

Some symbolic tools for the Fox H -function

Le Chen

Department of Mathematics and Statistics

Auburn University

le.chen@auburn.edu, chenle02@gmail.com

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In this note, we explain the code for checking the conditions of the Fox H -function [Fox61]. Here we follow the notation from Kilbas and Saigo [KS].

Let m, n, p, q be configure integers such that

$$0 \leq m \leq q \quad \text{and} \quad 0 \leq n \leq p.$$

Let $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}_+$ be the parameters given below:

$\in (\mathbb{C}, \mathbb{R}_+)$	Front list	Rear list	
p	$(a_1, \alpha_1), \dots, (a_n, \alpha_n)$	$(a_{n+1}, \alpha_{n+1}), \dots, (a_p, \alpha_p)$	Upper list
q	$(b_1, \beta_1), \dots, (b_m, \beta_m)$	$(b_{m+1}, \beta_{m+1}), \dots, (b_q, \beta_q)$	Lower list

and denote

$$\mathcal{H}_{p,q}^{m,n}(s) := \frac{\prod_{i=1}^n \Gamma(1 - a_i - \alpha_i s)}{\prod_{j=n+1}^p \Gamma(a_j + \alpha_j s)} \times \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \alpha_j s)}. \quad (1)$$

Then the Fox H -function $H_{2,3}^{2,1} \left(z \left| \begin{array}{c} \cdots \\ \cdots \end{array} \right. \right)$ is defined by a Mellin-Barnes type integral of the form

$$H_{m,n}^{p,q} \left(z \left| \begin{array}{c} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{array} \right. \right) := \frac{1}{2\pi i} \int_{\mathcal{L}} H_{p,q}^{m,n}(s) z^{-s} ds. \quad (2)$$

The basic assumption for the well-posedness of the Fox H -function is that two sets of poles do not overlap, i.e.,

$$\left\{ b_{j\ell} = \frac{-b_j - \ell}{\beta_j}, \ell = 0, 1, \dots \right\} \cap \left\{ a_{ik} = \frac{1 - a_i + k}{\alpha_i}, k = 0, 1, \dots \right\} = \emptyset. \quad (3)$$

The contour \mathcal{L} in (2) is given by one of the following three cases:

1. $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at point $-\infty + i\phi_1$ and terminating at point $-\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;
2. $\mathcal{L} = \mathcal{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at point $+\infty + i\phi_1$ and terminating at point $+\infty + i\phi_2$ for some $-\infty < \phi_1 < \phi_2 < \infty$;
3. $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ is a contour starting at point $\gamma - i\infty$ and terminating at point $\gamma + i\infty$ for some $\gamma \in (-\infty, \infty)$.

We need a set of conditions to ensure the convergence of the integral in (2). To explain this, we need to introduce some notation (following p. 2 of [KS]). First denote

$$a_1^* := \sum_{j=1}^m \beta_j - \sum_{i=n+1}^p \alpha_i,$$

$$a_2^* := \sum_{i=1}^n \alpha_i - \sum_{j=m+1}^q \beta_j.$$

The following two parameters play the most important role:

$$a^* := a_2^* + a_1^* = \sum_{i=1}^n \alpha_i - \sum_{i=n+1}^p \alpha_i + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j;$$

$$\Delta := a_2^* - a_1^* = \sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i.$$

Similar to a^* , we define

$$\xi := \sum_{i=1}^n a_i - \sum_{i=n+1}^p a_i + \sum_{j=1}^m b_j - \sum_{j=m+1}^q b_j.$$

Additionally, set

$$c^* := m + n - \frac{p+q}{2}.$$

In the critical cases, we need to use the following two parameters:

$$\delta := \prod_{i=1}^p \alpha_i^{-\alpha_i} \prod_{j=1}^q \beta_j^{\beta_j};$$

$$\mu := \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p-q}{2}$$

The well-posedness of the Fox H -function is given by Theorems 1.1 and 1.2 of [KS], which are summarized in the following figure 2:

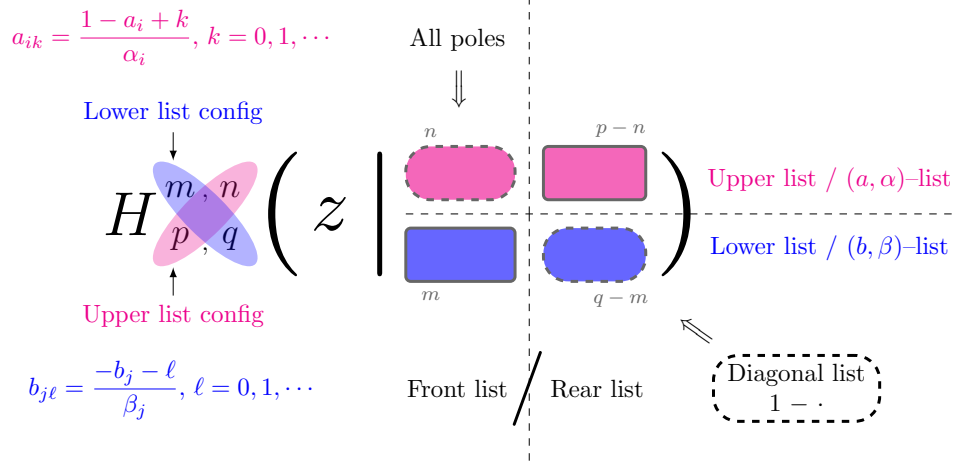


Figure 1: Diagram for the parameterization of the Fox H -function.

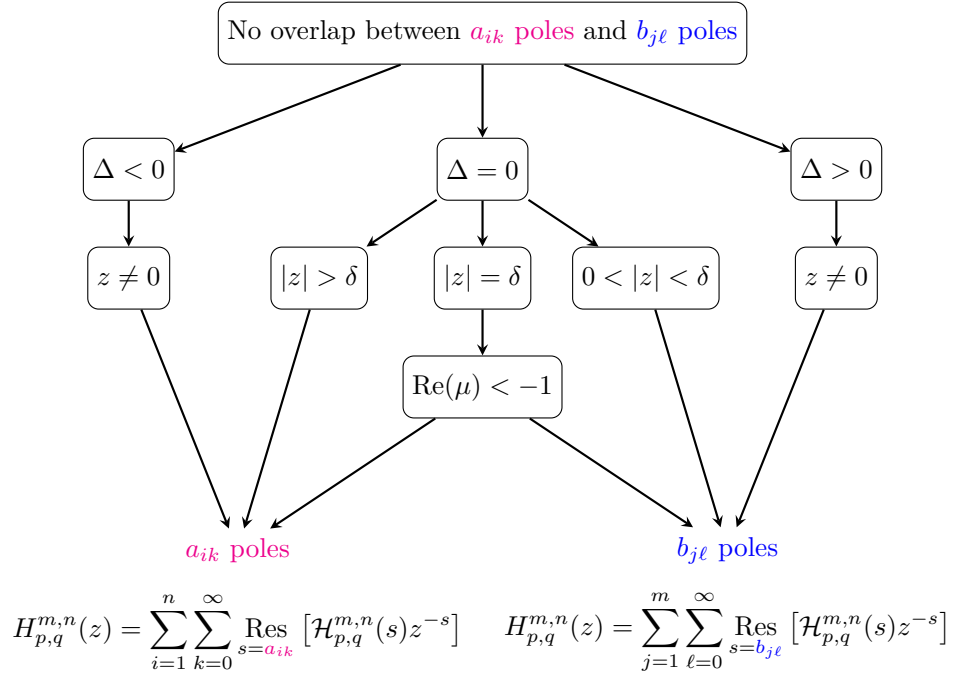


Figure 2: Well-posedness of the Fox H -function.

