EE206
Assignment 8

Due by next Tutorial, November 23rd. Starred questions will be done out in tutorials and do NOT need to be handed in.

1. Determine whether the given function is even, odd, or neither.

   *(a) $f(x) = x\cos x$

   $$f(x) = x\cos x$$
   $$f(-x) = -x\cos(-x) = -x\cos x = -f(x)$$

   $f(x)$ is an odd function.

   (b) $f(x) = x^4 - 4x$ **[1]**

   $$f(x) = x^4 - 4x$$
   $$f(-x) = x^4 + 4x \neq -(x^4 - 4x) = -f(x)$$

   $f(x)$ is neither odd nor an even function.

   (c) $f(x) = e^x - e^{-x}$ **[1]**

   $$f(x) = e^x - e^{-x}$$
   $$f(-x) = e^{-x} - e^x = -(e^x - e^{-x}) = -f(x)$$

   $f(x)$ is an odd function.

   (d) $f(x) = |x^5|$ **[1]**

   $$f(x) = |x^5|$$
   $$f(-x) = |(-x)^5| = |-x^5| = |x^5| = f(x)$$

   $f(x)$ is an even function.

   (e)
   $$f(x) = \begin{cases} x+5, & -2 < x < 0 \\ -x+5, & 0 \leq x < 2 \end{cases}$$

   [2]

   $$f(x) = \begin{cases} x+5, & -2 < x \leq 0 \\ -x+5, & 0 \leq x < 2 \end{cases}$$
   $$f(-x) = \begin{cases} (-x)+5, & -2 < (-x) \leq 0 \\ -(-x)+5, & 0 \leq (-x) < 2 \end{cases} = \begin{cases} -x+5, & 2 > x \geq 0 \\ x+5, & 0 \geq x > -2 \end{cases} = f(x)$$

   $f(x)$ is an even function.

2. Expand the given function in an appropriate cosine or sine series.

   *(b) $f(x) = x|x|, \quad -1 < x < 1$

   $$f(x) = x|x|$$
   $$f(-x) = (-x)|-x| = -x|x|$$

$f(x)$ is an odd function.

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}x\right)\\
b_n &= \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi}{p}x\right)dx\\
&= 2\int_0^1 f(x)\sin(n\pi x)dx\\
&\text{For} f(x) > 0, f(x) = x|x| = x(x) = x^2\\
b_n &= 2\int_0^1 x^2\sin(n\pi x)dx\\
&u = x^2 \qquad dv = \sin(n\pi x)dx\\
&du = 2xdx \quad v = -\frac{1}{n\pi}\cos(n\pi x)\\
&= 2\left[-\frac{x^2}{n\pi}\cos(n\pi x)\right]_0^1 + \frac{4}{n\pi}\int_0^1 x\cos(n\pi x)dx\\
&u = x \qquad dv = \cos(n\pi x)dx\\
&du = dx \quad v = \frac{1}{n\pi}\sin(n\pi x)\\
&= 2\left[-\frac{1}{n\pi}\cos(n\pi) + 0\right] + \frac{4}{n\pi}\left[[\frac{x}{n\pi}\sin(n\pi x)]_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\right]\\
&= -2\frac{(-1)^n}{n\pi} + \frac{4}{n\pi}\left(\frac{1}{n\pi}\sin(n\pi) - 0 + [\frac{1}{n^2\pi^2}\cos(n\pi x)]_0^1\right)\\
&= 2\frac{(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}(\cos(n\pi) - 1)\\
&= 2\frac{(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}((-1)^n - 1)\\
f(x) &= \sum_{n=1}^{\infty}\left(2\frac{(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}((-1)^n - 1)\right)\sin(n\pi x)
\end{aligned}
$$

(c)

$$
f(x) = \begin{cases} 1, & -2 < x < -1\\ 0, & -1 < x < 1\\ 1, & 1 < x < 2 \end{cases}
$$

[**4**] $f(x)$ is an even function $\Rightarrow$ cosine series.

