Representation homology and some computations with unipotent coefficients

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Representations

Let V be a k-vector space. When we say "a representation V", there are generally three settings:

1. a representation of a discrete group π (later we shall use G to denote an algebraic group, and here we use Greek letters to denote a discrete group) is a group homomorphism
$$\pi \to GL(V),$$
2. representation of a Lie algebra, and
3. representation of an associative algebra.

For any case, we can construct a space (or one could call it a variety), universally parameterizing all representations. We only use the first definition here.

Coefficients (group schemes)

An affine group scheme G ovr k is a functor
$$G : k-\mathbf{CommAlg} \to \mathbf{Gp}$$
that is representable.

For example, GL_2 is the affine group scheme
$$\begin{aligned} k-\mathbf{CommAlg} &\to \mathbf{Gp} \\ A &\mapsto GL_2(A) \end{aligned}$$
where $GL_2(A)$ is the group of 2×2 matrices with determinant invertible. The representative is $k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, (\det X)^{-1}]$, i.e.
$$GL_2 \cong \mathrm{Hom}(k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, (\det X)^{-1}], -).$$

Parametrizing representations

Given a (discrete) group π and an affine group scheme G over k, the functor
$$\begin{aligned} \mathrm{Rep}_G(\pi) : k-\mathbf{CommAlg} &\to \mathbf{Set} \\ A &\mapsto \mathrm{Hom}_{\mathbf{Gp}}(\pi, G(A)) \end{aligned}$$
is representable. The representative is denoted by $(\pi)_G$. In other words, there is a natural isomorphism
$$\mathrm{Hom}_{k-\mathbf{CommAlg}}((\pi)_G, A) \cong \mathrm{Hom}_{\mathbf{Gp}}(\pi, G(A)) \tag{1}$$
for any commutative k-algebra A.

The affine scheme $(\mathrm{Spec}\ (\pi)_G)$ associated to the ring $(\pi)_G$ is called the **representation scheme**.

This gives a functor
$$(-)_G : \mathbf{Gp} \to k-\mathbf{CommAlg},$$
which is the left adjunction of $G : k-\mathbf{CommAlg} \to \mathbf{Gp}$.

A running example

Let T^2 be the topological torus, and let $\pi = \pi_1(T^2) = \mathbb{Z}\times\mathbb{Z} = \langle x, y \mid xy = yx\rangle$, and let $G = U_3 = \mathrm{Spec}\ k\left[\begin{pmatrix} 1 & x_{1,2} & x_{1,3} \\ & 1 & x_{2,3} \\ & & 1 \end{pmatrix}\right]$. Then
$$(\pi)_{U_3} \cong \frac{k[x_{1,2}, x_{1,3}, x_{2,3}, y_{1,2}, y_{1,3}, y_{2,3}]}{XYX^{-1}Y^{-1} = I}.$$
This can be interpreted as all pairs of matrices $(X, Y) \in U_3$ s.t. $XY = YX$.

The scheme associated to this is called the commuting variety, denoted by $C(U_n)$.

Let's derive things

One could extend the adjunction
$$(-)_G : \mathbf{Gp} \to k-\mathbf{CommAlg},$$
levelwise to simplicial setting, then one has adjunction
$$(-)_G : s\mathbf{Gp} \to s(k-\mathbf{CommAlg}) : G$$
between two model categories, and

Theorem, [BRY22]

For the extended adjunction
$$(-)_G : s\mathbf{Gp} \to s(k-\mathbf{CommAlg}) : G,$$
$(-)_G$ admits a (total) left derived functor $\mathbb{L}(-)_G$, commuting with homotopy colimits, which could be computed by applying $(-)_G$ to a cofibrant replacement.

More settings

There is a chain of Quillen equivalences
$$\mathbf{Top}_{*,0} \simeq s\mathbf{Set}_0 \simeq s\mathbf{Gp}$$
where $s\mathbf{Set}_0$ is the (sub)category consisting of all reduced simplicial sets and $\mathbf{Top}_{*,0}$ is the (sub)category of all pointed, connected topological spaces.

So given an affine group scheme G over k, and a pointed, connected topological space X, the homology of
$$\mathbb{L}(X)_G \in \mathrm{Ho}\ s(k-\mathbf{CommAlg})$$
is called the *representation homology* of X with coefficient G, denoted by $HR_*(X, G)$.

Intepretation

One can show that $HR_0(X, G) = (\pi_1(X))_G$, thus the representation homology could be viewed as the higher order information of representation schemes.

Take a topological space

Let T^2 be the topological torus, then there is a homotopy colimit diagram
$$T^2 \simeq \mathrm{hocof}\left(S^1_a \xrightarrow{a\mapsto[b,c]} S^1_b \vee S^1_c\right).$$
The diagram will give a way of computing the representation homology of torus. A bit more generally,

Theorem ([Li24])

Let U_n be the affine group scheme consisting of all upper triangular unipotent matrices, $\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}$, and Σ_g the (oriented) topological surface of genus g. Then the following are equivalent:

1. The vanishing
$$HR_i(\Sigma_g, U_n) = 0 \quad \forall i \geq n$$
is true.
2. There is an isomorphism of graded algebras
$$HR_*(\Sigma_g, U_n) \cong HR_0(\Sigma_g, U_n) \otimes \mathrm{Sym}_k(T_1, \cdots, T_{n-1})$$
where Sym_k is the graded symmetric product over k and T_i is of homological degree 1.
3. The commuting variety of higher genus $C_g(U_n)$ is a complete intersection.

Corollary

- The commuting variety $C(U_n)$ is a complete intersection if $n = 2, 3, 4, 5$.
- The commuting variety $C(U_n)$ is NOT a complete intersection if $n \geq 6$.

Summary

The higher order information captures the geometric properties of the non-derived object. The theorem is an example of this principle, where the vanishing properties characterise complete intersection property of certain representation schemes.

References

[BRY22] Yuri Berest, Ajay C. Ramadoss, and Wai-Kit Yeung. Representation homology of topological spaces. *Int. Math. Res. Not. IMRN*, (6):4093–4180, 2022.

[Li24] Guanyu Li. Commuting varieties of upper triangular matrices and representation homology, 2024.