Lecture 6: Frequency Analysis

EE213 - Introduction to Signal Processing

Semester 1, 2021

Outline

- Frequency analysis of continuous and discrete signals.
  → Fourier series and Fourier transforms.
- Calculate Fourier series and transforms of elementary signals.

Frequency Analysis

- Frequency analysis

  → Decomposes a signal into its frequency (sinusoidal) components which are also known as spectrum.

  →Different signals have different spectra. Thus, spectrum provides an identity or a signature for the signal.

  →The process of obtaining the spectrum of a given signal using basic mathematical tools is known as frequency or spectral analysis.

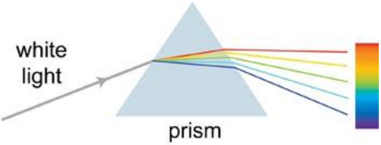
- Tools for frequency analysis

  →Fourier series

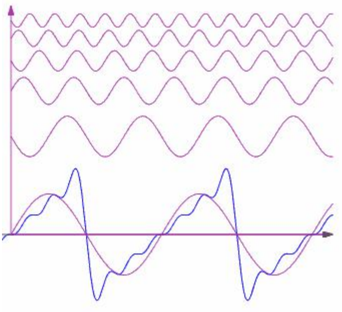
  → Fourier transform

Frequency Analysis...

- A good way to think about frequency analysis of a signal: As a mathematical version of a prism: breaking up a signal into different frequencies, just as a prism (or diffraction grating) breaks up light into different colours (which are light at different frequencies).



Prism breaks up light into different colours



Fourier series and transforms breaks up a signal into different frequencies

Fourier Series

- Let $x(t)$ be any real-valued periodic signal having period = T seconds, i.e.,

$$x(t) = x(t+T) \qquad (1)$$

- Then $x(t)$ can be expanded as a sum of sinusoids having frequencies that are integer multiples of $f_0 = \frac{1}{T}$ Hertz

$$x(t) = c_0 + c_1\cos(\omega_0 t - \theta_1) + (c_2\cos(2\omega_0 t - \theta_2)$$
$$+ c_3\cos(3\omega_0 t - \theta_3) + c_4\cos(4\omega_0 t - \theta_4) + \ldots \qquad (2)$$

- Recall that

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y) \qquad (3)$$

- We can rewrite x(t) as

$$x(t) = a_0 + a_1\cos(\omega_0 t) + a_2\cos(2\omega_0 t) + a_3\cos(3\omega_0 t) + \cdots$$
$$+ b_1\sin(\omega_0 t) + b_2\sin(2\omega_0 t) + b_3\sin(3\omega_0 t) + \cdots \qquad (4)$$

Fourier Series...

- Further, recall that

$$\cos = \frac{e^{jx} + e^{-jx}}{2} \qquad \sin = \frac{e^{jx} - e^{-jx}}{2j} \qquad (5)$$

- We can also rewrite x(t) as

$$\begin{aligned} x(t) \quad = \quad & x_0 + x_1 e^{j2\pi f_0 t} + x_2 e^{j4\pi f_0 t} + x_3 e^{j6\pi f_0 t} + x_4 e^{j8\pi f_0 t} + \cdots \\ & + x_1^* e^{-j2\pi f_0 t} + x_2^* e^{-j4\pi f_0 t} + x_3^* e^{-j6\pi f_0 t} + x_4^* e^{-j8\pi f_0 t} \ldots \quad (6) \end{aligned}$$

- Note that xn in the above equation is a complex number and X_n^* is the complex conjugate of xn. That is, if $x_n a_n$ = + jb_n then $x_n^* = a_n - jb_n$
- We can see that there are three different types of Fourier series, which expand the periodic signal x(t) as a sum of
  - → Phase-shifted sinusoids (Equation (2)).
  - → Sines plus cosines (Equation (4)).
  - → Complex exponentials (Equation (6)).
- They are equivalent but each has a different ease of computation.

Fourier Series...

Definition (Fourier Series)

We use the FS as sum of complex exponentials

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi n f_0 t}} \tag{7}$$

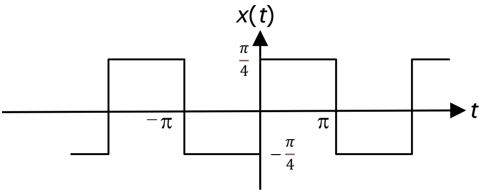
$$\boxed{x_n = \frac{1}{T}\int_0^T x(t) e^{-j2\pi n f_0 t} dt} \tag{8}$$

where x_n are the Fourier series coefficient and n = 0, ±1, ±2,. . ..

