Engineering Mathematics 1 (Fall 2021)

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Students should be able to (after learning)

- Add, subtract and multiply complex numbers
- Convert complex numbers between Cartesian and polar forms
- Differentiate all commonly occurring functions including polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of a derivative, namely the derivative as a tangent and the derivative as a rate of change
- Integrate certain standard functions, constructed from polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of integration, namely the integral as the inverse of the derivative and the integral as the area under a curve
- Apply Taylor series to numerically approximate functions
- Apply Simpson's rule to numerically evaluate integrals
- Solve simple first and second order ordinary differential equations
- Apply and select the appropriate mathematical techniques to solve a variety of associated engineering problems

Lecture 9: Differentiation-Part 1

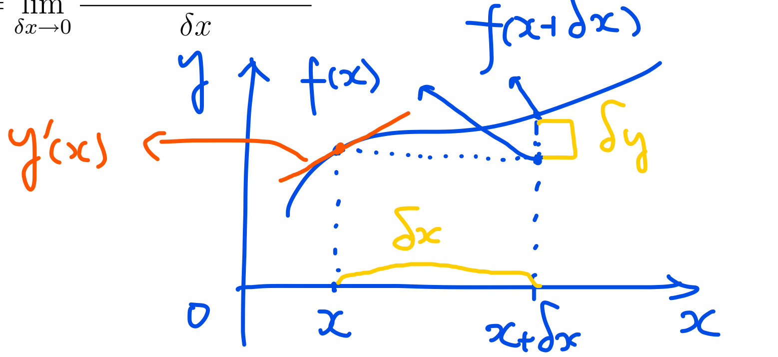
1. Definition of derivative

Given $y = f(x)$, derivative of y is defined as

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$\delta y = f(x+\delta x) - f(x)$

$\delta x \to 0, \ \delta y \to 0$



Understand $y'(x)$:

being the gradient at a point for a curve

OR, being the gradient of the tangent for a curve at a point

Second derivatives, third derivatives read as: dee two y by x squared

$$y'' = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right), \quad y''' = \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)$$

double prime y''

triple prime y'''

read as dee three y by x cubed

Ex1: $y = c$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Sol: $\because \delta y = c - c = 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{0}{\delta x} = 0 \quad \therefore \frac{dy}{dx} = 0 \quad \text{OR, } \boxed{c' = 0.}$$

Ex2: $y = x^2$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x$.

Sol: $y(x) = x^2$, $y(x+\delta x) = (x+\delta x)^2 = x^2 + 2x\delta x + (\delta x)^2$

$$\delta y = y(x+\delta x) - y(x) = 2x\delta x + (\delta x)^2$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{2x\delta x + (\delta x)^2}{\delta x} = \lim_{\delta x \to 0} (2x + \delta x) = 2x + 0 = 2x.$$

$\dfrac{dy}{dx} = 2x$, OR $\boxed{(x^2)' = 2x.}$

Ex3: $y = x^3$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$.

Sol: $\delta y = (x+\delta x)^3 - x^3 = 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} 3x^2 + \lim_{\delta x\to 0} 3x(\delta x) + \lim_{\delta x\to 0}(\delta x)^2 = 3x^2 + 3x\lim_{\delta x\to 0}\delta x + \lim_{\delta x\to 0}(\delta x)^2$$

$$= 3x^2 + 0 + 0 = 3x^2$$

$\dfrac{dy}{dx} = 3x^2$, OR $\boxed{(x^3)' = 3x^2}$

Ex4: $y = x^n$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$.

Sol: $\delta y = (x+\delta x)^n - x^n = C_n^1 x^{n-1}\delta x + C_n^2 x^{n-2}(\delta x)^2 + \dots + (\delta x)^n$

$(a+b)^n = a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \dots + b^n$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[C_n^1 x^{n-1} + C_n^2 x^{n-2}\delta x + \dots + (\delta x)^{n-1}\right] = \lim_{\delta x\to 0} n x^{n-1} + 0 + \dots + 0$$

$$= n x^{n-1}.$$

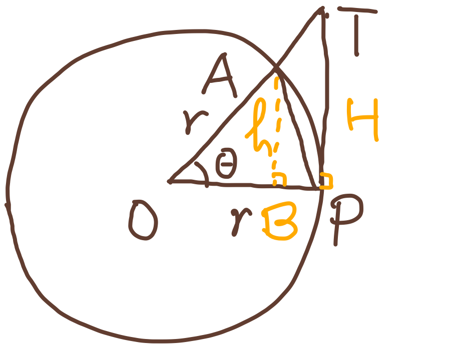
Ex5: $y = 3x^3 - 7x - 6$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 9x^2 - 7$.

