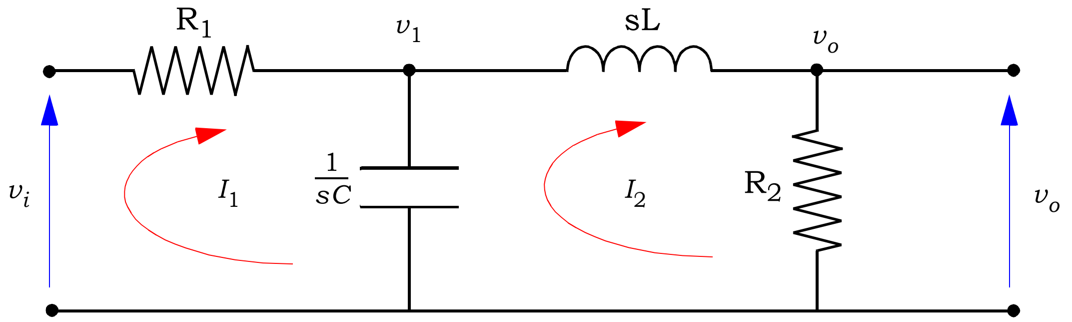
Tutorial Sheet 3 - Solutions

Q1



Circuit A

(i) **Mesh analysis** ...

$$I_1R_1 + (I_1 - I_2)\frac{1}{sC} = v_i \text{ and } (I_2 - I_1)\frac{1}{sC} + I_2 sL + I_2 R_2 = 0 \tag{1}$$

Converting these back to the time domain we get:

$$CR_1\dot{I}_1 + I_1 - I_2 = C\dot{v}_i \text{ and } I_2 - I_1 + CL\ddot{I}_2 + CR_2\dot{I}_2 = 0 \tag{2}$$

Expressing the equations in terms of the highest derivatives we get:

$$\dot{I}_1 - \frac{1}{R_1}\dot{v}_i = -\frac{1}{CR_1}I_1 + \frac{1}{CR_1}I_2 \text{ and } \ddot{I}_2 = -\frac{R_2}{L}\dot{I}_2 - \frac{1}{CL}I_2 + \frac{1}{CL}I_1 \tag{3}$$

Note that because the derivative of the input v_i appears in the first equation we bring the highest derivative of v_i to the LHS as well.

To get a state space model which does not involve the derivative of v_i we have to define our current state as:

$$x_1 = I_1 - \frac{1}{R_1}v_i,\ x_2 = I_2 \text{ and } x_3 = \dot{I}_2 \tag{4}$$

Therefore the state equations are:

$$\dot{x}_1 = \dot{I}_1 - \frac{1}{R_1}\dot{v}_i = -\frac{1}{CR_1}I_1 + \frac{1}{CR_1}I_2 = -\frac{1}{CR_1}\left(x_1 + \frac{1}{R_1}v_i\right) + \frac{1}{CR_1}x_2 \tag{5}$$

$$\dot{x}_2 = \dot{I}_2 = x_3 \tag{6}$$

$$\dot{x}_3 = \ddot{I}_2 = -\frac{R_2}{L}\dot{I}_2 - \frac{1}{CL}I_2 + \frac{1}{CL}I_1 = -\frac{R_2}{L}x_3 - \frac{1}{CL}x_2 + \frac{1}{CL}\left(x_1 + \frac{1}{R_1}v_i\right) \tag{7}$$

Note that from our definition of the states we have that $I_1 = x_1 + \frac{1}{R_1}v_i$.

$$\dot{x}_1 = -\frac{1}{CR_1}x_1 + \frac{1}{CR_1}x_2 - \frac{1}{CR_1^2}v_i \tag{8}$$

$$\dot{x}_2 = x_3 \tag{9}$$

$$\dot{x}_3 = \frac{1}{CL}x_1 - \frac{1}{CL}x_2 - \frac{R_2}{L}x_3 + \frac{1}{CLR_1}v_i \tag{10}$$

or in matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{CR_1} & \frac{1}{CR_1} & 0 \\ 0 & 0 & 1 \\ \frac{1}{CL} & -\frac{1}{CL} & -\frac{R_2}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -\frac{1}{CR_1^2} \\ 0 \\ \frac{1}{CLR_1} \end{bmatrix} v_i \tag{11}$$

