Analyzing Chaotic Inverted Driven Pendulum through Fourier Transformation

Note: I made two versions of Fourier transformation, one continuously reiterating a periodic curve within its first period, therefore producing a spectrum with more accuracy. And the other one, the continuous version, keeps adding new data into the Fourier solver, therefore being capable of calculating the dominant frequencies in a long spectrum of chaos.

Inverted pendulum with a vertically driven pivot is one of the less studied chaotic systems. This project models a typical Duffing's oscillator, a well-known laboratory demonstration equipment of chaotic motions. In the Lagrangian equation I used to obtain an ODE solution for such a system, however, I assumed a perfect periodic movement of the oscillator's pivot with a sine function, which is impossible in reality due to the mass and inertia of the driving spring. I have implemented this chaotic model by using Easy Java Simulation and tested its accuracy by comparing my implementation against published parameters. Then I implemented Fast Time Fourier transformation model onto Ejs.'s Ordinary Differential Equation solver to acquire a periodically-updating power spectrum (SPD). The purpose of this study is to examine the changes in the patterns of inverted pendulum's movement, which varies back and forth from chaotic to periodic, through the lenses of Poincare analysis and Fourier transformation.

Previous experiments have found that the motion of a vertically-driven planar pendulum vary back and forth from chaotic to periodic depending sensitively on the initial conditions of the system. Phenomena like sub-harmonic motion and period doubling were also observed during this transition. To create the model, I needed a linear ODE that describes the periodically driven pendulum. I first solved the Lagrangian:

$$L = ml(\frac{1}{2}l\dot{\theta}^2 + d_0\omega\dot{\theta}sin\theta\, sin(\omega t) - gcos\theta + \frac{1}{2l}\omega^2 d_0^2\, sin^2(\omega t)\ - \frac{gd_0}{l}cos\,(\omega t))$$

Then, by simplifying the Euler-Lagragian Equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$, we can obtain a linear solution of the system's motion $\ddot{\theta} + \left(\frac{g}{l} + \frac{d_0\omega^2}{l}\cos(\omega t)\right)\sin(\theta)$ =0.

At the same time, the effective energy of the pendulum is given by $U_{eff} = -\frac{g}{l}\cos(\theta) + \frac{1}{4}\frac{{d_0}^2\omega^2}{l^2}\sin^2(\theta)$, which indicates that there are four stability points in this inverted pendulum system at: $\theta_s = 0, \pi, \cos^{-1}\left(\frac{-2gl}{{d_0}^2\omega^2}\right), 2\pi - \cos^{-1}\left(\frac{-2gl}{{d_0}^2\omega^2}\right)$.

For the Fourier Transformation, I implemented the Fast Fourier Transformation (FFT) package provided by Opensource Physics.

Implementations of the FFT on a simple cosine waves, saw curve and Gaussian give:

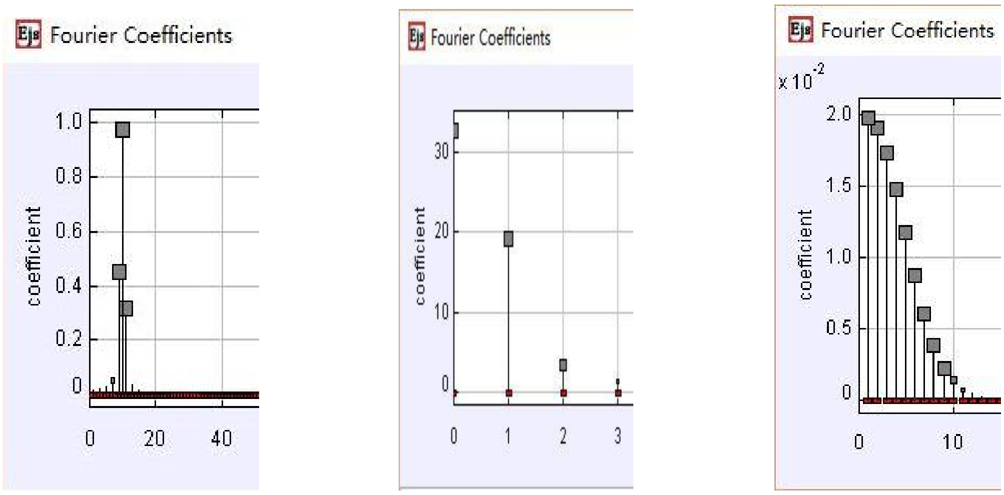


Fig.1 The Fourier Transformation of (from left to right): cosine, saw and Gaussian in Periodic Solver

