8. Transient and Steady-state Characteristics

8.1 Introduction

- Here**,** we are going to examine the transient and steady-state characteristics of our first and second order systems.

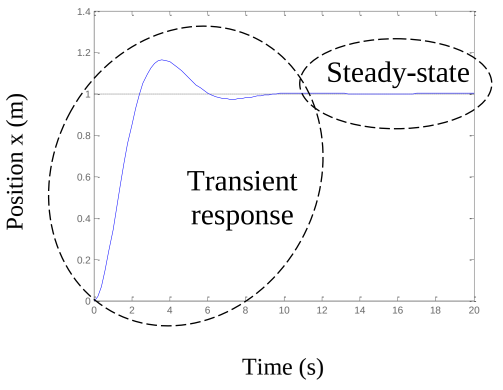
- The **transient response** refers to the initial part of the output response of the system, i.e. that part of the response before it settles to a steady condition.

- When the response settles, it is said to be in **steady-state**.

- For example, consider the response from the mass-spring-damper system studied previously, shown below.

- The steady-state value in this case is 1, as this is where the output finally settles.

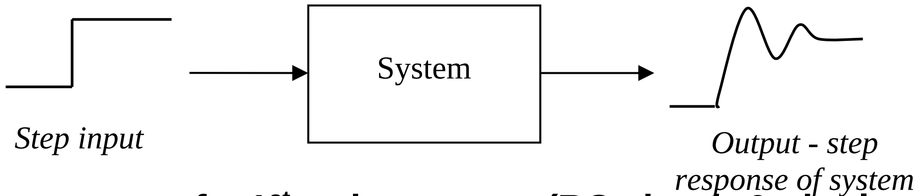
- The transient response refers to the initial part of the response which is changing prior to settling at the final value of 1 (in this example).



- In order to generate such a response, we typically apply standard test inputs as stimuli to the dynamical system.

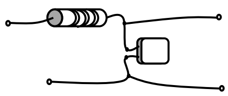
- Typical inputs include step, ramp and parabola, but **we are only going to consider the step input** for now.

- Hence, the typical system setup looks like:



8.2 Step response of a 1st order system (RC circuit & single tank)

- Recall that the transfer function of the RC circuit is: $\dfrac{1}{1+sRC}$



while that of the single tank system is: $\dfrac{1}{sA+k} \equiv \dfrac{\frac{1}{k}}{1+s\left(\frac{A}{k}\right)}$

- We can represent both these systems, and first-order systems in general, by the following standard transfer function:

$$U(s) \longrightarrow \boxed{\dfrac{K}{1+s\tau}} \longrightarrow Y(s)$$

where K and τ are constant values.

- Consider the output for this system for a step input with amplitude A, i.e. $U(s)=\dfrac{A}{s}$:

$$Y(s)=\frac{K}{1+s\tau}U(s) \quad = \frac{K}{1+s\tau}.\frac{A}{s} \quad = \frac{KA}{s(1+s\tau)}$$

- Using the partial fraction method, as before, we can obtain the inverse Laplace transform as follows:

$$\frac{KA}{s(1+s\tau)}=\frac{X}{s}+\frac{Y}{1+s\tau} \quad = \frac{X(1+s\tau)+Ys}{s(1+s\tau)}$$

Comparing numerators: $\quad KA = X(1+s\tau)+Ys$

Setting $s = 0$: $\quad X = KA$

Setting $s = -\dfrac{1}{\tau}$: $\quad Y = -KA\tau$

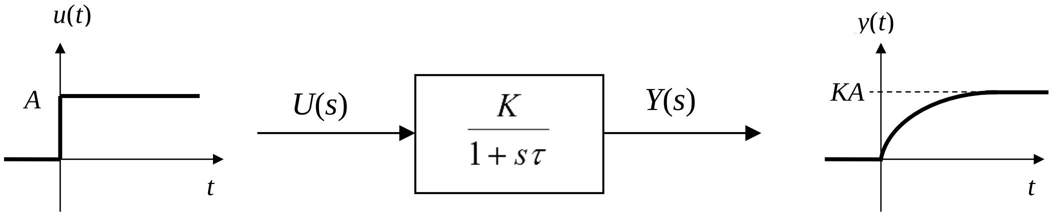
Therefore: $\quad Y(s)=\dfrac{KA}{s}-\dfrac{KA\tau}{1+s\tau} \quad = \dfrac{KA}{s}-\dfrac{KA}{s+\frac{1}{\tau}}$