Since the output, v_0 is the voltage across R_2 we can write the output equation as:

$$v_0 = I_2R_2 = x_2R_2 \tag{12}$$

hence the output equation in matrix form is:

$$v_0 = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{13}$$

(ii) Using **nodal analysis** with nodes v_1 and v_o:

$$\frac{v_1 - v_i}{R_1} + \frac{v_1}{\frac{1}{sC}} + \frac{v_1 - v_o}{sL} = 0 \text{ and } \frac{v_o - v_1}{sL} + \frac{v_o}{R_2} = 0 \tag{14}$$

$$Lsv_1 - Lsv_i + s^2LCR_1v_1 + R_1v_1 - R_1v_o = 0 \quad \text{and } R_2v_o - R_2v_1 + sLv_o = 0 \tag{15}$$

Converting these back to the time domain we get:

$$L\dot{v}_1 - L\dot{v}_i + LCR_1\ddot{v}_1 + R_1v_1 - R_1v_o = 0 \text{ and } R_2v_o - R_2v_1 + L\dot{v}_o = 0 \tag{16}$$

Expressing the equations in terms of the highest derivatives we get:

$$\ddot{v}_1 - \frac{1}{CR_1}\dot{v}_i = -\frac{1}{CR_1}\dot{v}_1 - \frac{1}{LC}v_1 + \frac{1}{LC}v_o \text{ and } \dot{v}_o = -\frac{R_2}{L}v_o + \frac{R_2}{L}v_1 \tag{17}$$

Note that because the derivative of the input v_i appears in the first equation we bring the highest derivative of v_i to the LHS as well.

To get a state space model which does not involve the derivative of v_i we have to define our node voltage state as:

$$x_1 = v_1\text{, } x_2 = v_o \text{ and } x_3 = \dot{v}_1 - \frac{1}{CR_1}v_i \tag{18}$$

Therefore the state equations are:

$$\dot{x}_1 = \dot{v}_1 = x_3 + \frac{1}{CR_1}v_i \tag{19}$$

$$\dot{x}_2 = \dot{v}_o = -\frac{R_2}{L}v_o + \frac{R_2}{L}v_1 = -\frac{R_2}{L}x_2 + \frac{R_2}{L}x_1 \tag{20}$$

$$\dot{x}_3 = \ddot{v}_1 - \frac{1}{CR_1}\dot{v}_i = -\frac{1}{CR_1}\dot{v}_1 - \frac{1}{LC}v_1 + \frac{1}{LC}v_o = -\frac{1}{CR_1}\left(x_3 + \frac{1}{CR_1}v_i\right) - \frac{1}{LC}x_1 + \frac{1}{LC}x_2 \tag{21}$$

Note that from our definition of the states we have that $v_1 = x_3 + \frac{1}{CR_1}v_i$.

$$\dot{x}_1 = x_3 + \frac{1}{CR_1}v_i \tag{22}$$

$$\dot{x}_2 = -\frac{R_2}{L}x_2 + \frac{R_2}{L}x_1 \tag{23}$$

$$\dot{x}_3 = -\frac{1}{LC}x_1 + \frac{1}{LC}x_2 - \frac{1}{CR_1}x_3 - \frac{1}{(CR_1)^2}v_i \tag{24}$$

or in matrix form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{R_2}{L} & -\frac{R_2}{L} & 0 \\ -\frac{1}{CL} & \frac{1}{CL} & -\frac{1}{CR_1}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}\frac{1}{CR_1}\\ 0\\ -\frac{1}{(CR_1)^2}\end{bmatrix}v_i \tag{25}$$

Since the output, $x_2 = v_o$ we can write the output equation in matrix form as:

$$v_0 = \begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \tag{26}$$

(iii) From Q2, Tutorial 2:

$$\frac{v_o}{v_i} = \frac{R_2}{R_1 CLs^2 + (CR_1R_2 + L)s + (R_1 + R_2)} \tag{27}$$

Notice that the highest power of s in the denominator is 2. Therefore only 2 states are needed to describe the system (i.e the minimal state realisation).

