Some symbolic tools for the Fox H-function

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1 Introduction

In this note, we explain the code for checking the conditions of the Fox H-function [Fox61]. Here we follow the notation from Kilbas and Saigo [KS04].

Let m, n, p, q be configure integers such that

$$0 \le m \le q \quad \text{and} \quad 0 \le n \le p.$$

Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+$ be the parameters given below:

$\in (\mathbb{C}, \mathbb{R}_+)$	Front list	Rear list	
p	$(a_1, \alpha_1), \cdots, (a_n, \alpha_n)$	$(a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p)$	Upper list
q	$(b_1, \beta_1), \cdots, (b_m, \beta_m)$	$(b_{m+1}, \beta_{m+1}), \cdots, (b_q, \beta_q)$	Lower list

and denote

$$\mathcal{H}_{p,q}^{m,n}(s) := \frac{\prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_j + \alpha_i s)} \times \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - \alpha_j s)}. \tag{1}$$

Then the Fox H-function $H_{2,3}^{2,1}\left(z \,\middle|\, \begin{matrix} \cdots \\ \\ \cdots \end{matrix}\right)$ is defined by a Mellin-Barnes type integral of the form

$$H_{m,n}^{p,q}\left(z \,\middle|\, \begin{matrix} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ \\ (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{matrix}\right) := \frac{1}{2\pi i} \int_{\mathcal{L}} H_{p,q}^{m,n}(s) z^{-s} \mathrm{d}s. \tag{2}$$

It is more convenient to use the following notation:

$$H_{m,n}^{p,q}\left(z \,\middle|\, \begin{array}{c|c} (a_1, \alpha_1), \cdots, (a_n, \alpha_n) & (a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p) \\ \hline (b_1, \beta), \cdots, (b_m, \beta_m) & (b_{m+1}, \beta_{m+1}), \cdots, (b_q, \beta_q) \end{array}\right),$$

in order to emphasize the front and rear lists.

The basic assumption for the well-posedness of the Fox H-function is that two sets of poles do not overlap, i.e.,

$$\left\{ b_{j\ell} = \frac{-b_j - \ell}{\beta_j}, \ell = 0, 1, \cdots \right\} \bigcap \left\{ a_{ik} = \frac{1 - a_i + k}{\alpha_i}, k = 0, 1, \cdots \right\} = \emptyset. \tag{3}$$

The contour $\mathcal{L}$ in (2) is given by one of the following three cases:

1. $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at point $-\infty + i\phi_1$ and terminating at point $-\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;

2. $\mathcal{L} = \mathcal{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at point $+\infty + i\phi_1$ and terminating at point $\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;

3. $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ is a contour starting at point $\gamma - i\infty$ and terminating at point $\gamma + i\infty$ for some $\gamma \in (-\infty, \infty)$.

We need a set of conditions to ensure the convergence of the integral in (2). To explain this, we need to introduce some notation (following p. 2 of [KS04]). First denote

$$a_1^* := \sum_{j=1}^{m} \beta_j - \sum_{i=n+1}^{p} \alpha_i,$$
$$a_2^* := \sum_{i=1}^{n} \alpha_i - \sum_{j=m+1}^{q} \beta_j.$$

The following two parameters play the most important role:

$$a^* := a_2^* + a_1^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j;$$
$$\Delta := a_2^* - a_1^* = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i.$$

Similar to a^*, we define

$$\xi := \sum_{i=1}^{n} a_i - \sum_{i=n+1}^{p} a_i + \sum_{j=1}^{m} b_j - \sum_{j=m+1}^{q} b_j.$$

Additionally, set

$$c^* := m + n - \frac{p+q}{2}.$$

In the critical cases, we need to use the following two parameters:

$$\delta := \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_i};$$
$$\mu := \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}$$

The well-posedness of the Fox H-function is given by Theorems 1.1 and 1.2 of [KS04], which are summarized in the following figure 2:

We will need the following two parameters (see (1.5.15)/(1.5.16) and (1.8.10)/(1.8.11), respectively, of [KS04] for ρ and ρ^*):

$$\rho := \max_{1\le i\le n} \left[\frac{\Re(a_i) - 1}{\alpha_i}\right] \quad \text{and} \quad \rho^* := \min_{1\le j\le m} \left[\frac{\Re(b_i)}{\beta_j}\right].$$

