Arithmetic & Algebraic Geometry

Ann Arbor, Michigan
Notes Taken by Guanyu Li
gl479 at cornell.edu

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1 Aise Johan de Jong: Local Picard groups

Abstract

We will discuss a local Lefschetz type theorem for Picard groups. In particular, we extend a theorem of Kollar on injectivity of the restriction map to the mixed characteristic case. The proofs use only classical commutative algebra.

2 Hélène Esnault: Frobenius invariant Subloci of formal Lie groups of multiplicative type over an l-adic ring, and applications

Abstract

(All joint with Moritz Kerz.) One application of the understanding of those loci is the Hard Lefschetz theorem in rank 1 in positive characteristic. Another one is on the codimension of the cohomological subloci defined by the Mellin transform in special towers. We shall review Delignes purity concept and the application of L. Lafforgues theorem to purity, then the notions defined in the title, then the method to find torsion points of those loci, then the applications.

2.1 August 7th

3 Wei Ho: Hessian constructions for genus one curves

4 Daniel Huybrechts: Algebraic and arithmetic aspects of twistor spaces (August 6th)

Goal: suppose $\zeta \to \mathbb{P}^1$ is a twistor space, and L_1, L_2 are two fibers, we want to know the relation of the fibers.

5 Daniel Krashen: Field patching, local-global principles and rationality (August 5th)

Abstract

This talk will present a brief survey of local-global principles for torsors for algebraic groups over higher dimensional arithmetic fields via field patching techniques. In particular, I'll discuss new work which makes a connection between obstructions to such local-global principles and obstructions to rationality of algebraic groups.

reference

Goal: Understanding the arithmetical fields as structure of algebraic objects. Suspicion: if K is a number field, X/K is a curve on the plane Ω_K, $F = K(X)$, G is a linear algebraic group then $\mathrm{Ker}(H^1(F,G) \to \prod_{v\in\Omega_K} H^1(K_v(X),G))$. Evidence: E/K is an elliptic curve, $G = PGL_n$, then $\mathrm{Ker}() \subseteq \#(K,E)[n]$.

Given a curve X over some complete discrete valuation field, we choose a regular model $\mathscr{X}/\mathcal{O}_K$

6 Jacob Lurie, Tamagawa Numbers in the Function Field Case

Abstract

Let G be a connected semisimple algebraic group over a global field K, and let A denote the ring of adeles of K. Tamagawa observed that the locally compact group G(A) is equipped with a canonical translation-invariant measure. A celebrated conjecture of Weil asserts that if G is simply connected, then the measure of the quotient space G(A)/G(K) is equal to 1. When K is a number field, this conjecture was proven Kottwitz (following earlier work of Langlands and Lai). In these talks, I'll discuss joint work with Dennis Gaitsgory about the function field case, exploiting ideas from algebraic topology.

6.1 August 5th

First take $K = \mathbb{Q}$ and G is an algebraic group of dimension d. The adele ring

$$\mathbb{A} = \mathbb{R} \times \prod_p^{res} \mathbb{Q}_p \subseteq \mathbb{R} \times \prod_p \mathbb{Q}_p$$

which is locally compact. Then we can talk about $G(\mathbb{A})$, which contains $G(\mathbb{Q})$ where $G(\mathbb{A})$ is locally compact and $G(\mathbb{Q})$ is discrete.

$G(\mathbb{A}) = G(\mathbb{R}) \times \prod_p^{res} G(\mathbb{Q}_p)$ has a left canonical invariant measure which is callen Tamagawa measure. $G(\mathbb{R})$ is a Lie group, let $V_\mathbb{R}$ be the space of translation invariant d-forms on G. Let $V_\mathbb{Q} \subseteq V_\mathbb{R}$ be the algebraic differential forms defined over $\mathbb{Q}$. So for one $\omega \in V_\mathbb{Q}$ we have a Haar measure $\mu_{\omega,\mathbb{R}}$ on $V_\mathbb{R}$. ω also determines a Haar measure $\mu_{\omega,\mathbb{Q}_p}$ on $G(\mathbb{Q})$, so we have a construction

$$\mu := \mu_{\omega,\mathbb{R}} \times \prod_p \mu_{\omega,\mathbb{Q}_p}.$$

Conjecture 1 (Weil)**.** If G is semisimple and simply connected, then

$$\mu(G(\mathbb{A})/G(\mathbb{R})) = 1.$$

For example, take $V = \{\text{rational }\}$. This conjecture was proved by Kottwits following Langlans, Lai.

