

Some symbolic tools for the Fox H -function

Le Chen

Guannan Hu

Department of Mathematics & Statistics

Department of Mathematics & Statistics

Auburn University

Washburn University

le.chen@auburn.edu, chenle02@gmail.com

guannan.hu@washburn.edu

December 18, 2023

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1 Introduction

In this note, we explain the code for checking the conditions of the Fox H -function [Fox61]. Here we follow the notation from Kilbas and Saigo [KS04].

Let m, n, p, q be configure integers such that

$$0 \leq m \leq q \quad \text{and} \quad 0 \leq n \leq p.$$

Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+$ be the parameters given below:

$\in (\mathbb{C}, \mathbb{R}_+)$	Front list	Rear list	
p	$(a_1, \alpha_1), \dots, (a_n, \alpha_n)$	$(a_{n+1}, \alpha_{n+1}), \dots, (a_p, \alpha_p)$	Upper list
q	$(b_1, \beta_1), \dots, (b_m, \beta_m)$	$(b_{m+1}, \beta_{m+1}), \dots, (b_q, \beta_q)$	Lower list

and denote

$$\mathcal{H}_{p,q}^{m,n}(s) := \frac{\prod_{i=1}^n \Gamma(1 - a_i - \alpha_i s)}{\prod_{j=n+1}^p \Gamma(a_j + \alpha_j s)} \times \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \alpha_j s)}. \quad (1)$$

Then the Fox H -function $H_{2,3}^{2,1} \left(z \left| \begin{array}{c} \dots \\ \dots \end{array} \right. \right)$ is defined by a Mellin-Barnes type integral of the form

$$H_{m,n}^{p,q} \left(z \left| \begin{array}{c} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{array} \right. \right) := \frac{1}{2\pi i} \int_{\mathcal{L}} H_{p,q}^{m,n}(s) z^{-s} ds. \quad (2)$$

It is more convenient to use the following notation:

$$H_{m,n}^{p,q} \left(z \left| \frac{(a_1, \alpha_1), \dots, (a_n, \alpha_n)}{(b_1, \beta_1), \dots, (b_m, \beta_m)} \middle| \frac{(a_{n+1}, \alpha_{n+1}), \dots, (a_p, \alpha_p)}{(b_{m+1}, \beta_{m+1}), \dots, (b_q, \beta_q)} \right. \right),$$

in order to emphasize the front and rear lists.

The basic assumption for the well-posedness of the Fox H -function is that two sets of poles do not overlap, i.e.,

$$\left\{ b_{j\ell} = \frac{-b_j - \ell}{\beta_j}, \ell = 0, 1, \dots \right\} \cap \left\{ a_{ik} = \frac{1 - a_i + k}{\alpha_i}, k = 0, 1, \dots \right\} = \emptyset. \quad (3)$$

The contour \mathcal{L} in (2) is given by one of the following three cases:

1. $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at point $-\infty + i\phi_1$ and terminating at point $-\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;
2. $\mathcal{L} = \mathcal{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at point $+\infty + i\phi_1$ and terminating at point $+\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;
3. $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ is a contour starting at point $\gamma - i\infty$ and terminating at point $\gamma + i\infty$ for some $\gamma \in (-\infty, \infty)$.

We need a set of conditions to ensure the convergence of the integral in (2). To explain this, we need to introduce some notation (following p. 2 of [KS04]). First denote

$$a_1^* := \sum_{j=1}^m \beta_j - \sum_{i=n+1}^p \alpha_i,$$

$$a_2^* := \sum_{i=1}^n \alpha_i - \sum_{j=m+1}^q \beta_j.$$

The following two parameters play the most important role:

$$a^* := a_2^* + a_1^* = \sum_{i=1}^n \alpha_i - \sum_{i=n+1}^p \alpha_i + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j;$$

$$\Delta := a_2^* - a_1^* = \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i.$$

Similar to a^* , we define

$$\xi := \sum_{i=1}^n a_i - \sum_{i=n+1}^p a_i + \sum_{j=1}^m b_j - \sum_{j=m+1}^q b_j.$$

Additionally, set

$$c^* := m + n - \frac{p+q}{2}.$$

In the critical cases, we need to use the following two parameters:

$$\delta := \prod_{i=1}^p \alpha_i^{-\alpha_i} \prod_{j=1}^q \beta_j^{\beta_j};$$

$$\mu := \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p-q}{2}$$

The well-posedness of the Fox H -function is given by Theorems 1.1 and 1.2 of [KS04], which are summarized in the following figure 2:

