Different Constructions and Geometric Properties of Blow-ups

Abstract

In this thesis, we would like to discuss the definition of blow-up, its different constructions, some of its geometric properties and some applications. It should give a panoramic picture of blow-ups including why we need it and how can we understand it.

First we will give the geometric construction of blow-up, starting from the simplest situation, that blowing up a affine space at the origin, to the most generalized ones, blowing up a complex manifold along a submanifold. Then there are algebraic construction, using the Proj construction of graded ring, obtaining a projective scheme actually. When it comes to algebraic varieties, we have both the constructions and it should coincide with each other. Therefore in Section 4 we give the comparison, that is, giving the universal property and showing that they both satisfy the property. Hence by ”Abstract nonsense”, they must be the same. It is a rather complicated way, but there are some illustrations to show the picture of them and the universal property is also of vital importance.

Blow-up is a really important example in both complex geometry and algebraic geometry, and there are many applications. It can be used to prove the Kodaira embedding theorem. More interesting, it is one of the most significant method to resolve singularities. And when it is a variety over a field of characteristic zero, Hironaka proved that all of the singularities can be resolved by blow-ups. We will give some examples to illustrate these.

Birational equivalence is an essential notion in algebraic geometry and blow-up is one of the most crucial instance. To do some concrete computation of blow-ups, one can really understand the language of algebraic geometry. It is a major example connecting almost all parts of algebraic geometry. And writing this thesis I have known much deeper about algebraic geometry.

[Keywords] Blow-up, construction, $\mathcal{O}_X$-module, desingularization

Contents

1 Introduction **3**

2 Geometric Construction **3**

2.1 Blow-up of $\mathbb{C}^n$ at Origin . . . 3

2.2 A Little Further: Surgery . . . 4

2.3 Blow-up of $\mathbb{C}^n$ along a Affine Variety . . . 6

2.4 Ultimate Generalization: Blowing up along a Submanifold . . . 8

3 Algebraic Construction **8**

3.1 Blow-up Algebra . . . 8

3.2 Proj Construction . . . 9

3.3 Example . . . 11

4 Comparison: Algebraic Variety and Scheme **12**

5 Geometric Properties of Blow-ups **16**

5.1 Local Properties . . . 16

5.2 Global Properties . . . 17

5.3 Desingularity . . . 18

6 Conclusion **19**

1 Introduction

In algebraic geometry, we have to deal with singularities of varieties. The problem of **resolution of singularities** asks whether every algebraic variety X has a resolution, a non-singular variety Y with a proper birational map $f : Y \to X$. We already have different ways to resolve a singularity, and blowing up is one of the most important methods. By using blow-up, one can actually separate lines with different directions, hence it can tells apart curves with different tangent lines. And the construction can be derived from this, adding all of the *direction* of the lines into the space.

It is really amazing that blow-up can solve the problem over a field of characteristic zero (e.g. $\mathbb{C}$) and it has *algebraic structure*. Hence there are supposed to be two different construction, one of these is described above, the other can be seen from the algebraic structure. These two should be the same since they represent the same structure.

To construct blow-up and do the calculation, we would encounter many basic notions of algebraic geometry and complex geometry. Furthermore it is an excellent instance for the comparison between algebraic and complex geometry. Although Serre has demonstrated when these two areas coincide, blow-up is still helpful for beginners to understand the similarities and differences.

Usually $\mathbb{C}^n$ is a denotation geometrically, while $\mathbb{A}^n_{\mathbb{C}}$ is more algebraically. However we will regard them equivalently through this paper. $\mathbb{A}^n, \mathbb{P}^n$ are $\mathbb{A}^n_{\mathbb{C}}$ and $\mathbb{CP}^n$ respectively except indicated explicitly. All of the construction can be transplanted over another (algebraic closed) field.

In order to get a full understand of this paper, the readers are supposed to know the results from commutative algebra, properties of projective spaces, some results from geometry and topology, basic construction of algebraic geometry, complex geometry and sheaves. All of the definition will be mentioned later in the text however it may be extremely confusing if it is the first time to encounter these terminologies.

2 Geometric Construction

2.1 Blow-up of $\mathbb{C}^n$ at Origin

We will start from the simplest case, i.e. blowing up the origin of $\mathbb{C}^n$.

Consider a subset of $\mathbb{C}^n \times \mathbb{P}^{n-1}$

$$\widetilde{\mathbb{C}^n} := \{((a_1, \cdots, a_n), [b_1, \cdots, b_n]) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid a_i b_j = a_j b_i, 1 \leq i, j \leq n\}.$$

It is easy to see that we have a natural projection from $\widetilde{\mathbb{C}^n}$ to $\mathbb{C}^n$:

$$\pi : ((a_1, \cdots, a_n), [b_1, \cdots, b_n]) \mapsto (a_1, \cdots, a_n).$$

Definition. The set $\widetilde{\mathbb{C}^n}$ along with the map $\pi : \widetilde{\mathbb{C}^n} \to \mathbb{C}^n$ is called the *blow-up* of $\mathbb{C}^n$ at the origin.

One can see that the blow-up of $\mathbb{C}^n$ is exactly $\mathcal{O}(-1)$ of $\mathbb{P}^{n-1}$.

Immediately we have these properties of $\widetilde{\mathbb{C}^n}$:

For any point $O \neq z = (a_1, \cdots, a_n) \in \mathbb{C}^n$, there is at least a $1 \leq i_0 \leq n$ s.t. $a_{i_0} \neq 0$ hence $b_{i_0} \neq 0$. Thus the equations can be written as $\frac{b_j}{b_{i_0}} = \frac{a_j}{a_{i_0}}, 1 \leq j \leq n$. Therefore there is only **ONE** point x in $\widetilde{\mathbb{C}^n}$ with $\pi(x) = z$, i.e. $x = ((a_1, \cdots, a_n), [a_1, \cdots, a_n])$.

$\pi^{-1}(O) \cong \mathbb{P}^{n-1}$. Indeed, any point $[b_1, \cdots, b_n] \in \mathbb{P}^{n-1}$ satisfies the equations if $(a_1, \cdots, a_n) = 0$.

The points of $\pi^{-1}(O)$ are in 1-1 corresponding to the lines through the origin in $\mathbb{C}^n$. We usually denote it as E. A line through the origin l can be given by the parametric equation $x_i = a_i t$ where $1 \leq i \leq n$ and $a_i \in \mathbb{C}$ are not all zero, thus $[a_1, \cdots, a_n]$ is a point in $\mathbb{P}^{n-1}$. Now consider the line $\widetilde{l} = \pi^{-1}(l - O)$ in $\widetilde{\mathbb{C}^n} - \pi^{-1}(O)$, it is $\{((a_1 t, \cdots, a_n t), [a_1, \cdots, a_n]) \mid t \in \mathbb{C}^*\}$. These equations also make sense for $t = 0$ and give the closure $\bar{l}$ of $\widetilde{l}$ in $\widetilde{\mathbb{C}^n}$. $\widetilde{l}$ meets $\pi^{-1}(O)$ in the point $[a_1, \cdots, a_n] \in \mathbb{P}^{n-1}$, so we see that sending l to Q gives us the correspondence between lines through origin in $\mathbb{C}$ and points of $\pi^{-1}(O)$.

