Trigonometric Fourier Series

Given a periodic function

$$f(t+T) = f(t)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right]$$

where,

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)\,dt, \\ a_n &= \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi t}{L}\right) f(t)\,dt, \\ b_n &= \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi t}{L}\right) f(t)\,dt \end{aligned}$$

Note: L is half the functions period: $L = \frac{1}{2}T$.

Useful Identities

$$\begin{aligned} \cos(n\pi) &= (-1)^n \qquad \text{for } n = 0, 1, 2, 3, \ldots \\ \sin(n\pi) &= 0 \qquad \text{for } n = 0, 1, 2, 3, \ldots \end{aligned}$$

Complex Fourier Series

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T} \text{ and } T = \text{period}.$$

where,

$$c_k = \frac{1}{T}\int_T f(t) e^{-ik\omega_0 t}\,dt,$$

and

$$a_0 = 2c_0, \quad a_k = 2\mathrm{Re}[c_k] \quad \text{and} \quad b_k = -2\mathrm{Im}[c_k].$$

Fourier Transform

The fourier transform of the function $f(t)$ denoted $\mathscr{F}[f(t)] = X(\omega)$ is given by

$$X(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\,dt$$

Useful Identities

$$\begin{aligned} e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\ e^{-i\theta} &= \cos(\theta) - i\sin(\theta) \end{aligned}$$

Table of Fourier Transforms

f(t)	X(ω)
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-i\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{i\omega t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$
$\sin(\omega_0 t)$	$-i\pi\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$
$U(t)$	$\pi\delta(\omega)+\frac{1}{i\omega}$
$e^{-at}U(t)$	$\frac{1}{i\omega+a}$, $a>0$
$te^{-at}U(t)$	$\frac{1}{(i\omega+a)^2}$, $a>0$
$e^{-a\|t\|}$	$\frac{2a}{\omega^2+a^2}$, $a>0$
e^{-at^2}	$\sqrt{\frac{\pi}{a}e^{-\frac{\omega^2}{4a}}}$, $a>0$

Integrals

1. Integrals of Polynomial functions

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$

2. Integrals of Exponential functions

$$\int e^x\,dx = e^x + C$$
$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

3. Integrals of Trigonometric functions

$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \sec x \tan x\,dx = \sec x + C$$

4. Integrals of Hyperbolic functions

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$
$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$
$$\int \tanh ax\,dx = \frac{1}{a}\ln|\cosh ax| + C$$

5. Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Derivatives

1. Powers

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

2. Exponentials and logs

$$\frac{d}{dx}[e^{kx}] = ke^{kx}$$
$$\frac{d}{dx}[\ln kx] = \frac{1}{x}$$

3. Trigonometric functions

$$\frac{d}{dx}[\sin(kx)] = k\cos(kx)$$
$$\frac{d}{dx}[\cos(kx)] = -k\sin(kx)$$
$$\frac{d}{dx}[\tan(kx)] = k\sec^2(kx)$$
$$\frac{d}{dx}[\sec(kx)] = k\sec(kx)\tan(kx)$$

4. Hyperbolic functions

$$\frac{d}{dx}[\sinh(kx)] = k\cosh(kx)$$
$$\frac{d}{dx}[\cosh(kx)] = k\sinh(kx)$$
$$\frac{d}{dx}[\tanh(kx)] = k(1-\tanh^2(kx))$$
$$\frac{d}{dx}[\text{sech}(kx)] = -k\,\text{sech}(kx)\tanh(kx)$$

5. Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f\frac{dg}{dx} + g\frac{df}{dx}$$

6. Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

Trigonometric Formula

Even Odd Property

$$
\begin{aligned}
\sin(-A) &= -\sin(A), \quad \text{Odd function} \\
\cos(-A) &= \cos(A), \quad \text{Even function} \\
\tan(-A) &= -\tan(A), \quad \text{Odd function}
\end{aligned}
$$

Radian Angle Shift

$$
\begin{aligned}
\sin(A \pm \pi) &= -\sin A \\
\cos(A \pm \pi) &= -\cos A \\
\tan(A \pm \pi) &= \tan A
\end{aligned}
$$

