EE213 Assignment 1: Fourier Analysis

In this assignment, you will:

- Use MATLAB to calculate the Fourier Series (FS) of periodic continuous signals.
- Synthesize a signal from its Fourier Series.
- Explore the Fourier Transform (FT) of continuous signals.
- Use MATLAB to compute and plot the FT.
- Explore the Discrete Time Fourier Series (DTFS) and Discrete Time Fourier Transform (DTFT).

The prerequisite theory for completing this assignment can be found in **Lecture 6: Frequency Analysis**.

This description of the Assignment includes code that you can borrow and/or use to complete the tasks. You should run the sample codes in MATLAB so that you can look at the output and ensure you understand how the code works. If you understand the operation of the code samples, you will find it easier to extend this to complete the tasks in the assignment.

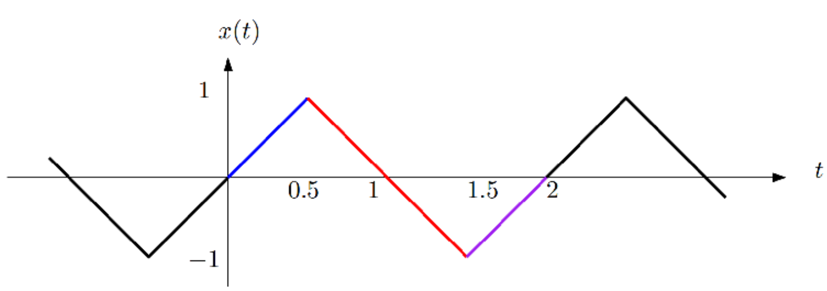
You should upload **a single PDF file** to Moodle. This PDF file should include your MATLAB code, figures, and answers to all the tasks and the questions therein.

Task 1: Fourier Series (FS) Coefficients (Analytical Method)

In Lab 1, we saw how a periodic square wave can be generated (or synthesized) by combining various complex exponential functions having different frequencies and weights. We used the Fourier series synthesis equation to achieve this.

$$x(t) = \sum_{n=-N}^{N} X_n e^{j2\pi n f_0 t}$$

In this assignment, we will explore the Fourier series of the periodic triangular wave shown in the following figure. The blue/red/purple portion of the signal constitutes one full period.



Recall that the n^th FS coefficient for a function $x(t)$ is given by

$$X_n = \frac{1}{T}\int_0^T x(t)\, e^{-j2\pi n f_0 t} dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

where T and $f_0 = \frac{1}{T}$ are the fundamental period and frequency of $x(t)$, respectively. This is referred to as the analysis equation. Note that the weight X_n represents the contribution that the complex exponential function $e^{-j2\pi n f_0 t}$ with the frequency nf_0 makes to the composition of $x(t)$.
Clearly, the fundamental period and fundamental frequency of $x(t)$ are $T = 2$s and $f_0 = \frac{1}{2} = 0.5$ Hz, respectively.

Mathematical model of $x(t)$
Before we can use the analysis equation, we need to write down a description of one period of $x(t)$, which will be necessarily piecewise. Each coloured section is described by a straight line over the relevant domain.

$$x(t) = \begin{cases} 2t & 0 \le t \le 0.5 \\ -2(t-1) & 0.5 \le t \le 1.5 \\ 2(t-2) & 1.5 \le t \le 2 \end{cases}$$

Calculating FS Coefficients
Trivial Case: $n = 0$

- We now take the analysis equation from above and step through values of n. The simplest case is when $n = 0$.
- This makes the exponent of the complex exponential 0 (because $-j2\pi n f_0 t$ is 0 when $n = 0$). Thus, we have

$$X_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

- Recall that the purpose of the complex exponential is to describe the oscillatory (or sinusoidal, or AC) components of $x(t)$.
- If there is no exponential term in the integral, then we must be referring to the DC component (or offset) of the signal.
- Intuitively, it should be clear to you that this signal does not have a DC offset. Thus, if we work out the integral, it will evaluate to zero.

$$X_0 = \frac{1}{T}\int_0^T x(t)\, dt = 0$$

Non-trivial Case: $n \neq 0$

Calculation of X_n is slightly more involved for non-zero values of n.