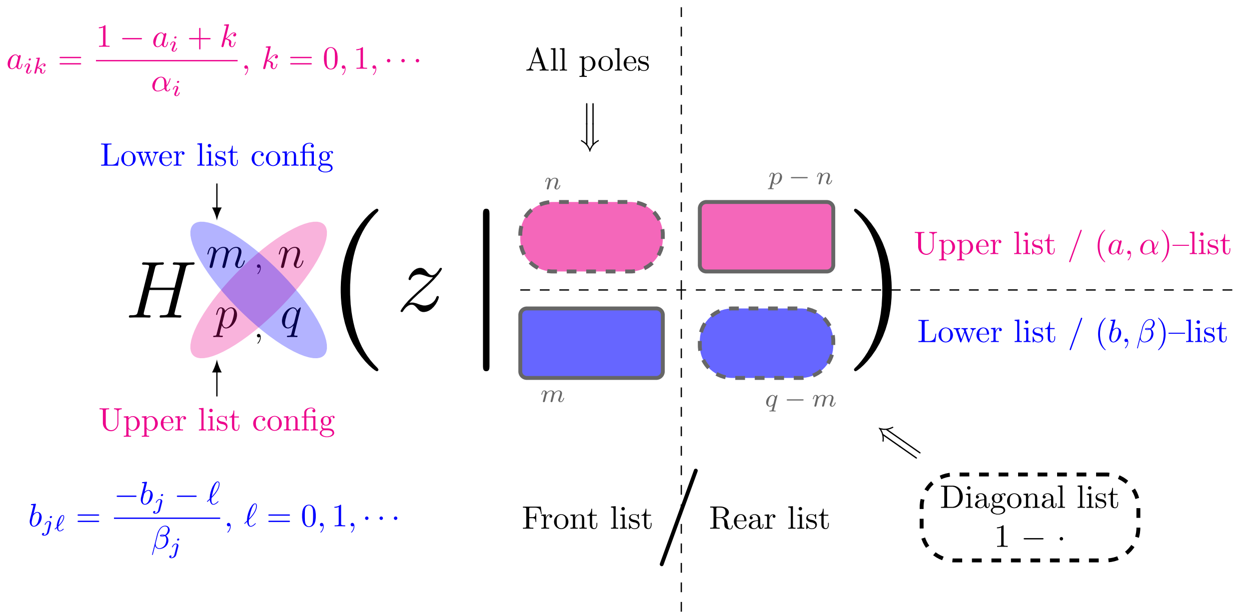
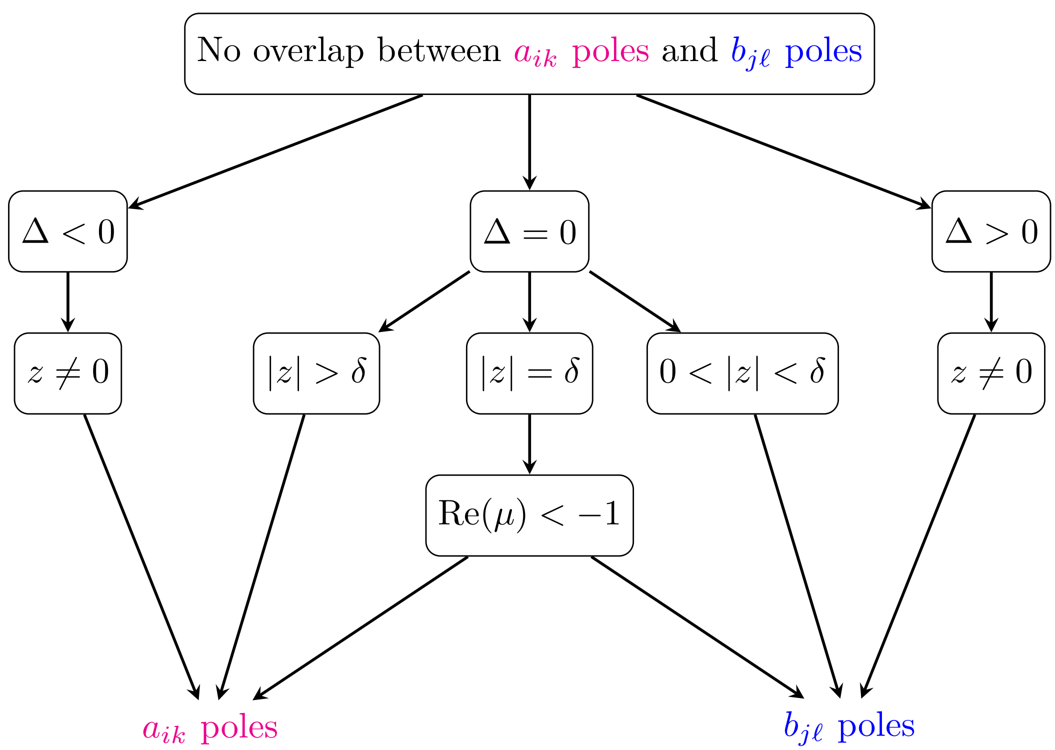


