Low-Rank Tensor Regression

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Tensor Regression

- Classical regression loss: $$\frac{1}{2}\sum_{i=1}^{N} \|y_i - <\hat{\mathscr{X}}_i, \hat{\mathscr{W}}> \|_F^2$$

Inner product:

$$< \hat{\mathscr{X}}, \hat{\mathscr{Y}} > \; = \sum_{i_0=0}^{I_0}\sum_{i_1=1}^{I_1}\cdots\sum_{i_N=0}^{I_N} x_{i_0 i_1 \cdots i_N} y_{i_0 i_1 \cdots i_N}$$

$$= vec(\hat{\mathscr{X}})^\top vec(\hat{\mathscr{Y}})$$

Tensor Regression

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- Note that this is equivalent to linear regression as:

$$< \hat{\mathscr{X}}_i, \hat{\mathscr{W}} > = vec(\hat{\mathscr{X}}_i)^\top vec(\hat{\mathscr{W}})$$

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$$\hat{\mathscr{W}} = [\![\mathbf{U}^{(0)}, \mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \cdots, \mathbf{U}^{(N)}]\!]$$

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- Add regularisation on the weights

Tensor Regression

$$\frac{1}{2}\sum_{i=1}^{N} \|y_i - < \hat{\mathscr{X}}_i, [\,|\,\mathbf{U}^{(0)}, \mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \cdots, \mathbf{U}^{(N)}\,|\,] > \|_F^2$$

- Add regularisation on the weights

$$\frac{\lambda}{2}\|\hat{\mathscr{W}}\|_F^2$$

Tensor Regression

$$\frac{1}{2}\sum_{i=1}^{N}\|y_i - < \hat{\mathscr{X}}_i, [\,|\,\mathbf{U}^{(0)}, \mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \cdots, \mathbf{U}^{(N)}\,|\,] > \|_F^2$$

- Add regularisation on the weights

$$\cancel{\frac{\lambda}{2}\|\hat{\mathscr{W}}\|_F^2} \qquad \frac{\lambda}{2}\sum_{k=0}^{N}\|\mathbf{U}^{(\mathbf{k})}\|_F^2$$

  - Separable regularisation term
  - Closed-form solution

Tensor Regression

$$\frac{1}{2}\sum_{i=1}^{N} \|y_i - < \hat{\mathscr{X}}_i, [\,|\,\mathbf{U}^{(0)}, \mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \cdots, \mathbf{U}^{(N)}\,|\,] > \|_F^2 + \frac{\lambda}{2}\sum_{k=0}^{N} \|\mathbf{U}^{(\mathbf{k})}\|_F^2$$

Tensor Regression

$$\frac{1}{2}\sum_{i=1}^{N} \|y_i - < \hat{\mathscr{X}}_i, [\,|\,\mathbf{U}^{(0)}, \mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \cdots, \mathbf{U}^{(N)}\,|\,] > \|_F^2 + \frac{\lambda}{2}\sum_{k=0}^{N} \|\mathbf{U}^{(\mathbf{k})}\|_F^2$$

$$\frac{1}{2}\sum_{i=1}^{N} \|y_i - Trace\left(\mathbf{U}^{(n)} \underbrace{\left(\mathbf{U}^{(0)} \odot \mathbf{U}^{(1)} \odot \cdots \odot \mathbf{U}^{(n-1)} \odot \mathbf{U}^{(n+1)} \odot \cdots \odot \mathbf{U}^{(N)}\right)^{\top} (\hat{\mathscr{X}}_i)^T_{[n]}}_{\boldsymbol{\Phi}^{\top}}\right) \|_F^2$$

$$+\frac{\lambda}{2}\sum_{k=0}^{N} \|\mathbf{U}^{(\mathbf{k})}\|_F^2$$

$$\boxed{\left(\,\textbf{Trace}(\mathbf{A}\mathbf{B}^{\top}) = vec(\mathbf{A})^{\top} vec(\mathbf{B})\,\right)}$$

Tensor Regression

$$\frac{1}{2}\sum_{i=1}^{N} \|y_i - < \hat{\mathscr{X}}_i, [\,|\,\mathbf{U}^{(0)}, \mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \cdots, \mathbf{U}^{(N)}\,|\,] > \|_F^2 + \frac{\lambda}{2}\sum_{k=0}^{N} \|\mathbf{U}^{(\mathbf{k})}\|_F^2$$

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$$+\frac{\lambda}{2}\sum_{k=0}^{N} \|\mathbf{U}^{(\mathbf{k})}\|_F^2$$

- Build $\mathbf{\Phi}$ so that row k contains

$$(\hat{\mathscr{X}}_k)_{[n]}\left(\mathbf{U}^{(0)} \odot \mathbf{U}^{(1)} \odot \quad \cdots \quad \odot \mathbf{U}^{(n-1)} \odot \mathbf{U}^{(n+1)} \odot \quad \cdots \quad \odot \mathbf{U}^{(N)}\right)$$

