

# Different Constructions and Geometric Properties of Blow-ups

## 1 Introduction

In algebraic geometry, we have to deal with singularities of varieties. The problem of **resolution of singularities** asks whether every algebraic variety  $X$  has a resolution, a non-singular variety  $Y$  with a proper birational map  $f : Y \rightarrow X$ . We already have different ways to resolve a singularity, and blowing up is one of the most important methods.

Usually  $\mathbb{C}^n$  is a denotation geometrically, while  $\mathbb{A}_{\mathbb{C}}^n$  is more algebraically. However we will regard them equivalently through this paper.  $\mathbb{A}^n, \mathbb{P}^n$  are  $\mathbb{A}_{\mathbb{C}}^n$  and  $\mathbb{CP}^n$  respectively except indicated explicitly. All of the construction can be transplanted over another (algebraic closed) field.

In order to get a full understand of this paper, the readers are supposed to know the results from commutative algebra, properties of projective spaces, some results from geometry and topology, basic construction of algebraic geometry, complex geometry and sheaves. All of the definition will be mentioned later in the text however it may be extremely confusing if it is the first time to encounter these terminologies.

### 1.1 Acknowledgements

## 2 Geometric Construction

### 2.1 Blow-up of $\mathbb{C}^n$ at Origin

We will start from the simplest case, i.e. blowing up the origin of  $\mathbb{C}^n$ .

Consider a subset of  $\mathbb{C}^n \times \mathbb{P}^{n-1}$

$$\widetilde{\mathbb{C}^n} := \{((a_1, \dots, a_n), [b_1, \dots, b_n]) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid a_i b_j = a_j b_i, 1 \leq i, j \leq n\}.$$

It is easy to see that we have a natural projection from  $\widetilde{\mathbb{C}^n}$  to  $\mathbb{C}^n$ :

$$\pi : ((a_1, \dots, a_n), [b_1, \dots, b_n]) \mapsto (a_1, \dots, a_n).$$

**Definition.** The set  $\widetilde{\mathbb{C}^n}$  along with the map  $\pi : \widetilde{\mathbb{C}^n} \rightarrow \mathbb{C}^n$  is called the *blow-up* of  $\mathbb{C}^n$  at the origin.

One can see that the blow-up of  $\mathbb{C}^n$  is exactly  $\mathcal{O}(-1)$  of  $\mathbb{P}^{n-1}$ .

Immediately we have these properties of  $\widetilde{\mathbb{C}^n}$ :

For any point  $O \neq z = (a_1, \dots, a_n) \in \mathbb{C}^n$ , there is at least a  $1 \leq i_0 \leq n$  s.t.  $a_{i_0} \neq 0$  hence  $b_{i_0} \neq 0$ . Thus the equations can be written as  $\frac{b_j}{b_{i_0}} = \frac{a_j}{a_{i_0}}, 1 \leq j \leq n$ . Therefore there is only **ONE** point  $x$  in  $\widetilde{\mathbb{C}^n}$  with  $\pi(x) = z$ , i.e.  $x = ((a_1, \dots, a_n), [a_1, \dots, a_n])$ .

$\pi^{-1}(O) \cong \mathbb{P}^{n-1}$ . Indeed, any point  $[b_1, \dots, b_n] \in \mathbb{P}^{n-1}$  satisfies the equations if  $(a_1, \dots, a_n) = 0$ .

The points of  $\pi^{-1}(O)$  are in 1-1 corresponding to the lines through the origin in  $\mathbb{C}^n$ . We usually denote it as  $E$ . A line through the origin  $l$  can be given by the parametric equation  $x_i = a_i t$  where  $1 \leq i \leq n$  and  $a_i \in \mathbb{C}$  are not all zero, thus  $[a_1, \dots, a_n]$  is a point in  $\mathbb{P}^{n-1}$ . Now consider the line  $\tilde{l} = \pi^{-1}(l - O)$  in  $\widetilde{\mathbb{C}^n} - \pi^{-1}(O)$ , it is  $\{((a_1 t, \dots, a_n t), [a_1, \dots, a_n]) \mid t \in \mathbb{C}^*\}$ . These equations also make sense for  $t = 0$  and give the closure  $\bar{\tilde{l}}$  of  $\tilde{l}$  in  $\widetilde{\mathbb{C}^n}$ .  $\bar{\tilde{l}}$  meets  $\pi^{-1}(O)$  in the point  $[a_1, \dots, a_n] \in \mathbb{P}^{n-1}$ , so we see that sending  $l$  to  $Q$  gives us the correspondence between lines through origin in  $\mathbb{C}$  and points of  $\pi^{-1}(O)$ .

$\widetilde{\mathbb{C}^n}$  is irreducible. Indeed  $\widetilde{\mathbb{C}^n} = \widetilde{\mathbb{C}^n} - \pi^{-1}(O) \cup \pi^{-1}(O)$ , where the first piece is isomorphic to  $\mathbb{C}^n - O$  which is obviously irreducible, and every point of the second part  $\pi^{-1}(O)$  is in the closure of some line of  $\mathbb{C}^n - \pi^{-1}(O)$ . Hence  $\mathbb{C}^n - \pi^{-1}(O)$  is dense in  $\widetilde{\mathbb{C}^n}$  and  $\widetilde{\mathbb{C}^n}$  is irreducible.

Here we give an example to illustrate how this process works.

The blow-up of  $\mathbb{R}^2$  at the origin can be visualized as follows: we have known that points not the origin are in a 1-1 correspondence by  $\pi$ , so we leave them stable. However we need to replace the origin by  $\mathbb{RP}^1$ , i.e. a circle. For any line  $l$  parameterized by  $x = a_1 t, y = a_2 t$ ,  $\pi^{-1}(l - O)$  lies in  $\widetilde{\mathbb{R}^2} - \mathbb{RP}^1$ , and the closure of  $\pi^{-1}(l - O)$  consist of the point  $[a_1, a_2]$  in  $\mathbb{RP}^1$ . So the disconnected part of  $\pi^{-1}(l - O)$  are glued by the point. This operation works like we glue the antipodal points together. Topologically, this is regarding the missing point as a disk, gluing the boundary of the disk with the boundary of a Möbius band, since both of the boundary are circles.

