Tutorial Sheet 5 - Solutions

Q1 (i) $\dfrac{1}{sL+R} = \dfrac{1/R}{1+s(L/R)} \equiv \dfrac{K}{1+s\tau}$ $\therefore \tau = \dfrac{L}{R}$

(ii) $\tau = 0.5s$

Q2 (i) We want to find t such that: $u\left(1-e^{-\frac{t}{RC}}\right) = 0.63u$

(Time constant is the time at which output has reached 63% of its final value)

Hence: $u\left(1-e^{-\frac{t}{RC}}\right) = 0.63u \quad \Rightarrow 1-e^{-\frac{t}{RC}} = 0.63 \quad \Rightarrow -e^{-\frac{t}{RC}} = -0.37$

$$\Rightarrow -\frac{t}{RC} = \log_e(0.37) \approx -1 \qquad \Rightarrow t = RC$$

(ii) C = 2 x 10^{-6}F, R = 100,000Ω $\Rightarrow t = 0.2s$

(iii) 98% occurs at 4 time constants $\Rightarrow t = 0.8s$

Alternatively:

$$u\left(1-e^{-\frac{t}{0.2}}\right) = 0.98u \quad \Rightarrow 1-e^{-\frac{t}{0.2}} = 0.02 \quad \Rightarrow -e^{-\frac{t}{0.2}} = -0.02$$

$$\Rightarrow -\frac{t}{0.2} = \log_e(0.02) \Rightarrow t = -0.2\log_e(0.02) \approx 0.8s$$

Q3 (i) $\dfrac{20}{(s+2)(s+5)}$ *Note: poles are real and distinct, therefore overdamped (>1)*

$$= \frac{20}{s^2+7s+10} \equiv \frac{K\omega_n}{s^2+2\zeta\omega_n s+\omega_n^{\ 2}}$$

Hence: $\omega_n^{\ 2} = 10 \Rightarrow \omega_n = \sqrt{10}$ and $2\zeta\omega_n = 7 \Rightarrow \zeta = \dfrac{7}{2\sqrt{10}} = 1.11$

Hence, the typical step response is an overdamped response (no oscillations).

Q3 (ii) $\dfrac{8}{s^2+2s+5}$ *Note: poles are complex, therefore underdamped (<1)*

$$\equiv \frac{K\omega_n}{s^2+2\zeta\omega_n s+\omega_n{}^2}$$

Hence: $\omega_n{}^2 = 5 \Rightarrow \omega_n = \sqrt{5}$ and $2\zeta\omega_n = 2 \Rightarrow \zeta = \dfrac{1}{\sqrt{5}} = 0.44$

Hence, the typical step response is an underdamped response (i.e. with oscillations).

Q4 (i) $\dfrac{1}{(s+2)(s+\alpha)}$

For critical damping (=1), we need both poles to be real and the same. We already have one pole at –2, therefore we need the other pole to be at –2 also, i.e. α = 2.

Alternatively:

$$\frac{1}{s^2+(2+\alpha)s+2\alpha} \equiv \frac{K\omega_n}{s^2+2\zeta\omega_n s+\omega_n{}^2}$$

Hence: $\omega_n{}^2 = 2\alpha \Rightarrow \omega_n = \sqrt{2\alpha}$ and $2\zeta\omega_n = 2+\alpha$

We want $\zeta = 1$, hence: $2\sqrt{2\alpha} = 2+\alpha \Rightarrow 8\alpha = (2+\alpha)^2$

$$\Rightarrow 8\alpha = \alpha^2 + 4\alpha + 4$$
$$\Rightarrow \alpha^2 - 4\alpha + 4 = 0 \quad \Rightarrow (\alpha-2)^2 = 0$$
$$\Rightarrow \alpha = 2$$

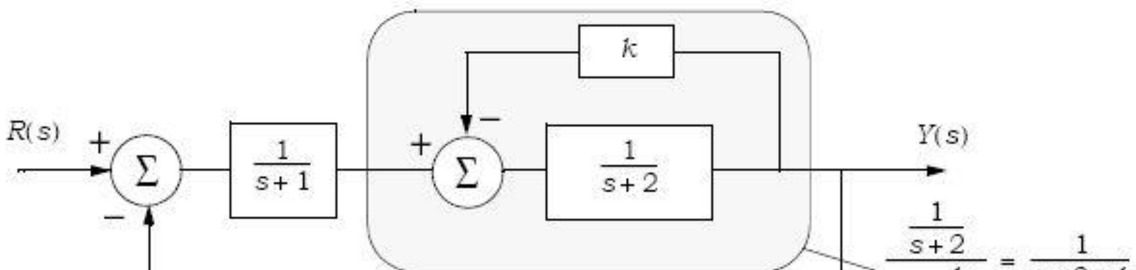
(ii) $\zeta = 1,\ \omega_n = \sqrt{2(2)} = 2$