Hence: $\quad y(t)=KA\left(1-e^{-\frac{t}{\tau}}\right)$

- This is the **characteristic response for standard first-order systems**.
- Note: $\quad y(t=0)=KA\left(1-e^{0}\right) \quad = KA(1-1) \quad = 0$

$$y(t \to \infty) = KA\left(1 - e^{-\infty}\right) \quad = KA(1-0) \quad = KA$$

- Graphically, we have:



Step input with amplitude A

Characteristic step response for a 1st order system

- The response is characterized by two important parameters, namely the **steady-state gain *K*** and the **time constant *τ***.

- The steady-state output is the value at which the output settles and we will refer to this as y_{SS}.

- We can determine this value by calculating $y(t)$ as time t approaches ∞, as above, i.e.

$$y_{SS} = y(t \to \infty) \quad = KA\left(1 - e^{-\infty}\right) \quad = KA$$

- We know that the input signal has a final steady-state value of A, as given earlier. We will refer to the steady-state input as u_{SS}.

- Hence: $$\frac{y_{ss}}{u_{ss}} = \frac{KA}{A} = K$$

- In other words, the parameter K defines the magnification (or magnitude change) between steady state input and the steady-state output of the first order system.

- This is known as the gain of the system, i.e. ***K* is the steady-state gain of the first order system**.

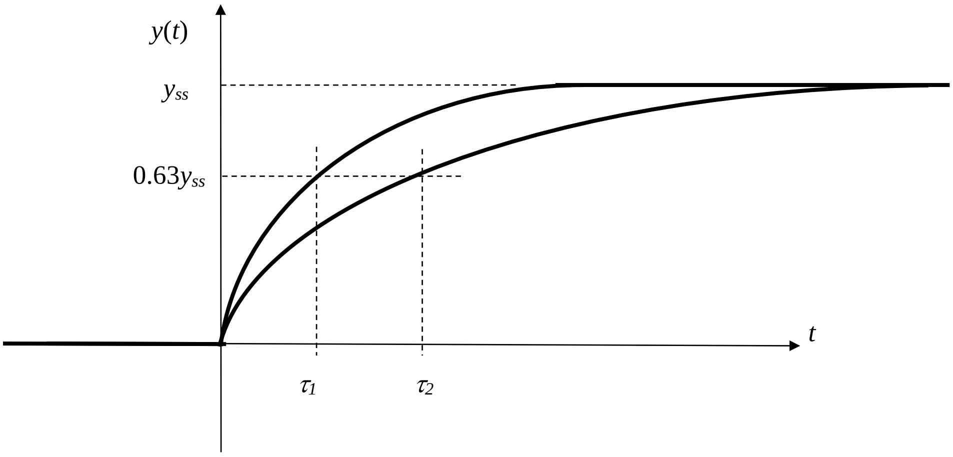
- Now let us consider the time constant τ parameter.

- When $t = \tau$, we get: $$y(t=\tau) = KA\left(1 - e^{-1}\right) = 0.63KA$$

$$y(t=\tau) = 0.63 y_{ss} \quad \text{or} \quad 63\% \text{ of } y_{ss}$$

- It is this relationship that defines τ. In other words, the **time constant τ is defined as the time at which the output has reached 63% of its final value**.
- The **time constant effectively determines the responsiveness of the output**.

- The *smaller the time constant, the faster the response,* as illustrated below:



- In the above diagram, the response with the smallest time constant τ_1 has the fastest response and settles quicker.

- In contrast, the response with the larger time constant τ_2 has the slower response and takes longer to settle as a result.

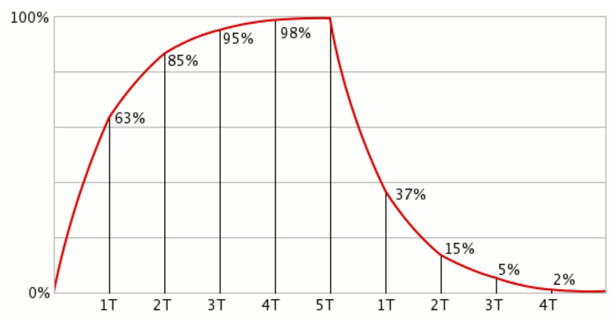
- Note, while one time constant corresponds to 63% of the final value, we can obtain alternative useful figures for multiple time constants as follows:

