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# Online Optimization with Memory and Competitive Control

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## Abstract

This paper presents competitive algorithms for a novel class of online optimization problems with memory. We consider a setting where the learner seeks to minimize the sum of a hitting cost and a switching cost that depends on the previous  $p$  decisions. This setting generalizes Smoothed Online Convex Optimization. The proposed approach, Optimistic Regularized Online Balanced Descent, achieves a constant, dimension-free competitive ratio. Further, we show a connection between online optimization with memory and online control with adversarial disturbances. This connection, in turn, leads to a new constant-competitive policy for a rich class of online control problems.

## 1 Introduction

This paper studies the problem of Online Convex Optimization (OCO) with *memory*, a variant of classical OCO [25] where an online learner iteratively picks an action  $y_t$  and then suffers a convex loss  $g_t(y_{t-p}, \dots, y_t)$ , depending on current and *previous* actions. Incorporating memory into OCO has seen increased attention recently, due to both its theoretical implications, such as to convex body chasing problems [11, 8, 37, 12], and its wide applicability to settings such as data centers [32], power systems [30, 9, 26], and electric vehicle charging [26, 17]. Of particular relevance to this paper is the considerable recent effort studying connections between OCO with memory with online control in dynamical systems, leading to online algorithms that enjoy sublinear static regret [4, 5], low dynamic regret [29, 31], constant competitive ratio [23], and the ability to boost weak controllers [3].

Prior work on OCO with memory is typically limited in one of two ways. First, algorithms with the strongest guarantees, a constant *competitive ratio*, are restricted to a special case known as Smoothed Online Convex Optimization (SOCO), or OCO with switching costs [16, 32, 24], which considers only one step of memory and assumes cost functions can be observed before actions are chosen. Second, algorithms proposed for the general case typically only enjoy sublinear *static regret* [6], which is a much weaker guarantee, because static regret compares to the offline optimal static solution while competitive ratio directly compares to the true offline optimal. It is known that algorithms that achieve sublinear static regret can be arbitrarily worse than the true offline optimal [22], and also may have unbounded competitive ratios [7]. The pursuit of general-purpose constant-competitive algorithms for OCO with memory remains open.

Our work is also motivated by establishing theoretical connections between online optimization and control. Recently a line of work has shown the applicability of tools from online optimization

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for control, albeit in limited settings [4, 5, 28, 23]. Deepening these connections can potentially be impactful since most prior work studies how to achieve sublinear regret compared to the best static linear controller [20, 5, 4, 19]. However, the best static linear controller is a weak benchmark compared to the true optimal controller [22], which may be neither linear nor static. To achieve stronger guarantees, one must seek to bound either the competitive ratio [23] or dynamic regret [29, 31], and connections to online optimization can provide such results. However, prior attempts either have significant caveats (e.g., bounds depend on the path length of the instance [29, 31]) or only apply to very restricted control systems (e.g., invertible control actuation matrices and perfect knowledge of disturbances [23]). As such, the potential to obtain constant-competitive policies for general control systems via online optimization remains unrealized.

**Main contributions.** We partially bridge the two gaps highlighted above. First, we propose a novel setting, OCO with *structured* memory, where the cost function depends on the previous  $p$  decisions and is not known precisely before determining the action. This setting generalizes SOCO to include more than one step of memory and to eliminate the assumption that the cost function must be perfectly known before choosing the action. Second, we propose a novel algorithm, Optimistic Regularized Online Balanced Descent, that has a constant and dimension-free competitive ratio for OCO with structured memory. This is the first algorithm with a constant competitive ratio for online optimization with memory longer than one step. Third, we provide a nontrivial reduction from a rich class of online control problems to OCO with structured memory and, via the reduction, show that a constant-competitive policy exists for this class of control problems. While not completely general, the class of problems is considerably more general than existing settings where competitive policies are known, e.g., the control matrix must be invertible and the disturbances are known in advance [23]. Finally, we use examples to (i) demonstrate the gap between the best offline linear policy and the true optimal offline policy can be arbitrarily large, and (ii) highlight that our algorithms can significantly outperform the best offline linear controller, which serves as the benchmark of no-regret policies.

## 2 Background and model

In this section, we formally present the problem setting in this paper. We first survey prior work on OCO with memory and then introduce our new model of OCO with structured memory. Throughout this paper,  $M_{i:j}$  denotes either  $\{M_i, M_{i+1}, \dots, M_j\}$  if  $i \leq j$ , or  $\{M_i, M_{i-1}, \dots, M_j\}$  if  $i > j$ .

### 2.1 Online convex optimization with memory

Online convex optimization (OCO) with memory is a variation of classical OCO that was first introduced by Anava et al. [6]. In contrast to classical OCO, in OCO with memory, the loss function depends on previous actions in addition to the current action. At time step  $t$ , the online agent picks  $y_t \in \mathcal{K} \subset \mathbb{R}^d$  and then a loss function  $g_t : \mathcal{K}^{p+1} \rightarrow \mathbb{R}$  is revealed. The agent incurs a loss of  $g_t(y_{t-p:t})$ . Thus,  $p$  quantifies the length of the memory in the loss function. Within this general model of OCO with memory, Anava et al. [6] focus on developing policies with small *policy regret*, which is defined as:

$$\text{PolicyRegret} = \sum_{t=p}^T g_t(y_{t-p:t}) - \min_{y \in \mathcal{K}} \sum_{t=0}^T g_t(y, \dots, y).$$

The main result presents a memory-based online gradient descent algorithm that achieves  $O(\sqrt{T})$  regret under some moderate assumptions on the diameter of  $\mathcal{K}$  and the gradient of the loss functions.

**Online convex optimization with switching costs (SOCO).** While the general form of OCO with memory was introduced only recently, specific forms of OCO problems involving memory have been studied for decades. Perhaps the most prominent example is OCO with switching costs, often termed Smoothed Online Convex Optimization (SOCO) [32, 14, 16, 23, 30, 24]. In SOCO, the loss function is separated into two pieces: (i) a *hitting cost*  $f_t$ , which depends on only the current action  $y_t$ , and a *switching cost*  $c(y_t, y_{t-1})$ , which penalizes big changes in the action between rounds. Often the hitting cost is assumed to be of the form  $\|y_t - v_t\|$  for some (squared) norm, motivated by tracking some unknown trajectory  $v_t$ , and the switching cost  $c$  is a (squared) norm motivated by penalizing switching in proportion to the (squared) distance between the actions, e.g., a common choice  $c(y_t, y_{t-1}) = \frac{1}{2} \|y_t - y_{t-1}\|_2^2$  [23, 30]. The goal of the online learner is to minimize its total cost over  $T$  rounds:  $\text{cost}(\text{ALG}) = \sum_{t=1}^T f_t(y_t) + c(y_t, y_{t-1})$ .

Under SOCO, results characterizing the policy regret are straightforward, and the goal is instead to obtain stronger results that characterize the *competitive ratio*. The competitive ratio is the worst-case ratio of total cost incurred by the online learner and the offline optimal. The cost of the offline optimal is defined as the minimal cost of an algorithm if it has full knowledge of the sequence  $\{f_t\}$ , i.e.:  $\text{cost}(OPT) = \min_{y_1 \dots y_T} \sum_{t=1}^T f_t(y_t) + c(y_t, y_{t-1})$ . Using this, the *competitive ratio* is defined as:

$$\text{CompetitiveRatio}(ALG) = \sup_{f_{1:T}} \frac{\text{cost}(ALG)}{\text{cost}(OPT)}.$$

Bounds for competitive ratio are stronger than for policy regret, since the dynamic offline optimal can change its decisions on different time steps [6].

In the context of SOCO, the first results bounding the competitive ratio focused on one-dimensional action sets [33, 10], but after a long series of papers there now exist algorithms that provide constant competitive ratios in high dimensional settings [16, 23, 24]. Among different choices of switching cost  $c$ , we are particularly interested in  $c(y_t, y_{t-1}) = \frac{1}{2} \|y_t - y_{t-1}\|_2^2$  due to the connection to quadratic costs in control problems. The state-of-the-art algorithm for this switching cost is Regularized Online Balanced Descent (ROBD), introduced by Goel et al. [24], which achieves the lowest possible competitive ratio of any online algorithm. Other recent results study the case where  $c(y_t, y_{t-1}) = \|y_t - y_{t-1}\|$  [11, 8, 37, 12]. Variations of the problem with predictions [14, 15, 30], non-convex cost functions [35], and constraints [34, 40] have been studied as well.

## 2.2 OCO with structured memory

Though competitive algorithms have been proposed for many SOCO instances, the SOCO setting has two limitations. First, the hitting cost  $f_t$  is revealed before making action  $y_t$ , i.e., SOCO requires one step exact prediction of  $f_t$ . Second, the switching cost in SOCO only depends on one previous action in the form  $c(y_t, y_{t-1})$ , so only one step of memory is considered. In this paper, our goal is to derive competitive algorithms (as exist for SOCO) in more general settings where more than one step of memory is considered. Working with the general model of OCO with memory is too ambitious for this goal. Instead, we introduce a model of OCO with *structured* memory that generalizes SOCO, and is motivated by a nontrivial connection with online control (as shown in Section 4.2).

Specifically, we consider a loss function  $g_t$  at time step  $t$  that can be decomposed as the sum of a hitting cost function  $f_t : \mathbb{R}^d \rightarrow \mathbb{R}^+ \cup \{0\}$  and a switching cost function  $c : \mathbb{R}^{d \times (p+1)} \rightarrow \mathbb{R}^+ \cup \{0\}$ . Additionally, we assume that the switching cost has the form:

$$c(y_{t:t-p}) = \frac{1}{2} \left\| y_t - \sum_{i=1}^p C_i y_{t-i} \right\|_2^2,$$

with known  $C_i \in \mathbb{R}^{d \times d}$ ,  $i = 1, \dots, p$ . Note that SOCO is a special case  $p = 1$  and  $C_1 = I$ . As we show in Section 4.2, this form connects online optimization with online control. Intuitively, this connection results from the fact that the hitting cost penalizes the agent for deviating from an optimal point sequence, while the switching cost captures the cost of implementing a control action. Specifically, suppose  $y_t$  is a robot's position at  $t$ , and then the classical SOCO switching cost  $\|y_t - y_{t-1}\|_2$  is approximately its velocity. Under our new switching cost, we can represent acceleration by  $\|y_t - 2y_{t-1} + y_{t-2}\|_2$ , and many other higher-order dynamics.

To summarize, we consider an online agent and an offline adversary interacting as follows in each time step  $t$ , and we assume  $y_i$  is already fixed for  $i = -p, -(p-1), \dots, 0$ .

1. The adversary reveals a function  $h_t$  and a convex *estimation set*  $\Omega_t \subseteq \mathbb{R}^d$ . We assume  $h_t$  is both  $m$ -strongly convex and  $l$ -strongly smooth, and that  $\arg \min_y h_t(y) = \mathbf{0}$ .
2. The agent picks  $y_t \in \mathbb{R}^d$ .
3. The adversary picks  $v_t \in \Omega_t$ .
4. The agent incurs *hitting cost*  $f_t(y_t) = h_t(y_t - v_t)$  and *switching cost*  $c(y_{t:t-p})$ .

Notice that the hitting cost  $f_t$  is revealed to the online agent in two separate steps. The geometry of  $f_t$  (given by  $h_t$  whose minimizer is at  $\mathbf{0}$ ) is revealed before the agent picks  $y_t$ . After  $y_t$  is picked, the minimizer  $v_t$  of  $f_t$  is revealed.

Unlike SOCO, due to the uncertainty about  $v_t$ , the agent cannot determine the exact value of the hitting cost it incurs at time step  $t$  when determining its action  $y_t$ . To keep the problem tractable, we

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**Algorithm 1:** Regularized OBD (ROBD), Goel et al. [24]

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**Parameter:**  $\lambda_1 \geq 0, \lambda_2 \geq 0$

**for**  $t = 1$  **to**  $T$  **do**

**Input:** Hitting cost function  $f_t$ , previous decision points  $y_{t-p}, \dots, y_{t-1}$

$v_t \leftarrow \arg \min_y f_t(y)$

$y_t \leftarrow \arg \min_y f_t(y) + \lambda_1 c(y, y_{t-1:t-p}) + \frac{\lambda_2}{2} \|y - v_t\|_2^2$

**Output:**  $y_t$

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assume an estimation set  $\Omega_t$ , which contains all possible  $v_t$ 's, is revealed to bound the uncertainty. The agent can leverage this information when picking  $y_t$ . SOCO is a special case where  $\Omega_t$  contains only one point, i.e.,  $\Omega_t = \{v_t\}$ , and then the agent has a precise estimate of the minimizer  $v_t$  when choosing its action [23, 24]. Like SOCO, the offline optimal cost in the structured memory model is defined as  $\text{cost}(OPT) = \min_{y_1 \dots y_T} \sum_{t=1}^T f_t(y_t) + c(y_{t:t-p})$  given the full sequence  $\{f_t\}_{t=1}^T$ .

### 3 Algorithms for OCO with memory

In OCO with structured memory, there is a key differentiation depending on whether the agent has knowledge of the hitting cost function (both  $h_t$  and  $v_t$ ) when choosing its action or not, i.e., whether the estimation set  $\Omega_t$  is a single point,  $v_t$ , or not. We deal with each case in turn in the following.

#### 3.1 Case 1: exact prediction of $v_t$ ( $\Omega_t = \{v_t\}$ )

We first study the simplest case where  $\Omega_t = \{v_t\}$ . Recall that  $\Omega_t$  is the convex set which contains all possible  $v_t$  and so, in this case, the agent has exact knowledge of the hitting cost when picking action. This assumption, while strict, is standard in the SOCO literature, e.g., [23, 24]. It is appropriate for situations where the cost function can be observed before choosing an action, e.g., [30, 26, 23].

Our main result in this setting is the following theorem, which shows that the ROBD algorithm (Algorithm 1), which is the state-of-the-art algorithm for SOCO, performs well in the more general case of structured memory. Note that, in this setting, the smoothness parameter  $l$  of hitting cost functions is not involved in the competitive ratio bound.

