Some symbolic tools for the Fox H-function

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Contents

1 Introduction

In this note, we explain the code for checking the conditions of the Fox H-function [Fox61]. Here we follow the notation from Kilbas and Saigo [KS].

Let m, n, p, q be configure integers such that

$$0 \leq m \leq q \quad \text{and} \quad 0 \leq n \leq p.$$

Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+$ be the parameters given below:

$\in (\mathbb{C}, \mathbb{R}_+)$	Front list	Rear list	
p	$(a_1, \alpha_1), \cdots, (a_n, \alpha_n)$	$(a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p)$	Upper list
q	$(b_1, \beta_1), \cdots, (b_m, \beta_m)$	$(b_{m+1}, \beta_{m+1}), \cdots, (b_q, \beta_q)$	Lower list

and denote

$$\mathcal{H}_{p,q}^{m,n}(s) := \frac{\prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_j + \alpha_i s)} \times \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - \alpha_j s)}. \tag{1}$$

Then the Fox H-function $H_{2,3}^{2,1}\left(z \left| \begin{array}{c} \cdots \\ \cdots \end{array} \right. \right)$ is defined by a Mellin-Barnes type integral of the form

$$H_{m,n}^{p,q}\left(z \left| \begin{array}{c} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ \\ (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{array} \right. \right) := \frac{1}{2\pi i} \int_{\mathcal{L}} H_{p,q}^{m,n}(s) z^{-s} \mathrm{d}s. \tag{2}$$

The basic assumption for the well-posedness of the Fox H-function is that two sets of poles do not overlap, i.e.,

$$\left\{ b_{j\ell} = \frac{-b_j - \ell}{\beta_j}, \ell = 0, 1, \cdots \right\} \bigcap \left\{ a_{ik} = \frac{1 - a_i + k}{\alpha_i}, k = 0, 1, \cdots \right\} = \emptyset. \tag{3}$$

The contour $\mathcal{L}$ in (2) is given by one of the following three cases:

1. $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at point $-\infty + i\phi_1$ and terminating at point $-\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;

2. $\mathcal{L} = \mathcal{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at point $+\infty + i\phi_1$ and terminating at point $\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;

3. $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ is a contour starting at point $\gamma - i\infty$ and terminating at point $\gamma + i\infty$ for some $\gamma \in (-\infty, \infty)$.

We need a set of conditions to ensure the convergence of the integral in (2). To explain this, we need to introduce some notation (following p. 2 of [KS]). First denote

$$a_1^* := \sum_{j=1}^{m} \beta_j - \sum_{i=n+1}^{p} \alpha_i,$$

$$a_2^* := \sum_{i=1}^{n} \alpha_i - \sum_{j=m+1}^{q} \beta_j.$$

The following two parameters play the most important role:

$$a^* := a_2^* + a_1^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j;$$

$$\Delta := a_2^* - a_1^* = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i.$$

Similar to a^*, we define

$$\xi := \sum_{i=1}^{n} a_i - \sum_{i=n+1}^{p} a_i + \sum_{j=1}^{m} b_j - \sum_{j=m+1}^{q} b_j.$$

Additionally, set

$$c^* := m + n - \frac{p+q}{2}.$$

In the critical cases, we need to use the following two parameters:

$$\delta := \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_i};$$

$$\mu := \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}$$

The well-posedness of the Fox H-function is given by Theorems 1.1 and 1.2 of [KS], which are summarized in the following figure 2:

