EE206
Assignment 10

1. Find the Z transform of the following sequence $\{x_k\}$

   *(a) $x_k = (-1)^k$

$$
\begin{aligned}
F(z) =& \sum_{k=0}^{\infty} \frac{(-1)^k}{z^k} \\
=& \sum_{k=0}^{\infty} \left(\frac{-1}{z}\right)^k
\end{aligned}
$$

   Now for $|x| < 1$:

$$
\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}
$$

   So, for $|\frac{-1}{z}| < 1 \Rightarrow 1 < |z|$

$$
\begin{aligned}
F(z) =& \sum_{k=0}^{\infty} \left(\frac{-1}{z}\right)^k \\
=& \frac{1}{1-(-\frac{1}{z})} \\
=& \frac{z}{z+1}
\end{aligned}
$$

   (b) $x_k = 3k + (-4)^{2k+1}$ **[2]**

$$
\begin{aligned}
F(z) =& 3\mathcal{Z}(k) - 4\mathcal{Z}((16)^k) \\
&= 3\left(\frac{z}{(z-1)^2}\right) - 4\left(\frac{z}{z-16}\right) \\
&= \frac{3z}{(z-1)^2} - \frac{4z}{z-16}
\end{aligned}
$$

2. Find the inverse Z transform of:

   *(a) $F(z) = \frac{z^2}{(z+1)(z+2)(z+3)}$

$$
\begin{aligned}
F(z) &= \frac{z^2}{(z+1)(z+2)(z+3)} \\
\frac{F(z)}{z} &= \frac{z}{(z+1)(z+2)(z+3)} \\
&= -\left(\frac{1}{2}\right)\frac{1}{z+1} + (2)\frac{1}{z+2} - \left(\frac{3}{2}\right)\frac{1}{z+3} \\
F(z) &= -\left(\frac{1}{2}\right)\frac{z}{z+1} + (2)\frac{z}{z+2} - \left(\frac{3}{2}\right)\frac{z}{z+3} \\
\mathcal{Z}\{a^k\} &= \frac{z}{z-a} \\
x_k &= -\frac{1}{2}(-1)^k + 2(-2)^k - \frac{3}{2}(-3)^k
\end{aligned}
$$

   This gives the sequence:

$$
\mathcal{Z}^{-1}\{F(z)\} = \left\{\frac{1}{2}(-1)^{k+1} - (-2)^{k+1} + \frac{1}{2}(-3)^{k+1}\right\}
$$

(b) $F(z) = \frac{z(3z+1)}{(z-3)^2}$ **[2]**

$$\begin{aligned}
F(z) &= \frac{z(3z+1)}{(z-3)^2} \\
\frac{F(z)}{z} &= \frac{3z+1}{(z-3)^2} = \frac{A}{z-3} + \frac{B}{(z-3)^2} \\
A(z-3)+B &= 3z+1 \\
z = 3;& \quad B = 10 \\
z \text{ term};& \quad A = 3. \\
\frac{F(z)}{z} &= (3)\frac{1}{z-3} + (10)\frac{1}{(z-3)^2} \\
F(z) &= (3)\frac{z}{z-3} + (10)\frac{z}{(z-3)^2} \\
\mathcal{Z}\{a^k\} &= \frac{z}{z-a}; \quad \mathcal{Z}\{ka^k\} = \frac{az}{(z-a)^2} \\
x_k &= 3(3^k) + \frac{10}{3}(k3^k)
\end{aligned}$$

This gives the sequence:

$$\mathcal{Z}^{-1}\{F(z)\} = \{3^{k+1} + 10k(3^{k-1})\}$$

3. Find the final value of the sequence $\{x_k\}$ with Z transform:

(a) $F(z) = \frac{3z^2-z}{2z^2-3z+1}$ **[2]**

$$\begin{aligned}
\lim_{x\to\infty} x_k = \lim_{z\to 1}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} \\
&= \lim_{z\to 1}\left\{\left(\frac{z-1}{z}\right)\frac{3z^2-z}{2z^2-3z+1}\right\} \\
&= \lim_{z\to 1}\left\{\left(\frac{z-1}{z}\right)\frac{z(3z-1)}{(2z-1)(z-1)}\right\} \\
&= \lim_{z\to 1}\left\{\frac{(3z-1)}{2z-1}\right\} \\
&= \frac{(3(1)-1)}{2(1)-1} \\
&= 2
\end{aligned}$$