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right)\\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right)
\end{aligned}
$$

$$
\begin{aligned}
a_0 &= \frac{2}{p}\int_0^p f(x)dx \\
&= \frac{2}{2}\int_0^2 f(x)dx = \int_1^2 dx = [x]_1^2 = 2-1 = 1 \\
a_n &= \frac{2}{p}\int_0^p f(x)\cos(\frac{n\pi}{p}x)dx \\
&= \frac{2}{2}\int_0^2 f(x)\cos(\frac{n\pi}{2}x)dx = \int_1^2 \cos(\frac{n\pi}{2}x)dx \\
&= \frac{2}{n\pi}[\sin(\frac{n\pi}{2}x)]_1^2 = \frac{2}{n\pi}[\sin(n\pi) - sin(\frac{n\pi}{2})] \\
&= \frac{2}{n\pi}[0 - \sin(\frac{n\pi}{2})] = -\frac{2}{n\pi}\sin(\frac{n\pi}{2}) \\
\Rightarrow f(x) &= \frac{1}{2} - \sum_{n=1}^{\infty}(\frac{2}{n\pi})\sin(\frac{n\pi}{2})\cos(\frac{n\pi}{2}x)
\end{aligned}
$$

(a) $f(x) = x, \qquad -\pi < x < \pi$ **[4]**

$f(x)$ is an odd function $\Rightarrow$ sine series.

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}x\right) \\
&= \sum_{n=1}^{\infty} b_n \sin(nx) \\
b_n &= \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi}{p}x\right)dx \\
&= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx
\end{aligned}
$$

$$
u = x \qquad dv = \sin(nx)dx
$$

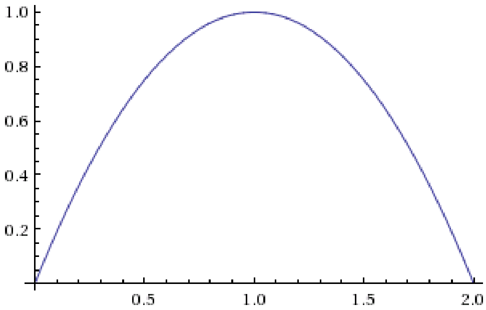
$$
du = dx \qquad v = -\frac{1}{n}\cos(nx)
$$

$$
\begin{aligned}
\Rightarrow \int x\sin(nx)dx &= -\frac{x}{n}\cos(nx) + \frac{1}{n}\int \cos(nx)dx \\
&= -\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx) \\
b_n &= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_0^{\pi} \\
&= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi) + 0 + \frac{1}{n^2}\sin(0)\right] \\
&= \frac{2}{\pi}\left[-\frac{\pi}{n}(-1)^n + 0 + 0\right] \\
&= -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1} \\
f(x) &= \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n}\right)\sin(nx)
\end{aligned}
$$

3. Find the half-range cosine and sine expansions of the given function, and graph each case.
   $f(x) = x(2-x), \qquad 0 < x < 2$ **[6]**