The diagram below shows the visualization above.

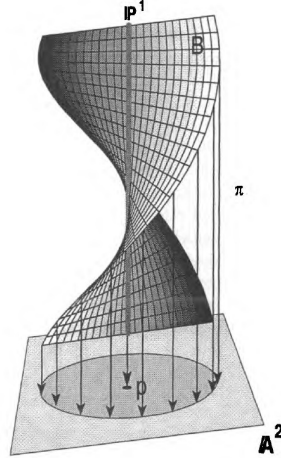


Figure 1: Visualization of Blowing up a Point

Using the technique above, it is easy to blow up an algebraic set at the origin.

**Definition.** If  $V$  is a closed algebraic set of  $\mathbb{C}$  passing through the origin, the *blow-up* of  $V$  at the origin is  $\widetilde{V} = \pi^{-1}(V - O)$ , where  $\pi : \widetilde{\mathbb{C}^n} \rightarrow \mathbb{C}^n$  is the blow-up of  $\mathbb{C}^n$  as above. We denote also by  $\pi : \widetilde{V} \rightarrow V$  the morphism by the restriction of  $\pi : \widetilde{\mathbb{C}^n} \rightarrow \mathbb{C}^n$  to  $\widetilde{V}$ .

## 2.2 A Little Further: Surgery

Consider every point on a complex manifold have a local Euclidean coordinate, so to blow up a (complex) manifold at a point, it suffices to "cut off" a small neighborhood of the point, blow-up the small piece at the origin and then glue it back up to the manifold. Yet we need some extra technique to do this surgery-like modification.

Let  $M$  be a complex manifold, and  $S \subset M$  a compact submanifold of  $M$ . We construct a new manifold  $\widetilde{M} = (M - S) \cup \widetilde{S}$  replacing  $S$  by another complex manifold  $\widetilde{S}$  as follows:

Take domains  $W, W_1$  s.t.  $S \subset W_1 \subset \overline{W_1} \subset W \subset M$  where  $\overline{W}$  is compact. Let  $\widetilde{S}$  be another compact manifold s.t.  $\widetilde{S} \subset \widetilde{W_1} \subset \widetilde{(\overline{W_1})} \subset \widetilde{W}$ , and a biholomorphic surjective map  $\psi : \widetilde{W} - \widetilde{S} \rightarrow W - S$  s.t.  $\psi : (\widetilde{W} - \widetilde{S}) = W - S$ . Let  $\widetilde{M}$  be the manifold obtained by gluing  $M - \overline{W_1}$  identifying  $P \in W - \overline{W_1}$  with  $\widetilde{P} = \psi^{-1}(P) \in \widetilde{W} - \widetilde{W_1}$  via  $\psi$ :

$$\widetilde{M} = (M - \overline{W_1}) \cup \widetilde{W}.$$

Since  $\psi$  is biholomorphic,  $\widetilde{M}$  becomes a manifold. Thus (see Figure 1)

$$\widetilde{M} = (M - W) \cup \widetilde{W}.$$

We would like to use this to construct the blow-up of a manifold at a point.

For example, let  $M = \mathbb{P}^2$ , and let  $[z_0, z_1, z_2]$  be its homogeneous coordinate and  $P = [1, 0, 0]$ . We denote by  $\mathbb{P}_\infty^1$  the projective line  $z_0 = 0$ . Then

$$\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}_\infty^1$$

where

$$U_0 = \{[z_0, z_1, z_2] \mid z_0 \neq 0\} = \{[1, \frac{z_1}{z_0}, \frac{z_2}{z_0}] \mid z_0 \neq 0\} = \{(w_1, w_2)\} = \mathbb{C}^2.$$

We call the line  $\mathbb{P}_\infty^1$  the line at infinity. Since we have known how to blow up  $\mathbb{C}^2$  at the origin, i.e. blow up  $U_0$  at the point  $[1, 0, 0]$ , and  $\pi|_{\widetilde{U_0} - \pi^{-1}(O)} : \widetilde{U_0} - \pi^{-1}(O) \rightarrow U_0 - O$  is biholomorphic, we can replace  $U_0$  by  $\widetilde{U_0}$ . That is blowing up  $\mathbb{P}^2$  at the point  $P$ .

## 2.3 Blow-up of $\mathbb{C}^n$ along a Affine Variety

For an *affine variety*, we mean an **irreducible** algebraic subset of  $\mathbb{A}^n$ , denoted as

$$Z(f_1, \dots, f_k) = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f_i(a_1, \dots, a_n) = 0 \ \forall 1 \leq i \leq k\},$$

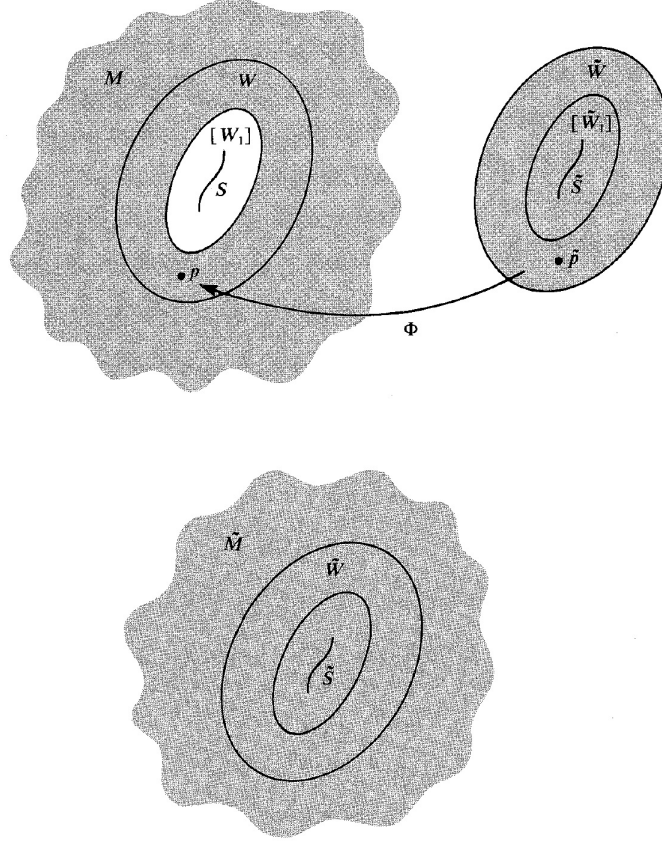


Figure 2: The Surgery of a Manifold

where  $(f_1, \dots, f_k)$  is a prime ideal. We can always find finitely many generators as a result of Hilbert basis theorem.