We will need the following two parameters (see (1.5.15)/(1.5.16) and (1.8.10)/(1.8.11), respectively, of [KS04] for ρ and ρ^*):

$$\rho := \max_{1 \leq i \leq n} \left\lceil \frac{\Re(a_i) - 1}{\alpha_i} \right\rceil \quad \text{and} \quad \rho^* := \min_{1 \leq j \leq m} \left\lceil \frac{\Re(b_j)}{\beta_j} \right\rceil.$$

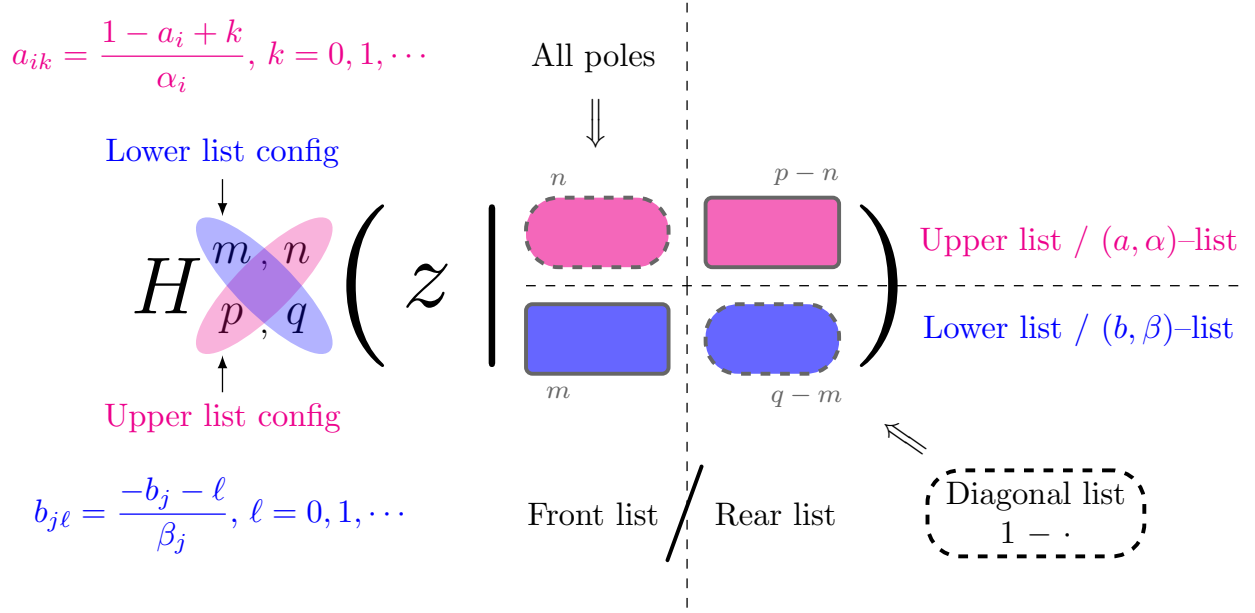


Figure 1: Diagram for the parameterization of the Fox H -function.

