Mathematical Analysis

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Preface

MATHEMATICAL Analysis is essentially a more rigorous and theoretical exploration of the concepts introduced in Calculus. It delves deeply into functions, sequences, limits, continuity, differentiation, and integration.

We study Mathematical Analysis because it provides a rigorous framework for understanding other mathematical disciplines, such as differential equations, topology, functional analysis, and more. Additionally, learning Mathematical Analysis enhances the ability to think critically, construct proofs, and work with abstract concepts, which are crucial in mathematics.

Acknowledgment

To be written ......

Contents

Chapter 1

Truth Table and Proof by Contradiction

1.1 Truth Table

We use the symbol $\neg$ to denote logical negation ("not"). We use the symbol $\wedge$ to denote logical conjunction ("and"). We use the symbol $\vee$ to denote the inclusive logical "or," meaning the proposition $A \vee B$ is true if at least one of the propositions A or B is true. We use the symbol $\implies$ to denote logical implication ("implies"). In addition, we let P and Q be propositions in this chapter.

Example 1.1.1

The conditional statement $P \implies Q$ means if P is true then Q **must** be true.

Solution:

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1.1: Truth table of $P \implies Q$

$\square$

Note: The implication $P \implies Q$ is false only when P is true and Q is false, as this violates the conditional. When P is false, the implication is true regardless of Q's truth value, often referred to as being **vacuously true**. This is because a false premise cannot lead to a false implication, as the condition P is not satisfied. Another way to understand this is that having P false does not contradict that Q must be true if P is true.

Example 1.1.2

The conditional statement $P \wedge Q$ is true only when P and Q are both true.

Solution:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 1.2: Truth table of $P \wedge Q$

$\square$

Example 1.1.3

The truth tables of $P \vee Q$, $\neg P \vee Q$, and $\neg Q \implies \neg P$.

Solution:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 1.3: Truth table of $P \vee Q$

P	Q	$\neg P \vee Q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1.4: Truth table of $\neg P \vee Q$

P	Q	$\neg P$	$\neg Q$	$\neg Q \implies \neg P$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Table 1.5: Truth table of $\neg Q \implies \neg P$

$\square$

In examples 1.1.1 and 1.1.3, we have shown the following theorem.

Theorem 1.1.1

Suppose P and Q are propositions, we have

1. $P \implies Q$ is equivalent to $\neg P \vee Q$,
2. $P \implies Q$ is equivalent to $\neg Q \implies \neg P$.

Definition 1.1.1 ***Contrapositive and Converse***

We call the proposition $\neg Q \implies \neg P$ the **contrapositive** of the proposition $P \implies Q$. The proposition $Q \implies P$ is called the **converse** of the proposition $P \implies Q$.

Proposition 1.1.2

Suppose P is a proposition. Then $P \iff \neg(\neg P)$.

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

Truth table of $\neg(\neg P)$

The next proposition shows that $\wedge$ and $\vee$ are “associative”.

Proposition 1.1.3

Suppose P, Q, R are propositions.

1. $(P \wedge Q) \wedge R \iff P \wedge (Q \wedge R)$,
2. $(P \vee Q) \vee R \iff P \vee (Q \vee R)$.

One can show the following useful equivalences.

Proposition 1.1.4
Suppose P, Q, R are propositions. Then:

1. $P \wedge Q \wedge R \iff (P \wedge Q) \wedge (P \wedge R)$,
2. $P \vee Q \vee R \iff (P \vee Q) \vee (P \vee R)$.

Proposition 1.1.5
Suppose P, Q, R are propositions.

1. $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$,
2. $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$.

Theorem 1.1.6
Suppose P, Q, R are propositions.

1. $\neg(P \wedge Q) \iff \neg P \vee \neg Q$,
2. $\neg(P \vee Q) \iff \neg P \wedge \neg Q$,
3. $\neg(P \implies Q) \iff (P \wedge \neg Q)$.

Proposition 1.1.7
Suppose P, Q, R are propositions. Then $P \vee Q \iff \neg P \implies Q$.

1.2 Proof by Contradiction

The proof by contradiction is as follows. Suppose R and S are propositions, where we aim to prove R true and know S is false. We construct the following truth table for $\neg R \implies S$:

S	R	$\neg R$	$\neg R \implies S$
F	T	F	T

Truth table of "Proof by Contradiction"

If we prove that $\neg R \implies S$ is true, then $\neg R$ is false, so R is true. To make sense of this, see Example 1.1.1.

Example 1.2.1
Suppose $x, y \in \mathbb{Z}^+$ with $x + y < 99$. We claim that $x < 50$ or $y < 50$.

Proof:
For the sake of contradiction, suppose $\neg[(x < 50) \vee (y < 50)]$; that is, suppose $x \geq 50$ and $y \geq 50$. Then $x + y \geq 100$, contradicting the assumption that $x + y < 99$. Hence, we must have $(x < 50) \vee (y < 50)$. $\square$

Note: In the example, $\neg R$ is $\neg[(x < 50) \vee (y < 50)]$, and S is $x + y \geq 99$. We know S is false because it contradicts the given condition $x + y < 99$. Since $\neg R \implies S$ is true, $\neg R$ must be false. Thus, R is true.

Proposition 1.2.1
For propositions P and Q, we have $[\neg(P \implies Q) \implies \neg P] \iff (P \implies Q)$.

Proposition 1.2.1 provides an alternative approach for proving statements by contradiction: Suppose we aim to prove $P \implies Q$. For contradiction, assume $\neg(P \implies Q)$, which is equivalent to $\neg(\neg P \vee Q)$. Thus, we assume $P \wedge \neg Q$. If we can show that $P \wedge \neg Q \implies \neg P$, then, since this leads to a contradiction (i.e., $P \wedge \neg P$), it follows that $P \wedge \neg Q$ is false. Consequently, $P \implies Q$ is true.

1.3 Exercises

Question 1.3.1. Write the truth table for $P \implies Q$.

Question 1.3.2. Prove from Proposition 1.1.2 to Proposition 1.1.5, and Theorem 1.1.6.

Question 1.3.3. Give two different examples of proof by contradiction with distinct logic shown in Section 1.2.

Chapter 2

Sets and Mappings

2.1 Sets

Definition 2.1.1 ***Set***
A **set** refers to a group of specific or abstract objects that share certain properties.

Definition 2.1.2 ***Elements***
The objects within a set are called the **elements** of the set.

Note: A **set** is typically denoted by a capital letter, such as S, while an **element** of the set is denoted by a lowercase letter, such as s. If x is an element of a set X, then we write $x \in X$. Here, the symbol $\in$ means "is an element of" and is read as "in". If x is not an element of a set X, then we write $x \notin X$. Here, the symbol $\notin$ means "is not an element of" and is read as "not in".

Example 2.1.1
Lets see some common examples of sets:

- The set of integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$.
- The set of rational numbers: $\mathbb{Q} = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\right\}$.
- The set of real numbers: $\mathbb{R}$.
- The set of complex numbers: $\mathbb{C} = \left\{a + b\sqrt{-i} \mid a, b \in \mathbb{R}\right\}$.
- The empty set: $\emptyset$. There is no element in the set.

We uses two set representations:

Roster Method, which lists all the elements of the set explicitly, i.e.,

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\},$$

and **Set-Builder Notation**, which defines the set by describing the properties or rules that its elements satisfy, i.e.,

$$\mathbb{C} = \left\{a + b\sqrt{-i} \mid a, b \in \mathbb{R}\right\}.$$

Note:

1. There is no order relationship in the representation of a set. That is $\{x, y\} = \{y, x\}$.
2. Repetition of elements in a set is meaningless. That is $\{x, y\} = \{y, x\} = \{x, y, x\} = \{x, y, x, x\}$.

Definition 2.1.3 ***Subset***

Suppose X and Y are two sets, if every element of X is an element of Y, that is

$$x \in X \Rightarrow x \in Y,$$

then we say X is a **subset** of Y and we denote this as $X \subseteq Y$.

Note: The $\Rightarrow$ symbol means "imply" and reads "if ... then ...".

Example 2.1.2

Suppose we have a set $T = \{1, 2, 3\}$, then T has 2^3 subsets:

$$\begin{aligned}&\emptyset;\\&\{1\}, \{2\}, \{3\};\\&\{1, 2\}, \{1, 3\}, \{2, 3\};\\&\{1, 2, 3\}.\end{aligned}$$

If there exists at least one $x \in X$ such that $x \notin Y$, then X is not a subset of Y. We denote it as $X \nsubseteq Y$.

Definition 2.1.4 ***Proper subset***

If X is a subset of Y, but $X \neq Y$, then we say X is a **proper subset** of Y and we denote it as

$$X \subset Y.$$

Example 2.1.3

$$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Example 2.1.4

If a set $T = \left\{t_1, t_2, t_3, \ldots, t_n\right\}$, then set T has 2^n subsets (including the empty set and the set itself) and $2^n - 1$ proper subsets.

If all the elements of X and Y are the same, then we say X and Y are equal, denoted as $X = Y$. That is

$$X = Y \iff X \subseteq Y \quad \text{and} \quad Y \subseteq X.$$

Note: The $\iff$ symbol means "if and only if" and is read as "is equivalent to" or "exactly when".

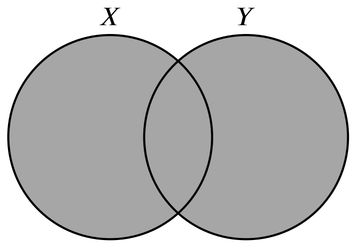
2.1.1 Set Operations

We will introduce five common set operations, including: Union ($\cup$), Intersection ($\cap$), Difference ($\setminus$), Complement (^{c}), and Cartesian Product ($\times$).

Suppose we have two sets X and Y:

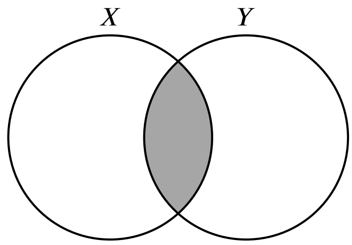
1. The **union** of two sets X and Y refers to the combination of all elements from both sets.

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$$



2. The **intersection** of two sets X and Y refers to the elements that are common to both sets.

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$$



The union and intersection of sets satisfy certain operational laws:

a. Commutative Law:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

b. Associative Law:

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

c. Distributive Law:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Example 2.1.5

For sets A, B, and C, prove the distributive law that is to show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof:

To show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, first we need to show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Second we need to show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.
Step 1: Show $x \in A \cap (B \cup C) \Rightarrow x \in (A \cap B) \cup (A \cap C) \iff A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

$$x \in A \cap (B \cup C) \Rightarrow$$
$$x \in A \text{ and } x \in (B \cup C) \Rightarrow$$
$$x \in A \text{ and } (x \in B \text{ or } x \in C) \Rightarrow \text{x in A for sure (no graph)}$$
$$(x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \Rightarrow \text{think it as two possibilities}$$
$$\text{(x in A and x in B) or (x in A and x in C)} \Rightarrow (A \cap B) \cup (A \cap C)$$

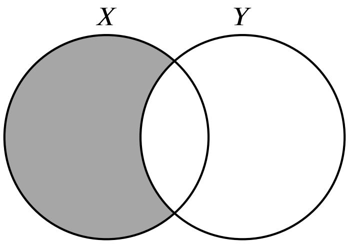
Therefore, we have $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.
Step 2: Show $x \in (A \cap B) \cup (A \cap C) \Rightarrow x \in A \cap (B \cup C) \iff (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

$$x \in (A \cap B) \cup (A \cap C) \Rightarrow$$
$$\text{(x in A and x in B) or (x in A and x in C)} \Rightarrow \text{graph}$$
$$\text{x in A and (x in B or x in C)} \Rightarrow x \in A \cap (B \cup C)$$

Therefore, we have $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$
Thus, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

3. The **difference** of two sets X and Y refers to the set of elements that are in X but not in Y. That is

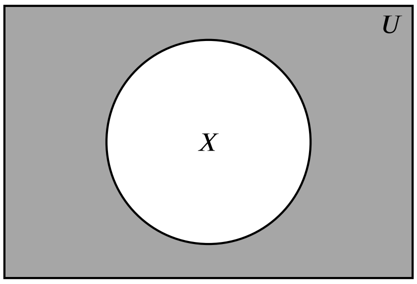
$$X \backslash Y = \{x \mid x \in X \text{ and } x \notin Y\}.$$



Note: here X does not need to be a subset of Y. For example, if $X = \{1, 2, 3\}$ and $Y = \{3, 4, 5\}$, then $X \backslash Y = \{1, 2\}$.

4. The **complement** of a set X is the set of all elements in the universal set U that are not in X. That is

$$X^c = \{x \mid x \in U \text{ and } x \notin X\} = U \backslash X.$$



De Morgan Rule:

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

2.1.2 Finite and Infinite Sets

Definition 2.1.5

If a set X consists of n elements, where n is a fixed non-negative integer, then X is called a **finite set**. A set that is not finite is called an **infinite set**.

Example 2.1.6

For example, $X = \{1, 2, 3\}, Y = \{y_1, y_2, y_3, \dots, y_{99}\}, \emptyset$ are finite sets, , while, $\mathbb{Q}, \mathbb{R}$ are infinite sets.

Definition 2.1.6

An infinite set is called a **countable set** if its elements can be arranged in a sequence according to some rule.

Example 2.1.7

For example, $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\{x \mid x = n\pi, n \in \mathbb{N}\}$ are countable sets.

Note: It is obvious that any infinite set must contain a countable subset (proof required), but an infinite set is not necessarily countable. (In later course, we will show that $\mathbb{R}$ is an uncountable set.)

Clearly, to prove that an infinite set is countable, the key is to design a rule for arranging the elements in such a way that all elements of the set can be listed without <u>repetition</u> or <u>omission</u>.

Example 2.1.8

Show that $\mathbb{Z}$ is a countable set.

Proof:

We can list $\mathbb{Z}$ as

$$\mathbb{Z} = \{0, \pm 1, \pm, 2, \pm 3, \dots, \pm n, \dots\}, \quad n \in \mathbb{N}.$$

Following this rule, we know that all elements in $\mathbb{Z}$ are listed without repetition or omission. Therefore $\mathbb{Z}$ is a countable set.

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

$\square$

Example 2.1.9

Suppose we have $n \in \mathbb{N}$ number of countable sets, A_n. We have an important countable set built from these A_n. That is

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots = \left\{x \mid \exists\, n \in \mathbb{N}^+, \text{ such that } x \in A_n\right\}$$

In other words, we want to prove that the union of countably many countable sets is also countable.

Theorem 2.1.1

The union of countably many countable sets is also countable.

Proof:

For any $n \in \mathbb{N}$, since A_n is a countable set, then W.O.L.G, we can list A_n as $A_n = \{a_{x1}, a_{x2}. \dots, a_{xk}, \dots\}$, for $k \in \mathbb{N}$. That is we have

$$\begin{array}{cccccccc} A_1 & : & a_{11} & a_{12} & a_{13} & a_{14} & \dots \\ A_2 & : & a_{21} & a_{22} & a_{23} & a_{24} & \dots \\ A_3 & : & a_{31} & a_{32} & a_{33} & a_{34} & \dots \\ A_4 & : & a_{41} & a_{42} & a_{43} & a_{44} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

There are various ways to list the elements in a sequence, one commonly used method is the **diagonalization method**. That is

$$\bigcup_{n=1}^{\infty} A_n = \left\{x_{11}, x_{12}, x_{21}, x_{13}, x_{22}, x_{31}, x_{14}, x_{23}, x_{32}, x_{41}, \dots\right\}.$$

By listing the elements this way, we ensure that there is no omission. For different sets A_i and A_j, with $i \neq j$, the intersection of A_i and A_j may not be the empty set. Therefore, we might have repetitions. We set the rule that if there is a repetition, we only include it once to ensure no duplicates. $\square$

Theorem 2.1.2

The set of rationals, $\mathbb{Q}$, is a countable set.

Proof:

The goal is to show that each element in $\mathbb{Q}$ is included and each element in $\mathbb{Q}$ is shown only once. We want to show that $(-\infty, \infty)$ is a composite of countably many, $(n, n+1]$, $n \in \mathbb{Z}$, small intervals. Therefore, we only need to show that all rational numbers in $(0, 1]$ is countable. Then we apply Theorem theorem 1.1.1 to complete the proof.

Each rational number in $\mathbb{Q}$ can be represented by $\frac{q}{p}$ with $p, q \in \mathbb{N}$, p and q are co-prime, and $q \leq p$.

For $p = 1$, we have $x_{11} = \frac{1}{1} = 1$

For $p = 2$, we have $x_{21} = \frac{1}{2}$

For $p = 3$, we have $x_{31} = \frac{1}{3}, x_{32} = \frac{2}{3}$

For $p = 4$, we have $x_{41} = \frac{1}{4}, x_{42} = \frac{3}{4}$ ensure no repetition

$\vdots$

For $p = n$, we have $x_{n1} = \frac{1}{n}, x_{n2}, \dots, x_{nk(n)}$ ensure no repetition

$\vdots$

Therefore, all rationals in $(0, 1]$ can be listed as $x_{11}, x_{21}, x_{31}, x_{32}, x_{41}, x_{42}, \dots, x_{n1}, x_{n2}, \dots, x_{nk(n)}$. This way we ensure that each element is included and each element is shown only once. $\square$

2.1.3 Cartesian Product of Sets

Definition 2.1.7

Let X and Y be two sets. For any element x chosen from set X and any element y chosen from set Y, the ordered pair (x, y) is formed. The collection of all such ordered pairs is called the **Cartesian product** of sets of X and Y. That is

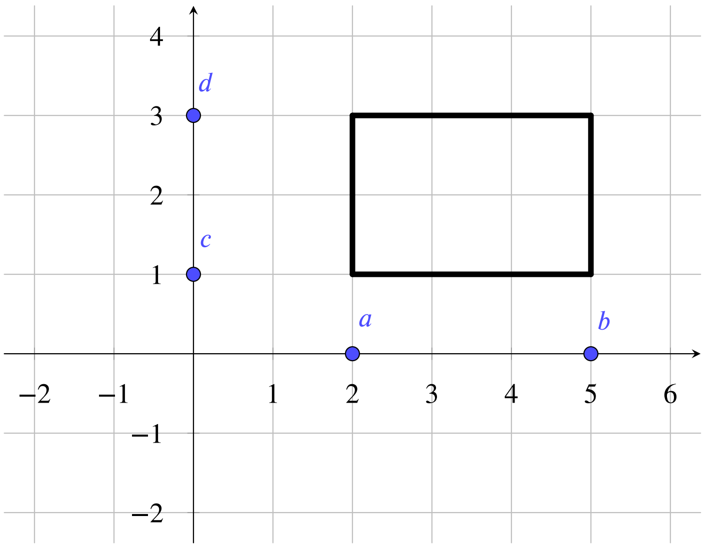
$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}.$$

Example 2.1.10

A special case is when $A = B = \mathbb{R}$, in which case we have the **Cartesian coordinate system in two dimensions**, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. The **Cartesian coordinate system in three dimensions** is $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Example 2.1.11

Say we have $A = \{x \mid a \leq x \leq b\}$, $B = \{y \mid c \leq y \leq d\}$, and $C = \{z \mid e \leq z \leq f\}$, then $A \times B$ is composite of all points in a rectangle.



It is easy to know that $A \times B \times C$ are all points in a cuboid.

2.1.4 Exercises

Question 2.1.1. Prove that the set $T = \left\{t_1, t_2, \dots, t_n\right\}$ has 2^n subsets.

Proof:

We prove the statement by listing subsets with $0, 1, \dots, n$ elements and add them up.

There is ${}^nC_0$ subset with 0 element: $\emptyset$,

There are ${}^nC_1$ subsets with 1 elements: $\{t_1\}, \{t_2\}, \dots, \{t_n\}$,

There are ${}^nC_2$ subsets with 2 elements,

…

There are ${}^nC_k$ subsets with k elements, with $k \in \mathbb{N}$,

…

There is ${}^nC_n$ subset with n elements: T.

Therefore the total number of subsets is ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_k + \cdots + {}^nC_n = \sum_{k=0}^{n} \binom{n}{k}$. Recall that the binomial theorem states that

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

It is obvious that by setting $a = b = 1$ we have that $2^n = \sum_{k=0}^{n} \binom{n}{k}$. □

Question 2.1.2. Prove:

1. Any infinite set must have a countable subset.

2. Assume two sets A and B are countable sets, show that $A \cup B$ is a countable set.

Proof:

For 1, □

2.2 Mappings

2.2.1 Mappings

A **mapping** refers to the correspondence between sets.

Definition 2.2.1

Let X and Y be two given sets. If there exists a rule f such that for every element x in the set X, a uniquely determined element y in the set Y can be found corresponding to it, then this rule f is called a **mapping** from set X to set Y. It is denoted as

$$f : X \to Y;$$
$$x \mapsto y = f(x).$$

Definition 2.2.2

If f is a mapping from X to Y. then y is called the **image** of x under the mapping f, and x is called the **preimage** (or **inverse image**) of y under the mapping f.

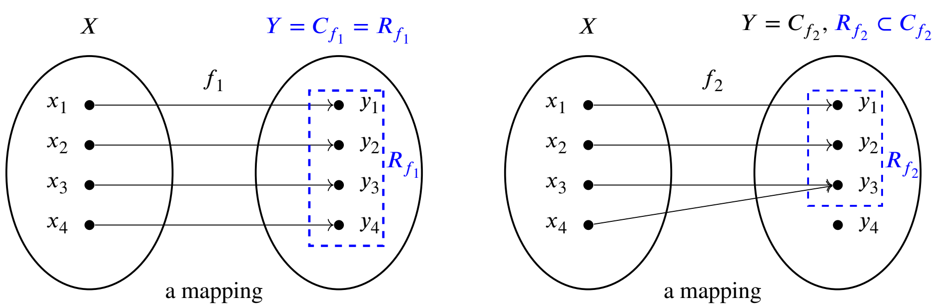
Definition 2.2.3

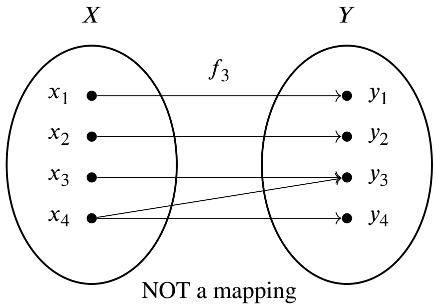
If f is a mapping from X to Y. The set X is called the **domain** of the mapping f, denoted as $D_f = X$, and the set of all images y of the elements x in X under the mapping f is called the **range** of the mapping f, denoted as $R_f = \{y \mid y \in Y \text{ and } y = f(x), \forall x \in X\}$, with $R_f \subseteq Y$. The set Y consists of the possible outcome values under the mapping f, so Y is called the **codomain** of the mapping f, denoted as C_f.

Note: Three key elements of a mapping:

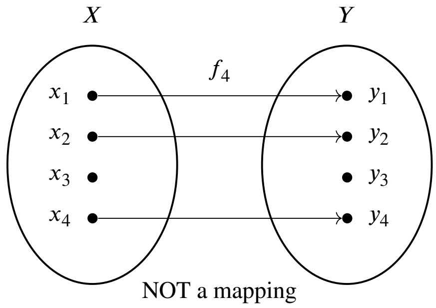
1. $X = D_f$;
2. $Y = C_f$;
3. $R_f \subseteq Y$;
4. f has the property: The **uniqueness of the image** and the **non-uniqueness of the preimage**. The **uniqueness of the image** means that in the set X, any element cannot have more than one outgoing arrow.

Example 2.2.1





NOT a mapping



NOT a mapping

In the figure above, f_1 and f_2 are mapping since the uniqueness of image, while f_3 is not a mapping because the uniqueness of image is violated. Specifically, x_4 is mapped to two different elements in the set Y. In addition, f_4 is not a mapping since one of the element, x_3, in the set X is not mapped to any element in the set Y.

Example 2.2.2

Let $X = Y = \mathbb{R}$ and $f : X \to Y$ with $f(x) = x^2$. It should be obvious that f is a mapping.

Example 2.2.3

Let $X = \mathbb{R}^+, Y = \mathbb{R}$ with $f : X \to Y, x \mapsto y : y^2 = x$. Then it is obvious that f is not a mapping, since when $x = 4$, $y = \pm 2$, meaning the image is not unique.

Example 2.2.4

Let $X = \mathbb{R}^+, Y = \mathbb{R}^-$ with $f : X \to Y, x \mapsto y : y^2 = x$. Then it is obvious that f is a mapping.

Definition 2.2.4

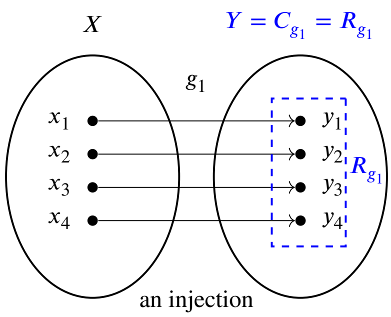
If f is a mapping from X to Y and f has the uniqueness of pre-image property, then we say f is an **injection** or f is **injective** (or **one-to-one**). To show f is injective, we only need to show:

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

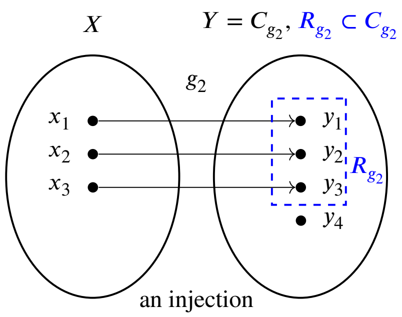
Note:

1. For a mapping to be qualified as an injection, it needs to have the uniqueness property for the preimage.
   The **uniqueness of the preimage** means that in the set Y, any element cannot have more than one ingoing arrow.
2. A mapping already has the uniqueness property for the image. So for a mapping to be a injection only uniqueness property for the preimage is needed.

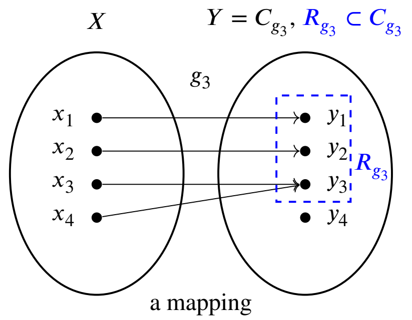
Example 2.2.5



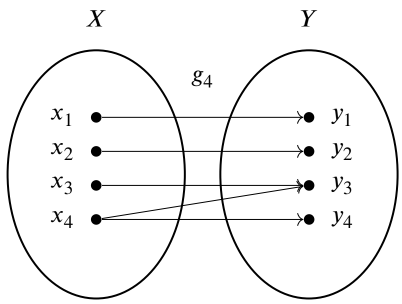
an injection



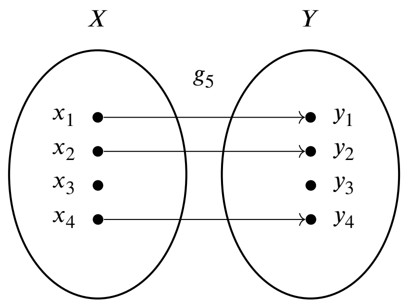
an injection



a mapping
NOT an injection



NOT a mapping
NOT an injection



NOT a mapping
NOT an injection

In the figures above, g_1 is an injection since both image and preimage has the uniqueness property, while g_3 is not an injection due to the violation of the uniqueness property of preimage. For g_4 and g_5, they are not even mappings, so they cannot be injections.

Definition 2.2.5

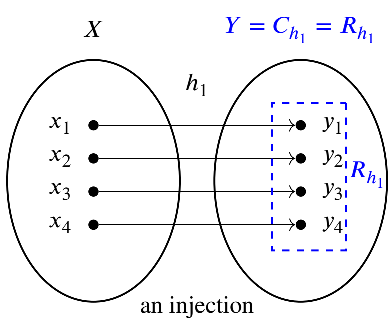
If f is a mapping from X to Y and f has the property, $R_f = Y$, then we say f is a **surjection** or f is **surjective**. A function f is **surjective** (or **onto**), if

$$\forall\, y \in Y, \exists\, x \in X \text{ such that } f(x) = y$$

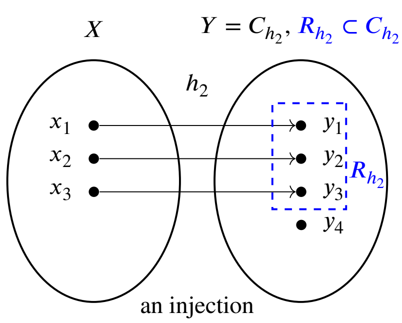
Note:

1. For a mapping to be qualified as a surjection, its range needs to be the same as its codomain.
2. For a mapping to be qualified as a surjection, the uniqueness of the preimage is NOT needed.

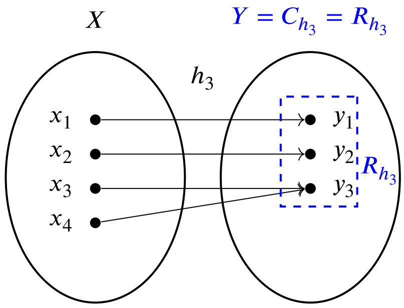
Example 2.2.6



an injection
a surjection



an injection
NOT a surjection



NOT an injection
a surjection

In the figures above, obviously h_1 is a surjection, while h_2 is not a surjection since one of the element, y_4, in the set Y is not an image of any element in the set X. Therefore, even h_2 is an injection, it is not a surjection. However, h_3 is a surjection even it is not an injection.

Definition 2.2.6

If a mapping f is both injective and surjective, then we say this mapping f is a **bijection** or f is **bijective** (or **One-to-One correspondence**).

Example 2.2.7

In the graph above, h_1 is a bijection.

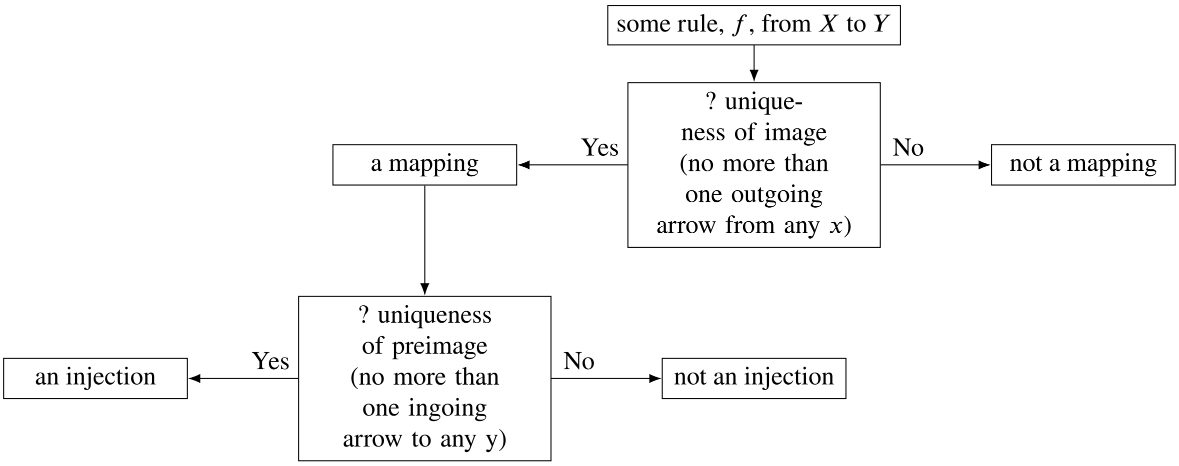


Figure 2.1: Injection Check

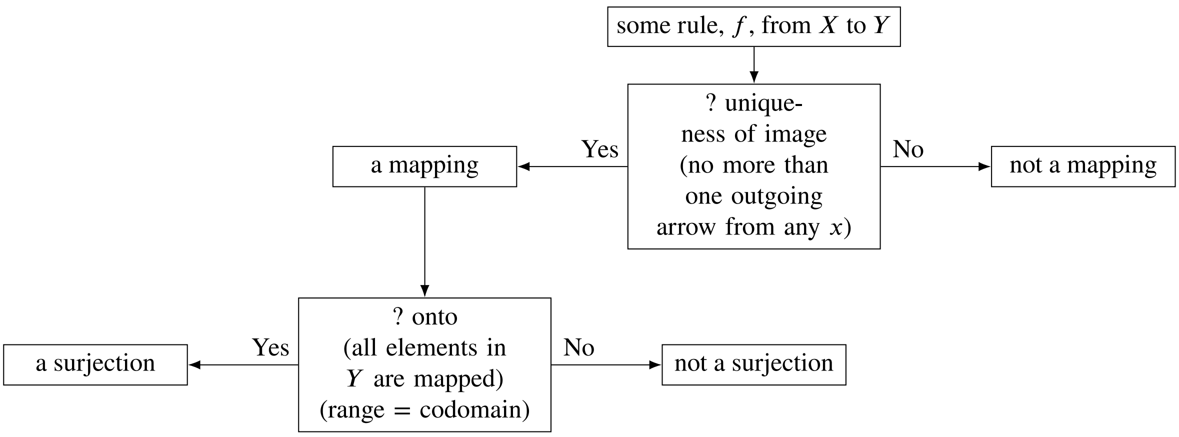


Figure 2.2: Surjection Check

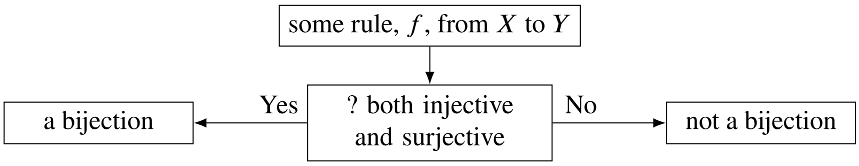


Figure 2.3: Surjection Check

2.2.2 Inverse Mappings

Definition 2.2.7

Let $f : X \to Y$ be an injection. For any $y \in R_f \subseteq Y$, the preimage $x \in X$ (that is, the unique x satisfying $f(x) = y$) is well-defined. Then the mapping $g : R_f \to X$, defined by $y \mapsto x$ such that $f(x) = y$, constitutes a mapping from R_f to X. This mapping g is called the **inverse mapping** of f, denoted by $g = f^{-1}$. Its domain is $D_{f^{-1}} = R_f$, and its range is $R_{f^{-1}} = X$.

$$
\begin{aligned}
g : R_f &\to X, \\
y &\mapsto x : f(x) = y.
\end{aligned}
$$

Note:

1. For an inverse mapping, its range equals its codomain.
2. Only an injective map has an inverse map.

Example 2.2.8

For $y = \sin x$, this is a mapping from $[-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$. It's inverse mapping $x = \arcsin y : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

Example 2.2.9

Kaoyan WangGe.

If we have a mapping f and its inverse mapping f^1:

$$
\begin{aligned}
f &: X \to Y \\
f^{-1} &: R_f \to X
\end{aligned}
$$

then we have $f^{-1} \circ f(x) = x, x \in X$ and $f \circ f^{-1}(y) = y, y \in R_f$.

2.2.3 Composition of Mappings

Definition 2.2.8

Suppose we have a mapping $g : X \to U_1, x \mapsto u : g(x) = u$ and another mapping $f : U_2 \to Y, u \mapsto y : f(u) = y$. If we have $R_g \subseteq U_2 = D_f$, then we have a composition of mappings of g and f, denoted as $f \circ g : X \to Y, x \mapsto y : f(g(x)) = y$.