We define  $A(V) = k[x_1, \dots, x_n]/(f_1, \dots, f_n)$  as the *coordinate ring* of affine variety  $V = Z(f_1, \dots, f_k)$ , and let  $k(V)$  be the quotient field, which is called the *field of fractional functions* on  $V$ . If  $P$  is a point of  $V$ , we define

$$\mathcal{O}_{V,P} = \left\{ \frac{f}{g} \mid f, g \in A(V) \text{ and } g(P) \neq 0 \right\}$$

as the *local ring* of  $V$  at point  $P$ . If  $U \subset V$  is a nonempty open set, we set

$$\mathcal{O}_V(U) := \bigcap_{P \in U} \mathcal{O}_{V,P}.$$

This is a subring of  $k(V)$ , and we call this the *the ring of regular function* on  $U$ . Note that the local ring of  $V$  at point  $P$  is actually a local ring.

**Lemma.** Let  $U$  be an open set in an affine variety  $X$ . A set theory map  $\varphi : U \rightarrow k$  is a rational function at the point  $P$  if and only if there is an open neighborhood of  $P$  in  $U$  s.t. there are polynomials  $f, g \in k[x_1, \dots, x_n]$ , with  $g(Q) \neq 0$  and  $\varphi(Q) = \frac{f(Q)}{g(Q)}$  for all  $Q \in U$ , and  $\varphi$  is a regular function on  $U$  if it is regular at every point in  $U$ .

A morphism  $\varphi : X \rightarrow Y$  between two affine variety is a continuous map s.t.  $f^*(\mathcal{O}_Y(U)) \subseteq \mathcal{O}_X(f^{-1}(U))$ , i.e. for all regular function  $f$  on an open set  $U$  of  $Y$ ,  $\varphi^\#(f) = f \circ \varphi$  is still regular. One can see that the definition of morphism is actually the definition of morphism of ringed spaces.

Generally, a *variety* means a **quasi-projective** variety. Since all of the varieties are ringed spaces, the morphism can be defined as morphism of ringed spaces. From another point of view, variety can be glued by affine varieties, and we usually study the properties locally, it is useful and easier just to consider the affine things.

**Lemma.** Let  $X, Y$  be (affine) varieties and let  $U$  be a nonempty open set in  $X$ . Suppose  $\varphi, \psi$  are morphism from  $X$  to  $Y$  and  $\varphi|_U = \psi|_U$ , then  $\varphi = \psi$ .

By the lemma above, we can define the *rational map* from  $X$  to  $Y$ , which is the equivalence class of the morphisms from a nonempty set  $U \subset X$  to  $Y$ , where two representatives equal if and only if they are identical on the intersection of the open sets. A *birational map* is a rational map admitting an inverse. Notice that for the blow-up at the origin,  $\pi : \widetilde{\mathbb{C}^n} - O \rightarrow \mathbb{C}^n - O$  is a birational map. This example of blow-up reminds us one of the reason we have rational maps is that sometimes it is difficult to construct a morphism from  $X$  to  $Y$ , but rather easy to define one on an open subset.

The blow-up of  $\mathbb{A}^n$  with respect to the subvariety  $X = Z(f_1, \dots, f_k)$  is given by

$$\widetilde{X} := \{((a_1, \dots, a_n), [b_1, \dots, b_k]) \mid b_i f_j(a_1, \dots, a_n) = b_j f_i(a_1, \dots, a_n)\},$$

which is a subset of  $\mathbb{A}^n \times \mathbb{P}^{k-1}$ , along with the projection  $\pi : \widetilde{X} \rightarrow \mathbb{A}^n$ . This can be characterized as the following commutative diagram:

$$\begin{array}{ccccc} X & \longrightarrow & \mathbb{A}^n & \longleftarrow & (\mathbb{A}^n - X) \cup E \\ \downarrow & & \downarrow & & \downarrow \\ X \times \{0\} & \longrightarrow & \mathbb{A}^n \times \mathbb{A}^k & \xleftarrow{\pi} & \mathbb{A}^n \times \widetilde{\mathbb{A}^k} \end{array}$$

As the definition, the blow-up of  $\mathbb{A}^n$  along a variety (thus at a point) is a variety since it is characterized by polynomials in  $\mathbb{A}^n \times \mathbb{P}^{k-1}$ . Generally, a blow-up is the closure of a birational map. Indeed, the restriction of the projection is identity, hence birational. And a simple example is the blow-up of  $\mathbb{C}^n$  along a linear subspace  $\mathbb{C}^m$  satisfying  $z_{m+1} = \dots = z_n = 0$  which will be used next. Since linear space is also a variety, the blow-up can be derived as

$$\text{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) := \{((z_1, \dots, z_n), [x_{m+1}, \dots, x_n]) \mid z_i x_j = z_j x_i, i, j = m+1, \dots, n\},$$

with a projection  $\pi : \text{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) \rightarrow \mathbb{C}^n$ .

## 2.4 Ultimate Generalization: Blowing up along a Submanifold

Finally we can construct the blow-up of an  $n$ -dimensional complex manifold  $X$  along an arbitrary submanifold  $Y \subset X$  of dimension  $m$ . In order to do so, we choose an atlas  $X = \bigcup \varphi(U_i)$ ,  $\varphi(U_i) \rightarrow X$  where  $U_i$  are open in  $\mathbb{C}^n$  and  $\varphi(U_i \cap \mathbb{C}^m) = \varphi(U_i) \cap Y$  for  $\mathbb{C}^m = \{(z_1, \dots, z_n) \mid z_{m+1} = \dots = z_n = 0\}$ .

