List of Papers

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1 Space of Unitary Vector Bundles on a Compact Riemann Surface [S2]

1.1 Basic Information

1. **Reading time:** 2019-Nov to
2. **Classification:** AG, GIT, Construction of Muduli spaces
3. **Content:**
4. **Main background:** Jordan-Hölder in an Abelian category, The Functor Qout.

We will assume the following:

Theorem 1.1. *Let $\mathcal{A}$ be an abelian category. If $M \in \operatorname{ob} \mathcal{A}$ has a Jordan-Hölder series, then its cycle of simple components is determined uniquely up to isomorphism. If $\mathcal{A}$ is both Artinian and Noetherian, then every object in $\mathcal{A}$ has a Jordan-Hölder series.*

Let X be a Riemann surface with genus $g \geq 2$, where $\mathscr{O}_X(1)$ is a fixed very ample invertible sheaf on X. For any coherent sheaf $\mathscr{F}$ over X, we denote by $\mathscr{F}(m)$ the coherent sheaf $\mathscr{F} \otimes \mathscr{O}_X(m)$. Similarly for F a holomorphic bundle over X.

For a coherent sheaf $\mathscr{E}$ on X, we denote by $Q(\mathscr{E}/p(x))$ the family of all coherent sheaves $\mathscr{F}$ on X such that $\mathscr{F}$ is a quotient of $\mathscr{F}$ and the Hilbert polynomial of $\mathscr{F}$ is $p(x)$, where $p(x)$ is a given linear polynomial, i.e. $p(x) = \dim H^0(\mathscr{F}(m)) - \dim H^1(\mathscr{F}(m))$ for m sufficiently large.

We now have the following theorem due to Grothendieck:

Theorem 1.2. *There is a (unique) structure of a projective algebraic scheme on $Q = Q(\mathscr{E}/p(x))$ and a surjective homomorphism $\theta : p_1^*(\mathscr{E}) \to \mathscr{F}$ of coherent sheaves on $X \times Q$ such that*

1. *$\mathscr{F}$ is flat over Q.*
2. *the restriction of homomorphism θ to $X \times \{q\} \cong X$ corresponds to the element of $Q(\mathscr{E}/p(x))$ represented by q.*
3. *given a surjective homomorphism $\varphi : p_1^*(\mathscr{E}) \to \mathscr{G}$ of coherent sheaves on $X \times T$, where T is an algebraic (resp. analytic) scheme such that $\mathscr{G}$ is flat over T, and the Hilbert polynomial of $\mathscr{G}_t$-restriction of $\mathscr{G}$ to $X \times \{t\}$ is $p(x)$, there exists a (unique) morphism $f : T \to Q$ such that $\varphi : p_1^*(\mathscr{E}) \to \mathscr{G}$ is the inverse image of $\theta : p_1^*(\mathscr{E}) \to \mathscr{F}$ by the morphism f.*

Proposition 1.3. *Let $R_1 = R_1(\mathscr{E}/p(x))$ be the subset of $Q = Q(\mathscr{E}/p(x))$ consisting of points $q \in Q$ such that $\mathscr{F}_q$ is locally free on X. Then R_1 is an open subset of Q, and the restriction $\mathscr{F}_1$ of $\mathscr{F}$ to $X \times R_1$ is locally free of constant rank.*

To prove this we need a lemma from commutative algebra:

Lemma 1.1. *Let $\varphi : A \to B$ be a homomorphism of commutative rings such that A is local with the maximal ideal $\mathfrak{m}$, and B is Noetherian. Suppose that for every maximal ideal $\mathfrak{n}$ of B, $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$, i.e. $\varphi(\mathfrak{m})B$ is contained in the radical of B. For a B-module M of finite type such that it is flat over A, then M is free over B if $M/\mathfrak{m}M$ is free over $B/\mathfrak{m}B$.*

For G the automorphism group of the coherent sheaf $\mathscr{E}$, we have an action of G on $Q = Q(\mathscr{E}/p(x))$, and $R_1(\mathscr{E}/p(x))$ is a G-invariant subset of Q. Let d be the degree and r be the rank of any F_q, $q \in R_1(E/p)$. We note that

$$\dim H^0(F_q) \geq d - r(g-1)$$

by Riemann-Roch.

1.2 Categories of Vector Bundles on a Riemann Surface

Let $\mathcal{V}$ be the additive category of vector bundles on a compact Riemann surface X, and let $\mathcal{V}^0$ be the full subcategory of vector bundles of degree 0. (Here the degree of a line bundle is defined to be the degree of its determinant bundle.)

Definition. A vector bundle $V \in \mathcal{V}$ is said to be *semi-stable* (resp. *stable*) if for every proper holomorphic subbundle W of V, we have

$$\frac{d(W)}{r(W)} \leq \frac{d(V)}{r(V)}$$

where $\frac{d(V)}{r(V)}$ is called the *slope* of V.

Let $\mathcal{S}$ be the full subcategory of $\mathcal{V}^0$ consisting of semi-stable vector bundles of degree 0.

Proposition 1.4. *The category $\mathcal{S}$ is abelian, Artinian, and Noetherian. Furthermore, if $\alpha \in \mathrm{Hom}(V, W)$, then α is of constant rank on the fibres of V.*

Proof. It suffices to show that ker α, coker α, and coim α are all of degree 0. By semi-stability, all degrees are ≤ 0. If $d(\text{ker } \alpha) < 0$, then by $0 = d(V) = d(\text{ker } \alpha) + d(\text{coker } \alpha)$ we get a contradiction. Similarly for others. ☐

By GAGA [S1], the compact Riemann surface X is uniquely determined by its underlying structure of a non-singular algebraic variety, and a holomorphic vector bundle V on X has a unique underlying structure of an algebraic vector bundle.

Definition. A subcategory $\mathcal{B}$ of $\mathcal{V}$ is said to be *bounded* if there is an algebraic family of vector bundles $\{V_t\}_{t \in T}$ parametrized by an algebraic scheme T such that given $V \in \mathcal{B}$, there is a $t \in T$ for $V \cong V_t$.

Proposition 1.5. *The subcategory $\mathcal{S}_n$ of $\mathcal{S}$ consisting of semi-stable vector bundles of degree 0 and rank $\leq n$, n being a fixed positive integer, is bounded.*

Proof. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bounded subcategories of $\mathcal{V}$, then the subcategory $\mathcal{B}$ consisting of vector bundles which are extensions of an object in $\mathcal{B}_1$ by an object in $\mathcal{B}_2$ is again bounded. Hence it suffices to prove that the stable bundles are bounded. But a stable bundle is indecomposable [NS], whence we can use a result by Atiyah [A]. ☐

Let $R(E/p)$ denote the subset of $R_1(E/p)$ (as in Theorem 1.2) consisting of points $q \in R_1(E/p)$ such that the canonical mapping $H^0(E) \to H^0(F_q)$ is an isomorphism and $\dim H^0(F_q) = d - r(g-1)$. It can be proved that $R(E/p)$ is invariant under the action of Aut E, and further it is open (in which the semi-continuity theorem will be used).

