Z-Transform

The Laplace transform deals with continuous functions and can be used to solve differential equations.

Similarly, the **Z-transform** deals with discrete sequences and the recurrence relations - or difference equations.

Sequences

Consider the sequence $..., 3^{-2}, 3^{-1}, 3^0, 3^1, 3^2....$ It has a general form 3^k and using a shorthand notation we can write the sequence as $\left\{3^k\right\}_{-\infty}^{\infty}$ indicating also that the powers range from $-\infty$ to ∞.

The sum

$$\sum_{k=-\infty}^{\infty}\left(\frac{3}{z}\right)^k = \ldots\left(\frac{3}{z}\right)^{-1}+\left(\frac{3}{z}\right)^{0}+\left(\frac{3}{z}\right)^{1}+\left(\frac{3}{z}\right)^{2}\ldots \tag{1}$$

is called the **Z-transform** of the sequence, $\mathcal{Z}\left\{3^k\right\}_{-\infty}^{\infty}$ and is denoted $F(z)$ where the complex number z is chosen to ensure that the sum is finite.

We say that $\left\{3^k\right\}_{-\infty}^{\infty}$ and $\mathcal{Z}\left\{3^k\right\}_{-\infty}^{\infty} = F(z) = \sum_{k=-\infty}^{\infty}\left(\frac{3}{z}\right)^k$ form a **Z-transform pair**.

$s = a + bi \quad e^{-st} = e^{-at-ibt} = e^{-at} e^{-ibt}$

$|e^{-st}| = |e^{-at}||e^{-ibt}| = e^{-at} = e^{-Re(s)\cdot t}$

For our purposes we shall consider only **causal sequences** of the form $\{x_k\}_0^\infty$ where $x_k = 0$ for $k < 0$:

Let $f(x)$ be a complex function $f(x) \in \mathbb{C}$

$$\mathcal{Z}\{x_k\} = F(z) = \sum_{k=0}^{\infty} \frac{x_k}{z^k} \tag{2}$$

$\int_0^\infty f(x)\,dx \Leftarrow \underline{\int_0^\infty |f(x)|\,dx} < \infty \quad e^{-Re(s)\cdot t}$

$|f(x)| \Leftarrow e$

Example 1: $\{\delta_k\} = \{1, 0, 0, \ldots\}$

$-Re(s) < 0$

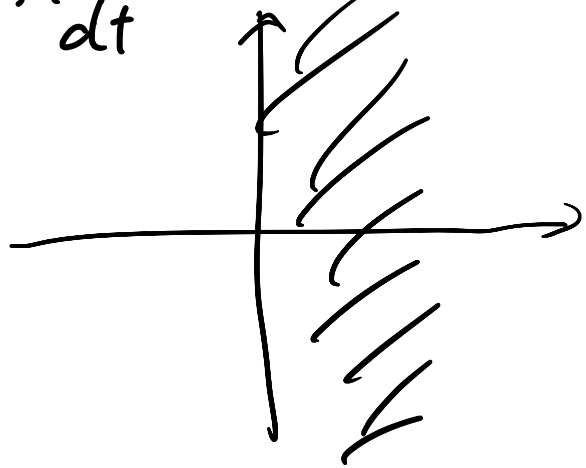
$$\mathcal{Z}\{\delta_k\} = F(z) = 1 + \frac{0}{z} + \frac{0}{z^2} + \frac{0}{z^3} + \ldots = 1$$

$f(t) = 1$

$Re(s) > 0$

$\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st}\,dt \quad = \int_0^\infty e^{-st}\,dt$

$\mathcal{F}\{f(t)\} = \int_0^\infty f(t) e^{-i\omega t}\,dt.$



Example 2: The **unit step sequence**: $\{u_k\} = \{1, 1, 1, \ldots\}$

$$\mathcal{Z}\{u_k\} = F(z) = \sum_{k=0}^{\infty} \frac{1}{z^k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots$$

Comparing this to the series expansion of $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ which is valid for $|x| < 1$ we get

$$F(z) = \frac{1}{1 - \frac{1}{z}} \quad \text{provided} \quad \left|\frac{1}{z}\right| < 1$$

or

$$F(z) = \frac{z}{z - 1} \quad \text{provided} \quad |z| > 1$$

$\mathcal{C}$

Example 3: $\{x_k\} = \{1, a, a^2, a^3, \ldots\} = \{a^k\}$

$$\mathcal{Z}\left\{a^k\right\} = \sum_{k=0}^{\infty} \frac{a^k}{z^k} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ldots$$

