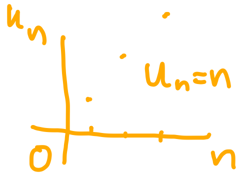
Engineering Mathematics 1 (Fall 2021)

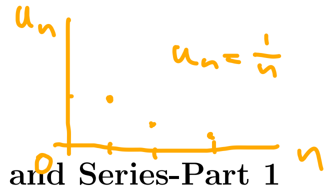
Prof. Fengying Wei
E-mail: weifengying@fzu.edu.cn
College of Mathematics and Statistics, Fuzhou University

October 8, 2021

Students should be able to (after learning)

- Add, subtract and multiply complex numbers
- Convert complex numbers between Cartesian and polar forms
- Differentiate all commonly occurring functions including polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of a derivative, namely the derivative as a tangent and the derivative as a rate of change
- Integrate certain standard functions, constructed from polynomials, exponential functions, trigonometric functions, hyperbolic functions and combinations thereof
- Distinguish between the two main meanings of integration, namely the integral as the inverse of the derivative and the integral as the area under a curve
- Apply Taylor series to numerically approximate functions
- Apply Simpson's rule to numerically evaluate integrals
- Solve simple first and second order ordinary differential equations
- Apply and select the appropriate mathematical techniques to solve a variety of associated engineering problems





Lecture 6: Sequences and Series-Part 1

u_n = general term

1. Infinity and limits

limits of n / n^2

$u_n=n$ "∞" $1, 2, 3, 4, 5, \dots, n, \dots$ $>$ "∞"

$u_n=n^2$ $1^2, 2^2, 3^2, \dots, n^2, \dots$ $n\to\infty$

$u_n=\frac{1}{n}$ $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$ "0" $n\to\infty$

2. Rules of limits

$u_n=\frac{1}{n^2}$, $u_n=\frac{1}{n^3}$

Multiplication by a constant

$$\lim_{n\to\infty} f(n) = F \Rightarrow \lim_{n\to\infty} a f(n) = aF$$

$u_n=\frac{2}{n}$ Take $f(n)=\frac{2}{n}$. We have known $\lim_{n\to\infty}\frac{1}{n}=0$, which gives

$$\lim_{n\to\infty}\frac{2}{n} = 2\lim_{n\to\infty}\frac{1}{n} = 2\cdot 0 = 0$$

Sums and differences

$$\lim_{n\to\infty}[f(n)\pm g(n)] = \lim_{n\to\infty} f(n) \pm \lim_{n\to\infty} g(n)$$

$$\lim_{n\to\infty}\frac{n+1}{n} = 1. \quad \lim_{n\to\infty}\frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty}\left(\frac{4n+3}{5n}+\frac{4}{n}\right) = \lim_{n\to\infty}\frac{4n+3}{5n} + \lim_{n\to\infty}\frac{4}{n}$$

$$= \frac{4}{5}\lim_{n\to\infty}\frac{n+\frac{3}{4}}{n} + 4\lim_{n\to\infty}\frac{1}{n} = \frac{4}{5}\cdot 1 + 4\cdot 0 = \frac{4}{5}.$$

Products and quotients

$$\lim_{n\to\infty} f(n)g(n) = \lim_{n\to\infty} f(n)\cdot\lim_{n\to\infty} g(n),$$

$$\lim_{n\to\infty}\frac{n+1}{n^2} = \lim_{n\to\infty}\frac{n+1}{n}\cdot\lim_{n\to\infty}\frac{1}{n} = 1\cdot 0 = 0 \checkmark$$

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \frac{\lim_{n\to\infty} f(n)}{\lim_{n\to\infty} g(n)} \quad \text{for } \lim_{n\to\infty} g(n) \neq 0.$$

$$\lim_{n\to\infty}\frac{n+1}{n^2} = \lim_{n\to\infty}\frac{\frac{n+1}{n^2}}{\frac{n^2}{n^2}} = \frac{\lim_{n\to\infty}\frac{n+1}{n^2}}{\lim_{n\to\infty}\frac{n^2}{n^2}} = 0$$

1

Indeterminate limits

$f(n)=u_n=n$ $1, 2, 3, 4, \dots, n, \dots$ $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} n = \infty$

$f(n)=u_n=n^2$ $1, 4, 9, 16, \dots, n^2, \dots$ $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} n^2 = \infty$

u_4 : 4^{th} term

$$f(n)=u_n=\frac{n^2+1}{n} \qquad \lim_{n\to\infty}\frac{n^2+1}{n} = \lim_{n\to\infty}\left(\frac{n^2}{n}+\frac{1}{n}\right) = \lim_{n\to\infty} n + \lim_{n\to\infty}\frac{1}{n} = \infty + 0 = \infty$$

Series

1. **Arithmetic series and mean:**

general term: $u_n = a + (n-1)d$ 2d

$$\underbrace{a}_{1}+\underbrace{(a+d)}_{2}+\underbrace{(a+2d)}_{3}+\cdots+\underbrace{(a+(n-1)d)}_{n} = \sum_{r=0}^{n-1}(a+rd) = \boxed{na + \frac{n(n-1)d}{2}}.$$

Given P, Q, mean of P, Q is $A = \frac{P+Q}{2}$. $\Rightarrow P, A, Q \Rightarrow a, a+d, a+2d$ (d, d; 2d)

Ex1 Given $10+6+2-2-6\cdots$, find first 20 terms.