Proposition 1.6. *Let $\mathcal{B}$ be a bounded subcategory of $\mathcal{V}$ such that every object of $\mathcal{B}$ has the same rank r and degree d. Then there is a positive integer m_0 such that, if $m \geq m_0$ and p the Hilbert polynomial of $V(m)$, $V \in \mathcal{B}$, we have*

1. *For every $V \in \mathcal{B}$, $H^1(V(m)) = 0$ and $H^0(V(m))$ generates $V(m)$ (i.e., $H^0(V(m))$generates the fibre of $V(m)$ at every $x \in X$). In particular, the rank of $H^0(V(m))$ is independent of $V \in \mathcal{B}$. Let this be p.*

2. *If E is the trivial vector bundle of rank p, the set of all $q \in R(E/p)$ such that $F_q \cong V(m)$ for a given $V \in \mathcal{B}$ is non-empty and is precisely an orbit for the operation of the group $G =$ Aut E.*

3. *If $\{V_i\}$ is an algebraic (resp. analytic) family of vector bundles on X parametrized by an algebraic (resp. analytic) scheme T such that for every $t \in T$, $V_t(m) \in \mathcal{B}$, then given $t_0 \in T$, there is a neighbourhood T_0 of t_0 and a morphism $f : T_0 \to R(E/p)$ such that if $q = f(t)$, $t \in T_0$, $F_q \cong V_t(m)$.*

4. *There is an open, irreducible, non-singular subvariety U of $R(E/p)$ invariant under the action of G such that, given $V \in \mathcal{B}$, there is a $q \in U$ with $F_q \cong V(m)$.*

1.3 Category of Points of N-folded Grassmannians

Through out this section, we shall use $Gr_{p,r}(E)$ denoting the Grassmannian of p dimensional sub-spaces of E which is a $\mathbb{C}$-vector space of rank r, and use $Gr^{p,r}(E)$ denoting the Grassmannian of p dimensional quotient spaces of E which is a $\mathbb{C}$-vector space of rank r. Hence there is a canonical isomorphism $Gr_{p,r}(E) \cong Gr^{r-p,r}(E)$. Let $Gr^N_{p,r}(E)$ denote the N-fold product of $Gr_{p,r}(E)$.

Definition. Let N be a fixed positive integer. We denote $\mathcal{G}^N$ the category whose objects are points of $Gr^N_{p,r}(E)$, where E is any vector space of rank $r \geq 0$ and $0 \leq p \leq r$.

A morphism $\alpha : Y \to X$, for $Y = \{F_i\}_{1\leq i\leq N} \in Gr^N_{q,s}(F)$ and $X = \{E_i\}_{1\leq i\leq N}$ in this category is a linear map $\bar{\alpha} : F \to E$ (called the underlying linear map) such that $\bar{\alpha}(F_i) \subseteq E_i$.

It is not hard to see $\mathcal{G}^N$ is an additive category, and satisfies these properties:

1. α is a monomorphism (resp. epi) if and only if $\bar{\alpha}$ is injective (resp. surjective).
2. α has a kernel if and only if the rank of $K_i = \ker\,\alpha \cap F_i$ is independent for i, and then $\{K_i\}_{1\leq i\leq N}$ is the kernel of α. If α has kernel, then its coimage exists.
3. α has a cokernel if and only if the rank of $M_i = \pi(E_i)$ is independent for i where $\pi : E \to \operatorname{coker}\,\alpha$ is the canonical projection, and then $\{M_i\}_{1\leq i\leq N}$ is the cokernel of α. If α has kernel, then its image exists.
4. If α has both kernel and cokernel, then the image and coimage of α exist and the canonical morphism from the coimage to the image is an isomorphism if and only if $r(F_i) - r(K_i) = r(E_i) - r(M_i)$ for all $1 \leq i \leq N$.

If α is a monomorphism (resp. epi) and has a cokernel (resp. kernel), then we say that $0 \to Y \xrightarrow{\alpha} X$ (resp. $Y \xrightarrow{\alpha} X \to 0$) is exact. In this case, let Z be the cokernel (resp. kernel) of α and $\beta : X \to Z$ (resp. $\beta : Z \to Y$) be the canonical morphism, we see by the previous comments that α is the kernel of β. Thus we write that $0 \to Y \to X \to Z \to 0$ (resp. $0 \to Z \to Y \to X \to 0$) is exact.

Let n be a integer ≥ 2, then we denote by $\mathcal{G}^{N,n}$ the full subcategory of $\mathcal{G}^N$ consisting of objects which are points of $Gr^N_{r(n-1),rn}(E)$, where E is any vector space of rank $r \geq 0$ and $0 \leq p \leq r$. It is not hard to show that a morphism in $\mathcal{G}^{N,n}$ is a monomorphism (resp. epimorphism) if and only if it is so in $\mathcal{G}^N$.

Definition. An object $X = \{E_i\}_{1\leq i\leq N} \in Gr^N_{p,r}(E)$ is said to be *semi-stable* (resp. *stable*) if, for every subspace F of E (resp. proper subspace) we have

$$\frac{\frac{1}{N}\sum_{i=1}^N r(F \cap E_i)}{p} \leq \frac{r(F)}{r}.$$

Also, for $X = \{E_i\}_{1\leq i\leq N} \in Gr^{p,r}_N(E)$, the canonical image of X in $Gr^N_{r-p,r}(E)$ is semi-stable (resp.) if and only if, for every subspace F of E,

$$\frac{\frac{1}{N}\sum_{i=1}^N r(F_i)}{p} \geq \frac{r(F)}{r}.$$

Proposition 1.7. *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence in $\mathcal{G}^N$ with Y, Z, X in $\mathcal{G}^{N,n}$. Then X is semi-stable if and only if both Y and Z are semi-stable.*

We also denote by $\mathcal{K}^{N,n}$ the full subcategory of $\mathcal{G}^{N,n}$ consisting of the semi-stable objects of $\mathcal{G}^{N,n}$. It is not hard to show that a morphism in $\mathcal{K}^{N,n}$ is a monomorphism (resp. epimorphism) if and only if it is so in $\mathcal{G}^N$.

Proposition 1.8. *Let $\alpha : Y \to X$ (resp. $\alpha : X \to Y$) be a monomorphism (resp. epi) in $\mathcal{G}^N$ with $X, Y \in \mathcal{G}^{N,n}$. Then if X is semi-stable, $0 \to Y \to X$ (resp. $X \to Y \to 0$) is exact, and Y is semi-stable.*

Proposition 1.9. *Let X be a stable object of $\mathcal{G}^{N,n}$. Then if $\alpha : X \to Y$ is a morphism in $\mathcal{K}^{N,n}$, then either α is 0, or $0 \to Y \to X$ is exact.*

Definition. An object $X \in \mathcal{G}^{N,n}$ is said to have a *stable series* S if there is an increasing sequence $S = \{X_i\}_{q\leq i\leq m}$

$$X_1 \subset X_2 \subset \cdots \subset X_m = X$$

of subobjects of X such that every one of the canonical monomorphisms $X_i \to X_{i+1}$ has a cokernel X_{i+1}/X_i, and $X_1, \cdots, X_m/X_{m-1}$ are all stable objects of $\mathcal{G}^{N,n}$.

By an application of Proposition 1.7, it follows that $X \in \mathcal{K}^{N,n}$ if X has a stable series S. We denote by $\mathcal{A}^{N,n}$ the full subcategory of $\mathcal{K}^{N,n}$ consisting of those objects which possess stable series.

Proposition 1.10. *The category $\mathcal{A}^{N,n}$ is abelian, artinian, and noetherian, and the simple object in it are precisely the stable objects.*

1.4 Connecting Two Categories

Let X be a Riemann surface with genus $g \geq 2$. Let $\mathcal{V}_1$ be the full subcategory of the additive category $\mathcal{V}$ of holomorphic vector bundles on X, having the property that all objects are globally generated. Let $\mathfrak{n}$ be an ordered set of N distinct points $P_1, \cdots, P_N$ on X. Then if V is an object of $\mathcal{V}_1$ of rank r, let E be the vector space $H^0(V)$ and p be its dimension.

We define a functor $\tau(\mathfrak{n})$, which to V associates the point $x \in Gr_N^{r,p}(E) \cong Gr_{p-r,p}^N(E)$ such that the i-th coordinate of x is precisely the quotient vector space of E represented by the fibre of V at P_i. Thus $\tau(\mathfrak{n})$ is an additive functor from $\mathcal{V}_1$ into $\mathcal{G}^N$.

It is not hard to verify that if $0 \to V_1 \to V_2 \to V_3 \to 0$ is exact in $\mathcal{V}$ such that V_i are in $\mathcal{V}_1$ and $H^1(V_i) = 0$, then

$$0 \to \tau(\mathfrak{n})(V_1) \to \tau(\mathfrak{n})(V_2) \to \tau(\mathfrak{n})(V_3) \to 0$$

is exact. If $\mathcal{D}$ is any subcategory of $\mathcal{V}$, we denote by $\mathcal{D}(m)$ the subcategory of $\mathcal{V}$ of all objects $V(m)$ where $V \in \mathcal{D}$.

