5. Linearisation

5.1 Introduction

- Most systems are nonlinear in reality over their complete range of operation.
- However, they are usually operated over a range which is reasonably linear (for example transistor amplifiers, etc.).
- Linear analysis techniques are well developed and understood, with the complete (LTI) theory readily available.
- As such, although we work with nonlinear systems, we often tend to obtain linear approximations of these systems about points of interest, known as **operating points**. This then allows us to easily apply our linear analysis techniques.
- The process of obtaining the linear approximation of a nonlinear system, about an operating point, is known as **linearisation**.

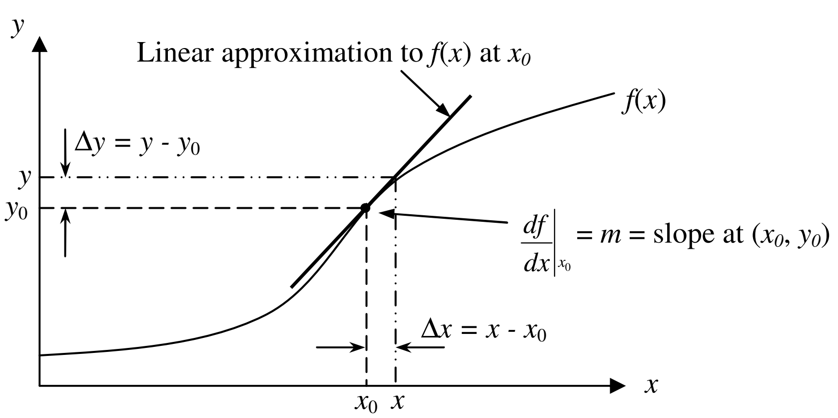
5.2 Linearisation of static systems

- Consider a simple static system:

$$y = f(x)$$

- A Taylor series expansion of the function $f(x)$ around the operating point x_0 gives:

$$y = f(x) = f(x_0) + (x - x_0)\left.\frac{df}{dx}\right|_{x_0} + \frac{(x-x_0)^2}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x_0} + \ldots$$



- For small changes in x about the operating point, i.e.:

$$\Delta x = (x - x_0)$$

the higher order terms in the Taylor series are negligible and we can approximate $f(x)$ as:

$$y \approx f(x_0) + (x - x_0)\left.\frac{df}{dx}\right|_{x_0}$$

$$\Rightarrow y = y_0 + m(x - x_0)$$ where m is the slope of the tangent line.

- Note that $y = y_0 + m(x - x_0)$ is the equation of a straight line, but mathematically it is not a linear system (rather, it is an *affine* system)

- By definition, a linear system must satisfy the property that

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2)$$

- Straight line equations of the form $y = mx + c$ do not satisfy this property. Basically, the straight line equation needs to pass through the origin, i.e. *zero in should give zero out*!

- However, equations of the form $y = mx$ satisfy this property.

- Hence, if we define the change in output from the operating point output, $y_0 = f(x_0)$ as:

$$\Delta y = (y - y_0)$$

we can get a linear model as follows:

$$\Delta y = m\Delta x \qquad \text{or} \qquad (y - y_0) = \left.\frac{df}{dx}\right|_{x_0}(x - x_0) \qquad \text{or} \qquad \Delta y = \left.\frac{df}{dx}\right|_{x_0}\Delta x$$

- Linear models of multi-input/multi-output (MIMO) systems can be obtained in a similar fashion.

- For each output in the model we will, in general, have an equation of the form:

$$y = f(\mathbf{x}) = f(x_1, x_2, \ldots, x_m)$$

- The linear approximation to this equation at an operating point ($\mathbf{x}_0$, y_0) is given by:

$$\Delta y = \left.\frac{df}{dx_1}\right|_{\mathbf{x_0}}\Delta x_1 + \left.\frac{df}{dx_2}\right|_{\mathbf{x_0}}\Delta x_2 + \ldots + \left.\frac{df}{dx_m}\right|_{\mathbf{x_0}}\Delta x_m$$

where:

$$\Delta x_i = x_i - x_{i0} \quad \text{and} \quad \Delta y = y - y_0$$

- **Example 5.1**: An electric light bulb consists of a filament in an evacuated tube. Due to the power dissipated, i^2R, the filament reaches a temperature of the order of 2500 K. However, as its temperature increases, the resistance of the filament also changes considerably, leading to a nonlinear *v*-*i* characteristic of the form:

$$i = f(v) = kv^2$$

where k is constant. Determine an approximate linear model for the light bulb, given that it normally operates at a voltage of 220 ± 20 volts.

