Some symbolic tools for the Fox H-function

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In this note, we explain the code for checking the conditions of the Fox H-function [Fox61]. Here we follow the notation from Kilbas and Saigo [KS].

Let m, n, p, q be configure integers such that

$$0 \le m \le q \quad \text{and} \quad 0 \le n \le p.$$

Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+$ be the parameters given below:

$\in (\mathbb{C}, \mathbb{R}_+)$	Front list	Rear list	
p	$(a_1, \alpha_1), \cdots, (a_n, \alpha_n)$	$(a_{n+1}, \alpha_{n+1}), \cdots, (a_p, \alpha_p)$	Upper list
q	$(b_1, \beta_1), \cdots, (b_m, \beta_m)$	$(b_{m+1}, \beta_{m+1}), \cdots, (b_q, \beta_q)$	Lower list

and denote

$$\mathcal{H}_{p,q}^{m,n}(s) \coloneqq \frac{\displaystyle\prod_{i=1}^{n} \Gamma\left(1 - a_i - \alpha_i s\right)}{\displaystyle\prod_{i=n+1}^{p} \Gamma\left(a_j + \alpha_i s\right)} \times \frac{\displaystyle\prod_{j=1}^{m} \Gamma\left(b_j + \beta_j s\right)}{\displaystyle\prod_{j=m+1}^{q} \Gamma\left(1 - b_j - \alpha_j s\right)}. \tag{1}$$

Then the Fox H-function $H^{2,1}_{2,3}\left(z \,\middle|\, \begin{matrix} \cdots \\ \\ \cdots \end{matrix}\right)$ is defined by a Mellin-Barnes type integral of the form

$$H^{p,q}_{m,n}\left(z \,\middle|\, \begin{matrix} (a_1,\alpha_1),\cdots,(a_p,\alpha_p) \\ \\ (b_1,\beta_1),\cdots,(b_q,\beta_q) \end{matrix}\right) := \frac{1}{2\pi i}\int_{\mathcal{L}} H^{m,n}_{p,q}(s) z^{-s} \mathrm{d}s. \tag{2}$$

The basic assumption for the well-posedness of the Fox H-function is that two sets of poles do not overlap, i.e.,

$$\left\{b_{j\ell} = \frac{-b_j-\ell}{\beta_j}, \ell = 0,1,\cdots\right\} \bigcap \left\{a_{ik} = \frac{1-a_i+k}{\alpha_i}, k = 0,1,\cdots\right\} = \emptyset. \tag{3}$$

The contour $\mathcal{L}$ in (2) is given by one of the following three cases:

1. $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at point $-\infty + i\phi_1$ and terminating at point $-\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;

2. $\mathcal{L} = \mathcal{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at point $+\infty + i\phi_1$ and terminating at point $\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;

3. $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ is a contour starting at point $\gamma - i\infty$ and terminating at point $\gamma + i\infty$ for some $\gamma \in (-\infty, \infty)$.

We need a set of conditions to ensure the convergence of the integral in (2). To explain this, we need to introduce some notation (following p. 2 of [KS]). First denote

$$a_1^* := \sum_{j=1}^{m} \beta_j - \sum_{i=n+1}^{p} \alpha_i,$$
$$a_2^* := \sum_{i=1}^{n} \alpha_i - \sum_{j=m+1}^{q} \beta_j.$$

The following two parameters play the most important role:

$$a^* \coloneqq a_2^* + a_1^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j;$$
$$\Delta \coloneqq a_2^* - a_1^* = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i.$$

Similar to a^*, we define

$$\xi \coloneqq \sum_{i=1}^{n} a_i - \sum_{i=n+1}^{p} a_i + \sum_{j=1}^{m} b_j - \sum_{j=m+1}^{q} b_j.$$

Additionally, set

$$c^* \coloneqq m + n - \frac{p+q}{2}.$$

In the critical cases, we need to use the following two parameters:

$$\delta \coloneqq \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_i};$$
$$\mu \coloneqq \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}$$