**Theorem 1.** *Suppose the hitting cost functions are  $m$ -strongly convex and the switching cost is given by  $c(y_{t:t-p}) = \frac{1}{2} \|y_t - \sum_{i=1}^p C_i y_{t-i}\|_2^2$ , where  $C_i \in \mathbb{R}^{d \times d}$  and  $\sum_{i=1}^p \|C_i\|_2 = \alpha$ . The competitive ratio of ROBD with parameters  $\lambda_1$  and  $\lambda_2$  is upper bounded by:*

$$\max \left\{ \frac{m + \lambda_2}{m\lambda_1}, \frac{\lambda_1 + \lambda_2 + m}{(1 - \alpha^2)\lambda_1 + \lambda_2 + m} \right\},$$

*if  $\lambda_1 > 0$  and  $(1 - \alpha^2)\lambda_1 + \lambda_2 + m > 0$ . If  $\lambda_1$  and  $\lambda_2$  satisfy  $m + \lambda_2 = \frac{m + \alpha^2 - 1 + \sqrt{(m + \alpha^2 - 1)^2 + 4m}}{2} \cdot \lambda_1$ , then the competitive ratio is:*

$$\frac{1}{2} \left( 1 + \frac{\alpha^2 - 1}{m} + \sqrt{\left( 1 + \frac{\alpha^2 - 1}{m} \right)^2 + \frac{4}{m}} \right).$$

The proof of Theorem 1 is given in Appendix C. To get insight into Theorem 1, first consider the case when  $\alpha$  is a constant. In this case, the competitive ratio is of order  $O(1/m)$ , which highlights that the challenging setting is when  $m$  is small. It is easy to see that this upper bound is in fact tight. To see this, note that the case of SOCO with  $\ell_2$  squared switching cost considered in Goel and Wierman [23], Goel et al. [24] is a special case where  $p = 1, C_1 = I, \alpha = 1$ . Substituting these parameters into Theorem 1 gives exactly the same upper bound (including constants) as Goel et al. [24], which has been shown to match a lower bound on the achievable cost of any online algorithm, including constant factors. On the other hand, if we instead assume that  $m$  is a fixed positive constant. The competitive ratio can be expressed as  $1 + O(\alpha^2)$ . Therefore, the competitive ratio gets worse quickly as  $\alpha$  increases. This is also the best possible scaling, achievable via any online algorithm, as we show in Appendix D.

Perhaps surprisingly, the memory length  $p$  does not appear in the competitive ratio bound, which contradicts the intuition that the online optimization problem should get harder as the memory length increases. However, it is worth noting that  $\alpha$  becomes larger as  $p$  increases, so the memory length implicitly impacts the competitive ratio. For example, an interesting form of switching cost is

$$c(y_{t:t-p}) = \frac{1}{2} \left\| \sum_{i=0}^p (-1)^i \binom{p}{i} y_{t-i} \right\|_2^2,$$

which corresponds to the  $p^{\text{th}}$  derivative of  $y$  and generalizes SOCO ( $p = 1$ ). In this case, we have  $\alpha = 2^p - 1$ . Hence  $\alpha$  grows exponentially in  $p$ .

### 3.2 Case 2: inexact prediction of $v_t$ ( $v_t \in \Omega_t$ )

For general  $\Omega_t$ , ROBD is no longer enough. It needs to be adapted to handle the uncertainty that results from the estimation set  $\Omega_t$ . Note that this uncertainty set is crucial for many applications, such as online control with adversarial disturbances.

To handle this additional complexity, we propose Optimistic ROBD (Algorithm 2). Optimistic ROBD is based on two key ideas. The first is to ensure that the algorithm tracks the sequence of actions it would have made if given observations of the true cost functions before choosing an action. To formalize this, we define the *accurate sequence*  $\{\hat{y}_1, \dots, \hat{y}_T\}$  to be the choices of ROBD (Algorithm 1) with  $\lambda_1 = \lambda$ ,  $\lambda_2 = 0$  when each hitting cost  $f_t$  is revealed before picking  $\hat{y}_t$ . The goal of Optimistic ROBD (Algorithm 2) is to approximate the accurate sequence. In order to track the accurate sequence, the first step is to recover it up to time step  $t - 1$  at time step  $t$ . To do this, after we observe the previous minimizer  $v_{t-1}$ , we can compute the accurate choice of ROBD as if both  $h_{t-1}$  and  $v_{t-1}$  are observed before picking  $y_{t-1}$ . Therefore, Algorithm 2 can compute the *accurate subsequence*  $\{\hat{y}_1, \dots, \hat{y}_{t-1}\}$  at time step  $t$ . Picking  $y_t$  based on the accurate sequence  $\{\hat{y}_1, \dots, \hat{y}_{t-1}\}$  instead of the noisy sequence  $\{y_1, \dots, y_{t-1}\}$  ensures that the actions do not drift too far from the accurate sequence.

The second key idea is to be optimistic by assuming the adversary will give it  $v \in \Omega_t$  that minimizes the cost it will experience. Specifically, before  $v_t$  is revealed, the algorithm assumes it is the point in  $\Omega_t$  which minimizes the weighted sum  $h_t(y - v) + \lambda c(y, \hat{y}_{t-1:t-p})$  if ROBD is implemented with parameter  $\lambda$  to pick  $y$ . This ensures that additional cost is never taken unnecessarily, which could be exploited by the adversary. Note that  $\min_y h_t(y - v) + \lambda c(y)$  is strongly convex with respect to  $v$  (proof in Appendix E), so it is tractable even if  $\Omega_t$  is unbounded.

Our main result in this paper (Theorem 2) bounds the competitive ratio of Optimistic ROBD.

**Theorem 2** (Main result). *Suppose the hitting cost functions are both  $m$ -strongly convex and  $l$ -strongly smooth and the switching cost is given by  $c(y_{t:t-p}) = \frac{1}{2} \|y_t - \sum_{i=1}^p C_i y_{t-i}\|_2^2$ , where  $C_i \in \mathbb{R}^{d \times d}$  and  $\sum_{i=1}^p \|C_i\|_2 = \alpha$ . For arbitrary  $\eta > 0$ , the cost of Optimistic ROBD with parameter  $\lambda > 0$ , is upper bounded by  $K_1 \text{cost}(\text{OPT}) + K_2$ , where:*

$$K_1 = (1 + \eta) \max \left\{ \frac{1}{\lambda}, \frac{\lambda + m}{(1 - \alpha^2)\lambda + m} \right\}, K_2 = \lambda \left( \frac{l}{1 + \eta - \lambda} + \frac{4\alpha^2}{\eta} - \frac{m}{\lambda + m} \right) \sum_{t=1}^T \frac{\|v_t - \tilde{v}_t\|_2^2}{2}.$$

The proof of Theorem 2 is given in Appendix E. This proof is nontrivial and relies on the two key ideas we mentioned before. Although Theorem 2 does not apply to the case  $\lambda = 0$ , we discuss it separately in Appendix F. Also, note that we can choose  $\eta$  to balance  $K_1$  and  $K_2$  and obtain a competitive ratio, in particular the smallest  $\eta$  such that:

$$\lambda \left( \frac{l}{1 + \eta - \lambda} + \frac{4\alpha^2}{\eta} - \frac{m}{\lambda + m} \right) \leq 0.$$

Therefore, we have  $\eta = O(l + \alpha^2)$  and  $K_2 \leq 0$ . So the competitive ratio is upper bounded by:

$$O\left((l + \alpha^2) \max\left\{\frac{1}{\lambda}, \frac{\lambda + m}{(1 - \alpha^2)\lambda + m}\right\}\right).$$

However, the reason we present Theorem 2 in terms of  $K_1$  and  $K_2$  is that, when the diameter of  $\Omega_t$  is small, we can pick a small  $\eta$  so that the ratio coefficient  $K_1$  will be close to the competitive ratio of ROBD when  $v_t$  is known before picking  $y_t$ . This “beyond-the-worst-case” analysis is useful in many applications and we discuss it more in Section 4.3.

## 4 Application to competitive online control

Goel and Wierman [23] show a connection between SOCO and online control in the setting where disturbance is perfectly known at time step  $t$  and the control actuation matrix  $B$  is invertible, which leads to the only constant-competitive control policy as far as we know. Since the new proposed OCO with structured memory generalizes SOCO, one may expect its connects to more general dynamical systems. In this section, we present a nontrivial reduction from *Input-Disturbed Squared Regulators* (IDSRs) to OCO with structured memory, leading to the first constant-competitive policy in online control with adversarial disturbance.

### 4.1 Control setting

**Input-disturbed systems.** We focus on systems in controllable canonical form defined by:

$$x_{t+1} = Ax_t + B(u_t + w_t), \quad (1)$$

where  $x_t \in \mathbb{R}^n$  is the state,  $u_t \in \mathbb{R}^d$  is the control, and  $w_t \in \mathbb{R}^d$  is a potentially adversarial disturbance to the system. We further assume that  $(A, B)$  is in controllable canonical form (see the right equation), where each  $*$  represents a (possibly) non-zero entry, and the rows of  $B$  with 1 are the same rows of  $A$  with  $*$  [36]. It is well-known that any controllable system can be linearly transformed to the canonical form. This system is more restrictive than the general form in linear systems. We call these *Input-Disturbed* systems, since the disturbance  $w_t$  is in the control input/action space. There are many corresponding real-world applications that are well-described by Input-Disturbed systems, e.g., external/disturbance force in robotics [38, 39, 18].

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ * & * & * & \dots & * \\ & & & & 0 & 1 & 0 & \dots & 0 \\ & & & & 0 & 0 & 1 & \dots & 0 \\ & & & & \vdots & & & \ddots & \vdots \\ & & & & 0 & 0 & 0 & \dots & 1 \\ * & * & * & \dots & * & * & * & \dots & * \\ & & & & \vdots & & & & \vdots \\ & & & & & & 0 & 1 & 0 & 0 \\ & & & & & & 0 & 0 & 1 & 0 \\ & & & & & & \vdots & & & \ddots \\ & & & & & & 0 & 0 & 0 & \dots & 1 \\ * & & & & \dots & & * & * & * & \dots & * \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 \end{bmatrix},$$

**Squared regulator costs.** We consider the following cost model for the controller:

$$c_t(x_t, u_t) = \frac{q_t}{2} \|x_t\|_2^2 + \frac{1}{2} \|u_t\|_2^2, \quad (2)$$

where  $q_t$  is a positive scalar. The sequence  $q_{0:T}$  is picked by the adversary and revealed online. The objective of the controller is to minimize the total control cost  $\sum_{t=0}^T c_t(x_t, u_t)$ . We call this cost model the *Squared Regulator* model, which is a restriction of the classical quadratic cost model. This class of costs is general enough to address a fundamental trade-off in optimal control: the trade-off between the state cost and the control effort [27].

**Disturbances.** In the online control literature, a variety of assumptions have been made about the noise  $w_t$ . In most works, the assumption is that the exact noise  $w_t$  is not known before  $u_t$  is taken. Many assume  $w_t$  is drawn from a certain known distribution, e.g., Agarwal et al. [5]. Others assume  $w_t$  is chosen adversarially subject to  $\|w_t\|_2$  being upper bounded by a constant  $W$ , e.g., Agarwal et al. [4]. In a closely related paper, Goel and Wierman [23] connect SOCO with online control under the assumption that  $w_t$  can be observed before picking the control action  $u_t$ . In contrast, in this paper we assume that the exact  $w_t$  is not observable before the agent picks  $u_t$ . Instead, we assume a convex estimation set  $W_t$  (not necessarily bounded) that contains all possible  $w_t$  is revealed to the online agent to help the agent decide  $u_t$ . Our assumption is a generalization of Goel and Wierman [23], where  $W_t$  is a one-point set, and Agarwal et al. [4], where  $W_t$  is a ball of radius  $W$  centered at  $0$ . Our setting can also naturally model time-Lipschitz noise, where  $w_t$  is chosen adversarially subject to  $\|w_t - w_{t-1}\|_2 \leq \epsilon$ , by picking  $W_t$  as a sphere of radius  $\epsilon$  centered at  $w_{t-1}$ , which has many

real-applications such as smooth disturbances in robotics [38, 39]. Moreover, note that our setting is naturally adaptive because of the estimation set  $W_t$  (e.g., controller may choose more aggressive action if  $W_t$  is small), which is different from the classic  $\mathcal{H}_\infty$  control setting [41].

**Competitive ratio.** Our goal is to develop policies with constant (small) competitive ratios. This is a departure from the bulk of the literature [5, 4, 20, 19], which focuses on designing policies that have low regret compared to the optimal linear controller. We show the optimal linear controller can have cost arbitrarily larger than the offline optimal, via an analytic example (Appendix B). We again denote the offline optimal cost, with full knowledge of the sequence  $w_{0:T}$ , as  $\text{cost}(OPT) = \min_{u_{0:T}} \sum_{t=0}^T c_t(x_t, u_t)$ . For an online algorithm  $ALG$ , let  $\text{cost}(ALG)$  be its cost on the same disturbance sequence  $w_{0:T}$ . The competitive ratio is then the worst-case ratio of  $\text{cost}(ALG)$  and  $\text{cost}(OPT)$  over any disturbance sequence, i.e.  $\sup_{w_{0:T}} \text{cost}(ALG)/\text{cost}(OPT)$ . We show in Section 4.2 an exact correspondence between this  $\text{cost}(OPT)$  and the one defined in Section 2.2, so that the competitive ratio guarantees directly translate.

To the best of our knowledge, the only prior work that studies competitive algorithms for online control is Goel and Wierman [23], which considers a very restricted system with invertible  $B$  and known  $w_t$  at step  $t$ . A related line of online optimization research studies *dynamic regret*, or *competitive difference*, defined as the difference between online algorithm cost and the offline optimal [31, 29]. For example, Li et al. [31] bound the dynamic regret of online control with time-varying convex costs with no noise. However, results for the dynamic regret depend on the path-length or variation budget, not just system properties. Bounding the competitive ratio is typically more challenging.