Proposition 1.11. *Let $\mathcal{B}$ be a bounded subcategory of $\mathcal{V}$. Then one can find a positive integer m and an ordered set $\mathfrak{n}$ of N distinct points $P_1, \cdots, P_N$ on X such that*

1. *$\mathcal{B}(m) \subseteq \mathcal{V}_1$ and*
2. *$\tau(\mathfrak{n})(V_1)$ and $\tau(\mathfrak{n})(V_2)$ for $V_1, V_2 \in \mathcal{B}(m)$ are isomorphic if and only if $V_1 \cong V_2$.*

Definition. A vector bundle $V \in \mathcal{V}$ is said to be *generically generated* by a linear subspace F of $H^0(V)$ if there is at least one $x \in X$ such that F generates the fibre of V at x.

Let $\mathcal{W}$ be the full subcategory of v consisting of objects $V \in \mathcal{V}$ such that

1. if G is any sub-bundle of V with the property
$$\frac{d(G)}{r(G)} \geq \frac{d(V)}{r(V)}$$
then $H^1(G) = 0$ and $H^0(G)$ generates G, and
2. if G is a proper sub-bundle such that $H^0(G)$ generates G generically, then
$$\frac{r(H^0(G))}{r(G)} > \frac{r(H^0(V))}{r(V)}$$
if $\frac{d(G)}{r(G)} < \frac{d(V)}{r(V)}$.

We see that $\mathcal{W} \subseteq \mathcal{V}_1$. Let $V \in \mathcal{W}$ and G a sub-bundle of V such that

$$\frac{d(G)}{r(G)} > \frac{d(V)}{r(V)} \qquad \text{resp.} \frac{d(G)}{r(G)} = \frac{d(V)}{r(V)}.$$

Then since $H^1(V) = H^1(G) = 0$, we have by the Riemann-Roch

$$\frac{r(H^0(V))}{r(V)} = \frac{d(V)}{r(V)} - g + 1 \qquad \frac{r(H^0(G))}{r(G)} = \frac{d(G)}{r(G)} - g + 1.$$

Therefore it follows that

$$\frac{r(H^0(G))}{r(G)} > \frac{r(H^0(V))}{r(V)} \qquad \text{resp.} \frac{r(H^0(G))}{r(G)} = \frac{r(H^0(V))}{r(V)}.$$

Proposition 1.12. *Let $\mathcal{B}$ be a bounded subcategory of $\mathcal{W}$. Then given any ordered set $\mathfrak{n}$ of distinct points $P_1, \cdots, P_N$ on X such that N is sufficiently large, the functor has the following property that, for $V \in \mathcal{B}$, $\tau(\mathfrak{n})(V)$ is a stable (resp. semi-stable) object of $\mathcal{G}^N$ if and only if V is a stable (resp. semi-stable) object of $\mathcal{V}$.*

To prove this, we need this lemma:

Lemma 1.2. *Let $V \in \mathcal{V}$ and F a subspace of $H^0(V)$ which generates V generically. Let μ be the number of distinct points such that F does not generate the fibre of V ate y, then*

$$\mu \leq d(V).$$

Proposition 1.13. *Let $\mathcal{B}$ be a bounded subcategory of $\mathcal{V}$. Then we can find an integer m_0 such that if $m \geq m_0$, then $\mathcal{B}(m) \subseteq \mathcal{W}$.*

We also need this lemma:

Lemma 1.3. *Let $V \in \mathcal{V}$ and $H^0(V)$ generate V generically. Then we have*

$$r(H^0(V)) \leq d(V) + r(V).$$

Theorem 1.14. *Let $\mathcal{B}$ be a bounded subcategory of $\mathcal{V}$. Then we can find an integer m and an ordered set $\mathfrak{n}$ of distinct points $P_1, \cdots, P_N$ on X such that*

1. *If $V \in \mathcal{B}$, then $H^1(V(m)) = 0$ and $H^0(V(m))$ which generates $V(m)$ so that, in particular $\mathcal{B}(m) \subseteq \mathcal{V}_1$.*
2. *$\tau(\mathfrak{n})(V_1)$ and $\tau(\mathfrak{n})(V_2)$ for $V_1, V_2 \in \mathcal{B}(m)$ are isomorphic if and only if $V_1 \cong V_2$.*
3. *If $V \in \mathcal{B}(m)$, $\tau(\mathfrak{n})(V)$ is stable (resp. semi-stable) if and only if V is stable (resp. semi-stable).*

Corollary 1.14.1. *Take for $\mathcal{B}$ the abelian sub-category $\mathcal{S}_r$ of $\mathcal{V}$ consisting of the semi-stable bundles of degree 0 and rank r. Let m and $\mathfrak{n}$ be chosen as in Theorem 1.14, and $n = d(L) - g + 1$, where L is the line bundle defined by the invertible sheaf $\mathscr{O}_X(m)$. Let $\tau_1 : \mathcal{S}_r \to \mathcal{G}^N$ be the functor $\tau_1(V) := \tau(\mathfrak{n})(V(m))$. Then we have*

1. *$\tau_1(\mathcal{S}_r)$ is contained in the abelian sub-category $\mathcal{A}^{N,n}$ of $\mathcal{G}^N$, and the functor $\tau_1 : \mathcal{S}_r \to \mathcal{A}^{N,n}$ is an exact functor.*
2. *For $V \in \mathcal{S}_r$, $\tau(\mathfrak{n})(V)$ is stable (resp. semi-stable) if and only if V is stable (resp. semi-stable).*
3. *For $V_1, V_2 \in \mathcal{S}_r$, $\tau_1(V_1)$ and $\tau_1(V_2)$ for $V_1, V_2 \in \mathcal{B}(m)$ are isomorphic if and only if $V_1 \cong V_2$. In particular,* $\mathrm{gr}\ V_1 \cong \mathrm{gr}\ V_2$ *if and only if* $\mathrm{gr}\ \tau_1(V_1) = \mathrm{gr}\ \tau_1(V_2)$.

Corollary 1.14.2. *Let $\{V_t\}_{t \in T}$ be an algebraic family of vector bundles parametrized by an algebraic scheme T. Then subset T_s (resp. T_{ss}) of points $t \in T$ such that V_t is stable (resp. semi-stable) is open in T. Similarly for an analytic family, then $T - T_s$ (resp. $T - T_{ss}$) is an analytic subset of T.*

Theorem 1.15. *Let $\{z_i\}_{1 \leq i \leq m}$ be a sequence of stable objects in $G^{N,n}$. Let T be the subset of $Gr^N_{r(n-1),rn}(E)_{ss}$ consisting of the points x in $\mathcal{A}^{N,n}$, i.e. those having a stable series*

$$x_1 \subset x_2 \subset \cdots \subset x_m = x$$

such that $x_1 \cong z_1, x_2/x_1 \cong z_2, \cdots, x_m/x_{m-1} = z_m$. Then T is closed and $GL(E)$ invariant.

If else, T is the subset of $Gr^N_{r(n-1),rn}(E)_{ss}$ consisting of the points having a stable series with a fixed cycle of stable components C. Then T is closed and $GL(E)$ invariant. If T_1 and T_2 are two such subsets associated to two distinct cycles of stable components, then $T_1 \cap T_2 = \emptyset$.

Corollary 1.15.1. *Let $X = Gr^N_{r(n-1),rn}(E)$. Then the categorical quotient Y of X_{ss} modulo $PGL(E)$ exists, and is a projective variety. Further if $\varphi : X_{ss} \to Y$ is the canonical morphism and x_1 and x_2 are two points of X_{ss} such that they belong to $\mathcal{A}^{N,n}$ and* $\mathrm{gr}\ x_1 \neq \mathrm{gr}\ x_2$, *then $\varphi(x_1) \neq \varphi(x_2)$.*

1.5 The Main Theorem and Its Proof

Let X as before be a Riemann surface with genus $g \geq 2$, with $p : \tilde{X} \to X$ the covering space of X. Let $\mathcal{S}$ be the category of semi-stable vector bundles of degree 0.

Definition. A holomorphic vector bundle is said to be *unitary* if it is the vector bundle associated to a unitary representation of $\pi_1(X)$.