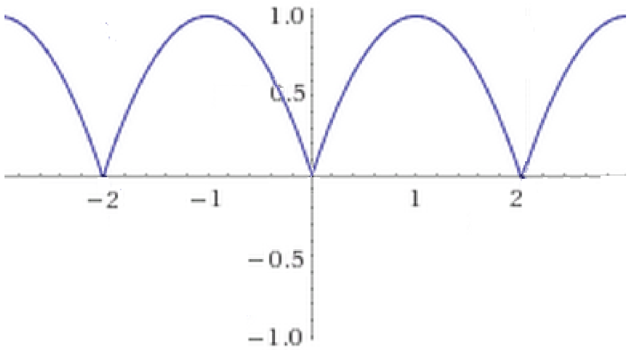
$$
f(x) = x(2-x) = 2x - x^2
$$

求给定函数的半值域余弦和正弦展开式，并画出每种情况



cosine expansion

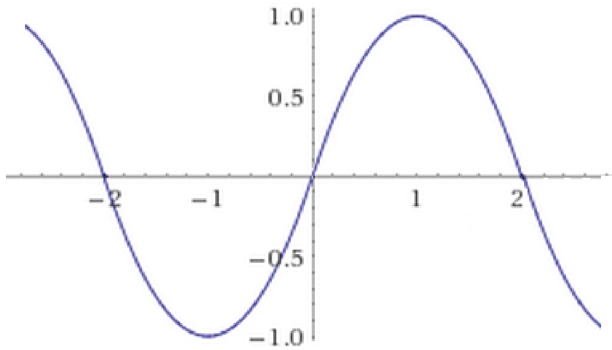
$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p}x\right) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}x\right) \\
a_0 &= \frac{2}{p}\int_0^p f(x)dx \\
&= \int_0^2 (2x - x^2)dx \\
&= \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 \\
&= \left[x^2 - \frac{x^3}{3}\right]_0^2 \\
&= [4 - \frac{8}{3} - 0]_0^2 \\
&= \frac{4}{3} \\
a_n &= \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi}{p}x\right)dx \\
&= \int_0^2 (2x - x^2)\cos\left(\frac{n\pi}{2}x\right)dx = 2\int_0^2 x\cos\left(\frac{n\pi}{2}x\right)dx - \int_0^2 x^2\cos\left(\frac{n\pi}{2}x\right)dx \\
&= 2\left[\frac{2x}{\pi n}\sin\left(\frac{n\pi}{2}x\right) + \frac{4}{\pi^2 n^2}\cos\left(\frac{n\pi}{2}x\right)\right]_0^2 - \left[\frac{8x}{\pi^2 n^2}\cos\left(\frac{n\pi}{2}x\right) + \frac{2x^2}{\pi n}\sin\left(\frac{n\pi}{2}x\right)\right. \\
&\left.- \frac{16}{\pi^3 n^3}\sin\left(\frac{n\pi}{2}x\right)\right]_0^2 \\
&= 2\left[\frac{4}{\pi n}\sin(n\pi) + \frac{4}{\pi^2 n^2}\cos(n\pi) - 0 - \frac{4}{\pi^2 n^2}\cos(0)\right] - \left[\frac{16}{\pi^2 n^2}\cos(n\pi) + \frac{8}{\pi n}\sin(n\pi)\right. \\
&\left.- \frac{16}{\pi^3 n^3}\sin(n\pi) - 0 - 0 + \frac{16}{\pi^3 n^3}\sin(0)\right] \\
&= 0 + \frac{8}{\pi^2 n^2}(-1)^n - 0 - \frac{8}{\pi^2 n^2} - \frac{16}{\pi^2 n^2}(-1)^n + 0 - 0 + 0 \\
&= -\frac{8}{\pi^2 n^2}(-1)^n - \frac{8}{\pi^2 n^2} \\
&= \frac{8}{\pi^2 n^2}(-1)^{n+1} - \frac{8}{\pi^2 n^2} \\
f(x) &= \frac{2}{3} + \sum_{n=1}^{\infty}\left(\frac{8}{\pi^2 n^2}(-1)^{n+1} - \frac{8}{\pi^2 n^2}\right)\cos\left(\frac{n\pi}{2}x\right)
\end{aligned}
$$



sine expansion

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p}x\right) \\
&= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{2}x\right)
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi}{p}x\right)dx \\
&= \int_0^2 (2x-x^2)\sin\left(\frac{n\pi}{2}x\right)dx = 2\int_0^2 x\sin\left(\frac{n\pi}{2}x\right)dx - \int_0^2 x^2\sin\left(\frac{n\pi}{2}x\right)dx \\
&= 2\left[-\frac{2x}{\pi n}\cos\left(\frac{n\pi}{2}x\right) + \frac{4}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}x\right)\right]_0^2 - [\frac{8x}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}x\right) - \frac{2x^2}{\pi n}\cos\left(\frac{n\pi}{2}x\right) \\
&+ \frac{16}{\pi^3 n^3}\cos\left(\frac{n\pi}{2}x\right)\Big]_0^2 \\
&= 2\left[-\frac{4}{\pi n}\cos(n\pi) + \frac{4}{\pi^2 n^2}\sin(n\pi) + 0 - \frac{4}{\pi^2 n^2}\sin(0)\right] - [\frac{16}{\pi^2 n^2}\sin(n\pi) - \frac{8}{\pi n}\cos(n\pi) \\
&+ \frac{16}{\pi^3 n^3}\cos(n\pi) - 0 + 0 - \frac{16}{\pi^3 n^3}\cos(0)\Big] \\
&= 2\left[-\frac{4}{\pi n}(-1)^n + 0 - 0\right] - \left[0 - \frac{8}{\pi n}(-1)^n + \frac{16}{\pi^3 n^3}(-1)^n - \frac{16}{\pi^3 n^3}\right] \\
&= -\frac{8}{\pi n}(-1)^n + \frac{8}{\pi n}(-1)^n - \frac{16}{\pi^3 n^3}(-1)^n + \frac{16}{\pi^3 n^3} \\
&= \frac{16}{\pi^3 n^3}(-1)^{n+1} + \frac{16}{\pi^3 n^3} \\
&= \frac{16((-1)^{n+1}+1)}{\pi^3 n^3} \\
f(x) &= \sum_{n=1}^{\infty} \frac{16((-1)^{n+1}+1)}{\pi^3 n^3}\sin(\frac{n\pi}{2}x)
\end{aligned}
$$