- When $t = 2\tau$: $\quad y(t) = KA\left(1 - e^{-2}\right) = 0.86KA$

- When $t = 3\tau$: $\quad y(t) = KA\left(1 - e^{-3}\right) = 0.95KA$

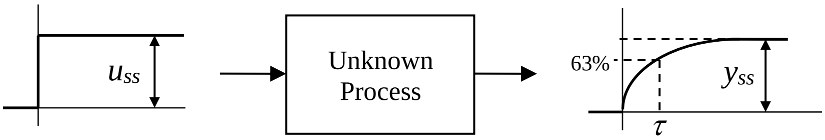
- When $t = 4\tau$: $\quad y(t) = KA\left(1 - e^{-4}\right) = 0.98KA$

- Hence, we now also know that the first order response reaches 98% of its final value after 4 time constants.



- Understanding the significance of K and τ allows us to ***quickly estimate a first order model of an unknown process from experimental data*** as follows:

    - Pass a step input into the unknown process and monitor the output response.



- Measure u_{ss} and y_{ss} and hence calculate gain K using: $$K = \frac{y_{ss}}{u_{ss}}$$

- Locate the time at which the output reached 63% of the final value. This time corresponds to the time constant τ.

- Hence, the first-order model for the unknown process is: $$\frac{K}{1+s\tau}$$

Do you think this will be a good model for the unknown process?

- ***Ex. 8.1 Determine the number of time constants at which the output of a standard first order system has reached 99% of its final value.***

Solution:

We want: $y(t) = KA\left(1-e^{-\frac{t}{\tau}}\right) = 0.99KA$ (*recall that* $y_{ss} = KA$)

This implies:

$$1-e^{-\frac{t}{\tau}} = 0.99$$
$$\Rightarrow -e^{-\frac{t}{\tau}} = -0.01$$
$$\Rightarrow e^{-\frac{t}{\tau}} = 0.01$$

Taking the natural log (i.e. log_e) of both sides gives:

$$-\frac{t}{\tau} = \log_e(0.01) = -4.6$$
$$\Rightarrow \frac{t}{\tau} = 4.6$$
$$\Rightarrow t = 4.6\,\tau$$

Hence, it takes **4.6 time constants** to reach 99% of the final value.

- ***Ex. 8.2 Determine the number of time constants at which the output of a standard first order system has reached half of its final value.***

Solution:

Here, we want: $y(t) = KA\left(1-e^{-\frac{t}{\tau}}\right) = 0.5KA$

This implies:
$$1-e^{-\frac{t}{\tau}} = 0.5$$
$$\Rightarrow e^{-\frac{t}{\tau}} = 0.5$$

Taking the natural log of both sides gives:

$$-\frac{t}{\tau} = \log_e(0.5) = -0.69$$
$$\Rightarrow \frac{t}{\tau} = 0.69$$
$$\Rightarrow t = 0.69\,\tau$$

Hence, it takes **0.69 time constants** to reach 50% of the final value.

- ***Ex. 8.3(a) Calculate the time constant for an RC circuit given the R = 1Ω and C = 1F.***

Solution:

From our previous work, we know that the solution for the RC circuit is given by

$$y(t) = u\left(1-e^{\frac{-t}{RC}}\right) = u\left(1-e^{-t}\right)$$

From this equation, taking u as a constant value, we determine the steady-state value of $y(t)$ as:

$$y(\infty) = u\left(1-e^{-\infty}\right) = u\,(1-0) = u$$

The time constant τ is the time at which $y(t)$ has reached 63% of its final value, i.e. $0.63u$:

$$u\left(1-e^{-\tau}\right) = 0.63u \quad \Rightarrow 1-e^{-\tau} = 0.63 \quad \Rightarrow -e^{-\tau} = -0.37$$