It was proved in [NS] that a holomorphic vector bundle V on X is isomorphic to a unitary vector bundle on X if and only if V is a direct sum of stable vector bundles of degree 0. Therefore for $V \in \mathcal{S}$, gr V represents an isomorphic class of unitary vector bundle.

Definition. For two elements in $\mathcal{S}$, we say they are *strongly equivalent* if gr $V_1 =$ gr V_2.

Theorem 1.16. *Let U_r denote the set of isomorphic classes of unitary vector bundles of rank r, or equivalently the set of equivalence classes of semi-stable vector bundles of rank r and degree 0.*

Then there is a unique structure of a normal projective variety on U_r such that, if $\{V_t\}_{t\in T}$ is an algebraic (resp. analytic) family of semi-stable vector bundles of rank r and degree 0, then the mapping $T \to U_r$ defined by $t \to \mathrm{gr}\ V_t$ is a morphism.

Proof. We shall use the notations mentioned in Theorem 1.14. Let $p(x)$ be the Hilbert polynomial of $V(m)$ where $V \in \mathcal{S}_r$, $r = r(V)$. Let $R = R(E/p)$ be the scheme as in Proposition 1.6, E being the trivial bundle of rank p, $p = \dim H^0(V(m))$. Consider the canonical morphism $\tau = \tau(\mathfrak{n}) : R \to Gr^N_{p,p-r}(E) \cong Gr^{p,r}_N(E)$ where $E = H^0(E)$ which to $q \in R$ associates the point x of $Gr^{p,r}_N(E)$ such that $p_i(x)$, the i-th canonical projection on to $Gr^{p,r}(E)$, is precisely the fibre of the vector bundle at P_i.

Let R_{ss} be the subset of R consisting of points $q \in R$ such that F_q is semi-stable. Then R_{ss} is an open, non-singular and irreducible subset of R invariant under $G = \mathrm{Aut}\ E$. Further given $V \in \mathcal{S}_r$, $r = r(V)$, there is a $q \in R$ such that $F_q \cong V$, and the set of such q constitutes precisely one orbit under G.

Let $Z = Gr^N_{r(n-1),rn}(E)_{ss}$ ($n := d(L) - g + 1$, where L is the line bundle associated to the invertible sheaf $\mathscr{O}_X(m)$). Then $\tau(R_{ss}) \subseteq Z$. In fact, if $q \in R$, $\tau(q) \in \mathcal{A}^{N,n}$; further if $q_1, q_2 \in R$ such that gr $F_{q_1} \neq$ gr F_{q_2}, then gr $\tau(q_1) \neq$ gr $\tau(q_2)$.

Let G be the automorphism group of $H^0(E)$, then we see that τ is a G-morphism. Let

$$\varphi : Z \to Y$$

be the categorical quotient of Z by G, then Y is projective and if $q_1, q_2 \in R$ such that gr $F_{q_1} \neq$ gr F_{q_2}, we have $\varphi \circ \tau(q_1) \neq \varphi \circ \tau(q_2)$ by Corollary 1.15.1.

Let Y_1' be the closure of $\varphi \circ \tau(R_{ss})$. Then the canonical morphism $\psi' : R_{ss} \to Y_1'$ is dominant and G-invariant, i.e. two points in the same orbit are mapped onto the same point by ψ'. Let Y_1 be the normalization of Y_1' and $p : Y_1 \to Y_1'$ be the canonical morphism.

Since R_{ss} is non-singular in particular normal, we have a morphism $\psi : R_{ss} \to Y_1$ such that $\psi' = p \circ \psi$. Since p is an isomorphism on a non-empty open subset of R_{ss}, it follows that ψ is G-invariant on a non-empty G-invariant open subset of R_{ss}, which implies that ψ is G-invariant on the entire R_{ss}. We note also that, if $q_1, q_2 \in R$ such that gr $F_{q_1} \neq$ gr F_{q_2}, then $\psi(q_1) \neq \psi(q_2)$.

It remains to prove that Y_1 is actually what we want. So we will define a set-theoretical bijection $U_r \to Y_1$. Let now U denote the space of all unitary representations of $\pi_1(X)$ of rank r. Then we know U is a compact subspace of the analytic space $R(r)$ of all representations of π of rank r, and we know that the canonical family of vector bundles on X parametrized by $R(r)$ (namely the one which assigns to each point θ of $R(r)$ the holomorphic vector bundle on X associated to the representation θ of π) is an analytic family. By the property (3) of the Proposition 1.6, we see that given a point θ of $R(r)_{ss}$ there is a neighbourhood K of θ in $R(r)_{ss}$ and an analytic morphism $f : K \to R_{ss}$ such that if $k \in K$ and $q = f(k)$, F_q is isomorphic to the vector bundle associated to the representation of π given by k.

Now if $f_1 : K_1 \to R_{ss}$ and $f_2 : K_2 \to R_{ss}$ are two such morphisms, K_1, K_2 being open in R_{ss}, then if $k \in K_1 \cap K_2$, we see that $f_1(k)$ and $f_2(k)$ lie in the same orbit under G. Hence we conclude that $\psi \circ f_1$ and $\psi \circ f_1$ coincide in $K_1 \cap K_2$. From these considerations, we get an analytic morphism from $R(r)_{ss}$ to R_{ss}, and the restriction of this morphism defines a *continuous* mapping $g : U \to Y_1$.

By the compactness of U, we know $g(U)$ is closed (in the usual Hausdorff sense) in Y. Further if U_0 is the subspace of U consisting of the irreducible unitary representations of π, we have $g(U_0) = \psi(R)$, since a vector bundle of degree 0 is stable if and only if it is isomorphic to a vector bundle associated to an irreducible unitary of π. Since ψ is dominant, $\psi(R_s)$ contains a non-empty (Zariski) open subset of Y by a theorem of Chevalley. This implies that $g : U \to Y_1$ is surjective, and we note that if θ_1 and θ_2 are in U, $g(\theta_1) = g(\theta_2)$ if and only if the unitary bundles on X defined by the representations θ_1 and θ_2 of π are isomorphic □

Proof Idea. For paratramizing semi-stable vector bundles, we take any $V \in \mathcal{S}_r$. If V is globally generated, then V is a quotient of $E \otimes \mathscr{O}_X$, where $E = H^0(V)$. By doing this, we can use the Quot scheme (1.2) to parametrize the vector bundles. However, there are immediately several issues:

1. The quotients may only be coherent sheaves, instead of vector bundles.
2. The parametrizing map may not be surjective.
3. The parametrizing map may not be injective.

For 1, we first take $R_1 = R_1(E/p)$ the subset of $Q = Q(\mathscr{E}/p(x))$ consisting of points $q \in Q$ such that $\mathscr{F}_q$ is locally free on X (Proposition 1.3).

But before getting into the bijectiveness, there is still some issue: what if V is not globally generated? By proposition 1.6, we should twist it sufficiently. And also the first part of this property tells us we sort of get all vector bundles of X.

There is also a natural way to study the quotient scheme: we map (a subscheme of) it to the Grassmannian, or N-folded (dual) Grassmannian. Loosely, this process is taking N points on the Riemann surface X, get N quotient space of E which turns out to be a point in Gr^N. □

2 Géométrie algébrique et géométrie analytique [S1]

2.1 Basic Information

1. **Reading time:** 2020-Aug to
2. **Classification:** AG, Complex Geometry
3. **Content:**
4. **Main background:**

1. For the subset $U \subseteq \mathbb{C}^n$ we say that U is **analytic** if for each $x \in U$, there are functions $f_1, \cdots, f_k$ holopmorphic in a neighborhood W of x, such that $U \cap W$ is identical to the set of points $z \in W$ satisfying the equations $f_i(z) = 0, i = 1, \cdots, k$. The subset U is then locally closed. We equip it with the topology induced by $\mathbb{C}^n$.