4. What is the initial value of the sequence whose Z transform is given by:

*(a) $F(z) = \frac{2z^2-z+1}{5-3z-7z^2}$

$$\begin{aligned}
x_0 &= \lim_{z\to\infty}\{F(z)\} \\
&= \lim_{z\to\infty}\{\frac{2z^2-z+1}{5-3z-7z^2}\} \\
&= \lim_{z\to\infty}\{\frac{2-\frac{1}{z}+\frac{1}{z^2}}{\frac{5}{z^2}-\frac{3}{z}-7}\} \\
&= \frac{2-0+0}{0-0-7} \\
&= -\frac{2}{7}
\end{aligned}$$

(b) $F(z) = \frac{2z^3+5z^2+2z-1}{6z^3-4z+2}$ **[2]**

$$\begin{aligned}
x_0 &= \lim_{z\to\infty}\{F(z)\} \\
&= \lim_{z\to\infty}\{\frac{2z^3+5z^2+2z-1}{6z^3-4z+2}\} \\
&= \lim_{z\to\infty}\{\frac{2+\frac{5}{z}+\frac{2}{z^2}-\frac{1}{z^3}}{6-\frac{4}{z^2}+\frac{2}{z^3}}\} \\
&\frac{2+0+0-0}{6-0+0} \\
&= \frac{1}{3}
\end{aligned}$$

5. Write out the first five terms in the sequence which satisfies the following conditions.

*(a) $x_{k+2} = 2x_k + x_{k+1} \qquad x_0 = 2, \quad x_1 = 5$

$$\begin{aligned}
x_2 &= 2x_0 + x_1 = 2(2) + 5 = 4 + 5 = 9 \\
x_3 &= 2x_1 + x_2 = 2(5) + 9 = 10 + 9 = 19 \\
x_4 &= 2x_2 + x_3 = 2(9) + 19 = 18 + 19 = 37
\end{aligned}$$

First 5 terms of the sequence: $\{2, 5, 9, 19, 37\}$

(b) $x_{k+2} = 3x_k - 2x_{k+1} \qquad x_0 = 1, \quad x_1 = 1$ **[2]**

$$\begin{aligned}
x_2 &= 3x_0 - 2x_1 = 3(1) - 2(1) = 3 - 2 = 1 \\
x_3 &= 3x_1 - 2x_2 = 3(1) - 2(1) = 3 - 2 = 1 \\
x_4 &= 3x_2 - 2x_3 = 3(1) - 2(1) = 3 - 2 = 1
\end{aligned}$$

First 5 terms of the sequence: $\{1, 1, 1, 1, 1\}$

6. Find the Z transform of the following sequence.

*(a) $\{0, 1, 0, 1, 0, 1, ...\}$ First we need to define the sequence:

$$x_{k+2} = x_k \qquad x_0 = 0, \;\; x_1 = 1$$

Taking the Z transform of this:

$$\begin{aligned}
z^2F(z) - z^2x_0 - zx_1 &= F(z) \\
(z^2-1)F(z) - z^2(0) - z(1) &= 0 \\
(z^2-1)F(z) &= z \\
F(z) &= \frac{z}{z^2-1}
\end{aligned}$$

(b) $\{2, 0, 4, 0, 8, 0, ...\}$ **[2]**
First we need to define the sequence:

$$x_{k+2} = 2x_k \qquad x_0 = 2, \;\; x_1 = 0$$

Taking the Z transform of this:

$$\begin{aligned}
z^2F(z) - z^2x_0 - zx_1 &= 2F(z) \\
(z^2-2)F(z) - z^2(2) - z(0) &= 0 \\
(z^2-2)F(z) &= 2z^2 \\
F(z) &= \frac{2z^2}{z^2-2}
\end{aligned}$$