Example 2.2.10

Suppose we have two mappings $g : \mathbb{R} \to \mathbb{R}, x \mapsto u : 1 - x^2 = u$ and $f : \mathbb{R}^+ \to \mathbb{R}, u \mapsto y : \log u = y$. We observe that $R_g = (-\infty, 1] \nsubseteq D_f$, therefore $f \circ g$ is not defined.

2.3 Functions

2.3.1 Functions

A function is a special case of mapping.

Definition 2.3.1

Suppose we have a mapping $f : X \to Y, x \mapsto y = f(x)$, if we have $X \subset \mathbb{R}$ and $Y = \mathbb{R}$ then we say the mapping f is a **real-valued function of one variable** or **function**. Since a function represents a correspondence between real number sets, we only need to write the function representation as

$$y = f(x), x \in X = D_f.$$

2.3.2 Elementary Functions

Elementary functions are composed of:

1. Linear functions: $y = ax + b, a, b \in \mathbb{R}$
2. Power functions: $y = x^p, p \in \mathbb{R}$
3. Exponential functions: $y = a^x, a > 0, a \neq 1$
4. Logarithmic functions: $y = \log_a x, a > 0, a \neq 1$
5. Trigonometric functions: $y = \sin x, \cos x, \ldots$
6. Inverse trigonometric functions: $y = \arcsin x, \arccos x, \ldots$

Functions are generated by performing a finite number of operations involving addition, subtraction, multiplication, division, and composition on elementary functions.

Example 2.3.1

For example, $y = a_0x^3 + a_1x^2 + a_2x + a_3$, with $a_1, a_2, a_3 \in \mathbb{R}$; $y = \frac{1+x}{\sqrt{x^2+1}}$; $y = \frac{\sin x}{x}$ are functions.

Definition 2.3.2

The **natural domain** of a function refers to the largest possible range of values for its independent variable.

Note: If we are given a function without a specified domain, we assume its domain is the natural domain.

2.3.3 Representation of Functions

There are various ways to represent a function, including Analytical Representation, Piecewise Representation, Implicit Representation, Parametric Representation, Graphical Representation, Numerical Representation, and Mapping Representation.

Definition 2.3.3

Analytical representation of a function refers to expressing the function using a mathematical formula or equation that explicitly defines the relationship between the input (independent variable) and the output (dependent variable).

$$y = f(x)$$

Definition 2.3.4

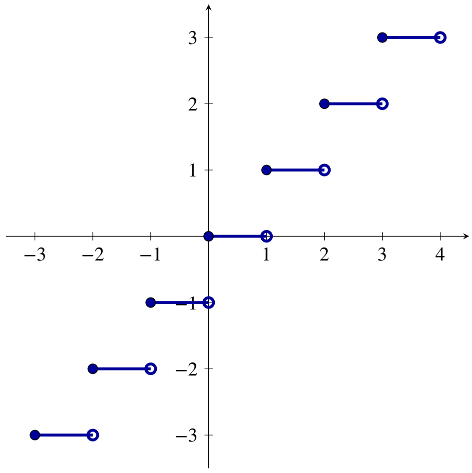
Piecewise representation of a function refers to expressing the function using different formulas or rules for different intervals of its domain.

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Example 2.3.2

The **floor function** $y = \lfloor x \rfloor$, which gives the greatest integer less than or equal to x.

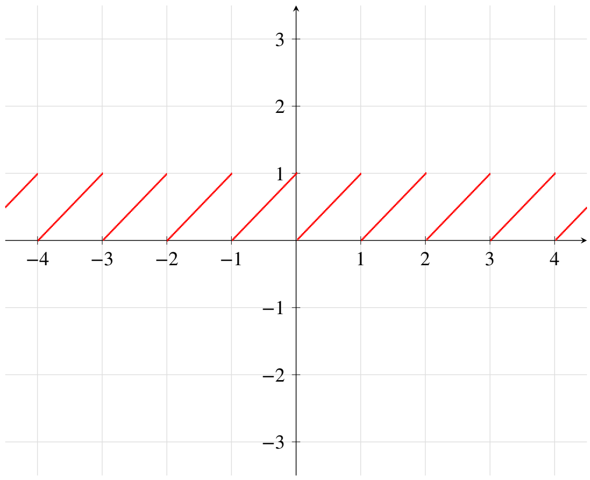
$$y = f(x) = \lfloor x \rfloor = n, \quad n \le x < n+1, \forall n \in \mathbb{Z}, \quad D_f = (-\infty, \infty), R_f = \mathbb{Z},$$



Example 2.3.3

The **non-negative fractional part function** gives the fractional part of a real number, but only the non-negative part. It can be expressed as:

$$y = f(x) = (x) = x - \lfloor x \rfloor, \quad D_f = (-\infty, \infty), R_f = [0, 1),$$



Definition 2.3.5

Implicit representation of a function refers to defining the relationship between the variables without explicitly solving for one variable in terms of the other.

$$\sin(xy) - x^2 + y = 1$$

Definition 2.3.6

Parametric representation of a function refers to expressing the dependent and independent variables of a function using one or more parameters.

$$x(t) = \cos t, \quad y(t) = \sin(t), \quad t \in [0, 2\pi]$$

Definition 2.3.7

Graphical representation of a function refers to visualizing the relationship between the independent variable and the dependent variable on a coordinate plane or graph.

Definition 2.3.8

Numerical representation of a function refers to expressing the function using a table of values that show specific input-output pairs.

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

Definition 2.3.9

Mapping representation of a function refers to describing the function as a relationship that assigns each element from the domain (input set) to a unique element in the codomain (output set).

$$f : \{1,2,3\} \to \{a,b,c\}, \quad f(1)=a, \quad f(2)=b, \quad f(3)=c$$

$$\text{Domain} \quad 1 \longrightarrow a, \quad 2 \longrightarrow b, \quad 3 \longrightarrow c \quad \text{(Codomain)}$$

2.3.4 Basics Properties of Functions

There are various properties of functions, including boundedness, monotonicity, parity, continuity, differentiability, periodicity, convexity and concavity, and etc.

1. **Boundedness**

   Definition 2.3.10

   A function f is **bounded** if and only if there exists $m < M$, $m, M \in \mathbb{R}$ such that $m \le f(x) \le M$, for all $x \in D_f$. Here, m is called a **lower bound** and M is called an **upper bound**.

   The following definition is equivalent to the Definition 2.3.10.

   Definition 2.3.11

   A function f is **bounded** if and only if $\exists\, X \in \mathbb{R}$ such that $|f(x)| \le X$, for all $x \in D_f$.

   Note: If a function f is bounded, its lower and upper bounds are non-unique.

2. **Monotonicity**

   Definition 2.3.12

   For any $x_1, x_2 \in D_f$, if we have $x_1 < x_2 \implies f(x_1) \le f(x_2)$ or $f(x_1) < f(x_2)$, then we say the function f is **monotonic increasing** or **strictly increasing**. We denoted it as $f(x) \uparrow$.

   Definition 2.3.13

   For any $x_1, x_2 \in D_f$, if we have $x_1 < x_2 \implies f(x_1) \ge f(x_2)$ or $f(x_1) > f(x_2)$, then we say the function f is **monotonic decreasing** or **strictly decreasing**. We denoted it as $f(x) \downarrow$.

Example 2.3.4
give some simple examples

3. **Parity**

Definition 2.3.14
A function f is called an **even function** if it satisfies the condition:

$$f(x) = f(-x), \forall x \in D_f.$$

Note: This means the function is symmetric with respect to the y-axis.

Definition 2.3.15
A function f is called an **odd function** if it satisfies the condition:

$$f(x) = -f(-x), \forall x \in D_f.$$

Note: This means the function has rotational symmetry about the origin (180−degree rotation).

Example 2.3.5
Give some simple examples.

4. **Periodicity**

Definition 2.3.16
A function f is said to be **periodic** if there exists a positive constant T such that:

$$f(x + T) = f(x), \forall x \in D_f.$$

The smallest, positive constant T is called the **period** of the function.

Example 2.3.6
give some simple examples.

Note: This means that the function repeats its values in regular intervals, and T is the smallest such interval where the function's values begin to repeat.

Example 2.3.7
Do all periodic functions have a smallest period?
The **Dirichlet function** is

$$D(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1, & \text{if } x \in \mathbb{Q}. \end{cases}$$

It should be obvious that any $T \in \mathbb{Q}^+$ makes the Dirichlet function a periodic function, so it is a period function without a period (smallest, positive T).

2.4 Some Useful Inequalities and Identities

Proposition 2.4.1

For all $x \in \mathbb{R}$, we have that

1. $|x| \geq 0$ with $|x| = 0$, iff $x = 0$.
2. $|xy| = |x||y|$.
3. $|x^2| = |x|^2 = x^2$.

Proposition 2.4.2

The Triangle Inequality. For any $a, b \in \mathbb{R}$, we have

$$\left||a| - |b|\right| \leq |a + b| \leq |a| + |b|.$$

Proof:

$$\begin{aligned} -2|a||b| &\leq 2ab \leq 2|a||b| \\ a^2 - 2|a||b| + b^2 &\leq a^2 + 2ab + b^2 \leq a^2 + 2|a||b| + b^2 \\ (|a| - |b|)^2 &\leq (a + b)^2 \leq (|a| + |b|)^2 \\ \left||a| - |b|\right| &\overset{(1)}{\leq} |a + b| \overset{(2)}{\leq} |a| + |b|. \end{aligned}$$

Here, the inequality (1) means that one side of the triangle is greater than the difference between the other two sides, and the inequality (2) means that the sum of two sides of a triangle is greater than the third side. □

The triangle inequality is used extensively in analysis.

Definition 2.4.1

The **Arithmetic Mean (AM)** is:

$$\frac{a_1 + a_2 + \cdots + a_n}{n}$$

Definition 2.4.2

The **Geometric Mean (GM)** is:

$$\sqrt[n]{a_1 a_2 \ldots a_n}$$

Definition 2.4.3

The **Harmonic Mean (HM)** is:

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

Proposition 2.4.3

AM-GM-HM Inequality:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \overset{(1)}{\geq} \sqrt[n]{a_1 a_2 \ldots a_n} \overset{(2)}{\geq} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

Proof:

To prove inequality (1). It is obvious that $n = 1, 2$ the inequality (1) holds. For $n = 4$, $\frac{a_1+a_2+a_3+a_4}{4} \geq \frac{2\cdot\sqrt{a_1a_2}+2\cdot\sqrt{a_3a_4}}{4} = \frac{\sqrt{a_1a_2}+\sqrt{a_3a_4}}{2} \overset{(3)}{\geq} \sqrt[4]{a_1a_2a_3a_4}$. In the inequality (3), we apply the AM $\geq$ GM for $n = 2$. In other words, we view $\sqrt{a_1a_2}$ and $\sqrt{a_3a_4}$ as b_1, b_2, respectively. Thus, we have shown that when $n = 2^k, k \in \mathbb{N}$, the inequality (1) holds. Now, if $n \neq 2^k$, then $\exists\, l \in \mathbb{N}$ s.t. $2^{l-1} < n < 2l$. Let $\sqrt[n]{a_1a_2a_3\dots a_n} = \bar{a}$, we composite the sequence below

$$a_1, a_2, a_3, \dots, a_n, \underbrace{\bar{a}, \bar{a}, \dots, \bar{a}}_{2^l - n \text{ number of } \bar{a}}.$$

Then, we have

$$\begin{aligned}
\frac{a_1 + a_2 + \cdots + a_n + (2^l - n)\bar{a}}{2^l} &\geq \sqrt[2^l]{a_1a_2\dots a_n\bar{a}\bar{a}\dots\bar{a}} \\
&= \sqrt[2^l]{\bar{a}^n \cdot \bar{a}^{2^l - n}} \\
&= \sqrt[2^l]{\bar{a}^{2^l}} \\
&= \bar{a}.
\end{aligned}$$

That is

$$\begin{aligned}
\frac{a_1 + a_2 + a_3 + \cdots + a_n + (2^l - n)\cdot\bar{a}}{2^l} &\geq \bar{a} \\
a_1 + a_2 + a_3 + \cdots + a_n + (2^l - n)\cdot\bar{a} &\geq 2^l\bar{a} \\
a_1 + a_2 + a_3 + \cdots + a_n &\geq n\bar{a} \\
\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} &\geq \bar{a} \\
\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} &\geq \sqrt[n]{a_1a_2a_3\dots a_n}.
\end{aligned}$$

This completes the proof of the inequality (1).

To prove inequality (2), we replace a_i, with $\frac{1}{a_i}$, for $i \in \{1, 2, 3, \dots, n\}$ and repeat the process above. □

Proposition 2.4.4

Pascal's Triangle Identity. For $n, k \in \mathbb{N}$ with $2 \leq k \leq n$ we have that

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Proof:

We have

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{\big(n-(k-1)\big)!(k-1)!} + \frac{n!}{(n-k)!k!} \\
&= \frac{n(n-1)\cdots\big(n-(k-1)+1\big)}{(k-1)!} + \frac{n(n-1)\cdots(n-k+1)}{k!} \\
&= \frac{n(n-1)\cdots(n-k+2)}{(k-1)!} + \frac{n(n-1)\cdots(n-k+1)}{k!} \\
&= \frac{kn(n-1)\cdots(n-k+2)}{k!} + \frac{n(n-1)\cdots(n-k+1)}{k!} \\
&= \frac{kn(n-1)\cdots(n-k+2)}{k!} + \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{k!} \\
&= \frac{n(n-1)\cdots(n-k+2)}{k!}\cdot\big(k + (n-k+1)\big) \\
&= \frac{n(n-1)\cdots(n-k+2)}{k!}\cdot(n+1) \\
&= \frac{(n+1)n(n-1)\cdots(n-k+2)}{k!} \\
&= \frac{(n+1)n(n-1)\cdots\big((n+1)-k+1\big)}{k!} \\
&= \frac{(n+1)!}{\big((n+1)-k\big)!k!} = \binom{n+1}{k}
\end{aligned}$$

□

Proposition 2.4.5

Binomial Theorem. For $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, we have that

$$(x+y)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} y^k.$$

Proof:

We prove the binomial theorem by Mathematical Induction.
Base case: $n = 0$, it is easy to see that $LHS = RHS$.
Hypothesis: As long as

$$(x+y)^m = \sum_{k=0}^{k=m} \binom{m}{k} x^{m-k} y^k, \quad \forall m \in \mathbb{N}, m \leq n-1,$$

then we prove that

$$(x+y)^{m+1} = \sum_{k=0}^{k=m+1} \binom{m+1}{k} x^{m+1-k} y^k$$

still holds.

$$\begin{aligned}
(x+y)^{m+1} &= (x+y)(x+y)^m \\
&= (x+y) \sum_{k=0}^{k=m} \binom{m}{k} x^{m-k} y^k. \\
&= \sum_{k=0}^{m} \binom{m}{k} x^{m-k+1} y^k + \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^{k+1} \\
&= \sum_{k=0}^{m} \binom{m}{k} x^{m-k+1} y^k + \sum_{k-1=0}^{k-1=m} \binom{m}{k-1} x^{m-(k-1)} y^{(k-1)+1} \\
&= \sum_{k=0}^{m} \binom{m}{k} x^{m-k+1} y^k + \sum_{k=1}^{k=m+1} \binom{m}{k-1} x^{m-k+1} y^k \\
&= x^{m+1} + \sum_{k=1}^{m} \binom{m}{k} x^{m-k+1} y^k + \sum_{k=1}^{k=m} \binom{m}{k-1} x^{m-k+1} y^k + y^{m+1} \\
&= x^{m+1} + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1} \right] x^{m-k+1} y^k + y^{m+1} \\
&= x^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} x^{m-k+1} y^k + y^{m+1} \\
&= \binom{m+1}{0} x^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} x^{m-k+1} y^k + \binom{m+1}{m+1} y^{m+1} \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} x^{m+1-k} y^k.
\end{aligned}$$

□

Proposition 2.4.6

Show that $0.\dot{9} = 0.999\cdots = 1$.

Proof:

We treat $0.999\ldots$ as the sum of a geometric series:

$$0.999\ldots = 0.9 + 0.09 + 0.009 + \ldots$$

This series has a sum given by:

$$S = \frac{a}{1-r} = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1.$$

□

Chapter 3

Limits of Sequences

Definition 3.0.1

A set S is **closed under an operation**, $\circ$, (e.g., addition, multiplication, etc.) if, for any $a, b \in S$, the result of $a \circ b$ is also in S.

Example 3.0.1

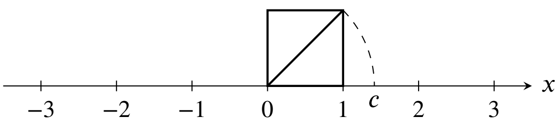
simple examples

3.1 Continuity of the Real Number System

The understanding of number systems by humanity has been a progressive journey. Around 3000 BCE, humans first recognized the natural number system $\mathbb{N}$, which is closed under addition and multiplication. By 2000 BCE, with the development of trade and bookkeeping, the need to represent deficits and zero arose. It was realized that the natural number system is not closed under subtraction, leading to its extension to the integer set $\mathbb{Z}$. The integers are closed under addition, multiplication, and subtraction, but not under division. Around 500 BCE, to meet the need for representing fractions, humanity extended the integers to the rational number set $\mathbb{Q}$. The set of rational numbers is closed under addition, subtraction, multiplication, and division (excluding division by zero). At this stage, rational numbers were considered sufficient, as they are closed under all four basic arithmetic operations. However, around 500 BCE, the Pythagoreans in Ancient Greece discovered that $\mathbb{Q}$ is not closed under square root operations, as evidenced by the proof that $\sqrt{2} \notin \mathbb{Q}$.

Example 3.1.1

Prove $\sqrt{2}$ is not rational.



Proof:

Let $c^2 = 2$. For the purpose of contradiction, let $c \in \mathbb{Q}$ that is $c = \sqrt{2} = \frac{q}{p}$, $p, q \in \mathbb{N}$, and p, q are coprime. Then $2 = \frac{q^2}{p^2} \iff 2p^2 = q^2$. Since *the square of an odd number is always odd*, we have that q is even, which we assume to be $2k$. That is $q = 2k, k \in \mathbb{N}$. Then we have $2 = \frac{4k^2}{p^2} \iff p^2 = 2k^2$, that means p is even as well. But if both p and q are even, this contradict the assumption that p and q are coprime. Therefore, $c = \sqrt{2}$ is not a rational number. In other words, the set of rational numbers is not closed under square root operations. $\square$

Note: The graph shows that The set of integers $\mathbb{Z}$ is **discrete** on the number line, whereas the set of rational numbers $\mathbb{Q}$ is **dense** on the number line but has **gaps** (i.e., c).

Later, in the 17th century, Descartes introduced the concept of the real number line, and in the 19th century, Dedekind and Cantor rigorously defined real numbers using methods such as Dedekind cuts and Cauchy sequences. This led to the construction of the real number system $\mathbb{R}$, which is closed under addition, subtraction, multiplication, division (excluding division by zero), square roots, and limits, thereby completing a significant milestone in the understanding of number systems.

Definition 3.1.1

A **real number** is any number that can be represented on the number line. It includes all the rational and irrational numbers.

$$\mathbb{R} = \{x \mid x \text{ is a rational or irrational number }\}.$$

Note: The **continuity of real numbers** refers to the continuous distribution of elements in the real number set $\mathbb{R}$ on the number line, without any gaps or jumps.

Question: How do we know for sure that there is no gap in the set of real numbers?

3.2 Infimum and Supremum

Assume we have a finite set $S \subset \mathbb{R}$, $S \neq \emptyset$, then the minimum and maximum of S exist and are elements of S. However, if S is an infinite set, it does not necessarily have a minimum or maximum.

Notation: The symbol $\forall$ means "for any" or "for all". The symbol $\exists$ means "exists" or "we can find".

Definition 3.2.1

We have a set $S \subset \mathbb{R}$, $S \neq \emptyset$. If $\exists\, \xi \in S$ such that $\forall\, x \in S$, we have $x \leq \xi$, then ξ is called a **maximum** of S, denoted $\max S$. If $\exists\, \eta \in S$ such that $\forall\, x \in S$, we have $\eta \leq x$, then η is called a **minimum** of S, denoted $\min S$.

Example 3.2.1

Some simple examples

Example 3.2.2

Let $B = \{x \mid 0 \leq x < 1\}$, then show that B does not have a maximum.

Proof:

We use prove by contradiction. Let $\beta \in B, \beta = \max B$, then $\beta \in [0, 1)$. Let $\beta' = \frac{1+\beta}{2} \in B$. Since β is the maximum, then we should have $\beta' \leq \beta$, but obviously $\beta' > \beta$. □

Definition 3.2.2

Suppose we have a set $S \subset \mathbb{R}, S \neq \emptyset$. If $\exists\, M \in \mathbb{R}$, such that $\forall\, x \in S$, we have $x \leq M$, then M is called an **upper bound** of S, or S is **bounded above**. (Here, since $M \in \mathbb{R}$, so there could be a lot of M.)

Definition 3.2.3

Suppose U is the set of all upper bounds of S, then U does not have a maximum. But U must have a minimum, (proof needed), denoted β. We say that β is the **supremum** of S. That is $\beta = \sup S$. In other words, $\beta = \sup S$ conveys two idea:

1. $\sup S$ is an upper bound; that is $\forall\, x \in S$, we have $x \leq \beta$.
2. $\sup S$ is the smallest upper bound, that is $\forall\, \epsilon > 0, \exists\, x \in S$ such that $x > \beta - \epsilon$.

Definition 3.2.4

Suppose L is the set of all lower bounds of S, then L does not have a minimum. But L must have a maximum, (proof needed), denoted α. We say that α is the **infimum** of S. That is $\alpha = \inf S$. In other words, $\alpha = \inf S$

conveys two idea:

1. $\inf S$ is a lower bound; that is $\forall\, x \in S$, we have $x \geq \alpha$.
2. $\inf S$ is the greatest lower bound; that is $\forall\, \epsilon > 0, \exists\, x \in S$ such that $x < \alpha + \epsilon$.

3.2.1 Completeness Axiom

Theorem 3.2.1 ***Completeness Axiom / Least Upper Bounde Axiom / Supremum Property***
Every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum (least upper bound) in $\mathbb{R}$.

Proof:
Outline of the proof:
We treat every $x \in \mathbb{R}$ as the sum of its integer part, $\lfloor x \rfloor$, and its decimal part (x). For the decimal part, we use $a_i, i \in \mathbb{N}$ to represent the i-th digit after the decimal point. If (x) is an infinite decimal, then it continues indefinitely. If (x) is a finite decimal, then we assume it stops at the p-th digit after the decimal point, $a_p \neq 0$, then we add 0 to the end of a_p. Then we using $0.999\cdots = 1$ to get $(x) = 0.a_1a_2\ldots a_p000\cdots = 0.a_1a_2\ldots(a_p-1)999\ldots$. For all $x \in \mathbb{R}$, We have $(x) = x - \lfloor x \rfloor \iff x = \lfloor x \rfloor + (x)$. That is

1. If (x) is an infinite decimal, $(x) = 0.a_1a_2\ldots a_n\ldots$
2. If (x) is a finite decimal. W.L.O.G, we assume (x) stops at, $a_p \neq 0$, the p-th digit after the decimal point, with $p \in \mathbb{N}$. That is $(x) = 0.a_1a_2\ldots a_p000\cdots = 0.a_1a_2\ldots(a_p-1)999\ldots$. (Here, we use the fact $0.\dot{9} = 1$.)

Assume there is an non-empty set $S \subseteq \mathbb{R}$, which is bounded above.

$$S = \left\{ a_0 + 0.a_1a_2\ldots a_n\ldots \mid a_0 = \lfloor x \rfloor, 0.a_1a_2\ldots a_n\ldots = (x),\ x \in S \right\}.$$

Since S is bounded above, we can find the largest $a_0 = \lfloor x \rfloor$ in the S and denote it as α_0. Then we construct a new set S_0:

$$S_0 = \left\{ x \mid x \in S,\ \text{and } \lfloor x \rfloor = \alpha_0 \right\}.$$

This means that all elements in S_0 have the largest units digit, α_0. In other words, if $x \in S$ and $x \notin S_0$ then $x < \alpha_0$.
Then, from elements in S_0, we choose the elements with the largest digit in the tenth place, and denote that largest tenth digit as α_1. Then we construct another new set S_1:

$$S_1 = \left\{ x \mid x \in S_0,\ \text{and the tenths place of } x \text{ is } \alpha_1 \right\}.$$

This means that if $x \in S$ and $X \notin S_1$, then $x < \alpha_0 + 0.\alpha_1$.
We repeat this process by following the general rule, that is: From S_{n-1}, take the largest value of the n-th decimal place of the fractional part of $x \in S$, and denote it as α_n. Then, we construct a new set S_n:

$$S_n = \left\{ x \mid x \in S_{n-1} \text{ and the } n\text{-th decimal place of } x \text{ is } \alpha_n \right\}.$$

Therefore, we have $S \subset S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_n \subset \ldots$. Since the latest set S_n is built up on the previous set S_{n-1}, by picking fewer elements from S_{n-1}. (For elements from S_{n-1}, Only the elements with the largest n-th decimal place will be included in to S_n.) We also have $a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots, a_n, \cdots \in \{0, 1, 2, \ldots, 9\}$.
Let, $\beta = \alpha_0 + 0.\alpha_1\alpha_2\ldots\alpha_n\ldots$. We just need to show that β is the supremum of the set S.

1) We show β is an upper bound.
For all $x \in S$, x is either in S but not in S_n, or x in S_n. If $x \in S$, and $x \notin S_n$, then $x < \alpha_0 + 0.\alpha_1\alpha_2\ldots\alpha_n000\cdots \leq \beta$. If $x \in S_n, \forall\, n \in \mathbb{N}$, we know not only the integer part of x but also each decimal places of x are the largest. That is $x = \alpha_0 + 0.\alpha_1\alpha_2\ldots\alpha_n\cdots = \beta$. Therefore, β is an upper bound.
2) We show that β is the least upper bound. That is equivalent to find a small ϵ such that $\beta - \epsilon$ is not longer an upper bound.
We pick $k \in \mathbb{N}$ such that $\epsilon > \frac{1}{10^k}$. Then choose $x \in S_k$, the integer part of this x is α_0, the first k decimal parts this x is $\alpha_1, \alpha_2, \ldots, \alpha_k$. Then we have:

$$\beta - x = \alpha_0 + 0.\alpha_1\alpha_2\ldots\alpha_n\cdots - (\alpha_0 + 0.\alpha_1\alpha_2\ldots\alpha_k) \leq \frac{1}{10^k} < \epsilon \implies x > \beta - \epsilon.$$

The red inequality is because the difference between β and x occurs after α_k. For example, if $k = 2$ then the largest difference between β and x could be 0.009, while $\frac{1}{10^2} = 0.01$.Therefore, we have show β is the least upper bound.

Thus, we have shown that for any non-empty set $S \subset \mathbb{R}$ that is bounded above has a supremum. □

Moreover, if the real number system has gaps (that is at some point, it is neither a rational number nor a irrational number), then the set to the left of that point has no supremum (within the real number system) and the set to the right of the point has no infimum (within the real number system). **The Supremum Existence Theorem** shows that there are no gaps in the real number system.

Proposition 3.2.2

Every non-empty subset of $\mathbb{R}$ that is bounded below has an infimum (greatest lower bound) in $\mathbb{R}$.

Proof:

Just for fun, let's prove this with a new method. Let $A \in \mathbb{R}$ be bounded blow with lower bound C. Consider the set

$$B = \{-a \mid a \in A\}.$$

For any $x \in B$ we have that $-x \in A$ and so $-x \geq C$. Then we have $x \leq -C$. Thus B is bounded above and non-empty and so by the completeness axiom (we just proved) has a least upper bound U. If we let $L = -U$ then for all $a \in A$ and $-a \in B$ we have $-a \leq U \iff a \geq -U = L$. This shows that L is a lower bound of A. We now show that L is the least lower bound. Since U is the least upper bound of the set B we know that if we take a $y \in A$ with $y > L$, then $-y \in B$ with $-y < -L = U$. Thus $-y$ is not an upper bound for B so we can find another element $b \in B$ such that $b > -y$. However, this means that $-b \in A$ and $-b < y$. So, y cannot be a lower bound for A (and we assumed $y > A$) so L is the greatest lower bound for A. $\square$

Though this proof is shorter, it stands on the proof of the completeness Axiom. The following example shows that $\mathbb{Q}$ has no completeness property.

Example 3.2.3

Assume $T\left\{x \mid x \in \mathbb{Q}, x > 0, \text{ and } x^2 < 2\right\}$, then T has no supremum in $\mathbb{Q}$.

Proof:

Assume T has a supremum in $\mathbb{Q}$, denoted as $\sup T = \frac{n}{m}$, m, n coprime and $m, n \in \mathbb{N}$. Then

$$1 < \left(\frac{n}{m}\right)^2 < 3$$

Since $\frac{n}{m} \notin \mathbb{Q}$, we have two situations:

1). $1 < \left(\frac{n}{m}\right)^2 < 2$. We want to show that $\frac{n}{m}$ is not an upper bound. That is to find an $r > 0, r \in \mathbb{Q}$, such that $\frac{n}{m} + r \in T$. In other words, we need to show $\left(\frac{n}{m} + r\right)^2 < 2$. That is

$$r^2 + \frac{2n}{m}r + \left(\frac{n}{m}\right)^2 < 2$$
$$r^2 + \frac{2n}{m}r + \underbrace{\frac{n^2}{m^2} - 2}_{=-t} < 0$$
$$r^2 + \frac{2n}{m}r - t < 0.$$

Here, we let $t = 2 - \frac{n^2}{m^2}$, thus $t \in (0, 1)$, for the purpose of applying $t^2 < t$. We also have the inequality $1 < \frac{n^2}{m^2} < 2$. We observe the possible form for r should be $r = \frac{an}{bm}t > 0, a, b, \in \mathbb{N}$. We substitute it into what we left to get

$$\frac{a^2n^2}{b^2m^2}t^2 + \frac{2n^2}{bm^2}t - t < 0$$
$$\frac{a^2n^2}{b^2m^2}t^2 + \frac{2n^2}{bm^2}t < t$$

Since $t \in (0, 1)$, then $\frac{a^2n^2}{b^2m^2}t^2 < \frac{a^2n^2}{b^2m^2}t$, then we have

$$\frac{a^2n^2}{b^2m^2}t + \frac{2n^2}{bm^2}t < t$$

We observe that as long as $\frac{a^2+2b}{b^2}\frac{n^2}{m^2} < 1$, the inequality above holds for sure. Since $\frac{n^2}{m^2} < 2$. We need to ensure $\frac{a^2+2b}{b^2} < \frac{1}{2}$. To keep it simple, we choose $a = 1b = 5$ so that we let $r = \frac{n}{5m}t > 0$, this will make $\left(\frac{n}{m} + r\right)^2 < 2$, meaning $\frac{n}{m}$ is not an upper bound. Thus, contradiction.

2). $2 < \left(\frac{n}{m}\right)^2 < 3$ We want to show that $\frac{n}{m}$ is not the least upper bound. That is to find an $r > 0, r \in \mathbb{Q}$, such that $\frac{n}{m} - r$ is an upper bound. Let $\frac{n^2}{m^2} - 2 = t$, then $t \in (0, 1)$. W.O.L.G.let $r = \frac{n}{5m}t > 0$. Obviously $\frac{n}{m} - t > 0$ and $\frac{n}{m} - r \in \mathbb{Q}$. We also know that $\frac{2n}{m}r = \frac{2n^2}{5m^2}t < \frac{4}{5}t < t$. Therefor,

$$\left(\frac{n}{m} - r\right)^2 - 2 = r^2 - \frac{2n}{m}r + t > 0.$$

So, we have find a $r > 0$ such that $\left(\frac{n}{m} - r\right)^2 - 2 > 0$, meaning that $\frac{n}{m}$ is not the lest upper bound.
Therefore, we have shown that T has no supremum in $\mathbb{Q}$. In other words, $\mathbb{Q}$ has no completeness property. □

Proposition 3.2.3 ***Archimedean property***
For any $x \in \mathbb{R}$ there exists $k \in \mathbb{N}$ such that $k \geq x$.

Proof:

Fix $x \in \mathbb{R}$ and suppose that no such integer k exists. This means that for all $z \in \mathbb{N}$ we have $z < x$ and so the natural numbers are bounded above by x. Thus by the completeness axiom the integers must have a least upper bound α. Then $\alpha - \frac{1}{2}$ is not an upper bound for $\mathbb{N}$. So we can conclude that there is a natural number z between $\alpha - \frac{1}{2}$ and α. However, in this case $z + 1$ is an integer greater than α and this contradicts with $alpha$ being the least upper bound. □

3.3 Limits of Sequences

A sequence is a function from a subset of the integers (usually the natural numbers) to a set. It is an ordered list of elements, where the order matters and repetition of elements is allowed.

$$x_1, x_2, \ldots, x_n, \ldots$$

Definition 3.3.1
Formally, a **sequence** can be defined as: a function $f : \mathbb{N} \to X$, where X is a set. The elements $a(n) = a_n$ are called the terms of the sequence, and $n \in \mathbb{N}$ represents their position in the sequence.

Example 3.3.1
The following are some examples of sequences:

$$\begin{aligned} \left\{\frac{1}{n}\right\} &: \quad 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots. \\ \{(-1)^n\} &: 1, -1, 1, -1, \ldots, (-1)^n \ldots. \end{aligned}$$

Definition 3.3.2
(**ϵ-N game**). For a sequence $\{x_n\}$ and a real constant A. If for any (given) $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that when $n > N$, we have $|x_n - a| < \epsilon$, then we say the sequence $\{x_n\}$ **converges** to a, denoted $\lim_{n\to\infty} x_n = A$, or $x_n \xrightarrow{n\to\infty} A$. If such an A does not exist, then we say $\{x_n\}$ **diverges**.

Note: The condition $|x_n - a| < \epsilon$ is equivalent to **the ϵ neighbourhood of point** A, denoted as $x_n \in O(A, \epsilon) = (A - \epsilon, A + \epsilon) = \{x \mid A - \epsilon < x < A + \epsilon\}$.

Remark. Whether a sequence converges and, if so, to which value it converges, is independent of the first finitely many terms of the sequence.

Example 3.3.2

one simple example to address the remark.

Example 3.3.3

Use the definite of limit of a sequence to show the limit of $\left\{\frac{n}{n+3}\right\}$ is 1.

Proof:

For any (given) $\epsilon > 0$, we want $|\frac{n}{n+3} - 1| < \epsilon \iff n > \frac{3}{\epsilon} - 3$. If we choose $N = \frac{3}{\epsilon} - 1$, we only need to ensure the RHS of is a positive integer. Since $\frac{3}{\epsilon} - 1 < \frac{3}{\epsilon}$ and to ensure it is an integer we choose $\lfloor \frac{3}{\epsilon} \rfloor$. To ensure it is positive (non-zero) we have $\lfloor \frac{3}{\epsilon} \rfloor < \lfloor \frac{3}{\epsilon} \rfloor + 1$. So, we choose $N = \lfloor \frac{3}{\epsilon} \rfloor + 1$. When $n > N$, we have $|\frac{n}{n+3} - 1| < \epsilon$. □

Remark. We enlarge the choice of N several times to satisfy the condition for N.