- Substituting $T = 2$s and $f_0 = \frac{1}{2} = 0.5$ Hz in the analysis equation, we have

$$X_n = \frac{1}{2}\int_0^2 x(t)\, e^{-j\pi n t} dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots$$

- This equation can be expanded using the piecewise function $x(t)$ as

$$X_n = \frac{1}{2}\int_0^{0.5} 2t\, e^{-j\pi n t} dt + \frac{1}{2}\int_{0.5}^{1.5} -2(t-1)\, e^{-j\pi n t} dt + \frac{1}{2}\int_{1.5}^{2} 2(t-2)\, e^{-j\pi n t} dt$$

- To simplify the evaluation of the integral slightly, we can make the following substitution: $a = -j\pi n$.

$$X_n = \frac{1}{2}\int_0^{0.5} 2t\, e^{at} dt + \frac{1}{2}\int_{0.5}^{1.5} -2(t-1)\, e^{at} dt + \frac{1}{2}\int_{1.5}^{2} 2(t-2)\, e^{at} dt$$

- We take the following result as given

$$\int x e^{ax} dx = e^{ax}\left(\frac{ax-1}{a^2}\right)$$

- **In your assignment, you should use this result to mathematically evaluate X_n as per the piecewise integral above, for odd and even values of n.**
- **Ensure you show all workings.**
- **You should arrive at the following result:**

$$\boldsymbol{X_n = \begin{cases} 0 & n \text{ Even} \\ j(-1)^{\frac{n+1}{2}} \dfrac{4}{\pi^2 n^2} & n \text{ Odd} \end{cases}}$$

Task 2: Fourier Series Coefficients (Numerical Method)

Task 1 concludes with an analytical solution to the integral. Clearly, solving the integral analytically can be arduous and time consuming, and requires broad knowledge of integrals. Conveniently, MATLAB has a suite of built in functions for numerical integration. While the numerical integration will solve the integral for us, the result will be an approximation to the analytical result (this is an inevitable result of the algorithm).

In this task, you will use one such function to compute the FS coefficients of the triangle wave numerically, and compare this to the analytical result.

Numerical Integration

The function you should use to compute the integrals is 'q = quadgk(fun,a,b)'.

- 'quadgk' stands for Gauss-Kronrod Quadrature (the name of the algorithm).
- For the purposes of this lab, you do not need to know the details of the algorithm, but feel free to look it up if you are particularly interested.
- The 'quadgk' function accepts three arguments: a function object, the lower limit of the integral, and the upper limit of the integral.

For example, the definite integral $\int_1^2 2te^t\, dt$ can be computed as follows.

```
f = (@(t)(2*t).*exp(t))     % This defines the function 2t*exp(t)
q = quadgk(f,1,2)           % This calculates the integral
```

Note the syntax for defining the function:

`@(t)` means that t is the independent variable.

`(2*t).*exp(t)` defines the function itself.

Calculating FS Coefficients Numerically

Recall the integral from earlier

$$X_n = \frac{1}{2}\int_0^{0.5} 2t\, e^{at} dt + \frac{1}{2}\int_{0.5}^{1.5} -2(t-1)\, e^{at} dt + \frac{1}{2}\int_{1.5}^{2} 2(t-2)\, e^{at} dt$$

where $a = -j\pi n$. We can evaluate this integral numerically using code similar to that described above. We must create a function for each section of the piecewise integral.

```
N = 11;                                    % number of FS coefficients
n = -N:N;
for k = 1:length(n)
    a = -1i*pi*n(k);                       % as per substitution

    f1 = (@(t)(2*t).*exp(a*t));              % blue interval
    f2 = (@(t)(-2*(t-1)).*exp(a*t));       % red interval
    f3 = (@(t)(2*(t-2)).*exp(a*t));        % purple interval

    % Note that this matches the integral specified above.
    x(k) = 0.5*quadgk(f1,0,0.5) + 0.5*quadgk(f2,0.5,1.5) + 0.5*quadgk(f3,1.5,2);
end
x
```

- Compare the analytical result from Task 1 with this numerical result.
- Hint: You may want to write code to investigate the difference between the two sets of coefficients.
- Comment on the level of agreement (or disagreement) between the coefficients generated by the analytical and numerical approaches.

