数值分析内部课程习题解答

第五章

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1 The interpolation problem

Solution 1.1. 代码见 Gitee 仓库.

Solution 1.2. 代码见 Gitee 仓库. 一般是样条插值比多项式插值可信，因为多项式插值在两端附近的振荡会加剧，从而导致不可信.

Solution 1.3. 代码见 Gitee 仓库.

Solution 1.4. 这题启发我们先在简单的区间上考虑问题，再线性变换到一个小区间上.

(a) 代入验证即可.

(b) 要求节点从 $s=-1,0,1$ 变换到 $x=x_0-h,\ x_0,\ x_0+h$，列出方程组并求解，有

$$\begin{cases} A(x_0-h)+B=-1, \\ Ax_0+B=0, \end{cases} \quad\implies\quad A=\frac{1}{h},\ B=-\frac{x_0}{h}.$$

故线性变换为

$$s=\frac{x-x_0}{h}.$$

(c) 题目本身给的 $q(s)$ 就是一个分段二次插值，所以我们只需要将线性变换作用到 s 上，即可得到在新的小区间上的分段二次插值函数.

将上一小问得到的线性变换 $s=(x-x_0)/h$ 代入 q，有

$$\tilde{q}(x)=a\frac{(x-x_0)(x-x_0-h)}{2h^2}-b\frac{(x-x_0-h)(x-x_0+h)}{h^2}+c\frac{(x-x_0)(x-x_0+h)}{2h^2}.$$

容易验证这就是小区间节点 $x=x_0-h,\ x_0,\ x_0+h$ 对应的分段二次插值

$$\tilde{q}(x_0-h)=a,\qquad \tilde{q}(x_0)=b,\qquad \tilde{q}(x_0+h)=c.$$

Solution 1.5. 根据定理 5.1.7，先算基函数的范数，再得到插值算子的条件数的界.

在标准区间下，$s\in[-1,1]$，基函数为

$$\phi_{-1}(s)=\tfrac{1}{2}s(s-1),\qquad \phi_0(s)=1-s^2,\qquad \phi_1(s)=\tfrac{1}{2}s(s+1)$$

在区间 $[-1,1]$ 的无穷范数皆为 1。根据定理 5.1.7 计算线性插值算子的条件数界

$$\underbrace{\max_k\|\phi_k\|_\infty}_{=1}\le\kappa\le\underbrace{\sum_k\|\phi_k\|_\infty}_{=3\times 1},$$

于是

$$1\ \le\ \kappa\ \le\ 3.$$

由于从 s 到 x 的变换只是平移与缩放，不改变基函数的最大绝对值，故此不等式同样适用于任意 $h>0$、中心节点 x_0 的二次插值.

2 Piecewise linear interpolation

Solution 2.1. 代码见 Gitee 仓库.

Solution 2.2. 理解帽函数的定义及其积分方法.

(a) 题中给定节点的帽函数为

$$H(x) = \begin{cases} x+1, & x \in [-1,0], \\ 1-x, & x \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

(b) 记 $Q(x) = \int_{x-1}^{x} H(t)\,\mathrm{d}t$，那么计算积分可分为如下几类讨论

$$x \leq 1, \qquad -1 \leq x \leq 0, \qquad 0 \leq x \leq 1, \qquad 1 \leq x \leq 2, \qquad x \geq 2,$$

分别积分，得到

$$Q(x) = \begin{cases} 0, & x \leq -1 \\ \frac{1}{2}(x+1)^2, & -1 \leq x \leq 0 \\ -x^2 + x + \frac{1}{2}, & 0 \leq x \leq 1 \\ \frac{1}{2}(x-2)^2, & 1 \leq x \leq 2 \\ 0, & x \geq 2 \end{cases}.$$

(c) 容易画出分段二次函数的草图.

(d) Q 连续. 检查 Q'，分段对其求导，发现在节点 $\{-1,0,1,2\}$ 也是连续的，故 Q' 连续. 但是进一步求导发现 Q'' 有跳跃，所以不连续.

Solution 2.3. 利用定理 5.2.7 估计误差上界.