Tensor Regression

$$\frac{1}{2}\sum_{i=1}^{N} \|y_i - < \hat{\mathscr{X}}_i, [\,|\,\mathbf{U}^{(0)}, \mathbf{U}^{(1)}, \mathbf{U}^{(2)}, \cdots, \mathbf{U}^{(N)}\,|\,] > \|_F^2 + \frac{\lambda}{2}\sum_{k=0}^{N}\|\mathbf{U}^{(\mathbf{k})}\|_F^2$$

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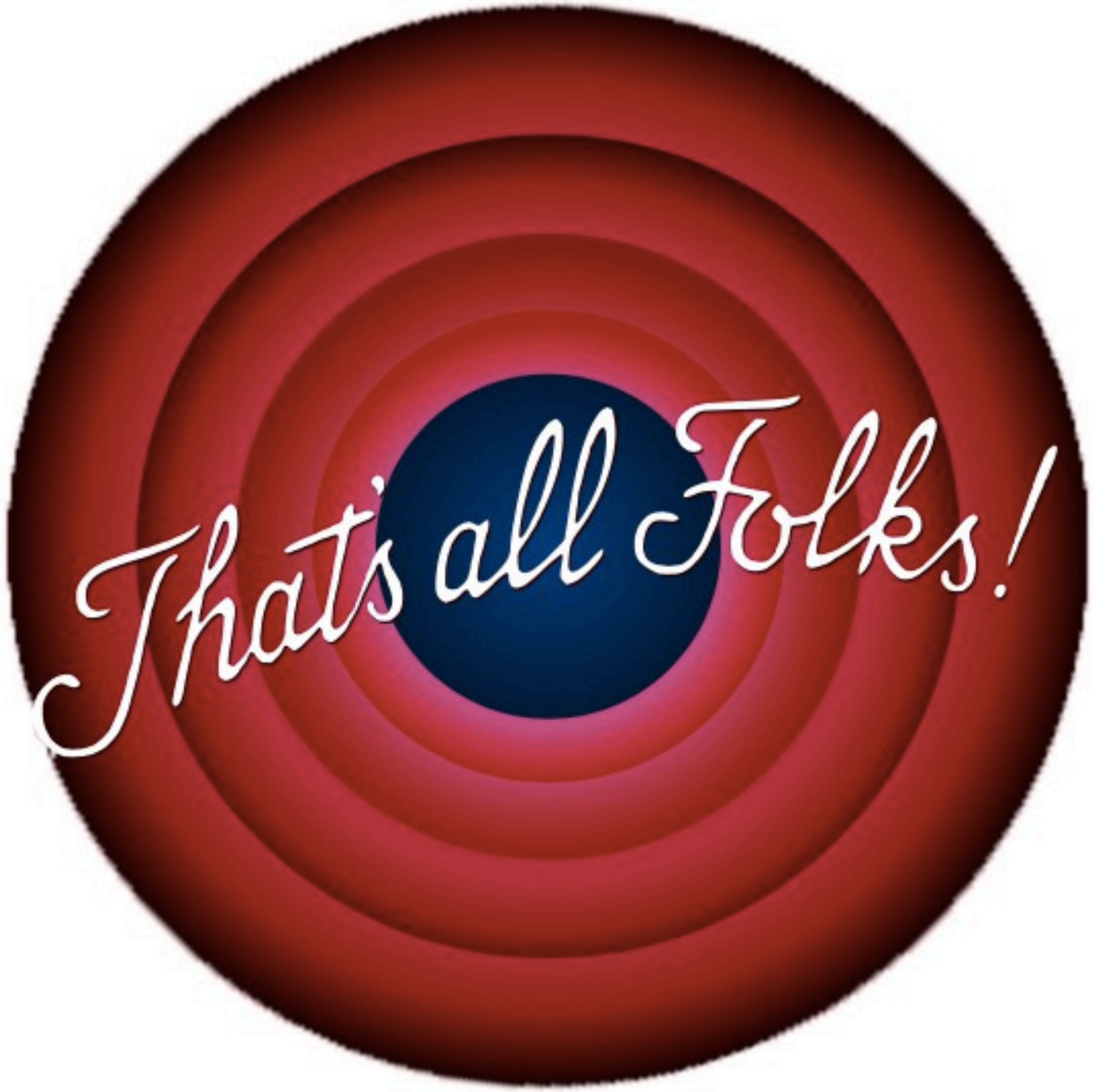
$$+\frac{\lambda}{2}\sum_{k=0}^{N}\|\mathbf{U}^{(\mathbf{k})}\|_F^2$$

- Build $\mathbf{\Phi}$ so that row k contains

$$(\hat{\mathscr{X}}_i)_{[n]}\left(\mathbf{U}^{(0)} \odot \mathbf{U}^{(1)} \odot \quad \cdots \quad \odot \mathbf{U}^{(n-1)} \odot \mathbf{U}^{(n+1)} \odot \quad \cdots \quad \odot \mathbf{U}^{(N)}\right)$$

$$vec(\mathbf{U}^{(n)}) = (\mathbf{\Phi}^{\top}\mathbf{\Phi} + \lambda\mathbf{I})^{-1}\mathbf{\Phi}^{\top}\mathbf{y}$$



Any questions?

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