## 4.2 A reduction to OCO with structured memory

We now present a reduction from *IDS*R, introduced in Section 4.1, to OCO with structured memory. This reduction allows us to inherit the competitive ratio bounds on Optimistic ROBD for this class of online control problems. Before presenting the reduction, we first introduce some important notations. The indices of non-zero rows in matrix  $B$  in (1) are denoted as  $\{k_1, \dots, k_d\} := \mathcal{I}$ . We define operator  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^d$  as:

$$\psi(x) = \left( x^{(k_1)}, \dots, x^{(k_d)} \right)^\top,$$

which extracts the dimensions in  $\mathcal{I}$ . Moreover, let  $p_i = k_i - k_{i-1}$  for  $1 \leq i \leq n$ , where  $k_0 = 0$ . The controllability index of the canonical-form  $(A, B)$  is defined as  $p = \max\{p_1, \dots, p_d\}$ . We assume that the initial state is zero, i.e.,  $x_0 = \mathbf{0}$ . In the reduction,

we also need to use matrices  $C_i \in \mathbb{R}^{d \times d}$ ,  $i = 1, \dots, p$ , which regroup the columns of  $A(\mathcal{I}, :)$ . We define  $C_i$  for  $i = 1, \dots, p$  formally by constructing each of its columns. For  $j = 1, \dots, d$ , if  $i \leq p_j$ , the  $j$ th column of  $C_i$  is the  $(k_j + 1 - i)$ th column of  $A(\mathcal{I}, :)$ ; otherwise, the  $j$ th column of  $C_i$  is  $\mathbf{0}$ . Formally, for  $i \in \{1, \dots, p\}$ ,  $j \in \{1, \dots, d\}$ , we have:

$$C_i(:, j) = \begin{cases} A(\mathcal{I}, k_j + 1 - i) & \text{if } i \leq p_j \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Based on coefficients  $q_{0:T}$ , we define:

$$q_{\min} = \min_{0 \leq t \leq T-1, 1 \leq i \leq d} \sum_{j=1}^{p_i} q_{t+j}, \quad q_{\max} = \max_{0 \leq t \leq T-1, 1 \leq i \leq d} \sum_{j=1}^{p_i} q_{t+j},$$

where we assume  $q_t = 0$  for all  $t > T$ .

**Theorem 3.** Consider *IDS*R where the cost function and dynamics are specified by (2) and (1). We assume the coefficients  $q_{t:t+p-1}$  are observable at step  $t$ . Any instance of *IDS*R in controllable canonical form can be reduced to an instance of OCO with structured memory by Algorithm 3.

---

### Algorithm 3: Reduction to OCO with memory

---

**Input:** Transition matrix  $A$  and control matrix  $B$

**Solver:** OCO with structured memory algorithm ALG

**for**  $t = 0$  **to**  $T - 1$  **do**

**Observe:**  $x_t$ ,  $W_t$ , and  $q_{t:t+p-1}$

**if**  $t > 0$  **then**

$w_{t-1} \leftarrow \psi(x_t - Ax_{t-1} - Bu_{t-1})$

$\zeta_{t-1} \leftarrow w_{t-1} + \sum_{i=1}^p C_i \zeta_{t-1-i}$

$v_{t-1} \leftarrow -\zeta_{t-1}$

    Define  $h_t(y) = \frac{1}{2} \sum_{i=1}^d \left( \sum_{j=1}^{p_i} q_{t+j} \right) (y^{(i)})^2$

    Define  $\Omega_t = \{-w - \sum_{i=1}^p C_i \zeta_{t-i} \mid w \in W_t\}$

    Feed  $v_{t-1}, h_t, \Omega_t$  into ALG

    Obtain ALG's output  $y_t$

$u_t \leftarrow y_t - \sum_{i=1}^p C_i y_{t-i}$

**Output:**  $u_t$

**Output:**  $u_T = 0$

---

A proof and an example of Theorem 3 are given in Appendix G. Notably,  $\text{cost}(\text{OPT})$  and  $\text{cost}(\text{ALG})$  remain unchanged in the reduction described by Algorithm 3. In fact, Algorithm 3, when instantiated with Optimistic ROBD, provides an efficient algorithm for online control. It only requires  $O(p)$  memory to compute the recursive sequences. As stated in Algorithm 3 the recursive computation of  $y_t$  and  $\zeta_t$  may have numerical issues. However this can be addressed in a straightforward manner when the algorithm is instantiated with Optimistic ROBD (see Appendix H).

### 4.3 Competitive policy

The reduction in Section 4.2 immediately translates the competitive ratio guarantees in Section 3 into competitive policies. As Theorem 2 suggests, we can tune  $\eta$  in Optimistic ROBD based on the quality of prediction. As a result, we present two forms of upper bounds for Algorithm 3 in Corollaries 1 and 2. Notably, Corollary 1 gives a tighter bound where good estimations are available, while Corollary 2 gives a bound that does not depend on the quality of the estimations.

In the first case, we assume that a good estimation of  $w_t$  is available before picking  $u_t$ . Specifically, we assume the diameter of set  $W_t$  is upper bounded by  $\epsilon_t$  at time step  $t$ , where  $\epsilon_t$  is a small positive constant. We derive Corollary 1 by setting  $\eta = 1 + \lambda$  in Theorem 2.

**Corollary 1.** *In IDSR, assume that coefficients  $q_{t:t+p-1}$  are observable at time step  $t$ . Let  $\alpha = \sum_{i=1}^q \|C_i\|_2$ , where  $C_i, i = 1, \dots, p$  are defined as in Section 4.2. When the diameter of  $W_t$  is upper bounded by  $\epsilon_t$  at time step  $t$ , the total cost incurred by Algorithm 3 (using Optimistic ROBD with parameter  $\lambda$ ) in the online control problem is upper bounded by  $K_1 \text{cost}(\text{OPT}) + K_2$ , where:*

$$K_1 = (2 + \lambda) \cdot \max \left\{ \frac{1}{\lambda}, \frac{\lambda + q_{\min}}{(1 - \alpha^2)\lambda + q_{\min}} \right\}, K_2 = \lambda \left( \frac{q_{\max}}{2} + \frac{4\alpha^2}{1 + \lambda} - \frac{q_{\min}}{\lambda + q_{\min}} \right) \cdot \sum_{t=0}^{T-1} \frac{1}{2} \epsilon_t^2.$$

The residue term  $K_2$  in Corollary 1 becomes negligible when the total estimation error  $\sum_{t=0}^{T-1} \epsilon_t^2$  is small, leading to a pure competitive ratio guarantee. Further, if we ignore  $K_2$ , the coefficient  $K_1$  is only constant factor worse than the ratio we obtain when exact prediction of  $w_t$  is available.

However, the bound in Corollary 1 can be significantly worse than the case where exact prediction is available when the diameter of  $W_t$  is large or unbounded. Hence we introduce a second corollary that does not use any information about  $w_t$  when picking  $u_t$ . Specifically, we assume the diameter of set  $W_t$  cannot be bounded, so the upper bound given in Corollary 1 is meaningless. By picking the parameter  $\eta$  such that  $\lambda \left( \frac{l}{1+\eta-\lambda} + \frac{4\alpha^2}{\eta} - \frac{m}{\lambda+m} \right) \leq 0$  in Theorem 2, we obtain the following result.

**Corollary 2.** *In IDSR, assume that coefficients  $q_{t:t+p-1}$  are observable at time step  $t$ . Let  $\alpha = \sum_{i=1}^q \|C_i\|_2$ , where  $C_i, i = 1, \dots, p$  are defined as in Section 4.2. The competitive ratio of Algorithm 3, using Optimistic ROBD with  $\lambda$ , is upper bounded by:*

$$O \left( (q_{\max} + 4\alpha^2) \max \left\{ \frac{1}{\lambda}, \frac{\lambda + q_{\min}}{(1 - \alpha^2)\lambda + q_{\min}} \right\} \right).$$

Compared with Corollary 1, Corollary 2 gives an upper bound that is independent of the size of  $W_t$ . It is also a pure constant competitive ratio, without any additive term. However, the ratio is worse than the case where exact prediction of  $w_t$  is available, especially when  $q_{\max}$  or  $\alpha$  is large.

**Contrasting no-regret and constant-competitive guarantees.** The predominant benchmark used in previous work on online control via learning is *static regret* relative to the best linear controller in hindsight, i.e.,  $u_t = -K^* x_t$  [19, 2, 4, 5, 20, 21, 1]. For example, Agarwal et al. [5] achieve logarithmic regret under stochastic noise and strongly convex loss, and Agarwal et al. [4] achieve  $O(\sqrt{T})$  regret under adversarial noise and convex loss. However, the cost of the optimal linear controller may be far from the true offline optimal cost. Goel and Hassibi [22] recently show that there is a linear regret between the optimal offline linear policy and the true offline optimal policy in online LQR control. Thus, achieving small regret may still mean having a significantly larger cost than optimal. We illustrate this difference and our algorithm's performance by a 1-d analytic example (Appendix B), and also numerical experiments in higher dimensions (Section 4.4). In particular, we see that the optimal linear controller can be significantly more costly than the offline optimal controller and that Optimistic ROBD can significantly outperform the optimal linear controller.



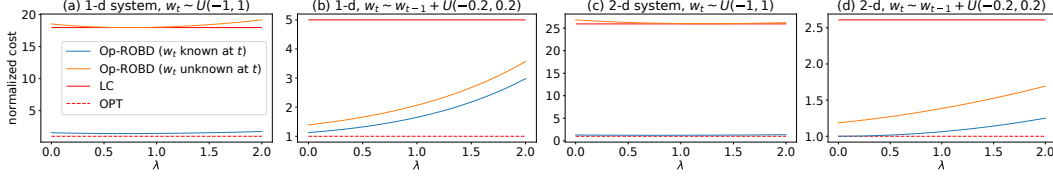


Figure 1: Numerical results of Optimistic ROBD in 1-d and 2-d systems, with different  $\lambda$ . LC means the best linear controller in hindsight and OPT means the global optimal controller in hindsight. LC is numerically searched in stable linear controller space. We consider two different types of  $w_t$ :  $w_t$  is i.i.d. random/random walk, and also two different settings:  $w_t$  is known/unknown at step  $t$ .

#### 4.4 Numerical results

In this section we use simple numerical examples to illustrate the contrast between the best linear controller in hindsight and the optimal offline controller. We also test our algorithm, Optimistic ROBO, and then numerically illustrate that Optimistic ROBD can obtain near-optimal cost and outperform the offline optimal linear controller.

In the first example we consider a simple 1-d system, where the object function is  $\sum_{t=0}^{200} 8|x_t|^2 + |u_t|^2$  and the dynamics is  $x_{t+1} = 2x_t + u_t + w_t$ . For the sequence  $\{w_t\}_{t=0}^T$ , we consider two cases, in the first case  $\{w_t\}_{t=0}^T$  is generated by  $w_t \sim \mathcal{U}(-1, 1)$  i.i.d., and in the second case the sequence is generated by  $w_{t+1} = w_t + \psi_t$  where  $\psi_t \sim \mathcal{U}(-0.2, 0.2)$  i.i.d.. The first case corresponds to unpredictable disturbances, where the estimation set  $W_t = (-1, 1)$ , and the second to smooth disturbances (i.e., a random walk), where  $W_t = w_{t-1} + (-0.2, 0.2)$ . For both types of  $\{w_t\}_{t=0}^T$ , we test Optimistic ROBD algorithms in two settings:  $w_t$  is known/unknown at step  $t$ . In the first setting,  $w_t$  is directly given to the algorithm, and in the latter setting, only  $W_t$  is given at time step  $t$ .

The results are shown in Figure 1 (a-b). We see that if  $w_t$  is known at step  $t$ , Optimistic ROBD is much better than the best linear controller in hindsight, and almost matches the true optimal when  $w_t$  is smooth. In fact, when  $w_t$  is smooth, Optimistic ROBD is much better than the best linear controller even if it does not know  $w_t$  at step  $t$ . Even in the case when  $w_t \sim \mathcal{U}(-1, 1)$ , and so is extremely unpredictable, Optimistic ROBD's performance still matches the best linear controller, which uses perfect hindsight.

Our second example considers a 2-d system with the following objective and dynamics:

$$\min_{u_t} \sum_{t=0}^{200} 8\|x_t\|_2^2 + \|u_t\|_2^2, \quad \text{s.t. } x_{t+1} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_t,$$

where  $(A, B)$  is the canonical form of double integrator dynamics. For this 2-d system, similarly, we test the performance of Optimistic ROBD with two types of  $w_t$ .

The results are shown in Figure 1 (c-d) and reinforce the same observations we observed in the 1-d system. In particular, we see that the optimal linear controller can be significantly more costly than the offline optimal controller and that Optimistic ROBD can outperform the optimal linear controller, sometimes by a significant margin.

## 5 Concluding remarks

We conclude with several open problems and potential future research directions. Our results show the existence of constant-competitive algorithms in a novel class of online optimization with memory, which generalizes SOCO. We also show the existence of constant-competitive control policies in *Input-Disturbed Squared Regulators (IDSRs)*, which is more general than prior work [23]. Following on our work, it will be interesting to understand the breadth of the class of online optimization problems that admit constant-competitive algorithms, and the breath of the class of online control problems where constant-competitive policies exist. Obtaining results (positive or negative) is an important and challenging future direction.

## Broader Impact

Online convex optimization with switching cost (SOCO) has been widely used in commercial and industrial applications such as data centers, power systems, and vehicle charging. By proposing a generalization of SOCO together with new algorithms with competitive ratio guarantees in this setting, this paper opens a new set of applications for online optimization. Additionally, the results provide new fundamental insights about the connection between online optimization and control. However, like many other theoretical contributions, this paper’s results are limited to its assumptions, e.g., strongly convex cost functions.

We see no ethical concerns related to the results in this paper.

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## Appendices

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### A Preliminaries

The appendices that follow provide the proofs of the results in the body of the paper. Throughout the proofs we use the following notation to denote the hitting and movement costs of the online learner:  $H_t := f_t(y_t)$  and  $M_t := c(y_{t:t-p})$ , where  $y_t$  is the point chosen by the online algorithm at time  $t$ . Similarly, we denote the hitting and movement costs of the offline optimal as  $H_t^* := f_t(y_t^*)$  and  $M_t^* := c(y_{t:t-p}^*)$ , where  $y_t^*$  is the point chosen by the offline optimal at time  $t$ .

Before moving to the proofs, we summarize a few standard definitions that are used throughout the paper.

**Definition 1.** A function  $f : \mathcal{X} \rightarrow \mathbb{R}$  is  $m$ -strongly convex with respect to a norm  $\|\cdot\|$  if for all  $x, y$  in the relative interior of the domain of  $f$  and  $\lambda \in (0, 1)$ , we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \frac{m}{2}\lambda(1 - \lambda) \|x - y\|^2.$$

**Definition 2.** A function  $f : \mathcal{X} \rightarrow \mathbb{R}$  is  $l$ -strongly smooth with respect to a norm  $\|\cdot\|$  if  $f$  is everywhere differentiable and if for all  $x, y$  we have

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{l}{2} \|y - x\|^2.$$

Finally, Lemma 13 in Goel et al. [24] will be useful, and so we restate it here.