Angle sums

$$
\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}
$$

Complex Form

$$
\begin{aligned}
e^{ix} &= \cos(x) + i\sin(x) \\
e^{-ix} &= \cos(x) - i\sin(x) \\
e^{i\pi} &= -1 \\
\cos(x) &= \frac{e^{ix} + e^{-ix}}{2} \\
\sin(x) &= \frac{e^{ix} - e^{-ix}}{2i}
\end{aligned}
$$

Identities Involving Squares

$$
\begin{aligned}
\sin^2(A) + \cos^2(A) &= 1 \\
\sec^2(A) - \tan^2(A) &= 1
\end{aligned}
$$

Multiples of π

$$
\begin{aligned}
\cos(n\pi) &= (-1)^n, \qquad \text{for } n = 0, 1, 2, 3, \ldots \\
\sin(n\pi) &= 0, \qquad \text{for } n = 0, 1, 2, 3, \ldots
\end{aligned}
$$

Common Angles

A	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
$\cos(A)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
$\sin(A)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
$\tan(A)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0

Note: π radians is equivalent to $180°$

Linear First Order ODE

Integrating Factor

Given the first order ODE of the form

$$y' + P(x)y = Q(x)$$

the solution is given by

$$y = \frac{\int F(x)Q(x)\,dx + C}{F(x)}$$

where,

$$F(x) = e^{\int P(x)\,dx}$$

Homogeneous Second Order ODE

Given the second order ODE

$$y'' + ay' + by = 0, \qquad a, b \text{ constants}$$

the characteristic equation is given by

$$\lambda^2 + a\lambda + b = 0$$

1: λ_1 and λ_2 are real and **Distinct**. The complimentary solution is

$$y_c = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

2: λ_1 and λ_2 are real and **Equal**. The complimentary solution is

$$y_c = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_2 x}$$

3: λ_1 and λ_2 are **Complex** conjugates. The complimentary solution is

$$y_c = e^{\alpha x}[k_1 \cos(\omega x) + k_2 \sin(\omega x)]$$

Note: $\lambda = \alpha \pm i\omega$

Inhomogeneous Second Order ODE

Given the second order ODE

$$y'' + ay' + by = R(x), \qquad a, b \text{ constants}$$

The trials for a particular solution are given by

$\boldsymbol{R(x)}$	$\boldsymbol{y_p(x)}$
$ke^{\omega x}$	$Ae^{\omega x}$
$a_0 + a_1 x + \cdots + a_n x^n$	$A_0 + A_1 x + \cdots + A_n x^n$
$a_1 \cos(\omega x)$	$A_1 \cos(\omega x) + A_2 \sin(\omega x)$
$a_1 \sin(\omega x)$	$A_1 \cos(\omega x) + A_2 \sin(\omega x)$
$e^{\omega x}[a_1 \cos(\omega x)]$	$e^{\omega x}[A_1 \cos(\omega x) + A_2 \sin(\omega x)\ ]$
$e^{\omega x}[a_1 \sin(\omega x)]$	$e^{\omega x}[A_1 \cos(\omega x) + A_2 \sin(\omega x)\ ]$

Further trial functions

1: Given

$$R(x) = e^{\omega x} f(x)$$

where $f(x)$ is already given in the table the new trial is

$$y_p(x) = e^{\omega x} \times \text{ The trial for } f(x)$$

2: Given

$$R(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

where the $f_i(x)$ are already given in the table the new trial is the sum of the trails

$$\begin{aligned} y_p(x) = (&\text{ The trial for } f_1(x)) + \\ &+ \cdots + (\text{ The trial for } f_n(x)) \end{aligned}$$

Modification Rule

If any of the terms in the trial solution for $y_p(x)$ occurs in the complementary solution, $y_c(x)$, then the correct form for $y_p(x)$ is found by multiplying the trial solution by the smallest power of x so that no term of the trail solution occurs in $y_c(x)$.

