

Some symbolic tools for the Fox H -function

Le Chen

Department of Mathematics and Statistics

Auburn University

le.chen@auburn.edu, chenle02@gmail.com

December 14, 2023

Contents

| | | |
|----------|--|-----------|
| 1 | Introduction | 1 |
| 2 | Examples | 5 |
| 2.1 | Example FoxH-2_9_11.wls | 5 |
| 2.2 | Example FoxH-2_9_12.wls | 6 |
| 2.3 | Example FoxH-2_9_13.wls | 7 |
| 2.4 | Example FoxH-2_9_4.wls | 8 |
| 2.5 | Example FoxH-2_9_5.wls | 9 |
| 2.6 | Example FoxH-2_9_6.wls | 10 |
| 2.7 | Example FoxH32-21.wls | 11 |
| 2.8 | Example FoxH32-21-Y.wls | 12 |
| 2.9 | Example FoxH32-21-Z.wls | 13 |
| 2.10 | Example FoxH32-21-Z-Star.wls | 15 |
| 2.11 | Example FoxH-Bessel-J_2_9_18.wls | 16 |
| 2.12 | Example FoxH-Cos.wls | 17 |
| 2.13 | Example FoxH-Mittag-Leffler.wls | 18 |
| 2.14 | Example FoxH-Sin.wls | 19 |
| | References | 20 |

1 Introduction

In this note, we explain the code for checking the conditions of the Fox H -function [Fox61]. Here we follow the notation from Kilbas and Saigo [KS].

Let m, n, p, q be configure integers such that

$$0 \leq m \leq q \quad \text{and} \quad 0 \leq n \leq p.$$

Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+$ be the parameters given below:

| $\in (\mathbb{C}, \mathbb{R}_+)$ | Front list | Rear list | |
|----------------------------------|---|---|------------|
| p | $(a_1, \alpha_1), \dots, (a_n, \alpha_n)$ | $(a_{n+1}, \alpha_{n+1}), \dots, (a_p, \alpha_p)$ | Upper list |
| q | $(b_1, \beta_1), \dots, (b_m, \beta_m)$ | $(b_{m+1}, \beta_{m+1}), \dots, (b_q, \beta_q)$ | Lower list |

and denote

$$\mathcal{H}_{p,q}^{m,n}(s) := \frac{\prod_{i=1}^n \Gamma(1 - a_i - \alpha_i s)}{\prod_{j=n+1}^p \Gamma(a_j + \alpha_j s)} \times \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \alpha_j s)}. \quad (1)$$

Then the Fox H -function $H_{2,3}^{2,1} \left(z \left| \begin{array}{c} \dots \\ \dots \end{array} \right. \right)$ is defined by a Mellin-Barnes type integral of the form

$$H_{m,n}^{p,q} \left(z \left| \begin{array}{c} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{array} \right. \right) := \frac{1}{2\pi i} \int_{\mathcal{L}} H_{p,q}^{m,n}(s) z^{-s} ds. \quad (2)$$

It is more convenient to use the following notation:

$$H_{m,n}^{p,q} \left(z \left| \frac{(a_1, \alpha_1), \dots, (a_n, \alpha_n)}{(b_1, \beta_1), \dots, (b_m, \beta_m)} \middle| \frac{(a_{n+1}, \alpha_{n+1}), \dots, (a_p, \alpha_p)}{(b_{m+1}, \beta_{m+1}), \dots, (b_q, \beta_q)} \right. \right),$$

in order to emphasize the front and rear lists.

The basic assumption for the well-posedness of the Fox H -function is that two sets of poles do not overlap, i.e.,

$$\left\{ b_{j\ell} = \frac{-b_j - \ell}{\beta_j}, \ell = 0, 1, \dots \right\} \cap \left\{ a_{ik} = \frac{1 - a_i + k}{\alpha_i}, k = 0, 1, \dots \right\} = \emptyset. \quad (3)$$

The contour \mathcal{L} in (2) is given by one of the following three cases:

1. $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at point $-\infty + i\phi_1$ and terminating at point $-\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;
2. $\mathcal{L} = \mathcal{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at point $+\infty + i\phi_1$ and terminating at point $+\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;
3. $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ is a contour starting at point $\gamma - i\infty$ and terminating at point $\gamma + i\infty$ for some $\gamma \in (-\infty, \infty)$.