2 Examples

2.1 Example `FoxH-2_9_11.wls`

File content

```
(* (2.9.11) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {{1,1},{1,1}}
```
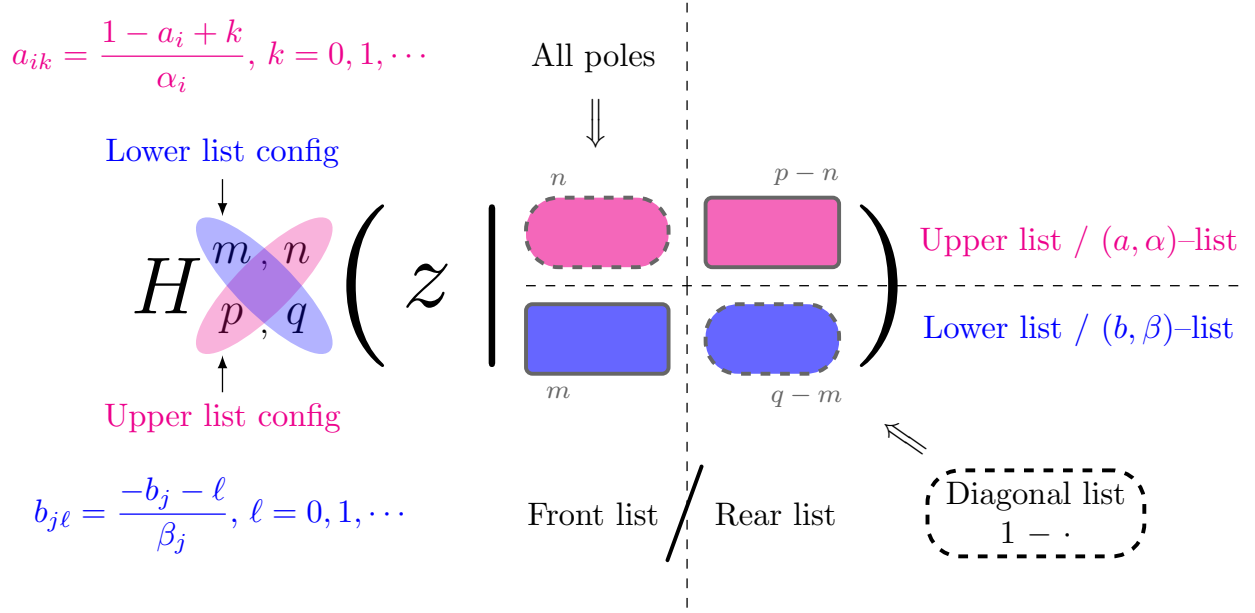
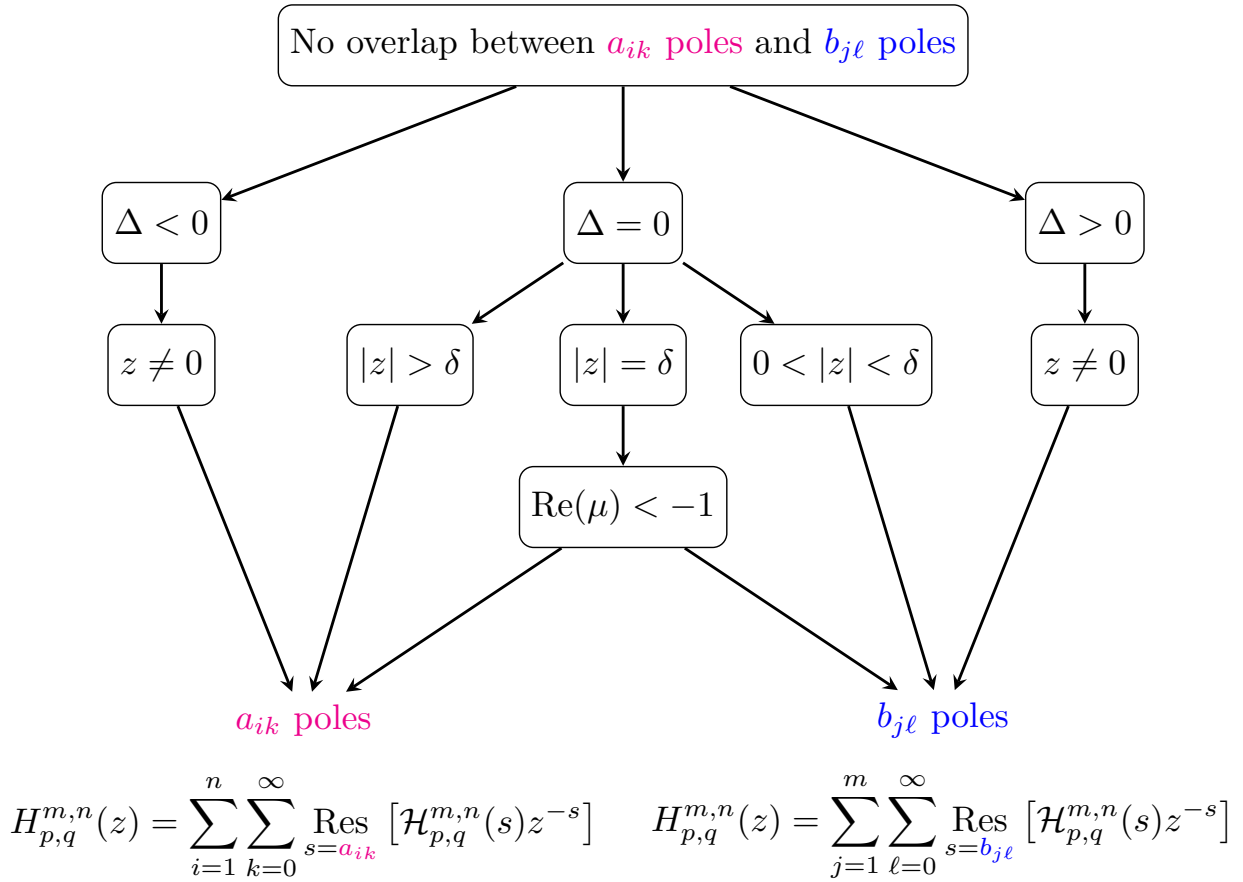Figure 1: Diagram for the parameterization of the Fox H-function.

