6. Solving the state equations

6.1 Discrete-time solution

- The general linear discrete-time state-space model is given by:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \end{aligned}$$

- Here, **we will restrict our attention to single-input single-output (SISO) systems**, i.e. the input and output are scalar quantities:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) \\ y(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}u(k) \end{aligned}$$

where:

$$\mathbf{A} \in R^{n\,x\,n},\quad \mathbf{B} \in R^{n\,x\,1},\quad \mathbf{C} \in R^{1\,x\,n},\quad \mathbf{D} \in R^{1\,x\,1},\quad \mathbf{x}(k) \in R^{n},\quad u(k), y(k) \in R^{1}$$

- The dynamics of the system are described by the state equation.

- Just like other difference/differential equations, it is solved by considering the unforced and forced responses separately.

Unforced response:

- This is where the input is set to zero for all time, i.e.:

$$u(k) = 0,\ \forall k$$

- The state equation then becomes:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k)$$

- Given an initial condition $\mathbf{x}(0)$ we can calculate the response of the system as follows:

$$\begin{aligned} &\mathbf{x}(1) = \mathbf{A}\mathbf{x}(0) \\ &\mathbf{x}(2) = \mathbf{A}\mathbf{x}(1) = \mathbf{A}^2\mathbf{x}(0) \\ &\mathbf{x}(3) = \mathbf{A}\mathbf{x}(2) = \mathbf{A}^3\mathbf{x}(0) \\ &\text{etc...} \end{aligned}$$

- Therefore, in general: $\mathbf{x}(k) = \mathbf{A}^k\mathbf{x}(0)$

- The matrix A^k is known as the (***discrete-time***) **state transition matrix**.

Forced response:

- Now, consider the situation when the input is not zero:

$$
\begin{aligned}
\boldsymbol{x}(1) &= \boldsymbol{A}\boldsymbol{x}(0) + \boldsymbol{B}u(0) \\
\boldsymbol{x}(2) &= \boldsymbol{A}\boldsymbol{x}(1) + \boldsymbol{B}u(1) \\
&= \boldsymbol{A}\big(\boldsymbol{A}\boldsymbol{x}(0) + \boldsymbol{B}u(0)\big) + \boldsymbol{B}u(1) \\
&= \boldsymbol{A}^2\boldsymbol{x}(0) + \boldsymbol{A}\boldsymbol{B}u(0) + \boldsymbol{B}u(1) \\
\boldsymbol{x}(3) &= \boldsymbol{A}\boldsymbol{x}(2) + \boldsymbol{B}u(2) \\
&= \boldsymbol{A}\big(\boldsymbol{A}^2\boldsymbol{x}(0) + \boldsymbol{A}\boldsymbol{B}u(0) + \boldsymbol{B}u(1)\big) + \boldsymbol{B}u(2) \\
&= \boldsymbol{A}^3\boldsymbol{x}(0) + \boldsymbol{A}^2\boldsymbol{B}u(0) + \boldsymbol{A}\boldsymbol{B}u(1) + \boldsymbol{B}u(2)
\end{aligned}
$$

etc...

- In general, we get:

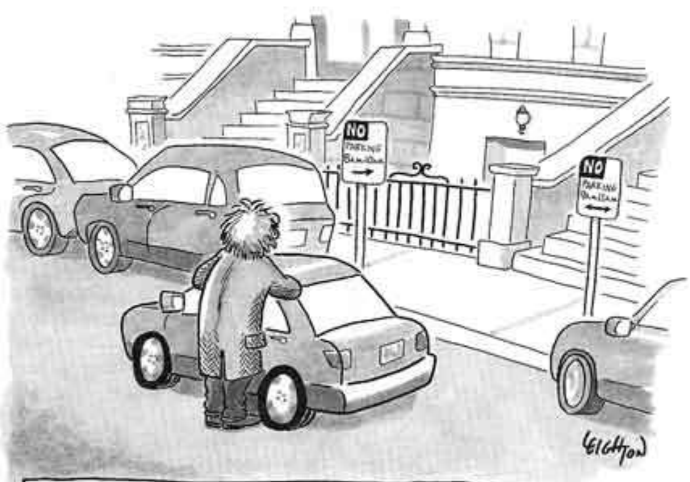
$$
\begin{aligned}
\boldsymbol{x}(k) &= \boldsymbol{A}^k\boldsymbol{x}(0) + \boldsymbol{A}^{k-1}\boldsymbol{B}u(0) + \boldsymbol{A}^{k-2}\boldsymbol{B}u(1) + \ldots + \boldsymbol{A}^0\boldsymbol{B}u(k-1) \\
&= \underbrace{\boldsymbol{A}^k\boldsymbol{x}(0)}_{\text{unforced response}} + \underbrace{\sum_{i=1}^{k}\boldsymbol{A}^{k-i}\boldsymbol{B}u(i-1)}_{\text{forced response when } \mathbf{x}(0) = 0}
\end{aligned}
$$

