Notes: Introduction to Copula Theory

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CHAPTER 1

Preliminaries

THEOREM **1.0.1**. Prove that $\mathbb{E}[X] = a$ when a is a point of symmetry of X.

PROOF. We have two method,

(1) Suppose that $\mathbb{E}[X]$ exists, then

$$\begin{aligned}
\mathbb{E}[X] &= \int_{\mathbb{R}} xf(x)dx \\
&= \int_{\mathbb{R}} (a - a + x)f(x)dx \\
&= a + \int_{-\infty}^{\infty} (x - a)f(x)dx \\
&= a + \int_{-\infty}^{\infty} yf(a + y)dy \\
&= a + \int_{-\infty}^{0} yf(a + y)dy + \int_{0}^{\infty} yf(a + y)dy \\
&= a + \int_{\infty}^{0} (-1)(-z)f(a - z)dz + \int_{0}^{\infty} yf(a + y)dy \\
&= a + \int_{0}^{\infty} (-z)f(a + z)dz + \int_{0}^{\infty} yf(a + y)dy = a.
\end{aligned}$$

(2) Assume that the expectation exists, then by definition of symmetry, $X - a$ and $a - X$ have the same distribution, then

$$\mathbb{E}[X - a] = \mathbb{E}[a - X] \implies \mathbb{E}[X] - a = a - \mathbb{E}[X] \implies \mathbb{E}[X] = a.$$

$\square$

EXAMPLE **1.0.2**. Let $X_{(1)}, X_{(2)}$ denote the minimum and maximum of $\{X_1, X_2\}$. Given that $F_{X_{(1)}}(x) = 1 - (1 - F(x))^2$ and $F_{X_{(2)}}(x) = F(x)^2$. For $x_1 \geq x_2$, we have

$$\begin{aligned} P(X_{(1)} \leq x_1, X_{(2)} \leq x_2) &= P(X_{(2)} \leq x_2) - P(X_{(1)} > x_1, X_{(2)} \leq x_2) \\ &= F(x_2)^2. \end{aligned}$$

For $x_1 < x_2$, we have

$$\begin{aligned} P(X_{(1)} \leq x_1, X_{(2)} \leq x_2) &= P(X_{(2)} \leq x_2) - P(X_{(1)} > x_1, X_{(2)} \leq x_2) \\ &= F(x_2)^2 - (F(x_2) - F(x_1))^2 \\ &= 2F(x_2)F(x_1) - F(x_1)^2. \end{aligned}$$

Thus

$$P(X_{(1)} \leq x_1, X_{(2)} \leq x_2) = 2F(\min\{x_1, x_2\})F(x_2) - F(\min\{x_1, x_2\})^2.$$

□

PROPOSITION **1.0.3**. Assume F_X is continuous and increasing, define $Y = F_X(x)$ and note that Y takes values in $[0, 1]$. Then Y follows Uniform $[0, 1]$.

PROOF. Write

$$F_Y(x') = P(F_X(x) \leq x') = P(x \leq F_X^{-1}(x')) = F_X(F_X^{-1}(x')) = x'.$$

On the other hand, if U is a uniform random variable that takes values in $[0, 1]$,

$$F_U(x') = \int_{\mathbb{R}} f_U(u)du = \int_0^{x'} 1du = x',$$

thus, $F_Y(x') = F_U(x')$ for every $x \in [0, 1]$. □

EXERCISE **1.0.4**. The first-order derivative of joint CDF is

$$\begin{aligned} \frac{\partial}{\partial x} F_{XY}(x, y) &= \frac{\partial}{\partial x} \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(t_1, t_2)dt_2dt_1 \\ &= \int_{-\infty}^{y} f_{XY}(x, t_2)dt_2 \end{aligned}$$

$$= \int_{-\infty}^{y} f_{Y|X}(t_2|x) f_X(x) dt_2$$

$$= \int_{-\infty}^{y} f_{Y|X} f(t_2|x) dt_2 \cdot f_X(x)$$

$$= P(Y \leq y | X = x) \cdot f_X(x).$$

$\square$

CHAPTER 2

Exercise

Exercise 2.1 (1) Let H be the function defined on I^2 by $H(x, y) = \max(x, y)$. Clearly, let $x_1 \leq x_2, y_1 \leq y_2$,

$$H(x_2, y) - H(x_1, y) = \max(x_2, y) - \max(x_1, y) \geq 0,$$

and

$$H(x, y_2) - H(x, y_1) = \max(x, y_2) - \max(x, y_1) \geq 0,$$

Thus H is non-decreasing in each argument. However,

$$\begin{aligned} V_H(\mathrm{I}^2) &= H(1,1) + H(0,0) - H(0,1) - H(1,0) \\ &= \max(1,1) + \max(0,0) - \max(1,0) - \max(0,1) \\ &= -1 < 0. \end{aligned}$$

Therefore, H is not 2-increasing.

(2) Let H be the function defined on I^2 by $H(x, y) = (2x-1)(2y-1)$. WLOG, let $B \in \mathrm{I}^2$ with vertices $(x_1, y_1), (x_2, y_2) \in \mathrm{I}^2$ such the $x_1 \leq x_2, y_1 \leq y_2$,

$$\begin{aligned} V_H(B) &= H(x_1, y_1) + H(x_2, y_2) - H(x_1, y_2) - H(x_2, y_1) \\ &= (2x_1 - 1)(2y_1 - 1) + (2x_2 - 1)(2y_2 - 1) - (2x_1 - 1)(2y_2 - 1) - (2x_2 - 1)(2y_1 - 1) \\ &= (2x_1 - 1)(2y_1 - 2y_2) + (2x_2 - 1)(2y_2 - 2y_1) \\ &= (2y_2 - 2y_1)(2x_2 - 2x_1) \geq 0. \end{aligned}$$

Thus H is 2-increasing, but

$$\frac{\partial H(x,y)}{\partial x} = 4y - 2.$$

It indicates that $\frac{\partial H(x,y)}{\partial x} < 0$ when $y \in [0, 1/2)$. Thus H is decreasing in x when $y \in [0, 1/2)$. Similarly, H is decreasing in y when $x \in [0, 1/2)$. □

Exercise 2.2 Show that $M(u, v) = \min(u, v), W(u, v) = \max(u + v - 1, 0)$ and $\Pi(u, v) = uv$ are indeed copulas.

Solution. Let's check the two axioms of a copula. ⊛

$$M(0, v) = \min(0, v) = 0 = \min(u, 0) = M(u, 0),$$

$$M(1, v) = \min(1, v) = v, \quad M(u, 1) = \min(u, 1) = u.$$

and

$$W(0, v) = \max(v - 1, 0) = 0 = \max(u - 1, 0) = W(u, 0),$$

$$W(1, v) = \max(v, 0) = v, \quad W(u, 1) = \max(u, 0) = u.$$

and

$$\Pi(0, v) = 0 \cdot v = 0 = u \cdot 0 = \Pi(u, 0),$$

$$\Pi(1, v) = 1 \cdot v = v, \quad \Pi(u, 1) = u \cdot 1 = u.$$

⊛ Then check the 2-increase of them. For every $(u_1, v_1), (u_2, v_2) \in \mathrm{I}^2$ such that $u_1 \leq u_2, v_1 \leq v_2$,

$$\begin{aligned} &M(u_1, v_1) + M(u_2, v_2) - M(u_1, v_2) - M(u_2, v_1) \\ &\quad = \min(u_1, v_1) + \min(u_2, v_2) - \min(u_1, v_2) - \min(u_2, v_1) = \beta. \end{aligned}$$

If $u_1 \leq v_1$,

$$\beta = u_1 + \min(u_2, v_2) - \min(u_2, v_1) - u_1 \geq 0.$$

If $u_1 > v_1$,

$$\beta = v_1 + \min(u_2, v_2) - \min(u_1, v_2) - v_1 \geq 0.$$

And

$$\alpha = W(u_1, v_1) + W(u_2, v_2) - W(u_1, v_2) - W(u_2, v_1)$$

$$= \max(u_1 + v_1 - 1, 0) + \max(u_2 + v_2 - 1, 0) - \max(u_1 + v_2 - 1, 0) - \max(u_2 + v_1 - 1, 0).$$

If $u_1 + v_2 \geq 1$,

$$\alpha = \max(u_1 + v_1 - 1, 0) + \max(u_2 + v_2 - 1, 0) - \max(u_2 + v_1 - 1, 0) - 0 \geq 0.$$

If $u_1 + v_2 < 1$,

$$\alpha = 0 + \max(u_2 + v_2 - 1, 0) - \max(u_2 + v_1 - 1, 0) + 0 \geq 0.$$

And

$$\begin{aligned}\gamma &= \Pi(u_1, v_1) + \Pi(u_2, v_2) - \Pi(u_1, v_2) - \Pi(u_2, v_1)\\ &= u_1 v_1 + u_2 v_2 - u_1 v_2 - u_2 v_1\\ &= (u_2 - u_1)(v_2 - v_1) \geq 0.\end{aligned}$$

□

Exercise 2.3 (a) Let C_0 and C_1 be copulas, and let θ be any number in I. Show that the weighted arithmetic mean $(1-\theta)C_0 + \theta C_1$ is also a copula. Hence conclude that *any convex linear combination of copulas is a copula.*

(b) Show that *the geometric mean of two copulas may fail to be a copula.*

Solution. (a) Check the two axioms of copula. ⊛ The groundedness and uniform margins are trivial. ⊛ The 2-increase is also trivial as $(1-\theta)V_{C_0}(B) + \theta V_{C_1}(B) \geq 0$, where B is a rectangle in I^2.

(b) Let C be the geometric mean of Π and W, then the C-volume of rectangle $[1/2, 3/4] \times [1/2, 3/4]$ is

$$\begin{aligned}&\sqrt{\Pi(1/2,1/2)W(1/2,1/2)} + \sqrt{\Pi(3/4,3/4)W(3/4,3/4)} - \sqrt{\Pi(1/2,3/4)W(1/2,3/4)}\\ &- \sqrt{\Pi(3/4,1/2)W(3/4,1/2)} = \sqrt{1/4 \cdot 0} + \sqrt{9/16 \cdot 1/2} - \sqrt{3/8 \cdot 1/4} - \sqrt{3/8 \cdot 1/4}\\ &= 1/2(\sqrt{9/8} - \sqrt{12/8}) < 0.\end{aligned}$$

Therefore C is not a copula. □

Exercise 2.4 The Fréchet and Mardia families of copulas. (a) Let α, β be in I with $\alpha + \beta \leq 1$. Set

$$C_{\alpha,\beta}(u,v) = \alpha M(u,v) + (1-\alpha-\beta)\Pi(u,v) + \beta W(u,v).$$

Show that $C_{\alpha,\beta}$ is a copula (**Fréchet**). A family of copulas that includes M, Π, W is called **comprehensive**.

(b) Let θ be in $[-1,1]$ and set

$$C_\theta(u,v) = \frac{\theta^2(1+\theta)}{2} M(u,v) + (1-\theta^2)\Pi(u,v) + \frac{\theta^2(1-\theta)}{2} W(u,v).$$

Show that C_θ is a copula (**Mardia**).

Solution. (a) Since convex combination of copulas is a copula. And $\alpha, \beta, 1-\alpha-\beta \in$ I with sum to 1. Thus $C_{\alpha,\beta}$ is a convex combination of the three copulas, it is indeed a copula.

(b) Similarly to part (a), C_θ is a convex combination of copulas. □

Exercise 2.5 The **Cuadras-Augé** family of copulas. Let $\theta \in$ I, and set

$$C_\theta(u,v) = [\min(u,v)]^\theta [uv]^{1-\theta} = \begin{cases} uv^{1-\theta}, & u \leq v, \\ u^{1-\theta}v, & u \geq v. \end{cases}$$

Show that C_θ is a copula. Note that, $C_0 = \Pi$ and $C_1 = M$. This family is weighted geometric mean of M and Π.