Let  $\pi : \text{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) \rightarrow \mathbb{C}^n$  be the canonical projection and let  $\pi_i : Z_i \rightarrow U_i$  be its restriction to the open sets  $U_i$ , i.e.  $Z_i = \pi^{-1}(U_i)$  and  $\pi_i = \pi|_{Z_i}$ . The blow-up can be naturally glued. Consider arbitrary open sets  $U, V \subset \mathbb{C}^n$ , and a holomorphic map  $\varphi : U \cong V$  with the property that  $\varphi(U \cap \mathbb{C}^m) = V \cap \mathbb{C}^m$ . Write  $\varphi = (\varphi_1, \dots, \varphi_n)$ , then for  $k > m$  one has  $\varphi_k = \sum_{j=m+1}^n z_j \varphi_{k,j}$ . Indeed,

$$\varphi_k$$

Thus we have a biholomorphic map

$$\hat{\varphi}(x, z) := ((\varphi_{k,j}(z))_{k,j=m+1,\dots,n} \cdot x, \varphi(z)).$$

It is clear that  $\hat{\varphi}(x, z)$  is contained in the blow-ups of the subspaces. In order to obtain the global blow-up  $\pi : \text{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) \rightarrow X$ , we have to ensure these gluing are compatible. This is obvious over  $X - Y$ . Over  $Y$ , the matrices we obtained are by definition the cocycle of normal bundle  $\mathcal{N}_{Y|X}$ . Thus they satisfy the cocycle condition, which also proves that  $\pi^{-1}(Y) \cong \mathbb{P}(\mathcal{N}_{Y|X})$ , the **projective bundle associated to  $\mathcal{N}_{Y|X}$** . We summarize the discussion by following

**Proposition 2.1.** *Let  $Y$  be a complex submanifold of  $X$ . Then there exists a complex manifold  $\tilde{X} = \text{Bl}_Y X$ , the blow-up of  $X$  along  $Y$ , together with a holomorphic map  $\pi : \tilde{X} \rightarrow X$  such that  $\pi : \tilde{X} - \pi^{-1}(Y) \rightarrow X - Y$  is identity and  $\pi : \pi^{-1}(Y) \rightarrow Y$  is isomorphic to  $\mathbb{P}(\mathcal{N}_{Y|X}) \rightarrow Y$ .*

## 3 Algebraic Construction

### 3.1 Blow-up Algebra

**Definition.** Let  $R$  be a ring and let  $I \subset R$  be an ideal of  $R$ . The *blow-up algebra* or *Rees algebra*, associated with the pair  $(R, I)$ , is the graded  $R$ -algebra

$$\text{Bl}_I(R) := \bigoplus_{n \geq 0} I^n = R \oplus I \oplus I^2 \oplus \dots$$

For a *graded ring*, we mean a ring  $R$  that is the direct sum of abelian groups  $R_i$  s.t.  $R_i R_j \subseteq R_{i+j}$ . We would write

$$R = \bigoplus_{n \in \mathbb{N}_0} R_n,$$

where the elements in any factor  $R_n$  of the decomposition are called the *homogeneous elements* of degree  $n$ . Every elements  $a \in R$  can be written as a sum  $a = a_{i_1} + \cdots + a_{i_k}$  where  $a_{i_j}$  lying different  $R_{i_j}$  are called the *homogeneous components* of  $a$ . A graded module  $M$  over a graded ring  $R$  can be written as

$$M = \bigoplus_{n \in \mathbb{N}_0} M_n,$$

where  $R_i M_j \subseteq M_{i+j}$ . A graded algebra  $A$  over a ring  $R$  is an algebra if it is graded as a ring. It is clear that the blow-up algebra is a graded  $R$ -algebra. An *homogeneous ideal* is an ideal generated by homogeneous elements. An ideal  $I$  is homogeneous if and only if  $I = \bigoplus_{n \geq 0} I \cap S_n$ . And this is the key property we will use following. A *homogeneous prime ideal* is a homogeneous ideal that is prime.

### 3.2 Projective Construction

Having the corresponding ring of blow-up, we still need to realize the geometrical object by this blow-up algebra. This moment, the simple spectrum does not work since the blow-up should be 'projective'. The process can be described as Proj construction. First we give two examples to illustrate the construction. This is exactly the analogy of the construction of projective spaces.

Here we start with two examples, that how we construct the projective spaces by the coordinate ring, then the generalization.

Consider affine lines

$$\begin{aligned} U_0 &= (\text{Spec } \mathbb{C}[x], \mathcal{O}_{\text{Spec } \mathbb{C}[x]}) \\ U_1 &= (\text{Spec } \mathbb{C}[y], \mathcal{O}_{\text{Spec } \mathbb{C}[y]}), \end{aligned}$$

one can define an affine scheme structure on an open set  $X_x$  of  $X = \text{Spec } \mathbb{C}[x]$  as follows:

$$U_{01} = (\text{Spec } \mathbb{C}[x, \frac{1}{x}], \mathcal{O}_{\text{Spec } \mathbb{C}[x, \frac{1}{x}]}).$$

The points of  $\text{Spec } \mathbb{C}[x, \frac{1}{x}]$  are the maximal ideals  $(x - c)$  where  $c \neq 0$  together with  $(0)$ . It is obvious that  $\mathcal{O}_{\text{Spec } \mathbb{C}[x, \frac{1}{x}]} = \mathcal{O}_X|_{X_x}$ . Similarly we have another affine scheme structure as

$$U_{10} = (\text{Spec } \mathbb{C}[y, \frac{1}{y}], \mathcal{O}_{\text{Spec } \mathbb{C}[y, \frac{1}{y}]}).$$

The isomorphism

$$\begin{aligned} \varphi &: \mathbb{C}[y, \frac{1}{y}] \rightarrow \mathbb{C}[x, \frac{1}{x}] \\ f(y, \frac{1}{y}) &\mapsto f(x, \frac{1}{x}) \end{aligned}$$

induces an isomorphism of affine schemes  $(\varphi^*, \varphi^\#) : U_{01} \rightarrow U_{10}$ . Through this isomorphism,  $U_0$  and  $U_1$  can be glued, yielding the scheme

$$\mathbb{P}^1 = (Z, \mathcal{O}_Z)$$

where

$$Z = X \cup_{\varphi^*} Y$$

and  $\mathcal{O}_Z|_X = \mathcal{O}_X$  and  $\mathcal{O}_Z|_Y = \mathcal{O}_Y$ . Hence  $\mathcal{O}_Z$  is obtained by identifying  $\mathcal{O}_X|_{X_x}$  and  $\mathcal{O}_Y|_{Y_y}$  through  $\varphi^\#$ .

