Foundations of Computing

Recursion

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February, 9th: On this day...(random facts)

- **1539** – The first recorded race is held on Chester Racecourse, known as the Roodee.
- **1822** – Haiti attacks the newly established Dominican Republic on the other side of the island of Hispaniola.
- **1895** – William G. Morgan creates a game called Mintonette, which soon comes to be referred to as volleyball.
- **1971** – Apollo program: Apollo 14 returns to Earth after the third manned Moon landing.
- **2018** – Winter Olympics: Opening ceremony is performed in Pyeongchang County in South Korea.

Lecture Agenda

- Last lecture:
  - Iterators and Itertools
- This lecture:
  - Recursion

Recursion

https://en.wikipedia.org/wiki/Recursive_islands_and_lakes

Contents [hide]

A Recursive Mindset I

- Imagine there was no iteration
  - No `for` or `while` loops (or `iter` and `next`)
  - No list comprehensions
  - No builtins like `min`, `sum`, `len`
- Count the area of The Great Barrier Reef that is bleached. You have data as a list of True and False for a 1 km^2 grid of The Reef.

```python
data = [True, False, False, True, ...]

def count(lst):
    '''Return the number of True's in a list.'''
```

No loops...

```python
def count(lst):
    '''Return the number of True's in a list.'''
    n = len(lst)
    if n==0: return 0
    if n==1: return lst[0]
    if n==2: return lst[0] + lst[1]
    if n==3: return lst[0] + lst[1] + lst[2]
    if n==4: return lst[0] + lst[1] + lst[2] + lst[3]
    ...
```

- Q: when you see repeated code, what do you do?
- A: ???

No loops...

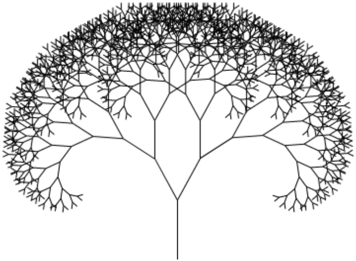
```
def count(lst):
  '''Return the number of True's in a list.'''
  n = len(lst)
  if n==0: return 0
  if n==1: return lst[0]
  if n==2: return lst[0] + lst[1]
  if n==3: return lst[0] + lst[1] + lst[2]
  if n==4: return lst[0] + lst[1] + lst[2] + lst[3]
  ...
```

- Q: when you see repeated code, what do you do?
- A: make a function ("modularise" or "factorise")

What is Recursion

Recursion, see Recursion.

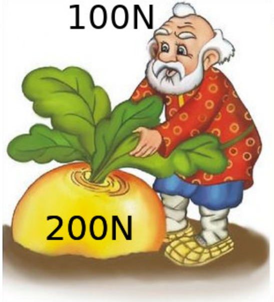
In order to understand recursion, you need to understand recursion.



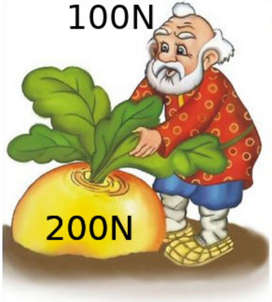
Fractal Tree

The Gigantic Turnip

The Gigantic Turnip



The Gigantic Turnip



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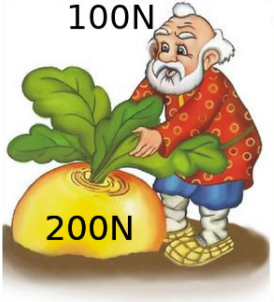

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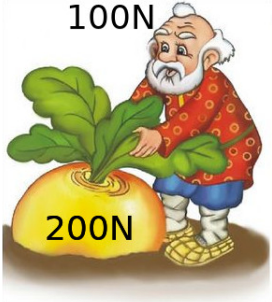

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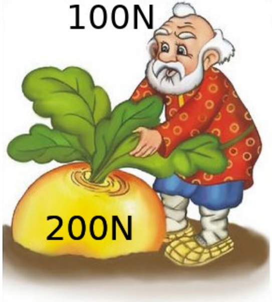

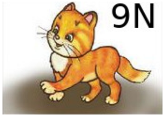

The Gigantic Turnip

The Gigantic Turnip

A mouse is a base case! We start moving backwards!