在区间 $[3.1,4]$ 等距节点，小区间长度 $h=0.1$. 若采用分段线性插值，最大误差满足

$$\|f-p_n\|_\infty = \max_{x\in[3.1,4]} |f(x)-p_n(x) \leq \|f''\|_\infty h^2.$$

对 $f(x)=\ln x$，有 $f''(x) = -1/x^2$，最大绝对值出现在最小节点 $x=3.1$，即

$$|f''(x)| \leq 1/3.1^2,$$

所以

$$\|f-p\|_\infty \leq 1/3.1^2 \times 0.1^2,$$

因此线性插值表查 $\ln x$ 的误差上界约为 $1/3.1^2 \times 0.1^2$.

Solution 2.4. 根据式 5.2.4 和式 5.2.5，并由线性插值函数的性质可证.

设 H_k 是分片线性的基函数，且满足

$$H_k(t_i) = \begin{cases} 1, & i = k \\ 0, & \text{otherwise} \end{cases}$$

那么分片线性插值函数可表示为

$$p(x) = \sum_{k=0}^{n} y_k H_k(x),$$

且满足 $p(t_i) = y_i$. 令 $y_k = 1, k = 0, 1, \cdots, n$，则插值得到的函数为

$$p(x) = \sum_{k=0}^{n} H_k(x), \qquad p(t_i) = 1,$$

由线性插值的性质，得到 $p(x) \equiv 1$，即

$$\sum_{k=0}^{n} H_k(x) = 1, \qquad \forall x \in [t_0, t_n]$$

这就是分区统一性 (partition of unity).

Solution 2.5. 根据中值定理证明 (a) 和 (b)，再代入定理

(a) 考虑 $x \in (t_k, t_{k+1})$，由微分中值定理

$$f'(s) = \frac{f(x) - f(t_k)}{x - t_k}, \qquad \exists s \in (t_k, x)$$

于是

$$f(x) = y_k + (x - t_k) f'(s). \qquad \exists s \in (t_k, t_{k+1})$$

(b) 由 (a) 知存在 $u \in (t_k, t_{k+1})$ 使得

$$f'(u) = \frac{f(y_{k+1}) - f(y_k)}{t_{k+1} - t_k},$$

取 $s \in (t_k, t_{k+1})$，进而在 s 和 u 之间存在一点 v 使得

$$f''(v) = \frac{f'(s) - f'(u)}{s - u},$$

即

$$f'(s) = \frac{y_{k+1} - y_k}{t_{k+1} - t_k} + (s - u) f''(v). \qquad \exists u, v \in (t_k, t_{k+1})$$

(c) 利用插值公式 (式 5.2.1)

$$p(x) = y_k + \frac{y_{k+1} - y_k}{t_{k+1} - t_k}(x - t_k)$$

代入式 5.2.7

$$\begin{aligned}
\|f - p_n\| &\leq \max_{k} \max_{x\in[t_k,t_{k+1}]} \left| f(x) - y_k - \frac{y_{k+1} - y_k}{t_{k+1} - t_k}(x - t_k) \right| \\
&= \max_{k} \max_{x,s\in[t_k,t_{k+1}]} \left| f'(s)(x - t_k) - \frac{y_{k+1} - y_k}{t_{k+1} - t_k}(x - t_k) \right| \\
&\leq \max_{k} \max_{x,s\in[t_k,t_{k+1}]} |(x - t_k)| \left| f'(s) - \frac{y_{k+1} - y_k}{t_{k+1} - t_k} \right| \\
&\leq \max_{k} \max_{x,s,u,v\in[t_k,t_{k+1}]} |(x - t_k)||(s - u)f''(v)| \\
&= h^2 \max_{v\in[a,b]} f''(v),
\end{aligned}$$

定理 5.2.7 得证.