Definition 3.3.3

A variable (in this case, a sequence) with 0 as its limit is an **infinitesimal**.

Example 3.3.4

If $\lim_{n\to\infty} x_n = a$ then immediately $x_n - a$ is an infinitesimal.

Example 3.3.5

Assume $0 < |q| < 1$ prove that $\{q^n\}$ is an infinitesimal.

Proof:

For any (given) $\epsilon > 0$, we have

$$\begin{aligned} |q^n - 0| &< \epsilon \\ |q^n| &< \epsilon \\ n \ln |q| &< \ln \epsilon \\ n &> \frac{\ln \epsilon}{\ln |q|}. \end{aligned}$$

Since $\ln |q|$ can be positive or negative, even with $\left\lfloor \frac{\ln \epsilon}{\ln |q|} \right\rfloor$, the positivity is not ensured so there is a trick to make sure it is positive. That is let $N = \max\left\{\left\lfloor \frac{\ln \epsilon}{\ln |q|} \right\rfloor, 1\right\}$, when $n > N$, we have $|q^n - 0| < \epsilon$. □

Remark. When we are discussing something (i.e., a sequence) is getting very close to a value, it is obvious that we are talking about very small value of ϵ. W.O.L.G., we often use $\epsilon \in (0, 1)$ instead of $\epsilon > 0$.

For example, if we let $0 < \epsilon < |q| < 1$, then $\frac{\ln \epsilon}{\ln |q|} > 1$. Then N would be $\left\lfloor \frac{\ln \epsilon}{\ln |q|} \right\rfloor$. This tells us that if we restrict ϵ to a smaller interval, the choice of N could be simpler.

Suppose we have two students who find two different valid values of N, one being 100 and another being 10000. Which one is better? Both are correct answers, because in the definition of the limit, we only need to find an N instead of the best (smallest) N. As we know to find the smallest (best) N could sometimes be very challenging. Why make our life harder. For example, in the inequality $n > RHS$, we often enlarge the RHS to make calculation easier, which leads to non-optimal but valid N.

Example 3.3.6

Prove $\lim_{n\to\infty} \sqrt[n]{a} = 1$, for $(a > 1)$.

Proof:
For any (given) $\epsilon > 0$, let $\sqrt[n]{a} - 1 = y_n > 0$

$$\sqrt[n]{a} = 1 + y_n$$
$$a = 1 + ny_n + \binom{n}{2} y_n^2 + \cdots + y^n > 1 + y_n^n$$
$$y_n < \frac{a-1}{n}$$

We want to show $\left|\sqrt[n]{a} - 1\right| < \epsilon$ that is $\sqrt[n]{a} - 1 = y_n < \frac{a-1}{n} < \epsilon$. We easily got $n > \frac{a-1}{\epsilon}$. Thus, we choose $N = \left\lfloor \frac{a-1}{\epsilon} \right\rfloor + 1$. When $n > N$, we have $\left|\sqrt[n]{a} - 1\right| = y_n < \frac{a-1}{n} < \epsilon$. □

Note: In the red inequality, we make the RHS smaller, since we do not need to find the best N. By doing so the proof is much easier.

For the previous example we do not need to use the method to make the RHS larger or smaller to solve the problem, but this is not true for the following example.

Example 3.3.7
Show $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

Proof:
For any (given) $\epsilon > 0$, we want to show $\sqrt[n]{n} - 1 < \epsilon \iff \sqrt[n]{n} < 1 + \epsilon$. Let $\sqrt[n]{n} - 1 = y_n > 0$ for $n = 2, 3, \ldots$.
Then we have

$$n = 1 + ny_n + \binom{n}{2} y_n^2 + \cdots + y_n^n > 1 + \frac{n(n-1)}{2} y_n^2$$
$$n > 1 + \frac{n(n-1)}{2} y_n^2$$
$$y_n^2 < \frac{2}{n}$$
$$y_n < \sqrt{\frac{2}{n}} < \epsilon$$
$$n > \frac{2}{\epsilon^2}.$$

We choose $N = \left\lfloor \frac{2}{\epsilon^2} \right\rfloor$, when $n > N$, we have $\left|\sqrt[n]{n} - 1\right| < \epsilon$. □

Example 3.3.8
Prove $\lim\limits_{n\to\infty} \sqrt[n]{n^2} = 1$.

Proof:
We need $\binom{n}{3} y_n^3$ □

Example 3.3.9
Prove $\lim\limits_{n\to\infty} \sqrt[n]{n^k} = 1$.

Example 3.3.10
Prove $\lim\limits_{n\to\infty} \frac{n^2+1}{2n^2-7n} = \frac{1}{2}$.

Proof:

For any (given) $\epsilon > 0$, we want to show $\left|\frac{n^2}{2n^2-7n} - \frac{1}{2}\right| < \epsilon$. LHS is $\left|\frac{7n+2}{2n(2n-7)}\right|$, when $n > 3$ we get $\left|\frac{7n+2}{2n(2n-7)}\right| = \frac{7n+2}{2n(2n-7)} < \frac{8n}{2n(n-7)}$. This is still not simple enough. We keep enlarging it. That is $\frac{8n}{2n(n-7)} < \frac{8n}{4n^2} \cdot 2 = \frac{4}{n}$. To ensure LHS $= \frac{7n+2}{2n(2n-7)} < \frac{4}{n}$, we got $n \geq 6$. Therefore, $\frac{4}{n} < \epsilon$, we get $n > \frac{4}{\epsilon}$. We choose $N = \max\left\{\left\lfloor\frac{4}{\epsilon}\right\rfloor, 6\right\}$. When $n > N$, we have that $\left|\frac{n^2}{2n^2-7n} - \frac{1}{2}\right| < \epsilon$. □

Example 3.3.11

Suppose, $\lim\limits_{n\to\infty} a_n = a$, prove that $\lim\limits_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = a$.

Proof:

We discuss $a = 0$ and $a \neq 0$.
Case 1)
Suppose $a = 0$. Since $\lim\limits_{n\to\infty} a_n = a$, that is for any (given) $\epsilon > 0$, there exist N_1, when $n > N_1$, $|a_n| < \frac{\epsilon}{2}$.

$$\frac{a_1+a_2+\cdots+a_n}{n} = \frac{a_1+a_2+\cdots+a_{N_1}}{n} + \underbrace{\frac{a_{N_1+1}+a_{N_1+2}+\cdots+a_n}{n}}_{\left|\frac{a_{N_1+1}+a_{N_1+2}+\cdots+a_n}{n}\right| < \frac{(n-N_1+1)+1}{n}\cdot\frac{\epsilon}{2} < \frac{\epsilon}{2}}$$

$$\frac{a_1+a_2+\cdots+a_n}{n} < \frac{a_1+a_2+\cdots+a_{N_1}}{n} + \frac{\epsilon}{2}$$

For the blue term, since for a given ϵ, N_1 is fixed, then $a_1 + a_2 + \cdots + a_{N_1}$ is fixed, then we choose $N >> N_1$, such that when $n > N$, we have $\left|\frac{a_1+a_2+\cdots+a_{N_1}}{n}\right| < \frac{\epsilon}{2}$. Therefore, we have $\frac{a_1+a_2+\cdots+a_n}{n} < \epsilon$.
Case 2)
Suppose $a \neq 0$, then $a_n - a$ is an infinitesimal. We want to show

$$\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} - a = 0$$

or

$$\left|\frac{a_1+a_2+\cdots+a_n}{n} - a - 0\right| < \epsilon$$

W.O.L.G., we have

$$\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} - a = \lim_{n\to\infty} \frac{(a_1-a)+(a_2-a)+\cdots+(a_n-a)}{n} = 0.$$

The red equality is because now we view $(a_n - a)$ as new a_n in case 1), so that we apply case 1) to get the answer.
Therefore, we have $\lim\limits_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = a$. □

3.4 Properties of Limits of Sequences

1. **Uniqueness** of the limit of a sequence.

Example 3.4.1 S

ppose $\lim\limits_{n\to\infty} x_n = a$ and $\lim\limits_{n\to\infty} x_n = b$, then $a = b$.

Proof:

For $\lim\limits_{n\to\infty} x_n = a$, we have $\forall\, \epsilon > 0, \exists\, n > N_1$ such that $|x_n - a| < \frac{\epsilon}{2}$. For $\lim\limits_{n\to\infty} x_n = b$, we have $\forall\, \epsilon > 0, \exists n > N_2$ such that $|x_n - b| < \frac{\epsilon}{2}$. Therefore, we choose $N = \max\{N_1, N_2\}$, $\forall\, n > N$, $|a-b| \leq |x_n - a| + |x_n - b| < \epsilon$. □

2. **Convergent** $\implies$ **Bounded**.
For a sequence $\{x_n\}$, if $\exists\, M \in \mathbb{R}, \forall\, n \in \mathbb{N}$ such that $x_n \leq M$, then M is called an **upper bound** of the sequence

$\{x_n\}$ or $\{x_n\}$ is **bounded above**. If $\exists\, m \in \mathbb{R}, \forall\, n \in \mathbb{N}$ such that $x_n \geq m$, then m is called a **lower bound** of the sequence $\{x_n\}$, or $\{x_n\}$ is **bounded below**.
If $\{x_n\}$ has both upper bound and lower bound ($|x_n| \leq X, X \in \mathbb{R}^+$), then $\{x_n\}$ is a **bounded sequence**.

Theorem 3.4.1

A convergent sequence is always bounded.

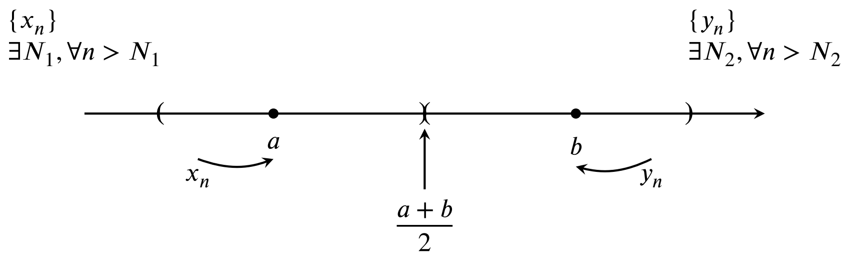
Proof:

Suppose $\{x_n\}$ converges to a, that is for a given $\epsilon = 1$, there exists N, such that $\forall\, n > N$, we have $a - 1 < x_n < a + 1$. Now we only need to show the elements before x_N are bounded. Let $M = \max\{x_1, x_2, \ldots, x_N, a+1\}$ and $m = \min\{x_1, x_2, \ldots, x_N, a-1\}$ then we have $\forall\, n \in \mathbb{N}$, we have $m \leq x_n \leq M$. $\square$

3. **Order Preservation** of limits of sequences.

Theorem 3.4.2

Suppose $\{x_n\}, \{y_n\}$ are two sequences and $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} x_n = b, a < b$, then there exists $N \in \mathbb{N}$ such that $\forall\, n > N, x_n < y_n$.



Proof:

We choose a (given) $\epsilon = \frac{b-a}{2} > 0$, there exists $N_1 \in \mathbb{N}$ such that if $\forall\, n > N_1$ then $|x_n - a| < \epsilon \implies x_n < \frac{a+b}{2}$. There exists $N_2 \in \mathbb{N}$ such that if $\forall\, n > N_2$ then $|y_n - b| < \epsilon \impliedby y_n > \frac{a+b}{2}$. Then we can choose $N = \max\{N_1, N_2\}$ such that if $n > N$ then $x_x < \frac{a+b}{2} < y_n$. $\square$

Note: It is obvious that the limit of $\{x_n\}$ ensure that for very large n, x_n will be in the neighborhood of a, $(a - \frac{b-a}{2}, a + \frac{b-a}{2})$. Similarly, for very large value of n, y_n will be in the neighborhood of b, $(b - \frac{b-a}{2}, b + \frac{b-a}{2})$. In the graph it is clear that $x_n < y_n$.

Remark. The converse is not true. That is $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b, x_n < y_n \not\Rightarrow a < b$. For example, $x_n = \frac{1}{n}, y_n = \frac{2}{n}$, indeed $x_n < y_n, \forall\, n \in \mathbb{N}$, but $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 0$. Actually we can only ensure that $a \leq b$ and the condition can be relaxed to $x_n \leq y_n$ for large value of n. This is the following proposition.

Proposition 3.4.3

Suppose $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$, if $\exists\, n > N$ such that $x_n \leq y_n$, then $a \leq b$.

Corollary 3.4.4

Suppose $\lim\limits_{n\to\infty} y_n = b > 0$, then $\exists\, N$ such that $\forall\, n > N$ we have $y_n > \frac{b}{2} > 0$. [A certain distance to the right of the origin.] similar graph To prove this we need two sequences to use order preservation theorem, then we let $\{x_n\}$ be a sequence of constants with value $\frac{b}{2}$.

Corollary 3.4.5

Suppose $\lim\limits_{n\to\infty} y_n = b < 0$, then $\exists\, N$ such that $\forall\, n > N$ we have $y_n < \frac{b}{2} < 0$. [A certain distance to the left of the origin.] similar graph

Corollary 3.4.6

Suppose $\lim_{n\to\infty} y_n \neq 0$, then $\exists\, N$ such that $\forall\, n > N$ we have $|y_n| > \frac{|b|}{2} > 0$. [A certain distance from the origin.] similar graph

4. The **squeeze theorem**

Theorem 3.4.7

Suppose we have three sequences $\{x_n\}, \{y_n\}, \{z_n\}$. If $\exists\, N$, when $n > N$ such that $x_n \le y_n \le z_n$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$, then $\lim_{n\to\infty} y_n = a$.

Proof:

Since $\lim_{n\to\infty} x_n = a$, for a (given) ϵ, there exists $N_1 \in \mathbb{N}$ such that when $n > N_1$, we have $|x_n - a| < \epsilon$. That is $a - \epsilon < x_n$. Similarly, for the same (given) ϵ, since $\lim_{n\to\infty} z_n = a$, there exists $N_2 \in \mathbb{N}$ such that when $n > N_2$, we have $|z_n - a| < \epsilon$. That is $z_n < a + \epsilon$. Now we choose $N = \max\{N_1, N_2\}$, when $n > N$, we have $a - \epsilon < x_n \le y_n \le z_n < a + \epsilon$. That is $a - \epsilon < y_n < a + \epsilon \iff |y_n - a| < \epsilon$. Therefore $\lim_{n\to\infty} y_n = a$. $\square$

Example 3.4.2

Compute $\lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right)$.

W.O.L.G., we have $0 < \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}}$. Since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, then by the squeeze theorem, we have $\lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = 0$.

Example 3.4.3

Proof that if $a_1, a_2, \dots, a_p > 0$ $\lim_{n\to\infty}\left(a_1^n + a_2^n + \cdots + a_p^n\right)^{\frac{1}{n}} = \max_{1\le i\le p}\{a_i\}$.

Proof:

W.O.L.G., let's assume that $a_1 = \max_{1\le i\le p}\{a_i\}$. Then we have

$$a_1 = \left(a_1^n\right)^{\frac{1}{n}} \le \left(a_1^n + a_2^n + \cdots + a_p^n\right)^{\frac{1}{n}} \le \left(pa_1^n\right)^{\frac{1}{n}} = a_1 p^{\frac{1}{n}}.$$

Recall that we have shown $a > 1$, $\lim_{n\to\infty} \sqrt[n]{a} = 1$, then it is obvious that $p \ge 1$. Therefore, by the squeeze theorem, we have $\lim_{n\to\infty}\left(a_1^n + a_2^n + \cdots + a_p^n\right)^{\frac{1}{n}} = a_1 = \max_{1\le i\le p}\{a_i\}$. $\square$

Example 3.4.4

Prove $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Proof:

Since $1 < \sqrt[n]{n} = \sqrt[n]{\sqrt{n}\cdot\sqrt{n}\cdot\underbrace{1\cdot 1\dots 1}_{n-2}} \le \frac{2\sqrt{n}+n-2}{n} = 1 + \frac{2\sqrt{n}-2}{n}$. Since $\lim_{n\to\infty}\frac{2\sqrt{n}-2}{n} = 0$ and the squeeze theorem We have the proof. (The red inequality is due to the geometric mean is less than or equal to the arithmetic mean.) $\square$

Example 3.4.5

Prove $\lim_{n\to\infty} \sqrt[n]{n^2} = 1$.

Proof:

Since $1 < \sqrt[n]{n} = \sqrt[n]{\sqrt{n}\cdot\sqrt{n}\sqrt{n}\cdot\sqrt{n}\cdot\underbrace{1\cdot 1\ldots 1}_{n-4}} \leq \frac{4\sqrt{n}+n-4}{n} = 1 + \frac{4\sqrt{n}-4}{n}$. Since $\lim_{n\to\infty}\frac{4\sqrt{n}-4}{n} = 0$ and the squeeze theorem We have the proof. □

3.5 Arithmetic operations on the limits of sequences

We start by stating the theorem.

Theorem 3.5.1

Suppose $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$, then we have

1. $\lim_{n\to\infty}\left(\alpha x_n + \beta y_n\right) = \alpha a + \beta b$; α, β are constants.
2. $\lim_{n\to\infty}\left(x_n y_n\right) = ab$;
3. $\lim_{n\to\infty}\frac{x_n}{y_n} = \frac{a}{b}, (b \neq 0)$.

Note: In 3., $b \neq 0$ ensures that when n is large enough, all y_n will be at some distance from zero. Although some early terms of the sequence $\left\{y_n\right\}$ can be zero, but for large n, this is not the case.

Proof:

Since $\lim_{n\to\infty} x_n = a$, then the sequence $\left\{x_n\right\}$ is bounded, that is $\exists\, X > 0$ such that $\left|x_n\right| < X, \forall\, n \in \mathbb{N}$. Also, we have

$$\forall, \epsilon > 0, \exists\, N_1, \forall\, n > N_1 \text{ such that } \left|x_n - a\right| < \epsilon$$

$$\exists\, N_2, \forall\, n > N_2 \text{ such that } \left|y_n - b\right| < \epsilon$$

let $N = \max\left\{N_1, N_2\right\}, \forall\, n > N$ we have

$$\left|\left(\alpha x_n + \beta y_n\right) - (\alpha a + \beta b)\right| = \left|\alpha\left(x_n - a\right) + \beta\left(y_n - b\right)\right| \leq |\alpha|\left|x_n - a\right| + |\beta|\left|y_n - b\right| < (|\alpha| + |\beta|)\,\epsilon.$$

We have proved 1.). To prove 2.)

$$\left|x_n y_n - ab\right| = \left|x_n y_n - x_n b + x_n b - ab\right| = \left|x_n\left(y_n - b\right) + \left(x_n - a\right)b\right| < (|X| + |b|)\,\epsilon.$$

To prove 3.), since $\lim_{n\to\infty} y_n = b \neq 0$, then $\exists\, N_3$ such that $\forall\, n > N_3$, we have $\left|y_n\right| > \frac{|b|}{2} > 0$. This is saying that y_n will be more than $\frac{|b|}{2}$ away from from 0, when n is larger than N_3. Let $N' = \max\left\{N_1, N_2, N_3\right\}$, for all $n > N'$, we have

$$\begin{aligned}
\left|\frac{x_n}{y_n} - \frac{a}{b}\right| = \frac{\left|bx_n - ay_n\right|}{\left|by_n\right|} = \frac{\left|bx_n - ab + ab - ay_n\right|}{\left|by_n\right|} &= \frac{\left|b\left(x_n - a\right) + a\left(b - y_n\right)\right|}{\left|by_n\right|} \\
&\leq \frac{2\left(|b|\left|x_n - a\right| + |a|\left|y_n - b\right|\right)}{|b|^2} \\
&< \frac{2\,(|a| + |b|)}{|b|^2}\epsilon.
\end{aligned}$$

The inequality step above is due to $\left|y_n\right| > \frac{|b|}{2}$ for $n > N_3$. □

Example 3.5.1

Compute $\lim_{n\to\infty} \frac{5^{n+1}-(-2)^n}{3\cdot 5^n+2\cdot 3^n}$.

We divide both the numerator and the denominator by 5^n to get $\lim_{n\to\infty} \frac{5-(-\frac{2}{5})^n}{3+2\cdot\left(\frac{3}{5}\right)^n}$, then apply 3.) to get $\frac{5}{3}$.

Example 3.5.2

Prove that when $a > 0$, $\lim_{n\to\infty} \sqrt[n]{a} = 1$.

Proof:

We have shown that for $a > 1$, the statement is true. For $a = 1$, the statement is obvious. For $a \in (0, 1)$, the $\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\frac{1}{a}}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{1}{a}}}$. Since $\frac{1}{a} > 1$, the denominator equals 1. $\square$

Example 3.5.3

Compute $\lim_{n\to\infty} n\left(\sqrt{n^2+1} - \sqrt{n^2-1}\right)$.

$$
\begin{aligned}
\lim_{n\to\infty} n\left(\sqrt{n^2+1} - \sqrt{n^2-1}\right) &= \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+1}+\sqrt{n^2-1}} \\
&= \lim_{n\to\infty} \frac{2}{\frac{\sqrt{n^2+1}}{n}+\frac{\sqrt{n^2-1}}{n}} \\
&= \lim_{n\to\infty} \frac{2}{\sqrt{1+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n^2}}} = 1.
\end{aligned}
$$

Example 3.5.4

Compute $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right)$.

Note: The wrong way to calculate is

$$
\begin{aligned}
&\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right) \\
=& \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} + \lim_{n\to\infty} \frac{1}{\sqrt{n^2+2}} + \cdots + \lim_{n\to\infty} \frac{1}{\sqrt{n^2+n}} \\
=& 0.
\end{aligned}
$$

The correct way is

$$\frac{n}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \le \frac{n}{\sqrt{n^2+1}}$$

Since $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1 = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}}$ then by the squeeze theorem

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right) = 1.$$

Remark. Why we get a wrong answer before? Because the wrong way use 1.) for n different limits, however 1.) only holds for $k \in \mathbb{N}$ different limits with k being fixed. Obviously here for n different limits, this n is not fixed but increases to infinity. So we could not apply 1.) directly.

Example 3.5.5

Suppose $a_n > 0$ and $\lim_{n\to\infty} a_n = a$, prove $\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \dots a_n} = a$.

Proof:

Since $a_n > 0$ and $\lim_{n\to\infty} a_n = a$, then $a > 0$ or $a = 0$.

For $a > 0$, we have $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$. We also have

$$\underbrace{\frac{a_1 + a_2 + \dots + a_n}{n}}_{(1)} \ge \sqrt[n]{a_1 a_2 \dots a_n} \ge \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}} = \underbrace{\frac{1}{\frac{\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}}{n}}}_{(2)}$$

We have shown that the limit of (1) equals a. For term (2), the limit of the denominator is $\frac{1}{a}$. Why? Since $\frac{\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}}{n}$ is the arithmetic mean of the sequence $\left\{\frac{1}{a_n}\right\}$ and $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$. Therefore, the $\lim_{n\to\infty} \frac{\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}}{n} = \frac{1}{a}$. By the squeeze theorem, $\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \dots a_n} = a$.

For $a = 0$, the proof is shorter. Since $a = 0$, we have $\lim_{n\to\infty} a_n = a = 0$. Also, we have

$$\frac{a_1 + a_2 + \dots + a_n}{n} \ge \sqrt[n]{a_1 a_2 \dots a_n} > 0$$

We have shown that $\lim_{n\to\infty} a_n = a = 0$ then $\lim_{n\to\infty} \frac{a_1+a_2+\dots+a_n}{n} = a = 0$, Again, we get the result by applying the squeeze theorem. □

Note: This example states that if the limit of a sequence with all positive terms is A, then the limits of its arithmetic and geometric means are also A.

3.6 Infinity

3.6.1 Definition of Infinity

For intuition, suppose that we have a sequence $\{n^3\}$, it should be obvious that as n increases, the value of n^3 increases as well. Moreover, the value of n^3 will not be bounded.

Definition 3.6.1

If for any give $G > 0$, there exists an $N \in \mathbb{N}$ such that when $n > N$, $|x_n| > G$ holds, then the sequence $\{x_n\}$ diverges to **infinity**, denoted $\lim_{n\to\infty} x_n = \infty$.

Symbolic representation, $\forall G > 0, \exists N \in \mathbb{N}, \forall n > N : |x_n| > G$.

Remark. Here, we use $|x_n| > G$ with $G > 0$. If $x_n > G$, then the sequence $\{x_n\}$ diverges to **positive infinity**. If $x_n < -G$, then the sequence $\{x_n\}$ diverges to **negative infinity**.

Example 3.6.1

We have three different types of infinities.

1. $\lim_{n\to\infty} n^2 = +\infty$;
2. $\lim_{n\to\infty} (10^n) = -\infty$;
3. $\lim_{n\to\infty} (-2)^n = \infty$.

Example 3.6.2

Suppose $|q| > 1$, prove that $\{q^n\}$ diverges to infinity.

Proof:

We want to show $|q^n| = |q|^n > G$, that is $n > \frac{\lg G}{\lg|q|}$. To prove some quantity is infinity, we only need to consider a very large value for G. For all $G > |q|$, let $N = \lfloor \frac{\lg G}{\lg|q|} \rfloor$, $\forall\, n > N$ we have $|q^n| = |q|^n > G$. $\square$

Example 3.6.3

Prove $\left\{ \frac{n^2-1}{n+5} \right\}$ diverges to infinity.

Proof:

We want to show $\left|\frac{n^2-1}{n+5}\right| > G$. To ensure this inequality, we want to find something smaller than $\left|\frac{n^2-1}{n+5}\right|$ but is still larger than G. That is $\left|\frac{n^2-1}{n+5}\right| > \frac{n^2}{n} \cdot \frac{1}{2} = \frac{n}{2}$, only holds when $n > 5$. Why do we want $\frac{n}{2}$ here? It is because it is really easy to find N, with $\frac{n}{2} > G \iff n > 2G$. For all $G > 0$, and when $n > 5$, let $N = \max\{\lfloor 2G \rfloor, 5\}$, for all $n > N$ we have $\left|\frac{n^2-1}{n+5}\right| > \frac{n}{2} > G$. $\square$

Theorem 3.6.1

Suppose $x_n \neq 0$, then the necessary and sufficient condition for the sequence $\{x_n\}$ to diverge to infinity is that the sequence $\left\{\frac{1}{x_n}\right\}$ is infinitesimal.

Proof:

($\Rightarrow$)

Suppose the sequence $\{x_n\}$ diverges to infinity, we want to show the sequence $\left\{\frac{1}{x_n}\right\}$ is infinitesimal.

That is $\forall\, \epsilon > 0$ we want to find some threshold N, when $n > N$ we have that $\left|\frac{1}{x_n}\right| < \epsilon$. We only know that $\lim\limits_{n\to\infty} x_n = \infty$, that is we only have a G-N relation, but we want an ϵ-N relation. W.O.L.G., let $G = \frac{1}{\epsilon} > 0$, $\exists\, N$, $\forall\, n > N$ we have $|x_n| > G = \frac{1}{\epsilon}$ then we get $\left|\frac{1}{x_n}\right| < \epsilon$.

($\Leftarrow$)

Suppose the sequence $\left\{\frac{1}{x_n}\right\}$ is infinitesimal, we want to show that the sequence $\{x_n\}$ diverges to infinity.

That is $\forall\, G > 0$, let $\epsilon = \frac{1}{G} > 0$, $\exists\, N$, $\forall\, n > N$ we have $\left|\frac{1}{x_n}\right| < \epsilon$, then we get $|x_n| > G$. $\square$

Theorem 3.6.2

Suppose a sequence $\{x_n\}$ diverges to infinity and another sequence $\{y_n\}$ has the property: $\exists\, N_0, \forall\, n > N_0$, we have $|y_n| \geq \delta > 0$, then the sequence $\{x_n y_n\}$ diverges to infinity.

Proof:

The proof will be left as an exercise. $\square$

Remark. This theorem is saying that given a sequence $\{x_n\}$ that diverges to infinity, the condition for the new sequence $\{x_n y_n\}$ also diverges to infinity is that the sequence $\{y_n\}$ will be some distance away from 0 for large value of n. (The earlier terms in the sequence $\{y_n\}$ could be 0.)

Note: Compare this proof with the Q7 in the exercise.

Corollary 3.6.3

Suppose a sequence $\{x_n\}$ diverges to infinity and another sequence $\{y_n\}$ has $\lim\limits_{n\to\infty} y_n = b \neq 0$, then the sequences

$\{x_n y_n\}$ and $\left\{\frac{x_n}{y_n}\right\}$ diverge to infinity.

Proof:

Sketch of the proof.
Since $\lim\limits_{n\to\infty} y_n = b \neq 0$, then $\exists\, N, \forall\, n > N : |y_n| \geq \frac{|b|}{2} > 0$. Then apply the theorem above we know that $\{x_n\}$ diverges to infinity. Also, $\left\{\frac{x_n}{y_n}\right\} = \left\{x_n \cdot \frac{1}{y_n}\right\}$ and $\lim\limits_{n\to\infty} \frac{1}{y_n} = \frac{1}{b} \neq 0$. Then apply the theorem again, we know that $\left\{\frac{x_n}{y_n}\right\}$ diverges to infinity. □

Example 3.6.4

Consider the sequence $\left\{\frac{n}{\sin x}\right\}$

Proof:

Sketch of the proof.
It is obvious that $\left|\frac{1}{\sin x}\right| \geq 1$, then apply the theorem. □

Example 3.6.5

Consider the sequence $\{n \arctan n\}$.

Proof:

Sketch of the proof:
It is obvious that $\lim\limits_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$, then we apply the theorem. □

Example 3.6.6

Discuses the limit of $\lim\limits_{n\to\infty} \frac{a_0 n^k + a_1 n^{k-1} + \cdots + a_{k-1} n + a_k}{b_0 n^l + b_1 n^{l-1} + \cdots + b_{l-1} n + b_l}$, $a_0, b_0 \neq 0; k, l \in \mathbb{N}$.

We have

$$\lim_{n\to\infty} \frac{a_0 n^k + a_1 n^{k-1} + \cdots + a_{k-1} n + a_k}{b_0 n^l + b_1 n^{l-1} + \cdots + b_{l-1} n + b_l} = \lim_{n\to\infty} \left(n^{k-l} \cdot \underbrace{\frac{a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots + \frac{a_k}{n^k}}{b_0 + \frac{b_1}{n} + \frac{b_2}{n^2} + \cdots + \frac{b_l}{n^l}}}_{(*)} \right)$$
$$= \begin{cases} \infty & \text{if } k > l, \\ \frac{a_n}{b_n} & \text{if } k = l, \\ 0 & \text{if } k < l. \end{cases}$$

It is obvious that the limit of $(*)$ is $\frac{a_0}{b_0} \neq 0$. The key to solve this limit lies in n^{k-l}.

3.6.2 Operations with Infinity

1. $(+\infty) + (+\infty) = +\infty$,
2. $(+\infty) - (-\infty) = +\infty$,
3. $\infty \pm$ bounded quantity $= \infty$,
4. $(+\infty) \cdot (+\infty) = +\infty$,
5. $(+\infty) \cdot (-\infty) = -\infty$.

The results of the above operations are deterministic, but there are also cases where the results are indeterminate.

$$
\begin{aligned}
(+\infty) - (+\infty) &= ?, \\
(+\infty) + (-\infty) &= ?, \\
\infty + \infty &= ?, \\
0 \cdot \infty &= ?, \\
\frac{0}{\infty} &= ?, \\
\frac{\infty}{\infty} &= ?, \\
&\vdots
\end{aligned}
$$

These indeterminate forms are known as **indeterminate forms**. The following theorems are very useful for solving limit problems involving indeterminate forms.

Definition 3.6.2 ***Monotonically Increasing Sequence***
A **monotonically increasing sequence** is a sequence $\{x_n\}$ such that $\forall n \in \mathbb{N}$, the terms satisfy $x_n \leq x_{n+1}$. Denoted $\{x_n\} \uparrow$.

Definition 3.6.3 ***Strictly Increasing Sequence***
A **strictly increasing sequence** is a sequence $\{x_n\}$ such that $\forall n \in \mathbb{N}$, the terms satisfy $x_n < x_{n+1}$. Denoted $\{x_n\}$ (strictly) $\uparrow$.

Definition 3.6.4 ***Monotonically Decreasing Sequence***
A **monotonically decreasing sequence** is a sequence $\{x_n\}$ such that $\forall n \in \mathbb{N}$, the terms satisfy $x_n \geq x_{n+1}$. Denoted $\{x_n\} \downarrow$.

Definition 3.6.5 ***Strictly Decreasing Sequence***
A **strictly decreasing sequence** is a sequence $\{x_n\}$ such that $\forall n \in \mathbb{N}$, the terms satisfy $x_n > x_{n+1}$. Denoted $\{x_n\}$ (strictly) $\downarrow$.

Theorem 3.6.4 ***Stolz Theorem***
Suppose we have a strictly increasing sequence $\{y_n\}$ and $\lim\limits_{n\to\infty} y_n = +\infty$. If $\lim\limits_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = A$, ($A$ is a finite quantity, or $\pm\infty$, but not ∞) then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = A$.