More generally, we can do the same things on a global field. If $K/\mathbb{Q}$ is a finite extension, G is some algebraic group over K. $G(K) \subseteq G(\mathbb{A}_K)$, $H := \mathrm{Res}_G^K(???)$.

In the rest of this lecture, $x \in X$ is the closed point of a smooth projective connected variety over $\mathbb{F}_q$, K_X is the function field of X, $\kappa(x)$ is the residue field, $\mathcal{O}_x$ is the complete local ring at x, $= \kappa(x)$, K_x is the fraction field of $\mathcal{O}_x$. We have

$$K_X \subseteq \mathbb{A}_X = \prod_{x\in X}^{res} K_x.$$

Suppose G_0 is an algebraic group over K_X, then $G(K_X) \subseteq G(\mathbb{A}_X)$. To define the Tamagawa measure, we choose a translation invariant top form ω in G_0, then this gives a measure $\mu_{\omega,x}$. Then we want to choose an integral model of G_0. Suppose $G \to X$ is a smooth affine group scheme with connected fibers. Then

$$G(K_x) \supseteq G(\mathcal{O}_x) \to G(\kappa(x))$$

where the latter ????

For most of $x \in X$, ω has no zeros or poles at x. In this case,

$$\mu_{\omega,x}(G(\mathcal{O}_x)) = \frac{|\kappa(x)|}{|\kappa(x)|^d}.$$

This was not quite right, since there are still finitely many point where ω has zeros or poles. So we consider $\prod_{x\in X} G(\mathcal{O}_x)$ acting on $G(\mathbb{A}_X)/G(K_X)$. Then we have

$$\{\text{principle } G \text{ bundle on } X\}/\text{isomorphisms} \simeq \frac{G(\mathbb{A}_X)/G(K_X)}{\prod_{x\in X} G(\mathcal{O}_x)}.$$

Theorem. *Let $\mathcal{P}$ be a G-bundle on X, assume G_0 is simply connected, then*

1. *(Hurder) $\mathcal{P}$ is trivial at the generic point of X.*
2. *(Larey) $\mathcal{P}$ is trivial at each* $\operatorname{Spec} \mathcal{O}_x$.

We can choose some trivialization $\mathcal{P}|_{\text{Spce } K_X}$ and $\mathcal{P}|_{\text{Spce } \mathcal{O}_x}$, and we differ $\text{Spce } K_X$ by an element of $G(K_X)$. Therefore, we have a naive guess

$$\mu(G(\mathbb{A}_X)/G(K_X)) = (\#\text{ of } G \text{ bundles on } X)\, q^{-D} \prod_{x\in X}^{res} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}.$$

This is true only for when the group action is free. We need to know the multiplicity. The correct conjecture is

$$\mu(G(\mathbb{A}_X)/G(K_X)) = \left(\sum_{\mathcal{P} \text{ a } G \text{ bundle on } X} |\frac{1}{\operatorname{Aut}(\mathcal{P})}|\right) q^{-D} \prod_{x\in X}^{res} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}.$$

Hence we can restate the Weil's conjecture

$$\frac{\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}}{q^D} = \prod_{x\in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|}.$$

Then we consider a stack $\operatorname{Bun}_G(X)$, sending R to G-bundles over $X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} R$. Then we have that $\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|} = |\operatorname{Bun}_G(X)(\mathbb{F}_q)|$. For each $x \in X$, we can also consider $\operatorname{Bun}_G(\{x\}) = \operatorname{Bun}(\operatorname{Res}_{\mathbb{F}_q}^{\kappa(x)} G_x)$. We have $|\operatorname{Bun}_G(\{x\})(\mathbb{F}_q)| = \frac{1}{G(\kappa(x))}$, so we can restate the Weil's conjecture

$$\frac{|\operatorname{Bun}_G(X)(\mathbb{F}_q)|}{q^{\dim \operatorname{Bun}_X(G)}} = \frac{\sum_{\mathcal{P}} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}}{q^D} = \prod_{x\in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|} = \prod_{x\in X} \frac{|\operatorname{Bun}_G(\{x\})(\mathbb{F}_q)|}{q^{\dim \operatorname{Bun}_{\{x\}}(G)}}.$$

6.2 Aug 6th

Recap: Suppose X be a projective scheme over $\operatorname{Spec} \mathbb{F}_q$, G is a smooth affine algebraic group with connected fibers, and generically it is semisimple simply connected. The Weil conjecture is

$$\frac{|\operatorname{Bun}_G(X)(\mathbb{F}_q)|}{q^{\dim \operatorname{Bun}_X(G)}} = \prod_{x\in X} \frac{|\operatorname{Bun}_G(\{x\})(\mathbb{F}_q)|}{q^{\dim \operatorname{Bun}_{\{x\}}(G)}}.$$

Recall if Y is an algebraic variety over $\mathbb{F}_q$, there is a formula for $|Y(\mathbb{F}_q)|$, the $\mathbb{F}_q$ points in terms of étale fundamental group of Y. This lecture is to count points via $\pi_*(Y)$.

