# Tutorial Sheet 1 - Solutions

Q1 & Q2 Refer to notes.

Q3 (i) $\frac{dy(t)}{dt} = 3u(t) - 2y(t)$

(ii) $\frac{dy(t)}{dt} = 3u(t) - 2\sqrt{y(t)}$

(iii) $y(t) = 3u(t)$

(iv) $y(t) = 3\sqrt{u(t)}$

(v) $\frac{dy(t)}{dt} = 3u(t) - a(t)y(t)$, where $a(t)$ is a constant that varies with time!

For all examples, the dependent variable is *y*, the independent variable is *t* and the parameters are the constants used.

Q4 (i) This is a system that obeys the principle of superposition and homogeneity, i.e.:

$$Af(x_1) + Bf(x_2) = f(Ax_1 + Bx_2) \text{ for any constants } A \text{ and } B$$

(ii) $y = 2u \rightarrow Af(u_1) + Bf(u_2) = A(2u_1) + B(2u_2) = 2Au_1 + 2Bu_2$

$y = 2u \rightarrow f(Au_1 + Bu_2) = 2(Au_1 + Bu_2) = 2Au_1 + 2Bu_2$

Hence: $Af(u_1) + Bf(u_2) = f(Au_1 + Bu_2) \Rightarrow$ Linear

(iii) $y = 2\sqrt{u} \rightarrow Af(u_1) + Bf(u_2) = A(2\sqrt{u_1}) + B(2\sqrt{u_2}) = 2A\sqrt{u_1} + 2B\sqrt{u_2}$

$y = 2\sqrt{u} \rightarrow f(Au_1 + Bu_2) = 2\sqrt{Au_1 + Bu_2}$

Take A = 1, B = 1 for example:

$$Af(u_1) + Bf(u_2) = 2\sqrt{u_1} + 2\sqrt{u_2}$$

$$f(Au_1 + Bu_2) = 2\sqrt{u_1 + u_2}$$

Hence: $Af(u_1) + Bf(u_2) \neq f(Au_1 + Bu_2) \Rightarrow$ Nonlinear

(iv) $y = 2u + 1 \rightarrow Af(u_1) + Bf(u_2) = A(2u_1 + 1) + B(2u_2 + 1) = 2(Au_1 + Bu_2) + A + B$

$y = 2u + 1 \rightarrow f(Au_1 + Bu_2) = 2(Au_1 + Bu_2) + 1$

Hence: $Af(u_1) + Bf(u_2) \neq f(Au_1 + Bu_2) \Rightarrow$ Nonlinear

Q5 (i) KVL: $v_i = v_R + v_L$

Now, $v_R = iR$ and $v_L = L\frac{di}{dt}$

Hence the first equation becomes: $v_i = iR + L\frac{di}{dt}$ (*relating $v_i$ to i*)

Q5 (ii) KVL: $v_i = v_R + v_L$

Here, we want out the relationship between $v_L$ and $v_i$.

Now, $v_R = iR$ and hence the equation becomes: $v_i = iR + v_L$

We need to eliminate $i$. We know that: $v_L = L\frac{di}{dt} \Rightarrow i = \frac{1}{L}\int v_L$

Hence: $v_i = iR + v_L \rightarrow v_i = \frac{R}{L}\int v_L + v_L$ (*relating* $v_i$ *to* $v_L$)

Differentiating once to give a differential equation model:

$$\frac{dv_i}{dt} = \frac{R}{L}v_L + \frac{dv_L}{dt} \qquad (\textit{relating } v_i \textit{ to } v_L)$$

Q6 KVL: $v_i = v_R + v_L + v_C$

Here, we want out the relationship between $v_L$ and $v_C$.

We know that: $v_R = iR$ and $v_L = L\frac{di}{dt}$

Hence the equation becomes: $v_i = iR + L\frac{di}{dt} + v_C$

We need to eliminate $i$. We know that: $i = C\frac{dv_C}{dt}$

Hence:

$$v_i = RC\frac{dv_C}{dt} + L\frac{d}{dt}\left(C\frac{dv_C}{dt}\right) + v_C$$

$$\Rightarrow v_i = RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} + v_C$$

We normally express this as:

$$LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = v_i$$

or

$$\frac{d^2v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_C = \frac{1}{LC}v_i$$

Q7 (i) A mass $M$ (the skydiver) in free fall experiences a force due to gravity $Mg$, which causes it to accelerate towards the ground.