3 Cubic splines

Solution 3.1. 我们按照教材 spinterp 程序代码，展示样条插值矩阵的组装. 这里仅给出 (a) 的过程，剩余的答案可以用 Gitee 仓库的代码验证. 我们看到条件

$$f(x) = \cos\big(\pi^2 x^2\big), \qquad \mathbf{t} = \big[t_0, t_1, t_2\big] = [-1, 1, 4], \qquad \mathbf{h} = \big[h_1, h_2\big] = \big[2, 3\big],$$

设样条线性系统形为 $\mathbf{Az} = \mathbf{v}$，根据教材，有

$$\mathbf{A} = \left[\begin{array}{c|cccccccc}
\text{(左端点插值)} & \mathbf{I} & & \mathbf{0} & & \mathbf{0} & & \mathbf{0} & \\
 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\text{(右端点插值)} & \mathbf{I} & & \mathbf{H} & & \mathbf{H}^2 & & \mathbf{H}^3 & \\
 & 1 & 0 & 2 & 0 & 4 & 0 & 8 & 0 \\
 & 0 & 1 & 0 & 3 & 0 & 9 & 0 & 27 \\
\hline
\text{(一阶导连续)} & \mathbf{0} & & \mathbf{EJ} & & 2\mathbf{EH} & & 3\mathbf{EH}^2 & \\
 & 0 & 0 & 1 & -1 & 4 & 0 & 12 & 0 \\
\hline
\text{(二阶导连续)} & \mathbf{0} & & \mathbf{0} & & \mathbf{EJ} & & 3\mathbf{EH} & \\
 & 0 & 0 & 0 & 0 & 1 & -1 & 6 & 0 \\
\hline
\text{(Not-a-knot 条件，}d_1 + d_2 = 0\text{)} & & & & & & & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{array}\right].$$

$$
\mathbf{z} = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ c_1 \\ c_2 \\ d_1 \\ d_2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} y_0 \\ y_1 \\ y_1 \\ y_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f(-1) \\ f(1) \\ f(1) \\ f(4) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
$$

这里需要注意，上面 **A** 的最后两行是一样的，原因是仅有三个节点，即两个小区间. not-a-knot 条件的意思是首尾两个小区间上的三阶导与其相邻区间的相等，即

$$
d_1 = d_2, \qquad d_{n-1} = d_n,
$$

而当仅有两个小区间时，$d_1 = d_2$ 和 $d_{n-1} = d_n$ 是一样的. 如此一来就会导致方程组的解不唯一，我们在实际应用时可以考虑采用自然边界 $S_1''(t_0) = 0$ 或 $S_n''(t_n) = 0$，又或者退化到整段只考虑一个三次多项式插值 (章节 5.1 的第 4 题). 当节点数 ≥ 4 时则不会有该问题.

Solution 3.2. 代码见 Gitee 仓库.

Solution 3.3. 代码见 Gitee 仓库.

Solution 3.4. 代码见 Gitee 仓库.

Solution 3.5. 代码见 Gitee 仓库.

Solution 3.6. 参考教材的推导. 现在我们希望推导分段二次插值的线性系统，对给定的 $n+1$ 个节点，我们有已知条件 $\{t_k, y_k\}$, $k = 0, \cdots n$. 分片二次多项式形如

$$
S_k(x) = a_k + b_k(x - t_{k-1}) + c_k(x - t_{k-1})^2, \qquad k = 1, \cdots, n
$$

这意味着我们需要 $3n$ 个约束条件才能求解所有的系数. 考虑每个小区间的左端点和右端点，有 $2n$ 个约束条件

$$
a_k = y_{k-1}, \qquad a_k + b_k h_k + c_k h_k^2 = y_k. \qquad k = 1, \cdots, n
$$

考虑每个小区间之间的连接，要求一阶导连续，有 $n-1$ 个约束条件

$$
b_k + 2c_k h = b_{k+1}, \qquad k = 1. \cdots, n-1
$$

至此还缺 1 个条件，只需考虑边界即可.

Solution 3.7. 参考教材处理边界的做法.

(a) 设 $S(x)$ 满足周期边界条件，则有

$$
S(t_0) = S(t_n), \qquad S'(t_0) = S'(t_n), \qquad S''(t_0) = S''(t_n),
$$

由于第一个条件已经在考虑每个节点的取值时使用过了，所以我们关注到后两个条件，展开有

$$b_0 = b_n + 2c_n h_n + 3d_n h_n^2, \qquad 2c_0 = 2c_n + 6d_n h_n.$$

(b) 代码见 Gitee 仓库.