Comparing this to the series expansion of $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ which is valid for $|x| < 1$ then

$$\begin{aligned} F(z) &= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ldots \\ &= \frac{1}{1 - \frac{a}{z}} \quad \text{provided} \quad \left|\frac{a}{z}\right| < 1 \end{aligned}$$

or

$$F(z) = \frac{z}{z - a} \quad \text{provided} \quad |z| > |a|$$

Example 4: $\{x_k\} = \{0, 1, 2, 3, 4, \ldots\} = \{k\}$

$$\mathcal{Z}\{k\} = F(z) = \sum_{k=0}^{\infty} \frac{k}{z^k} = 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} +$$

Comparing this with the derivative of $(1-x)^{-1}$ and its series expansion

$$1 + 2x + 3x^2 + 4x^3 + \ldots = \frac{\mathrm{d}}{\mathrm{d}x}(1 + x + x^2 + x^3 + x^4 + \ldots) = \frac{\mathrm{d}}{\mathrm{d}x}(1-x)^{-1} = \frac{1}{(1-x)^2}$$

we can write

$$zF(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \ldots = \frac{1}{(1-1/z)^2}$$

and dividing both sides by z we obtain

$$F(z) = \frac{1}{z(1-1/z)^2} = \frac{z}{(z-1)^2}$$

Table of Z transforms

Sequence	Transform $F(z)$	Permitted values of z
$\{\delta_k\} = \{1, 0, 0, 0, \ldots\}$	1	All values of z
$\{u_k\} = \{1, 1, 1, 1, \ldots\}$	$\frac{z}{z-1}$	$\|z\| > 1$
$\{k\} = \{0, 1, 2, 3, \ldots\}$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
$\{k^2\} = \{0, 1, 4, 9, \ldots\}$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
$\{k^3\} = \{0, 1, 8, 27, \ldots\}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\|z\| > 1$
$\{a^k\} = \{1, a, a^2, a^3, \ldots\}$	$\frac{z}{(z-a)}$	$\|z\| > \|a\|$
$\{ka^k\} = \{0, a, 2a^2, 3a^3, \ldots\}$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$

Properties of Z transforms

1 Linearity

The Z transform is a linear transform, that is

$$\mathcal{Z}\{ax_k + by_k\} = a\mathcal{Z}\{x_k\} + b\mathcal{Z}\{y_k\} \tag{3}$$

where a and b are constants.

$$Z\{ax_k + by_k) = \sum_{k=0}^{\infty} (ax_k + by_k) z^{-k}$$

$$= a \sum_{k=0}^{\infty} x_k z^{-k} + b \sum_{k=0}^{\infty} y_k z^{-k}$$

$$= a Z\{x_k\} + b Z\{y_k\}$$

$$\mathcal{Z}\{k\} = \frac{z}{(z-1)^2} \qquad \mathcal{Z}\{e^{-2k}\} = \frac{z}{z-e^{-2}}$$

Example 5: $3\{k\} - 5\left\{e^{-2k}\right\}$ $\quad \frac{3z}{(z-1)^2} - \frac{5z}{z-e^{-2}}$

$$\mathcal{Z}\{k\} = \frac{z}{(z-1)^2}$$

$$\mathcal{Z}\left\{a^k\right\} = \frac{z}{z-a} \quad \Rightarrow \quad \mathcal{Z}\left\{e^{-2k}\right\} = \frac{z}{z-e^{-2}}$$

Consequently

$$\begin{aligned}\mathcal{Z}\left(3\{k\} - 5\left\{e^{-2k}\right\}\right) &= \frac{3z}{(z-1)^2} - \frac{5z}{(z-e^{-2})} \\ &= \frac{-5z^3 + 13z^2 - z(3e^{-2}+5)}{(z-1)^2(z-e^{-2})}\end{aligned}$$

$$Z\{x_{k+m}\} = \sum_{k=0}^{\infty} x_{k+m} z^{-k}$$

$n = k+m \quad k = n-m$

when $k=0 \quad n=m$

$$= \sum_{n=m}^{\infty} x_n z^{m-n}$$

2 First shifting theorem

$$= z^m \sum_{n=m}^{\infty} x_n z^{-n} = z^m \left\{ \sum_{n=0}^{\infty} x_n z^{-n} - \sum_{n=0}^{m-1} x_n z^{-n} \right\}$$