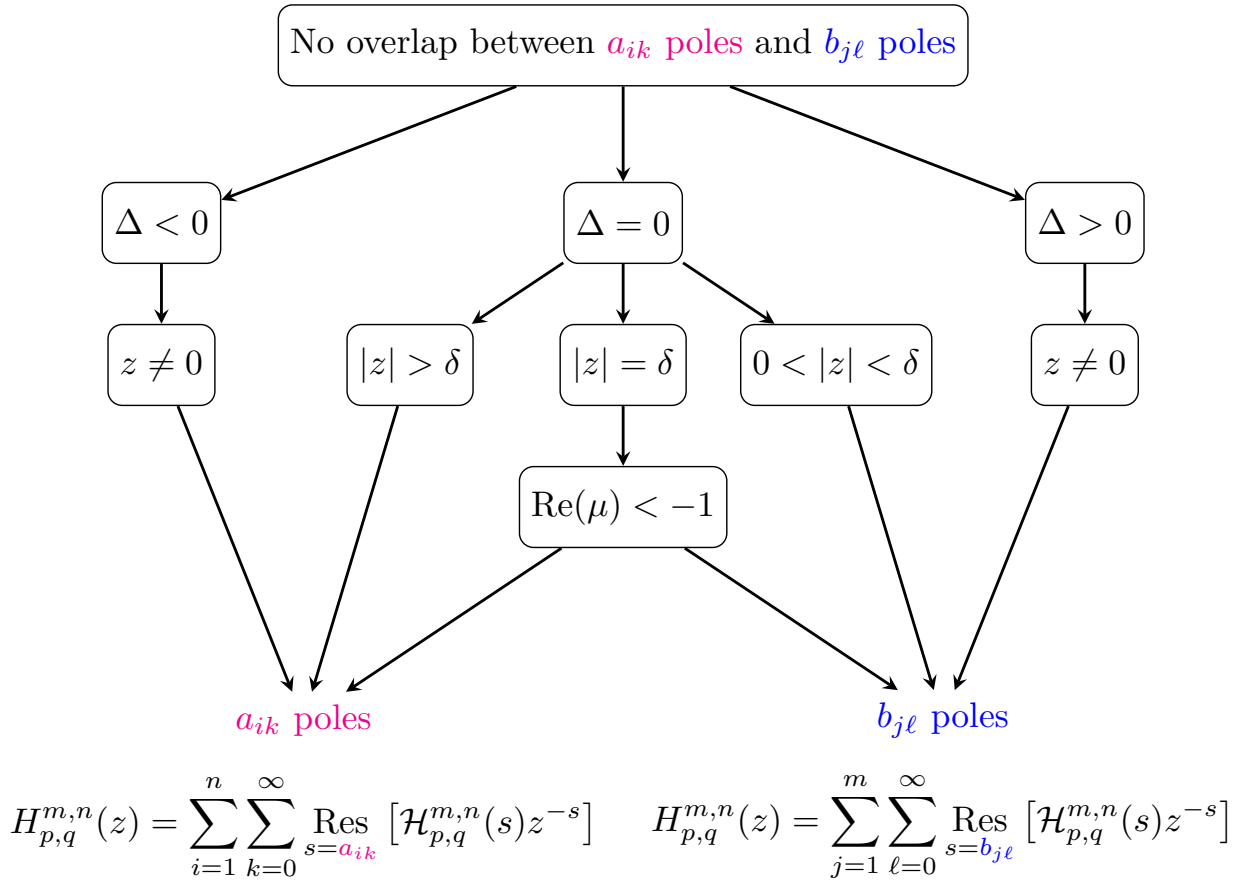


Figure 2: Well-posedness of the Fox H -function.

2 Examples

2.1 Example FoxH-2_9_11.wls

File content

```
(* (2.9.11) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {{1,1},{1,1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1,1}},
    (* Lower Rear List *) {{0,0}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,0} \left(\cdot \left| \begin{array}{c} (1,1), (1,1) \\ (1,1), (0,0) \end{array} \right. \right)$$

$$H_{2,2}^{1,0} \left(\cdot \left| \begin{array}{c|c} & (1,1), (1,1) \\ \hline (1,1) & (0,0) \end{array} \right. \right)$$

Summary

$$\begin{aligned} a^* &= -1 \\ \Delta &= -1 \\ \delta &= \text{Indeterminate} \\ \mu &= -1 \\ a_1^* &= -1 \\ a_2^* &= 0 \\ \xi &= -1 \\ c^* &= -1 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{ \}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{cccccccc} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{array} \right)$$

2.2 Example FoxH-2_9_12.wls

File content

```
(* (2.9.12) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1/2,1},{1/2,1}},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0,1}},
    (* Lower Rear List *) {{-1/2,1}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,2} \left(. \left| \begin{array}{c} \left(\frac{1}{2}, 1 \right), \left(\frac{1}{2}, 1 \right) \\ (0, 1), \left(-\frac{1}{2}, 1 \right) \end{array} \right. \right)$$

$$H_{2,2}^{1,2} \left(. \left| \frac{\left(\frac{1}{2}, 1 \right), \left(\frac{1}{2}, 1 \right)}{(0, 1)} \right| \frac{}{\left(-\frac{1}{2}, 1 \right)} \right)$$

Summary

$$\begin{aligned} a^* &= 2 \\ \Delta &= 0 \\ \delta &= \text{ComplexInfinity} \\ \mu &= -\frac{3}{2} \\ a_1^* &= 1 \\ a_2^* &= 1 \\ \xi &= \frac{3}{2} \\ c^* &= 1 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \left(\begin{array}{cccccccc} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} \\ \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} \end{array} \right)$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

2.3 Example FoxH-2_9_13.wls

File content

```
(* (2.9.13) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1/2,1},{1,1}},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2,1}},
    (* Lower Rear List *) {{0,1}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,2} \left(\cdot \left| \begin{array}{c} (\frac{1}{2}, 1), (1, 1) \\ (\frac{1}{2}, 1), (0, 1) \end{array} \right. \right)$$

$$H_{2,2}^{1,2} \left(\cdot \left| \begin{array}{c|c} (\frac{1}{2}, 1), (1, 1) & \\ \hline (\frac{1}{2}, 1) & (0, 1) \end{array} \right. \right)$$

Summary

$$\begin{aligned} a^* &= 2 \\ \Delta &= 0 \\ \delta &= \text{ComplexInfinity} \\ \mu &= -1 \\ a_1^* &= 1 \\ a_2^* &= 1 \\ \xi &= 2 \\ c^* &= 1 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \left(\begin{array}{cccccccc} \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} & \frac{15}{2} \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \right)$$

2. First eight poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{cccccccc} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} & -\frac{7}{2} & -\frac{9}{2} & -\frac{11}{2} & -\frac{13}{2} & -\frac{15}{2} \end{array} \right)$$

2.4 Example FoxH-2_9_4.wls

File content