4 Finite differences

Solution 4.1. 直接代入验证即可，过程略.

Solution 4.2. 按照题目提示即可.

(a) 令 $g(x) = f(-x)$，那么对 g 做后向差分 (形如 $g(0) - g(-h)$) 就等价于对 f 做前向差分 (形如 $f(0) - f(h)$). 于是容易由表格 5.4.2 得到

order	0	$-h$	$-2h$	$-3h$	$-4h$
1	1	-1			
2	$\frac{1}{2}$	-2	$\frac{3}{2}$		
3	$-\frac{1}{3}$	$\frac{3}{2}$	-3	$\frac{11}{6}$	
4	$\frac{1}{4}$	$-\frac{4}{3}$	3	-4	$\frac{25}{12}$

(b) 代码见 Gitee 仓库.

Solution 4.3. 代码见 Gitee 仓库.

Solution 4.4. 代码见 Gitee 仓库.

Solution 4.5. 代码见 Gitee 仓库.

Solution 4.6. 写出由一阶导推二阶导的差分公式，然后直接代入前向和后向的差分公式近似一阶导，即可得到二阶中心差分公式，过程略.

Solution 4.7. (a) For the centered formula

$$R(h) = \big[f(h) - f(-h)\big]/(2h),$$

apply L’Hôpital’s Rule:

$$\lim_{h\to 0} R(h) = \lim_{h\to 0} \frac{f(h) - f(-h)}{2h} = \lim_{h\to 0} \frac{f'(h) + f'(-h)}{2} = \frac{f'(0) + f'(0)}{2} = f'(0),$$

which proves convergence as $h \to 0$.

(b) Writing the general stencil (5.4.1)

$$D(h) = \frac{1}{h} \sum_{k=-p}^{q} a_k f(kh),$$

and assuming $f \in C^1$, we expand each $f(kh)$ about $h = 0$:

$$D(h) = \frac{1}{h}\left[\sum a_k f(0) + h \sum k a_k f'(0) + O(h^2)\right].$$

For $D(h) \to f'(0)$ as $h \to 0$ we require the two conditions as follow

$$\sum_{k=-p}^{q} a_k = 0, \qquad \sum_{k=-p}^{q} k\, a_k = 1,$$

which ensure that the $1/h$ term cancels while the constant term equals $f'(0)$. Higher-order accuracy enforces additional conditions on the weights.

5 Convergence of finite differences

Solution 5.1. Please see the codes in Gitee.

Solution 5.2. Note that the context $h = 0$ should be $x = 0$, which is a typo. We first write the truncation error of the formula (5.4.3) as follows

$$\begin{aligned}
\tau_f(h) &= f'(0) - \frac{f(0) - f(-h)}{h} \\
&= f'(0) - h^{-1}[f(0) - (f(0) - hf'(0) + \frac{1}{2}h^2 f''(0) - \frac{1}{6}h^3 f^{(3)}(0) + \cdots)] \\
&= f'(0) - h^{-1}[f(0) - (f(0) - hf'(0) + \frac{1}{2}h^2 f''(0) - \frac{1}{6}h^3 f^{(3)}(0) + O(h^4))] \\
&= \frac{1}{2}hf''(0) - \frac{1}{6}h^2 f'''(0) + O(h^3).
\end{aligned}$$

Then the term $\frac{1}{2}hf''(0) - \frac{1}{6}h^3 f'''(0)$ is first two nonzero terms of $\tau_f(h)$.