Solution:

$$\Delta i = \left.\frac{df}{dv}\right|_{v_0} \Delta v, \quad \text{where } v_0 = 220$$

$$\left.\frac{df}{dv}\right|_{v=v_0} = 2kv\big|_{v=v_0} = 2kv_0$$

Hence: $\Delta i = 2kv_0 \Delta v$

$$R = \frac{\Delta v}{\Delta i} = \frac{1}{2kv_0} = \frac{1}{440k}$$

- **Example 5.2:** Determine the linear model for the following two input function, $f(x_1, x_2)$, about the operating points (1, 1), (1, -1) and (0,0):

$$y = 2 + 3x_1x_2 - x_1^2$$

Solution:

$$\Delta y = \left.\frac{df}{dx_1}\right|_{\mathbf{x_0}} \Delta x_1 + \left.\frac{df}{dx_2}\right|_{\mathbf{x_0}} \Delta x_2 \qquad \text{where } \mathbf{x}_0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$$

So: $\frac{df}{dx_1} = 3x_2 - 2x_1$ and $\frac{df}{dx_2} = 3x_1$

Hence: $\Delta y = (3x_2^0 - 2x_1^0)\Delta x_1 + (3x_1^0)\Delta x_2$

If $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\Delta y = (3(1) - 2(1))\Delta x_1 + (3(1))\Delta x_2 \qquad \Rightarrow \Delta y = \Delta x_1 + 3\Delta x_2$

If $\mathbf{x}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then $\Delta y = -5\Delta x_1 + 3\Delta x_2$ and if $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $\Delta y = 0$

5.3 Linearisation of nonlinear dynamic systems

- The form of the general nonlinear system is given as:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$$
$$\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u})$$

- We want to obtain an approximate linear model that represents the system about some operating point ($\boldsymbol{x}_0$, $\boldsymbol{u}_0$), that is:

$$\Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta\mathbf{u}$$
$$\Delta\mathbf{y} = \mathbf{C}\Delta\mathbf{x} + \mathbf{D}\Delta\mathbf{u}$$

where: $\Delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_0$, $\Delta\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}_0$ and $\Delta\boldsymbol{y} = \boldsymbol{y} - \boldsymbol{y}_0$.

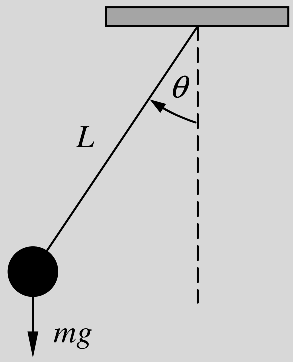
- Note that the operating point $\boldsymbol{y}_0$ is given by $\boldsymbol{y}_0 = \boldsymbol{g}(\boldsymbol{x}_0, \boldsymbol{u}_0)$.

- While it is possible to linearise a dynamical system about any operating point ($\boldsymbol{x}_0$, $\boldsymbol{u}_0$), it is usually best to choose a point which is **a steady-state operating point**, which is referred to as an **equilibrium point** of the system.

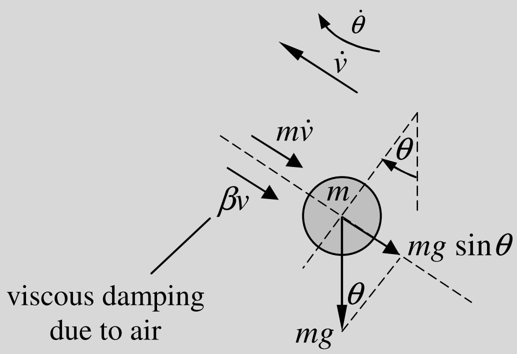
- Equilibrium points are those points where the state of the system does not change for a constant input and are obtained by solving the state equation when $\dot{\boldsymbol{x}} = 0$, i.e.:

$$\boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{u}_0) = 0$$

- **Example 5.3:** Determine the **equilibrium points** of a simple pendulum of mass *m* suspended by a rod of length *L*, as shown below:



Physical diagram



Free body diagram
(in terms of forces perpendicular to rod)

Note: $v = \omega L = \dot{\theta} L$

Solution:

Newton law gives: $m\dot{v}+\beta v+mg\sin\theta=0$ where $v=\dot{\theta}L$ and $\dot{v}=\ddot{\theta}L$

Rewriting in terms of θ only gives: $mL\ddot{\theta}+\beta L\dot{\theta}+mg\sin\theta=0$

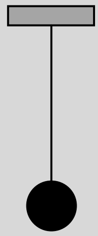
Setting the state variables $x_1=\theta, x_2=\dot{\theta}$ gives the state equations:

$$\dot{x}_1=x_2$$
$$\dot{x}_2=-\frac{g}{L}\sin x_1-\frac{\beta}{m}x_2$$

Equilibrium is when $\dot{x}_1=\dot{x}_2=0$.