$\widetilde{\mathbb{C}^n}$ is irreducible. Indeed $\widetilde{\mathbb{C}^n} = \widetilde{\mathbb{C}^n} - \pi^{-1}(O) \cup \pi^{-1}(O)$, where the first piece is isomorphic to $\mathbb{C}^n - O$ which is obviously irreducible, and every point of the second part $\pi^{-1}(O)$ is in the closure of some line of $\widetilde{\mathbb{C}^n - \pi^{-1}}(O)$. Hence $\widetilde{\mathbb{C}^n - \pi^{-1}}(O)$ is dense in $\widetilde{\mathbb{C}^n}$ and $\widetilde{\mathbb{C}^n}$ is irreducible.

Here we give an example to illustrate how this process works.

The blow-up of $\mathbb{R}^2$ at the origin can be visualized as follows: we have known that points not the origin are in a 1-1 correspondence by π, so we leave them stable. However we need to replace the origin by $\mathbb{RP}^1$, i.e. a circle. For any line l parameterized by $x = a_1 t, y = a_2 t$, $\pi^{-1}(l - O)$ lies in $\widetilde{\mathbb{R}^2} - \mathbb{RP}^1$, and the closure of $\pi^{-1}(l - O)$ consist of the point $[a_1, a_2]$ in $\mathbb{RP}^1$. So the disconnected part of $\pi^{-1}(l - O)$ are glued by the point. This operation works like we glue the antipodal points together. Topologically, this is regarding the missing point as a disk, gluing the boundary of the disk with the boundary of a Möbius band, since both of the boundary are circles.

The diagram below shows the visualization above.

Using the technique above, it is easy to blow up an algebraic set at the origin.

Definition. If V is a closed algebraic set of $\mathbb{C}$ passing through the origin, the *blow-up* of V at the origin is $\widetilde{V} = \overline{\pi^{-1}(V - O)}$, where $\pi : \widetilde{\mathbb{C}^n} \to \mathbb{C}^n$ is the blow-up of $\mathbb{C}^n$ as above. We denote also by $\pi : \widetilde{V} \to V$ the morphism by the restriction of $\pi : \widetilde{\mathbb{C}^n} \to \mathbb{C}^n$ to $\widetilde{V}$.

2.2 A Little Further: Surgery

Consider every point on a complex manifold have a local Euclidean coordinate, so to blow up a (complex) manifold at a point, it suffices to "cut off" a small neighborhood of the point, blow-up the small piece at the origin and then glue it back up to the manifold. Yet we need some extra technique to do this surgery-like modification.

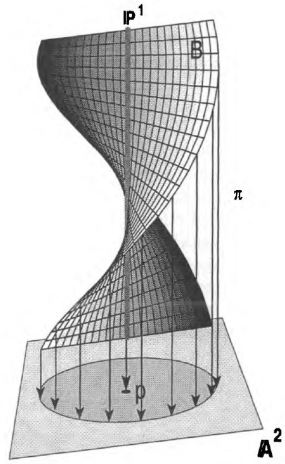

Figure 1: Visualization of Blowing up a Point

Let M be a complex manifold, and $S \subset M$ a compact submanifold of M. We construct a new manifold $\widetilde{M} = (M - S) \cup \widetilde{S}$ replacing S by another complex manifold $\widetilde{S}$ as follows:

Take domains W, W_1 s.t. $S \subset W_1 \subset \overline{W_1} \subset W \subset M$ where $\overline{W}$ is compact. Let $\widetilde{S}$ be another compact manifold s.t. $\widetilde{S} \subset \widetilde{W_1} \subset \overline{(\widetilde{W_1})} \subset \widetilde{W}$, and a biholomorphic surjective map $\psi : \widetilde{W} - \widetilde{S} \to W - S$ s.t. $\psi : (\widetilde{W} - \widetilde{S}) = W - S$. Let $\widetilde{M}$ be the manifold obtained by gluing $M - \overline{W_1}$ identifying $P \in W - \overline{W_1}$ with $\widetilde{P} = \psi^{-1}(P) \in \widetilde{W} - \overline{\widetilde{W_1}}$ via ψ:

$$\widetilde{M} = \left(M - \overline{W_1}\right) \cup \widetilde{W}.$$

Since ψ is biholomorphic, $\widetilde{M}$ becomes a manifold. Thus (see Figure 1)

$$\widetilde{M} = (M - W) \cup \widetilde{W}.$$

We would like to use this to construct the blow-up of a manifold at a point.

For example, let $M = \mathbb{P}^2$, and let $[z_0, z_1, z_2]$ be its homogeneous coordinate and $P = [1, 0, 0]$. We denote by $\mathbb{P}^1_\infty$ the projective line $z_0 = 0$. Then

$$\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1_\infty$$

where

$$U_0 = \{[z_0, z_1, z_2] \mid z_0 \neq 0\} = \{[1, \frac{z_1}{z_0}, \frac{z_2}{z_0}] \mid z_0 \neq 0\} = \{(w_1, w_2)\} = \mathbb{C}^2.$$

We call the line $\mathbb{P}^1_\infty$ the line at infinity. Since we have known how to blow up $\mathbb{C}^2$ at the origin, i.e. blow up U_0 at the point $[1, 0, 0]$, and $\pi|_{\widetilde{U_0} - \pi^{-1}(O)} : \widetilde{U_0} - \pi^{-1}(O) \to U_0 - O$ is biholoporphic, we can replace U_0 by $\widetilde{U_0}$. That is blowing up $\mathbb{P}^2$ at the point P.