Solution 5.3. We first write the truncation error of the formula as follows

$$\begin{aligned}
\tau_f(h) &= f'(0) - \frac{-\frac{3}{2}f(0) + 2f(h) - \frac{1}{2}f(2h)}{h} \\
&= f'(0) - h^{-1}[-\tfrac{3}{2}f(0) + 2(f(0) + hf'(0) + \frac{1}{2}h^2 f''(0) + \cdots) \\
&- \frac{1}{2}(f(0) + 2hf'(0) + \frac{1}{2}4h^2 f''(0) + \cdots)] \\
&= f'(0) - h^{-1}[-\tfrac{3}{2}f(0) + 2(f(0) + hf'(0) + \frac{1}{2}h^2 f''(0) + O(h^3)) \\
&- \frac{1}{2}(f(0) + 2hf'(0) + \frac{1}{2}4h^2 f''(0) + O(h^3))] \\
&= f'(0) - h^{-1}[2(hf'(0) + \frac{1}{2}h^2 f''(0) + O(h^3)) - \frac{1}{2}(2hf'(0) + \frac{1}{2}4h^2 f''(0) + O(h^3))] \\
&= \frac{1}{2}h^2 f''(0) + O(h^2).
\end{aligned}$$

Then the term $\frac{1}{2}h^2 f''(0)$ is the first nonzero term of $\tau_f(h)$.

Solution 5.4. We first write the truncation error of the formula as follows

$$\begin{aligned}
\tau_f(h) &= f'(0) - \frac{-\frac{11}{6}f(0) + 3f(h) - \frac{3}{2}f(2h) + \frac{1}{3}f(3h)}{h} \\
&= f'(0) - h^{-1}[-\tfrac{11}{6}f(0) + 3(f(0) + hf'(0) + \frac{1}{2}h^2 f''(0) + \frac{1}{6}h^3 f^{(3)}(0) + \frac{1}{24}h^4 f^{(4)}(0) + \cdots) \\
&- \frac{3}{2}(f(0) + 2hf'(0) + \frac{1}{2}4h^2 f''(0) + \frac{1}{6}8h^3 f^{(3)}(0) + \frac{1}{24}16h^4 f^{(4)}(0) + \cdots) \\
&+ \frac{1}{3}(f(0) + 3hf'(0) + \frac{1}{2}9h^2 f''(0) + \frac{1}{6}27h^3 f^{(3)}(0) + \frac{1}{24}81h^4 f^{(4)}(0) + \cdots)] \\
&= -h^{-1}[3(\frac{1}{24}h^4 f^{(4)}(0) + O(h^5)) - \frac{3}{2}(\frac{1}{24}16h^4 f^{(4)}(0) + O(h^5)) + \frac{1}{3}(\frac{1}{24}81h^4 f^{(4)}(0) + O(h^5))] \\
&= -\frac{1}{4}h^3 f^{(4)}(0) + O(h^4).
\end{aligned}$$

Then the term $-\frac{1}{4}h^3 f^{(4)}(0)$ is the first nonzero term of $\tau_f(h)$.

Solution 5.5. The approximation formula of $f''(0)$ is

$$f''(0) \approx \frac{f(-h) - 2f(0) + f(h)}{h^2}. \tag{1}$$

By Taylor expansion at $x = 0$, we have

$$\begin{aligned}
f(-h) &= f(0) - hf'(0) + \frac{h^2}{2}f''(0) - \frac{h^3}{6}f^{(3)}(0) + \frac{h^4}{24}f^{(4)}(0) + O(h^5), \\
f(h) &= f(0) + hf'(0) + \frac{h^2}{2}f''(0) + \frac{h^3}{6}f^{(3)}(0) + \frac{h^4}{24}f^{(4)}(0) + O(h^5).
\end{aligned} \tag{2}$$

Combining Eqs. (1) and (2), we obtain the truncation error

$$\begin{aligned}
\tau_f(h) &= f''(0) - \frac{f(-h) - 2f(0) + f(h)}{h^2} \\
&= -\frac{h^2}{12}f^{(4)}(0) + O(h^3).
\end{aligned}$$

Thus, we conclude that the formula (5.4.7) is second order accurate.