Suppose k be an algebraically closed field of characteristic $p > 0$ ($k = \bar{\mathbb{F}}_q$), l is another prime number, Y is a smooth variety over k ($\operatorname{Bun}_G(X) \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} k$) with base point $y \in Y(k)$. Grothendieck defined the étale fundamental group which is a profinite group. We want $\pi_1(Y, y)_l$.

Artin and Mazur had a construction, with a connected algebraic variety, output a simply connected topological space called the étale homotopy type of Y, denoted by $EHT(Y)$. The construction has properties:

1. Each $\pi_n(EHT(Y))$ is a finitely generated $\mathbb{Z}_l$ module.
2. $H^*(EHT(Y), \mathbb{F}_l) \cong$

Then we can define that

$$\pi_n(Y) := \pi_n(EHT(Y))$$

which is a finitely generated $\mathbb{Z}_l$-module, and

$$(\pi_n(Y))_{\mathbb{Q}_l} := (\pi_n(Y))[\frac{1}{l}],$$

which is a finite dimensional $\mathbb{Q}_l$-vector space. We have the relationship: the bilinear map

$$H^n(Y, \mathbb{Q}_l) \times \pi_n(Y)_{\mathbb{Q}_l} \to \mathbb{Q}_l,$$

factors through $I \wedge^2 \times \pi_n(Y)_{\mathbb{Q}_l}$ with $I = \bigoplus_{n>0} H^n(Y, \mathbb{Q}_l)$.

Special case: If $H^*(Y, \mathbb{Q}_l) \cong$

Examples: (i) $Y = \mathbb{G}_m$, then $H^*_{et}(Y, \mathbb{Q}_l) \cong \mathbb{Q}_l[v]/(v^2)$ with $\deg\ v = 1$, then the only nontrivial fundamental group is $\pi_1 = \mathbb{Q}_l$.
(ii) $Y = SL_n$, with nontrivial at $n = 3, 5, \cdots, 2n-1$.

Here our goal is to compute $|Y_0(\mathbb{F}_q)|$, where Y_0 is a variety over $\mathbb{F}_q$. Suppose Y is the base change of Y_0 to k. We also have $\mathbb{F}_q$ acting on Y, then

Theorem (Grothendieck-Lefschetz trace formula)**.**

$$|Y_0(\mathbb{F}_q)| = \sum_i (-1)^i \mathrm{Tr}(\mathcal{Q}|H^i_c(Y, \mathbb{Q}_l)|).$$

Suppose Y is smooth, then we have the Poincare duality, there is a perfect pairing $H^i_c(Y, \mathbb{Q}_l) \times H^{2d-i}(Y, \mathbb{Q}_l) \to H^{2d}(Y, \mathbb{Q}_l) \cong \mathbb{Q}_l(-d)$, the upshot is

$$Tr(\mathcal{Q}|H^i_c(Y, \mathbb{Q}_l)) = q^d Tr(\mathcal{Q}^{-1}|H^{2d-i}(Y, \mathbb{Q}_l)).$$

So the dual version of Grothendieck-Lefschetz trace formula is

$$\frac{|Y_0(\mathbb{F}_q)|}{q^d} = \sum_i (-1)^i \mathrm{Tr}(\mathcal{Q}|H^i(Y, \mathbb{Q}_l)|).$$

Assume $H^*(Y, \mathbb{Q}_l) \cong \wedge^*(V)$, and $\mathbb{Q}_l \hookrightarrow \mathbb{C}$. Set $V = I/I^2$ has an action of Frobenius with generalized eigenvalues $\lambda_1, \cdots, \lambda_r$. On $H^*(Y, \mathbb{Q}_l)$, $\mathcal{Q}^{-1}$ has generalized eigenvalues $\lambda_{i_1}, \cdots, \lambda_{i_m}$, then

$$\frac{|Y_0(\mathbb{F}_q)|}{q^d} = \sum_{1 \le i_1 < \cdots < i_m \le r} (-1^m) \lambda_{i_1} \cdots \lambda_{i_m}$$

