Financial Mathematics
金融数学

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目录

1 Portfolio Theory

1.1 Utility theory & Portfolio choice

定义 1.1 Non-satiated

An investor's utility function, U, should reflect that the investor is non-satiated (which we assume) and the investor's risk appetite.

$$\left.\frac{\mathrm{d}U}{\mathrm{d}W}\right|_{w} > 0, \quad \forall w.$$

We will also use the notation $U'(w) > 0$.

定义 1.2 Convex (凸性)

$f(x)$ is strictly convex if

$$f(px_1 + (1-p)x_2) < pf(x_1) + (1-p)f(x_2), \quad \forall p \in (0,1).$$

That is, $f''(x) > 0$ or $f\left(\mathbb{E}\left[X\right]\right) < \mathbb{E}\left[f(X)\right]$.

定义 1.3 Concave (凹性)

$f(x)$ is strictly concave if

$$f(px_1 + (1-p)x_2) > pf(x_1) + (1-p)f(x_2), \quad \forall p \in (0,1).$$

That is, $f''(x) < 0$ or $f\left(\mathbb{E}\left[X\right]\right) > \mathbb{E}\left[f(X)\right]$.

Remember: con**V**ex (has a minimum) and con**CAVE** (has a maximum).

推论 1.1 Risk appetite

Let U be the utility function reflecting an investor's preferences. Then the investor is,

- risk-averse $\Leftrightarrow$ $U''(W) < 0$;
- risk-neutral $\Leftrightarrow$ $U''(W) = 0$;
- risk-seeking $\Leftrightarrow$ $U''(W) > 0$.

1.2 Risk-aversion coefficients

定义 1.4 Coefficient of Absolute Risk Aversion (ARA)

$$\mathcal{A}(w) := -\frac{U''(w)}{U'(w)}$$

定义 1.5 Coefficient of Relative Risk Aversion (RRA)

$$\mathcal{R}(w) := -\frac{wU''(w)}{U'(w)} = w \cdot \mathcal{A}(w)$$

定义 1.6

An investor can exhibit several types of risk-averse behaviour. An investor displays:

1. Increasing Absolute Risk-Aversion (IARA) if, $\mathcal{A}'(w) > 0$.
2. Constant Absolute Risk-Aversion (CARA) if, $\mathcal{A}'(w) = 0$.
3. Decreasing Absolute Risk-Aversion (DARA) if, $\mathcal{A}'(w) < 0$.

We have similar definitions for Relative Risk-Aversion. An investor displays:

1. Increasing Relative Risk-Aversion (IRRA) if, $\mathcal{R}'(w) > 0$.
2. Constant Relative Risk-Aversion (CRRA) if, $\mathcal{R}'(w) = 0$.
3. Decreasing Relative Risk-Aversion (DRRA) if, $\mathcal{R}'(w) < 0$.

Notion:

- x – initial wealth;
- ϕ – quantity of risky assets purchased;
- S – initial price (for risky assets);
- Y – the random return (for risky assets).

For a strategy (x, ϕ), the amount of wealth invested in the money market account

$$Z = x - \phi S,$$

which should use continuous compound interest rate to calculate interest.

The portfolio (ϕ, Z) results in (random) wealth W at the end of the single period where,

$$\begin{aligned} W &= \phi \cdot (S \cdot Y) + Ze^r = \phi \cdot (S \cdot Y) + (x - \phi S)e^r \\ &= \phi \cdot S(Y - e^r) + x \cdot e^r. \end{aligned}$$

Hence, $U(W) = U(\phi \cdot S(Y - e^r) + x \cdot e^r)$. The question now is: what portfolio maximises $\mathbb{E}[U(W)]$?

定义 1.7

A portfolio (ϕ_*, Z_*) is optimal for initial wealth x if

$$\mathbb{E}\left[U(W_*)\right] = \max_{M \in \mathbb{R}} \mathbb{E}\left[U(W)\right].$$

In the above definition the optimal portfolio (ϕ_*, Z_*) is associated with the random wealth outcome W_*.

2 Mean-Variance Analysis

2.1 Assumptions on the market and on returns

定义 2.1

Let R be the return on a portfolio

$$\begin{aligned} \text{minimise:} \quad & \mathrm{Var}[R] \\ \text{subject to:} \quad & \mathbb{E}[R] = \mu \end{aligned}$$

where $\mu \in \mathbb{R}$ is fixed (note that μ is the expected return on the whole portfolio, not a particular risky asset).