Solution 5.6. (a) If $p = q = 2$, then the approximation of $f'(x)$ is as follows

$$f'(x) \approx h^{-1}(a_{-2}f(x-2h) + a_{-1}f(x-h) + a_0 f(x) + a_1 f(x+h) + a_2 f(x+2h). \tag{3}$$

Using Taylor expansion, we have

$$\begin{aligned}
f(x-2h) &= f(x) - 2hf'(x) + \frac{4h^2}{2}f''(x) - \frac{8h^2}{6}f^{(3)}(x) + \frac{16h^4}{24}f^{(4)}(x) + O(h^5), \\
f(x+2h) &= f(x) + 2hf'(x) + \frac{4h^2}{2}f''(x) + \frac{8h^2}{6}f^{(3)}(x) + \frac{16h^4}{24}f^{(4)}(x) + O(h^5), \\
f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^2}{6}f^{(3)}(x) + \frac{h^4}{24}f^{(4)}(x) + O(h^5), \\
f(x-h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^2}{6}f^{(3)}(x) + \frac{h^4}{24}f^{(4)}(x) + O(h^5),
\end{aligned} \tag{4}$$

Combining Eqs. (3) and (4), we obtain the system of equations as follows

$$\begin{cases} a_{-2}+a_{-1}+a_0+a_1+a_2 & = b_0 \\ -2a_{-2}-a_{-1}+a_1+2a_2 & = b_1 \\ 2a_{-2}+\frac{1}{2}a_{-1}+\frac{1}{2}a_1+2a_2 & = b_2 \\ -\frac{4}{3}a_{-2}-\frac{1}{6}a_{-1}+\frac{1}{6}a_1+\frac{4}{3}a_2 & = b_3 \\ \frac{2}{3}a_{-2}+\frac{1}{24}a_{-1}+\frac{1}{24}a_1+\frac{2}{3}a_2 & = b_4 \end{cases}$$

Then we set

$$b_0=0, \quad b_1=1, \quad b_2=0, \quad b_3=0, \quad b_4=0.$$

Therefore, we have the following system

$$\begin{cases} a_{-2}+a_{-1}+a_0+a_1+a_2 & = 0 \\ -2a_{-2}-a_{-1}+a_1+2a_2 & = 1 \\ 2a_{-2}+\frac{1}{2}a_{-1}+\frac{1}{2}a_1+2a_2 & = 0 \\ -\frac{4}{3}a_{-2}-\frac{1}{6}a_{-1}+\frac{1}{6}a_1+\frac{4}{3}a_2 & = 0 \\ \frac{2}{3}a_{-2}+\frac{1}{24}a_{-1}+\frac{1}{24}a_1+\frac{2}{3}a_2 & = 0 \end{cases}$$

(b) We can do the verification by using Taylor expansion.

(c) We can compute a_k by Taylor expansion.

6 Numerical integration

Solution 6.1. Please see the codes in Gitee.

Solution 6.2. Note that the finite difference formula used here should be *line 2,3 in Example 5.4.4*, rather than Equation 5.4.8. By using the finite difference, we can approximate $f'(a)$, $f'(b)$ as follows

$$f'(a)=f'(t_0)=\frac{1}{h}[-\frac{3}{2}f(t_0)+2f(t_1)-\frac{1}{2}f(t_2)],$$
$$f'(b)=f'(t_0)=\frac{1}{h}[\frac{3}{2}f(t_n)-2f(t_{n-1})+\frac{1}{2}f(t_{n-2})].$$

Then we have that

$$f(b)-f'(a)=\frac{1}{24}[3(f(t_n)+f(t_0))-4(f(t_{n-1})+f(t_1))+(f(t_{n-2}+f(t_2))]. \tag{5}$$

Hence, combining (5) and the Euler-Maclaurin error expansion (5.6.6), we have the Gregory integration formula $G_f(h)$ as

$$G_f(h)=T_f(h)-\frac{1}{24}[3(f(t_n)+f(t_0))-4(f(t_{n-1})+f(t_1))+(f(t_{n-2}+f(t_2))]. \tag{6}$$

Solution 6.3. Please see the codes in Gitee.

Solution 6.4. (a) We only need to substitute the values a, b, and c into the function f, and verify whether $f(x)$ equals the given value.