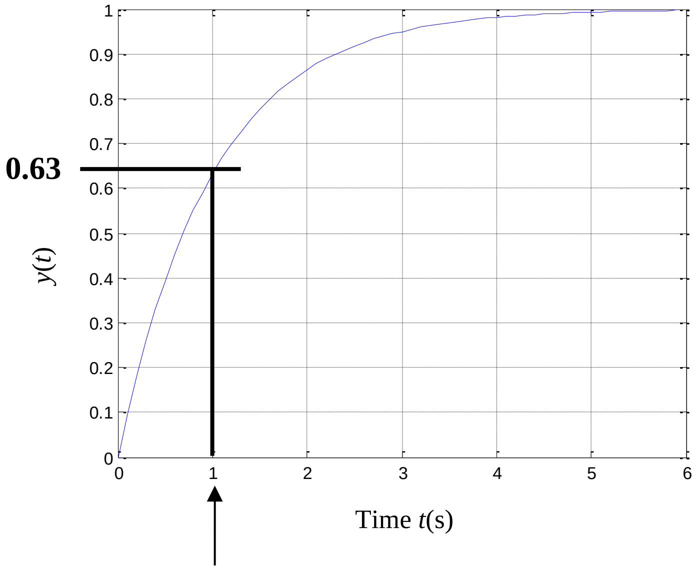
$$\Rightarrow -\tau = \log_e(0.37) = -0.99$$

$$\Rightarrow \boldsymbol{\tau \approx 1s}$$

Aside: While we solved this question analytically, we could also have taken a graphical approach as follows:

Plot the output $u\left(1-e^{-t}\right)$ for u = 1 (this does not affect our solution) over a suitable period of time.

Then, graphically determine the time at which the output has reached 63% of its final output, as shown below:



From the graph we can clearly see that it takes 1 second for the output to reach at 63% of its final value.

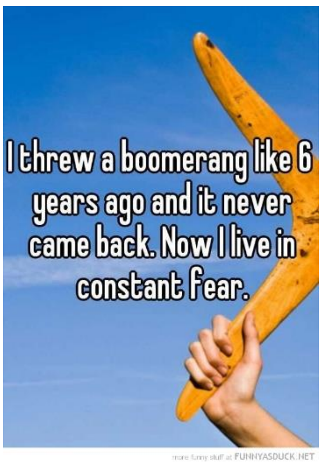
- ***Ex. 8.3(b) Determine the length of time (in seconds) it takes the output of the RC circuit to reach 98% of its final value.***

Solution:

Graphically, we can read this off the above graph … so at 0.98 of the output, the time is approximately 4 seconds.

Analytically, we know that the output reaches 98% of its final output after 4 time constants, i.e. 4τ.

Here, $\tau = 1$, hence the time taken to reach 98% of the steady-state value is **4s**.



First-order system – significance of pole

- Recall from the previous section that the poles of a system are obtained from the denominator of its transfer function.

- A first-order system has only one pole, as we know, and for our standard system this pole is given by:

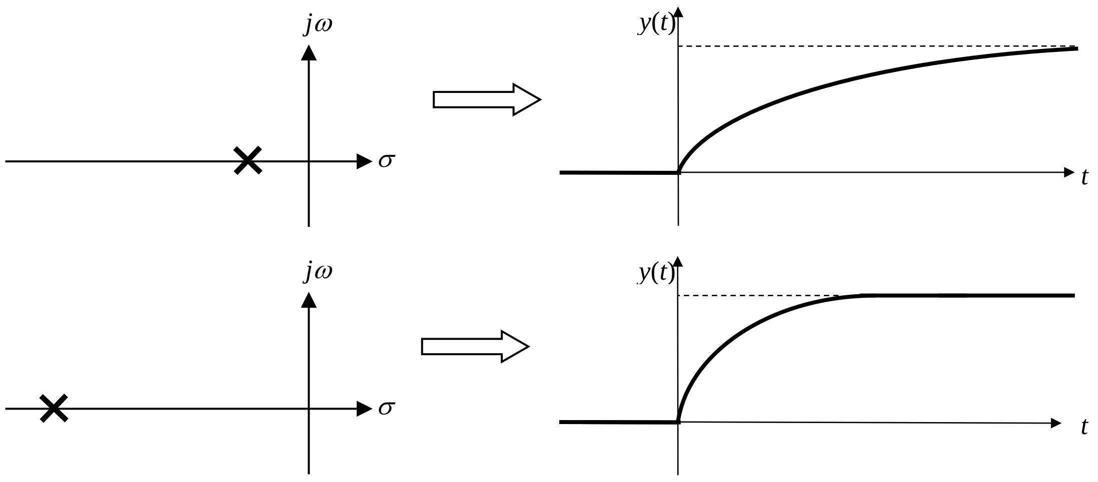
$$1 + s\tau = 0 \Rightarrow s = -\frac{1}{\tau}$$

- Hence, this expression allows us to easily relate the time constant of the first-order system to its pole location.