Solution. Check the two axioms of copula. ⊛

$$C_\theta(u,0) = 0 = C_\theta(0,v), \quad C_\theta(u,1) = u, \quad C_\theta(1,v) = v.$$

⊛ And for every $(u_1,v_1),(u_2,v_2) \in \mathrm{I}^2$ with $u_1 \leq u_2, v_1 \leq v_2$,

$$\alpha = [\min(u_1,v_1)]^\theta [u_1v_1]^{1-\theta} + [\min(u_2,v_2)]^\theta [u_2v_2]^{1-\theta} - [\min(u_1,v_2)]^\theta [u_1v_2]^{1-\theta} - [\min(u_2,v_1)]^\theta [u_2v_1]^{1-\theta}.$$

If $u_2 \leq v_1$,

$$\alpha = u_1v_1^{1-\theta} + u_2v_2^{1-\theta} - u_1v_2^{1-\theta} - u_2v_1^{1-\theta} = (u_2-u_1)(v_2^{1-\theta} - v_1^{1-\theta}) \geq 0.$$

If $v_2 > u_2 > v_1 > u_1$,

$$\begin{aligned}\alpha &= u_1 v_1^{1-\theta} + u_2 v_2^{1-\theta} - u_1 v_2^{1-\theta} - u_2^{1-\theta} v_1 \\ &= [u_2 v_2^{1-\theta} - u_1 v_2^{1-\theta}] + v_1(u_1 v_1^{-\theta} - u_2 u_2^{-\theta}) \\ &\geq [u_2 v_2^{1-\theta} - u_1 v_2^{1-\theta}] + v_1(u_1 u_2^{-\theta} - u_2 u_2^{-\theta}) \\ &= (u_2 - u_1)(v_2 v_2^{-\theta} - v_1 u_2^{-\theta}) \\ &\geq (u_2 - u_1)(v_2 v_2^{-\theta} - v_1 v_1^{-\theta}) \geq 0.\end{aligned}$$

If $v_2 > u_2 > u_1 > v_1$,

$$\begin{aligned}\alpha &= u_1^{1-\theta} v_1 + u_2 v_2^{1-\theta} - u_1 v_2^{1-\theta} - u_2^{1-\theta} v_1 \\ &= [u_2 v_2^{1-\theta} - u_1 v_2^{1-\theta}] + v_1(u_1 u_1^{-\theta} - u_2 u_2^{-\theta}) \\ &\geq [u_2 v_2^{1-\theta} - u_1 v_2^{1-\theta}] + v_1(u_1 u_2^{-\theta} - u_2 u_2^{-\theta}) \\ &= (u_2 - u_1)(v_2 v_2^{-\theta} - v_1 u_2^{-\theta}) \\ &\geq (u_2 - u_1)(v_2 v_2^{-\theta} - v_1 v_1^{-\theta}) \geq 0.\end{aligned}$$

If $u_2 > v_2 > v_1 > u_1$,

$$\begin{aligned}\alpha &= u_1 v_1^{1-\theta} + u_2^{1-\theta} v_2 - u_1 v_2^{1-\theta} - u_2^{1-\theta} v_1 \\ &= (v_2 - v_1) u_2 u_2^{-\theta} + u_1(v_1 v_1^{-\theta} - v_2 v_2^{-\theta}) \\ &\geq (v_2 - v_1) u_2 u_2^{-\theta} + u_1(v_1 v_2^{-\theta} - v_2 v_2^{-\theta}) \\ &= (v_2 - v_1)(u_2 u_2^{-\theta} - u_1 v_2^{-\theta}) \\ &\geq (v_2 - v_1)(u_2 u_2^{-\theta} - u_1 u_1^{-\theta}) \geq 0.\end{aligned}$$

If $u_2 > v_2 > u_1 > v_1$,

$$\begin{aligned}\alpha &= u_1^{1-\theta} v_1 + u_2^{1-\theta} v_2 - u_1 v_2^{1-\theta} - u_2^{1-\theta} v_1 \\ &= (v_2 - v_1) u_2 u_2^{-\theta} + u_1(v_1 u_1^{-\theta} - v_2 v_2^{-\theta}) \\ &\geq (v_2 - v_1) u_2 u_2^{-\theta} + u_1(v_1 v_2^{-\theta} - v_2 v_2^{-\theta})\end{aligned}$$

$$
\begin{aligned}
&= (v_2 - v_1)(u_2 u_2^{-\theta} - u_1 v_2^{-\theta}) \\
&\geq (v_2 - v_1)(u_2 u_2^{-\theta} - u_1 u_1^{-\theta}) \geq 0.
\end{aligned}
$$

□

Exercise 2.6 Let C be a copula, and let (a, b) be any point in I^2. For (u, v) in I^2, define

$$
K_{a,b}(u, v) = V_C\left([a(1-u), u + a(1-u)] \times [b(1-v), v + b(1-v)]\right).
$$

Show that $K_{a,b}$ is a copula. Note that,

$$
\begin{aligned}
K_{0,0}(u, v) &= C(u, v), \\
K_{0,1}(u, v) &= u - C(u, 1 - v), \\
K_{1,0}(u, v) &= v - C(1 - u, v), \\
K_{1,1}(u, v) &= u + v - 1 + C(1 - u, 1 - v).
\end{aligned}
$$

Solution. ⊛ Write

$$
\begin{aligned}
K_{a,b}(u, v) = {} & C(a(1-u), b(1-v)) + C(u + a(1-u), v + b(1-v)) \\
& - C(a(1-u), v + b(1-v)) - C(u + a(1-u), b(1-v)).
\end{aligned}
$$

Then

$$
\begin{aligned}
K_{a,b}(0, v) &= C(a, b(1-v)) + C(a, v + b(1-v)) - C(a, v + b(1-v)) - C(a, b(1-v)) \\
&= 0 \\
&= C(a(1-u), b) + C(u + a(1-u), b) - C(a(1-u), b) - C(u + a(1-u), b) \\
&= K_{a,b}(u, 0).
\end{aligned}
$$

And

$$
K_{a,b}(u, 1) = C(a(1-u), 0) + C(u + a(1-u), 1) - C(a(1-u), 1) - C(u + a(1-u), 0)
$$

$$= u + a(1-u) - a(1-u) = u$$

$$K_{a,b}(1,v) = C(0, b(1-v)) + C(1, v + b(1-v)) - C(0, v + b(1-v)) - C(1, b(1-v))$$
$$= v + b(1-v) - b(1-v) = v.$$

⊛ Then for every $(u_1, v_1), (u_2, v_2) \in \mathrm{I}^2$ such that $u_1 \le u_2, v_1 \le v_2$,

$$\begin{aligned}
\alpha &= K_{a,b}(u_1,v_1) + K_{a,b}(u_2,v_2) - K_{a,b}(u_2,v_1) - K_{a,b}(u_1,v_2) \\
&= C(a(1-u_1), b(1-v_1)) + C(u_1 + a(1-u_1), v_1 + b(1-v_1)) \\
&\quad - C(a(1-u_1), v_1 + b(1-v_1)) - C(u_1 + a(1-u_1), b(1-v_1)) \\
&\quad + C(a(1-u_2), b(1-v_2)) + C(u_2 + a(1-u_2), v_1 + b(1-v_2)) \\
&\quad - C(a(1-u_2), v_2 + b(1-v_2)) - C(u_2 + a(1-u_2), b(1-v_2)) \\
&\quad - C(a(1-u_1), b(1-v_2)) - C(u_1 + a(1-u_1), v_2 + b(1-v_2)) \\
&\quad + C(a(1-u_1), v_2 + b(1-v_2)) + C(u_1 + a(1-u_1), b(1-v_2)) \\
&\quad - C(a(1-u_2), b(1-v_1)) - C(u_2 + a(1-u_2), v_1 + b(1-v_1)) \\
&\quad + C(a(1-u_2), v_1 + b(1-v_1)) + C(u_2 + a(1-u_2), b(1-v_1)) \\
&= A + B + C + D,
\end{aligned}$$

where

$$\begin{aligned}
A &= C(a(1-u_1), b(1-v_1)) + C(a(1-u_2), b(1-v_2)) \\
&\quad - C(a(1-u_2), b(1-v_1)) - C(a(1-u_1), b(1-v_2)) \\
&= V_C\left([a(1-u_2, a(1-u_1))] \times [b(1-v_2), b(1-v_1)]\right) \ge 0.
\end{aligned}$$

Similarly for B, C, D. Therefore, $\alpha \ge 0$. □

Exercise 2.7 Let f be a function from I^2 into I which is non-decreasing in each variable and has margins given by $f(t,1) = t = f(1,t)$ for all $t \in \mathrm{I}$. Prove that f is grounded.

Solution. We need to prove that on I^2 "non-decreasingness + uniform margins $\Longrightarrow$ groundedness". For every $(x, y) \in \mathrm{I}^2$,

$$0 \le f(x, 0) \le f(1, 0) = 0, \quad 0 \le f(0, x) \le f(0, 1) = 0.$$

$\square$

Exercise 2.8 (a) Show that for any copula C, $\max(2t-1, 0) \le \delta_C(t) \le t$ for all $t \in \mathrm{I}$.

(b) Show that $\delta_C(t) = \delta_M(t)$ for all $t \in \mathrm{I}$ implies $C = M$.

(c) Show $\delta_C(t) = \delta_W(t)$ for all $t \in \mathrm{I}$ does not imply that $C = W$.

Solution. (a) Write

$$W(t, t) \le C(t, t) \le M(t, t) \Leftrightarrow \max(2t-1, 0) \le \delta_C(t) \le \min(t, t) = t.$$

(b) Since for all $t \in \mathrm{I}$,

$$\delta_M(t) = M(t, t) = \min(t, t) = t \implies \delta_C(t) = C(t, t) = t.$$

Assume that $C \ne M$, then there exists $(u, v) \in \mathrm{I}^2$ with $u \le v$ such that

$$C(u, v) \ne M(u, v) \implies C(u, v) < M(u, v) = u.$$

Then by non-decreasingness,

$$u > C(u, v) \ge C(u, u) = u,$$

which is a contradiction.

(c) We just need to show that $\delta_C = \delta_W$ with $C \ne W$ holds. $\square$

Exercise 2.9 The **secondary diagonal section** of C is given by $C(t, 1-t)$. Show that $C(t, 1-t) = 0$ for all $t \in \mathrm{I}$ implies $C = W$.

Solution. Assume that $C \ne W$, then for all $t \in \mathrm{I}$, we have

$$C(t, 1-t) > W(t, 1-t) = \max(t + 1 - t, 0) = 0,$$

which contradicts $C(t, 1-t) = 0$. $\square$

Exercise 2.10 Let t be in $[0,1)$, and let C_t be the function from I^2 into I given by

$$C_t(u,v) = \begin{cases} \max(u+v-1,t), & (u,v) \in [t,1]^2, \\ \min(u,v), & \text{o.w..} \end{cases}$$

(a) Show that C_t is a copula.

(b) Show that the level set $\{(u,v) \in \mathrm{I}^2 | \mathrm{C_t(u,v) = t}\}$ is the set of points in the triangle with vertices $(t,1),(1,t)$ and (t,t).

Solution. (a) ⊛ Uniform margins and groundedness are trivial. ⊛ For every $(u_1,v_1),(u_2,v_2) \in [t,1]^2$ with $u_1 \le u_2, v_1 \le v_2$,

$$\alpha = \max(u_1+v_1-1,t)+\max(u_2+v_2-1,t)-\max(u_1+v_2-1,t)-\max(u_2+v_1-1,t).$$

If $u_2+v_1 \ge 1+t$, then

$$\alpha = \max(u_1+v_1-1,t)+u_2+v_2-1-\max(u_1+v_2-1,t)-(u_2+v_1-1) \ge 0.$$

If $u_2+v_1 < 1+t$, then

$$\alpha = t+\max(u_2+v_2-1,t)-\max(u_1+v_2-1,t)-t \ge 0.$$

Therefore C_t is a copula.

(b) Using the Fréchet-Hoeffding bounds,

$$W(u,v) \le C_t(u,v) \le M(u,v).$$

Since C_t is non-decreasing in each argument. The level sets $L = \{(u,v) \in \mathrm{I}^2 : \mathrm{C_t(u,v) = t}\}$ have bounds

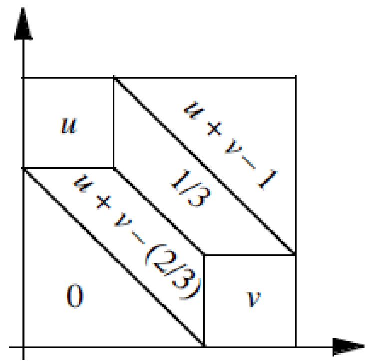
$$\{(u,v) \in \mathrm{I}^2 : \mathrm{W(u,v) = t}\} \le \mathrm{L} \le \{\mathrm{(u,v)} \in \mathrm{I}^2 : \mathrm{M(u,v) = t}\}.$$

□

Exercise 2.11 *This exercise shows that the 2-increasing condition for copulas is not a consequence of simpler properties.* Let Q be the function form I^2 to I given by

$$Q(u,v)=\begin{cases}\min\left(u,v,1/3,u+v-2/3\right), & 2/3\le u+v\le 4/3,\\ \max(u+v-1,0), & o.w..\end{cases}$$

That is, Q is given as following figure.



(a) Show that for every $u,v\in \mathrm{I}^2, \mathrm{Q(u,0)=0=Q(0,v),Q(u,1)=u,Q(1,v)=v}$; $W(u,v)\le Q(u,v)\le M(u,v)$; and that Q is continuous, satisfies the Lipschitz condition, and is non-decreasing in each variable.

(b) Show that Q fails to be 2-increasing, and hence is not a copula.

Solution. (a) The uniform margins and groundedness are trivial. The upper bounded is clear since $\min(u,v,1/3,u+v-2/3)\le\min(u,v)$. The lower bounded is clear from the figure, in the 1/3 region, 1/3 is the largest value of W in this region, thus $W\le Q$. The non-decreasingness is obvious. The Lipschitz condition, let $(u_1,v_1),(u_2,v_2)\in\{2/3\le u+v\le 4/3\}$ with $u_1\le u_2, v_1\le v_2$. If $(u_2,v_2),(u_1,v_1)$ are in the same region, then $A=0$. Consider different region, if (u_1,v_1) is in u region, (u_2,v_2) is in 1/3 region, then

$$Q(u_2,v_2)-Q(u_1,v_1)=1/3-u_1\le u_2-u_1\le u_2-u_1+v_2-v_1.$$

Similarly, if (u_1,v_1) is in $u+v-2/3$ region, (u_2,v_2) is in 1/3 region, then

$$Q(u_2,v_2)-Q(u_1,v_1)=1/3-u_1-v_1+2/3\le u_2-u_1+v_2-v_1.$$

If (u_1, v_1) is in $u + v - 2/3$ region, (u_2, v_2) is in u region, then

$$Q(u_2, v_2) - Q(u_1, v_1) = u_2 - u_1 - v_1 + 2/3 \leq u_2 - u_1 + v_2 - v_1.$$

If (u_1, v_1) is in v region, (u_2, v_2) is in $1/3$ region, then

$$Q(u_2, v_2) - Q(u_1, v_1) = 1/3 - v_1 \leq v_2 - v_1 \leq u_2 - u_1 + v_2 - v_1.$$

If (u_1, v_1) is in $u + v - 2/3$ region, (u_2, v_2) is in v region, then

$$Q(u_2, v_2) - Q(u_1, v_1) = v_2 - u_1 - v_1 + 2/3 \leq u_2 - u_1 + v_2 - v_1.$$

Therefore, the Lipschitz condition holds.

