Tutor6. Hanlin Cai

Q1.

(i) $\dot{x} = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t).$

$\begin{vmatrix} \lambda-1 & 0 \\ -2 & \lambda-2 \end{vmatrix} = 0 \Rightarrow \begin{cases} \lambda = 1 \\ \lambda = 2 \end{cases}$

$Re(\lambda_2) > 0$ Hence unstable

(ii) $\dot{x} = \begin{bmatrix} 0 & 2 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t).$

$\begin{vmatrix} \lambda-0 & 2 \\ -2 & \lambda+3 \end{vmatrix} = 0 \Rightarrow \lambda(\lambda+3)+4 = 0$

So $\lambda^2 + 3\lambda + 4 = 0$

$\cancel{(\lambda+4)}$

$\therefore \begin{cases} \lambda_1 = -1.5 + (1.3229)i < 0 \\ \lambda_2 = \cancel{-1.5+(\ \ )\lambda_1}\ -1.5 - 1.3229i < 0 \end{cases}$

$\therefore$ Hence asymptotically stable.

(iii) $\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t).$

$\begin{vmatrix} \lambda & 1 \\ -1 & \lambda+5 \end{vmatrix} = 0 \Rightarrow \lambda^2 + 5\lambda + 1 = 0$

$\therefore \lambda_1 = -0.2087 < 0$

$\lambda_2 = -4.7913 < 0$

$\therefore$ Hence asymptotically stable

No.

Date

(iv) $x(k+1) = \begin{bmatrix} 0.4 & 0.8 \\ -0.4 & 0.2 \end{bmatrix} \cdot x(k) + \begin{bmatrix} 0.3 \\ 1 \end{bmatrix} \cdot u(k)$,

$\begin{vmatrix} \lambda - 0.4 & 0.8 \\ 0.4 & \lambda - 0.2 \end{vmatrix} = 0 \Rightarrow \lambda_1 = \frac{2}{5} \quad \lambda_2 = \frac{1}{5}$

All $|\lambda| < 1 \Rightarrow$ asymptotically stable.

(i) $\dot{x} = \begin{bmatrix} 1 & 0 \\ -2 & \alpha \end{bmatrix} x + \begin{bmatrix} \beta \\ 1 \end{bmatrix} u(t)$

Q2. $(\lambda - 1)(\lambda - \alpha) = 0$

$\begin{cases} \lambda_1 = 1 \\ \lambda_2 = \alpha \end{cases}$

Because $Re_1(\lambda_1) > 1$ already.
Hence no matter α, β are, the system is unstable

(ii) $\dot{x} = \begin{bmatrix} 0 & \beta \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t)$

$\begin{vmatrix} \lambda & \beta \\ -2 & \lambda + 3 \end{vmatrix} = 0 \Rightarrow \lambda^2 + 3\lambda + 2\beta = 0 \qquad \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore \begin{cases} \lambda_1 = \frac{-3 + \sqrt{9 - 8\beta}}{2} & ① \\ \lambda_2 = \frac{-3 - \sqrt{9 - 8\beta}}{2} & ② \end{cases}$

$\lambda_2 < 0$ already

① If system is asymptotically stable
then $-3 + \sqrt{9 - 8\beta} < 0 \quad \therefore \beta > 0$

② If system is marginally stable
then $-3 + \sqrt{9 - 8\beta} = 0 \quad \therefore \beta = 0$

③ If system is unstable
then $\beta < 0$

Campus

Tutor 6.

Q2 (iii) $\dot{x} = \begin{bmatrix} 0 & 1 \\ -\beta & \alpha \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t).$

$$\begin{vmatrix} \lambda & 1 \\ -\beta & \lambda-\alpha \end{vmatrix} = 0 \iff \lambda(\lambda-\alpha)+\beta = 0$$

So $\lambda^2 - \alpha\lambda + \beta = 0$ $\therefore \begin{cases} \lambda_1 = \dfrac{\alpha + \sqrt{\alpha^2-4\beta}}{2} \\ \lambda_2 = \dfrac{\alpha - \sqrt{\alpha^2-4\beta}}{2}. \end{cases}$

① If asymptotically stable, $\sqrt{\alpha^2-4\beta} \geq 0$

then $\begin{cases} \alpha + \sqrt{\alpha^2-4\beta} < 0 \\ \alpha - \sqrt{\alpha^2-4\beta} < 0 \end{cases}$ $\therefore \begin{cases} \alpha < 0 \\ \beta < 0 \end{cases}$

② If marginally stable

then $\begin{cases} \alpha + \sqrt{\alpha^2-4\beta} = 0 \\ \cdots\cdots \end{cases} \Rightarrow \alpha = 0 \quad \beta = 0$

But In this situation $\lambda_1 = \lambda_2 = 0 \Rightarrow$ unstable

then $\alpha - \sqrt{\alpha^2-4\beta} = 0 \Rightarrow \beta = 0$

Hence There is no suitable α and β to match.