To derive the minimal state-space model from this transfer function we begin by making the coefficient of the highest power of *s* on the denominator equal to 1, i.e.:

$$\frac{v_o}{v_i} = \frac{\left[\frac{R_2}{R_1 CL}\right]}{s^2 + \left[\frac{CR_1R_2 + L}{R_1 CL}\right]s + \left[\frac{(R_1 + R_2)}{(R_1 CL)}\right]} \tag{28}$$

Converting this into a differential equation gives:

$$\ddot{v}_o = -\left[\frac{CR_1R_2 + L}{R_1CL}\right]\dot{v}_o - \left[\frac{(R_1 + R_2)}{(R_1CL)}\right]v_o + \left[\frac{R_2}{R_1CL}\right]v_i \tag{29}$$

Defining the states as $x_1 = v_o$ and $x_2 = \dot{v}_o$ the state equations are:

$$\dot{x}_1 = \dot{v}_o = x_2 \tag{30}$$

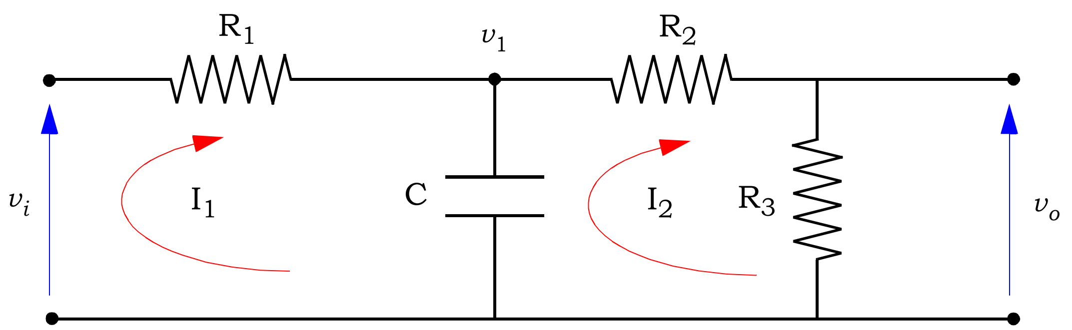
$$\begin{aligned}\dot{x}_2 = \ddot{v}_o &= -\left[\frac{CR_1R_2 + L}{R_1CL}\right]\dot{v}_o - \left[\frac{(R_1 + R_2)}{(R_1CL)}\right]v_o + \left[\frac{R_2}{R_1CL}\right]v_i \\ &= -\left[\frac{CR_1R_2 + L}{R_1CL}\right]x_2 - \left[\frac{(R_1 + R_2)}{(R_1CL)}\right]x_1 + \left[\frac{R_2}{R_1CL}\right]v_i\end{aligned} \tag{31}$$

Writing the state equations in matrix form we get:

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\left[\frac{(R_1 + R_2)}{(R_1CL)}\right] & -\left[\frac{CR_1R_2 + L}{R_1CL}\right]\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ \frac{R_2}{R_1CL}\end{bmatrix}v_i \tag{32}$$

and since $v_o = x_1$ the output equation in matrix form is:

$$v_o = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} \tag{33}$$



Circuit B

(i) Using **mesh analysis**:

$$I_1R_1 + (I_1 - I_2)\frac{1}{sC} = v_i \text{ and } (I_2 - I_1)\frac{1}{sC} + I_2(R_2 + R_3) = 0 \tag{34}$$

Converting these back to the time domain we get:

$$CR_1\dot{I}_1 + I_1 - I_2 = C\dot{v}_i \text{ and } I_2 - I_1 + C(R_2 + R_3)\dot{I}_2 = 0 \tag{35}$$

Expressing the equations in terms of the highest derivatives we get:

$$\dot{I}_1 - \frac{1}{R_1}\dot{v}_i = -\frac{1}{CR_1}I_1 + \frac{1}{CR_1}I_2 \text{ and } \dot{I}_2 = -\frac{1}{C(R_2+R_3)}I_2 + \frac{1}{C(R_2+R_3)}I_1 \tag{36}$$

Note that because the derivative of the input v_i appears in the first equation we bring the highest derivative of v_i to the LHS as well.

To get a state space model which does not involve the derivative of v_i we have to define our current state as:

$$x_1 = I_1 - \frac{1}{R_1}v_i \text{ and } x_2 = I_2 \tag{37}$$

Therefore the state equations are:

$$\dot{x}_1 = -\frac{1}{CR_1}\left(x_1 + \frac{1}{R_1}v_i\right) + \frac{1}{CR_1}x_2 \tag{38}$$

$$\dot{x}_2 = -\frac{1}{C(R_2+R_3)}x_2 + \frac{1}{C(R_2+R_3)}\left(x_1 + \frac{1}{R_1}v_i\right) \tag{39}$$

Note that from our definition of the states we have that $I_1 = x_1 + \frac{1}{R_1}v_i$.

