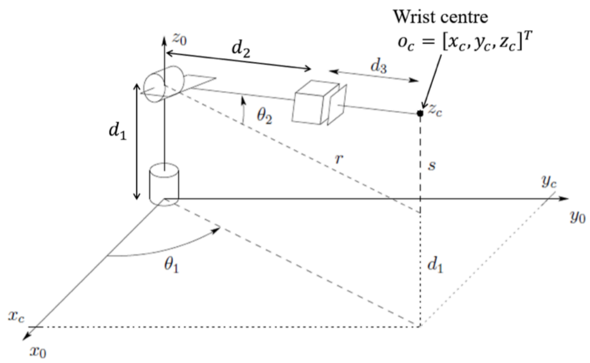
这里再强调一下，对于Homogeneous transformation, from U to H是: T_H^U

对于Rotation Matrix, from 1 to 2是：A_2^1 , from U to H是：A_H^U

3 Q3 Inverse Kinematics

Question 3 [15 marks]

For the spherical manipulator



Given the position of wrist centre $o_c = \begin{bmatrix} x_c & y_c & z_c \end{bmatrix}^T$, Solve the inverse kinematics problem for θ_1^*, θ_2^* and d_3^*. (To remove ambiguity, you should express θ_1^* and θ_2^* using inverse Trigonometric function atan2(x, y))

$\theta_1^* =$
$\theta_2^* =$
$d_3^* =$

$$
\begin{aligned}
&r = \sqrt{x_c^2 + y_c^2}, \quad s = z_c - d_1 \\
&\tan\theta_1 = \frac{y_c}{x_c}, \quad \tan\theta_2 = \frac{s}{r} = \frac{z_c - d_1}{\sqrt{x_c^2 + y_c^2}}, \quad \cos\theta_2 = \frac{r}{d_2 + d_3^*} \\
\\
&\theta_1 = a\tan 2(x_c, y_c) \quad or \quad \pi + a\tan 2(x_c, y_c) \\
&\theta_2 = a\tan 2(\sqrt{x_c^2 + y_c^2}, z_c - d_1) \\
&d_3^* = \frac{r}{\cos\theta_2} - d_2 = \sqrt{s^2 + r^2} - d_2
\end{aligned}
\tag{24}
$$

21年给分情况：给出一种给12分，2种14，3种15

本题可参考Slides，复习时建议把Spherical manipulator (Page 96, Fig. 3.21), Example 3.9 Elbow Manipulator, Example 3.10 SCARA manipulator都过一遍！

最后的最后，祝你好运！——Lance于2022/12/29.