- For example, a first-order system with a pole at $s = -4$ has a time constant $\tau = 0.25$s. Furthermore, we can state that such a system will reach 98% of its final output after 1s.

- A first-order system with a pole at $s = -10$ has a time constant $\tau = 0.1$s and will reach 98% of its final output after 0.4s.

- Clearly **for a first-order system**, we can observe that **the further the pole is from the imaginary axis** in the *s*-plane, **the smaller the time constant** becomes and **the faster the system output settles**.



- ***Ex. 8.4 A first-order system has a pole located at –5. What is the time constant of this system?***

  Solution:

$$\tau = \frac{1}{5} = \mathbf{0.2s}$$

8.3 Step response of a 2nd order system (mass-spring-damper)

- Recall the transfer function model for the mass-spring-damper (bicycle) system:

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

- The more general, and standard, form of a second order system is given by a transfer function of the form:

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

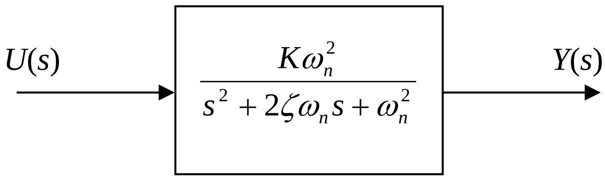
where:

- K is the steady-state gain,
- ζ is referred to as the damping ratio, and
- ω_n is the undamped natural frequency.

- The steady-state gain K has the same meaning as that of the first order system.

- The other two parameters (ζ and ω_n) offer important insights into second-order systems and we will examine the significance of each.

- We now analyze the following setup:



where K, ζ and ω_n are all constant values.

- Let us consider the output for this system for a step input with amplitude A, i.e. $U(s) = \frac{A}{s}$:

$$Y(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{A}{s} \qquad = \frac{KA\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$= \frac{KA\omega_n^2}{s(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})}$$

- The noteworthy point here is (not the mat

- Hence the following results:

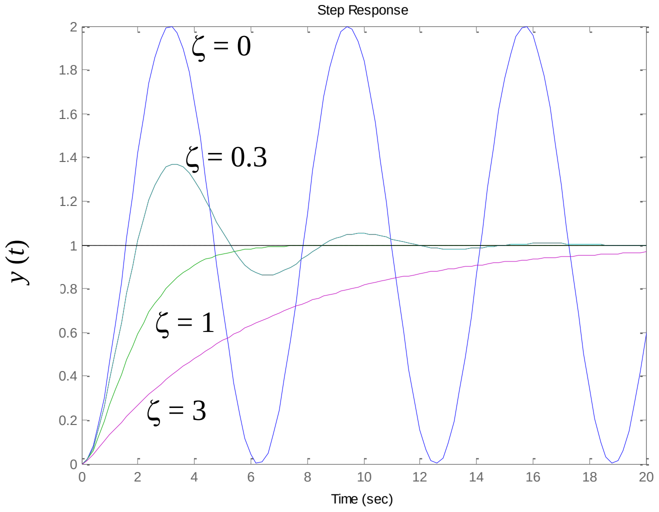
$\zeta > 1$ - **poles** of $Y(s)$ from quadratic term **are real and distinct**

$\zeta < 1$ - **poles** of $Y(s)$ from quadratic term **are complex**

$\zeta = 1$ - **poles** of $Y(s)$ from quadratic term **are equal**

$\zeta = 0$ - **poles** of $Y(s)$ from quadratic term **are imaginary**

- Plotting the solution $y(t)$ for different values of the damping ratio ζ gives ($K=1$, $A=1$):



Significance of ζ

- The **damping factor or damping ratio** ζ indicates the type of transient response of the system (both in terms of speed and possible oscillations).

- It has the following relevance with respect to the transient performance of a 2nd order system:

$\zeta = 1$	-	critical damping, just no overshoot
$0 < \zeta < 1$	-	underdamped (*and hence, some decaying oscillations*)
$\zeta > 1$	-	overdamped (*no oscillations, similar to a first order response*)
$\zeta = 0$	-	undamped, oscillates (*constant oscillations*)

- In general, the preferred damping range would be $0.5 < \zeta \leq 1$, as this gives a *fast response* consistent with only a *small overshoot*.

- Of course, this is application-specific! For example, a damping factor of $\zeta > 1$ might be more suitable for a system that involves controlling the level of water in a tank, or controlling the smoothness of an airplane's flight, so as to avoid unwanted oscillations!