(b) Consider the Q-volume of the rectangle $[1/3, 2/3]^2$. Then

$$V_Q([1/3, 2/3]^2) = 0 + 1/3 - 1/3 - 1/3 = -1/3 < 0.$$

Q is not 2-increasing. □

Exercise 2.12 *Gumbel's bivariate logistic distribution.* Let X and Y be random variables with a joint distribution function given by

$$H(x, y) = (1 + e^{-x} + e^{-y})^{-1}$$

for all $x, y \in \overline{\mathbb{R}}$. (a) Show that X and Y have standard univariate logistic distribution, *i.e.*,

$$F(x) = (1 + e^{-x})^{-1}, \quad G(y) = (1 + e^{-y})^{-1}.$$

(b)Show that the copula of X and Y is the copula given by

$$C(u, v) = \frac{uv}{u + v - uv}.$$

Solution. (a) It is obvious.

(b) Since

$$C(u, v) = H(F^{-1}(u), G^{-1}(v)),$$

and

$$F^{-1}(u) = -\ln(u^{-1} - 1), \quad G^{-1}(v) = -\ln(v^{-1} - 1).$$

We have

$$C(u,v) = (1 + u^{-1} - 1 + v^{-1} - 1)^{-1} = \frac{uv}{u + v - uv}.$$

□

Exercise 2.13 *Type B bivariate extreme value distributions.* Let X and Y be random variables with a joint distribution function given by

$$H_\theta(x,y) = \exp\left(-\left(e^{-\theta x} + e^{-\theta y}\right)^{1/\theta}\right)$$

for all $x, y \in \overline{\mathbb{R}}$, where $\theta \geq 1$. Show that the copula of X and Y is given by

$$C_\theta(u,v) = \exp\left(-\left[-(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}\right).$$

This parametric family of copulas is known as **Gumbel-Hougaard family**.

Solution. The margins are

$$F(x) = H_\theta(x,\infty) = \exp(-e^{-x}), \quad G(y) = H_\theta(\infty, y) = \exp(-e^{-y}).$$

The reverses are

$$F^{-1}(u) = -\ln(-\ln u), \quad G^{-1}(v) = -\ln(-\ln v).$$

Therefore,

$$C_\theta(u,v) = \exp\left(-\left[-(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}\right).$$

□

Exercise 2.14 Note that Gumbel's bivariate logistic distribution suffers from the defect that it lacks a parameter, which limits its usefulness in applications. This can be corrected in a number of ways, one of which is to define H_θ as

$$H_\theta(x,y) = \left(1 + e^{-x} + e^{-y} + (1-\theta)e^{-x-y}\right)^{-1}$$

for all $x, y \in \overline{\mathbb{R}}$, where $\theta \in [-1, 1]$. Show that

(a) the margins are standard logistic distributions;

(b) when $\theta = 1$, we have Gumbel's bivariate logistic distribution;

(c) when $\theta = 0$, X, Y are independent;

(d) the copula of X, Y is given by

$$C_\theta(u, v) = \frac{uv}{1 - \theta(1-u)(1-v)}.$$

This is the **Ali-Mikhail-Haq family** of copulas.

Solution. (a) The margins are standard logistic distribution.

(b) This is obvious.

(c) When $\theta = 0$,

$$H_\theta(x, y) = \left(1 + e^{-x} + e^{-y} + e^{-x-y}\right)^{-1} = \left((1 + e^{-x})(1 + e^{-y})\right)^{-1}.$$

(d) The reverses are

$$F^{-1}(u) = -\ln(u^{-1} - 1), \quad G^{-1}(v) = -\ln(v^{-1} - 1).$$

Then

$$\begin{aligned} C_\theta(u, v) = H_\theta(F^{-1}(u), G^{-1}(v)) &= (u^{-1} + v^{-1} - 1 + (1-\theta)(u^{-1} - 1)(v^{-1} - 1))^{-1} \\ &= \frac{uv}{1 - \theta(1-u)(1-v)}. \end{aligned}$$

□

Exercise 2.15 Let X_1, Y_1 be random variables with continuous distribution functions F_1, G_1, and copula C. Let F_2, G_2 be another pair of continuous distribution functions, and set $X_2 = F_2^{(-1)}(F_1(X_1)), Y_2 = G_2^{(-1)}(G_1(Y_1))$. Prove that

(a) the distribution functions of X_2, Y_2 are F_2, G_2;

(b) the copula of X_2, Y_2 is C.

Solution. (a) The CDF of X_2 is

$$P(X_2 \le x_2) = P(F_2^{(-1)}(F_1(X_1)) \le x_2)$$

$$= P(F_1(X_1) \le F_2(x_2))$$
$$= P(X_1 \le F_1^{(-1)}(F_2(x_2)))$$
$$= F_1(F_1^{(-1)}(F_2(x_2))) = F_2(x_2).$$

And similar for Y_2.

(b) Write

$$\begin{aligned} C_{X_2,Y_2}(F_2(x), G_2(y)) &= P(X_2 \le x, Y_2 \le y) \\ &= P(F_2^{(-1)}(F_1(X_1)) \le x, G_2^{(-1)}(G_1(Y_1)) \le y) \\ &= P(X_1 \le F_1^{(-1)}(F_2(x)), Y_1 \le G_1^{(-1)}(G_2(y))) \\ &= C_{X_1,Y_1}(F_1[F_1^{(-1)}(F_2(x))], G_1[G_1^{(-1)}(G_2(y))]) \\ &= C_{X_1,Y_1}(F_2(x), G_2(y)). \end{aligned}$$

□

Exercise 2.16 (a) Let X and Y be continuous random variables with copula C and univariate distribution functions F and G, respectively. The random variables $\max(X, Y)$ and $\min(X, Y)$ are the order statistics for X, Y. Prove that the distribution functions of the order statistics are given by

$$P(\max(X, Y) \le t) = C(F(t), G(t))$$

and

$$P(\min(X, Y) \le t) = F(t) - G(t) - C(F(t), G(t)),$$

so that when $F = G$,

$$P(\max(X, Y) \le t) = \delta_C(F(t)), \quad P(\min(X, Y) \le t) = 2F(t) - \delta_C(F(t)).$$

(b) Show that bounds on the distribution functions of the order statistics are given by

$$\max(F(t) + G(t) - 1, 0) \le P(\max(X, Y) \le t) \le \min(F(t), G(t))$$

and

$$\max(F(t), G(t)) \leq P(\min(X, Y) \leq t) \leq \min(F(t) + G(t), 1).$$

Solution. (a) One have

$$P(\max(X, Y) \leq t) = P(X \leq t, Y \leq t) = H(t, t) = C(F(t), G(t)).$$

And

$$\begin{aligned}
P(\min(X, Y) \leq t) &= 1 - P(\min(X, Y) > t) \\
&= 1 - P(X > t, Y > t) \\
&= 1 - [1 - P(X \leq t) - P(Y \leq t) + P(X \leq t, Y \leq t)] \\
&= P(X \leq t) + P(Y \leq t) - P(X \leq t, Y \leq t) \\
&= F(t) + G(t) - C(F(t), G(t)).
\end{aligned}$$

(b) We only show the bounds for $P(\min(X, Y) \leq t)$. Write

$$\begin{aligned}
F(t) + G(t) - \min(F(t), G(t)) &\leq F(t) + G(t) - C(F(t), G(t)) \\
&\leq F(t) + G(t) - \max(F(t) + G(t) - 1, 0),
\end{aligned}$$

which is equal to

$$\begin{aligned}
\max(F(t), G(t)) &\leq F(t) + G(t) - C(F(t), G(t)) \\
&\leq \min(F(t) + G(t), 1).
\end{aligned}$$

□

Exercise 2.17 [Theorem 2.4.4.] Let X and Y be continuous random variables with copula C_{XY}. Let α, β be strictly monotone on $\operatorname{ran} X$ and $\operatorname{ran} Y$, respectively.

(1) If α is strictly increasing and β is strictly decreasing, then

$$C_{\alpha(X)\beta(Y)}(u, v) = u - C_{XY}(u, 1 - v).$$

(2) If α is strictly decreasing and β is strictly increasing, then

$$C_{\alpha(X)\beta(Y)}(u,v) = v - C_{XY}(1-u,v).$$

(3) If α and β are both strictly decreasing, then

$$C_{\alpha(X)\beta(Y)}(u,v) = u + v - 1 + C_{XY}(1-u,1-v).$$

PROOF. (1) Write, for any $x, y \in \overline{\mathbb{R}}$,

$$\begin{aligned}
C_{\alpha(X)\beta(Y)}(F_{\alpha(X)}(x), G_{\beta(Y)}(y)) &= P(\alpha(X) \le x, \beta(Y) \le y) \\
&= P(X \le \alpha^{-1}(x), Y > \beta^{-1}(y)) \\
&= P(X \le \alpha^{-1}(x)) - P(X \le \alpha^{-1}(x), Y \le \beta^{-1}(y)) \\
&= F_X(\alpha^{-1}(x)) - C_{XY}(F_X(\alpha^{-1}(x)), G_Y(\beta^{-1}(y))) \\
&= F_{\alpha(X)}(x) - C_{XY}(F_{\alpha(X)}(x), 1 - G_{\beta(Y)}(y)).
\end{aligned}$$

(2) Similarly, this part is obvious.

(3) Write

$$\begin{aligned}
C_{\alpha(X)\beta(Y)}(F_{\alpha(X)}(x), G_{\beta(Y)}(y)) &= P(\alpha(X) \le x, \beta(Y) \le y) \\
&= P(X > \alpha^{-1}(x), Y > \beta^{-1}(y)) \\
&= P(X > \alpha^{-1}(x)) - P(X > \alpha^{-1}(x), Y \le \beta^{-1}(y)) \\
&= P(X > \alpha^{-1}(x)) - P(Y \le \beta^{-1}(y)) + P(X \le \alpha^{-1}(x), Y \le \beta^{-1}(y)) \\
&= F_{\alpha(X)}(x) + G_{\beta(Y)}(y) - 1 + C_{XY}(F_X(\alpha^{-1}(x)), G_Y(\beta^{-1}(y))) \\
&= F_{\alpha(X)}(x) + G_{\beta(Y)}(y) - 1 + C_{XY}(1 - F_{\alpha(X)}(x), 1 - G_{\beta(Y)}(y)).
\end{aligned}$$

$\square$

Exercise 2.18 Let S be a subset of $\overline{\mathbb{R}}^2$. Then S is non-increasing if and only if for each (x,y) in $\overline{\mathbb{R}}^2$, either

(1) for all (u,v) in S, $u \le x$ implies $v > y$; or

(2) for all (u, v) in S, $v > y$ implies $u \leq x$.

Solution. "$\Rightarrow$": Assume S is non-increasing and neither (1) nor (2) holds. Then there exists points $(a, b), (c, d)$ in S such that $a \leq x, b \leq y$ and $d > y, c > x$. Hence $a \leq c$ and $b \leq d$, contradict the non-increasingness.

"$\Leftarrow$": Assume that S is not non-increasing. Then there exists points $(a, b), (c, d)$ in S with $a \leq c$ and $b \geq d$. For $(x, y) = ((a + c)/2, (b + d)/2)$, neither (1) nor (2) holds.

□

Exercise 2.19 Let X, Y be random variables whose joint distribution function H is equal to its Fréchet-Hoeffding lower bound. Then for every $(x, y) \in \overline{\mathbb{R}}^2$, either $P(X > x, Y > y) = 0$ or $P(X \leq x, Y \leq y) = 0$.

Solution. Since

$$P(X > x, Y > y) = 1 - F(x) - G(y) + H(x, y), \quad P(X \leq x, Y \leq y) = H(x, y),$$

$H(x, y) = \max(F(x) + G(y) - 1, 0)$ if and only if either $P(X > x, Y > y) = 0$ or $P(X \leq x, Y \leq y) = 0$. Since if $P(X \leq x, Y \leq y) = 0$, $F(x) + G(y) - 1 < 0$ and $H(x, y) = 0$. □

Exercise 2.20 [Theorem 2.5.5]

Solution. Let S denote the support of H, and let (x, y) be any point in $\overline{\mathbb{R}}^2$. Then (1) holds in Exercise 2.18 if and only if

$$\{(u, v) : u \leq x, v \leq y\} \cap S = \varnothing.$$

That is $P(X \leq x, Y \leq y) = 0$. By Exercise 2.19, the proof is completed.

□

Exercise 2.21 Let X, Y be non-negative random variables whose survival function is $\overline{H}(x, y) = (e^x + e^y - 1)^{-1}$ for $x, y \geq 0$.

(a) Show that X, Y are standard exponential random variables.

(b) Show that the copula of X, Y is the copula given by

$$C(u,v) = \frac{uv}{u+v+uv}.$$

Solution. (a) The univariate survival margins are

$$\overline{F}(x) = \overline{H}(x, -\infty) = (e^x - 1)^{-1}, \quad \overline{G}(y) = \overline{H}(-\infty, y) = (e^y - 1)^{-1}.$$

(b) The inverse functions are

$$\overline{F}^{(-1)}(u) = \ln(u^{-1} + 1), \quad \overline{G}^{(-1)}(v) = \ln(v^{-1} + 1).$$

Then

$$\hat{C}(u,v) = \overline{H}(\overline{F}^{(-1)}(u), \overline{G}^{(-1)}(v)) = \frac{uv}{u+v+uv}.$$

□

Exercise 2.22 Let X, Y be continuous random variables whose joint distribution function is given by $C(F(x), G(y))$, where C is the copula of X, Y, and F, G are the distribution functions of X, Y respectively. Verify

$$P(X \le x \cup Y \le y) = \tilde{C}(F(x), G(y)), \quad P(X > x \cup Y > y) = C^*(\overline{F}(x), \overline{G}(y)).$$

Solution. We have

$$\begin{aligned} P(X \le x \cup Y \le y) &= P(X \le x) + P(Y \le y) - P(X \le x, Y \le y) \\ &= F(x) + G(y) - C(F(x), G(y)) \\ &= \tilde{C}(F(x), G(y)). \end{aligned}$$

And

$$\begin{aligned} P(X > x \cup Y > y) &= 1 - P(X \le x, Y \le y) \\ &= 1 - C(F(x), G(y)) \\ &= C^*(\overline{F}(x), \overline{G}(y)). \end{aligned}$$

□

Exercise 2.23 Let $X_1, Y_1, F_1, G_1, F_2, G_2$ and C be as usual. Set $X_2 = F_2^{(-1)}(1 - F_1(X_1))$ and $Y_2 = G_2^{(-1)}(1 - G_1(Y_1))$. Prove that

(a) The distribution functions of X_2, Y_2 are F_2, G_2, and

(b) The copula of X_2, Y_2 is $\hat{C}$.