Fourier Series...

Example

Find the FS coefficient for the following period signal.



Zero-mean square wave

- Fundamental period: T = 2 π

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Example (continued)

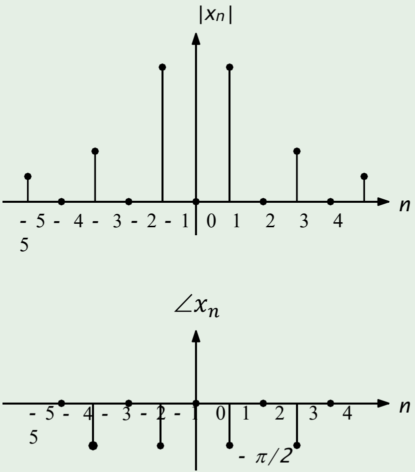
- DC component :

$$x_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-\frac{\pi}{4}) + \int_0^{\pi}(\frac{\pi}{4})\right)dt = 0$$

- Recall that $\int e^{at}dt = \frac{1}{a}e^{at} + c$, where c is a constant. Thus

$$\begin{aligned} x_n &= \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-\frac{\pi}{4})e^{-j2\pi nf_0 t}dt + \int_0^{\pi}(\frac{\pi}{4})e^{-j2\pi nf_0 t}dt\right) && (9)\\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-\frac{\pi}{4})e^{-jnt}dt + \int_0^{\pi}(\frac{\pi}{4})e^{-jnt}dt\right) && (10)\\ &= \begin{cases} 0 & n \text{ even} \\ -\frac{j}{2n} & n \text{ odd} \end{cases} && (11) \end{aligned}$$

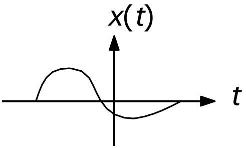
Example (continued)



Amplitude and phase representation of the Fourier series of zero-mean square wave

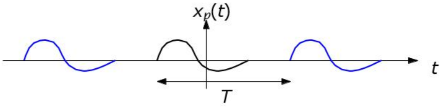
Fourier Transform

- Consider a continuous nonperiodic signal x(t)



A nonperiodic signal

- Let us consider a periodic signal $x_p(t)$ obtained by repeating $x(t)$ after a period of T seconds.



The expanded periodic signal

Fourier Transform...

- We can say that when T $\rightarrow\infty$, x_p(t) $\rightarrow$x(t)
- Fourier transform of x(t) can be obtained as a limiting case of Fourier series of x_p (t) as period T $\rightarrow\infty$.

Definition (Fourier Transform)

Fourier Transform of x(t) is given by

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt} \tag{12}$$

and inverse FT is given by

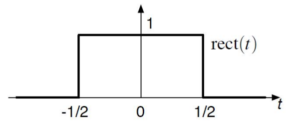
$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega} \tag{13}$$

- FT is also known as continuous time FT (CTFT).

Fourier Transform...

Example

Find the FT of the unit rectangular signal



FT of unit rectangular signal

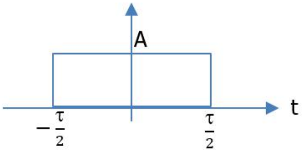
$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-1/2}^{1/2} (1)e^{-j\omega t}dt && (14) \\ &= \frac{1}{-j\omega}\left(e^{-j\omega t}\Big|_{-1/2}^{1/2} \right) && (15) \\ &= \frac{\sin(\omega/2)}{\omega/2} = \mathrm{sinc}(\omega/2) && (16) \end{aligned}$$

Fourier Transform...

Example (continued)

Time and frequency domain representation of the rectangular signal.