Denote the covariance of return on the ith and jth stocks by $\mathrm{Cov}[R_i, R_j]$ for all $i \neq j \in \{1, \cdots, n\}$ and hence, we have that

$$R = \sum_{i=1}^{n} \omega_i R_i + \left(1 - \sum_{i=1}^{n} \omega_i\right) R_0$$

so that

$$\begin{aligned} \mathbb{E}[R] &= \sum_{i=1}^{n} \omega_i \mathbb{E}[R_i] + \left(1 - \sum_{i=1}^{n} \omega_i\right) R_0 \\ \mathrm{Var}[R] &= \sum_{i=1}^{n} \omega_i^2 \mathrm{Var}[R_i] + 2 \sum_{i<j} \omega_i \omega_j \mathrm{Cov}[R_i, R_j] \\ &= \sum_{i=1}^{n} \omega_i^2 \sigma_i^2 + 2 \sum_{i<j} \omega_i \omega_j \rho_{i,j} \sigma_i \sigma_j. \end{aligned}$$

Then we can rewrite the minimum-variance problem, as follows:

定义 2.2

$$\begin{aligned} \min \quad & \sigma^2 = \mathrm{Var}[R] = \sum_{i=1}^{n} \omega_i^2 \mathrm{Var}[R_i] + 2 \sum_{i<j} \omega_i \omega_j \mathrm{Cov}[R_i, R_j] \\ \text{s.t.} \quad & \mathbb{E}[R] = \sum_{i=1}^{n} \omega_i \mathbb{E}[R_i] + \left(1 - \sum_{i=1}^{n} \omega_i\right) R_0 = \mu \end{aligned}$$

This is an example of a "quadratic programming problem".

It is possible to plot such solutions as a curve on an μ-σ^2 diagram (or μ-σ diagram) called the portfolio diagram. This is a graph with σ^2 (or σ) on the x-axis and μ on the y-axis.

2.2 The method of Lagrange multipliers

定义 2.3 The method of Lagrange multipliers

To solve the following type of problem (which should be rewritten in **standard Karush-Kuhn-Tucker (KKT) format**):

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & f(x_1, x_2, \cdots, x_n) \\ \text{s.t.} \quad & g_i(x_1, x_2, \cdots, x_n) = 0 \quad i = 1, 2, \cdots m \end{aligned}$$

Hence, the **Lagrangian function** is,

$$\mathcal{L}(x_1, \cdots, x_n, \lambda_1, \cdots, \lambda_m) = f(x_1, \cdots, x_n) + \sum_{i=1}^{m} \lambda_i g_i(x_1, x_2, \cdots, x_n),$$

where λ_i are called **Lagrange multipliers**.

定理 2.1

For a convex function f and convex equality constraints g_i, the vector $\boldsymbol{x}_* = (x_{1*}, x_{2*}, \cdots, x_{n*})$ is the optimal solution if and only if

$$\nabla \mathcal{L} = 0 \quad \Leftrightarrow \quad \begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_1} = 0, \\ \quad \vdots \\ \dfrac{\partial \mathcal{L}}{\partial x_n} = 0, \\ \dfrac{\partial \mathcal{L}}{\partial \lambda_1} = 0, \\ \quad \vdots \\ \dfrac{\partial \mathcal{L}}{\partial \lambda_m} = 0. \end{cases} \quad \xRightarrow{\text{solve}} \quad \begin{cases} x_1 = x_{1*}, \\ x_2 = x_{2*}, \\ \quad \vdots \\ x_n = x_{n*}. \end{cases}$$

Given a collection of risky assets, the **feasible set (可行集)** (or **feasible region (可行域)**) is the set of all points in the μ-σ^2 diagram (or μ-σ diagram) for which there exists a portfolio with the corresponding expected return and variance of return.

The left boundary of the feasible region is called the **mean-variance frontier (均值-方差前沿)** or **minimum-variance set (最小方差集)**.

The points on the minimum-variance set with the highest value of μ for a given σ^2 (i.e. the upper boundary of the minimum-variance set) is called the **efficient frontier (有效前沿)** of the feasible region.

The **global minimum variance (全局最小方差)** is the minimum value that the variance can take.

2.3 Capital asset pricing model

定义 2.4 Capital market line

We refer to the unique efficient portfolio of risky assets as the “market portfolio”, M.
We define the **capital market line** to be the straight line in the μ-σ-diagram which passes through portfolio entirely invested in the risk-free money market account (with return R_0) and the market portfolio M.