If $\mathcal{Z}\{x_k\} = F(z)$ then

$$\mathcal{Z}\{x_{k+m}\} = z^m F(z) - [z^m x_0 + z^{m-1} x_1 + \ldots + z x_{m-1}] \tag{4}$$

is the Z transform of the sequence that has been shifted by m places to the left.

$$= z^m F(z) - \sum_{n=0}^{m-1} x_n z^{m-n}$$

Example:

$$\mathcal{Z}\{x_{k+1}\} = zF(z) - zx_0$$
$$\mathcal{Z}\{x_{k+2}\} = z^2 F(z) - z^2 x_0 - zx_1$$

$$Z\{x_{k+1}\} = zF(z) - \sum_{n=0}^{0} x_n z^{m-n} = zF(z) - x_0 z^1$$

$$Z\{x_{k+2}\} = z^2 F(z) - \sum_{n=0}^{1} x_n z^{m-n}$$

$$= z^2 F(z) - x_0 z^2 - x_1 z$$

$n=3 \qquad \mathcal{Z}\{4^k\} = \dfrac{z}{z-4}$

$\mathcal{Z}\{4^{k+3}\} = z^3 \dfrac{z}{z-4} - \sum_{n=0}^{2} x_n z^{3-n}$

Example 6: $\left\{4^{k+3}\right\}$

$= z^3 \dfrac{z}{z-4} - x_0 z^3 - x_1 z^2 - x_2 z$

Given

$$\mathcal{Z}\left\{4^k\right\} = \frac{z}{z-4}$$

$$\begin{aligned}
\mathcal{Z}\left\{4^{k+3}\right\} &= z^3\mathcal{Z}\left\{4^k\right\} - [z^3 4^0 + z^2 4^1 + z4^2] \\
&= z^3\frac{z}{z-4} - [z^3 + 4z^2 + 16z] = \frac{z^4}{z-4} - [z^3 + 4z^2 + 16z] \\
&= \frac{z^4 - [z^3 + 4z^2 + 16z](z-4)}{z-4} = \frac{z^4 - (z^4 - 64z)}{z-4} \\
&= \frac{64z}{z-4}
\end{aligned}$$

We have just derived the Z transform of the sequence $\{64, 256, 1024, ...\}$ by shifting $\{1, 4, 16, 64, 256, ...\}$ three places to the left and loosing the first three terms.

$$\{k\} + \{1\} = \frac{z}{(z-1)^2} + \frac{z}{z-1}$$

$$= \frac{z + z^2 - z}{(z-1)^2} = \frac{z^2}{(z-1)^2}$$

Example 7: $\underbrace{\{k+1\}}$

$$\mathcal{Z}\{k\} = \frac{z}{(z-1)^2}$$

$$\mathcal{Z}\{k+1\} = z\frac{z}{(z-1)^2} - [z \times 0]$$

$$= \frac{z^2}{(z-1)^2}$$

$$Z\{k\} = \frac{z}{(z-1)^2}$$

$$Z\{k+1\} = z^m Z\{k\} - \sum_{n=0}^{\substack{m-1 \\ 0}} x_n z^{m-n} \qquad m=1$$

$$= z\frac{z}{(z-1)^2} - x_0 z = \frac{z^2}{(z-1)^2}$$

$$Z\{x_{k-m}\} = \sum_{k=0}^{\infty} x_{k-m} z^{-k} = \sum_{n=-m}^{\infty} x_n z^{-(n+m)}$$

let $n = k-m$ $$= z^{-m} \sum_{n=0}^{\infty} x_n z^{-n} + \sum_{n=-m}^{-1} x_n z^{-(n+m)}$$

3 Second shift theorem

If $\mathcal{Z}\{x_k\} = F(z)$ then   Assume $x_n = 0$ for $n < 0$

$$\mathcal{Z}\{x_{k-m}\} = z^{-m}F(z) \tag{5}$$

T_s : sampling time

is the Z transform of the sequence that has been shifted by m places to the right.