Figure 1: Diagram for the parameterization of the Fox H-function.



$$H^{m,n}_{p,q}(z) = \sum_{i=1}^{n} \sum_{k=0}^{\infty} \operatorname*{Res}_{s=a_{ik}} \left[\mathcal{H}^{m,n}_{p,q}(s) z^{-s}\right] \qquad H^{m,n}_{p,q}(z) = \sum_{j=1}^{m} \sum_{\ell=0}^{\infty} \operatorname*{Res}_{s=b_{j\ell}} \left[\mathcal{H}^{m,n}_{p,q}(s) z^{-s}\right]$$

Figure 2: Well-posedness of the Fox H-function.

2 Examples

2.1 Example `FoxH-2_9_11.wls`

File content

```
(* (2.9.11) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {{1,1},{1,1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1,1}},
    (* Lower Rear List *)  {{0,0}}
  }
}
```

Fox H-function

$$H^{1,0}_{2,2}\left(\cdot\ \middle|\ \begin{matrix}(1,1)\,,(1,1)\\ \\ (1,1)\,,(0,0)\end{matrix}\right)$$

$$H^{1,0}_{2,2}\left(\cdot\ \middle|\ \begin{array}{c|c} & (1,1)\,,(1,1)\\ \hline (1,1) & (0,0)\end{array}\right)$$

Summary

$$\begin{aligned} a^* &= -1\\ \Delta &= -1\\ \delta &= \text{Indeterminate}\\ \mu &= -1\\ a_1^* &= -1\\ a_2^* &= 0\\ \xi &= -1\\ c^* &= -1 \end{aligned}$$

Poles **1. First eight poles from upper front list**

$$a_{i,k} = \{\}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{pmatrix}$$

2.2 Example `FoxH-2_9_12.wls`

File content

```
(* (2.9.12) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1/2,1},{1/2,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0,1}},
    (* Lower Rear List *)  {{-1/2,1}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,2}\left(\cdot \left| \begin{matrix} \left(\frac{1}{2},1\right),\left(\frac{1}{2},1\right) \\ \\ (0,1),\left(-\frac{1}{2},1\right) \end{matrix} \right.\right)$$

$$H_{2,2}^{1,2}\left(\cdot \left| \begin{array}{c|c} \left(\frac{1}{2},1\right),\left(\frac{1}{2},1\right) & \\ \hline (0,1) & \left(-\frac{1}{2},1\right) \end{array} \right.\right)$$

Summary

$$\begin{aligned} a^* &= 2 \\ \Delta &= 0 \\ \delta &= \text{ComplexInfinity} \\ \mu &= -\frac{3}{2} \\ a_1^* &= 1 \\ a_2^* &= 1 \\ \xi &= \frac{3}{2} \\ c^* &= 1 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} \end{pmatrix}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

2.3 Example `FoxH-2_9_13.wls`

File content

```
(* (2.9.13) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1/2,1},{1,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2,1}},
    (* Lower Rear List *)  {{0,1}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,2}\left(\cdot \,\middle|\, \begin{matrix} \left(\frac{1}{2},1\right),(1,1) \\ \\ \left(\frac{1}{2},1\right),(0,1) \end{matrix}\right)$$

$$H_{2,2}^{1,2}\left(\cdot \,\middle|\, \begin{array}{c|c} \left(\frac{1}{2},1\right),(1,1) & \\ \hline \left(\frac{1}{2},1\right) & (0,1) \end{array}\right)$$

Summary

$$\begin{aligned} a^* &= 2 \\ \Delta &= 0 \\ \delta &= \text{ComplexInfinity} \\ \mu &= -1 \\ a_1^* &= 1 \\ a_2^* &= 1 \\ \xi &= 2 \\ c^* &= 1 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} & -\frac{7}{2} & -\frac{9}{2} & -\frac{11}{2} & -\frac{13}{2} & -\frac{15}{2} \end{pmatrix}$$

2.4 Example `FoxH-2_9_4.wls`

File content

```
(* (2.9.4) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{b, β}},
    (* Lower Rear List *)  {}
  }
}
```

