Lifting for block sensitivity

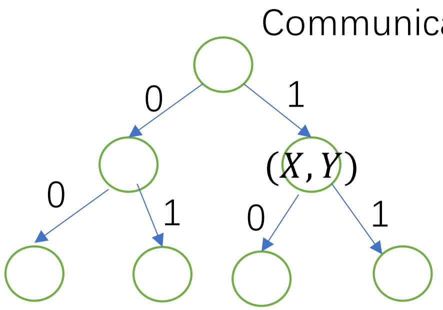
There exists a constant $c \geq 0$ *such that the following holds. Let* $f:\{0,1\}^n \to \{0,1\}$ *be any boolean function, and let* $g:[q]\times[q] \to \{0,1\}$ *be low discrepancy gadget function such that* $q \geq c$*. Then*

$$P^{CC}(f \circ g^n) = \Omega(bs(f) \cdot \log q)$$

Previous approach: Information complexity [HN12] Reduction [Zha09][GP14]

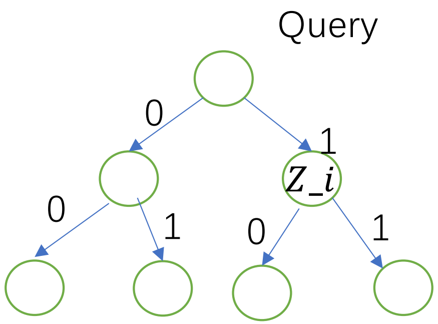
Our approach: Simulation !

Raz-McKenzie's simulation [RM99,GPW15]



Convert communication protocol of $f \circ \mathrm{g}^n$

to decision tree for $f: \{0,1\}^n \to \{0,1\}$



To show the simulation is correct,

1) **Thickness Lemma:** Average (degree) to worst (degree) reduction (or sunflower lemma [LMMPZ20]) to keep the disperser property:

$$\mathrm{g}^n(X_I, Y_I) = \{0,1\}^I$$ (g is gadget function and I is unfixed coordinates)

To show the simulation is efficient,

2) **Projection Lemma:** Potential function argument and probabilistic method to ensure

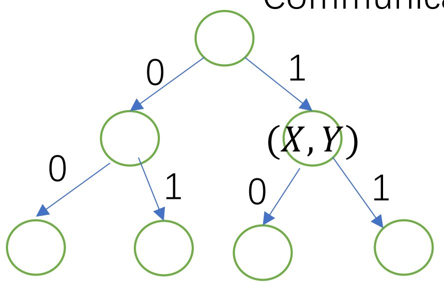
$\mathrm{g}(X_i, Y_i) = Z_i$ and potential function decreases by at least Ω(log q) in each "query iteration"

Raz-McKenzie’ s simulation [RM99,GPW15]

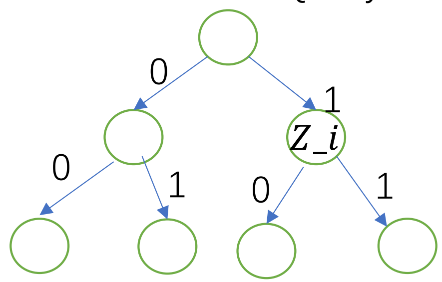
$f \circ g^n$

$f : \{0,1\}^n \to \{0,1\}$

Communication by Alice or Bob



Query



However, **Thickness Lemma** has gadget size barrier:

Average (degree) to worst (degree) reduction (or sunflower lemma [LMMPZ20]) need q= **Ω(n)**

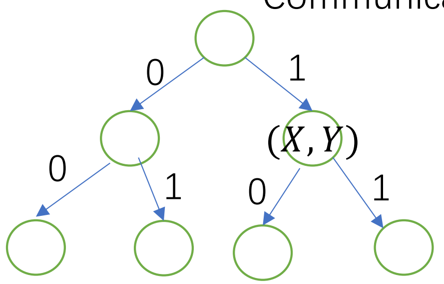
Gadget size is a fundamental parameter in lifting theorems and their applications [GP16, GJW16, GR18].

Our simulation

$f \circ \mathrm{g}^n$

$f: \{0,1\}^n \rightarrow \{0,1\}$

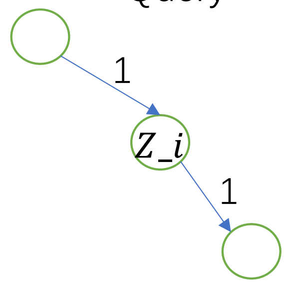
Communication by Alice or Bob



Find a communication path of $f \circ \mathrm{g}^n$

by a decision tree path of $f: \{0,1\}^n \rightarrow \{0,1\}$

Query



To show the simulation is correct, we follow the decision tree path in **each "query iteration"**

We don't need Thickness Lemma !