Further Investigation of FS Coefficients

- Plot the magnitude and phase of X_n to verify that $X_n = X_{-n}^*$ (where X_{-n}^* is the complex conjugate of X_{-n}).
- Recall that the coefficients X_n are complex numbers, and therefore can be expressed in terms of magnitude and phase (polar coordinates).

```
% Magnitude
subplot(211)
stem(n,abs(x))

% Phase
subplot(212)
stem(n,angle(x))
```

- In your report, add appropriate titles, labels, gridlines, etc. to the plots.
- Comment on how the plots verify that $X_n = X_{-n}^*$.
- Hint: You may want to generate further plots that can assist your explanations.

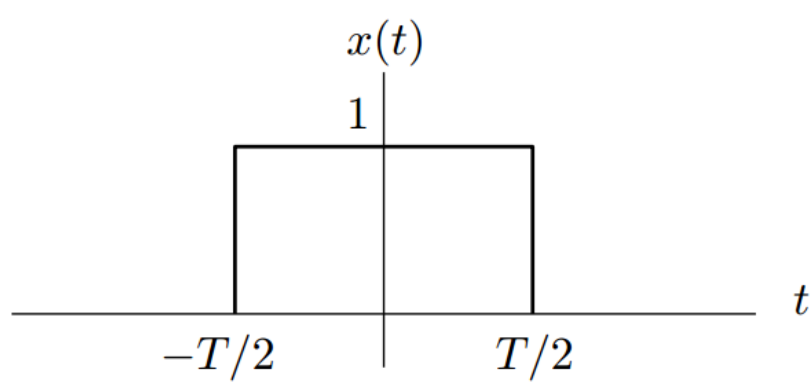
Task 3: Synthesis

Write a MATLAB code to synthesize the triangular signal from its FS coefficients.

- You can modify the code from Lab 1 to achieve this.
- Use the coefficients given at the end of Task 1.
- Comment on the FS coefficients with respect to how the magnitude and phase of the coefficients change as a function of *n*.

Task 4: Fourier Transform (Analytical Method)

Consider the signal given below



For $T = 1$ (i.e. the rectangular signal), we know from the lecture notes that the Fourier Transform is as follows

$$X(\omega) = \frac{\sin(\omega/2)}{\omega/2} = \text{sinc}(\omega/2)$$

We can plot the Fourier Transform of the signal for $T = 1$ and $\omega \in [-4\pi, 4\pi]$ as follows

```
freqs = linspace(-4*pi,4*pi,100);
FT_rect = sin(freqs/2)./(freqs/2);
plot(freqs,FT_rect)
```

Derive the Fourier Transform of the signal for general T. Show all workings in your report.

Hint: recall that time scaling a function has a predictable effect on its Fourier Transform.

Modify the MATLAB code given above to compute and plot the Fourier Transform of the signal for

- $T = 0.1$
- $T = 5$
- $T = 10$

Comment on how the Fourier Transform of the signal changes as a function of T.

Task 5: Fourier Transform (Numerical Method)

In a similar fashion to Task 2, we can compute the Fourier Transform numerically using 'quadgk'. Recall that the Fourier Transform is given by:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

We can evaluate this integral numerically as follows:

```
freqs = linspace(-4*pi,4*pi,100);
T = 1;

for k = 1:length(freqs)
    f = (@(t)exp(-1i*freqs(k)*t));
    FT_rect(k) = quadgk(f,-T/2, T/2);
end

plot(freqs, FT_rect)
```

Again write MATLAB code to compute (numerically) and plot the Fourier Transform of the signal for

- $T = 0.1$
- $T = 5$
- $T = 10$

Compare the numerical results with the analytical results from Task 4.

Task 6

Consider the following signal

$$x_a(t) = e^{-a|t|}$$

- Plot $x_a(t)$ for $a = 1$, $a = 10$, $a = 100$.
- Comment on how $x_a(t)$ changes as a function of a.

Task 7

- Using a numerical approach (similar to Task 5), plot the Fourier Transform of $x_a(t)$ for $a = 1$, $a = 10$, $a = 100$.
- In this instance, consider a wider frequency range: $\omega \in [-100\pi, 100\pi]$.
- Comment on how the Fourier Transform of $x_a(t)$ changes as a function of a.

