Tutorial Sheet 6 - Solutions

Q1(i) $\dot{\underline{x}} = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}\underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(t)$, $\begin{vmatrix} 1-\lambda & 0 \\ -2 & 2-\lambda \end{vmatrix} = 0$, $\Rightarrow (1-\lambda)(2-\lambda) = 0$, $\Rightarrow \lambda = 1, 2$.

$Re(\lambda) > 0 \Rightarrow$ system is unstable.
Eigenvalues are not complex. Therefore transient response will be non-oscillatory.

(ii) $\dot{\underline{x}} = \begin{bmatrix} 0 & 2 \\ -2 & -3 \end{bmatrix}\underline{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}u(t)$, $\begin{vmatrix} -\lambda & 2 \\ -2 & -3-\lambda \end{vmatrix} = 0$, $\Rightarrow \lambda^2 + 3\lambda + 4 = 0$, $\Rightarrow \lambda = -\frac{3}{2} \pm j\frac{\sqrt{7}}{2}$

$Re(\lambda) < 0 \Rightarrow$ system is asymptotically stable.
Eigenvalues are complex. Therefore the transient response will be oscillatory (under damped).

(iii) $\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -5 \end{bmatrix}\underline{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}u(t)$, $\begin{vmatrix} -\lambda & 1 \\ -1 & -5-\lambda \end{vmatrix} = 0$, $\Rightarrow \lambda^2 + 5\lambda + 1 = 0$,

$$\Rightarrow \lambda = -\frac{5}{2} \pm \frac{\sqrt{21}}{2} = -0.21, -4.79$$

$Re(\lambda) < 0 \Rightarrow$ system is asymptotically stable.
Eigenvalues are real. Therefore the transient response will be non-oscillatory (over damped).

(iv) $\underline{x}(k+1) = \begin{bmatrix} 0.4 & 0.8 \\ -0.4 & 0.2 \end{bmatrix}\underline{x}(k) + \begin{bmatrix} 0.3 \\ 1 \end{bmatrix}u(k)$, $\begin{vmatrix} 0.4-\lambda & 0.8 \\ -0.4 & 0.2-\lambda \end{vmatrix} = 0$, $\Rightarrow \lambda^2 - 0.6\lambda + 0.4 = 0$,

$$\Rightarrow \lambda = -\frac{0.6}{2} \pm \frac{\sqrt{0.36 - 1.6}}{2} = 0.3 \pm j0.557$$

This is a discrete system so the stability rules are different, i.e asymptotically stable if $|\lambda| < 1$

$$|\lambda| = \sqrt{0.3^2 + 0.557^2} = 0.6327 < 1 \quad \Rightarrow \text{system is asymptotically stable.}$$

Eigenvalues are complex. Therefore the transient response will be oscillatory (under damped).

Q2(i) $\dot{\underline{x}} = \begin{bmatrix} 1 & 0 \\ -2 & \alpha \end{bmatrix}\underline{x} + \begin{bmatrix} \beta \\ 1 \end{bmatrix}u(t)$, $\begin{vmatrix} 1-\lambda & 0 \\ -2 & \alpha-\lambda \end{vmatrix} = 0$, $\Rightarrow (1-\lambda)(\alpha-\lambda) = 0$, $\Rightarrow \lambda = 1, \alpha$.

Since $\lambda_1 = 1 \Rightarrow Re(\lambda) > 0$ then system is unstable for all values of α.
Also since stability depends only on the eigenvalues of the A matrix β does affect stability.

(ii) $\dot{\underline{x}} = \begin{bmatrix} 0 & \beta \\ -2 & -3 \end{bmatrix}\underline{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}u(t)$, $\begin{vmatrix} -\lambda & -\beta \\ -2 & -3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 3\lambda - 2\beta = 0 \Rightarrow \lambda = \frac{-3 \pm \sqrt{9+8\beta}}{2}$

System is stable provided $Re(\lambda) < 0$. Thus we require $\sqrt{9+8\beta} < 3 \Rightarrow 9 + 8\beta < 9 \Rightarrow \beta < 0$.
Therefore system is asymptotically stable for $\beta < 0$, marginally stable for $\beta = 0$ and unstable for $\beta > 0$.

(iii) $\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -\beta & \alpha \end{bmatrix}\underline{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}u(t)$, $\begin{vmatrix} -\lambda & 1 \\ -\beta & \alpha-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \alpha\lambda + \beta = 0 \Rightarrow \lambda = \dfrac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$

$Re(\alpha \pm \sqrt{\alpha^2 - 4\beta}) < 0$ to be asymptotically stable (A.S).

