Arithmetic & Algebraic Geometry

Ann Arbor, Michigan
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1 Hélène Esnault: Frobenius invariant Subloci of formal Lie groups of multiplicative type over an l-adic ring, and applications

2 Wei Ho: Hessian constructions for genus one curves

3 Daniel Krashen: Field patching, local-global principles and rationality (August 5th)

Abstract

This talk will present a brief survey of local-global principles for torsors for algebraic groups over higher dimensional arithmetic fields via field patching techniques. In particular, I'll discuss new work which makes a connection between obstructions to such local-global principles and obstructions to rationality of algebraic groups.

reference

Goal: Understanding the arithmetical fields as structure of algebraic objects. Suspicion: if K is a number field, X/K is a curve on the plane Ω_K, $F = K(X)$, G is a linear algebraic group then $\mathrm{Ker}(H^1(F, G) \to \prod_{v \in \Omega_K} H^1(K_v(X), G))$. Evidence: E/K is an elliptic curve, $G = PGL_n$, then $\mathrm{Ker}() \subseteq \#(K, E)[n]$.

Given a curve X over some complete discrete valuation field, we choose a regular model $\mathscr{X}/\mathcal{O}_K$

4 Jacob Lurie, Tamagawa Numbers in the Function Field Case

Abstract

Let G be a connected semisimple algebraic group over a global field K, and let A denote the ring of adeles of K. Tamagawa observed that the locally compact group G(A) is equipped with a canonical translation-invariant measure. A celebrated conjecture of Weil asserts that if G is simply connected, then the measure of the quotient space G(A)/G(K) is equal to 1. When K is a number field, this conjecture was proven Kottwitz (following earlier work of Langlands and Lai). In these talks, I'll discuss joint work with Dennis Gaitsgory about the function field case, exploiting ideas from algebraic topology.

4.1 August 5th

First take $K = \mathbb{Q}$ and G is an algebraic group of dimension d. The adele ring

$$\mathbb{A} = \mathbb{R} \times \prod_p^{res} \mathbb{Q}_p \subseteq \mathbb{R} \times \prod_p \mathbb{Q}_p$$

which is locally compact. Then we can talk about $G(\mathbb{A})$, which contains $G(\mathbb{Q})$ where $G(\mathbb{A})$ is locally compact and $G(\mathbb{Q})$ is discrete.

$G(\mathbb{A}) = G(\mathbb{R}) \times \prod_p^{res} G(\mathbb{Q}_p)$ has a left canonical invariant measure which is callen Tamagawa measure. $G(\mathbb{R})$ is a Lie group, let $V_\mathbb{R}$ be the space of translation invariant d-forms on G. Let $V_\mathbb{Q} \subseteq V_\mathbb{R}$ be the algebraic differential forms defined over $\mathbb{Q}$. So for one $\omega \in V_\mathbb{Q}$ we have a Haar measure $\mu_{\omega,\mathbb{R}}$ on $V_\mathbb{R}$. ω also determines a Haar measure $\mu_{\omega,\mathbb{Q}_p}$ on $G(\mathbb{Q})$, so we have a construction

$$\mu := \mu_{\omega,\mathbb{R}} \times \prod_p \mu_{\omega,\mathbb{Q}_p}.$$

Conjecture 1 (Weil)**.** If G is semisimple and simply connected, then

$$\mu(G(\mathbb{A})/G(\mathbb{R})) = 1.$$

For example, take $V = \{\text{rational }\}$. This conjecture was proved by Kottwits following Langlans, Lai.

More generally, we can do the same things on a global field. If $K/\mathbb{Q}$ is a finite extension, G is some algebraic group over K. $G(K) \subseteq G(\mathbb{A}_K)$, $H := \mathrm{Res}_G^K(???)$.

In the rest of this lecture, $x \in X$ is the closed point of a smooth projective connected variety over $\mathbb{F}_q$, K_X is the function field of X, $\kappa(x)$ is the residue field, $\mathcal{O}_x$ is the complete local ring at x, $= \kappa(x)$, K_x is the fraction field of $\mathcal{O}_x$. We have

$$K_X \subseteq \mathbb{A}_X = \prod_{x\in X}^{res} K_x.$$

Suppose G_0 is an algebraic group over K_X, then $G(K_X) \subseteq G(\mathbb{A}_X)$. To define the Tamagawa measure, we choose a translation invariant top form ω in G_0, then this gives a measure $\mu_{\omega,x}$. Then we want to choose an integral model of G_0. Suppose $G \to X$ is a smooth affine group scheme with connected fibers. Then

$$G(K_x) \supseteq G(\mathcal{O}_x) \to G(\kappa(x))$$

where the latter ????