$PO(\%) = 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \quad = 100e^{-\infty} \quad = 0\%$ (no overshoot, as expected)

$$T_s = \frac{4}{\zeta\omega_n} \quad = \frac{4}{1(2)} \quad = 2s$$

$$G_{ss} = \lim_{s\to 0} G(s) \quad = \lim_{s\to 0}\frac{1}{s^2+4s+4} \quad = \frac{1}{4} \quad = 0.25$$

Q5

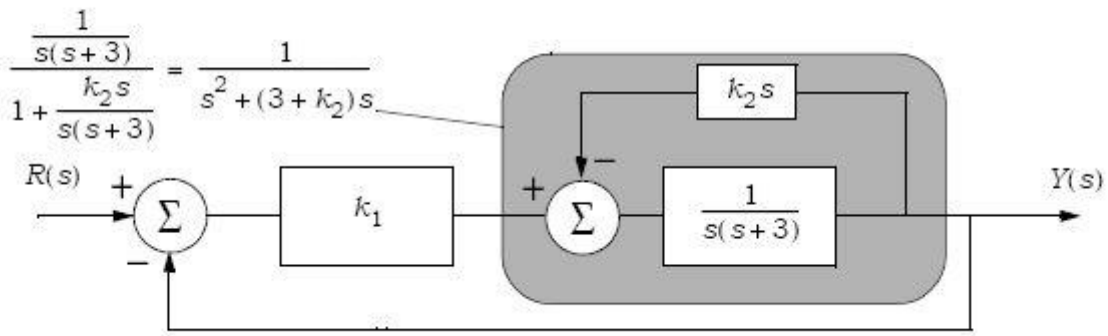


This is a second order system of the form: $\dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

Comparing terms gives $\omega_n = \sqrt{3+k}$ and $\zeta = \dfrac{(3+k)}{2\omega_n} = \dfrac{(3+k)}{2\sqrt{3+k}} = \dfrac{1}{2}\sqrt{3+k}$.

A critically damped system is one where $\zeta = 1 \quad \Rightarrow \dfrac{1}{2}\sqrt{3+k} = 1 \Rightarrow k = 1$

Q6



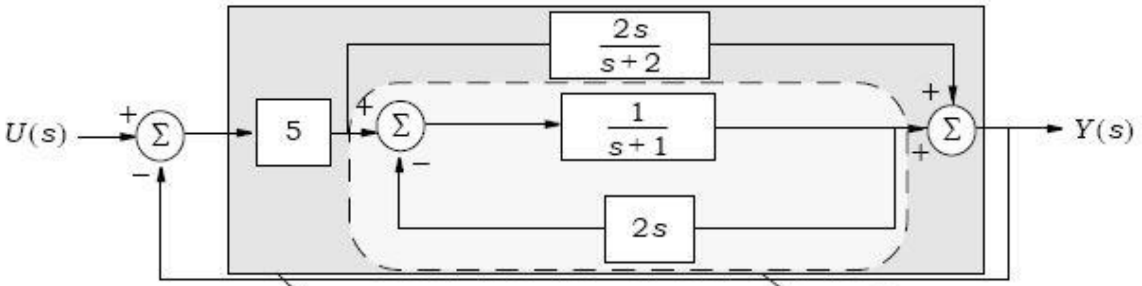
$$G(s) = \frac{\dfrac{k_1}{s^2 + (3+k_2)s}}{1 + \dfrac{k_1}{s^2 + (3+k_2)s}} = \frac{k_1}{s^2 + (3+k_2)s + k_1}$$

Here $\omega_n = \sqrt{k_1}$ and $\zeta = \dfrac{3+k_2}{2\omega_n} = \dfrac{3+k_2}{2\sqrt{k_1}}$.

The settling time is given by: $\tau_s = \dfrac{4}{\zeta\omega_n} = 0.5 \Rightarrow \zeta\omega_n = 8$

Since $2\zeta\omega_n = 3 + k_2$ we can write: $2\zeta\omega_n = 3 + k_2 = 16 \Rightarrow k_2 = 13$

Q7



$$5\left[\frac{1}{3s+1} + \frac{2s}{s+2}\right] = \frac{30s^2+15s+10}{3s^2+7s+2} \qquad \frac{\frac{1}{s+1}}{1+\frac{1}{s+1}\cdot 2s} = \frac{1}{3s+1}$$

$$G(s) = \frac{\frac{30s^2+15s+10}{3s^2+7s+2}}{1+\frac{30s^2+15s+10}{3s^2+7s+2}} = \frac{30s^2+15s+10}{3s^2+7s+2+30s^2+15s+10} = \frac{30s^2+15s+10}{33s^2+22s+12}$$

System stability depends on the poles of the transfer function:

$$33s^2+22s+12 = 0 \rightarrow s = \frac{-22 \pm \sqrt{22^2-4(33)(12)}}{66} = -0.3333 \pm j0.5025$$

$Re(s) < 0 \Rightarrow$ system is asymptotically stable.

Poles are complex so transient response will be oscillatory.