We need a set of conditions to ensure the convergence of the integral in (2). To explain this, we need to introduce some notation (following p. 2 of [KS]). First denote

$$a_1^* := \sum_{j=1}^m \beta_j - \sum_{i=n+1}^p \alpha_i,$$

$$a_2^* := \sum_{i=1}^n \alpha_i - \sum_{j=m+1}^q \beta_j.$$

The following two parameters play the most important role:

$$a^* := a_2^* + a_1^* = \sum_{i=1}^n \alpha_i - \sum_{i=n+1}^p \alpha_i + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j;$$

$$\Delta := a_2^* - a_1^* = \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i.$$

Similar to a^* , we define

$$\xi := \sum_{i=1}^n a_i - \sum_{i=n+1}^p a_i + \sum_{j=1}^m b_j - \sum_{j=m+1}^q b_j.$$

Additionally, set

$$c^* := m + n - \frac{p+q}{2}.$$

In the critical cases, we need to use the following two parameters:

$$\delta := \prod_{i=1}^p \alpha_i^{-\alpha_i} \prod_{j=1}^q \beta_j^{\beta_j};$$

$$\mu := \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p-q}{2}$$

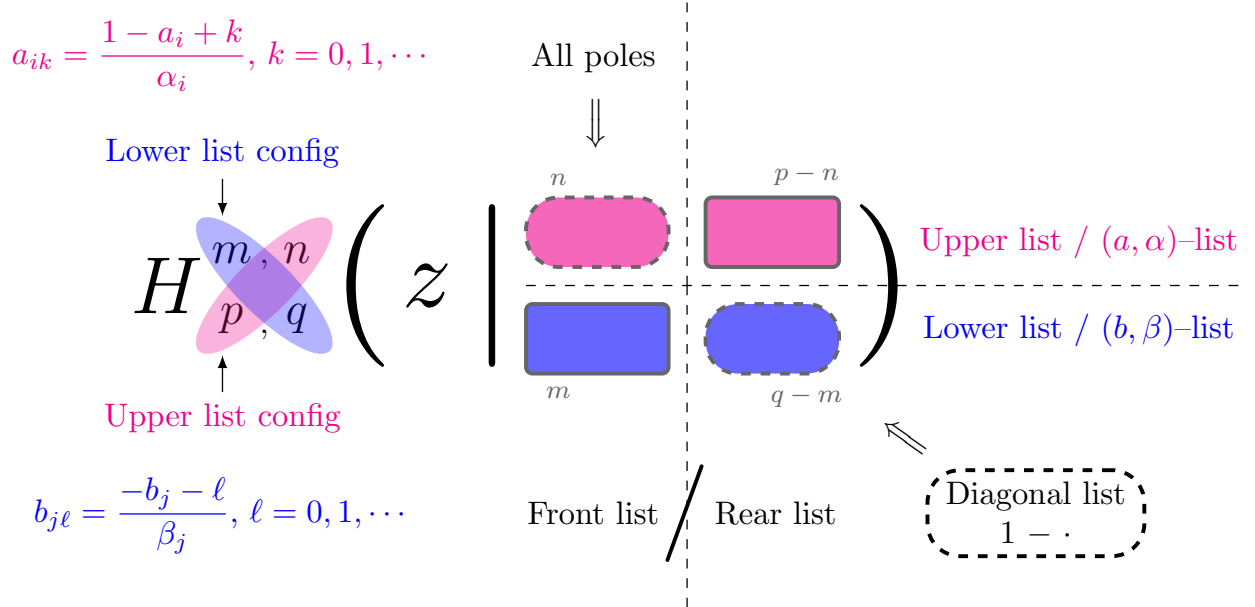


Figure 1: Diagram for the parameterization of the Fox H -function.

The well-posedness of the Fox H -function is given by Theorems 1.1 and 1.2 of [KS], which are summarized in the following figure 2:

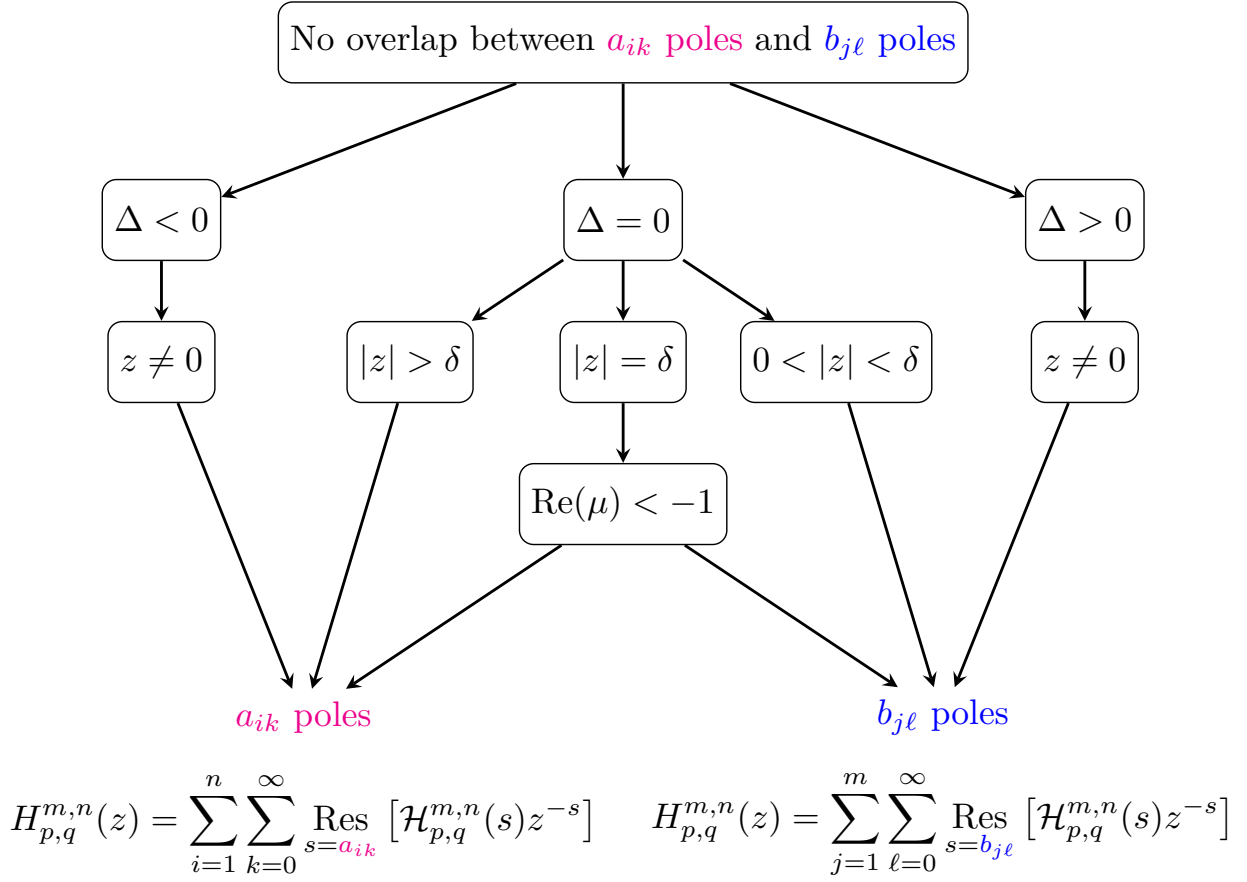


Figure 2: Well-posedness of the Fox H -function.

2 Examples

2.1 Example FoxH-2_9_11.wls

File content

```
(* (2.9.11) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {{1,1},{1,1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1,1}},
    (* Lower Rear List *) {{0,0}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,0} \left(\cdot \left| \begin{array}{c} (1,1), (1,1) \\ (1,1), (0,0) \end{array} \right. \right)$$

$$H_{2,2}^{1,0} \left(\cdot \left| \begin{array}{c|c} & (1,1), (1,1) \\ (1,1) & (0,0) \end{array} \right. \right)$$

Summary

$$\begin{aligned} a^* &= -1 \\ \Delta &= -1 \\ \delta &= \text{Indeterminate} \\ \mu &= -1 \\ a_1^* &= -1 \\ a_2^* &= 0 \\ \xi &= -1 \\ c^* &= -1 \end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \{\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{cccccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{array} \right)$$

2.2 Example FoxH-2_9_12.wls

File content

```
(* (2.9.12) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1/2,1},{1/2,1}},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0,1}},
    (* Lower Rear List *) {{-1/2,1}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,2} \left(. \left| \begin{array}{c} \left(\frac{1}{2}, 1 \right), \left(\frac{1}{2}, 1 \right) \\ (0, 1), \left(-\frac{1}{2}, 1 \right) \end{array} \right. \right)$$

$$H_{2,2}^{1,2} \left(. \left| \frac{\left(\frac{1}{2}, 1 \right), \left(\frac{1}{2}, 1 \right)}{(0, 1)} \right| \frac{}{\left(-\frac{1}{2}, 1 \right)} \right)$$