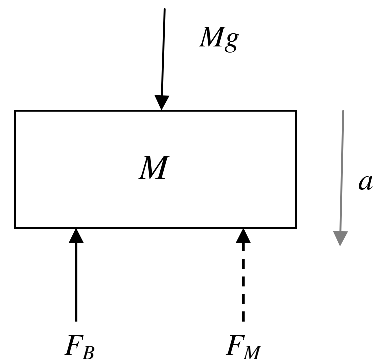
It also experiences a viscous friction force which opposes the motion. This is proportional to the velocity squared. Given that the constant of proportionality is $B$ we get:

$$F_B = Bv^2$$

Newton's second law states that the total applied force is:

$$F = Ma.$$

We can represent these force relationships in the free body diagram for the skydiver as shown:




Hence:

$$F_M + F_B = Mg$$
$$\Rightarrow Ma + Bv^2 = Mg$$

Since $a = \dfrac{dv}{dt} \quad \Rightarrow M\dfrac{dv}{dt} + Bv^2 = Mg$

Terminal velocity occurs when the mass stops accelerating, i.e. $a = \dfrac{dv}{dt} = 0$.

Hence: $M(0) + Bv^2 = Mg \quad \Rightarrow v = \sqrt{\dfrac{Mg}{B}}$

(ii) Given $M = 90$ kg, $v_T = 55\text{m/s}$.

Hence: $v_T = \sqrt{\dfrac{Mg}{B}} \quad \Rightarrow 55 = \sqrt{\dfrac{(90)(9.81)}{B}} \quad \Rightarrow B = \dfrac{90(9.81)}{55^2} = 0.2919$

(iii) Since $B$ is proportional to the surface area, a reduction in surface area by a factor of 4, i.e. 0.2/0.8 means that $B$ is also reduced by a factor of 4.

Hence: $v_T = \sqrt{\dfrac{Mg}{B}} \rightarrow \sqrt{\dfrac{Mg}{B/4}} = \sqrt{\dfrac{4Mg}{B}} = 2\sqrt{\dfrac{Mg}{B}}$

Thus, a factor of 4 reduction in $B$ results in a factor of 2 increase in terminal velocity.

In this case: $v_T = 2(55) = 110\text{m/s}$

(iv) With the parachute open, the coefficient of viscous friction in (i) changes to:

$$B + B_p$$

Hence: $$v_T = \sqrt{\frac{Mg}{B + B_p}}$$

Thus: $$4 = \sqrt{\frac{90(9.81)}{0.2919 + B_p}} \Rightarrow B_p = \frac{90(9.81)}{4^2} - 0.2919 \Rightarrow B_p = 54.89$$

Finally: $$B_p = 2.5A = 54.89 \quad \Rightarrow A = 21.96\,\mathrm{m}^2$$

Q8 The free body diagram for this system is as shown.

From the free body diagram showing only the forces acting on *M* we obtain the Force equilibrium equation:

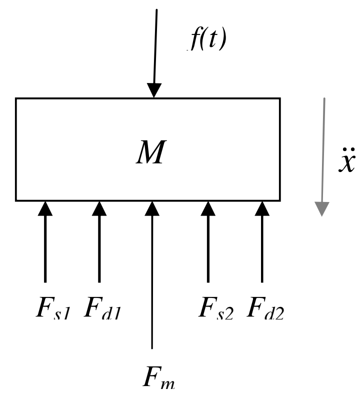
$$F_m + F_{d1} + F_{s1} + F_{d2} + F_{s2} = f(t)$$

Using the physical force-geometry relations this becomes:

$$M\frac{d^2x}{dt^2} + B_1\frac{dx}{dt} + K_1 x + B_2\frac{dx}{dt} + K_2 x = f(t)$$

or:

$$M\frac{d^2x}{dt^2} + (B_1 + B_2)\frac{dx}{dt} + (K_1 + K_2)x = f(t)$$



Free body diagram

Q9 (i) Writing flow balance equation for the tank gives: $\frac{dV}{dt} = F_{in} - F_{out}$

Since volume $V = Ah$: $$\frac{dV}{dt} = A\frac{dh}{dt} = F_{in} - F_{out}$$

Given that $F_{out} = \frac{h}{R}$ (for laminar flow) then the flow balance equation is:

$$A\frac{dh}{dt} = F_{in} - \frac{h}{R} \quad \text{or} \quad A\frac{dh}{dt} + \frac{h}{R} = F_{in}$$