Solution. (a) The distribution function of X_2 is

$$\begin{aligned}
P(X_2 \le x) &= P(F_2^{(-1)}(1 - F_1(X_1)) \le x) \\
&= P((1 - F_1(X_1)) \le F_2(x)) \\
&= P(1 - F_2(x) \le F_1(X_1)) \\
&= 1 - P(F_1(X_1) \le 1 - F_2(x)) \\
&= 1 - P(X_1 \le F_1^{(-1)}(1 - F_2(x))) \\
&= 1 - F_1[F_1^{(-1)}(1 - F_2(x))] = F_2(x).
\end{aligned}$$

Similarly for G_2.

(b) Write

$$\begin{aligned}
P(X_2 \le x, Y_2 \le y) &= P(F_2^{(-1)}(1 - F_1(X_1)) \le x, G_2^{(-1)}(1 - G_1(Y_1)) \le y) \\
&= P((1 - F_1(X_1)) \le F_2(x), (1 - G_1(Y_1)) \le G_2(y)) \\
&= P((1 - F_2(x)) \le F_1(X_1), (1 - G_2(y)) \le G_1(Y_1)) \\
&= 1 - [1 - F_2(x) + 1 - G_2(y) - P(F_1(X_1) \le 1 - F_2(x), G_1(Y_1) \le 1 - G_2(y))] \\
&= F_2(x) + G_2(y) - 1 + C(F_1(F_1^{(-1)}(1 - F_2(x))), G_1(G_1^{(-1)}(1 - G_2(y)))) \\
&= F_2(x) + G_2(y) - 1 + C(1 - F_2(x), 1 - G_2(y)) \\
&= \hat{C}(F_2(x), G_2(y)).
\end{aligned}$$

□

Exercise 2.24 Let X, Y be continuous random variables with copula C and a common univariate distribution function F. Show that the distribution and survival functions of the order statistics are given by

Order statistic	Distribution function	Survival function
$\max(X,Y)$	$\delta(F(t))$	$\delta^*(\overline{F}(t))$
$\min(X,Y)$	$\tilde{\delta}(F(t))$	$\hat{\delta}(\overline{F}(t))$

where $\delta, \hat{\delta}, \tilde{\delta}$ and δ^* denote the diagonal sections of $C, \hat{C}, \tilde{C}, C^*$, respectively.

Solution. Write

$$\begin{aligned} P(\max(X,Y) \leq t) &= P(X \leq t, Y \leq t) \\ &= C(F(t), F(t)) = \delta(F(t)). \end{aligned}$$

And

$$P(\max(X,Y) > t) = 1 - C(F(t), F(t)) = \delta^*(\overline{F}(t)).$$

Further,

$$\begin{aligned} P(\min(X,Y) \leq t) &= 1 - P(\min(X,Y) > t) \\ &= 1 - P(X > t, Y > t) \\ &= P(X \leq t) + P(Y \leq t) - P(X \leq t, Y \leq t) \\ &= F(t) + F(t) - C(F(t), F(t)) \\ &= \tilde{\delta}(F(t)). \end{aligned}$$

And

$$\begin{aligned} P(\min(X,Y) > t) &= 1 - F(t) - F(t) + C(F(t), F(t)) \\ &= \hat{\delta}(\overline{F}(t)). \end{aligned}$$

□

Exercise 2.25 Show that under composition, the set of operations of forming the survival copula, the dual of a copula, and the co-copula of a given copula. along with the identity yields the dihedral group:

$\circ$	i	$\wedge$	$\sim$	$*$
i	i	$\wedge$	$\sim$	$*$
$\wedge$	$\wedge$	i	$*$	$\sim$
$\sim$	$\sim$	$*$	i	$\wedge$
$*$	$*$	$\sim$	$\wedge$	i

Solution. Clearly,

$$\begin{aligned}\wedge(\wedge(C(u,v))) &= \wedge(u+v-1+C(1-u,1-v)) \\ &= u+v-1+(1-u)+(1-v)-1+C(u,v) = C(u,v).\end{aligned}$$

And

$$\begin{aligned}\sim(\sim(C(u,v))) &= \sim(u+v-C(u,v)) \\ &= u+v-u-v+C(u,v) = C(u,v).\end{aligned}$$

And

$$\begin{aligned}\wedge(\sim(C(u,v))) &= \wedge(u+v-C(u,v)) \\ &= u+v-u-v+1-C(1-u,1-v) = C^*(u,v).\end{aligned}$$

Others are similar. □

Exercise 2.26 Prove for any $(u,v) \in \mathrm{I}^2$,

$$\hat{C}(u,v) = \overline{H}(\overline{F}^{(-1)}(u), \overline{G}^{(-1)}(v)).$$

Solution. One has

$$\begin{aligned}\overline{H}(\overline{F}^{(-1)}(u), \overline{G}^{(-1)}(v)) &= P(X > \overline{F}^{(-1)}(u), Y > \overline{G}^{(-1)}(v)) \\ &= 1 - F(\overline{F}^{(-1)}(u)) - G(\overline{G}^{(-1)}(v)) + H(\overline{F}^{(-1)}(u), \overline{G}^{(-1)}(v)) \\ &= u+v-1+C(F(\overline{F}^{(-1)}(u))), G(\overline{G}^{(-1)}(v))) \\ &= u+v-1+C(1-u,1-v) = \hat{C}(u,v).\end{aligned}$$

□

Exercise 2.27 Let X, Y be continuous random variables symmetric about a and b with marginal distribution function F, G, and with copula C. Is (X, Y) is radially symmetric (or jointly symmetric) about (a, b) if C is

(a) a member of the Fréchet family in Exercise 2.4?

(b) a member of the Cuadras-Augé family in Exercise 2.5?

Solution. (a) The Fréchet family is

$$C_{\alpha,\beta}(u,v) = \alpha M(u,v) + (1-\alpha-\beta)\Pi(u,v) + \beta W(u,v).$$

Then

$$\begin{aligned}
\hat{C}_{\alpha,\beta}(u,v) &= u + v - 1 + C_{\alpha,\beta}(1-u, 1-v) \\
&= u + v - 1 + \alpha \min(1-u, 1-v) + (1-\alpha-\beta)(1-u)(1-v) \\
&\quad + \beta \max(1-u+1-v+1, 0) \\
&= \alpha[\min(1-u, 1-v) - 1 + u + v] + (1-\alpha-\beta)uv \\
&\quad + \beta[\max(1-u-v, 0) - 1 + u + v] \\
&= C_{\alpha,\beta}(u,v).
\end{aligned}$$

(b) The Cuadras-Augé family is

$$C_\theta(u,v) = [\min(u,v)]^\theta [uv]^{1-\theta} = \begin{cases} uv^{1-\theta}, & u \le v, \\ u^{1-\theta}v, & u \ge v. \end{cases}$$

Then when $u \le v$,

$$\hat{C}_\theta(u,v) = u + v - 1 + (1-u)(1-v)^{1-\theta}.$$

And similarly for $u \ge v$. Clearly, $\hat{C}_0(u,v) = C_0(u,v)$. And $\hat{C}_1(u,v) = C_1(u,v)$. □

Exercise 2.28 Suppose X, Y are identically distributed continuous random variables, each symmetric about a. Show that "exchangeability" does not imply "radial symmetry", nor does "radial symmetry" imply "exchangeability".

Solution. Suppose that X, Y are exchangeable, then

$$C(u,v) = C(v,u).$$

Thus

$$\begin{aligned}\hat{C}(u,v) &= u + v - 1 + C(1-u, 1-v) \\ &= u + v - 1 + C(1-v, 1-u) \neq C(u,v).\end{aligned}$$

Conversely, assume "radial symmetry", then

$$\hat{C}(u,v) = C(u,v).$$

Thus

$$C(v,u) = v + u - 1 + C(1-v, 1-u) \neq u + v - 1 + C(1-u, 1-v) = C(u,v).$$

$\square$

Exercise 2.29 Let X, Y be continuous random variables with joint distribution function H and margins F, G. Let (a,b) be a point in $\mathbb{R}^2$. Then (X,Y) is jointly symmetric about (a,b) if and only if

$$H(a+x, b+y) = F(a+x) - H(a+x, b-y), \quad H(a+x, b+y) = G(b+y) - H(a-x, b+y)$$

for all $(x,y) \in \overline{\mathbb{R}}^2$.

Solution. According to the definition,

$$\begin{aligned}P(X-a \leq x, Y-b \leq y) &= P(X-a \leq x, b-Y \leq y) \\ &= P(X \leq a+x) - P(X \leq a+x, Y \leq b-y) \\ &= F(x+a) - H(a+x, b-y). \qquad (2.0.1)\end{aligned}$$

Similarly,

$$P(X-a \leq x, Y-b \leq y) = P(a-X \leq x, Y-b \leq y) = G(b+y) - H(a-x, b+y). \qquad (2.0.2)$$

Let $y = \infty$, (2.0.1) and (2.0.2) imply that (X, Y) is marginally symmetric about (a, b),

$$G(b+y) = \overline{G}(b-y), \quad F(a+x) = \overline{F}(a-x).$$

Then

$$G(b+y) - H(a+x, b+y) = \overline{G}(b-y) - P(X \le a+x, Y > b-y),$$

which implies

$$P(X > a+x, Y > b-y) = P(X > a+x, Y \le b+y) = P(X \le a-x, Y \le b+y).$$

It follows that (draw a picture)

$$P(X > a+x, Y \le b+y) = P(X \le a-x, Y > b-y).$$

It ends the proof. □

Exercise 2.30 Let X, Y be continuous random variables with joint distribution function H and margins F, G and copula C. Further suppose that X, Y are symmetric about a and b. Then (X, Y) is jointly symmetric about (a, b) if and only if C satisfies

$$C(u, v) = u - C(u, 1-v), \quad C(u, v) = v - C(1-u, v)$$

for all $(u, v) \in \mathrm{I}^2$.

Solution. Write

$$\begin{aligned}
&H(a+x, b+y) = F(x+a) - H(a+x, b-y) \\
&\Leftrightarrow C(F(a+x), G(b+y)) = F(x+a) - C(F(a+x), G(b-y)) \\
&\Leftrightarrow C(F(a+x), G(b+y)) = F(x+a) - C(F(a+x), \overline{G}(b+y)) \\
&\Leftrightarrow C(u, v) = u - C(u, 1-v).
\end{aligned}$$

Similar for the other equation. □

Exercise 2.31 (a) Show that $C_1 \prec C_2$ if and only if $\overline{C}_1 \prec \overline{C}_2$.

(b) Show that $C_1 \prec C_2$ if and only if $\hat{C}_1 \prec \hat{C}_2$.

Solution. (a) We have for all $(u,v) \in \mathrm{I}^2$,

$$\begin{aligned} C_1 \prec C_2 &\Leftrightarrow C_1(u,v) \leq C_2(u,v) \\ &\Leftrightarrow 1-u-v+C_1(u,v) \leq 1-u-v+C_2(u,v) \\ &\Leftrightarrow \overline{C}_1 \leq \overline{C}_2 \end{aligned}$$

(b) Similarly, for all $(u,v) \in \mathrm{I}^2$,

$$\begin{aligned} C_1 \prec C_2 &\Leftrightarrow C_1(u,v) \leq C_2(u,v) \\ &\Leftrightarrow C_1(1-u,1-v) \leq C_2(1-u,1-v) \\ &\Leftrightarrow u+v-1+C_1(u,v) \leq u+v-1+C_2(u,v) \\ &\Leftrightarrow \hat{C}_1 \leq \hat{C}_2 \end{aligned}$$

□

Exercise 2.32 Show that Ali-Mikhail-Haq family of copulas from Exercise 2.14 is positively ordered.

Solution. The copula of X, Y is given by

$$C_\theta(u,v) = \frac{uv}{1-\theta(1-u)(1-v)},$$

where $\theta \in [-1,1]$. For $0 \leq \alpha \leq \beta \leq 1$ and $u, v \in (0,1)$,

$$\frac{C_\alpha(u,v)}{C_\beta(u,v)} = \frac{1-\beta(1-u)(1-v)}{1-\alpha(1-u)(1-v)} \leq 1,$$

thus $C_\alpha(u,v) \leq C_\beta(u,v)$. □

Exercise 2.33 Show that the Mardia family from Exercise 2.4 is neither positively nor negatively ordered.

Solution. Let θ be in $[-1,1]$ and set

$$C_\theta(u,v) = \frac{\theta^2(1+\theta)}{2}M(u,v) + (1-\theta^2)\Pi(u,v) + \frac{\theta^2(1-\theta)}{2}W(u,v).$$

Note that, let $(u,v)=(3/4,1/4)$,

$$C_0(3/4,1/4)=\frac{3}{16}=\frac{96}{512},\quad C_{1/4}(3/4,1/4)=\frac{95}{512},\quad C_{1/2}(3/4,1/4)=\frac{96}{512}.$$

□

Exercise 2.34 (a) Show that the $(n-1)$-margins of an n-copula are $(n-1)$-copulas.

(b) Show that if C is an n-copula, $n\geq 3$, then for any $k, 2\leq k<n$, all $\binom{n}{k}$ k-margins of C are k-copulas.