Proof:

Step (1), suppose $A = 0 = \lim\limits_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$, then $\forall \epsilon > 0$, $\exists N_1$ such that $\forall n > N_1$ we have $\left|\frac{x_n - x_{n-1}}{y_n - y_{n-1}}\right| < \epsilon \Rightarrow |x_n - x_{n-1}| < \epsilon(y_n - y_{n-1})$. We have

$$x_n - x_{N_1} = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_{N_1+1} - x_{N_1}),$$

then

$$\left|x_n - x_{N_1}\right| \leq \underbrace{|x_n - x_{n-1}|}_{<\epsilon(y_n - y_{n-1})} + \underbrace{|x_{n-1} - x_{n-2}|}_{<\epsilon(y_{n-1} - y_{n-2})} + \cdots + \underbrace{\left|x_{N_1+1} - x_{N_1}\right|}_{<\epsilon(y_{N_1+1} - y_{N_1})} = \epsilon(y_n - y_{N_1}),$$

divide both side by y_n to get

$$\left|\frac{x_n}{y_n} - \frac{x_{N_1}}{y_n}\right| \leq \epsilon\left(1 - \frac{y_{N_1}}{y_n}\right),$$

Here, $n > N_1$ and the sequence $\{y_n\}$ is strictly increasing, so $1 - \frac{y_{N_1}}{y_n} \in (0,1)$. Therefore we have

$$\left|\frac{x_n}{y_n} - \frac{x_{N_1}}{y_n}\right| \leq \epsilon\left(1 - \frac{y_{N_1}}{y_n}\right) < \epsilon,$$
$$\left|\frac{x_n}{y_n}\right| < \epsilon + \left|\frac{x_{N_1}}{y_n}\right|,$$

Here, since ϵ is chosen, then the corresponding N_1 is fixed. So x_{N_1} is fixed. Since $\lim\limits_{n\to\infty} y_n = +\infty$, for any (given), $\exists N > N_1, \forall n > N$ we have $\left|\frac{x_{N_1}}{y_n}\right| < \epsilon$. Then

$$\left|\frac{x_n}{y_n}\right| < \epsilon + \epsilon = 2\epsilon.$$

That is $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = 0 = A$.
Step (2), suppose $A \neq 0$.
Let $x'_n = x_n - ay_n$, then

$$\lim_{n\to\infty} \frac{x'_n - x'_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty}\left(\frac{x_n - x_{n-1}}{y_n - y_{n-1}} - a\right) = 0,$$

then

$$\lim_{n\to\infty} \frac{x'_n}{y_n} = 0 \Rightarrow \lim_{n\to\infty} \frac{x_n - ay_n}{y_n} = \lim_{n\to\infty}\left(\frac{x_n}{y_n} - a\right) = 0 \Rightarrow \lim_{n\to\infty}\frac{x_n}{y_n} = A.$$

Step (3), suppose $A = +\infty$.
We have $\lim\limits_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = +\infty$. That is $\forall G > 0, \exists N, \forall n > N$ we have $\frac{x_n - x_{n-1}}{y_n - y_{n-1}} > G$. W.O.L.G., let $G = 1$, we have

$$\frac{x_n - x_{n-1}}{y_n - y_{n-1}} > 1 \Rightarrow x_n - x_{n-1} > y_n - y_{n-1} > 0,$$

Here, $y_n - y_{n-1} > 0$ is because the sequence $\{y_n\}$ is strictly increasing. This tells us that the sequence $\{x_n\}$ is also strictly increasing. Then

$$x_n - x_N = \underbrace{(x_n - x_{n-1})}_{> y_n - y_{n-1}} + \underbrace{(x_{n-1} - x_{n-2})}_{> y_{n-1} - y_{n-2}} + \cdots + \underbrace{(x_{N+1} - x_N)}_{> y_{N+1} - y_N} > y_n - y_N,$$

Now, since N is fixed, so $y_n - y_N \xrightarrow{n\to\infty} +\infty$, therefore

$$x_n - x_N = \underbrace{(x_n - x_{n-1})}_{> y_n - y_{n-1}} + \underbrace{(x_{n-1} - x_{n-2})}_{> y_{n-1} - y_{n-2}} + \cdots + \underbrace{(x_{N+1} - x_N)}_{> y_{N+1} - y_N} > y_n - y_N \xrightarrow{n\to\infty} +\infty,$$

So the sequence $\{x_n\}$ also diverges to infinity. (So x_n will not be zero for large n.) More importantly the sequence $\{x_n\}$ goes to infinity faster than the sequence $\{y_n\}$. Thus,

$$\lim_{n\to\infty} \frac{y_n}{x_n} = 0 \Rightarrow \lim_{n\to\infty} \frac{x_n}{y_n} = +\infty = A.$$

Step (3), suppose $A = -\infty$.
The proof is similar. □

Example 3.6.7
Using the Stolz theorem to show if $\lim\limits_{n\to\infty} a_n = A$, then $\lim\limits_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = A$.

Proof:
Let $x_n = a_1 + a_2 + \cdots + a_n$ and $y_n = n$. Since $\lim\limits_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim\limits_{n\to\infty} \frac{a_n}{1} = A$, then by the Stolz theorem the proof is done. □

Remark. Compare the proof of this example by the Stolz theorem with the proof by ϵ-δ game.

Example 3.6.8

Compute $\lim\limits_{n\to\infty} \frac{1^k+2^k+\cdots+n^k}{n^{k+1}}$.

Let $x_n = 1^k + 2^k + \cdots + n^k$ and $y_n = n^{k+1}$,

$$\lim_{n\to\inf} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} \frac{n^k}{n^{k+1} - (n-1)^{k+1}} = \lim_{n\to\infty} \frac{n^k}{\binom{k+1}{1}n^k - \binom{k+1}{2}n^{k-1} + \dots} = \frac{1}{\binom{k+1}{1}} = \frac{1}{k+1}.$$

Example 3.6.9

Suppose $\lim\limits_{n\to\infty} a_n = A$, compute $\lim\limits_{n\to\infty} \frac{a_1+2a_2+3a_3+\cdots+na_n}{n^2}$.

Let $x_n = a_1 + 2a_2 + 3a_3 + \cdots + na_n$ and $y_n = n^2$,

$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} \frac{na_n}{n^2 - (n-1)^2} = \lim_{n\to\infty} \left(\frac{n}{2n-1} \cdot a_n\right) = \frac{A}{2}.$$

3.7 Convergence Criteria for Sequences

3.7.1 Monotone Convergence Theorem

We have discussed sequences for a while. For now we know that for a sequence

$$\text{convergent} \implies \text{bounded}$$

but not the other way around. In other words, a convergent sequence is bounded, but a bounded sequence does not necessarily converge.

We want to ask two questions:

1. What kind of condition is needed to make a bounded sequence converge?
2. What kind of weak conclusion do we have for a bounded sequence?

Theorem 3.7.1 ***Monotone Convergence Theorem***

A monotonic and bounded sequence is convergent.

Proof:

W.O.L.G., $\{x_n\}$ is monotonically increasing and is bounded above. Then a set with elements from the sequence $\{x_n\}$ is bounded above, which means there exists a supremum, denoted β. For β being a supremum, it has two meaning:

1.) β is an upper bound: $\forall\, n, x_n \le \beta$;
2.) β is the smallest upper bound: $\forall\, \epsilon > 0$, $\exists\, n_0$ such that $x_{n_0} > \beta - \epsilon$.
Let $N = n_0$, $\forall\, n > N$ we have $\beta - \epsilon < x_{n_0} \le x_n \le \beta$. That is

$$|x_n - \beta| < \epsilon, \text{ or } \lim_{n\to\infty} x_n = \beta.$$

□

Remark. We have answered the first question by stating the Monotone Convergence Theorem. That is if a bounded sequence is monotonic then this sequence convergences.

Remark. Different from the ϵ-δ definition of the limit of a sequence we studied before, the Monotone Convergence Theorem is important because when using the ϵ-δ definition we are kind of verifying that the limit of a sequence, say $\{x_n\}$ is A. That is if we do not know $\lim\limits_{n\to\infty} x_n = A$, or A is unknown, we will not be able to find ϵ, N, and δ. The beauty of the Monotone Convergence Theorem is that it allows us to prove or disprove that a sequence is convergent without knowing its limit.

Note: The red inequality is due to monotonically increasing, and we find N by the definition of supremum, which comes from the condition that the sequence is bounded. It should be clear why monotonic and bounded sequence is convergent.

Example 3.7.1

Suppose $x_1 > 0$, $x_{n+1} = 1 + \frac{x_n}{1+x_n}$, $n = 2, 3, \ldots$, prove that the sequence $\{x_n\}$ converges and find the limit.

Proof:

By mathematical induction, we know that $x_n > 0$, then it is obvious that $1 < x_n < 2$, for $n = 2, 3, \ldots$. This is saying that the sequence $\{x_n\}$ is bounded. For monotonicity, we have

$$x_{n+1} - x_n = \left(1 + \frac{x_n}{1+x_n}\right) - \left(1 + \frac{x_{n-1}}{1+x_{n-1}}\right) = \frac{x_n - x_{n-1}}{(1+x_{n-1})(1+x_n)}.$$

Since $(1 + x_{n-1})(1 + x_n) > 0$, we observe that $x_{n+1} - x_n$ and $x_n - x_{n-1}$ have the same sign. This tells us that the sequence $\{x_n\}$ is monotonic. Therefore, the sequence $\{x_n\}$ converges.
To find the limit, since we know the sequence $\{x_n\}$ converges. Let $\lim\limits_{n\to\infty} x_n = A$. We take the limit of both sides of the given condition

$$x_{n+1} = 1 + \frac{x_n}{1+x_n} \implies \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(1 + \frac{x_n}{1+x_n}\right)$$

to get

$$A = 1 + \frac{A}{1+A}.$$

We get $A = \frac{1\pm\sqrt{5}}{2}$, but $\frac{1-\sqrt{5}}{2} < 0$, which is not possible since $1 < x_n < 2$, so $\lim\limits_{n\to\infty} x_n = A = \frac{1+\sqrt{5}}{2}$. □

Example 3.7.2

Suppose $0 < x_1 < 1$ and $x_{n+1} = x_n(1 - x_n)$, prove that the sequence $\{x_n\}$ converges and find the limit.

Proof:

By mathematical induction, we know that $0 < x_n < 1$. Also

$$x_{n+1} - x_n = x_n(1 - x_n) - x_n = -x_n^2 < 0,$$

so $\{x_n\}$ ↓ and is bounded below. By the Monotone Convergence theorem, the sequence $\{x_n\}$ converges. W.O.L.G., let $\lim\limits_{n\to\infty} x_n = A$.

$$x_{n+1} = x_n(1 - x_n) \implies \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(x_n(1 - x_n)\right)$$

to get

$$A = A(1 - A) \implies A = 0 \implies \lim_{n\to\infty} x_n = 0.$$

Here, $A = 1$ is impossible since $0 < x_n < 1$. □

Remark. From the example above, we know that $\lim\limits_{n\to\infty} x_n = 0$, which means that the sequence $\{x_n\}$ is infinitesimal. Can we know more about this infinitesimal sequence? In other words, can we know how fast the rate approaches zero as n approaches infinity, compared with other sequences such as $\left\{\frac{1}{n}\right\}$, $\left\{\frac{1}{n^2}\right\}$, $\left\{\frac{1}{\sqrt{n}}\right\}$, and $\left\{\frac{1}{\log n}\right\}$?

Let's compute $\lim\limits_{n\to\infty} \frac{x_n}{\frac{1}{n}} = \lim\limits_{n\to\infty} nx_n = \lim\limits_{n\to\infty} \frac{n}{\frac{1}{x_n}}$. It should be obvious that the sequence $\left\{\frac{1}{x_n}\right\}$ is monotonically increasing and its limit is infinity so we could apply Stolz theorem.

$$\lim_{n\to\infty}\frac{n}{\frac{1}{x_n}}=\lim_{n\to\infty}\frac{n-(n-1)}{\frac{1}{x_n}-\frac{1}{x_{n-1}}}=\lim_{n\to\infty}\frac{x_{n-1}x_n}{x_{n-1}-x_n}=\lim_{n\to\infty}\frac{x_{n-1}^2(1-x_{n-1})}{x_{n-1}-x_{n-1}(1-x_{n-1})}=\lim_{n\to\infty}\frac{x_{n-1}^2(1-x_{n-1})}{x_{n-1}^2}$$
$$=\lim_{n\to\infty}(1-x_{n-1})1-0=1.$$

Thus, $\lim\limits_{n\to\infty}\frac{x_n}{\frac{1}{n}}=1$. This tells us that the rate at which the sequence $\{x_n\}$ approaches zero is the same as the rate at which the sequence $\left\{\frac{1}{n}\right\}$ approaches zero. In other words, these two sequences $\{x_n\}$ and $\left\{\frac{1}{n}\right\}$ are asymptotically equivalent infinitesimals.

Example 3.7.3

Suppose $x_1=\sqrt{2}, x_{n+1=\sqrt{3+2x_n}}, n\in\mathbb{N}$, prove the sequence $\{x_n\}$ converges and compute the limit.

Proof:

We have $0<x_1=\sqrt{2}<3$. If $0<x_n<3$, then $0<x_{n+1}=\sqrt{3+2x_n}<3$. We know that by mathematical induction we can shown that $0<x_n<3$. Also,

$$x_{n+1}-x_n=\sqrt{3+2x_n}-x_n=\frac{(3-x_n)(1+x_n)}{\sqrt{3+2x_n}+x_n}>0.$$

So, the sequence $\{x_n\}$ $\uparrow$. By Monotone Convergence Theorem, we know the sequence $\{x_n\}$ converge. W.O.L.G., let $\lim\limits_{n\to\infty}x_n=A$, we have

$$x_{n+1}=\sqrt{3+2x_n}\implies\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}\sqrt{3+2x_n}\implies A=\sqrt{3+2A}\implies A=3\text{ or }A=-1.$$

Since $0<x_n<3$, we have $\lim\limits_{n\to\infty}x_n=A=3$. □

Example 3.7.4

The Fibonacci sequence $a_{n+1}=an+a_{n-1}, a_1=a_2=1$. Let $b_n=\frac{a_{n+1}}{a_n}$, then b_n-1 is the rate of increase. Let's discuss the sequence $\{b_n\}$.

$$b_n=\frac{a_{n+1}}{a_n}=\frac{a_n+a_{n-1}}{a_n}=1+\frac{a_{n-1}}{a_n}=1+\frac{1}{b_{n-1}}.$$

$$\text{When } b_{n-1}<\frac{\sqrt{5}+1}{2}, b_n>1+\frac{1}{\frac{\sqrt{5}+1}{2}}=\frac{\sqrt{5}+1}{2},$$
$$\text{when } b_{n-1}>\frac{\sqrt{5}+1}{2}, b_n<1+\frac{1}{\frac{\sqrt{5}+1}{2}}=\frac{\sqrt{5}+1}{2}.$$

Therefore, the sequence $\{b_n\}$ is non-monotonic. We now create two subsequences from the sequence $\{b_n\}$, one composes of odd-indexed terms and another composes of even-indexed terms. We then observe that

$$b_{2n-1}\in\left(0,\frac{\sqrt{5}+1}{2}\right),$$
$$b_{2n}\in\left(\frac{\sqrt{5}+1}{2},+\infty\right).$$

We now check the monotonicity of these two subsequences.

$$b_{2k+1} - b_{2k-1} = 1 + \frac{1}{1+\frac{1}{b_{2k-1}}} - b_{2k-1} = \frac{1 + b_{2k-1} - b_{2k-1}^2}{1 + b_{2k-1}}$$
$$= \frac{\left(\frac{\sqrt{5}+1}{2} - b_{2k-1}\right)\left(\frac{\sqrt{5}+1}{2} + b_{2k-1}\right)}{1 + b_{2k-1}} > 0.$$

Therefore, the sequence $\{b_{2n-1}\} \uparrow$. Also the sequence $\{b_{2n-1}\}$ is bounded then by Monotone Convergence Theorem, we know that the sequence $\{b_{2n-1}\}$ converges.

$$b_{2k+2} - b_{2k} = 1 + \frac{1}{1+\frac{1}{b_{2k-1}}} - b_{2k} = \frac{\left(\frac{\sqrt{5}+1}{2} - b_{2k}\right)\left(\frac{\sqrt{5}+1}{2} + b_{2k}\right)}{1 + b_{2k}} < 0.$$

Therefore, the sequence $\{b_{2n}\} \downarrow$. Also the sequence $\{b_{2n}\}$ is bounded then by Monotone Convergence Theorem, we know that the sequence $\{b_{2n}\}$ converges. W.O.L.G., let $\lim_{n\to\infty} b_{2n-1} = A$ and $\lim_{n\to\infty} b_{2n} = B$.

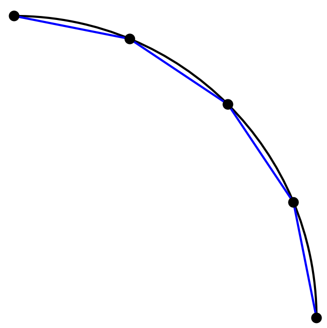
$$b_{2n+1} = 1 + \frac{1}{1+\frac{1}{b_{2n-1}}} = \frac{1 + 2b_{2n-1}}{1 + b_{2n-1}} \implies A = \frac{1+2A}{1+A} \implies A^2 - A - 1 = 0,$$
$$A = \frac{1 \pm \sqrt{5}}{2}.$$

Since $b_{2n-1} \in \left(0, \frac{\sqrt{5}+1}{2}\right)$, so $\lim_{n\to\infty} b_{2n-1} = A = \frac{1+\sqrt{5}}{2}$. Similarly, $\lim_{n\to\infty} b_{2n} = B = \frac{1+\sqrt{5}}{2}$. Therefore, $\lim_{n\to\infty} b_n = \frac{1+\sqrt{5}}{2}$. Then $\lim_{n\to\infty} b_n - 1 = \frac{1+\sqrt{5}}{2} - 1 = \frac{\sqrt{5}-1}{2} \simeq 0.618$.

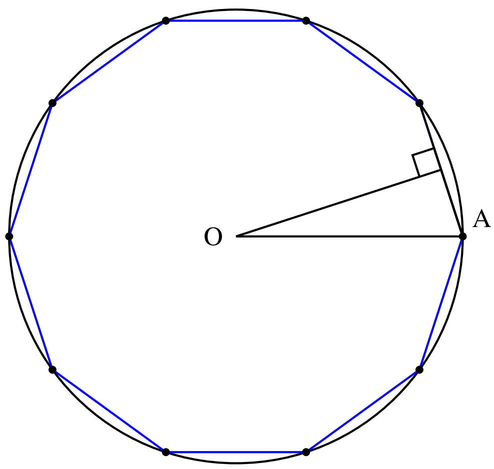
Remark. Some readers may ask that where we get $\frac{\sqrt{5}+1}{2}$ at the first place? Consider $b_n = 1 + \frac{1}{b_{n-1}}$, if we assume $\{b_n\}$ converges then we could compute the limit, which is $\frac{\sqrt{5}+1}{2}$.

3.7.2 From Monotone Convergence Theorem to π and e

We begin this section with a question: "How do we calculate the length of a curve?". A logical approach is to mark points at equal intervals along the curve, connect these points with straight lines, and sum the lengths of these lines together to estimate the curve's length. For greater accuracy, we can decrease the distance between the points. When the distance between the points becomes sufficiently small, we can be confident that we have a close approximation of the curve's length.



The procedure of decreasing the distance between the points to estimate the curve's length is essentially calculating the limit of the sum of the lengths of these (blue) straight lines as the distance between the points approaches zero. In other words, the length of the curve is the limit of the sum of the lengths of the blue line segments.



The perimeter of a regular n-gon inscribed in the unit circle is $2n \sin \frac{180°}{n}$. Let the half perimeter be $L_n = n \sin \frac{180°}{n}$. We then have a sequence $\{L_n\}$.

Example 3.7.5

Prove that the sequence $\{L_n\}$ with $L_n = n \sin \frac{180°}{n}$ converges.

Proof:

Let $t = \frac{180°}{n(n+1)}$ and $n \geq 3$ is obvious, so $t \leq 15°, nt \leq 45°$. Then

$$\tan nt = \tan\left((n-1)\,t + t\right) = \frac{\tan(n-1)t + \tan t}{1 - \tan(n-1)t \cdot \tan t} > \tan(n-1)t + \tan t,$$

In the red inequality, we can verify that $(n-1)t = \frac{(n-1)180°}{n(n+1)}$, which decreases for $n \geq 3$ and takes value in $(0°, 30°]$ Then $\tan(n-1)t \cdot \tan t \leq \tan 30° \cdot \tan 45° < 1$. Thus we have

$$\tan nt = \tan\left((n-1)\,t + t\right) > \tan(n-1)t + \tan t.$$

For the blue term we have

$$\tan(n-1)t = \tan\left((n-2)\,t + t\right) > \tan(n-2)t + \tan t.$$

Therefore we have

$$\tan(n-1)t + \tan t > \tan(n-2)t + 2\tan t.$$

The pattern should be obvious now, that is

$$\tan nt > \tan(n-1)t + \tan t > \tan(n-2)t + 2\tan t > \cdots > \tan(n-n)t + n\tan t = n\tan t.$$

That is we have shown $\tan nt > n \tan t$.
Also,

$$\begin{aligned}
\sin(n+1)t &= \sin nt \cos t + \cos nt \sin t \\
&= \sin nt \cos t + \sin nt \cos t \frac{\tan t}{\tan nt} \\
&= \sin nt \underbrace{\cos t}_{\leq 1} \cdot \underbrace{\left(1 + \frac{\tan t}{\tan nt}\right)}_{<\frac{1}{n}} \\
&< \sin nt \cdot \left(\frac{n+1}{n}\right)
\end{aligned}$$

That is $\sin(n+1)t < \sin nt \cdot \left(\frac{n+1}{n}\right)$. Substitute $t = \frac{180°}{n(n+1)}$ to get

$$\begin{aligned}
\sin\frac{180°}{n} &< \sin\frac{180°}{n+1} \cdot \left(\frac{n+1}{n}\right) \\
n \sin\frac{180°}{n} &< (n+1)\sin\frac{180°}{n+1}
\end{aligned}$$

We get

$$L_n = n \sin \frac{180°}{n} < (n+1) \sin \frac{180°}{n+1} = L_{n+1}.$$

Therefore, we have shown the sequence $\{L_n\}$ is monotonically increasing. For the boundedness, lets start with the area of a regular n-gon inscribed in the unit circle. Let this area be S_n, we know

$$S_n = 2n \cdot \frac{1}{2} \sin \frac{180°}{n} \cdot \cos \frac{180°}{n} < 4.$$

That is

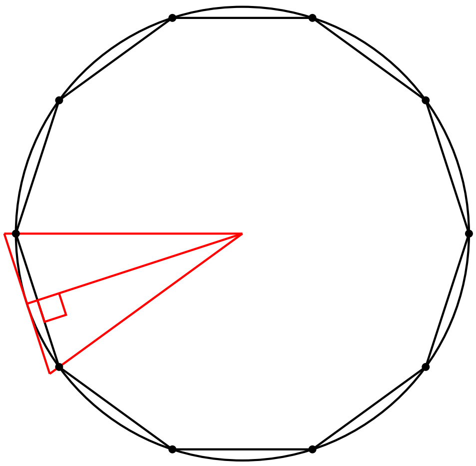
$$n \sin \frac{180°}{n} < \frac{4}{\cos \frac{180°}{n}} < 8.$$

The LHS equals L_n therefore we have $L_n < 8, \forall\, n \geq 3$. Thus the sequence $\left\{L_n\right\}$ is bounded. By the Monotone Convergence Theorem, the sequence $\left\{L_n\right\}$ is convergent. Therefore, we know that the limit of the sequence $\left\{L_n\right\}$ exist and is the half of the perimeter of the unit circle which is π. That is

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} n \sin \frac{180°}{n} = \pi.$$

$\square$

Remark. We can keep going to calculate the area of the unit circle. We have calculated the area of a regular n-gon inscribed in the unit circle, S_n. Now we want to calculated the are of a regular n-gon circumscribed about a unit circle, S'_n.



We have $S'_n = n \tan \frac{180°}{n} = \frac{n \sin \frac{180°}{n}}{\cos \frac{180°}{n}}$. We also have this inequality for the area of the unit circle S:

$$n \sin \frac{180°}{n} \cos \frac{180°}{n} = S_n < S < S'_n = n \tan \frac{180°}{n} = \frac{n \sin \frac{180°}{n}}{\cos \frac{180°}{n}}$$

Since $n \sin \frac{180°}{n} \xrightarrow{n\to\infty} \pi$ and $\cos \frac{180°}{n} \xrightarrow{n\to\infty} 1$, therefore when $n \to \infty$ we have

$$\pi = \lim_{n\to\infty} S_n < \lim_{n\to\infty} S < \lim_{n\to\infty} S'_n = \pi$$

By the squeeze theorem, we have $\lim\limits_{n\to\infty} S = \pi$, which verifies that the area of a unit circle is indeed π.

Remark. We have shown, in the example above, that if the perimeter of a circle is 2π by definition, then the area of the circle is π.

Now let's talk about e.

Example 3.7.6

We have two sequences $x_n = \left(1 + \frac{1}{n}\right)^n$ and $y_n = \left(1 + \frac{1}{n}\right)^{n+1}$. Do the sequences $\{x_n\}$ and $\{y_n\}$ converge? If so, find their respective limits.

Proof:

For the sequence x_n we have

$$x_n = \left(1+\frac{1}{n}\right)^n \cdot 1 \leq \left(\frac{n\cdot\left(1+\frac{1}{n}\right)+1}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = x_{n+1}$$

That is the sequence $\{x_n\}$ is increasing, therefore $\{x_n\}$ has upper limit. For $\{y_n\}$, we have

$$\frac{1}{y_n} = \left(\frac{n}{n+1}\right)^{n+1} \cdot 1 \leq \left(\frac{(n+1)\cdot\left(\frac{n}{n+1}\right)+1}{n+2}\right)^{n+2} = \left(\frac{n+1}{n+2}\right)^{n+2} = \frac{1}{y_{n+1}}$$

The sequence $\{y_n\}$ is decreasing, therefore $\{y_n\}$ has lower limit. It is obvious that

$$2 < x_1 < x_n < y_n \leq y_1 = 4.$$

Therefore, we know $\lim\limits_{x\to\infty} x_n$ and $\lim\limits_{n\to\infty} y_n$ converge. Also,

$$y_n = x_n\left(1+\frac{1}{n}\right)$$

Taking limit on both side, we get $\lim\limits_{x\to\infty} x_n = \lim\limits_{n\to\infty} y_n$. Using the binomial theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} x_n &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\binom{n}{k}\frac{1}{n^k} \\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{n\cdot(n-1)\ldots(n-k+1)}{k!}\cdot\frac{1}{n^k} \\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{n\cdot(n-1)\ldots(n-k+1)}{n^k}\cdot\frac{1}{k!} \\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{(n-0)\cdot(n-1)\ldots(n-(k-1))}{n^k}\cdot\frac{1}{k!} \\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\prod_{j=0}^{k-1}\frac{n-j}{n}\cdot\frac{1}{k!} \\
&= \lim_{n\to\infty}\sum_{k=0}^{\infty}\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)\cdot\frac{1}{k!} \\
&= \sum_{k=0}^{\infty}\frac{1}{k!} \\
&= e
\end{aligned}$$

□

Note: The red inequality is due to the geometric mean being less than or equal to the arithmetic mean. There are $n+1$ terms multiplied together on the LHS of the red inequality.

Example 3.7.7

Let $a_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p}$, with $p > 0$. It is obvious that $\{a_n\}$ is increasing. Prove that when $p > 1$ the sequence $\{a_n\}$ converges and when $p \leq 1$ it diverges to infinity.

Proof:

For $p > 1$, let $\frac{1}{2^{p-1}} = r$. It is obvious that $0 < r < 1$. Then we have

$$\begin{aligned}
&\frac{1}{2^p} + \frac{1}{3^p} \leq \frac{2}{2^p} = r \\
&\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \leq \frac{4}{4^p} = r^2 \\
&\frac{1}{8^p} + \frac{1}{9^p} + \cdots + \frac{1}{15^p} \leq \frac{8}{8^p} = r^3 \\
&\vdots \\
&\frac{1}{(2^k)^p} + \frac{1}{(2^k+1)^p} + \cdots + \frac{1}{(2^{k+1}-1)^p} \leq \frac{2^k}{(2^k)^p} = \frac{1}{(2^k)^{p-1}} = r^k
\end{aligned}$$

It should now be obvious that $\forall\, n,\ a_n < a_{2^n-1} < 1 + r^2 + \cdots + r^{n-1} = \frac{1}{1-r}$. Therefore, the sequence $\{a_n\}$ is bounded above so as to have an upper limit. That is $\lim\limits_{n\to\infty} a_n$ exists.

For $0 < p \leq 1$, we have

$$\begin{aligned}
&\frac{1}{2^p} \geq \frac{1}{2} \\
&\frac{1}{3^p} + \frac{1}{4^p} \geq \frac{1}{4^p} + \frac{1}{4^p} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\
&\frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \frac{1}{8^p} \geq \frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \\
&\vdots \\
&\frac{1}{(2^k+1)^p} + \frac{1}{(2^k+2)^p} + \cdots + \frac{1}{(2^{k+1})^p} \geq \frac{2^k}{(2^{k+1})^p} > \frac{2^k}{2^{k+1}} = \frac{1}{2}
\end{aligned}$$

It should now be obvious that as $n \to \infty$, the sequence $\{a_n\}$ is not bounded above. □

Note: Be aware that

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \ldots$$

are different.

Let us compare two quantities, $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ and $\ln n$, which both tend to positive infinity, by subtracting one from the other.

Example 3.7.8

Let $b_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$ and show that the sequence $\{b_n\}$ converges.

Proof:

Recall that we have $\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}$. From the right inequality, we get

$$1 < (n+1)\ln\frac{n+1}{n} \implies \frac{1}{n+1} < \ln\frac{n+1}{n}.$$

From the left inequality, we get

$$n\ln\frac{n+1}{n} < 1 \implies \ln\frac{n+1}{n} < \frac{1}{n}.$$

That is

$$\frac{1}{n+1} < \ln\frac{n+1}{n} < \frac{1}{n}.$$

Next we will verify if $\{b_n\}$ converges or not.

$$b_{n+1} - b_n = \frac{1}{n+1} - \ln(n+1) + \ln n = \frac{1}{n+1} - \ln\frac{n+1}{n}$$

Using the red inequality, we know that $b_{n+1} - b_n < 0$. That is the sequence $\{b_n\}$ is strictly decreasing. We also have

$$\begin{aligned} b_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \\ &> \ln\frac{2}{1} + \ln\frac{3}{2} + \ln\frac{4}{3} + \cdots + \ln\frac{n+1}{n} - \ln n \\ &= \ln(n+1) - \ln n = \ln\frac{n+1}{n} > 0 \end{aligned}$$

Therefore, we conclude that the sequence $\{b_n\}$ converges. □

Note: We briefly touch the limit of the sequence $\{b_n\}$ here, which is called the **Euler Constant** or **Euler-Mascheroni constant** and it is denoted $\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \ln n \simeq 0.5772156649....$, an irrational number.

Remark. As we can see that the difference between these two infinite quantities, the sequence $\{a_n\}$ and $\ln n$, are not too big as n goes to infinity. In other words, the difference between these two infinite quantities is the **Euler Constant**, γ, as n approaches infinity. We conclude that the two infinite quantities are **asymptotically equivalent** or **equivalent**.

Example 3.7.9

Prove $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}\right) = \ln 2$.

Proof:

Recall in the last example $b_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$ and $b_n \xrightarrow{n\to\infty} \gamma$. It should be obvious that $b_{2n} \xrightarrow{n\to\infty} \gamma$. The limit of $\{b_n\}$ is γ, so no matter how large n is the limit of the sequence $\{b_n\}$, $\{b_{2n}\}$, or $\{b_{5n}\}$ will always be γ.

$$\begin{aligned} b_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \\ b_{2n} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n} - \ln 2n \\ b_{2n} - b_n &= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} - \ln 2n + \ln n \\ &= \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} - \ln 2 \xrightarrow{n\to\infty} 0 \end{aligned}$$

It is obvious that we take the limit we have what we want to show. □

Note: Before proving, it is worth to point out that

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}\right) \neq \lim_{n\to\infty} \frac{1}{n+1} + \lim_{n\to\infty} \frac{1}{n+2} + \lim_{n\to\infty} \frac{1}{n+3} + \cdots + \lim_{n\to\infty} \frac{1}{2n}$$

We have discussed this reason why it is not the case in previous lectures. In short, the number of terms inside the limit increases as n goes to infinity.

Example 3.7.10

Find out if the sequence $d_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1}\frac{1}{n}$ converges.

Proof:

Recall,

$$
\begin{aligned}
b_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \\
b_{2n} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n} - \ln 2n \\
b_{2n} - b_n &= 1 + (\frac{1}{2} - 1) + \frac{1}{3} + (\frac{1}{4} - \frac{1}{2}) + \cdots + (\frac{1}{2n} - \frac{1}{n}) - \ln 2n + \ln n \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n} - \ln 2 \\
&= d_n - \ln 2 \xrightarrow{n\to\infty} 0
\end{aligned}
$$

Thus, we have $d_n \xrightarrow{n\to\infty} \ln 2$. □

It is a good time to have a summary of the previous several examples and to summarize it yourself and try to learn the techniques used in proving these problems.

Before talking about a very important theorem, **The Nested Intervals Theorem**, let us introduce a definition.

3.7.3 The Nested Intervals Theorem

Definition 3.7.1 ***Nested Closed Intervals***

If a sequence of closed intervals $I_n = \{[a_n, b_n]\}$ satisfy:

1. $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$ for $n = 1, 2, 3, \ldots$; and
2. $b_n - a_n \xrightarrow{n\to\infty} 0$, then

then the sequence of intervals I_n is called the Nested Closed Intervals.

Theorem 3.7.2 ***The Nested Intervals Theorem***

For a nested closed intervals $\{[a_n, b_n]\}$, there exists an unique real number ξ such that $\xi \in [a_n, b_n], \forall\, n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \xi$.

Proof:

By the definition of nested closed intervals, it is obvious that

$$a_1 \leq \cdots \leq a_{n-1} \leq a_n < b_n \leq b_{n-1} \leq \cdots \leq b_1.$$

Hint: as n increases the closed intervals are nested or the intervals are shrinking.
It should also be obvious that the sequence $\{a_n\}$ is monotonically increasing with an upper bound b_1, and the sequence $\{b_n\}$ is monotonically decreasing with a lower bound a_1. Therefore $\{a_n\}$ and $\{b_n\}$ converge. Naturally, we can assume $\lim_{n\to\infty} a_n = \xi \in \mathbb{R}$, or ξ is the supremum of $\{a_n\}$, then $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \big(a_n + (b_n - a_n)\big) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} (b_n - a_n) = \xi - 0 = \xi$. That is $\lim_{n\to\infty} b_n = \xi$ as well. Also, we know ξ is the infimum of $\{b_n\}$, that is $a_n \leq \xi \leq b_n$; equivalently, $\xi \in [a_n, b_n], \forall\, n \in \mathbb{N}$. We only need to show the uniqueness of ξ.
To show the uniqueness of ξ, we assume $\exists\, \xi' \in [a_n, b_n], \forall\, n \in \mathbb{N}$ such that $a_n \leq \xi' \leq b_n$, then we take the limit of the inequalities and apply the squeeze theorem to get $\xi \leq \xi' \leq \xi$. It is straightforward that $\xi' = \xi$. □

Theorem 3.7.3

The real number set, $\mathbb{R}$, is uncountable.

Proof:

We use prove by contradiction and apply the Nested Intervals Theorem to prove this theorem.
Suppose $\mathbb{R}$ is countable, for the sake of contradiction, that is we can list the set of real numbers as $\mathbb{R} = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}, \forall\, n \in \mathbb{N}$. Then to show this is not the case, we only need to find one real number that is not listed. To do so, we first construct a closed interval $[a_1, b_1]$ such that $x_1 \notin [a_1, b_1]$, then we construct a new closed interval by dividing $[a_1, b_1]$ into three parts, $\left[a_1, \frac{2a_1+b_1}{3}\right]$, $\left[\frac{2a_1+b_1}{3}, \frac{a_1+2b_1}{3}\right]$, $\left[\frac{2a_1+b_1}{3}, b_1\right]$. It

is obvious that x_2 will be in only one of the three. So, pick any interval from the three that does not contain x_2 and name it $[a_2, b_2]$. In other words, we have $x_2 \notin [a_2, b_2]$, and more importantly, $[a_1, b_1] \supseteq [a_2, b_2]$. Next, we divide $[a_2, b_2]$ into three parts and name it $[a_3, b_3]$ which does not have x_3 in it, that is $x_3 \notin [a_3, b_3]$ and $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3]$. We apply this method and every time we construct a new interval $[a_n, b_n]$, we know that

1. $[a_{n-1}, b_{n-1}] \supseteq [a_n, b_n]$
2. $b_n - a_n \xrightarrow{n\to\infty} 0$. This is because the way we create these new intervals.
3. $x_n \notin [a_n, b_n]$.