Output calculation:

- Once the state has been determined, the output is easily computed as:

$$
\begin{aligned}
y(k) &= \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}u(k) \\
&= \boldsymbol{C}\boldsymbol{A}^k\boldsymbol{x}(0) + \sum_{i=1}^{k}\left(\boldsymbol{C}\boldsymbol{A}^{k-i}\boldsymbol{B}u(i-1)\right) + \boldsymbol{D}u(k)
\end{aligned}
$$

- Note, that ***D*** is usually zero.



EINSTEIN PONDERS THE MYSTERIES OF SPACE AND TIME

- **Example 6.1:** Determine the output $y(k)$ for the system:

$$\boldsymbol{A} = \begin{bmatrix} 0.5 & 0 \\ 0.25 & 0.25 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 0.5 & 0 \end{bmatrix}, \quad \boldsymbol{D} = [0]$$

given that the input is $u(k) = \delta(k)$ and $\boldsymbol{x}(0) = [0\ 0]^T$.

Solution:

$$y(k) = \underbrace{\begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0.25 & 0.25 \end{bmatrix}^k \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0} + \sum_{i=1}^{k} \begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0.25 & 0.25 \end{bmatrix}^{k-i} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \delta(i-1)$$

$\delta(i-1) = 1$ when $i = 1$
$= 0\ \forall\ i \neq 1$

$$\Rightarrow y(k) = \begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0.25 & 0.25 \end{bmatrix}^{k-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \boldsymbol{C}\boldsymbol{A}^{k-1}\boldsymbol{B}$$

$$y(1) = \begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0.5$$

$$y(2) = \begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = 0.25$$

$$y(3) = \begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0.25 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix} = 0.125$$

In general:

$$y(k) = \underbrace{\begin{bmatrix} 0.5 & 0 \end{bmatrix}}_{\boldsymbol{C}} \underbrace{\begin{bmatrix} 0.5^{k-1} \\ 0.5^{k-1} \end{bmatrix}}_{\boldsymbol{A}^{k-1}\boldsymbol{B} = \boldsymbol{x}(k)} = 0.5(0.5)^{k-1} = 0.5^k$$

- In this example we have succeeded in obtaining the solution of $y(k)$ (and $\boldsymbol{x}(k)$) in ***closed form***.

- This implies that the solution is in terms of k only and, as a result, it is easy to work out answers for any value of k.

Notes:

- It is worth noting that we are fortunate in being able to get the general '*closed-form solution*' for the elements of the state vector.

- The special case of zero initial conditions and (unit) impulse input helps to simplify things considerably.

- More computationally attractive methods are needed for calculating the transition matrix for general problems.

- We would like a solution that is easy to implement using computers.

- Two popular methods exist – one uses canonical state transformations (to diagonalise ***A***), the other uses the Cayley-Hamilton theorem.

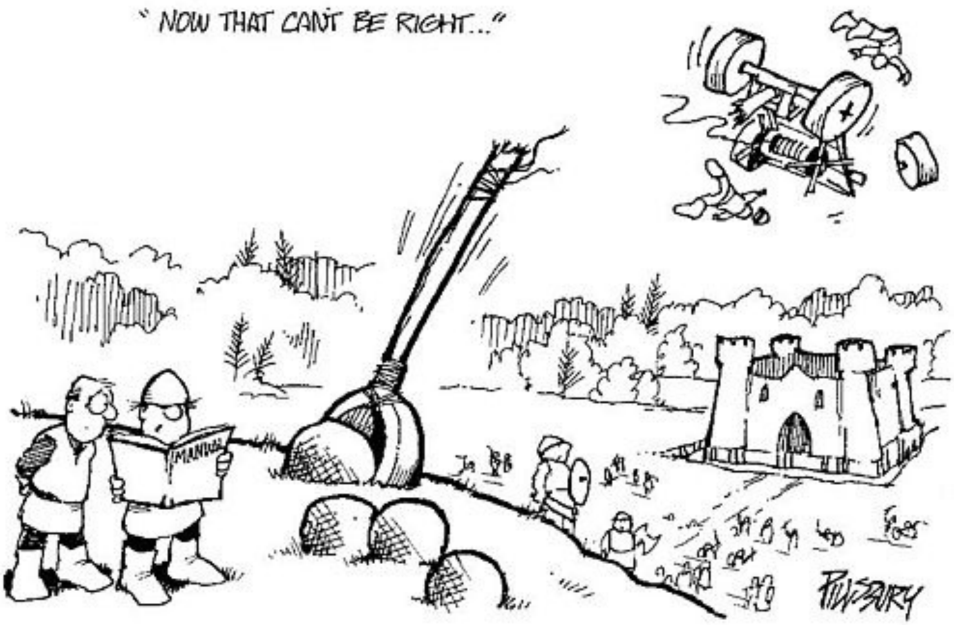
- In this module, we will only consider the method of state transformation (and, in particular, using the modal matrix).

