EE211 Tutor 3

Q1 (i) sol

(A) Leading to the state-space model:

$x_1 = I_1 \quad x_2 = I_2 \quad x_3 = u_c$

then we got:

$$\begin{cases} x_1 - x_2 - C\dot{x}_3 = 0 & ① \checkmark \\ u_i - R_1 x_1 - x_3 = 0 & ② \\ x_3 - L\dot{x}_2 - R_2 x_2 = 0 & ③ \end{cases}$$

Hence
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cancel{0}\, -\frac{1}{R_1} \\ 0 & -\frac{R_2}{L} & \frac{1}{C} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1} \\ 0 \\ 0 \end{bmatrix} u_i$$

(B)
$$\begin{cases} x_1 - x_2 - C\dot{x}_3 = 0 & ① \\ u_i - R_1 x_1 - \cancel{x_3} = 0 & ② \\ x_3 - R_2 x_2 - R_3 x_2 = 0 & ③ \end{cases}$$

Hence
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{R_1} \\ 0 & 0 & \frac{1}{R_2+R_3} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1} \\ 0 \\ 0 \end{bmatrix} u_i$$

Q1 (ii) sol

(A)
$$\begin{cases} \frac{L}{R_1}\dot{v}_1 + v_1 + LC\ddot{v}_1 - v_0 - \frac{L}{R_1}\dot{v}_{in} = 0 & ① \\ v_1 = \frac{L}{R_2}\dot{v}_0 + v_0 & ② \end{cases}$$

$$\therefore \begin{cases} \ddot{v}_1 = -\frac{1}{CR_1}\dot{v}_1 - \frac{1}{LC}v_1 + \frac{1}{LC}v_0 + \frac{1}{CR_1}\dot{v}_{in} & ③ \\ \dot{v}_0 = \frac{R_2}{L}v_1 - \frac{R_2}{L}v_0 & ④ \end{cases}$$

Let $X_1 = v_1 \quad x_2 = \dot{v}_1 \quad x_3 = v_0$

$$\begin{cases} x_1 = v_1 \\ x_2 = \dot{v}_1 - v_{in} \\ v_3 = v_0 \end{cases}$$

$$\begin{cases} \dot{x}_1 = x_2 = \\ \cancel{\dot{x}_2 = x_2 - x_1 + x_3 + \dot{v}_{in}} \\ \dot{x}_2 = -\frac{1}{CR_1}x_2 - \frac{1}{LC}x_1 + \frac{1}{LC}x_3 + \frac{1}{CR_1}v_{in} \\ \dot{x}_3 = \frac{R_2}{L}x_1 - \frac{R_2}{L}x_3 \end{cases}$$

& $V_{in} = \dot{v}_1 - x_2$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{LC} & -\frac{1}{CR_1} & \frac{1}{LC} \\ \frac{R_2}{L} & 0 & -\frac{R_2}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{1}{CR_1} \\ 0 \end{bmatrix} V_{in}$$

Q1 (ii). $x_1 = v_1 \quad x_2 = v_1 - v_i \quad x_3 = v_0$

(B) $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{c} & -\frac{R_2}{c} & \frac{R_1}{c} \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(iii) ① $$\frac{v_0}{v_i} = \frac{R_2}{R_1 CLs^2 + (cR_1R_2 + L)s + (R_1 + R_2)}$$

② $$\frac{v_0}{v_i} = \frac{R_3}{(R_1 + R_2 + R_3) + scR_1(R_2 + R_3)}$$

① the system is 2-order and we only need 2 states to describe it. Thus, It's not a minimal state space realisation.

② It's a minimal state space realization.

Q2 sol (i) $$\begin{cases} M\ddot{x}_1 + B(\dot{x}_1 - \dot{x}_2) + K(x_1 - x_2) = f(t) & ① \\ m\ddot{x}_2 + B_t\dot{x}_2 + K_t x_2 = B(\dot{x}_1 - \dot{x}_2) + K(x_1 - x_2) & ② \end{cases}$$

$$\begin{cases} \ddot{x}_1 = -\frac{B}{M}\dot{x}_1 + \frac{B}{M}\dot{x}_2 - \frac{K}{M}x_1 + \frac{K}{M}x_2 + \frac{f(t)}{M} \\ \ddot{x}_2 = -(\frac{B}{m} + \frac{B_t}{m})\dot{x}_2 + \frac{B}{m}\dot{x}_1 - (\frac{K}{m} + \frac{K_t}{m})x_2 + \frac{K}{m}x_1 \end{cases}$$

Let $X_1 = x_1 \quad X_2 = x_2 \quad X_3 = \dot{x}_1 \quad X_4 = \dot{x}_2$

$$\begin{cases} \dot{X}_1 = \dot{x}_1 = X_3 \\ \dot{X}_2 = \dot{x}_2 = X_4 \\ \dot{X}_3 = -\frac{B}{M}X_3 + \frac{B}{M}X_4 - \frac{K}{M}X_1 + \frac{K}{M}X_2 + \frac{f(t)}{M} \\ \dot{X}_4 = -(\frac{B}{m} + \frac{B_t}{m})X_4 + \frac{B}{m}X_3 - (\frac{K}{m} + \frac{K_t}{m})X_2 + \frac{K}{m}X_1 \end{cases}$$