**Lemma 1.** If  $f : \mathcal{X} \rightarrow \mathbb{R}$  is a  $m$ -strongly convex function with respect to some norm  $\|\cdot\|$ , and  $v$  is the minimizer of  $f$  (i.e.  $v = \arg \min_{y \in \mathcal{X}} f(y)$ ), then we have  $\forall y \in \mathcal{X}$ ,

$$f(y) \geq f(v) + \frac{m}{2} \|y - v\|^2.$$

### B Analytic 1-d example

In this section we use simple examples to illustrate the contrast between the best linear controller in hindsight, which is the predominant benchmark, and the optimal offline controller, which is not necessarily linear or static. We highlight analytically that the optimal linear controller can be arbitrarily worse than the optimal offline controller, and then illustrate that analytically that Optimistic ROBD can obtain near-optimal cost.

**Example: a scalar system.** Consider the following scalar system:

$$\begin{aligned} \min_{u_t} \quad & \sum_{t=0}^T q|x_t|^2 + |u_t|^2 \\ \text{s.t.} \quad & x_{t+1} = ax_t + u_t + w_t \end{aligned}$$

where  $a > 1$ ,  $x_0 = 0$  and  $w_t$  is the disturbance. For this system, we have:

$$\frac{\text{cost}(LC)}{\text{cost}(OPT)} > \frac{q + (a-1)^2}{4}, \forall \{w_t\}_{t=0}^T,$$

where  $\text{cost}(LC)$  is the cost of the optimal linear controller in hindsight. Hence,  $\text{cost}(LC)/\text{cost}(OPT)$  is arbitrarily large as  $q$  and  $a$  increase. We emphasize that this lower bound holds for any disturbance sequence, and there exist many sequences making this lower bound even bigger. For example, if  $w_t$  is a constant ( $w_t = w, \forall t$ ):

$$\frac{\text{cost}(LC)}{\text{cost}(OPT)} \geq \frac{q + (a-1)^2}{4} \cdot \frac{q + (a-1)^2}{q}.$$

Alternatively, if  $w_t = (-1)^t \cdot w$ :

$$\frac{\text{cost}(LC)}{\text{cost}(OPT)} \geq \frac{q + (a-1)^2}{4} \cdot \frac{q + (a+1)^2}{q}.$$

Proofs are given in Appendix I. This example highlights that the gap between  $\text{cost}(LC)$  and  $\text{cost}(OPT)$  can be arbitrarily large for strongly convex costs. Thus, even if an algorithm has no regret compared to the optimal linear controller, it has an unbounded competitive ratio.

Further, we can contrast the competitive ratio of the optimal linear controller derived above with that of Optimistic ROBD. For convenience, assume  $\text{cost}(OPT) = T$ . First, notice that there exists  $\{w_t\}_{t=0}^T$  such that  $\text{cost}(LC) \geq O(\max\{q, a^4/q\} \cdot T)$  for big enough  $a$  and  $q$ . From Corollary 1, in the case exact prediction of  $w_t$  is possible, Optimistic ROBD has  $\text{cost}(ALG) \leq O(\max\{1, a^2/q\} \cdot T), \forall \{w_t\}_{t=0}^T$ , which is orders-of-magnitude lower than  $\text{cost}(LC)$ .

In the case exact prediction is impossible and the estimation error is  $\epsilon_t = w_t - \tilde{w}_t$ , Optimistic ROBD guarantees  $\text{cost}(ALG) \leq O(\max\{1, a^2/q\} \cdot T + \max\{a^2, q\} \cdot \sum_{t=0}^{T-1} \epsilon_t^2)$  by Corollary 1. Moreover, Corollary 2 gives a constant competitive ratio,  $\text{cost}(ALG) \leq O(\max\{q, a^4/q\} \cdot T)$  for any  $\{w_t\}_{t=0}^T$ , which is the same as the lower bound of  $\text{cost}(LC)$  we found. Thus, even without any estimate of the noise, our *upper* bound on the cost of Optimistic ROBD matches the *lower* bound on the cost of the optimal linear controller.

## C Proof of Theorem 1

Our approach is to make use of strong convexity and properties of the hitting cost, the switching cost, and the regularization term to derive an inequality in the form of  $H_t + M_t + \Delta\phi_t \leq C(H_t^* + M_t^*)$  for some positive constant  $C$ , where  $\Delta\phi_t$  is the change in potential, which satisfies  $\sum_{t=1}^T \Delta\phi_t \geq 0$ . We will give the formal definition of  $\Delta\phi_t$  later. The constant  $C$  is then an upper bound for the competitive ratio.

We use  $\|\cdot\|$  to denote  $\ell_2$  norm or matrix norm induced by  $\ell_2$  norm throughout the proof.

By assumption, we have  $y_i = y_i^*$  for  $i = 0, -1, \dots, -(p-1)$ .

For convenience, we define

$$\phi_t = \frac{\lambda_1 + \lambda_2 + m}{2} \|y_t - y_t^*\|^2.$$

Recall that we define  $v_t = \arg \min_y f_t(y)$ . Since the function

$$g_t(y) = f_t(y) + \frac{\lambda_1}{2} \left\| y - \sum_{i=1}^p C_i y_{t-i} \right\|^2 + \frac{\lambda_2}{2} \|y - v_t\|^2$$

is  $(m + \lambda_1 + \lambda_2)$ -strongly convex and ROBD selects  $y_t = \arg \min_y g_t(y)$ , we see that

$$g_t(y_t) + \frac{m + \lambda_1 + \lambda_2}{2} \|y_t - y_t^*\|^2 \leq g_t(y_t^*),$$

which implies

$$\begin{aligned} & H_t + \lambda_1 M_t + \left( \phi_t - \sum_{i=1}^p \frac{\|C_i\|}{\alpha} \phi_{t-i} \right) \\ & \leq \left( H_t^* + \frac{\lambda_2}{2} \|y_t^* - v_t\|^2 \right) + \left( \frac{\lambda_1}{2} \left\| y_t^* - \sum_{i=1}^p C_i y_{t-i} \right\|^2 - \sum_{i=1}^p \frac{\|C_i\|}{\alpha} \phi_{t-i} \right). \end{aligned} \quad (3)$$

In the following steps, we bound the second term in the right-hand side of (3) by the switching cost of the offline optimal.

$$\begin{aligned} & \sum_{i=1}^p \frac{\|C_i\|}{\alpha} \phi_{t-i} \\ & = \frac{\lambda_1 + \lambda_2 + m}{2\alpha} \sum_{i=1}^p \|C_i\| \cdot \|y_{t-i} - y_{t-i}^*\|^2 \\ & \geq \frac{\lambda_1 + \lambda_2 + m}{2\alpha^2} \left( \sum_{i=1}^p \|C_i\| \cdot \|y_{t-i} - y_{t-i}^*\| \right)^2 \end{aligned} \quad (4a)$$

$$\geq \frac{\lambda_1 + \lambda_2 + m}{2\alpha^2} \left( \sum_{i=1}^p \|C_i y_{t-i} - C_i y_{t-i}^*\| \right)^2 \quad (4b)$$

$$\geq \frac{\lambda_1 + \lambda_2 + m}{2\alpha^2} \left\| \sum_{i=1}^p C_i y_{t-i} - \sum_{i=1}^p C_i y_{t-i}^* \right\|^2, \quad (4c)$$

where we use Jensen's Inequality in (4a); the definition of the matrix norm in (4b); the triangle inequality in (4c).

For notation convenience, we define

$$\delta_t = \sum_{i=1}^p C_i y_{t-i} - \sum_{i=1}^p C_i y_{t-i}^*.$$

Therefore, we obtain that

$$\begin{aligned} & \frac{\lambda_1}{2} \left\| y_t^* - \sum_{i=1}^p C_i y_{t-i} \right\|^2 - \sum_{i=1}^p \frac{\|C_i\|}{\alpha} \phi_{t-i} \\ & \leq \frac{\lambda_1}{2} \left\| y_t^* - \sum_{i=1}^p C_i y_{t-i} \right\|^2 - \frac{\lambda_1 + \lambda_2 + m}{2\alpha^2} \cdot \|\delta_t\|^2 \end{aligned} \quad (5a)$$

$$\begin{aligned} & = \frac{\lambda_1}{2} \left\| \left( y_t^* - \sum_{i=1}^p C_i y_{t-i}^* \right) - \delta_t \right\|^2 - \frac{\lambda_1 + \lambda_2 + m}{2\alpha^2} \cdot \|\delta_t\|^2 \\ & \leq \frac{\lambda_1}{2} \left\| y_t^* - \sum_{i=1}^p C_i y_{t-i}^* \right\|^2 + \lambda_1 \left\| y_t^* - \sum_{i=1}^p C_i y_{t-i}^* \right\| \cdot \|\delta_t\| \\ & \quad + \frac{\lambda_1}{2} \|\delta_t\|^2 - \frac{\lambda_1 + \lambda_2 + m}{2\alpha^2} \|\delta_t\|^2 \end{aligned} \quad (5b)$$

$$\begin{aligned} & = \frac{\lambda_1}{2} \left\| y_t^* - \sum_{i=1}^p C_i y_{t-i}^* \right\|^2 + \lambda_1 \left\| y_t^* - \sum_{i=1}^p C_i y_{t-i}^* \right\| \cdot \|\delta_t\| \\ & \quad - \frac{(1 - \alpha^2)\lambda_1 + \lambda_2 + m}{2\alpha^2} \|\delta_t\|^2 \\ & \leq \frac{\lambda_1}{2} \left\| y_t^* - \sum_{i=1}^p C_i y_{t-i}^* \right\|^2 + \frac{\alpha^2 \lambda_1^2}{2((1 - \alpha^2)\lambda_1 + \lambda_2 + m)} \left\| y_t^* - \sum_{i=1}^p C_i y_{t-i}^* \right\|^2 \\ & \quad + \frac{(1 - \alpha^2)\lambda_1 + \lambda_2 + m}{2\alpha^2} \|\delta_t\|^2 - \frac{(1 - \alpha^2)\lambda_1 + \lambda_2 + m}{2\alpha^2} \|\delta_t\|^2 \\ & = \frac{\lambda_1(\lambda_1 + \lambda_2 + m)}{(1 - \alpha^2)\lambda_1 + \lambda_2 + m} M_t^*, \end{aligned} \quad (5c)$$

where we use (4) in (5a); the triangle inequality in (5b); the AM-GM inequality in (5c).

We also notice that since  $f_t$  is  $m$ -strongly convex, the first term in the right-hand side of (3) can be bounded by

$$H_t^* + \frac{\lambda_2}{2} \|y_t^* - v_t\|^2 \leq \frac{m + \lambda_2}{m} H_t^*. \quad (6)$$

Substituting (5) and (6) into (3), we obtain that

$$\begin{aligned} & H_t + \lambda_1 M_t + \phi_t - \sum_{t=1}^q \frac{\|C_i\|}{\alpha} \phi_{t-i} \\ & \leq \frac{m + \lambda_2}{m} H_t^* + \frac{\lambda_1(\lambda_1 + \lambda_2 + m)}{(1 - \alpha^2)\lambda_1 + \lambda_2 + m} M_t^*. \end{aligned} \quad (7)$$

Define  $\Delta\phi_t = \phi_t - \sum_{t=1}^q \frac{\|C_i\|}{\alpha} \phi_{t-i}$ . We see that

$$\sum_{t=1}^T \Delta\phi_t = \frac{1}{\alpha} \sum_{i=0}^{q-1} \left( \sum_{j=i+1}^q \|C_j\| \right) \phi_{T-i} - \frac{1}{\alpha} \sum_{i=0}^{q-1} \left( \sum_{j=i+1}^q \|C_j\| \right) \phi_{-i}.$$

Since  $\phi_t \geq 0, \forall t$  and  $\phi_0 = \phi_{-1} = \dots = \phi_{-q+1} = 0$ , we have

$$\sum_{t=1}^T \Delta\phi_t \geq 0. \quad (8)$$

Summing (7) over timesteps  $t = 1, 2, \dots, T$ , we see that

$$\sum_{t=1}^T (H_t + \lambda_1 M_t) + \sum_{t=1}^T \Delta\phi_t \leq \sum_{t=1}^T \left( \frac{m + \lambda_2}{m} H_t^* + \frac{\lambda_1(\lambda_1 + \lambda_2 + m)}{(1 - \alpha^2)\lambda_1 + \lambda_2 + m} M_t^* \right).$$



Using (8), we obtain that

$$\sum_{t=1}^T (H_t + \lambda_1 M_t) \leq \sum_{t=1}^T \left( \frac{m + \lambda_2}{m} H_t^* + \frac{\lambda_1(\lambda_1 + \lambda_2 + m)}{(1 - \alpha^2)\lambda_1 + \lambda_2 + m} M_t^* \right), \quad (9)$$

which implies

$$\sum_{t=1}^T (H_t + M_t) \leq \sum_{t=1}^T \left( \frac{m + \lambda_2}{m\lambda_1} H_t^* + \frac{\lambda_1 + \lambda_2 + m}{(1 - \alpha^2)\lambda_1 + \lambda_2 + m} M_t^* \right).$$

## D Lower bound of online optimization with structured memory

Theorem 1 considers the problem setting where the hitting cost functions are  $m$ -strongly convex in feasible set  $\mathcal{X}$  and the switching cost is given by  $c(y_{t:t-p}) = \frac{1}{2} \|y_t - \sum_{i=1}^p C_i y_{t-i}\|_2^2$ , where  $C_i \in \mathbb{R}^{d \times d}$  and  $\sum_{i=1}^p \|C_i\|_2 = \alpha$ . We prove that the competitive ratio provided in Theorem 1 is optimal in parameters  $\alpha$  and  $m$  by showing a lower bound for a specific sequence of hitting costs and a specific form of switching cost,  $c(y_t, y_{t-1}) = \frac{1}{2} \|y_t - \alpha y_{t-1}\|_2^2$ .

Notice that making improvements on the competitive ratio is still possible if we consider more specific matrix  $C_i$  or adding more assumptions on the hitting cost functions.

**Theorem 4.** *When the hitting cost functions are  $m$ -strongly convex in feasible set  $\mathcal{X}$  and the switching cost is given by  $c(y_t, y_{t-1}) = \frac{1}{2} \|y_t - \alpha y_{t-1}\|_2^2$  for a constant  $\alpha \geq 1$ , the competitive ratio of any online algorithm is lower bounded by*

$$\frac{1}{2} \left( 1 + \frac{\alpha^2 - 1}{m} + \sqrt{\left( 1 + \frac{\alpha^2 - 1}{m} \right)^2 + \frac{4}{m}} \right).$$

Theorem 4 is a generalization of [24][Theorem 1], which only considers the case when  $\alpha = 1$ . Our proof uses a parallel approach but extends it to general  $\alpha$ . Before giving the proof of Theorem 4, we first prove the generalization of [24][Lemma 7]. To simplify presentation in the proofs, we use  $\mathcal{K}(n, y)$  to denote the set  $\{y \in \mathbb{R}^{n+2} \mid y_i \in \mathbb{R}, y_0 = 0, y_{n+1} = y\}$ .