By the Nested Intervals Theorem, we know that there exists a unique real number $\xi \in [a_n, b_n]$ and we know $\xi \neq x_n$. This shows that ξ is not listed in $\mathbb{R} = \{x_1, x_2, x_3, \dots, x_n, \dots\}, \forall\, n \in \mathbb{N}$. Thus, $\mathbb{R}$ is uncountable. □

Note: It is not possible that $x_2 \notin [a_2, b_2]$ but $x_1 \in [a_2, b_2]$, since the new interval, $[a_2, b_2]$, is smaller than the previous one, $[a_1, b_1]$, which does not have x_1.

3.7.4 Subsequence

We give a definition of the subsequence.

Definition 3.7.2

Given a sequence $\{x_n\}$ and a list of strictly increasing integers

$$n_1 < n_2 < n_3 < \cdots < n_k < \dots, \forall\, k \in \mathbb{N},$$

then we say $x_{n_1}, x_{n_2}, x_{n_3}, \dots, x_{n_k}, \dots$ is a **subsequence** of the sequence $\{x_n\}$ and denoted as $\{x_{n_k}\}, \forall\, k \in \mathbb{N}$.

Note: We observe that k indicates that x_{n_k} is the k-th term in the subsequence and n_k shows that x_{n_k} is the n_k-th term in the original sequence. Then it is obvious to have two properties:

1. $n_k \geq k, \forall\, k \in \mathbb{N}$,
2. $n_j > n_k, \forall\, j > k$.

Theorem 3.7.4

If a sequence $\{x_n\}$ converges to a then any of its subsequence $\{x_{n_k}\}$ converges to a.

Proof:

Let $\lim_{n\to\infty} x_n = a$. We want to show $\lim_{k\to\infty} x_{n_k} = a$ as well. That is equivalent to show that there $\forall\, \epsilon > 0$ we can find a K such that $\forall\, k > K$ there is $|x_{n_k} - a| < \epsilon$.
Rewriting $\lim_{n\to\infty} x_n = a$, we have $\forall\, \epsilon > 0$ we can find an N such that $\forall\, n > N$ there is $|x_n - a| < \epsilon$. By letting $K = N$, we can always find $k > K = N$ and using the property of subsequence we know that we have $n_k \geq N$. So, we know that $\forall\, \epsilon > 0$, we can find an $N = K$ such that $\forall\, n_k \geq k > K = N$ there is $|x_{n_k} - a| < \epsilon$. □

Remark. This theorem is powerful in proving some sequence diverges. As long as we can find two subsequences (of a sequence) converges to different limit then we know the sequence does not converge.

Corollary 3.7.5

If a sequence $\{x_n\}$ has two subsequences that converge to two different limits, then the sequence $\{x_n\}$ diverges.

Example 3.7.11

Show that the sequence $\{\sin \frac{n\pi}{4}\}$ diverges.

Proof:

Let the first sequence of integers be $n_k^{(1)} = 4k$ and another sequence of integers be $n_k^{(2)} = 8k + 2$. Then it should be obvious that $\{x_{n_k^{(1)}}\}$ has a limit 0, while $\{x_{n_k^{(2)}}\}$ has a limit 1. So, $\{\sin \frac{n\pi}{4}\}$ diverges. □

3.7.5 Bolzano-Weierstrass Theorem

Theorem 3.7.6 ***Bolzano-Weierstrass Theorem***

Every bounded sequence has a convergent subsequence.

Proof:

Given a sequence $\{x_n\}$ is bounded that is $a_1 \le x_n \le b_1 \Leftrightarrow x_n \in [a_1, b_1], \forall\, n \in \mathbb{N}$. Now, we divide $[a_1, b_1]$ into two intervals, $[a_1, \frac{a_1+b_1}{2}]$ and $[\frac{a_1+b_1}{2}, b_1]$. In the two intervals, there must be at least one contains infinitely many terms of $\{x_n\}$ and we call it $[a_2, b_2]$. Now we divide $[a_2, b_2]$ into two smaller interval, $[a_2, \frac{a_2+b_2}{2}]$ and $[\frac{a_2+b_2}{2}, b_2]$. It is obvious that at least one of the two intervals contains infinitely many terms in $\{x_n\}$. We keep doing this by spiting the interval into two intervals and choose the one that contains infinitely many terms from $\{x_n\}$ to be $[a_n, b_n]$. Thus, we have a Nested Closed Intervals $\{[a_n, b_n]\}$, there exists a real number ξ such that $\xi \in [a_n, b_n]$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \xi$. Now we construct a subsequence and show its limit is ξ. In $[a_1, b_1]$, we can find x_{n_1}. In $[a_2, b_2]$, we can find x_{n_2}, with $n_2 > n_1$. (Why we can always find $n_2 > n_1$? Since the interval contains infinitely many terms from $\{x_n\}$) .In $[a_3, b_3]$, we can find x_{n_3}, with $n_3 > n_2$. The pattern follows, thus in $[a_k, b_k]$, we can find x_{n_k}, with $n_k > n_{k-1}$. Thus, we constructed a subsequence $\{x_{n_k}\}$ from the original sequence $\{x_n\}$. It should be obvious that $a_k \le x_{n_k} \le b_k$ and since $\lim\limits_{k\to\infty} a_k = \xi = \lim\limits_{k\to\infty} b_k$, then by the squeeze theorem, we arrive at $\lim\limits_{k\to\infty} x_{n_k} = \xi$. □

Note: An unbounded sequence is not necessarily an infinite quantity. For example the sequence

$$\{2, 3, 2, 4, 2, 5, 2, 6, \dots\}$$

is increasing but the odd number of terms in this sequence stay at 2. This contradicts with the definition of the infinite quantity as n goes to infinity.

Theorem 3.7.7

An unbounded sequence has a subsequence that diverges to infinite.

Proof:

Let $\{x_n\}$ be an unbounded sequence, thus we can say $\forall\, M > 0$ we can always find infinitely many terms in $\{x_n\}$ such that

$$|x_n| > M.$$

(Think about this. If we could only find finitely many terms in $\{x_n\}$ that is greater than M, then $\{x_n\}$ must be bounded.) Let

$$\begin{aligned}
&M_1 = 1, \exists\, |x_{n_1}| > 1;\\
&M_2 = 2, \exists\, |x_{n_2}| > 2; n_2 > n_1\\
&M_3 = 3, \exists\, |x_{n_3}| > 3; n_3 > n_2\\
&\dots\\
&M_k = k, \exists\, |x_{n_k}| > k; n_k > n_{k-1}
\end{aligned}$$

Note, we can always find $n_k > n_{k-1}$ since there are infinitely many terms in $\{x_n\}$ such that $|x_{n_k}| > M_k$. Then, we have constructed a subsequence $\{x_{n_k}\}$, with $|x_{n_k}| > k$. Therefore, x_{n_k} diverges to infinity. □

3.7.6 Cauchy Convergence Theorem

We have studied **Monotone Convergence Theorem for Sequences** and this is a sufficiency theorem. In other words, the Monotone Convergence Theorem provides a sufficient condition for the convergence of a sequence, namely that it is monotone and bounded. That means if a sequence is bounded and not monotone then it may converge or diverge.

The **Cauchy Convergence Theorem** is sufficient and necessary condition for the convergence of a sequence.

Definition 3.7.3 ***The Cauchy Sequence / Fundamental Sequence***

If a sequence $\{x_n\}$ satisfies: $\forall\,\epsilon > 0$, $\exists\, N$ such that $\forall\, m, n > N$ we have $|x_n - x_m| < \epsilon$, then the sequence $\{x_n\}$ is called the **Cauchy Sequence**.

Note: In the definition above, we can safely change the condition $\forall\, m, n > N$ to $\forall\, m > n > N$, without altering anything. In other words, we just assume $m > n$.

Example 3.7.12

Tell if $x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$ is a Cauchy sequence?

Proof:

We have

$$\begin{aligned}
|x_m - x_n| &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{m^2} - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right) \\
&= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{m^2} < \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(m-1)m} \\
&= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\
&= \frac{1}{n} - \frac{1}{m} < \frac{1}{n}.
\end{aligned}$$

Let $N = \left\lfloor \frac{1}{\epsilon} \right\rfloor$, we can show that $\forall\,\epsilon > 0$, we can find an N such that $\forall\, m > n > N$, we have $|x_m - x_n| < \epsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence. □

Example 3.7.13

Tell if $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is a Cauchy sequence?

Proof:

We have

$$|x_m - x_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m}$$

Let $m = 2n$, then we get

$$|x_{2n} - x_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > n \cdot \frac{1}{2n} = \frac{1}{2}.$$

Choose $\epsilon = \frac{1}{2}$, $\forall\, N, \exists\, m = 2n > n > N$ such that $|x_{2n} - x_n| > \frac{1}{2}$. In other words, the dynamic in the ϵ-N game is lost (n is canceled when calculating the difference between x_m and x_n), so no matter how large the N we choose the difference between x_m and x_n will not be less than $\frac{1}{2}$, as long as $m = 2n$. Therefore, x_n is not a Cauchy sequence. □

Theorem 3.7.8 ***Cauchy Convergence Theorem / The Completeness Theorem of the Real Numbers***

The sufficient and necessary condition for a sequence, $\{x_n\}$, to converge is that it is a Cauchy sequence.

Proof:

(Cauchy sequence $\Leftarrow$ convergent sequence)
Assume $\lim\limits_{n\to\infty} x_n = a$, that is $\forall\,\epsilon > 0$, we can find an N such that $\forall\, n > N$, we have $|x_n - a| < \frac{\epsilon}{2}$. For the same ϵ and N, $\forall\, m > N$, we have $|x_m - a| < \frac{\epsilon}{2}$. Therefore, we know $\forall\,\epsilon > 0$, we can find an N, such that $\forall\, m, n > N$, we have $|x_m - x_n| = |(x_m - a) - (x_n - a)| \le |x_m - a| + |x_n - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
(Cauchy sequence $\Rightarrow$ convergent sequence) Briefly, we need to show that a Cauchy sequence is bounded then use Bolzano-Weierstrass Theorem to show that a bounded sequence has a convergent subsequence. Then show that

the Cauchy sequence converge to that limit, so it is a convergent sequence.
Since $\{x_n\}$ is a Cauchy sequence, that is $\forall\, \epsilon > 0$ we can find an N_0 such that $\forall\, m, n > N_0$, we have $|x_n - x_m| < \epsilon \Leftrightarrow |x_n| < |x_m + \epsilon|$. Here we only need $m > N_0$ so we let $m = N_0 + 1$. Then we have $|x_n| < |x_{N_0+1} + \epsilon|$. Let $M = \max\left\{|x_1|, |x_2|, \ldots, |x_{N_0+1}|, |x_{N_0+1} + \epsilon|\right\}$, we have $|x_n| \le M$. So $\{x_n\}$ is bounded. By Bolzano-Weierstrass Theorem, we know that there is a convergent subsequence $\{x_{n_k}\}$ and we let $\lim\limits_{k\to\infty} x_{n_k} = \xi$. (Now, we only need to show that $\{x_n\}$ indeed convergent to ξ).
Again, by the definition of Cauchy sequence, we have $\forall\, \epsilon > 0$, we can find an N such that $\forall\, m, n > N$ we have $|x_n - x_m| < \frac{\epsilon}{2}$. We can replace m with n_k as long as k is large enough to make $n_k > N$. That is for very large k so that $n_k > N$ (just like $m > N$), we have

$$|x_n - x_{n_k}| < \frac{\epsilon}{2}$$

then

$$\lim_{k\to\infty} |x_n - x_{n_k}| < \frac{\epsilon}{2}$$
$$|x_n - \xi| \le \frac{\epsilon}{2} < \epsilon.$$

Therefore, $\lim\limits_{n\to\infty} x_n = \xi$. This is $\{x_n\}$ is convergent. $\square$

The Geometric Contraction Condition

The Geometric Contraction Condition of a sequence $\{x_n\}$ is that it satisfies

$$|x_{n+1} - x_n| \le k|x_n - x_{n-1}|, \quad 0 < k < 1.$$

Note: It is called **Geometric** Contraction Condition because

$$|x_{n+1} - x_n| \le k|x_n - x_{n-1}| \le k^2|x_{n-1} - x_{n-2}| \le \cdots \le k^n|x_1 - x_0|, \quad 0 < k < 1.$$

If a sequence satisfies the Geometric Contraction Condition, then it converges.

Example 3.7.14

Prove that if a sequence $\{x_n\}$ satisfies the Geometric Contraction Condition, then it converges.

Proof:

We just need to show a sequence satisfies the Geometric Contraction Condition is a Cauchy Sequence.
W.O.L.G., let us assume $m > n$. We have

$$\begin{aligned}
|x_m - x_n| &= |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)| \\
&\le |(x_m - x_{m-1})| + |(x_{m-1} - x_{m-2})| + \cdots + |(x_{n+1} - x_n)| \\
&< k^{m-1}|x_1 - x_0| + k^{m-2}|x_1 - x_0| + \cdots + k^n|x_1 - x_0| \\
&= k^n|x_1 - x_0|\left(k^{m-1-n} + k^{m-2-n} + \cdots + k + 1\right) \\
&= k^n|x_1 - x_0|\frac{1}{1-k} \xrightarrow{n\to\infty} 0
\end{aligned}$$

Thus, $\forall\, \epsilon > 0$, we have an N such that $\forall\, m > n > N$ we have $|x_m - x_n| < \epsilon$. Therefore, the sequence $\{x_n\}$ satisfying the Geometric Contraction Condition is a Cauchy sequence. (So, it is convergent.) $\square$

3.7.7 Summary

We need a summary!

1. The Supremum Existence Theorem / (The Continuity Theorem of the Real Numbers)

$\Downarrow$

2. The Monotone (Bounded Sequence) Convergence Theorem

$\Downarrow$

3. The Nested Closed Interval Theorem

$\Downarrow$

4. The Bolzano-Weierstrass Theorem

$\Downarrow$

5. The Cauchy Convergence Theorem / (The Completeness Theorem of the Real Numbers)

Note: The set of rational numbers is not complete. A quick example: the set $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is a set of rational numbers, but $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, which is an irrational number.

Surprisingly or not so surprise, these five theorems above are equivalent, meaning assuming any one of these theorems as true, we can prove the other four.

We will show their equivalence by using The Cauchy Convergence Theorem to prove The Nested Closed Interval Theorem and then to The Supremum Existence Theorem.

Example 3.7.15

Using the Cauchy Convergence Theorem, we can prove the Nested Closed Interval Theorem.

Proof:

To prove the Nested Closed Intervals Theorem, we at least need to have a nested closed intervals. Given a nested closed interval $\{[a_n, b_n]\}$. That is it has two properties: $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ and $b_n - a_n \xrightarrow{n\to\infty} 0$. We need to show that there is a unique limit $\xi \in [a_n, b_n]$.
Assuming $m > n$, by the definition of a nested closed interval, it is obvious that $[a_m, b_m] \subseteq [a_n, b_n]$. Equivalently we have

$$a_m - a_n < b_n - a_n \xrightarrow{n\to\infty} 0$$

We proved the sequence $\{a_n\}$ is a Cauchy sequence. W.O.L.G., assuming $\lim\limits_{n\to\infty} a_n = \xi$, then we know $\lim\limits_{n\to\infty} b_n = \xi$. Since $\{a_n\}$ is monotone increasing, thus ξ is an upper bound of $\{a_n\}$, which ξ is the lower bound for $\{b_n\}$. So $\xi \in [a_n, b_n]$. Again, since $b_n - a_n \xrightarrow{n\to\infty} 0$, we know ξ is unique. $\square$

Example 3.7.16

Using the Nested Closed Intervals to prove the Supremum Existence Theorem.

Proof:

Given a non-empty set S, which is up bounded, we need to show that the set S has a supremum. Let the set of upper bounds of S be T and it should be obvious that there is no upper bound for T. Whether there is a lower bound for T is for us to prove.
Choose $a_1 \notin T$ and $b_1 \in T$, then we construct $\left[a_2, b_2\right]$

$$\left[a_2, b_2\right] = \begin{cases} \left[a_1, \dfrac{a_1+b_1}{2}\right], & \text{if } \dfrac{a_1+b_1}{2} \in T; \\ \left[\dfrac{a_1+b_1}{2}, b_1\right], & \text{if } \dfrac{a_1+b_1}{2} \notin T. \end{cases}$$

Then we construct $[a_3, b_3]$

$$[a_3, b_3] = \begin{cases} \left[a_2, \dfrac{a_2+b_2}{2}\right], & \text{if } \dfrac{a_2+b_2}{2} \in T; \\ \left[\dfrac{a_2+b_2}{2}, b_2\right], & \text{if } \dfrac{a_2+b_2}{2} \notin T. \end{cases}$$

Following this process, we can construct $[a_n, b_n]$.

$$[a_n, b_n] = \begin{cases} \left[a_{n-1}, \dfrac{a_{n-1}+b_{n-1}}{2}\right], & \text{if } \dfrac{a_{n-1}+b_{n-1}}{2} \in T; \\ \left[\dfrac{a_{n-1}+b_{n-1}}{2}, b_{n-1}\right], & \text{if } \dfrac{a_{n-1}+b_{n-1}}{2} \notin T. \end{cases}$$

Now, we have a nested closed intervals $[a_n, b_n]$ and it is always true that $a_n \notin T$ and $b_n \in T$. By the Nested Closed Intervals theorem, we know that there exists an unique real number $\xi \in [a_n, b_n]$. Now, we need to show ξ is the supremum. (2 steps).
(1) Show that ξ is an upper bound:
If ξ is not an upper bound (or $\xi \notin T$), that is $\exists\, x \in S$ such that $x > \xi$. However, by the Nested Closed Interval Theorem, we know $\lim_{n\to\infty} b_n = \xi$. That is when n is large, we have $x > b_n$. But $b_n \in T$. We reach a contradiction, therefore ξ is an upper bound (or $\xi \in T$).
(2) Show that ξ is the least upper bounded:
If $\exists\, \eta \in T$, then $\eta < \xi$. By the Nested Closed Intervals theorem, $\lim_{n\to\infty} a_n = \xi$. Thus, when n is large, we have $\eta < a_n$. However, since $a_n \notin T$, then $\exists\, y \in S$ such that $\eta < y \in S$. This is saying that η is not an upper bound. Contradiction. Therefore, we have shown that if something is less then ξ, then it is not an upper bound. So ξ is the least upper bound (supremum). $\square$

Finally, we have the equivalence of these five theorems.

1. The Supremum Existence Theorem / (The Continuity Theorem of the Real Numbers)

$\Updownarrow$

2. The Monotone (Bounded Sequence) Convergence Theorem

$\Updownarrow$

3. The Nested Closed Interval Theorem

$\Updownarrow$

4. The Bolzano-Weierstrass Theorem

$\Updownarrow$

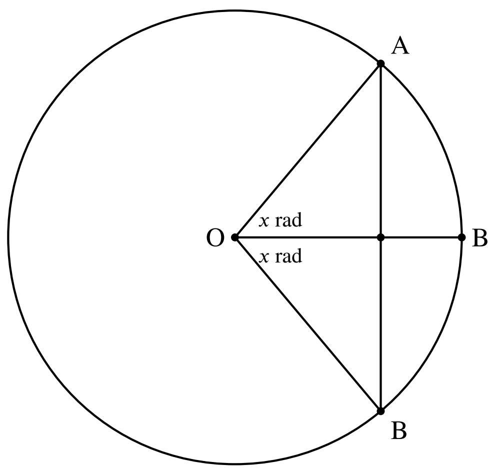
5. The Cauchy Convergence Theorem / (The Completeness Theorem of the Real Numbers)

Chapter 4

Limits and Continuity of Functions

4.1 Limits of Functions

Let's look at the graph below, (x is in radius):



The quantity of our interest is the ratio of the length of the chord $AB = 2r\sin x$ to the length of the arc $\overset{\frown}{AB} = 2rx$. Let $y = \frac{2r\sin x}{2rx} = \frac{\sin x}{x}$. Specifically, we are interested in the value of y as $x \to 0$.

$x =$	0.5	0.1	0.05	0.01
$y =$	0.96	0.998	0.9996	0.99998

Numerically, we can see from the table as $y \xrightarrow{x\to 0} 1$, then this is our best guess $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

Definition 4.1.1 ***The Limit of a Function***

A function $f(x)$ is well defined on the open interval $O(x_0, \rho)\backslash x_0$. If $\forall\, \epsilon > 0$, there exists $\delta > 0$ such that for all x satisfying

$$0 < |x - x_0| < \delta \implies |f(x) - A| < \epsilon,$$

then we say that the **limit of $\boldsymbol{f(x)}$**, as x approaches x_0, is A, denoted as $\lim\limits_{x\to x_0} f(x) = A$. Otherwise, we say that the limit of $f(x)$, as x approaches x_0, does not exist.

Remark. Mathematically it is saying

$$\forall\,\epsilon > 0, \exists\,\delta > 0, \quad \forall\, x \qquad \underbrace{\big(0 < |x - x_0| < \delta\big)}_{x \text{ lives in the Punctured } \delta\text{-neighbourhood of } x_0} \qquad \implies \qquad \underbrace{|f(x) - f(x_0)| < \epsilon}_{f(x) \text{ is not too far away from } f(x_0)} \quad .$$

Note: Here, $O(x_0, \rho) = (x_0 - \rho, x_0 + \rho)$. In the above Definition 4.1.1, $f(x)$ does not have to be defined on x_0.

Definition 4.1.2 ***The δ-Neighborhood of x_0***

The **δ-neighborhood of x_0** is $O(x_0, \delta) \iff |x - x_0| < \delta$.

Remark. Think about whey the two expression are equivalent.

Definition 4.1.3 ***The Punctured δ-Neighborhood of x_0***

The **punctured δ-neighborhood of x_0** is $O(x_0, \delta)\backslash x_0 \iff 0 < |x - x_0| < \delta$.

Remark. Think about whey the two expression are equivalent.

Example 4.1.1

Use the definiton of the limit of a function to prove $\lim_{x\to 0} e^x = 1$.

Proof:

[Hint]: In the limit, x approaches 0. So we know x lives in the punctured δ-neighborhood of 0. For all $\epsilon > 0$, we need to find a $\delta > 0$ such that $\forall\, x\, (0 < |x - 0| < \delta) \implies |e^x - 1| < \epsilon$.
To find δ, again we need to work the back way around.

$$\begin{aligned} |e^x - 1| &< \epsilon \\ -\epsilon < e^x - 1 &< \epsilon \\ 1 - \epsilon < e^x &< \epsilon + 1 \\ \ln(1 - \epsilon) < x &< \ln(1 + \epsilon) \end{aligned}$$

So, we find $\delta = \min\left\{\ln \frac{1}{1-\epsilon}, \ln(1 + \epsilon)\right\}$.

Note: Here, $\ln(1 - \epsilon)$ is negative, but the δ, we want, has to be positive. Since $\ln(1 - \epsilon) = -\ln \frac{1}{1-\epsilon}$ and indeed $\ln \frac{1}{1-\epsilon}$ is positive.

Therefore, we have shown that $\forall\, x\, (0 < |x - 0| < \delta)$, we have $|e^x - 1| < \epsilon$. We proved $\lim_{x\to 0} e^x = 1$. $\square$

Example 4.1.2

Prove $\lim_{x\to 2} x^2 = 4$ using the definition.

Proof:

[Hint]: Where does x live? How to find a **positive** δ for all $\epsilon > 0$.
For all $\epsilon > 0$, we need to find a positive δ such that $\forall\, x$ in punctured δ-neighborhood of 2, x^2 will not be too far away from 4.
Again, we work backward to find a δ.

$$|x^2 - 4| = |(x - 2)(x + 2)| = |x - 2| \cdot |x + 2| < |x - 2| \cdot 5 < \epsilon \implies |x - 2| < \frac{\epsilon}{5}$$

We find $\delta = \min\left\{\frac{\epsilon}{5}, 1\right\}$. $\forall\, x\, (0 < |x - 2| < \delta)$: $|x^2 - 4| < 5|x - 2| < \epsilon$. Thus, $\lim_{x\to 2} x^2 = 4$. $\square$

Note: The red inequality is because $x \in O(2, \delta)\backslash 2$ meaning x will not be too far away from 2, i.e., $|x - 2| < 1$, meaning $|x + 2| < 5$. More importantly, we kee $|x - 2|$ but expand $|x + 2|$ to 5, because $|x - 2|$ is the δ-neighborhood of 2, where x lives.

Note: Here, we include 1 when find δ is because we used an extra (but valid and reasonable) condition $|x - 2| < 1$ to expand $|x + 2|$ to make it easier for us to find a δ. Remember we just need to find a δ, NOT the best δ.

Remark. In Example 4.1.1, the δ we find is indeed the best δ.

Remark. In Example 4.1.2, think about why we expand $|x+2|$ instead of $|x-2|$.

Example 4.1.3

Prove $\lim\limits_{x\to 1} \frac{x(x-1)}{x^2-1} = \frac{1}{2}$.

Proof:

For all $\epsilon > 0$, we need to find a positive δ such that $\forall\, x\,(0<|x-1|<\delta)$ we have $\left|\frac{x(x-1)}{x^2-1} - \frac{1}{2}\right| < \epsilon$. We work backwards to find δ.

$$\begin{aligned}\left|\frac{x(x-1)}{x^2-1} - \frac{1}{2}\right| &= \left|\frac{2x(x-1)-x^2+1}{2(x^2-1)}\right| \\ &= \left|\frac{x^2-2x+1}{2(x^2-1)}\right| = \frac{|x-1|}{2\,|x+1|} < \frac{|x-1|}{2} < \epsilon\end{aligned}$$

We find $\delta = \min\left\{\frac{\epsilon}{2}, 1\right\}$. That is $\forall\,(0<|x-1|<\delta)$, we have $\left|\frac{x(x-1)}{x^2-1} - \frac{1}{2}\right| < \frac{|x-1|}{2} < \epsilon$. So, we proved $\lim\limits_{x\to 1} \frac{x(x-1)}{x^2-1} = \frac{1}{2}$. Obviously, the δ we found is not the best one and it does not have to be. □

Note: The red inequality is because an extra but reasonable condition $|x-1|<1 \iff |x+1|>1$.

Remark. In Example 4.1.3, think about why we expand $|x+1|$ instead of $|x-1|$.

4.2 The Properties of the Limits of Functions

4.2.1 The Uniqueness of the Limit of a Function

Theorem 4.2.1 ***The Uniqueness of the Limit of a Function (at a given point)***

Suppose both A and B are limits of $f(x)$ at x_0, then $A = B$.

Proof:

Since we are given that both A and B are limits to $f(x)$ at x_0, that is

$$\begin{aligned}\text{For all } \epsilon > 0, \exists\, \delta_1 > 0, \forall\, x\left(0<|x-x_0|<\delta_1\right) &: |f(x)-A| < \frac{\epsilon}{2} \\ \exists\, \delta_2 > 0, \forall\, x\left(0<|x-x_0|<\delta_2\right) &: |f(x)-B| < \frac{\epsilon}{2}\end{aligned}$$

We find $\delta = \min\left\{\delta_1, \delta_2\right\}$. We observe

$$\begin{aligned}|A-B| = |A-B+f(x)-f(x)| &= |f(x)-B+A-f(x)| \\ &\le |f(x)-B| + |f(x)-A| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Therefore, we have shown $A = B$. □

4.2.2 The Local Order-Preserving Property of a Limit of a Function

Theorem 4.2.2

Suppose $\lim\limits_{x\to x_0} f(x) = A$, $\lim\limits_{x\to x_0} g(x) = B$, and $A > B$. Then $\exists\, \delta > 0$, when $0 < |x-x_0| < \delta$, we have $f(x) > g(x)$.

Proof:
Let $\epsilon_0 = \frac{A-B}{2}$.

$$\lim_{x\to x_0} f(x) = A: \quad \exists\, \delta_1, \forall\, x \left(0 < |x - x_0| < \delta_1\right) : |f(x) - A| < \epsilon_0 \implies f(x) > A - \epsilon_0 = \frac{A+B}{2}.$$
$$\lim_{x\to x_0} g(x) = B: \quad \exists\, \delta_2, \forall\, x \left(0 < |x - x_0| < \delta_2\right) : |g(x) - B| < \epsilon_0 \implies g(x) < B + \epsilon_0 = \frac{A+B}{2}.$$

Let $\delta = \min\left\{\delta_1, \delta_2\right\}$, we have shown that $\forall\, x \left(0 < |x - x_0| < \delta\right)$, we have $f(x) > \frac{A+B}{2} > g(x)$. □

Corollary 4.2.3
Suppose $\lim_{x\to x_o} f(x) = A \neq 0$, then $\exists\, \delta > 0$ when $0 < |x - x_0| < \delta$ we have $|f(x)| > \frac{|A|}{2} > 0$.

Proof:
Since $\lim_{x\to x_0} f(x) = A$, we have $\forall\, \epsilon > 0, \exists\, \delta > 0, \forall\, x \left(0 < |x - x_0| < \delta\right) : |f(x) - A| < \epsilon$. Note we also have

$$|f(x)| - |A| \leq |f(x) - A|$$
$$\Big||f(x)| - |A|\Big| \leq |f(x) - A|$$

That is $\forall\, \epsilon > 0, \exists\, \delta > 0, \forall\, x \left(0 < |x - x_0| < \delta\right) : \Big||f(x)| - |A|\Big| < \epsilon$. We have shown $\lim_{x\to x_0} |f(x)| = |A|$.
Let $g(x) = \frac{|A|}{2}$, it is obvious that $|A| > \frac{|A|}{2}$. By Theorem 4.2.2, we have that $0 < |x - x_0| < \delta : |f(x)| > g(x) = \frac{|A|}{2}$. □

Remark. The converse of Theorem 4.2.2 is incorrect. That is suppose $\lim_{x\to x_0} f(x) = A$, $\lim_{x\to x_0} g(x) = B$, and $\exists\, \delta > 0$, $0 < |x - x_0| < \delta$, if $f(x) > g(x) \;\not\Longrightarrow\; A > B$. A counter example is $f(x) = 2x^2, g(x) = x^2$ and $x_0 = 0$. It is obvious that in the punctured δ-neighborhood of 0, we do have $f(x) > g(x)$, but $A = B$. So, the puncture δ-neighborhood is the key. Next we give a valid converse of Theorem 4.2.2.

Corollary 4.2.4
Suppose $\lim_{x\to x_0} f(x) = A$, $\lim_{x\to x_0} g(x) = B$. If $\exists\, \delta > 0$, for $0 < |x - x_0| < \delta$ such that $f(x) > g(x)$, then $A \geq B$.

Proof:
If $A < B$, then

$$\exists\, \delta_1 > 0, \forall\, x \left(0|x - x_0| < \delta_1\right) : f(x) < g(x).$$

Now, let $\delta^* = \min\left\{\delta, \delta_1\right\}, \forall\, x \left(0 < |x - x_0| < \delta^*\right) : f(x) \geq g(x)$ and $f(x) < g(x)$. Contradiction. □

4.2.3 The Local Boundedness Property of a Limit of a Function

Theorem 4.2.5
Suppose $\lim_{x\to x_0} f(x) = A$, then

$$\exists\, \delta > 0, \forall\, x \left(0 < |x - x_0| < \delta\right) : m \leq f(x) \leq M,$$

with m, and M are two fixed real numbers.

Proof:
W.O.L.G., let $m \leq A \leq M$, and let $g(x) = m$ and $h(x) = M$. Immediately, we have $\lim_{x\to x_0} g(x) = m$ and $\lim_{x\to x_0} h(x) = M$, since both $g(x)$ and $h(x)$ are constant functions. Then by Theorem 4.2.2,

$$\exists\, \delta > 0, \forall\, x \left(0 < |x - x_0| < \delta\right) : m = g(x) < f(x) < h(x) = M.$$

□

Remark. If $f(x)$ is defined at x_0, then $|x - x_0| < \delta$: $\min\{m, f(x_0)\} \le f(x) \le \max\{M, f(x_0)\}$.

4.2.4 The Squeeze Theorem

Theorem 4.2.6 ***The Squeeze Theorem***

Suppose $\exists\, r > 0, \forall\, x \left(0 < |x - x_0| < r\right), g(x) \le f(x) \le h(x)$ and $\lim\limits_{x\to x_0} g(x) = \lim\limits_{x\to x_0} h(x) = A$, then $\lim\limits_{x\to x_0} f(x) = A$.

Proof:

Immediately, we have

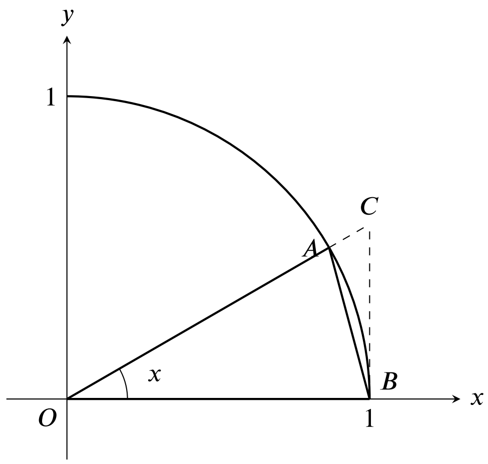
$$\forall\, \epsilon > 0, \exists\, \delta_1 > 0, \forall\, x \left(0 < |x - x_0| < \delta_1\right) : |g(x) - A| < \epsilon \implies g(x) > A - \epsilon \implies -\epsilon < g(x) - A.$$
$$\exists\, \delta_2 > 0, \forall\, x \left(0 < |x - x_0| < \delta_2\right) : |h(x) - A| < \epsilon \implies h(x) < A + \epsilon \implies h(x) - A < \epsilon.$$

We find $\delta = \min\{r, \delta_1, \delta_2\}, \forall\, x\left(0 < |x - x_0| < \delta\right) : g(x) \le f(x) \le h(x) \implies g(x) - A \le f(x) - A \le h(x) - A$, then $-\epsilon < f(x) - A < \epsilon \implies |f(x) - A| < \epsilon$. Thus, we proved $\lim\limits_{x\to x_0} f(x) = A$. □

Note: The Theorem 4.2.6 is extremely useful when the limit of $f(x)$ is difficult to find, but we observe that if we loose and expand $f(x)$ to $g(x)$ and $h(x)$, respectively, whose limits are the same and easy to find.

Example 4.2.1

Prove $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.



Proof:

From the graph, it is obvious that $S_{\triangle 0AB} < S_{\overset{\frown}{ABC}} < S_{\triangle OBC}$. That is

$$\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x, x \in (0, \frac{\pi}{2}), x \text{ in radius,}$$
$$\implies \sin x < x < \tan x,$$
$$\implies \frac{\sin x}{x} < 1, \text{ and } \cos x < \frac{\sin x}{x},$$
$$\implies \cos x < \frac{\sin x}{x} < 1.$$

By the Squeeze Theorem (Theorem 4.2.6), we immediately have $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$, since $\lim\limits_{x\to 0} \cos x = 1$.

Note: In this proof, it still remains to show that $\lim\limits_{x\to 0} \cos x = 1$, which is left as an exercise.