Let Y be any algebraic group G defined over $\mathbb{F}_q$, we have the Steinbergs formula

$$|G(\mathbb{F}_q)| = q^{\dim\ G}(\det(1 - \mathcal{Q}|_{\pi_*(\bar{G})_{\mathbb{Q}_l}})).$$

In general, there is spectral sequence

$$\mathrm{Sym}^*_{gr}(\pi_*(Y)^\vee_{\mathbb{Q}_l}) \to H^*(Y, \mathbb{Q}_l),$$

modolo convergence we conclude that

$$\frac{|Y_0(\mathbb{F}_q)|}{q^d} = \frac{\det(1 - \mathcal{Q}|_{\pi_{all}(Y)_{\mathbb{Q}_l}})}{\det(1 - \mathcal{Q}|_{\pi_{even}(Y)_{\mathbb{Q}_l}})}$$

Example: $Y_0 := B\bar{G}$, by some topology we should have

$$\pi_*(B\bar{G})_{\mathbb{Q}_l} = \pi_{*-1}(\bar{G})_{\mathbb{Q}_l} = \frac{|G(\mathbb{F}_q|^{-1}}{q^{-d}} = \det(1 - \mathcal{Q}|_{\pi_*(\bar{G})_{\mathbb{Q}_l}})$$

and by Weil's conjecture we have

$$\det(1 - \mathcal{Q}|_{\pi_*(B\bar{G})_{\mathbb{Q}_l}})^{-1} = \frac{|BG(\mathbb{F}_q)|}{q^{\dim\ BG}}.$$

6.3 August 7th

7 Davesh Maulik: Topology of Higgs moduli spaces via abelian surfaces (August 5th)

Abstract

In this talk, we study cases of the $P = W$ conjecture for Higgs bundles on a curve, using techniques from compact hyperkähler geometry. This is joint work in progress with Mark de Cataldo and Junliang Shen.

Setup: let C be a smooth projective curve over $\mathbb{C}$ with genus $g \geq 2$. Suppose a moduli problem

$$\mathcal{M}_{\text{Higgs}} = \{\text{vector bundle } E \text{ over } C \text{ with rank } r \text{ and degree } d.\}$$

where $E \xrightarrow{\varphi} E \otimes K_{\mathbb{C}}$ has stability. The functor sends a Higgs bundle to pure 1-dimensional sheaf on T^*C with proper support

$$[\text{supp } \mathcal{E}] = r[C].$$

For smooth variety with holomorphic symplectic, we have another space, which is the twisted character variety of C:

$$\mathcal{M}_{\text{Betti}} = \{\text{smooth affine varieties } A_1, \cdots, A_g, B_1, \cdots, B_g, \text{s.t. } \prod [A_i, B_i] = e^{2\pi i d/r}\}//GL_r(\mathbb{C})$$

The talk starts with non-abelian Hodge theory, i.e. there is a diffeomorphism

$$\mathcal{M}_{\text{Higgs}} \simeq \mathcal{M}_{\text{Betti}}.$$

For example, when $r = 1$, we have

$$\text{Pic}_d(\mathbb{C}) \times \mathbb{C}^g \xrightarrow{\sim} (\mathbb{C}^*)^{2g},$$

and so

$$H^*(\mathcal{M}_{\text{Higgs}}) \cong H^*(\mathcal{M}_{\text{Betti}}).$$

Conjecture 2 ($P = W$)**.** $\mathcal{M}_{\text{Betti}}$ carries a mixed Hodge structure

$$W_K H^* \subseteq H^*,$$

then what is the meaning of W_K on $H^*(\mathcal{M}_{\text{Higgs}})$?

We need some extra structure. Suppose we have the Hitchin map $\mathcal{M}_{\text{Higgs}} \xrightarrow{\pi} \mathbb{A}^N$, and hence use π to define a filtration $\mathcal{E} \mapsto \text{supp } \mathcal{E}$, whose fiber is $\text{Pic}_d(C)$. For example suppose $X \xrightarrow{\pi} Y$ is a proper filtration. We have a decomposition theorem

$$R\pi_* \mathbb{Q}_X[\dim \ X - a] \cong \bigoplus_{k=0}^{2a} P_k[-k],$$

where the right hand side preserve the sheaf on Y. If X_y is smooth, then $P_k|_y = H^k(X_y)[-]$. Hence we define

$$P_k H^d(X) = \text{Im}(H^{d-(\dim \ X-a)}(Y, \bigoplus_{i \leq k} P_i[-i]) \to H^d(X)).$$

In our case, $P_k H^d(X) = \text{Ker}(H^d(X) \to H^d(\pi^{-1}(Y')))$ (which is hard by dC, Mig) where Y' is a generic plane of dimension $d - k - 1$.

