# Some symbolic tools for the Fox $H$-function

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In this note, we explain the code for checking the conditions of the Fox $H$-function [Fox61]. Here we follow the notation from Kilbas and Saigo [KS].

Let $m, n, p, q$ be configure integers such that

$$0 \le m \le q \quad \text{and} \quad 0 \le n \le p.$$

Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+$ be the parameters given below:

| $\in (\mathbb{C}, \mathbb{R}_+)$ | Front list | Rear list | |
|---|---|---|---|
| $p$ | $(a_1, \alpha_1), \cdots, (a_n, \alpha_n)$ | $(a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p)$ | Upper list |
| $q$ | $(b_1, \beta_1), \cdots, (b_m, \beta_m)$ | $(b_{m+1}, \beta_{m+1}), \cdots, (b_q, \beta_q)$ | Lower list |

and denote

$$\mathcal{H}_{p,q}^{m,n}(s) := \frac{\displaystyle\prod_{i=1}^{n} \Gamma\left(1 - a_i - \alpha_i s\right)}{\displaystyle\prod_{i=n+1}^{p} \Gamma\left(a_j + \alpha_i s\right)} \times \frac{\displaystyle\prod_{j=1}^{m} \Gamma\left(b_j + \beta_j s\right)}{\displaystyle\prod_{j=m+1}^{q} \Gamma\left(1 - b_j - \alpha_j s\right)}. \tag{1}$$

Then the Fox $H$-function $H^{2,1}_{2,3}\left(z \,\middle|\, \begin{matrix} \cdots \\ \\ \cdots \end{matrix}\right)$ is defined by a Mellin-Barnes type integral of the form

$$H^{p,q}_{m,n}\left(z \,\middle|\, \begin{matrix} (a_1,\alpha_1),\cdots,(a_p,\alpha_p) \\ \\ (b_1,\beta_1),\cdots,(b_q,\beta_q) \end{matrix}\right) := \frac{1}{2\pi i}\int_{\mathcal{L}} H^{m,n}_{p,q}(s) z^{-s}\mathrm{d}s. \tag{2}$$

The basic assumption for the well-posedness of the Fox H-function is that two sets of poles do not overlap, i.e.,

$$\left\{b_{j\ell} = \frac{-b_j-\ell}{\beta_j}, \ell = 0,1,\cdots\right\} \bigcap \left\{a_{ik} = \frac{1-a_i+k}{\alpha_i}, k = 0,1,\cdots\right\} = \emptyset. \tag{3}$$

The contour $\mathcal{L}$ in (2) is given by one of the following three cases:

1. $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at point $-\infty + i\phi_1$ and terminating at point $-\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;

2. $\mathcal{L} = \mathcal{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at point $+\infty + i\phi_1$ and terminating at point $\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;

3. $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ is a contour starting at point $\gamma - i\infty$ and terminating at point $\gamma + i\infty$ for some $\gamma \in (-\infty,\infty)$.

We need a set of conditions to ensure the convergence of the integral in (2). To explain this, we need to introduce some notation (following p. 2 of [KS]). First denote

$$a_1^* := \sum_{j=1}^{m}\beta_j - \sum_{i=n+1}^{p}\alpha_i,$$
$$a_2^* := \sum_{i=1}^{n}\alpha_i - \sum_{j=m+1}^{q}\beta_j.$$

The following two parameters play the most important role:

$$a^* \coloneqq a_2^* + a_1^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j;$$

$$\Delta \coloneqq a_2^* - a_1^* = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i.$$

Similar to $a^*$, we define

$$\xi \coloneqq \sum_{i=1}^{n} a_i - \sum_{i=n+1}^{p} a_i + \sum_{j=1}^{m} b_j - \sum_{j=m+1}^{q} b_j.$$

Additionally, set

$$c^* \coloneqq m + n - \frac{p+q}{2}.$$

In the critical cases, we need to use the following two parameters:

$$\delta \coloneqq \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_i};$$

$$\mu \coloneqq \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}$$

The well-posedness of the Fox $H$-function is given by Theorems 1.1 and 1.2 of [KS], which are summarized in the following figure 2:

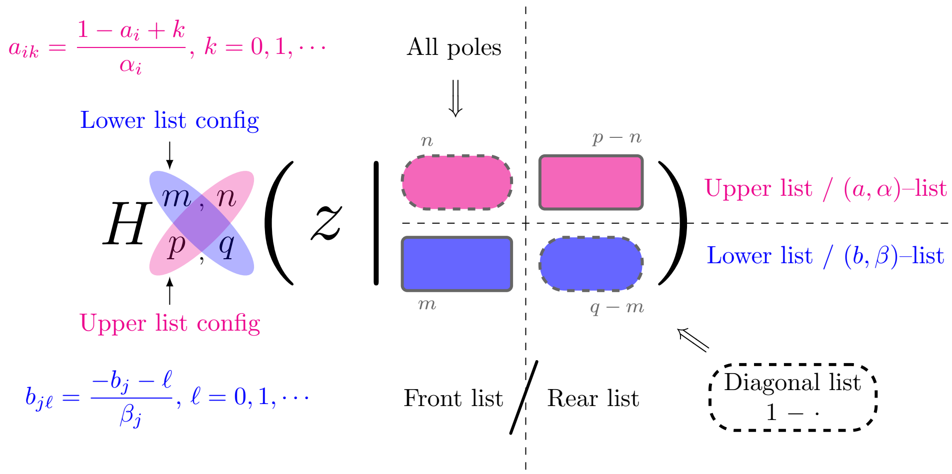


Figure 1: Diagram for the parameterization of the Fox $H$-function.

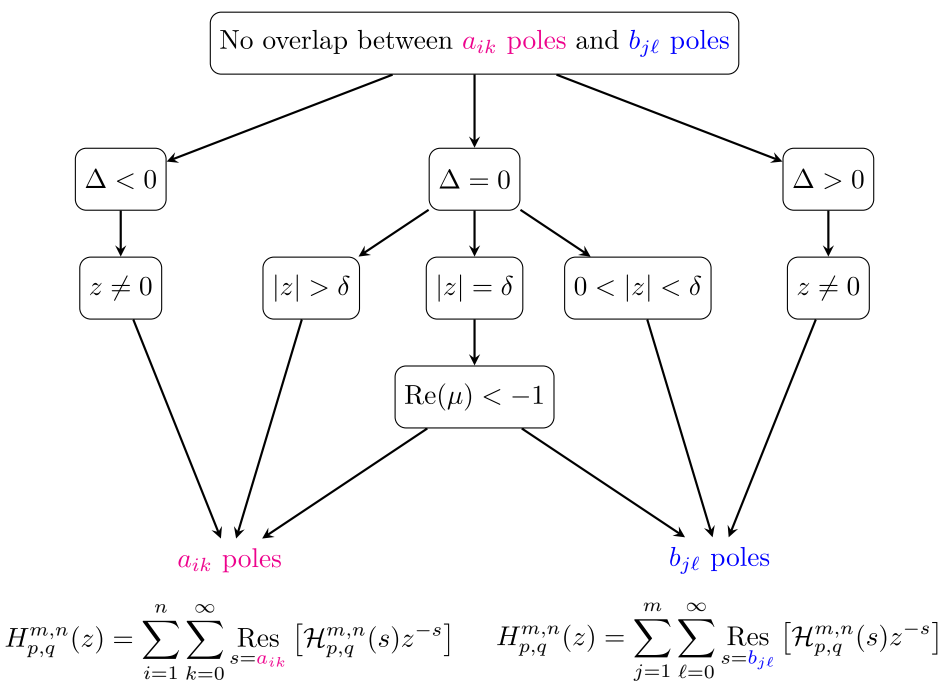


Figure 2: Well-posedness of the Fox $H$-function.