When we have n risky assets and a risk-free money market account, then the efficient frontier is the capital market line and has equation:

$$\mu = R_0 + \frac{\mu_M - R_0}{\sigma_M}\sigma$$

in the μ-σ-diagram where μ_M and σ_M are the expected return and standard deviation of return of the market portfolio, M.

定义 2.5 Sharpe ratio (夏普比率)

The gradient (slope) of the capital market line

$$\lambda_M = \frac{\mu_M - R_0}{\sigma_M}$$

is called the market price of risk or **Sharpe ratio**.

The Sharpe ratio essentially indicates the increase in expected return of an efficient portfolio if the standard deviation of return on that portfolio increases by 1 unit.

定义 2.6 Capital Asset Pricing Model, CAPM

Suppose that R_M is a solution to the mean variance problem for $\mu_M \geqslant R_0$, corresponding to the efficient market portfolio M. Then suppose that R_i is the return of an arbitrary asset i with expectation equal to μ_i. Then the expected excess return of asset i is given by

$$\mathbb{E}\left[R_i\right] - R_0 = \beta_{i,M}\left(\mathbb{E}\left[R_M\right] - R_0\right),$$

or

$$\mu_i - R_0 = \beta_{i,M}\left(\mu_i - R_0\right),$$

where

$$\beta_{i,M} = \frac{\mathrm{Cov}\left[R_i, R_M\right]}{\mathrm{Var}\left[R_M\right]},$$

is the so-called “beta of the asset i”, solved using OLS.

3 Risk measures

定义 3.1 Value at Risk, VaR

Value at Risk with level of confidence α is a loss on a portfolio such that there is a probability $p = 1 - \alpha$ of losing more than or equal to VaR in a given trading period.

In order to express the above concept in mathematical terms we denote the loss of the portfolio by L (and the returns are $R = -L$). Then, the above definition implies that the VaR is

$$\Pr(L > \mathrm{VaR}) = p = 1 - \alpha, \quad \Leftrightarrow \quad \Pr(L < \mathrm{VaR}) = 1 - p = \alpha.$$

定义 3.2 Expected Shortfall

This is the expected loss of an investor or company at a confidence level $\alpha = 1 - p$ if the loss exceeds VaR. In other words, it is the expected return on the portfolio over the worst cases, which occur with total probability p.

$$\mathrm{ES} = \mathbb{E}\left[L | L > \mathrm{VaR}\right] = \frac{\mathbb{E}\left[L \mathbb{1}_{\{L > \mathrm{VaR}\}}\right]}{\Pr(L > \mathrm{VaR})} = \begin{cases} \dfrac{1}{1-\alpha} \displaystyle\int_{\mathrm{VaR}}^{\infty} L \times f_L(x)\, \mathrm{d}x, & \text{continuous,} \\ \dfrac{1}{1-\alpha} \displaystyle\sum_i L_i \times \Pr(L_i > \mathrm{VaR}), & \text{discrete.} \end{cases}$$

4 Single Period Market Models

4.1 The elementary one-period market model

定义 4.1 Arbitrage

A trading strategy (x, ϕ) in our elementary market model is called an **arbitrage**, if

- $x = V_0(x, \phi) = 0$.
- $V_1(x, \phi) \geqslant 0$.
- $\mathbb{E}\left[V_1(x, \phi)\right] = pV_1(x, \phi)(H) + (1-p)V_1(x, \phi)(T) > 0$.

命题 4.1

The elementary single period market model discussed above is **arbitrage free**, if and only if

$$d < 1 + r < u.$$

定义 4.2 European option

A **European call option** is a contract between two investors which gives its owner the right (but not the obligation!) to **buy** a specific asset at a specific time T in the future at a specific price K (called the strike price).
A European call option is equivalent to an asset which has a payoff at time T of

$$(S_T - K)^+ = \max(S_T - K, 0).$$

A **European put option** is a contract which gives its owner the right (but not the obligation) to **sell** a specific asset at a specific time T in the future at a specific price K.
A European put option is equivalent to an asset which has a payoff at time T of

$$(K - S_T)^+ = \max(K - S_T, 0).$$

4.2 Replication principle

定义 4.3 Replicating Strategy (Hedge)