- *Aside – the Cayley-Hamilton theorem states that an n x n matrix* ***A*** *satisfies its own characteristic equation:*

$$p(\lambda) = |\lambda \mathbf{I} - \mathbf{A}| = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$$

In other words, if we replace λ with ***A****, then p(****A****) = 0.*

This theorem is the basis of an alternative method used to calculate the state transition matrix – please refer to a suitable text book for detail.



6.2 State transformations

- State-space representations are not unique. There are many selections of state variables which can describe a system.

- State variables can be real or fictitious.

- Some state representations lead to computationally attractive forms.

- Thus, by using the appropriate state transformation, we can obtain a state-space representation that leads to significantly easier computation.

- Consider the following state transformation:

$$\mathbf{x}(k) = \mathbf{Tz}(k)$$

- Here, $\boldsymbol{T}$ is any constant non-singular n x n matrix. Since $\boldsymbol{T}$ is constant, we can write:

$$\mathbf{x}(k+1) = \mathbf{Tz}(k+1)$$

- Substituting these expressions into the state equation $\mathbf{x}(k+1) = \mathbf{Ax}(k) + \mathbf{B}u(k)$ gives:

$$\mathbf{Tz}(k+1) = \mathbf{ATz}(k) + \mathbf{B}u(k)$$

- Finally, premultiplying by $\boldsymbol{T}^{-1}$ gives us the new state equation:

$$\begin{aligned}\mathbf{z}(k+1) &= \mathbf{T}^{-1}\mathbf{ATz}(k) + \mathbf{T}^{-1}\mathbf{B}u(k) \\ &= \mathbf{A}_Z\mathbf{z}(k) + \mathbf{B}_Z u(k)\end{aligned}$$

- The new output equation is:

$$\begin{aligned}y(k) &= \mathbf{CTz}(k) + \mathbf{D}u(k) \\ &= \mathbf{C}_Z\mathbf{z}(k) + \mathbf{D}u(k)\end{aligned}$$

- Note – the only condition on $\boldsymbol{T}$ is that it must be non-singular (i.e. invertible).

- Hence, there are an infinite number of state representations for the system.

- **We are interested in finding a representation that leads to a diagonal** A **matrix** as this will make our computation significantly easier.

- So the issue becomes one of finding a suitable $\boldsymbol{T}$ so that $\boldsymbol{A}_Z = \boldsymbol{T}^{-1}\boldsymbol{AT}$ is diagonal.

- Now consider a matrix $\boldsymbol{A}$ with distinct eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ and let $\boldsymbol{m}_1, \boldsymbol{m}_2, \ldots \boldsymbol{m}_n$ be the corresponding eigenvectors.

- The ***modal matrix***, $\boldsymbol{M}$, is formed from these eigenvectors: $\boldsymbol{M} = \begin{bmatrix} \boldsymbol{m}_1 & \boldsymbol{m}_2 & \cdots & \boldsymbol{m}_n \end{bmatrix}$

- Note that by definition, $\boldsymbol{A}\boldsymbol{m}_i = \lambda_i \boldsymbol{m}_i$.

- Hence:

$$\begin{aligned} \boldsymbol{AM} &= \boldsymbol{A}\begin{bmatrix} \boldsymbol{m}_1 & \boldsymbol{m}_2 & \cdots & \boldsymbol{m}_n \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{A}\boldsymbol{m}_1 & \boldsymbol{A}\boldsymbol{m}_2 & \cdots & \boldsymbol{A}\boldsymbol{m}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\boldsymbol{m}_1 & \lambda_2\boldsymbol{m}_2 & \cdots & \lambda_n\boldsymbol{m}_n \end{bmatrix} \\ &= \boldsymbol{M}\Lambda \end{aligned}$$

  where:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

- Since: $\boldsymbol{AM} = \boldsymbol{M}\Lambda \;\Rightarrow\; \Lambda = \boldsymbol{M}^{-1}\boldsymbol{AM}$

- Thus, we can conclude that in order to obtain a diagonal state matrix, **we choose the modal matrix $\boldsymbol{M}$ as the state transformation matrix.**