```
(* (2.9.4) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{b, β}},
    (* Lower Rear List *) {}
  }
}
```

Fox H-function

$$H_{0,1}^{1,0} \left(\cdot \left| \begin{array}{c} \\ (b, \beta) \end{array} \right. \right)$$

$$H_{0,1}^{1,0} \left(\cdot \left| \begin{array}{c} \hline (b, \beta) \end{array} \right. \right)$$

Summary

$$\begin{aligned} a^* &= \beta \\ \Delta &= \beta \\ \delta &= \text{Indeterminate} \\ \mu &= b - \frac{1}{2} \\ a_1^* &= \beta \\ a_2^* &= 0 \\ \xi &= b \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{ \}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \left(-\frac{b}{\beta} \quad -\frac{b+1}{\beta} \quad -\frac{b+2}{\beta} \quad -\frac{b+3}{\beta} \quad -\frac{b+4}{\beta} \quad -\frac{b+5}{\beta} \quad -\frac{b+6}{\beta} \quad -\frac{b+7}{\beta} \right)$$

Source This example is from (2.9.4) of [\[KS04\]](#):

$$H_{0,1}^{1,0} \left(z \middle| \begin{array}{c} \\ (b, \beta) \end{array} \right) = \frac{1}{\beta} z^{b/\beta} \exp \left(-z^{1/\beta} \right).$$

2.5 Example FoxH-2_9_5.wls

File content

```
(* (2.9.5) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1-a,1}},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *) {}
  }
}
```

Fox H-function

$$H_{1,1}^{1,1} \left(\cdot \left| \begin{array}{c} (1-a, 1) \\ (0, 1) \end{array} \right. \right)$$

$$H_{1,1}^{1,1} \left(\cdot \left| \frac{(1-a, 1)}{(0, 1)} \right| \right)$$

Summary

$$\begin{aligned} a^* &= 2 \\ \Delta &= 0 \\ \delta &= \text{Indeterminate} \\ \mu &= a - 1 \\ a_1^* &= 1 \\ a_2^* &= 1 \\ \xi &= 1 - a \\ c^* &= 1 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \left(\begin{array}{cccccccc} a & a+1 & a+2 & a+3 & a+4 & a+5 & a+6 & a+7 \end{array} \right)$$

2. First eight poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{cccccccc} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{array} \right)$$

2.6 Example FoxH-2_9_6.wls

File content