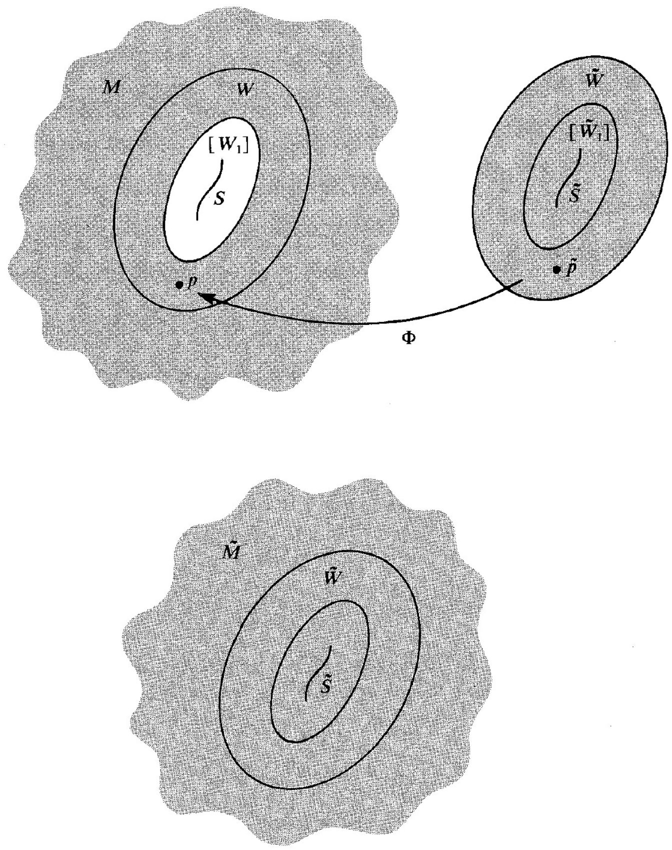

Figure 2: The Surgery of a Manifold

2.3 Blow-up of $\mathbb{C}^n$ along a Affine Variety

For an *affine variety*, we mean an **irreducible** algebraic subset of $\mathbb{A}^n$, denoted as

$$Z(f_1, \cdots, f_k) = \{(a_1, \cdots, a_n) \in \mathbb{A}^n \mid f_i(a_1, \cdots, a_n) = 0 \ \ \forall 1 \leq i \leq k\},$$

where $(f_1, \cdots, f_k)$ is a prime ideal. We can always find finitely many generators as a result of Hilbert basis theorem.

We define $A(V) = k[x_1, \cdots, x_n]/(f_1, \cdots, f_n)$ as the *coordinate ring* of affine variety $V = Z(f_1, \cdots, f_k)$, and let $k(V)$ be the quotient field, which is called the *field of fractional functions* on V. If P is a point of V, we define

$$\mathcal{O}_{V,P} = \big\{\frac{f}{g} \mid f, g \in A(V) \text{ and } g(P) \neq 0\big\}$$

as the *local ring* of V at point P. If $U \subset V$ is a nonempty open set, we set

$$\mathcal{O}_V(U) := \bigcap_{P \in U} \mathcal{O}_{V,P}.$$

This is a subring of $k(V)$, and we call this the *the ring of regular function* on U. Note that the local ring of V at point P is actually a local ring.

Lemma. *Let U be an open set in an affine variety X. A set theory map $\varphi : U \to k$ is a rational function at the point P if and only if there is an open neighborhood of P in U s.t. there are polynomials $f, g \in k[x_1, \cdots, x_n]$, with $g(Q) \neq 0$ and $\varphi(Q) = \frac{f(Q)}{g(Q)}$ for all $Q \in V$, and φ is a regular function on U if it is regular at every point in U.*

A *morphism* $\varphi : X \to Y$ between two affine variety is a continuous map s.t. $f^*(\mathcal{O}_Y(U)) \subseteq \mathcal{O}_X(f^{-1}(U))$, i.e. for all regular function f on an open set U of Y, $\varphi^{\#}(f) = f \circ \varphi$ is still regular. One can see that the definition of morphism is actually the definition of morphism of ringed spaces.

Generally, a *variety* means a **quasi-projective** variety. Since all of the varieties are ringed spaces, the morphism can be defined as morphism of ringed spaces. From another point of view, variety can be glued by affine varieties, and we usually study the properties locally, it is useful and easier just to consider the affine things.

Lemma. *Let X, Y be (affine) varieties and let U be a nonempty open set in X. Suppose φ, ψ are morphism from X to Y and $\varphi|_U = \psi|_U$, then $\varphi = \psi$.*

By the lemma above, we can define the *rational map* from X to Y, which is the equivalence class of the morphisms from a nonempty set $U \subset X$ to Y, where two representatives equal if and only if they are identical on the intersection of the open sets. A *birational map* is a rational map admitting an inverse. Notice that for the blow-up at the origin, $\pi : \widetilde{\mathbb{C}^n} - O \to \mathbb{C}^n - O$ is a birational map. This example of blow-up reminds us one of the reason we have rational maps is that sometimes it is difficult to construct a morphism from X to Y, but rather easy to define one on an open subset.

The blow-up of $\mathbb{A}^n$ with respect to the subvariety $X = Z(f_1, \cdots, f_k)$ is given by

$$\widetilde{X} := \{((a_1, \cdots, a_n), [b_1, \cdots, b_k]) \mid b_i f_j(a_1, \cdots, a_n) = b_j f_i(a_1, \cdots, a_n)\},$$

which is a subset of $\mathbb{A}^n \times \mathbb{P}^{k-1}$, along with the projection $\pi : \widetilde{X} \to \mathbb{A}^n$. This can be characterized as the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathbb{A}^n \arrow[d] & (\mathbb{A}^n - X) \cup E \arrow[l] \arrow[d] \\
X \times \{0\} \arrow[r] & \mathbb{A}^n \times \mathbb{A}^k & \mathbb{A}^n \times \widetilde{\mathbb{A}^k} \arrow[l, "\pi"']
\end{tikzcd}$$

As the definition, the blow-up of $\mathbb{A}^n$ along a variety (thus at a point) is a variety since it is characterized by polynomials in $\mathbb{A}^n \times \mathbb{P}^{k-1}$. Generally, a blow-up is the closure of a birational map. Indeed, the restriction of the projection is identity, hence birational. And a simple example is the blow-up of $\mathbb{C}^n$ along a linear subspace $\mathbb{C}^m$ satisfying $z_{m+1} = \cdots = z_n = 0$ which will be used next. Since linear space is also a variety, the blow-up can be derived as

$$\mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) := \{((z_1, \cdots, z_n), [x_{m+1}, \cdots, z_n]) \mid z_i x_j = z_j x_i,\ i, j = m+1, \cdots, n\},$$

with a projection $\pi : \mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) \to \mathbb{C}^n$.

2.4 Ultimate Generalization: Blowing up along a Submanifold

Finally we can construct the blow-up of an n-dimensional complex manifold X along an arbitrary submanifold $Y \subset X$ of dimension m. In order to do so, we choose an atlas $X = \bigcup \varphi(U_i),\ \varphi(U_i) \to X$ where U_i are open in $\mathbb{C}^n$ and $\varphi(U_i \bigcap \mathbb{C}^m) = \varphi(U_i) \bigcap Y$ for $\mathbb{C}^m = \{(z_1, \cdots, z_n) \mid z_{m+1} = \cdots = z_n = 0\}$.