$$\dot{x}_1 = -\frac{1}{CR_1}x_1 + \frac{1}{CR_1}x_2 - \frac{1}{CR_1^2}v_i \tag{40}$$

$$\dot{x}_2 = \frac{1}{C(R_2+R_3)}x_1 - \frac{1}{C(R_2+R_3)}x_2 + \frac{1}{CR_1(R_2+R_3)}v_i \tag{41}$$

or in matrix form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{1}{CR_1} & \frac{1}{CR_1}\\ \frac{1}{C(R_2+R_3)} & -\frac{1}{C(R_2+R_3)}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}-\frac{1}{CR_1^2}\\ \frac{1}{CR_1(R_2+R_3)}\end{bmatrix}v_i \tag{42}$$

Since the output, v_0 is the voltage across R_3 we can write the output equation as:

$$v_0 = I_2R_3 = x_2R_3 \tag{43}$$

hence the output equation in matrix form is:

$$v_0 = \begin{bmatrix}0 & R_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{44}$$

(ii) Using **nodal analysis** with nodes v_1 and v_o:

$$\frac{v_1 - v_i}{R_1} + \frac{v_1}{\frac{1}{sC}} + \frac{v_1 - v_o}{R_2} = 0 \text{ and } \frac{v_o - v_1}{R_2} + \frac{v_o}{R_3} = 0 \tag{45}$$

$$\frac{v_1}{R_1} - \frac{v_i}{R_1} + sCv_1 + \frac{v_1}{R_2} - \frac{v_o}{R_2} = 0 \text{ and } \frac{v_o}{R_2} + \frac{v_o}{R_3} = \frac{v_1}{R_2} \tag{46}$$

Converting these back to the time domain we get:

$$\dot{v}_1 = -\left[\frac{1}{CR_1} + \frac{1}{CR_2}\right]v_1 + \frac{v_o}{R_2 C} + \frac{v_i}{R_1 C} \text{ and } v_o = \frac{R_3}{R_2 + R_3}v_1 \tag{47}$$

Note that the second equation has no derivative terms $\Rightarrow$ it is not a dynamic equation and $\Rightarrow$ only need one state, $x_1 = v_1$.

Therefore the state equations are:

$$\dot{x}_1 = -\left[\frac{1}{CR_1} + \frac{1}{CR_2}\right]x_1 + \frac{v_o}{R_2 C} + \frac{v_i}{R_1 C} \tag{48}$$

and using the second equation to substitute for v_o:

$$\dot{x}_1 = -\left[\frac{1}{CR_1} + \frac{1}{CR_2}\right]x_1 + \frac{R_3}{R_2 C(R_2 + R_3)}x_1 + \frac{v_i}{R_1 C} \tag{49}$$

$$\dot{x}_1 = \left[\frac{R_3}{R_2 C(R_2 + R_3)} - \frac{1}{CR_1} - \frac{1}{CR_2}\right]x_1 + \left[\frac{1}{R_1 C}\right]v_i \tag{50}$$

Since $v_o = \frac{R_3}{R_2 + R_3}v_1$ the output equation is simply:

$$v_o = \left[\frac{R_3}{R_2 + R_3}\right]x_1 \tag{51}$$

(iii) From Q2, Tutorial 2:

$$\frac{v_o}{v_i} = \frac{R_3}{(R_1 + R_2 + R_3) + sCR_1(R_2 + R_3)} \tag{52}$$

Notice that the highest power of s in the denominator is 1. Therefore only 1 state is needed to describe the system (i.e the minimal state realisation). Therefore the state space model obtained using nodal analysis is a minimum state realisation.

To derive a minimal state-space model from the transfer function we begin by making the coefficient of the highest power of *s* on the denominator equal to 1, i.e.:

$$\frac{v_o}{v_i} = \frac{\left[\frac{R_3}{CR_1(R_2 + R_3)}\right]}{s + \left[\frac{(R_1 + R_2 + R_3)}{CR_1(R_2 + R_3)}\right]} \tag{53}$$

Converting this into a differential equation gives:

$$\dot{v}_o = -\left[\frac{(R_1 + R_2 + R_3)}{CR_1(R_2 + R_3)}\right]v_o + \left[\frac{R_3}{CR_1(R_2 + R_3)}\right]v_i \tag{54}$$