```
(* (2.9.6) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {{α+β+1,1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{α, 1}},
    (* Lower Rear List *) {}
  }
}
```

Fox H-function

$$H_{1,1}^{1,0} \left(\begin{matrix} . \\ (\alpha, 1) \end{matrix} \middle| \begin{matrix} (\alpha + \beta + 1, 1) \end{matrix} \right)$$

$$H_{1,1}^{1,0} \left(\begin{matrix} . \\ (\alpha, 1) \end{matrix} \middle| \frac{(\alpha + \beta + 1, 1)}{(\alpha, 1)} \right)$$

Summary

$$\begin{aligned} a^* &= 0 \\ \Delta &= 0 \\ \delta &= \text{Indeterminate} \\ \mu &= -\beta - 1 \\ a_1^* &= 0 \\ a_2^* &= 0 \\ \xi &= -\beta - 1 \\ c^* &= 0 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{ \}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \left(\begin{matrix} -\alpha & -\alpha - 1 & -\alpha - 2 & -\alpha - 3 & -\alpha - 4 & -\alpha - 5 & -\alpha - 6 & -\alpha - 7 \end{matrix} \right)$$

2.7 Example FoxH32-21-Y.wls

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *) {{β + γ, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *) {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c} (1, 1), (\beta + \gamma, \beta) \\ (\frac{d}{2}, \frac{\alpha}{2}), (1, 1), (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c|c} (1, 1) & (\beta + \gamma, \beta) \\ \hline (\frac{d}{2}, \frac{\alpha}{2}), (1, 1) & (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

Summary

$$\begin{aligned} a^* &= 2 - \beta \\ \Delta &= \alpha - \beta \\ \delta &= 2^{-\alpha} \left(2^{\alpha/2} \alpha^{\alpha/2} + \alpha^\alpha \right) \beta^{-\beta} \\ \mu &= \frac{1}{2} (-2\beta - 2\gamma + d + 1) \\ a_1^* &= \frac{1}{2} (\alpha - 2\beta + 2) \\ a_2^* &= 1 - \frac{\alpha}{2} \\ \xi &= \frac{1}{2} (d - 2(\beta + \gamma - 1)) \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \left(\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \right)$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{pmatrix}$$

2.8 Example FoxH32-21-Z.wls

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *) {{Ceiling[β], β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *) {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c} (1, 1), (\lceil \beta \rceil, \beta) \\ (\frac{d}{2}, \frac{\alpha}{2}), (1, 1), (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left(\cdot \left| \frac{(1, 1)}{(\frac{d}{2}, \frac{\alpha}{2}), (1, 1)} \right| \frac{(\lceil \beta \rceil, \beta)}{(1, \frac{\alpha}{2})} \right)$$

Summary

$$\begin{aligned} a^* &= 2 - \beta \\ \Delta &= \alpha - \beta \\ \delta &= 2^{-\alpha} \left(2^{\alpha/2} \alpha^{\alpha/2} + \alpha^\alpha \right) \beta^{-\beta} \\ \mu &= \frac{1}{2} (-2\lceil \beta \rceil + d + 1) \\ a_1^* &= \frac{1}{2} (\alpha - 2\beta + 2) \\ a_2^* &= 1 - \frac{\alpha}{2} \\ \xi &= \frac{1}{2} (-2\lceil \beta \rceil + d + 2) \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \left(\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \right)$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{pmatrix}$$

Source This is the fundamental solution to the fractional diffusion equation used, e.g., in [Che+17; CHN19; CE22; CGS22].

2.9 Example FoxH32-21-Z-Star.wls

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{1, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c} (1, 1), (1, \beta) \\ (\frac{d}{2}, \frac{\alpha}{2}), (1, 1), (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left(\cdot \left| \begin{array}{c|c} (1, 1) & (1, \beta) \\ \hline (\frac{d}{2}, \frac{\alpha}{2}), (1, 1) & (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

Summary

$$\begin{aligned} a^* &= 2 - \beta \\ \Delta &= \alpha - \beta \\ \delta &= 2^{-\alpha} \left(2^{\alpha/2} \alpha^{\alpha/2} + \alpha^\alpha \right) \beta^{-\beta} \\ \mu &= \frac{d-1}{2} \\ a_1^* &= \frac{1}{2}(\alpha - 2\beta + 2) \\ a_2^* &= 1 - \frac{\alpha}{2} \\ \xi &= \frac{d}{2} \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \left(\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \right)$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -\frac{d+2}{\alpha} & -\frac{d+4}{\alpha} & -\frac{d+6}{\alpha} & -\frac{d+8}{\alpha} & -\frac{d+10}{\alpha} & -\frac{d+12}{\alpha} & -\frac{d+14}{\alpha} \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \end{pmatrix}$$

2.10 Example FoxH-Bessel-J_2_9_18.wls

File content

```
(* (2.9.18) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(a+η)/2, 1}},
    (* Lower Rear List *) {{(a-η)/1, 1}}
  }
}
```

Fox H-function

$$H_{0,2}^{1,0} \left(. \left| \begin{array}{c} \\ \left(\frac{a+\eta}{2}, 1 \right), (a-\eta, 1) \end{array} \right. \right)$$

$$H_{0,2}^{1,0} \left(. \left| \begin{array}{c} \\ \left(\frac{a+\eta}{2}, 1 \right) \end{array} \right| \begin{array}{c} \\ (a-\eta, 1) \end{array} \right)$$

Summary

$$\begin{aligned} a^* &= 0 \\ \Delta &= 2 \\ \delta &= \text{ComplexInfinity} \\ \mu &= \frac{1}{2}(3a - \eta - 2) \\ a_1^* &= 1 \\ a_2^* &= -1 \\ \xi &= \frac{1}{2}(3\eta - a) \\ c^* &= 0 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{\}^T$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} \frac{1}{2}(-a - \eta) \\ \frac{1}{2}(-a - \eta - 2) \\ \frac{1}{2}(-a - \eta - 4) \\ \frac{1}{2}(-a - \eta - 6) \\ \frac{1}{2}(-a - \eta - 8) \\ -\frac{a}{2} - \frac{\eta}{2} - 5 \\ -\frac{a}{2} - \frac{\eta}{2} - 6 \\ -\frac{a}{2} - \frac{\eta}{2} - 7 \end{pmatrix}^T$$

2.11 Example FoxH-Bessel-K_2_9_19.wls

File content