Solution. (a) The groundedness and uniform margins are directly from the property of this n-copula. Further, evaluate the C^n-volume of box

$$[a_1,b_1]\times\cdots\times[a_{k-1},b_{k-1}]\times[0,1]\times[a_{k+1},b_{k+1}]\times\cdots\times[a_n,b_n].$$

Clearly, it is greater than 0, thus the C^{n-1}-volume of

$$[a_1,b_1]\times\cdots\times[a_{k-1},b_{k-1}]\times[a_{k+1},b_{k+1}]\times\cdots\times[a_n,b_n]$$

is also greater than 0.

(b) This is similar to part (a). □

Exercise 2.35 Let M^n and Π^n be multivariate copula, and let $[\boldsymbol{a},\boldsymbol{b}]$ be an n-box in $\mathrm{I^n}$. Prove that

$$V_{M^n}([\boldsymbol{a},\boldsymbol{b}])=\max(\min(b_1,b_2,\ldots,b_n)-\max(a_1,a_2,\ldots,a_n),0)$$

and

$$V_{\Pi^n}([\boldsymbol{a},\boldsymbol{b}])=(b_1-a_1)(b_2-a_2)\cdots(b_n-a_n),$$

and hence conclude that M^n and Π^n are n-copulas for all $n\geq 2$.

Solution. We only prove for M^n, for bivariate M, if $a_1\leq a_2$,

$$V_M([\boldsymbol{a},\boldsymbol{b}])=\min(b_1,b_2)-\min(b_1,a_2)\geq 0.$$

If $a_1>a_2$,

$$V_M([\boldsymbol{a},\boldsymbol{b}])=\min(b_1,b_2)-\min(a_1,b_2)\geq 0.$$

That is

$$V_M([\boldsymbol{a},\boldsymbol{b}]) = \max(\min(b_1,b_2) - \max(a_1,a_2),0).$$

Then similarly, for M^n,

$$V_{M^n}([\boldsymbol{a},\boldsymbol{b}]) = \max(\min(b_1,b_2,\dots,b_n) - \max(a_1,a_2,\dots,a_n),0).$$

$\square$

Exercise 2.36 Show that

$$V_{W^n}([\mathbf{1/2},\mathbf{1}]) = 1 - n/2,$$

where $\mathbf{1} = (1,1,\dots,1)$ and $\mathbf{1/2} = (1/2,1/2,\dots,1/2)$, and hence W^n fails to be an n-copula whenever $n > 2$.

Solution. We have for all vertices $\boldsymbol{c}$, that is c_i is either $1/2$ or 1, for all $i = 1,\dots,n$,

$$V_{W^n}([\mathbf{1/2},\mathbf{1}]) = \sum(-1)^{\{\#:c_i=1/2\}}W^n(\boldsymbol{c}) = -\max(1/2,0)\cdot n + \max(1,0) = 1 - n/2.$$

Thus W^n fails to be an n-copula whenever $n > 2$. $\square$

Exercise 2.37 Let $\{X_1,\dots,X_n\}$ be continuous random variables with copula C and distribution functions $\{F_1,\dots,F_n\}$, respectively. Let $X_{(1)}$ and $X_{(n)}$ denote the extreme order statistics for $\{X_1,\dots,X_n\}$. Prove that the distribution functions $F_{(1)}, F_{(n)}$ of $X_{(1)}, X_{(n)}$ satisfy

$$\max(F_1(t),\dots,F_n(t)) \le F_{(1)}(t) \le \min\left(\sum_{k=1}^n F_k(t), 1\right)$$

and

$$\max\left(\sum_{k=1}^n F_k(t) - n + 1, 0\right) \le F_{(n)}(t) \le \min(F_1(t),F_2(t),\dots,F_n(t)).$$

Solution. Note that,

$$\begin{aligned} F_{(1)}(t) &= 1 - P(X_1 > t,\dots,X_n > t) \\ &= (-1)^{1-1}\sum_{i=1}^n P(X_i \le t) + (-1)^{2-1}\sum_{i_1\ne i_2}^n P(X_{i_1} \le t, X_{i_2} \le t) \end{aligned}$$

$$+\cdots+(-1)^{n-1}P(X_1\le t,\ldots,X_n\le t),$$

which implies that

$$F_{(1)}(t)\le \min\left(\sum_{k=1}^{n}F_k(t),1\right).$$

Clearly, for $i=1,\ldots,n$,

$$F_{(1)}(t)=1-P(X_1>t,\ldots,X_n>t)\ge 1-P(X_i>t)\implies F_{(1)}\ge \max(F_1,\ldots,F_n).$$

Further,

$$\begin{aligned}F_{(n)}(t)&=P(X_1\le t,\ldots,X_n\le t)\\&=1+(-1)^1\sum_{i=1}^{n}P(X_i>t)+(-1)^2\sum_{i_1\ne i_2}^{n}P(X_{i_1}>t,X_{i_2}>t)\\&\quad+\cdots+(-1)^nP(X_1>t,\ldots,X_n>t),\end{aligned}$$

which implies that

$$F_{(n)}(t)\ge\max\left(1+(-1)^1\sum_{i=1}^{n}P(X_i>t),0\right)=\max\left(\sum_{k=1}^{n}F_k(t)-n+1,0\right).$$

Clearly, for $i=1,\ldots,n$,

$$F_{(n)}(t)=P(X_1\le t,\ldots,X_n\le t)\le P(X_i\le t)\implies F_{(n)}\le\min(F_1,\ldots,F_n).$$

□

Exercise 3.1 Show that when either of the parameters α or β is equal to 0 or 1, the function

$$C_{\alpha,\beta}=\min(u^{1-\alpha}v,uv^{1-\beta})=\begin{cases}u^{1-\alpha}v, & u^\alpha\ge v^\beta,\\ uv^{1-\beta}, & u^\alpha\le v^\beta.\end{cases}$$

is a copula.

Solution. Clearly,

$$C_{0,0}=C_{\alpha,0}=C_{0,\beta}=\Pi,\quad C_{1,1}=M.$$

And

$$C_{\alpha,1} = \min(u^{1-\alpha}v, u), \quad C_{1,\beta} = \min(v, uv^{1-\beta}).$$

The groundedness and uniform margins are obvious. Further, let $(u_1, v_1), (u_2, v_2) \in \mathrm{I}^2$ with $u_1 \le u_2, v_1 \le v_2$, then

$$\begin{aligned} V_{C_{\alpha,1}}([u_1, u_2] \times [v_1, v_2]) &= \min(u_1^{1-\alpha}v_1, u_1) + \min(u_2^{1-\alpha}v_2, u_2) \\ &\quad - \min(u_2^{1-\alpha}v_1, u_2) - \min(u_1^{1-\alpha}v_2, u_1). \end{aligned}$$

If $v_2/u_1^{\alpha} < 1$,

$$\begin{aligned} V_{C_{\alpha,1}}([u_1, u_2] \times [v_1, v_2]) &= u_1^{1-\alpha}v_1 + u_2^{1-\alpha}v_2 - u_2^{1-\alpha}v_1 - u_1^{1-\alpha}v_2 \\ &= (v_2 - v_1)(u_2^{1-\alpha} - u_1^{1-\alpha}) \ge 0. \end{aligned}$$

If $v_2/u_1^{\alpha} > 1, v_1/u_1^{\alpha} < 1, v_2/u_2^{\alpha} < 1$,

$$\begin{aligned} V_{C_{\alpha,1}}([u_1, u_2] \times [v_1, v_2]) &= u_1^{1-\alpha}v_1 + u_2^{1-\alpha}v_2 - u_2^{1-\alpha}v_1 - u_1 \\ &\ge u_1^{1-\alpha}v_1 + u_2^{1-\alpha}v_2 - u_2^{1-\alpha}v_1 - u_1^{1-\alpha}v_2 \\ &= (v_2 - v_1)(u_2^{1-\alpha} - u_1^{1-\alpha}) \ge 0. \end{aligned}$$

If $v_2/u_1^{\alpha} > 1, v_1/u_1^{\alpha} < 1, v_2/u_2^{\alpha} > 1$,

$$\begin{aligned} V_{C_{\alpha,1}}([u_1, u_2] \times [v_1, v_2]) &= u_1^{1-\alpha}v_1 + u_2 - u_2^{1-\alpha}v_1 - u_1 \\ &= u_1(v_1/u_1^{\alpha} - 1) + u_2(1 - v_1/u_2^{\alpha}) \\ &\ge u_1(v_1/u_1^{\alpha} - 1) + u_1(1 - v_1/u_2^{\alpha}) \\ &= v_1/u_1^{\alpha} - v_1/u_2^{\alpha} \ge 0. \end{aligned}$$

If $v_2/u_1^{\alpha} > 1, v_1/u_1^{\alpha} > 1, v_2/u_2^{\alpha} < 1$,

$$\begin{aligned} V_{C_{\alpha,1}}([u_1, u_2] \times [v_1, v_2]) &= u_1^{1-\alpha}v_1 + u_2 - u_2 - u_1 \\ &\ge u_1^{1-\alpha}v_1 + u_2 - u_2^{1-\alpha}v_1 - u_1 \ge 0. \end{aligned}$$

If $v_2/u_1^\alpha > 1, v_1/u_1^\alpha > 1, v_2/u_2^\alpha > 1, v_1/u_2^\alpha > 1$,

$$V_{C_{\alpha,1}}([u_1, u_2] \times [v_1, v_2]) = u_1 + u_2 - u_2 - u_1 = 0$$

If $v_2/u_1^\alpha > 1, v_1/u_1^\alpha > 1, v_2/u_2^\alpha > 1, v_1/u_2^\alpha < 1$,

$$V_{C_{\alpha,1}}([u_1, u_2] \times [v_1, v_2]) = u_1 + u_2 - u_2^{1-\alpha} v_1 - u_1 \geq 0.$$

The other $C_{1,\beta}$ is similar. □

Exercise 3.2 Show that a version of the Marshall-Olkin bivariate distribution with Pareto margins has joint survival functions given by

$$\overline{H}(x, y) = (1 + x)^{-\theta_1}(1 + y)^{-\theta_2}[\max(1 + x, 1 + y)]^{-\theta_{12}},$$

for $x, y \geq 0$, where $\theta_1, \theta_2, \theta_{12}$ are positive parameters.

Solution. Note that the Pareto margins are

$$\overline{F}(x) = \begin{cases} (1+x)^{-\theta}, & x \geq 0, \\ 1, & x < 0. \end{cases} \qquad \overline{G}(y) = \begin{cases} (1+y)^{-\theta}, & y \geq 0, \\ 1, & y < 0. \end{cases}$$

Then

$$\begin{aligned} \overline{H}(x, y) &= \hat{C}(\overline{F}(x), \overline{G}(y)) \\ &= \begin{cases} (1+x)^{-\theta}(1+y)^{-\theta} \min((1+x)^{\alpha\theta}, (1+y)^{\beta\theta}), & x \geq 0, y \geq 0, \\ (1+x)^{-\theta} \min((1+x)^{\alpha\theta}, 1), & x \geq 0, y < 0, \\ (1+y)^{-\theta} \min(1, (1+y)^{\beta\theta}), & x < 0, y \geq 0, \\ 1, & x \geq 0, y \geq 0, \end{cases} \end{aligned}$$

Therefore, for $x, y \geq 0$,

$$\begin{aligned} \overline{H}(x, y) &= (1+x)^{-\theta}(1+y)^{-\theta} \min((1+x)^{\alpha\theta}, (1+y)^{\beta\theta}), \\ &= (1+x)^{-\theta+\alpha\theta}(1+y)^{-\theta} \left[\min\left(1, \frac{(1+y)^\beta}{(1+x)^\alpha}\right)\right]^\theta, \end{aligned}$$

$$= (1+x)^{-\theta+\alpha\theta}(1+y)^{-\theta}\left[\max\left(1, \frac{(1+x)^\alpha}{(1+y)^\beta}\right)\right]^{-\theta},$$
$$= (1+x)^{-\theta+\alpha\theta}(1+y)^{-\theta-\beta}\left[\max\left((1+y)^\beta, (1+x)^\alpha\right)\right]^{-\theta},$$
$$= (1+x)^{-\theta_1}(1+y)^{-\theta_2}[\max(1+x, 1+y)]^{-\theta_{12}}.$$

□

Exercise 3.3 Prove the following generalization of the Marshall-Olkin family of copulas: Suppose that a, b are increasing functions defined on I such that $a(0) = b(0) = 0$ and $a(1) = b(1) = 1$. Further suppose that the functions $u \mapsto a(u)/u$ and $v \mapsto b(v)/v$ are both increasing on $(0, 1]$. Then the function C defined on I^2 by

$$C(u, v) = \min(va(u), ub(v))$$

is a copula. $a(u) = u^{1-\alpha}, b(v) = v^{1-\beta}$ is a special case.

Solution. The groundness and uniform margins are obvious. Let $(u_1, v_1), (u_2, v_2) \in \mathrm{I}^2$ with $u_1 \leq u_2, v_1 \leq v_2$, then

$$\alpha = \min(v_1a(u_1), u_1b(v_1)) + \min(v_2a(u_2), u_2b(v_2)) - \min(v_1a(u_2), u_2b(v_1)) - \min(v_2a(u_1), u_1b(v_2)).$$

If $a(u_1)/u_1 \leq b(v_1)/v_1, a(u_2)/u_2 > b(v_1)/v_1, a(u_2)/u_2 \leq b(v_2)/v_2$,

$$\begin{aligned}\alpha &= v_1a(u_1) + v_2a(u_2) - u_2b(v_1) - v_2a(u_1)\\ &\geq v_1a(u_1) + v_2a(u_2) - v_1a(u_2) - v_2a(u_1)\\ &= (v_2 - v_1)(a(u_2) - a(u_1)) \geq 0.\end{aligned}$$

Other circumstance are similar. □

Exercise 3.4 (a) Show that the following algorithm generates random variates (x, y) from Marshall-Olkin bivariate-exponential distribution with parameters $\lambda_1, \lambda_2, \lambda_{12}$:

1. Generate three independent uniform $(0, 1)$ variates r, s, t;
2. Set $x = \min\left(\frac{-\ln r}{\lambda_1}, \frac{-\ln t}{\lambda_{12}}\right), y = \min\left(\frac{-\ln s}{\lambda_2}, \frac{-\ln t}{\lambda_{12}}\right)$;
3. The desired pair is (x, y).