# 1 Example `FoxH32-21.wls`

### File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, α^(-1)}},
    (* Upper Rear List *)  {{Ceiling[β], β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

### Fox H-function

$$H_{2,3}^{2,1}\left( \cdot \;\middle|\; \begin{matrix} \left(1, \frac{1}{\alpha}\right), (\lceil\beta\rceil, \beta) \\ \\ \left(\frac{1}{2}, \frac{\alpha}{2}\right), (1,1), \left(1, \frac{\alpha}{2}\right) \end{matrix} \right)$$

$$H_{2,3}^{2,1}\left( \cdot \;\middle|\; \begin{array}{c|c} \left(1, \frac{1}{\alpha}\right) & (\lceil\beta\rceil, \beta) \\ \hline \left(\frac{1}{2}, \frac{\alpha}{2}\right), (1,1) & \end{array} \right)$$

**Summary**

$$
\begin{aligned}
a^* &= \frac{1}{\alpha} - \beta + 1 \\
\Delta &= \alpha - \frac{1}{\alpha} - \beta + 1 \\
\delta &= 2^{-\alpha} \left(\frac{1}{\alpha}\right)^{-1/\alpha} \left(2^{\alpha/2}\alpha^{\alpha/2} + \alpha^{\alpha}\right) \beta^{-\beta} \\
\mu &= 1 - \lceil \beta \rceil \\
a_1^* &= \frac{1}{2}(\alpha - 2\beta + 2) \\
a_2^* &= \frac{1}{\alpha} - \frac{\alpha}{2} \\
\xi &= \frac{3}{2} - \lceil \beta \rceil \\
c^* &= \frac{1}{2}
\end{aligned}
$$

**Poles 1. First ten poles from upper front list**

$$a_{i,k} = \begin{pmatrix} 0 \\ \alpha \\ 2\alpha \\ 3\alpha \\ 4\alpha \\ 5\alpha \\ 6\alpha \\ 7\alpha \\ 8\alpha \\ 9\alpha \\ 10\alpha \end{pmatrix}$$

**2. First ten poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} -\frac{1}{\alpha} & -1 \\ -\frac{3}{\alpha} & -2 \\ -\frac{5}{\alpha} & -3 \\ -\frac{7}{\alpha} & -4 \\ -\frac{9}{\alpha} & -5 \\ -\frac{11}{\alpha} & -6 \\ -\frac{13}{\alpha} & -7 \\ -\frac{15}{\alpha} & -8 \\ -\frac{17}{\alpha} & -9 \\ -\frac{19}{\alpha} & -10 \\ -\frac{21}{\alpha} & -11 \end{pmatrix}$$

## 2 Example **FoxH32-21-Y.wls**

### File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{β + γ, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

### Fox H-function

$$H^{2,1}_{2,3}\left(\cdot\;\middle|\;\begin{array}{c}(1,1),(\beta+\gamma,\beta)\\ \\ \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1),\left(1,\frac{\alpha}{2}\right)\end{array}\right)$$

$$H^{2,1}_{2,3}\left(\cdot\;\middle|\;\begin{array}{c|c}(1,1) & (\beta+\gamma,\beta)\\ \hline \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1) & \end{array}\right)$$

**Summary**

$$
\begin{aligned}
a^* &= 2 - \beta \\
\Delta &= \alpha - \beta \\
\delta &= 2^{-\alpha}\left(2^{\alpha/2}\alpha^{\alpha/2} + \alpha^{\alpha}\right)\beta^{-\beta} \\
\mu &= \frac{1}{2}(-2\beta - 2\gamma + d + 1) \\
a_1^* &= \frac{1}{2}(\alpha - 2\beta + 2) \\
a_2^* &= 1 - \frac{\alpha}{2} \\
\xi &= \frac{1}{2}(d - 2(\beta + \gamma - 1)) \\
c^* &= \frac{1}{2}
\end{aligned}
$$

**Poles 1. First ten poles from upper front list**

$$
a_{i,k} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{pmatrix}
$$

## 2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -1 \\ -\frac{d+2}{\alpha} & -2 \\ -\frac{d+4}{\alpha} & -3 \\ -\frac{d+6}{\alpha} & -4 \\ -\frac{d+8}{\alpha} & -5 \\ -\frac{d+10}{\alpha} & -6 \\ -\frac{d+12}{\alpha} & -7 \\ -\frac{d+14}{\alpha} & -8 \\ -\frac{d+16}{\alpha} & -9 \\ -\frac{d+18}{\alpha} & -10 \\ -\frac{d+20}{\alpha} & -11 \end{pmatrix}$$