For most of $x \in X$, ω has no zeros or poles at x. In this case,

$$\mu_{\omega,x}(G(\mathcal{O}_x)) = \frac{|\kappa(x)|}{|\kappa(x)|^d}.$$

This was not quite right, since there are still finitely many point where ω has zeros or poles. So we consider $\prod_{x\in X} G(\mathcal{O}_x)$ acting on $G(\mathbb{A}_X)/G(K_X)$. Then we have

$$\{\text{principle } G \text{ bundle on } X\}/\text{isomorphisms} \simeq \frac{G(\mathbb{A}_X)/G(K_X)}{\prod_{x\in X} G(\mathcal{O}_x)}.$$

Theorem. *Let $\mathcal{P}$ be a G-bundle on X, assume G_0 is simply connected, then*

1. *(Hurder) $\mathcal{P}$ is trivial at the generic point of X.*
2. *(Larey) $\mathcal{P}$ is trivial at each* $\mathrm{Spec}\ \mathcal{O}_x$.

We can choose some trivialization $\mathcal{P}|_{\mathrm{Spce}\ K_X}$ and $\mathcal{P}|_{\mathrm{Spce}\ \mathcal{O}_x}$, and we differ $\mathrm{Spce}\ K_X$ by an element of $G(K_X)$. Therefore, we have a naive guess

$$\mu(G(\mathbb{A}_X)/G(K_X)) = (\#\text{ of } G \text{ bundles on } X)\, q^{-D} \prod_{x\in X}^{res} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}.$$

This is true only for when the group action is free. We need to know the multiplicity. The correct conjecture is

$$\mu(G(\mathbb{A}_X)/G(K_X)) = \left(\sum_{\mathcal{P} \text{ a } G \text{ bundle on } X} |\frac{1}{\mathrm{Aut}(\mathcal{P})}|\right) q^{-D} \prod_{x\in X}^{res} \frac{|G(\kappa(x))|}{|\kappa(x)|^d}.$$

Hence we can restate the Weil's conjecture

$$\frac{\sum_{\mathcal{P}} \frac{1}{|\mathrm{Aut}(\mathcal{P})|}}{q^D} = \prod_{x\in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|}.$$

Then we consider a stack $\mathrm{Bun}_G(X)$, sending R to G-bundles over $X \times_{\mathrm{Spec}\ \mathbb{F}_q} \mathrm{Spec}\ R$. Then we have that $\sum_{\mathcal{P}} \frac{1}{|\mathrm{Aut}(\mathcal{P})|} = |\mathrm{Bun}_G(X)(\mathbb{F}_q)|$. For each $x \in X$, we can also consider $\mathrm{Bun}_G(\{x\}) = \mathrm{Bun}(\mathrm{Res}_{\mathbb{F}_q}^{\kappa(x)} G_x)$. We have $|\mathrm{Bun}_G(\{x\})(\mathbb{F}_q)| = \frac{1}{G(\kappa(x))}$, so we can restate the Weil's conjecture

$$\frac{|\mathrm{Bun}_G(X)(\mathbb{F}_q)|}{q^{\dim\ \mathrm{Bun}_X(G)}} = \frac{\sum_{\mathcal{P}} \frac{1}{|\mathrm{Aut}(\mathcal{P})|}}{q^D} = \prod_{x\in X} \frac{|\kappa(x)|^d}{|G(\kappa(x))|} = \prod_{x \in X} \frac{|\mathrm{Bun}_G(\{x\})(\mathbb{F}_q)|}{q^{\dim\ \mathrm{Bun}_{\{x\}}(G)}}.$$

4.2 Aug 6th

5 Davesh Maulik: Topology of Higgs moduli spaces via abelian surfaces (August 5th)

Abstract

In this talk, we study cases of the $P = W$ conjecture for Higgs bundles on a curve, using techniques from compact hyperkähler geometry. This is joint work in progress with Mark de Cataldo and Junliang Shen.

Setup: let C be a smooth projective curve over $\mathbb{C}$ with genus $g \geq 2$. Suppose a moduli problem

$$\mathcal{M}_{\text{Higgs}} = \{\text{vector bundle } E \text{ over } C \text{ with rank } r \text{ and degree } d.\}$$

where $E \xrightarrow{\varphi} E \otimes K_{\mathbb{C}}$ has stability. The functor sends a Higgs bundle to pure 1-dimensional sheaf on T^*C with proper support

$$[\mathrm{supp}\ \mathcal{E}] = r[C].$$

For smooth variety with holomorphic symplectic, we have another space, which is the twisted character variety of C:

$$\mathcal{M}_{\text{Betti}} = \{\text{smooth affine varieties } A_1, \cdots, A_g, B_1, \cdots, B_g, \text{s.t. } \prod [A_i, B_i] = e^{2\pi i d/r}\}//GL_r(\mathbb{C})$$