(c) $\{1, 1, 2, 3, 5, 8, ...\}$ **[2]**

First we need to define the sequence:

$$x_{k+2} = x_k + x_{k+1} \qquad x_0 = 1, \;\; x_1 = 1$$

Taking the Z transform of this:

$$\begin{aligned}
z^2F(z) - z^2x_0 - zx_1 &= F(z) + zF(z) - zx_0 \\
(z^2 - z - 1)F(z) - z^2(1) - z(1) &= -z(1) \\
(z^2 - z - 1)F(z) &= z^2 \\
F(z) &= \frac{z^2}{z^2 - z - 1}
\end{aligned}$$

7. Solve the following recurrence relation.

*(a) $x_{k+2} - 4x_{k+1} + 4x_k = 3 \qquad$ where $x_0 = 1, \quad x_1 = 0$

$$\begin{aligned}
z^2F(z) - z^2x_0 - zx_1 - 4zF(z) + 4zx_0 + 4F(z) &= \frac{3z}{z-1} \\
(z^2 - 4z + 4)F(z) - z^2(1) - z(0) + 4z(1) &= \frac{3z}{z-1} \\
(z^2 - 4z + 4)F(z) - z^2 + 4z &= \frac{3z}{z-1} \\
(z-2)F(z)^2 &= \frac{3z}{z-1} + z^2 - 4z \\
F(z) &= \frac{3z + z^3 - z^2 - 4z^2 + 4z}{(z-1)(z-2)^2} \\
&= \frac{z^3 - 5z^2 + 7z}{(z-1)(z-2)^2} \\
\frac{F(z)}{z} &= \frac{z^2 - 5z + 7}{(z-1)(z-2)^2} \\
&= \frac{A}{(z-2)^2} + \frac{B}{z-2} + \frac{C}{z-1} \\
A(z-1) + B(z-2)(z-1) + C(z-2)^2 &= z^2 - 5z + 7 \\
z = 2 \quad &\Rightarrow A = 4 - 10 + 7 = 1 \\
z = 1 \quad &\Rightarrow C = 1 - 5 + 7 = 3 \\
-A + 2B + 4C = 7 \quad &\Rightarrow -1 + 2B + 12 = 7 \quad \Rightarrow B = -2 \\
\frac{F(z)}{z} &= \frac{1}{(z-2)^2} - \frac{2}{z-2} + \frac{3}{z-1} \\
F(z) &= \frac{z}{(z-2)^2} - \frac{2z}{z-2} + \frac{3z}{z-1}
\end{aligned}$$

Using $\{a^k\} = \frac{z}{z-a}$ and $\{ka^k\} = \frac{az}{(z-a)^2}$ we obtain the sequence $\{x_k\}$ where:

$$\begin{aligned}
x_k &= \frac{1}{2}k2^k - 2(2^k) + 3(1^k) \\
&= k2^{k-1} - 2^{k+1} + 3
\end{aligned}$$

(b) $x_{k+2} + 5x_{k+1} + 6x_k = 2^{k+2}$ where $x_0 = 0, \quad x_1 = 2$ **[2]**

$$\begin{aligned}
z^2F(z) - z^2x_0 - zx_1 + 5zF(z) - 5zx_0 + 6F(z) &= \frac{4z}{z-2} \\
(z^2+5z+6)F(z) - z^2(0) - z(2) - 5z(0) &= \frac{4z}{z-2} \\
(z+3)(z+2)F(z) - 2z &= \frac{4z}{z-2} \\
(z+3)(z+2)F(z) &= \frac{4z}{z-2} + 2z \\
F(z) &= \frac{4z + 2z^2 - 4z}{(z-2)(z+3)(z+2)} \\
&= \frac{2z^2}{(z-2)(z+3)(z+2)} \\
\frac{F(z)}{z} &= \frac{2z}{(z-2)(z+3)(z+2)} \\
&= \frac{A}{z-2} + \frac{B}{z+2} + \frac{C}{z+3}
\end{aligned}$$

