# Some Open Problems I am Interested in

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**Problem 1** (Kervaire Invariant Problem in 126, Geometrical Topology)**.** In 1960, Kervaire proved that there is a manifold homotopic to a sphere $S^2$ without any smooth structure [4]. What he did was actually constructing a "Kervaire invariant" that lived in $\pi_2^s(S^0)$. In 1969, Browder proved that the Kervaire invariant could only live in $\pi_{2^n-2}^s(S^0)$ [1] and soon Barratt, Jones, Mahowald proved for $n = 2, 3, 4, 5, 6$ the fundamental group was not trivial and the element existed [2]. Since then 126 has become an open problem for the existence. In 2016, Hill, Hopkins and Ravenel proved $\pi_{256k-2}^s(S^0) = 0$ so that the invariant does not exist for $n \geq 8$. Thus the only left one is $n = 7$ [3].

**Problem 2** (Derived Algebraic Geometry, Algebraic Geometry)**.**

**Problem 3** (Class Number Problem, Number Theory)**.** When we were solving number theoretical equations like

$$x^2 + dy^2 = n$$

for some integer $n$, it is helpful for consider the primes in $\mathbb{Z}[\sqrt{-d}]$. Since generally $\mathbb{Z}[\sqrt{-d}]$ is not a U.F.D., it is better to consider its integral closure $\mathcal{O}_K$ where $K = \mathbb{Q}(\sqrt{-d})$. This ring $\mathcal{O}_K$ is a Dedekind domain, Gauss, in his famous book *Disquisitiones Arithmeticae* of 1801 (Section V, Articles 303 and 304), posed these questions

**Problem 4** (The Section Conjecture, Algebraic Geometry)**.** When we were dealing with a scheme $X$, the usual construction of the fundamental group fails since it is difficult to have a path in $X$. However by the relations between the covering spaces and their deck transformation groups, we are able to have a notion of étale fundamental group $\pi_1^{\text{et}}(X)$.

## References

[1] Browder, William., *The Kervaire invariant of framed manifolds and its generalization*. Annals of Mathematics. **90** (1) (1969): 157186. JSTOR 1970686.

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[3] Hill, Michael A.; Hopkins, Michael J.; Ravenel, Douglas C., *On the nonexistence of elements of Kervaire invariant one*. Annals of Mathematics. **184** (1) (2016).: 1262. arXiv:0908.3724.

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