□

4.3 The Arithmetic Operations on Limits of Functions

Theorem 4.3.1

Suppose $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} g(x) = B$. Then:

$$1)\quad \lim_{x\to x_0} (\alpha f(x) + \beta g(x)) = \alpha A + \beta B;$$

$$2)\quad \lim_{x\to x_0} (f(x)g(x)) = AB;$$

$$3)\quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}, B \neq 0.$$

Proof:

From $\lim_{x\to x_0} f(x) = A$, we have:

$$\exists\, \delta_0 > 0, \forall\, x \left(0 < |x - x_0| < \delta_0\right) : f(x) \leq X.$$
$$\forall\, \epsilon, \exists\, \delta_1 > 0, \forall\, x \left(0 < |x - x_0| < \delta_1\right) : |f(x) - A| < \epsilon.$$

From $\lim_{x\to x_0} g(x) = B$, we have:

$$\forall\, \epsilon, \exists\, \delta_2 > 0, \forall\, x \left(0 < |x - x_0| < \delta_2\right) : |g(x) - B| < \epsilon.$$

To prove 1) in Theorem 4.3.1, we find $\delta = \min\left\{\delta_0, \delta_1, \delta_2\right\}, \forall\, x \left(0 < |x - x_0| < \delta\right)$, we need to show:

$$|\alpha f(x) + \beta g(x) - (\alpha A + \beta B)| < \epsilon.$$

We start from the LHS.

$$\begin{aligned} |\alpha f(x) + \beta g(x) - (\alpha A + \beta B)| &= \left|\alpha\big(f(x) - A\big) + \beta\big(g(x) - B\big)\right| \\ &\leq |\alpha|\, |f(x) - A| + |\beta|\, |g(x) - B| \\ &< (|\alpha| + |\beta|)\, \epsilon \end{aligned}$$

We have proved 1) in Theorem 4.3.1.

To prove 2) from Theorem 4.3.1, we find $\delta = \min\left\{\delta_0, \delta_1, \delta_2\right\}, \forall\, x \left(0 < |x - x_0| < \delta\right)$, we need to show $|f(x)g(x) - AB| < \epsilon$.

$$\begin{aligned} |f(x)g(x) - AB| &= |f(x)g(x) - Bf(x) + Bf(x) - AB| \\ &= \left|f(x)\big(g(x) - B\big) + B\big(f(x) - A\big)\right| \\ &\leq |f(x)|\, |g(x) - B| + |B|\, |f(x) - A| \\ &< \big(\, |f(x)| + |B|\, \big)\epsilon \\ &< (X + |B|)\, \epsilon \end{aligned}$$

We have proved 2.) in Theorem 4.3.1.

To prove 3) in Theorem 4.3.1, we find $\delta = \min\left\{\delta_0, \delta_1, \delta_2\right\}, \forall\, x \left(0 < |x - x_0| < \delta\right)$, we need to show $\left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| < \epsilon$. By Corollary 4.2.3, we know $\exists\, \delta^* > 0, \forall\, x \left(0 < |x - x_0| < \delta^*\right) : |g(x)| > \frac{|B|}{2}$.

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| &= \frac{|Bf(x) - AB + AB - Ag(x)|}{|Bg(x)|} \\ &= \frac{\left|B\big(f(x) - A\big) + A\big(B - g(x)\big)\right|}{|Bg(x)|} \\ &\leq \frac{(|B| + |A|)\, \epsilon}{|B||g(x)|} \\ &< 2\frac{|B| + |A|}{|B|^2}\epsilon \end{aligned}$$

We have proved 3) in Theorem 4.3.1. □

Example 4.3.1

For $\alpha \neq 0$, $\lim\limits_{x\to 0} \frac{\sin \alpha x}{x} = \alpha \lim\limits_{x\to 0} \frac{\sin \alpha x}{\alpha x} = \alpha$.

Example 4.3.2

For $\alpha \neq 0, \beta \neq 0$, $\lim\limits_{x\to 0} \frac{\sin \alpha x}{\sin \beta x} = \lim\limits_{x\to 0} \frac{\alpha x}{\beta x} \frac{\frac{\sin \alpha x}{\alpha x}}{\frac{\sin \beta x}{\beta x}} = \frac{\alpha x}{\beta x} = \frac{\alpha}{\beta}$.

4.4 The Relationship Between the Limit of a Function and the Limit of a Sequence

4.4.1 The Mathematical Analysis Expression of the Negation Proposition

Before we dive into the main content for this chapter, we need to make sure that we know how to express the **negation** of a given proposition in mathematical analysis way.

The proposition P:

A sequence $\{x_n\}$ converges to a: $\forall\, \epsilon > 0,\ \exists\, N, \forall\, n > N\ :\ |x_n - a| < \epsilon.$

It's **negation** is $\neg P$:

A sequence $\{x_n\}$ does not converge to a: $\exists\, \epsilon > 0, \forall\, N,\ \exists\, n > N\ :\ |x_n - a| \geq \epsilon.$

4.4.2 The Heine Theorem

The Heine theorem provides a crucial link between the limit of a function and the limit of a sequence, offering an alternative way to define and understand function limits using sequences.

Theorem 4.4.1 ***The Heine Theorem***

The necessary and sufficient condition for $\lim\limits_{x\to x_0} f(x) = A$ is that for any arbitrary sequence $\{x_n\}$ satisfies $x_n \neq x_0$ and $\lim\limits_{n\to\infty} x_n = x_0$, the sequence $\{f(x_n)\}$ converges to A.

Proof:

($\Rightarrow$). Given $\lim\limits_{x\to x_0} f(x) = A$, we have

$$\forall\, \epsilon > 0,\ \exists\, \delta > 0, \forall\, x \left(0 < |x - x_0| < \delta\right)\ :\ |f(x) - A| < \epsilon.$$

Give $\lim\limits_{n\to\infty} x_n = x_0$, we have

$$\forall\, \delta > 0,\ \exists\, N > 0, \forall\, n > N\ :\ 0 < |x_n - x_0| < \delta.$$

Therefore, we have $|f(x_n) - A| < \epsilon$, that is $\lim\limits_{n\to\infty} f(x_n) = A$.

($\Leftarrow$). We will use **proof by contrapositive** to prove the sufficient condition.

The sufficient condition ($\Leftarrow$), call it S, is:

For any sequence $\{x_n\}$ satisfies $x_n \neq x_0$, $\lim\limits_{n\to\infty} x_n = x_0$, and the sequence $\{f(x_n)\}$ converges to A, **then** $\lim\limits_{x\to x_0} f(x) = A$.

It's **contrapositive**, $\neg S$, is:

If $\lim\limits_{x\to x_0} f(x) \neq A$, **then** for any sequence $\{x_n\}$ satisfies $x_n \neq x_0$, $\lim\limits_{n\to\infty} x_n = x_0$, the sequence $\{f(x_n)\}$ does NOT converge to A.

Now, we just need to prove the **contrapositive**. We write the contrapositive in the mathematical analysis expression.

$\lim\limits_{x\to x_0} f(x) \neq A$: $\exists\, \epsilon_0 > 0, \forall\, \delta > 0, \forall\, x \left(0 < |x - x_0| < \delta\right)\ :\ |f(x) - A| \geq \epsilon_0.$

Let $\delta_n = \frac{1}{n}$, this is for the sake of $\frac{1}{n} \xrightarrow{n\to\infty} 0$:

$$
\begin{aligned}
\delta_1 = 1 &: \quad \exists\, x_1, 0 < |x_1 - x_0| < \delta_1 \,:\, |f(x_1) - A| \geq \epsilon_0, \\
\delta_2 = \frac{1}{2} &: \quad \exists\, x_2, 0 < |x_2 - x_0| < \delta_2 \,:\, |f(x_2) - A| \geq \epsilon_0, \\
&\vdots \\
\delta_3 = \frac{1}{3} &: \quad \exists\, x_2, 0 < |x_3 - x_0| < \delta_3 \,:\, |f(x_2) - A| \geq \epsilon_0, \\
&\vdots \\
\delta_k = \frac{1}{k} &: \quad \exists\, x_2, \underbrace{0 < |x_k - x_0| < \delta_k = \frac{1}{k}}_{x_n \xrightarrow{n\to\infty} x_0} : |f(x_k) - A| \geq \epsilon_0, \\
&\vdots
\end{aligned}
$$

We find a sequence $\{x_n\}$, $x_n \neq x_0$, $\lim_{n\to\infty} x_n = x_0$, but $\{f(x_n)\}$ does not converge to A. We have prove the **contrapositive**, $\neg S$, so the proposition S is true. □

Remark. The Heine Theorem 4.4.1 tells us that finding only one sequence $\{x_n\}, x_n \neq x_0, \lim_{n\to\infty} x_n = x_0$ and the sequence $\{f(x_n)\}$ converges to A is not enough to show $\lim_{x\to x_0} f(x) = A$.

Example 4.4.1

Find the limit of $\sin\frac{1}{x}$ at $x_0 = 0$.

Proof:

We pick the fist sequence $\left\{x_n^{(1)}\right\}$:

$$x_n^{(1)} = \frac{1}{n\pi}, x_n \neq x_0, \lim_{n\to\infty} x_n = x_0 = 0 : \lim_{x\to\infty} \sin\frac{1}{x_n^{(1)}} = \lim_{x\to\infty} \sin n\pi = 0.$$

We pick the second sequence $\left\{x_n^{(2)}\right\}$:

$$x_n^{(2)} = \frac{1}{2n\pi + \frac{\pi}{2}}, x_n \neq x_0, \lim_{n\to\infty} x_n = x_0 = 0 : \lim_{x\to\infty} \sin\frac{1}{x_n^{(2)}} = \lim_{x\to\infty} \sin(2n\pi + \frac{\pi}{2}) = 1.$$

So, by Heine Theorem 4.4.1, we know the limit of $\sin\frac{1}{x}$, at $x_0 = 0$ does not exist. □

Theorem 4.4.2

The necessary and sufficient condition for $\lim_{x\to x_0} f(x)$ exists and being finite (or converges) is that for any sequence $\{x_n\}$ satisfies $x_n \neq x_0$ and $\lim_{n\to\infty} x_n = x_0$, the sequence $\{f(x_n)\}$ converges.

Remark. Compare Theorem 4.4.2 with Theorem 4.4.1. Actually we do not need to know the limit, we only need to know they converge.

4.5 One-sided Limit

Definition 4.5.1 ***Left-hand Limit***

Suppose $f(x)$ is defined on $(x_0 - \rho, x_0)$. If there exists a B, $\forall\,\epsilon > 0$, $\exists\,\delta > 0, \forall\,x\,\left(-\delta < x - x_0 < 0\right)$: $|f(x) - B| < \epsilon$. Then, B is called the **Left-hand Limit** of $f(x)$ at x_0. Denoted as

$$\lim_{x\to x_0^-} f(x) = B, \qquad \text{or} \qquad f(x) \xrightarrow{x\to x_0^-} B.$$

Note: Here, δ has to be smaller than ρ.

Definition 4.5.2 ***Right-hand Limit***

Suppose $f(x)$ is defined on $(x_0, x_0+\rho)$. If there exists a C, $\forall\, \epsilon > 0$, $\exists\, \delta > 0, \forall\, x\, \big(0 < x - x_0 < \delta\big) : |f(x) - C| < \epsilon$. Then, C is called the **Right-hand Limit** of $f(x)$ at x_0. Denoted as

$$\lim_{x\to x_0^+} f(x) = C, \quad \text{or} \quad f(x) \xrightarrow{x\to x_0^+} C.$$

Note: Again, δ has to be smaller than ρ.

Proposition 4.5.1

$$\lim_{x\to x_0} f(x) = A \quad \Longleftrightarrow \quad \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = A.$$

Example 4.5.1

For $\operatorname{sgn} x$,

$$\operatorname{sgn} x = \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0. \end{cases}$$

$$\lim_{x\to 0^-} \operatorname{sgn} x = -1, \quad \text{or} \quad \lim_{x\to 0^+} \operatorname{sgn} x = 1,$$

Example 4.5.2

Find the limit of $f(x)$ at $x_0 = 0$.

$$f(x) = \begin{cases} \frac{\sin 2x}{x}, & \text{if } x < 0; \\ 2\cos x^2, & \text{if } x \geq 0. \end{cases}$$

Solution:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2\cos x^2 = 2,$$
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin 2x}{x} = 2$$

Therefore, $\lim_{x\to 0} f(x) = 2$. □

4.6 The Extension of the Definition of a Limit of a Function

For $\lim_{x\to x_0} f(x) = A$, it contains two messages $x \to x_0$ and $f(x) \to A$.

$$\forall\, \epsilon > 0, \exists\, \delta > 0, \forall\, x\, \big(0 < |x - x_0| < \delta\big), |f(x) - A| < \epsilon.$$

It is obvious that the blue terms are about x and orange terms about $f(x)$. Since we have introduced left-and right-limits in last section, we know have more mathematical analysis expression for x.

$$\begin{aligned}
x \to x_0 &: \quad \exists\, \delta > 0, \forall\, x,\ (0 < |x - x_0| < \delta); \\
x \to x^+ &: \quad \exists\, \delta > 0, \forall\, x,\ (0 < x - x^+ < \delta); \\
x \to x^- &: \quad \exists\, \delta > 0, \forall\, x,\ (-\delta < x - x^- < 0); \\
x \to +\infty &: \quad \exists\, X > 0, \forall\, x,\ (x > X); \\
x \to -\infty &: \quad \exists\, X > 0, \forall\, x,\ (x < -X); \\
x \to \infty &: \quad \exists\, X > 0, \forall\, x,\ (|x| > X).
\end{aligned}$$

Now, let's look at $f(x)$.

$$\begin{aligned}
f(x) \to A &: \quad \forall\, \epsilon > 0, \dots, |f(x) - A| < \epsilon; \\
f(x) \to +\infty &: \quad \forall\, G > 0, \dots, f(x) > G; \\
f(x) \to -\infty &: \quad \forall\, G > 0, \dots, f(x) < -G; \\
f(x) \to \infty &: \quad \forall\, G > 0, \dots, |f(x)| > G.
\end{aligned}$$

Example 4.6.1

$$\begin{aligned}
\lim_{x \to x_0} f(x) = \infty &: \quad \forall\, G > 0,\ \exists\, \delta > 0,\ \forall\, x,\ (0 < |x - x_0| < \delta) : |f(x)| > G. \\
\lim_{x \to +\infty} f(x) = A &: \quad \forall\, \epsilon > 0,\ \exists X > 0,\ \forall\, x,\ (x > X) : |f(x) - A| < \epsilon. \\
\lim_{x \to -\infty} f(x) = +\infty &: \quad \forall\, G > 0,\ \exists\, X > 0,\ \forall\, x,\ (x < -X) : f(x) > G.
\end{aligned}$$

Example 4.6.2

Prove $\lim_{x \to -\infty} e^x = 0$.

Proof:

That is $\forall\, \epsilon > 0$, we need to find $X > 0,\ \forall\, x,\ (x < -X)$:

$$|e^x - 0| < \epsilon \iff 0 < e^x < \epsilon \iff x < \ln \epsilon = -\ln \frac{1}{\epsilon}.$$

We find $X = \ln \frac{1}{\epsilon} > 0,\ \forall\, x,\ (x < -X) : \quad |e^x - 0| < \epsilon$. □

Example 4.6.3

Prove $\lim_{x \to -1} \frac{x^2}{x-1} = -\infty$.

Proof:

That is $\forall\, G > 0$, we want to find a $\delta > 0,\ \forall\, x,\ (-\delta < x - 1 < 0)$:

$$\frac{x^2}{x-1} < \frac{M}{x-1} < -G \iff x - 1 > -\frac{M}{G}$$

Note: The red inequality is because $x - 1 < 0$.

Now, we only need to contract x^2 to M, again since $x - 1 < 0$. Let $-\frac{1}{2} < x - 1 < 0 \implies x > \frac{1}{2} \implies x^2 > \frac{1}{4}$. Therefore,

$$x - 1 > -\frac{1}{4G}.$$

We find $\delta = \min\left\{\frac{1}{2}, \frac{1}{4G}\right\},\ \forall\, x,\ (-\delta < x - 1 < 0)$: $\frac{x^2}{x-1} < -G$. We proved $\lim_{x \to -1} \frac{x^2}{x-1} = -\infty$. □

Remark. When talking about The Properties of the Limits of Functions (Section (4.2)), we were talking about

$$\lim_{x \to x_0} f(x) = A.$$

Since we have now extended A to $-\infty, +\infty, \infty$, we need to be careful with ∞. Specifically, the Local Order-Preserving Property (Theorem 4.2.2) and the Squeeze Theorem 4.2.6 works for all situations except for ∞.

Remark. When talking about The Arithmetic Operations on Limits of Functions (Section (4.3)), the arithmetic operations do not apply to the **indeterminate forms** in function limit operations.

Example 4.6.4

To rewrite the Heine Theorem 4.4.1, with $-\infty, +\infty, \infty$ included:

$$\lim_{x\to+\infty} f(x) = A \iff \text{For any } \{x_n\}, x_n \xrightarrow{n\to+\infty} +\infty : \{f(x_n)\} \text{ converges to } A.$$

$$\lim_{x\to+\infty} f(x) \text{ exists and converges} \iff \text{For any } \{x_n\}, x_n \xrightarrow{n\to+\infty} +\infty : \{f(x_n)\} \text{ converges.}$$

Example 4.6.5

Given $f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_k x^k}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_j x^j}$, with $a_n, a_k, b_m, b_j \neq 0$. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to 0} f(x)$.

Solution:

$(x \to \infty)$

i.) $n = m$:

$$f(x) = \frac{a_n + a_{n-1}\frac{1}{x} + \ldots + a_k \frac{1}{x^{n-k}}}{b_m + b_{m-1}\frac{1}{x} + \ldots + b_j \frac{1}{x^{m-j}}} \implies \lim_{x\to\infty} f(x) = \frac{a_n}{b_m},$$

ii.) $n > m$:

$$f(x) = x^{n-m} \cdot \frac{a_n + a_{n-1}\frac{1}{x} + \ldots + a_k \frac{1}{x^{n-k}}}{b_m + b_{m-1}\frac{1}{x} + \ldots + b_j \frac{1}{x^{m-j}}} \implies \lim_{x\to\infty} f(x) = \infty,$$

iii.) $n < m$:

$$f(x) = \frac{1}{x^{m-n}} \cdot \frac{a_n + a_{n-1}\frac{1}{x} + \ldots + a_k \frac{1}{x^{n-k}}}{b_m + b_{m-1}\frac{1}{x} + \ldots + b_j \frac{1}{x^{m-j}}} \implies \lim_{x\to\infty} f(x) = 0.$$

$(x \to 0)$

i.) $k = j$:

$$f(x) = \frac{a_n x^{n-j} + a_{n-1}x^{n-j-1} + \ldots + a_k}{b_m x^{m-j} + b_{m-1}x^{m-j-1} + \ldots + b_j} \implies \lim_{x\to 0} f(x) = \frac{a_k}{b_j},$$

ii.) $k > j$:

$$f(x) = \frac{a_n x^{n-k} + a_{n-1}x^{n-k-1} + \ldots + a_k}{b_m x^{m-j} + b_{m-1}x^{m-j-1} + \ldots + b_j} x^{k-j} \implies \lim_{x\to 0} f(x) = 0,$$

iii.) $k < j$:

$$f(x) = \frac{a_n x^{n-k} + a_{n-1}x^{n-k-1} + \ldots + a_k}{b_m x^{m-j} + b_{m-1}x^{m-j-1} + \ldots + b_j} \frac{1}{x^{j-k}} \implies \lim_{x\to 0} f(x) = \infty.$$

□

Note: This example tells us that when $x \to \infty$ the dominant terms are x with the highest exponent, while $x \to 0$ the dominant terms are x with the lowest exponent.

Example 4.6.6

Show that $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$.

Solution:

We have prove the limit of the sequence $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$. We cannot use the Heine Theorem 4.4.1 here just

because only one sequence $\left\{x_n = 1 + \frac{1}{n}\right\}$ converges to e. We will use the Squeeze Theorem 4.2.6.
First, we show that $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$. Since $\forall\, x \geq 1$ we have $[x] \leq x < [x]+1$, then

$$\underbrace{\left(1+\frac{1}{[x]+1}\right)^{[x]}}_{\text{a sequence}} < \left(1+\frac{1}{x}\right) < \underbrace{\left(1+\frac{1}{[x]}\right)^{[x]+1}}_{\text{a sequence}}.$$

It is obvious that two sequences have e as their limits. Therefore, $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$.
Now, we show that $\lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$. Let $y = -x$, then $y \to +\infty$. Then

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to+\infty}\left(1-\frac{1}{y}\right)^{-y} = \lim_{y\to+\infty}\left(\frac{y}{y-1}\right)^y = \lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^y = e.$$

So, $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$. □

Note: From the red equal sign, we know that $\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)^x = \frac{1}{e}$.

4.7 The Cauchy Convergence Principle for The Limits of Functions

Let's review the Cauchy Convergence Principle for the Limits of Sequence:

For a sequence $\{x_n\}$: $\lim\limits_{x\to\infty} x_n$ converges $\Longleftrightarrow \forall\,\epsilon > 0,\ \exists\, N,\ \forall\, m,n > N\ :\ |x_m - x_n| < \epsilon.$

For a function $f(x)$: $\lim\limits_{x\to+\infty} f(x)$ converges $\Longleftrightarrow \forall\,\epsilon > 0,\ \exists\, X > 0,\ \forall\, x', x'' > X\ :\ |f(x') - f(x'')| < \epsilon.$

Note: As in the Section (4.6), here we have six different situations for x to approach.

Example 4.7.1
Let's prove the Cauchy Convergence Principle for The Limits of Functions.

Proof:
($\Rightarrow$)
Let $\lim\limits_{x\to+\infty} f(x) = A$, then

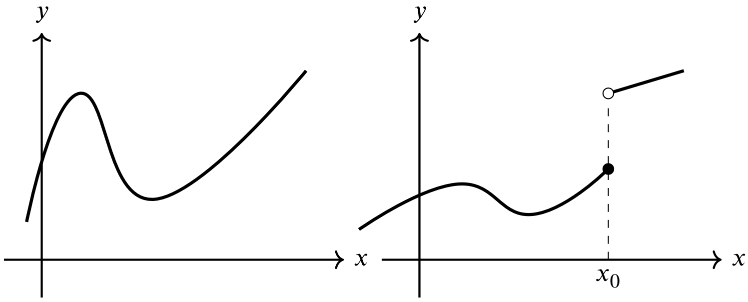
$$\forall\,\epsilon > 0,\ \exists\, X > 0,\ \forall\, x' > X\ :\ |f(x') - A| < \frac{\epsilon}{2},$$
$$\forall\,\epsilon > 0,\ \exists\, X > 0,\ \forall\, x'' > X\ :\ |f(x'') - A| < \frac{\epsilon}{2}.$$

Therefore, we have

$$|f(x') - f(x'')| = |f(x') - A + A - f(x'')| \leq |f(x') - A| + |f(x'') - A| < \epsilon.$$

($\Leftarrow$)
[Hint: We use the Heine Theorem.]
We have $\forall\,\epsilon > 0. \exists\, X > 0,\ \forall\, x', x'' > X\ :\ |f(x') - f(x'')| < \epsilon$. Now, for any $\{x_n\}$, $\lim\limits_{n\to\infty} x_n = +\infty$, For the given $X > 0$, $\exists\, N$, $\forall\, m,n > N, x_m > X, x_n > X$, [this is because the sequence we choose has this property.] then $|f(x_m) - f(x_n)| < \epsilon.$, thus $\{f(x_n)\}$ converges. By the Heine Theorem, we know $\lim\limits_{x\to+\infty} f(x)$ exists and is finite. □

4.8 Continuous Functions



Analytically, we say a function $f(x)$ is continuous at x_0 : when $x \to x_0$ we have $f(x) \to f(x_0)$.

Definition 4.8.1 ***Continuity of a Function at a Point***
Suppose a function $f(x)$ is defined in the neighborhood of x_0 and $\lim_{x \to x_0} f(x) = f(x_0)$, then we say that the function $f(x)$ is continuous at x_0.

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, (|x - x_0| < \delta) : |f(x) - f(x_0)| < \epsilon.$$

Definition 4.8.2 ***Continuity of a Function on an Open Interval***
A function f is continuous on an open interval (a, b) if it is continuous at every point $x \in (a, b)$.

Example 4.8.1
For $f(x) = \frac{1}{x}$, show that $f(x)$ is continuous on $(0, 1)$.

Proof:

Let x_0 be any point on the interval $(0, 1)$. For all $\epsilon > 0$, we need to find a $\delta > 0$, $\forall x$, $(|x - x_0| < \delta)$: $\left|\frac{1}{x} - \frac{1}{x_0}\right| < \epsilon$.

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{x x_0}\right| < \frac{2|x - x_0|}{|x_0|^2} < \epsilon$$

The red inequality is due to $|x - x_0| < \frac{x_0}{2} \iff x > \frac{x_0}{2}$. We choose $\delta = \min\left\{\frac{x_0}{2}, \frac{|x_0|^2 \epsilon}{2}\right\}$. □

Definition 4.8.3 ***Left- and Right- Continuity of a Function at a Point***
If $\lim_{x \to x_0^-} f(x) = f(x_0)$, we say that $f(x)$ is **left-continuous** at point x_0.

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, \left(-\delta < x - x_0 \leq 0\right) : \left|f(x) - f(x_0)\right| < \epsilon$$

If $\lim_{x \to x_0^+} f(x) = f(x_0)$, we say that $f(x)$ is **right-continuous** at point x_0.

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, \left(0 \leq x - x_0 < \delta\right) : \left|f(x) - f(x_0)\right| < \epsilon$$

Note: Note the difference in red to the left- and right-limit.

Definition 4.8.4 ***Continuity of a Function on an Closed Interval***
If a function $f(x)$ is continuous on (a, b), right-continuous at point a, and left-continuous at point b, then we say that function $f(x)$ is continuous on $[a, b]$.

Example 4.8.2

For $f(x) = \sqrt{x(1-x)}$, prove $f(x)$ is continuous on $[0,1]$.
[Hint:] We need to prove three things: 1.) $f(x)$ is continuous on $(0,1)$; 2.) $f(x)$ is right-continuous at 0; and 3.) $f(x)$ is left-continuous at 1.

Proof:

First, let x_0 be any point in $(0,1)$. Let $\eta = \min\left\{x_0, 1-x_0\right\} > 0$. When $\left|x-x_0\right| < \eta$, we want to find a δ such that $\forall\,\epsilon > 0$, $\forall\,x$, $(|x-x_0| < \delta) : |\sqrt{x(1-x)} - \sqrt{x_0(1-x_0)}| < \epsilon$.

$$\begin{aligned}
\left|\sqrt{x(1-x)} - \sqrt{x_0(1-x_0)}\right| &= \frac{\left|x(1-x) - x_0(1-x_0)\right|}{\sqrt{x(1-x)} + \sqrt{x_0(1-x_0)}} \\
= \frac{\left|x - x^2 - x_0 + x_0^2\right|}{\sqrt{x(1-x)} + \sqrt{x_0(1-x_0)}} &= \frac{\left|(x-x_0)(1-x-x_0)\right|}{\sqrt{x(1-x)} + \sqrt{x_0(1-x_0)}} \\
= \frac{\left|1-x-x_0\right|}{\sqrt{x(1-x)} + \sqrt{x_0(1-x_0)}}\left|(x-x_0)\right| &< \frac{1}{\sqrt{x_0(1-x_0)}}\left|(x-x_0)\right|
\end{aligned}$$

We choose $\delta = \min\left\{\eta, \sqrt{x_0(1-x_0)}\cdot\epsilon\right\}$, such that $\forall\,x$, $(\left|x-x_0\right| < \delta)$ we have $\left|\sqrt{x(1-x)} - \sqrt{x_0(1-x_0)}\right| < \epsilon$.
Secondly, for $x_0 = 0$, we want to show $\lim\limits_{x\to0+}\sqrt{x(1-x)} = 0$. For all $\epsilon > 0$, we want to find a $\delta > 0$, $\forall\,x$, $(0 \le x - 0 < \delta) : |\sqrt{x(1-x)} - 0| < \epsilon$.

$$\sqrt{x(1-x)} < \sqrt{x} < \epsilon.$$

We find $\delta = \epsilon^2$, $\forall\,x$, $(0 \le |x-0| < \delta) : \left|\sqrt{1-x} - 0\right| < \epsilon$.
Thirdly, for $x_0 = 1$, we want to show $\lim\limits_{x\to1^-}\sqrt{x(1-x)} = 0$. For all $\epsilon > 0$, we want to find a $\delta > 0$, $\forall\,x$, $(-\delta < |x-1| \le 0) : \left|\sqrt{x(1-x)} - 0\right| < \epsilon$.

$$\left|\sqrt{x(1-x)}\right| < \sqrt{1-x} < \epsilon.$$

We find $\delta = \epsilon^2$, $\forall\,x$, $(-\delta < x - 1 \le 0) : \left|\sqrt{x(1-x)} - 0\right| < \epsilon$.
Therefore, we have shown that $f(x) = \sqrt{x(1-x)}$ is continuous on $[0,1]$. □

Remark. It seems a bit tedious to show three things. A more efficient way is given below, which eliminates the need for three different definitions.

Definition 4.8.5 ***Continuity of a Function on Some Interval***

Suppose $f(x)$ is defined on some interval X, if $x_0 \in X$, $\forall\epsilon > 0$, $\exists\,\delta > 0$, $\forall\,x \in X$, $\left(\left|x-x_0\right| < \delta\right) : \left|f(x) - f(x_0)\right| < \epsilon$, then we say $f(x)$ is continuous on the interval X.

Example 4.8.3

Suppose $f(x) = \sin x$, show that $f(x)$ is continuous on $(-\infty, +\infty)$.

Proof:

Let x_0 be any point in $(-\infty, \infty)$, for all $\epsilon > 0$, we want to find a $\delta > 0$, $\forall\,x \in (-\infty, \infty)$, $\left(\left|x-x_0\right| < \delta\right) : \left|\sin x - \sin x_0\right| < \epsilon$.

$$\left|\sin x - \sin x_0\right| = \left|2\cos\frac{x+x_0}{2}\sin\frac{x-x_0}{2}\right| \le 2\left|\sin\frac{x-x_0}{2}\right| \le \left|x-x_0\right| < \epsilon.$$

We find $\delta = \epsilon$.
We have shown that $f(x) = \sin x$ is continuous on $(-\infty, +\infty)$. □

Note: The blue inequality is due to $\cos x \le 1$, the red inequality is because $\lim\limits_{x\to0}\frac{\sin x}{x} \le 1$.

Example 4.8.4

For $f(x) = a^x, a > 0, a \neq 1$, show $f(x)$ is continuous on $(-\infty, +\infty)$.

Proof:

Let $x_0 \in (-\infty, +\infty)$, we want to show

$$\lim_{x\to x_0} a^x = a^{x_0} \iff \lim_{x\to x_0} a^{x-x_0} = 1 \iff \lim_{t\to 0} a^t = 1.$$

For $t \to 0^+$:

1.) $a > 1$: We note that $\frac{1}{t} \geq \left\lfloor \frac{1}{t} \right\rfloor$ and $a^t = a^{\frac{1}{\frac{1}{t}}} \leq a^{\frac{1}{\left\lfloor \frac{1}{t} \right\rfloor}}$. Also, $\frac{1}{t} \xrightarrow{t\to 0} +\infty$, view $\left\lfloor \frac{1}{t} \right\rfloor$ as n, and recall that we have shown $\lim_{n\to+\infty} \sqrt[n]{a} = \lim_{n\to+\infty} a^{\frac{1}{n}} = 1$. We have

$$1 < a^t \leq a^{\frac{1}{\left\lfloor \frac{1}{t} \right\rfloor}}.$$

Thus, by the Squeeze theorem, we have $\lim_{t\to 0} a^t = 1$.

2.) $0 < a < 1$: We have $\lim_{t\to 0^+} a^t = \lim_{t\to 0+} \frac{1}{\left(\frac{1}{a}\right)^t} = 1$.

For $t \to 0^-$:

Let $u = -t$, then $u \to 0^+$. Then

$$\lim_{t\to 0^-} a^t = \lim_{u\to 0^+} a^{-u} = \lim_{u\to 0^+} \frac{1}{a^u} = 1.$$

Since $\lim_{t\to 0^-} a^t = \lim_{t\to 0^+} a^t$, we have shown $\lim_{t\to 0} a^t = \lim_{x\to x_0} a^x = 1$. □

4.9 The Arithmetic Operations on Limits of The Continuous Functions

Theorem 4.9.1

Suppose $f(x)$ and $g(x)$ are two continuous functions at the point x_0. That is $\lim_{x\to x_0} f(x) = f(x_0)$ and $\lim_{x\to x_0} g(x) = g(x_0)$.

1. $\lim_{x\to x_0} (\alpha f(x) + \beta g(x)) = \alpha f(x_0) + \beta g(x_0)$;
2. $\lim_{x\to x_0} (f(x)g(x)) = f(x_0)g(x_0)$;
3. $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{f(x_0)}{g(x_0)}, \quad g(x_0) \neq 0.$

Example 4.9.1

Find the value of $\lim_{x\to 2} \frac{x^2+\sin x}{3^x+2x}$.

Solution:

It is obvious that the answer is $\frac{4+\sin 2}{13}$, since all four functions are continuous at $x = 2$. □

Example 4.9.2

The polynomial $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is continuous on $(-\infty, +\infty)$.
The polynomial $Q_m(x) = \frac{P_n(x)}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_0}$ is continuous on its domain.

Example 4.9.3

We know that $\sin x$ and $\cos x$ are continuous on $(-\infty, +\infty)$, then it is obvious that

1.) $\tan x = \frac{\sin x}{\cos x}$ is continuous on $\left\{x \mid x \neq k\pi + \frac{\pi}{2}, k \in \mathbb{Z}\right\}$;

2.) $\cot x = \frac{\cos x}{\sin x}$ is continuous on $\{x \mid x \neq k\pi, k \in \mathbb{Z}\}$.

4.10 Types of Discontinuities

Before we discuss different types of discontinuities, let's have a closer look at $\lim_{x\to x_0} f(x) = f(x_0)$. What does it mean by saying $\lim_{x\to x_0} f(x) = f(x_0)$?

1. $f(x)$ is defined at x_0;
2. $\lim_{x\to x_0^+} f(x) = f(x_0)$;
3. $\lim_{x\to x_0^-} f(x) = f(x_0)$.

Missing any one of these three conditions at the point x_0, $f(x)$ is considered discontinuous at x_0.

