A Decision Tree Based Analysis of communication complexity for FORK Game

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1 Preliminaries

Definition 1.1 (FORK game)**.** *Define a relation* $\boldsymbol{FORK}_{w,l} \subseteq [w]^l \times [w]^l \times [l] \cup \{0\}$*, we implicitly pad* $x \in [w]^l$ *and* $y \in [w]^l$ *with additional* 0 *and* $l+1$ *positions such that* $x_0 = y_0 = 1$*,* $x_{l+1} \neq y_{l+1}$*. This ensures that* $\forall x, y \in [w]^l, \exists i \in \{0, ..., l\}$ *such that* $(x, y, i) \in \boldsymbol{FORK}_{w,l}$ *where* $(x, y, i) \in \boldsymbol{FORK}_{w,l} \iff (x_i = y_i$ *and* $x_{i+1} \neq y_{i+1})$*.*

Definition 1.2 (Switch search problem)**.** *Define the "Switch" relation:*

$$\boldsymbol{Switch}_l = \{(z, i) \in \{0,1\}^l \times \{0, 1, ..., l\} | z_i = 1, z_{i+1} = 0\}$$

where we use $z_0 = 1$ *and* $z_{l+1} = 0$*, i.e., we are given* l *bits and wish to find a "switching point", a position* i *where a* 1*-bit flips into a* 0*-bit. If* $z = 0^l$ *we must output* $i = 0$ *and if* $z = 1^l$ *we must output* $i = l$*.*

We note that the FORK game $\mathbf{FORK}_{w,l} = \mathbf{Switch}_l \circ \mathbf{Eq}_w$ where $z_i = \mathbf{Eq}(x_i, y_i)$.

2 Communication complexity of FORK game

2.1 Query complexity of switch

Let $z \in \{0, 1, *\}^l$ be string. we call i active coordinate if $z_i = *$. Let $I \in [l]$ be the set of active coordinates. and I_0 be the the first half coordinates of I and I_1 be the last half coordinates of I.

Claim 2.1. $\mathbf{P}^{\mathrm{dt}}(\boldsymbol{Switch}) = \log l$.

Proof. Let T be a decision tree for **Switch**, we show there is a path of length $\Omega(\log l)$ in T. Starting from the root of T, we find a path step by step. In each step t, we reach a node v_t in T of depth t. At the beginning, we set $z = *^l$ and $|I| = l$

Find long path in decision tree:

1. If the decision query a coordinate $i \in I_0$, we fix $z_i = 1$ for all coordinates in I_0 and $I = I_1$.
2. If the decision query a coordinate $i \in I_1$, we fix $z_i = 0$ for all coordinates in I_1 and $I = I_0$.
3. Let $v_{t+1} = v_{t,z_i}$.

In each query round, we fix at most $|I|/2$ coordinates. At the end, $|I| \leq 1$, the query rounds at least $\log l$. ☐

Let S be the set of long query paths we could find in above proof. In next section, we will show that we could find a long communication path which is consistent with a long query path in S.

2.2 Communication complexity of FORK game

Let $z \in \{0,1,*\}^n$ be string. we call i unfixed coordinate if $z_i = *$. Let I be the set of unfixed coordinates and I_0 be the the first half coordinates of I and I_1 be the last half coordinates of I.

We get z^1 by set $z^1_i = z_i$ if $z_i \in \{0,1\}$ and $z^1_i = 1$ if $z_i = *$.

We get z^0 by set $z^0_i = z_i$ if $z_i \in \{0,1\}$, $z^0_i = 1$ if $i \in I_0$ and $z_i = *$ and $z^0_i = 0$ if $i \in I_1$ and $z_i = *$.