Features of chaotic motion are very well studies. Obviously, the non-periodic and constantly-changing diagram of the pendulum's displacement from origin is one of them. The other one is the formation of attractors in the phase space diagram and Poincare sections at positions of low potential energy, i.e. at potential wells. (Although we have aforementioned that numerically there are four points in our system where the ∂ (Potential)=0, two of them that are not upright or downright are local maxima if all four stable points exist). In the following section, I compare the simper phenomena produced by my model with that of commonly-accepted findings.

Attractors: Trajectories centered at potential wells

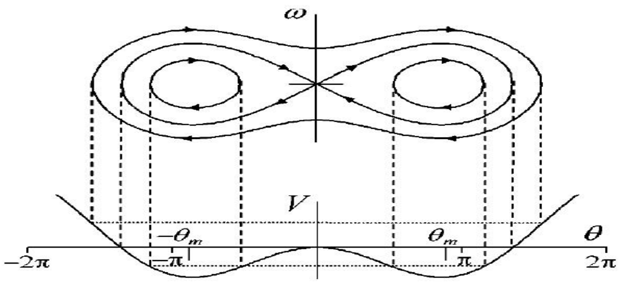
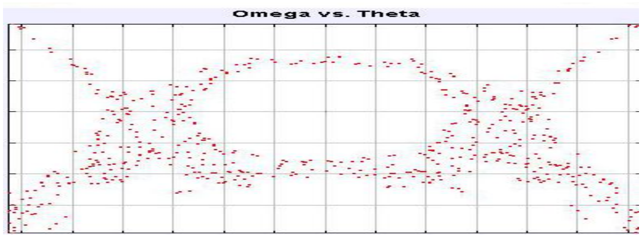


Fig.2 The theoratical illustration of attractors and potential well.



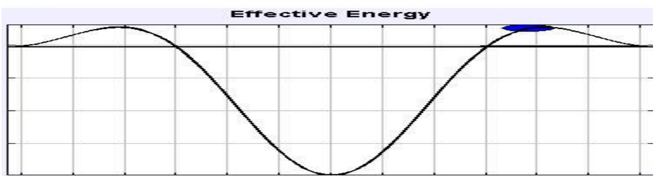


Fig.3. The simulation of this phenomenon with Poincare section.

(The scale for both graphs is from –PI to PI, please refer to the folder pics)

Chaotic Diagram of the Amplitude

Another feature of chaotic motion is its ununiform displacemen diagram. However, as I will illustrate later in Fourier transformation section, chaotic motion could also exhibit a single frequency with varying amplitude.

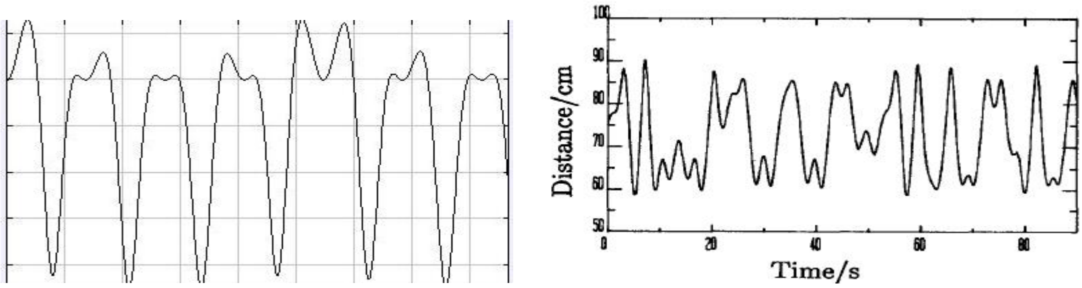


Fig.4. The simulation (left) and experimental diagram (right) of the amplitude of chaotic motion.

Formation of Attractors at the two stable points on the side.

As the amplitude and/or frequency increases, the two numerically stable points at:

$$\cos^{-1}\left(\frac{-2gl}{{d_0}^2\omega^2}\right), 2\pi - \cos^{-1}\left(\frac{-2gl}{{d_0}^2\omega^2}\right)$$

Increases in strength. Additionally, in our controlled experiment, the starting position of the pendulum is fixed, but the two stable positions shifts closer and closer to our starting point, further increasing the size and strength of the attractors as driving frequency increases. The following is a illustration of these attractors' formation.