**Lemma 2.** *For  $m > 0$  and  $\alpha \geq 1$ , define*

$$a_n = 2 \min_{y^* \in \mathcal{K}(n, 1)} \left( \sum_{i=1}^n \frac{m}{2} (y_i^*)^2 + \sum_{i=1}^{n+1} \frac{1}{2} (y_i^* - \alpha y_{i-1}^*)^2 \right).$$

*Then we have  $\lim_{n \rightarrow \infty} a_n = \frac{-m - \alpha^2 + 1 + \sqrt{(m + \alpha^2 - 1)^2 + 4m}}{2}$ .*

*Proof of Lemma 2.* Using a parallel approach to [24][Lemma 7], we can show that sequence  $\{a_n\}$  satisfies the recursive relationship

$$a_{n+1} = \frac{a_n + m}{a_n + m + \alpha^2}.$$

Solving the equation  $y = \frac{y+m}{y+m+\alpha^2}$ , we find the two fixed points of the recursive relationship  $a_{n+1} = \frac{a_n+m}{a_n+m+\alpha^2}$  are

$$y_1 = \frac{-m - \alpha^2 + 1 + \sqrt{(m + \alpha^2 - 1)^2 + 4m}}{2},$$

and

$$y_2 = \frac{-m - \alpha^2 + 1 - \sqrt{(m + \alpha^2 - 1)^2 + 4m}}{2}.$$

Notice that for  $i = 1, 2$ , we have

$$m - (m + \alpha^2)y_i = -(1 - y_i)y_i.$$

Using this property, we obtain

$$a_{n+1} - y_1 = \frac{a_n + m}{a_n + m + \alpha^2} - y_1 = \frac{(1 - y_1)a_n + m - (m + \alpha^2)y_1}{a_n + m + \alpha^2} = \frac{(1 - y_1)(a_n - y_1)}{a_n + m + \alpha^2}, \quad (10)$$

and

$$a_{n+1} - y_2 = \frac{a_n + m}{a_n + m + \alpha^2} - y_2 = \frac{(1 - y_2)a_n + m - (m + \alpha^2)y_2}{a_n + m + \alpha^2} = \frac{(1 - y_2)(a_n - y_2)}{a_n + m + \alpha^2}. \quad (11)$$

Notice that  $a_{n+1} - y_2 > 0$ . By dividing equations (10) and (11), we obtain

$$\left( \frac{a_{n+1} - y_1}{a_{n+1} - y_2} \right) = \frac{1 - y_1}{1 - y_2} \cdot \left( \frac{a_n - y_1}{a_n - y_2} \right), \forall n \geq 0.$$

Solving this in a parallel way to [24][Lemma 7], we get

$$a_n = \left( 1 - \left( \frac{1 - y_1}{1 - y_2} \right)^{n+1} \right)^{-1} \left( y_1 - y_2 \cdot \left( \frac{1 - y_1}{1 - y_2} \right)^{n+1} \right).$$

Since  $0 < \left( \frac{1 - y_1}{1 - y_2} \right) < 1$ , we have

$$\lim_{n \rightarrow \infty} a_n = y_1 = \frac{-m - \alpha^2 + 1 + \sqrt{(m + \alpha^2 - 1)^2 + 4m}}{2}. \quad (12)$$

□

Now we come back to the proof of Theorem 4.

*Proof of Theorem 4.* We consider the counterexample where the starting point of the algorithm and the offline adversary is  $y_0 = y_0^* = 0$ , and the hitting cost functions are

$$f_t(y) = \begin{cases} \frac{m}{2}y^2 & t \in \{1, 2, \dots, n\} \\ \frac{m'}{2}(y - 1)^2 & t = n + 1 \end{cases}$$

for some large parameter  $m'$  that we choose later.

By a parallel approach to [24][Theorem 1], we can show the cost incurred by any online algorithm has the lower bound

$$\text{cost}(ALG) \geq \min_y \left( \frac{1}{2}y^2 + \frac{m'}{2}(y - 1)^2 \right) = \frac{1}{2\left(1 + \frac{1}{m'}\right)}. \quad (13)$$

In contrast to the case when  $\alpha = 1$ , the offline adversary can leverage the factor  $\alpha$  to approach 1 quicker if  $\alpha > 1$ .

Let the sequence of points the adversary chooses be  $y^* = (y_0^*, y_1^*, \dots, y_{n+1}^*) \in \mathbb{R}^{n+2}$ . We compute the cost incurred by the adversary as follows.

$$\begin{aligned} a_n &= 2 \min_{y^* \in \mathcal{K}(n, 1)} \sum_{i=1}^{n+1} (H_i^* + M_i^*) \\ &= 2 \min_{y^* \in \mathcal{K}(n, 1)} \left( \sum_{i=1}^n \frac{m}{2}(y_i^*)^2 + \sum_{i=1}^{n+1} \frac{1}{2}(y_i^* - \alpha y_{i-1}^*)^2 \right). \end{aligned}$$

In words,  $a_n$  is twice the minimal offline cost subject to the constraints  $y_0^* = 0, y_{n+1}^* = 1$ . Recall that we have derived the limiting behavior of the offline costs as  $n \rightarrow \infty$  for general  $\alpha$  in the Lemma 2. Given Lemma 2, the total cost of the offline adversary will be  $\frac{a_n}{2}$ . Finally, applying (13), we know  $\forall n$  and  $\forall m' > 0$ ,

$$\frac{\text{cost}(ALG)}{\text{cost}(ADV)} \geq \frac{\frac{1}{2\left(1 + \frac{1}{m'}\right)}}{\frac{a_n}{2}} = \frac{1}{\left(1 + \frac{1}{m'}\right)a_n}.$$

By taking the limit  $n \rightarrow \infty$  and  $m' \rightarrow \infty$  and using Lemma 2, we obtain

$$\frac{\text{cost}(ALG)}{\text{cost}(OPT)} = \lim_{n, m' \rightarrow \infty} \frac{\text{cost}(ALG)}{\text{cost}(ADV)} \geq \frac{1}{2} \left( 1 + \frac{\alpha^2 - 1}{m} + \sqrt{\left( 1 + \frac{\alpha^2 - 1}{m} \right)^2 + \frac{4}{m}} \right).$$

□

## E Proof of Theorem 2

We use  $\|\cdot\|$  to denote  $\ell_2$  norm or matrix norm induced by  $\ell_2$  norm throughout the proof. Before giving the proof of Theorem 2, we first prove three lemmas that we use later.

Recall that ROBD with parameters  $\lambda_1 = \lambda, \lambda_2 = 0$  minimizes a weighted sum of the *hitting cost*  $f_t$  and the *switching cost*  $c$ . To pick the appropriate estimation of  $v_t$  from the set  $\Omega_t$ , we want to study when the previous decision points  $\hat{y}_{t-p:t-1}$  is fixed, how the position of  $v_t$  will affect the minimum of this weighted sum. By a change of variable, we see this is equivalent to study when the hitting cost function is fixed, how the sum  $\sum_{i=1}^p C_i \hat{y}_{t-i}$  will affect the weighted sum. We use  $x$  to denote the sum  $\sum_{i=1}^p C_i \hat{y}_{t-i}$  in Lemma 3.

**Lemma 3.** Suppose function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is  $m$ -strongly convex. Define function  $g : \mathbb{R}^d \rightarrow \mathbb{R}$  as

$$g(x) = \min_y f(y) + \frac{\lambda}{2} \|y - x\|^2.$$

Then  $g$  is  $\frac{\lambda m}{\lambda + m}$ -strongly convex.

*Proof of Lemma 3.* Due to the definition of strongly convexity, we only need to show that for all  $x_1, x_2 \in \mathbb{R}^d$  and  $\eta \in (0, 1)$ , we have

$$g(\eta x_1 + (1 - \eta)x_2) \leq \eta g(x_1) + (1 - \eta)g(x_2) - \frac{\lambda m}{2(\lambda + m)} \eta(1 - \eta) \|x_1 - x_2\|^2.$$

For convenience, we define

$$y_1 := \arg \min_y f(y) + \frac{\lambda}{2} \|y - x_1\|^2,$$

and

$$y_2 := \arg \min_y f(y) + \frac{\lambda}{2} \|y - x_2\|^2.$$

We have that

$$\begin{aligned} & \eta g(x_1) + (1 - \eta)g(x_2) - \frac{\lambda m}{2(\lambda + m)} \eta(1 - \eta) \|x_1 - x_2\|^2 \\ &= \eta f(y_1) + (1 - \eta)f(y_2) + \frac{\eta \lambda}{2} \|y_1 - x_1\|^2 + \frac{(1 - \eta)\lambda}{2} \|y_2 - x_2\|^2 - \frac{\lambda m}{2(\lambda + m)} \eta(1 - \eta) \|x_1 - x_2\|^2 \end{aligned} \quad (14a)$$

$$\begin{aligned} & \geq f(\eta y_1 + (1 - \eta)y_2) + \frac{m}{2} \eta(1 - \eta) \|y_1 - y_2\|^2 - \frac{\lambda m}{2(\lambda + m)} \eta(1 - \eta) \|x_1 - x_2\|^2 \\ & \quad + \frac{\eta \lambda}{2} \|y_1 - x_1\|^2 + \frac{(1 - \eta)\lambda}{2} \|y_2 - x_2\|^2 \end{aligned} \quad (14b)$$

$$\begin{aligned} & \geq g(\eta x_1 + (1 - \eta)x_2) + \frac{m}{2} \eta(1 - \eta) \|y_1 - y_2\|^2 - \frac{\lambda m}{2(\lambda + m)} \eta(1 - \eta) \|x_1 - x_2\|^2 \\ & \quad + \frac{\eta \lambda}{2} \|y_1 - x_1\|^2 + \frac{(1 - \eta)\lambda}{2} \|y_2 - x_2\|^2 - \frac{\lambda}{2} \|\eta(y_1 - x_1) + (1 - \eta)(y_2 - x_2)\|^2 \end{aligned} \quad (14c)$$

$$\begin{aligned} & \geq g(\eta x_1 + (1 - \eta)x_2) + \frac{m}{2} \eta(1 - \eta) \|y_1 - y_2\|^2 - \frac{\lambda m}{2(\lambda + m)} \eta(1 - \eta) \|x_1 - x_2\|^2 \\ & \quad + \frac{\eta(1 - \eta)\lambda}{2} \|(y_1 - y_2) - (x_1 - x_2)\|^2, \end{aligned}$$

where in (14a) and (14c) we use the definition of function  $g$ ; in (14b) we use the fact that  $f$  is  $m$ -strongly convex; in (14c) we use function  $\frac{\lambda}{2} \|\cdot\|^2$  is  $\lambda$ -strongly convex.

Notice that

$$\begin{aligned}
& m \|y_1 - y_2\|^2 - \frac{\lambda m}{\lambda + m} \|x_1 - x_2\|^2 + \lambda \|(y_1 - y_2) - (x_1 - x_2)\|^2 \\
& \geq m \|y_1 - y_2\|^2 - \frac{\lambda m}{\lambda + m} \|x_1 - x_2\|^2 + \lambda \|y_1 - y_2\|^2 + \lambda \|x_1 - x_2\|^2 - 2\lambda \|y_1 - y_2\| \cdot \|x_1 - x_2\| \\
& = (m + \lambda) \|y_1 - y_2\|^2 + \frac{\lambda^2}{m + \lambda} \|x_1 - x_2\|^2 - 2\lambda \|y_1 - y_2\| \cdot \|x_1 - x_2\| \\
& \geq 0.
\end{aligned} \tag{15}$$

Substituting (15) into (14) finishes the proof.  $\square$

In the second lemma, we show that if a function  $f$  is strongly smooth, the function value  $f(y)$  at point  $y$  can be upper bounded by a weighted sum of the function value  $f(x)$  at another point  $x$  and the squared distance between  $x$  and  $y$ .

**Lemma 4.** *If  $f : \mathbb{R}^d \rightarrow \mathbb{R}^+ \cup \{0\}$  is convex and  $l$ -strongly smooth, we have for all  $x, y \in \mathbb{R}^d$ , the inequality*

$$f(y) \leq (1 + \eta)f(x) + \left(1 + \frac{1}{\eta}\right) \cdot \frac{l}{2} \|x - y\|^2$$

holds for all  $\eta > 0$ .

*Proof of Lemma 4.* Let  $v := \arg \min_z f(z)$ .

Using the property of  $l$ -strongly smoothness, we see that

$$f(x) \geq f(v) + \langle \nabla f(v), x - v \rangle + \frac{1}{2l} \|\nabla f(x) - \nabla f(v)\|^2 \tag{16a}$$

$$\geq \frac{1}{2l} \|\nabla f(x)\|^2, \tag{16b}$$

where we use [13][Lemma 3.5] in (16a); we use  $f(v) \geq 0, \nabla f(v) = 0$  in (16b).

Therefore, we obtain that

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{l}{2} \|y - x\|^2 \tag{17a}$$

$$\leq f(x) + \|\nabla f(x)\| \cdot \|y - x\| + \frac{l}{2} \|y - x\|^2 \tag{17b}$$

$$\leq f(x) + \frac{\eta}{2l} \|\nabla f(x)\|^2 + \frac{l}{2\eta} \|y - x\|^2 + \frac{l}{2} \|y - x\|^2 \tag{17c}$$

$$\leq f(x) + \eta f(x) + \left(1 + \frac{1}{\eta}\right) \cdot \frac{l}{2} \|y - x\|^2 \tag{17d}$$

$$= (1 + \eta)f(x) + \left(1 + \frac{1}{\eta}\right) \cdot \frac{l}{2} \|y - x\|^2,$$

where we use that  $f$  is  $l$ -strongly smooth in (17a); Cauchy-Schwarz Inequality in (17b); AM-GM inequality in (17c); (16) in (17d).  $\square$

Recall that  $\hat{y}_t$  is the decision point of ROBD which knows the exact  $v_t$  before picking  $\hat{y}_t$ .  $y_t$  is the decision point of Optimistic ROBD which cannot observe the exact  $v_t$  before picking  $y_t$ . In the third lemma, we show that  $y_t$  and  $\hat{y}_t$  will be close to each other once the estimated minimizer  $\tilde{v}_t$  computed by Optimistic ROBD is close to the true minimizer  $v_t$ .