Figure 2: Well-posedness of the Fox H-function.

```
  },
  (* Lower List *) {
    (* Lower Front List *) {{1,1}},
    (* Lower Rear List *)  {{0,0}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,0} \left(\cdot \left| \begin{array}{c} (1,1),(1,1) \\ \\ (1,1),(0,0) \end{array} \right. \right)$$

$$H_{2,2}^{1,0} \left(\cdot \left| \begin{array}{c|c} & (1,1),(1,1) \\ \hline (1,1) & (0,0) \end{array} \right. \right)$$

Summary

$$a^* = -1$$
$$\Delta = -1$$
$$\delta = \text{Indeterminate}$$
$$\mu = -1$$
$$a_1^* = -1$$
$$a_2^* = 0$$
$$\xi = -1$$
$$c^* = -1$$

Poles   1. First ten poles from upper front list

$$a_{i,k} = \{\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{ccccccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{array} \right)$$

2.2   Example `FoxH-2_9_12.wls`

File content

```
(* (2.9.12) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
```

```
    (* Upper Front List *) {{1/2,1},{1/2,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0,1}},
    (* Lower Rear List *)  {{-1/2,1}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,2}\left(\cdot \left| \begin{array}{c} \left(\frac{1}{2},1\right), \left(\frac{1}{2},1\right) \\ \\ (0,1), \left(-\frac{1}{2},1\right) \end{array} \right.\right)$$

$$H_{2,2}^{1,2}\left(\cdot \left| \begin{array}{c|c} \left(\frac{1}{2},1\right), \left(\frac{1}{2},1\right) & \\ \hline (0,1) & \left(-\frac{1}{2},1\right) \end{array} \right.\right)$$

Summary

$$a^* = 2$$
$$\Delta = 0$$
$$\delta = \mathrm{ComplexInfinity}$$
$$\mu = -\frac{3}{2}$$
$$a_1^* = 1$$
$$a_2^* = 1$$
$$\xi = \frac{3}{2}$$
$$c^* = 1$$

Poles   1. First ten poles from upper front list

$$a_{i,k} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} & \frac{17}{2} & \frac{19}{2} & \frac{21}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} & \frac{17}{2} & \frac{19}{2} & \frac{21}{2} \end{pmatrix}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{pmatrix}$$

2.3 Example `FoxH-2_9_13.wls`

File content

```
(* (2.9.13) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1/2,1},{1,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2,1}},
    (* Lower Rear List *)  {{0,1}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,2}\left(\cdot\left|\begin{array}{c}\left(\frac{1}{2},1\right),(1,1)\\[2mm]\left(\frac{1}{2},1\right),(0,1)\end{array}\right.\right)$$

$$H_{2,2}^{1,2}\left(\cdot\left|\begin{array}{c|c}\left(\frac{1}{2},1\right),(1,1)&\\\hline\left(\frac{1}{2},1\right)&(0,1)\end{array}\right.\right)$$

Summary

$$a^* = 2$$
$$\Delta = 0$$
$$\delta = \text{ComplexInfinity}$$
$$\mu = -1$$
$$a_1^* = 1$$
$$a_2^* = 1$$
$$\xi = 2$$
$$c^* = 1$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} & \frac{17}{2} & \frac{19}{2} & \frac{21}{2} \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} & -\frac{7}{2} & -\frac{9}{2} & -\frac{11}{2} & -\frac{13}{2} & -\frac{15}{2} & -\frac{17}{2} & -\frac{19}{2} & -\frac{21}{2} \end{pmatrix}$$

2.4 Example `FoxH-2_9_4.wls`

File content

```
(* (2.9.4) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{b, β}},
    (* Lower Rear List *)  {}
  }
}
```

Fox H-function

$$H_{0,1}^{1,0}\left(\cdot \left| \begin{array}{c} \\ (b,\beta) \end{array} \right. \right)$$

$$H_{0,1}^{1,0}\left(\cdot \left| \begin{array}{c|c} \quad & \\ \hline (b,\beta) & \end{array} \right. \right)$$

Summary

$$a^* = \beta$$
$$\Delta = \beta$$
$$\delta = \text{Indeterminate}$$
$$\mu = b - \frac{1}{2}$$
$$a_1^* = \beta$$
$$a_2^* = 0$$
$$\xi = b$$
$$c^* = \frac{1}{2}$$

Poles   1. First ten poles from upper front list

$$a_{i,k} = \{\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\quad -\frac{b}{\beta} \quad -\frac{b+1}{\beta} \quad -\frac{b+2}{\beta} \quad -\frac{b+3}{\beta} \quad -\frac{b+4}{\beta} \quad -\frac{b+5}{\beta} \quad -\frac{b+6}{\beta} \quad -\frac{b+7}{\beta} \quad -\frac{b+8}{\beta} \quad -\frac{b+9}{\beta} \quad -\frac{b+10}{\beta} \quad \right)$$

2.5   Example `FoxH-2_9_5.wls`

File content