Definition 2.2 (Potential function)**.** *Let $X \times Y \subseteq [w]^l \times [w]^l$ be a rectangle and z be a string. The potential function is defined by*

$$\alpha(X,Y,z) = \frac{|\{(x_I, y_I) : (x,y) \in X \times Y, \boldsymbol{Eq}_w(x,y) = z^1\}|}{w^I}$$

Claim 2.3. *Let $X \times Y$ be a rectangle and $I \subseteq [n]$ be the unfixed coordinates, then for any partition of $Y = Y_0 \cup Y_1$, there is a Y_b such that $\alpha(X, Y_b) \geq \alpha(\mathcal{X}, \mathcal{Y})/2$.*

Claim 2.4. *Let $X \times Y$ be a rectangle and $I \subseteq [n]$ be the unfixed coordinates, then for any partition of $X = X_0 \cup X_1$, there is a X_b such that $\alpha(X_b, Y) \geq \alpha(\mathcal{X}, \mathcal{Y})/2$.*

Lemma 2.5 (Disperser property)**.** *Let $X \times Y$ be a rectangle and I be the set of unfixed coordinates, If $\alpha(X,Y,z) \geq \frac{16}{w}$, then*

$$\{\boldsymbol{Eq}^I_w(x,y) : x \in X, y \in Y\} = \{0,1\}^I$$

Proof. We do an induction on $|I| = l$. The base case of $l = 1$ follows from equality function. For the induction step, assume the lemma holds for all integers at most $l-1$. We need prove that for any string $z \in \{0,1\}^l$, $z \in \{\mathbf{Eq}^l_w(x,y) : x \in X, y \in Y\}$ when $\alpha(X,Y,z) \geq \frac{16}{w}$.

let $I_0 \subseteq I$ be the set of coordinates that $z_i = 0$ and $I_1 \subseteq I$ be the set of coordinates that $z_i = 1$. If $|I_0| \leq l-1$, by average argument, there is a sub-rectangle $X' \times Y'$ and $q^{I_1} \in [w]^{|I_1|}$ with $X_{I_1} = Y_{I_1} = q^{I_1}$ such that I_0 be the set of unfixed coordinates and $\alpha(X',Y',z) \geq \frac{16}{w}$. Since the lemma holds for all integers at most $l-1$, $z \in \{\mathbf{Eq}^l_w(x,y) : x \in X, y \in Y\}$.

If $|I_0| = l$, we prove $0^l \in \{\mathbf{Eq}^l_w(x,y) : x \in X, y \in Y\}$ by probabilistic method. Let $A = A_1 \times ...A_l$ where each A_i is subset of $[w]$ with size $w/2$. if we ensure that exist a set A such that there is a $x \in X \cap A$ and $y \in Y \cap A^c$ for $\mathbf{Eq}^l_w(x,y) = 0^l$ then the lemma immediately holds.

So it remains to show that there exist such set A. We choose A_is as follows: first choose at random $w/2$ strings $v^1, ..., v^{w/2}$ each of length l. Then we define A_i to include the i-th letter in each of these $w/2$ strings and extend it into a set of size $w/2$ randomly((Note that the resulting sets $A_1, ...A_l$ are indeed random and independent.). For any $x \in A$, $\Pr[x \in X] \geq \alpha$ and any $y \in A^c$, $\Pr[y \in Y] \geq \alpha$. We call A X-good if $X \cap A \neq \emptyset$ and Y-good if $Y \cap A^c \neq \emptyset$. We note that if there is a A both X-good and Y-good the lemma hold. We could prove that:

$$\Pr[A \text{ is not } X\text{-good }] \leq (1-\alpha)^{w/2} \leq e^{-\alpha w/2}$$

and

$$\Pr[A \text{ is not } Y\text{-good }] \leq (1-\alpha)^{w/2} \leq e^{-\alpha w/2}$$

Therefore, the probability that either A is not X-good or A is not Y-good is at most $2e^{-\alpha w/2} < 1$. There is a A both X-good and Y-good, the lemma holds. ☐

Lemma 2.6 (Projection lemma)**.** *[GS95] Let $X \times Y$ be a rectangle and I be the set of unfixed coordinates, If $\alpha(X,Y,z) \geq 4w^{-2/3}$ then there is a $b \in \{0,1\}$, we could fix $z_i = b$ for all coordinates in I_{1-b} to get a sub-rectangle $X' \times Y'$ and $I = I_b$ such that $\alpha(X',Y',z) \geq w^{-1/3}$.*

Theorem 2.7. $\mathbf{P}^{cc}(\boldsymbol{FORK}_{w,l}) = \Omega(\log l \log w)$.