1 Example FoxH32-21.wls

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1,  $\alpha^{(-1)}$ }},
    (* Upper Rear List *) {{Ceiling[ $\beta$ ],  $\beta$ }}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2,  $\alpha/2$ }, {1, 1}},
    (* Lower Rear List *) {{1,  $\alpha/2$ }}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1} \left(. \left| \begin{array}{c} (1, \frac{1}{\alpha}), (\lceil \beta \rceil, \beta) \\ (\frac{1}{2}, \frac{\alpha}{2}), (1, 1), (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left(. \left| \frac{(1, \frac{1}{\alpha})}{(\frac{1}{2}, \frac{\alpha}{2}), (1, 1)} \right| \frac{(\lceil \beta \rceil, \beta)}{} \right)$$

Summary

$$a^* = \frac{1}{\alpha} - \beta + 1$$

$$\Delta = \alpha - \frac{1}{\alpha} - \beta + 1$$

$$\delta = 2^{-\alpha} \left(\frac{1}{\alpha} \right)^{-1/\alpha} \left(2^{\alpha/2} \alpha^{\alpha/2} + \alpha^\alpha \right) \beta^{-\beta}$$

$$\mu = 1 - \lceil \beta \rceil$$

$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$

$$a_2^* = \frac{1}{\alpha} - \frac{\alpha}{2}$$

$$\xi = \frac{3}{2} - \lceil \beta \rceil$$

$$c^* = \frac{1}{2}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \begin{pmatrix} 0 \\ \alpha \\ 2\alpha \\ 3\alpha \\ 4\alpha \\ 5\alpha \\ 6\alpha \\ 7\alpha \\ 8\alpha \\ 9\alpha \\ 10\alpha \end{pmatrix}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{1}{\alpha} & -1 \\ -\frac{3}{\alpha} & -2 \\ -\frac{5}{\alpha} & -3 \\ -\frac{7}{\alpha} & -4 \\ -\frac{9}{\alpha} & -5 \\ -\frac{11}{\alpha} & -6 \\ -\frac{13}{\alpha} & -7 \\ -\frac{15}{\alpha} & -8 \\ -\frac{17}{\alpha} & -9 \\ -\frac{19}{\alpha} & -10 \\ -\frac{21}{\alpha} & -11 \end{pmatrix}$$

2 Example FoxH32-21-Y.wls

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{β + γ, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1} \left(. \left| \begin{array}{c} (1, 1), (\beta + \gamma, \beta) \\ (\frac{d}{2}, \frac{\alpha}{2}), (1, 1), (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left(. \left| \frac{(1, 1)}{(\frac{d}{2}, \frac{\alpha}{2}), (1, 1)} \right| \frac{(\beta + \gamma, \beta)}{} \right)$$

Summary

$$a^* = 2 - \beta$$

$$\Delta = \alpha - \beta$$

$$\delta = 2^{-\alpha} \left(2^{\alpha/2} \alpha^{\alpha/2} + \alpha^\alpha \right) \beta^{-\beta}$$

$$\mu = \frac{1}{2}(-2\beta - 2\gamma + d + 1)$$

$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$

$$a_2^* = 1 - \frac{\alpha}{2}$$

$$\xi = \frac{1}{2}(d - 2(\beta + \gamma - 1))$$

$$c^* = \frac{1}{2}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{pmatrix}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -1 \\ -\frac{d+2}{\alpha} & -2 \\ -\frac{d+4}{\alpha} & -3 \\ -\frac{d+6}{\alpha} & -4 \\ -\frac{d+8}{\alpha} & -5 \\ -\frac{d+10}{\alpha} & -6 \\ -\frac{d+12}{\alpha} & -7 \\ -\frac{d+14}{\alpha} & -8 \\ -\frac{d+16}{\alpha} & -9 \\ -\frac{d+18}{\alpha} & -10 \\ -\frac{d+20}{\alpha} & -11 \end{pmatrix}$$