Fox H-function

$$H^{1,0}_{0,1}\left(\cdot \,\middle|\, \begin{array}{c} \\ (b,\beta) \end{array}\right)$$

$$H^{1,0}_{0,1}\left(\cdot \,\middle|\, \begin{array}{c|c} & \\ \hline (b,\beta) & \end{array}\right)$$

Summary

$$\begin{aligned}
a^* &= \beta \\
\Delta &= \beta \\
\delta &= \text{Indeterminate} \\
\mu &= b - \frac{1}{2} \\
a_1^* &= \beta \\
a_2^* &= 0 \\
\xi &= b \\
c^* &= \frac{1}{2}
\end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{\}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{b}{\beta} & -\frac{b+1}{\beta} & -\frac{b+2}{\beta} & -\frac{b+3}{\beta} & -\frac{b+4}{\beta} & -\frac{b+5}{\beta} & -\frac{b+6}{\beta} & -\frac{b+7}{\beta} \end{pmatrix}$$

Source This example is from (2.9.4) of [KS04]:

$$H_{0,1}^{1,0}\left(z \,\middle|\, \begin{matrix} \\ (b,\beta) \end{matrix}\right) = \frac{1}{\beta} z^{b/\beta} \exp\left(-z^{1/\beta}\right).$$

2.5 Example `FoxH-2_9_5.wls`

File content

```
(* (2.9.5) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1-a,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *)  {}
  }
}
```

Fox H-function

$$H_{1,1}^{1,1}\left(\cdot\;\middle|\;\begin{matrix}(1-a,1)\\ \\ (0,1)\end{matrix}\right)$$

$$H_{1,1}^{1,1}\left(\cdot\;\middle|\;\begin{array}{c|c}(1-a,1) & \\ \hline (0,1) & \end{array}\right)$$

Summary

$$\begin{aligned}a^* &= 2\\ \Delta &= 0\\ \delta &= \text{Indeterminate}\\ \mu &= a-1\\ a_1^* &= 1\\ a_2^* &= 1\\ \xi &= 1-a\\ c^* &= 1\end{aligned}$$

Poles

1. First eight poles from upper front list

$$a_{i,k} = \begin{pmatrix} a & a+1 & a+2 & a+3 & a+4 & a+5 & a+6 & a+7 \end{pmatrix}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

2.6 Example `FoxH-2_9_6.wls`

File content

```
(* (2.9.6) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {{α+β+1,1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{α, 1}},
    (* Lower Rear List *)  {}
  }
}
```

Fox H-function

$$H_{1,1}^{1,0}\left(\cdot \,\middle|\, \begin{array}{l} (\alpha+\beta+1,1) \\ \\ (\alpha,1) \end{array}\right)$$

$$H_{1,1}^{1,0}\left(\cdot \,\middle|\, \begin{array}{c|c} & (\alpha+\beta+1,1) \\ \hline (\alpha,1) & \end{array}\right)$$

Summary

$$\begin{aligned} a^* &= 0 \\ \Delta &= 0 \\ \delta &= \text{Indeterminate} \\ \mu &= -\beta-1 \\ a_1^* &= 0 \\ a_2^* &= 0 \\ \xi &= -\beta-1 \\ c^* &= 0 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{\}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\alpha & -\alpha-1 & -\alpha-2 & -\alpha-3 & -\alpha-4 & -\alpha-5 & -\alpha-6 & -\alpha-7 \end{pmatrix}$$

2.7 Example `FoxH32-21-Y.wls`

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{β + γ, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1}\left(\cdot\ \middle|\ \begin{matrix} (1,1)\,,(\beta+\gamma,\beta) \\ \\ \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1)\,,\left(1,\frac{\alpha}{2}\right) \end{matrix}\right)$$

$$H_{2,3}^{2,1}\left(\cdot\ \middle|\ \begin{array}{c|c} (1,1) & (\beta+\gamma,\beta) \\ \hline \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1) & \left(1,\frac{\alpha}{2}\right) \end{array}\right)$$

Summary

$$\begin{aligned} a^* &= 2-\beta \\ \Delta &= \alpha-\beta \\ \delta &= 2^{-\alpha}\left(2^{\alpha/2}\alpha^{\alpha/2}+\alpha^{\alpha}\right)\beta^{-\beta} \\ \mu &= \frac{1}{2}(-2\beta-2\gamma+d+1) \\ a_1^* &= \frac{1}{2}(\alpha-2\beta+2) \\ a_2^* &= 1-\frac{\alpha}{2} \\ \xi &= \frac{1}{2}(d-2(\beta+\gamma-1)) \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{pmatrix}$$