- A state transformation by the modal matrix produces a set of n independent first order difference equations:

$$\begin{gathered} z_1(k+1) = \lambda_1 z_1(k) + b_1 u(k) \\ \vdots \\ z_n(k+1) = \lambda_n z_n(k) + b_n u(k) \end{gathered}$$

6.2.1 Determining the modal matrix

- Recall that eigenvalues are calculated as the roots of the **matrix characteristic equation**:

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = |\lambda \boldsymbol{I} - \boldsymbol{A}| = 0$$

- The eigenvectors are then determined either by:

  - solving $\boldsymbol{A}\boldsymbol{m}_i = \lambda_i \boldsymbol{m}_i$

  or

  - evaluating the cofactors of a row of $(\lambda \boldsymbol{I} - \boldsymbol{A})$

- **Example 6.2:** Determine the modal matrix for $\boldsymbol{A}=\begin{bmatrix}-1 & 2\\ -1 & -4\end{bmatrix}$

Solution: Eigenvalues:

$$|\lambda \boldsymbol{I}-\boldsymbol{A}|=0\Rightarrow\left|\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}-\begin{bmatrix}-1 & 2\\ -1 & -4\end{bmatrix}\right|=0\Rightarrow\begin{vmatrix}\lambda+1 & -2\\ 1 & \lambda+4\end{vmatrix}=0$$

$$\Rightarrow(\lambda+1)(\lambda+4)+2=0$$

$$\Rightarrow\lambda^2+5\lambda+6=0$$

$$\Rightarrow(\lambda+2)(\lambda+3)=0$$

$$\Rightarrow\lambda_1=-2,\ \lambda_2=-3$$

Eigenvectors – ***method 1:***

$$\boldsymbol{Am}=\lambda\boldsymbol{m}\Rightarrow\begin{bmatrix}-1 & 2\\ -1 & -4\end{bmatrix}\begin{bmatrix}m_1\\ m_2\end{bmatrix}=\lambda\begin{bmatrix}m_1\\ m_2\end{bmatrix}\Rightarrow\begin{matrix}-m_1+2m_2=\lambda m_1\\ -m_1-4m_2=\lambda m_2\end{matrix}$$

$$\text{Solving for } \lambda=-2\Rightarrow\begin{matrix}-m_1+2m_2=-2m_1\\ -m_1-4m_2=-2m_2\end{matrix}\Rightarrow\left.\begin{matrix}m_1=-2m_2\\ m_1=-2m_2\end{matrix}\right\rangle\Rightarrow\begin{bmatrix}-2\\ 1\end{bmatrix}$$

$$\text{Solving for } \lambda=-3\Rightarrow\begin{matrix}-m_1+2m_2=-3m_1\\ -m_1-4m_2=-3m_2\end{matrix}\Rightarrow\left.\begin{matrix}m_1=-m_2\\ m_1=-m_2\end{matrix}\right\rangle\Rightarrow\begin{bmatrix}1\\ -1\end{bmatrix}$$

$$\Rightarrow\boldsymbol{M}=\begin{bmatrix}-2 & 1\\ 1 & -1\end{bmatrix}$$

eigenvector for $\lambda=-3$

eigenvector for $\lambda=-2$

Eigenvectors – ***method 2 (cofactors)*** – here, the eigenvectors are obtained by writing the cofactors of **any** row of ($\lambda\boldsymbol{I}-\boldsymbol{A}$) in column format (note – make sure to allow for the correct sign, as shown below):

$$(\lambda\boldsymbol{I}-\boldsymbol{A})=\begin{bmatrix}\lambda+1 & -2\\ 1 & \lambda+4\end{bmatrix}\leftrightarrow\begin{bmatrix}+ & -\\ - & +\end{bmatrix}$$

Take row 1, for example. The cofactor = $\begin{bmatrix}\lambda+4\\ -1\end{bmatrix}$: $\lambda=-2\Rightarrow\begin{bmatrix}2\\ -1\end{bmatrix}$, $\lambda=-3\Rightarrow\begin{bmatrix}1\\ -1\end{bmatrix}$

Take row 2. The cofactor = $\begin{bmatrix}2\\ \lambda+1\end{bmatrix}$: $\lambda=-2\Rightarrow\begin{bmatrix}2\\ -1\end{bmatrix}$, $\lambda=-3\Rightarrow\begin{bmatrix}2\\ -2\end{bmatrix}$

Here, each cofactor is an eigenvector and combining these for either row will give the modal matrix, as before.

- Note, in the last example, that the actual eigenvector values are not the same but this is not important.