Let $\pi : \mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) \to \mathbb{C}^n$ be the canonical projection and let $\pi_i : Z_i \to U_i$ be its restriction to the open sets U_i, i.e. $Z_i = \pi^{-1}(U_i)$ and $\pi_i = \pi|_{Z_i}$. The blow-up can be naturally glued. Consider arbitrary open sets $U, V \subset \mathbb{C}^n$, and a holomorphic map $\varphi : U \cong V$ with the property that $\varphi(U \bigcap \mathbb{C}^m) = V \bigcap \mathbb{C}^m$. Write $\varphi = (\varphi_1, \cdots, \varphi_n)$, then for $k > m$ one has $\varphi_k = \sum_{j=m+1}^n z_j \varphi_{k,j}$. Indeed,

$$\varphi_k$$

Thus we have a biholomorphic map

$$\hat{\varphi}(x, z) := \left((\varphi_{k,j}(z))_{k,j=m+1,\cdots,n} \cdot x, \varphi(z)\right).$$

It is clear that $\hat{\varphi}(x, z)$ is contained in the blow-ups of the subspaces. In order to obtain the global blow-up $\pi : \mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) \to X$, we have to ensure these gluing are compatible. This is obvious over $X - Y$. Over Y, the matrices we obtained are by definition the cocycle of normal bundle $\mathcal{N}_{Y|X}$. Thus they satisfy the cocycle condition, which also proves that $\pi^{-1}(Y) \cong \mathbb{P}(\mathcal{N}_{Y|X})$, the **projective bundle associated to** $\mathcal{N}_{Y|X}$. We summarize the discussion by following

Proposition 2.1. *Let Y be a complex submanifold of X. Then there exists a complex manifold $\widetilde{X} = \mathrm{Bl}_Y X$, the blow-up of X along Y, together with a holomorphic map $\pi : \widetilde{X} \to X$ such that $\pi : \widetilde{X} - \pi^{-1}(Y) \to X - Y$ is identity and $\pi : \pi^{-1}(Y) \to Y$ is isomorphic to $\mathbb{P}(\mathcal{N}_{Y|X}) \to Y$.*

3 Algebraic Construction

3.1 Blow-up Algebra

Definition. Let R be a ring and let $I \subset R$ be an ideal of R. The *blow-up algebra* or *Rees algebra* , associated with the pair (R, I), is the graded R-algebra

$$\mathrm{Bl}_I(R) := \bigoplus_{n \geq 0} I^n = R \oplus I \oplus I^2 \oplus \cdots$$

For a *graded ring*, we mean a ring R that is the direct sum of abelian groups R_i s.t. $R_iR_j \subseteq R_{i+j}$. We would write

$$R = \bigoplus_{n \in \mathbb{N}_0} R_n,$$

where the elements in any factor R_n of the decomposition are called the *homogeneous elements* of *degree* n. Every elements $a \in R$ can be written as a sum $a = a_{i_1} + \cdots + a_{i_k}$ where a_{i_j} lying different R_{i_j} are called the *homogeneous components* of a. A graded module M over a graded ring R can be written as

$$M = \bigoplus_{n \in \mathbb{N}_0} M_n,$$

where $R_iM_j \subseteq M_{i+j}$. A graded algebra A over a ring R is an algebra if it is graded as a ring. It is clear that the blow-up algebra is a graded R-algebra. An *homogeneous ideal* is an ideal generated by homogeneous elements. An ideal I is homogeneous if and only if $I = \bigoplus_{n \geq 0} I \bigcap S_n$. And this is the key property we will use following. A *homogeneous prime ideal* is a homogeneous ideal that is prime.

3.2 Proj Construction

Having the corresponding ring of blow-up, we still need to realize the geometrical object by this blow-up algebra. This moment, the simple spectrum does not work since the blow-up should be 'projective'. The process can be described as Proj construction. First we give two examples to illustrate the construction. This is exactly the analogy of the construction of projective spaces.

Here we start with two examples, that how we construct the projective spaces by the coordinate ring, then the generalization.

Consider affine lines

$$\begin{aligned} U_0 &= (\text{Spec } \mathbb{C}[x], \mathcal{O}_{\text{Spec } \mathbb{C}[x]}) \\ U_1 &= (\text{Spec } \mathbb{C}[y], \mathcal{O}_{\text{Spec } \mathbb{C}[y]}), \end{aligned}$$

one can define an affine scheme structure on an open set X_x of $X = \text{Spec } \mathbb{C}[x]$ as follows:

$$U_{01} = (\text{Spec } \mathbb{C}[x, \frac{1}{x}], \mathcal{O}_{\text{Spec } \mathbb{C}[x, \frac{1}{x}]}).$$

The points of Spec $\mathbb{C}[x, \frac{1}{x}]$ are the maximal ideals $(x - c)$ where $c \neq 0$ together with (0). It is obvious that $\mathcal{O}_{\text{Spec } \mathbb{C}[x, \frac{1}{x}]} = \mathcal{O}_X|_{X_x}$. Similarly we have another affine scheme structure as

$$U_{10} = (\text{Spec } \mathbb{C}[y, \frac{1}{y}], \mathcal{O}_{\text{Spec } \mathbb{C}[y, \frac{1}{y}]}).$$

The isomorphism

$$\begin{aligned}\varphi \quad : \quad & \mathbb{C}[y, \frac{1}{y}] \to \mathbb{C}[x, \frac{1}{x}] \\ & f(y, \frac{1}{y}) \mapsto f(x, \frac{1}{x})\end{aligned}$$

induces an isomorphism of affine schemes $(\varphi^*, \varphi^{\#}) : U_{01} \to U_{10}$. Through this isomorphism, U_0 and U_1 can be glued, yielding the scheme

$$\mathbb{P}^1 = (Z, \mathcal{O}_Z)$$

where

$$Z = X \cup_{\varphi^*} Y$$

and $\mathcal{O}_Z|_X = \mathcal{O}_X$ and $\mathcal{O}_Z|_Y = \mathcal{O}_Y$. Hence $\mathcal{O}_Z$ is obtained by identifying $\mathcal{O}_X|_{X_x}$ and $\mathcal{O}_Y|_{Y_y}$ through $\varphi^{\#}$.

It is obviously similar to the construction of projective spaces.

The next example show some generalization of the previous one.

Suppose

$$S = \bigoplus_{n=0}^{\infty} S_n$$

is a graded ring, and I is a homogeneous ideal of S. Let

$$S_+ := \bigoplus_{n=1}^{\infty} S_n,$$

define

$$X = \mathrm{Proj}\, S := \{\mathfrak{p} \mid \mathfrak{p} \text{ is a homogeneous prime ideal of } S,\ S \nsubseteq \mathfrak{p}\}$$

called the *homogeneous prime spectrum* of the graded ring S. For a homogeneous ideal I, put

$$V(I) := \{\mathfrak{p} \in \mathrm{Proj}\, S \mid I \subseteq \mathfrak{p}\},$$

then the Zariski topology can be define on Proj S by taking $\{V(I)\}$ as closed sets. For any homogeneous element f, put

$$D_+(f) := \{\mathfrak{p} \in \mathrm{Proj}\, S \mid f \in \mathfrak{p}\},$$

then $D_+(f)$ is an open set in Proj S, and $\{D_+(f)\}$ form a base of the topology. Indeed, if I is a homogeneous ideal, $V(I)^c = \bigcup_{f \in I, f \text{ is homogeneous}} D_+(f)$.