```
(* (2.9.19) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(a-η)/2, 1},{(a+η)/2, 1}},
    (* Lower Rear List *) {}
  }
}
```

Fox H-function

$$H_{0,2}^{2,0} \left(\cdot \left| \begin{array}{c} \\ \left(\frac{a-\eta}{2}, 1 \right), \left(\frac{a+\eta}{2}, 1 \right) \end{array} \right. \right)$$

$$H_{0,2}^{2,0} \left(\cdot \left| \frac{\cdot}{\left(\frac{a-\eta}{2}, 1 \right), \left(\frac{a+\eta}{2}, 1 \right)} \right| \right)$$

Summary

$$\begin{aligned} a^* &= 2 \\ \Delta &= 2 \\ \delta &= \text{Indeterminate} \\ \mu &= a - 1 \\ a_1^* &= 2 \\ a_2^* &= 0 \\ \xi &= a \\ c^* &= 1 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{\}^T$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} \frac{\eta-a}{2} & \frac{1}{2}(-a-\eta) \\ \frac{1}{2}(-a+\eta-2) & \frac{1}{2}(-a-\eta-2) \\ \frac{1}{2}(-a+\eta-4) & \frac{1}{2}(-a-\eta-4) \\ \frac{1}{2}(-a+\eta-6) & \frac{1}{2}(-a-\eta-6) \\ \frac{1}{2}(-a+\eta-8) & \frac{1}{2}(-a-\eta-8) \\ \frac{1}{2}(-a+\eta-10) & -\frac{a}{2}-\frac{\eta}{2}-5 \\ \frac{1}{2}(-a+\eta-12) & -\frac{a}{2}-\frac{\eta}{2}-6 \\ \frac{1}{2}(-a+\eta-14) & -\frac{a}{2}-\frac{\eta}{2}-7 \end{pmatrix}^T$$

2.12 Example FoxH-Bessel-Y_2_9_20.wls

File content

```
(* (2.9.20) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {{(a-η-1)/2, 1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(a-η)/2, 1},{(a+η)/2, 1}},
    (* Lower Rear List *) {{(a-η-1)/2, 1}}
  }
}
```

Fox H-function

$$H_{1,3}^{2,0} \left(. \left| \begin{array}{c} (\frac{1}{2}(a-\eta-1), 1) \\ (\frac{a-\eta}{2}, 1), (\frac{a+\eta}{2}, 1), (\frac{1}{2}(a-\eta-1), 1) \end{array} \right. \right)$$

$$H_{1,3}^{2,0} \left(. \left| \frac{\quad}{(\frac{a-\eta}{2}, 1), (\frac{a+\eta}{2}, 1)} \right| \begin{array}{c} (\frac{1}{2}(a-\eta-1), 1) \\ (\frac{1}{2}(a-\eta-1), 1) \end{array} \right)$$

Summary

$$\begin{aligned} a^* &= 0 \\ \Delta &= 2 \\ \delta &= \text{ComplexInfinity} \\ \mu &= a - 1 \\ a_1^* &= 1 \\ a_2^* &= -1 \\ \xi &= \eta + 1 \\ c^* &= 0 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{\}^T$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} \frac{\eta-a}{2} & \frac{1}{2}(-a-\eta) \\ \frac{1}{2}(-a+\eta-2) & \frac{1}{2}(-a-\eta-2) \\ \frac{1}{2}(-a+\eta-4) & \frac{1}{2}(-a-\eta-4) \\ \frac{1}{2}(-a+\eta-6) & \frac{1}{2}(-a-\eta-6) \\ \frac{1}{2}(-a+\eta-8) & \frac{1}{2}(-a-\eta-8) \\ \frac{1}{2}(-a+\eta-10) & -\frac{a}{2}-\frac{\eta}{2}-5 \\ \frac{1}{2}(-a+\eta-12) & -\frac{a}{2}-\frac{\eta}{2}-6 \\ \frac{1}{2}(-a+\eta-14) & -\frac{a}{2}-\frac{\eta}{2}-7 \end{pmatrix}^T$$

2.13 Example FoxH-Cos.wls

File content

```
(* (2.9.8) and (2.9.10) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *) {{1/2, 1}}
  }
}
```

Fox H-function

$$H_{0,2}^{1,0} \left(\cdot \left| \begin{array}{c} (0, 1), (\frac{1}{2}, 1) \end{array} \right. \right)$$

$$H_{0,2}^{1,0} \left(\cdot \left| \begin{array}{c|c} (0, 1) & (\frac{1}{2}, 1) \end{array} \right. \right)$$

Summary

$$\begin{aligned} a^* &= 0 \\ \Delta &= 2 \\ \delta &= \text{ComplexInfinity} \\ \mu &= -\frac{1}{2} \\ a_1^* &= 1 \\ a_2^* &= -1 \\ \xi &= -\frac{1}{2} \\ c^* &= 0 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{ \}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \left(\begin{array}{cccccccc} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{array} \right)$$

2.14 Example FoxH-H.G_2_9_14.wls

File content