$d = 6-10 = -4$, $a = 10$.

$\sum_{r=0}^{20-1}(a+rd) = 20 \times 10 + \frac{20\times 19}{2} \times (-4) = -560$.

Ex2 $U_7 = 22$, $U_{12} = 37$, find series.

$U_7 = a + 6d = 22$

$U_{12} = a + 11d = 37$

$\therefore 5d = 15 \quad \therefore d = 3$ $\quad \therefore a = 4$. $\quad \therefore 4+7+10+13+16+\cdots$

Ex3 Inserting 3 arithemetic means between 8 and 18.

$8 + \Delta + \Delta + \Delta + 18$

10.5 13 15.5

Let $a = 8$, ✓

$a + 4d = 18$ ✓

$\therefore d = 2.5$

$\therefore a+d = 10.5$, $a+2d = 13$, $a+3d = 15.5$

$\therefore$ they are 10.5, 13, 15.5.

2. **Geometric series and mean:**

general term: $u_n = ar^{n-1}$

① $\boxed{\underbrace{a}_{1} + \underbrace{\cancel{ar}}_{2} + \underbrace{\cancel{ar^2}}_{3} + \cancel{\cdots} + \underbrace{\cancel{ar^{n-1}}}_{n} = \sum_{k=0}^{n-1} ar^k} = \boxed{\frac{a(1-r^n)}{1-r}}.$

$\times r$

Given P, Q, mean of P, Q is $A = \sqrt{PQ}$.

$P + A + Q$ (r, r)

$a \quad ar \quad ar^2$

② $\cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cancel{\cdots} + ar^n = a\sum_{k=0}^{n-1} ar^k$

②−① $ar^n - a = (r-1)\sum_{k=0}^{n-1} ar^k \xrightarrow[a \neq 1]{} \sum_{k=0}^{n-1} ar^k = \frac{ar^n - a}{r-1} = \frac{a(1-r^n)}{1-r}$

Ex1 Given $8+4+2+1+\frac{1}{2}+\cdots$ find sum of first 8 terms. $\because r = \frac{1}{2}$, $a = 8$

$\sum_{k=0}^{8-1} ar^k = \frac{8(1-(\frac{1}{2})^8)}{1-\frac{1}{2}} = \frac{8(1-\frac{1}{256})}{\frac{1}{2}} = 16(1-\frac{1}{256}) = 16 \cdot \frac{255}{256} = \frac{255}{16} = 15\frac{15}{16}$.

Ex2 $U_5 = 162$, $U_8 = 4374$. find series.

$U_5 = ar^4 = 162$,

$U_8 = ar^7 = 4374$ $> r^3 = 27 \quad \therefore r = 3$, $r^4 = 81 \quad \therefore a = 2$, $\therefore 2+6+18+54+\cdots$

Ex3 Inserting 4 geometric means between 5 and 1215.

$5 + \Delta + \Delta + \Delta + \Delta + 1215$

$a = 5$. $ar^5 = 1215$ $\therefore r^5 = 243$ $\therefore r = 3$

$\therefore A_1 = ar = 5\times 3 = 15$, $A_2 = ar^2 = 5\times 3^2 = 45$, $A_3 = ar^3 = 5\times 3^3 = 135$, $A_4 = 405$

they are 15, 45, 135, 405.

3. Series of power of natural number:

Sum of natural numbers $1 + 2 + 3 + \cdots + n = \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

Sum of squares $1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

Sum of squares $1^3 + 2^3 + 3^3 + \cdots + n^3 = \sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$

4. **Infinite series and limiting values:**

$$\lim_{n\to\infty}\sum_{r=0}^{n-1}(a+rd) = \lim_{n\to\infty}\left[na+\frac{n(n-1)d}{2}\right] = \infty \text{ or } -\infty, \quad \text{depends on } a \text{ and } d.$$

$$\lim_{n\to\infty}\sum_{k=0}^{n-1} ar^k = \lim_{n\to\infty}\frac{a(1-r^n)}{1-r} = \frac{a}{1-r}, \qquad \text{as } |r|<1.$$

5. **Convergent and divergent series:**

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

If $\sum_{n=1}^{\infty} u_n$ is a definite value, then $\sum_{n=1}^{\infty} u_n$ is convergent.

If $\sum_{n=1}^{\infty} u_n$ is NOT a definite value, then $\sum_{n=1}^{\infty} u_n$ is divergent.

6. **Tests for convergency:**

If $\lim_{n\to\infty} u_n = 0$, then $\sum_{n=1}^{\infty} u_n$ may be convergent.

If $\lim_{n\to\infty} u_n \neq 0$, then $\sum_{n=1}^{\infty} u_n$ is certainly divergent.

Comparison test-Useful standard series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

$p > 1, \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent,

$p \leq 1, \sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent.

D'Alembert's ratio test for positive terms

If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$, then $\sum_{n=1}^{\infty} u_n$ is convergent.

If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$, then $\sum_{n=1}^{\infty} u_n$ is divergent.

If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, then $\sum_{n=1}^{\infty} u_n$ is inconclusive.

For general series

If $\sum_{n=1}^{\infty} |u_n|$ is convergent, then $\sum_{n=1}^{\infty} u_n$ is absolutely convergent.

If $\sum_{n=1}^{\infty} |u_n|$ is divergent, but $\sum_{n=1}^{\infty} u_n$ is convergent, $\sum_{n=1}^{\infty} u_n$ is conditionally convergent.