For the structure sheaf $\mathcal{O}_X$, it suffices to construct $\mathcal{O}_X(D_+(f))$. For $f \in S_n$, define

$$\Gamma(D_+(f), \mathcal{O}_X) := \{\frac{g}{f^m} \mid g \in S_{mn}\} = S_f^{(0)},$$

which consists of all the elements of degree 0 in S_f.

Proposition 3.1. *Suppose X is a topological space and $\mathscr{B}$ is a base of open sets. If a collection of groups $\mathscr{F}(U)$ and maps res_V^U for open sets $V \subset U$ of X form a $\mathscr{B}$-**sheaf** if they satisfy the sheaf axiom with respect to the inclusions of basic open sets. Then*

1. *Every $\mathscr{B}$-sheaf on X extends to a unique sheaf on X;*
2. *Given sheaves $\mathscr{F}$ and $\mathscr{G}$ and a collection of maps*

$$\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$$

commuting with restrictions, there is a unique sheaf morphism $\widetilde{\varphi}$ s.t. $\widetilde{\varphi}(U) = \varphi(U)$ for all $U \in \mathscr{B}$.

That is we have rings $\mathcal{O}_X(U)$ consisting of all the maps

$$f : U \to \bigcup_{p \in U} S_{(p)}$$

(where $S_{(p)}$ denotes the subring of the ring S_p of degree 0) such that for each prime ideal $p \in U$:

1. $f(p) \in S_{(p)}$;
2. There exist an open subset of U containing p and homogeneous elements $a, b \in S$ of the same degree for all the point $q \in V$, $t \notin q$ and $f(q) = \frac{s}{t}$.

One can actually see that it is the inductive limit. Hence here we get a local ringed space $(X, \mathcal{O}_X)$. From the construction, we see an isomorphism

$$\begin{aligned}(\varphi^*, \varphi^\#) : (D_+(f), \mathcal{O}_X|_{D_+(f)}) &\to (\mathrm{Spec}\ S_f^{(0)}, \mathcal{O}_{\mathrm{Spec}\ S_f^{(0)}}) \\ \varphi^* \ : \ \mathfrak{p} &\mapsto (\mathfrak{p} \cap S^{(0)})_f\end{aligned}$$

Consequently, $(X, \mathcal{O}_X)$ is a scheme and it is called a *projective scheme* determined by the graded ring S.

This construction can be easily generalized to a Noetherian scheme with a quasi-coherent graded $\mathcal{O}_X$-algebras.

3.3 Example

We want to see the process of the construction. Let $R = k[x, y]$ where k is a algebraically closed field. Then R can be naturally graded and $R_+ = (x, y)$. Hence all of the points in $X = \mathrm{Proj}\ R$

can be categorized into two (joint) parts: $D_+(x) = \{\mathfrak{p} \in \text{Proj } R \mid x \in \mathfrak{p}\}$ and $D_+(y) = \{\mathfrak{p} \in \text{Proj } R \mid y \in \mathfrak{p}\}$. We can construct ringed space morphism as

$$\begin{array}{rcl} (\varphi^*, \varphi^\#) : (D_+(x), \mathcal{O}_X|_{D_+(x)}) & \to & (\text{Spec } k[y], \mathcal{O}_{\text{Spec } k[y]}) \\ \varphi^* : \mathfrak{p} & \mapsto & \varphi^*(\mathfrak{p}) = \{f(1,y) \mid f(x,y) \in \varphi^*\mathfrak{p}\} \\ \varphi^\# : \dfrac{g(x,y)}{x^m} & \mapsto & g(1,y), \end{array}$$

therefore we identify $D_+(x)$ with Spec $k[y]$. Similarly $D_+(y) \cong$ Spec $k[x]$. On the other hand, $D_+(xy)$ can be identified with Spec $k[x, \frac{1}{x}]$ and Spec $k[y, \frac{1}{y}]$, hence $D_+(x)$ and $D_+(y)$ are glued along $D_+(xy)$, and that is exactly the example at the beginning of Section 3.2.

4 Comparison: Algebraic Variety and Scheme

In order to demonstrate the two definition are equivalent, we would give a universal property of blow-up, and show that these two construction both satisfying the property.

Let X be a scheme, and let $\mathcal{I}$ be a coherent sheaf of ideals on X. We say the blow-up of X with respect to $\mathcal{I}$ is a scheme $\widetilde{X}$ along with a morphism $\pi : \widetilde{X} \to X$, such that $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\widetilde{X}}$ is a invertible sheaf, with the universal property: for any scheme and morphism $f : Z \to X$ such that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$ is a invertible sheaf, there is a unique factorization:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\pi"] & X \\ & Z \arrow[ul, "\widetilde{f}"] \arrow[u, "f"'] \end{tikzcd}$$

Here we need to explain some terminologies.

Suppose X is a scheme and $\mathcal{O}_X$ is its structure sheaf. A sheaf $\mathscr{F}$ is said to be an $\mathcal{O}_X$-*module* if the following condition is satisfied: for each open set $U \subset X$, $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$-module s.t. for any open set $V \subset U$ the diagram

$$\begin{tikzcd} \mathcal{O}_X(U) \times \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[d] \\ \mathcal{O}_X(V) \times \mathscr{F}(V) \arrow[r] & \mathscr{F}(V) \end{tikzcd}$$

commutes, where the horizontal maps indicate $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$ structures on $\mathscr{F}(U)$ and $\mathscr{F}(V)$ respectively. Note that the stalk $\mathscr{F}_x$ at point x of $\mathscr{F}$ is actually an $\mathcal{O}_{X,x}$-module.

For $\mathcal{O}_X$-module $\mathscr{F}, \mathscr{G}$, let $\varphi : \mathscr{F} \to \mathscr{G}$ be a homomorphism of sheaves of additive groups. When φ is compatible with the $\mathcal{O}_X$-module structure of $\mathscr{F}$ and $\mathscr{G}$, namely, for any open set $U \subset X$, the diagram

$$\begin{array}{ccc} \mathscr{O}_X(U) \times \mathscr{F}(U) & \longrightarrow & \mathscr{O}_X(U) \times \mathscr{G}(U) \\ \downarrow & & \downarrow \\ \mathscr{F}(U) & \longrightarrow & \mathscr{G}(U) \end{array}$$

commutes, then φ is said to be a *homomorphism of $\mathcal{O}_X$-module*. It is clear that $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is an $\mathcal{O}_{X,x}$-module homomorphism. We denote the totality of the $\mathcal{O}_X$-module homomorphism from $\mathscr{F}$ to $\mathscr{G}$ by $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$. It is a $\Gamma(X, \mathcal{O}_X)$-module. Besides, we have the following isomorphism for any open set $U \subset X$:

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X|_U, \mathscr{F}|_U) \cong \mathscr{F}(U).$$

Lemma. *Let $\mathscr{F}$ be a presheaf of additive groups over a scheme $(X, \mathcal{O}_X)$. A presheaf of $\mathcal{O}_X$-module is just a presheaf satisfying the axioms for sheaf of $\mathcal{O}_X$-modules. Then the shefification $\widetilde{\mathscr{F}}$ is a sheaf of $\mathcal{O}_X$-modules.*

We often write $\mathcal{O}_X^{\oplus n}$ for $\mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X$, or simply $\mathcal{O}_X^n$. If an $\mathcal{O}_X$-module $\mathscr{F}$ is isomorphic to $\mathcal{O}_X^n$ as $\mathcal{O}_X$-module, then $\mathscr{F}$ is said to be a *free module of rank n*. An $\mathcal{O}_X$-module $\mathscr{F}$ is called a *locally free module of rank n* if there is an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of X s.t. the restriction $\mathscr{F}|_{U_\lambda}$ is a free module of rank n. A locally free sheaf of rank 1 is called *invertible sheaf*.