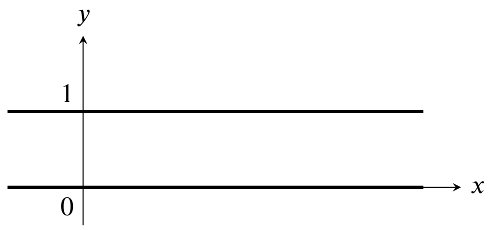
Definition 4.10.1 ***Types of Discontinuities***
There are four types of discontinuities:

1. **Jump Discontinuity**: $\lim_{x\to x_0^+} f(x) \neq \lim_{x\to x_0^-} f(x)$;
2. **Oscillatory Discontinuity**: As x approaches x_0, the function $f(x)$ fluctuates between different values (e.g., between -1 and 1) with increasing frequency, preventing the limit from settling on a single value.
3. **Infinite Discontinuity**: At least one of the $\lim_{x\to x_0^+} f(x)$ and $\lim_{x\to x_0^-} f(x)$ does not converge;
4. **Removable Discontinuity**: Removable discontinuity occurs at a point x_0 in the domain of a function $f(x)$ when the function is not defined at x_0 or the value of the function at x_0 does not match the limit, but the limit $\lim_{x\to x_0} f(x)$ exists and is finite. This means the discontinuity can be "removed" by redefining the function at x_0 to equal the limit, making the function continuous at that point. That is

$$\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) \begin{cases} \neq f(x_0), \\ \text{or } f(x) \text{ is not defined at } x_0. \end{cases}$$

Example 4.10.1 ***Dirichlet Function***
Prove that the Dirichlet function, $D(x)$, is not continuous on $(-\infty, +\infty)$.

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$



Proof:
Let $x_0 \in (-\infty, +\infty)$. Let x_n' be any rational, $x_n' > x_0, x_n' \to x_0$, then

$$\lim_{x_n'\to x_0} D(x_n') = 1.$$

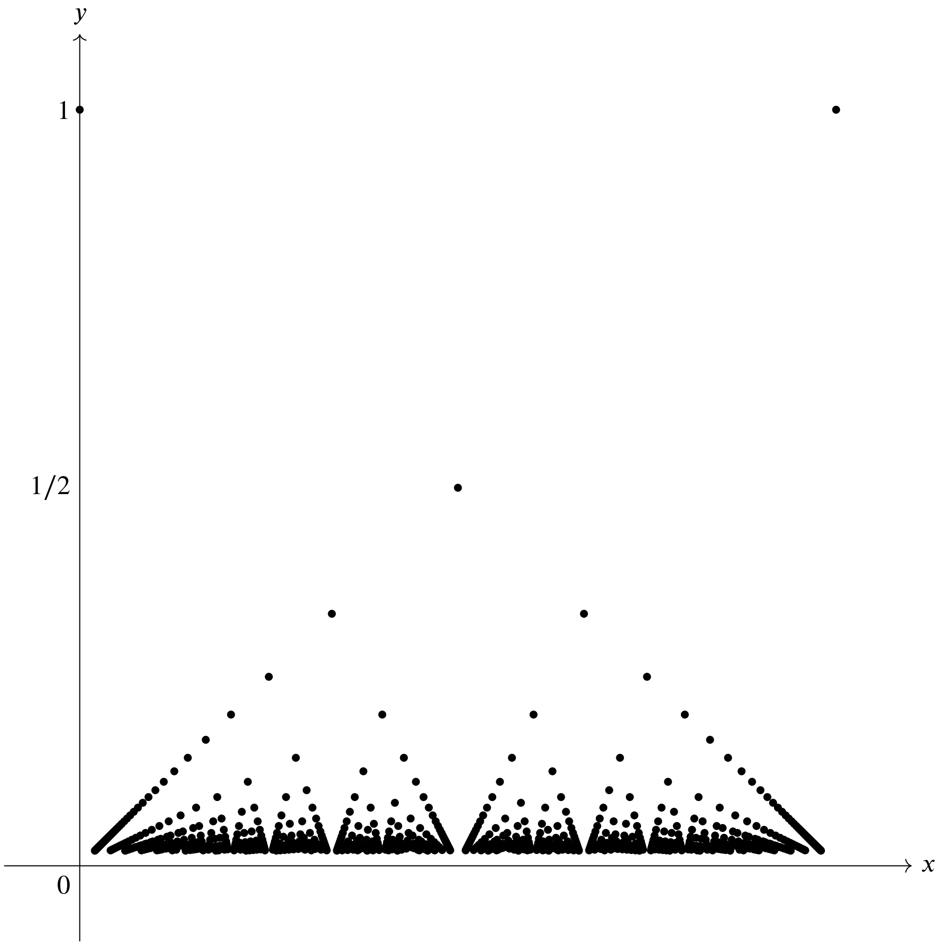
Let x_n'' be any irrational, $x_n'' > x_0, x_n'' \to x_0$, then

$$\lim_{x_n'' \to x_0} D(x_n'') = 0.$$

By Heine theorem, we know that $\lim_{x \to x_0} D(x)$ does not exists. Since $x_0 \in (-\infty, +\infty)$, we know that the Dirichlet Function, $D(x)$, is not continuous on $(-\infty, +\infty)$. □

Example 4.10.2 ***Popcorn Function***
Prove for all $x_0 \in (-\infty, +\infty)$, $\lim_{x \to x_0} R(x) = 0$. That is the Popcorn function is continuous at all irrationals and it is discontinuous at all rationals.

$$R(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q}, \\ \frac{1}{p} & \text{if } p \in \mathbb{N}^+, q \in \mathbb{Z}\backslash\{0\}, p \text{ and } q \text{ co-prime}, \\ 1 & \text{if } x = 0. \end{cases}$$



Here, $x = 0$, $R(x) = 1$ is to maintain the periodicity of the Popcorn function.

Proof:
Consider the periodicity, we only need to consider $x_0 \in [0, 1]$. That is we only need to show for $x_0 \in [0, 1]$, $\lim_{x \to x_0} R(x) = 0\ldots$

1. In $[0, 1]$, the number of numbers that have 1 as their denominator is 2: $\frac{0}{1}, \frac{1}{1}$
2. In $[0, 1]$, the number of numbers that have 2 as their denominator is 1: $\frac{1}{2}$;
3. In $[0, 1]$, the number of numbers that have 3 as their denominator is 2: $\frac{1}{3}, \frac{2}{3}$;

4. ……
5. In $[0, 1]$, the number of numbers that have k as their denominator is: at most k.

For any $k \in \mathbb{Z}^+$, there are finitely many rational numbers on $[0, 1]$ with denominators less than or equal to k. For all $\epsilon > 0$, find $\delta > 0$. Let $k = \left\lfloor \frac{1}{\epsilon} \right\rfloor$. In $[0, 1]$, denote the rationals whose denominator is less than or equal to k as $r_1, r_2, \ldots, r_n$. Let $\delta = \min\limits_{\substack{1 \le i \le n \\ r_i \ne x_0}} \left\{ |r_i - x_0| \right\}$, $\forall\, x \in [0, 1]$, $\left(0 < |x - x_0| < \delta\right)$.

1. x is irrational, $R(x) = 0$;

2.) x is rational, the denominator of x has to be greater than k. Since $k = \left\lfloor \frac{1}{\epsilon} \right\rfloor$, that means the denominator is greater than or equal to $\left\lfloor \frac{1}{\epsilon} \right\rfloor + 1$.

$$R(x) - 0 = \frac{1}{k} \le \frac{1}{\left\lfloor \frac{1}{\epsilon} \right\rfloor + 1} < \frac{1}{\frac{1}{\epsilon}} = \epsilon.$$

□

Remark. The key difference between the Dirichlet function and Popcorn function is that the Popcorn function has the following property:

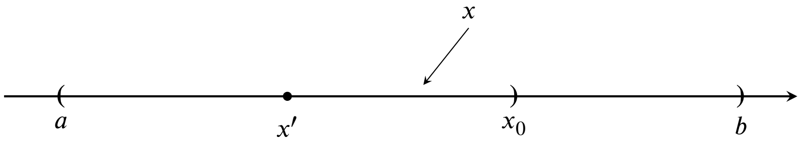
$$\forall\, \epsilon > 0, \text{ there are at most limited number of points that makes } R(x) > \epsilon, \text{ on the interval } [0, 1].$$

Example 4.10.3

The discontinuities of a monotonic function on an interval (a, b) must be the Jump discontinuity.

Proof:

W.O.L.G., let $f(x)$ be monotonically increasing on (a, b). Let $x_0 \in (a, b)$. The set $\left\{ f(x) \mid x \in (a, x_0) \right\}$ has a upper bound and must have a supremum, denoted $\alpha = \sup \left\{ f(x) \mid x \in (a, x_0) \right\}$. For all $x \in (a, x_0)$ we have $f(x) \le \alpha$. That is $\forall\, \epsilon > 0, \exists\, x' \in (a, x_0)$ such that $f(x') > \alpha - \epsilon$. Let $\delta = x_0 - x'$, $\forall\, x$, $(-\delta < x - x_0 < 0)$. That is saying $x \in (x', x_0)$. Since $f(x)$ is monotonically increasing, we have



$$\alpha - \epsilon < f(x') \le f(x) \le \alpha$$

Therefore, we have $|f(x) - \alpha| < \epsilon$, that is $\lim\limits_{x \to x_0^-} f(x) = \alpha$.

Similarly,

$$\lim_{x \to x_0^+} f(x) = \beta, \beta = \inf \left\{ f(x) \mid x \in (x_0, b) \right\}$$

.

□

4.11 Inverse Function

We will discuss two properties of the inverse function: the *existence* and *continuity*. We will differ the third property, *differentiability*, to later sections.

Theorem 4.11.1 ***Inverse of a Monotone (or Monotonic) Function Theorem***

If a function $f(x)$ is strictly monotonic (increasing or decreasing) over D_f, then there exists an inverse function $x = f^{-1}(y), y \in R_f$ and the inverse function f^{-1} is also strictly monotonic (increasing or decreasing).

Proof:

For $x_1, x_2 \in D_f$ with $x_1 < x_2$, it is obvious that $f(x_1) < f(x_2)$ or $y_1 < y_2$. Moreover, x_1 and x_2 can swap, meaning we choose them randomly. In other words, for $x_1, x_2 \in D_f$ with $x_1 > x_2$, it is obvious that $f(x_1) > f(x_2)$ or $y_1 > y_2$. Then it is easy to see that if $x_1 \neq x_2$ then $y_1 \neq y_2$ (Injection). Therefore, there exists an inverse function $x = f^{-1}(y)$. Now $\forall\, y_1 < y_2$, if $x_1 > x_2$ is a contradiction to strictly monotonic increasing for the function $f(x)$. If $x_1 = x_2$, this is a contradiction to the definition of a function. So $x_1 < x_2$ that is $f^{-1}(y)$ is strictly monotonic increasing. □

Note: In short: If f is strictly monotonic, then f^{-1} exists and is also strictly monotonic. This takes care of the *existence*.

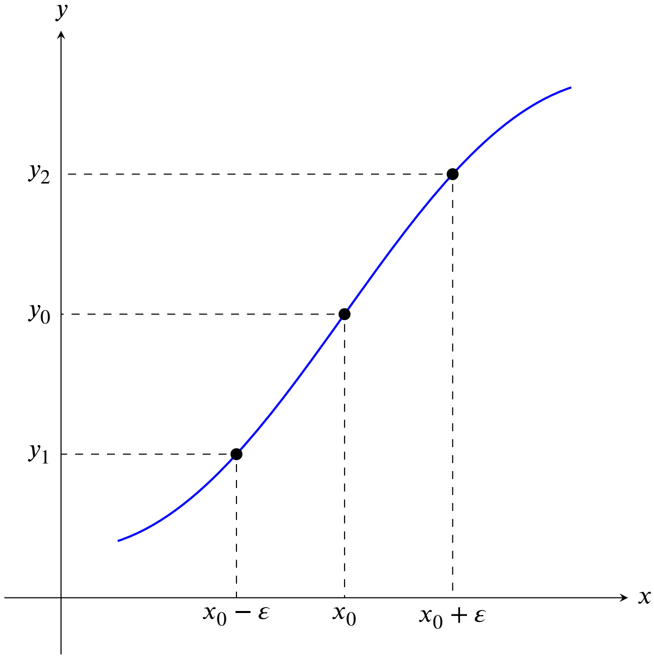
Theorem 4.11.2 ***Continuity of the Inverse Function Theorem for Monotonic Functions***

Suppose $y = f(x)$ is continuous and strictly increasing on the interval $[a, b]$. If $f(a) = \alpha$, $f(b) = \beta$, then $f^{-1}(y)$ is continuous on $[\alpha, \beta]$.

Proof:

[Hint:] First, we need to show $R_f = [\alpha, \beta]$. Since $f(a), f(b)$ have to be in R_f we only need to find a point say $\gamma \in [\alpha, \beta]$ and a point $x_0 \in [a, b]$ such that $f(x_0) = \gamma$, then we are done. For continuity, we need to show f^{-1} is continuous at $y_0 \in (\alpha, \beta)$ and left- and right-continuous at β and α.
First, we need to show $R_f = [\alpha, \beta]$. Now, $\forall\, \gamma \in [\alpha, \beta]$, let $S = \{x \mid x \in [a, b], f(x) < \gamma\}$. Let $\sup S = x_0$. (We know S has the supremum because S has an upper bound.) When $x < x_0$, we have $f(x) < \gamma$ and when $x > x_0$, we have $f(x) > \gamma$, by f is strictly increasing. $\lim\limits_{x \to x_0^-} f(x) \leq \gamma$, $\lim\limits_{x \to x_0^+} f(x) \geq \gamma$. Also, f is continuous in $[a, b]$ so f is continuous at x_0, therefore $\lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x) = f(x_0)$. Therefore, $f(x_0) = \gamma$. Thus, $R_f = [\alpha, \beta]$.

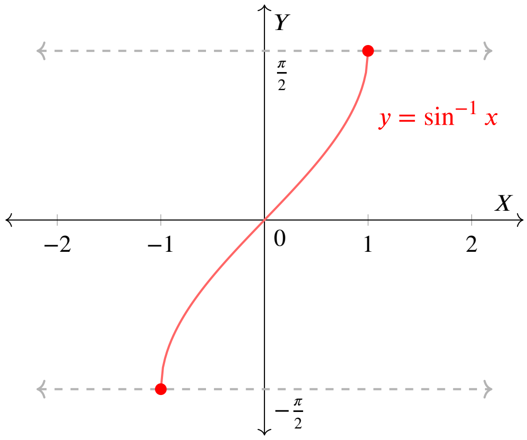
Now, we deal with the continuity. For all $y_0 \in (\alpha, \beta)$, we need to show that f^{-1} is continuous at y_0 and is left- and right- continuous at β and α.



Here, we only prove one of them. Let $f(x_0) = y_0$ that is $f^{-1}(y_0) = x_0$. We need to show that $\forall\, \epsilon > 0$, we need to find $\delta > 0, \forall\, y$, $\left(|y - y_0| < \delta\right)$: $\underbrace{\left|f^{-1}(y) - f^{-1}(y_0)\right|}_{|x - x_0| < \epsilon} < \epsilon$. Let $\delta = \min\left\{y_0 - y_1, y_2 - y_0\right\}$, when $|y - y_0| < \delta$, there is $|x - x_0| < \epsilon$. The left- and right-continuity at β and α are left as exercises. □

Example 4.11.1

For $y = \sin x$, which is continuous and strictly increasing on it domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and its range is $[-1, 1]$. Its inverse function $y = \sin^{-1} x = \arcsin x$ has domain $D = [-1, 1]$ (continuous and strictly increasing over $[-1, 1]$) and range $R = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



$$\text{Domain}= [-1, 1], \quad \text{Range}= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Now, let us talk about the composite function. If we compose two continuous functions together, is it still continuous.

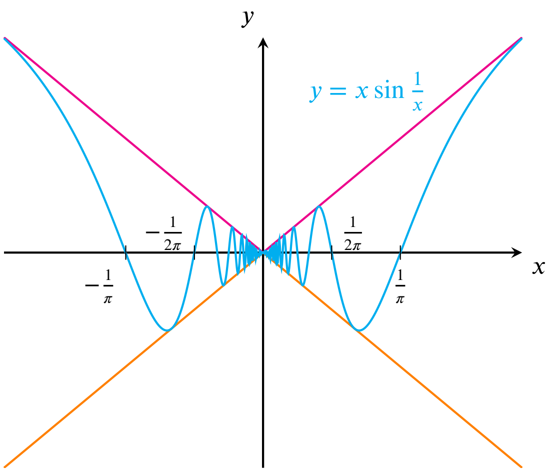
Example 4.11.2

If $\lim\limits_{u\to u_0} f(u) = A$, $\lim\limits_{x\to x_0} g(x) = u_0$, does $\lim\limits_{x\to x_0} f\circ g(x) = A$ hold?

Solution:

The answer is No! A counter example would be

$$f(u) = \begin{cases} 0 & \text{if } u = 0, \\ 1 & \text{if } u \neq 0. \end{cases} \quad g(x) = x \sin \frac{1}{x}$$



It is obvious that $\lim\limits_{u\to 0} f(u) = 1$ and $\lim\limits_{x\to 0} g(x) = 0$. Then we have

$$f\circ g(x) = \begin{cases} 0 & \text{if } x = \frac{1}{n\pi}, \\ 1 & \text{if } x \neq \frac{1}{n\pi}. \end{cases}$$

Now, we want to use the Heine Theorem 4.4.1. We need to find two sequences that five two different limits for $f\circ g(x)$. Let the first sequence be $\{x'_n\}, x'_n = \frac{1}{n\pi}$, it is obvious that $x'_n \xrightarrow{n\to+\infty} 0$, and $\lim\limits_{n\to+\infty} f\circ g(x'_n) = 0$. Let the second sequence be $\{x''_n\}, x''_n \neq \frac{1}{k\pi}$, it is obvious that $x''_n \xrightarrow{n\to+\infty} 0$, but $\lim\limits_{n\to+\infty} f\circ g(x''_n) = 1$. So we find two difference sequences that are not zero, but approaches zero give two difference limits to $f \circ g(x)$. Therefore, we know that $\lim_{x \to 0} f \circ g(x)$ does not exist. $\square$

Note: In example 4.11.2, it is because $f(u)$ is discontinuous at $u = 0$ that makes the statement in example 4.11.2 invalid.

Theorem 4.11.3

If $u = g(x)$ is continuous at x_0 that is $\lim_{x \to x_0} g(x) = u_0$, and $g(x_0) = u_0$. Also, if $f(u)$ is continuous at u_0 that is $\lim_{u \to u_0} f(u) = f(u_0)$, then $f \circ g$ is continuous at x_0 that is $\lim_{x \to x_0} f \circ g(x) = f \circ g(x_0) = f(u_0)$.

Proof:

For all $\epsilon > 0$, $\exists \eta > 0$, $\forall u$, $(|u - u_0| < \eta) : |f(u) - f(u_0)| < \epsilon$. For all $\eta > 0$, $\exists \delta > 0$, $\forall x$, $(|x - x_0| < \delta)$: $|g(x) - g(x_0)| < \eta$. That is $|g(x) - u_0| < \eta$. That is $|f \circ g(x) - f \circ g(x_0)| < \epsilon$. $\square$

Note: For $f \circ g$ to be a continuous, functions f and g both have to be continuous. Also, do not forget that the range of g has to sit within the domain of f.

Example 4.11.3

Given $y = \sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$. Let $u = e^x$, then $y = \frac{u - u^{-1}}{2}$. Applying the theorem above we know that $\sinh x$ is continuous. It is similar to show $\cosh x$ is continuous.

Example 4.11.4

Prove for any real number α, $f(x) = x^\alpha$ is continuous on $(0, +\infty)$.

Proof:

W.O.L.G., $f(x) = x^\alpha = e^{\alpha \ln x}$. That is $y = e^u$ and $u = \alpha \ln x$. First, the range of u, which is $R_u = (0, +\infty)$, is in the domain of y, which is $D_y = (-\infty, +\infty)$. Both u and y are continuous in their domain, so their composite $f(x) = y \circ u(x)$ is continuous by applying theorem 4.11.3. $\square$

Note: Several things to point out:

1. When $\alpha = n, n \in \mathbb{Z}^+$, $f(x) = x^n$ is continuous on $(-\infty, +\infty)$.
2. When $\alpha = -n, n \in \mathbb{Z}^+$, $f(x) = x^{-n}$ is continuous on $(-\infty, 0) \cup (0, +\infty)$.
3. When $\alpha = \frac{q}{p}$, p, q are co-primes,
   (a) if p is odd, $f(x) = x^{\frac{q}{p}}$ is continuous on $(-\infty, +\infty)$.
   (b) if p is even $f(x) = x^{\frac{q}{p}}$ is continuous on $[0, +\infty)$.

Definition 4.11.1 ***Basic Elementary Functions***

Basic elementary functions are functions obtained from constants, powers, exponentials, logarithms, trigonometric, and inverse trigonometric functions.

Definition 4.11.2 ***Elementary Functions***

Elementary functions are the functions obtained from **basic elementary functions**, using a *finite* number of operations of addition, subtraction, multiplication, division, and composition.

Theorem 4.11.4

All **elementary functions** are continuous on their domains.

Note: Theorem 4.11.4 can help us find limit of certain types of functions.

Example 4.11.5

Compute $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$.

Solution:

[Hint:] By theorem 4.11.4, we know $(\cos x)^{\frac{1}{x^2}}$ is continuous on its domain, which means that we only need to plugin $x = 0$ into the function to find the answer. So, when $x = 0$, $\cos x = 1$ then you may think the answer is 1. Sadly, it is wrong. The limit has the **indeterminate form**, 1^∞. So, we need to work more to work this out. Recall, this is very similar to $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 1$. Because 1^∞ is an **indeterminate form** of limit.
We have:

$$\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0}\left(1 - 2\sin^2\frac{x}{2}\right)^{\frac{1}{x^2}}$$

Also,

$$(\cos x)^{\frac{1}{x^2}} = \exp\left\{\ln(\cos x)^{\frac{1}{x^2}}\right\} = \exp\left\{\frac{1}{x^2}\ln(\cos x)\right\}$$

Combining them together, we have

$$(\cos x)^{\frac{1}{x^2}} = \exp\left\{\frac{1}{x^2}\ln(\cos x)\right\} = \exp\left\{\frac{1}{x^2}\ln\left(1 - 2\sin^2\frac{x}{2}\right)\right\} =$$

Let's focus on the blue part, $\frac{1}{x^2}\ln\left(1 - 2\sin^2\frac{x}{2}\right) = \frac{2\sin^2\frac{x}{2}}{x^2}\ln\left(1 - 2\sin^2\frac{x}{2}\right)^{\frac{1}{2\sin^2\frac{x}{2}}}$. Also, we know

$$\lim_{x\to 0}(1-x)^{\frac{1}{x}} = \frac{1}{e}, \qquad \lim_{x\to 0}\left(1 - 2\sin^2\frac{x}{2}\right)^{\frac{1}{2\sin^2\frac{x}{2}}} = \frac{1}{e}$$

$$\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{4\cdot\left(\frac{x}{2}\right)^2} = \frac{1}{2}$$

Putting everything together, we have

$$\begin{aligned}
\lim_{x\to 0}\frac{1}{x^2}\ln\left(1 - 2\sin^2\frac{x}{2}\right) &= -\frac{1}{2}\\
\lim_{x\to 0}\frac{1}{x^2}\ln\cos x &= -\frac{1}{2}\\
\lim_{x\to 0}\ln\,(\cos x)^{\frac{1}{x^2}} &= -\frac{1}{2}\\
\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} &= e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}.
\end{aligned}$$

□

Note: The big takeaway here is that when we calculating the limit of a continuous function by plugin the limit of x directly, if the answer we got is indeterminate form, we know we have more work to do.

Example 4.11.6

The mass variation of a radioactive substance. At time $t = 0$, the initial total mass of the substance is $M = M(0)$, and the decay constant is k. Find the mass of the substance $M(t)$ at time t.

Solution:

[Hint]: The key idea is to divide time into very small intervals so that the decay differs little in each small time interval.

We divide $(o, t]$ into n small intervals $\Delta_i = \left(\frac{(i-1)t}{n}, \frac{it}{n}\right]$, $i = 1, 2, \ldots, n$. That is $(0, t] = \bigcup_{i=1}^{n} \Delta_i$. Then we have

$$
\begin{aligned}
\Delta_1: \quad & M\left(\frac{t}{n}\right) \simeq M - kM \cdot \frac{t}{n} = M\left(1 - \frac{kt}{n}\right) \\
\Delta_2: \quad & M\left(\frac{2t}{n}\right) \simeq M\left(1 - \frac{kt}{n}\right) - kM\left(1 - \frac{kt}{n}\right)\frac{t}{n} = M\left(1 - \frac{kt}{n}\right)^2 \\
& \vdots \\
\Delta_n: \quad & M\left(\frac{nt}{n}\right) = M(t) \simeq M\left(1 - \frac{kt}{n}\right)^n.
\end{aligned}
$$

They key idea is that to make each time interval smaller and smaller, n needs to approach infinity. This is where the limit kicks in.

$$
\begin{aligned}
M(t) = \lim_{n\to\infty} M\left(1 - \frac{kt}{n}\right)^n &= \lim_{n\to\infty} M \exp\left\{n \ln\left(1 - \frac{kt}{n}\right)\right\} \\
&= \lim_{n\to\infty} M \exp\left\{n \cdot \frac{kt}{n} \ln\left(1 - \frac{kt}{n}\right)^{\frac{n}{kt}}\right\} \\
&= Me^{-kt}
\end{aligned}
$$

□

Chapter 5

Orders of infinitesimals and infinities

5.1 Order of an infinitesimal

We have studied the infinitesimal of a sequence, $\lim_{n\to\infty} x_n = 0$. Now let us have a look at the infinitesimal of a function, $\lim_{x\to x_0} f(x) = 0$.

Definition 5.1.1 ***Infinitesimal of a Function***
When $x \to x_0$, we have $f(x) \to 0$, that is $\lim_{x\to x_0} f(x) = 0$, we say when $x \to x_0$, $f(x)$ is an infinitesimal.

Note: When we talk about a function infinitesimal, we cannot omit the underlined part. It is meaningless by only saying a function $f(x)$ is an infinitesimal.

Different infinitesimals approach zero with different speeds. How do we compare infinitesimals? In other words, how can we know which infinitesimal approaches zero in the fastest manner?

Definition 5.1.2 ***Higher-order Infinitesimal, Lower-order Infinitesimal, Bounded Quantity, Infifnitesimals of The Same Order, and Equivalent Infinitesimals***
When $x \to x_0$, $u(x), v(x)$ are two infinitesimals.

1. If $\lim_{x\to x_0} \frac{u(x)}{v(x)} = 0$, we say when $x \to x_0$, $u(x)$ is the **higher-order infinitesimal** of $v(x)$, denoted as
$$u(x) = o\big(v(x)\big), \quad (x \to x_0).$$
Similarly, $v(x)$ is the **lower-order infinitesimal** of $u(x)$.
2. If there exists $A > 0$, when $x \in \big\{x \mid 0 < |x - x_0| < \rho\big\}$, we have $\left|\frac{u(x)}{v(x)}\right| \le A$, then we say when $x \to x_0$, $\left|\frac{u(x)}{v(x)}\right|$ is a **bounded quantity**, denoted as
$$u(x) = \mathcal{O}\big(v(x)\big), \quad (x \to x_0).$$
3. If there exists $0 < a < A < +\infty$, when $x \big\{x \mid 0 < |x - x_0| < \rho\big\}$, we have $0 < a < \left|\frac{u(x)}{v(x)}\right| < A < +\infty$, then we say when $x \to x_0$, $u(x)$ and $v(x)$ are **infinitesimals of the same order**, denoted as
$$\lim_{x\to x_0} \frac{u(x)}{v(x)} = c \neq 0.$$
4. If $\lim_{x\to x_0} \frac{u(x)}{v(x)} = 1$, then we say when $x \to x_0$, $u(x)$ and $v(x)$ are **equivalent infinitesimals**, denoted as
$$u(x) \sim v(x), \quad (x \to x_0).$$

Remark. Higher-order infinitesimal just means that the term goes to zero faster. A lower-order infinitesimal just means that the term goes to zero slower.

Example 5.1.1 ***Higher-order Infinitesimal***

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x}=\lim_{x\to 0}\frac{x\cdot 2\sin^2\frac{x}{2}}{4\cdot\left(\frac{x}{2}\right)^2}=0.$$

Therefore, we say $1-\cos x$ is an higher-order infinitesimal of x, denoted as

$$1-\cos x=o\big(x\big),\quad (x\to 0).$$

Example 5.1.2 ***Higher-order Infinitesimal***

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^2}=\lim_{x\to 0}\frac{\sin x}{\cos x}\cdot\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{x\cdot\cos x}\cdot\frac{1-\cos x}{x}=0.$$

Therefore, we say $\tan x-\sin x$ is an higher-order infinitesimal of x^2, denoted at

$$\tan x-\sin x=o\big(x^2\big),\quad (x\to 0).$$

Example 5.1.3 ***Bounded Quantity***

We have $u(x)=x\sin\frac{1}{x}, v(x)=x,(x\to 0)$.

$$\left|\frac{u(x)}{v(x)}\right|=\left|\sin\frac{1}{x}\right|\leq 1.$$

Therefore, we say when $x\to 0$, $\left|\frac{x\sin\frac{1}{x}}{x}\right|$ is a bounded quantity, denoted as

$$x\sin\frac{1}{x}=\mathcal{O}(x),\quad (x\to 0).$$

Example 5.1.4 ***Equivalent Infinitesimals***

We all know that $\lim_{x\to 0}\frac{\sin x}{x}=1$ and it can be denoted as

$$\sin x\sim x,\quad (x\to 0);\quad \text{or}$$

$$\sin x=x+o(x),\quad (x\to 0).$$

Remark. Recall that this $o(x)$, higher-order infinitesimal of x, as $x\to 0$, means something that goes to zero faster than x.

Example 5.1.5 ***Equivalent Infinitesimals***

We know that $\lim_{x\to 0}\frac{1-\cos x}{\frac{1}{2}x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2\cdot\left(\frac{x}{2}\right)^2}=1$, and it can be denoted as

$$1-\cos x\sim\frac{1}{2}x^2,\quad (x\to 0);\ \text{or}$$

$$1-\cos x=\frac{1}{2}x^2+o\left(x^2\right).$$

Example 5.1.6 ***Equivalent Infinitesimals***

We know that $\lim_{x\to 0}\frac{\tan x-\sin x}{\frac{1}{2}x^3}=\lim_{x\to 0}\frac{\sin x}{x\cos x}\cdot\frac{1-\cos x}{\frac{1}{2}x^2}=1$, and it can be denoted as

$$\tan x-\sin x\sim\frac{1}{2}x^3,\quad (x\to 0);\ \text{or}$$

$$\tan x-\sin x=\frac{1}{2}x^3+o\left(x^3\right),\quad (x\to 0).$$

Note: Though the *equivalent relation* between infinitesimals is very useful, but in some cases we need to be very careful when using. In example 5.1.6, indeed we have $\tan x\sim x,(x\to 0)$ and $\sin\sim x,(x\to 0)$, but they are not equal (missing some higher-order terms). So, if you substitute $\tan x$ and $\sin x$ by x into the limit, the answer will be wrong.

Remark. W.O.L.G. function $v(x)$ will have the form of $\left(x-x_0\right)^k$, when discussing infinitesimal. That is we have $v(x)=\left(x-x_0\right)^k$. Because writing $v(x)$ in this form allows us to determine the order of the infinitesimal $u(x)$.

Example 5.1.7 ***Infinitesimal***

As $x\to 0^+$, $\ln x\to-\infty$, then $\frac{-1}{\ln x}$ is an (positive) infinitesimal. What is the order of this infinitesimal $\frac{-1}{\ln x}$? We observe that for any $\alpha>0$, $-\frac{1}{\ln x}$ is the **lower-order infinitesimal** of x^α that is $\lim_{x\to 0^+}-\frac{\frac{1}{\ln x}}{x^\alpha}=+\infty$. This means that no matter how small the α is, $-\frac{1}{\ln x}$ always goes to zero slower than x^α. Denoted $-\frac{1}{\ln x}=o(1),\left(x\to 0^+\right)$, meaning $-\frac{1}{\ln x}$ is an infinitesimal.

Example 5.1.8 ***Self-bounded Quantity***

When $x\to 0$, $u(x)=\sin\frac{1}{x}$ is a bounded equality, denoted $u(x)=\mathcal{O}(1),\ (x\to 0)$.

5.2 Comparison of Infinities

Definition 5.2.1 ***Infinity***

If $\lim_{x\to x_0} f(x)=\infty;(\pm\infty)$, then we say when $x\to x_0$, $f(x)$ is a (positive or negative) **infinity**.

Definition 5.2.2 ***Higher-order Infinity, Lower-order Infinity, Bounded Quantity, Infinities of The Same Order, and Equivalent Infinities***

When $x\to x_0$ $u(x)$, $v(x)$ are both infinity:

1. If $\lim_{x\to x_0}\frac{u(x)}{v(x)}=\infty$, we say when $x\to x_0$, $u(x)$ is the **higher-order infinity** of $v(x)$. Also, $v(x)$ is the **lower-order infinity** of $u(x)$.
   When study sequences, we have the following result
   $$n^n>>n!>>a^n>>n^\alpha>>\ln^\beta n,\qquad a>1,\alpha>0,\beta>0$$
2. If $\exists A>0$, for $\left\{x\mid 0<|x-x_0|<\rho\right\}$, we have $\left|\frac{u(x)}{v(x)}\right|\le A$, then we say when $x\to x_0$, $\left|\frac{u(x)}{v(x)}\right|$ is a **bounded quantity**, denoted as
   $$u(x)=\mathcal{O}(v(x)),\quad (x\to x_0).$$
3. If there exists $0<a<A<+\infty$, for $\left\{x\mid 0<\left|x-x_0\right|<\rho\right\}$, we have $0<a<\left|\frac{u(x)}{v(x)}\right|<A<+\infty$, then we say when $x\to x_0$, $u(x)$ and $v(x)$ are **infinities of the same order**, denoted as
   $$\lim_{x\to x_0}\frac{u(x)}{v(x)}=c\ne 0,\quad (x\to x_0).$$

4. If $\lim_{x \to x_0} \frac{u(x)}{v(x)} = 1$, then we say when $x \to x_0$, $u(x)$ are $v(x)$ are **equivalent infinities**, denoted as

$$u(x) \sim v(x), \quad (x \to x_0).$$

Remark. Higher-order infinity means a term goes to infinity faster. Lower-order infinity means a term goes to infinity slower.

Remark. W.O.L.G. function $v(x)$ will have the form of $\left(\frac{1}{x}\right)^k$, when discussing infinity.

Note: It is a conversion that we do not use lower case o, when discussing infinities.

Example 5.2.1

Given $u(x) = x^3 \sin \frac{1}{x}$, $v(x) = x^2$, $x \to +\infty$.

Solution:

It is easy to show that both $u(x)$ and $v(x)$ approaches infinity as x goes to infinity.

$$\lim_{x \to +\infty} \frac{u(x)}{v(x)} = \lim_{x \to +\infty} \frac{x^3 \sin \frac{1}{x}}{x^2} = \lim_{x \to +\infty} \frac{x^2 \sin \frac{1}{x}}{x^2 \cdot \frac{1}{x}} = 1$$

Therefore, $x^3 \sin \frac{1}{x} \sim x^2, (x \to +\infty)$. □

Example 5.2.2

Find $\lim_{x \to \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right) \tan x$.