- It is the relationship/ratio between each of the values in each eigenvector that is of key importance and this is the same in all cases.

- *This effectively relates to direction, and it is the direction itself that is important and not the actual distance in that direction!*

- **Example 6.3:** Determine the modal matrix for $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

Solution:

Eigenvalues:

$$|\lambda \boldsymbol{I} - \boldsymbol{A}| = 0 \Rightarrow \begin{vmatrix} \lambda-1 & -2 & -1 \\ -6 & \lambda+1 & 0 \\ 1 & 2 & \lambda+1 \end{vmatrix} = 0$$

$$\begin{aligned}
&\Rightarrow (\lambda-1)\big((\lambda+1)^2\big) - (-2)\big((-6)(\lambda+1)\big) + (-1)\big(-12-(\lambda+1)\big) = 0 \\
&\Rightarrow (\lambda-1)(\lambda^2+2\lambda+1) - 12\lambda - 12 + 12 + \lambda + 1 = 0 \\
&\Rightarrow (\lambda^3+\lambda^2-\lambda-1) - 11\lambda + 1 = 0 \\
&\Rightarrow \lambda^3 + \lambda^2 - 12\lambda = 0 \quad \Rightarrow \lambda(\lambda^2+\lambda-12) = 0 \\
&\Rightarrow \lambda(\lambda-3)(\lambda+4) = 0 \quad \Rightarrow \lambda = 0,\ 3,\ -4
\end{aligned}$$

Eigenvectors – using cofactor method:

$$(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{bmatrix} \lambda-1 & -2 & -1 \\ -6 & \lambda+1 & 0 \\ 1 & 2 & \lambda+1 \end{bmatrix} \leftrightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Take row 1: cofactor = $\boldsymbol{M}(\lambda) = \begin{bmatrix} (\lambda+1)^2 \\ 6(\lambda+1) \\ -12-(\lambda+1) \end{bmatrix} = \begin{bmatrix} (\lambda+1)^2 \\ 6(\lambda+1) \\ -13-\lambda \end{bmatrix}$:

Hence: $\lambda = 0 \Rightarrow \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix}$, $\lambda = 3 \Rightarrow \begin{bmatrix} 16 \\ 24 \\ -16 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$, $\lambda = -4 \Rightarrow \begin{bmatrix} 9 \\ -18 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$

Note – we have scaled the eigenvectors for convenience!

Finally:

$$\boldsymbol{M}=\begin{bmatrix}1 & 2 & 1\\ 6 & 3 & -2\\ -13 & -2 & -1\end{bmatrix} \qquad \text{for} \qquad \Lambda=\begin{bmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & -4\end{bmatrix}$$

must be in the same order !

- By choosing $\boldsymbol{T}=\boldsymbol{M}$ we get:

$$z(k+1)=\boldsymbol{M}^{-1}\boldsymbol{AM}z(k)+\boldsymbol{M}^{-1}\boldsymbol{B}u(k) \qquad \equiv \boldsymbol{A}_Z z(k)+\boldsymbol{B}_Z u(k)$$

$$=\Lambda z(k)+\boldsymbol{B}_Z u(k)$$

- The state transition matrix for this system is given as:

$$\boxed{\Lambda^k=\begin{bmatrix}\lambda_1^k & 0 & \cdots & 0\\ 0 & \lambda_2^k & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n^k\end{bmatrix}}$$

- We can also easily obtain a closed form for the transition matrix $\boldsymbol{A}^k$ of the original system $\boldsymbol{x}(k+1)=\boldsymbol{Ax}(k)+\boldsymbol{B}u(k)$ as follows.

- Since:

$$\Lambda=\boldsymbol{M}^{-1}\boldsymbol{AM}\Rightarrow \boldsymbol{A}=\boldsymbol{M}\Lambda\boldsymbol{M}^{-1}$$

- Then:

$$\boldsymbol{A}^k=\left(\boldsymbol{M}\Lambda\boldsymbol{M}^{-1}\right)^k=\boldsymbol{M}\Lambda\boldsymbol{M}^{-1}\boldsymbol{M}\Lambda\boldsymbol{M}^{-1}\ldots\boldsymbol{M}\Lambda\boldsymbol{M}^{-1}=\boldsymbol{M}\Lambda^k\boldsymbol{M}^{-1}$$

- Hence:

$$\boxed{\boldsymbol{A}^k=\boldsymbol{M}\Lambda^k\boldsymbol{M}^{-1}}$$

- **Example 6.4:** Determine the state transition matrix for the autonomous (i.e. no input) system:

$$\boldsymbol{x}(k+1)=\begin{bmatrix}-1 & 2\\ -1 & -4\end{bmatrix}\boldsymbol{x}(k)$$

and hence evaluate its response at k = 10, for the initial state $\boldsymbol{x}(0) = [1\ 0]^T$.

