Chapter 7: Z Transform*

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A continuous time function/signal $f(t)$ of time t progresses from $t = 0$

1 From CTFT to DTFT

If a continuous function $f(t)$ of time t progresses from $t = -\infty$ to ∞ and is measured at every time interval T_s by a series of delta functions, then the result is

$$f^*(t) = \sum_{k=-\infty}^{\infty} f(kT_s)\delta(t - kT_s) \tag{1}$$

The Fourier transform of (1) is given as

$$\begin{aligned} \mathscr{F}(f^*(t)) &= \sum_{k=-\infty}^{\infty} f(kT_s)\mathscr{F}\left(\delta(t - kT_s)\right) &(2)\\ &= \sum_{k=-\infty}^{\infty} f(kT_s)e^{-i\omega kT_s} = F^*(\omega) &(3)\end{aligned}$$

Define $n = kT_s$

$$F^*(\omega) = \sum_{k=-\infty}^{\infty} f^*(n)e^{-i\omega n} \tag{4}$$

Recall the Fourier complex series with period $T = 2\pi$, $\omega_0 = 1$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt.$$

Set $t \to -\omega$, $\omega \to t$

$$f(-\omega) = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-\omega)e^{in\omega}d\omega.$$

Set $c_n \to f^*(n)$, $f(-\omega) \to F^*(\omega)$

$$F^*(\omega) = \sum_{n=-\infty}^{\infty} f^*(n)e^{-i\omega n}, \qquad \text{(Fourier Transform)} \tag{5}$$

$$f^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^*(\omega)e^{i\omega n}d\omega, \qquad \text{(Inverse Fourier Transform)} \tag{6}$$

*the notes were written for EE206FZ differential equations and transform method by Dr Siyuan Zhan, Maynooth University, Autumn 2021

2 From Laplace transform to Z-transform

Causal sequence: A causal continuous function $f(t)$ of time t progresses from $t = 0$ to ∞ and is measured at every time interval T_s by a series of delta functions, then the result is

$$f^*(t) = \sum_{k=0}^{\infty} f(kT_s)\delta(t - kT_s) \tag{7}$$

The Laplace transform of (7) is given as

$$\mathscr{L}(f^*(t)) = \sum_{k=0}^{\infty} f(kT_s)\mathscr{L}\left(\delta(t - kT_s)\right) \tag{8}$$

$$= \sum_{k=0}^{\infty} f(kT_s)e^{-skT_s} \tag{9}$$

Define a new variable $z = e^{sT_s}$ and we can see that

$$\mathscr{L}(f^*(t)) = \sum_{k=0}^{\infty} f(kT_s)z^{-k} \tag{10}$$

which is the Z-transform of the sequence $\{f(kT_s)\}$