2.8 Example `FoxH32-21-Z.wls`

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{Ceiling[β], β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1}\left(\cdot\ \middle|\ \begin{array}{l} (1,1)\,,(\lceil\beta\rceil,\beta) \\ \\ \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1)\,,\left(1,\frac{\alpha}{2}\right) \end{array}\right)$$

$$H_{2,3}^{2,1}\left(\cdot\ \middle|\ \begin{array}{c|c} (1,1) & (\lceil\beta\rceil,\beta) \\ \hline \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1) & \left(1,\frac{\alpha}{2}\right) \end{array}\right)$$

Summary

$$\begin{aligned} a^* &= 2-\beta \\ \Delta &= \alpha-\beta \\ \delta &= 2^{-\alpha}\left(2^{\alpha/2}\alpha^{\alpha/2}+\alpha^{\alpha}\right)\beta^{-\beta} \\ \mu &= \frac{1}{2}(-2\lceil\beta\rceil+d+1) \\ a_1^* &= \frac{1}{2}(\alpha-2\beta+2) \\ a_2^* &= 1-\frac{\alpha}{2} \\ \xi &= \frac{1}{2}(-2\lceil\beta\rceil+d+2) \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{pmatrix}$$

Source This is the fundamental solution to the fractional diffusion equation used, e.g., in [Che+17; CHN19; CE22; CGS22].

2.9 Example FoxH32-21-Z-Star.wls

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{1, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1}\left(\cdot\;\middle|\;\begin{array}{l}(1,1)\,,(1,\beta)\\ \\ \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1)\,,\left(1,\frac{\alpha}{2}\right)\end{array}\right)$$

$$H_{2,3}^{2,1}\left(\cdot\;\middle|\;\begin{array}{c|c}(1,1) & (1,\beta)\\ \hline \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1) & \left(1,\frac{\alpha}{2}\right)\end{array}\right)$$

Summary

$$\begin{aligned}
a^* &= 2-\beta\\
\Delta &= \alpha-\beta\\
\delta &= 2^{-\alpha}\left(2^{\alpha/2}\alpha^{\alpha/2}+\alpha^{\alpha}\right)\beta^{-\beta}\\
\mu &= \frac{d-1}{2}\\
a_1^* &= \frac{1}{2}(\alpha-2\beta+2)\\
a_2^* &= 1-\frac{\alpha}{2}\\
\xi &= \frac{d}{2}\\
c^* &= \frac{1}{2}
\end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k}=\begin{pmatrix}0 & 1 & 2 & 3 & 4 & 5 & 6 & 7\end{pmatrix}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{pmatrix}$$

2.10 Example `FoxH-Bessel-J_2_9_18.wls`

File content

```
(* (2.9.18) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(a+η)/2, 1}},
    (* Lower Rear List *)  {{(a-η)/1, 1}}
  }
}
```

Fox H-function

$$H_{0,2}^{1,0}\left(\cdot\ \middle|\ \begin{array}{c} \\ \left(\frac{a+\eta}{2},1\right),(a-\eta,1)\end{array}\right)$$

$$H_{0,2}^{1,0}\left(\cdot\ \middle|\ \begin{array}{c|c} & \\ \hline \left(\frac{a+\eta}{2},1\right) & (a-\eta,1)\end{array}\right)$$

Summary

$$\begin{aligned}
a^* &= 0\\
\Delta &= 2\\
\delta &= \text{ComplexInfinity}\\
\mu &= \frac{1}{2}(3a-\eta-2)\\
a_1^* &= 1\\
a_2^* &= -1\\
\xi &= \frac{1}{2}(3\eta-a)\\
c^* &= 0
\end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{\}^T$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} \frac{1}{2}(-a-\eta) \\ \frac{1}{2}(-a-\eta-2) \\ \frac{1}{2}(-a-\eta-4) \\ \frac{1}{2}(-a-\eta-6) \\ \frac{1}{2}(-a-\eta-8) \\ -\frac{a}{2}-\frac{\eta}{2}-5 \\ -\frac{a}{2}-\frac{\eta}{2}-6 \\ -\frac{a}{2}-\frac{\eta}{2}-7 \end{pmatrix}^T$$

2.11 Example `FoxH-Bessel-K_2_9_19.wls`

File content

```
(* (2.9.19) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(a-η)/2, 1},{(a+η)/2, 1}},
    (* Lower Rear List *)  {}
  }
}
```