Thus when $\beta > \dfrac{\alpha^2}{4}$ eigenvalues are complex and system is A.S. when $\alpha < 0$. For real roots system is A.S. provided $\sqrt{\alpha^2 - 4\beta} < \alpha \Rightarrow \alpha^2 - 4\beta < \alpha^2 \Rightarrow \beta > 0$. Thus:

For A.S. we require that $\alpha < 0, \beta > 0$, for marginally stable we require $\alpha = 0$ or $\beta = 0$ and for unstable we require $\alpha > 0$ or $\beta < 0$.

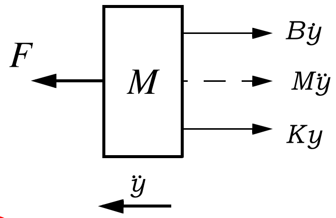
(iv) $\underline{x}(k+1) = \begin{bmatrix} \alpha-1 & 0 \\ -2 & \beta \end{bmatrix}\underline{x}(k) + \begin{bmatrix} 3 \\ 1 \end{bmatrix}u(k)$

$$\Rightarrow \lambda = \beta, \alpha - 1$$

For discrete case $|\lambda| < 1$. Thus $|\beta| < 1$ and $0 < \alpha < 2$ for asymptotic stability and marginally stable if $\beta = \pm 1$ or $\alpha = 0$ or 2. Unstable otherwise.

Q3 From the free body diagram

$$F = M\ddot{y} + B\dot{y} + Ky$$



(i) Letting $x_1 = y$, $x_2 = \dot{y}$ gives

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}F, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix}\underline{x}$$

Substituting for M=2 and K=10 gives $\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -5 & -\frac{B}{2} \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}F$

(ii) Stability determined by $|A - \lambda I| = 0 \rightarrow \begin{vmatrix} -\lambda & 1 \\ -5 & -\frac{B}{2} - \lambda \end{vmatrix} = \lambda^2 + \frac{B}{2}\lambda + 5 = 0$

$$\lambda = \frac{-\frac{B}{2} \pm \sqrt{\frac{B^2}{4} - 20}}{2} = -\frac{B}{4} \pm \frac{1}{4}\sqrt{B^2 - 80}$$

$|B| < \sqrt{80}$ gives complex eigenvalues and a stable system if $B > 0$.

If $|B| \geq \sqrt{80}$ roots are real and system is stable if $\sqrt{B^2 - 80} \leq B$ which is true for all $B \geq \sqrt{80}$.

Therefore system is marginally stable when $B = 0$, unstable when $B < 0$ and asymptotically stable when $B > 0$.

Q4.(i)(a) $\underline{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} \underline{x}_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k,\ y_k = \begin{bmatrix} 2 & 1 \end{bmatrix} \underline{x}_k + u_k$

$$G(z) = C(zI - A)^{-1}B + D = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} z & -1 \\ -2 & z+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 = \frac{\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} z+2 & 1 \\ 2 & z \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{z^2 + 2z - 2} + 1$$

$$= \frac{\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} z+3 \\ z+2 \end{bmatrix}}{z^2 + 2z - 2} + \frac{z^2 + 2z - 2}{z^2 + 2z - 2} = \frac{3z + 8 + z^2 + 2z - 2}{z^2 + 2z - 2} = \frac{z^2 + 5z + 6}{z^2 + 2z - 2}$$

(b) $\quad \dot{\underline{x}} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t) \quad , y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x}(t)$

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ -1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 1 \\ 1 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{(s+1)(s+2) - 1}$$

$$= \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+4 \\ 2s+3 \end{bmatrix}}{s^2 + 3s + 1} = \frac{2s + 3}{s^2 + 3s + 1}$$

(ii) (a) $\begin{vmatrix} -\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = 0 \quad , \qquad \Rightarrow \lambda^2 + 2\lambda - 2 = 0 \quad ,$

$$\Rightarrow \lambda = -\frac{2}{2} \pm \frac{\sqrt{4+8}}{2} = 0.732, -2.732$$

System is unstable as $|\lambda| > 1$

Denominator of transfer function is $z^2 + 2z - 2 = 0 \qquad \equiv \lambda^2 + 2\lambda - 2 = 0$

Hence, one of the poles lies outside the unit circle and hence the system is unstable.

(b) $\begin{vmatrix} -1-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0 \ , \qquad \Rightarrow \lambda^2 + 3\lambda + 1 = 0 \quad ,$

$$\Rightarrow \lambda = -\frac{3}{2} \pm \frac{\sqrt{9-4}}{2} = -2.618 \text{ or } -0.382$$

Both eigenvalues are < 0, hence system is asympototically stable.

Denominator of transfer function is $s^2 + 3s + 1 = 0 \qquad \equiv \lambda^2 + 3\lambda + 1 = 0$

Hence, both poles are negative - they lie of the LHS of the s-plane and this system is asympototically stable.