Solution:

From example 6.2:

$$\boldsymbol{A}=\begin{bmatrix}-1 & 2\\ -1 & -4\end{bmatrix} \rightarrow \lambda=-2,-3 \rightarrow \boldsymbol{M}=\begin{bmatrix}-2 & 1\\ 1 & -1\end{bmatrix}$$

Now: $\boldsymbol{x}(k)=\boldsymbol{A}^k\boldsymbol{x}(0)$

$$\begin{aligned}\boldsymbol{A}^k &= \boldsymbol{M}\Lambda^k\boldsymbol{M}^{-1}\\
&=\begin{bmatrix}-2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}-2 & 0\\ 0 & -3\end{bmatrix}^k\begin{bmatrix}-2 & 1\\ 1 & -1\end{bmatrix}^{-1}\\
&=\begin{bmatrix}-2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}(-2)^k & 0\\ 0 & (-3)^k\end{bmatrix}\left(\frac{1}{2-1}\right)\begin{bmatrix}-1 & -1\\ -1 & -2\end{bmatrix}\\
&=\begin{bmatrix}-2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}(-2)^k & 0\\ 0 & (-3)^k\end{bmatrix}\begin{bmatrix}-1 & -1\\ -1 & -2\end{bmatrix}\\
&=\begin{bmatrix}-2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}-(-2)^k & -(-2)^k\\ -(-3)^k & -2(-3)^k\end{bmatrix}\end{aligned}$$

$$\Rightarrow \boldsymbol{A}^k=\begin{bmatrix}2(-2)^k-(-3)^k & 2(-2)^k-2(-3)^k\\ -(-2)^k+(-3)^k & (-2)^k+2(-3)^k\end{bmatrix}$$

Hence: $\boldsymbol{x}(k)=\boldsymbol{A}^k\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2(-2)^k-(-3)^k\\ -(-2)^k+(-3)^k\end{bmatrix}$

For k = 10: $\boldsymbol{x}(10)=\begin{bmatrix}2(-2)^{10}-(-3)^{10}\\ -(-2)^{10}+(-3)^{10}\end{bmatrix}=\begin{bmatrix}-57001\\ 58025\end{bmatrix}$

6.3 Continuous-time solution

- The general linear SISO continuous-time state-space model is given by:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t)$$

- Once again, we will consider the unforced and forced responses separately.

Unforced response:

- This is where the input is set to zero for all time, i.e.: $u(t) = 0, \ \forall t$
- The state equation then becomes: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$
- Solving this first order differential equation gives: $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$
- In this case, the matrix e^{At} is known as the **(*continuous-time*) state transition matrix**.

Forced response:

- Now, consider the situation when the input is not zero: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$
- The solution for this can be written as: $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$

Output calculation:

- Once the state has been determined, the output is easily computed as:

$$\begin{aligned} y(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}u(t) \\ &= \boldsymbol{C}e^{At}\boldsymbol{x}(0) + \int_{t_0}^{t} \boldsymbol{C}e^{A(t-\tau)}\boldsymbol{B}u(\tau)d\tau \ + \boldsymbol{D}u(t) \end{aligned}$$

- *Aside note – these equations are similar in form to those obtained for the discrete-time state-space models but* $\sum \rightarrow \int$ *and* $\mathbf{A}^k \rightarrow e^{\mathbf{A}t}$.
- Just as in the discrete case, we need efficient ways of computing the state transition matrix.
- The Modal matrix and Cayley-Hamilton methods described for $\boldsymbol{A}^k$ can also be applied to $e^{\mathbf{A}t}$ with some minor modifications.

6.3.1 The Modal matrix (or state transformation) method

- Consider $e^{\boldsymbol{A}t}$ when $\boldsymbol{A}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$.

- Expanding $e^{\boldsymbol{A}t}$ as a power series gives:

$$\exp\left(\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}t\right)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}t+\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}^2\frac{t^2}{2!}+\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}^3\frac{t^3}{3!}+\ldots$$

$$=\begin{bmatrix}1+\lambda_1 t+\lambda_1{}^2\frac{t^2}{2!}+\lambda_1{}^3\frac{t^3}{3!}+\ldots & 0\\ 0 & 1+\lambda_2 t+\lambda_2{}^2\frac{t^2}{2!}+\lambda_2{}^3\frac{t^3}{3!}+\ldots\end{bmatrix}$$

$$=\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}$$

- Thus, when $\boldsymbol{A}$ is a diagonal matrix, calculating $e^{\boldsymbol{A}t}$ is straightforward.