Fox H-function

$$H_{0,2}^{2,0}\left(\cdot \left| \begin{matrix} \\ \left(\frac{a-\eta}{2},1\right),\left(\frac{a+\eta}{2},1\right) \end{matrix} \right.\right)$$

$$H_{0,2}^{2,0}\left(\cdot \left| \begin{array}{c|} \\ \hline \left(\frac{a-\eta}{2},1\right),\left(\frac{a+\eta}{2},1\right) \end{array} \right.\right)$$

Summary

$$\begin{aligned} a^* &= 2 \\ \Delta &= 2 \\ \delta &= \text{Indeterminate} \\ \mu &= a-1 \\ a_1^* &= 2 \\ a_2^* &= 0 \\ \xi &= a \\ c^* &= 1 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{\}^T$$

2. First eight poles from lower front list

$$
b_{j,\ell} = \begin{pmatrix}
\frac{\eta-a}{2} & \frac{1}{2}(-a-\eta) \\
\frac{1}{2}(-a+\eta-2) & \frac{1}{2}(-a-\eta-2) \\
\frac{1}{2}(-a+\eta-4) & \frac{1}{2}(-a-\eta-4) \\
\frac{1}{2}(-a+\eta-6) & \frac{1}{2}(-a-\eta-6) \\
\frac{1}{2}(-a+\eta-8) & \frac{1}{2}(-a-\eta-8) \\
\frac{1}{2}(-a+\eta-10) & -\frac{a}{2}-\frac{\eta}{2}-5 \\
\frac{1}{2}(-a+\eta-12) & -\frac{a}{2}-\frac{\eta}{2}-6 \\
\frac{1}{2}(-a+\eta-14) & -\frac{a}{2}-\frac{\eta}{2}-7
\end{pmatrix}^T
$$

2.12 Example `FoxH-Bessel-Y_2_9_20.wls`

File content

```
(* (2.9.20) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {{(a-η-1)/2, 1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(a-η)/2, 1},{(a+η)/2, 1}},
    (* Lower Rear List *)  {{(a-η-1)/2, 1}}
  }
}
```

Fox H-function

$$H_{1,3}^{2,0}\left(\cdot\ \middle|\ \begin{array}{l} \left(\frac{1}{2}(a-\eta-1),1\right) \\ \\ \left(\frac{a-\eta}{2},1\right),\left(\frac{a+\eta}{2},1\right),\left(\frac{1}{2}(a-\eta-1),1\right) \end{array}\right)$$

$$H_{1,3}^{2,0}\left(\cdot\ \middle|\ \begin{array}{c|c} & \left(\frac{1}{2}(a-\eta-1),1\right) \\ \hline \left(\frac{a-\eta}{2},1\right),\left(\frac{a+\eta}{2},1\right) & \left(\frac{1}{2}(a-\eta-1),1\right) \end{array}\right)$$

Summary

$$\begin{aligned} a^* &= 0 \\ \Delta &= 2 \\ \delta &= \text{ComplexInfinity} \\ \mu &= a-1 \\ a_1^* &= 1 \\ a_2^* &= -1 \\ \xi &= \eta+1 \\ c^* &= 0 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{\}^T$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} \frac{\eta-a}{2} & \frac{1}{2}(-a-\eta) \\ \frac{1}{2}(-a+\eta-2) & \frac{1}{2}(-a-\eta-2) \\ \frac{1}{2}(-a+\eta-4) & \frac{1}{2}(-a-\eta-4) \\ \frac{1}{2}(-a+\eta-6) & \frac{1}{2}(-a-\eta-6) \\ \frac{1}{2}(-a+\eta-8) & \frac{1}{2}(-a-\eta-8) \\ \frac{1}{2}(-a+\eta-10) & -\frac{a}{2}-\frac{\eta}{2}-5 \\ \frac{1}{2}(-a+\eta-12) & -\frac{a}{2}-\frac{\eta}{2}-6 \\ \frac{1}{2}(-a+\eta-14) & -\frac{a}{2}-\frac{\eta}{2}-7 \end{pmatrix}^T$$

2.13 Example `FoxH-Cos.wls`

File content

```
(* (2.9.8) and (2.9.10) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *)  {{1/2,1}}
  }
}
```

Fox H-function

$$H^{1,0}_{0,2}\left(\cdot\,\middle|\,\begin{array}{c} \\ (0,1)\,,\left(\tfrac{1}{2},1\right)\end{array}\right)$$

$$H^{1,0}_{0,2}\left(\cdot\,\middle|\,\begin{array}{c|c} & \\ \hline (0,1) & \left(\tfrac{1}{2},1\right)\end{array}\right)$$