(b) We now compute the integral

$$\begin{aligned}\int_{-h}^{h} p(s)\,ds &= \beta h + \frac{\gamma-\alpha}{4h}h^2 + \frac{\alpha-2\beta+\gamma}{6h^2}h^3 \\ &\quad - (-\beta h + \frac{\gamma-\alpha}{4h}(-h)^2 + \frac{\alpha-2\beta+\gamma}{6h^2} - h^3) \\ &= 2\beta h + \frac{\alpha-2\beta+\gamma}{3}h \\ &= \frac{\alpha+4\beta+\gamma}{3}h.\end{aligned}$$

(c) We can approximate $\int_{t_i}^{t_{i+1}} f(x)\,dx$ as follows

$$\begin{aligned}\int_{t_{i-1}}^{t_{i+1}} f(x)\,dx &\approx \int_{t_{i-1}}^{t_{i+1}} p(x)\,dx \\ &= \int_{-h}^{h} p(s+t_i)\,ds \\ &= \frac{1}{3}[p(t_i-h)+4p(t_i)+p(t_i+h)]h \\ &= \frac{1}{3}[f(t_{i-1})+4f(t_i)+f(t_{i+1})].\end{aligned}$$

Here, we assume that $\alpha = t_{i-1}, \beta = t_i, \gamma = t_{i+1}$.

(d) Note that $n = 2m$ is even, then we can derive the Simpson's formula as follows

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \sum_{i=1}^{2n-1}\int_{t_{i-1}}^{t_{i+1}} f(x)\,dx \\ &= \sum_{i=1}^{2n-1}\frac{1}{3}[f(t_{i-1})+4f(t_i)+f(t_{i+1})] \\ &= \frac{1}{3}(f(t_0)+4f(t_1)+2f(t_2)+4f(t_3)+2f(t_4)+\cdots \\ &\qquad +2f(t_{n-2})+4f(t_{n-1})+f(t_n)).\end{aligned}$$

Here, $\alpha = t_{2k-2}, \beta = t_{2k-1}, \gamma = t_{2k}$.

Solution 6.5. We first compute $T_f(n), T_f(\frac{1}{2}n)$ as follows

$$\begin{aligned}T_f(n) = T_f(2m) &= h\left[\frac{1}{2}f(t_0) + \sum_{k=1}^{2m-1} f(t_k) + \frac{1}{2}f(t_{2m})\right], \\ T_f(\frac{1}{2}n) = T_f(m) &= 2h\left[\frac{1}{2}f(t_0) + \sum_{k=1}^{m-1} f(t_{2k}) + \frac{1}{2}f(t_{2m})\right].\end{aligned}$$

Next, we compute $S_f(n) = S_f(2m)$ as follows

$$\begin{aligned}
S_f(2m) &= \frac{1}{3}(4T_f(2m) - T_f(m)) \\
&= \frac{1}{3}\left(4h\left[\frac{1}{2}f(t_0) + \sum_{k=1}^{2m-1} f(t_k) + \frac{1}{2}f(t_{2m})\right] - 2h\left[\frac{1}{2}f(t_0) + \sum_{k=1}^{m-1} f(t_{2k}) + \frac{1}{2}f(t_{2m})\right]\right), \\
&= \frac{1}{3}\left(h\left[f(t_0) + 4\sum_{k=1}^{m} f(t_{2k-1}) + 2\sum_{k=1}^{m-1} f(t_{2k}) + f(t_{2m})\right]\right),
\end{aligned}$$

which equals the approximation formula in (d) when $n = 2m$.

Solution 6.6. Please see the codes in Gitee.

Solution 6.7. Please see the codes in Gitee.

Solution 6.8. Please see the codes in Gitee.

Solution 6.9. We first compute the $R_f(8n)$ approximation of I as follows

$$I = R_f(8n) + \frac{1}{64}c_6 n^{-6} + \frac{1}{256}c_8 n^{-8} + \cdots .$$

Then combining $64R_f(8n)$ and $R_f(4n)$, we obtain that

$$I = \frac{1}{63}(64R_f(8n) - R_f(4n)) + d_8 n^{-8} + d_{10} n^{-10} + \cdots ,$$

which can achieve an eighth-order accurate.

7 Adaptive integration

Solution 7.1. Please see the codes in Gitee.