So $$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K}{M} & \frac{K}{M} & -\frac{B}{M} & \frac{B}{M} \\ \frac{K}{m} & -(\frac{K+K_t}{m}) & \frac{B}{m} & -\frac{(B+B_t)}{m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ 0 \end{bmatrix} f(t)$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} X, \quad \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

Q3 (i) sol

2^{nd} transfer function : $\dfrac{Y(z)}{U(z)} = \dfrac{0.65z + 2.55}{z^2 + 3.1z \cancel{+} -2.51}$

$$\begin{cases} \dfrac{Q(z)}{U(z)} = \dfrac{1}{(z^2+3.1z \cancel{+} -2.51)} & ① \\[2ex] \dfrac{Y(z)}{Q(z)} = \dfrac{(0.65z + 2.55)}{1} & ② \end{cases} \Longrightarrow \begin{cases} q_k + 3.1 q_{k-1} \cancel{+} -2.51 q_{k-2} = u_k \\ 0.65 q_{k-1} + 2.55 q_{k-2} = y_k \end{cases}$$

Let $\begin{cases} x_1(k) = q_{k-1} \\ x_2(k) = q_{k-2} \end{cases} \Longrightarrow \begin{cases} x_1(k+1) = q_k = -3.1 q_{k-1} + 2.51 q_{k-2} + u_k \\ x_2(k+1) = q_{k-1} = x_1(k) \end{cases}$

$$\therefore \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -3.1 & 2.51 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_{k-1} \\ q_{k-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\therefore y_k = 0.65 x_1(k) + 2.55 x_2(k).$$

(ii) $\begin{cases} \dot{x}(t) = \begin{bmatrix} -3.1 & 2.51 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0.65 & 2.55 \end{bmatrix} x(t). \end{cases}$

$$G(s) = C \cdot (zI - A)^{-1} \cdot B$$

$$= \begin{bmatrix} 0.65 & 2.55 \end{bmatrix} \left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} -3.1 & 2.51 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \cdots$$

$$= \frac{0.65z + 2.55}{z^2 + 3.1z - 2.51}.$$

$$\therefore G(z) = \frac{Y(z)}{U(z)} = \frac{0.65z + 2.55}{z^2 + 3.1z - 2.51}$$

Q4 Sol (i) $m = 0.1kg$ $K_1 = 6N/m$ $K_2 = 4\,N/m$ $B = 0.4\,Ns/m$

$$\begin{cases} b\frac{dz}{dt} + k_2(z-y) = 0 & ① \\ F(t) = m\frac{dy^2}{dt} + k_1 y + K_2(y-z) & ② \end{cases}$$

$$\begin{cases} Sb\,Z(s) + k_2[Z(s) - y(s)] = 0 & ③ \\ F(s) = s^2 m Z(s) + K_1 y(s) + K_2(-Z(s) + y(s)) & ④ \end{cases}$$

$$\therefore \begin{cases} y(s) = \frac{(sb+k_2)Z(s)}{K_2} \quad \& \quad Z(s) = \frac{K_2}{sb+K_2} y(s) & ⑤ \\ F(s) = (s^2 m - K_2)Z(s) + (K_1 + K_2) y_s & ⑥ \end{cases}$$

$$\frac{y(s)}{F(s)} = \frac{(sb+k_2)\,Z(s)}{K_2\left[(s^2 m - k_2)Z(s) + (K_1+K_2)y_s\right]} \quad ⑦$$

Hence, $$\frac{y(s)}{F(s)} = \frac{0.4s + 4}{0.4s^2 + 4s + 24} \quad ⑧$$

(ii) $x_1 = y$ $x_2 = \dot{y}$ $x_3 = z$

$$\begin{cases} b\dot{x}_3 + k_2(x_3 - x_1) = 0 & ⑧ \\ F = m\dot{x}_2 + k_1 x_1 + k_2(x_1 - x_3) & ⑨ \end{cases}$$

$$\dot{x}_1 = \dot{y} = x_2$$

$$\therefore \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{-(k_1+k_2)}{m} x_1 + \frac{K_2}{m} x_3 + \frac{F}{m} \\ \dot{x}_3 = \frac{K_2}{b} x_1 - \frac{K_2}{b} x_3 \end{cases}$$

$$\therefore \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{(k_1+k_2)}{m} & 0 & \frac{K_2}{m} \\ \frac{K_2}{b} & 0 & \frac{-K_2}{b} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} F$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -60 & 0 & 40 \\ 10 & 0 & -10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} F$$

Q4 (iii) $\frac{Y(s)}{F(s)} = \frac{0.4s + 4}{0.4s^2 + 4s + 24}$

(iv) $\begin{cases} \frac{Z(s)}{F(s)} = \frac{1}{0.4s^2 + 4s + 24} \\ \frac{Y(s)}{Z(s)} = 0.4s + 4 \end{cases}$

$\begin{cases} F = 0.4\ddot{z} + 4\dot{z} + 24z \\ Y = 0.4\dot{z} + 4z \end{cases}$

$x_1 = z \quad x_2 = \dot{z} \quad x_3 = \ddot{z}$

$\therefore \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = 2.5F - 10x_2 + 60x_1 \end{cases}$

$\therefore \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 60 & -10 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2.5 \end{bmatrix} F$

$y = \begin{bmatrix} 4 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$G(s) = \frac{Y(s)}{F(s)} = \frac{0.4s + 4}{0.4s^2 + 4s + 24}$

(PS: It's corresponding