Conjecture 3.

$$P_k H^*(\mathcal{M}_{\text{Higgs}}) = W_{2k} H^*(\mathcal{M}_{\text{Betti}}) = W_{2k+1}.$$

We already have

1. all genus for $r = 2$.
2. Hard Lefschetz for $gr_W H^*$ by Mellit.

The main results are

Theorem. *$P = W$ holds for $g = 2$ curves for all rank r, degree d, with $(r, d) = 1$. When $g > 2$, for some $E \to C \times \mathcal{M}_{Higgs}$, twist: for any $\alpha \in H^2(C) \oplus H^2(\mathcal{M}_{Higgs})$, let*

$$ch^\alpha(E) = ch(E) \cup e^\alpha \in H^*(C) \oplus H^*(\mathcal{M}_{Higgs}).$$

Pick α s.t. $ch_1^\alpha(E) \in H^1 \oplus H^1$. For any $\gamma \in H^*(C)$, $ch(k, \gamma) = \int_\gamma ch_k^\alpha(E) \in H^*(\mathcal{M})$

Idea of the proof. □

8 Bjorn Poonen: The local-global principle for stacky curves (August 5th)

Abstract

For smooth projective curves of genus g over a number field, the local-global principle holds when g=0 and can fail for g=1, as has been known since the 1940s. Stacky curves, however, can have fractional genus. We construct stacky curves of genus 1/2 that violate the local-global principle, and show that 1/2 cannot be reduced. This is joint work with Manjul Bhargava.

Local-global principle: fix some genus g, if a smooth projective geometrically integral curve X of genus g over a number field k, has a k_v point for every place v, must it have a k-point?

1. Yes, if $g = 0$.
2. No, if $g \geq 1$. E.g. $X : 2y^2 = 1 - 17x^4$.

We want to ask

1. What if X is a stack and $0 < g < 1$?
2. What is the smallest g for which the local-global principle fails?

Root Stacks: Problem: given a scheme V, an effective Cartier divisor, $n \in \mathbb{N}$, how can we modify V so that we can replace D by $\frac{1}{n}D$? The solution is to assume $V = \text{Spec } A$, D is principle workably, choose $f \in A$ s.t. $D = (f)$, then

$$X := [\text{Spec } A[y]/(y^n - f)/\mu_n].$$

Suppose k is algebraically closed field of characteristic 0, we define a stacky curve over k is a smooth irreducible 1 dimensional Deligne-Munford stack X containing a nonempty open substack isomorphic to a scheme. Fact is that X is a smooth integral curve over X_{curve} with $P_1, \cdots, P_n$ replaced by $\frac{1}{e_1}P_1, \cdots, \frac{1}{e_n}P_n$.

Next we define Euler characteristic

$$\chi := \chi_{\text{curve}} - \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} \frac{1}{e_1}$$

and genus to be $2 - 2g = \chi$. If k is not algebraically closed, a stacky curve over k is some algebraic stack X s.t. $X_{\bar{k}} := X \times \text{Spec } \bar{k}$ is a stacky curve.

$$X(A) := \{\text{morphismsSpec } A \to X\}/\text{isomorphisms}.$$

Notice that a stacky curve of genus 0 has k-point if and only if the coarse space has a k-point. So we want to study the integral points.

Example: pick three positive natural number p, q, r and let

$$S = \text{Spec} \frac{\mathbb{Z}[x, y, z]}{x^p + y^q - z^r} - (0, 0, 0) \subseteq \mathbb{A}^3,$$

and $\mathbb{G}_m^3$ has an action on it. Then

$$S(\mathbb{Z}) = \{\text{gcd 1 integer solutions to } x^p + y^q - z^r = 0\}.$$

Let H be the subgroup of $\mathbb{G}_m^3$ preserving S, which leads to a fact $S(\mathbb{Z})$ is

1. finite if $\chi < 0$;
2. infinite if $\chi > 0$.