- *What damping would you expect the bicycle (mass-spring-damper) system to have?*

Significance of ω_n

- The **undamped natural frequency** ω_n is defined as the frequency at which the system oscillates in the absence of any damping.

- In addition, the term $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is referred to as the **damped natural frequency**.

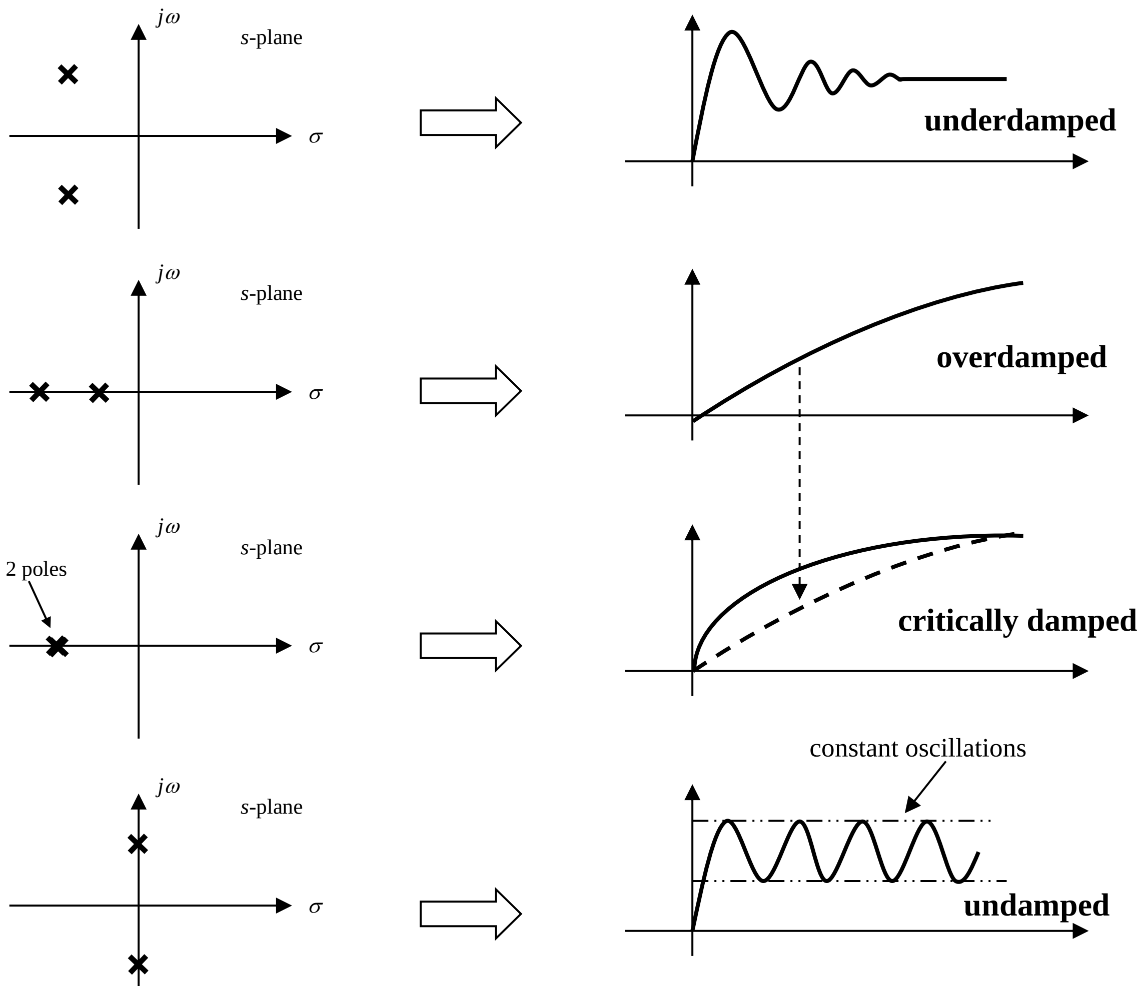
- Note, the terms ω_d and ω_n determine the frequency of oscillation of the output response.

