# Some symbolic tools for the Fox $H$-function

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## Contents

## 1 Introduction

In this note, we explain the code for checking the conditions of the Fox $H$-function [Fox61]. Here we follow the notation from Kilbas and Saigo [KS].

Let $m, n, p, q$ be configure integers such that

$$0 \leq m \leq q \quad \text{and} \quad 0 \leq n \leq p.$$

Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+$ be the parameters given below:

| $\in (\mathbb{C}, \mathbb{R}_+)$ | Front list | Rear list | |
|---|---|---|---|
| $p$ | $(a_1, \alpha_1), \cdots, (a_n, \alpha_n)$ | $(a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p)$ | Upper list |
| $q$ | $(b_1, \beta_1), \cdots, (b_m, \beta_m)$ | $(b_{m+1}, \beta_{m+1}), \cdots, (b_q, \beta_q)$ | Lower list |

and denote

$$\mathcal{H}_{p,q}^{m,n}(s) := \frac{\displaystyle\prod_{i=1}^{n} \Gamma\left(1 - a_i - \alpha_i s\right)}{\displaystyle\prod_{i=n+1}^{p} \Gamma\left(a_j + \alpha_i s\right)} \times \frac{\displaystyle\prod_{j=1}^{m} \Gamma\left(b_j + \beta_j s\right)}{\displaystyle\prod_{j=m+1}^{q} \Gamma\left(1 - b_j - \alpha_j s\right)}. \tag{1}$$

Then the Fox $H$-function $H_{2,3}^{2,1}\left(z \left| \begin{array}{c} \cdots \\ \cdots \end{array} \right.\right)$ is defined by a Mellin-Barnes type integral of the form

$$H_{m,n}^{p,q}\left(z \left| \begin{array}{c} (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ \\ (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{array} \right.\right) := \frac{1}{2\pi i} \int_{\mathcal{L}} H_{p,q}^{m,n}(s) z^{-s} \mathrm{d}s. \tag{2}$$

The basic assumption for the well-posedness of the Fox H-function is that two sets of poles do not overlap, i.e.,

$$\left\{ b_{j\ell} = \frac{-b_j - \ell}{\beta_j}, \ell = 0, 1, \cdots \right\} \bigcap \left\{ a_{ik} = \frac{1 - a_i + k}{\alpha_i}, k = 0, 1, \cdots \right\} = \emptyset. \tag{3}$$

The contour $\mathcal{L}$ in (2) is given by one of the following three cases:

1. $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at point $-\infty + i\phi_1$ and terminating at point $-\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;

2. $\mathcal{L} = \mathcal{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at point $+\infty + i\phi_1$ and terminating at point $\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;

3. $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ is a contour starting at point $\gamma - i\infty$ and terminating at point $\gamma + i\infty$ for some $\gamma \in (-\infty, \infty)$.

We need a set of conditions to ensure the convergence of the integral in (2). To explain this, we need to introduce some notation (following p. 2 of [KS]). First denote

$$a_1^* := \sum_{j=1}^{m} \beta_j - \sum_{i=n+1}^{p} \alpha_i,$$

$$a_2^* := \sum_{i=1}^{n} \alpha_i - \sum_{j=m+1}^{q} \beta_j.$$

The following two parameters play the most important role:

$$a^* := a_2^* + a_1^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j;$$

$$\Delta := a_2^* - a_1^* = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i.$$

2

Similar to $a^*$, we define

$$\xi := \sum_{i=1}^{n} a_i - \sum_{i=n+1}^{p} a_i + \sum_{j=1}^{m} b_j - \sum_{j=m+1}^{q} b_j.$$

Additionally, set

$$c^* := m + n - \frac{p+q}{2}.$$

In the critical cases, we need to use the following two parameters:

$$\delta := \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_j};$$

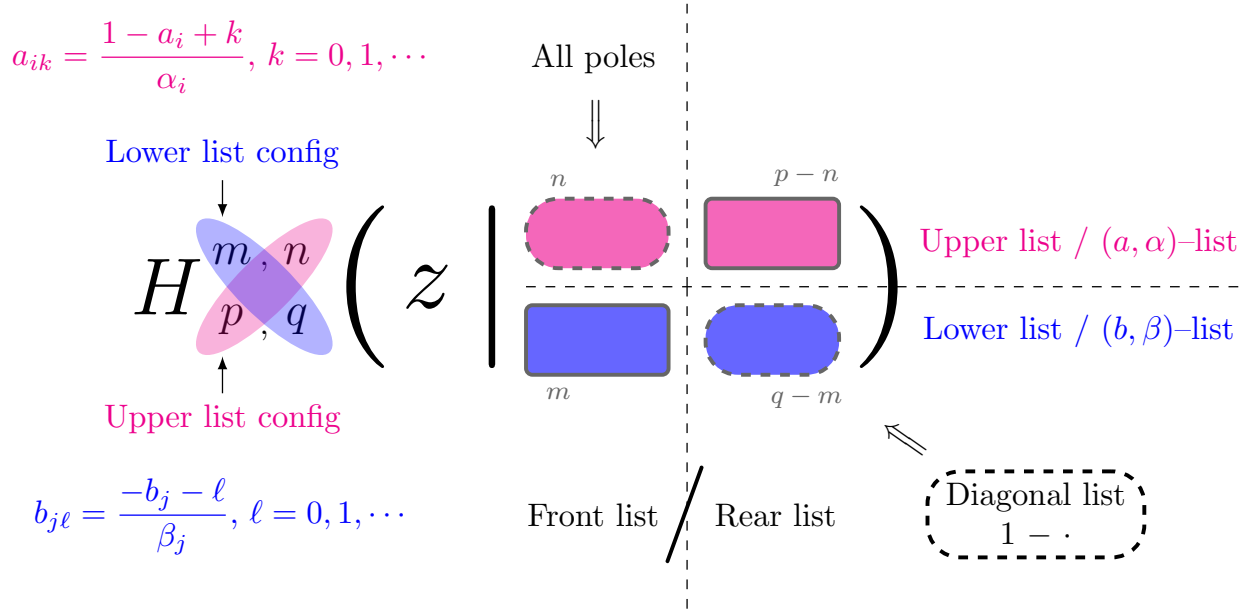$$\mu := \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}$$



Figure 1: Diagram for the parameterization of the Fox $H$-function.

The well-posedness of the Fox $H$-function is given by Theorems 1.1 and 1.2 of [KS], which are summarized in the following figure 2:

## 2 Examples

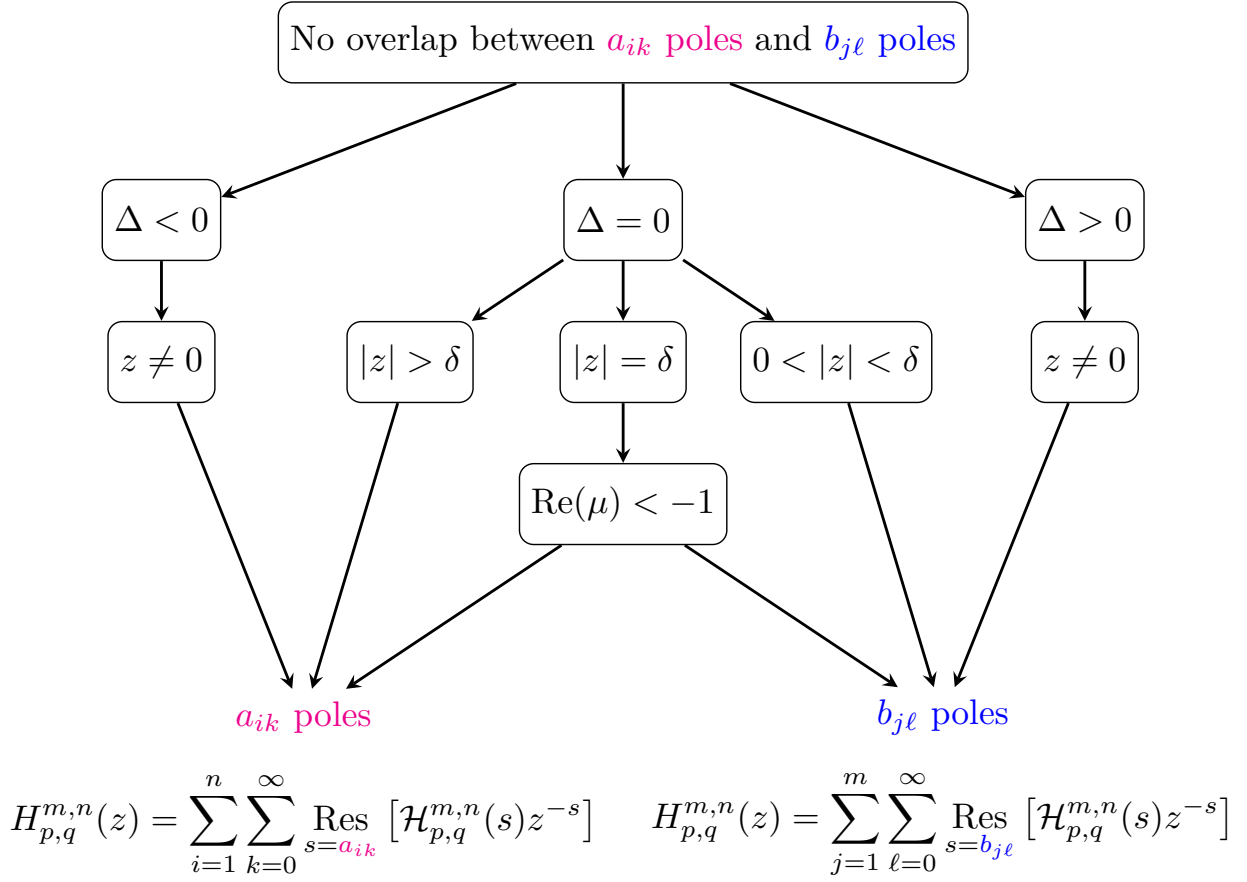### 2.1 Example `FoxH32-21.wls`

**File content**

No overlap between $a_{ik}$ poles and $b_{j\ell}$ poles

$\Delta < 0$

$\Delta = 0$

$\Delta > 0$

$z \neq 0$

$|z| > \delta$

$|z| = \delta$

$0 < |z| < \delta$

$z \neq 0$

$\mathrm{Re}(\mu) < -1$

$a_{ik}$ poles

$b_{j\ell}$ poles

$$H_{p,q}^{m,n}(z) = \sum_{i=1}^{n} \sum_{k=0}^{\infty} \operatorname*{Res}_{s=a_{ik}} \left[ \mathcal{H}_{p,q}^{m,n}(s) z^{-s} \right] \qquad H_{p,q}^{m,n}(z) = \sum_{j=1}^{m} \sum_{\ell=0}^{\infty} \operatorname*{Res}_{s=b_{j\ell}} \left[ \mathcal{H}_{p,q}^{m,n}(s) z^{-s} \right]$$

Figure 2: Well-posedness of the Fox $H$-function.

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, α^(-1)}},
    (* Upper Rear List *)  {{Ceiling[β], β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

**Fox H-function**

$$H_{2,3}^{2,1}\left(\cdot \left| \begin{array}{c} \left(1,\frac{1}{\alpha}\right),\left(\lceil\beta\rceil,\beta\right) \\ \\ \left(\frac{1}{2},\frac{\alpha}{2}\right),(1,1),\left(1,\frac{\alpha}{2}\right) \end{array} \right.\right)$$

$$H_{2,3}^{2,1}\left(\cdot \left| \begin{array}{c|c} \left(1,\frac{1}{\alpha}\right) & \left(\lceil\beta\rceil,\beta\right) \\ \hline \left(\frac{1}{2},\frac{\alpha}{2}\right),(1,1) & \end{array} \right.\right)$$

**Summary**

$$a^* = \frac{1}{\alpha} - \beta + 1$$

$$\Delta = \alpha - \frac{1}{\alpha} - \beta + 1$$

$$\delta = 2^{-\alpha}\left(\frac{1}{\alpha}\right)^{-1/\alpha}\left(2^{\alpha/2}\alpha^{\alpha/2} + \alpha^\alpha\right)\beta^{-\beta}$$

$$\mu = 1 - \lceil\beta\rceil$$

$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$

$$a_2^* = \frac{1}{\alpha} - \frac{\alpha}{2}$$

$$\xi = \frac{3}{2} - \lceil\beta\rceil$$

$$c^* = \frac{1}{2}$$

**Poles** **1. First ten poles from upper front list**

$$a_{i,k} = \left(\begin{array}{ccccccccccc} 0 & \alpha & 2\alpha & 3\alpha & 4\alpha & 5\alpha & 6\alpha & 7\alpha & 8\alpha & 9\alpha & 10\alpha \end{array}\right)$$