For any space X, denote by $\mathscr{C}(X)$ the sheaf of germs of functions (not necessiraly continuous) on X with values in $\mathbb{C}$. $\mathscr{C}(\mathbb{C}^n)$ has a subsheaf $\mathscr{H}$ consisting of all holomorphic functions defined on an open subset. Then for an analytic subset U, there is a restriction

$$\epsilon_x : \mathscr{C}(\mathbb{C}^n)_x \to \mathscr{C}(U)_x$$

for any $x \in U$. The image $\mathscr{H}_{U,x}$ of ϵ_x restricted on the subring $\mathscr{H}_{\mathbb{C}^n,x}$ is a subring of $\mathscr{C}(U)_x$, which will be denoted by $\mathscr{H}_{U,x}$. The $\mathscr{H}_{U,x}$ form a subsheaf $\mathscr{H}_U$ of $\mathscr{C}(U)$, which we call the **sheaf of germs of holomorphic functions** on U. We denote $\mathscr{A}_{U,x}$ the kernel of ϵ_x, then it is the set of $f \in \mathscr{H}_{\mathbb{C}^n,x}$ whose restriction to U is zero in a neighborhood o x. $\mathscr{H}_{U,x} \cong \mathscr{H}_{\mathbb{C}^n,x}/\mathscr{A}_{U,x}$.

For two analytic subsets U, V of $\mathbb{C}^r$ and $\mathbb{C}^s$. A continuous map $\varphi : U \to V$ is said to be holomorphic if for each $f \in \mathscr{H}_{U,x}$, $f \circ \varphi \in \mathscr{H}_{V,\varphi(x)}$.

2. We call an analytic space a set X equipped with a topology and a subsheaf $\mathscr{H}_X$ of the sheaf $\mathscr{C}(X)$, subject to the following axions:

(H1) There exists open cover $\{V_i\}$ of X such that each V_i, equipped with the topology and sheaf induced by those of X, is isomorphic to an analytic subset U_i of an affine space.

(H2) The topology of X is Hausdorff.

We can similarly define a morphism and the product.

3. For an analytic space X, an analytic sheaf $\mathscr{F}$ is simply a sheaf modules over the sheaf of rings $\mathscr{H}_X$. Let Y be a closed analytic subspace, let $\mathscr{A}_{Y,x}$ be the set of $f \in \mathscr{H}_{X,x}$ whose restriction onto Y is zero in a neighborhood of x. The $\mathscr{A}_{Y,x}$ form a sheaf of ideals $\mathscr{A}_Y$ of the sheaf $\mathscr{H}_X$. The quotient sheaf $\mathscr{H}_X/\mathscr{A}_Y$ is zero outside of Y, and its restriction to Y is none other than $\mathscr{H}_Y$.

Proposition 2.1. *1. The sheaf $\mathscr{H}_X$ is a coherent sheaf of rings.*

2. If Y is a closed analytic subspace of X, then the sheaf $\mathscr{A}_Y$ is a coherent analytic sheaf.

2.2

Let X be an algebraic variety, and let X^{an} be the analytic space associated with X, then we have

3 On the periodicity theorem for complex vector bundles [AB]

3.1 Basic Information

1. **Reading time:** 2020-Nov to 2020-Dec
2. **Classification:** Algebraic Topology
3. **Content:** The proof of Bott Periodicity
4. **Main background:** Vector Bundles, The Definition and Basic Properties of K-groups

The Bott-Periodicity theorem for the infinite unitary group is a statement about complex vector bundles. It describes the relation between vector bundles over X and $X \times S^2$:

$$K^0(X \times S^2) \cong K^0(X) \otimes K(S^2)$$

where X is a compact space. This paper talks about an elementary proof using the tool called clutching functions.

For a vector bundle E over X, if Y is a subspace of X, then $E|_Y := \coprod_{y \in Y} E_y$ has a natural vector bundle structure. We call it the restriction of E on Y.

Lemma 3.1. *Let Y be a closed subspace of a compact (Hausdorff) space X and let E be a vector bundle over X. Then any section of $E|_Y$ extends to a section of E.*

Proof. ☐

Lemma 3.2. *Let Y be a closed subspace of a compact space X, E and F two vector bundles over X. Then any isomorphism $s : E|_Y \to F|_Y$ extends to an isomorphism $t : E|_U \to F|_U$ for some open set U containing Y.*

Proof. ☐

Proposition 3.1. *Let Y be a compact space, $f_t : Y \to X$ a homotopy and E a vector bundle over X. Then*

$$f_0^*(E) \cong f_1^*(E).$$

Proof. ☐

Definition. A *projection operator* P for a vector bundle $E \to X$ is an endomorphism $P : E \to E$ with $P^2 = P$.

If one is given a projection operator P on E, then PE and $(\mathrm{id} - P)E$ inherit from E a topology, a projection so that they are naturally subbundles. To see this, we locally choose for each $x \in X$, a sufficiently small neighborhood U_x with local sections $s_1, \cdots, s_n : U_x \to \pi^{-1}(U)$ such that

1. $s_1, \cdots, s_r$ form a base of $(PE)_x$.
2. $s_{r+1}, \cdots, s_n$ form a base of $((\mathrm{id} - P)E)_x$.

Thus we have a vector bundle isomorphism

$$\varphi : U \times \mathbb{C}^n \to PE|_U \oplus (\mathrm{id} - P)E|_U$$
$$(y, (a_1, \cdots, a_n)) \mapsto \sum_{i=1}^{r} a_i P_y s_i(y) + \sum_{j=r+1}^{n} a_j (\mathrm{id} - P)_y s_i(y).$$

This establishes

Lemma 3.3. *If P is a projection operator for the vector bundle E, then PE and $(\mathrm{id} - P)E$ have an induced vector bundle structure and*

$$E \cong PE \oplus (\mathrm{id} - P)E.$$

Vector bundles are frequently constructed by a glueing or clutching construction. Let

$$X = X_1 \cup X_2, \; A = X_1 \cap X_2,$$

all being compact. Suppose that E_i is a vector bundle over X_i and that $\varphi : E_1|_A \to E_2|_A$ is a bundle isomorphism. Then we can define a vector bundle $E_1 \cup_\varphi E_2$ as follows.

Elementary properties of this construction:

1. If E is a bundle over X and $E_i := E|_{X_i}$,then the identity defines an isomorphism $\mathrm{id}_A : E_1|_A \to E_2|_A$, and
$$E_1 \cup_{\mathrm{id}_A} E_2 \cong E.$$
2. If $\beta_i : E_i \to E_i'$ are isomorphisms on X_i and $\varphi'\beta_1 = \beta_2\varphi$, then
$$E_1 \cup_\varphi E_2 \cong E_1' \cup_{\varphi'} E_2'.$$
3. If (E_i, φ) and (E_i', φ') are two "clutching data" on X_i, then
$$(E_1 \cup_\varphi E_2) \oplus (E_1' \cup_{\varphi'} E_2') \cong (E_1 \oplus E_1') \cup_{\varphi \oplus \varphi'} (E_2 \oplus E_2'),$$
$$(E_1 \cup_\varphi E_2) \otimes (E_1' \cup_{\varphi'} E_2') \cong (E_1 \otimes E_1') \cup_{\varphi \otimes \varphi'} (E_2 \otimes E_2'),$$
$$(E_1 \cup_\varphi E_2)^* \cong E_1^* \cup_{\varphi^{-1}} E_2^*.$$

Lemma 3.4. *The isomorphism class of $E_1 \cup_\varphi E_2$ depends only on the homotopy class of the isomorphism $\varphi : E_1 \to E_2$.*

Proof. A homotopy of isomorphisms $E_1|_A \to E_2|_A$ means an isomorphism

$$\Phi : \pi^* E_1|_{A \times I} \to \pi^* E_2|_{A \times I}$$

where π is the projection $X \times I \to X$. Let

$$f_t : X \to X \times I$$

be defined by $f_t(x) = (x, t)$ and denote by

$$\varphi_t : E_1|_A \to E_2|_A$$

the isomorphism induced by Φ and f_t, i.e.

$$\varphi_t = (f_t|_A)^*(\Phi) : (\pi \circ f_t)^* E_1|_{A \times I} \to (\pi \circ f_t)^* E_2|_{A \times I}.$$

Then

$$E_1 \cup_{\varphi_1} E_2 \cong f_t^*(\pi^* E_1 \cup_\Phi \pi^* E_2).$$

Since f_0 and f_1 are homotopic it follows from Proposition 3.1 that

$$E_1 \cup_{\varphi_2} E_2 \cong E_1 \cup_{\varphi_1} E_2$$

as required. ∎

Definition. Let E be a vector bundle over X. By deleting the zero section and dividing a $\mathbb{C}^*$-action, one is given a space $P(E) \to X$, with each fibre over $x \in X$ is a projective space $P(E_x)$. If we further assign to each $y \in P(E)$ **the** 1-dimensional subspace of E_x which corresponds to it (since y itself is a 1-dimensional subspace), we obtain a line bundle over $P(E)$. It is denoted by H^* whose dual is H (the choice of convention here is dictated by algebra-geometric considerations which we do not discuss here). The projection $\mathrm{pr} : P(E) \to X$ gives us a ring homomorphism

$$\mathrm{pr}^* : K^0(X) \to K^0(P(E))$$

so that $K^0(P(E))$ becomes a $K^0(X)$ algebra.