**Lemma 5.** *Under the same assumptions as Theorem 2, the distance between  $y_t$  and  $\hat{y}_t$  can be upper bounded by*

$$\|y_t - \hat{y}_t\| \leq 2 \|\zeta_t\|,$$

where  $\zeta_t = v_t - \tilde{v}_t$ .

*Proof of Lemma 5.* Recall that by definition, the real hitting cost function which we used to pick  $\hat{y}_t$  is  $f_t(y) = h_t(y - v_t)$ , and the estimated hitting cost function which we used to pick  $y_t$  is given by  $\tilde{f}_t(y) = h_t(y - \tilde{v}_t)$ . Therefore, we have  $\tilde{f}_t(y) = f_t(y + \zeta_t)$ .

Since  $\hat{y}_t = \text{ROBD}(f_t, \hat{y}_{t-1:t-q}) = \arg \min_y f_t(y) + \lambda c(y, \hat{y}_{t-1:t-p})$ , by strongly convexity, we have that

$$\begin{aligned} & f_t(\hat{y}_t) + \frac{\lambda}{2} \left\| \hat{y}_t - \sum_{i=1}^p C_i \hat{y}_{t-i} \right\|^2 + \frac{m+\lambda}{2} \|\hat{y}_t - y_t - \zeta_t\|^2 \\ & \leq f_t(y_t + \zeta_t) + \frac{\lambda}{2} \left\| y_t + \zeta_t - \sum_{i=1}^p C_i \hat{y}_{t-i} \right\|^2. \end{aligned} \quad (18)$$

Similarly, using  $y_t = \text{ROBD}(\tilde{f}_t, \hat{y}_{t-1:t-q}) = \arg \min_y \tilde{f}_t(y) + \lambda c(y, \hat{y}_{t-1:t-p})$ , we obtain that

$$\begin{aligned} & \tilde{f}_t(y_t) + \frac{\lambda}{2} \left\| y_t - \sum_{i=1}^p C_i \hat{y}_{t-i} \right\|^2 + \frac{m+\lambda}{2} \|\hat{y}_t - y_t - \zeta_t\|^2 \\ & \leq \tilde{f}_t(\hat{y}_t) + \frac{\lambda}{2} \left\| \hat{y}_t - \zeta_t - \sum_{i=1}^p C_i \hat{y}_{t-i} \right\|^2. \end{aligned} \quad (19)$$

Adding (18) and (19) together, we obtain that

$$\begin{aligned} & (m+\lambda) \|\hat{y}_t - y_t - \zeta_t\|^2 \\ & \leq \frac{\lambda}{2} \left( \left\| y_t + \zeta_t - \sum_{i=1}^p C_i \hat{y}_{t-i} \right\|^2 - \left\| y_t - \sum_{i=1}^p C_i \hat{y}_{t-i} \right\|^2 + \left\| \hat{y}_t - \zeta_t - \sum_{i=1}^p C_i \hat{y}_{t-i} \right\|^2 - \left\| \hat{y}_t - \sum_{i=1}^p C_i \hat{y}_{t-i} \right\|^2 \right) \\ & = \lambda \zeta_t^\top (y_t + \zeta_t - \hat{y}_t) \\ & \leq \lambda \|\zeta_t\| \cdot \|\hat{y}_t - y_t - \zeta_t\|. \end{aligned} \quad (20)$$

Therefore, we see that

$$\|\hat{y}_t - y_t - \zeta_t\| \leq \|\zeta_t\|,$$

which implies

$$\|y_t - \hat{y}_t\| \leq 2\|\zeta_t\|.$$

□

Now we come back to the proof of Theorem 2.

Define function  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^+ \cup \{0\}$  as

$$\psi(v) = \min_y h_t(y - v) + \lambda c(y, \hat{y}_{t-1:t-q}).$$

By a change of variable  $y \leftarrow z + v$ , we can rewrite function  $\psi$  as

$$\psi(v) = \min_z h_t(z) + \frac{\lambda}{2} \left\| z - \left( -v + \sum_{i=1}^p C_i \hat{y}_{t-i} \right) \right\|^2. \quad (21)$$

By Lemma 3, we see that function  $\psi$  is  $\frac{\lambda m}{\lambda + m}$ -strongly convex.

Recall that

$$y_t = \text{ROBD}(\tilde{f}_t, \hat{y}_{t-1:t-q}) = \arg \min_y h_t(y - \tilde{v}_t) + \lambda c(y, \hat{y}_{t-1:t-q}), \quad (22)$$

and

$$\hat{y}_t = \text{ROBD}(f_t, \hat{y}_{t-1:t-q}) = \arg \min_y h_t(y - v_t) + \lambda c(y, \hat{y}_{t-1:t-q}). \quad (23)$$

Since  $\tilde{v}_t$  minimizes  $\psi$  and  $\psi$  is  $\frac{\lambda m}{\lambda+m}$ -strongly convex, using (22) and (23), we obtain that

$$\begin{aligned} h_t(y_t - \tilde{v}_t) + \frac{\lambda}{2} \left\| y_t - \sum_{i=1}^p C_i \hat{y}_{t-i} \right\|^2 + \frac{1}{2} \cdot \frac{m\lambda}{\lambda+m} \|v_t - \tilde{v}_t\|^2 \\ \leq h_t(\hat{y}_t - v_t) + \frac{\lambda}{2} \left\| \hat{y}_t - \sum_{i=1}^p C_i \hat{y}_{t-i} \right\|^2. \end{aligned} \quad (24)$$

Using Lemma 4, we see that for any  $\eta_1 > 0$ ,

$$\frac{1}{1+\eta_1} h_t(y_t - v_t) \leq h_t(y_t - \tilde{v}_t) + \frac{l}{2\eta_1} \|v_t - \tilde{v}_t\|^2. \quad (25)$$

Since function  $\frac{\lambda}{2} \|y_t - y\|^2$  is  $\lambda$ -strongly smooth in  $y$ , by Lemma 4, we see that for any  $\eta_2 > 0$ ,

$$\frac{1}{1+\eta_2} \cdot \frac{\lambda}{2} \left\| y_t - \sum_{i=1}^p C_i y_{t-i} \right\|^2 \leq \frac{\lambda}{2} \left\| y_t - \sum_{i=1}^p C_i \hat{y}_{t-i} \right\|^2 + \frac{\lambda}{2\eta_2} \left\| \sum_{i=1}^p C_i (y_{t-i} - \hat{y}_{t-i}) \right\|^2. \quad (26)$$

Notice that

$$\frac{1}{2} \left\| \sum_{i=1}^p C_i (y_{t-i} - \hat{y}_{t-i}) \right\|^2 \leq \frac{1}{2} \left( \sum_{i=1}^p \|C_i\| \cdot \|y_{t-i} - \hat{y}_{t-i}\| \right)^2 \quad (27a)$$

$$\leq \frac{\alpha}{2} \left( \sum_{i=1}^p \|C_i\| \cdot \|y_{t-i} - \hat{y}_{t-i}\|^2 \right) \quad (27b)$$

$$\leq 2\alpha \left( \sum_{i=1}^p \|C_i\| \cdot \|\tilde{v}_{t-i} - v_{t-i}\|^2 \right), \quad (27c)$$

where we use the triangle inequality and the definition of matrix norm in (27a); Jensen's inequality in (27b); Lemma 5 in (27c).

Substituting (27) into (26) gives

$$\frac{1}{1+\eta_2} \cdot \frac{\lambda}{2} \left\| y_t - \sum_{i=1}^p C_i y_{t-i} \right\|^2 \leq \frac{\lambda}{2} \left\| y_t - \sum_{i=1}^p C_i \hat{y}_{t-i} \right\|^2 + \frac{2\alpha\lambda}{\eta_2} \left( \sum_{i=1}^p \|C_i\| \cdot \|\tilde{v}_{t-i} - v_{t-i}\|^2 \right). \quad (28)$$

Substituting (25) and (28) into (24), we obtain that

$$\begin{aligned} \frac{1}{1+\eta_1} h_t(y_t - v_t) + \frac{\lambda}{2(1+\eta_2)} \left\| y_t - \sum_{i=1}^p C_i y_{t-i} \right\|^2 \\ \leq h_t(\hat{y}_t - v_t) + \frac{\lambda}{2} \left\| \hat{y}_t - \sum_{i=1}^p C_i \hat{y}_{t-i} \right\|^2 + \left( \frac{l}{\eta_1} - \frac{m\lambda}{\lambda+m} \right) \cdot \frac{1}{2} \|v_t - \tilde{v}_t\|^2 + \frac{2\alpha\lambda}{\eta_2} \left( \sum_{i=1}^p \|C_i\| \cdot \|\tilde{v}_{t-i} - v_{t-i}\|^2 \right). \end{aligned} \quad (29)$$

Summing up (29) over all time steps, we see that

$$\begin{aligned} \min\left\{ \frac{1}{1+\eta_1}, \frac{\lambda}{1+\eta_2} \right\} \sum_{t=1}^T (H_t + M_t) \\ \leq \sum_{t=1}^T \left( \hat{H}_t + \lambda \hat{M}_t \right) + \left( \frac{l}{\eta_1} + \frac{4\alpha^2\lambda}{\eta_2} - \frac{m\lambda}{\lambda+m} \right) \cdot \sum_{t=1}^T \frac{1}{2} \|v_t - \tilde{v}_t\|^2. \end{aligned} \quad (30)$$

We pick  $\eta_2 = \eta$  and  $\eta_1 = \frac{1+\eta-\lambda}{\lambda}$  so that  $\frac{1}{1+\eta_1} = \frac{\lambda}{1+\eta}$ . Substituting into (30) gives

$$\sum_{t=1}^T (H_t + M_t) \leq \frac{1+\eta}{\lambda} \sum_{t=1}^T \left( \hat{H}_t + \lambda \hat{M}_t \right) + \lambda \left( \frac{l}{1+\eta-\lambda} + \frac{4\alpha^2}{\eta} - \frac{m}{\lambda+m} \right) \cdot \sum_{t=1}^T \frac{1}{2} \|v_t - \tilde{v}_t\|^2. \quad (31)$$

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**Algorithm 4:** Optimistic ROBD with  $\lambda = 0$ 


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**for**  $t = 1$  **to**  $T$  **do**

    **Observe:**  $v_{t-1}, h_t, \Omega_t$ 

     $s_t \leftarrow \sum_{i=1}^p C_i v_{t-i}$ 

    Let  $y_t$  be the projection of  $s_t$  on  $\Omega_t$ 

    **Output:**  $y_t$  (the decision at time step  $t$ )

---

Recall that the point sequence  $\{\hat{y}_t\}_{1 \leq t \leq T}$  is identical with the one picked by ROBD, which has parameters  $\lambda_1 = \lambda, \lambda_2 = 0$  and has access to the exact  $v_t$  before picking  $\hat{y}_t$ . Therefore, the same upper bound of  $\sum_{t=1}^T (\hat{H}_t + \lambda \hat{M}_t)$  given in (9) in the proof of Theorem 1 also applies here. It shows that

$$\sum_{t=1}^T (\hat{H}_t + \lambda \hat{M}_t) \leq \sum_{t=1}^T \left( H_t^* + \frac{\lambda(\lambda + m)}{(1 - \alpha^2)\lambda + m} M_t^* \right). \quad (32)$$

Substituting (32) into (31) finishes the proof.

## F Optimistic ROBD with $\lambda = 0$

Although Theorem 2 does cover the case when  $\lambda = 0$ , it is possible to extend the analysis to cover this setting. Notice that the agent may choose any point in  $\Omega_t$  in Algorithm 2 when  $\lambda = 0$ . Thus, a tiebreaking rule is needed to cover the case of  $\lambda = 0$ . We break the tie by choosing the projection of  $\sum_{i=1}^p C_i v_{t-i}$  on  $\Omega_t$ , which is natural if we consider  $\lambda \rightarrow 0^+$ . We give the pseudo for this specific case in Algorithm 4.

As in Section 3, we first consider the case when  $\Omega_t$  is a one-point set, i.e.  $\Omega_t = \{v_t\}$ .

**Theorem 5.** Suppose the hitting cost functions are  $m$ -strongly convex and the switching cost is given by  $c(y_{t:t-p}) = \frac{1}{2} \|y_t - \sum_{i=1}^p C_i y_{t-i}\|_2^2$ , where  $C_i \in \mathbb{R}^{d \times d}$  and  $\sum_{i=1}^p \|C_i\|_2 = \alpha$ . When  $\Omega_t = \{v_t\}$ , the competitive ratio of Algorithm 4 is upper bounded by  $1 + \frac{(1+\alpha)^2}{m}$ .

*Proof of Theorem 5.* Notice that when  $\Omega_t = \{v_t\}$ , Algorithm 4 will pick  $y_t = v_t$  for all time step  $t$ .

Since  $v_t = \arg \min_y f_t(y)$  and  $f_t$  is  $m$ -strongly convex, we have that

$$f_t(v_t) + \frac{m}{2} \|y_t^* - v_t\|^2 \leq f_t(y_t^*). \quad (33)$$

On the other hand, we can bound the switching cost of Algorithm 4 by

$$\begin{aligned} & \frac{1}{2} \left\| v_t - \sum_{i=1}^p C_i v_{t-i} \right\|^2 \\ &= \frac{1}{2} \left\| y_t^* - \sum_{i=1}^p C_i y_{t-i}^* \right\|^2 + \left\langle y_t^* - \sum_{i=1}^p C_i y_{t-i}^*, v_t - \sum_{i=1}^p C_i v_{t-i} \right\rangle + \frac{1}{2} \left\| (v_t - y_t^*) - \sum_{i=1}^p C_i (v_{t-i} - y_{t-i}^*) \right\|^2 \\ &\leq \frac{1}{2} \left\| y_t^* - \sum_{i=1}^p C_i y_{t-i}^* \right\|^2 + \left\| y_t^* - \sum_{i=1}^p C_i y_{t-i}^* \right\| \cdot \left\| v_t - \sum_{i=1}^p C_i v_{t-i} \right\| + \frac{1}{2} \left\| (v_t - y_t^*) - \sum_{i=1}^p C_i (v_{t-i} - y_{t-i}^*) \right\|^2 \end{aligned} \quad (34a)$$

$$\leq \left( 1 + \frac{(1+\alpha)^2}{m} \right) \cdot \frac{1}{2} \left\| y_t^* - \sum_{i=1}^p C_i y_{t-i}^* \right\|^2 + \left( 1 + \frac{m}{(1+\alpha)^2} \right) \cdot \frac{1}{2} \left\| (v_t - y_t^*) - \sum_{i=1}^p C_i (v_{t-i} - y_{t-i}^*) \right\|^2, \quad (34b)$$

where we use Cauchy-Schwartz inequality in (34a); we use AM-GM inequality in (34b).