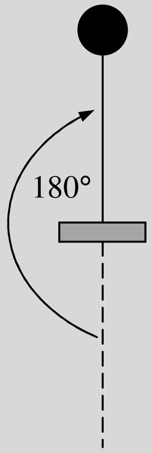
Hence:

$$x_2=0 \text{ and } \sin x_1=0$$
$$\Rightarrow x_1=0° \text{ or } 180°$$
$$\text{or } x_1=0 \text{ or } \pi \text{ radians}$$



$x_1=0, x_2=0$

Stable equilibrium point



$x_1=\pi, x_2=0$

Unstable equilibrium point

- **Example 5.4:** Determine the **equilibrium operating point** of the coupled tank system state equations for a constant input flow f_0:

$$\dot{h}_1=\frac{1}{A_1}f_{in}-\frac{k_1}{A_1}\sqrt{h_1-h_2}$$
$$\dot{h}_2=\frac{k_1}{A_2}\sqrt{h_1-h_2}-\frac{k_2}{A_2}\sqrt{h_2-h_3}$$

Solution: Input flow $f_{in}=f_0$. At equilibrium $\dot{h}_1=\dot{h}_2=0$. Hence:

$$\left.\begin{aligned}\frac{k_1}{A_1}\sqrt{h_1^0-h_2^0}&=\frac{1}{A_1}f_0\\ \frac{k_1}{A_2}\sqrt{h_1^0-h_2^0}&=\frac{k_2}{A_2}\sqrt{h_2^0-h_3}\end{aligned}\right\}\Rightarrow k_2\sqrt{h_2^0-h_3}=f_0$$

Squaring both sides gives:

$$h_2^0-h_3=\left(\frac{f_0}{k_2}\right)^2\Rightarrow h_2^0=\left(\frac{f_0}{k_2}\right)^2+h_3$$

Note, h_3 is a constant value

Also: $k_1\sqrt{h_1^0-h_2^0}=f_0\Rightarrow h_1^0=\left(\frac{f_0}{k_1}\right)^2+h_2^0 \qquad \Rightarrow h_1^0=\left(\frac{f_0}{k_1}\right)^2+\left(\frac{f_0}{k_2}\right)^2+h_3$

5.4 Linearising the state-space model

5.4.1 Linearising the state equations

- Consider the general form of the state equations:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
\dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m)
\end{aligned}$$

- By taking a Taylor series expansion of each of these equations in turn about the equilibrium operating point $(\boldsymbol{x}_0, \boldsymbol{u}_0)$ and dropping the higher order terms, we can obtain the following linear approximations:

$$\begin{aligned}
\Delta\dot{x}_1 &= \left.\frac{df_1}{dx_1}\right|_0 \Delta x_1 + \ldots + \left.\frac{df_1}{dx_n}\right|_0 \Delta x_n + \left.\frac{df_1}{du_1}\right|_0 \Delta u_1 + \ldots + \left.\frac{df_1}{du_m}\right|_0 \Delta u_m \\
&\vdots \\
&\vdots \\
\Delta\dot{x}_n &= \left.\frac{df_n}{dx_1}\right|_0 \Delta x_1 + \ldots + \left.\frac{df_n}{dx_n}\right|_0 \Delta x_n + \left.\frac{df_n}{du_1}\right|_0 \Delta u_1 + \ldots + \left.\frac{df_n}{du_m}\right|_0 \Delta u_m
\end{aligned}$$

- This can be written in matrix form as:

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{A}\Delta\boldsymbol{x} + \boldsymbol{B}\Delta\boldsymbol{u}$$

where:

$$\boldsymbol{A} = \begin{bmatrix} \frac{df_1}{dx_1} & \cdots & \frac{df_1}{dx_n} \\ \vdots & \ddots & \vdots \\ \frac{df_n}{dx_1} & \cdots & \frac{df_n}{dx_n} \end{bmatrix}_0 \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} \frac{df_1}{du_1} & \cdots & \frac{df_1}{du_m} \\ \vdots & \ddots & \vdots \\ \frac{df_n}{du_1} & \cdots & \frac{df_n}{du_m} \end{bmatrix}_0$$

- Vectors $\Delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_0, \Delta\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}_0$ are given by:

$$\Delta\boldsymbol{x} = \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix} \quad \text{and} \quad \Delta\boldsymbol{u} = \begin{bmatrix} \Delta u_1 \\ \vdots \\ \Delta u_m \end{bmatrix}$$

5.4.2 Linearising the output equation

- Consider the general nonlinear output equations:

$$\begin{aligned} y_1 &= g_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\ y_2 &= g_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\ &\vdots \\ y_p &= g_p(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \end{aligned}$$

- Using a similar approach, these can be linearised to give:

$$\Delta\mathbf{y} = \boldsymbol{C}\Delta\boldsymbol{x} + \boldsymbol{D}\Delta\boldsymbol{u}$$

where:

$$\boldsymbol{C} = \begin{bmatrix} \frac{dg_1}{dx_1} & \cdots & \frac{dg_1}{dx_n} \\ \vdots & \ddots & \vdots \\ \frac{dg_p}{dx_1} & \cdots & \frac{dg_p}{dx_n} \end{bmatrix}_0 \quad \text{and} \quad \boldsymbol{D} = \begin{bmatrix} \frac{dg_1}{du_1} & \cdots & \frac{dg_1}{du_m} \\ \vdots & \ddots & \vdots \\ \frac{dg_p}{du_1} & \cdots & \frac{dg_p}{du_m} \end{bmatrix}_0$$

and $\Delta\mathbf{y} = \mathbf{y} - \mathbf{y}_0$.

- **Example 5.5:** Linearise the coupled tanks system model about the operating point $f_0 = 12$, when $A_1 = A_2 = 5$, $k_1 = 6$, $k_2 = 10$ and $h_3 = 0.2$.

Solution:

$$\begin{aligned} \dot{h}_1 &= \frac{1}{A_1} f_{in} - \frac{k_1}{A_1}\sqrt{h_1 - h_2} \\ \dot{h}_2 &= \frac{k_1}{A_2}\sqrt{h_1 - h_2} - \frac{k_2}{A_2}\sqrt{h_2 - h_3} \end{aligned} \quad \rightarrow \quad \begin{aligned} \dot{h}_1 &= 0.2 f_{in} - 1.2\sqrt{h_1 - h_2} \\ \dot{h}_2 &= 1.2\sqrt{h_1 - h_2} - 2\sqrt{h_2 - 0.2} \end{aligned}$$

From example 5.4:

$$h_1^0 = \left(\frac{f_0}{k_1}\right)^2 + \left(\frac{f_0}{k_2}\right)^2 + h_3 \quad \rightarrow \quad h_1^0 = \left(\frac{12}{6}\right)^2 + \left(\frac{12}{10}\right)^2 + 0.2 = 5.64$$

$$h_2^0 = \left(\frac{f_0}{k_2}\right)^2 + h_3 \quad \rightarrow \quad h_2^0 = \left(\frac{12}{10}\right)^2 + 0.2 = 1.64$$

Now: $\dot{h}_1 = f_1(h_1, h_2, f_{in}) = 0.2 f_{in} - 1.2\sqrt{h_1 - h_2}$

$$\Delta\dot{h}_1 = \frac{df_1}{dh_1}\bigg|_0 \Delta h_1 + \frac{df_1}{dh_2}\bigg|_0 \Delta h_2 + \frac{df_1}{df_{in}}\bigg|_0 \Delta f_{in}$$

subscript 0 represents operating point (h_1^0, h_2^0, f_0) = (5.64, 1.64, 12) in this case

Evaluating each of the derivatives gives:

$$\frac{df_1}{dh_1}\bigg|_0 = \left(0.5(-1.2)(h_1 - h_2)^{-0.5}\right)\Big|_{h_1=5.64, h_2=1.64} = 0.5(-1.2)(4)^{-0.5} = -0.3$$

$$\frac{df_1}{dh_2}\bigg|_0 = \left(0.5(-1.2)(h_1 - h_2)^{-0.5}(-1)\right)\Big|_{h_1=5.64, h_2=1.64} = 0.5(1.2)(4)^{-0.5} = 0.3$$

$$\frac{df_1}{df_{in}}\bigg|_0 = 0.2$$

$$\therefore \Delta\dot{h}_1 = -0.3\Delta h_1 + 0.3\Delta h_2 + 0.2\Delta f_{in}$$