$$A(z+2)(z+3) + B(z-2)(z+3) + C(z-2)(z+2) = 2z$$

$$\begin{aligned}
z = 2 \quad &\Rightarrow A(4)(5) = 4 \quad \Rightarrow A = \frac{1}{5} \\
z = -2 \quad &\Rightarrow B(-4)(1) = -4 \quad \Rightarrow B = 1 \\
z = -3 \quad &\Rightarrow C(-5)(-1) = -6 \quad \Rightarrow C = -\frac{6}{5}
\end{aligned}$$

$$\begin{aligned}
\frac{F(z)}{z} &= \left(\frac{1}{5}\right)\frac{1}{z-2} + \frac{1}{z+2} - \left(\frac{6}{5}\right)\frac{1}{z+3} \\
F(z) &= \left(\frac{1}{5}\right)\frac{z}{z-2} + \frac{z}{z+2} - \left(\frac{6}{5}\right)\frac{z}{z+3}
\end{aligned}$$

Using $\{a^k\} = \frac{z}{z-a}$ we obtain the sequence $\{x_k\}$ where:

$$\begin{aligned}
x_k &= \frac{1}{5}(2)^k + (-2)^k - \frac{6}{5}(-3)^k \\
&= \frac{1}{5}(2)^k + (-2)^k + \frac{2}{5}(-3)^{k+1}
\end{aligned}$$

(c) $x_{k+2} - 9x_k = 2k$ where $x_0 = 1, \quad x_1 = 1$ **[2]**

$$\begin{aligned}
z^2F(z) - z^2x_0 - zx_1 - 9F(z) &= \frac{2z}{(z-1)^2} \\
(z^2-9)F(z) - z^2(1) - z(1) &= \frac{2z}{(z-1)^2} \\
(z^2-9)F(z) - z^2 - z &= \frac{2z}{(z-1)^2} \\
(z^2-9)\frac{F(z)}{z} &= \frac{2}{(z-1)^2} + z + 1 \\
(z^2-9)\frac{F(z)}{z} &= \frac{2 + z^2 - 2z + 1 + z^3 - 2z^2 + z}{(z-1)^2} \\
\frac{F(z)}{z} &= \frac{z^3 - z^2 - z + 3}{(z-1)^2(z^2-9)} \\
&= \frac{z^3 - z^2 - z + 3}{(z-1)^2(z-3)(z+3)} \\
&= \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-3} + \frac{D}{z+3}
\end{aligned}$$

$$
\begin{aligned}
z &= 1 \qquad \Rightarrow A(-2)(4) = 1 - 1 - 1 + 3 = 2 \qquad \Rightarrow A = -\frac{1}{4} \\
z &= 3 \qquad \Rightarrow C(4)(6) = 27 - 9 - 3 + 3 = 18 \qquad \Rightarrow C = \frac{3}{4} \\
z &= -3 \qquad \Rightarrow D(16)(-6) = -27 - 9 + 3 + 3 = -30 \qquad \Rightarrow D = \frac{5}{16} \\
-9A + 9B + 3C - 3D &= 3 \qquad \Rightarrow -9\left(-\frac{1}{4}\right) + 9B + 3\left(\frac{3}{4}\right) - 3\left(\frac{5}{16}\right) = 3 \qquad \Rightarrow B = -\frac{1}{16} \\
\frac{F(z)}{z} &= \left(-\frac{1}{4}\right)\frac{1}{(z-1)^2} + \left(-\frac{1}{16}\right)\frac{1}{z-1} + \left(\frac{3}{4}\right)\frac{1}{z-3} + \left(\frac{5}{16}\right)\frac{1}{z+3} \\
F(z) &= -\left(\frac{1}{4}\right)\frac{z}{(z-1)^2} - \left(\frac{1}{16}\right)\frac{z}{z-1} + \left(\frac{3}{4}\right)\frac{z}{z-3} + \left(\frac{5}{16}\right)\frac{z}{z+3}
\end{aligned}
$$

Using $\{a^k\} = \frac{z}{z-a}$ and $\{k\} = \frac{z}{(z-a)^2}$ we obtain the sequence $\{x_k\}$ where:

$$
\begin{aligned}
x_k &= -\frac{1}{4}k - \frac{1}{16}(1)^k + \frac{3}{4}(3)^k + \frac{5}{16}(-3)^k \\
&= -\frac{1}{4}k - \frac{1}{16} + \frac{1}{4}(3)^{k+1} + \frac{5}{16}(-3)^k
\end{aligned}
$$

