W

Assignment 4. Page 1-2

EX1. $y'' + 4y' + 4y = \cos x + 3\sin 2x$

Sol.

① $y^2 + 4y + 4 = 0$

$\rightarrow (\lambda+2)^2 = 0 \quad \lambda_1 = \lambda_2 = -2$

$\therefore \bar{y} = (c_1 + c_2 x)e^{-2x}$

② Let $y^* = A\cos x + B\sin x$

$y^{*\prime} = -A\sin x + B\cos x$

$y^{*\prime\prime} = -A\cos x - B\sin x$

(1) $y_I^{*\prime\prime} + 4y_I^{*\prime} + 4y_I^* = \cos x$

(2) $y_{II}^{*\prime\prime} + 4y_{II}^{*\prime} + 4y_{II}^* = 3\sin 2x$

$y_{II}^* = C\cos 2x + D\sin 2x$

$y_{II}^{*\prime} = -2C\sin 2x + 2D\cos 2x$

$y_{II}^{*\prime\prime} = -4C\cos 2x - 4D\sin 2x$

$\therefore \begin{cases} A = \frac{3}{25} \\ B = \frac{4}{25} \end{cases} \qquad \begin{cases} C = -\frac{3}{8} \\ D = 0 \end{cases}$

$\therefore y^* = y^*_I + y^*_{II}$

$= \frac{3}{25}\cos x + \frac{4}{25}\sin x - \frac{3}{8}\cos 2x$

$\therefore y = \bar{y} + y^*$

$= (c_1 + c_2 x)e^{-2x} + \frac{3}{25}\cos x + \frac{4}{25}\sin x - \frac{3}{8}\cos 2x$

that's all.

(a) $y = (c_1 + c_2 x)e^{-2x} + \frac{3}{25}\cos x + \frac{4}{25}\sin x - \frac{3}{8}\cos 2x$

(b) $y'' - 10y' + 25y = 40x + 3$

① Sol.

Let $\lambda^2 - 10\lambda + 25 = 0$

so $\lambda_1 = \lambda_2 = 5$.

$\therefore \bar{y} = (c_1 + c_2 x)e^{5x}$

② Let $y^* = ax + b$

$y^{*\prime} = a \qquad y^{*\prime\prime} = 0$

$\therefore 25(ax+b) - 10a = 40x + 3$

$\therefore \begin{cases} a = \frac{8}{5} \\ b = \frac{19}{25} \end{cases}$

$\therefore y^* = \frac{8}{5}x + \frac{29}{15}$.

$\therefore y = \bar{y} + y^* = (c_1 + c_2 x)e^{5x} + \frac{8}{5}x + \frac{29}{15}$

(b) $y = (c_1 + c_2 x)e^{5x} + \frac{8}{5}x + \frac{29}{15}$

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(c). $\frac{d^2x}{dt^2} + w^2 x = F_0 \sin wt$,

Sol.

①. let $\lambda^2 + 1 \cdot w^2 = 0$

$\therefore \lambda = \pm wi$

$\therefore \bar{y} = c_1 \cos w \cdot x + c_2 \sin wx$

② Let $y^* = t \cdot (a \cos wt + b \sin wt) = x_p$

$(y^*)' = a \cos wt + b \sin wt + awt(-\sin wt) + bwt \cos wt$

$(y^*)'' = -tw^2 (a \cos wt + b \sin wt) + 2w(-a \sin wt + b \cos wt)$

and $x_p'' + w^2 x_p = F_0 \sin wt$

So $-2w \cdot a \sin wt + 2wb \cos wt = F_0 \sin wt$

$\begin{cases} a = \frac{F_0}{-2w} \\ b = 0 \end{cases}$

$\therefore x = x_c + x_p = C_1 \cos wt + C_2 \sin wt - \frac{F_0 t}{2w} \cos wt$.

(3) $x = c_1 \cdot \cos wt + c_2 \sin wt - \frac{F_0 t}{2w} \cos wt$

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2(a)

① let $4\lambda^2 - 1 = 0$

$\lambda^2 = \frac{1}{4}$ $\quad \lambda_1 = \frac{1}{2}$ $\quad \lambda_2 = -\frac{1}{2}$

$\therefore \bar{y} = c_1 e^{\frac{1}{2}x} + c_2 \cdot e^{-\frac{1}{2}x}$.