Let $\mathscr{F}$ and $\mathscr{G}$ are $\mathcal{O}_X$-modules, then for any open set U, the presheaf $\mathrm{Hom}_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$. It is denoted as $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$. For invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ over a scheme X, $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ is also an invertible sheaf. Put $\mathcal{L}^{-1} := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$, then one have a natural $\mathcal{O}_X$-homomorphism

$$\begin{array}{rcl} \varphi : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} & \to & \mathcal{O}_X \\ a \otimes f & \mapsto & f(a) \end{array}$$

For an affine open set U satisfying $\mathcal{L}|_U \cong \mathcal{O}_X$, we get

$$\mathcal{L}^{-1}|_U \cong \underline{\mathrm{Hom}}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{O}_U) \cong \mathcal{O}_U.$$

Therefore, over U, φ is an $\mathcal{O}_U$-isomorphism. Namely, we have $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \cong \mathcal{O}_X$. Hence, under the tensor product, isomorphic classes of invertible sheaves form a group, called the *Picard group*, denoted as $\mathrm{Pic}\, X$.

Definition. Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. If for each point x in X there is an open neighborhood U of x so that the sequence of $\mathcal{O}_X$-module

$$\mathcal{O}_U^{\oplus I} \to \mathcal{O}_U^{\oplus J} \to \mathscr{F}|_U \to 0$$

is exact, then $\mathscr{F}$ is said to be *quasicoherent*. The index set I, J need not be finite. If for each $x \in X$, there is an open set U of x s.t. the following sequence of $\mathcal{O}_X$-module is exact

$$\mathcal{O}_U^{\oplus n} \to \mathscr{F}|_U \to 0$$

then $\mathscr{F}$ is said to be a *finitely generated* $\mathcal{O}_X$-module.

Suppose M is an R-module, we would like to construct a sheaf on the affine scheme $(\mathrm{Spec}\ R, \mathcal{O}_{\mathrm{Spec}\ R})$ associated to M. Let M_f be the localization of M with respect to $\{1, f, f^2, \cdots\}$ where $f \in R$. It is easy to verify that the family is a $\mathscr{B}$-sheaf where $\mathscr{B} := \{D(f)\}_{f \in R}$. Hence by Proposition 3.1, the family of R-modules forms a sheaf of $\mathcal{O}_X$-modules. It is called the *associated sheaf* with an R-module M and it is denoted as $\widetilde{M}$.

Proposition 4.1. *Suppose $(X, \mathcal{O}_X)$ is the affine scheme determined by a commutative ring R.*

1. *The $\mathcal{O}_X$-module $\widetilde{M}$ determined by an R-module is quasicoherent, and*

$$\Gamma(X, \widetilde{M}) = M.$$

2. *For any R-module homomorphism $\varphi : M \to N$, the map*

$$\Phi : \mathrm{Hom}_R(M, N) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$$

assigning an $\mathcal{O}_X$-module homomorphism $\widetilde{\varphi}$ is an isomorphism of R-modules.

3. *For R-modules M and N, we have isomorphisms of $\mathcal{O}_X$-modules $\widetilde{M} \oplus \widetilde{N} \cong \widetilde{(M \oplus N)}$, $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{(M \otimes_R N)}$. Furthermore, if M is finitely presented, we have*

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \cong \widetilde{\mathrm{Hom}_R(M}, N)$$

Theorem 4.2. *An $\mathcal{O}_X$-module $\mathscr{F}$ over an affine scheme $X = \mathrm{Spec}\ R$ is quasicoherent if and only if $\mathscr{F}$ is isomorphic to the associated sheaf $\widetilde{M}$ and $\Gamma(X, \mathscr{F})$ is isomorphic to M as R-module.*

We have seen that quasicoherent sheaves are analogous to R-modules. On the other hand, we want to see the analogy of finitely generated R-modules, which is a coherent sheaf.

Definition. An $\mathcal{O}_X$-module $\mathscr{F}$ is said to be *coherent* if the following conditions are satisfied:
(i) $\mathscr{F}$ is a finitely generated $\mathcal{O}_X$-module;
For an arbitrary open set U of X and for an arbitrary $\mathcal{O}_U$-module homomorphism

$$\varphi : \mathcal{O}_U^{\oplus n} \to \mathscr{F}|_U$$

$\mathrm{Ker}\ \varphi$ is a finitely generated $\mathcal{O}_X$-module.

It is easy to prove that a coherent sheaf is quasicoherent. When R is a Noethrian ring, we have the following fact:

Theorem 4.3. *A quasicoherent $\mathcal{O}_X$-module $\mathscr{F}$ over an affine scheme $X = \mathrm{Spec}\ R$ determined by a Noetherian ring is coherent if and only if $\Gamma(X, \mathscr{F})$ is a finitely generated R-module. Conversely, a finitely generated $\mathcal{O}_X$-module is coherent.*

Furthermore, we have

Theorem 4.4. *Let*

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

be an exact sequence of $\mathcal{O}_X$-modules over a scheme X. If any of two sheaves among $\mathscr{F}$,$\mathscr{G}$ and $\mathscr{H}$ are coherent, then the third is coherent.

All of the results we have discussed are schemes instead of what we want to prove, i.e. varieties. However, there exists an 1-1 correspondence between varieties and schemes, and hence we can omit the differences.

Theorem 4.5. *There exists a fully faithful functor functor F from $(\mathrm{Var})/k$ the category of varieties over field k, to $(\mathrm{Sch})/k$ the category of schemes over k.*

We say that $\mathrm{Proj}\ \mathrm{Bl}_I(X)$ is what we want. Before actually proving it, we first give an example to illustrate it. Let $R = k[x, y]$ where k is a algebraically closed field, and let $\mathfrak{m} = (x, y)$, then the blow-up algebra $\mathrm{Bl}_{\mathfrak{m}}(R) = R \oplus \mathfrak{m} \oplus \mathfrak{m}^2 \oplus \cdots$ is a graded algebra over R. First, all of the maximal ideals $(x - a, y - b)$ of R belong to $\mathrm{Proj}\ \mathrm{Bl}_{\mathfrak{m}}(R)$, where $(a, b) \neq (0, 0)$.That is $\mathbb{A}^2 - 0$. On the other hand, the rest of the points in $\mathrm{Bl}_{\mathfrak{m}}(R)$ correspond to the homogeneous prime ideals in $k[x, y]$, and from the discussion above that is $\mathbb{P}^1$. We see this is the blow-up of plane at the origin.