```
(* (2.9.14) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1-a,1}},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0,1}},
    (* Lower Rear List *) {{1-c,1}}
  }
}
```

Fox H-function

$$H_{1,2}^{1,1} \left(\cdot \left| \begin{array}{c} (1-a, 1) \\ (0, 1), (1-c, 1) \end{array} \right. \right)$$

$$H_{1,2}^{1,1} \left(\cdot \left| \frac{(1-a, 1)}{(0, 1) \mid (1-c, 1)} \right. \right)$$

Summary

$$\begin{aligned} a^* &= 1 \\ \Delta &= 1 \\ \delta &= \text{ComplexInfinity} \\ \mu &= a - c - \frac{1}{2} \\ a_1^* &= 1 \\ a_2^* &= 0 \\ \xi &= c - a \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \left(\begin{array}{cccccccc} a & a+1 & a+2 & a+3 & a+4 & a+5 & a+6 & a+7 \end{array} \right)$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

2.15 Example FoxH-H.G_2_9_15.wls

File content

```
(* (2.9.15) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{1-a,1},{1-b,1}},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0,1}},
    (* Lower Rear List *) {{1-c,1}}
  }
}
```

Fox H-function

$$H_{2,2}^{1,2} \left(\cdot \left| \begin{array}{c} (1-a, 1), (1-b, 1) \\ (0, 1), (1-c, 1) \end{array} \right. \right)$$

$$H_{2,2}^{1,2} \left(\cdot \left| \frac{(1-a, 1), (1-b, 1)}{(0, 1)} \right| (1-c, 1) \right)$$

Summary

$$\begin{aligned} a^* &= 2 \\ \Delta &= 0 \\ \delta &= \text{ComplexInfinity} \\ \mu &= a + b - c - 1 \\ a_1^* &= 1 \\ a_2^* &= 1 \\ \xi &= -a - b + c + 1 \\ c^* &= 1 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \begin{pmatrix} a & a+1 & a+2 & a+3 & a+4 & a+5 & a+6 & a+7 \\ b & b+1 & b+2 & b+3 & b+4 & b+5 & b+6 & b+7 \end{pmatrix}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

2.16 Example FoxH-Lommel_2_9_22.wls

File content

```
(* (2.9.22) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{(1+μ)/2,1}},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{(1+μ)/2,1},{η/2,1},{-η/2,1}},
    (* Lower Rear List *) {}
  }
}
```

Fox H-function

$$H_{1,3}^{3,1} \left(. \left| \begin{array}{c} \left(\frac{\mu+1}{2}, 1 \right) \\ \left(\frac{\mu+1}{2}, 1 \right), \left(\frac{\eta}{2}, 1 \right), \left(-\frac{\eta}{2}, 1 \right) \end{array} \right. \right)$$

$$H_{1,3}^{3,1} \left(. \left| \frac{\left(\frac{\mu+1}{2}, 1 \right)}{\left(\frac{\mu+1}{2}, 1 \right), \left(\frac{\eta}{2}, 1 \right), \left(-\frac{\eta}{2}, 1 \right)} \right| \right)$$

Summary

$$\begin{aligned} a^* &= 4 \\ \Delta &= 2 \\ \delta &= \text{Indeterminate} \\ \mu &= -1 \\ a_1^* &= 3 \\ a_2^* &= 1 \\ \xi &= \mu + 1 \\ c^* &= 2 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \begin{pmatrix} \frac{1-\mu}{2} \\ \frac{3-\mu}{2} \\ \frac{5-\mu}{2} \\ \frac{7-\mu}{2} \\ \frac{9-\mu}{2} \\ \frac{11-\mu}{2} \\ \frac{13-\mu}{2} \\ \frac{15-\mu}{2} \end{pmatrix}^T$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} \frac{1}{2}(-\mu-1) & -\frac{\eta}{2} & \frac{\eta}{2} \\ \frac{1}{2}(-\mu-3) & -\frac{\eta}{2}-1 & \frac{\eta-2}{2} \\ \frac{1}{2}(-\mu-5) & -\frac{\eta}{2}-2 & \frac{\eta-4}{2} \\ \frac{1}{2}(-\mu-7) & -\frac{\eta}{2}-3 & \frac{\eta-6}{2} \\ \frac{1}{2}(-\mu-9) & -\frac{\eta}{2}-4 & \frac{\eta-8}{2} \\ \frac{1}{2}(-\mu-11) & -\frac{\eta}{2}-5 & \frac{\eta}{2}-5 \\ \frac{1}{2}(-\mu-13) & -\frac{\eta}{2}-6 & \frac{\eta}{2}-6 \\ \frac{1}{2}(-\mu-15) & -\frac{\eta}{2}-7 & \frac{\eta}{2}-7 \end{pmatrix}^T$$

2.17 Example FoxH-Mittag-Leffler.wls

File content