Second-order system – significance of poles

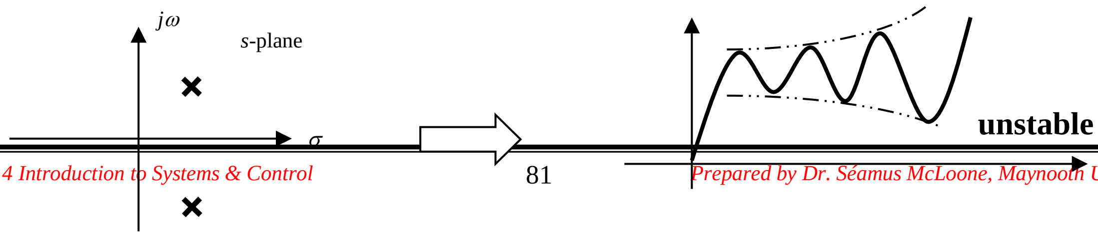
- As we know, a second order system has two poles and, as we have seen already, these poles can be real and distinct, real and equal, complex or entirely imaginary.

- Each of these equated to a corresponding damping scenario. Hence, we can determine the damping characteristics of a second-order system from knowledge of the location of the poles.

- We can capture this relationship graphically as follows:



- We can also include the following set of pole locations, for the sake of completeness, in relation to the issue of stability, as covered in previous section for the notes:



- Pole/zero diagrams play a central role in the design of control systems.

- Transient performance specifications are often expressed in terms of desired ‘closed-loop’ pole positions.

- The objective of controllers is to move the system poles to these desired positions. *This will be studied in much more detail in future Control related modules.*

- Finally, it should be noted that analyzing a second-order system may appear quite specialized at first glance.

- However, many systems can be reasonably approximated as 2nd order and so these results have a wider application than first appears.

8.4 Measures of performance for a 2nd order system

- In order to compare and analyze the outputs of different systems, we need to have some way of capturing or measuring key features of the response.

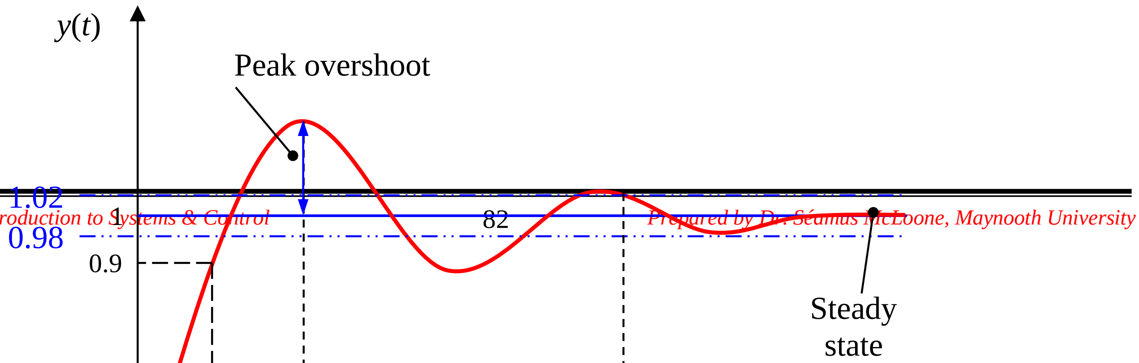
- Typical features include:

  - the **rise time**,
  - the **settling time**,
  - the **peak overshoot** and
  - the **steady-state value**.

Students excuses through out the ages.

- These can be determined analytically, using the appropriate equations, or can be measured experimentally by plotting the step response of the system and graphically determining the relevant values, as will now be illustrated.

- A typical underdamped unit step response is shown below (K = 1, A = 1):



- The **rise time T_r** is the time taken for the response to rise from 10% to 90% of its final value.

- The **time to peak T_p** is the time it takes the step response to reach its peak-overshoot value.

- Analytically we can determine the time to peak using the following equation:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

- The **peak overshoot *PO*** is the amount by which the (underdamped) response overshoots its final value. It is typically expressed as a percentage.

- The PO can be calculated using the following equation:

$$PO = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

or

$$PO(\%) = 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

- The **settling time T_s** is the time for the response to settle within 2% of its final value and can be measured for all responses.