Summary

$$\begin{aligned} a^* &= 2 \\ \Delta &= 0 \\ \delta &= \text{ComplexInfinity} \\ \mu &= -\frac{3}{2} \\ a_1^* &= 1 \\ a_2^* &= 1 \\ \xi &= \frac{3}{2} \\ c^* &= 1 \end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \left(\begin{array}{cccccccccccc} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} & \frac{17}{2} & \frac{19}{2} & \frac{21}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} & \frac{17}{2} & \frac{19}{2} & \frac{21}{2} \end{array} \right)$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{pmatrix}$$

2.3 Example FoxH-2_9_13.wls

File content

```
(* (2.9.13) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1/2,1},{1,1}},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2,1}},
    (* Lower Rear List *) {{0,1}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,2} \left(\cdot \left| \begin{array}{c} (\frac{1}{2}, 1), (1, 1) \\ (\frac{1}{2}, 1), (0, 1) \end{array} \right. \right)$$

$$H_{2,2}^{1,2} \left(\cdot \left| \begin{array}{c|c} (\frac{1}{2}, 1), (1, 1) & \\ \hline (\frac{1}{2}, 1) & (0, 1) \end{array} \right. \right)$$

Summary

$$\begin{aligned} a^* &= 2 \\ \Delta &= 0 \\ \delta &= \text{ComplexInfinity} \\ \mu &= -1 \\ a_1^* &= 1 \\ a_2^* &= 1 \\ \xi &= 2 \\ c^* &= 1 \end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} & \frac{17}{2} & \frac{19}{2} & \frac{21}{2} \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} & -\frac{7}{2} & -\frac{9}{2} & -\frac{11}{2} & -\frac{13}{2} & -\frac{15}{2} & -\frac{17}{2} & -\frac{19}{2} & -\frac{21}{2} \end{pmatrix}$$

2.4 Example FoxH-2_9_4.wls

File content

```
(* (2.9.4) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{b, β}},
    (* Lower Rear List *) {}
  }
}
```

Fox H-function

$$H_{0,1}^{1,0} \left(\begin{matrix} \cdot \\ (b, \beta) \end{matrix} \right)$$

$$H_{0,1}^{1,0} \left(\begin{matrix} \cdot \\ (b, \beta) \end{matrix} \right)$$

Summary

$$\begin{aligned}
a^* &= \beta \\
\Delta &= \beta \\
\delta &= \text{Indeterminate} \\
\mu &= b - \frac{1}{2} \\
a_1^* &= \beta \\
a_2^* &= 0 \\
\xi &= b \\
c^* &= \frac{1}{2}
\end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \{\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(-\frac{b}{\beta} \quad -\frac{b+1}{\beta} \quad -\frac{b+2}{\beta} \quad -\frac{b+3}{\beta} \quad -\frac{b+4}{\beta} \quad -\frac{b+5}{\beta} \quad -\frac{b+6}{\beta} \quad -\frac{b+7}{\beta} \quad -\frac{b+8}{\beta} \quad -\frac{b+9}{\beta} \quad -\frac{b+10}{\beta} \right)$$

2.5 Example FoxH-2_9_5.wls

File content

```
(* (2.9.5) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1-a,1}},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *) {}
  }
}
```

Fox H-function

$$H_{1,1}^{1,1} \left(\cdot \left| \begin{array}{c} (1-a, 1) \\ (0, 1) \end{array} \right. \right)$$

$$H_{1,1}^{1,1} \left(\cdot \left| \frac{(1-a, 1)}{(0, 1)} \right. \right)$$

Summary

$$\begin{aligned}
a^* &= 2 \\
\Delta &= 0 \\
\delta &= \text{Indeterminate} \\
\mu &= a - 1 \\
a_1^* &= 1 \\
a_2^* &= 1 \\
\xi &= 1 - a \\
c^* &= 1
\end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \begin{pmatrix} a & a+1 & a+2 & a+3 & a+4 & a+5 & a+6 & a+7 & a+8 & a+9 & a+10 \end{pmatrix}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{pmatrix}$$

2.6 Example FoxH-2_9_6.wls

File content