```
(* (2.9.27) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {{0, 1}},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *) {{1 - μ, ρ}}
  }
}
```

Fox H-function

$$H_{1,2}^{1,1} \left(\cdot \left| \begin{array}{c} (0, 1) \\ (0, 1), (1 - \mu, \rho) \end{array} \right. \right)$$

$$H_{1,2}^{1,1} \left(\cdot \left| \frac{(0, 1)}{(0, 1) \mid (1 - \mu, \rho)} \right. \right)$$

Summary

$$\begin{aligned} a^* &= 2 - \rho \\ \Delta &= \rho \\ \delta &= \text{ComplexInfinity} \\ \mu &= \frac{1}{2} - \mu \\ a_1^* &= 1 \\ a_2^* &= 1 - \rho \\ \xi &= \mu - 1 \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \left(\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \right)$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 & -1 & -2 & -3 & -4 & -5 & -6 & -7 \end{pmatrix}$$

2.18 Example FoxH-Sin.wls

File content

```
(* (2.9.7) and (2.9.9) of Kilbas & Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2, 1}},
    (* Lower Rear List *) {{0, 1}}
  }
}
```

Fox H-function

$$H_{0,2}^{1,0} \left(\cdot \left| \begin{array}{c} \\ (\frac{1}{2}, 1), (0, 1) \end{array} \right. \right)$$

$$H_{0,2}^{1,0} \left(\cdot \left| \begin{array}{c|c} & \\ \hline (\frac{1}{2}, 1) & (0, 1) \end{array} \right. \right)$$

Summary

$$\begin{aligned} a^* &= 0 \\ \Delta &= 2 \\ \delta &= \text{ComplexInfinity} \\ \mu &= -\frac{1}{2} \\ a_1^* &= 1 \\ a_2^* &= -1 \\ \xi &= \frac{1}{2} \\ c^* &= 0 \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{ \}$$

2. First eight poles from lower front list

$$b_{j,\ell} = \left(-\frac{1}{2} \quad -\frac{3}{2} \quad -\frac{5}{2} \quad -\frac{7}{2} \quad -\frac{9}{2} \quad -\frac{11}{2} \quad -\frac{13}{2} \quad -\frac{15}{2} \right)$$

2.19 Example FoxH-Whittaker_2_9_21.wls

File content

```
(* (2.9.21) of Kilbas and Saigo 04 *)
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {{a-λ+1,1}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{a+μ+1/2,1},{a-μ+1/2,1}},
    (* Lower Rear List *) {}
  }
}
```

Fox H-function

$$H_{1,2}^{2,0} \left(\cdot \left| \begin{array}{c} (a - \lambda + 1, 1) \\ (a + \mu + \frac{1}{2}, 1), (a - \mu + \frac{1}{2}, 1) \end{array} \right. \right)$$

$$H_{1,2}^{2,0} \left(\cdot \left| \frac{}{(a + \mu + \frac{1}{2}, 1), (a - \mu + \frac{1}{2}, 1)} \right| (a - \lambda + 1, 1) \right)$$

Summary

$$\begin{aligned} a^* &= 1 \\ \Delta &= 1 \\ \delta &= \text{Indeterminate} \\ \mu &= a + \lambda - \frac{1}{2} \\ a_1^* &= 1 \\ a_2^* &= 0 \\ \xi &= a + \lambda \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First eight poles from upper front list

$$a_{i,k} = \{\}^T$$

2. First eight poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -a - \mu - \frac{1}{2} & -a + \mu - \frac{1}{2} \\ -a - \mu - \frac{3}{2} & -a + \mu - \frac{3}{2} \\ -a - \mu - \frac{5}{2} & -a + \mu - \frac{5}{2} \\ -a - \mu - \frac{7}{2} & -a + \mu - \frac{7}{2} \\ -a - \mu - \frac{9}{2} & -a + \mu - \frac{9}{2} \\ -a - \mu - \frac{11}{2} & -a + \mu - \frac{11}{2} \\ -a - \mu - \frac{13}{2} & -a + \mu - \frac{13}{2} \\ -a - \mu - \frac{15}{2} & -a + \mu - \frac{15}{2} \end{pmatrix}^T$$

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