3 Example FoxH32-21-Z.wls

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *) {{Ceiling[β], β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *) {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1} \left(. \left| \begin{array}{c} (1, 1), (\lceil \beta \rceil, \beta) \\ (\frac{d}{2}, \frac{\alpha}{2}), (1, 1), (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left(. \left| \frac{(1, 1)}{(\frac{d}{2}, \frac{\alpha}{2}), (1, 1)} \right| \frac{(\lceil \beta \rceil, \beta)}{\phantom{(\frac{d}{2}, \frac{\alpha}{2}), (1, 1)}} \right)$$

Summary

$$a^* = 2 - \beta$$

$$\Delta = \alpha - \beta$$

$$\delta = 2^{-\alpha} \left(2^{\alpha/2} \alpha^{\alpha/2} + \alpha^\alpha \right) \beta^{-\beta}$$

$$\mu = \frac{1}{2}(-2\lceil\beta\rceil + d + 1)$$

$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$

$$a_2^* = 1 - \frac{\alpha}{2}$$

$$\xi = \frac{1}{2}(-2\lceil\beta\rceil + d + 2)$$

$$c^* = \frac{1}{2}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{pmatrix}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -1 \\ -\frac{d+2}{\alpha} & -2 \\ -\frac{d+4}{\alpha} & -3 \\ -\frac{d+6}{\alpha} & -4 \\ -\frac{d+8}{\alpha} & -5 \\ -\frac{d+10}{\alpha} & -6 \\ -\frac{d+12}{\alpha} & -7 \\ -\frac{d+14}{\alpha} & -8 \\ -\frac{d+16}{\alpha} & -9 \\ -\frac{d+18}{\alpha} & -10 \\ -\frac{d+20}{\alpha} & -11 \end{pmatrix}$$

4 Example FoxH32-21-Z-Star.wls

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1}},
    (* Upper Rear List *)  {{1, β}}
  },
  (* Lower List *) {
    (* Lower Front List *) {{d/2, α/2}, {1, 1}},
    (* Lower Rear List *)  {{1, α/2}}
  }
}
```

Fox H-function

$$H_{2,3}^{2,1} \left(. \left| \begin{array}{c} (1, 1), (1, \beta) \\ (\frac{d}{2}, \frac{\alpha}{2}), (1, 1), (1, \frac{\alpha}{2}) \end{array} \right. \right)$$

$$H_{2,3}^{2,1} \left(. \left| \frac{(1, 1)}{(\frac{d}{2}, \frac{\alpha}{2}), (1, 1)} \right| \frac{(1, \beta)}{} \right)$$

Summary

$$a^* = 2 - \beta$$

$$\Delta = \alpha - \beta$$

$$\delta = 2^{-\alpha} \left(2^{\alpha/2} \alpha^{\alpha/2} + \alpha^\alpha \right) \beta^{-\beta}$$

$$\mu = \frac{d-1}{2}$$

$$a_1^* = \frac{1}{2}(\alpha - 2\beta + 2)$$

$$a_2^* = 1 - \frac{\alpha}{2}$$

$$\xi = \frac{d}{2}$$

$$c^* = \frac{1}{2}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{pmatrix}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{d}{\alpha} & -1 \\ -\frac{d+2}{\alpha} & -2 \\ -\frac{d+4}{\alpha} & -3 \\ -\frac{d+6}{\alpha} & -4 \\ -\frac{d+8}{\alpha} & -5 \\ -\frac{d+10}{\alpha} & -6 \\ -\frac{d+12}{\alpha} & -7 \\ -\frac{d+14}{\alpha} & -8 \\ -\frac{d+16}{\alpha} & -9 \\ -\frac{d+18}{\alpha} & -10 \\ -\frac{d+20}{\alpha} & -11 \end{pmatrix}$$

5 Example FoxH-Cos.wls

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{1, 1/2}},
    (* Upper Rear List *)  {{1/2, 1/2}}
  },
  (* Lower List *) {
    (* Lower Front List *) {},
    (* Lower Rear List *) {}
  }
}
```