Example 8: Given $$\mathcal{L}\{\delta(t - nT_s)\} = \begin{cases} 0 & \text{if } n < 0 \\ e^{-nT_s s} & \text{if } n > 0 \end{cases} \qquad e^{-nT_s s} \cdot 0$$

$$\mathcal{Z}\{x_k\} = \frac{z}{z-1}$$

then

$$\mathcal{Z}\{x_{k-3}\} = z^{-3}\frac{z}{z-1} = \frac{1}{z^2(z-1)} \qquad \boxed{e^{T_s \cdot s} = z}$$

We thus derived the Z transform of the sequence $\{0, 0, 0, 1, 1, 1, ...\}$ by shifting $\{1, 1, 1, ...\}$ three places to the right and defining the first three terms as zeros.

$$\sum_{n=-\infty}^{\infty} f(nT_s) \cdot \delta(t - nT_s) \xrightarrow{\mathcal{L}} \sum_{n=-\infty}^{\infty} \underline{f(nT_s)\beta} z^{-n} \quad \begin{cases} \beta = 1 & n \geq 0 \\ \beta = 0 & n < 0 \end{cases}$$

$$= \sum_{n=-\infty}^{-1} f(nT_s)\delta(t - nT_s) + \sum_{n=0}^{\infty} f(nT_s)\delta(t - nT_s)$$

Example 9:

Given the Z transform

$$\mathcal{Z}\{x_k\} = \frac{1}{z-a}$$

$$= \frac{z}{z-a} \cdot \frac{1}{z} \qquad z^{-1}$$

$$a^k \qquad \left\{a^{k-1}\right\}$$

where a is a constant. The sequence $\{x_k\}$ is

$$\left\{a^{k-1}\right\}$$

because

$$\frac{1}{z-a} = \frac{1}{z} \times \frac{z}{z-a} = z^{-1}F(z)$$

where $F(z) = \mathcal{Z}\left\{a^k\right\}$ and so

$$\frac{1}{z-a} = \mathcal{Z}\left\{a^{k-1}\right\}$$

$$Z\{x_k\} = F(z) \qquad Z\{a^k x_k\} = F(z/a)$$

4 Translation

If the sequence $\{x_k\}$ has the Z transform $\mathcal{Z}\{x_k\} = F(z)$ then the sequence $\left\{a^k x_k\right\}$ has the Z transform $\mathcal{Z}\left\{a^k x_k\right\} = F(a^{-1}z)$.

$$Z\{a^k x_k\} = \sum_{k=0}^{\infty} a^k x_k z^{-k}$$

$$= \sum_{k=0}^{\infty} x_k \left(\frac{z}{a}\right)^{-k}$$

Example 10:

$$F(z) \quad \mathcal{Z}\{k\} = \frac{z}{(z-1)^2}$$

so

$$\mathcal{Z}\left\{2^k k\right\} = F\left(2^{-1}z\right) = \frac{2^{-1}z}{\left(2^{-1}z-1\right)^2} = \frac{2z}{(z-2)^2}$$

$$Z\{2^k k\} = F\left(\frac{z}{2}\right) = \frac{\frac{z}{2}}{\left(\frac{z}{2}-1\right)^2} = \frac{2z}{(z-2)^2}$$

$\{x_{k+1}\}$ $\quad \{x_k\} \xrightarrow{Z} F(z)$

$\{x_{k+1}\} \xrightarrow{Z} zF(z) - zx_0$

5 Final value theorem

$Z\{x_{k+1} - x_k\} = zF(z) - F(z) - zx_0$

For the sequence $\{x_k\}$ with Z transform $F(z)$

$$\lim_{k\to\infty} x_k = \lim_{z\to 1}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} \tag{6}$$

provided that $\lim_{k\to\infty} x_k$ exists.

$Z\{x_{k+1} - x_k\} = \sum_{n=0}^{\infty}(x_{n+1} - x_n)z^{-n}$

Example: The sequence $\left\{\left(\frac{1}{2}\right)^k\right\}$ has the Z transform

$$F(z) = \frac{z}{z-\frac{1}{2}} = \frac{2z}{2z-1}$$ $z \to 1$

so

$$\lim_{k\to\infty}\left\{\left(\frac{1}{2}\right)^k\right\} = \lim_{z\to 1}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} = \lim_{z\to 1}\left\{\frac{2(z-1)}{2z-1}\right\} = 0$$

$$\lim_{z\to 1} Z\{x_{k+1} - x_k\} = \lim_{m\to\infty} \underbrace{\sum_{n=0}^{m}(x_{n+1} - x_n)}_{} = \lim_{m\to\infty} x_{m+1} - x_0$$

$$= \lim_{z\to 1} zF(z) - F(z) - zx_0 = \lim_{z\to 1}(z-1)F(z) - x_0$$

$$\lim_{m\to\infty} x_m = \lim_{m\to\infty} x_{m+1} = \lim_{z\to 1} \frac{(z-1)}{z} F(z)$$

$$\{1, 1, 1, \ldots\}$$

$$F(z) = \frac{z}{z-1}$$

Example 11:

Using the final value theorem, the final value of the sequence with the Z transform

$$F(z) = \frac{10z^2 + 2z}{(z-1)(5z-1)^2} \tag{7}$$

is calculated as follows

$$\begin{aligned}
\lim_{z\to 1}\left\{\left(\frac{z-1}{\cancel{z}}\right)F(z)\right\} &= \lim_{z\to 1}\left\{\left(\frac{z-1}{\cancel{z}}\right)\frac{10z^2+2z}{(z-1)(5z-1)^2}\right\}\\
&= \lim_{z\to 1}\left\{\frac{10z+2}{(5z-1)^2}\right\}\\
&= \frac{12}{16}\\
&= 0.75
\end{aligned}$$

$$x_\infty = \lim_{z\to 1} (z-1) F(z)$$

$$= \lim_{z\to 1} \frac{\cancel{(z-1)}(10z^2+2z)}{\cancel{(z-1)}(5z-1)^2} = \frac{12}{4^2} = \frac{12}{16} = 0.75$$

$$F(z) = \sum_{n=0}^{\infty} x_k z^{-n} = x_0 + x_1 z^{-1} + x_2 z^{-2} + \cdots + \cdots$$

$z \to \infty \quad z^{-1} \to 0$

$x_0 = \lim_{z\to\infty} F(z)$

6 The initial value theorem

For the sequence $\{x_k\}$ with the Z transform $F(z)$

$$x_0 = \lim_{z\to\infty} \{F(z)\} \tag{8}$$

Example: $F(z) = \frac{z}{z-a} \Rightarrow \{a^n\}$

The sequence $\left\{a^k\right\}$ has the Z transform $F(z) = \frac{z}{z-a}$ and, using the l'Hospital rule,

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{z}{z-a} = \lim_{z\to\infty} \frac{1}{1} = 1$$

Furthermore $x_0 = a^0 = 1$.

$$x_0 = \lim_{z\to\infty} \frac{z}{z-a} = \lim_{z\to\infty} \frac{z'}{(z-a)'} = 1$$

$$F(z) = \sum_{n=0}^{\infty} x_k z^{-n} \qquad F'(z) = \sum_{n=0}^{\infty} -n\, x_n z^{-n-1}$$

$$zF'(z) = -\sum_{n=0}^{\infty} n x_n z^{-n} = -\mathcal{Z}\{n x_n\}$$

7 The derivative of the transform

If $\mathcal{Z}\{x_k\} = F(z)$ then

$$-zF'(z) = \mathcal{Z}\{kx_k\} \tag{9}$$

<u>Proof:</u>

$$F(z) = \sum_{k=0}^{\infty} x_k z^{-k}$$

and so

$$\begin{aligned} F'(z) &= \sum_{k=0}^{\infty} x_k(-k)z^{-k-1} = -\frac{1}{z}\sum_{k=0}^{\infty} x_k k z^{-k} \\ &= -\frac{1}{z}\mathcal{Z}\{kx_k\} \end{aligned}$$

Example 12:

The sequence $\left\{a^k\right\}$ has the Z transform $F(z) = \frac{z}{z-a}$ and so the sequence $\left\{ka^k\right\}$ has Z transform

$$\mathcal{Z}\left\{kx_k\right\} = -zF'(z) = -z\left(\frac{z}{z-a}\right)' = -z\left(\frac{z-a-z}{(z-a)^2}\right) = \frac{az}{(z-a)^2}$$

Notice that this is in agreement with the table of transforms.

Inverse Z transforms

If the sequence $\{x_k\}$ has Z transform $\mathcal{Z}\{x_k\} = F(z)$, the inverse transform is defined as

$$\mathcal{Z}^{-1}F(z) = \{x_k\}$$

To carry out the inverse Z transform, we will usually need to perform some manipulation, the most often using the partial fraction decomposition.