## 3 Example `FoxH32-21-Z.wls`

### File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{Ceiling[β], β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

### Fox H-function

$$H_{2,3}^{2,1}\left(\cdot\,\middle|\,\begin{array}{l}(1,1)\,,(\lceil\beta\rceil,\beta)\\ \\ \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1)\,,\left(1,\frac{\alpha}{2}\right)\end{array}\right)$$

$$H_{2,3}^{2,1}\left(\cdot\,\middle|\,\begin{array}{c|c}(1,1) & (\lceil\beta\rceil,\beta)\\ \hline \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1) & \end{array}\right)$$

**Summary**

$$\begin{aligned}
a^* &= 2-\beta \\
\Delta &= \alpha-\beta \\
\delta &= 2^{-\alpha}\left(2^{\alpha/2}\alpha^{\alpha/2}+\alpha^{\alpha}\right)\beta^{-\beta} \\
\mu &= \frac{1}{2}(-2\lceil\beta\rceil+d+1) \\
a_1^* &= \frac{1}{2}(\alpha-2\beta+2) \\
a_2^* &= 1-\frac{\alpha}{2} \\
\xi &= \frac{1}{2}(-2\lceil\beta\rceil+d+2) \\
c^* &= \frac{1}{2}
\end{aligned}$$

**Poles 1. First ten poles from upper front list**

$$a_{i,k} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{pmatrix}$$

## 2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -1 \\ -\frac{d+2}{\alpha} & -2 \\ -\frac{d+4}{\alpha} & -3 \\ -\frac{d+6}{\alpha} & -4 \\ -\frac{d+8}{\alpha} & -5 \\ -\frac{d+10}{\alpha} & -6 \\ -\frac{d+12}{\alpha} & -7 \\ -\frac{d+14}{\alpha} & -8 \\ -\frac{d+16}{\alpha} & -9 \\ -\frac{d+18}{\alpha} & -10 \\ -\frac{d+20}{\alpha} & -11 \end{pmatrix}$$

## 4 Example `FoxH32-21-Z-Star.wls`

### File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{1, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

### Fox H-function

$$H_{2,3}^{2,1}\left(\cdot \;\middle|\; \begin{array}{l} (1,1)\,,(1,\beta) \\ \\ \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1)\,,\left(1,\frac{\alpha}{2}\right) \end{array}\right)$$

$$H_{2,3}^{2,1}\left(\cdot \;\middle|\; \begin{array}{c|c} (1,1) & (1,\beta) \\ \hline \left(\frac{d}{2},\frac{\alpha}{2}\right),(1,1) & \end{array}\right)$$

**Summary**

$$
\begin{aligned}
a^* &= 2-\beta \\
\Delta &= \alpha-\beta \\
\delta &= 2^{-\alpha}\left(2^{\alpha/2}\alpha^{\alpha/2}+\alpha^{\alpha}\right)\beta^{-\beta} \\
\mu &= \frac{d-1}{2} \\
a_1^* &= \frac{1}{2}(\alpha-2\beta+2) \\
a_2^* &= 1-\frac{\alpha}{2} \\
\xi &= \frac{d}{2} \\
c^* &= \frac{1}{2}
\end{aligned}
$$

**Poles 1. First ten poles from upper front list**

$$
a_{i,k} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{pmatrix}
$$

**2. First ten poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -1 \\ -\frac{d+2}{\alpha} & -2 \\ -\frac{d+4}{\alpha} & -3 \\ -\frac{d+6}{\alpha} & -4 \\ -\frac{d+8}{\alpha} & -5 \\ -\frac{d+10}{\alpha} & -6 \\ -\frac{d+12}{\alpha} & -7 \\ -\frac{d+14}{\alpha} & -8 \\ -\frac{d+16}{\alpha} & -9 \\ -\frac{d+18}{\alpha} & -10 \\ -\frac{d+20}{\alpha} & -11 \end{pmatrix}$$

# 5 Example `FoxH-Cos.wls`

### File content

```
(* (2.9.8) and (2.9.10) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *)  {{1/2,1}}
  }
}
```