The talk starts with non-abelian Hodge theory, i.e. there is a diffeomorphism

$$\mathcal{M}_{\text{Higgs}} \simeq \mathcal{M}_{\text{Betti}}.$$

For example, when $r = 1$, we have

$$\mathrm{Pic}_d(\mathbb{C}) \times \mathbb{C}^g \xrightarrow{\sim} (\mathbb{C}^*)^{2g},$$

and so

$$H^*(\mathcal{M}_{\text{Higgs}}) \cong H^*(\mathcal{M}_{\text{Betti}}).$$

Conjecture 2 ($P = W$)**.** $\mathcal{M}_{\text{Betti}}$ carries a mixed Hodge structure

$$W_K H^* \subseteq H^*,$$

then what is the meaning of W_K on $H^*(\mathcal{M}_{\text{Higgs}})$?

We need some extra structure. Suppose we have the Hitchin map $\mathcal{M}_{\text{Higgs}} \xrightarrow{\pi} \mathbb{A}^N$, and hence use π to define a filtration $\mathcal{E} \mapsto \mathrm{supp}\ \mathcal{E}$, whose fiber is $\mathrm{Pic}_d(C)$. For example suppose $X \xrightarrow{\pi} Y$ is a proper filtration. We have a decomposition theorem

$$R\pi_* \mathbb{Q}_X[\dim\ X - a] \cong \bigoplus_{k=0}^{2a} P_k[-k],$$

where the right hand side preserve the sheaf on Y. If X_y is smooth, then $P_k|_y = H^k(X_y)[-]$. Hence we define

$$P_k H^d(X) = \mathrm{Im}(H^{d-(\dim\ X-a)}(Y, \bigoplus_{i\leq k} P_i[-i]) \to H^d(X)).$$

In our case, $P_k H^d(X) = \mathrm{Ker}(H^d(X) \to H^d(\pi^{-1}(Y')))$ (which is hard by dC, Mig) where Y' is a generic plane of dimension $d - k - 1$.

Conjecture 3.

$$P_k H^*(\mathcal{M}_{\text{Higgs}}) = W_{2k} H^*(\mathcal{M}_{\text{Betti}}) = W_{2k+1}.$$

We already have

1. all genus for $r = 2$.
2. Hard Lefschetz for $gr_W H^*$ by Mellit.

The main results are

Theorem. *$P = W$ holds for $g = 2$ curves for all rank r, degree d, with $(r, d) = 1$. When $g > 2$, for some $E \to C \times \mathcal{M}_{Higgs}$, twist: for any $\alpha \in H^2(C) \oplus H^2(\mathcal{M}_{Higgs})$, let*

$$ch^\alpha(E) = ch(E) \cup e^\alpha \in H^*(C) \oplus H^*(\mathcal{M}_{Higgs}).$$

Pick α s.t. $ch_1^\alpha(E) \in H^1 \oplus H^1$. For any $\gamma \in H^*(C)$, $ch(k, \gamma) = \int_\gamma ch_k^\alpha(E) \in H^*(\mathcal{M})$

Idea of the proof. □

6 Bjorn Poonen: The local-global principle for stacky curves (August 5th)

Abstract

For smooth projective curves of genus g over a number field, the local-global principle holds when g=0 and can fail for g=1, as has been known since the 1940s. Stacky curves, however, can have fractional genus. We construct stacky curves of genus 1/2 that violate the local-global principle, and show that 1/2 cannot be reduced. This is joint work with Manjul Bhargava.

Local-global principle: fix some genus g, if a smooth projective geometrically integral curve X of genus g over a number field k, has a k_v point for every place v, must it have a k-point?

1. Yes, if $g = 0$.
2. No, if $g \geq 1$. E.g. $X : 2y^2 = 1 - 17x^4$.

We want to ask

1. What if X is a stack and $0 < g < 1$?
2. What is the smallest g for which the local-global principle fails?

Root Stacks: Problem: given a scheme V, an effective Cartier divisor, $n \in \mathbb{N}$, how can we modify V so that we can replace D by $\frac{1}{n}D$? The solution is to assume $V = \text{Spec } A$, D is principle workably, choose $f \in A$ s.t. $D = (f)$, then

$$X := [\text{Spec } A[y]/(y^n - f)/\mu_n].$$

Suppose k is algebraically closed field of characteristic 0, we define a stacky curve over k is a smooth irreducible 1 dimensional Deligne-Munford stack X containing a nonempty open substack isomorphic to a scheme. Fact is that X is a smooth integral curve over X_{curve} with $P_1, \cdots, P_n$ replaced by $\frac{1}{e_1}P_1, \cdots, \frac{1}{e_n}P_n$.