```
(* (2.9.6) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {{\alpha+\beta+1,1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{\alpha, 1}},
    (* Lower Rear List *) {}
  }
}
```

Fox H-function

$$H_{1,1}^{1,0} \left(\begin{matrix} \cdot \\ (\alpha, 1) \end{matrix} \middle| \begin{matrix} (\alpha + \beta + 1, 1) \end{matrix} \right)$$

$$H_{1,1}^{1,0} \left(\begin{matrix} \cdot \\ (\alpha, 1) \end{matrix} \middle| \frac{(\alpha + \beta + 1, 1)}{(\alpha, 1)} \right)$$

Summary

$$\begin{aligned}
a^* &= 0 \\
\Delta &= 0 \\
\delta &= \text{Indeterminate} \\
\mu &= -\beta - 1 \\
a_1^* &= 0 \\
a_2^* &= 0 \\
\xi &= -\beta - 1 \\
c^* &= 0
\end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \{\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{cccccccccccc} -\alpha & -\alpha - 1 & -\alpha - 2 & -\alpha - 3 & -\alpha - 4 & -\alpha - 5 & -\alpha - 6 & -\alpha - 7 & -\alpha - 8 & -\alpha - 9 & -\alpha - 10 \end{array} \right)$$

2.7 Example FoxH32-21.wls

File content

```

{
  (* Upper List *) {
    (* Upper Front List *) {{1,  $\alpha^{(-1)}$ }},
    (* Upper Rear List *) {{Ceiling[ $\beta$ ],  $\beta$ }}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2,  $\alpha/2$ }, {1, 1}},
    (* Lower Rear List *) {{1,  $\alpha/2$ }}
  }
}

```

Fox H-function

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c} (1, \frac{1}{\alpha}), (\lceil \beta \rceil, \beta) \\ (\frac{1}{2}, \frac{\alpha}{2}), (1, 1), (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c|c} (1, \frac{1}{\alpha}) & (\lceil \beta \rceil, \beta) \\ \hline (\frac{1}{2}, \frac{\alpha}{2}), (1, 1) & (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

Summary

$$\begin{aligned}
a^* &= \frac{1}{\alpha} - \beta + 1 \\
\Delta &= \alpha - \frac{1}{\alpha} - \beta + 1 \\
\delta &= 2^{-\alpha} \left(\frac{1}{\alpha} \right)^{-1/\alpha} \left(2^{\alpha/2} \alpha^{\alpha/2} + \alpha^\alpha \right) \beta^{-\beta} \\
\mu &= 1 - \lceil \beta \rceil \\
a_1^* &= \frac{1}{2} (\alpha - 2\beta + 2) \\
a_2^* &= \frac{1}{\alpha} - \frac{\alpha}{2} \\
\xi &= \frac{3}{2} - \lceil \beta \rceil \\
c^* &= \frac{1}{2}
\end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \begin{pmatrix} 0 & \alpha & 2\alpha & 3\alpha & 4\alpha & 5\alpha & 6\alpha & 7\alpha & 8\alpha & 9\alpha & 10\alpha \end{pmatrix}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{1}{\alpha} & -\frac{3}{\alpha} & -\frac{5}{\alpha} & -\frac{7}{\alpha} & -\frac{9}{\alpha} & -\frac{11}{\alpha} & -\frac{13}{\alpha} & -\frac{15}{\alpha} & -\frac{17}{\alpha} & -\frac{19}{\alpha} & -\frac{21}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{pmatrix}$$

2.8 Example FoxH32-21-Y.wls

File content

```

{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *) {{β + γ, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *) {{1, α/2}}
  }
}

```

Fox H-function

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c} (1, 1), (\beta + \gamma, \beta) \\ (\frac{d}{2}, \frac{\alpha}{2}), (1, 1), (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left(\cdot \left| \frac{(1, 1)}{(\frac{d}{2}, \frac{\alpha}{2}), (1, 1)} \right| \frac{(\beta + \gamma, \beta)}{(1, \frac{\alpha}{2})} \right)$$

Summary

$$\begin{aligned} a^* &= 2 - \beta \\ \Delta &= \alpha - \beta \\ \delta &= 2^{-\alpha} \left(2^{\alpha/2} \alpha^{\alpha/2} + \alpha^\alpha \right) \beta^{-\beta} \\ \mu &= \frac{1}{2} (-2\beta - 2\gamma + d + 1) \\ a_1^* &= \frac{1}{2} (\alpha - 2\beta + 2) \\ a_2^* &= 1 - \frac{\alpha}{2} \\ \xi &= \frac{1}{2} (d - 2(\beta + \gamma - 1)) \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \left(\begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \right)$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{cccccccccccc} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} & -\frac{d+16}{\alpha} & -\frac{d+18}{\alpha} & -\frac{d+20}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{array} \right)$$

2.9 Example FoxH32-21-Z.wls

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
```

```

(* Upper Rear List *) {{Ceiling[β], β}}
},
(* Lower List *) {
(* Lower Front List *) {{d/2, α/2}, {1, 1}},
(* Lower Rear List *) {{1, α/2}}
}
}

