5. Block Diagram Analysis

5.1 Introduction

- In section 3 of the notes, we examined the modelling of various static and dynamical systems and ended up with mathematical models in the form of linear differential equations.

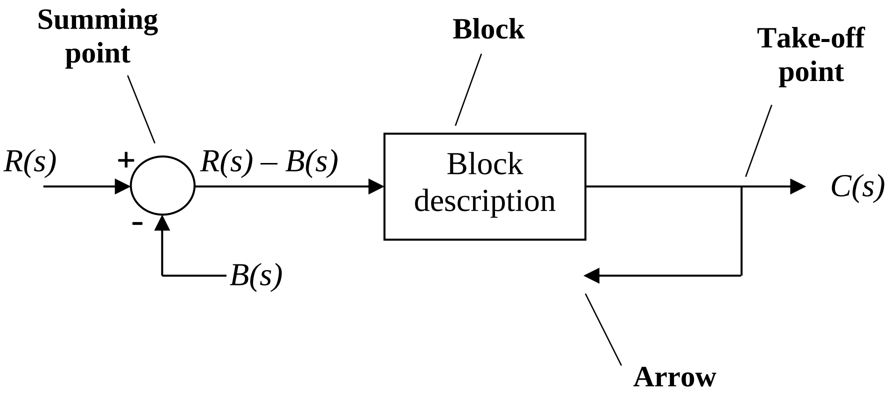
- In section 4 we effectively described the same systems as transfer function models, using the Laplace transform.

- Large, complicated systems may consist of many components, each of which may be represented by a transfer function.

- A **block diagram** shows these transfer functions and illustrates the functional relationship between them.

- In general, a block diagram consists of a specific configuration of four types of elements:

  - blocks,
  - summing points,
  - take-off points and
  - arrows representing unidirectional signal flow.



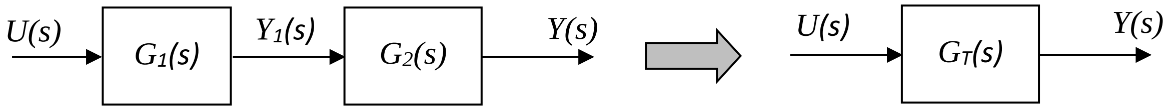
5.2 Block diagram algebra

- Complicated block diagrams can be simplified by combining different blocks together in a step-by-step fashion using block diagram algebra.

- There are 3 basic operations, depending on the type of connection between blocks.

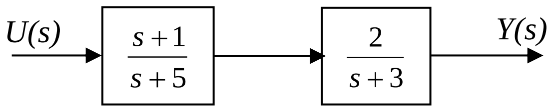
- Series, parallel and feedback connections are possible. We will now look at each of these in turn.
- **Series (or cascade) connection** – the blocks are connected as shown below.

- **Blocks in series** are simply combined by **multiplying** them together.



$$\mathbf{G_T(s) = G_1(s).G_2(s)}$$

- ***Ex. 5.1 Simplify the following system:***



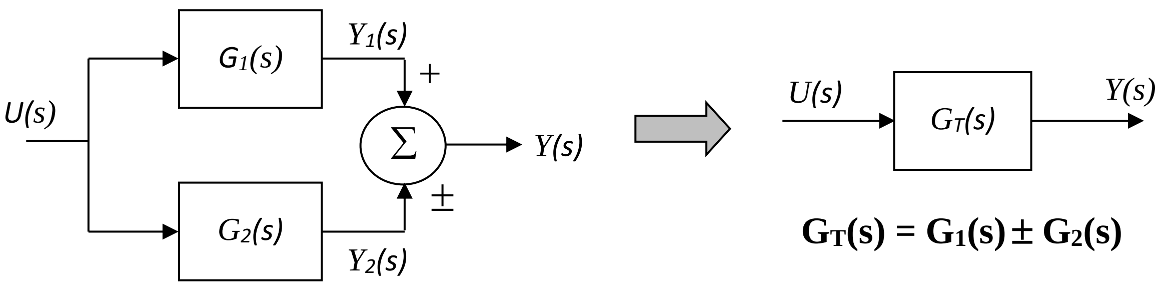
Solution:

$$\frac{Y(s)}{U(s)} = \left(\frac{s+1}{s+5}\right)\left(\frac{2}{s+3}\right)$$

$$= \frac{2(s+1)}{(s+3)(s+5)}$$

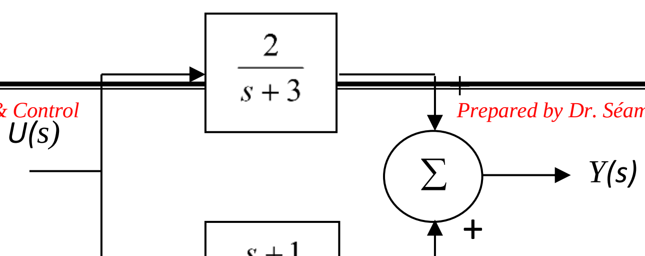
$$= \frac{2s+2}{s^2+8s+15}$$

- **Parallel connection** – the blocks are connected as shown below.
- **Blocks in parallel** are simply combined by **adding** them together.



$$\mathbf{G_T(s) = G_1(s) \pm G_2(s)}$$

- ***Ex. 5.2 Simplify the following system:***



Solution:

$$\frac{Y(s)}{U(s)} = \left(\frac{s+1}{s+5}\right) + \left(\frac{2}{s+3}\right) \qquad = \frac{(s+1)(s+3)+2(s+5)}{(s+3)(s+5)}$$

$$= \frac{s^2+4s+3+2s+10}{s^2+8s+15} \qquad = \frac{s^2+6s+13}{s^2+8s+15}$$

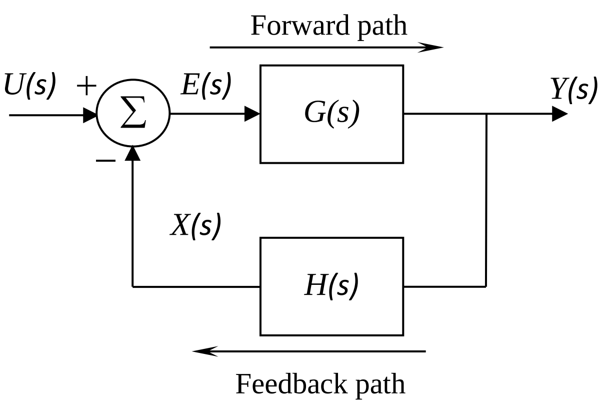
- **Feedback connection** – when the output of a block is fed back to the input of an earlier block in the block diagram, a feedback mechanism is introduced.

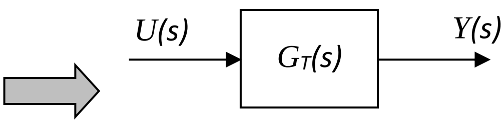
"The good news is we're getting a lot of feedback. The bad news is we're getting a lot of feedback."

- This is usually referred to as ***closing the loop***.

- The resulting transfer function is called the **closed loop transfer function**.

- The fundamental closed loop diagram for a feedback system is as follows:





$$\mathbf{G_T(s)} = \frac{\mathbf{G}}{\mathbf{1 + GH}}$$

- This is known as the **canonical block diagram** and is characterised by:

  - *G(s)* is the direct or forward path transfer function.

- $H(s)$ is the feedback transfer function (a unity feedback system has $H(s) = 1$).

- $\dfrac{X(s)}{U(s)} = G(s)H(s)$, **with feedback disconnected**, is the **open-loop transfer function (OLTF).**

- $\dfrac{Y(s)}{U(s)} = \dfrac{G(s)}{1+G(s)H(s)}$ is the **closed-loop transfer function (CLTF).**

- Looking at the block diagram, we can derive the CLTF as follows:

$$Y(s) = G(s)E(s)$$

$$E(s) = U(s) - X(s) \qquad \Rightarrow Y(s) = G(s)\big(U(s) - X(s)\big)$$

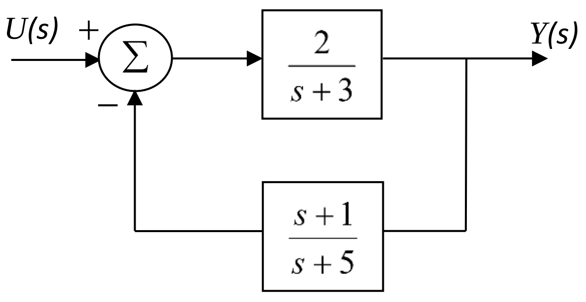
$$X(s) = H(s)Y(s) \qquad \Rightarrow Y(s) = G(s)\big(U(s) - H(s)Y(s)\big)$$

$$\Rightarrow Y(s) = G(s)U(s) - G(s)H(s)Y(s)$$

$$\Rightarrow Y(s)\big(1 + G(s)H(s)\big) = G(s)U(s)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{G(s)}{1+G(s)H(s)} = \text{CLTF}$$