The well-posedness of the Fox H-function is given by Theorems 1.1 and 1.2 of [KS], which are summarized in the following figure 2:

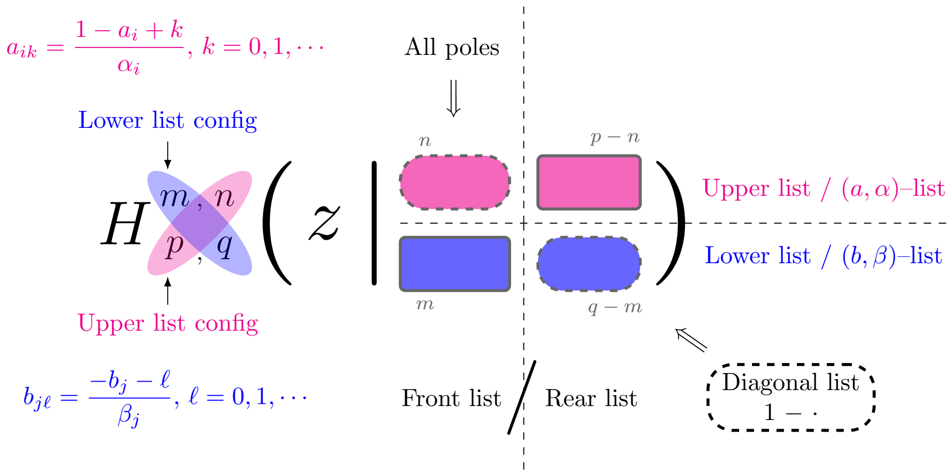

Figure 1: Diagram for the parameterization of the Fox H-function.

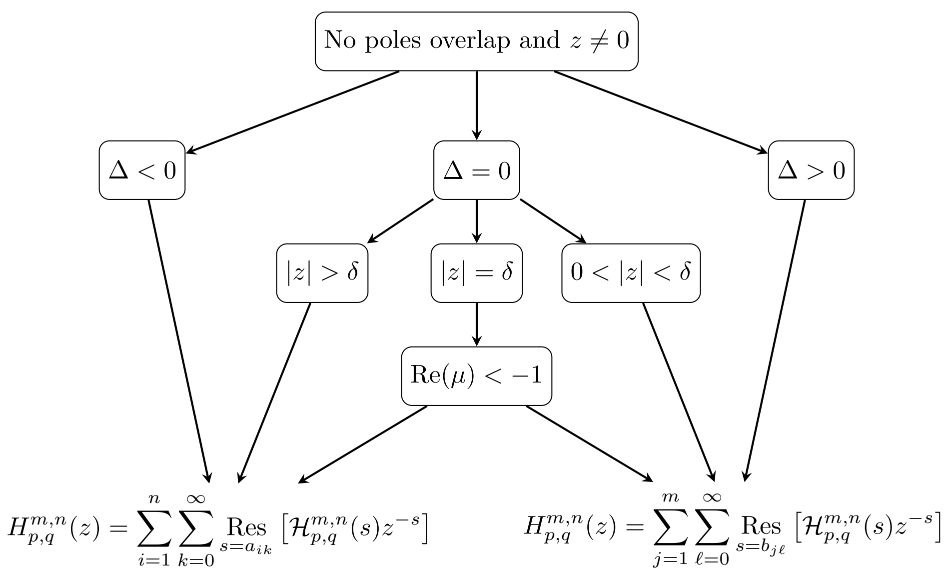

Figure 2: Well-posedness of the Fox H-function.

References

[Fox61] Charles Fox. The G and H functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, 98:395–429, 1961.

[KS] Anatoly A. Kilbas and Megumi Saigo. *H-transforms*, volume 9 of *Analytical Methods and Special Functions.* Chapman & Hall/CRC, Boca Raton, FL. Theory and applications.