```
(* (2.9.5) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1-a,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *)  {}
  }
}
```

Fox H-function

$$H_{1,1}^{1,1} \left(\cdot \left| \begin{array}{c} (1-a,1) \\ \\ (0,1) \end{array} \right. \right)$$

$$H_{1,1}^{1,1} \left(\cdot \left| \begin{array}{c|c} (1-a,1) & \\ \hline (0,1) & \end{array} \right. \right)$$

Summary

$$a^* = 2$$
$$\Delta = 0$$
$$\delta = \text{Indeterminate}$$
$$\mu = a - 1$$
$$a_1^* = 1$$
$$a_2^* = 1$$
$$\xi = 1 - a$$
$$c^* = 1$$

Poles   1. First ten poles from upper front list

$$a_{i,k} = \left(\quad a \quad a+1 \quad a+2 \quad a+3 \quad a+4 \quad a+5 \quad a+6 \quad a+7 \quad a+8 \quad a+9 \quad a+10 \quad \right)$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\quad 0 \quad -1 \quad -2 \quad -3 \quad -4 \quad -5 \quad -6 \quad -7 \quad -8 \quad -9 \quad -10 \quad \right)$$

2.6   Example `FoxH-2_9_6.wls`

File content

```
(* (2.9.6) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {{α+β+1,1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{α, 1}},
    (* Lower Rear List *)  {}
  }
}
```

Fox H-function

$$H_{1,1}^{1,0} \left(\cdot \left| \begin{array}{c} (\alpha + \beta + 1, 1) \\ \\ (\alpha, 1) \end{array} \right. \right)$$

$$H_{1,1}^{1,0} \left(\cdot \left| \begin{array}{c|c} & (\alpha + \beta + 1, 1) \\ \hline (\alpha, 1) & \end{array} \right. \right)$$

Summary

$$a^* = 0$$
$$\Delta = 0$$
$$\delta = \text{Indeterminate}$$
$$\mu = -\beta - 1$$
$$a_1^* = 0$$
$$a_2^* = 0$$
$$\xi = -\beta - 1$$
$$c^* = 0$$

Poles   1. First ten poles from upper front list

$$a_{i,k} = \{\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{ccccccccccc} -\alpha & -\alpha-1 & -\alpha-2 & -\alpha-3 & -\alpha-4 & -\alpha-5 & -\alpha-6 & -\alpha-7 & -\alpha-8 & -\alpha-9 & -\alpha-10 \end{array} \right)$$

2.7   Example `FoxH32-21.wls`

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, α^(-1)}},
    (* Upper Rear List *)  {{Ceiling[β], β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c} \left(1, \frac{1}{\alpha}\right), \left(\lceil\beta\rceil, \beta\right) \\[2ex] \left(\frac{1}{2}, \frac{\alpha}{2}\right), (1,1), \left(1, \frac{\alpha}{2}\right) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c|c} \left(1, \frac{1}{\alpha}\right) & \left(\lceil\beta\rceil, \beta\right) \\ \hline \left(\frac{1}{2}, \frac{\alpha}{2}\right), (1,1) & \left(1, \frac{\alpha}{2}\right) \end{array} \right. \right)$$

Summary

$$a^* = \frac{1}{\alpha} - \beta + 1$$

$$\Delta = \alpha - \frac{1}{\alpha} - \beta + 1$$

$$\delta = 2^{-\alpha} \left(\frac{1}{\alpha}\right)^{-1/\alpha} \left(2^{\alpha/2} \alpha^{\alpha/2} + \alpha^\alpha\right) \beta^{-\beta}$$

$$\mu = 1 - \lceil \beta \rceil$$

$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$

$$a_2^* = \frac{1}{\alpha} - \frac{\alpha}{2}$$

$$\xi = \frac{3}{2} - \lceil \beta \rceil$$

$$c^* = \frac{1}{2}$$

Poles   1. First ten poles from upper front list

$$a_{i,k} = \begin{pmatrix} 0 & \alpha & 2\alpha & 3\alpha & 4\alpha & 5\alpha & 6\alpha & 7\alpha & 8\alpha & 9\alpha & 10\alpha \end{pmatrix}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{1}{\alpha} & -\frac{3}{\alpha} & -\frac{5}{\alpha} & -\frac{7}{\alpha} & -\frac{9}{\alpha} & -\frac{11}{\alpha} & -\frac{13}{\alpha} & -\frac{15}{\alpha} & -\frac{17}{\alpha} & -\frac{19}{\alpha} & -\frac{21}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{pmatrix}$$

2.8   Example `FoxH32-21-Y.wls`

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{β + γ, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c} (1,1), (\beta+\gamma, \beta) \\ \\ \left(\frac{d}{2}, \frac{\alpha}{2}\right), (1,1), \left(1, \frac{\alpha}{2}\right) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c|c} (1,1) & (\beta+\gamma, \beta) \\ \hline \left(\frac{d}{2}, \frac{\alpha}{2}\right), (1,1) & \left(1, \frac{\alpha}{2}\right) \end{array} \right. \right)$$

Summary

$$a^* = 2 - \beta$$
$$\Delta = \alpha - \beta$$
$$\delta = 2^{-\alpha} \left(2^{\alpha/2} \alpha^{\alpha/2} + \alpha^\alpha \right) \beta^{-\beta}$$
$$\mu = \frac{1}{2}(-2\beta - 2\gamma + d + 1)$$
$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$
$$a_2^* = 1 - \frac{\alpha}{2}$$
$$\xi = \frac{1}{2}(d - 2(\beta + \gamma - 1))$$
$$c^* = \frac{1}{2}$$

Poles   1. First ten poles from upper front list

$$a_{i,k} = \left(\begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \right)$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{ccccccccccc} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} & -\frac{d+16}{\alpha} & -\frac{d+18}{\alpha} & -\frac{d+20}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{array} \right)$$

2.9   Example `FoxH32-21-Z.wls`

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
```

```
    (* Upper Rear List *)  {{Ceiling[β], β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1}\left(\cdot \left|\begin{array}{c} (1,1),(\lceil\beta\rceil,\beta) \\ \\ \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1),\left(1,\frac{\alpha}{2}\right) \end{array}\right.\right)$$

$$H_{2,3}^{2,1}\left(\cdot \left|\begin{array}{c|c} (1,1) & (\lceil\beta\rceil,\beta) \\ \hline \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1) & \left(1,\frac{\alpha}{2}\right) \end{array}\right.\right)$$

Summary

$$a^* = 2 - \beta$$
$$\Delta = \alpha - \beta$$
$$\delta = 2^{-\alpha}\left(2^{\alpha/2}\alpha^{\alpha/2} + \alpha^\alpha\right)\beta^{-\beta}$$
$$\mu = \frac{1}{2}(-2\lceil\beta\rceil + d + 1)$$
$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$
$$a_2^* = 1 - \frac{\alpha}{2}$$
$$\xi = \frac{1}{2}(-2\lceil\beta\rceil + d + 2)$$
$$c^* = \frac{1}{2}$$

Poles   1. First ten poles from upper front list

$$a_{i,k} = \left(\begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}\right)$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{ccccccccccc} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} & -\frac{d+16}{\alpha} & -\frac{d+18}{\alpha} & -\frac{d+20}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{array}\right)$$

2.10 Example `FoxH32-21-Z-Star.wls`

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{1, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1}\left(\cdot \left| \begin{array}{c} (1,1),(1,\beta) \\ \\ \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1),\left(1,\frac{\alpha}{2}\right) \end{array} \right.\right)$$

$$H_{2,3}^{2,1}\left(\cdot \left| \begin{array}{c|c} (1,1) & (1,\beta) \\ \hline \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1) & \left(1,\frac{\alpha}{2}\right) \end{array} \right.\right)$$

Summary

$$a^* = 2 - \beta$$
$$\Delta = \alpha - \beta$$
$$\delta = 2^{-\alpha}\left(2^{\alpha/2}\alpha^{\alpha/2} + \alpha^{\alpha}\right)\beta^{-\beta}$$
$$\mu = \frac{d-1}{2}$$
$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$
$$a_2^* = 1 - \frac{\alpha}{2}$$
$$\xi = \frac{d}{2}$$
$$c^* = \frac{1}{2}$$

Poles   1. First ten poles from upper front list

$$a_{i,k} = \left(\begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \right)$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} & -\frac{d+16}{\alpha} & -\frac{d+18}{\alpha} & -\frac{d+20}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{pmatrix}$$

2.11   Example `FoxH-Bessel-J_2_9_18.wls`

File content