It is obviously similar to the construction of projective spaces.

The next example show some generalization of the previous one.

Suppose

$$S = \bigoplus_{n=0}^{\infty} S_n$$

is a graded ring, and  $I$  is a homogeneous ideal of  $S$ . Let

$$S_+ := \bigoplus_{n=1}^{\infty} S_n,$$

define

$$X = \text{Proj } S := \{\mathfrak{p} \mid \mathfrak{p} \text{ is a homogeneous prime ideal of } S, S \not\subseteq \mathfrak{p}\}$$

called the *homogeneous prime spectrum* of the graded ring  $S$ . For a homogeneous ideal  $I$ , put

$$V(I) := \{\mathfrak{p} \in \text{Proj } S \mid I \subseteq \mathfrak{p}\},$$

then the Zariski topology can be define on  $\text{Proj } S$  by taking  $\{V(I)\}$  as closed sets. For any homogeneous element  $f$ , put

$$D_+(f) := \{\mathfrak{p} \in \text{Proj } S \mid f \notin \mathfrak{p}\},$$

then  $D_+(f)$  is an open set in  $\text{Proj } S$ , and  $\{D_+(f)\}$  form a base of the topology. Indeed, if  $I$  is a homogeneous ideal,  $V(I)^c = \bigcup_{f \in I, f \text{ is homogeneous}} D_+(f)$ .

For the structure sheaf  $\mathcal{O}_X$ , it suffices to construct  $\mathcal{O}_X(D_+(f))$ . For  $f \in S_n$ , define

$$\Gamma(D_+(f), \mathcal{O}_X) := \left\{ \frac{g}{f^m} \mid g \in S_{mn} \right\} = S_f^{(0)},$$

which consists of all the elements of degree 0 in  $S_f$ .

**Proposition 3.1.** *Suppose  $X$  is a topological space and  $\mathcal{B}$  is a base of open sets. If a collection of groups  $\mathcal{F}(U)$  and maps  $\text{res}_V^U$  for open sets  $V \subset U$  of  $X$  form a  **$\mathcal{B}$ -sheaf** if they satisfy the sheaf axiom with respect to the inclusions of basic open sets. Then*

1. *Every  $\mathcal{B}$ -sheaf on  $X$  extends to a unique sheaf on  $X$ ;*
2. *Given sheaves  $\mathcal{F}$  and  $\mathcal{G}$  and a collection of maps*

$$\varphi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$$

*commuting with restrictions, there is a unique sheaf morphism  $\tilde{\varphi}$  s.t.  $\tilde{\varphi}(U) = \varphi(U)$  for all  $U \in \mathcal{B}$ .*



Hence here we get a local ringed space  $(X, \mathcal{O}_X)$ . From the construction, we see an isomorphism

$$\begin{aligned} (\varphi^*, \varphi^\#) : (D_+(f), \mathcal{O}_X|_{D_+(f)}) &\rightarrow (\operatorname{Spec} S_f^{(0)}, \mathcal{O}_{\operatorname{Spec} S_f^{(0)}}) \\ 1 &\mapsto 1 \end{aligned}$$

Consequently,  $(X, \mathcal{O}_X)$  is a scheme and it is called a *projective scheme* determined by the graded ring  $S$ .

### 3.3 Example

We want to see the process of the construction. Let  $R = k[x, y]$  where  $k$  is a algebraically closed field.

## 4 Comparison: Algebraic Variety and Scheme

In order to demonstrate the two definition are equivalent, we would give a universal property of blow-up, and show that these two construction both satisfying the property.

We first give the universal property of blow-up. Let  $X$  be a scheme, and let  $\mathcal{I}$  be a coherent sheaf of ideals on  $X$ . We say the blow-up of  $X$  with respect to  $\mathcal{I}$  is a scheme  $\tilde{X}$  along with a morphism  $\pi : \tilde{X} \rightarrow X$ , such that  $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\tilde{X}}$  is a invertible sheaf, with the universal property: for any scheme and morphism  $f : Y \rightarrow X$  such that  $f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$  is a invertible sheaf, there is a unique factorization:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi} & X \\ & \nwarrow \tilde{f} & \uparrow f \\ & & Y \end{array}$$

Here we need to explain some terminologies.

Suppose  $X$  is a scheme and  $\mathcal{O}_X$  is its structure sheaf. A sheaf  $\mathcal{F}$  is said to be an  $\mathcal{O}_X$ -*module* if the following condition is satisfied: for each open set  $U \subset X$ ,  $\mathcal{F}(U)$  is an  $\mathcal{O}_X(U)$ -module s.t. for any open set  $V \subset U$  the diagram

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

commutes, where the horizontal maps indicate  $\mathcal{O}_X(U)$  and  $\mathcal{O}_X(V)$  structures on  $\mathcal{F}(U)$  and  $\mathcal{F}(V)$  respectively. Note that the stalk  $\mathcal{F}_x$  at point  $x$  of  $\mathcal{F}$  is actually an  $\mathcal{O}_{X,x}$ -module.