3.2 Laurent Cluching Functions

Suppose L is a line bundle over X, then for any $x \in X$ there is a natural embedding

$$L_x \to P(L \oplus \epsilon^1)_x$$
$$y \mapsto (y, 1)$$

which exhibits $P(L \oplus \epsilon^1)_x$ as the compactification of L_x obtained by adding a "section at infinity". Now let us choose a definite metric in L and let $S \subseteq L$ be the unit circle bundle in this metric. Thus we identify L a subspace of $P := P(L \oplus \epsilon^1)$ so that

$$P = P^0 \cup P^\infty, \ S = P^0 \cap P^\infty,$$

where P^0 is the closed disc bundle interior to S (i.e. containing the 0-section) and P^∞ is the closed disc bundle interior to S (i.e. containing the ∞-section). The projections $S \to X$, $P^0 \to X$, and $P^\infty \to X$ will be denoted by π, π_0 and π_∞ respectively.

Suppose now E^0, E^∞ are two vector bundles over X and that $f \in \mathrm{Iso}(\pi^* E^0, \pi^* E^\infty)$. Then we can form a vector bundle

$$\pi_0^* E^0 \cup_f \pi_\infty^* E^\infty$$

over P. We denote this bundle for brevity (E^0, f, E^∞), and we say that f is a clutching function for (E^0, E^∞).

Lemma 3.5. *Let E be any bundle over P and let E^0, E^∞ be the vector bundles over X induced by the 0-section and the ∞-section respectively. Then there exists $f \in \mathrm{Iso}(\pi^* E^0, \pi^* E^\infty)$ such that*

$$E \cong (E^0, f, E^\infty),$$

where the isomorphism is the obvious one on the 0-section and the ∞-section. Moreover f is uniquely determined, up to homotopy by these properties.

Proof. Let $s_0 : X \to P^0$ be the 0-section, then $s_0 \pi_0$ is homotopic to the identity map, and so by Proposition 3.1 we have an isomorphism

$$f_0 : E|_{P^0} \to \pi_0^* E^0.$$

The same argument applies to $E|_{P^\infty}$ and the lemma then follows, taking

$$f := f_\infty f_0^{-1}.$$

□

If F is a vector bundle over X then (F, id, F) is isomorphic to the pullback of F along $P \to X$. In $K(P)$ this is the equation

$$[(F, \mathrm{id}, F)] = [F][1] = [(P \to X)^* F].$$

When L is the trivial line bundle $X \times \mathbb{C}$, S is the trivial circle bundle $X \times S^1$ so that the points of S can be represented by pairs (x, z) with $x \in X$ and $z \in \mathbb{C}$ with $|z| = 1$. In this case, z and z^{-1} are functions on S. We consider functions on S which are finite Laurent series in z:

$$\sum_{k=-n}^{n} a_k(x) z^k.$$

When L is not trivial, we notice that the inclusion $S \hookrightarrow L$ defines a (tautologous) section of $\pi^* L \to S$. More precisely, the projection $p : L \to X$ restricted onto S gives the diagram

$$\begin{tikzcd} \pi^*L \arrow[r] \arrow[d] & L \arrow[d, "p"] \\ S \arrow[ur, hook] \arrow[r, "p|_S"] & X \end{tikzcd}$$

where π^*L is all the $(l,s) \in L \times S$ s.t. $p(l) = p|_S(s)$. *Thus the section, denoted by* z, is defined by

$$\begin{aligned} z : S &\to \pi^*L \\ s &\mapsto (s,s). \end{aligned}$$

When $L = \epsilon^1$, $S = X \times S^1$. Then an element in S is a pair (x,z) with $|z| = 1$. For Corollary 3.5.1, one only needs to think the section as functions defined above. To obtain the more general Theorem 3.5, however we have to consider z as a section.

By the canonical isomorphism

$$\pi^*L \cong \pi^*\mathscr{H}\mathrm{om}(\epsilon^1, L)$$

we may also regard z as a section of $\pi^*L\mathscr{H}\mathrm{om}(\epsilon^1, L)$ and as such, it has an inverse z^{-1} which is a section of

$$\pi^*\mathscr{H}\mathrm{om}(L, \epsilon^1) \cong \pi^*L^{-1}.$$

More generally, for any integer k, we may regard z^k as a section of π^*L^k. If now $a_k \in \Gamma(L^{\otimes -k})$ then

$$\pi^*(a_k) \otimes z^k \in \Gamma\pi^*(\epsilon^1),$$

i.e. it is a function on S. For simplicity, we write $a_k z^k$ instead of $\pi^*(a_k) \otimes z^k$.

Finally, suppose that E^0, E^∞ are two vector bundles on X and that

$$a_k \in \Gamma\mathscr{H}\mathrm{om}(L^k \otimes E^0, E^\infty),$$

then

$$a_k z^k \in \Gamma\mathscr{H}\mathrm{om}(\pi^*E^0, \pi^*E^\infty).$$

A finite sum

$$f = \sum_{k=-n}^{n} a_k z^k \in \Gamma\mathscr{H}\mathrm{om}(\pi^*E^0, \pi^*E^\infty)$$

with the a_k as above will be called a finite Laurent series for (E^0, E^∞). If $f \in \mathrm{Iso}(\pi^*E^0, \pi^*E^\infty)$ then it defines a clutching function and we call this a *Laurent clutching function* for (E^0, E^∞).

Example 3.1. Using the same notations above, take $E^0 := \epsilon^1, E^\infty := L$, then z itself is a Laurent clutching function. Recall that the line bundle H^* over P as a subbundle of $\pi^*(L \oplus \epsilon^1)$. For each $y \in P(L \oplus \epsilon^1)$, H^*_y is a subspace of $(L \oplus \epsilon^1)_x$ and

1. $H^*_y = L_x \oplus 0$ is equivalent to $y = \infty$.

2. $H^*_y = 0 \oplus \epsilon^1_x$ is equivalent to $y = 0$.

Thus the composition

$$H^* \hookrightarrow \pi^*(L \oplus \epsilon^1) \to \pi^*(\epsilon^1)$$

induced by the projection $L \oplus \epsilon^1 \to \epsilon^1$, defines an isomorphism

$$f_0 : H^*|_{P^0} \to \pi_0^*(\epsilon^1)$$

and similarly the composition

$$H^* \hookrightarrow \pi^*(L \oplus \epsilon^1) \to \pi^*(L)$$

defines another isomorphism

$$f_\infty : H^*|_{P^\infty} \to \pi_\infty^*(L).$$

Hence

$$f = f_\infty f_0^{-1} : \pi^*(\epsilon^1) \to \pi^*(L)$$

is a clutching function for H^*.