Solution:

Let $y = \frac{\pi}{2} - x$, then $\lim_{x \to \frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right) \tan x = \lim_{y \to 0^+} (y \cdot \cot y) = \lim_{y \to 0^+} \left(y \cdot \frac{\cos y}{\sin y}\right) = 1$. Since $\lim_{y \to 0^+} \frac{y}{\sin y} = 1$. This tells us, $\tan x \sim \frac{1}{\frac{\pi}{2} - x}, \left(x \to \frac{\pi}{2}^-\right)$. □

Next, we will prove that when $x \to 0^+$, $\frac{-1}{\ln x}$ is an lower-order infinitesimal to $x^\alpha, \forall \alpha > 0$.

Example 5.2.3

Show that when $x \to 0^+$ for any $k \in \mathbb{Z}^+$, $\left(\frac{-1}{\ln x}\right)^k$ is an lower-order infinitesimal to x.

Proof:

Let $y = -\ln x$, then $\lim_{x \to 0^+} \frac{x}{\left(\frac{-1}{\ln x}\right)^k} = \lim_{y \to +\infty} \frac{y^k}{e^y} = 0$. Since e^y is an higher-order infinitesimal of y^k. □

Example 5.2.4

When $x \to 0$, $e^{-\frac{1}{x}}$ is an higher-order infinitesimal of x^k.

Proof:

Let $y = \frac{1}{x}$, then $\lim_{x \to 0^+} \frac{e^{-\frac{1}{x}}}{x^k} = \lim_{y \to +\infty} \frac{y^k}{e^y} = 0$. □

Remark. We shall remember by heart some of the conclusions such as $x \to +\infty$,

$$e^x >> x^k >> \ln^m x.$$

5.3 Equivalent Asymptotics

We introduce some important **equivalent asymptotics** by examples, other then example 5.1.4, which is $\sin x \sim x, (x \to 0)$.

Example 5.3.1

Show $\ln(1+x) \sim x$, when $x \to 0$.

Proof:

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} = 1.$$

□

Example 5.3.2

Show that $e^x - 1 \sim x$, when $x \to 0$.

Proof:

Let $e^x - 1 = y$,

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{y\to 0} \frac{y}{\ln(1+y)} = 1.$$

□

Example 5.3.3

Show $(1+x)^\alpha - 1 \sim \alpha x, \forall \alpha \in \mathbb{R}$, when $x \to 0$.

Proof:

Let $y = (1+x)^\alpha - 1$,

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \lim_{y\to 0} \frac{y}{\ln(1+y)} \cdot \lim_{x\to 0} \frac{\alpha \ln(1+x)}{x} = \alpha.$$

□

Example 5.3.4

For $u(x) = \sqrt{x + \sqrt{x}}$, when $x \to +\infty$, it is infinity and when $x \to 0^+$, it is infinitesimal, but we need to know the order.

Solution:

$$\lim_{x\to +\infty} \frac{\sqrt{x+\sqrt{x}}}{\sqrt{x}} = \lim_{x\to +\infty} \sqrt{1 + \frac{1}{\sqrt{x}}} = 1.$$

Therefore, $\sqrt{x+\sqrt{x}} \sim \sqrt{x}$, when $(x \to +\infty)$. Why we divided by $\sqrt{x}$? Because, compared to x, $\sqrt{x}$ is NOT the dominant component when $x + \sqrt{x}$ approaches positive infinity.

$$\lim_{x\to 0^+} \frac{\sqrt{x+\sqrt{x}}}{x^{\frac{1}{4}}} = \lim_{x\to 0^+} \sqrt{1+\sqrt{x}} = 1.$$

Therefore, $\sqrt{x+\sqrt{x}} \sim \sqrt[4]{x}$, when $x \to 0^+$. □

Example 5.3.5

Given $v(x) = 2x^3 + 3x^5$,

1. when $x \to +\infty$, $v(x) \sim 3x^5$,
2. when $x \to 0$, $v(x) \sim 2x^3$.

5.4 Calculating Limits Using Equivalent Asymptotics

Theorem 5.4.1

Three functions $u(x), v(x), w(x)$ are defined on some punctured neighbourhood at x_0 and $\lim\limits_{x \to x_0} = \frac{v(x)}{w(x)} = 1 \iff v(x) \sim w(x), x \to x_0$, then:

1. $\lim\limits_{x \to x_0} u(x)w(x) = A \iff \lim\limits_{x \to x_0} u(x)v(x) = A$,
2. $\lim\limits_{\to x_0} \frac{u(x)}{w(x)} = A \iff \lim\limits_{x \to x)_0} \frac{u(x)}{v(x)} = A$.

Example 5.4.1

Calculate $\lim\limits_{x \to 0} \frac{\ln(1+x^2)}{(e^{2x}-1)\tan x}$

Solution:

Assume we know that $\ln\left(1+x^2\right) \sim x^2, x \to 0$. Alos, $e^{2x}-1 \sim 2x, x \to 0$, and $\tan x \sim x, x \to 0$, (These results will be left as exercises.). Then:

$$\lim_{x \to 0} \frac{\ln\left(1+x^2\right)}{\left(e^{2x}-1\right)\tan x} = \lim_{x \to 0} \frac{x^2}{2x \cdot x} = \frac{1}{2}.$$

□

Example 5.4.2

Calculate $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-e^{\frac{x}{3}}}{\ln(1+2x)}$.

Solution:

$$\begin{aligned}\lim_{x \to 0} \frac{\left(\sqrt{1+x}-1\right)\left(e^{\frac{x}{3}}-1\right)}{2x} &= \lim_{x \to 0} \frac{\left(\frac{x}{2}+o\left(x\right)\right)-\left(\frac{x}{3}+o\left(x\right)\right)}{2x} \\ &= \lim_{x \to 0} \frac{\frac{1}{6}x+o\left(x\right)}{2x} = \frac{1}{12}\end{aligned}$$

□

Remark. The red equality is because:

$$\begin{aligned} x \to 0, &\quad \sqrt{1+x}-1 \sim \frac{x}{2} \iff \sqrt{1+x}-1 = \frac{x}{2}+o\left(x\right) \\ x \to 0, &\quad e^{\frac{x}{3}}-1 \sim \frac{x}{3} \iff e^{\frac{x}{3}}-1 = \frac{x}{3}+o\left(x\right)\end{aligned}$$

Example 5.4.3

Calculate $\lim\limits_{x \to \infty} x\left(\sqrt[3]{x^3+x}-\sqrt[3]{x^3-x}\right)$.

Solution:

$$
\begin{aligned}
\lim_{x\to\infty} x\left(\sqrt[3]{x^3+x}-\sqrt[3]{x^3-x}\right) &= \lim_{x\to\infty} x^2\left(\left(\sqrt[3]{1+\frac{1}{x^2}}-1\right)-\left(\sqrt[3]{x^3-\frac{1}{x^2}}-1\right)\right)\\
&\color{red}{=} \lim_{x\to\infty}\left[\left(\frac{1}{3x^2}+o\left(\frac{1}{x^2}\right)\right)-\left(-\frac{1}{3x^2}+o\left(\frac{1}{x^2}\right)\right)\right]\\
&= \lim_{x\to\infty}\left(\frac{2}{3x^2+o\left(\frac{1}{x^2}\right)}\right)=\frac{2}{3}
\end{aligned}
$$

□

Remark. The red equality is due to:

$$
\begin{aligned}
x\to\infty,\quad & \sqrt[3]{1+x}-1=\frac{x}{3}+o(x)\\
x\to\infty,\quad & \sqrt[3]{1-x}+1=-\frac{x}{3}+o(x)
\end{aligned}
$$

and view $\frac{1}{x^2}$ as x.

Example 5.4.4

Calculate $\lim\limits_{x\to0}\cos(x)^{\frac{1}{x^2}}$.

Solution:

$$
\begin{aligned}
\lim_{x\to0}\cos(x)^{\frac{1}{x^2}} &= \lim_{x\to0}\left[1-\left(1-\cos(x)\right)\right]^{\frac{1}{x^2}}\\
&\color{red}{=} \lim_{x\to0}\left[\left(1-\frac{x^2}{2}\right)^{\frac{2}{x^2}}\right]^{\frac{1}{2}}\\
&= \left(\frac{1}{e}\right)^{\frac{1}{2}}=\frac{1}{\sqrt{e}}.
\end{aligned}
$$

□

Remark. The red equality is due to:

$$
x\to0,\quad 1-\cos(x)=2\sin^2\frac{x}{2}\sim\frac{x^2}{2}.
$$

We have encountered this question before, but we did not use equivalence to easily solve it.

Remark. In example 5.4.2 we use $\frac{x}{2}+o(x)=\sqrt{1+x}-1$, also in example 5.4.3 we replace an infinitesimal with its equivalent and we always keep a $o(x)$ term. Why is this? We use next example to address this.

Example 5.4.5

Calculate $\lim\limits_{x\to0}\frac{\tan(x)-\sin(x)}{x^2}$.

Solution:

The wrong way to solve this is using the equivalence without the $o(x)$ term. It goes like:
Since $\sin(x)\sim x$ and $\tan(x)\sim x$ then we have

$$
\lim_{x\to0}\frac{\tan(x)-\sin(x)}{x^2}=\lim_{x\to0}\frac{x-x}{x^3}=0.
$$

We have calculated this limit before in example 5.1.6, we know $\lim\limits_{x\to 0}\frac{\tan(x)-\sin(x)}{x^2}=\frac{1}{2}$. The problem lies in omitting the $o\,(x)$ terms and the operation is subtraction. Regiously,

$$\begin{aligned}
\sin(x) &= x + o\,(x)\,,\\
\tan(x) &= x + o\,(x)\,.\\
\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{x^2} &= \lim_{x\to 0}\frac{(x-o\,(x))-(x-o\,(x))}{x^3}\\
&= \frac{o\,(x)}{x^3}\\
&= ???.
\end{aligned}$$

Though this method, keeping the $o\,(x)$ terms in calculation, we do not know the answer, but at least we avoid making mistakes. □

Remark. In the red equality step, the $o\,(x)$ terms will not be cancled, since they are different higher order infinitesimals of x.

Example 5.4.6

Calculate $\lim\limits_{x\to 0}\frac{\left(\sqrt{1+x}-1\right)-\frac{x}{2}}{x^2}$.

Solution:
Since $\sqrt{1+x}-1\sim\frac{x}{2}$, then:

$$\lim_{x\to 0}\frac{\left(\sqrt{1+x}-1\right)-\frac{x}{2}}{x^2}=\lim_{x\to 0}\frac{\left(\frac{x}{2}+o\,(x)-\frac{x}{2}\right)}{x^2}=\lim_{x\to 0}\frac{o\,(x)}{x^2}.$$

To this step, we know it is undetermined by using this method.

$$\begin{aligned}
\lim_{x\to 0}\frac{\left(\sqrt{1+x}-1\right)-\frac{x}{2}}{x^2} &= \lim_{x\to 0}\frac{(1+x)-\left(1+\frac{x}{2}\right)^2}{x^2\left(\sqrt{1+x}+1+\frac{x}{2}\right)}\\
&= \lim_{x\to 0}\frac{-\frac{1}{4}x^2}{x^2\left(\sqrt{1+x}+1+\frac{x}{2}\right)}=-\frac{1}{8}
\end{aligned}$$

□

Note: Later, we will see example 5.4.6 tells us that $\sqrt{1+x}=1+\frac{x}{2}-\frac{1}{8}x^2+o\left(x^2\right),\quad x\to 0.$

5.5 Continuous Function on A Closed Interval

Continuous functions on a closed interval process certain properties that continuous functions on an open interval do not necessarily have.

Theorem 5.5.1 ***Boundedness Theorem***

If a function $f\,(x)$ is continuous in $[a,b]$, then $f\,(x)$ is bounded in $[a,b]$.
Remark. A function $f\,(x)$ is continuous in $[a,b]$ means:

$$\lim_{x\to x_0} f\,(x)=f\left(x_0\right),\quad \forall\, x_0\in[a,b]\,.$$

The *Boundedness Theorem* means that if $f\,(x)$ is continuous in $[a,b]$, then there exists a real number $M>0$, such that $\forall\, x\in[a,b]$, we have $|f\,(x)|\le M$.

Proof:

We assume $f(x)$ is unbouned in $[a,b]$ and we want to find a contradiction. We divide $[a,b]$ into two parts, $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$, then $f(x)$ is at least unbounded in one of them, denoted that interval as $\left[a_1, b_1\right]$. We divide $\left[a_1, b_1\right]$ into two parts, $\left[a_1, \frac{a_1+b_1}{2}\right]$ and $\left[\frac{a_1+b_1}{2}, b_1\right]$, then $f(x)$ is at least unbounded in one of them, denoted that interval as $\left[a_2, b_2\right]$. Keep repeating this process, we can get a sequence of *nested intervals*, $\left\{\left[a_n, b_n\right]\right\}$, then $\exists\, \xi \in \left[a_n, b_n\right], \forall\, n$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{x\to\infty} b_n = \xi$. Since $\xi \in [a,b]$, $f(x)$ is continuous at ξ, by *Local Boundedness Property*, theorem 4.2.5, $\exists\, \delta > 0, B > 0, \forall\, x \in O(\xi,\delta) \cap [a,b]$, we have $|f(x)| \le B$. When n is large enough, $\left[a_n, b_n\right] \subset O(\xi,\delta) \cap [a,b]$. Contradiction. □

Note: In other words, since $f(x)$ is continuous at ξ, which is the limit and is inside a sequence of *nested intervals*, $\left\{\left[a_n, b_n\right]\right\}$, by *Local Boundedness Property*, theorem 4.2.5, we know $f(x)$ is (at least) bounded in the neighbourhood of $\xi \in \left[a_n, b_n\right]$. But we assumed $f(x)$ is unbounded in all $\left[a_n, b_n\right]$. Here, $O(\xi,\delta) = (\xi-\delta, \xi+\delta)$.

Remark. An example that a function is countinous in an open interval is unbounded: $f(x) = \frac{1}{x}$ is continuous in $(0,1)$. Obviously $f(x)$ is unbounded in $(0,1)$.

Theorem 5.5.2 ***Extereme Value Theorem (EVT)***

If $f(x)$ is continuous in $[a,b]$, then $\exists\, \xi, \eta \in [a,b]$ such that $f(\xi) \le f(x) \le f(\eta), \forall\, x \in [a,b]$.
Remark. In other words, if $f(x)$ is continuous in $[a,b]$, then $f(x)$ has minimum and maximum values in $[a,b]$.

Proof:

Since $f(x)$ is continuous in $[a,b]$, we know $f(x)$ is bounded. The *Boundedness Theorem*, theorem 5.5.1, tells us that $f(x)$ is bounded above and below. That is $R_f = \{f(x) \mid x \in [a,b]\}$ is a bounded set. By the *Completeness Axiom / Least Upper Bound Axiom / Supremum Property*, theorem 3.2.1, we can let $\alpha = \inf R_f, \beta = \sup R_f$. Now we just need to show that $\alpha, \beta, \in R_f$. We first prove $\exists\, \xi \in [a,b]$ such that $f(\xi) = \alpha$. Since $\alpha = \inf R_f, \forall\, x \in [a,b], f(x) \ge \alpha$. $\forall\, \epsilon > 0, \exists\, x \in [a,b]$ such that $f(x) < \alpha + \epsilon$. Let $\epsilon_n = \frac{1}{n}, \exists\, x_n \in [a,b]$ such taht $\alpha \le f\left(x_n\right) < \alpha + \frac{1}{n}$. Here, x_n is a bounded sequence which has a convergent subsequence (by Bolzano-Weierstrass Theorem 3.7.6). Let $x_{n_k} \to \xi \in [a,b]$, such that $\alpha \le f\left(x_{n_k}\right) < \alpha + \frac{1}{n_k}$. As long as $k \to \infty$, we have $f(\xi) = \alpha$. Similarly, we can prove $\exists\, \eta \in [a,b]$ such that $f(x) = \beta$. □

Remark. An example: $f(x) = \frac{1}{x}, x \in (0,1), \alpha = \inf R_f = 0, \beta = \sup R_f = 1$, but there is NO $\xi, \eta \in (0,1)$ such that $f(\xi) = 0, f(\eta) = 1$.

Theorem 5.5.3 ***Bolzano's Theorem***

If $f(x)$ is continuous in $[a,b]$ and $f(a)f(b) < 0$, then $\exists\, \xi \in (a,b)$ such that $f(\xi) = 0$.

Proof:

W.O.L.G., let $f(a) < 0, f(b) > 0$. Let $V = \{x \mid f(x) < 0, x \in [a,b]\}$, so $a \in V, b \notin V; \xi = \sup V$. First we need to show $\xi \in (a,b)$. Since $f(a) < 0$, by continuity, $\exists\, \delta_1 > 0$ such that $f(x) < 0, \forall\, x \in \left[a, a+\delta_1\right)$, also since $f(b) > 0$, by continuity, $\exists\, \delta_2 > 0$ such that $\forall\, x \in \left(b-\delta_2, b\right]$. This is saying that numbers close to a give f negative value, while numbers close to b gives f positive value. So, $\xi \in (a,b)$. Second we need to prove $f(\xi) = 0$. Let $x_n \in V, f\left(x_n\right) < 0$. Alos, let $x_n \to \xi$, taht is $f(\xi) = \lim\limits_{n\to\infty} f\left(x_n\right) \le 0$. [Of course $f(\xi)$ will not be greater than zero.] If $f(\xi) < 0, \exists\, \delta > 0$ such that $f(x) < 0$ for $x \in (\xi-\delta, \xi+\delta)$. A contradiction with the definiton of ξ being the supremum of the set V. Therefore, $f(\xi) = 0$. □

Note: We can let $\xi = \sup V$, is due to the *Completness Axiom*, since the set V is bounded above.

Remark. The *Bolzano's theorem* is a special case (or corollary) of the *Intermediat Value Theorem (IVT)*, which we will cover next.

Example 5.5.1

Give a polynomial $p(x) = 2x^3 - 3x^2 - 3x + 2$, what can we say about its root(s)?

Solution:

It is obvious that $p(-2) < 0, p(0) > 0, p(1) < 0, p(3) > 0$. It is not difficult to see that $p(x)$ has three real roots ξ_1, ξ_2, ξ_3 and by the *Intermediate Value Theorem (IVT)*, theorem 5.5.3, we know that $\xi_1 \in (-2, 0), \xi_2 \in (0, 1), \xi_3 \in (1, 3)$. □

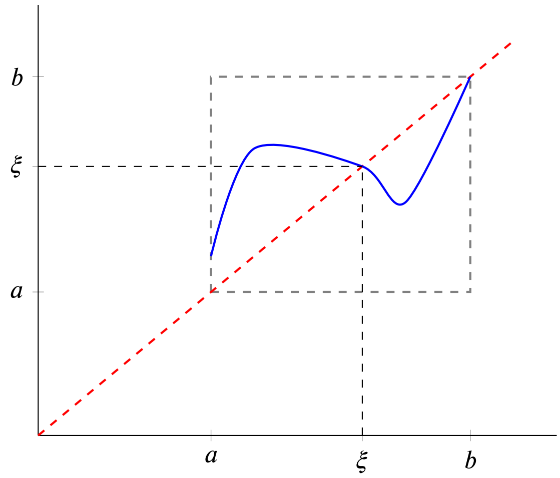
Note: Actually, $p(x) = 2(x+1)\left(x - \frac{1}{2}\right)(x-2)$.

Example 5.5.2

Suppose $f(x)$ is continuous in $[a, b]$ and $f([a, b]) \subset [a, b]$, then prove that $\exists\, \xi \in [a, b]$ such that $f(\xi) = \xi$. Here, ξ is the **Fixed Point** of the continuous function $f(x)$.

Proof:

Let $g(x) = f(x) - x$. To find the **fixed point** of f is to find the zero of g. Since $f([a, b]) \subset [a, b], a \leq f(x) \leq b, \forall\, x \in [a, b]$, then $g(a) \geq 0, g(b) \leq 0$. First case, if $g(a) = 0$, then $\xi = a$. Second case, if $g(b) = 0$, then $\xi = b$. Third case, if $g(a) > 0, g(b) < 0$, then $\exists\, \xi \in (a, b)$ such that $g(\xi) = 0$, that is $f(\xi) = \xi$. Let's see the graph of this question.



□

Remark. If $f(x)$ is continuous in (a, b) and $f((a, b)) \subset (a, b)$, determine whether f has a **finxed point** in (a, b)? The answer is No. The counter example is $f(x) = \frac{x}{2}$, which is continuous on $(0, 1)$ and $f((0, 1)) = \left(0, \frac{1}{2}\right) \subset (0, 1)$, but $f(x) = \frac{x}{2}$ has no **fixed point** in $(0, 1)$.

Theorem 5.5.4 ***Intermediate Value Theorem (IVT)***

If $f(x)$ is continuous on $[a, b]$, then it takes on every value between its minimum m and its maximum M.

Proof:

By the *Extereme Value Theorem (EVT)*, theorem 5.5.2, we know $\exists\, \xi, \eta \in [a, b], f(\xi) = m, f(\eta) = M$. W.O.L.G., let $\xi < \eta, \forall\, c \in (m, M)$. Let $g(x) = f(x) - c$, then $g(x)$ is continuous on $[\xi, \eta]$. Since $g(\xi) = f(\xi) - c = m - c < 0$ and $g(\eta) = f(\eta) - c = M - c > 0$, then $\exists\, \zeta \in (\xi, \eta) \subset [a, b]$ such that $g(\zeta) = 0$ that is $f(\zeta) = c$. □

Remark. The *IVT* can make the proof of the *Continuity of The Inverse Function Theorem*, theorem 4.11.2, much easier.

5.6 Uniform Continuity

First let's review the concept of continuity. Suppose X is an interval and $f(x)$ is continuous on X that is $f(x)$ is continuous on every point in X (right-continuous at the left ennd and left-continuous at the right end). Mathematicall,

$$\forall x_0 \in X, \forall \epsilon > 0, \forall x \in |x - x_0| < \delta : \quad \left|f(x) - f\left(x_0\right)\right| < \epsilon.$$

Remark. Here, δ is a function of x_0 and δ that is $\delta = \delta\left(x_0, \epsilon\right)$. The first question is can we find a δ (positive of course) that works for all x_0? In other words, can we find a δ that only depends on ϵ that is $\delta = \delta(\epsilon)$? (The answer is yes, we can.) If we can find that δ (only depends on ϵ), then we can write:

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0, \forall x', x'' \in |x' - x''| < \delta : \quad \left|f\left(x'\right) - f\left(x''\right)\right| < \epsilon.$$

The difference between the new mathematical form and the previous one is x', x'' and x, x_0. Though they are all any arbitrary points on X, the key difference is whether the choise of δ depends on the arbitrary pick of x_0.

Remark. The second question is that can we always find that δ (only depends on ϵ)? The answer is NO. Actually, whether we can find this $\delta(\epsilon)$ depends on the interval X and also the function f.

Previously, when we dealing with $\delta\left(x_0, \epsilon\right)$, we say there is no need to find the largest $\delta\left(x_0, \epsilon\right)$, which is true. But for $\delta(\epsilon)$, indeed we need to find the largest $\delta\left(x_0, \epsilon\right)$. That is

$$\exists \delta(\epsilon) \iff \inf_{x_0 \in X} \delta^*\left(x_0, \epsilon\right) > 0.$$

Here, we define the largest $\delta\left(x_0, \epsilon\right)$ or its supremum to be $\delta^*\left(x_0, \epsilon\right)$. Also, we are taking the infimum over x_0, so when the condition is satisfied δ will only dependes on ϵ.

Definition 5.6.1 ***Uniform Continuity***

Let $f(x)$ be a function defined on an interval $X \subseteq \mathbb{R}$. We say $f(x)$ is **uniformly continuous** on X. if $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x', x'' \in X$, the condition $|x' - x''| < \delta \implies \left|f\left(x'\right) - f\left(x''\right)\right| < \epsilon$.

Remark. If we replace x' or x'' with x_0, we get the definition of continuity of $f(x)$ on x_0. Therefore, we know:

$$f(x) \text{ is } \textbf{uniformly continuous} \text{ on } X \implies f(x) \text{ is } \textbf{continuous} \text{ on } X.$$

Example 5.6.1

Prove that $y = \sin(x)$ is uniformly continuous on $(-\infty, +\infty)$.

Proof:

We have

$$\left|\sin\left(x'\right) - \sin\left(x''\right)\right| = 2\left|\cos\frac{x' + x''}{2}\sin\frac{x' - x''}{2}\right| \leq |x' - x''|.$$

Then $\forall \epsilon > 0$, let $\delta(\epsilon) = \epsilon$, $\forall x', x''$, the conditoin $|x' - x''| < \delta \implies |\sin x' - \sin x''| \leq |x' - x''| < \epsilon$. Therefore, $y = \sin x$ is uniformly continuous in $\mathbb{R}$. ☐

Note: The red inequality is due to $|\cos x| \leq 1$, $|\sin x| \leq |x|$ (using graph to see this).

Example 5.6.2

Is $f(x) = \frac{1}{x}$ uniformly continuous on $X = (0, 1)$?

Solution:

Let $x_0 \in (0, 1)$ then we want to find $\delta^*(x_0, \epsilon)$.

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < \epsilon \iff -\epsilon + \frac{1}{x_0} < \frac{1}{x} < \epsilon + \frac{1}{x_0} \iff \frac{x_0}{1 + x_0\epsilon} < x < \frac{x_0}{1 - x_0\epsilon}$$

$$\iff -\frac{x_0^2\epsilon}{1 + x_0\epsilon} < x - x_0 < \frac{x_0^2\epsilon}{1 - x_0\epsilon}.$$

Then $\delta^*(x_0, \epsilon) = \min\left\{\frac{x_0^2\epsilon}{1+x_0\epsilon}, \frac{x_0^2\epsilon}{1-x_0\epsilon}\right\} = \frac{x_0^2\epsilon}{1+x_0\epsilon}$. Obviously, the $\inf\limits_{x_0\in(0,1)} \frac{x_0^2\epsilon}{1+x_0\epsilon} = 0$, so we can not find a $\delta(\epsilon) > 0$ that is irrevalent to x_0. Therefore $f(x) = \frac{1}{x}$ is not unifromly continuous on $(0, 1)$. □

Remark. The readers may also feel that to tell if a function is uniformly continuous is not easy, even when the function $f(x) = \frac{1}{x}$ is not complicated at all. Imagine we had a complicated funciton ... Next thereom comes handy in telling if a function is uniformly continuous.

Theorem 5.6.1 ***Sequential Criterion for Uniform Continuity***

Let f be a function defined on a set $X \subseteq \mathbb{R}$. Then f is uniformly continuous on X if and only if for every pair of sequences $\{x_n'\}$ and $\{x_n''\}$ in X,

$$\lim_{n\to\infty}(x_n' - x_n'') = 0 \implies \lim_{n\to\infty}(f(x_n') - f(x_n'')) = 0.$$

Proof:

Necessity (⇒)

Since $f(x)$ is uniformly continuous on X, then we have

$$\forall \underset{\sim\sim\sim}{\epsilon > 0}, \exists \delta > 0, \forall x', x'' \in X, \quad |x' - x''| < \delta \implies \left|f(x') - f(x'')\right| < \epsilon.$$

Given $\lim\limits_{x\to\infty}(x_n' - x_n'') = 0$, then

$$\text{to the same } \delta > 0, \underset{\sim\sim\sim\sim\sim\sim}{\exists N, \forall n > N} : \underset{\sim\sim\sim\sim\sim\sim}{|x_n' - x_n''| < \delta} \implies \underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\left|f(x_n') - f(x_n'')\right| < \epsilon}.$$

Therefore, $\lim\limits_{n\to\infty}\left(f(x_n') - f(x_n'')\right) = 0$.

Sufficiency (⇐)

We prove the statement by prove its *countrapositive* statement, or *proof by contraposition*.

For the sufficiency direction, the original statement is if

$$\lim_{n\to\infty}(x_n' - x_n'') = 0 \implies \lim_{n\to\infty}(f(x_n') - f(x_n'')) = 0.$$

then $f(x)$ is uniformly continuous on X. Its countrapositive statement is:

If $f(x)$ is NOT uniformly continuous on X, then

$$\exists \underset{\sim\sim\sim}{\epsilon_0 > 0}, \forall \delta > 0, \exists x', x'' \in X, \quad |x' - x''| < \delta \implies \left|f(x') - f(x'')\right| \geq \epsilon_0.$$

Let $\delta = \delta_n = \frac{1}{n}, \exists x_n', x_n'' \in X$, the condition

$$|x_n' - x_n''| \leq \frac{1}{n} \to 0 \implies \left|f(x_n') - f(x_n'')\right| \geq \epsilon_0.$$

Therefore, indeed the condition $\lim\limits_{n\to\infty}(x' - x'') = 0$ holds, but $\lim\limits_{n\to\infty}\left(f(x_n') - f(x_n'')\right) \neq 0$. The contrapostive statement is true so the original sufficieny direction holds.

For example $f(x) = \frac{1}{x}$, $X = (0, 1)$, let $x_n' = \frac{1}{n}$, $x_n'' = \frac{1}{2n}$ then $x_n' - x_n'' = \frac{1}{2n} \to 0$, but $f(x_n') - f(x_n'') = -n \not\to 0$. □

Remark. The proof of theorem 5.6.1, in both directions, is just the rewrite of the definition of limit, so it is common to feel that we proved nothing, when we finishe our proof.

Note:

1. One thing to point out is that for funciton $f(x) = \frac{1}{x}$, the point 0 is the problem, preventing it to be uniformly continuous.
2. *Proof by contraposition* is different from *proof by contradiction*.

Example 5.6.3

Let $f(x) = \frac{1}{x}, X = [\eta, 1), \eta \in (0, 1)$. Prove $f(x)$ is uniform continuous on X.

Proof:

$$\forall \epsilon > 0, \left|\frac{1}{x'} - \frac{1}{x''}\right| = \frac{|x' - x''|}{|x'x''|} \leq \frac{|x' - x''|}{\eta^2}$$

Let $\delta = \eta^2\epsilon > 0, \forall x', x'' \in [\eta, 1), |x' - x''| < \delta$ then

$$\left|\frac{1}{x'} - \frac{1}{x''}\right| = \frac{|x' - x''|}{|x'x''|} \leq \frac{|x' - x''|}{\eta^2} < \frac{\eta^2\epsilon}{\eta^2} < \epsilon.$$

Here, $\delta = \eta^2\epsilon$ is NOT related to x_0. □

Remark. The use of η here is to make sure that the domain X is not too close to 0, which causes major problem for uniform continuity. By not too close to 0, we mean that for some fixed (or choosen) η, it can be very close to 0, since $\eta \in (0, 1)$, but it will not be asymptotically close to 0, when it is choosen. In other words, Because η is a fixed (though potentially very small) positive number, the function $f(x)$ now has a maximum possible slope on that specific interval. Because the domain is $[\eta, 1)$: Once η is chosen, the "danger zone" is capped. The function is "prevented" from getting asymptotically close to its singularity during the proof of that specific set.

Example 5.6.4

Prove $f(x) = x^2$ is NOT uinformly continuous on $[0, +\infty)$.

Proof:

Let $x'_n = \sqrt{n+1}, x''_n = \sqrt{n}$, then $x'_n - x''_n = \sqrt{n+1} - \sqrt{n} \to 0$, but $f(x'_n) - f(x''_n) = 1 \not\to 0$. □

Example 5.6.5

Prove $f(x) = x^2$ is uniformly continuous on $[0, A], A \in \mathbb{R}^+$.

Proof:

$$\forall \epsilon > 0, \left|(x')^2 - (x'')^2\right| = |x' - x''|\,|x' + x''| \leq 2A\,|x' - x''|.$$

Let $\delta = \frac{\epsilon}{2A} > 0, \forall x', x'' \in [0, A]$, the condition $|x' - x''| < \delta \implies \left|(x')^2 - (x'')^2\right| < \epsilon$. □

Theorem 5.6.2 ***Cantor Theorem***

If a function $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ is uniformly continuous on $[a, b]$.

Proof:

We proof theorem 5.6.2 by contradiction. That is if a function $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ is NOT uniformly continuous on $[a, b]$. Mathematically, it is

$$\exists\, x'_n, x''_n \in [a, b], |x'_n - x''_n| < \frac{1}{n}, \text{ but } \left|f(x'_n) - f(x''_n)\right| \geq \epsilon_0.$$

Since $\{x'_n\}$ is a bounded sequence, it must has a convergent subsequence, let $\lim_{k\to\infty} x'_{n_k} = \xi \in [a, b]$. From $\left|x'_{n_k} - x''_{n_k}\right| < \frac{1}{x_{n_k}}$, we know $\lim_{k\to\infty} x''_{n_k} = \xi$. Since $f(x)$ is continuous on ξ, then $\lim_{k\to\infty} f\left(x'_{n_k}\right) = \lim_{k\to\infty} f\left(x''_{n_k}\right) = f(\xi)$. Thus, $f\left(x'_{n_k}\right) - f\left(x''_{n_k}\right) \xrightarrow{k\to\infty} 0$. Contradiction. □

Theorem 5.6.3

Let $f(x)$ be a fucntion that is continuous on an finite open interval (a, b), then $f(x)$ is uniformly continuous on the open interval (a, b) if and only if:

$$\lim_{x\to a^+} f(x) = f\left(a^+\right), \text{ and } \lim_{x\to b^-} f(x) = f\left(b^-\right) \text{ both exist and are finite.}$$

Proof:

Sufficiency ($\Leftarrow$)
Let $f\left(a^+\right) = A$, $f\left(b^-\right) = B$. Define a new function:

$$\tilde{f}(x) = \begin{cases} A, & x = a \\ f(x), & x \in (a, b) \\ B. & x = b \end{cases}$$

Obviously $\tilde{f}(x)$ is continuous on $[a, b]$, by *Cantor theorem*, theorem 5.6.2, we know $\tilde{f}(x)$ is unformly continuous on $[a, b]$. Thus, $\tilde{f}(x)$ is uniformly continuous on (a, b). Therefore, $f(x)$ is uniformly continuous on (a, b).
Necessity ($\Rightarrow$)
Since $f(x)$ is uniformly continuous on (a, b), that is $\forall\, \epsilon > 0, \exists\, \delta > 0, \forall\, x', x'' \in (a, b)$, the condition $\left|x' - x''\right| < \delta$ implies $\left|f\left(x'\right) - f\left(x''\right)\right| < \epsilon$. Now, in (a, b) choose an arbitrary *Cauchy sequence*, $\left\{x_n\right\}, x_n \xrightarrow{n\to\infty} a^+$, for the same $\delta > 0, \exists\, N, \forall\, n, m > N$ the condition $\left|x_n - x_m\right| < \delta$ implies $\left|f\left(x_n\right) - f\left(x_m\right)\right| < \epsilon$. So, $\left\{f\left(x_n\right)\right\}$ is a *Cauchy sequence* that converges. By *Heine theorem*, theorem 4.4.1, $f\left(a^+\right)$ exists. Similarly, we can show $f\left(b^-\right)$ exists. $\square$

Example 5.6.6
Let $f(x) = \sin\frac{1}{x}$ is continuous on $(0, 1)$, but NOT uniformly continuous. Because when $x \to 0^+$, $f(x) = \sin\frac{1}{x}$ has no limit.

Chapter 6

Differentials

6.1 Differentials and Derivatives