4. Find the complex Fourier series of f on the given interval.

*(a)

$$f(x) = \begin{cases} 0, & -2 < x < 0 \\ 1, & 0 < x < 2 \end{cases}$$

$$\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{p}} \\
c_n &= \frac{1}{2p}\int_{-p}^{p} f(x)e^{-\frac{in\pi x}{p}}\,dx \\
&= \frac{1}{2(2)}\int_{-2}^{2} f(x)e^{-\frac{in\pi x}{2}}\,dx \\
&= \frac{1}{4}\int_{0}^{2} e^{-\frac{in\pi x}{2}}\,dx \\
&= \frac{1}{4}\frac{(-2)}{in\pi}[e^{-\frac{in\pi x}{2}}]_0^2 \\
&= -\frac{1}{2in\pi}\left[e^{-in\pi} - 1\right] \\
&= \frac{1}{2in\pi}\left[1 - (-1)^n\right] \\
c_0 &= \frac{1}{4}\int_0^2 1\,dx \\
&= \frac{1}{2} \\
f(x) &= \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{2in\pi}\left[1-(-1)^n\right] e^{\frac{in\pi x}{2}}
\end{aligned}$$

(b)

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

[6]

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{p}} \\
c_n &= \frac{1}{2p}\int_{-p}^{p} f(x) e^{\frac{-in\pi x}{p}} dx \\
&= \frac{1}{2(\pi)}\int_{-\pi}^{\pi} f(x) e^{-inx} dx \\
&= \frac{1}{2\pi}\int_{0}^{\pi} x e^{-inx} dx
\end{aligned}
$$

$$
\begin{aligned}
u &= x \qquad dv = e^{-inx}dx \\
du &= dx \qquad v = -\frac{1}{in}e^{-inx}
\end{aligned}
$$

$$
\begin{aligned}
\int xe^{-inx}dx &= -\frac{x}{in}e^{-inx} + \frac{1}{in}\int e^{-inx}dx \\
&= -\frac{x}{in}e^{-inx} + \frac{1}{n^2}e^{-inx} \\
c_n &= \frac{1}{2\pi}\left[-\frac{x}{in}e^{-inx} + \frac{1}{n^2}e^{-inx}\right]_0^{\pi} \\
&= \frac{1}{2\pi}\left[-\frac{\pi}{in}e^{-in\pi} + \frac{1}{n^2}e^{-in\pi} + 0 - \frac{1}{n^2}\right] \\
&= -\frac{1}{2in}e^{-in\pi} + \frac{1}{2\pi n^2}e^{-in\pi} - \frac{1}{2\pi n^2} \\
&= -\frac{1}{2in}(-1)^n + \frac{((-1)^n - 1)}{2\pi n^2} \\
&= \frac{i}{2n}(-1)^n + \frac{((-1)^n - 1)}{2\pi n^2} \\
c_0 &= \frac{1}{2\pi}\int_0^{\pi} x\,dx \\
&= \frac{1}{2\pi}\frac{x^2}{2}\bigg|_0^{\pi} \\
&= \frac{\pi}{4} \\
f(x) &= \frac{\pi}{4} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\frac{i}{2n}(-1)^n + \frac{((-1)^n - 1)}{2\pi n^2}\right)e^{-inx}
\end{aligned}
$$