② Let $y_1 = u(x)\, e^{\frac{1}{2}x}$

$y_2 = u(x)\, x e^{\frac{1}{2}x}$

$\therefore \begin{cases} u_1 e^{\frac{1}{2}x} + u_2 x e^{\frac{1}{2}x} = 0 \\ u_1 \frac{1}{2} e^{\frac{1}{2}x} + u_2 (e^{\frac{1}{2}x} + \frac{x}{2} e^{\frac{1}{2}x}) = x e^{\frac{x}{2}} \end{cases}$

$W = \begin{vmatrix} e^{\frac{1}{2}x} & x e^{\frac{1}{2}x} \\ \frac{1}{2} e^{\frac{1}{2}x} & e^{\frac{1}{2}x} + \frac{x}{2} e^{\frac{1}{2}x} \end{vmatrix}$

$= e^x + \frac{x}{2} e^x - \frac{x}{2} e^x$

$= e^x$

$W_1 = \begin{vmatrix} 0 & x e^{\frac{1}{2}x} \\ x e^{\frac{x}{2}} & \cdots \end{vmatrix} = -x^2 e^x$

$W_2 = \begin{vmatrix} e^{\frac{1}{2}x} & 0 \\ \cdots & x e^{\frac{x}{2}} \end{vmatrix} = x e^x$

$u_1'(x) = \frac{W_1}{W} = -x^2$

$u_1(x) = -\frac{1}{3} x^3$

$u_2'(x) = \frac{W_2}{W} = x$

$u_2(x) = \frac{1}{2} x^2$

So $y^* = -\frac{1}{3} x^3 e^{\frac{1}{2}x} + \frac{1}{2} x^2 x e^{\frac{1}{2}x}$

$= \frac{1}{6} x^3 e^{\frac{1}{2}x}$

$\therefore y = c_1 e^{\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x} + \frac{1}{6} x^3 e^{\frac{1}{2}x}$

and $\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases}$

$\Rightarrow \begin{cases} c_1 = 1 \\ c_2 = -1 \end{cases}$

$\therefore y = e^{\frac{1}{2}x} - e^{-\frac{1}{2}x} + \frac{1}{6} x^3 e^{\frac{1}{2}x}$

$y = \frac{5}{4} e^{\frac{x}{2}} - \frac{5}{4} e^{-\frac{x}{2}} + \frac{1}{8} x^2 e^{\frac{x}{2}} - \frac{x}{4} e^{\frac{x}{2}}$,

(b) sol

① Let $\lambda^2 + 2\lambda - 8 = 0$

$\therefore (\lambda + 4)(\lambda - 2) = 0$ $\quad \begin{cases} \lambda_1 = -4 \\ \lambda_2 = 2 \end{cases}$

$\therefore \bar{y} = (c_1 e^{-4x} + c_2 e^{2x})$

② Let $y_1 = u_1(x) e^{2x} + u_2(x) e^{-4x}$

$\begin{cases} e^{2x} u_1 + e^{-4x} u_2 = 0 \\ 2e^{2x} u_1 - 4e^{-4x} u_2 = 2e^{-3x} - e^{-x} \end{cases}$

$\therefore u_1' = \frac{1}{6}(2e^{-5x} - e^{-3x})$

$u_1 = \frac{1}{15} e^{-5x} - \frac{1}{18} e^{-3x}$

and $u_2' = \frac{1}{6}(2e^x - e^{3x})$

$u_2 = \frac{1}{3} e^x - \frac{1}{18} e^{3x}$

$y^* = (\frac{1}{15} e^{-5x} - \frac{1}{18} e^{-3x}) e^{2x} + (\frac{1}{3} e^x - \frac{1}{18} e^{3x}) e^{-4x}$

$\begin{cases} y = \bar{y} + y^* & ① \\ y(0) = 0 & ② \\ y'(0) = 0 & ③ \end{cases}$

$\therefore y = \frac{61}{90} e^{2x} + \frac{11}{18} e^{-4x} - \frac{2}{5} e^{-3x} + \frac{1}{9} e^{-x}$

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(3) sol

① let $\lambda^2 + 1 = 0$ $\quad \lambda = \pm i$

So $\bar{y} = c_1 \cdot \cos x + c_2 \sin x$

② let $y_1 = \cos x \quad y_2 = \sin x$

$$\begin{cases} \cos x \, u_1' + \sin x \, u_2' = 0 \\ -\sin x \, u_1' + \cos x \, u_2' = \sec x \tan x \end{cases}$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x \tan x & \cos x \end{vmatrix} = -\sec x \tan x \sin x$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \tan x \end{vmatrix} = \sec x \sin x$$

$\therefore u_1'(x) = \frac{W_1}{W} = -\sec x \cdot \tan x \sin x$

$u_1(x) = x - \tan x$

$u_2'(x) = \sec x \sin x$

$u_2(x) = -\ln(\cos x)$

$\therefore y^* = (x - \tan x) \cos x - \ln(\cos x) \sin x$

$\therefore y = y^* + \bar{y}$

$= (x - \tan x + c_1) \cos x - \ln(\cos x) \sin x + c_2 \sin x$

that's all.