Driving Frequency=7:

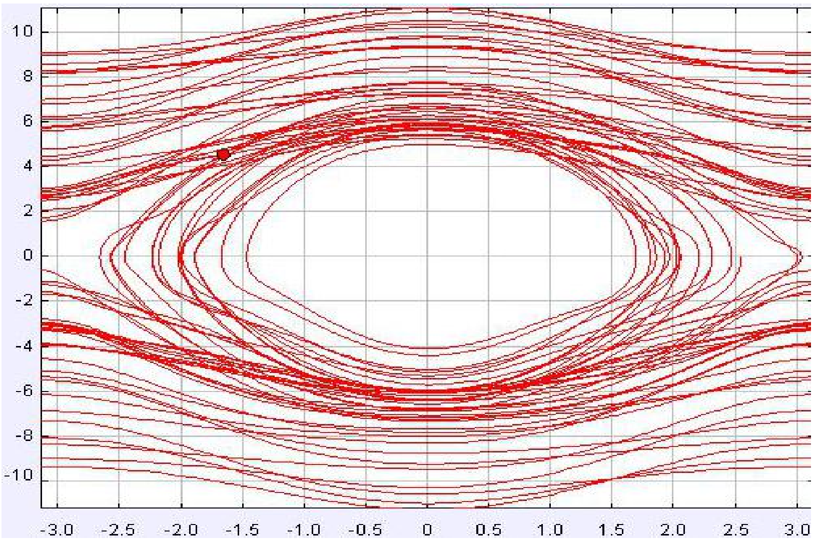


Fig.5..wd=7, no sign of attractor on the side can be observed.

Driving Frequency=22:

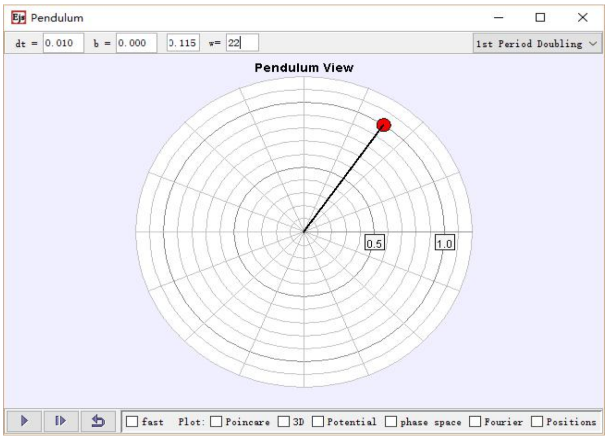


Fig.6..wd=22, no numerical stable point.

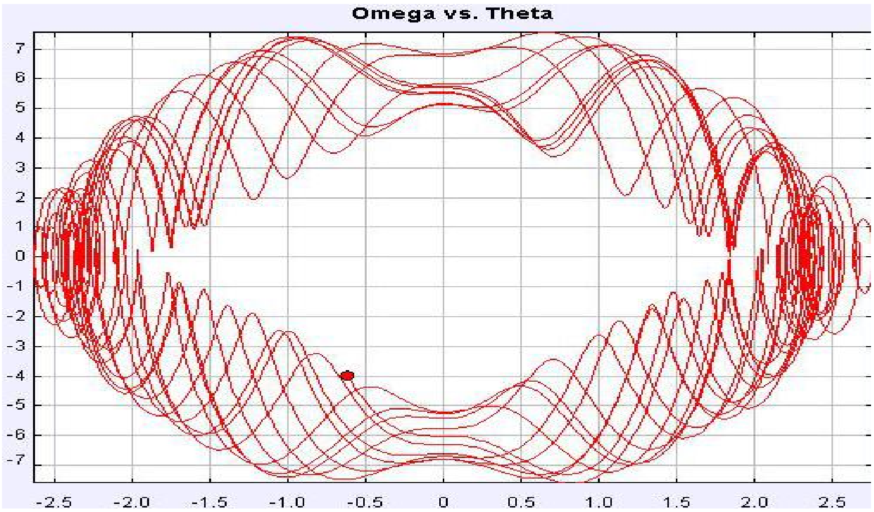


Fig.7..wd=22, however, weak signs of attactors on the side could already been observed.