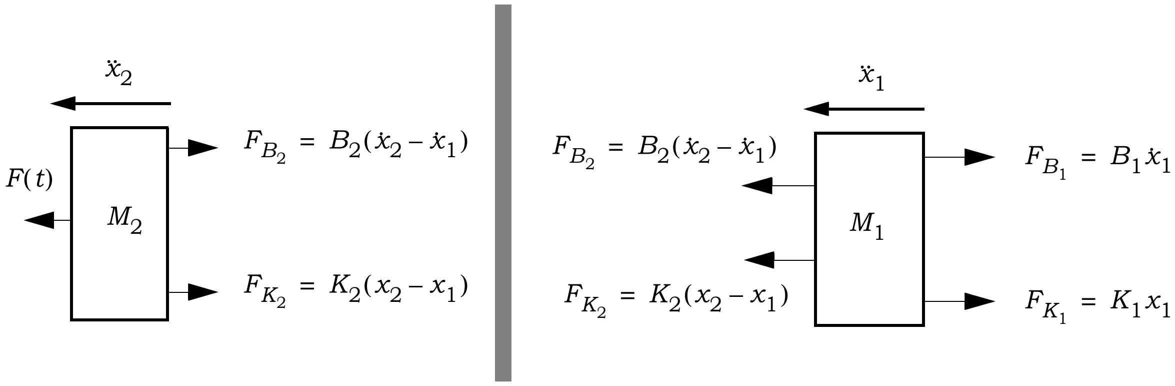
Defining the state as $x_1 = v_o$ we get the state equation:

$$\dot{x}_1 = -\left[\frac{(R_1 + R_2 + R_3)}{CR_1(R_2 + R_3)}\right]x_1 + \left[\frac{R_3}{CR_1(R_2 + R_3)}\right]v_i \tag{55}$$

and the output equation is:

$$v_o = \left[1\right]x_1 \tag{56}$$

Q2 (i) The free body diagram for the masses in the system are as follows.



Therefore for M_2, Newton's 2nd law gives:

$$F(t) - B_2(\dot{x}_2 - \dot{x}_1) - K_2(x_2 - x_1) = M_2\ddot{x}_2 \tag{57}$$

$$\Rightarrow \ddot{x}_2 = -\frac{B_2}{M_2}\dot{x}_2 - \frac{K_2}{M_2}x_2 + \frac{B_2}{M_2}\dot{x}_1 + \frac{K_2}{M_2}x_1 + \frac{F(t)}{M_2} \tag{58}$$

and for M_1:

$$B_2(\dot{x}_2 - \dot{x}_1) + K_2(x_2 - x_1) - B_1\dot{x}_1 - K_1x_1 = M_1\ddot{x}_1 \tag{59}$$

$$\Rightarrow \ddot{x}_1 = -\frac{B_1 + B_2}{M_1}\dot{x}_1 - \frac{K_1 + K_2}{M_1}x_1 + \frac{B_2}{M_1}\dot{x}_2 + \frac{K_2}{M_1}x_2 \tag{60}$$

Defining the states as:

$$X_1 = x_1,\ X_2 = \dot{x}_1,\ X_3 = x_2 \text{ and } X_4 = \dot{x}_2 \tag{61}$$

we obtain the following state equations:

$$\dot{X}_1 = X_2 \tag{62}$$

$$\dot{X}_2 = -\frac{(B_1 + B_2)}{M_1}X_2 - \frac{(K_1 + K_2)}{M_1}X_1 + \frac{B_2}{M_1}X_4 + \frac{K_2}{M_1}X_3 \tag{63}$$

$$\dot{X}_3 = X_4 \tag{64}$$

$$\dot{X}_4 = -\frac{B_2}{M_2}X_4 - \frac{K_2}{M_2}X_3 + \frac{B_2}{M_2}X_2 + \frac{K_2}{M_2}X_1 + \frac{F(t)}{M_2} \tag{65}$$

Writing these in matrix form gives:

$$\begin{bmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{X}_3\\ \dot{X}_4\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{(K_1+K_2)}{M_1} & -\dfrac{(B_1+B_2)}{M_1} & \dfrac{K_2}{M_1} & \dfrac{B_2}{M_1} \\ 0 & 0 & 0 & 1 \\ \dfrac{K_2}{M_2} & \dfrac{B_2}{M_2} & -\dfrac{K_2}{M_2} & -\dfrac{B_2}{M_2} \end{bmatrix} \begin{bmatrix} X_1\\ X_2\\ X_3\\ X_4\end{bmatrix} + \begin{bmatrix}0\\0\\0\\ \dfrac{1}{M_2}\end{bmatrix} F(t) \tag{66}$$