Laplace Transforms

Laplace Transform Table

$f(t)$	$F(s) = \mathscr{L}[f(t)]$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$, n a positive integer
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
e^{at}	$\frac{1}{s-a}$
$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$, n a positive integer
$\delta(t-a)$	e^{-as}
$U(t-a)$	$\frac{e^{-as}}{s}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
Laplace Transform of Derivatives	
$y(t)$	$Y(s)$
$\frac{dy(t)}{dt}$	$sY(s) - y(0)$
$\frac{d^2y(t)}{dt^2}$	$s^2Y(s) - sy(0) - y'(0)$

Translation Theorems

The First Translation Theorem

$$\mathscr{L}[e^{at}f(t)] = F(s-a)$$

The Second Translation Theorem

$$\mathscr{L}[f(t-a)U(t-a)] = e^{-as}F(s)$$

Laplace Transform of a Periodic Function

Given a piecewise continuous function $f(t)$, for $t \geq 0$ that is of exponential order and has a period of T, the Laplace transform of $f(t)$ is given by

$$\mathscr{L}[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

Convolution

Given $f(t)$ and $g(t)$ are piecewise continuous for $t \geq 0$. Then the convolution of f and g, denoted $f \star g$ is defined to be

$$f \star g = \int_0^t f(\tau)g(t-\tau)\, d\tau$$

Laplace Transform Convolution

Given $f(t)$ and $g(t)$ are piecewise continuous for $t \geq 0$. Then the convolution of $f \star g$ has the Laplace Transform

$$\mathscr{L}[f \star g] = F(s)G(s)$$

where $\mathscr{L}[f(t)] = F(s)$ and $\mathscr{L}[g(t)] = G(s)$.

Laplace Transform of an Integral

$$\mathscr{L}\left[\int_0^t f(\tau)\, d\tau\right] = \frac{F(s)}{s}$$

Partial Fractions

1: The degree of the numerator is less than the degree of the denominator. If it is not then divide out the expression

2: A linear factor $(s+a)$ in the denominator contributes a partial fraction term of the form

$$\frac{A}{s+a}$$

where A is a constant to be determined.

3: A repeated factor $(s+a)^n$ contributes partial fraction terms

$$\frac{A_1}{(s+a)} + \frac{A_2}{(s+a)^2} + \cdots + \frac{A_n}{(s+a)^n}$$

where the A_i are constants to be determined.

4: A quadratic factor (s^2+as+b) contributes a partial fraction term

$$\frac{A_1 s + A_2}{s^2+as+b}$$

Difference Equations

Trial Solutions

To construct a particular solutions to the difference equation

$$x_{n+2} + ax_{n+1} + bx_n = R(n)$$

one can construct a trial particular solution using the table

Terms in $R(n)$	**Form of trial**
β^n	$A\beta^n$
$a_1\cos(\alpha n) + a_2\sin(\alpha n)$	$A_1\cos(\alpha n) + A_2\sin(\alpha n)$
Degree k Polynomial $P(n)$	$A_0 + A_1 n + \cdots + A_k n^k$
$\beta^n P(n)$	β^n times $P(n)$ trial

Z-Transform Formulae

The Z-Transform of x_n, $\mathscr{Z}[x_n] = X(z)$ is

$$\mathscr{Z}[x_n] = X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$

The Z-Transform of $x_{(n+1)}$ is

$$\mathscr{Z}[x_{(n+1)}] = z(X(z) - x_0)$$

The Z-Transform of $x_{(n+2)}$ is

$$\mathscr{Z}[x_{(n+2)}] = z^2(X(z) - x_0 - x_1 z^{-1})$$

Z-Transform Table

$\{x_n\}$	$\mathscr{Z}\{x_n\}$	R.O.C
$\{\delta\}$	1	All z
$\{u_n\}$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$\{n\}$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$\{n^2\}$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
$\{n^3\}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\lvert z\rvert > 1$
$\{a^n\}$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$\{na^n\}$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$\{\cos(\omega n)\}$	$\frac{z^2 - z\cos(\omega)}{z^2 - 2z\cos(\omega) + 1}$	$\lvert z\rvert > 1$
$\{\sin(\omega n)\}$	$\frac{z\sin(\omega)}{z^2 - 2z\cos(\omega) + 1}$	$\lvert z\rvert > 1$
$\{a^n\cos(\omega n)\}$	$\frac{z^2 - az\cos(\omega)}{z^2 - 2az\cos(\omega) + a^2}$	$\lvert z\rvert > \lvert a\rvert$
$\{a^n\sin(\omega n)\}$	$\frac{az\sin(\omega)}{z^2 - 2az\cos(\omega) + a^2}$	$\lvert z\rvert > \lvert a\rvert$

Note u_n is the Unit-Step sequence defined by

$$u_n = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$