Statistical Inference

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1 极大似然估计 (Maximum Likelihood Estimate, MLE)

1.1 One Parameter

1.1.1 用极大似然法估计参数

随机变量 X 服从关于参数 θ 的某分布, 假设它的 p.m.f. (discrete) 或 p.d.f. (continuous) 为

$$\Pr(X = x) \quad \text{or} \quad f(\theta; x),$$

那么:

- The likelihood function:

$$L(\theta) = \prod_{i=1}^{n} \Pr(X = x_i) \quad \text{or} \quad L(\theta) \propto \prod_{i=1}^{n} f(\theta; x_i);$$

- The log-likelihood function:

$$\ell(\theta) = \log(L(\theta)) = \log\left(\prod_{i=1}^{n} \Pr(X = x_i)\right) = \sum_{i=1}^{n} \log\left(\Pr(X = x_i)\right);$$

- Set $\ell'(\theta) = 0$ to find a turning point $\widehat{\theta}_{\text{MLE}}$;
- Check to ensure it is a maximum turning point:

$$\ell''(\theta) < 0, \ \forall\theta.$$

1.1.2 Relative likelihood

- The relative likelihood function:

$$R(\theta) = \frac{L(\theta)}{L(\widehat{\theta}_{\text{MLE}})};$$

- The relative likelihood function:

$$r(\theta) = \log(R(\theta)) = \ell(\theta) - \ell(\widehat{\theta}_{\text{MLE}}).$$

- A p likelihood interval:

$$R(\theta) \geqslant p \quad \text{or} \quad r(\theta) \geqslant \log(p).$$

1.1.3 Newton-Raphson Method

The Newton-Raphson approach can be used to find the roots on an equation ($f(x) = 0$).

The following is iterated (for iteration n) until convergence (收敛) for each bound B:

$$x_B^{(n+1)} = x_B^{(n)} - \frac{f\left(x_B^{(n)}\right)}{f'\left(x_B^{(n)}\right)}, \quad \text{until } x_B^{(n+1)} = x_B^{(n)}.$$

In the case of likelihood interval, $f(\theta_B) = r(\theta_B) - \log(p)$.

1.1.4 Generalised Likelihood Ratio Test (GLRT)

The GLRT test statistic is

$$\lambda = 2(\ell(\widehat{\theta}_{H_1}) - \ell(\widehat{\theta}_{H_0})) \sim \chi^2_{df}(0.95),$$

We compare this to the χ^2_{df} distribution whose upper α point is $\chi^2_{df}(1-\alpha)$, since there is $[df]$ restriction on the parameters under H_0.

1.2 Two Parameters

1.2.1 用极大似然法估计参数

- The joint likelihood function:

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i; \boldsymbol{\theta});$$

例 1.1 (via. 2022-2023 T1-(a)) Assume that the data follow independent Binomial probability models, so that:

$$Y_1 \sim \mathrm{Bi}(30, \theta_1), 0 < \theta < 1$$
$$Y_2 \sim \mathrm{Bi}(25, \theta_2), 0 < \theta < 1$$

The observed values of the number of samples above the threshold for each reservoir are $y_1 = 16$ and $y_2 = 13$. 由于这里只有两个独立的观测值 Y_1, Y_2, 因此联合似然函数写作

$$
\begin{aligned}
L(\theta_1, \theta_2) &= \Pr(Y_1 = y_1 \mid 30, \theta_1) \cdot \Pr(Y_2 = y_2 \mid 25, \theta_2) \\
&= \binom{30}{y_1} \theta_1^{y_1} (1-\theta_2)^{30-y_1} \cdot \binom{25}{y_2} \theta_2^{y_2} (1-\theta_2)^{25-y_2} \\
&= K \cdot \theta_1^{16} (1-\theta_2)^{14} \cdot \theta_2^{13} (1-\theta_2)^{12} \qquad\qquad K = \binom{30}{16} \cdot \binom{25}{13}
\end{aligned}
$$