- As we have seen, this can be computed for the first order system as follows:

$$y(t) = KA\left(1-e^{-\frac{T_s}{\tau}}\right) = 0.98KA \Rightarrow e^{-\frac{T_s}{\tau}} = 0.02 \quad \Rightarrow T_s = 4\tau$$

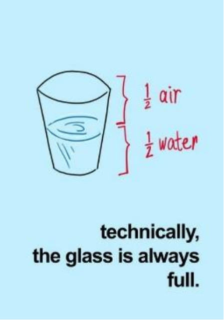
- For the second order underdamped response, the settling time T_s is given by:

$$T_s = \frac{4}{\zeta\omega_n}$$

- Finally, the **steady-state gain G_{ss}** is a measure of the ratio of the steady-state output to the steady-state input for a given system.
- The **steady-state output** is the value at which the system's response finally settles.
- In the 1st and 2nd order models, the steady state gain is *K*.
- For a general transfer function, *G*(*s*), the steady-state gain can be computed as:

$$G_{ss} = \lim_{s \to 0} G(s)$$

- The above measure of performances can be read directly from a plot of the system's response or can be determined using the appropriate equations given a model of the system.
- **You do not need to commit these equations to memory - but you do need to know how to apply them**.
- *For those interested, derivation of the various equations can be found in a suitable textbook.*



- ***Ex. 8.5 Determine the peak overshoot, settling time and steady-state output for a second-order system with the following transfer function:***

$$G(s) = \frac{2}{s^2 + s + 4}$$

Solution:

Compare the denominator with that of the standard second-order transfer function, i.e.:

$$s^2 + s + 4 \quad \Leftrightarrow \quad s^2 + 2\zeta\omega_n s + \omega_n^2$$

This gives: $\omega_n^2 = 4 \Rightarrow \omega_n = 2$ and $2\zeta\omega_n = 1 \Rightarrow \zeta = 0.25$

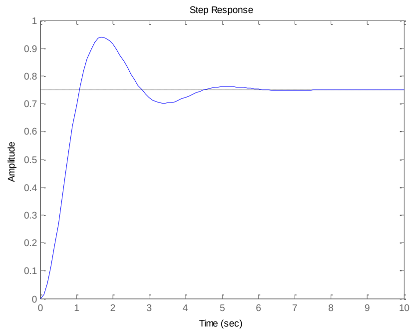
Hence, the **peak overshoot** is:

$$PO(\%) = 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 100e^{\frac{-0.25\pi}{\sqrt{1-(0.25)^2}}} = 44.34\%$$

The **settling time** is: $T_s = \dfrac{4}{\zeta\omega_n} = \dfrac{4}{0.25\,(2)} = 8s$

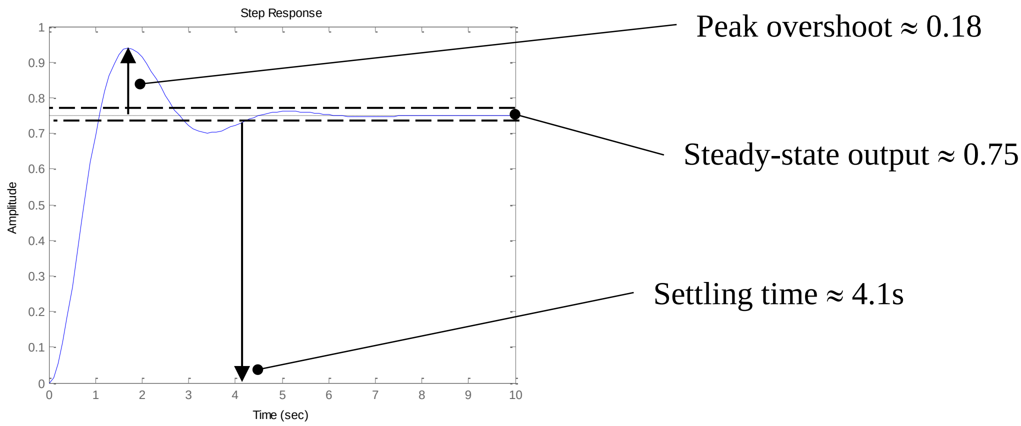
The **steady-state output** is: $G_{ss} = \lim\limits_{s\to 0} G(s) = \lim\limits_{s\to 0} \dfrac{2}{s^2+s+4} = \dfrac{2}{4} = 0.5$

- ***Ex. 8.6 Determine the peak overshoot, settling time and steady-state output for a second-order system given its step response as follows:***



Solution:

In this case, we simply read the necessary values from the graph, as shown below:



To determine the percentage PO (if required), we need to compare the PO to the final value as follows:

$$PO(\%) = 100\left(\frac{0.18}{0.75}\right) = 24\%$$