- Consider: $\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)$

- Let $\boldsymbol{x}(t)=\boldsymbol{M}\mathbf{z}(t)$, where $\boldsymbol{M}$ is the modal matrix for $\boldsymbol{A}$.

- This gives: $\dot{z}(t)=\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}z(t)=\Omega\, z(t)$

- Solving this equation, we obtain: $\mathbf{z}(t)=e^{\Omega t}\mathbf{z}(0)$

- Note: $\Omega=\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$

- But $\mathbf{z}(t)=\boldsymbol{M}^{-1}\boldsymbol{x}(t)$, hence: $\boldsymbol{x}(t)=\boldsymbol{M}z(t)=\boldsymbol{M}e^{\Omega t}z(0)=\boldsymbol{M}e^{\Omega t}\boldsymbol{M}^{-1}\boldsymbol{x}(0)$

- Previously, we noted that $\boldsymbol{x}(t)=e^{At}\boldsymbol{x}(0)$, hence: $\boldsymbol{M}e^{\Omega t}\boldsymbol{M}^{-1}\boldsymbol{x}(0)=e^{At}\boldsymbol{x}(0)$

- Thus, the state transition matrix $e^{\boldsymbol{A}t}$ can be computed as:

$$\boxed{e^{At}=\boldsymbol{M}e^{\Omega t}\boldsymbol{M}^{-1}}$$

- Note the similarity with the discrete-time equivalent.

- **Example 6.5 (a):** Determine the **state transition matrix** for the autonomous system:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix} \boldsymbol{x}$$

Solution:

Eigenvalues: $\lambda = -2, \ -3$ (… *verify for yourself*)

Modal matrix: $\boldsymbol{M} = \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}$ (… *verify for yourself*)

Thus: $\boldsymbol{M}^{-1} = \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix}$

Hence:

$$\begin{aligned} e^{At} &= \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix} e^{\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} t} \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 2e^{-2t} - 2e^{-3t} \\ -3e^{-2t} + 3e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix} \end{aligned}$$

- **Example 6.5 (b):** Determine **the evolution of the state vector** for the system:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix} \boldsymbol{x}(\mathrm{t}) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

if the input $u(t)$ is a step and the initial state $\boldsymbol{x}(0) = [1 \ -1]^T$.

Solution:

We need to determine: $\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)} \boldsymbol{B} u(\tau) d\tau$

$$e^{\boldsymbol{A}t}\boldsymbol{x}(0) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & 2e^{-2t} - 2e^{-3t} \\ -3e^{-2t} + 3e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}$$

$$\int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau \rightarrow e^{\boldsymbol{A}t}\int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}u(\tau)d\tau$$

$$= e^{\boldsymbol{A}t}\int_0^t \begin{bmatrix} 3e^{2\tau}-2e^{3\tau} & 2e^{2\tau}-2e^{3\tau} \\ -3e^{2\tau}+3e^{3\tau} & -2e^{2\tau}+3e^{3\tau} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}(1)d\tau$$

$$= e^{\boldsymbol{A}t}\int_0^t \begin{bmatrix} 2e^{2\tau}-2e^{3\tau} \\ -2e^{2\tau}+3e^{3\tau} \end{bmatrix}d\tau$$

$$= e^{\boldsymbol{A}t}\left(\left.\begin{bmatrix} e^{2\tau}-\tfrac{2}{3}e^{3\tau} \\ -e^{2\tau}+e^{3\tau} \end{bmatrix}\right|_0^t\right)$$

$$= e^{\boldsymbol{A}t}\left(\begin{bmatrix} e^{2t}-\tfrac{2}{3}e^{3t}-1+\tfrac{2}{3} \\ -e^{2t}+e^{3t}+1-1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 3e^{-2t}-2e^{-3t} & 2e^{-2t}-2e^{-3t} \\ -3e^{-2t}+3e^{-3t} & -2e^{-2t}+3e^{-3t} \end{bmatrix}\begin{bmatrix} e^{2t}-\tfrac{2}{3}e^{3t}-\tfrac{1}{3} \\ -e^{2t}+e^{3t} \end{bmatrix}$$

$$= \begin{bmatrix} \left(3-2e^{t}-e^{-2t}-2e^{-t}+\tfrac{4}{3}+\tfrac{2}{3}e^{-3t}\right)+\left(-2+2e^{t}+2e^{-t}-2\right) \\ \left(-3+2e^{t}+e^{-2t}+3e^{-t}-2-e^{-3t}\right)+\left(2-2e^{t}-3e^{-t}+3\right) \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-2t}+\tfrac{2}{3}e^{-3t}+\tfrac{1}{3} \\ e^{-2t}-e^{-3t} \end{bmatrix}$$