```

Fox H-function

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c} (1,1), (\lceil \beta \rceil, \beta) \\ (\frac{d}{2}, \frac{\alpha}{2}), (1,1), (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c|c} (1,1) & (\lceil \beta \rceil, \beta) \\ \hline (\frac{d}{2}, \frac{\alpha}{2}), (1,1) & (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

Summary

$$\begin{aligned}
a^* &= 2 - \beta \\
\Delta &= \alpha - \beta \\
\delta &= 2^{-\alpha} \left(2^{\alpha/2} \alpha^{\alpha/2} + \alpha^\alpha \right) \beta^{-\beta} \\
\mu &= \frac{1}{2} (-2\lceil \beta \rceil + d + 1) \\
a_1^* &= \frac{1}{2} (\alpha - 2\beta + 2) \\
a_2^* &= 1 - \frac{\alpha}{2} \\
\xi &= \frac{1}{2} (-2\lceil \beta \rceil + d + 2) \\
c^* &= \frac{1}{2}
\end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \left(\begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \right)$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{cccccccccccc} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} & -\frac{d+16}{\alpha} & -\frac{d+18}{\alpha} & -\frac{d+20}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{array} \right)$$

2.10 Example FoxH32-21-Z-Star.wls

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{1, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c} (1, 1), (1, \beta) \\ (\frac{d}{2}, \frac{\alpha}{2}), (1, 1), (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c|c} (1, 1) & (1, \beta) \\ \hline (\frac{d}{2}, \frac{\alpha}{2}), (1, 1) & (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

Summary

$$a^* = 2 - \beta$$

$$\Delta = \alpha - \beta$$

$$\delta = 2^{-\alpha} \left(2^{\alpha/2} \alpha^{\alpha/2} + \alpha^\alpha \right) \beta^{-\beta}$$

$$\mu = \frac{d-1}{2}$$

$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$

$$a_2^* = 1 - \frac{\alpha}{2}$$

$$\xi = \frac{d}{2}$$

$$c^* = \frac{1}{2}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \left(\begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \right)$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} & -\frac{d+16}{\alpha} & -\frac{d+18}{\alpha} & -\frac{d+20}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{pmatrix}$$

2.11 Example FoxH-Bessel-J_2_9_18.wls

File content

```
(* (2.9.18) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(a+η)/2, 1},{(a-η)/2, 1}},
    (* Lower Rear List *) {}
  }
}
```

Fox H-function

$$H_{0,2}^{2,0} \left(\cdot \left| \begin{array}{c} \\ \left(\frac{a+\eta}{2}, 1 \right), \left(\frac{a-\eta}{2}, 1 \right) \end{array} \right. \right)$$

$$H_{0,2}^{2,0} \left(\cdot \left| \frac{\quad}{\left(\frac{a+\eta}{2}, 1 \right), \left(\frac{a-\eta}{2}, 1 \right)} \right| \right)$$