**2. First ten poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} -\frac{1}{\alpha} & -\frac{3}{\alpha} & -\frac{5}{\alpha} & -\frac{7}{\alpha} & -\frac{9}{\alpha} & -\frac{11}{\alpha} & -\frac{13}{\alpha} & -\frac{15}{\alpha} & -\frac{17}{\alpha} & -\frac{19}{\alpha} & -\frac{21}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{pmatrix}$$

## 2.2 Example `FoxH32-21-Y.wls`

**File content**

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{β + γ, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

**Fox H-function**

$$H_{2,3}^{2,1}\left(\,\cdot\,\left|\begin{array}{l} (1,1),(\beta+\gamma,\beta) \\[2ex] \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1),\left(1,\frac{\alpha}{2}\right) \end{array}\right.\right)$$

$$H_{2,3}^{2,1}\left(\,\cdot\,\left|\begin{array}{c|c} (1,1) & (\beta+\gamma,\beta) \\ \hline \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1) & \end{array}\right.\right)$$

**Summary**

$$a^* = 2 - \beta$$
$$\Delta = \alpha - \beta$$
$$\delta = 2^{-\alpha}\left(2^{\alpha/2}\alpha^{\alpha/2} + \alpha^\alpha\right)\beta^{-\beta}$$
$$\mu = \frac{1}{2}(-2\beta - 2\gamma + d + 1)$$
$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$
$$a_2^* = 1 - \frac{\alpha}{2}$$
$$\xi = \frac{1}{2}(d - 2(\beta + \gamma - 1))$$
$$c^* = \frac{1}{2}$$

**Poles**  **1. First ten poles from upper front list**

$$a_{i,k} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

**2. First ten poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} & -\frac{d+16}{\alpha} & -\frac{d+18}{\alpha} & -\frac{d+20}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{pmatrix}$$

### 2.3   Example `FoxH32-21-Z.wls`

**File content**

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{Ceiling[β], β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

**Fox H-function**

$$H_{2,3}^{2,1} \left( \cdot \left| \begin{array}{c} (1,1), (\lceil \beta \rceil, \beta) \\ \\ \left(\frac{d}{2}, \frac{\alpha}{2}\right), (1,1), \left(1, \frac{\alpha}{2}\right) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left( \cdot \left| \begin{array}{c|c} (1,1) & (\lceil \beta \rceil, \beta) \\ \hline \left(\frac{d}{2}, \frac{\alpha}{2}\right), (1,1) & \end{array} \right. \right)$$

**Summary**

$$a^* = 2 - \beta$$
$$\Delta = \alpha - \beta$$
$$\delta = 2^{-\alpha} \left( 2^{\alpha/2} \alpha^{\alpha/2} + \alpha^{\alpha} \right) \beta^{-\beta}$$
$$\mu = \frac{1}{2} (-2 \lceil \beta \rceil + d + 1)$$
$$a_1^* = \frac{1}{2} (\alpha - 2\beta + 2)$$
$$a_2^* = 1 - \frac{\alpha}{2}$$
$$\xi = \frac{1}{2} (-2 \lceil \beta \rceil + d + 2)$$
$$c^* = \frac{1}{2}$$

**Poles   1. First ten poles from upper front list**

$$a_{i,k} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

**2. First ten poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} & -\frac{d+16}{\alpha} & -\frac{d+18}{\alpha} & -\frac{d+20}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{pmatrix}$$

## 2.4   Example `FoxH32-21-Z-Star.wls`

**File content**

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{1, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

**Fox H-function**

$$H_{2,3}^{2,1} \left( \cdot \left| \begin{array}{c} (1,1), (1,\beta) \\ \\ \left(\frac{d}{2}, \frac{\alpha}{2}\right), (1,1), \left(1, \frac{\alpha}{2}\right) \end{array} \right. \right)$$

$$H_{2,3}^{2,1}\left(\cdot\left|\begin{array}{c|c}(1,1) & (1,\beta)\\\hline\left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1) & \end{array}\right.\right)$$

**Summary**

$$a^* = 2 - \beta$$
$$\Delta = \alpha - \beta$$
$$\delta = 2^{-\alpha}\left(2^{\alpha/2}\alpha^{\alpha/2} + \alpha^{\alpha}\right)\beta^{-\beta}$$
$$\mu = \frac{d-1}{2}$$
$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$
$$a_2^* = 1 - \frac{\alpha}{2}$$
$$\xi = \frac{d}{2}$$
$$c^* = \frac{1}{2}$$

