

# Some symbolic tools for the Fox $H$ -function

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In this note, we explain the code for checking the conditions of the Fox  $H$ -function [Fox61]. Here we follow the notation from Kilbas and Saigo [KS].

Let  $m, n, p, q$  be configure integers such that

$$0 \leq m \leq q \quad \text{and} \quad 0 \leq n \leq p.$$

Let  $a_i, b_j \in \mathbb{C}$  and  $\alpha_i, \beta_j \in \mathbb{R}_+$  be the parameters given below:

$\in (\mathbb{C}, \mathbb{R}_+)$	Front list	Rear list	
$p$	$(a_1, \alpha_1), \dots, (a_n, \alpha_n)$	$(a_{n+1}, \alpha_{n+1}), \dots, (a_p, \alpha_p)$	Upper list
$q$	$(b_1, \beta_1), \dots, (b_m, \beta_m)$	$(b_{m+1}, \beta_{m+1}), \dots, (b_q, \beta_q)$	Lower list

and denote

$$\mathcal{H}_{p,q}^{m,n}(s) := \frac{\prod_{i=1}^n \Gamma(1 - a_i - \alpha_i s)}{\prod_{j=n+1}^p \Gamma(a_j + \alpha_j s)} \times \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \alpha_j s)}. \quad (1)$$

Then the Fox  $H$ -function  $H_{2,3}^{2,1} \left( z \left| \begin{array}{c} \cdots \\ \cdots \end{array} \right. \right)$  is defined by a Mellin-Barnes type integral of the form

$$H_{m,n}^{p,q} \left( z \left| \begin{array}{c} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{array} \right. \right) := \frac{1}{2\pi i} \int_{\mathcal{L}} H_{p,q}^{m,n}(s) z^{-s} ds. \quad (2)$$

The basic assumption for the well-posedness of the Fox  $H$ -function is that two sets of poles do not overlap, i.e.,

$$\left\{ b_{j\ell} = \frac{-b_j - \ell}{\beta_j}, \ell = 0, 1, \dots \right\} \cap \left\{ a_{ik} = \frac{1 - a_i + k}{\alpha_i}, k = 0, 1, \dots \right\} = \emptyset. \quad (3)$$

The contour  $\mathcal{L}$  in (2) is given by one of the following three cases:

1.  $\mathcal{L} = \mathcal{L}_{-\infty}$  is a left loop situated in a horizontal strip starting at point  $-\infty + i\phi_1$  and terminating at point  $-\infty + i\phi_2$  for some  $-\infty < \phi_1 < \phi_2 < \infty$ ;
2.  $\mathcal{L} = \mathcal{L}_{+\infty}$  is a right loop situated in a horizontal strip starting at point  $+\infty + i\phi_1$  and terminating at point  $+\infty + i\phi_2$  for some  $-\infty < \phi_1 < \phi_2 < \infty$ ;
3.  $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$  is a contour starting at point  $\gamma - i\infty$  and terminating at point  $\gamma + i\infty$  for some  $\gamma \in (-\infty, \infty)$ .

We need a set of conditions to ensure the convergence of the integral in (2). To explain this, we need to introduce some notation (following p. 2 of [KS]). First denote

$$a_1^* := \sum_{j=1}^m \beta_j - \sum_{i=n+1}^p \alpha_i,$$

$$a_2^* := \sum_{i=1}^n \alpha_i - \sum_{j=m+1}^q \beta_j.$$

The following two parameters play the most important role:

$$a^* := a_2^* + a_1^* = \sum_{i=1}^n \alpha_i - \sum_{i=n+1}^p \alpha_i + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j;$$

$$\Delta := a_2^* - a_1^* = \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i.$$

Similar to  $a^*$ , we define

$$\xi := \sum_{i=1}^n a_i - \sum_{i=n+1}^p a_i + \sum_{j=1}^m b_j - \sum_{j=m+1}^q b_j.$$

Additionally, set

$$c^* := m + n - \frac{p+q}{2}.$$

In the critical cases, we need to use the following two parameters:

$$\delta := \prod_{i=1}^p \alpha_i^{-\alpha_i} \prod_{j=1}^q \beta_j^{\beta_j};$$

$$\mu := \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p-q}{2}$$

The well-posedness of the Fox  $H$ -function is given by Theorems 1.1 and 1.2 of [KS], which are summarized in the following figure 2:

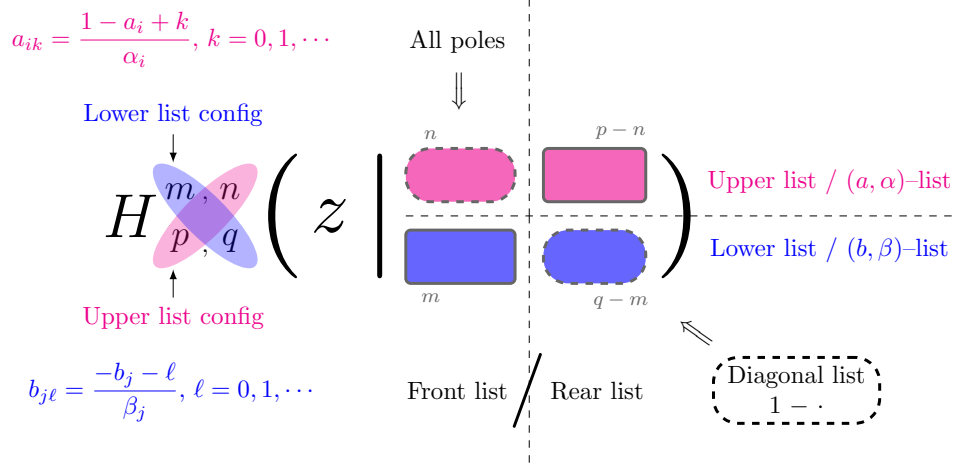


Figure 1: Diagram for the parameterization of the Fox  $H$ -function.

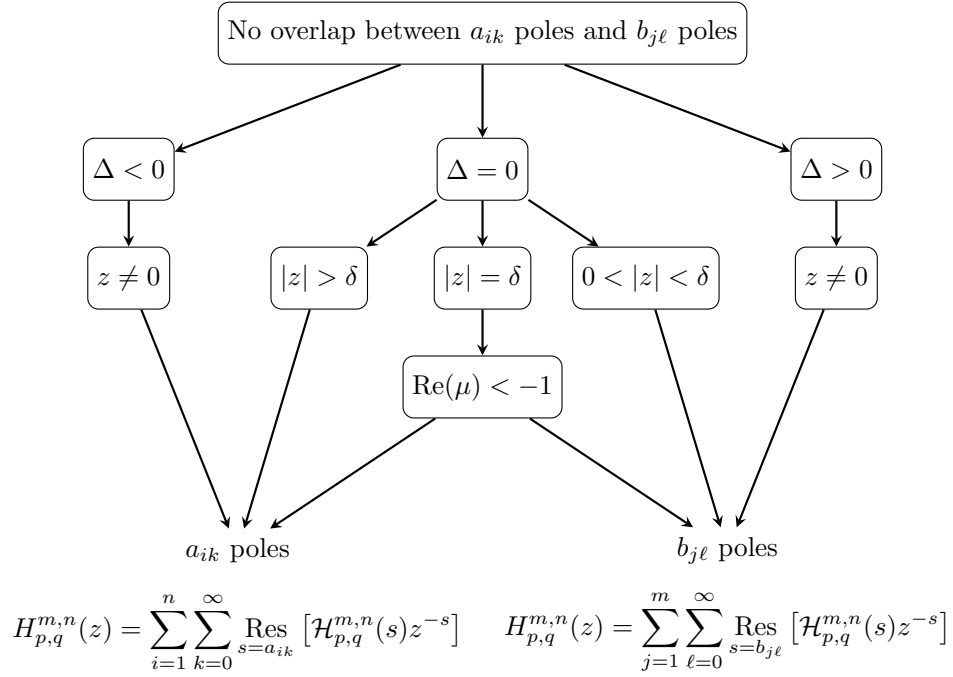


Figure 2: Well-posedness of the Fox  $H$ -function.

## References

- [Fox61] Charles Fox. The  $G$  and  $H$  functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, 98:395–429, 1961.
- [KS] Anatoly A. Kilbas and Megumi Saigo.  $H$ -transforms, volume 9 of *Analytical Methods and Special Functions*. Chapman & Hall/CRC, Boca Raton, FL. Theory and applications.