Theorem 4.6. *Let X be a Noetherian scheme, $\mathcal{I}$ a coherent sheaf of ideals and $\pi : \mathrm{Proj} \oplus_{n \geq 0} \mathcal{I}^n \to X$ the projective scheme of the blow-up algebra. Then it satisfies the universal property demonstrated this section.*

Proof We say that the question is local on X thus we may assume that $X = \mathrm{Spec}\ R$ is affine for some Noetherian commutative ring R, and that $\mathcal{I}$ corresponds to an ideal $I \subseteq R$. Then $\widetilde{X} = \mathrm{Proj}\ \mathrm{Bl}_I(R)$. Let $a_0, \cdots, a_n$ be the generators for the ideal I. Then we can define a surjective map of graded rings $\varphi : R[x_0, \cdots, x_n] \to S = \mathrm{Bl}_I(R)$ by sending x_i to $a_i \in I$, considered as an element of degree 1 in S. The homomorphism gives us a closed immersion $\widetilde{X} \hookrightarrow \mathbb{P}^n$. The kernel is the homogeneous ideal in $R[x_0, \cdots, x_n]$ generated by all homogeneous polynomials $F(x_0, \cdots, x_n)$ s.t. $F(a_0, \cdots, a_n) = 0$ in R.

Now let $f : Z \to X$ be a morphism such that the inverse image ideal sheaf $f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$ is an invertible sheaf $\mathcal{L}$ on Z. Since I is generated by the elements $a_0, \cdots, a_n$, the inverse images considered as the global sections $s_0, \cdots, s_n$ generating $\mathcal{L}$. By the properties of projective morphisms, there is a unique morphism $g : Z \to \mathbb{P}^n$ with $\mathcal{L} \cong g^*\mathcal{O}(1)$ and $s_i = g^{-1}(x_i)$. This g factors through the closed subscheme $\widetilde{X}$ of $\mathbb{P}^n$ since if F is a homogeneous element of degree d of $\mathrm{Ker}\ \varphi$, then $F(a_0, \cdots, a_n)$ in R and hence $F(s_0, \cdots, s_n) = 0$ in $\Gamma(Z, \mathcal{L}^d)$.

Thus we have constructed a morphism $g : Z \to \widetilde{X}$ factoring f. For any such morphism, we have $f^{-1}\mathcal{I} \cdot \mathcal{O}_Z = g^{-1}(\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\widetilde{X}}) \cdot \mathcal{O}_Z$ which is just $g^{-1}(\mathcal{O}_{\widetilde{X}}(1)) \cdot \mathcal{O}_Z$. Hence there is a surjective map $g^*\mathcal{O}_{\widetilde{X}}(1) \to f^{-1}\mathcal{I} \cdot \mathcal{O}_Z = \mathcal{L}$. But a surjective map of invertible sheaves on a locally ringed space is an isomorphism so $g^*\mathcal{O}_{\widetilde{X}}(1) \cong \mathcal{L}$. Clearly the sections s_i of $\mathcal{L}$ must be the pull-backs of the sections x_i of $\mathcal{O}(1)$ on $\mathbb{P}^n$. Hence the uniqueness follows from the uniqueness of projective morphisms.[Ha] ■

If X is a variety, we can also consider the affine local coordinates. Thus without loss of generality, we assume X is an affine variety. The proof of the universal property is really similar to Theorem 4.6. A invertible sheaf on X is actually a Weil divisor, and so is the pull-back. We get the factorization by the embedding of $\widetilde{X}$ to a projective space and the uniqueness can also be derived from the properties of projective morphism of varieties.

5 Geometric Properties of Blow-ups

5.1 Local Properties

In this section we would discuss the geometry near E on $\widetilde{\mathbb{C}^n}$ in detail. Let $(z_1, \cdots, z_n)$ be a local coordinate of U with center x. Hence

$$\widetilde{U} := \{(z, l) \in U \times \mathbb{P}^{n-1} \mid z_i l_j = z_j l_i\}$$

is an open set of $\widetilde{\mathbb{C}^n}$ and

$$\widetilde{U}_i := \{l_i \neq 0\} \subset \widetilde{U}.$$

In this way we obtain an open cover of the neighborhood $\widetilde{U}$ of E. And it can be proved that $\mathcal{O}(-1) = \widetilde{\mathbb{C}^n}$ is a vector bundle over $\mathbb{P}^{n-1}$. Let $p : \mathcal{O}(-1) \to \mathbb{P}^{n-1}$ be the canonical projection and let $\mathbb{P}^{n-1} = \bigcup_{i=1}^{n-1} U_i$ be the standard open covering. A canonical trivialization of $\mathcal{O}(-1)$ over U_i is given by

$$\begin{array}{rcl} \varphi : p^{-1}(U_i) & \cong & \mathbb{C} \times U_i \\ (z, l) & \mapsto & (z_i, l) \end{array}$$

and hence the transition maps $\varphi_{ij} : \mathbb{C} \to \mathbb{C}$ are $w \mapsto \frac{z_i}{z_j} w$. Therefore the blow-up at a point is naturally a manifold and by the open covering, it is of dimension n. Using the same technique, the blow-up of $\mathbb{C}^n$ along a linear subspace is also a manifold.

In the first section, we have seen that E is naturally identified with $\mathbb{P}(\mathbb{C}^n)$, or precisely, $\mathbb{P}(T_0(\mathbb{C}^n))$ where the T is the holomorphic tangent space. So that the global sections of $[-E]$ over E correspond exactly to the linear functionals on the tangent space, i.e.

$$H^0(E, \mathcal{O}_E(-E)) \cong T_0(\mathbb{C}^n)^*.$$

On the other hand, given a function f on U vanishing at the origin, the function $\pi^* f$ vanishes along E so can be considered as a section of $[-E]$ over U. Let $\mathcal{O}(U)_x$ be the sheaf of holomorphic functions on U vanishing at the Point x, hence we get the following diagram:

$$\begin{tikzcd} H^0(\widetilde{U}, \mathcal{O}_X(-E)) \arrow[r] & H^0(E, \mathcal{O}_E(-E)) \arrow[d, equal] \\ H^0(U, \mathcal{O}_X(U)_0) \arrow[u] \arrow[r, "d"] & T_0(\mathbb{C}^n)^* \end{tikzcd}$$

Using the blow-up at a point, we can actually prove the following important theorem:

Theorem 5.1 (Kodaira Embedding Theorem)**.** *A compact complex manifold M can be embedded in a projective if and only if it has a closed, positive (1,1)-form ω whose cohomology class $[\omega]$ is rational.*

The proof can be summarized as below: $[\omega] \in H^2(M, \mathbb{Q})$ if and only if $[k\omega] \in H^2(M, \mathbb{Z})$ for some natural number k, hence in the exact sequence

$$H^1(M, O^*) \to H^2(M, \mathbb{Z}) \xrightarrow{\iota_*} H^2(M, \mathcal{O})$$

$\iota_*([k\omega]) = 0$, then there is a holomorphic line bundle $L \to M$ with $c_1(L) = [k\omega]$, and it is positive.