Matryoshka the Recursive Doll

Modularise

```python
def count(lst):
    '''Return the number of True's in a list.'''
    n = len(lst)
    if n == 0: return 0
    return lst[0] + count_list(lst[1:])

def count_list(lst):
    '''Return the number of True's in a list.'''
```

- Hold on, `count_list` looks very familar...
- We already have a function that does this: `count`
- Can a function call itself?

Modularise

```python
def count(lst):
    '''Return the number of True's in a list.'''
    if len(lst) == 0:
        return 0

    return lst[0] + count(lst[1:])
```

- Python cannot tell the difference between a function calling itself, and calling some other function.
- But, because it is often a useful way of thinking about problems in computing/maths it has a special name: recursion.

A Recursive Mindset II

- How can we break the problem into an instance of the same problem, but on a smaller input?
- What happens on the smallest case (the "base case")?
- `len(lst) = 1 + len(lst[1:])`
- `my_max(lst) = max(lst[0], my_max(lst[1:]))`
- `my_min(lst) = min(lst[0], my_min(lst[1:]))`

Recursion

Base case: Anakin and Padme

Class Exercise

- Write a function to sum all elements in a list without using iteration.
- Hint: think recursively. How can you break down the problem of adding up n elements in a list into one of adding up one element and $n-1$ elements?

The Elements of Recursion

- "Recursive" function definitions are often use to solve problems in a "divide-and-conquer" manner, breaking the problem down into smaller sub-problems and solving them in the same way as the big problem
- They are generally made up of two parts:
  - recursive function call(s) on smaller inputs
  - a (reachable) base case to ensure the calculation halts
- Recursion is closely related to "mathematical induction"

Class Exercise

- Write a function to compute $n!$ without using iteration.
- Hint: think recursively. How can you compute $n!$ based on $(n-1)!$? What is the base case?

But why?

- Defining answers recursively (in terms of instances of the same problem on a smaller input) is common in maths
- Simple to translate to Python

$$F(n) = \begin{cases} F(n-1) + F(n-2) & \text{, if } n > 2 \\ 1 & \text{, otherwise} \end{cases}$$

$$Q(n) = \begin{cases} Q(n - Q(n-1)) + Q(n - Q(n-2)) & , n > 2 \\ 1 & , n \leq 2 \end{cases}$$

But why? II

- Consider the following problem...
  - Assuming an unlimited number of coins of each of the following denominations:
    $$(1, 2, 5, 10, 20)$$
    calculate the number of distinct coin combinations which make up a given amount N (in cents).
- We can answer this with 5 nested for loops

Coins I

```python
'''Count the number of combinations of
   (1,2,5,10,20) that sum to N
'''
answer = 0
for a in range(N+1):
  for b in range(N//2+1):
    for c in range(N//5+1):
      for d in range(N//10+1):
        for e in range(N//20+1):
          if a+2*b+5*c+10*d+20*e == N:
            answer += 1
```

An iterative solution. But what if there were 6 denominations, or 7, or 8, or k?

Coins I

- Think recursively. How many ways can we put in the first coin, and then work out all the combinations for the rest.

`answer(N, (1,2,5,10,20) )`

Put in zero 1's, then need `answer(N, (2,5,10,20) )`

Put in one 1, then need `answer(N-1, (2,5,10,20) )`

Put in two 1's, then need `answer(N-2, (2,5,10,20) )`

Put in three 1's, then need `answer(N-3, (2,5,10,20) )`

...

```
answer(N, coins) = sum answer(N-i*coins[0], coins[1:])
                   for i in 0,1,2,...N//coins[0]
```

Coins II

```
answer(N, coins) = sum answer(N-i*coins[0], coins[1:])
                   for i in 0,1,2,...N//coins[0]
```

What's the base case?

```
answer(N, single_coin) =
```

How many ways can you make up N with only one coin denomination?