```
(* (2.9.18) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(a+η)/2, 1}},
    (* Lower Rear List *)  {{(a-η)/1, 1}}
  }
}
```

Fox H-function

$$H^{1,0}_{0,2}\left(\cdot \left|\begin{array}{c} \\ \left(\frac{a+\eta}{2},1\right),(a-\eta,1) \end{array}\right.\right)$$

$$H^{1,0}_{0,2}\left(\cdot \left|\begin{array}{c|c} & \\ \hline \left(\frac{a+\eta}{2},1\right) & (a-\eta,1) \end{array}\right.\right)$$

Summary

$$a^* = 0$$
$$\Delta = 2$$
$$\delta = \text{ComplexInfinity}$$
$$\mu = \frac{1}{2}(3a - \eta - 2)$$
$$a^*_1 = 1$$
$$a^*_2 = -1$$
$$\xi = \frac{1}{2}(3\eta - a)$$
$$c^* = 0$$

Poles **1. First ten poles from upper front list**

$$a_{i,k} = \{\{\},\{\},\{\},\{\},\{\},\{\},\{\},\{\},\{\},\{\},\{\}\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} \frac{1}{2}(-a-\eta) \\ \frac{1}{2}(-a-\eta-2) \\ \frac{1}{2}(-a-\eta-4) \\ \frac{1}{2}(-a-\eta-6) \\ \frac{1}{2}(-a-\eta-8) \\ -\frac{a}{2}-\frac{\eta}{2}-5 \\ -\frac{a}{2}-\frac{\eta}{2}-6 \\ -\frac{a}{2}-\frac{\eta}{2}-7 \\ -\frac{a}{2}-\frac{\eta}{2}-8 \\ -\frac{a}{2}-\frac{\eta}{2}-9 \\ \frac{1}{2}(-a-\eta-20) \end{pmatrix}$$

2.12 Example `FoxH-Bessel-K_2_9_19.wls`

File content