- The joint log-likelihood function:

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{n} \log f(x_i; \boldsymbol{\theta});$$

- Derive the first partial derivatives and set equal to zero to find a turning point:

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \theta_i} = 0;$$

- Check to ensure it is a maximum turning point (Hessian matrix):

$$H = \begin{bmatrix} \dfrac{\partial^2 \ell}{\partial \theta_1^2} & \dfrac{\partial^2 \ell}{\partial \theta_1 \partial \theta_2} \\ \dfrac{\partial^2 \ell}{\partial \theta_2 \partial \theta_1} & \dfrac{\partial^2 \ell}{\partial \theta_2^2} \end{bmatrix}.$$

If H is a diagonal matrix and all entries on the main diagonal are negative, H will be negative definite for all values for θ_1 and θ_2 and hence the estimators θ_1 and θ_2 are maximum likelihood estimators.

1.2.2 Wald confidence interval

If we now wish to estimate $\boldsymbol{b}^\top \boldsymbol{\theta}$, we have

$$\boldsymbol{b}^\top \widehat{\boldsymbol{\theta}} \pm \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\sqrt{\boldsymbol{b}^\top K^{-1}\boldsymbol{b}},$$

where $K^{-1} = -H^{-1}$.

例 1.2 (via. 2022-2023 T1-(a)) If we now wish to estimate $\theta_1 - \theta_2$ then by setting $\boldsymbol{b} = (1, -1)^\top$, that is

$$\widehat{\theta}_1 - \widehat{\theta}_2 = \boldsymbol{b}^\top \widehat{\boldsymbol{\theta}} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \widehat{\theta}_1 \\ \widehat{\theta}_2 \end{bmatrix}.$$

Assume that

$$H = \begin{bmatrix} -h_1 & 0 \\ 0 & -h_2 \end{bmatrix}, h_1 > 0, h_2 > 0,$$

then

$$K^{-1} = -H^{-1} = \begin{bmatrix} \dfrac{1}{h_1} & 0 \\ 0 & \dfrac{1}{h_2} \end{bmatrix}, h_1 > 0, h_2 > 0.$$

so, $\boldsymbol{b}^\top K^{-1} \boldsymbol{b} = \dfrac{1}{h_1} + \dfrac{1}{h_2}$.

2 Bayesian Inference

For the case of one data point x and a parameter θ:

$$p(\theta \mid X) = \frac{p(\theta) \cdot L(\theta \mid X)}{p(x)} \propto p(\theta) \cdot L(\theta \mid X),$$

- $p(\theta \mid X)$: the posterior distribution (后验分布);
- $L(\theta \mid X) = p(X \mid \theta)$: the likelihood for a parameter θ given the data x (似然函数);
- $p(\theta)$: the prior distribution of the parameter θ (先验分布);
- $p(x)$: the conditional (marginal) distribution of the data.

通过似然函数找到共轭先验分布:

似然函数 $L(\theta \mid X)$	共轭 (conjugate) 先验分布 $p(\theta)$
$X \sim \text{Binomial}(\theta)$	$\theta \sim \text{Beta}(\alpha, \beta),\ p(\theta) = \dfrac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha, \beta)}$
$X \sim \text{Poisson}(\lambda)$	$\lambda \sim \text{Gamma}(\alpha, \beta),\ p(\lambda) = \dfrac{\lambda^{\alpha-1}e^{-\frac{1}{\beta}\lambda}}{\beta^{\alpha}\Gamma(\alpha)}$
$X \sim \text{Exp}(\lambda)$	
$X \sim \text{Normal}(\mu, \sigma^2)$	$\theta \sim \text{Normal}(\mu, \sigma^2)$

例 2.1 (Binomial Model) 用 Bayesian Inference 来估计成功比例, 因为若 $\theta \sim \mathrm{Beta}(\alpha, \beta)$, 则 $\mathbb{E}[\theta] = \dfrac{\alpha}{\alpha+\beta}$, 其中 α, β 分别为成功和失败次数的先验假设.