Counterexample of genus $\frac{1}{2}$: if $p, q, r \equiv 7 \pmod 8$ s.t. $\left(\frac{p}{q}\right) = \left(\frac{p}{r}\right) = \left(\frac{r}{q}\right) = 1$, $f(x,y) = ax^2 + bxy + cy^2$ is of discriminant $-pqr$ and $\left(\frac{a}{q}\right) = 1$, $\left(\frac{a}{p}\right) = \left(\frac{a}{r}\right) = -1$. Let $Y = \mathrm{Proj}\frac{\mathbb{Z}[x,y,z]}{z^2 - f(x,y)}$ with a μ_2 action (λ acting on $(x,y,z) \mapsto (x,y,\lambda z)$), and finally let $X := [Y/\mu_2]$, (E.g. $p = 7, q = 47, r = 31$.) then

1. $\chi_X = 1, g = \frac{1}{2}$.
2. $X(\mathbb{Z}_l) \neq \emptyset$ for all l, hence $X(\mathbb{R}) \neq \emptyset$.
3. $X(\mathbb{Z}) = \emptyset$

Theorem (Local-global principle for $\chi > 1$)**.** *Suppose k is a finite field over $\mathbb{Q}$, S is a finite set of places of k containing the Archimedean places, $\mathcal{O} = \mathcal{O}_{k,S}$, $k_v :=$ the completion of k at v, and $\mathcal{O}_v$ is the valuation ring of k if $v \notin S$ and k_v if $v \in S$. Then X is a separated finite type algebraic (Artin) stack over* $\mathrm{Spec}\ \mathcal{O}$ *s.t.*

1. *$X_{\bar{k}}$ is a stacky curve with $\chi > 1$.*
2. *If $X(\mathcal{O}_v) \neq \emptyset$ for every v, then $X(\mathcal{O})$ is not empty.*

Sketch of the proof. For simplicity, suppose $\mathcal{O} = \mathbb{Z}$. $\chi > 1$ means that $X_{\mathbb{Q}}$ is some smooth proper $\mathcal{O}$-curve (X_{curve}) of genus 0 and with at most 1 stacky point. $X(\mathbb{Z}_p) \neq \emptyset$ for all p implies $X_{\text{curve}}(\mathbb{Z}_p) = X_{\text{curve}}(\mathbb{Q}_p) \neq \emptyset$ for all p, implying $X_{\text{curve}} \simeq \mathbb{P}^1_{\mathbb{Q}}$, $X_{\mathbb{R}} \supseteq \mathbb{A}^1_{\mathbb{R}}$.

Hence $X_{\mathbb{Z}[\frac{1}{N}]}$ contains $\mathbb{A}_{\mathbb{Z}[\frac{1}{N}]}$ for some N. Again for simplicity suppose $N = p$ for some prime number. Thus $X(\mathbb{Z}_p) \neq \emptyset$ implies X has many $\mathbb{Z}_p$-points (*), the subset $X(\mathbb{Z}_p) \subseteq X(\mathbb{Q}_p)$ contains a nonempty open subset U of $\mathbb{A}^1(\mathbb{Q}_p)$. **By strong approximation**, there exists an $x \in \mathbb{A}^1(\mathbb{Z}[\frac{1}{p}])$ s.t. $x \in U$, then $x \in X(\mathbb{Z}[\frac{1}{p}]) \cap X(\mathbb{Z})$, and (*) tells us $x \in X(\mathbb{Z})$. $\square$

9 Rachel Pries: Infinite clutching systems and unlikely intersections with the Newton polygon stratification (August 6th)

Abstract

Clutching morphisms have been important for proving many results about the moduli space of curves. In this work, we study clutching systems for moduli spaces of cyclic covers of the projective line and PEL-type Shimura varieties. We focus on the Kottwitz sets and Newton polygon stratification for the moduli p reduction of these moduli spaces. We prove that the Newton polygon stratification cooperates well with the clutching morphisms under certain compatibility conditions. As an application, we find infinitely many situations when a conjecture of Oort is true and when the Newton polygon stratification of the moduli space of abelian varieties has an unlikely intersection with the Torelli locus. This is joint work with Li, Mantovan, and Tang.

Cyclic covers: $X \xrightarrow{\mathbb{Z}/m\mathbb{Z}} \mathbb{CP}^1$, with equation $y^m = \prod_{i=1}^{N}(x - b_i)^{a_i}$ s.t. $0 \leq a_i < m$, $\sum a_i \equiv 0 \pmod m$. $\gamma = (m, N, a = (a_1, \cdots, a_N))$ the monodromy datum determines the genus g. We have $H^0(X, \Omega^1) \cong \bigoplus_{i=1}^{m-1} L_i$,we also have the signature type $f = (f_1, \cdots, f_{m-1})$ where $f_i = \dim\ L_i$. Let Z_γ be the Hurwitz space which "is" the subspace of $\bar{\mathcal{M}}_g$ with dimension $N - 3$. We also have the Torelli morphism $T : Z_\gamma \to S_\sigma$ the Shiruma vatiery.