**Poles    1. First ten poles from upper front list**

$$a_{i,k} = \left(\begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}\right)$$

**2. First ten poles from lower front list**

$$b_{j,\ell} = \left(\begin{array}{ccccccccccc} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} & -\frac{d+16}{\alpha} & -\frac{d+18}{\alpha} & -\frac{d+20}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 & -11 \end{array}\right)$$

## 2.5    Example `FoxH-Sin.wls`

**File content**

```
(* (2.9.7) and (2.9.9) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2, 1}},
    (* Lower Rear List *)  {{0, 1}}
  }
}
```

**Fox H-function**

$$H_{0,2}^{1,0}\left(\cdot\ \middle|\ \begin{array}{c} \\ \left(\frac{1}{2},1\right),(0,1) \end{array}\right)$$

$$H_{0,2}^{1,0}\left(\cdot\ \middle|\ \begin{array}{c|c} & \\ \hline \left(\frac{1}{2},1\right) & \end{array}\right)$$

**Summary**

$$a^* = 0$$
$$\Delta = 2$$
$$\delta = \text{ComplexInfinity}$$
$$\mu = -\frac{1}{2}$$
$$a_1^* = 1$$
$$a_2^* = -1$$
$$\xi = \frac{1}{2}$$
$$c^* = 0$$

**Poles   1. First ten poles from upper front list**

$$a_{i,k} = \{\}$$

**2. First ten poles from lower front list**

$$b_{j,\ell} = \left(\ -\frac{1}{2}\quad -\frac{3}{2}\quad -\frac{5}{2}\quad -\frac{7}{2}\quad -\frac{9}{2}\quad -\frac{11}{2}\quad -\frac{13}{2}\quad -\frac{15}{2}\quad -\frac{17}{2}\quad -\frac{19}{2}\quad -\frac{21}{2}\ \right)$$

## 2.6   Example `FoxH-Cos.wls`

**File content**

```
(* (2.9.8) and (2.9.10) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *)  {{1/2,1}}
  }
}
```

**Fox H-function**

$$H_{0,2}^{1,0}\left(\cdot\left|\begin{array}{c} \\ (0,1),\left(\frac{1}{2},1\right)\end{array}\right.\right)$$

$$H_{0,2}^{1,0}\left(\cdot\left|\begin{array}{c|c} & \\ \hline (0,1) & \end{array}\right.\right)$$

**Summary**

$$a^* = 0$$
$$\Delta = 2$$
$$\delta = \text{ComplexInfinity}$$
$$\mu = -\frac{1}{2}$$
$$a_1^* = 1$$
$$a_2^* = -1$$
$$\xi = -\frac{1}{2}$$
$$c^* = 0$$

**Poles   1. First ten poles from upper front list**

$$a_{i,k} = \{\}$$

**2. First ten poles from lower front list**

$$b_{j,\ell} = \left(\begin{array}{ccccccccccc} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{array}\right)$$

## 2.7   Example `FoxH-Mittag-Leffler.wls`

**File content**

```
(* (2.9.27) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{0, 1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *)  {{1 - μ, ρ}}
  }
}
```

**Fox H-function**

$$H_{1,2}^{1,1} \left( \cdot \left| \begin{array}{c} (0,1) \\[2mm] (0,1), (1-\mu, \rho) \end{array} \right. \right)$$

$$H_{1,2}^{1,1} \left( \cdot \left| \begin{array}{c|c} (0,1) & \\ \hline (0,1) & \end{array} \right. \right)$$

**Summary**

$$a^* = 2 - \rho$$
$$\Delta = \rho$$
$$\delta = \text{ComplexInfinity}$$
$$\mu = \frac{1}{2} - \mu$$
$$a_1^* = 1$$
$$a_2^* = 1 - \rho$$
$$\xi = \mu - 1$$
$$c^* = \frac{1}{2}$$

**Poles   1. First ten poles from upper front list**

$$a_{i,k} = \left( \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \right)$$

**2. First ten poles from lower front list**

$$b_{j,\ell} = \left( \begin{array}{ccccccccccc} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 & -9 & -10 \end{array} \right)$$

# References

[Fox61] Charles Fox. The $G$ and $H$ functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, 98:395–429, 1961.

[KS] Anatoly A. Kilbas and Megumi Saigo. *H-transforms*, volume 9 of *Analytical Methods and Special Functions*. Chapman & Hall/CRC, Boca Raton, FL. Theory and applications.