Notice that

$$\left\| (v_t - y_t^*) - \sum_{i=1}^p C_i(v_{t-i} - y_{t-i}^*) \right\|^2 \leq \left( \|v_t - y_t^*\| + \sum_{i=1}^p \|C_i\| \cdot \|v_{t-i} - y_{t-i}^*\| \right)^2 \quad (35a)$$

$$\leq (1 + \alpha) \cdot \left( \|v_t - y_t^*\|^2 + \sum_{i=1}^p \|C_i\| \cdot \|v_{t-i} - y_{t-i}^*\|^2 \right), \quad (35b)$$

where we use the triangle inequality in (35a) and the Cauchy-Schwartz inequality in (35b).

Substituting (35) into (34) and summing up through time steps, we obtain that

$$\sum_{t=1}^T \frac{1}{2} \left\| v_t - \sum_{i=1}^p C_i v_{t-i} \right\|^2 \leq \sum_{t=1}^T \left( 1 + \frac{(1 + \alpha)^2}{m} \right) M_t^* + ((1 + \alpha)^2 + m) \cdot \frac{1}{2} \|v_t - y_t^*\|^2. \quad (36)$$

Substituting (33) gives that

$$\sum_{t=1}^T \frac{1}{2} \left\| v_t - \sum_{i=1}^p C_i v_{t-i} \right\|^2 \leq \sum_{t=1}^T \left( 1 + \frac{(1 + \alpha)^2}{m} \right) M_t^* + \left( 1 + \frac{(1 + \alpha)^2}{m} \right) \cdot (H_t^* - f_t(v_t)),$$

which implies

$$\sum_{t=1}^T \left( f_t(v_t) + \frac{1}{2} \left\| v_t - \sum_{i=1}^p C_i v_{t-i} \right\|^2 \right) \leq \left( 1 + \frac{(1 + \alpha)^2}{m} \right) \sum_{t=1}^T (H_t^* + M_t^*). \quad (37)$$

□

Now we consider the case when  $\Omega_t$  is a general convex set.

**Theorem 6.** Suppose the hitting cost functions are both  $m$ -strongly convex and  $l$ -strongly smooth and the switching cost is given by  $c(y_{t:t-p}) = \frac{1}{2} \|y_t - \sum_{i=1}^p C_i y_{t-i}\|_2^2$ , where  $C_i \in \mathbb{R}^{d \times d}$  and  $\sum_{i=1}^p \|C_i\|_2 = \alpha$ . For arbitrary  $\eta > 0$ , the cost of Algorithm 4 is upper bounded by  $K_1 \text{cost}(\text{OPT}) + K_2$ , where:

$$K_1 = (1 + \eta) \cdot \left( 1 + \frac{(1 + \alpha)^2}{m} \right),$$

$$K_2 = \left( l + \left( 1 + \frac{1}{\eta} \right) \alpha^2 - (1 + \eta) \right) \cdot \sum_{t=1}^T \frac{1}{2} \|y_t - v_t\|^2.$$

Like Theorem 2, we can choose  $\eta$  to balance  $K_1$  and  $K_2$  and obtain a competitive ratio, in particular the smallest  $\eta$  such that:

$$l + \left( 1 + \frac{1}{\eta} \right) \alpha^2 - (1 + \eta) \leq 0.$$

Therefore, we have  $\eta = O(l + \alpha^2)$  and  $K_2 \leq 0$ . So the competitive ratio is upper bounded by:

$$O \left( (l + \alpha^2) \cdot \left( 1 + \frac{(1 + \alpha)^2}{m} \right) \right).$$

*Proof of Theorem 6.* Since  $y_t$  is the projection of  $\sum_{i=1}^p C_i v_{t-i}$  on  $\Omega_t$ , and  $\Omega_t$  is a convex set, we have that

$$\frac{1}{2} \left\| y_t - \sum_{i=1}^p C_i v_{t-i} \right\|^2 \leq \frac{1}{2} \left\| v_t - \sum_{i=1}^p C_i v_{t-i} \right\|^2 - \frac{1}{2} \|v_t - y_t\|^2. \quad (38)$$

Because the hitting cost function  $f_t$  is  $l$ -strongly smooth, and  $v_t$  is the minimizer of  $f_t$ , we see that

$$\frac{1}{\eta_1} f_t(y_t) \leq \frac{l}{2\eta_1} \|v_t - y_t\|^2 + \frac{1}{\eta_1} f_t(v_t) \quad (39)$$



holds for any  $\eta_1 \geq 1$ .

Since function  $\frac{1}{2} \|y_t - y\|^2$  is 1-strongly smooth in  $y$ , by Lemma 4, we see that for any  $\eta_2 > 0$ ,

$$\frac{1}{1 + \eta_2} \cdot \frac{1}{2} \left\| y_t - \sum_{i=1}^p C_i y_{t-i} \right\|^2 \leq \frac{1}{2} \left\| y_t - \sum_{i=1}^p C_i v_{t-i} \right\|^2 + \frac{1}{2\eta_2} \left\| \sum_{i=1}^p C_i (v_{t-i} - y_{t-i}) \right\|^2. \quad (40)$$

Notice that

$$\frac{1}{2} \left\| \sum_{i=1}^p C_i (v_{t-i} - y_{t-i}) \right\|^2 \leq \frac{1}{2} \left( \sum_{i=1}^p \|C_i\| \cdot \|y_{t-i} - v_{t-i}\| \right)^2 \quad (41a)$$

$$\leq \frac{\alpha}{2} \left( \sum_{i=1}^p \|C_i\| \cdot \|y_{t-i} - v_{t-i}\|^2 \right), \quad (41b)$$

where we use the triangle inequality and the definition of matrix norm in (41a); Jensen's Inequality in (41b).

Substituting (41) into (40) gives

$$\frac{1}{1 + \eta_2} \cdot \frac{1}{2} \left\| y_t - \sum_{i=1}^p C_i y_{t-i} \right\|^2 \leq \frac{1}{2} \left\| y_t - \sum_{i=1}^p C_i v_{t-i} \right\|^2 + \frac{\alpha}{2\eta_2} \left( \sum_{i=1}^p \|C_i\| \cdot \|y_{t-i} - v_{t-i}\|^2 \right). \quad (42)$$

Substituting (39) and (42) into (38) gives

$$\begin{aligned} & \frac{1}{\eta_1} f_t(y_t) + \frac{1}{1 + \eta_2} \cdot \frac{1}{2} \left\| y_t - \sum_{i=1}^p C_i y_{t-i} \right\|^2 \\ & \leq \frac{1}{\eta_1} f_t(v_t) + \frac{1}{2} \left\| v_t - \sum_{i=1}^p C_i v_{t-i} \right\|^2 + \left( \frac{l}{\eta_1} - 1 \right) \cdot \frac{1}{2} \|v_t - y_t\|^2 + \frac{\alpha}{2\eta_2} \left( \sum_{i=1}^p \|C_i\| \cdot \|y_{t-i} - v_{t-i}\|^2 \right). \end{aligned} \quad (43)$$

Summing up (43) through time steps, we obtain that

$$\begin{aligned} & \min\left\{ \frac{1}{\eta_1}, \frac{1}{1 + \eta_2} \right\} \sum_{t=1}^T \left( f_t(y_t) + \frac{1}{2} \left\| y_t - \sum_{i=1}^p C_i y_{t-i} \right\|^2 \right) \\ & \leq \sum_{t=1}^T \left( f_t(v_t) + \frac{1}{2} \left\| v_t - \sum_{i=1}^p C_i v_{t-i} \right\|^2 \right) + \left( \frac{l}{\eta_1} + \frac{\alpha^2}{\eta_2} - 1 \right) \cdot \frac{1}{2} \|y_t - v_t\|^2. \end{aligned} \quad (44)$$

Let  $\eta_2 = \eta$  and  $\eta_1 = 1 + \eta$ . Combining with (37), we obtain that

$$\begin{aligned} & \sum_{t=1}^T \left( f_t(y_t) + \frac{1}{2} \left\| y_t - \sum_{i=1}^p C_i y_{t-i} \right\|^2 \right) \\ & \leq (1 + \eta) \cdot \left( 1 + \frac{(1 + \alpha)^2}{m} \right) \cdot \sum_{t=1}^T (H_t^* + M_t^*) + \left( l + \left( 1 + \frac{1}{\eta} \right) \alpha^2 - (1 + \eta) \right) \cdot \frac{1}{2} \|y_t - v_t\|^2. \end{aligned} \quad (45)$$

□

## G Proof and example of Theorem 3

The proof will proceed as follows. First, we extract the controllable dimensions in  $x_t$ ,  $\{x_t^{(k_1)}, \dots, x_t^{(k_d)}\}$ , to construct a new vector  $z_t$ . Then we can represent  $x_t$  by  $z_t, z_{t-1}, \dots, z_{t-p}$ .

Therefore, we can rewrite the dynamics in sequence  $\{z_t\}_{0 \leq t \leq T}$ , control action  $u_t$ , and noise  $w_t$ . By this approach, we can remove the control matrix  $B$  before  $(u_t + w_t)$  in the dynamics. Finally, we can convert the resulting dynamics to an OCO problem with structured memory.

We use  $\|\cdot\|$  to denote  $\ell_2$  norm throughout the proof.

Recall that the objective is given as

$$\frac{1}{2} \sum_{t=0}^T \left( q_t \|x_t\|^2 + \|u_t\|^2 \right), \quad (46)$$

where  $q_t > 0$  for all  $0 \leq t \leq T$ . Without loss of generality, we assume  $q_t = 0$  for all  $t > T$ .

Recall that we define operator  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^m$  as

$$\psi(x) = \left( x^{(k_1)}, \dots, x^{(k_d)} \right)^\top.$$

Using this notation, we define vector  $z_t$  as

$$z_t := \psi(x_t), t \geq 0.$$

Notice that  $z_t^j = x_t^{(k_j)}$  for  $j = 1, \dots, d$ . Since we have  $x_t^{(i)} = x_{t-1}^{(i+1)}$  for  $i \notin \mathcal{I}$ ,  $x_t$  can be represented by

$$x_t = \left( z_{t-p_1+1}^{(1)}, \dots, z_t^{(1)}, \dots, z_{t-p_d+1}^{(d)}, \dots, z_t^{(d)} \right)^\top. \quad (47)$$

Since  $x_0 = \mathbf{0}$ , we have  $z_t = \mathbf{0}$  for  $t \leq 0$ .

Using (47), we can rewrite the objective function as a function of sequence  $\{z_t\}$  and  $\{u_t\}$ . Notice that

$$\begin{aligned} \sum_{t=0}^T q_t \|x_t\|_2^2 &= \sum_{t=0}^T q_t \sum_{i=1}^d \sum_{j=1}^{p_i} \left( z_{t+1-j}^{(i)} \right)^2 \\ &= \sum_{t=0}^{T-1} \sum_{i=1}^d \left( \sum_{j=1}^{p_i} q_{t+j} \right) \left( z_{t+1}^{(i)} \right)^2, \end{aligned} \quad (48a)$$

where in (48a) we use  $z_t = \mathbf{0}$  for all  $t \leq 0$  and  $q_t = 0$  for all  $t > T$ .

Therefore, we define function  $h_t : \mathbb{R}^d \rightarrow \mathbb{R}^+ \cup \{0\}$  as

$$h_t(y) = \frac{1}{2} \sum_{i=1}^d \left( \sum_{j=1}^{p_i} q_{t+j} \right) \left( y^{(i)} \right)^2.$$

Using this definition, the objective (46) can be rewrite as

$$\frac{1}{2} \sum_{t=0}^T \left( q_t \|x_t\|^2 + \|u_t\|^2 \right) = \sum_{t=0}^{T-1} h_t(z_{t+1}) + \frac{1}{2} \|u_T\|^2, \quad (49)$$

where we notice that the optimal choice of control action  $u_T$  is always zero because it will not affect any state.

We also see that  $u_t$  can be determined by  $z_{t-p+1:t+1}$  because

$$u_t = z_{t+1} - w_t - A(\mathcal{I}, :)x_t, \quad (50)$$

where  $A(\mathcal{I}, :)$  consists of  $k_1, \dots, k_n$  rows of  $A$  and  $t \geq 0$ .

Notice that  $A(\mathcal{I}, :)x_t$  can be written as  $\sum_{i=1}^p C_i z_{t-i+1}$  by the definition of  $C_i, i = 1, \dots, p$ . Therefore, we can rewrite (50) as

$$u_t = z_{t+1} - w_t - \sum_{i=1}^p C_i z_{t-i+1}, \quad (51)$$

which is equivalent to

$$z_{t+1} = u_t + w_t + \sum_{i=1}^p C_i z_{t-i+1}.$$

We recursively define sequence  $\{y_t\}_{t \geq -p}$  as the accumulation of control actions, i.e.

$$y_t = u_t + \sum_{i=1}^p C_i y_{t-i}, \forall t \geq 0,$$

where  $y_t = \mathbf{0}$  for all  $t < 0$ . We also define sequence  $\{\zeta_t\}_{t \geq -p}$  as the accumulation of control noises, i.e.

$$\zeta_t = w_t + \sum_{i=1}^p C_i \zeta_{t-i}, \forall t \geq 0,$$

where  $\zeta_t = \mathbf{0}$  for all  $t < 0$ .

Recall that we have  $x_0 = \mathbf{0}$  by assumption. Therefore,

$$z_{t+1} = y_t + \zeta_t \quad (52)$$

holds for all  $t \geq -1$ .

Using (49) and (52), we can formalize the problem as *online optimization with memory*, where the hitting cost function is given by

$$f_t(y) = h_t(y + \zeta_t),$$

and the switching cost is  $\frac{1}{2} \|y_t - \sum_{i=1}^p C_i y_{t-i}\|^2$ .

Although  $h_t$  is revealed before the agent picks  $y_t$ , we need the knowledge of  $v_t = -\zeta_t$  to construct the hitting cost function  $f_t$ , which depends on previous noises  $w_{0:t}$ . At time step  $t$ , we know the exact  $w_\tau$  for all  $\tau \leq t-1$ , thus we can compute the exact  $\zeta_\tau$  for all  $\tau \leq t-1$ . Since the set  $W_t$  contains all possible noise  $w_t$ , we can construct the set  $\Omega_t = \{-w - \sum_{i=1}^p C_i \zeta_{t-i} \mid w \in W_t\}$  which contains all possible  $v_t$ .