First attempt: build singular curve with given Newton polygon σ then deform it without changing the Newton polygon. Strategy: part 1: work with Newton polygons in $\mathbb{Z}_\sigma$ that are μ-ordinary.

Set-up: two initial datum $\gamma_i = (m_i, N_i, a_i)$ with $d = \frac{m_2}{m_1}$, let $\kappa : Z_{\gamma_1} \times Z_{\gamma_2} \to \bar{Z}_{\gamma_3}$. We assume that it is admissible, $d \cdot a_1(N) + a_2(1) \equiv 0 \pmod m$), wichi is equivalent to that we can deform to a smooth curve with $\mathbb{Z}/m\mathbb{Z}$ action of type γ_3.

10 Karen Smith: Non-Commutative Resolution of Singularities and Frobenius (August 6th)

Abstract

Consider a finitely generated commutative algebra R over a field K. Roughly speaking, a non-commutative resolution of singularities of Spec R is a (non-commutative) R-algebra A with finite global dimension, meaning that (like a commutative regular local ring), every module over A has a finite projective resolution. Typically, the algebra A has the form End(M) where M is some finitely generated R-module. The existence of a non-commutative resolution for a commutative ring R places strong conditions on R, such as rational singularities. In this talk, we discuss how in prime characteristic, the Frobenius can be used to construct non-commutative resolutions of nice enough rings. We conjecture that for a strongly F-regular ring R, $End(F_*R)$ is a non-commutative resolution of R, where F_*R denotes R viewed as an R-module via restriction of scalars from Frobenius. We prove this conjecture when R is the coordinate ring of an affine toric variety. We also show that for toric rings, the ring of differential operators D(R) has finite global dimension (joint with Eleonore Faber and Greg Muller).

We assume our modules are right modules. Define the projective dimension as the length of shortest possible projective resolution. Define the global (homological) dimension of a ring R is the supremum of all projective dimensions of all right & left modules.

Theorem (Hilbert syzygies,1890)**.** *The global dimension of $R[x_1, \cdots, x_n]$ is n.*

Theorem (Serre, 1955)**.** *A commutative local ring R has finite global dimension if and only if R is regular.*

The idea is that we can generalize the notion of smoothness to a noncommutative ring.

Definition 1 (Van den Bergh)**.** Let R be a Noetherian commutative ring, a non-commutative resolution of singularities of R is a ring $\Lambda = \mathrm{End}_R M$ where

1. M is a finitely generated reflexive module.

2. Λ is of finite global dimension.

Furthermore, we say Λ is crepitant if all simple Λ modules have the same projective dimension.

Recall that the classical definition of a resolution of $\mathrm{Spec}\, R$ is a proper morphism from a smooth scheme $X \xrightarrow{\pi} \mathrm{Spec}\, R$ s.t. it is an isomorphism on a smooth locus of $\mathrm{Spec}\, R$. We would like to ask, when does R admit a noncommutative resolution? When can we construct cannonical noncommutative resolution?

Theorem. *A normal ring $(R, \mathfrak{m})$ over k with characteristic 0 has a noncommutative resolution only if $\mathrm{Spec}\ R$ admits a rational singularities.*

We consider the positive characteristic situation, then we have the Frobenius map $R \xrightarrow{F} R$.

Theorem (Kunz, 1964)**.** *R is regular if the pushforward R module by the Frobenius map F_*R is a flat R-module.*

Assume R is F-finite, i.e. F is a finite morphism.

Definition 2. Suppose R is a F-finite Noetherian commutative ring, then we say R is F-regular if there is some $c \in R$ and some integer e s.t. the map $R \to F_*^e R$ splits as a map of R modules.

Facts:

1. Regular inplies F-regular.

2. $R \hookrightarrow S$ splits as R-modules, then the F-regularity of S implies S.

3. If a finite group G acting on R, then R^G is F-regular.

Conjecture 4. Suppose $(R, \mathfrak{m})$ be a Noetherian F-finite, F-regular local ring with characteristic > 0, then $\mathrm{End}_R F_*^e R$ is of finite global dimension.