Proof. Let T be a protocol tree for $(S \circ \mathrm{Eq}^l_w)$, we show there is a path of length $\Omega(\log l \log w)$ in T. Starting from the root of T, we find a path step by step. In each step t, we reach a node v_t in T of depth t. At the beginning, We set $z = *^n$. Let (X_t, Y_t) be a rectangle which consistent with current node v_t and z. At the beginning, $\alpha(X,Y,z) = 1$.

Find long path in protocol tree:

1. In each communication round, If it is Bob's turn in the protocol, then Bob sends a bit $b \in \{0,1\}$ that maximizes the potential function $\alpha(X_t, Y_t)$. Set $\mathcal{X}_t = \mathcal{X}_{t-1}, \mathcal{Y}_t = \mathcal{Y}_{t-1,b}$. If it is Alice's turn in the protocol, then Bob sends a bit $b \in \{0,1\}$ that maximizes the potential function $\alpha(X_t, Y_t)$. Set $\mathcal{X}_t = \mathcal{X}_{t-1,b}, \mathcal{Y}_t = \mathcal{Y}_{t-1}$.

2. If $8w^{-2/3} \geq \alpha(X, Y, z) \geq 4w^{-2/3}$, we do projection. By projection lemma, there is a b such that we could fix $z_i = b$ for all coordinates in I_{1-b} to get a sub-rectangle (X_{t+1}, Y_{t+1}) and $\alpha(X_{t+1}, Y_{t+1}, z) \geq w^{-1/3}$. Set $I = I_b$.

By disperser property, we know that we would do projection at least $\log l$ times and by Claim 2.3 and Claim 2.4, the communication rounds between two projection is at least $\log(\frac{1}{4}w^{1/3})$. As a result, the communication complexity of FORK game is $\Omega(\log l \log w)$. □

3 Open problems

Problem 3.1 (Direct sum of FORK game by Or Meir)**.** *The m-fold direct sum of FORK relation on l bits,denoted $U_l^{\otimes m}$ is the communication problem in which Alice and Bob get matrices $X, Y \in [w]^{m \times l}$ that each row is a instance of FORK relation. They are required to output a tuple $(j_1, ..., j_m) \in [l]^m$ such that for every row $i \in [m]$ holds that $\boldsymbol{Eq}(x_{i,j_i}, y_{i,j_i}) \neq \boldsymbol{Eq}_w(x_{i,j_i+1}, y_{i,j_i+1})$.*

Remark: We note that when $w = 2$, $\mathbf{P}^{\mathrm{cc}}(U_l^{\otimes m}) = \Omega(m \log l)$. [Mei18] prove the Direct Sum of Universal Relations by Raz-McKenzie simulation and Rank argument.

Definition 3.2. *Given a deterministic decision-tree T over $\{0,1\}^l$, the 0-depth of T is the maximum number of queries which are answered 0, in any root-to-leaf path of T. The 0-query complexity of f, denoted $\mathbf{P}_0^{\mathrm{dt}}(f)$,to be the smallest 0-depth of T, taken over deterministic decision-trees T which solve the search problem associated with f.*

Theorem 3.3 (by [LM19])**.** *For any Boolean relation $f : \{0,1\}^l \times \mathcal{C}$, whenever $n \geq \cdot \log p$*

$$\mathbf{P}^{\mathrm{cc}}(f \circ \mathrm{Eq}_n^q) = \Omega(n \cdot \mathbf{P}_0^{\mathrm{dt}}(f))$$

Problem 3.4. *Improving the size of the gadget for the simulation theorems of [LM19]*

Lemma 3.5 (Disperser property of index gadget)**.** *[LMZ20] Let $X \times Y$ be a rectangle and $q \geq \log n$. Assume that X is $\log n$-spread and $Y \geq \frac{2^{qn}}{2^q}$ then*

$$\{\boldsymbol{Ind}(x, y) : x \in X, y \in Y\} = \{0,1\}^n$$

Problem 3.6. *If we could change the condition of projection in [LMZ20] to prove a P lifting theorem with $\log n$ gadget size ?*

Problem 3.7. *In [dRMN+20], They prove communication lower bound of $\boldsymbol{Search}(Peb_G) \circ \mathrm{Eq}^n$. Could we prove the same lower bound by disperser property of equality gadget and path simulation ?*

References

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