8. Find the Z transform of the sequence of values obtained when $f(t)$ is sampled at regular intervals of $t = T$ where

*(a) $f(t) = \sin(t)$

$$
\begin{aligned}
\sin(t) &= \frac{e^{it} - e^{-it}}{2i} \\
f(kT) &= \frac{e^{ikT} - e^{-ikT}}{2i} \\
\mathcal{Z}\{f(kT)\} &= \sum_{k=0}^{\infty} \frac{f(kT)}{z^k} \\
&= \frac{1}{2i}\sum_{k=0}^{\infty}\frac{e^{ikT}}{z^k} - \frac{1}{2i}\sum_{k=0}^{\infty}\frac{e^{-ikT}}{z^k} \\
&= \frac{1}{2i}\sum_{k=0}^{\infty}\left(\frac{e^{iT}}{z}\right)^k - \frac{1}{2i}\sum_{k=0}^{\infty}\left(\frac{e^{-iT}}{z}\right)^k \\
&= \frac{1}{2i}\left(\frac{1}{1 - \frac{e^{iT}}{z}}\right) - \frac{1}{2i}\left(\frac{1}{1 - \frac{e^{-iT}}{z}}\right) \\
&= \frac{z}{2i}\left(\frac{1}{z - e^{iT}}\right) - \frac{z}{2i}\left(\frac{1}{z - e^{-iT}}\right) \\
&= \frac{z}{2i}\left(\frac{1}{z - e^{iT}} - \frac{1}{z - e^{-iT}}\right) \\
&= \frac{z}{2i}\left(\frac{z - e^{-iT} - z + e^{iT}}{z^2 - ze^{iT} - ze^{-iT} + 1}\right) \\
&= \frac{z}{2i}\left(\frac{e^{iT} - e^{-iT}}{z^2 - z(e^{iT} + e^{-iT}) + 1}\right) \\
&= \frac{z}{2i}\left(\frac{2i\sin(T)}{z^2 - z(2\cos(T)) + 1}\right) \\
&= \frac{z\sin(T)}{z^2 - 2z\cos(T) + 1}
\end{aligned}
$$

(b) $f(t) = \sinh(t)$ **[2]**

$$\begin{aligned}
\sinh(t) &= \frac{e^t - e^{-t}}{2} \\
f(kT) &= \frac{e^{kT} - e^{-kT}}{2} \\
\mathcal{Z}\{f(kT)\} &= \sum_{k=0}^{\infty} \frac{f(kT)}{z^k}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\sum_{k=0}^{\infty} \frac{e^{kT}}{z^k} - \frac{1}{2}\sum_{k=0}^{\infty} \frac{e^{-kT}}{z^k} \\
&= \frac{1}{2}\sum_{k=0}^{\infty} \left(\frac{e^T}{z}\right)^k - \frac{1}{2}\sum_{k=0}^{\infty} \left(\frac{e^{-T}}{z}\right)^k \\
&= \frac{1}{2}\left(\frac{1}{1 - \frac{e^T}{z}}\right) - \frac{1}{2}\left(\frac{1}{1 - \frac{e^{-T}}{z}}\right) \\
&= \frac{z}{2}\left(\frac{1}{z - e^T}\right) - \frac{z}{2}\left(\frac{1}{z - e^{-T}}\right) \\
&= \frac{z}{2}\left(\frac{1}{z - e^T} - \frac{1}{z - e^{-T}}\right) \\
&= \frac{z}{2}\left(\frac{z - e^{-T} - z + e^T}{z^2 - ze^T - ze^{-T} + 1}\right) \\
&= \frac{z}{2}\left(\frac{e^T - e^{-T}}{z^2 - z(e^T + e^{-T}) + 1}\right) \\
&= \frac{z}{2}\left(\frac{2\sinh(T)}{z^2 - z(2\cosh(T)) + 1}\right) \\
&= \frac{z\sinh(T)}{z^2 - 2z\cosh(T) + 1}
\end{aligned}$$