```
(* (2.9.19) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(a-η)/2, 1},{(a+η)/2, 1}},
    (* Lower Rear List *)  {}
  }
}
```

Fox H-function

$$H_{0,2}^{2,0}\left(\cdot \,\middle|\, \left(\tfrac{a-\eta}{2},1\right),\left(\tfrac{a+\eta}{2},1\right)\right)$$

$$H_{0,2}^{2,0} \left(\cdot \left| \begin{array}{c} \\ \hline \left(\frac{a-\eta}{2}, 1\right), \left(\frac{a+\eta}{2}, 1\right) \end{array} \right| \right)$$

Summary

$$a^* = 2$$
$$\Delta = 2$$
$$\delta = \text{Indeterminate}$$
$$\mu = a - 1$$
$$a_1^* = 2$$
$$a_2^* = 0$$
$$\xi = a$$
$$c^* = 1$$

Poles   1. First ten poles from upper front list

$$a_{i,k} = \{\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} \frac{\eta-a}{2} & \frac{1}{2}(-a+\eta-2) & \frac{1}{2}(-a+\eta-4) & \frac{1}{2}(-a+\eta-6) & \frac{1}{2}(-a+\eta-8) & \frac{1}{2}(-a+\eta-10) & \frac{1}{2}(-a+\eta \\ \frac{1}{2}(-a-\eta) & \frac{1}{2}(-a-\eta-2) & \frac{1}{2}(-a-\eta-4) & \frac{1}{2}(-a-\eta-6) & \frac{1}{2}(-a-\eta-8) & -\frac{a}{2}-\frac{\eta}{2}-5 & -\frac{a}{2}-\frac{\eta}{2} \end{pmatrix}$$

2.13   Example `FoxH-Bessel-Y_2_9_20.wls`

File content

```
(* (2.9.20) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {{(a-η-1)/2, 1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(a-η)/2, 1},{(a+η)/2, 1}},
    (* Lower Rear List *)  {{(a-η-1)/2, 1}}
  }
}
```

Fox H-function

$$H_{1,3}^{2,0}\left(\cdot\,\middle|\, \begin{array}{c} \left(\frac{1}{2}(a-\eta-1),1\right) \\[2mm] \left(\frac{a-\eta}{2},1\right),\left(\frac{a+\eta}{2},1\right),\left(\frac{1}{2}(a-\eta-1),1\right) \end{array}\right)$$

$$H_{1,3}^{2,0}\left(\cdot\,\middle|\, \begin{array}{c|c} & \left(\frac{1}{2}(a-\eta-1),1\right) \\[2mm] \hline \left(\frac{a-\eta}{2},1\right),\left(\frac{a+\eta}{2},1\right) & \left(\frac{1}{2}(a-\eta-1),1\right) \end{array}\right)$$

Summary

$$a^* = 0$$
$$\Delta = 2$$
$$\delta = \text{ComplexInfinity}$$
$$\mu = a - 1$$
$$a_1^* = 1$$
$$a_2^* = -1$$
$$\xi = \eta + 1$$
$$c^* = 0$$

Poles   **1. First ten poles from upper front list**

$$a_{i,k} = \{\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} \frac{\eta-a}{2} & \frac{1}{2}(-a+\eta-2) & \frac{1}{2}(-a+\eta-4) & \frac{1}{2}(-a+\eta-6) & \frac{1}{2}(-a+\eta-8) & \frac{1}{2}(-a+\eta-10) & \frac{1}{2}(-a+\eta \\[2mm] \frac{1}{2}(-a-\eta) & \frac{1}{2}(-a-\eta-2) & \frac{1}{2}(-a-\eta-4) & \frac{1}{2}(-a-\eta-6) & \frac{1}{2}(-a-\eta-8) & -\frac{a}{2}-\frac{\eta}{2}-5 & -\frac{a}{2}-\frac{\eta}{2} \end{pmatrix}$$

2.14   Example `FoxH-Cos.wls`

File content

```
(* (2.9.8) and (2.9.10) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *)  {{1/2,1}}
  }
}
```

Fox H-function

$$H_{0,2}^{1,0}\left(\ \cdot\ \middle|\ \begin{array}{c} \\ (0,1),\left(\tfrac{1}{2},1\right) \end{array}\right)$$

$$H_{0,2}^{1,0}\left(\ \cdot\ \middle|\ \begin{array}{c|c} & \\ \hline (0,1) & \left(\tfrac{1}{2},1\right) \end{array}\right)$$

Summary

$$a^* = 0$$
$$\Delta = 2$$
$$\delta = \text{ComplexInfinity}$$
$$\mu = -\frac{1}{2}$$
$$a_1^* = 1$$
$$a_2^* = -1$$
$$\xi = -\frac{1}{2}$$
$$c^* = 0$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \{\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{ccccccccccc} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{array}\right)$$

2.15 Example `FoxH-H.G_2_9_14.wls`

File content