Summary

$$\begin{aligned} a^* &= 2 \\ \Delta &= 2 \\ \delta &= \text{Indeterminate} \\ \mu &= a - 1 \\ a_1^* &= 2 \\ a_2^* &= 0 \\ \xi &= a \\ c^* &= 1 \end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \{\{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} \frac{1}{2}(-a - \eta) & \frac{\eta - a}{2} \\ \frac{1}{2}(-a - \eta - 2) & \frac{1}{2}(-a + \eta - 2) \\ \frac{1}{2}(-a - \eta - 4) & \frac{1}{2}(-a + \eta - 4) \\ \frac{1}{2}(-a - \eta - 6) & \frac{1}{2}(-a + \eta - 6) \\ \frac{1}{2}(-a - \eta - 8) & \frac{1}{2}(-a + \eta - 8) \\ -\frac{a}{2} - \frac{\eta}{2} - 5 & \frac{1}{2}(-a + \eta - 10) \\ -\frac{a}{2} - \frac{\eta}{2} - 6 & \frac{1}{2}(-a + \eta - 12) \\ -\frac{a}{2} - \frac{\eta}{2} - 7 & \frac{1}{2}(-a + \eta - 14) \\ -\frac{a}{2} - \frac{\eta}{2} - 8 & \frac{1}{2}(-a + \eta - 16) \\ -\frac{a}{2} - \frac{\eta}{2} - 9 & \frac{1}{2}(-a + \eta - 18) \\ \frac{1}{2}(-a - \eta - 20) & \frac{1}{2}(-a + \eta - 20) \end{pmatrix}$$

2.12 Example FoxH-Cos.wls

File content

```
(* (2.9.8) and (2.9.10) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *) {{1/2, 1}}
  }
}
```

Fox H-function

$$H_{0,2}^{1,0} \left(\cdot \left| \begin{array}{c} (0, 1), \left(\frac{1}{2}, 1\right) \end{array} \right. \right)$$

$$H_{0,2}^{1,0} \left(\cdot \left| \frac{\quad}{(0,1)} \right| \frac{\quad}{(\frac{1}{2},1)} \right)$$

Summary

$$\begin{aligned} a^* &= 0 \\ \Delta &= 2 \\ \delta &= \text{ComplexInfinity} \\ \mu &= -\frac{1}{2} \\ a_1^* &= 1 \\ a_2^* &= -1 \\ \xi &= -\frac{1}{2} \\ c^* &= 0 \end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \{\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{cccccccccccc} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{array} \right)$$

2.13 Example FoxH-Mittag-Leffler.wls

File content

```
(* (2.9.27) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{0, 1}},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *) {{1 - μ, ρ}}
  }
}
```

Fox H-function

$$H_{1,2}^{1,1} \left(\begin{matrix} \cdot \\ (0,1) \end{matrix} \middle| \begin{matrix} (0,1) \\ (0,1), (1-\mu, \rho) \end{matrix} \right)$$

$$H_{1,2}^{1,1} \left(\begin{matrix} \cdot \\ (0,1) \end{matrix} \middle| \frac{(0,1)}{(0,1)} \middle| (1-\mu, \rho) \right)$$

Summary

$$\begin{aligned} a^* &= 2 - \rho \\ \Delta &= \rho \\ \delta &= \text{ComplexInfinity} \\ \mu &= \frac{1}{2} - \mu \\ a_1^* &= 1 \\ a_2^* &= 1 - \rho \\ \xi &= \mu - 1 \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \left(\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{matrix} \right)$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{matrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{matrix} \right)$$

2.14 Example FoxH-Sin.wls

File content

```
(* (2.9.7) and (2.9.9) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2, 1}},
    (* Lower Rear List *) {{0, 1}}
  }
}
```

Fox H-function

$$H_{0,2}^{1,0} \left(. \left| \begin{array}{c} \\ \left(\frac{1}{2}, 1 \right), (0, 1) \end{array} \right. \right)$$

$$H_{0,2}^{1,0} \left(. \left| \begin{array}{c|c} & \\ \hline \left(\frac{1}{2}, 1 \right) & (0, 1) \end{array} \right. \right)$$

Summary

$$\begin{aligned} a^* &= 0 \\ \Delta &= 2 \\ \delta &= \text{ComplexInfinity} \\ \mu &= -\frac{1}{2} \\ a_1^* &= 1 \\ a_2^* &= -1 \\ \xi &= \frac{1}{2} \\ c^* &= 0 \end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \{ \}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(-\frac{1}{2} \quad -\frac{3}{2} \quad -\frac{5}{2} \quad -\frac{7}{2} \quad -\frac{9}{2} \quad -\frac{11}{2} \quad -\frac{13}{2} \quad -\frac{15}{2} \quad -\frac{17}{2} \quad -\frac{19}{2} \quad -\frac{21}{2} \right)$$

References

- [Fox61] Charles Fox. The G and H functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, 98:395–429, 1961.
- [KS] Anatoly A. Kilbas and Megumi Saigo. H -transforms, volume 9 of *Analytical Methods and Special Functions*. Chapman & Hall/CRC, Boca Raton, FL. Theory and applications.