$\dfrac{dy}{dx} = nx^{n-1}$, OR, $\boxed{(x^n)' = nx^{n-1}}$

Sol: Direct application

$$\frac{dy}{dx} = (3x^2 - 7x - 6)' = (3x^2)' - (7x)' - 6' = 6x - 7 - 0 = 6x - 7.$$

2. Standard derivatives



Ex1: Show that $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$.

Sol: $S_{\triangle OAP} < \text{Sector } OAP < S_{\triangle OTP}$

$\frac{1}{2}OP\cdot AB < \frac{1}{2}r^2\theta < \frac{1}{2}OP\cdot TP$, $AB = r\sin\theta$, $TP = r\tan\theta$

$\frac{1}{2}r\cdot r\sin\theta < \frac{r^2}{2}\theta < \frac{1}{2}r\cdot r\tan\theta$ $\therefore \sin\theta < \theta < \tan\theta$

$\therefore \dfrac{1}{\sin\theta} < \dfrac{1}{\theta} < \dfrac{\cos\theta}{\sin\theta}$ $\therefore 1 < \dfrac{\sin\theta}{\theta} < \cos\theta$ $\therefore \lim\limits_{\theta\to 0} 1 < \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} < \lim\limits_{\theta\to 0}\cos\theta = 1$

$\therefore \boxed{\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1.}$

Ex2: $y = \sin x$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos x$. use $\sin\alpha - \sin\beta = 2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$

Sol: $\delta y = \sin(x+\delta x) - \sin x = 2\cos\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0} 2\cos\left(x+\frac{\delta x}{2}\right)\cdot\lim_{\delta x\to 0}\sin\left(\frac{\delta x}{2}\right)\cdot\frac{1}{\delta x} = \cos x\cdot\lim_{\delta x\to 0}\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

$= \cos x\cdot 1 = \cos x$ $\therefore \dfrac{dy}{dx} = \cos x$, OR, $\boxed{(\sin x)' = \cos x}$

Ex3: $y = \cos x$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\sin x$. use $\cos\alpha - \cos\beta = -2\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$

Sol: $\delta y = \cos(x+\delta x) - \cos x = -2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = -\lim_{\delta x\to 0}\sin\left(x+\frac{\delta x}{2}\right)\cdot\lim_{\delta x\to 0}\frac{\sin(\delta x/2)}{\delta x/2} = -\sin x\cdot 1 = -\sin x.$$

$\dfrac{dy}{dx} = -\sin x$, OR, $\boxed{(\cos x)' = -\sin x.}$

Ex4: $y = e^x$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^x$. use $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots = \displaystyle\sum_{k=0}^{\infty}\frac{x^k}{k!}$

Sol: Direct application:

$$\frac{dy}{dx} = \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots\right)' = 0+1+\frac{2x}{2!}+\frac{3x^2}{3!}+\frac{4x^3}{4!}+\dots$$
$$= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots = e^x$$

3. Product rule and quotient rule

Let $y = uv$ and u, v are functions of x, $y' = u'v + uv'$, $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x}$, $u' = \dfrac{\mathrm{d}u}{\mathrm{d}x}$, $v' = \dfrac{\mathrm{d}v}{\mathrm{d}x}$.

Let $y = \dfrac{u}{v}$ and u, v are functions of x, $y' = \dfrac{u'v - uv'}{v^2}$, $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x}$, $u' = \dfrac{\mathrm{d}u}{\mathrm{d}x}$, $v' = \dfrac{\mathrm{d}v}{\mathrm{d}x}$.

Sol: $\because \delta x\to 0 \therefore \delta u\to 0,\ \delta v\to 0$

$$\delta y = (u+\delta u)(v+\delta v) - uv = u\,\delta v + \delta u\cdot v + \delta u\cdot\delta v$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = u\lim_{\delta x\to 0}\frac{\delta v}{\delta x} + v\lim_{\delta x\to 0}\frac{\delta u}{\delta x} + \lim_{\delta x\to 0}\frac{\delta u\,\delta v}{\delta x}$$

$$= u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx} + \frac{du}{dx}\cdot\lim_{\delta x\to 0}\delta v = \boxed{u\frac{dv}{dx} + v\frac{du}{dx} = \frac{dy}{dx}}$$

$$\delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v} = \frac{\delta u\cdot v - u\cdot\delta v}{(v+\delta v)v} \quad [\delta x\to 0 \Rightarrow \delta u\to 0,\ \delta v\to 0]$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{v(v+\delta v)}\left(\frac{\delta u}{\delta x}\cdot v - u\frac{\delta v}{\delta x}\right) = \lim_{\delta x\to 0}\frac{1}{v(v+\delta v)}\left[\lim_{\delta x\to 0}\frac{\delta u}{\delta x}\cdot v - \lim_{\delta x\to 0}u\frac{\delta v}{\delta x}\right]$$

$$\boxed{= \frac{1}{v^2}\left[\frac{du}{dx}\cdot v - u\frac{dv}{dx}\right] = \frac{dy}{dx}}$$