Hence:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau$$

$$\Rightarrow \boldsymbol{x}(t) = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} + \begin{bmatrix} -e^{-2t}+\tfrac{2}{3}e^{-3t}+\tfrac{1}{3} \\ e^{-2t}-e^{-3t} \end{bmatrix}$$

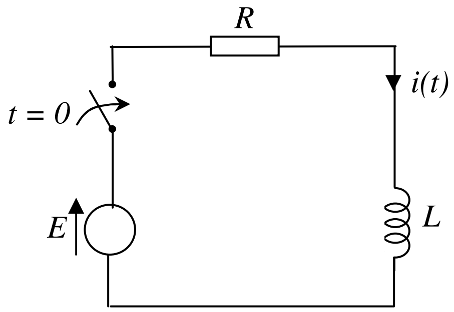
$$\Rightarrow \boldsymbol{x}(t) = \begin{bmatrix} \tfrac{2}{3}e^{-3t}+\tfrac{1}{3} \\ -e^{-3t} \end{bmatrix}$$

6.4 The *trilogy* of solutions

- In general linear systems can be represented using differential equations, transfer functions or state-space models.

- Each of these are solved in very different ways but will ultimately yield the same solution!

- By way of example, recall the series R-L circuit from EE214:



- The first order **differential equation model** for this simple circuit is given by:

$$L\frac{di}{dt}+Ri=E$$

- Solving this equation directly gives us the solution:

$$i(t)=\frac{E}{R}\left(1-e^{\frac{-R}{L}t}\right)$$

- Using Laplace Transforms, we obtain the **transfer function model** for this system as:

$$I(s)=\frac{E}{s(R+sL)}$$

- Solving this model using partial fractions and Inverse Laplace Transforms gives us the exact same solution, as expected, i.e.:

$$i(t)=\frac{E}{R}\left(1-e^{\frac{-R}{L}t}\right)$$

- *Refer to EE214 notes for details of the above solutions.*

- Here, we will complete the ‘trilogy’ of solutions, by obtaining a state space model for the R-L circuit and solving this model directly.

6.4.1 The state-space model and solution for the series R-L circuit

- We know the first-order differential equation is: $L\frac{di}{dt}+Ri=E$
- Here, we only have a ***first-order*** differential equation and hence only one state: $x_1=i$
- We now obtain an expression for $\dot{x}_1$:

$$\dot{x}_1=\frac{di}{dt}=\frac{E}{L}-\frac{R}{L}i$$

- Rewriting this equation in terms of state x_1, and letting input $u = E$, we get:

$$\dot{x}_1=\frac{-R}{L}x_1+\frac{1}{L}u$$

- The output $y = i(t) = x_1$. Hence, our simple **state-space model** is expressed, in full, as:

$$[\dot{x}_1]=\left[\frac{-R}{L}\right][x_1]+\left[\frac{1}{L}\right][u]$$

$$[y]=[1][x_1]$$

- Since $y = x_1$, to solve for the output we simply need to evaluate the expression:

$$\mathbf{x}(t)=e^{\mathbf{A}t}\mathbf{x}(0)+\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$

- Here, we have only one state and, also, the initial condition is $x_1(0) = 0$. Hence:

$$y=x(t)=0+\int_0^t e^{-\frac{R}{L}(t-\tau)}\left(\frac{1}{L}\right)Ed\tau$$

$$=\frac{E}{L}e^{-\frac{R}{L}t}\int_0^t e^{\frac{R}{L}\tau}d\tau \quad =\frac{E}{L}e^{-\frac{R}{L}t}\left(e^{\frac{R}{L}\tau}\left(\frac{L}{R}\right)\right)_0^t \quad =\frac{E}{R}e^{-\frac{R}{L}t}\left(e^{\frac{R}{L}t}-1\right)$$

$$=\frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)$$

- Once again, we obtain the same solution.

- Hence, we have shown that we can represent systems using three different models and the solutions to each of these models, as expected, are the same.

- The key benefit of using state-space models is that the model and its solution scales easily for higher-order problems.

- Differential equations, on the other hand, become more complex to solve for higher orders.

- Transfer function models do not allow for initial conditions and can only be written for an individual input-output relationship.

- State-space models can allow represent multi-input multi-output systems in the same model, caters for initial conditions and offers the same solution methodology irrespective of the order of the model.

- In the next section, we will examine the issue of stability from the state-space model viewpoint.

"IT JUST SEEMED LOGICAL TO HAVE A SCREEN FOR EACH DIMENSION"