Time domain



rectangular signal

$$F(w)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$=\frac{A}{-j\omega}e^{-j\omega t}\Big|_{-\tau/2}^{-\tau/2}$$

$$=\frac{A}{-j\omega}\left(e^{-j\frac{\tau}{2}w}-e^{j\frac{\tau}{2}w}\right)$$

$$=\frac{A}{j\omega}\sin\left(\frac{w\tau}{2}\right)\ 2j$$

$$=A\tau\sin\left(\frac{w\tau}{2}\right)/\frac{w\tau}{2}$$

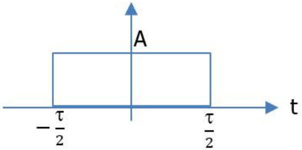
$$=A\tau\mathrm{Sa}\left(\frac{w\tau}{2}\right)$$

$$\mathrm{Sa}(x)=\frac{sinx}{x}$$

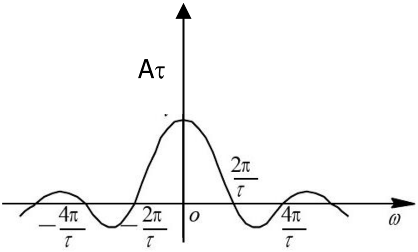
$$\mathrm{Sinc}(x)=\frac{sinx}{x}$$

Fourier Transform ...

$$A\,rect(t/\tau) \longleftrightarrow A\tau \mathrm{Sa}\left(\frac{w\tau}{2}\right)$$



$\longleftrightarrow$



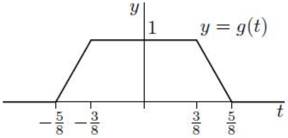
FT of rectangular signal

Fourier Transform...

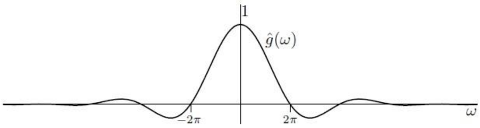
Example

Analyse the following signal in the frequency domain

Time domain



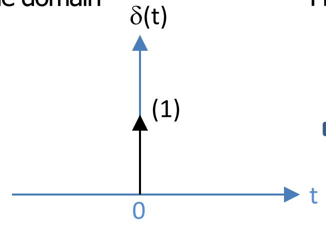
Frequency domain (FT)



- The FT of the signal given in this example looks somewhat like the Fourier transform in the previous example but exhibits faster decay for large ω. (what is the intuitive explanation?)

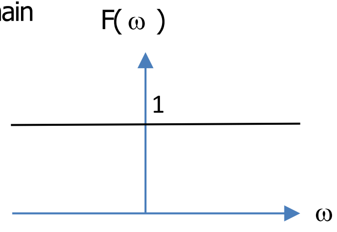
Fourier Transform ...

Time domain



Impulse signal

Frequency domain



FT of Impulse signal

$$\delta(t) \longleftrightarrow 1$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} dt$$

$$= 1$$

Fourier Transform...

- The Fourier transform is a major cornerstone in the analysis and representation of signals.
- However, computing FT of a signal may be a difficult task, especially when we have a very complex signal.
- Recall that a complex signal can be constructed from **elementary signals** using proper **basic operations**. This fact suggests that we can compute FT of a complex signal based on FT of basic signals. Two main ingredients for computing FT

  → Fourier transform table: show FT of common signals.
  → Fourier transform properties: relates the operations on time domain to the operations on frequency domain.

- From now on, we use notations $F()$ and F^{-1} ()to denote the FT and inverse FT, respectively.

Fourier Transform Properties

- Linearity: if

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \quad (17)$$

Then the FT of $x(t)$ is

$$F(x(t)) = \alpha_1 F(x_1(t)) + \alpha_2 F(x_2(t)) \quad (18)$$

- Time Shifting:

$$x(t - t_0) \overset{\mathcal{F}}{\leftrightarrow} e^{-j\omega t_0} X(\omega) \quad (19)$$

- Frequency shift:

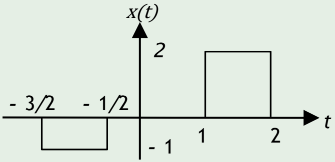
$$X(w-w_0) \overset{\mathcal{F}}{\leftrightarrow} e^{jw_0 t}\, x(t)$$

- Time and frequency scaling

$$x(at) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \quad (20)$$

Example

Find the FT of the following signal



Solution:

$$A\, rect(t/\tau) \longleftrightarrow A\tau Sa\left(\frac{\omega\tau}{2}\right)$$

$$x(t) = 2rect\left(t - \frac{3}{2}\right) - \mathrm{rect}(\mathrm{t}+1)$$

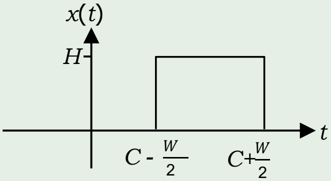
$$2rect\left(t - \frac{3}{2}\right) \longleftrightarrow 2e^{-j\frac{3}{2}\omega} Sa\left(\frac{\omega}{2}\right)$$

$$\mathrm{rect}(\mathrm{t}+1) \longleftrightarrow e^{j\omega} Sa\left(\frac{\omega}{2}\right)$$

$$x(t) \longleftrightarrow 2e^{-j\frac{3}{2}\omega} Sa\left(\frac{\omega}{2}\right) - e^{j\omega} Sa\left(\frac{\omega}{2}\right)$$