(b) Show that $u = \exp(-(\lambda_1+\lambda_{12})x)$ and $v = \exp(-(\lambda_2+\lambda_{12})y)$ are uniform $(0,1)$ variates whose joint distribution function is a Marshall-Olkin copula.

Solution. (a) We have

$$\begin{aligned}
&P\left(\min\left(\frac{-\ln r}{\lambda_1}, \frac{-\ln t}{\lambda_{12}}\right) > x, \min\left(\frac{-\ln s}{\lambda_2}, \frac{-\ln t}{\lambda_{12}}\right) > y\right)\\
&= P\left(\frac{-\ln r}{\lambda_1} > x, \frac{-\ln t}{\lambda_{12}} > x, \frac{-\ln s}{\lambda_2} > y, \frac{-\ln t}{\lambda_{12}} > y\right)\\
&= P(r < \exp(-\lambda_1 x), s < \exp(-\lambda_2 y), t < \exp(-\lambda_{12}\max(x,y)))\\
&= \exp(-\lambda_1 x)\exp(-\lambda_2 y)\exp(-\lambda_{12}\max(x,y)),
\end{aligned}$$

which is the Marshall-Olkin bivariate-exponential distribution.

(b) The quasi-inverses are

$$x = \frac{-\ln u}{\lambda_1+\lambda_{12}}, \quad y = \frac{-\ln v}{\lambda_2+\lambda_{12}}.$$

Then

$$\begin{aligned}
\hat{C}(u,v) &= \exp\left(-(\lambda_1+\lambda_{12})\frac{-\ln u}{\lambda_1+\lambda_{12}}\right)\exp\left(-(\lambda_2+\lambda_{12})\frac{-\ln v}{\lambda_2+\lambda_{12}}\right)\\
&\quad\cdot \min\left\{\exp\left(\lambda_{12}\frac{-\ln u}{\lambda_1+\lambda_{12}}\right), \exp\left(\lambda_{12}\frac{-\ln v}{\lambda_2+\lambda_{12}}\right)\right\}\\
&= uv\min(u^{-\alpha}, v^{-\beta}),
\end{aligned}$$

where $\alpha = \lambda_{12}/(\lambda_1+\lambda_{12}), \beta = \lambda_{12}/(\lambda_2+\lambda_{12})$. □

Exercise 3.5 Let (X,Y) be random variables with circular uniform distribution. Find the distribution of $\max(X,Y)$.

Solution. The distribution function of $\max(X,Y)$ is

$$P(\max(X,Y)\le t) = H(t,t) = \begin{cases} 3/4, & x^2+y^2 \le 1,\\ 1, & x^2+y^2 > 1, x, y \ge 0,\\ 1-\frac{\arccos t}{\pi}, & x^2+y^2 > 1, \min(x,y) < 0 \le \max(x,y),\\ 0, & x^2+y^2 > 1, x, y < 0.\end{cases}$$

□

Exercise 3.6 Let Z_1, Z_2, Z_3 be three mutually independent exponential random variables with parameter $\lambda > 0$, and let J be a Bernoulli random variable, independent with Z's, with parameter θ in $(0,1)$. Set

$$X = (1-\theta)Z_1 + JZ_3, \quad Y = (1-\theta)Z_2 + JZ_3.$$

Show that

(a) for $x, y \geq 0$, the joint survival function of X and Y is given by

$$\overline{H}(x,y) = \exp[-\lambda(x \vee y)] + \frac{1-\theta}{1+\theta} \exp\left[\frac{-\lambda(x+y)}{1-\theta}\right] \left(1 - \exp\left[\lambda \frac{1+\theta}{1-\theta}(x \vee y)\right]\right).$$

(b) X, Y are exponential with parameter λ;

(c) the survival copula of X, Y is given by

$$\hat{C}_\theta(u,v) = M(u,v) + \frac{1-\theta}{1+\theta}(uv)^{1/(1-\theta)} \left(1 - [\max(u,v)]^{-(1+\theta)/(1-\theta)}\right).$$

(d) $\hat{C}_\theta$ is absolutely continuous, $\hat{C}_0 = \Pi, \hat{C}_1 = M$.

Solution. (a)

(b) □

Exercise 3.7

Solution. □

Exercise 3.35 For Plackett family of copulas, show that

(a) $C_0(u,v) = \lim_{\theta \to 0^+} C_\theta(u,v) = \frac{(u+v-1)+|u+v-1|}{2} = W(u,v)$,

(b) $C_\infty(u,v) = \lim_{\theta \to \infty} C_\infty(u,v) = \frac{(u+v)-|u-v|}{2} = M(u,v)$.

Solution. (a) We have

$$\begin{aligned}
&\lim_{\theta \to 0} \frac{[1+(\theta-1)(u+v)] - \sqrt{[1+(\theta-1)(u+v)]^2 - 4uv\theta(\theta-1)}}{2(\theta-1)} \\
&= \frac{1-(u+v) - |1-(u+v)|}{-2} \\
&= \frac{(u+v-1)+|u+v-1|}{2} = W(u,v).
\end{aligned}$$

(b)

$$\begin{aligned}
&\lim_{\theta\to\infty}\frac{[1+(\theta-1)(u+v)]-\sqrt{[1+(\theta-1)(u+v)]^2-4uv\theta(\theta-1)}}{2(\theta-1)}\\
&=\lim_{\theta\to\infty}\frac{[1/(\theta-1)+(u+v)]-\sqrt{[1/(\theta-1)+(u+v)]^2-4uv\theta/(\theta-1)}}{2}\\
&=\frac{(u+v)-|u-v|}{2}=M(u,v).
\end{aligned}$$

□

Exercise 3.36 Let C_θ be a member of the Plackett family of copulas, where θ is in $(0,\infty)$.

(a) Show that $C_{1/\theta}(u,v)=u-C_\theta(u,1-v)=v-C_\theta(1-u,v)$.

(b) Conclude that C_θ satisfies the functional equation $C=\hat{C}$ for radial symmetry.

Solution. (a) Write

$$\begin{aligned}
&C_{1/\theta}(u,v)\\
&=\frac{[1+(1/\theta-1)(u+v)]-\sqrt{[1+(1/\theta-1)(u+v)]^2-4uv1/\theta(1/\theta-1)}}{2(1/\theta-1)}\\
&=\frac{[\theta+(1-\theta)(u+v)]-\sqrt{[\theta+(1-\theta)(u+v)]^2-4uv(1-\theta)}}{2(1-\theta)}\\
&=\frac{2u(1-\theta)+1-(1-\theta)(u+1-v)}{2(1-\theta)}\\
&\quad+\frac{\sqrt{[2u(1-\theta)+1-(1-\theta)(u+1-v)]^2-4uv(1-\theta)}}{2(1-\theta)}\\
&=\frac{2u(1-\theta)+1-(1-\theta)(u+1-v)}{2(1-\theta)}\\
&\quad+\frac{\sqrt{[1+(\theta-1)(u+1-v)]^2+4u^2(1-\theta)^2+4u(1-\theta)[1+(\theta-1)(u+1-v)]-4uv(1-\theta)}}{2(1-\theta)}\\
&=u-\frac{[1+(\theta-1)(u+1-v)]-\sqrt{[1+(\theta-1)(u+1-v)]^2-4u(1-v)\theta(\theta-1)}}{2(\theta-1)}.
\end{aligned}$$

Also fit for $\theta=1$.

(b) Plackett family is radially symmetric,

$$
\begin{aligned}
&C(u,v)\\
&=\frac{[1+(\theta-1)(u+v)]-\sqrt{[1+(\theta-1)(u+v)]^2-4uv\theta(\theta-1)}}{2(\theta-1)}\\
&=\frac{2(\theta-1)(u+v-1)+[1+(\theta-1)(2-u-v)]}{2(\theta-1)}\\
&\quad-\frac{\sqrt{[2(\theta-1)(u+v-1)+1+(\theta-1)(2-u-v)]^2-4uv\theta(\theta-1)}}{2(\theta-1)}\\
&=u+v-1+\frac{[1+(\theta-1)(2-u-v)]-\sqrt{[1+(\theta-1)(2-u-v)]^2-4(1-u)(1-v)\theta(\theta-1)}}{2(\theta-1)}\\
&=u+v-1+C(1-u,1-v)=\hat{C}(u,v).
\end{aligned}
$$

Also fit for $\theta=1$. □

Exercise 3.37 Show that the Plackett family is positively ordered.

Solution. Let $0<\theta_1\le\theta_2<1$,

$$C_\theta=\frac{[1+(\theta-1)(u+v)]-\sqrt{[1+(\theta-1)(u+v)]^2-4uv\theta(\theta-1)}}{2(\theta-1)},$$

then the derivative is

$$
\begin{aligned}
\frac{\partial C_\theta}{\partial\theta}&=\frac{\{(u+v)-1/2A^{-1/2}(2(u+v)[1+(\theta-1)(u+v)]-4uv(2\theta-1))\}2(\theta-1)}{[2(\theta-1)]^2}\\
&\quad-\frac{2\{[1+(\theta-1)(u+v)]-\sqrt{[1+(\theta-1)(u+v)]^2-4uv\theta(\theta-1)}\}}{[2(\theta-1)]^2}\\
&\ge 0.
\end{aligned}
$$

□

Exercise 3.38 Show that the following algorithm generates random variates (u,v) from Plackett distribution with parameter θ:

1. Generate two independent uniform $(0,1)$ variates u,t;

2. Set $a=t(1-t); b=\theta+a(\theta-1)^2; c=2a(u\theta^2+1-u)+\theta(1-2a)$; and $d=\sqrt{\theta}\cdot\sqrt{\theta+4au(1-u)(1-\theta)^2}$;

3. Set $v=[c-(1-2t)d]/2b$;

4. The desired pair is (u, v).

Solution. We only need to show that

$$[c-(1-2t)d]/2b = c_u^{(-1)}(t),$$

where $c_u(t) = P(V \leq t | U = u) = \frac{\partial C(u,v)}{\partial u}$. That is

$$\frac{\partial C(u,v)}{\partial u} = \frac{1}{2} - \frac{1}{\sqrt{[1+(\theta-1)(u+v)]^2 - 4uv\theta(\theta-1)}} \cdot [1+(\theta-1)(u+v)-2v\theta],$$

The associated quasi-inverse is

$$c_u^{(-1)}(v) =$$

□

Exercise 3.39

Solution. □

Exercise 3.40 Let C_θ denote a member of the Ali-Mikhail-Haq family. Show that

$$C_\theta(u,v) = uv\sum_{k=0}^{\infty}[\theta(1-u)(1-v)]^k$$

and hence

Solution. □

Exercise 3.41 (a) Show that the harmonic mean of two Ali-Mikhail-Haq copulas is again an Ali-Mikhail-Haq copula.

(b) Show that each Ali-Mikhail-Haq copula is a weighted harmonic mean of the two extreme members of the family, *i.e.*, for all $\theta \in [-1, 1]$,

$$C_\theta(u,v) = \frac{1}{\frac{1-\theta}{2}\cdot\frac{1}{C_{-1}(u,v)} + \frac{1+\theta}{2}\cdot\frac{1}{C_{+1}(u,v)}}.$$

Solution. (a) Let C_α, C_β be Ali-Mikhail-Haq copulas, then

$$\begin{aligned}\frac{2}{\frac{1}{C_\alpha(u,v)} + \frac{1}{C_\beta(u,v)}} &= \frac{2uv}{1-\alpha(1-u)(1-v)+1-\beta(1-u)(1-v)} \\ &= \frac{2uv}{2-(\alpha+\beta)(1-u)(1-v)} = C_{(\alpha+\beta)/2}.\end{aligned}$$

(b) Write

$$\frac{1}{\frac{1-\theta}{2}\cdot\frac{1}{C_{-1}(u,v)}+\frac{1+\theta}{2}\cdot\frac{1}{C_{+1}(u,v)}} = \frac{2uv}{(1-\theta)[1+(1-u)(1-v)]+(1+\theta)[1-(1-u)(1-v)]}$$
$$= \frac{uv}{1-\theta(1-u)(1-v)} = C_\theta(u,v).$$

□

Exercise 3.42 Show that the following algorithm generates random variates (u, v) from an Ali-Mikhail-Haq distribution with parameter θ:

1. Generate two independent uniform $(0, 1)$ variates u, t;
2. Set $a = 1-u; b = -\theta(2at+1)+2\theta^2a^2t+1$; and $c = \theta^2(4a^2t-4at+1)-\theta(4at-4t+2)+1$;
3. Set $v = 2t(a\theta-1)^2/(b+\sqrt{c})$;
4. The desired pair is (u, v).

Solution. Recall that Ali-Mikhail-Haq copula is

$$C_\theta(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}$$

for $\theta \in [-1, 1]$. Thus

$$c_u(v) = \frac{v - v\theta + v^2\theta}{(1-\theta(1-u)(1-v))^2}.$$

The associated quasi-inverse is

$$c_u^{(-1)}(v) =$$

□

Exercise 4.1 [Theorem 4.1.5] Let C be an Archimedean copula with generator φ. Then:

1. C is symmetric; *i.e.* $C(u, v) = C(v, u)$ for all $u, v \in \mathrm{I}$;
2. C is associative; *i.e.*, $C(C(u, v), w) = C(u, C(v, w))$ for all $u, v, w \in \mathrm{I}$;
3. If $c > 0$ is any constant, then $c\varphi$ is also a generator of C.