- ***Ex. 5.3 Simplify the following system:***



Solution:

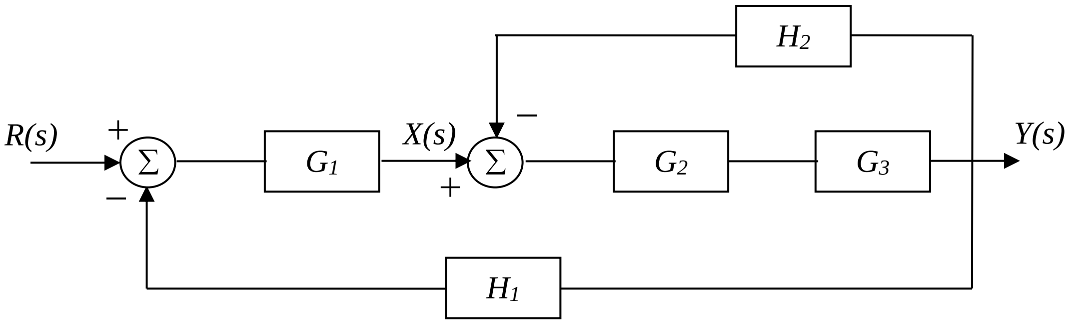
$$\frac{Y(s)}{U(s)} = \frac{\left(\dfrac{2}{s+3}\right)}{1+\left(\dfrac{2}{s+3}\right)\left(\dfrac{s+1}{s+5}\right)}$$

Multiplying up and down by (s + 3)(s + 5) gives:

$$\frac{2(s+5)}{(s+3)(s+5)+2(s+1)}$$

$$= \frac{2s+10}{s^2+8s+15+2s+2} \qquad = \frac{2s+10}{s^2+10s+17}$$

- ***Ex. 5.4 Find the closed-loop transfer function (CLTF) for the following system:***



Solution:

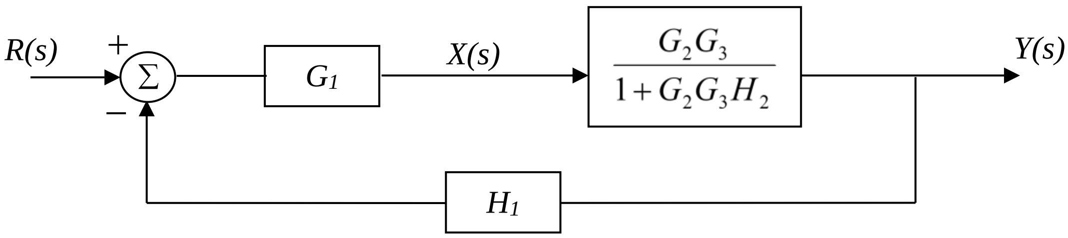
Firstly, we can combine the two transfer functions in cascade, $G_2(s)$ and $G_3(s)$ to give one transfer function:

$$G_2G_3$$

Then, the inner loop can be reduced using the closed-loop formula for a canonical block diagram to give:

$$\frac{Y(s)}{X(s)} = \frac{G_2G_3}{1+G_2G_3H_2}$$

Hence:



Finally, we can, once again, combine the two blocks in the forward path and use the canonical formula to find the closed-loop transfer function, as follows:

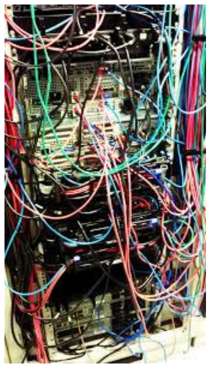
$$\frac{Y(s)}{R(s)} = \frac{\dfrac{G_1G_2G_3}{1+G_2G_3H_2}}{1+\left(\dfrac{G_1G_2G_3}{1+G_2G_3H_2}\right)H_1} = \frac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2G_3H_1}$$

5.3 Dealing with multiple inputs

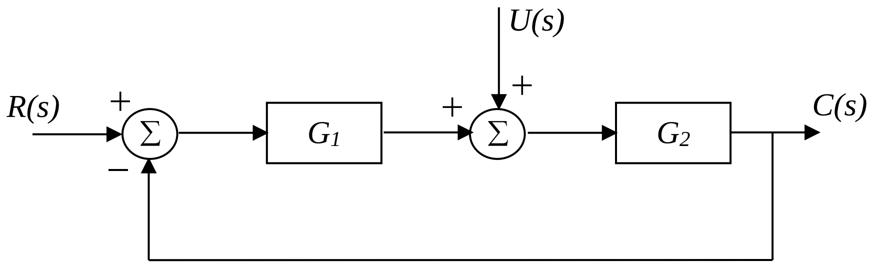
- When multiple inputs are present in a linear system, the *superposition theorem* is used.