Summary

$$\begin{aligned} a^* &= 0\\ \Delta &= 2\\ \delta &= \text{ComplexInfinity}\\ \mu &= -\frac{1}{2}\\ a_1^* &= 1\\ a_2^* &= -1\\ \xi &= -\frac{1}{2}\\ c^* &= 0 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{\}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

2.14 Example `FoxH-H.G_2_9_14.wls`

File content

```
(* (2.9.14) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1-a,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0,1}},
    (* Lower Rear List *)  {{1-c,1}}
  }
}
```

Fox H-function

$$H^{1,1}_{1,2}\left(\cdot \;\middle|\; \begin{matrix} (1-a,1) \\ \\ (0,1)\,,(1-c,1) \end{matrix}\right)$$

$$H^{1,1}_{1,2}\left(\cdot \;\middle|\; \begin{array}{c|c} (1-a,1) & \\ \hline (0,1) & (1-c,1) \end{array}\right)$$

Summary

$$\begin{aligned} a^* &= 1 \\ \Delta &= 1 \\ \delta &= \text{ComplexInfinity} \\ \mu &= a-c-\frac{1}{2} \\ a_1^* &= 1 \\ a_2^* &= 0 \\ \xi &= c-a \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \begin{pmatrix} a & a+1 & a+2 & a+3 & a+4 & a+5 & a+6 & a+7 \end{pmatrix}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

2.15 Example FoxH-H.G_2_9_15.wls

File content

```
(* (2.9.15) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1-a,1},{1-b,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0,1}},
    (* Lower Rear List *)  {{1-c,1}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,2}\left(\cdot \;\middle|\; \begin{matrix} (1-a,1)\,,(1-b,1) \\ \\ (0,1)\,,(1-c,1) \end{matrix}\right)$$

$$H_{2,2}^{1,2}\left(\cdot \;\middle|\; \begin{array}{c|c} (1-a,1)\,,(1-b,1) & \\ \hline (0,1) & (1-c,1) \end{array}\right)$$

Summary

$$\begin{aligned} a^* &= 2 \\ \Delta &= 0 \\ \delta &= \text{ComplexInfinity} \\ \mu &= a+b-c-1 \\ a_1^* &= 1 \\ a_2^* &= 1 \\ \xi &= -a-b+c+1 \\ c^* &= 1 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \begin{pmatrix} a & a+1 & a+2 & a+3 & a+4 & a+5 & a+6 & a+7 \\ b & b+1 & b+2 & b+3 & b+4 & b+5 & b+6 & b+7 \end{pmatrix}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

2.16 Example FoxH-Lommel_2_9_22.wls

File content

```
(* (2.9.22) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{(1+μ)/2,1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(1+μ)/2,1},{η/2,1},{-η/2,1}},
    (* Lower Rear List *)  {}
  }
}
```

Fox H-function

$$H_{1,3}^{3,1}\left(\cdot \,\middle|\, \begin{array}{l} \left(\frac{\mu+1}{2},1\right) \\ \\ \left(\frac{\mu+1}{2},1\right),\left(\frac{\eta}{2},1\right),\left(-\frac{\eta}{2},1\right) \end{array} \right)$$

$$H_{1,3}^{3,1}\left(\cdot \,\middle|\, \begin{array}{c|c} \left(\frac{\mu+1}{2},1\right) & \\ \hline \left(\frac{\mu+1}{2},1\right),\left(\frac{\eta}{2},1\right),\left(-\frac{\eta}{2},1\right) & \end{array} \right)$$

Summary

$$\begin{aligned} a^* &= 4 \\ \Delta &= 2 \\ \delta &= \text{Indeterminate} \\ \mu &= -1 \\ a_1^* &= 3 \\ a_2^* &= 1 \\ \xi &= \mu + 1 \\ c^* &= 2 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \begin{pmatrix} \frac{1-\mu}{2} \\ \frac{3-\mu}{2} \\ \frac{5-\mu}{2} \\ \frac{7-\mu}{2} \\ \frac{9-\mu}{2} \\ \frac{11-\mu}{2} \\ \frac{13-\mu}{2} \\ \frac{15-\mu}{2} \end{pmatrix}^T$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} \frac{1}{2}(-\mu-1) & -\frac{\eta}{2} & \frac{\eta}{2} \\ \frac{1}{2}(-\mu-3) & -\frac{\eta}{2}-1 & \frac{\eta-2}{2} \\ \frac{1}{2}(-\mu-5) & -\frac{\eta}{2}-2 & \frac{\eta-4}{2} \\ \frac{1}{2}(-\mu-7) & -\frac{\eta}{2}-3 & \frac{\eta-6}{2} \\ \frac{1}{2}(-\mu-9) & -\frac{\eta}{2}-4 & \frac{\eta-8}{2} \\ \frac{1}{2}(-\mu-11) & -\frac{\eta}{2}-5 & \frac{\eta}{2}-5 \\ \frac{1}{2}(-\mu-13) & -\frac{\eta}{2}-6 & \frac{\eta}{2}-6 \\ \frac{1}{2}(-\mu-15) & -\frac{\eta}{2}-7 & \frac{\eta}{2}-7 \end{pmatrix}^T$$

2.17 Example `FoxH-Mittag-Leffler.wls`

File content