11 Jason Starr: Symplectic invariants and rational points in positive characteristic (August 6th)

Abstract

Tsen's Theorem produces a rational point over a function field of a curve for every smooth complete intersection of type $(d_1, ...d_c)$ in projective n-space provided the Fano index $i = n - (d_1 + ... + d_c)$ is positive. Is there more than one rational point? Zhiyu Tian, Runhong Zong and I prove "weak approximation" by rational points at all places of potentially good reduction if $i > 1$ and if the characteristic$p > max(d_1, ..., d_c)$. This follows from a general theorem proving cohomology vanishing and separable uniruledness of Fano manifolds with cyclic Picard group whenever p is prime to certain Gromov-Witten invariants.

reference1 reference2

Set-up: $k(B)$ the field of rational functions on a smooth, connected, projective curve B over the field k. X is the common intersection in $\mathbb{P}^n_{k(B)}$ of c hypersurface of degrees $(d_1, \cdots, d_c)$. The Fano index is $i(X) = n + 1 - (d_1 + \cdots + d_c)$. X is smooth of dimension $n - c$, and $c_1(T_X) = i(X) \cdot c_1(\mathcal{O}_1)$.

Theorem (Tsen's). *For algebraically closed field k, if $i(X) > 0$ then $\#(X(k(B)) > 0$.*

Furthermore, if X is singular sometimes $\#(X(k(B)) = 1$. There are still some questions remaining open: if X is smooth of dimension $n - c$ and $i(X) > 0$, is $\#(X(k(B)) > 1$? Is $\#(X(k(B))$ infinite? Zariski dense? Dense for a finer topology?

Theorem (Starr-Tian-Zong). *For algebraically closed field k, if X is a smooth variety over k and $i(X) > 0$, $p > \max\{d_1, \cdots, d_c\}$, then $X(k(B))$ satisfies weak approximation at places of potentially good reduction, such reduction are separably rationally connected, and such reduction are separably uniruled by lines.*

The equality $p > \max\{d_1, \cdots, d_c\}$ is approximately sharp.

Theorem. *For Fano X over $k(B)$ with $\mathrm{Pic} = \mathbb{Z}$, potentially good reductions of X are separably rationally connected if and only if they are separably uniruled and $h^0(X, \Omega^r_X) = 0$ for all $r > 0$.*

Definition 3.

1. We say a variety has a potentially good reduction at t if there exists a tame stacky curve $\mathcal{B} \to \hat{\mathcal{O}}_{\mathcal{B},t}$ and smooth, proper, representable $\mathcal{X} \to \mathbb{B}$ with generic fiber $X \otimes_{k(B)} \mathrm{Frac}(\hat{\mathcal{O}}_{\mathcal{B},t})$.
2. We say X is weak approximation away from Σ if the image of $X(k(B))$ in the adelic points $X(\mathbb{A}_{\mathcal{B},\Sigma})$ is dense.
3. We say X is separably uniruled if the evaluation map $\mathrm{ev}^1 : \mathcal{M}_{0,1}(X) \to X$ is somewhere smooth.
4. The Uniruling index $u_1(X)$ is the g.c.d. of the degrees of all$\mathrm{ev}^1|_Z : Z \to X$ for every $Z \to \mathcal{M}_{0,1}(X)$ an integral $k(B)$ scheme with $\mathrm{ev}^1|_Z$ dominant and generically finite.
5. We say X is separably rationally connected if the evaluation morphism $\mathrm{ev}^1 : \mathcal{M}_{0,2}(X) \to X \times X$ is somewhere smooth.
6. The torsor order $\tau(X)$ is the denominator in the decomposition of diagonal.

12 Lenny Taelman: Derived equivalences of hyperkhler varieties (August 7th)

Abstract

In this talk we consider auto-equivalences of the bounded derived category D(X) of coherent sheaves on a smooth projective complex variety X. By a result of Orlov, any such auto-equivalence induces an (ungraded) automorphism of the singular cohomology $H(X, \mathbb{Q})$. If X is a K3 surface, then work of Mukai, Orlov, Huybrechts, Macrì and Stellari completely describes the image of the map ρ_X : Aut $D(X) \to$ Aut $(H(X, \mathbb{Q}))$. We will study the image of ρ_X for higher-dimensional hyperkähler varieties. An important tool is a certain Lie algebra acting on H(X, Q), introduced by Verbitsky, Looijenga and Lunts. We show that this Lie algebra is a derived invariant, and use this to study the image of ρ_X.