Example

Find the FT of the following signal



Solution:

$$A\, rect(t/\tau) \longleftrightarrow A\tau Sa\left(\frac{\omega\tau}{2}\right)$$

$$X(t) = H\, rect\left(\frac{t-C}{W}\right)$$

$$H\, rect\left(\frac{t-C}{W}\right) \longleftrightarrow HWe^{-j\omega C} Sa\left(\frac{W\omega}{2}\right)$$

Fourier Transform Properties...

- Differentiation:

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\leftrightarrow} j\omega X(\omega) \tag{21}$$

- Integration:

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega) \tag{22}$$

- The differentiation and integration properties are particularly useful when we use the Fourier transform to analyse RLC circuits.

Example:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, dt$$

$$\delta(t) \longleftrightarrow 1$$

$$u(t) \longleftrightarrow \frac{1}{jw} + \pi\delta(w)$$

Fourier Transform Properties...

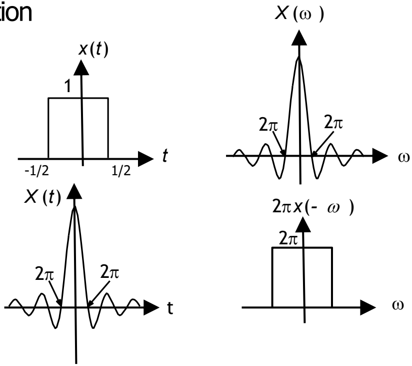
- Time/Frequency Duality: If

$$x(t) \overset{\mathcal{F}}{\leftrightarrow} X(\omega) \tag{23}$$

then

$$X(t) \overset{\mathcal{F}}{\leftrightarrow} 2\pi x(-\omega) \tag{24}$$

- Graphical illustration



Fourier Transform...

1. FT of DC signal

$$\delta(t) \longleftrightarrow 1$$

$$1 \longleftrightarrow 2\pi\delta(\omega)$$

2. FT of sinusoidal signal

$$e^{jw_0 t} \longleftrightarrow 2\pi(\omega - \omega_0)$$

$$e^{-jw_0 t} \longleftrightarrow 2\pi(\omega + \omega_0)$$

$$\cos(w_0 t) = \frac{1}{2}\left(e^{jw_0 t} + e^{-jw_0 t}\right)$$

$$\cos(w_0 t) \longleftrightarrow \pi[\delta(\omega\text{-}\omega_0) + \delta(\omega + \omega_0)]$$

$$\sin(w_0 t) = \frac{1}{2j}\left(e^{jw_0 t} - e^{-jw_0 t}\right)$$

$$\sin(w_0 t) \longleftrightarrow -j\pi[\delta(\omega\text{-}\omega_0) - \delta(\omega + \omega_0)]$$

Fourier Transform Table

$$
\begin{aligned}
f(t) &\leftrightarrow F(\omega) \\
\delta(t) &\leftrightarrow 1 \\
1 &\leftrightarrow 2\pi\delta(\omega) \\
e^{\pm j\omega_0 t} &\leftrightarrow 2\pi\delta(\omega \mp \omega_0) \\
\cos\omega_0 t &\leftrightarrow \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] \\
\sin\omega_0 t &\leftrightarrow -j\pi\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right] \\
rect(t/\tau) &\leftrightarrow \tau\,\mathrm{sinc}(\omega\tau/2) \\
\frac{W}{\pi}\mathrm{sinc}(Wt) &\leftrightarrow rect(\omega/(2W))
\end{aligned}
$$

Fourier transform table.

Discrete-time Fourier Series

DTFS applies to discrete-time periodic signals.