```
(* (2.9.14) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1-a,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0,1}},
    (* Lower Rear List *)  {{1-c,1}}
  }
}
```

Fox H-function

$$H_{1,2}^{1,1} \left(\cdot \left| \begin{array}{c} (1-a,1) \\ \\ (0,1),(1-c,1) \end{array} \right. \right)$$

$$H_{1,2}^{1,1} \left(\cdot \left| \begin{array}{c|c} (1-a,1) & \\ \hline (0,1) & (1-c,1) \end{array} \right. \right)$$

Summary

$$a^* = 1$$
$$\Delta = 1$$
$$\delta = \text{ComplexInfinity}$$
$$\mu = a - c - \frac{1}{2}$$
$$a_1^* = 1$$
$$a_2^* = 0$$
$$\xi = c - a$$
$$c^* = \frac{1}{2}$$

Poles   1. First ten poles from upper front list

$$a_{i,k} = \left(\begin{array}{ccccccccccc} a & a+1 & a+2 & a+3 & a+4 & a+5 & a+6 & a+7 & a+8 & a+9 & a+10 \end{array} \right)$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{ccccccccccc} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{array} \right)$$

2.16   Example `FoxH-H.G_2_9_15.wls`

File content

```
(* (2.9.15) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1-a,1},{1-b,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0,1}},
    (* Lower Rear List *)  {{1-c,1}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,2}\left(\cdot \left| \begin{array}{c} (1-a,1),(1-b,1) \\ \\ (0,1),(1-c,1) \end{array} \right. \right)$$

$$H_{2,2}^{1,2}\left(\cdot \left| \begin{array}{c|c} (1-a,1),(1-b,1) & \\ \hline (0,1) & (1-c,1) \end{array} \right. \right)$$

Summary

$$a^* = 2$$
$$\Delta = 0$$
$$\delta = \text{ComplexInfinity}$$
$$\mu = a + b - c - 1$$
$$a_1^* = 1$$
$$a_2^* = 1$$
$$\xi = -a - b + c + 1$$
$$c^* = 1$$

Poles **1. First ten poles from upper front list**

$$a_{i,k} = \begin{pmatrix} a & a+1 & a+2 & a+3 & a+4 & a+5 & a+6 & a+7 & a+8 & a+9 & a+10 \\ b & b+1 & b+2 & b+3 & b+4 & b+5 & b+6 & b+7 & b+8 & b+9 & b+10 \end{pmatrix}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{pmatrix}$$

2.17 Example `FoxH-Lommel_2_9_22.wls`

File content

```
(* (2.9.22) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{(1+µ)/2,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
```

```
    (* Lower Front List *) {{(1+µ)/2,1},{η/2,1},{-η/2,1}},
    (* Lower Rear List *)  {}
  }
}
```

Fox H-function

$$H_{1,3}^{3,1}\left(\cdot\left|\begin{array}{c}\left(\frac{\mu+1}{2},1\right)\\ \left(\frac{\mu+1}{2},1\right),\left(\frac{\eta}{2},1\right),\left(-\frac{\eta}{2},1\right)\end{array}\right.\right)$$

$$H_{1,3}^{3,1}\left(\cdot\left|\begin{array}{c|c}\left(\frac{\mu+1}{2},1\right)\\\hline\left(\frac{\mu+1}{2},1\right),\left(\frac{\eta}{2},1\right),\left(-\frac{\eta}{2},1\right)\end{array}\right|\right)$$

Summary

$$a^* = 4$$
$$\Delta = 2$$
$$\delta = \text{Indeterminate}$$
$$\mu = -1$$
$$a_1^* = 3$$
$$a_2^* = 1$$
$$\xi = \mu + 1$$
$$c^* = 2$$

Poles   1. First ten poles from upper front list

$$a_{i,k} = \left(\begin{array}{cccccccccc} \frac{1-\mu}{2} & \frac{3-\mu}{2} & \frac{5-\mu}{2} & \frac{7-\mu}{2} & \frac{9-\mu}{2} & \frac{11-\mu}{2} & \frac{13-\mu}{2} & \frac{15-\mu}{2} & \frac{17-\mu}{2} & \frac{19-\mu}{2} & \frac{21-\mu}{2} \end{array}\right)$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{ccccccccc} \frac{1}{2}(-\mu-1) & \frac{1}{2}(-\mu-3) & \frac{1}{2}(-\mu-5) & \frac{1}{2}(-\mu-7) & \frac{1}{2}(-\mu-9) & \frac{1}{2}(-\mu-11) & \frac{1}{2}(-\mu-13) & \frac{1}{2}(-\mu-15) & \frac{1}{2}(\\ -\frac{\eta}{2} & -\frac{\eta}{2}-1 & -\frac{\eta}{2}-2 & -\frac{\eta}{2}-3 & -\frac{\eta}{2}-4 & -\frac{\eta}{2}-5 & -\frac{\eta}{2}-6 & -\frac{\eta}{2}-7 \\ \frac{\eta}{2} & \frac{\eta-2}{2} & \frac{\eta-4}{2} & \frac{\eta-6}{2} & \frac{\eta-8}{2} & \frac{\eta}{2}-5 & \frac{\eta}{2}-6 & \frac{\eta}{2}-7 \end{array}\right.$$