To show the simulation is efficient,

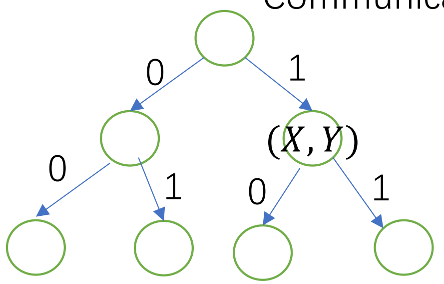
2) **Projection Lemma:** Potential function argument and low-discrepancy property [CFKMP19] to ensure

potential function decreases by at least Ω(log q) in each "query iteration"

Our simulation

$f \circ \mathrm{g}^n$

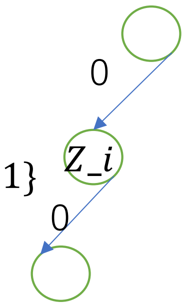
Communication by Alice or Bob



Find a long communication path of $f \circ \mathrm{g}^n$

by a long decision tree path of $f: \{0,1\}^n \to \{0,1\}$

Query



How to keep a long path ? Block sensitivity of $f: \{0,1\}^n \to \{0,1\}$ (a lower bound of query complexity)

WLOG Assume $bs(f, 0^n) = n,$ we could find a query path with $\Omega(n)$ length by always answer 0.

We need a new **Projection Lemma:**

1) Projection condition: $e_i \notin \mathrm{g}^n(X, Y\ )$

2) Potential function: density of 0^n

3) Keep $\mathrm{g}(X_i, Y_i) = 0$

Key Observation: Our projection can be delayed

Projection Lemma

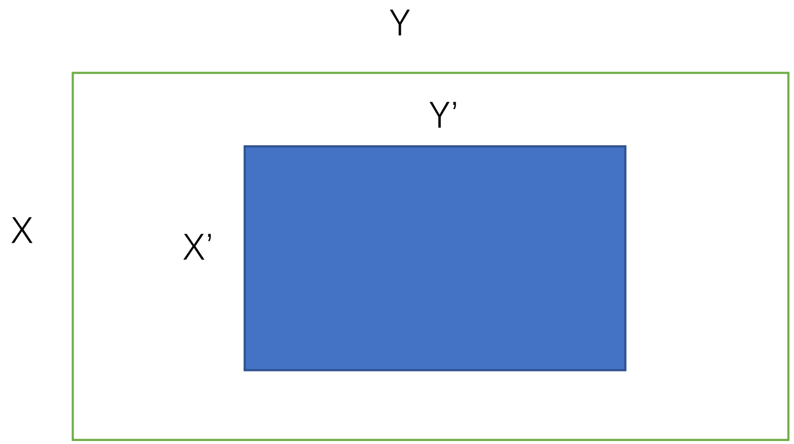
WLOG Assume $bs(f, 0^n) = n$,

1) Potential function: density of 0^n We define $D_0^I \coloneqq \{(x, y) \in [q]^I \times [q]^I : g^I(x, y) = 0^I\}$.

For each $I \subseteq [n]$ and $R = X \times Y \subseteq [q]^I \times [q]^I$. we define it's potential function as

$$E(R) := \log \frac{|R \cap D_0^I|}{|D_0^I|}$$

2) Projection condition: if $e_i \notin g^n(X, Y)$, we do projection on i and keep $g(X_i, Y_i) = 0$



Do projection on Alice' s side: Find a $u \in [q]$

$$X' = \{x \in X : x_i = u\} \quad and \quad Y' = \{y \in Y : g(u, y_i) = 0\}$$

Do projection on Bob' s side: Find a $v \in [q]$

$$Y' = \{y \in Y : y_i = v\} \quad and \quad X' = \{x \in X : g(x\_i, v) = 0\}$$

Potential function argument

Assume $bs(f, 0^n) = n$, $\quad E(R) := \log \frac{|R \cap D_0^I|}{|D_0^I|}$

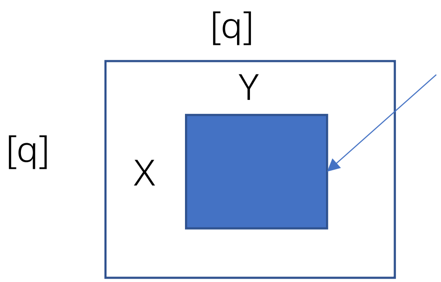
By average argument,

potential function decrease by at least O(1) in each "communication iteration"

if $e_i \notin \mathrm{g}^n(X, Y)$, we do projection on i and keep $\mathrm{g}(X_i, Y_i) = 0$

potential function increase by at least Ω(log q) in each "query iteration"

Low-discrepancy gadget [CFKMP19] **:** Fix $(x_{I\setminus i}, y_{I\setminus i})$ with $\mathrm{g}^{I\setminus i}(x_{I\setminus i}, y_{I\setminus i}) = 0^{\mathrm{I}\setminus i}$, If $e_i \notin \mathrm{g}^I(X_I, Y_I)$ then $|X_i||Y_i| \leq q$



$|\mathrm{X}||\mathrm{Y}| \leq q$ When $1 \notin \mathrm{g}(X, Y)$

By average argument, either we could do projection on Alice' side or Bob' s side to increase potential function.

Reference

[RM99] Separation of the monotone nc hierarchy.

[Zha09] On the Tightness of the Buhrman-Cleve-Wigderson Simulation

[HN12] On the virtue of succinct proofs: amplifying communication complexity hardness to time–space trade-offs in proof complexity.

[GP14] Communication Lower Bounds via Critical Block Sensitivity

[GPW15] Deterministic Communication vs. Partition Number

[CFKMP19] Query-to-communication lifting using low-discrepancy gadgets