For each $y \in S_x$,

Suppose now that $f \in \Gamma\mathscr{H}\mathrm{om}(\pi^* E^0, \pi^* E^\infty)$ is any section, we then can define its *Fourier* coefficients

$$a_k \in \Gamma\mathscr{H}\mathrm{om}(L^k \otimes E^0, E^\infty)$$

by

$$a_k(x) := \frac{1}{2\pi i} \int_{S_x} f_x z_x^{-k-1} dz_x,$$

where f_x and z_x denote the restrictions of f, z to S_x, and dz_x is therefore a differential in S_x with coefficients in L_x. Let s_n be the partial sum

$$s_n := \sum_{k=-n}^{n} a_k z^k$$

and define the Cesaro means

$$f_n := \frac{1}{n} \sum_{m=0}^{n} s_m.$$

Result from analysis gives

Lemma 3.6. *Let f be any clutching function for (E^0, E^∞), f_n the sequence of Cesaro means of the Fourier series of f. Then f_n converges uniformly to f and hence is a Laurent clutching function for all sufficiently large n.*

The uniformity can be defined by using metrics in E^0 and E^∞, but does not depend on the choice of metrics.

3.3 Linearization

By a *polynomial clutching function* we mean a Laurent clutching function without negative powers of z. Thus let

$$p = \sum_{k=0}^{n} a_k z^k \in \Gamma\mathscr{H}\mathrm{om}(\pi^* E^0, \pi^* E^\infty)$$

be a polynomial clutching function of degree $\leq n$ for (E^0, E^∞). Consider the homomorphism

$$\mathcal{L}^n(p) : \pi^* \left(\sum_{k=0}^{n} L^k \otimes E^0 \right) \to \pi^* \left(E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0 \right)$$

given by the matrix

$$\mathcal{L}^n(p) := \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ -z & 1 & & & \\ & -z & 1 & & \\ & & \ddots & \ddots & \\ & & & -z & 1 \end{bmatrix}.$$

It is clear that $\mathcal{L}^n(p)$ is linear in z. Now define the sequence $p_r(z)$ inductively by $p_0 = p$, and $zp_{r+1}(z) = p_r(z) - p_r(0)$. Then we have the following matrix identity

$$\mathcal{L}^n(p) = \begin{bmatrix} 1 & p_1 & p_2 & \cdots & p_n \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} p & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ -z & 1 & & & \\ & -z & 1 & & \\ & & \ddots & \ddots & \\ & & & -z & 1 \end{bmatrix},$$

or more briefly

$$\mathcal{L}^n(p) = (\mathrm{id} + N_1)(p \oplus \mathrm{id})(\mathrm{id} + N_2)$$

with N_1, N_2 nilpotent.

Since $\mathrm{id} + tN$ with $t \in [0,1]$ gives a homotopy of isomorphisms, if N is nilpotent, it follows from Lemma 3.4 that

Proposition 3.2. *$\mathcal{L}^n(p)$ and $p \oplus \epsilon$ define isomorphic bundles on P, i.e.*

$$(E^0, p, E^\infty) \oplus (\sum_{k=0}^{n} L^k \otimes E^0, \mathrm{id}, \sum_{k=0}^{n} L^k \otimes E^\infty) \cong \left(\sum_{k=0}^{n} L^k \otimes E^0, \mathcal{L}^n(p), E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0 \right).$$

For brevity we now write $\mathcal{L}^n(E^0, p, E^\infty)$ for the bundle

$$\left(\sum_{k=0}^{n} L^k \otimes E^0, \mathcal{L}^n(p), E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0 \right).$$

Lemma 3.7. *Let p be a polynomial clutching function of degree $\leq n$ for (E^0, E^∞). Then*

$$\mathcal{L}^{n+1}(E^0, p, E^\infty) \cong \mathcal{L}^n(E^0, p, E^\infty) \oplus (L^{n+1} \otimes E^0, \mathrm{id}, L^{n+1} \otimes E^0)$$

and

$$\mathcal{L}^{n+1}(L^{-1} \otimes E^0, zp, E^\infty) \cong \mathcal{L}^n(E^0, p, E^\infty) \oplus (L^{-1} \otimes E^0, z, E^0).$$

Proof.

$$\mathcal{L}^{n+1}(E^0, p, E^\infty) = \begin{bmatrix} \mathcal{L}^n(p) & 0 & \cdots & 0 \\ 0 & \cdots & -z & 1 \end{bmatrix}.$$

Similarly in

$$\mathcal{L}^{n+1}(p) := \begin{bmatrix} 0 & a_0 & a_1 & a_2 & \cdots & a_n \\ -z & 1 & & & & \\ & -z & 1 & & & \\ & & -z & 1 & & \\ & & & \ddots & \ddots & \\ & & & & -z & 1 \end{bmatrix}$$

we multiply 1 on the second row by t and get a homotopy from $\mathcal{L}^{n+1}(zp)$ to $\mathcal{L}^n(p) \oplus -z$. ☐

Proposition 3.3. *For any polynomial clutching function p for (E^0, E^∞), we have the identity*

$$([(E^0, p, E^\infty)] - [(E^0, \mathrm{id}, E^0)])([L][H] - [\epsilon^1]) = 0.$$

Proof. By the result above, we have

$$(L^{-1} \otimes E^0, zp, E^\infty) \oplus (\sum_{k=0}^{n} L^k \otimes E^0, \mathrm{id}, \sum_{k=0}^{n} L^k \otimes E^\infty)$$
$$\cong \left(\sum_{k=0}^{n} L^k \otimes E^0, \mathcal{L}^n(p), E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0 \right)$$

Passing to $K^0(P)$ this gives

$$[L^{-1}][H^{-1}][(E^0, p, E^\infty)] + [(E^0, \mathrm{id}, E^0)] = [(E^0, p, E^\infty)] + [L^{-1}][H^{-1}][(E^0, \mathrm{id}, E^0)],$$

from which the required result follows. ☐

Putting $E^0 = 1, p = z, E^\infty = L$ and using Example 3.1 we obtain the formula:

$$([H] - [1])([L][H] - [1]) = 0.$$

3.4 Linear Clutching Functions

Suppose T is an endomorphism of a finite-dimensional vector space E, and let S be a circle in the complex plane which does not pass any eigenvalue of T. Then

$$Q = \frac{1}{2\pi i}\int_S (z-T)^{-1}dz$$

is a projection operator in E which commutes with T. The decomposition

$$E = E_+ \oplus E_- := QE \oplus (\mathrm{id}-Q)E$$

is therefore invariant under T, so that we can write

$$T = T_+ \oplus T_-$$

where T_+ has all eigenvalues inside S and T_- has all eigenvalues outside S. This is the spectral decomposition of T corresponding to the two components of the complement of S.

To extend these results to vector bundles, we first make a remark on notation. So far z and hence $p(z)$ have been sections over S. However they extend in a natural way to sections over the whole of L. It will also be convenient to include the ∞-section of P in certain statements. Thus if we assert that $p(z) = az + b$ is an isomorphism outside S, we shall take this to include the statement that a is an isomorphism.

Proposition 3.4. *Let p be a linear clutching function for (E^0, E^∞) and define endomorphisms Q^0, Q^∞ of E^0, E^∞ b putting*

$$Q^0_x := \frac{1}{2\pi i}\int_{S_x} p_x^{-1}dp_x$$

and

????????????

Then Q^0 and Q^∞ are projection operators and

$$pQ_0 = Q^\infty p.$$

Write $E^i_+ = Q^iE$ and $E^i_- = (\mathrm{id}-Q)^iE$ $(i = 1, \infty)$ so that $E^i = E^i_+ \oplus E^i_-$. Then p is compatible with these decompositions so that

$$p = p_+ \oplus p_-.$$

Moreover, p_+ is an isomorphism outside S, and p_- is an isomorphism inside S.