Coins III

```
def answer(N, coins):
    if len(coins) == 1:
        if N % coins[0] == 0:
            return 1
        else:
            return 0

    c = coins[0]
    count = 0
    for i in range(0, N//c+1):
        count += answer(N-i*c, coins[1:])

    return(count)
```

The problem is difficult with iteration.

The Powerset Problem

Given a set, S, compute the powerset $\mathcal{P}(S)$ of that set (a set of all subsets, including $\{\}$).

Think recursively: construct the powerset of $n-1$ items, and add first item to each of them.

```python
def power_set(lst): # lists easier than sets
    if lst == []:
        return [[]]
    rest = power_set(lst[1:])
    result = []
    for item in rest:
        result.append(item)
        result.append([lst[0]] + item)
    return result
```

`index` - Linear Search

- Input: sorted `list` of numbers
- Output: the index of a given number `x`, or `None` if it's not in the list
- Thinking recursively:

$$index(x, lst) = \begin{cases} None & \text{if lst is empty} \\ 0 & \text{if lst[0] == x} \\ 1 + index(x, lst[1:]) & \text{otherwise} \end{cases}$$

`index` - Binary Search

- Input: sorted `list` of numbers
- Output: the index of a given number `x`, or `None` if it's not in the list
- Thinking recursively and cleverly (`n=len(lst)`):

$$index(x, lst) = \begin{cases} None & \text{if lst is empty} \\ n/2 & \text{if lst[n/2] is x} \\ index(x, lst[:n/2]) & \text{if x} < \text{lst[n/2]} \\ n/2 + index(x, lst[n/2:]) & \text{otherwise} \end{cases}$$

0	1	2	3	4	5	6	7
1	3	10	12	15	45	86	91

Binary Search: Recursive Solution

```
def bsearch(val,nlist):
    return bs_rec(val,nlist,0,len(nlist)-1)

def bs_rec(val,nlist,start,end):
    if start > end:
        return None
    mid = start+(end-start)//2
    if nlist[mid] == val:
        return mid
    elif nlist[mid] < val:
        return bs_rec(val,nlist,mid+1,end)
    else:
        return bs_rec(val,nlist,start,mid-1)
```

Binary Search: Iterative Solution

... but again, there's an equally elegant iterative solution:

```
def bs_it(val,nlist):
    start = 0
    end = len(nlist) - 1
    while start < end:
        mid = start+(end-start)//2
        if nlist[mid] == val:
            return mid
        elif nlist[mid] < val:
            start = mid + 1
        else:
            end = mid - 1
    return None
```

So When *Should* You Use Recursion?

Recursion comes to its fore when an iterative solution would involve a level of iterative nesting proportionate to the size of the input, e.g.:

- the powerset problem: given a list of items, return the list of unique groupings of those items (each in the form of a list)
- the change problem: given a list of different currency denominations (e.g. `[5,10,20,50,100,200]`), calculate the number of distinct ways of forming a given amount of money from those denominations

Making Head and Tail of Recursion

- Recursion occurs in two basic forms:
  1. **head recursion:** recurse first, then perform some local calculation

```
def counter_head(n):
    if n < 0: return
    counter_head(n-1)
    print n
```

  2. **tail recursion:** perform some local calculation, then recurse

```
def counter_tail(n):
    if n < 0: return
    print n
    counter_tail(n-1)
```

Recursion: A Final Word

- Recursion is very powerful, and should always be used with caution:
  - function calls are expensive, meaning deep recursion comes at a price
  - always make sure to catch the base case, and avoid infinite recursion!
  - there is often a more efficient iterative solution to the problem, although there may not be a general iterative solution (esp. in cases where the obvious solution involves arbitrary levels of nested iteration)
  - recursion is elegant, but elegance $\neq$ more readable or efficient

Lecture Summary

- What is recursion? What two parts make up a recursive function?
- What is the difference between head and tail recursion?
- What is binary search, and how does it work?
- In what cases is recursion particularly effective?
- Why should recursion be used with caution?