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Q1. sol

(a) $A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$|A - \lambda I| = \begin{bmatrix} 1-\lambda & 0 \\ 2 & 2-\lambda \end{bmatrix}$

$m(\lambda) = \begin{bmatrix} 2-\lambda \\ -2 \end{bmatrix}$

Then $m(1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $m(2) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$\Rightarrow M = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} \Rightarrow M^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & -\frac{1}{2} \end{bmatrix}$

$M^{-1} = \frac{1}{-2}\begin{bmatrix} -2 & 0 \\ +2 & 1 \end{bmatrix}$

Therefore $\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}^k = \frac{1}{-2}\begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^k\begin{bmatrix} -2 & 0 \\ 2 & 1 \end{bmatrix}$

$= -\frac{1}{2}\begin{bmatrix} -2 & 0 \\ 4-4(2^k) & -2(2^k) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2\cdot(2^k)-2 & 2^k \end{bmatrix}$

$e^{At} = -\frac{1}{2}\begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} e^t & 0 \\ (2e^{2t} - 2e^t) & e^{2t} \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8} \end{bmatrix}$ eigenvalue $= 0 \,\&\, \frac{9}{8}$ $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & \frac{9}{8} \end{bmatrix}$

$|A - \lambda I| = \begin{bmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8}-\lambda \end{bmatrix}$ $m(\lambda) = \begin{bmatrix} \frac{1}{8}-\lambda \\ -\frac{1}{4} \end{bmatrix}$ $\begin{cases} m(0) = \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{4} \end{bmatrix} \\ m(\frac{9}{8}) = \begin{bmatrix} -1 \\ -\frac{1}{4} \end{bmatrix} \end{cases}$

$\Rightarrow M = \begin{bmatrix} \frac{1}{8} & -1 \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}$ $M^{-1} = \frac{1}{9}\begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix}$.

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$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{8} \end{bmatrix}^k = \frac{1}{4}\begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \frac{9}{8} \end{bmatrix}^k\begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 8(\frac{9}{8})^k & 4(\frac{9}{8})^k \\ 2(\frac{9}{8})^k & (\frac{9}{8})^k \end{bmatrix} = \begin{bmatrix} \frac{8}{9}(\frac{9}{8})^k & \frac{4}{9}(\frac{9}{8})^k \\ \frac{2}{9}(\frac{9}{8})^k & \frac{1}{9}(\frac{9}{8})^k \end{bmatrix}.$$

$$e^{At} = \frac{1}{4}\begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{(\frac{9}{8}t)} \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{9}e^{(\frac{9}{8}t)} + \frac{1}{9} & \frac{4}{9}e^{\frac{9}{8}t} - \frac{4}{9} \\ \frac{2}{9}e^{\frac{9}{8}t} - \frac{2}{9} & \frac{1}{9}e^{\frac{9}{8}t} + \frac{8}{9} \end{bmatrix}.$$

(c)

$A = \begin{bmatrix} -2 & -1 \\ -4 & -5 \end{bmatrix}$ $\quad \lambda = -1$ & -6 $\quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}$

$m(\lambda) = \begin{bmatrix} -5-\lambda \\ 4 \end{bmatrix}$ $\quad m(-1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad m(-6) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

$\Rightarrow M = \begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}$ $\quad M^{-1} = -\frac{1}{5} = \frac{1}{5}\begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix}$

Therefore:

$$\begin{bmatrix} -2 & -1 \\ -4 & -5 \end{bmatrix}^k = \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} (-1)^k & 0 \\ 0 & (-6)^k \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{5}(-1)^k + \frac{1}{5}(-6)^k & -\frac{1}{5}(-1)^k + \frac{1}{5}(-6)^k \\ -\frac{4}{5}(-1)^k + \frac{4}{5}(-6)^k & \frac{1}{5}(-1)^k + \frac{4}{5}(-6)^k \end{bmatrix}.$$

$$e^{At} = \frac{1}{5}\begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-6t} \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{5}e^{-t} + \frac{1}{5}e^{-6t} & -\frac{1}{5}e^{-t} + \frac{1}{5}e^{-6t} \\ -\frac{4}{5}e^{-t} + \frac{4}{5}e^{-6t} & \frac{1}{5}e^{-t} + \frac{4}{5}e^{-6t} \end{bmatrix}.$$

Tutor 5

Q2. sol

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k).$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k).$$

(i) $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^k. \Leftrightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -2. \end{cases} \Rightarrow m(\lambda) = \begin{bmatrix} 1 \\ -\lambda \end{bmatrix}.$

$m(-1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad m(-2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad M^{+} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$

Hence $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^k = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}^k \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$

$$= \begin{bmatrix} 2(-1)^k - (-2)^k & (-1)^k - (-2)^k \\ -2(-1)^k + 2(-2)^k. & -(-1)^k + 2(-2)^k \end{bmatrix}$$

$$x(k) = A^k \cdot x(0) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3(-1)^k - 2(-2)^k \\ -3(-1)^k + 4(-2)^k \end{bmatrix}.$$

output : $y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 3(-1)^k - 2(-2)^k \\ -3(-1)^k + 4(-2)^k \end{bmatrix} = 3(-1)^k - 2(-2)^k.$$