Fox H-function

$$H_{2,0}^{0,1} \left(\cdot \left| \left(1, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right) \right. \right)$$

$$H_{2,0}^{0,1} \left(\cdot \left| \frac{\left(1, \frac{1}{2}\right)}{\quad} \left| \frac{\left(\frac{1}{2}, \frac{1}{2}\right)}{\quad} \right. \right)$$

Summary

$$\begin{aligned} a^* &= 0 \\ \Delta &= -1 \\ \delta &= 0 \\ \mu &= -\frac{1}{2} \\ a_1^* &= -\frac{1}{2} \\ a_2^* &= \frac{1}{2} \\ \xi &= \frac{1}{2} \\ c^* &= 0 \end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \\ 8 \\ 10 \\ 12 \\ 14 \\ 16 \\ 18 \\ 20 \end{pmatrix}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \{\{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}\}$$

6 Example FoxH-Mittag-Leffler.wls

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {{0, 1}},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{0, 1}},
    (* Lower Rear List *) {{1 - μ, ρ}}
  }
}
```

Fox H-function

$$H_{1,2}^{1,1} \left(\cdot \left| \begin{array}{c} (0,1) \\ (0,1), (1-\mu, \rho) \end{array} \right. \right)$$

$$H_{1,2}^{1,1} \left(\cdot \left| \frac{(0,1)}{(0,1)} \right| \right)$$

Summary

$$\begin{aligned} a^* &= 2 - \rho \\ \Delta &= \rho \\ \delta &= \text{ComplexInfinity} \\ \mu &= \frac{1}{2} - \mu \\ a_1^* &= 1 \\ a_2^* &= 1 - \rho \\ \xi &= \mu - 1 \\ c^* &= \frac{1}{2} \end{aligned}$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{pmatrix}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} 0 \\ -1 \\ -2 \\ -3 \\ -4 \\ -5 \\ -6 \\ -7 \\ -8 \\ -9 \\ -10 \end{pmatrix}$$

7 Example FoxH-Sin.wls

File content

```
{
  (* Upper List *) {
    (* Upper Front List *) {},
    (* Upper Rear List *) {}
  },
  (* Lower List *) {
    (* Lower Front List *) {{1/2, 1}},
    (* Lower Rear List *) {{0, 1}}
  }
}
```

Fox H-function

$$H_{0,2}^{1,0} \left(. \left| \begin{array}{c} \\ \left(\frac{1}{2}, 1 \right), (0, 1) \end{array} \right. \right)$$

$$H_{0,2}^{1,0} \left(. \left| \frac{\quad}{\left(\frac{1}{2}, 1 \right)} \right| \right)$$

Summary

$$a^* = 0$$

$$\Delta = 2$$

$$\delta = \text{ComplexInfinity}$$

$$\mu = -\frac{1}{2}$$

$$a_1^* = 1$$

$$a_2^* = -1$$

$$\xi = \frac{1}{2}$$

$$c^* = 0$$

Poles 1. First ten poles from upper front list

$$a_{i,k} = \{\{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}, \{\}\}$$

2. First ten poles from lower front list

$$b_{j,\ell} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ -\frac{5}{2} \\ -\frac{7}{2} \\ -\frac{9}{2} \\ -\frac{11}{2} \\ -\frac{13}{2} \\ -\frac{15}{2} \\ -\frac{17}{2} \\ -\frac{19}{2} \\ -\frac{21}{2} \end{pmatrix}$$

References

- [Fox61] Charles Fox. The G and H functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, 98:395–429, 1961.
- [KS] Anatoly A. Kilbas and Megumi Saigo. H -transforms, volume 9 of *Analytical Methods and Special Functions*. Chapman & Hall/CRC, Boca Raton, FL. Theory and applications.