Solution. 1. Since

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)) = \varphi^{[-1]}(\varphi(v) + \varphi(u)) = C(v,u).$$

2. Write

$$\begin{aligned} C(C(u,v),w) &= \varphi^{[-1]}(\varphi[C(u,v)] + \varphi(w)) \\ &= \varphi^{[-1]}(\varphi[\varphi^{[-1]}(\varphi(u) + \varphi(v))] + \varphi(w)) \\ &= \varphi^{[-1]}(\varphi(u) + \varphi(v) + \varphi(w)) \\ &= \varphi^{[-1]}(\varphi(u) + \varphi[\varphi^{[-1]}(\varphi(v) + \varphi(w))]) \\ &= C(u, C(v,w)). \end{aligned}$$

If $\varphi(u) + \varphi(v) \geq \varphi(0)$ or $\varphi(v) + \varphi(w) \geq \varphi(0)$, $C(C(u,v),w) = C(u,C(v,w)) = 0$.

3. Write

$$\begin{aligned} c\varphi^{[-1]}(c\varphi(u) + c\varphi(v)) &= c\varphi^{[-1]}[c(\varphi(u) + \varphi(v))] \\ &= \varphi^{[-1]}\left[\frac{1}{c} \cdot c(\varphi(u) + \varphi(v))\right] \\ &= \varphi^{[-1]}(\varphi(u) + \varphi(v)). \end{aligned}$$

If $\varphi(u) + \varphi(v) \geq \varphi(0)$, then $c(\varphi(u) + \varphi(v)) \geq c\varphi(0)$, it implies that $c\varphi^{[-1]}[c(\varphi(u) + \varphi(v))] = 0 = C(u,v)$. □

Exercise 4.2 The diagonal section of an Archimedean copula C with generator ϕ in Ω is given by $\delta_C(u) = \varphi^{[-1]}[2\varphi(u)]$. Prove that if C is Archimedean, then for $u \in (0,1), \delta_C(u) < u$. Conclude that M is not Archimedean copula.

Solution. If $2\varphi(u) \geq \varphi(0)$, then $\delta_C(u) = 0 < u$. If $2\varphi(u) < \varphi(0)$, since $\varphi^{[-1]}$ is decreasing, $\delta_C(u) = \varphi^{[-1]}[2\varphi(u)] < \varphi^{[-1]}[\varphi(u)] = u$. The diagonal section of M is

$$\delta_M(u) = \min(u,u) = u,$$

thus M is not Archimedean. □

Exercise 4.3 Show that $\varphi : \mathrm{I} \to [0, \infty]$ is in Ω iff $1-\varphi^{[-1]}(t)$ is a unimodal distribution function on $[0, \infty]$ with mode at zero.

Solution. The corresponding density function is

$$\frac{d(1-\varphi^{[-1]}(t))}{dt} = -\frac{1}{\varphi'(\varphi^{[-1]}(t))}. \tag{2.0.3}$$

This distribution is unimodal at zero iff the density function is decreasing on $[0, \infty]$ and

$$\frac{d(-1/\varphi'(\varphi^{[-1]}(t)))}{dt} = \frac{\varphi''(\varphi^{[-1]}(t))}{\varphi'(\varphi^{[-1]}(t))} < 0.$$

Sine $\varphi'(\varphi^{[-1]}(t)) < 0$ by (2.0.3), $\varphi''(\varphi^{[-1]}(t)) > 0$. Thus φ is convex. And we need $\varphi'(\varphi^{[-1]}(t))$ is a decreasing function of t. Thus it is iff φ is strictly decreasing and convex with $\varphi(1) = 0$. Since if $\varphi(1) = a > 0$, $1 - \varphi^{[-1]}(0) < 1 - \varphi^{[-1]}(a) = 0$, which is impossible. □

Exercise 4.4 Show that non-Archimedean copulas can have

(a) non-convex level curves;

(b) convex level curves.

Solution. □

Exercise 4.5 Let C be an Archimedean copula. Prove that C is strict if and only if $C(u, v) > 0$ for $(u, v) \in (0, 1]^2$.

Solution. We need to prove that $\varphi(0) = \infty$ iff $C(u, v) > 0$ for $(u, v) \in (0, 1]^2$. That is, for all $(u, v) \in (0, 1]^2$,

$$\begin{aligned}\varphi^{[-1]}(\varphi(u) + \varphi(v)) > 0 &\Leftrightarrow 0 \leq \varphi(u) + \varphi(v) < \varphi(0) \\ &\Leftrightarrow \varphi(0) = \infty.\end{aligned}$$

□

Exercise 4.6 This exercise shows that different Archimedean copulas can have the same zero set. Let

Solution. □

Exercise 4.7

Solution. □

Exercise 4.8

Solution. □

Exercise 4.9

Solution. □

Exercise 4.10

Solution. □

Exercise 4.11

Solution. □

Exercise 4.12

Solution. □

Exercise 4.13

Solution. □

Exercise 4.14

Solution. □

Exercise 4.15

Solution. □

Exercise 4.16

Solution. □

Exercise 4.17 Show that the following algorithm generates random variates (u, v) whose joint distribution function is the Clayton copula with parameter $\theta > 0$:

1.
2.
3.

Solution. □

Exercise 4.18

Solution. □

Exercise 4.19

Solution. □

Exercise 4.20

Solution. □

Exercise 4.21

Solution. □

Exercise 4.22

Solution. □

Exercise 4.23

Solution. □

Exercise 4.24

Solution. □

Exercise 4.25

Solution. □

Exercise 5.1 [Corollary 5.1.2.] 1. Q is symmetric in its arguments: $Q(C_1, C_2) = Q(C_2, C_1)$.

2. Q is non-decreasing in each argument: if $C_1 \prec C_1'$ and $C_2 \prec C_2'$ for all $(u,v) \in \mathrm{I}^2$, then $Q(C_1, C_2) \leq Q(C_1', C_2')$.

3. Copulas can be replaced by survival copulas in Q, *i.e.*, $Q(C_1, C_2) = Q(\hat{C}_1, \hat{C}_2)$.

Solution. 1. We have

$$\begin{aligned}
Q(C_1, C_2) &= 4 \iint_{\mathrm{I}^2} C_2(u,v) dC_1(u,v) - 1 \\
&= 2P((X_1 - X_2)(Y_1 - Y_2) > 0) - 1 \\
&= 2\left[P(X_1 < X_2, Y_1 < Y_2) + P(X_1 > X_2, Y_1 > Y_2)\right] - 1 \\
&= 2\left[\iint_{\mathbb{R}^2} P(X_1 < x, Y_1 < y) dC_2(F(x), G(y)) \right. \\
&\quad \left. + \iint_{\mathbb{R}^2} P(X_1 > x, Y_1 > y) dC_2(F(x), G(y))\right] - 1 \\
&= 2\left[\iint_{\mathbb{R}^2} P(X_1 < x, Y_1 < y) dC_2(F(x), G(y)) \right. \\
&\quad \left. + \iint_{\mathbb{R}^2} \{1 - F(x) - G(y) + P(X_1 < x, Y_1 < y)\} dC_2(F(x), G(y))\right] - 1
\end{aligned}$$

$$= 4 \iint_{\mathbf{I}^2} C_1(u,v) dC_2(u,v) - 1$$
$$= Q(C_2, C_1).$$

2. This is trivial by the definition.
3. For any H, it is true that

$$\iint H(x,y)dH(x,y) = \iint \overline{H}(x,y)dH(x,y) = \iint \overline{H}(x,y)d\overline{H}(x,y).$$

□

Exercise 5.2 Let X, Y be r.v.'s with the Marshall-Olkin bivariate exponential distribution with parameters $\lambda_1, \lambda_2, \lambda_{12}$, that is, the survival function is given by

$$\overline{H}(x,y) = \exp(-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x,y)).$$

(a) Show that the ordinary Pearson product-moment correlation coefficient of X, Y is given by

$$\frac{\lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}.$$

(b) Show that Kendall's tau and Pearson's product-moment correlation coefficient are numerically equal for members of this family.

Solution. (a) In Marshall's paper, these moments were calculated using MGF.

(b) From Example 5.5.,

$$\begin{aligned}
\tau_{\alpha,\beta} &= \frac{\alpha\beta}{\alpha - \alpha\beta + \beta} \\
&= \frac{\frac{\lambda_{12}}{\lambda_1+\lambda_{12}} \cdot \frac{\lambda_{12}}{\lambda_2+\lambda_{12}}}{\frac{\lambda_{12}}{\lambda_1+\lambda_{12}} - \frac{\lambda_{12}^2}{(\lambda_1+\lambda_{12})(\lambda_2+\lambda_{12})} + \frac{\lambda_{12}}{\lambda_2+\lambda_{12}}} \\
&= \frac{\lambda_{12}^2}{\lambda_{12}(\lambda_2+\lambda_{12}) - \lambda_{12}^2 + \lambda_{12}(\lambda_1+\lambda_{12})} \\
&= \frac{\lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}.
\end{aligned}$$

□

Exercise 5.3 Prove that an alternate expression for Kendall's tau for an Archimedean copula C with generator φ is

$$\tau_C = 1 - 4\int_0^\infty u\left[\frac{d}{du}\varphi^{[-1]}(u)\right]^2 du.$$

Solution. For Archimedean copula C,

$$\tau_C = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt.$$

Then let $\varphi(t) = u$, one has $d\varphi^{[-1]}(u)/du = 1/\varphi'(t)$ if $u \leq \varphi(0)$, if $u \geq \varphi(0)$, $d\varphi^{[-1]}(u)/du = 0$. Thus

$$\begin{aligned}
\tau_C &= 1 + 4\int_{\varphi(0)}^0 u\frac{d}{du}\varphi^{[-1]}(u)d\varphi^{[-1]}(u) \\
&= 1 - 4\int_0^{\varphi(0)} u\left[\frac{d}{du}\varphi^{[-1]}(u)\right]^2 du \\
&= 1 - 4\int_0^\infty u\left[\frac{d}{du}\varphi^{[-1]}(u)\right]^2 du.
\end{aligned}$$

□

Exercise 5.4 (a) Let $C_\theta, \theta \in [0,1]$ be a member of

(b)

Solution. □

Exercise 5.5 Let C be a diagonal copula, that is, $C(u,v) = \min(u, v, (1/2)[\delta(u) + \delta(v)])$.

(a) Show the Kendall's tau is given by

$$\tau_C = 4\int_0^1 \delta(t)dt - 1.$$

(b) For diagonal

Solution. (a) Write

$$C(u,u) = \min(u, \delta(u)) = \delta(u),$$

since □

Exercise 5.6

Solution. □

Exercise 5.7

Solution. □

Exercise 5.8 Let C_θ be a member of the Plackett family

$$a$$

for $\theta > 0$. Show that Spearman's rho for this C_θ is

$$\rho_\theta = \frac{\theta+1}{\theta-1} - \frac{2\theta}{(\theta-1)^2}\ln\theta.$$

There does not appear to be a closed form expression for Kendall's τ for members of this family.

Solution. □

Exercise 5.9 Let $C_\theta, \theta \in \overline{\mathbb{R}}$ be a member of the Frank family. Show that

$$\tau_\theta = 1 - \frac{4}{\theta}[1 - D_1(\theta)], \quad \rho_\theta = 1 - \frac{12}{\theta}[D_1(\theta) - D_2(\theta)],$$

where $D_k(x)$ is the Debye function, which is defined for any positive integer k by

$$D_k(x) = \frac{k}{x^k}\int_0^x \frac{t^k}{e^t - 1}dt.$$

Solution. □

Exercise 5.10

Solution. □

Exercise 5.11

Solution. □

Exercise 5.12

Solution. □

Exercise 5.13

Solution. □

Exercise 5.14

Solution. □

Exercise 5.15

Solution. □

Exercise 5.16 Let X, Y be continuous random variables with copula C. Show that an alternate expression for Spearman's rho for X, Y is

$$\rho = 3 \iint_{\mathbf{I}^2} \left([u+v-1]^2 - [u-v]^2\right) dC(u,v).$$

Solution. Write

$$\begin{aligned}
\rho &= 3 \iint_{\mathbf{I}^2} \left([u+v-1]^2 - [u-v]^2\right) dC(u,v) \\
&= 3 \iint_{\mathbf{I}^2} \left((v-1)^2 + 2uv - 2u - v^2 + 2uv\right) dC(u,v) \\
&= 3 \iint_{\mathbf{I}^2} (1 - 2v - 2u + 4uv)\, dC(u,v) \\
&= 12 \iint uv dC(u,v) - 3.
\end{aligned}$$

□

Exercise 5.17 Let X and Y be continuous random variables with copula C. Establish the following inequalities between Blomqvist's β and Kendall's τ, Spearman's ρ, and Gini's γ:

$$\begin{aligned}
\frac{1}{4}(1+\beta)^2 - 1 \le \tau \le 1 - \frac{1}{4}(1-\beta)^2, \\
\frac{3}{16}(1+\beta)^3 - 1 \le \rho \le 1 - \frac{3}{16}(1-\beta)^3, \\
\frac{3}{8}(1+\beta)^2 - 1 \le \gamma \le 1 - \frac{3}{8}(1-\beta)^2.
\end{aligned}$$

Solution. □

Exercise 5.18

Solution. □

Exercise 5.19

Solution. □

Exercise 5.20 Let X, Y be continuous random variables whose copula C satisfies one (or both) of the functional equations

$$C(u,v) = u - C(u,1-v), \quad C(u,v) = v - C(1-u,v).$$

for joint symmetry. Show that

$$\tau_{X,Y} = \rho_{X,Y} = \gamma_{X,Y} = \beta_{X,Y} = 0.$$

Solution. Clearly, $C(1/2,1/2) = 1/2 - C(1/2,1/2) \implies C(1/2,1/2) = 1/4$,

$$\beta_{X,Y} = 4C(1/2,1/2) - 1 = 1 - 1 = 0.$$

Since these four are measures of concordance,

$$\kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}.$$

From the monotonic transformation,

$$C(u,v) = u - C(u,1-v), \quad C(u,v) = v - C(1-u,v) \implies C_{X,Y} = C_{X,-Y}, \quad C_{X,Y} = C_{-X,Y}.$$

Thus

$$\kappa_{X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}, \quad \kappa_{X,Y} = \kappa_{-X,Y} = -\kappa_{X,Y}.$$

Therefore, $\kappa_{X,Y} = 0$ as its range is $[-1,1]$. □

Exercise 5.21 Another measure of association between two variates is **Spearman's foot-rule**, for which the sample version is

$$f = 1 - \frac{3}{n^2 - 1} \sum_{i=1}^{n} |p_i - q_i|,$$

where p_i, q_i denote the ranks of a sample of size n of two continuous random variables X, Y.