Definition (Discrete-time Fourier Series)

Consider x[n]= x[n + N], where N is the fundamental period. Then the DTFS representation of x[n] is given by

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n}, k = 0, 1, \ldots, N-1 \quad (25)$$

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jn\Omega_0 k} \quad (26)$$

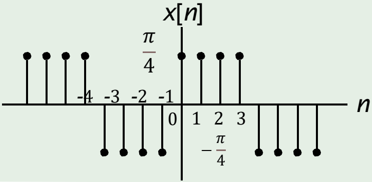
where $\Omega_0 = \frac{2\pi}{N}$ is the fundamental frequency.

- DTFS is similar to CTFS.
- Note that DTFS is a periodic sequence, i.e., X[k]= X[k + N].

Discrete-time Fourier Series...

Example

Determine DTFS of the following discrete periodic signal



Discrete-time Fourier Series...

$$X[-4] = \frac{1}{8}\ \sum_{n=-4}^{3}\ x[n]\, e^{j4\frac{2\pi}{8}n} = \frac{1}{8}\ \sum_{n=-4}^{3}\ x[n]\, e^{j\pi n}$$

$$= \frac{1}{8}\left(-\frac{\pi}{4}\right)\left(e^{-j4\pi} + e^{-j3\pi} + e^{-j2\pi} + e^{-j\pi} - 1 - e^{j\pi} - e^{j2\pi} - e^{j3\pi}\right)$$

$$= 0$$

$$X[-3] = \frac{1}{8}\ \sum_{n=-4}^{3}\ x[n]\, e^{j3\frac{2\pi}{8}n} = \frac{1}{8}\ \sum_{n=-4}^{3}\ x[n] e^{j\frac{3}{4}\pi n}$$

$$= \frac{1}{8}\left(-\frac{\pi}{4}\right)\left(e^{-j3\pi} + e^{-j\frac{9}{4}\pi} + e^{-j\frac{3}{2}\pi} + e^{-j\frac{3}{4}\pi} - 1 - e^{j\frac{3}{4}\pi} - e^{j\frac{3}{2}\pi} - e^{j\frac{9}{4}\pi}\right)$$

$$= \frac{1}{8}\left(-\frac{\pi}{4}\right)\left(-2 - 2j\sin\frac{9}{4}\pi - 2j\sin\frac{3}{2}\pi - 2j\sin\frac{3}{4}\pi\right)$$

$$= \frac{1}{8}\left(-\frac{\pi}{4}\right)\left(-2 + j2 - j2\sqrt{2}\right)$$

$$= \left(-\frac{\pi}{4}\right)\left(\frac{-1 + j(1-\sqrt{2})}{4}\right)$$

$$X[3] = \left(-\frac{\pi}{4}\right)\left(\frac{-1 - j(1-\sqrt{2})}{4}\right)$$

Discrete-time Fourier Series...

$$X[-2] = \frac{1}{8} \sum_{n=-4}^{3} x[n]\, e^{j2\frac{2\pi}{8}n} = \frac{1}{8} \sum_{n=-4}^{3} x[n]\, e^{j\frac{\pi}{2}n}$$

$$= \frac{1}{8}\left(-\frac{\pi}{4}\right)\left(e^{-j2\pi} + e^{-j\frac{3}{2}\pi} + e^{-j\pi} + e^{-j\frac{\pi}{2}} - 1 - e^{j\frac{\pi}{2}} - e^{j\pi} - e^{j\frac{3}{2}\pi}\right)$$

$$= 0$$

$$X[2] = 0$$

$$X[-1] = \frac{1}{8} \sum_{n=-4}^{3} x[n]\, e^{j\frac{2\pi}{8}n} = \frac{1}{8} \sum_{n=-4}^{3} x[n]\, e^{j\frac{\pi}{4}n}$$

$$= \frac{1}{8}\left(-\frac{\pi}{4}\right)\left(e^{-j\pi} + e^{-j\frac{3}{4}\pi} + e^{-j\frac{1}{2}\pi} + e^{-j\frac{\pi}{4}} - 1 - e^{j\frac{\pi}{4}} - e^{j\frac{1}{2}\pi} - e^{j\frac{3}{4}\pi}\right)$$

$$= \frac{1}{8}\left(-\frac{\pi}{4}\right)\left(-2 - 2j\sin\frac{3}{4}\pi - 2j\sin\frac{1}{4}\pi - 2j\sin\frac{1}{2}\pi \right)$$

$$= \left(-\frac{\pi}{4}\right)\left(\frac{-1-j(1+\sqrt{2})}{4}\right)$$

$$X[1] = \left(-\frac{\pi}{4}\right)\left(\frac{-1+j(1+\sqrt{2})}{4}\right)$$

$$X[0] = \frac{1}{8} \sum_{n=-4}^{3} x[n] = 0$$

Discrete-time Fourier Transform

- DTFT applies to discrete aperiodic signals.
- DTFT is similar to FT for continuous signals, except
  - → Integral is replaced by summation.
  - → x(t) is replaced by sampled values x[n].

