Some Open Problems I am Interested in

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I maintain this document only to keep a record of the problems that I was/am interested in. Once a problem was wrote, I would not mark it again. Hence there is possibility that the problem is not a problem anymore. Also, you can see a lot of chaos in this document, not in order, so unrelated, classified into numerous field. However, I hope this document can make me feel less aimless, and can probably help others find interesting problems.

Problem 1 (Kervaire Invariant Problem in 126, Geometrical Topology)**.** In 1960, Kervaire proved that there is a manifold homotopic to a sphere S^2 without any smooth structure [4]. What he did was actually constructing a "Kervaire invariant" that lived in $\pi_2^s(S^0)$. In 1969, Browder proved that the Kervaire invariant could only live in $\pi_{2^n-2}^s(S^0)$ [1] and soon Barratt, Jones, Mahowald proved for $n = 2, 3, 4, 5, 6$ the fundamental group was not trivial and the element existed [2]. Since then 126 has become an open problem for the existence. In 2016, Hill, Hopkins and Ravenel proved $\pi_{256k-2}^s(S^0) = 0$ so that the invariant does not exist for $n \geq 8$. Thus the only left one is $n = 7$ [3].

Problem 2 (Auto-equivalence of Derived Categories, Derived Algebraic Geometry)**.** Bridgeland, sm 3-folds

Problem 3 (Class Number Problem, Number Theory)**.** When we were solving number theoretical equations like

$$x^2 + dy^2 = n$$

for some integer n, it is helpful for consider the primes in $\mathbb{Z}[\sqrt{-d}]$. Since generally $\mathbb{Z}[\sqrt{-d}]$ is not a U.F.D., it is better to consider its integral closure $\mathcal{O}_K$ where $K = \mathbb{Q}(\sqrt{-d})$. This ring $\mathcal{O}_K$ is a Dedekind domain, Gauss, in his famous book *Disquisitiones Arithmeticae* of 1801 (Section V, Articles 303 and 304), posed these questions

Problem 4 (The Section Conjecture, Algebraic Geometry)**.** When we were dealing with a scheme X, the usual construction of the fundamental group fails since it is difficult to have a path in X. However by the relations between the covering spaces and their deck transformation groups, we are able to have a notion of étale fundamental group $\pi_1^{\mathrm{et}}(X)$.

Problem 5 (Moduli of Mixed Hodge Structure, Algebraic Geometry)**.** Kapranov

Problem 6 (Tannakian Reconstruction, Algebraic Geometry)**.** If X is a G-variety where G is a reductive group, and $x \in X$ is a point with a dense orbit. We would get (by beyond GIT) a complex $DF(X/G, x)_\bullet$ (degeneration fan). If X is toric, then DF recovers the fan of X. Then the question is to what extend does DF determine X?

Problem 7 (Quasi-smooth Hilbert Scheme, DAG)**.** We have the Hilbert scheme in the usual algebraic geometry. However, in the derived world, the analogy is to consider the moduli of (derived) l.c.i. families of subschemes.

References

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[2] Barratt, Michael G.; Jones, J. D. S.; Mahowald, Mark E., *Relations amongst Toda brackets and the Kervaire invariant in dimension 62*. Journal of the London Mathematical Society. 2. 30 (3) (1984): 533550. MR 0810962.

[3] Hill, Michael A.; Hopkins, Michael J.; Ravenel, Douglas C., *On the nonexistence of elements of Kervaire invariant one*. Annals of Mathematics. **184** (1) (2016).: 1262. arXiv:0908.3724.

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