(ii) Outputs are:

$$y_1 = x_1 \text{ and } y_2 = \dot{x}_2 - \dot{x}_1 \tag{67}$$

Therefore in terms of the states we have:

$$y_1 = X_1 \text{ and } y_2 = X_4 - X_2 \tag{68}$$

which in matrix form corresponds to:

$$\begin{bmatrix} y_1\\ y_2\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}\begin{bmatrix} X_1\\ X_2\\ X_3\\ X_4\end{bmatrix} \tag{69}$$

Q3 (i) The 2nd order transfer model (Q5, Tutorial 2) is:

$$\frac{y(z)}{u(z)} = \frac{0.65z^{-1} + 2.55z^{-2}}{1 + 3.1z^{-1} - 2.51z^{-2}} \tag{70}$$

Defining intermediate variable $q(z)$ we can write:

$$\frac{q(z)}{u(z)} = \frac{1}{1 + 3.1z^{-1} - 2.51z^{-2}} \text{ and } \frac{y(z)}{q(z)} = 0.65z^{-1} + 2.55z^{-2} \tag{71}$$

i.e.:

$$u_k = q_k + 3.1q_{k-1} - 2.51q_{k-2} \text{ and } y_k = 0.65q_{k-1} + 2.55q_{k-2} \tag{72}$$

Rearranging the first equation gives:

$$q_k = -3.1q_{k-1} + 2.51q_{k-2} + u_k \tag{73}$$

Defining the states as:

$$x_1(k) = q_{k-1} \text{ and } x_2(k) = q_{k-2} \tag{74}$$

we obtain the states equations:

$$x_1(k+1) = q_k = -3.1x_1(k) + 2.51x_2(k) + u_k \tag{75}$$

$$x_2(k+1) = q_{k-1} = x_1(k) \tag{76}$$

Therefore the state equation in matrix from is :

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -3.1 & 2.51 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k \tag{77}$$

or

$$\underline{X}(k+1) = \begin{bmatrix} -3.1 & 2.51 \\ 1 & 0 \end{bmatrix} \underline{X}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k \tag{78}$$

The output equation is:

$$y_k = 0.65 q_{k-1} + 2.55 q_{k-2} = 0.65 x_1(k) + 2.55 x_2(k)$$

In matrix form this is:

$$y_k = \begin{bmatrix} 0.65 & 2.55 \end{bmatrix} \underline{X}(k) \tag{79}$$

(ii)
Taking the z transform of the state equation we get:

$$z\underline{X}(z) = \begin{bmatrix} -3.1 & 2.51 \\ 1 & 0 \end{bmatrix} X(z) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(z) \tag{80}$$

$$\begin{aligned} &\Rightarrow \left\{ \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} -3.1 & 2.51 \\ 1 & 0 \end{bmatrix} \right\} X(z) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(z) \\ &\Rightarrow \begin{bmatrix} z+3.1 & -2.51 \\ -1 & z \end{bmatrix} X(z) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(z) \end{aligned} \tag{81}$$

$$\Rightarrow \underline{X}(z) = \begin{bmatrix} z+3.1 & -2.51 \\ -1 & z \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(z) \tag{82}$$

$$\Rightarrow \underline{X}(z) = \frac{1}{z(z+3.1)-2.51} \begin{bmatrix} z & 2.51 \\ 1 & z+3.1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(z) \tag{83}$$

$$\Rightarrow \underline{X}(z) = \frac{1}{z^2+3.1z-2.51} \begin{bmatrix} z \\ 1 \end{bmatrix} u(z) \tag{84}$$

The z transform of the output equation is:

$$y(z) = \begin{bmatrix} 0.65 & 2.55 \end{bmatrix} \underline{X}(z) \tag{85}$$

Substituting for $\underline{X}(z)$ gives:

$$y(z) = \frac{1}{z^2+3.1z-2.51} \begin{bmatrix} 0.65 & 2.55 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} u(z) \tag{86}$$

$$\Rightarrow y(z) = \frac{0.65z+2.55}{z^2+3.1z-2.51} u(z) \tag{87}$$

Hence the transfer function is:

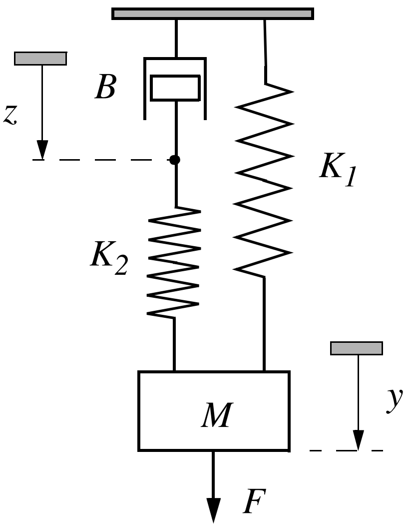
$$\frac{y(z)}{u(z)} = \frac{0.65z + 2.55}{z^2 + 3.1z - 2.51} \tag{88}$$

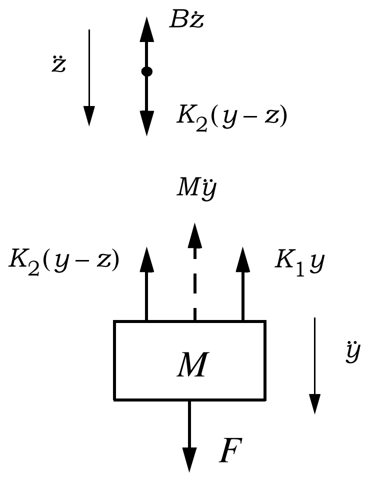
or

$$\frac{y(z)}{u(z)} = \frac{0.65z^{-1} + 2.55z^{-2}}{1 + 3.1z^{-1} - 2.51z^{-2}} \tag{89}$$

as before!

Q4





(i) Equations of motion for this system are obtained from the free body diagrams shown above, as follows:

$$F = M\ddot{y} + K_1 y + K_2(y - z)$$
$$K_2(y - z) = B\dot{z}$$

Rearranging in terms of the highest derivatives:

$$\ddot{y} = -\frac{(K_1 + K_2)}{M}y + \frac{K_2}{M}z + \frac{F}{M}, \ \dot{z} = \frac{K_2}{B}(y - z)$$

(ii) Choosing the states as $x_1 = y$, $x_2 = \dot{y}$ and $x_3 = z$ and the output as y gives:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{(K_1 + K_2)}{M}x_1 + \frac{K_2}{M}x_3 + \frac{F}{M}$$
$$\dot{x}_3 = \frac{K_2}{B}x_1 - \frac{K_2}{B}x_3$$

or in matrix form:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{(K_1+K_2)}{M} & 0 & \frac{K_2}{M} \\ \frac{K_2}{B} & 0 & -\frac{K_2}{B} \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \end{bmatrix}F,\ y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\underline{x}$$

Subbing in the numbers gives:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -100 & 0 & 40 \\ 10 & 0 & -10 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}F,\ y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\underline{x}$$

(iii)(a)

Taking the Laplace transform of the equations gives:

$$s^2 y(s) = -\frac{(K_1+K_2)}{M}y(s) + \frac{K_2}{M}z(s) + \frac{F(s)}{M} \Rightarrow \left(s^2 + \frac{(K_1+K_2)}{M}\right)y(s) = \frac{K_2}{M}z(s) + \frac{F(s)}{M},$$

$$sz(s) = \frac{K_2}{B}(y(s) - z(s)) \Rightarrow \left(s + \frac{K_2}{B}\right)z(s) = \frac{K_2}{B}y(s) \Rightarrow z(s) = \frac{\frac{K_2}{B}}{\left(s + \frac{K_2}{B}\right)}y(s)$$

Substituting for z(s) gives:

$$\left[s^2 + \frac{(K_1+K_2)}{M}\right]y(s) = \frac{K_2}{M}\left(\frac{\frac{K_2}{B}}{\left(s + \frac{K_2}{B}\right)}\right)y(s) + \frac{F(s)}{M}$$

$$\Rightarrow \left(s + \frac{K_2}{B}\right)\left[s^2 + \frac{(K_1+K_2)}{M}\right]y(s) = \frac{K_2}{M}\frac{K_2}{B}y(s) + \frac{F(s)}{M}\left(s + \frac{K_2}{B}\right)$$

$$\Rightarrow \left[s^3 + \frac{K_2}{B}s^2 + \frac{(K_1+K_2)}{M}s + \frac{K_2}{B}\frac{(K_1+K_2)}{M}\right]y(s) = \frac{K_2}{M}\frac{K_2}{B}y(s) + \frac{F(s)}{M}\left(s + \frac{K_2}{B}\right)$$

$$\Rightarrow \left[s^3 + \frac{K_2}{B}s^2 + \frac{(K_1+K_2)}{M}s + \frac{K_1K_2}{BM}\right]y(s) = \frac{F(s)}{M}\left(s + \frac{K_2}{B}\right)$$

$$\Rightarrow \frac{y(s)}{F(s)} = \frac{\frac{1}{M}\left(s + \frac{K_2}{B}\right)}{\left[s^3 + \frac{K_2}{B}s^2 + \frac{(K_1+K_2)}{M}s + \frac{K_1K_2}{BM}\right]}$$

$$\Rightarrow \frac{y(s)}{F(s)} = \frac{10(s+10)}{s^3 + 10s^2 + 100s + 600}$$

(b)

