1. Consider a first-order autoregressive process:
a. Determine $E(\varepsilon_t)$
b. Determine $Cov(\varepsilon_t, \varepsilon_{t+h})$
c. Determine $Corr(\varepsilon_t, \varepsilon_{t+h})$
d. Is ε_t a stationary process?

a. Since $Z_t \sim N(0, \sigma^2)$:

$$
\begin{aligned}
E(\varepsilon_t) &= E(\sum_{j=0}^{\infty} \phi^j Z_{t-j}) && (1)\\
&= \sum_{j=0}^{\infty}[E(\phi^j Z_{t-j})] && (2)\\
&= \sum_{j=0}^{\infty}[\phi^j \times E(Z_{t-j})] && (3)\\
&= E(Z_{t-j}) \times \sum_{j=0}^{\infty} \phi^j && (4)\\
&= 0 \times \sum_{j=0}^{\infty} \phi^j && (5)\\
&= 0 && (6)
\end{aligned}
$$

b. Since $E(Z_t) = 0, Var(Z_t) = \sigma^2$,
we have $E(Z_t^2) = Var(Z_t)^2 + E(Z_t)^2 = \sigma^2$,
we also know that $|\phi| < 1$,

and Z_t are $i.i.d$:

$$\begin{aligned}
Cov(\varepsilon_t, \varepsilon_{t+h}) &= E(\varepsilon_t\varepsilon_{t+h}) - E(\varepsilon_t)E(\varepsilon_{t+h}) && (7)\\
&= E(\varepsilon_t\varepsilon_{t+h}) && (8)\\
&= E[(\sum_{j=0}^{\infty}\phi^j Z_{t-j})(\sum_{i=0}^{\infty}\phi^i Z_{t+h-i})] && (9)\\
&= E[\overbrace{\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(\phi^j Z_{t-j}\phi^i Z_{t+h-i})}^{t-j\neq t+h-i} + \overbrace{\sum_{i=0}^{\infty}(\phi^{i-h+i} Z_{t+h-i}^2)]}^{t-j=t+h-i} && (10)\\
&= \overbrace{E(\sum_{j=0}^{\infty}\phi^j Z_{t-j})E(\sum_{i=0}^{\infty}\phi^i Z_{t+h-i})}^{t-j\neq t+h-i} + \overbrace{E(\sum_{i=0}^{\infty}\phi^{i-h+i} Z_{t+h-i}^2)]}^{t-j=t+h-i} && (11)\\
&= E(Z_{t-j})\sum_{j=0}^{\infty}\phi^j E(Z_{t+h-i})\sum_{i=0}^{\infty}\phi^i + E(Z_{t+h-i}^2))\sum_{i=0}^{\infty}\phi^{i-h+i} && (12)\\
&= 0 \quad + \quad \sigma^2(\frac{\phi^{-h}}{1-\phi^2}) && (13)\\
&= \phi^{-h}(1-\phi^2)^{-1}\sigma^2 && (14)
\end{aligned}$$

c.

$$\begin{aligned}
Corr(\varepsilon_t, \varepsilon_{t+h}) &= \frac{Cov(\varepsilon_t, \varepsilon_{t+h})}{\sqrt{Var(\varepsilon_t)Var(\varepsilon_{t+h})}} && (15)\\
Var(\varepsilon_t) &= Cov(\varepsilon_t, \varepsilon_t) && (16)\\
&= (1-\phi^2)^{-1}\sigma^2 && (17)\\
Var(\varepsilon_{t+h}) &= Cov(\varepsilon_{t+h}, \varepsilon_{t+h}) && (18)\\
&= (1-\phi^2)^{-1}\sigma^2 && (19)\\
\text{Thus, } Corr(\varepsilon_t, \varepsilon_{t+h}) &= \frac{\phi^{-h}(1-\phi^2)^{-1}\sigma^2}{(1-\phi^2)^{-1}\sigma^2} && (20)\\
&= \phi^{-h} && (21)
\end{aligned}$$

d. AR(1) is a weakly stationary process, since the mean and variance is the same for all t and the covariance between $\varepsilon_t, \varepsilon_{t+h}$ is the same for all t.