EE206 Differential Equations and Transform Methods

Tutorial 5

Problem 1a: Use the First Translation (Shift) Theorem to find either $F(s)$ or $f(t)$, as indicated. State in each case how the translation theorem applies.

$\mathscr{L}\{\sinh(t)\sin(t)\}$

We have that:

$$\mathscr{L}\{\sinh(t)\sin(t)\} = \frac{1}{2}\left(\mathscr{L}\{e^{t}\sin(t)\} - \mathscr{L}\{e^{-t}\sin(t)\}\right)$$

We can apply the shift theorem that $\mathscr{L}\{e^{at}\sin(t)\} = \frac{1}{s^2+1}{}_{s\to s-a}$ so the above becomes:

$$\frac{1}{2}\left(\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right)$$

Simplifying:

$$
\begin{aligned}
\mathscr{L}\{\sinh(t)\sin(t)\} &= \frac{1}{2}\frac{(s+1)^2+1-(s-1)^2-1}{[(s+1)(s-1)]^2+(s+1)^2+(s-1)^2+1} \\
&= \frac{1}{2}\frac{s^2+2s+1-s^2+2s-1}{[s^2-1]^2+s^2+2s+1+s^2-2s+1+1} \\
&= \frac{1}{2}\frac{4s}{[s^2-1]^2+2s^2+3} \\
&= \frac{2s}{s^4-2s^2+1+2s^2+3} \\
&= \frac{2s}{s^4+4}
\end{aligned}
$$

Problem 2c: Use the Second Translation (Shift) Theorem to find either $F(s)$ or $f(t)$, as indicated. State in each case how the translation theorem applies

$\mathscr{L}^{-1}\left\{\frac{se^{-\frac{\pi s}{2}}}{s^2+4}\right\}$

From the second shift theorem we will acquire a factor of $e^{-\frac{\pi}{2}s}$ from multiplying a function by $\mathscr{U}(t-\pi/2)$ so working backwards:

$$
\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{se^{-\frac{\pi s}{2}}}{s^2+4}\right\} &= \mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}_{t\to t-\frac{\pi}{2}} \mathscr{U}(t-\frac{\pi}{2}) \\
&= \cos(2t)_{t\to t-\frac{\pi}{2}}\mathscr{U}(t-\frac{\pi}{2}) \\
&= \cos\left(2(t-\frac{\pi}{2})\right)\mathscr{U}(t-\frac{\pi}{2})
\end{aligned}
$$

which if you want is:

$$= -\cos(2t)\mathscr{U}(t - \frac{\pi}{2})$$

Problem 3a: Use the relation between multiplication of $f(t)$ (by t^n) and differentiation of $F(s)$ to find the Laplace transforms of the following.

The relation we are going to use for all of these questions is:

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$f(t) = t^2 \cos t$

$$\begin{aligned}
\mathscr{L}\{t^2 \cos t\} &= (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right] \\
&= \frac{d}{ds}\left[\frac{(s^2+1)(1)-(s)(2s)}{(s^2+1)^2}\right] \\
&= \frac{d}{ds}\left[\frac{-s^2+1}{(s^2+1)^2}\right] \\
&= \frac{((s^2+1)^2)(-2s)-(-s^2+1)(2(2s)(s^2+1))}{(s^2+1)^4} \\
&= \frac{(s^2+1)(-2s)-(-s^2+1)(2(2s)}{(s^2+1)^3} \\
&= \frac{-2s^3-2s+4s^3-4s}{(s^2+1)^3} \\
&= \frac{2s^3-6s}{(s^2+1)^3}
\end{aligned}$$

Problem 4a: Use the Laplace transform to solve the given initial-value problems.

These problems will use $\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a)$, and $e^{at}f(t) = \mathscr{L}^{-1}\left\{F(s-a)\right\}$

$$y' - y = 1 + te^t, \qquad y(0) = 0$$

$$\begin{aligned}
\mathscr{L}\{y' - y\} &= \mathscr{L}\{1 + te^t\} \\
sY(s) - y(0) - Y(s) &= \frac{1}{s} + \frac{1}{(s-1)^2} \\
(s-1)Y(s) &= \frac{1}{s} + \frac{1}{(s-1)^2} \\
Y(s) &= \frac{1}{s(s-1)} + \frac{1}{(s-1)^3} \\
\frac{A}{s} + \frac{B}{s-1} &= \frac{1}{s(s-1)} \\
As - A + Bs &= 1 \\
A &= -1 \\
A + B &= 0 \ \Rightarrow B = -A \ \Rightarrow B = 1 \\
Y(s) &= -\frac{1}{s} + \frac{1}{s-1} + \frac{1}{(s-1)^3} \\
y(t) &= \mathscr{L}^{-1}\{Y(s)\} \\
&= -\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\}
\end{aligned}$$

Problem 5a: Use the Laplace transform to solve the given initial-value problems.

$y' + 2y = f(t), \quad y(0) = 0,$ with:

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ -1 & 1 \le t. \end{cases}$$

We can use the formula for piecewise functions:

$$f(t) := \begin{cases} g(t) & 0 \le t \le a \\ h(t) & a \le t \end{cases}$$

Then $f(t) = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a)$

$$f(t) = 1 - 1\mathcal{U}(t-1) - 1\mathcal{U}(t-1) = 1 - 2\mathcal{U}(t-1)$$

We have that:

$$\mathscr{L}\{y' + 2y\} = \mathscr{L}\{1 - 2\mathscr{U}(t - 1)\}$$

Then using the second shift theorem we have:

$$\begin{aligned}
sY(s) - y(0) + 2Y(s) &= \frac{1}{s} - \frac{2}{s}e^{-s} \\
(s+2)Y(s) &= \frac{1-2e^{-s}}{s} \\
Y(s) &= \frac{1-2e^{-s}}{s(s+2)} = \frac{1}{s(s+2)} - e^{-s}\frac{2}{s(s+2)} \\
\Rightarrow \frac{A}{s} + \frac{B}{s+2} &= \frac{1}{s(s+2)} \\
As + 2A + Bs &= 1 \\
A &= 1/2 \\
B &= -A \quad \Rightarrow B = -1/2
\end{aligned}$$

$$\Rightarrow \frac{C}{s} + \frac{D}{s+2} = \frac{2}{s(s+2)}$$

$$Cs + 2C + Ds = 2$$

$$C = 1$$

$$D = -C \quad \Rightarrow D = -1$$

$$\Rightarrow Y(s) = \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{1}{s+2} - e^{-s}\left(\frac{1}{s} - \frac{1}{s+2}\right)$$

$$\Rightarrow y(t) = \mathscr{L}^{-1}\{Y(s)\}$$

$$y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} - \mathscr{U}(t-1) + e^{-2t+2}\mathscr{U}(t-1)$$