Definition (Discrete Time Fourier Transform)

DTFT of x[n] is given by

$$\boxed{X(\omega) = \sum_{-\infty}^{\infty} x[n] e^{-j\omega n}} \tag{28}$$

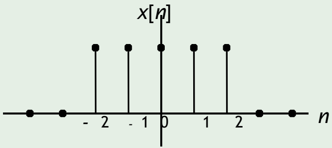
and inverse FT is given by

$$\boxed{x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega} \tag{29}$$

- Note that $X(\omega)$ is continuous and periodic (with a period of 2π, i.e., $X(\omega) = X(\omega + 2\pi)$).

Example

Find the DTFT of the following discrete-time signal

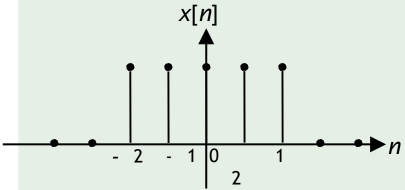


$$
\begin{aligned}
X(\omega) &= \sum_{-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-2}^{2} e^{-j\omega n} && (30)\\
&= e^{-j2\omega} + e^{-j\omega} + 1 + e^{j2\omega} + e^{j\omega} && (31)\\
&= e^{j2\omega}\left(1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega}\right) && (32)\\
&= e^{j2\omega}\left(\frac{1 - e^{-j5\omega}}{1 - e^{-j\omega}}\right) = \frac{e^{j\frac{5}{2}\omega} - e^{-j\frac{5}{2}\omega}}{e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega}} = \frac{\sin(\frac{5}{2}\omega)}{\sin(\frac{1}{2}\omega)} && (33)
\end{aligned}
$$

Discrete-time Fourier Transform...

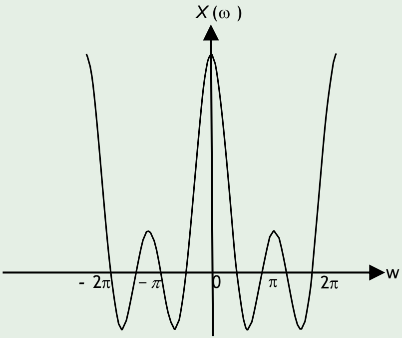
Example (continued)

Time domain



Discrete rectangular signal

Frequency domain (FT)



DTFT of discrete rectangular signal

Properties of DTFT

- Linearity: if

$$x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n] \tag{34}$$

Then the FT of x(t) is

$$F(x[n]) = \alpha_1 F(x_1[n]) + \alpha_2 F(x_2[n]) \tag{35}$$

- Time Shifting:

$$x[n - n_0] \stackrel{\mathcal{F}}{\leftrightarrow} e^{-j\omega n_0} X(\omega) \tag{36}$$

- Frequency shifting:

$$x[n] e^{j\omega_0 n} \stackrel{\mathcal{F}}{\leftrightarrow} X(\omega - \omega_0) \tag{37}$$

This property is also known as modulation property.

- Differentiation in frequency

$$\mathcal{F}^{-1}(j \frac{dX(\omega)}{d\omega}) = nx[n] \tag{38}$$

Summary of Frequency Analysis

Time domain	Periodic	Nonperiodic	
Continuous	Fourier series $x_n = \frac{1}{T}\int_0^T x(t)e^{-j2\pi n f_0 t}dt$ $x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi n t}$ $x(t)$ has period of T $f_0 = 1/T$	Fourier transform $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$	Nonperiodic
Discrete	DT Fourier series $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n}$ $x[n] = \sum_{k=0}^{N-1} X[k]e^{jn\Omega_0 k}$ $x[n]$ and $X[k]$ have period N $\Omega_0 = \frac{2\pi}{N}$	DT Fourier transform $X(\omega) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$ $x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega$ $X(\omega)$ has period 2π	Periodic
	Discrete	Continuous	Frequency domain