A **replicating strategy** (or **hedge**) for the option with payoff function $h(S_1)$ in the elementary single-period market model is a trading strategy (x, ϕ) which satisfies $V_1(x, \phi) = h(S_1)$, i.e.

$$(x - \phi S_0)(1 + r) + \phi S_1(H) = h(S_1(H)),$$
$$(x - \phi S_0)(1 + r) + \phi S_1(T) = h(S_1(T)).$$

Delta hedging formula:

$$\phi = \frac{h(S_1(H)) - h(S_1(T))}{S_1(H) - S_1(T)}.$$

4.3 A general single period market model

定义 4.4 Value process

The **value process** of the trading strategy (x, ϕ) in our general single period market model is given by $(V_0(x,\phi), V_1(x,\phi))$, where $V_0(x,\phi) = x$ and $V_1(x,\phi)$ is

$$V_1(x,\phi) = \left[x - \left(\sum_{i=1}^{n} \phi^i S_0^i\right)\right](1+r) + \left(\sum_{i=1}^{n} \phi^i S_1^i\right).$$

Since prices at $t = 1$ are random, $V_1(x,\phi)$ is also a random variable.

定义 4.5 Gain process

The gains process $G(x,\phi)$, is defined as

$$G(x,\phi) = \left[x - \left(\sum_{i=1}^{n} \phi^i S_0^i\right)\right] r + \left(\sum_{i=1}^{n} \phi^i \Delta S^i\right),$$

where ΔS^i represents the change in price of the i-th stock, i.e.

$$\Delta S^i \coloneqq S_1^i - S_0^i.$$

As the name indicates, G represents the gains (or losses) the agent obtains from his investment. With only a little work we can see that

$$V_1(x,\phi) = V_0(x,\phi) + G(x,\phi).$$

It is often convenient to study the prices of the stocks in relation to the money market account. For this reason we introduce the **discounted stock prices** $\widehat{S}_t^i$ defined as follows:

$$\begin{aligned} \widehat{S}_0^i &\coloneqq S_0^i, \\ \widehat{S}_1^i &\coloneqq \frac{1}{1+r} S_1^i, \end{aligned} \qquad i = 1, \cdots, n.$$

We also define the **discounted value process** corresponding to the trading strategy (x, ϕ) via

$$\begin{aligned} \widehat{V}_0(x,\phi) &\coloneqq x, \\ \widehat{V}_1(x,\phi) &\coloneqq \left[x - \left(\sum_{i=1}^{n} \phi^i S_0^i\right)\right] + \left(\sum_{i=1}^{n} \phi^i \widehat{S}_1^i\right), \end{aligned} \qquad i = 1, \cdots, n,$$

as well as the **discounted gains process** $\widehat{G}(x,\phi)$ via

$$\widehat{G}(x,\phi) \coloneqq \sum_{i=1}^{n} \phi^i \Delta \widehat{S}^i,$$

with $\Delta \widehat{S}^i = \widehat{S}^i_1 - \widehat{S}^i_0$. Hence,

$$\widehat{V}_t = \frac{V_t}{B_t}, \quad t \in \{0, 1\},$$

where $B_0 = 1$ and $B_1 = 1 + r$, as well as

$$\widehat{V}_1(x, \phi) = \widehat{V}_0(x, \phi) + \widehat{G}(x, \phi).$$

4.4 Risk neutral measures & Pricing

定义 4.6 Risk-neutral measure

The measure $\mathbb{Q}$ is often called a **risk-neutral measure**, since under this measure the option price depends only on the expectation of the payoff, not on its riskiness. Risk-neutral measures (also called **equivalent martingale measures**) are important as they can be used to compute prices for options in complete and incomplete markets.

定理 4.1

The probability under the risk-neutral measure $\mathbb{Q}$ is

$$q := \frac{1 + r - d}{u - d},$$

where $u = \dfrac{S_1(H)}{S_0}$ and $d = \dfrac{S_1(T)}{S_0}$.

It follows from the assumption $d < 1 + r < u$ that $0 < q < 1$.

定义 4.7 Complete market model

A market model where any payoff function, h, can always be replicated are called **complete**.

定理 4.2 Put-Call parity

$$C - P = S_0 - K(1 + r)^{-1},$$

where C is the price of call option and P is the price of put option.

定理 4.3 Pricing

$$V_0 = \frac{1}{(1 + r)^t} \mathbb{E}^{\mathbb{Q}}\left[V_t\right].$$

5 Multi-period market models

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