For  $\mathcal{O}_X$ -module  $\mathcal{F}, \mathcal{G}$ , let  $\varphi : \mathcal{F} \rightarrow \mathcal{G}$  be a homomorphism of sheaves of additive groups. When  $\varphi$  is compatible with the  $\mathcal{O}_X$ -module structure of  $\mathcal{F}$  and  $\mathcal{G}$ , namely, for any open set  $U \subset X$ , the diagram

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{O}_X(U) \times \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \end{array}$$

commutes, then  $\varphi$  is said to be a *homomorphism of  $\mathcal{O}_X$ -module*. It is clear that  $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$  is an  $\mathcal{O}_{X,x}$ -module homomorphism. We denote the totality of the  $\mathcal{O}_X$ -module homomorphism from  $\mathcal{F}$  to  $\mathcal{G}$  by  $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ . It is a  $\Gamma(X, \mathcal{O}_X)$ -module. Besides, we have the following isomorphism for any open set  $U \subset X$ :

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X|_U, \mathcal{F}|_U) \cong \mathcal{F}(U).$$

**Lemma.** *Let  $\mathcal{F}$  be a presheaf of additive groups over a scheme  $(X, \mathcal{O}_X)$ . A presheaf of  $\mathcal{O}_X$ -module is just a presheaf satisfying the axioms for sheaf of  $\mathcal{O}_X$ -modules. Then the shefification  $\tilde{\mathcal{F}}$  is a sheaf of  $\mathcal{O}_X$ -modules.*

We often write  $\mathcal{O}_X^{\oplus n}$  for  $\mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X$ , or simply  $\mathcal{O}_X^n$ . If an  $\mathcal{O}_X$ -module  $\mathcal{F}$  is isomorphic to  $\mathcal{O}_X^n$  as  $\mathcal{O}_X$ -module, then  $\mathcal{F}$  is said to be a *free module of rank  $n$* . An  $\mathcal{O}_X$ -module  $\mathcal{F}$  is called a *locally free module of rank  $n$*  if there is an open covering  $\{U_\lambda\}_{\lambda \in \Lambda}$  of  $X$  s.t. the restriction  $\mathcal{F}|_{U_\lambda}$  is a free module of rank  $n$ . A locally free sheaf of rank 1 is called *invertible sheaf*.

Let  $\mathcal{F}$  and  $\mathcal{G}$  are  $\mathcal{O}_X$ -modules, then for any open set  $U$ , the presheaf  $\text{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ . It is denoted as  $\underline{\text{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ . For invertible sheaves  $\mathcal{L}$  and  $\mathcal{M}$  over a scheme  $X$ ,  $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$  is also an invertible sheaf. Put  $\mathcal{L}^{-1} := \underline{\text{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ , then one have a natural  $\mathcal{O}_X$ -homomorphism

$$\begin{aligned} \varphi : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} &\rightarrow \mathcal{O}_X \\ a \otimes f &\mapsto f(a) \end{aligned}$$

For an affine open set  $U$  satisfying  $\mathcal{L}|_U \cong \mathcal{O}_U$ , we get

$$\mathcal{L}^{-1}|_U \cong \underline{\text{Hom}}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{O}_U) \cong \mathcal{O}_U.$$

Therefore, over  $U$ ,  $\varphi$  is an  $\mathcal{O}_U$ -isomorphism. Namely, we have  $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \cong \mathcal{O}_X$ . Hence, under the tensor product, isomorphic classes of invertible sheaves form a group, called the *Picard group*, denoted as  $\text{Pic } X$ .

**Definition.** Let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module. If for each point  $x$  in  $X$  there is an open neighborhood  $U$  of  $x$  so that the sequence of  $\mathcal{O}_X$ -module

$$\mathcal{O}_U^{\oplus I} \rightarrow \mathcal{O}_U^{\oplus J} \rightarrow \mathcal{F}|_U \rightarrow 0$$

is exact, then  $\mathcal{F}$  is said to be *quasicoherent*. The index set  $I, J$  need not be finite. If for each  $x \in X$ , there is an open set  $U$  of  $x$  s.t. the following sequence of  $\mathcal{O}_X$ -module is exact

$$\mathcal{O}_U^{\oplus n} \rightarrow \mathcal{F}|_U \rightarrow 0$$

then  $\mathcal{F}$  is said to be a *finitely generated  $\mathcal{O}_X$ -module*.

Suppose  $M$  is an  $R$ -module, we would like to construct a sheaf on the affine scheme  $(\text{Spec } R, \mathcal{O}_{\text{Spec } R})$  associated to  $M$ . Let  $M_f$  be the localization of  $M$  with respect to  $\{1, f, f^2, \dots\}$  where  $f \in R$ . It

is easy to verify that the family is a  $\mathcal{B}$ -sheaf where  $\mathcal{B} := \{D(f)\}_{f \in R}$ . Hence by Proposition 3.1, the family of  $R$ -modules forms a sheaf of  $\mathcal{O}_X$ -modules. It is called the *associated sheaf* with an  $R$ -module  $M$  and it is denoted as  $\widetilde{M}$ .

**Proposition 4.1.** *Suppose  $(X, \mathcal{O}_X)$  is the affine scheme determined by a commutative ring  $R$ .*

1. *The  $\mathcal{O}_X$ -module  $\widetilde{M}$  determined by an  $R$ -module is quasicoherent, and*

$$\Gamma(X, \widetilde{M}) = M.$$

2. *For any  $R$ -module homomorphism  $\varphi : M \rightarrow N$ , the map*

$$\Phi : \text{Hom}_R(M, N) \rightarrow \underline{\text{Hom}}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$$

*assigning an  $\mathcal{O}_X$ -module homomorphism  $\widetilde{\varphi}$  is an isomorphism of  $R$ -modules.*

3. *For  $R$ -modules  $M$  and  $N$ , we have isomorphisms of  $\mathcal{O}_X$ -modules  $\widetilde{M} \oplus \widetilde{N} \cong \widetilde{(M \oplus N)}$ ,  $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{(M \otimes_R N)}$ . Furthermore, if  $M$  is finitely presented, we have*

$$\underline{\text{Hom}}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \cong \widetilde{\text{Hom}_R(M, N)}$$

**Theorem 4.2.** *An  $\mathcal{O}_X$ -module  $\mathcal{F}$  over an affine scheme  $X = \text{Spec } R$  is quasicoherent if and only if  $\mathcal{F}$  is isomorphic to the associated sheaf  $\widetilde{M}$  and  $\Gamma(X, \mathcal{F})$  is isomorphic to  $M$  as  $R$ -module.*

We have seen that quasicoherent sheaves are analogous to  $R$ -modules. On the other hand, we want to see the analogy of finitely generated  $R$ -modules, which is a coherent sheaf.

