Representation homology and Lie algebra cohomology of nilpotent algebras

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- Given a discrete group $\pi_1(X)$, one has the representation scheme parameterizing all $\pi_1(X)$-representations.
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- This representation scheme also parametrizes all local systems over X provided X is a CW complex.
- Similarly for Lie algebra representations.

Space parameterizing Lie algebra representations

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Definition-Lemma

Given a finite dimensional Lie algebra $\mathfrak{g}$ and a Lie algebra $\mathfrak{a}$, the functor $\mathrm{Rep}_{\mathfrak{g}}(\mathfrak{a}) : \mathbf{CommAlg}_k \to \mathbf{Set}$

$$A \mapsto \mathrm{Hom}_{\mathbf{Lie}}(\mathfrak{a}, \mathfrak{g}(A))$$

is representable, where $\mathfrak{g}(A) := \mathfrak{g} \otimes A$ is "the Lie algebra with coefficient A", with Lie bracket

$$[\xi \otimes a, \eta \otimes b] := [\xi, \eta] \otimes (ab).$$

The representative is denoted by $(\mathfrak{a})_{\mathfrak{g}}$. The space $\mathrm{Spec}\ (\mathfrak{a})_{\mathfrak{g}}$ is called **representation scheme**.

Examples

Let $\mathfrak{a}$ be the two dimensional abelian Lie algebra, and let $\mathfrak{g} = \mathfrak{gl}_2$. Then

$$(\mathfrak{a})_{\mathfrak{g}} \cong \frac{k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}]}{[X, Y] = 0}. \quad (1)$$

The space $\mathrm{Spec}\,(\mathfrak{a})_{\mathfrak{g}}$ can be interpreted as all pairs of matrices $(X, Y) \in \mathfrak{gl}_2$ s.t. $[X, Y] = 0$. This example is the coordinate ring of (additive) commuting scheme of $\mathfrak{gl}_2$.

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This gives a functor

$$(-)_{\mathfrak{g}} : \mathbf{Lie} \to \mathbf{CommAlg}_k,$$

which is the left adjunction of $\mathbf{Lie} \leftarrow \mathbf{CommAlg}_k : \mathfrak{g}(-)$.

Character scheme

There is another scheme called the character scheme defined by "the orbits of the adjoint action". This corresponds to the commutative ring inclusion

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It is known that "there is a symplectic structure on the *smooth* locus of the character scheme".

Problems

1. The representation / character schemes are generally very singular. This makes it very difficult to study them.
2. If the Lie algebra (or the discrete group) comes from $\mathbf{Top}$, we lose all higher information of the topological space.

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Many people (a very incomplete list consists of Kontsevich, Kapranov, Töen, ...) suggest that we should consider the derived version of representation / character schemes.

Deriving the representation schemes

Berest-Felder-Patotski-Ramadoss-Willwacher, [BFP+17]

Given a finite dimensional Lie algebra $\mathfrak{g}$, the adjunction

$$(-)_{\mathfrak{g}} : \mathbf{DGLA}_k \leftrightarrows \mathbf{DGCommAlg}_k : \mathfrak{g}(-)$$

is an Quillen pair, so $(-)_{\mathfrak{g}}$ admits a (total) left derived functor $\mathbb{L}(-)_{\mathfrak{g}}$, which could be computed by $(Q\mathfrak{a})_{\mathfrak{g}}$ where $Q\mathfrak{a}$ is a cofibrant replacement of $\mathfrak{a}$.

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Then the homology of

$$\mathbb{L}(\mathfrak{a})_{\mathfrak{g}} \in \mathrm{Ho}(\mathbf{DGCommAlg}_k)$$

is called the **representation homology** of $\mathfrak{a}$ with coefficient $\mathfrak{g}$, denoted by $HR_*(\mathfrak{a}, \mathfrak{g})$.

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Some reasons

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- The Harish-Chandra homomorphism gives an isomorphism

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In the derived setting, there is a similar derived Harish-Chandra homomorphism

$$HR_*(\mathfrak{a}, \mathfrak{g})^{\mathfrak{g}} \to \mathrm{Sym}[\mathfrak{h} \oplus \mathfrak{h} \oplus \mathfrak{h}^*[1]]^W$$

which is conjectured to be true where $\mathfrak{a}$ is the two dimensional abelian Lie algebra [BFP+17].

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- (Time permitting) Strong Macdonald conjecture.

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- Nilpotent coefficient representation homology computation turns to be experimentally computationally easier than the semisimple cases.

Examples

Let $\mathfrak{n}_m$ be the maximal nilpotent subalgebra of $\mathfrak{gl}_m$ and $\mathfrak{a}$ be the abelian Lie algebra of dimension 2. Then the following are equivalent [L, analogue of [Li24]]:

1. The representation homology $HR_i(\mathfrak{a}, \mathfrak{n}_m)$ vanishes in dimension greater or equal than m, namely

$$HR_i(\mathfrak{a}, \mathfrak{n}_m) = 0 \quad \forall i \geq m.$$

2. There is an isomorphism of graded algebras

$$HR_*(\mathfrak{a}, \mathfrak{n}_m) \cong HR_0(\mathfrak{a}, \mathfrak{n}_m) \otimes \mathrm{Sym}_k(T_1, \cdots, T_{m-1}) \tag{2}$$

where Sym_k is the graded symmetric algebra over k and T_i is of homological degree 1.

