EE206 Differential Equations and Transform Methods Exam Preparation

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Autumn 2021

1 First-order differential equations

Find the general solutions/ solve the initial value problem (IVP)

- **Separate of variables**

$$
\begin{aligned}
&2e^{-x}\frac{dy}{dx} = x \\
&y' + 3x^2y = x^2 \\
&\cos(x)\sin(x)\frac{dy}{dx} + \sin(x)\cos(y) = 0 \\
&(1+x)dy - ydx = 0 \\
&(1-x^2)\frac{dy}{dx} + x(y-a) = 0
\end{aligned}
$$

- **Bournoulli equation**

$$
\begin{aligned}
&\frac{dy}{dx} = y(xy^4 - 1) \\
&y' + \frac{y}{x} - \sqrt{y} = 0 \\
&x\frac{dy}{dx} - (1+x)y = xy^2
\end{aligned}
$$

- **Substitution (reduction to separation of variables)**

$$
\begin{aligned}
&\frac{dy}{dx} = \cos(x+y) \\
&\frac{dy}{dx} = (x+y+1)^2 \\
&\frac{dy}{dx} = (-2x+y)^2 - 7, \qquad y(0) = 0
\end{aligned}
$$

- **Exact differential**

$$2xydx + (x^2-1)dy = 0$$

- **Homogeneous functions**

$$(x^2+y^2)dx + (x^2-xy)dy = 0$$

- **Linear constant coefficient**

$$\frac{dy}{dx} + x^2 y = x$$
$$x^2 \frac{dy}{dx} + xy = \frac{1}{x}$$
$$\frac{dy}{dx} + 2\sin(2x)y = 2e^{\cos(2x)}, \qquad y(0) = 0$$
$$y' + (\tan\ x)y = \cos^2\ x, \qquad y(\pi) = 1$$

2 Second-order differential equations

- **Wronskian determinant:**

  1. The functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear DE $y'' - 9y = 0$ on $I = (-\infty, +\infty)$. Calculate the Wronskian and determine whether the functions form a fundamental set of solutions. If yes, determine a general solution.

- **Reduction of order:**

  1. Given $y_1 = e^x$ is a solution of $y'' - y = 0$ on $(-\infty, \infty)$, use the reductions of order to find a second solution y_2
  2. $y_1 = x^2$ is a solution of $x^2 y'' - 3xy' + 4y = 0$. Find the general solution on $(0, \infty)$

- **Homogeneous:**

$$y'' + 5y' + 6y = 0$$
$$y'' + 10y' + 25y = 0$$

- **Nonhomogeneous:**

$$y'' + 5y' + 6y = 5x - 3$$
$$y'' + 4y' + 4y = 2x - 3e^{-2x}$$
$$y'' - 9y' + 14 = 3x^2 - 5\sin 2x + 7e^{6x}$$

3 Laplace Transform

- **Find the following Laplace Transform:**

  1. $f(t) = 4t \star \delta(t - 2\pi)$
  2. $\cos(\omega t) \star (\cos \omega t)$
  3. $f(t) = (1 - e^t + 3e^{-4t}) \cos 5t$
  4. $f(t) = t\, \mathcal{U}(t - 2)$
  5. $f(t) = (1 + e^{2t})^2$
  6. $f(t) = \int_0^t e^{-\tau}\ \cos\tau\ d\tau$
  7. $t \star e^t$

- **Find the following Laplace Transform:**

  1. $\dfrac{5.5}{(s + 1.5)(s - 4)}$

2. $\dfrac{9}{s(s+3)}$
3. $\dfrac{2\pi s}{(s^2+\pi^2)^2}$
4. $\dfrac{e^{-as}}{s(s-2)}$
5. $\dfrac{s^2+12}{(s^2+5s+6)(s+6)}$
6. $f(t) = \int_0^t e^{-\tau} \ \cos\tau \ d\tau$
7. $t \star e^t$

- **Solving IVP by Laplace Transform:**

1. Solving the following initial value problem (Hint $\gamma = \omega;\ \gamma \neq \omega$)

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin\gamma t, \qquad x(0) = 0,\ x'(0) = 0$$

2. Solving the following initial value problem

$$y'' + 2y' = \delta(t-1), \qquad y(0) = 0,\ y'(0) = 1$$

where $\delta(t - t_0)$ is the delta function.

3. Solving the following initial value problems using Laplace Transform

$$\begin{aligned} y' - y &= 1 + te^t, \qquad && y(0) = 0 \\ y'' + 2y &= 12e^{2t}, \qquad && y(0) = 0,\ y'(0) = 0 \end{aligned}$$

4. Find the solution of the initial value problem using the Laplace Transform

$$y' + y = f(t), \qquad y(0) = 0$$

for $t > 0$, where

$$f(t) = \begin{cases} 3, & 0 \le t \le 1 \\ 1, & 1 < t \end{cases}$$

5. Solving the following integral equations

$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau} d\tau$$

$$y(t) + 4\int_0^t y(\tau)(t-\tau)\ d\tau = 2t$$

$$y(t) - \int_0^t y(\tau)\sin\ 2(t-\tau)\ d\tau = \sin 2t$$

6. Find the solution of the integro-differential equation for $t \ge 0$

$$y'(t) - \int_0^t y(\tau)\sin(t-\tau)\ d\tau = y(t), \qquad y(0) = 0.$$