**Definition.** An  $\mathcal{O}_X$ -module  $\mathcal{F}$  is said to be *coherent* if the following conditions are satisfied:

- (i)  $\mathcal{F}$  is a finitely generated  $\mathcal{O}_X$ -module;

For an arbitrary open set  $U$  of  $X$  and for an arbitrary  $\mathcal{O}_U$ -module homomorphism

$$\varphi : \mathcal{O}_U^{\oplus n} \rightarrow \mathcal{F}|_U$$

$\text{Ker } \varphi$  is a finitely generated  $\mathcal{O}_X$ -module.

It is easy to prove that a coherent sheaf is quasicoherent. When  $R$  is a Noetherian ring, we have the following fact:

**Theorem 4.3.** *A quasicoherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  over an affine scheme  $X = \operatorname{Spec} R$  determined by a Noetherian ring is coherent if and only if  $\Gamma(X, \mathcal{F})$  is a finitely generated  $R$ -module. Conversely, a finitely generated  $\mathcal{O}_X$ -module is coherent.*

Furthermore, we have

**Theorem 4.4.** *Let*

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$$

*be an exact sequence of  $\mathcal{O}_X$ -modules over a scheme  $X$ . If any of two sheaves among  $\mathcal{F}, \mathcal{G}$  and  $\mathcal{H}$  are coherent, then the third is coherent.*

We say that  $\operatorname{Proj} \operatorname{Bl}_I(X)$  is what we want. Before actually proving it, we first give an example to illustrate it. Let  $R = k[x, y]$  where  $k$  is a algebraically closed field, and let  $\mathfrak{m} = (x, y)$ , then the blow-up algebra  $\operatorname{Bl}_{\mathfrak{m}}(R) = R \oplus \mathfrak{m} \oplus \mathfrak{m}^2 \oplus \cdots$  is a graded algebra over  $R$ . First, all of the maximal ideals  $(x - a, y - b)$  of  $R$  belong to  $\operatorname{Proj} \operatorname{Bl}_I(X)$ , where  $(a, b) \neq (0, 0)$ .

The first example is blowing up the maximal ideal  $(x, y)$  of ring  $\mathbb{C}[x, y]$ . Consider the map  $f : \mathbb{C} - V(x, y) \rightarrow \mathbb{C} \times \mathbb{P}^1, (a, b) \mapsto ((a, b), [a, b])$ .

Then is blowing up the ideal  $(x^2, y)$  of ring  $\mathbb{C}[x, y]$ .

## 5 Geometric Properties of Blow-ups

### 5.1 Local Properties

In this section we would discuss the geometry near  $E$  on  $\widetilde{\mathbb{C}^n}$  in detail. Let  $(z_1, \dots, z_n)$  be a local coordinate of  $U$  with center  $x$ . Hence

$$\widetilde{U} := \{(z, l) \in U \times \mathbb{P}^{n-1} \mid z_i l_j = z_j l_i\}$$

is an open set of  $\widetilde{\mathbb{C}^n}$  and

$$\widetilde{U}_i := \{l_i \neq 0\} \subset \widetilde{U}.$$

In this way we obtain an open cover of the neighborhood  $\widetilde{U}$  of  $E$ . And it can be proved that  $\mathcal{O}(-1) = \widetilde{\mathbb{C}^n}$  is a vector bundle over  $\mathbb{P}^{n-1}$ . Let  $p : \mathcal{O}(-1) \rightarrow \mathbb{P}^{n-1}$  be the canonical projection and let  $\mathbb{P}^{n-1} = \bigcup_{i=1}^{n-1} U_i$  be the standard open covering. A canonical trivialization of  $\mathcal{O}(-1)$  over  $U_i$  is given by

$$\begin{aligned} \varphi : p^{-1}(U_i) &\cong \mathbb{C} \times U_i \\ (z, l) &\mapsto (z_i, l) \end{aligned}$$

and hence the transition maps  $\varphi_{ij} : \mathbb{C} \rightarrow \mathbb{C}$  are  $w \mapsto \frac{z_i}{z_j} w$ . Therefore the blow-up at a point is naturally a manifold and by the open covering, it is of dimension  $n$ . Using the same technique, the blow-up of  $\mathbb{C}^n$  along a linear subspace is also a manifold.

In the first section, we have seen that  $E$  is naturally identified with  $\mathbb{P}(\mathbb{C}^n)$ , or precisely,  $\mathbb{P}(T_0(\mathbb{C}^n))$  where the  $T$  is the holomorphic tangent space. So that the global sections of  $[-E]$  over  $E$  correspond exactly to the linear functionals on the tangent space, i.e.

$$H^0(E, \mathcal{O}_E(-E)) \cong T_0(\mathbb{C}^n)^*.$$

On the other hand, given a function  $f$  on  $U$  vanishing at the origin, the function  $\pi^* f$  vanishes along  $E$  so can be considered as a section of  $[-E]$  over  $U$ . Let  $\mathcal{O}(U)_x$  be the sheaf of holomorphic functions on  $U$  vanishing at the Point  $x$ , hence we get the following diagram:

$$\begin{array}{ccc} H^0(\tilde{U}, \mathcal{O}_X(-E)) & \xrightarrow{res} & H^0(E, \mathcal{O}_E(-E)) \\ \uparrow & & \parallel \\ H^0(U, \mathcal{O}_X(U)_0) & \xrightarrow{d} & T_0(\mathbb{C}^n)^* \end{array}$$

Using the blow-up at a point, we can actually prove the following important theorem:

**Theorem 5.1** (Kodaira Embedding Theorem). *A compact complex manifold  $M$  can be embedded in a projective if and only if it has a closed, positive  $(1,1)$ -form  $\omega$  whose cohomology class  $[\omega]$  is rational.*

The proof can be summarized as below:  $[\omega] \in H^2(M, \mathbb{Q})$  if and only if  $[k\omega] \in H^2(M, \mathbb{Z})$  for some natural number  $k$ , hence in the exact sequence

$$H^1(M, \mathcal{O}^*) \rightarrow H^2(M, \mathbb{Z}) \xrightarrow{\iota_*} H^2(M, \mathbb{C})$$

$\iota_*([k\omega]) = 0$ , then there is a holomorphic line bundle  $L \rightarrow M$  with  $c_1(L) = [k\omega]$ , and it is positive.