Proof. Because of Lemma 3.3, □

Corollary 3.4.1. *Let p be the same as Proposition 3.4 and write*

$$p_+ = a_+z + b_+,\ p_- = a_-z + b_-.$$

Then putting $p(t,-) := p_+(t,-) + p_-(t,-)$ where

$$p_+(t,-) := a_+z + tb_+,\ p_-(t,-) := ta_-z + b_-$$

for $0 \le t \le 1$, we then obtain a homotopy a linear clutching functions connecting p with $a_+ \oplus b_-$. Thus

$$(E^0, p, E^\infty) \cong (E^0_+, z, L\otimes E^\infty_+) \oplus (E^0_-, z, E^\infty_-).$$

Proof. □

For p a polynomial clutching function of degree $\leq n$ for (E^0, E^∞), $\mathcal{L}^n(p)$ is a linear clutching function for (V^0, V^∞) where

$$V^0 = \sum_{k=0}^{n} L^k \otimes E^0,\ V^\infty = E^\infty \oplus \sum_{k=1}^{n} L^k \otimes E^0.$$

Hence it defines a decomposition

$$V^0 = V^0_+ \oplus V^0_-.$$

To express the dependence of V^0_+ on p, n we write

$$V^0_+ = V_n(E^0, p, E^\infty).$$

Note that this is a vector bundle on X (by definition). If p_t is a homotopy of polynomial clutching functions of degree $\leq n$ it follows by constructing V_n over $X \times I$ that

$$V_n(E^0, p_0, E^\infty) \cong V_n(E^0, p_1, E^\infty).$$

Hence by what we get

$$\begin{gathered} V_{n+1}(E^0, p, E^\infty) \cong V_n(E^0, p, E^\infty) \\ V_{n+1}(L^{-1} \otimes E^0, zp, E^\infty) \cong V_n(E^0, p, E^\infty) \oplus (L^{-1} \otimes E^0), \end{gathered}$$

or equivalently

$$V_{n+1}(E^0, zp, L \otimes E^\infty) \cong L \otimes V_n(E^0, p, E^\infty) \oplus E^0.$$

Finally put all together we obtain the following equation in $K^0(P)$

$$[E^0, p_0, E^\infty] + (\sum_{k=1}^{n} [L^k \otimes E^0])[\epsilon^1] = [V_n(E^0, p, E^\infty)][H^{-1}] + (\sum_{k=0}^{n} [L^k \otimes E^0] - [V_n(E^0, p, E^\infty)])[\epsilon^1],$$

and hence

$$[E^0, p_0, E^\infty] = [V_n(E^0, p, E^\infty)]([H^{-1}] - [\epsilon^1]) + [E^0][\epsilon^1].$$

3.5 The Main Theorem and Its Proof

The main theorem is:

Theorem 3.5. *Let L be a line bundle over the compact space X, H the line bundle over $P(L \oplus \epsilon^1)$ defined above. Then as a $K^0(X)$-algebra, $K^0(P(L \oplus \epsilon^1))$ is generated by $[H]$ subject to the single relation*

$$([H] - [1])([L][H] - [1]) = 0.$$

When $X = *$, $P(L \oplus \epsilon^1)$ is a projective line, i.e. S^2. Then the theorem implies that $K^0(S^2)$ is a free abelian group generated by $[1]$ and $[H]$ such that $([1] - [H])^2 = 0$.

When L is trivial (X nontrivial), we notice that

$$P(\epsilon^2) \cong X \times S^2,$$

so

Corollary 3.5.1. *Let $\pi_1 : X \times S^2 \to X, \pi_2 : X \times S^2 \to S^2$ denote the projections. Then the homomorphism*

$$\begin{gathered} f : K^0(X) \otimes_{\mathbb{Z}} K^0(S^2) \to K^0(X \times S^2) \\ [a \otimes b] \mapsto \pi_1^* a \cdot \pi_2^* b \end{gathered}$$

is a ring isomorphism.

The ideas can be summarized as follows:

The vector bundles over S^2 are well-known and are easily determined

Proof of Theorem 3.5. Let t be an indeterminate. Then because of Proposition 3.3 the mapping $t \mapsto [H]$ induces a $K(X)$-algebra homomorphism

$$\mu : K^0(X)[t]/(t-1)([L]t-1) \to K(P).$$

To prove the theorem we have to show that μ is an isomorphism, and we shall do this by explicitly constructing an inverse.

First let f be any clutching function for (E^0, E^∞). Let f_n be the sequence of Cesaro means of its Fourier series and put $p_n := z^n f_n$. Then if n is sufficiently large, Lemma 3.6 asserts that p_n is a polynomial clutching function (of degree $\leq 2n$) for $(E^0, L^n \otimes E^\infty)$. Define

$$\begin{aligned} \nu_n(f) &\in K^0(X)[t]/(t-1)([L]t-1) \\ f &\mapsto [V_{2n}(E^0, p_n, L^n \otimes E^\infty)](t^{n-1} - t^n) + [E^0]t^n. \end{aligned}$$

Now for sufficiently large n, the linear segment joining p_{n+1} and zp_n provides a homotopy of polynomial clutching functions of degree $\leq 2(n+1)$. Hence

$$\begin{aligned} V_{2n+2}(E^0, p_{n+1}, L^{n+1} \otimes E^\infty) &\cong V_{2n+2}(E^0, zp_n, L^{n+1} \otimes E^\infty) \\ &\cong V_{2n+1}(E^0, zp_n, L^{n+1} \otimes E^\infty) \\ &\cong L \otimes V_{2n}(E^0, p_n, L^n \otimes E^\infty) \oplus E^0, \end{aligned}$$

hence

$$\nu_{n+1}(f) = \{[L][V_{2n}(E^0, p_n, L^n \otimes E^\infty)] + [E^0]\}(t^n - t^{n+1}) + [E^0]t^{n+1} = \nu_n(f)$$

because $(t-1)([L]t-1) = 0$.

Thus $\nu_n(f)$, for large n, is independent of n and so depends only on f. We write it as $\nu(f)$. If now g is sufficiently close to f and n is sufficiently large then the linear segment joining f_n and g_n provides a homotopy of polynomial clutching functions of degree $\leq 2n$ and therefore

$$\nu(f) = \nu_n(f) = \nu_n(g) = \nu(g).$$

So $\nu(f)$ is a locally constant function of f and hence depends only on the homotopy class of f. Hence if E is any vector bundle over P and f a clutching function defining E, we can define

$$\nu(E) := \nu(f),$$

and $\nu(E)$ depends only on the isomorphism class of E. Since $\nu(E)$ is additive for $\oplus$ it induces a group homomorphism

$$\nu : K^0(P) \to K^0(X)[t]/(t-1)([L]t-1)$$

which is clearly a $K^0(X)$-module homomorphism.

Then on the one hand, in $K^0(P)$

$$\begin{aligned} \mu\nu([E]) &= ([V_{2n}(E^0, p_n, L^n \otimes E^\infty)](t^{n-1} - t^n) + [E^0]t^n) \\ &= [V_{2n}(E^0, p_n, L^n \otimes E^\infty)]([H]^{n-1} - [H]^n) + [E^0][H]^n \\ &= [(E^0, p_n, L^n \otimes E^\infty)][H]^n \\ &= [(E^0, f_n, E^\infty)] \\ &= [(E^0, f, E^\infty)] \\ &= [E], \end{aligned}$$

so $\mu\nu$ is the identity. On the other hand, to check $\nu\mu$ is the identity of $K^0(X)[t]/(t-1)([L]t-1)$, it suffices to check

$$\begin{aligned} \nu\mu(t^n) &= \nu([H^n]) \\ &= \nu([1, z^{-n}, L^{-n}]) \\ &= [V_{2n}(1,1,1)](t^{n-1}t^n) + [\epsilon^1]t^n \\ &= t^n \end{aligned}$$

since $[V_{2n}(1,1,1)] = 0$. This completes the proof. $\square$

4 Murphy's law in algebraic geometry: Badly-behaved deformation spaces [V]

4.1 Basic Information

1. **Reading time:** 2020-Dec
2. **Classification:** AG
3. **Content:**
4. **Main background:** Moduli spaces

The paper considers the question: "How band can the deformation space of an object be?" The answer seems to be: "Unless there is some a priori reason otherwise, the deformation space may be as bad as possible."

4.2 Mnëv' Universality Theorem

5 Vector Bundles Over an Elliptic Curve [A]

5.1 Basic Information

1. **Reading time:** 2020-Mar to
2. **Classification:** AG
3. **Content:**
4. **Main background:** Derived categories, Derived Functors

6 Grothendieck Duality Made Simple [N]

6.1 Basic Information

1. **Reading time:** 2020-Mar to
2. **Classification:** AG, Homological Algebra
3. **Content:**
4. **Main background:** Derived categories, Derived Functors

We will assume the following:

7 Stable and unitary vector bundles on a compact Riemann surface [NS]

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