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -100 & 0 & 40 \\ 10 & 0 & -10 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}F,\ y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\underline{x}$$

$$G(s) = C(sI-A)^{-1}B$$

$$(sI-A)^{-1} = \begin{bmatrix} s & -1 & 0 \\ 100 & s & -40 \\ -10 & 0 & s+10 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} \begin{vmatrix} s & -40 \\ 0 & s+10 \end{vmatrix} & -\begin{vmatrix} 100 & -40 \\ -10 & s+10 \end{vmatrix} & \begin{vmatrix} 100 & s \\ -10 & 0 \end{vmatrix} \\ -\begin{vmatrix} -1 & 0 \\ 0 & s+10 \end{vmatrix} & \begin{vmatrix} s & 0 \\ -10 & s+10 \end{vmatrix} & -\begin{vmatrix} s & -1 \\ -10 & 0 \end{vmatrix} \\ \begin{vmatrix} -1 & 0 \\ s & -40 \end{vmatrix} & -\begin{vmatrix} s & 0 \\ 100 & -40 \end{vmatrix} & \begin{vmatrix} s & -1 \\ 100 & s \end{vmatrix} \end{bmatrix}}{s\begin{vmatrix} s & -40 \\ 0 & s+10 \end{vmatrix} - (-1)\begin{vmatrix} 100 & -40 \\ -10 & s+10 \end{vmatrix}}$$

$$= \frac{\begin{bmatrix} s^2+10s & -100s-600 & 10s \\ s+10 & s^2+10s & 10 \\ 40 & 40s & s^2+100 \end{bmatrix}^T}{s^3+10s^2+100s+600} = \frac{\begin{bmatrix} s^2+10s & s+10 & 40 \\ -100s-600 & s^2+10s & 40s \\ 10s & 10 & s^2+100 \end{bmatrix}}{s^3+10s^2+100s+600}$$

Therefore:

$$G(s) = C(sI-A)^{-1}B = \frac{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} s^2+10s & s+10 & 40 \\ -100s-600 & s^2+10s & 40s \\ 10s & 10 & s^2+100 \end{bmatrix}\begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}}{s^3+10s^2+100s+600}$$

$$= \frac{10(s+10)}{s^3+10s^2+100s+600}$$

(iv)

$$G(s) = \frac{10s+100}{s^3+10s^2+100s+600}$$

Following the method given in the notes we get:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -600 & -100 & -10 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}F,\ y = \begin{bmatrix} 100 & 10 & 0 \end{bmatrix}\underline{x}$$

Verifying ...

$$(sI-A)^{-1} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 600 & 100 & s+10 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s^2+10s+100 & -600 & -600s \\ s+10 & s^2+10s & -100s-600 \\ 1 & s & s^2 \end{bmatrix}^T}{s\begin{vmatrix} s & -1 \\ 100 & s+10 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -1 \\ 600 & s+10 \end{vmatrix}}$$

$$= \frac{\begin{bmatrix} s^2+10s+100 & s+10 & 1 \\ -600 & s^2+10s & s \\ -600s & -100s-600 & s^2 \end{bmatrix}}{s(s^2+10s+100)+600} = \frac{\begin{bmatrix} s^2+10s+100 & s+10 & 1 \\ -600 & s^2+10s & s \\ -600s & -100s-600 & s^2 \end{bmatrix}}{s^3+10s^2+100s+600}$$

$$G(s) = C(sI-A)^{-1}B = \frac{\begin{bmatrix} 100 & 10 & 0 \end{bmatrix}\begin{bmatrix} s^2+10s+100 & s+10 & 1 \\ -600 & s^2+10s & s \\ -600s & -100s-600 & s^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{s^3+10s^2+100s+600}$$

$$= \frac{\begin{bmatrix} 100 & 10 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}}{s^3+10s^2+100s+600}$$

$$= \frac{10s+100}{s^3+10s^2+100s+600}$$