## 5.2 Global Properties

Using the same notation as the previous subsection, we put  $\mathbb{C}^{n*} = \mathbb{C}^n - 0$ ,

## 5.3 Desingularity

In 1964, Heisuke Hironaka proved a theorem saying that every variety can be desingularized, or equivalently, birational to a smooth quasi-projective variety. And the proof is based on the construction of blow-ups.

**Definition.** A morphism of varieties  $\pi : X \rightarrow V$  is called a *projective morphism* if  $X$  is a closed subvariety of a product variety

$$X \subset V \times \mathbb{P}^n$$

and  $\pi : X \rightarrow V$  is the restriction of the projection onto the first coordinate.

**Theorem 5.2** (Desingularization Theorem). *Let  $V$  be a quasi-projective variety. Then there exists a smooth quasi-projective variety  $X$  and a projective birational morphism  $\pi : X \rightarrow V$ . Furthermore,  $\pi$  may be assumed to be an isomorphism on the smooth locus of  $V$ , and if  $V$  is projective variety, then so is  $X$ .*

We would conclude the thesis by some examples of applications of blow-ups, showing how the construction works in desingularization.

Consider the curve  $C \subset \mathbb{C}^2$  given by  $z_1 \cdot z_2 = 0$ . Obviously the origin is not smooth. On the other hand, the closure of  $C$  in  $\widetilde{\mathbb{C}^2}$  is the union of two separated lines  $\overline{\pi^{-1}(l_1 - O)}$  and  $\overline{\pi^{-1}(l_2 - O)}$ , where  $l_i = \{(z_1, z_2) | z_i = 0\}$ ,  $i = 1, 2$ . Hence the closure is smooth.

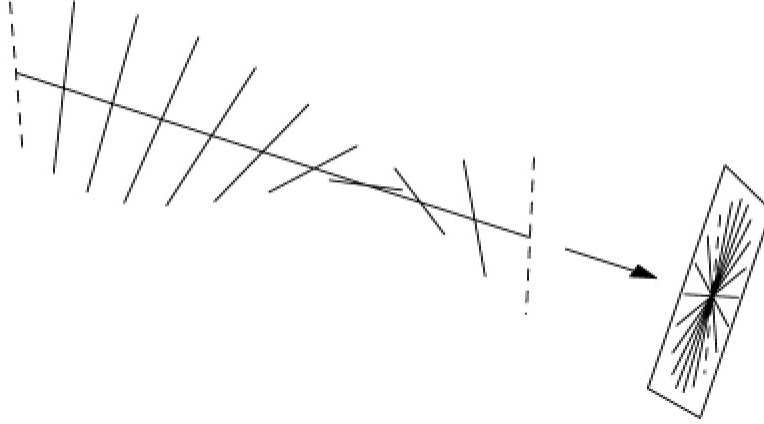


Figure 3: Desingularization of All the Lines through the Origin

Actually, from this point of view, we can say that the blow-up of  $\mathbb{C}^n$  at the origin is separating all the lines through the origin. So, there is supposed to be a projective line replacing the origin, corresponding each of a point of the exceptional bundle to a line. And this is what the figure above illustrating.

The next example is more interesting. It says by blowing up the singularity of a conic, we do

get a ruled surface. Consider

$$V = Z(z - x^2 - y^2)$$

is a affine algebraic set of  $\mathbb{C}^3$ , and it consists of all the lines  $l =$

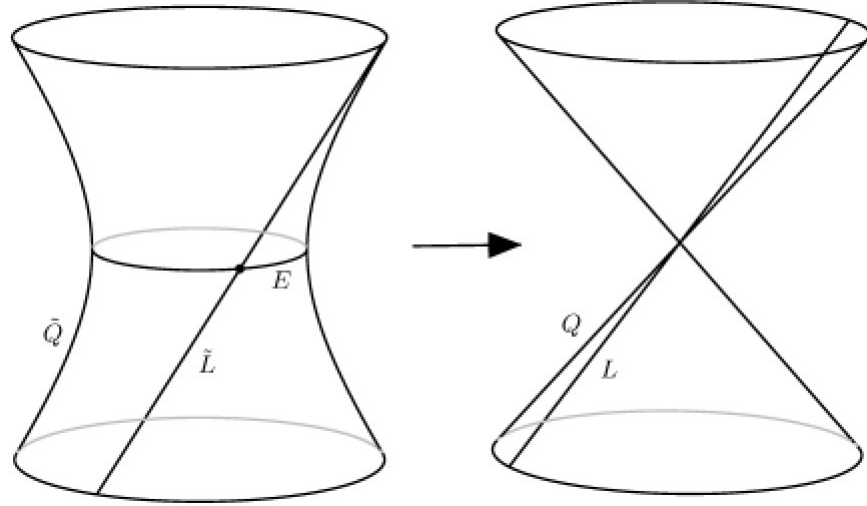


Figure 4: Desingularization of a Cone at the Origin

## 6 Conclusion

## References

- [GH] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.
- [Ha] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. 52, Springer-Verlag, New York-Heidelberg, 1977.
- [Huy] D. Huybrechts, *Complex Geometry: An Introduction*, Universitext, Springer-Verlag, Berlin-Heidelberg, 2005.
- [Ken] U. Kenji, *Algebraic Geometry 1,2*, Mathematical Monographs 185,197 - Iwanami Series in Modern Mathematics, Amer. Math. Soc., Providence, RI, 1999.
- [SKKT] K. E. Smith, L. Kahanpää, P. Kekäläinen and W. Traves *An Invitation to Algebraic Geometry*, Universitext, Springer-Verlag, New York, 2000.
- [Vak] R. Vakil, *The Rising Sea: Foundations of Algebraic Geometry*, preprint 2017.