- In other words, each input is treated independently of the others and the resultant output is given by the sum of the outputs due to each input separately.

- The procedure is as follows:
  - Set all inputs to zero, except for one.
  - Calculate the response due to the chosen input acting alone.
  - Repeat the above two steps for all inputs.
  - Algebraically add all of the responses (outputs) determined.

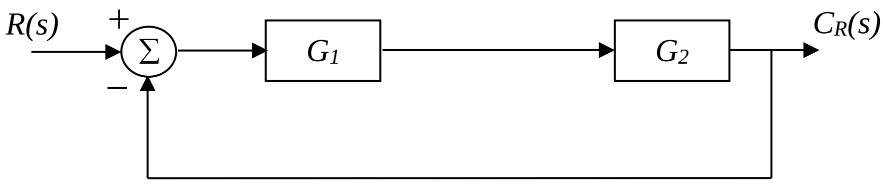


- ***Ex 5.5 Simplify the following system:***



Solution:

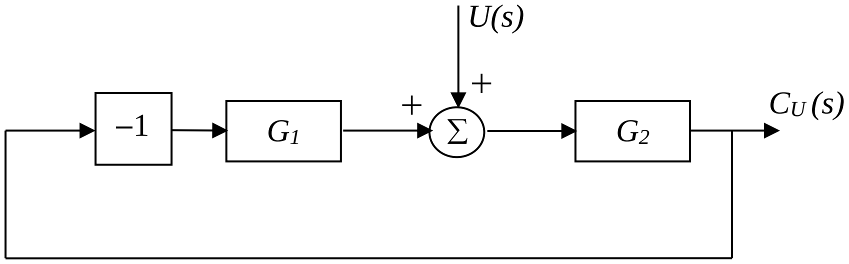
Firstly, set $U(s) = 0$. Thus, the system reduces to:



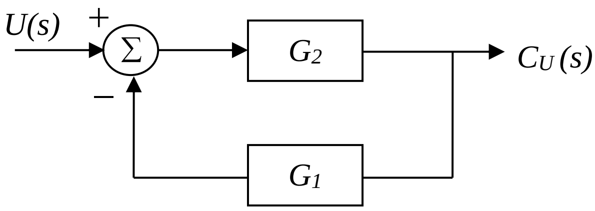
Hence, the output due to input R is given by:

$$C_R(s) = \frac{G_1G_2}{1+G_1G_2}R$$

Now, set $R(s) = 0$. Thus, the system reduces to:



This is equivalent to:



Hence, the output due to input U is given by:

$$C_U(s) = \frac{G_2}{1+G_1G_2}U$$

Finally, the total response is:

$$C(s) = C_R(s) + C_U(s)$$

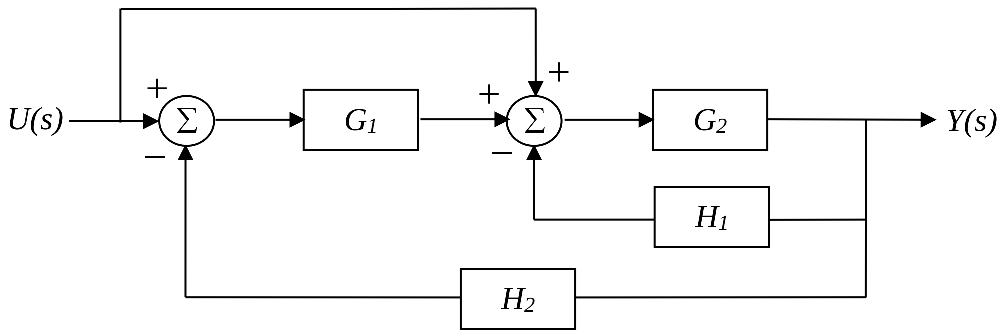
$$= \frac{G_1G_2}{1+G_1G_2}R + \frac{G_2}{1+G_1G_2}U$$