③ If unstable

$\begin{cases} (1) \text{ then } \alpha + \sqrt{\alpha^2-4\beta} > 0 \Rightarrow \alpha > 0 \quad (\beta \in R) \\ (2) \text{ or } \alpha - \sqrt{\alpha^2-4\beta} > 0 \Rightarrow \beta > 0 \quad (\alpha \in R). \\ (3) \text{ And } \alpha = 0 \ \& \ \beta = 0 \end{cases}$

Q2 (iv)

$$x(k+1) = \begin{bmatrix} \alpha-1 & 0 \\ -2 & \beta \end{bmatrix} x(k) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(k)$$

sol. $(\lambda-\alpha+1)(\lambda-\beta) = 0$

$$\begin{cases} \lambda_1 = \alpha-1 \\ \lambda_2 = \beta \end{cases}$$

① If asymptotically stable : $\alpha < 1 \quad \beta < 0$

② If marginally stable

$$\begin{cases} (1) \quad \alpha = 1 \quad \beta < 0 \\ (2) \quad \alpha < 1 \quad \beta = 0 \end{cases}$$

③ If unstable

$$\begin{cases} (1) \quad \alpha = 1 \quad \beta = 0 \\ (2) \quad \alpha > 1 \quad \beta \in R \\ (3) \quad \beta > 0 \quad \alpha \in R. \end{cases}$$

Q3 sol (i)

$[f(t) = K y + B \dot{y} + M \ddot{y}]$

$x_1 = x \quad x_2 = \dot{x} \overset{S_0}{\Longleftrightarrow} \dot{x}_1 = \dot{x}$ ① $\quad \dot{x}_2 = \ddot{x} = \frac{1}{M}(f(t) - B\dot{x} - Kx)$ ②

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{M}(f(t) - Bx_2 - Kx_1). \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} f(t).$$

$x \to y$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -\frac{B}{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} f(t).$$

$$\begin{cases} M = 2 \\ K = 10 \end{cases}$$

$$[z] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$ ②

(ii) $\dot{y} = \begin{bmatrix} 0 & 1 \\ -5 & -\frac{1}{2}B \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} f(t). \rightarrow \begin{cases} \lambda_1 = \dfrac{-\frac{B}{2} + \sqrt{\frac{1}{4}B^2 - 20}}{2} \\ \lambda_2 = \dfrac{-\frac{B}{2} - \sqrt{\frac{1}{4}B^2 - 20}}{2} \end{cases}$

① asymptotically $\begin{cases} \lambda_1 < 0 \\ \lambda_2 < 0 \end{cases} \Rightarrow B < 0$

② marginally $\cancel{B=0}$ not suitable B to match it.

③ unstable (1) $B > 0$ (2) $B = 0$

No. 3
Date

Tutor 6.

Q4 sol

(i) ① $x_{k+1} = \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k$

$y_k = [2 \quad 1] x_k + u_k.$

$G(z) = C(zI - A)^{-1} B.$

$= [2 \quad 0] \left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \dfrac{2}{z^2 + 2z - 2} + \dfrac{2(z+2)}{z^2 + 2z - 2} = \dfrac{2z+6}{(z^2 + 2z - 2)}$

② $\dot{x} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t) \qquad y(t) = [0 \quad 1] x$

$G(z) = [0 \quad 1] \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

$= \dfrac{2s+3}{s^2 + 3s + 1},$

(ii) For ① $\begin{vmatrix} z & 1 \\ 2 & z+2 \end{vmatrix} = 0 \iff z^2 + 2z - 2 = 0$

$\therefore \begin{cases} x_1 \approx 0.732 \\ x_2 \approx -2.73 \end{cases}$

$\therefore$ All $|\alpha| < 1$

$\therefore$ asymptotically stable

For ② $\begin{vmatrix} \lambda+1 & 1 \\ 1 & \lambda+2 \end{vmatrix} = 0 \iff \lambda^2 + 3\lambda + 1 = 0$

$\therefore \begin{cases} \alpha_1 \approx -0.382 < 0 \\ \alpha_2 \approx -2.62 < 0 \end{cases}$

$\because$ For all $Re(\lambda) < 0 \implies$ asymptotically stable.