Similarly:

$$\dot{h}_2 = f_2(h_1, h_2, f_{in}) = 1.2\sqrt{h_1 - h_2} - 2\sqrt{h_2 - 0.2}$$

$$\Delta\dot{h}_2 = \frac{df_2}{dh_1}\bigg|_0 \Delta h_1 + \frac{df_2}{dh_2}\bigg|_0 \Delta h_2 + \cancelto{0}{\frac{df_2}{df_{in}}\bigg|_0 \Delta f_{in}}$$

$$\frac{df_2}{dh_1}\bigg|_0 = 0.3$$

$$\frac{df_1}{dh_2}\bigg|_0 = \left(0.5(1.2)(h_1 - h_2)^{-0.5}(-1) - 0.5(2)(h_2 - 0.2)^{-0.5}\right)\Big|_{h_1=5.64, h_2=1.64}$$

$$= -0.5(1.2)(4)^{-0.5} - 0.5(2)(1.44)^{-0.5} = -0.3 - \tfrac{1}{1.2} = -1.1333$$

$$\therefore \Delta\dot{h}_2 = 0.3\Delta h_1 - 1.1333\Delta h_2$$

Hence, we obtain the following linear model approximation:

$$\therefore \begin{bmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \end{bmatrix} = \begin{bmatrix} -0.3 & 0.3 \\ 0.3 & -1.1333 \end{bmatrix} \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0 \end{bmatrix} \Delta f_{in}$$

$$\boldsymbol{A} = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{bmatrix}_0 \qquad \boldsymbol{B} = \begin{bmatrix} \frac{df_1}{df_{in}} \\ \frac{df_2}{df_{in}} \end{bmatrix}_0$$

5.5 Linearising discrete-time state-space equations

- Nonlinear discrete-time equations can be linearised in exactly the same way but instead of having $\Delta\dot{\boldsymbol{x}}$ in the states equation we have $\Delta\boldsymbol{x}(k+1)$, i.e:

$$\begin{aligned}\boldsymbol{x}(k+1) &= \boldsymbol{f}(\boldsymbol{x}(k),\boldsymbol{u}(k))\\ \boldsymbol{y}(k) &= \boldsymbol{g}(\boldsymbol{x}(k),\boldsymbol{u}(k))\end{aligned}$$

becomes:

$$\begin{aligned}\Delta\boldsymbol{x}(k+1) &= \boldsymbol{A}_D\Delta\boldsymbol{x}(k)+\boldsymbol{B}_D\Delta\boldsymbol{u}(k)\\ \Delta\boldsymbol{y}(k) &= \boldsymbol{C}_D\Delta\boldsymbol{x}(k)+\boldsymbol{D}_D\Delta\boldsymbol{u}(k)\end{aligned}$$

- There is however a difference in the way the equilibrium points are determined.

- If a discrete-time system is in steady-state, i.e. all the states have settled down to constant values for a given constant input, then:

$$\boldsymbol{x}(k+1) = \boldsymbol{x}(k) = \boldsymbol{f}(\boldsymbol{x}(k),\boldsymbol{u}(k))$$

- Hence, to obtain the equilibrium points, we have to solve the set of equations given by:

$$\boldsymbol{x}(k) = \boldsymbol{f}(\boldsymbol{x}(k),\boldsymbol{u}(k))$$

- **Example 5.6:** Determine the equilibrium points and corresponding linear models for the following discrete-time dynamical system when the steady-state input is $u_e = \frac{3}{4}$:

$$\begin{aligned}x_1(k+1) &= \tfrac{1}{2}x_1(k)+\tfrac{1}{4}x_2(k)\\ x_2(k+1) &= \tfrac{1}{3}x_1(k)x_2(k)+\tfrac{3}{2}u(k)\end{aligned}$$

Solution: To obtain the equilibrium points x_{1e} and x_{2e}, we solve:

$$\begin{aligned}x_{1e} &= \tfrac{1}{2}x_{1e}+\tfrac{1}{4}x_{2e}\\ x_{2e} &= \tfrac{1}{3}x_{1e}x_{2e}+\tfrac{3}{2}u_e\end{aligned}$$