3. The commuting scheme $C(\mathfrak{n}_m)$ is a complete intersection of codimension $\frac{(n-2)(n-1)}{2}$.

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- Nilpotent coefficient representation homology computation turns to be experimentally easier than the semisimple cases.
- For a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, we have the triangular decomposition

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  We have seen $HR_*(\mathfrak{a}, \mathfrak{g})^{\mathfrak{g}}$ and $HR_*(\mathfrak{a}, \mathfrak{h})^W$ are (conjecturally) related. It would be nice if we can say more things from the triangular decomposition.

Lie (co)homology

Recall

Given a Lie algebra $\mathfrak{g}$ over k and a $\mathfrak{g}$-module M. The complex $C_*(\mathfrak{g}; M) := (M \otimes_k \bigwedge_{i=1}^n \mathfrak{g}, \delta_n)$ called the Chavelley-Eilenberg chain complex computes the Lie homology

$$\mathrm{H}_*(\mathfrak{g}; M).$$

Dually, the cochain complex $C^*(\mathfrak{g}; M) := (\mathrm{Hom}_k(\bigwedge_{i=1}^n \mathfrak{g}, M), d^n)$ computes the Lie cohomology

$$\mathrm{H}^*(\mathfrak{g}; M).$$

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Theorem (Chavelley)

Let $\mathfrak{g}$ be the Lie algebra of the compact Lie group G (so $\mathfrak{g}$ is semisimple). Then there is a canonical isomorphism of groups

$$\mathrm{H}^*_{\mathrm{CE}}(\mathfrak{g}; \mathbb{R}) \cong \mathrm{H}^*_{\mathrm{dR}}(G; \mathbb{R}).$$

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Theorem (Bott-Kostant)

Let $\mathfrak{n}$ be the maximal nilpotent Lie algebra of a complex semisimple Lie algebra $\mathfrak{g}$. Let G be the complex Lie group of $\mathfrak{g}$. Then there is an isomorphism of groups

$$\mathrm{H}^*_{\mathrm{CE}}(\mathfrak{n};\mathbb{C}) \cong \mathrm{H}^{2*}_{\mathrm{Sing}}(G/B;\mathbb{C})$$

where B is the Borel subgroup of G.

Result

Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank r, with triangular decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}^{\alpha}$$

and $\mathfrak{g}^{\alpha} = \{x \in \mathfrak{g} \mid [H, x] = \alpha(H)x, \forall H \in \mathfrak{h}\}$, where R is its root system and W is its Weyl group. By *Bott-Kostant theorem*, a basis of $\mathrm{H}^1(\mathfrak{n}; k)$ corresponds to the elements in W of length 1. There is a characteristic pairing map

$$\chi_{\mathfrak{a}}(\mathfrak{n})_{2,1} : \mathrm{H}_2(\mathfrak{a}; k) \otimes \mathrm{H}^1(\mathfrak{n}; k) \to \mathrm{HR}_1(\mathfrak{a}, \mathfrak{n}) \tag{3}$$

giving nontrivial distinct homology classes $T_1, \cdots, T_r \in \mathrm{HR}_1(\mathfrak{a}, \mathfrak{n})$.

Linking Lie cohomology and representation homology

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Construction of the pairing map

1. We have the cobar-bar adjunction $\boldsymbol{\Omega}_{\mathtt{Comm}} : \mathbf{DGCC}_{k/k} \leftrightarrows \mathbf{DGLA}_k : \mathcal{C}$.
2. We know that $\mathcal{C}(\mathfrak{a})$ is the Chavelley-Eilenberg complex.

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3. The semifree Lie algebra $L := \Omega_{\mathtt{Comm}}(\mathcal{C}(\mathfrak{a}))$ is a resolution of $\mathfrak{a}$. Hence $L_{\mathfrak{g}}$ computes the representation homology $\mathrm{HR}_*(\mathfrak{a}, \mathfrak{g})$.

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4. There is a universal representation $\mathfrak{a} \to \mathfrak{g}(\mathfrak{a}_{\mathfrak{g}})$ lifted to the derived universal representation

$$\pi : L \to \mathfrak{g}(L_{\mathfrak{g}})$$

in $\mathbf{DGCC}_{k/k}$.

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$$\tau : \mathcal{C}(\mathfrak{a}) \to \mathcal{C}(\mathfrak{g}(L_{\mathfrak{g}})) = \mathcal{C}(\mathfrak{g}) \otimes L_{\mathfrak{g}}.$$

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6. At the homology level, we have

$$\mathrm{H}_n(\mathfrak{a}; k) \to \bigoplus_{p+q=n} \mathrm{H}_p(\mathfrak{a}, \mathfrak{g}) \otimes \mathrm{H}_q(\mathfrak{g}; k).$$

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7. Dualizing by the cohomology, we have

$$\bigoplus_{p+q=n} \mathrm{H}_n(\mathfrak{a}; k) \otimes \mathrm{H}^q(\mathfrak{g}; k) \to \mathrm{H}_p(\mathfrak{a}, \mathfrak{g}).$$

Thank you!

References

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