(ii) Turbulent $\Rightarrow F_{out} = \frac{\sqrt{h}}{R}$ and hence model becomes: $A\frac{dh}{dt} + \frac{\sqrt{h}}{R} = F_{in}$

Q10 From Q9(i), we can easily write out the system model for each tank as follows:

Tank 1: $$A_1 \frac{dh_1}{dt} + \frac{h_1}{R_1} = F_{in}$$

Tank 2: $$A_2 \frac{dh_2}{dt} + \frac{h_2}{R_2} = F_{12}$$

Since $F_{12}$ is the output flow of Tank 1, then we know that $F_{12} = \frac{h_1}{R_1}$ and hence the model for Tank 2 becomes:

$$A_2 \frac{dh_2}{dt} + \frac{h_2}{R_2} = \frac{h_1}{R_1}$$

Now, in order to obtain the desired model, i.e. relating $F_{in}$ to $h_2$, we need to eliminate $h_1$ as follows:

From the Tank 2 equation:

$$\frac{h_1}{R_1} = A_2 \frac{dh_2}{dt} + \frac{h_2}{R_2} \quad \Rightarrow h_1 = A_2 R_1 \frac{dh_2}{dt} + \frac{R_1 h_2}{R_2} \quad \Rightarrow \frac{dh_1}{dt} = A_2 R_1 \frac{d^2 h_2}{dt^2} + \frac{R_1}{R_2}\frac{dh_2}{dt}$$

Subbing into the Tank 1 equation gives:

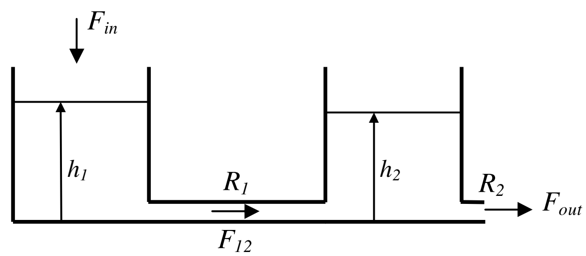
$$A_1\left( A_2 R_1 \frac{d^2 h_2}{dt^2} + \frac{R_1}{R_2}\frac{dh_2}{dt}\right) + \left( A_2 \frac{dh_2}{dt} + \frac{h_2}{R_2}\right) = F_{in}$$

$$\Rightarrow A_1\left( A_2 R_1 R_2 \frac{d^2 h_2}{dt^2} + R_1 \frac{dh_2}{dt}\right) + \left( A_2 R_2 \frac{dh_2}{dt} + h_2\right) = R_2 F_{in}$$

$$\Rightarrow \frac{d^2 h_2}{dt^2}\left(A_1 A_2 R_1 R_2\right) + \frac{dh_2}{dt}\left(A_1 R_1 + A_2 R_2\right) + h_2 = R_2 F_{in}$$

Q11 Based, on the solution to Q9(i), we know that the flow out of the tank is proportional to the height of the water in the tank and inversely proportional to the resistance to flow.

In the case of the flow between connected tanks, as in this question, we have to allow for both heights in our model of flow. In this case, the flow between tanks is proportional to the difference in heights on either side of the connection.




*Note, if $h_1 > h_2$ then the liquid flows from tank 1 to tank 2, if $h_2 > h_1$ then the liquid flows from tank 2 to tank 1 and if $h_1 = h_2$ then the liquid does not flow!*

Hence we can state the following: $F_{12} = \frac{h_1 - h_2}{R_1}$ and $F_{out} = \frac{h_2}{R_2}$

Our flow balance equations are:

$$A_1 \frac{dh_1}{dt} = F_{in} - F_{12}$$

$$\Rightarrow A_1 \frac{dh_1}{dt} = F_{in} - \left(\frac{h_1 - h_2}{R_1}\right)$$

$$\Rightarrow A_1 R_1 \frac{dh_1}{dt} = R_1 F_{in} - h_1 + h_2$$

and

$$A_2 \frac{dh_2}{dt} = F_{12} - F_{out}$$

$$\Rightarrow A_2 \frac{dh_2}{dt} = \left(\frac{h_1 - h_2}{R_1}\right) - \left(\frac{h_2}{R_2}\right)$$

$$\Rightarrow A_2 R_1 R_2 \frac{dh_2}{dt} = R_2 h_1 - h_2 (R_1 + R_2)$$