Task 8

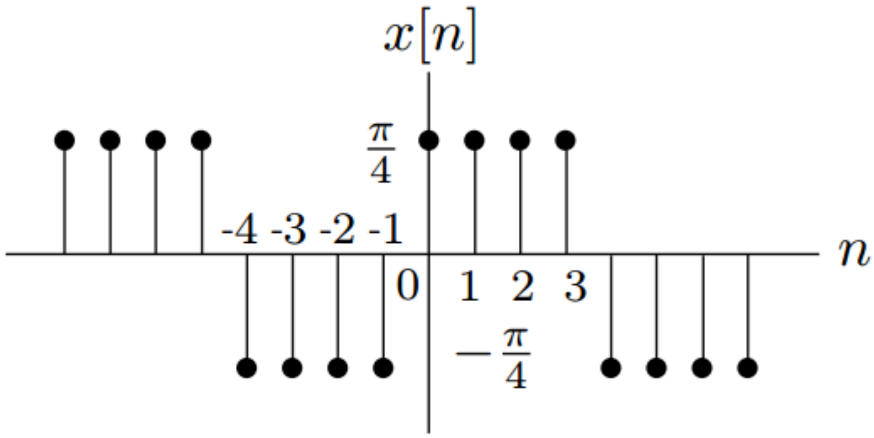
This task is similar to Task 2, but deals with discrete-time signals, as opposed to continuous-time signals.

If we have a periodic discrete signal $x[n]$ with fundamental period N such that $x[n] = x[n+N]$, then the Discrete Time Fourier Series representation of $x[n]$ is given by

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n}, \qquad k = 0,1,\dots,N-1$$

where $\Omega_0 = 2\pi/N$ is the fundamental frequency.

1. Use MATLAB to find the DTFS representation of the following discrete-time signal.
2. Compare your numerical results to the analytical results in Lecture 6.



Note: The DTFS representation for this signal is calculated analytically in Lecture 6 for $k = 0,1,2$. Your MATLAB code should compute the DTFS representation for $k \in [0, N-1]$.

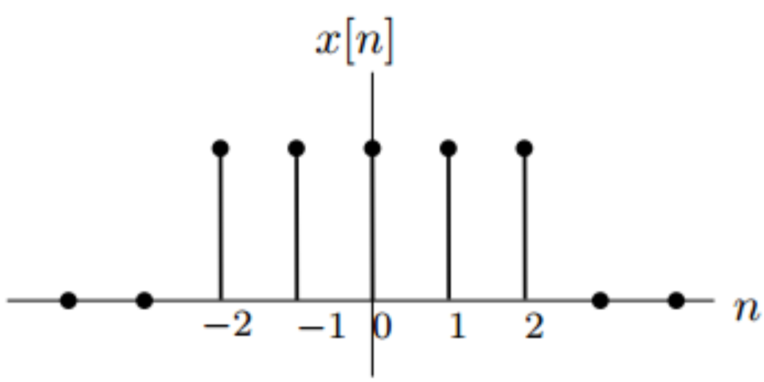
Task 9

This task is similar to Task 5, but deals with discrete-time signals, as opposed to continuous-time signals.

The Discrete Time Fourier Transform (DTFT) of a discrete signal $x[n]$ is given by

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

1. Use MATLAB to compute and **plot** the Discrete Time Fourier Transform (DTFT) of the following discrete-time signal.
2. Again, compare your numerical results with the analytical results from Lecture 6.



Recall that the DTFT is both continuous and periodic, having a period of 2π. As such, you should plot the DTFT over a sufficiently wide range of frequencies so that this periodicity is apparent i.e. the frequency range should be at least $\omega \in [-2\pi, 2\pi]$, preferably more.

Assignment Checklist

- Ensure that you do all the tasks, answer all the questions and include your matlab codes in a single PDF file.
- All figures should be clear, and should have appropriate labels, titles, units, etc.
- In some cases, the questions require you to write code. In other cases, the questions ask for a comment on some results. If you do not give your comments, you cannot get the marks.