### Fox H-function

$$H_{0,2}^{1,0}\left(\cdot\,\middle|\,\begin{matrix} \\ (0,1)\,,\left(\tfrac{1}{2},1\right)\end{matrix}\right)$$

$$H_{0,2}^{1,0}\left(\cdot\,\middle|\,\begin{array}{c|c} & \\ \hline (0,1) & \end{array}\right)$$

### Summary

$$\begin{aligned} a^* &= 0 \\ \Delta &= 2 \\ \delta &= \text{ComplexInfinity} \\ \mu &= -\frac{1}{2} \\ a_1^* &= 1 \\ a_2^* &= -1 \\ \xi &= -\frac{1}{2} \\ c^* &= 0 \end{aligned}$$

**Poles 1. First ten poles from upper front list**

$$a_{i,k} = \{\{\},\{\},\{\},\{\},\{\},\{\},\{\},\{\},\{\},\{\},\{\}\}$$

**2. First ten poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} 0 \\ -1 \\ -2 \\ -3 \\ -4 \\ -5 \\ -6 \\ -7 \\ -8 \\ -9 \\ -10 \end{pmatrix}$$

# 6 Example `FoxH-Mittag-Leffler.wls`

### File content

```
(* (2.9.27) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{0, 1}},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *)  {{1 - μ, ρ}}
  }
}
```

### Fox H-function

$$H_{1,2}^{1,1}\left( \cdot \left| \begin{array}{l} (0,1) \\ \\ (0,1)\,,(1-\mu,\rho) \end{array} \right. \right)$$

$$H_{1,2}^{1,1}\left( \cdot \left| \begin{array}{c|c} (0,1) & \\ \hline (0,1) & \end{array} \right. \right)$$

### Summary

$$\begin{aligned}
a^* &= 2-\rho \\
\Delta &= \rho \\
\delta &= \text{ComplexInfinity} \\
\mu &= \frac{1}{2}-\mu \\
a_1^* &= 1 \\
a_2^* &= 1-\rho \\
\xi &= \mu-1 \\
c^* &= \frac{1}{2}
\end{aligned}$$

**Poles 1. First ten poles from upper front list**

$$a_{i,k} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{pmatrix}$$

**2. First ten poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} 0 \\ -1 \\ -2 \\ -3 \\ -4 \\ -5 \\ -6 \\ -7 \\ -8 \\ -9 \\ -10 \end{pmatrix}$$

# 7 Example **`FoxH-Sin.wls`**

### File content

```
(* (2.9.7) and (2.9.9) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *)  {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2, 1}},
    (* Lower Rear List *)  {{0, 1}}
  }
}
```

### Fox H-function

$$H^{1,0}_{0,2}\left(\cdot \,\middle|\, \begin{matrix} \\ \left(\frac{1}{2},1\right),(0,1) \end{matrix}\right)$$

$$H^{1,0}_{0,2}\left(\cdot \,\middle|\, \begin{array}{c|c} & \\ \hline \left(\frac{1}{2},1\right) & \end{array}\right)$$

### Summary

$$\begin{aligned} a^* &= 0 \\ \Delta &= 2 \\ \delta &= \text{ComplexInfinity} \\ \mu &= -\frac{1}{2} \\ a_1^* &= 1 \\ a_2^* &= -1 \\ \xi &= \frac{1}{2} \\ c^* &= 0 \end{aligned}$$

**Poles 1. First ten poles from upper front list**

$$a_{i,k} = \{\{\},\{\},\{\},\{\},\{\},\{\},\{\},\{\},\{\},\{\},\{\}\}$$

**2. First ten poles from lower front list**

$$b_{j,\ell} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ -\frac{5}{2} \\ -\frac{7}{2} \\ -\frac{9}{2} \\ -\frac{11}{2} \\ -\frac{13}{2} \\ -\frac{15}{2} \\ -\frac{17}{2} \\ -\frac{19}{2} \\ -\frac{21}{2} \end{pmatrix}$$

## References


[Fox61] Charles Fox. The $G$ and $H$ functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, 98:395–429, 1961.

[KS] Anatoly A. Kilbas and Megumi Saigo. *$H$-transforms*, volume 9 of *Analytical Methods and Special Functions.* Chapman & Hall/CRC, Boca Raton, FL. Theory and applications.