**Example.** To illustrate the reduction, consider the following example:

$$\begin{bmatrix} x_{t+1}^{(1)} \\ x_{t+1}^{(2)} \\ x_{t+1}^{(3)} \\ x_{t+1}^{(4)} \\ x_{t+1}^{(5)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{bmatrix} \begin{bmatrix} x_t^{(1)} \\ x_t^{(2)} \\ x_t^{(3)} \\ x_t^{(4)} \\ x_t^{(5)} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \left( \begin{bmatrix} u_t^{(1)} \\ u_t^{(2)} \end{bmatrix} + \begin{bmatrix} w_t^{(1)} \\ w_t^{(2)} \end{bmatrix} \right). \quad (53)$$

Notice that since  $x_{t+1}^{(1)} = x_t^{(2)}$ ,  $x_{t+1}^{(3)} = x_t^{(4)}$ , we can rewrite (53) in a more compact form:

$$\underbrace{\begin{bmatrix} x_{t+1}^{(2)} \\ x_{t+1}^{(5)} \end{bmatrix}}_{z_{t+1}} = \underbrace{\begin{bmatrix} a_2 & a_5 \\ b_2 & b_5 \end{bmatrix}}_{C_1} \underbrace{\begin{bmatrix} x_t^{(2)} \\ x_t^{(5)} \end{bmatrix}}_{x_t} + \underbrace{\begin{bmatrix} a_1 & a_4 \\ b_1 & b_4 \end{bmatrix}}_{C_2} \underbrace{\begin{bmatrix} x_{t-1}^{(2)} \\ x_{t-1}^{(5)} \end{bmatrix}}_{x_{t-1}} + \underbrace{\begin{bmatrix} 0 & a_3 \\ 0 & b_3 \end{bmatrix}}_{C_3} \underbrace{\begin{bmatrix} x_{t-2}^{(2)} \\ x_{t-2}^{(5)} \end{bmatrix}}_{x_{t-2}} + \begin{bmatrix} u_t^{(1)} \\ u_t^{(2)} \end{bmatrix} + \begin{bmatrix} w_t^{(1)} \\ w_t^{(2)} \end{bmatrix}. \quad (54)$$

In this example  $p_1 = 2, p_2 = 3, \mathcal{I} = \{k_1, k_2\} = \{2, 5\}$  and thus  $p = 3$  and  $n = 2$ . From (54) we have

$$z_{t+1} = C_1 z_t + C_2 z_{t-1} + C_3 z_{t-2} + u_t + w_t. \quad (55)$$

Recall the definition of  $y_t$  and  $\zeta_t$ :

$$y_t = u_t + \sum_{i=1}^3 C_i y_{t-i}, \forall t \geq 0, \quad \zeta_t = w_t + \sum_{i=1}^3 C_i \zeta_{t-i}, \forall t \geq 0. \quad (56)$$

Then the original system could be translated to the compact form:

$$z_{t+1} = y_t + \zeta_t. \quad (57)$$

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**Algorithm 5:** Adaptive control via optimistic ROBD

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**Parameter:**  $\lambda > 0$

**Input:** Transition matrix  $A$  and control matrix  $B$

**for**  $t = 0$  **to**  $T - 1$  **do**

**Observe:**  $x_t, W_t$ , and  $q_{t:t+p-1}$

**if**  $t > 0$  **then**

$w_{t-1} \leftarrow \psi(x_t - Ax_{t-1} - Bu_{t-1})$

$\hat{z}_t \leftarrow \psi(x_t)$

    Define function  $h_t(z) = \frac{1}{2} \sum_{i=1}^d \left( \sum_{j=1}^{p_i} q_{t+j} \right) (z^{(i)})^2$

$\tilde{w}_t \leftarrow \arg \min_{w \in W_t} \min_z h_t(z) + \frac{\lambda}{2} \|z - w - \sum_{i=1}^p C_i \hat{z}_{t+1-i}\|^2$

$z_t \leftarrow \arg \min_z h_t(z) + \frac{\lambda}{2} \|z - \tilde{w}_t - \sum_{i=1}^p C_i \hat{z}_{t+1-i}\|^2$

$u_t \leftarrow z_t - \tilde{w}_t - \sum_{i=1}^p C_i \hat{z}_{t-i}$

**Output:**  $u_t$

**Output:**  $u_T = 0$

---

If the objective is given as (46), we have that

$$h_t(z) = \frac{1}{2}(q_{t+1} + q_{t+2}) \left( z^{(1)} \right)^2 + \frac{1}{2}(q_{t+1} + q_{t+2} + q_{t+3}) \left( z^{(2)} \right)^2.$$

Lastly, we want to point out that our reduction can work for more general forms of objectives than (46). Specifically, we only require that the objective can be transformed to

$$\sum_{t=0}^{T-1} h_t(z_{t+1}) + \frac{1}{2} \|u_t\|^2,$$

where  $h_t$  is a strongly convex and strongly smooth function that is observable before the agent picks  $u_t$ . Therefore, our reduction is more general than the reduction given in [24][Corollary 2], which considered the case when  $B = I$ . Notice that when  $B = I$ , we have  $p = 1$  and  $z_t = x_t$ .

## H A numerical issue in algorithm 3 and its solution

We have presented Algorithm 3 in as simple and intuitive a manner as possible but, as a result, there is a potential numerical issue that may arise for large horizon  $T$ . Although the sequence  $\{z_t\}$  is naturally bounded and we always have  $z_{t+1} = y_t + \zeta_t$ , the magnitudes of  $y_t$  and  $\zeta_t$  may grow exponentially since they accumulate the actions and the noises separately. However, this is not a fundamental problem, and there is a straightforward solution when the *Solver* in Algorithm 3 is Optimistic ROBD (Algorithm 2). The key insight is to solve optimization in  $\{u_t, w_t, z_t\}$  space, instead of  $\{y_t, \zeta_t, z_t\}$  space.

More specifically, when instantiated with Optimistic ROBD, we can rewrite the pseudo code of Algorithm 3 as Algorithm 5 so that variables  $y_t$  and  $\zeta_t$  are not involved. While equivalent to Algorithm 3 with Optimistic ROBD as the *Solver*, Algorithm 5 is numerically stable because we avoid the potentially unstable recursive calculation of  $\zeta_t$  and the sequence  $\{w_t\}$  is bounded.

## I Proofs for Appendix B

In this section, we establish the lower bound of the cost incurred by any linear controller and the upper bound of the offline optimal cost for different noise sequences. Specifically, we show a lower bound of the linear controller's cost on any noise sequence in Section I.1. We also give an upper bound of the offline optimal cost on any noise sequence in Section I.2. We further show that the upper bound of the offline optimal cost can be improved on two specific noise sequences in Section I.3 and I.4. Based on these results, we derive the lower bound of the competitive ratio for any linear control with respect to these noise sequences in Section I.5, I.6, and I.7.

### I.1 Lower bound of $\text{cost}(LC)$ for any noise sequence $\{w_t\}_{t=0}^T$

For any stable linear controller  $u_t = -kx_t$ , we have the following closed-loop dynamics

$$x_{t+1} = (a - k)x_t + w_t.$$

Our technique is to consider the sum of squares of two consecutive states  $x_{t+1}$  and  $x_t$ . Due to the constraints given by the dynamics and the linear controller itself,  $x_{t+1}$  and  $x_t$  cannot reach zero simultaneously. Specifically, we define  $\beta = a - k$ . Since the controller is stable, we have  $-1 < \beta < 1$ . Consider  $|x_{t+1}|^2 + |x_t|^2, \forall t \geq 0$ , we have:

$$\begin{aligned} & |x_{t+1}|^2 + |x_t|^2 \\ &= (\beta x_t + w_t)^2 + x_t^2 \\ &= (\beta^2 + 1)x_t^2 + 2\beta x_t w_t + w_t^2 \\ &= (\beta^2 + 1)\left(x_t + \frac{\beta}{\beta^2 + 1}w_t\right)^2 + \frac{1}{\beta^2 + 1}w_t^2 \\ &\geq \frac{1}{\beta^2 + 1}w_t^2 > \frac{w_t^2}{2}. \end{aligned}$$

Since  $\text{cost}(LC) = \sum_{t=0}^T qx_t^2 + u_t^2 = \sum_{t=0}^T (q + k^2)x_t^2$ ,  $\text{cost}(LC) \geq \sum_{t=0}^{T-1} (q + k^2)x_{t+1}^2$ . Then we will have

$$\text{cost}(LC) \geq \frac{1}{2} \sum_{t=0}^{T-1} (q + k^2)(x_{t+1}^2 + x_t^2) > \frac{q + k^2}{4} \sum_{t=0}^{T-1} w_t^2 > \frac{q + (a-1)^2}{4} \sum_{t=0}^{T-1} w_t^2, \quad (58)$$

where the last step comes from the fact  $-1 < a - k < 1$  and  $a > 1$ .

### I.2 Upper bound of $\text{cost}(OPT)$ for any $\{w_t\}_{t=0}^T$

When the controller has the full knowledge of the future noise sequence, the simplest strategy is to correct the noise greedily at the start of each time step so that the agent always stays at state 0.

Formally, for  $\text{cost}(OPT)$ , consider controller  $u_t = -w_t, \forall t \neq T$  and  $u_t = 0, t = T$ . Then we will have  $x_t = 0, \forall t \leq T$  so the cost would be  $\sum_{t=0}^{T-1} w_t^2$ . Therefore we have

$$\text{cost}(OPT) \leq \sum_{t=0}^{T-1} w_t^2.$$

### I.3 Upper bound of $\text{cost}(OPT)$ for $w_t = w$

Compared with Section I.2, since  $w_t$  is a constant case, we can balance the hitting cost and the switching cost by keeping the agent at non-zero stationary state that is close to the zero state.

Formally, we consider the following control strategy:

$$u_t = \begin{cases} \frac{u+w}{1-a} - w, & t = 0 \\ u, & t \geq 1, \end{cases}$$

where  $u$  is another constant. This controller yields  $x_t = \frac{u+w}{1-a}, t \geq 1$ . Then, we have

$$\text{cost}(u) = T\left(q\left(\frac{u+w}{1-a}\right)^2 + u^2\right) + \left(\frac{u+w}{1-a} - w\right)^2,$$

where the first part is a quadratic function w.r.t.  $u$  and the minimum is  $\frac{q}{q+(a-1)^2} \cdot Tw^2$  with minimizer  $u^* = \frac{-qw}{q+(a-1)^2}$ . Therefore we get

$$\text{cost}(OPT) \leq \frac{q}{q+(a-1)^2} Tw^2 + c_1,$$

where  $c_1 = \left(\frac{u^*+w}{1-a} - w\right)^2$  is a constant.

#### I.4 Upper bound of $\text{cost}(OPT)$ for $w_t = (-1)^t \cdot w$

Instead of keeping the noise  $w_t$  at a fixed value, we let it oscillate between two values  $w$  and  $-w$ . The resulting offline optimal controller will also oscillate between a positive state and a negative state. We show that in this case, the offline optimal cost can be even smaller than the one when  $w_t$  is fixed at  $w$  (Section I.3).

In this case the dynamics follows

$$\begin{cases} x_{2k+1} = ax_{2k} + u_{2k} + w, & k \geq 0 \\ x_{2k+2} = ax_{2k+1} + u_{2k+1} - w, & k \geq 0. \end{cases}$$

Consider controller class

$$u_t = \begin{cases} -\frac{u-w}{a+1} - w, & t = 0 \\ u, & t = 2k+1, k \geq 0 \\ -u, & t = 2k+2, k \geq 0. \end{cases}$$

Following this controller class, we have

$$x_t = \begin{cases} -\frac{u-w}{a+1}, & t = 2k+1, k \geq 0 \\ \frac{u-w}{a+1}, & t = 2k+2, k \geq 0. \end{cases}$$

For simplicity, assume  $T$  is an even number. Then, we have

$$\text{cost}(u) = T(q(\frac{u-w}{a+1})^2 + u^2) + (\frac{u-w}{a+1} + w)^2.$$

Similarly, the first part of  $\text{cost}(u)$  is a quadratic function and the minimum is  $\frac{q}{q+(a+1)^2} \cdot Tw^2$ . Therefore, we have

$$\text{cost}(OPT) \leq \frac{q}{q+(a+1)^2} Tw^2 + c_2,$$

where  $c_2$  is also a constant.

#### I.5 Lower bound of $\frac{\text{cost}(LC)}{\text{cost}(OPT)}$ for any $\{w_t\}_{t=0}^T$

Combining I.1 and I.2 we will have, for any  $\{w_t\}_{t=0}^T$ :

$$\frac{\text{cost}(LC)}{\text{cost}(OPT)} > \frac{\frac{q+(a-1)^2}{4} \sum_{t=0}^{T-1} w_t^2}{\sum_{t=0}^{T-1} w_t^2} = \frac{q+(a-1)^2}{4}.$$

#### I.6 Lower bound of $\frac{\text{cost}(LC)}{\text{cost}(OPT)}$ for $w_t = w$

Combining I.1 and I.3, we will have, if  $w_t = w$ :

$$\frac{\text{cost}(LC)}{\text{cost}(OPT)} > \frac{\frac{q+(a-1)^2}{4} Tw^2}{\frac{q}{q+(a-1)^2} Tw^2 + c_1}.$$

Therefore as  $T \rightarrow \infty$ ,  $\frac{\text{cost}(LC)}{\text{cost}(OPT)} \geq \frac{q+(a-1)^2}{4} \cdot \frac{q+(a-1)^2}{q}$ .

#### I.7 Lower bound of $\frac{\text{cost}(LC)}{\text{cost}(OPT)}$ for $w_t = (-1)^t \cdot w$

Combining I.1 and I.4, we will have, if  $w_t = (-1)^t \cdot w$ :

$$\frac{\text{cost}(LC)}{\text{cost}(OPT)} > \frac{\frac{q+(a-1)^2}{4} Tw^2}{\frac{q}{q+(a+1)^2} Tw^2 + c_2}.$$

Therefore as  $T \rightarrow \infty$ ,  $\frac{\text{cost}(LC)}{\text{cost}(OPT)} \geq \frac{q+(a-1)^2}{4} \cdot \frac{q+(a+1)^2}{q}$ .