Hence:

$$x_{1e} = \tfrac{1}{2}x_{1e}+\tfrac{1}{4}x_{2e} \quad \Rightarrow \tfrac{1}{2}x_{1e} = \tfrac{1}{4}x_{2e} \quad \Rightarrow x_{1e} = \tfrac{1}{2}x_{2e}$$

$$\begin{aligned}&x_{2e} = \tfrac{1}{3}x_{1e}x_{2e}+\tfrac{3}{2}u_e \quad = \tfrac{1}{3}\left(\tfrac{1}{2}x_{2e}\right)x_{2e}+\tfrac{3}{2}u_e\\ &\Rightarrow x_{2e} = \tfrac{1}{6}x_{2e}{}^2+\tfrac{3}{2}u_e\\ &\Rightarrow x_{2e}{}^2-6x_{2e}+9u_e = 0\end{aligned}$$

Solving this equation gives: $$x_{2e} = \frac{+6\pm\sqrt{36-36\,u_e}}{2} = \frac{6\pm 6\sqrt{1-u_e}}{2} = 3\pm 3\sqrt{1-u_e}$$

Hence, we have **two equilibrium points** (x_{1e}, x_{2e}) as follows:

$$x_{2e} = 3 + 3\sqrt{1-u_e} \qquad\qquad \text{or} \qquad\qquad x_{2e} = 3 - 3\sqrt{1-u_e}$$
$$x_{1e} = 1.5 + 1.5\sqrt{1-u_e} \qquad\qquad\qquad\qquad x_{1e} = 1.5 - 1.5\sqrt{1-u_e}$$

Given that $u_e = \frac{3}{4}$, then $\sqrt{1-u_e} = \sqrt{\frac{1}{4}} = \frac{1}{2}$ and hence:

$(x_{1e}, x_{2e}) = (2.25, 4.5)$ or $(0.75, 1.5)$

Linearise the system about (2.25, 4.5):

$$x_1(k+1) = f_1(x_1, x_2) = \tfrac{1}{2}x_1(k) + \tfrac{1}{4}x_2(k)$$

$$\Delta x_1(k+1) = \left.\frac{df_1}{dx_1}\right|_0 \Delta x_1(k) + \left.\frac{df_1}{dx_2}\right|_0 \Delta x_2(k) = \tfrac{1}{2}\Delta x_1(k) + \tfrac{1}{4}\Delta x_2(k)$$

… already a linear equation!

$$x_2(k+1) = f_2(x_1, x_2, u_k) = \tfrac{1}{3}x_1(k)x_2(k) + \tfrac{3}{2}u(k)$$

$$\Delta x_2(k+1) = \left.\frac{df_2}{dx_1}\right|_0 \Delta x_1(k) + \left.\frac{df_2}{dx_2}\right|_0 \Delta x_2(k) + \left.\frac{df_2}{du}\right|_0 \Delta u(k)$$

$$\left.\frac{df_2}{dx_1}\right|_0 = \tfrac{1}{3}x_2(k)\big|_{x_{2e}} = \tfrac{1}{3}(4.5) = \tfrac{3}{2}, \quad \left.\frac{df_2}{dx_2}\right|_0 = \tfrac{1}{3}x_1(k)\big|_{x_{1e}} = \tfrac{1}{3}(2.25) = \tfrac{3}{4}, \quad \left.\frac{df_2}{du}\right|_0 = \tfrac{3}{2}$$

Hence: $\quad \Delta x_2(k+1) = \tfrac{3}{2}\Delta x_1(k) + \tfrac{3}{4}\Delta x_2(k) + \tfrac{3}{2}\Delta u(k)$

Thus: $\quad \Delta \boldsymbol{x}(k+1) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{3}{2} & \frac{3}{4} \end{bmatrix} \Delta \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix} \Delta \boldsymbol{u}(k) \equiv \boldsymbol{A}\Delta \boldsymbol{x}(k) + \boldsymbol{B}\Delta \boldsymbol{u}(k)$

Note that this model is only valid for small changes around the operating point (2.25, 4.5), i.e.:

$$\Delta x_1(k) = x_1(k) - 2.25$$
$$\Delta x_2(k) = x_2(k) - 4.5$$
$$\Delta u(k) = u(k) - 0.75$$

The model for the other operating point can be found in the same way.