Driving Frequency=65:

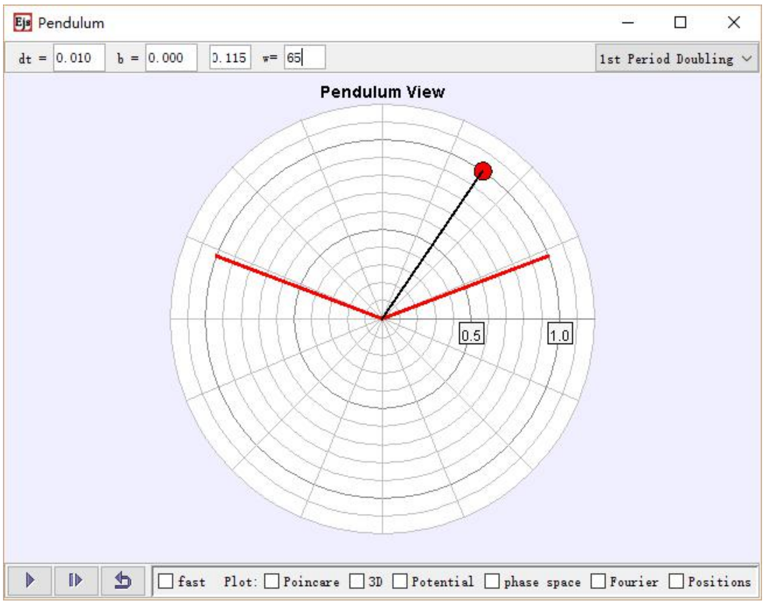


Fig.8..wd=65, two red bars are the two stable positions on the side.

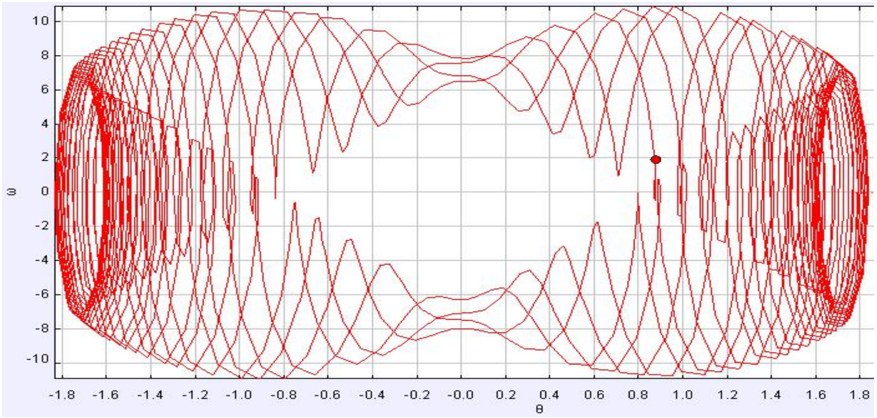


Fig.9..wd=65, two conspicuous attractos from the two stable positions are formed.

Period Doubling and 3D view of the trail:

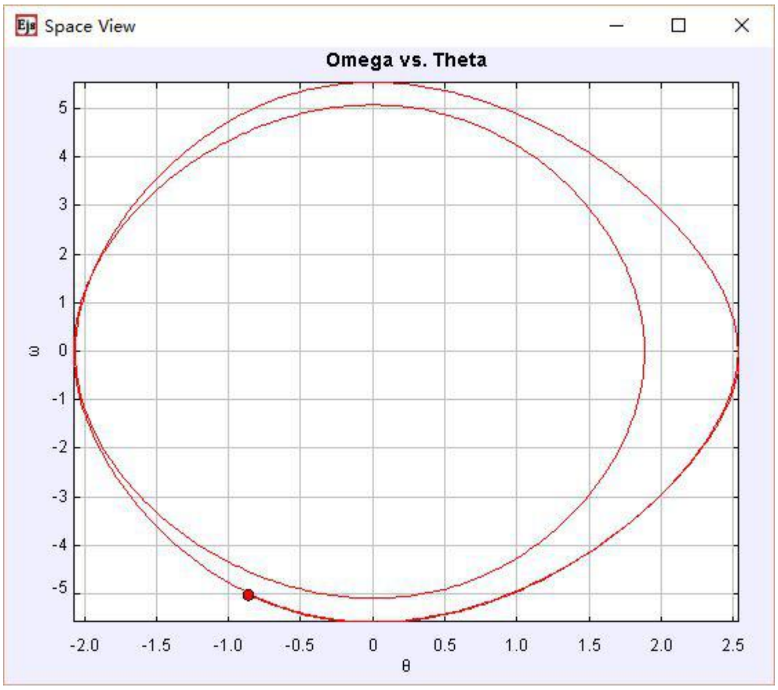
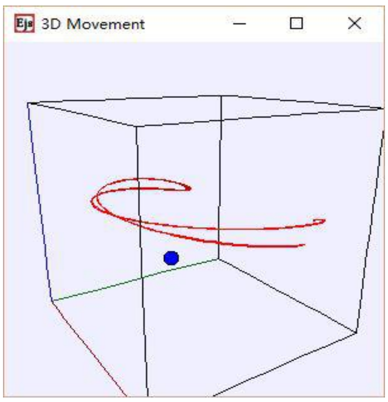