(a) Show that the population version of the foot-rule, which is

$$\phi = 1 - 3 \iint_{\mathrm{I}^2} |u - v| dC(u,v) = \frac{1}{2}\left[3Q(C,M) - 1\right]$$

(b) Show that ϕ fails to satisfy

$$-1 \leq \kappa_{X,Y} \leq 1, \kappa_{X,X} = 1, \kappa_{X,-X} = -1,$$

and

$$\kappa_{-X,Y} = \kappa_{X,-Y} = -\kappa_{X,Y}.$$

Hence it is not a "measure of concordance".

Solution. (a) Rewrite f as

$$\begin{aligned} f &= 1 - \frac{3n^2}{n^2-1}\left[\sum_{i=1}^{n}\left|\frac{p_i}{n} - \frac{q_i}{n}\right|\right] \cdot \frac{1}{n} \\ &= 1 - 3\mathbb{E}[|U-V|] \\ &= 1 - 3\iint_{\mathrm{I}^2} |u-v| dC(u,v). \end{aligned}$$

Recall that

$$\begin{aligned} Q(C,M) &= 4\iint_{\mathrm{I}^2} M(u,v)dC(u,v) - 1 \\ &= 2\iint_{\mathrm{I}^2} [u+v-|u-v|]dC(u,v) - 1 \\ &= 1 - 2\iint_{\mathrm{I}^2} |u-v|dC(u,v). \end{aligned}$$

Hence

$$f = 1 - \frac{3}{2}\left(1 - Q(C,M)\right) = \frac{1}{2}[3Q(C,M) - 1].$$

(b) Since $Q(C,M) \in [0,1]$, $f \in [-1/2, 1]$. We can never attain -1 for f. Besides,

$$\begin{aligned} \frac{3}{2}Q(C_{X,-Y}, M) - \frac{1}{2} &= 6\int_{\mathrm{I}} u - C(u,1-u)du - \frac{3}{2} - \frac{1}{2} \\ &= 1 - 6\int_{\mathrm{I}} C(u,1-u)du \\ &\neq \frac{1}{2} - 6\int_{\mathrm{I}} C(u,u)du + \frac{3}{2} \\ &= \frac{1}{2} - \frac{3}{2}Q(C_{X,Y}, M). \end{aligned}$$

□

Exercise 5.22 (a) Show that

$$P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y)$$

and

$$P(X > x, Y > y) \geq P(X > x)P(Y > y)$$

are equivalent.

(b) Show that

$$H(x, y) \geq F(x)G(y), \quad \forall (x, y) \in \mathbb{R}^2$$

is equivalent to

$$\overline{H}(x, y) \geq \overline{F}(x)\overline{G}(y), \quad \forall (x, y) \in \mathbb{R}^2.$$

Solution. (a) Write

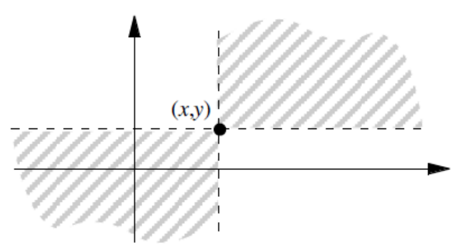
$$\begin{aligned}
&P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y) \\
&\quad \Leftrightarrow 1 - P(X > x) - P(Y > y) + P(X > x, Y > y) \geq (1 - P(X > x))(1 - P(Y > y)) \\
&\quad \Leftrightarrow P(X > x, Y > y) \geq P(X > x)P(Y > y).
\end{aligned}$$

(b) It is directly from part (a). □

Exercise 5.23 (a) Let X, Y be random variables with joint distribution function H and margins F and G. Show that PQD(X, Y) iff for any $(x, y) \in \mathbb{R}^2$,

$$H(x, y)[1 - F(x) - G(y) + H(x, y)] \geq [F(x) - H(x, y)][G(y) - H(x, y)],$$

that is the product of two probabilities corresponding to the two shaded quadrants is at least as great as the unshaded quadrant in the following figure:



(b) Give an interpretation of quadrant dependence in terms of the cross product ratio

$$\theta = \frac{H(x,y)[1 - F(x) - G(y) + H(x,y)]}{[F(x) - H(x,y)][G(y) - H(x,y)]}$$

for continuous random variables.

(c) In copula notation,

$$C(u,v)[1 - u - v + C(u,v)] \geq [u - C(u,v)][v - C(u,v)].$$

Solution. (a) From Exercise 5.22,

$$\begin{aligned} H(x,y)[1 - F(x) - G(y) + H(x,y)] &\geq [1 - F(x)][1 - G(y)]F(x)G(y) \\ &= [G(y) - F(x)G(y)][F(x) - F(x)G(y)] \\ &\geq [G(y) - H(x,y)][F(x) - H(x,y)]. \end{aligned}$$

(b) If $\theta \geq 1$, X, Y are positive quadrant dependent, if $\theta \leq 1$, X, Y are negative quadrant dependent.

(c) The interpretation is the product of two probabilities for (U, V) corresponding to the two shaded quadrants is at least as great as the unshaded quadrant. $\square$

Exercise 5.24 (a) Show that if X, Y are PQD, then $-X, Y$ are NQD, X and $-Y$ are NQD, and $-X, -Y$ are PQD.

(b) Show that if C is the copula of PQD random variables, then so is $\hat{C}$.

Solution. (a) If X, Y are PQD,

$$P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y).$$

One has

$$\begin{aligned} P(-X \leq x, Y \leq y) &= P(X \geq -x, Y \leq y) \\ &= P(Y \leq y) - P(X \leq -x, Y \leq y) \\ &= G(Y) - H(-x, y). \end{aligned}$$

Since PQD,

$$H(-x,y) \geq F(-x)G(y),$$

then

$$\begin{aligned}
P(-X \leq x, Y \leq y) &= G(y) - H(-x,y) \\
&\leq G(y) - F(-x)G(y) \\
&= (1 - F(-x))G(y) \\
&= P(-X \leq x)P(Y \leq y).
\end{aligned}$$

The others are in the same fashion.

(b) The PQD is

$$C(u,v) \geq uv.$$

From part (a), $-X, -Y$ are also PQD,

$$C_{-X,-Y}(u,v) \geq uv \Leftrightarrow \hat{C}_{X,Y}(u,v) \geq uv.$$

□

Exercise 5.25 Consider the random variable $Z = H(X,Y) - F(X)G(Y)$.

(a) Show that $\mathbb{E}[Z] = (3\tau_C - \rho_C)/12$.

(b) Show that $\omega_C = 6\mathbb{E}[Z] = (3\tau_C - \rho_C)/2$ can be interpreted as a measure of "expected" quadrant dependence for which $\omega_M = 1$, $\omega_\Pi = 0, \omega_W = -1$.

(c) Show that ω_C fails to be a measure of concordance.

Solution. (a) Write

$$\begin{aligned}
\mathbb{E}[Z] &= \mathbb{E}[C(U,V)] - E[UV] \\
&= \frac{\tau_C + 1}{4} - \frac{\rho_C + 3}{12} \\
&= \frac{3\tau_C - \rho_C}{12}.
\end{aligned}$$

(b) Write

$$6\mathbb{E}[Z] = 6\iint_{\mathrm{I}^2} [C(u,v) - uv]dC(u,v).$$

(c)

□

Exercise 5.26 *Hoeffding's lemma.* Let X, Y be random variables with joint distribution function H and margins F, G, such that $\mathbb{E}[|X|], \mathbb{E}[|Y|]$ and $\mathbb{E}[|XY|]$ are finite. Prove that

$$\mathbb{C}\mathrm{ov}(X,Y) = \iint_{\mathbb{R}^2} [H(x,y) - F(x)G(y)]dxdy.$$

Solution. □

Exercise 5.27 Let X, Y be random variables. Show that if $\mathrm{PQD}(X,Y)$, then $\mathbb{C}\mathrm{ov}(X,Y) \geq 0$, and hence Pearson's product-moment correlation coefficient is on-negative for positively quadrant dependent random variables.

Solution. This is directly from Exercise 5.26. □

Exercise 5.28 Show that X, Y are PQD iff $\mathbb{C}\mathrm{ov}[f(X), g(Y)] \geq 0$ for all functions f, g that are non-decreasing in each place and for which expectations $\mathbb{E}[f(X)], \mathbb{E}[g(Y)], \mathbb{E}[f(X)g(Y)]$ exist.

Solution. □

Exercise 5.29 Prove that if the copula of X, Y are max-stable, then $\mathrm{PQD}(X,Y)$.

Solution. □

Exercise 5.30

Solution. □

Exercise 5.31 Let X and Y be continuous random variables whose copula is C.

(a) Show that if $C = \hat{C}$, then $\mathrm{LTD}(Y|X)$ iff $\mathrm{RTI}(Y|X)$, and $\mathrm{LTD}(X|Y)$ iff $\mathrm{RTI}(X|Y)$.

(b) Show that if C is symmetric, then $\mathrm{LTD}(Y|X)$ iff $\mathrm{LTD}(X|Y)$, and $\mathrm{RTI}(Y|X)$ iff $\mathrm{RTI}(X|Y)$.

Solution. (a) $\mathrm{LTD}(Y|X)$, if and only if $C(u,v)/u$ is non-increasing in u. That is

$$\frac{\frac{\partial C(u,v)}{\partial u}u - C(u,v)}{u^2} \leq 0 \implies \frac{\partial C(u,v)}{\partial u}u \leq C(u,v). \tag{2.0.4}$$

RTI($Y|X$) iff $\hat{C}(1-u, 1-v)/(1-u)$ is non-decreasing in u. Assume $\hat{C} = C$, $\hat{C}(1-u, 1-v)/(1-u) = C(1-u, 1-v)/(1-u)$, then

$$\frac{\frac{\partial C(1-u,1-v)}{\partial(1-u)}(1-u)(-1) + C(1-u, 1-v)}{(1-u)^2} \geq 0$$

by (2.0.4).

(b) Assume $C(u, v) = C(v, u)$, $C(u, v)/v = C(v, u)/v$, the derivative

$$\frac{\frac{\partial C(v,u)}{\partial v}v - C(v, u)}{v^2} \leq 0 \Leftrightarrow \frac{\partial C(v, u)}{\partial v}v \leq C(v, u)$$

iff

$$\frac{\partial C(u, v)}{\partial u}u \leq C(v, u) \Leftrightarrow \frac{\frac{\partial C(u,v)}{\partial u}u - C(u, v)}{u^2} \leq 0.$$

□

Exercise 5.32

Solution. □

Exercise 5.33

Solution. □

Exercise 5.34

Solution. □

Exercise 5.35

Solution. □

Exercise 5.36 Show that (a) if the function $u - C(u, v)$ is TP_2, then LTD($Y|X$) and RTI($X|Y$);

(b) if the function $v - C(u, v)$ is TP_2, then LTD($X|Y$) and RTI($Y|X$);

(c)the function $1 - u - v + C(u, v)$ is TP_2 iff $\hat{C}$ is TP_2.

Solution. (a) $u - C(u, v)$ is TP_2, that is

$$F(x) - H(x, y) = P(X \leq x) - P(X \leq x, Y \leq y) = P(X \leq x, Y > y).$$

(b)

(c) □

Exercise 5.37

Solution. □

Exercise 5.38

Solution. □

Exercise 5.39

Solution. □

Exercise 5.40 Let X, Y be continuous random variables whose copula C is a member of a totally ordered family that include Π. Show that $\sigma_{X,Y} = |\rho_{X,Y}|$.

Solution. □

Exercise 5.41

Solution. □

Exercise 5.42

Solution. □

Exercise 5.43

Solution. □

Exercise 5.44 Show that k_p is given by

$$k_p = \frac{\Gamma(2p+3)}{2\Gamma^2(p+1)}.$$

Solution. □

Exercise 5.45 Show that the "ℓ_p" generalization of γ_C, ρ_C leads to measures of association given by

Solution. □

Exercise 5.46 Show that the "L_p" generalization of γ_C leads to measures of association given by

Solution. □

Exercise 5.47 Verify the entries for λ_U, λ_L.

Solution. □

Exercise 5.48 Write $\lambda_U(C), \lambda_L(C)$ to specify the copula under consideration. Prove that $\lambda_U(\hat{C}) = \lambda_L(C)$ and $\lambda_L(\hat{C}) = \lambda_U(C)$.

Solution. Write

$$\begin{aligned}\lambda_U(\hat{C}) &= \lim_{t\to 1^-} \frac{2-2t-1+[2t-1+C(1-t,1-t)]}{1-t} \\ &= \lim_{t\to 1^-} \frac{C(1-t,1-t)}{1-t} \\ &= \lim_{t\to 0^+} \frac{C(t,t)}{t} = \lambda_L(C).\end{aligned}$$

The other one is similar. □

Exercise 5.49

Solution. □

Exercise 5.50

Solution. □