If we don't substitute in the actual values, we can get the general form of $\boldsymbol{A}$ and $\boldsymbol{B}$ for any operating point, i.e.:

$$\boldsymbol{A} = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3}x_{2e} & \frac{1}{3}x_{1e} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} \frac{df_1}{du} \\ \frac{df_2}{du} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix}$$

where:

$$f_1 = x_{1e} = \tfrac{1}{2}x_{1e} + \tfrac{1}{4}x_{2e}$$
$$f_2 = x_{2e} = \tfrac{1}{3}x_{1e}x_{2e} + \tfrac{3}{2}u_e$$

Hence, for the second equilibrium point (0.75, 1.5), we get:

$$\boldsymbol{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix}, \ \Delta\boldsymbol{x}(k+1) = \boldsymbol{A}\Delta\boldsymbol{x}(k) + \boldsymbol{B}\Delta\boldsymbol{u}(k)$$

- **Example 5.7:** The Verhust population model (see section 2.4.3 of the notes) is given by:

$$p(k+1) = (1+\alpha-\beta p(k))p(k)$$

where *p(k)* is the population (in billions) in year *k*. Linearise this model about the operating point p_0.

Solution:

$$p_0 = (1+\alpha-\beta p_0)p_0 \quad \Rightarrow p_0 = (1+\alpha)p_0 - \beta {p_0}^2$$
$$\Rightarrow \beta {p_0}^2 - \alpha p_0 = 0 \quad \Rightarrow p_0(\beta p_0 - \alpha) = 0$$
$$\Rightarrow p_0 = 0 \quad \text{or} \quad p_0 = \frac{\alpha}{\beta}$$

Hence, the equilibrium points are $p_0 = 0$ (i.e. *extinction*!) or $p_0 = \dfrac{\alpha}{\beta}$.

Linearise as follows:

$$p(k+1) = f(p(k)) \qquad \therefore \Delta p(k+1) = \left.\frac{df}{dp_k}\right|_{p_o} \Delta p(k)$$

$$\frac{df}{dp_k} = (1+\alpha-\beta p_k)(1) + (-\beta)p_k = 1+\alpha-2\beta p_k$$

$$\left.\frac{df}{dp_k}\right|_{p_o} = 1+\alpha-2\beta p_0$$

$$\therefore \Delta p(k+1) = (1+\alpha-2\beta p_0)\Delta p(k)$$

If $p_0 = 0$ then linear model is: $\Delta p(k+1) = (1+\alpha)\Delta p(k)$

If $p_0 = \dfrac{\alpha}{\beta}$ then linear model is: $\Delta p(k+1) = (1-\alpha)\Delta p(k)$

- **Example 5.8:** Complete the linearization of the pendulum system (see example 5.3):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{L}\sin x_1 - \frac{\beta}{m}x_2$$

about its equilibrium points (x_{10}, x_{20}) = (0, 0) and (π, 0).

Solution:

The first equation is already linear. Hence:

$$\Delta\dot{x}_1 = \Delta x_2$$

Note that: $\Delta\dot{x}_1 = \left.\frac{df}{dx_2}\right|_{\mathbf{x}_0} \Delta x_2 = (1)\Delta x_2 = \Delta x_2$

The second equation is nonlinear:

$$\dot{x}_2 = f(x_1, x_2) = -\frac{g}{L}\sin x_1 - \frac{\beta}{m}x_2$$

$$\left.\frac{df}{dx_1}\right|_{\mathbf{x}_0} = \left.\left(-\frac{g}{L}\cos x_1\right)\right|_{\mathbf{x}_0} = -\frac{g}{L}\cos x_{10}$$
$$\left.\frac{df}{dx_2}\right|_{\mathbf{x}_0} = -\frac{\beta}{m}$$

Hence:

$$\Delta\dot{x}_2 = \left(-\frac{g}{L}\cos x_{10}\right)\Delta x_1 - \frac{\beta}{m}\Delta x_2$$

Therefore, the linear model about (x_{10}, x_{20}) is given by:

$$\begin{bmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{g}{L}\cos x_{10} & -\frac{\beta}{m}\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix}$$

Thus for (x_{10}, x_{20}) = (0, 0): $\begin{bmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{g}{L} & -\frac{\beta}{m}\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix}$ (stable)

and for (x_{10}, x_{20}) = (π, 0): $\begin{bmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{g}{L} & -\frac{\beta}{m}\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix}$ (unstable)