```
(* (2.9.27) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{0, 1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *)  {{1 - μ, ρ}}
  }
}
```

Fox H-function

$$H_{1,2}^{1,1}\left(\cdot \;\middle|\; \begin{array}{l} (0,1) \\ \\ (0,1)\,,(1-\mu,\rho) \end{array}\right)$$

$$H_{1,2}^{1,1}\left(\cdot \;\middle|\; \begin{array}{c|c} (0,1) & \\ \hline (0,1) & (1-\mu,\rho) \end{array}\right)$$

Summary

$$\begin{aligned} a^* &= 2-\rho \\ \Delta &= \rho \\ \delta &= \text{ComplexInfinity} \\ \mu &= \frac{1}{2}-\mu \\ a_1^* &= 1 \\ a_2^* &= 1-\rho \\ \xi &= \mu-1 \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

2.18 Example `FoxH-Sin.wls`

File content

```
(* (2.9.7) and (2.9.9) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2, 1}},
    (* Lower Rear List *)  {{0, 1}}
  }
}
```

Fox H-function

$$H_{0,2}^{1,0}\left(\cdot \,\middle|\, \begin{array}{c} \\ \left(\frac{1}{2},1\right),(0,1) \end{array}\right)$$

$$H_{0,2}^{1,0}\left(\cdot \,\middle|\, \begin{array}{c|c} & \\ \hline \left(\frac{1}{2},1\right) & (0,1) \end{array}\right)$$

Summary

$$\begin{aligned}
a^* &= 0\\
\Delta &= 2\\
\delta &= \text{ComplexInfinity}\\
\mu &= -\frac{1}{2}\\
a_1^* &= 1\\
a_2^* &= -1\\
\xi &= \frac{1}{2}\\
c^* &= 0
\end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{\}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} & -\frac{7}{2} & -\frac{9}{2} & -\frac{11}{2} & -\frac{13}{2} & -\frac{15}{2} \end{pmatrix}$$

2.19 Example `FoxH-Whittaker_2_9_21.wls`

File content

```
(* (2.9.21) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {{a-λ+1,1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{a+μ+1/2,1},{a-μ+1/2,1}},
    (* Lower Rear List *)  {}
  }
}
```

Fox H-function

$$H_{1,2}^{2,0}\left(\cdot\,\middle|\,\begin{array}{l}(a-\lambda+1,1)\\ \\ \left(a+\mu+\frac{1}{2},1\right),\left(a-\mu+\frac{1}{2},1\right)\end{array}\right)$$

$$H_{1,2}^{2,0}\left(\cdot\,\middle|\,\begin{array}{c|c} & (a-\lambda+1,1)\\ \hline \left(a+\mu+\frac{1}{2},1\right),\left(a-\mu+\frac{1}{2},1\right) & \end{array}\right)$$

Summary

$$\begin{aligned}a^* &= 1\\ \Delta &= 1\\ \delta &= \text{Indeterminate}\\ \mu &= a+\lambda-\frac{1}{2}\\ a_1^* &= 1\\ a_2^* &= 0\\ \xi &= a+\lambda\\ c^* &= \frac{1}{2}\end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{\}^T$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -a-\mu-\frac{1}{2} & -a+\mu-\frac{1}{2} \\ -a-\mu-\frac{3}{2} & -a+\mu-\frac{3}{2} \\ -a-\mu-\frac{5}{2} & -a+\mu-\frac{5}{2} \\ -a-\mu-\frac{7}{2} & -a+\mu-\frac{7}{2} \\ -a-\mu-\frac{9}{2} & -a+\mu-\frac{9}{2} \\ -a-\mu-\frac{11}{2} & -a+\mu-\frac{11}{2} \\ -a-\mu-\frac{13}{2} & -a+\mu-\frac{13}{2} \\ -a-\mu-\frac{15}{2} & -a+\mu-\frac{15}{2} \end{pmatrix}^T$$

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