2.18 Example `FoxH-Mittag-Leffler.wls`

File content

```
(* (2.9.27) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{0, 1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *)  {{1 - μ, ρ}}
  }
}
```

Fox H-function

$$
H_{1,2}^{1,1} \left(\cdot \, \middle| \, \begin{matrix} (0,1) \\ \\ (0,1) \,, (1-\mu,\rho) \end{matrix} \right)
$$

$$
H_{1,2}^{1,1} \left(\cdot \, \middle| \, \begin{array}{c|c} (0,1) & \\ \hline (0,1) & (1-\mu,\rho) \end{array} \right)
$$

Summary

$$
\begin{aligned}
a^* &= 2 - \rho \\
\Delta &= \rho \\
\delta &= \text{ComplexInfinity} \\
\mu &= \frac{1}{2} - \mu \\
a_1^* &= 1 \\
a_2^* &= 1 - \rho \\
\xi &= \mu - 1 \\
c^* &= \frac{1}{2}
\end{aligned}
$$

Poles **1. First ten poles from upper front list**

$$
a_{i,k} = \left(\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{matrix} \right)
$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{pmatrix}$$

2.19 Example `FoxH-Sin.wls`

File content

```
(* (2.9.7) and (2.9.9) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2, 1}},
    (* Lower Rear List *)  {{0, 1}}
  }
}
```

Fox H-function

$$H_{0,2}^{1,0}\left(\cdot \left|\begin{array}{c} \\ \left(\tfrac{1}{2},1\right),(0,1) \end{array}\right.\right)$$

$$H_{0,2}^{1,0}\left(\cdot \left|\begin{array}{c|c} & \\ \hline \left(\tfrac{1}{2},1\right) & (0,1) \end{array}\right.\right)$$

Summary

$$a^* = 0$$
$$\Delta = 2$$
$$\delta = \text{ComplexInfinity}$$
$$\mu = -\frac{1}{2}$$
$$a_1^* = 1$$
$$a_2^* = -1$$
$$\xi = \frac{1}{2}$$
$$c^* = 0$$

Poles   **1. First ten poles from upper front list**

$$a_{i,k} = \{\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{cccccccccc} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} & -\frac{7}{2} & -\frac{9}{2} & -\frac{11}{2} & -\frac{13}{2} & -\frac{15}{2} & -\frac{17}{2} & -\frac{19}{2} & -\frac{21}{2} \end{array} \right)$$

2.20   Example `FoxH-Whittaker_2_9_21.wls`

File content

```
(* (2.9.21) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {{a-λ+1,1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{a+μ+1/2,1},{a-μ+1/2,1}},
    (* Lower Rear List *)  {}
  }
}
```

Fox H-function

$$H_{1,2}^{2,0} \left(\cdot \left| \begin{array}{c} (a - \lambda + 1, 1) \\[2ex] \left(a + \mu + \frac{1}{2}, 1\right), \left(a - \mu + \frac{1}{2}, 1\right) \end{array} \right. \right)$$

$$H_{1,2}^{2,0} \left(\cdot \left| \frac{ \left| (a - \lambda + 1, 1) \right.}{\left(a + \mu + \frac{1}{2}, 1\right), \left(a - \mu + \frac{1}{2}, 1\right) \left| \right.} \right. \right)$$

Summary

$$a^* = 1$$
$$\Delta = 1$$
$$\delta = \text{Indeterminate}$$
$$\mu = a + \lambda - \frac{1}{2}$$
$$a_1^* = 1$$
$$a_2^* = 0$$
$$\xi = a + \lambda$$
$$c^* = \frac{1}{2}$$

Poles   1. First ten poles from upper front list

$$a_{i,k} = \{\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -a - \mu - \frac{1}{2} & -a - \mu - \frac{3}{2} & -a - \mu - \frac{5}{2} & -a - \mu - \frac{7}{2} & -a - \mu - \frac{9}{2} & -a - \mu - \frac{11}{2} & -a - \mu - \frac{13}{2} & -a - \mu - \\ -a + \mu - \frac{1}{2} & -a + \mu - \frac{3}{2} & -a + \mu - \frac{5}{2} & -a + \mu - \frac{7}{2} & -a + \mu - \frac{9}{2} & -a + \mu - \frac{11}{2} & -a + \mu - \frac{13}{2} & -a + \mu - \end{pmatrix}$$

References

[Fox61] Charles Fox. The G and H functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, 98:395–429, 1961.

[KS]   Anatoly A. Kilbas and Megumi Saigo. *H-transforms*, volume 9 of *Analytical Methods and Special Functions*. Chapman & Hall/CRC, Boca Raton, FL. Theory and applications.