Next we define Euler characteristic

$$\chi := \chi_{\text{curve}} - \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} \frac{1}{e_1}$$

and genus to be $2 - 2g = \chi$. If k is not algebraically closed, a stacky curve over k is some algebraic stack X s.t. $X_{\bar{k}} := X \times \text{Spec } \bar{k}$ is a stacky curve.

$$X(A) := \{\text{morphismsSpec } A \to X\}/\text{isomorphisms}.$$

Notice that a stacky curve of genus 0 has k-point if and only if the coarse space has a k-point. So we want to study the integral points.

Example: pick three positive natural number p, q, r and let

$$S = \mathrm{Spec}\frac{\mathbb{Z}[x,y,z]}{x^p + y^q - z^r} - (0,0,0) \subseteq \mathbb{A}^3,$$

and $\mathbb{G}_m^3$ has an action on it. Then

$$S(\mathbb{Z}) = \{\text{gcd 1 integer solutions to } x^p + y^q - z^r = 0\}.$$

Let H be the subgroup of $\mathbb{G}_m^3$ preserving S, which leads to a fact $S(\mathbb{Z})$ is

1. finite if $\chi < 0$;
2. infinite if $\chi > 0$.

Counterexample of genus $\frac{1}{2}$: if $p, q, r \equiv 7 \pmod 8$ s.t. $\left(\frac{p}{q}\right) = \left(\frac{p}{r}\right) = \left(\frac{r}{q}\right) = 1$, $f(x,y) = ax^2 + bxy + cy^2$ is of discriminant $-pqr$ and $\left(\frac{a}{q}\right) = 1$, $\left(\frac{a}{p}\right) = \left(\frac{a}{r}\right) = -1$. Let $Y = \mathrm{Proj}\frac{\mathbb{Z}[x,y,z]}{z^2 - f(x,y)}$ with a μ_2 action (λ acting on $(x,y,z) \mapsto (x,y,\lambda z)$), and finally let $X := [Y/\mu_2]$, (E.g. $p = 7, q = 47, r = 31$.) then

1. $\chi_X = 1, g = \frac{1}{2}$.
2. $X(\mathbb{Z}_l) \neq \emptyset$ for all l, hence $X(\mathbb{R}) \neq \emptyset$.
3. $X(\mathbb{Z}) = \emptyset$

Theorem (Local-global principle for $\chi > 1$)**.** *Suppose k is a finite field over $\mathbb{Q}$, S is a finite set of places of k containing the Archimedean places, $\mathcal{O} = \mathcal{O}_{k,S}$, $k_v :=$ the completion of k at v, and $\mathcal{O}_v$ is the valuation ring of k if $v \notin S$ and k_v if $v \in S$. Then X is a separated finite type algebraic (Artin) stack over* Spec $\mathcal{O}$ *s.t.*

1. *$X_{\bar{k}}$ is a stacky curve with $\chi > 1$.*
2. *If $X(\mathcal{O}_v) \neq \emptyset$ for every v, then $X(\mathcal{O})$ is not empty.*

Sketch of the proof. For simplicity, suppose $\mathcal{O} = \mathbb{Z}$. $\chi > 1$ means that $X_{\mathbb{Q}}$ is some smooth proper $\mathcal{O}$-curve (X_{curve}) of genus 0 and with at most 1 stacky point. $X(\mathbb{Z}_p) \neq \emptyset$ for all p implies $X_{\text{curve}}(\mathbb{Z}_p) = X_{\text{curve}}(\mathbb{Q}_p) \neq \emptyset$ for all p, implying ${}_{\text{curve}} \simeq \mathbb{P}^1_{\mathbb{Q}}$, $X_{\mathbb{R}} \supseteq \mathbb{A}^1_{\mathbb{R}}$.

Hence $X_{\mathbb{Z}[\frac{1}{N}]}$ contains $\mathbb{A}_{\mathbb{Z}[\frac{1}{N}]}$ for some N. Again for simplicity suppose $N = p$ for some prime number. Thus $X(\mathbb{Z}_p) \neq \emptyset$ implies X has many $\mathbb{Z}_p$-points (*), the subset $X(\mathbb{Z}_p) \subseteq X(\mathbb{Q}_p)$ contains a nonempty open subset U of $\mathbb{A}^1(\mathbb{Q}_p)$. **By strong approximation**, there exists an $x \in \mathbb{A}^1(\mathbb{Z}[\frac{1}{p}])$ s.t. $x \in U$, then $x \in X(\mathbb{Z}[\frac{1}{p}]) \cap X(\mathbb{Z})$, and (*) tells us $x \in X(\mathbb{Z})$. $\square$