Fig.10. Phase Space of period doubling

By clicking on selecting mode option and select “Period Doubling”, you can view this phenomenon.



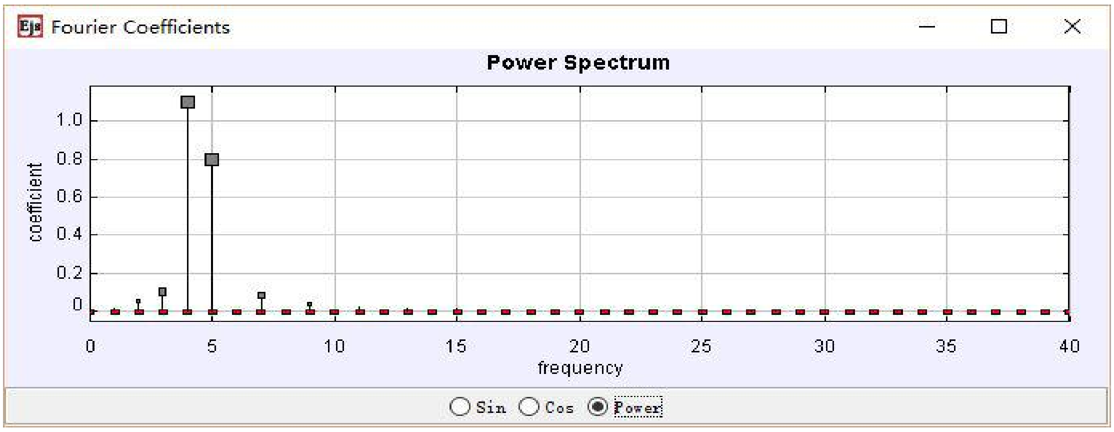


Fig.11. Fourier Transformation in the periodic solver version of Fourier Analysis

Fourier Analysis

After creating the main system of my pendulum and testing out my Fourier Transformation, which produces satisfactory Power Specturm (SPD) with a little noise, I begin my analysis of the evolution of the pendulum's motion across driving frequncies, strictly controlling all other parameters the same due to the the system's sensitivity. I start with the following parameters, where the system is at period doubling:

Initial Angular Displacement from Upright (rad)=2.542;
Driving Amplitude (cm)=0.115;
Driving Frequency (Hz)=0.7324 (wd=2πf=4.6);
Dampling Coefficient=0;
g=-9.8;

I first determined the natural frequency of the pendulum at the given inital position as 39, as given on the Power Spectrum. Then I incresed the driving frequency from period doubling at Δw=0.5 steps until the starting point of the pendulum, which is located at the first quadrant, is included into the potential well of the upright position. The observed driving w range from 4.6 to 43.

The following is a complete list of my observations. Broadband Spectrum indicates chaos. Although many of the

*wd(=2*PI*f)*	*Description of Fourier Power Spectrum*
5	50
5.5	82, 110, 130
6	Chaos with a spike at 40
6.5	Broadband spectrum with spikes at 40 (major) and 110 (minor)
7	7: 50
7.5	50
8	Broadband spectrum with a spike at 40
8.5	Broadband spectrum
9	Broadband spectrum