- Likelihood: $L(\theta \mid X)$;
- Prior (Beta): $p(\theta)$;
- Posterior: $p(\theta \mid X) \propto p(\theta) \cdot L(\theta \mid X) \Longrightarrow \theta \mid X \sim \mathrm{Beta}(\widetilde{\alpha}, \widetilde{\beta})$;
- The mean of the posterior distribution: $\mathbb{E}[\theta \mid X] = \dfrac{\widetilde{\alpha}}{\widetilde{\alpha}+\widetilde{\beta}}$.

3 Hypothesis Testing

- H_0: Null hypothesis;
- H_1: Alternative hypothesis (study hypothesis: 即题目中所研究的内容).

Notion:

- η: population median (总体中位数);
- μ: population mean (总体均值);
- θ: population proportion/probability (总体比例/概率).

结论要么是we reject H_0 in favour of H_1, 要么是we do not reject H_0, 绝不可能是prove H_0 from a sample.

3.1 Confidence Interval

$1-\alpha$ confidence interval:

$$\overline{x} \pm t_{1-\frac{\alpha}{2}}(n-1) \cdot \frac{s}{\sqrt{n}},$$

where:

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

$$s^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right].$$

3.2 Wilcoxon Signed Ranks Test (non-parametric test)

Test assumptions:

- Observations come from a symmetric distribution (对称分布);
- Observations are independent from one another (彼此独立).

Then H_0 :?? vs. H_1 :??.

Observed value	-1	2	5	3	2	-2	5	8
$\|d_i\|$	1	2	5	3	2	2	5	8
sign	-	+	+	+	+	-	+	+
rank	1	3	6.5	5	3	3	6.5	8

- Test statistic: $W = \min(W+, W-) = 4$, where (sum of rank)

$$\begin{aligned} W+ &= 3 + 6.5 + 5 + 3 + 6.5 + 8 = 32, \\ W- &= 1 + 3 = 4. \end{aligned}$$

- Rejection region: $RR = \{W : W \leqslant ?\}$

3.3 Generalised Likelihood Ratio Test (GLRT)

例 3.1 (via. 2022-2023 T2-(c)) Use the Generalised Likelihood Ratio Test to investigate the hypothesis.

- $H_0 : \theta_1 = \theta, \theta_2 = 2\theta, \theta_3 = 4\theta$;
- H_1 : the multinomial probabilities will not have the specified form in H_0.

The cell probabilities must sum to 1, i.e. $\sum_{i=1}^{3} \theta_i = 1 \implies \theta + 2\theta + 4\theta = 1 \implies \theta = \frac{1}{7}$.

In this case, $n = 89 + 210 + 414 = 713$.

No.	1	2	3
Observed frequency of orders Expected frequency of orders (under H_0)	89 101.8571	210 203.7143	414 407.4286
Reason	$713 \times \frac{1}{7} = 101.8571$	$713 \times \frac{2}{7} = 203.7143$	$713 \times \frac{4}{7} = 407.4286$

- The test statistic:

$$2\sum_{i=1}^{k} O_i \log\left(\frac{O_i}{E_i}\right).$$

- The observed value of the test statistic:

$$2\left(89\log\left(\frac{89}{101.8571}\right)+210\log\left(\frac{210}{203.7143}\right)+414\log\left(\frac{414}{407.4286}\right)\right)=2\times 0.9966=1.993.$$

- Rejection Region:

$$\begin{aligned} RR &= \left\{x : 2\sum_{i=1}^{k} O_i \log\left(\frac{O_i}{E_i}\right) > \chi^2_{k-1-p}(1-\alpha)\right\} \\ &= \left\{x : 2\sum_{i=1}^{k} O_i \log\left(\frac{O_i}{E_i}\right) > \chi^2_{3-1-0}(1-\alpha)\right\}, \end{aligned}$$

  where

  - k = the number of levels (or categories) (类别个数);
  - p = the number of parameters being estimated (待估参数个数).