Example 13:

The sequence $\{x_k\}$ has Z transform $F(z) = \frac{z}{z^2-5z+6}$. We first perform the partial fraction decomposition

$$F(z) = \frac{z}{z^2 - 5z + 6} = \frac{z}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3} = \frac{A(z-3) + B(z-2)}{(z-2)(z-3)}$$

Equating numerators and solving for A and B gives $A = -2$ and $B = 3$. So

$$F(z) = \frac{3}{z-3} - \frac{2}{z-2}$$

The nearest Z transform in the table to either of these two partial fractions is $\mathcal{Z}\left\{a^k\right\} = \frac{z}{z-a}$ so we write

$$
\begin{aligned}
F(z) &= \frac{3}{z-3} - \frac{2}{z-2} = \frac{3}{z} \times \frac{z}{z-3} - \frac{2}{z} \times \frac{z}{z-2} \\
&= 3 \times z^{-1} \mathcal{Z}\left\{3^k\right\} - 2 \times z^{-1} \mathcal{Z}\left\{2^k\right\}
\end{aligned}
$$

The inverse Z transform is then

$$
\begin{aligned}
\mathcal{Z}^{-1} F(z) &= 3 \times \left\{3^{k-1}\right\} - 2 \times \left\{2^{k-1}\right\} \\
&= \left\{3^k\right\} - \left\{2^k\right\} \\
&= \left\{3^k - 2^k\right\}
\end{aligned}
$$

giving $x_k = 3^k - 2^k$.

We can solve this problem also without using the second shift theorem. We consider instead the partial fraction decomposition of $\frac{F(z)}{z}$:

$$\begin{aligned}\frac{F(z)}{z} &= \frac{1}{z}\times\frac{z}{z^2-5z+6} = \frac{1}{z^2-5z+6} = \frac{1}{(z-2)(z-3)} \\ &= \frac{A}{z-2}+\frac{B}{z-3} = \frac{A(z-3)+B(z-2)}{(z-2)(z-3)}\end{aligned}$$

Equating numerators and solving for A and B yields $A=-1$ and $B=1$, so that

$$\frac{F(z)}{z} = \frac{1}{z-3}-\frac{1}{z-2} \qquad \Rightarrow \qquad F(z) = \frac{z}{z-3}-\frac{z}{z-2} = \mathcal{Z}\left\{3^k\right\} - \mathcal{Z}\left\{2^k\right\}$$

The final result is

$$\begin{aligned}\mathcal{Z}^{-1}F(z) &= \left\{3^k\right\} - \left\{2^k\right\} \\ &= \left\{3^k - 2^k\right\}\end{aligned}$$

Example 14:

The sequence $\{x_k\}$ has Z transform

$$F(z) = \frac{5z}{(z^2 - 4z + 4)(z + 2)}$$

We divide $F(z)$ by z and perform the partial fraction decomposition

$$\begin{aligned}
\frac{F(z)}{z} &= \frac{1}{z} \times \frac{5z}{(z^2 - 4z + 4)(z + 2)} = \frac{5}{(z-2)^2(z+2)} \\
&= \frac{A}{(z-2)^2} + \frac{B}{z-2} + \frac{C}{z+2} \\
&= \frac{A(z+2) + B(z-2)(z+2) + C(z-2)^2}{(z-2)^2(z+2)}
\end{aligned}$$

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Equating numerators and solving for A, B and C yields $A = 5/4$, $B = -5/16$ and $C = 5/16$, so

$$\frac{F(z)}{z} = \frac{5/4}{(z-2)^2} - \frac{5/16}{z-2} + \frac{5/16}{z+2}$$

giving

$$F(z) = \frac{5}{8} \times \frac{2z}{(z-2)^2} - \frac{5}{16} \times \frac{z}{z-2} + \frac{5}{16} \times \frac{z}{z+2}$$

The inverse Z transform is then

$$\begin{aligned} \mathcal{Z}^{-1}F(z) &= \frac{5}{8} \times \left\{k2^k\right\} - \frac{5}{16} \times \left\{2^k\right\} + \frac{5}{16} \times \left\{(-2)^k\right\} \\ &= \left\{\frac{5}{16}\left[(2k-1)2^k + (-2)^k\right]\right\} \end{aligned}$$