9.5	40 (minor) and 50 (major)
10	Broadband spectrum
10.5	110 and 140
11	120, 180 and 260
11.5	Broadband spectrum with a small spike at 40
12	22 (minor) and 40 (major)
12.5	Broadband spectrum
13	Initially 40 and 50, then broadband chaos
13.5	Mostly chaotic (broadband spectrum) with a small spike at 40
14	Uniform at 40-41
14.5	43
15	Broadband Spectrum
15.5	120, 150, 190
16	40, 80 and possibly 100, mostly chaotic
16.5	55
17	40
17.5	40 and 80
18	40, 50 and 90
18.5	40 and 120
19	40
19.5	40 and 80
20	40, 60 and 80
20.5	40
21	40, 60, 150
21.5	Broadband chaos with a spike at 40
22	40
22.5	35.5 (major) and 40 (minor)
23	30
23.5	Broadband chaos with a spike at 40
24	30
24.5	30
Between *wd*=24-wd=39	A periodic motion at SPD frequency=30-34, being sub-harmonic. From *wd*=20 the dominant frequency begins to shift to the left.
43	The motion is periodic at SPD frequency=22. A half of the natural frequency. This is where the one of the solutions of Mathieu's equation for effective energy well almost meets the starting position.
43.5	The starting position is on the brink of a stable point, SPD frequency=20. Which is less

	than 1/2 of the natural frequency.
43.6	Begin oscillating on the shallow plain of effective energy. SPD frequency=8, which is 1/5 of the natural frequency.
43.7	SPD frequency=11, which is 1/3 of the natural frequency.
44	The pendulum restores uniform harmonic motion.

From 4.6 to 25, I observed that the pendulum's motion is chaotic with a few short periods of chaotic but periodic motion: around wd=5,7,8,14-14.5, 16.5-17, 19, 20.5, 22, 22.7, 24-24.5. Most of the frequncies of the periodic motions are around 39, the natural frequency of the pendulum on the SPD. For the chaotic motions I observed, most of them exhibit a broadband spectrum with peaks at one or several of the following frequncies on the SPD: 50, 80, 100, 120, 160.

From w=25 to w=39, the stable point (where ∂Potential/∂t=0) approaches the initial position of the pendulum, and the attractors created by the two stable points becomes strong. The pendulum exhibits uniform periodic motion with frequency lower than its natural frequency, with the observed minimum at a third of the latter. But I have not successfully stablized the pendulum at the calculated stable points, possibly because they are the local maxima between two potential wells.

Two important aspects of this list to notice are:

1. Although the singular frequncies on the SPD indicates periodicity, it does not indicate uniformity of the motion. In all cases between period doubling and subharmonic oscillation, the motion is chaotic, as indicated by contantly varying positions of poincare section and its variance in amplitude. The following are pictures that illustrates this aspect.

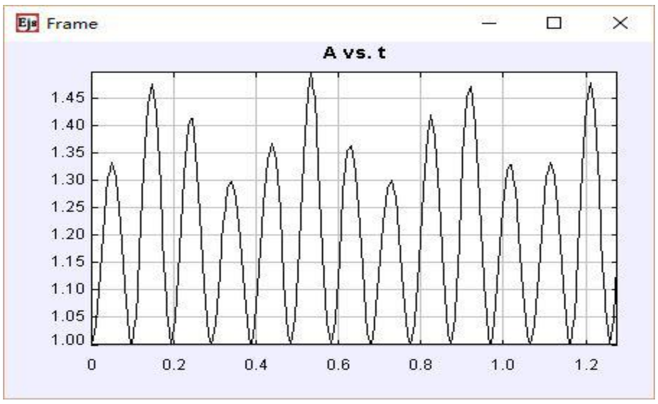


fig.Variation in amplitude even in a periodic motion

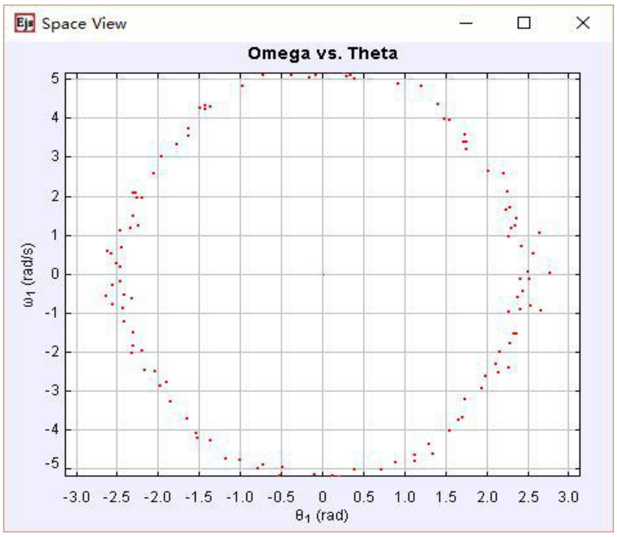


fig.Variation in Poincare section in a periodic motion

It can also be observed that even though the specific positions of the poincare section varies, they still follow a generally circular shape around (0,0), which is the downright position.