$$= \frac{G_1G_2R(s) + G_2U(s)}{1+G_1G_2}$$

5.4 Block diagram reduction

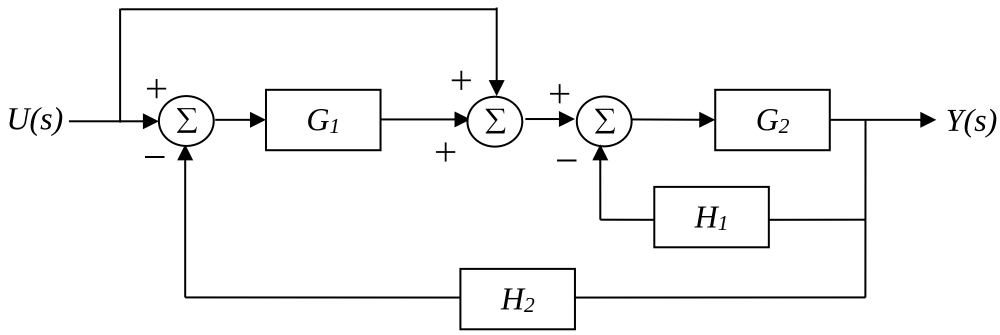
- The procedure for block diagram reduction is:
  - Assign intermediate variables to signals between blocks.
  - Write a complete set of equations for the system.
  - Eliminate the intermediate variables.
  - Where possible, use standard results (series, parallel, feedback) to simplify analysis.

- ***Ex 5.6 Determine the CLTF for the system represented by the following block diagram:***

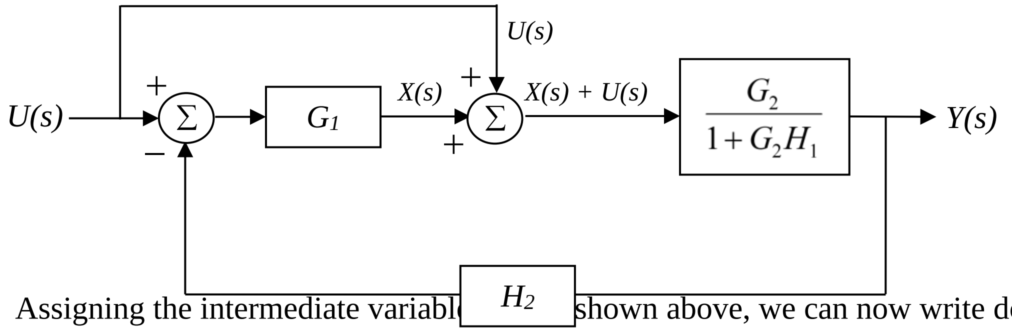


Solution:

For clarity, we can redraw this diagram as follows:



We can now replace the inner loop containing G_2 and H_1 as follows:



Assigning the intermediate variabl[illegible] shown above, we can now write down the following set of equations for the system:

$$Y(s) = \frac{G_2}{1+G_2H_1}\left(X(s)+U(s)\right)$$

and

$$X(s) = G_1\left(U(s) - H_2Y(s)\right)$$

Eliminating $X(s)$, by substituting the second equation into the first gives:

$$Y(s) = \frac{G_2}{1+G_2H_1}\left(G_1\left(U(s)-H_2Y(s)\right)+U(s)\right)$$

$$\Rightarrow Y(s) = \frac{G_1G_2}{1+G_2H_1}U(s) - \frac{G_1G_2H_2}{1+G_2H_1}Y(s) + \frac{G_2}{1+G_2H_1}U(s)$$

$$\Rightarrow Y(s)\left(1+\frac{G_1G_2H_2}{1+G_2H_1}\right) = U(s)\left(\frac{G_1G_2}{1+G_2H_1}+\frac{G_2}{1+G_2H_1}\right)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\left(\frac{G_1G_2}{1+G_2H_1}+\frac{G_2}{1+G_2H_1}\right)}{\left(1+\frac{G_1G_2H_2}{1+G_2H_1}\right)}$$

$$= \frac{G_1G_2+G_2}{1+G_2H_1+G_1G_2H_2} \qquad = \text{CLTF}$$

"Look, Bernie, all I'm saying is
I think you're riding the new guy pretty hard."