(ii) When $u(k) = (-1)^k$.

$$y(k) = \sum_{i=1}^{k} CA^{(k-i)} B \cdot u(i-1) = \sum_{i=1}^{k} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2(-1)^{k-i} - (-2)^{k-i} ; & (-1)^{k-i} - (-2)^{k-i} \\ -2(-1)^{k-i} + 2(-2)^{k-i} ; & -(-1)^{k-i} + 2(-2)^{k-i} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (-1)^{i-1}$$

$$= k(-1)^{k-1} + (-2)^k \sum_{i=1}^{k} \left[(-1)^i (-2)^{-i} \right].$$

$$\sum_{i=1}^{k} \left[(-1)^i (-2)^{-i} \right] = \sum_{i=1}^{k} \left[(-1)^i \left(-\tfrac{1}{2}\right)^i \right] = 1 - \left(\tfrac{1}{2}\right)^k.$$

Hence : $y(k) = k(-1)^{k-1} + (-2)^k \left[1 - \left(\tfrac{1}{2}\right)^k\right]$

$= k(-1)^{k-1} + (-2)^k - (-1)^k$

$= (-2)^k - (1+k)(-1)^k.$

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(iii) $T = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = M \qquad M^{-1} = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$

$$\Lambda = M^{-1} A M = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$B = M^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$C = [1 \quad 0]\, M = [-1 \quad 1]$$

Hence $\begin{cases} \underline{z}(k+1) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \underline{z}(k) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(k) \\ y(k) = [-1 \quad -1]\, \underline{z}(k). \end{cases}$

(iv) (i) Since $\underline{x} = M \underline{z} \Longrightarrow \underline{z} = M^{-1} \underline{x} \qquad \underline{z}(0) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$$\underline{z}(k) = \Lambda^k \underline{z}(0) = \begin{bmatrix} -3(-1)^k \\ 2(-2)^k \end{bmatrix}$$

$$y(k) = [-1 \ , \ 1]\, \underline{z}(k) = 3(-1)^k - 2(-2)^k.$$

(ii)

$$y(k) = \sum_{i=1}^{k} C \Lambda^{k-i} B u(i-1) = \sum_{i=1}^{k} \left[(-1)^{k-i} - (-2)^{k-i} \right] (-1)^{i-1}.$$

hence, $y(k) = (-2)^k - (1+k)(-1)^k.$

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Q3. sol

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underline{x}.$$

(i) $\underline{x} = e^{At}x(0)$ ① $\quad y(t) = Ce^{At}x(0)$ ②.

$\lambda = 0, -2$

$$m(\lambda) = \begin{bmatrix} -1 \\ -\lambda \end{bmatrix} \quad m(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad m(-2) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

So $M = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} (-\frac{1}{2})$

$$e^{At} = \begin{bmatrix} 1 & \frac{1}{2} - \frac{1}{2}e^{(-2t)} \\ 0 & e^{-2t} \end{bmatrix}$$

$$y(t) = C \cdot e^{At}x(0) = \frac{1}{2} - \frac{1}{2}e^{(-2t)}.$$

(ii) $\underline{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$

$$y(t) = \int_0^t Ce^{A(t-\tau)} B u(\tau) d\tau$$

$$= \int_0^t \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cdots & \cdots \\ & e^{At} \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 \, d\tau$$

$$= \int_0^t \left[\frac{1}{2} - \frac{1}{2}e^{-2(t-\tau)}\right] d\tau.$$

$$\Longleftrightarrow y(t) = \frac{1}{2}t + \frac{1}{4}\left[e^{-2t} - 1\right].$$

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Q4 sol

(i) $\lambda = -1$ & -5

$m(\lambda) = \begin{bmatrix} \lambda+2 \\ -3 \end{bmatrix} \quad m(-1) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad m(-6) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $M = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \quad M^{-1} = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} (\frac{1}{4})$.

$$\begin{cases} \dot{\underline{z}} = M^{-1} A M \underline{z} + M^{-1} B u, \\ y = C M \underline{z}. \end{cases}$$

$A_z = M^{-1} A M = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}$

$B_z = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \end{bmatrix}$.

$$\begin{cases} \dot{\underline{z}} = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix} \underline{z} + \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix} u(t). \\ \\ y(t) = \begin{bmatrix} 4 & 0 \end{bmatrix} \underline{z}. \end{cases}$$

$\underline{z}(0) = M^{-1} \underline{x}(0) = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$.

$y(t) = C e^{At} \underline{z}(0) + \int_0^t C e^{A(t-\tau)} B u(\tau) d\tau$.

where $C e^{At} \underline{z}(0) = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-5t} \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} = e^{-t}$

$$\begin{cases} \int_0^t C \cdot e^{A(t-\tau)} B u(\tau) d\tau \\ = (e^{-t} - 1) \\ y(t) = (2e^{-t} - 1). \end{cases}$$