2. The Power Spectrum evloves, sometimes significantly, across time at specific driving frequncies. I tried my best to capture each fourier spectrum reported at the most stable positions.
   I have also observed that for some driving frequncies, the SPD varies significantly with the evolution of the pendulum's motion. For example, at *wd*=21, the pendulum's motion showed a major frequency over the general broadband chaos at SPD frequency=40, as shown below:

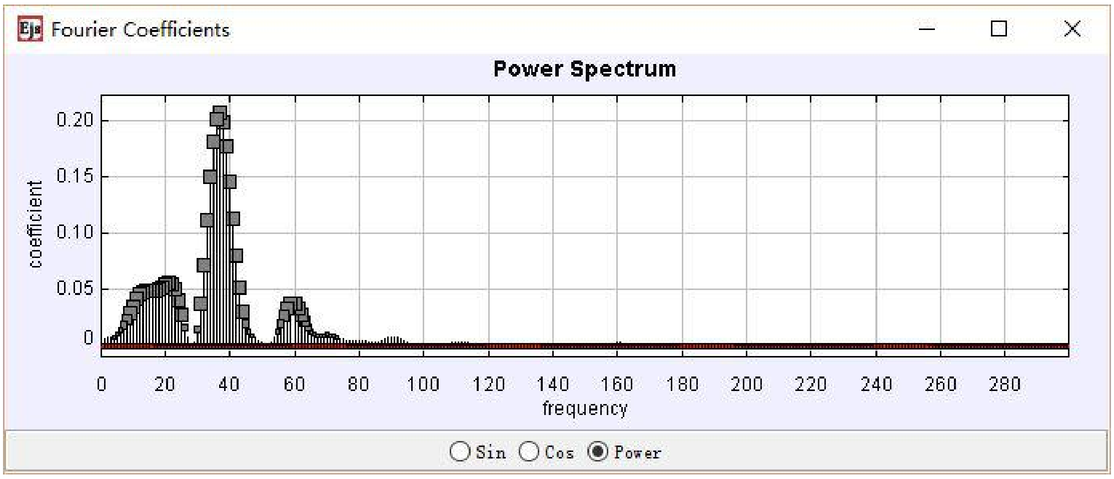


fig.Initial Fouirer Spctrum

But as the chaos evolves, new major frequencies emerge as old ones fade. And this change corresponds to a change in the Angular Displacement vs. time diagram, as shown below:

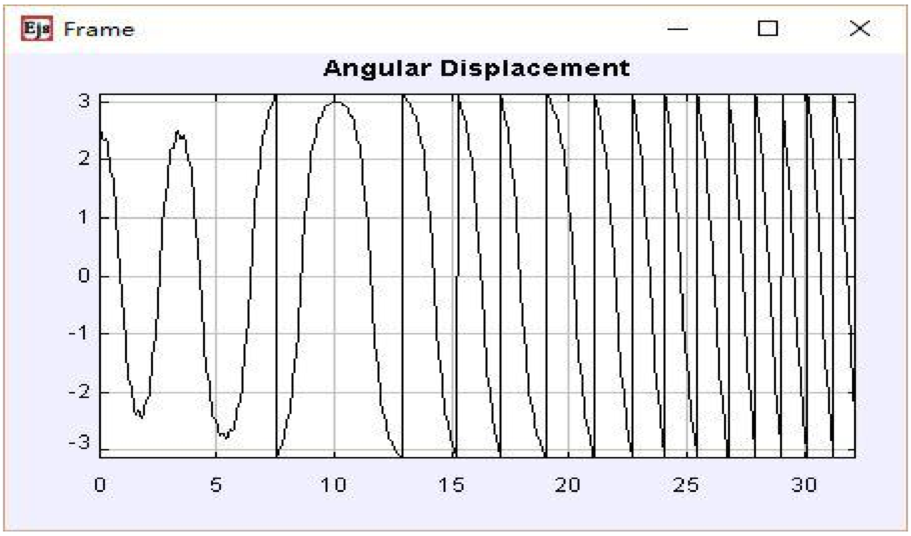


fig. Rate of Change of Angular Displacement increases

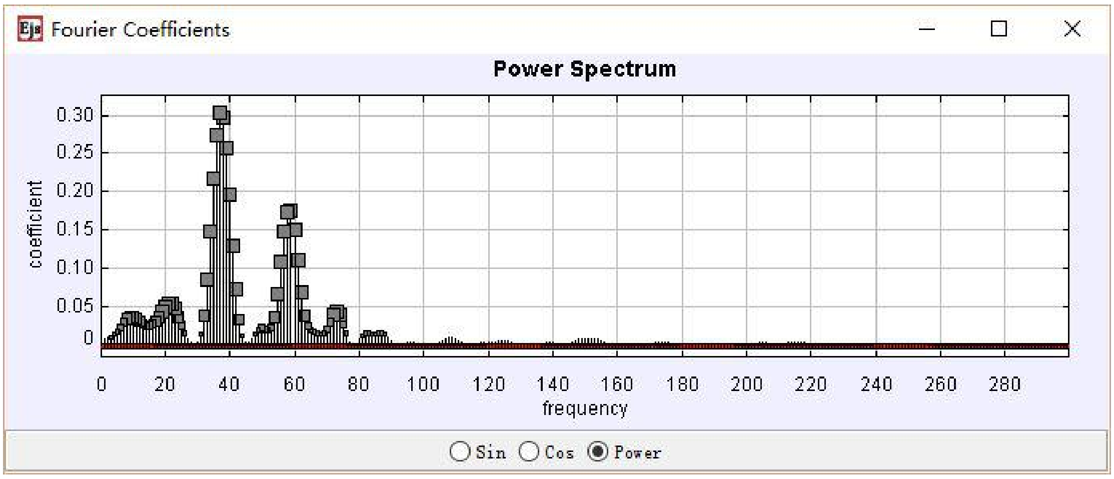


fig. A second major frequency at 60 emerges.

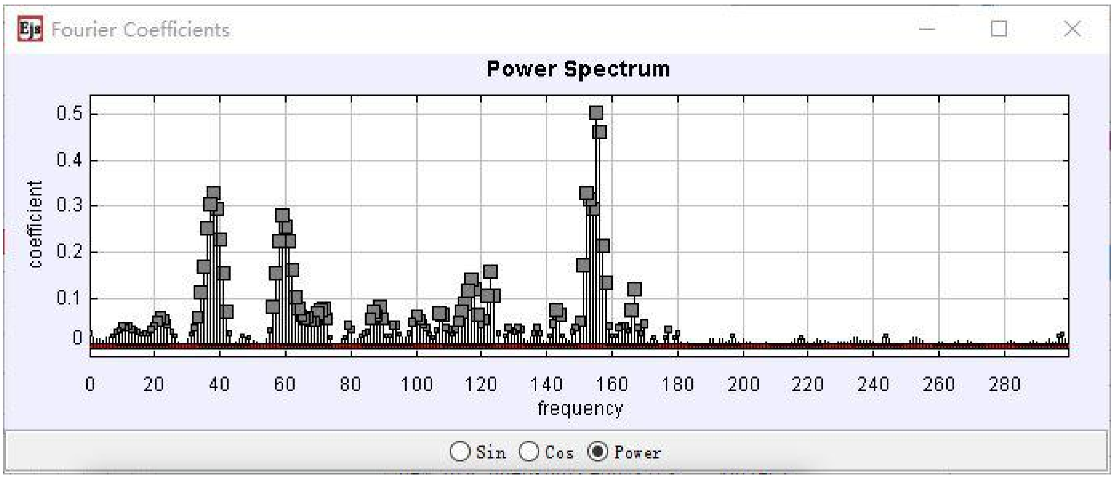


fig. Finally, a third major frequency at 160 emerges.

Credits:

Gollub, J.P. Chaotic Dynamics: an Introduction. Cambridge ; New York : Cambridge

University Press, 1996.
Gould, H., Tobochnik, J., & Christian, W. (2007). An Introduction to Computer Simulation Methods Third Edition (draft).Retrieved February 13, 2016
Christian, W. “FFT Real Analysis”. Davidson College Ejs. Library.
Chaos in the motion of an inverted pendulum: An undergraduate laboratory Experiment. B. Duchesne, C. W.
Fischer, C. G. Gray, and K.R. Jeffrey. American Journal of Physics 59, 987 (1991); doi: 10.1119/1.16657.