5.2 Global Properties

Using the same notation as the previous subsection, we put $\mathbb{C}^{n*} = \mathbb{C}^n - 0$, $\widetilde{\mathbb{C}^n}^* = \pi^{-1}\mathbb{C}^{n*} = \widetilde{\mathbb{C}^n} - E$, $U^* = U - 0$ and $\widetilde{U}^* = \pi^{-1}U^* = \widetilde{U} - E$. Compare the Mayer-Victoris sequences of $\mathbb{C}^n = \mathbb{C}^{n*} \cup U$ and $\widetilde{\mathbb{C}^n} = \widetilde{\mathbb{C}^n}^* \cup \widetilde{U}$, we have

$$\begin{tikzcd}
H_i(\widetilde{U}^*) \arrow[r] \arrow[d, "\pi_*"] & H_i(\widetilde{U}) \oplus H_i(\widetilde{\mathbb{C}^n}^*) \arrow[r] \arrow[d, "\pi_*"] & H_i(\widetilde{\mathbb{C}^n}) \arrow[r] \arrow[d, "\pi_*"] & H_{i+1}(\widetilde{U}^*) \arrow[d, "\pi_*"] \\
H_i(U^*) \arrow[r] & H_i(U) \oplus H_i(\mathbb{C}^{n*}) \arrow[r] & H_i(\mathbb{C}^n) \arrow[r] & H_{i+1}(U^*).
\end{tikzcd}$$

Now π_* is an isomorphism between $H_*(\widetilde{U}^*)$ and $H_*(U^*)$, and between $H_*(\widetilde{\mathbb{C}^n}^*)$ and $H_*(\mathbb{C}^{n*})$. On the other hand, we may choose our open set U a ball around the origin, and the contraction $z \mapsto tz$ of U onto 0 induces a contraction $\widetilde{U}$ onto E via π. Thus

$$H_i(\widetilde{\mathbb{C}^n}) = H_i(\mathbb{C}^n) \oplus H_i(E), \quad i > 0.$$

We now suppose S is a complex surface and $\widetilde{S}$ is its blow-up at point $p \in S$. If C is any curve on $\widetilde{S}$ not containing the exceptional bundle E, then C is just the closure of image of $\pi^{-1}(C) - p$ via π, thus

$$\mathrm{Div}(\widetilde{S}) = \pi^*\mathrm{Div}(S) \oplus \mathbb{Z}\{E\}.$$

Therefore the intersection number

$$(E \cdot E) = \deg([E]|_E) = \deg(N_E) = -1.$$

Since π has degree 1, i.e. the image under π_* of the fundamental class $[\widetilde{S}] \in H_4(\widetilde{S}, \mathbb{Z})$ of $\widetilde{S}$ is just the fundamental class of M, for any divisors D_1, D_2 on M,

$$\pi^*(D_1) \cdot \pi^*(D_2) = D_1 \cdot D_2.$$

And since the class of the exceptional bundle of the blow-up is in the kernel of π_*,

$$\pi^*(D) \cdot E = \#(D \cdot \pi_*(E)) = 0$$

for any divisor D on S. In particular, if D_1, D_2 are two divisors on S intersecting transversely at p, and if the closure of their image except p under π are $\widetilde{D}_1, \widetilde{D}_2$, then we have

$$\begin{aligned} \widetilde{D}_1 \cdot \widetilde{D}_2 &= (\pi^*(D_1) - E) \cdot (\pi^*(D_2) - E) \\ &= \pi^*(D_1) \cdot \pi^*(D_2) + E \cdot E \\ &= D_1 \cdot D_2 - 1. \end{aligned}$$

That is, as long as D_1 and D_2 have distinct tangent lines at p, $\widetilde{D}_1$ and $\widetilde{D}_2$ will not meet with each other at any point of E. This is what we would like to discuss in the next subsection.

We have seen that the blow-up of a surface is closely related to the surface itself. If we ask the question conversely, that given a surface S and a curve C on it, when can we realize them as a blow-up $\tilde{T}$ of some surface T? Clearly there are supposed to be conditions C is rational and $C \cdot C = -1$, the fact is that it is also sufficient.

Theorem 5.2 (Castelnuovo-Enriques Criterion)**.** *Let S be a algebraic surface and $C \subset S$ a smooth rational curve on S of self-intersection -1. Then exists a smooth algebraic surface T and a map $\pi : S \to T$ such that $S \xrightarrow{\pi} T$ is the blow-up of T at p and $\pi^{-1}(p) = C$.*

5.3 Desingularity

In 1964, Heisuke Hironaka proved a theorem saying that every variety can be desingularized, or equivalently, birational to a smooth quasi-projective variety. And the proof is based on the construction of blow-ups.

Definition. A morphism of varieties $\pi : X \to V$ is called a *projective morphism* if X is a closed subvariety of a product variety

$$X \subset V \times \mathbb{P}^n$$

and $\pi : X \to V$ is the restriction of the projection onto the first coordinate.

Theorem 5.3 (Desingularization Theorem)**.** *Let V be a quasi-projective variety. Then there exists a smooth quasi-projective variety X and a projective birational morphism $\pi : X \to V$. Furthermore, π may be assumed to be an isomorphism on the smooth locus of V, and if V is projective variety, then so is X.*

What's more, all of the birational projective morphism can be get from blowing up. This is

Theorem 5.4. *Let V be a quasi-projective variety over k. If X is another variety and $f : X \to V$ is any birational projective morphism, then there exists a coherent sheaf of ideals $\mathcal{I}$ on V such that X is isomorphic to the blow-up $\widetilde{V}$ of V with respect to $\mathcal{I}$, and f corresponds to $\pi : \widetilde{V} \to V$ under this isomorphism.*

We would conclude the thesis by some examples of applications of blow-ups, showing how the construction works in disingularization.

Consider the curve $C \subset \mathbb{C}^2$ given by $z_1 \cdot z_2 = 0$. Obviously the origin is not smooth. On the other hand, the closure of C in $\widetilde{\mathbb{C}^2}$ is the union of two separated lines $\overline{\pi^{-1}(l_1 - O)}$ and $\overline{\pi^{-1}(l_2 - O)}$, where $l_i = \{(z_1, z_2)|z_i = 0\},\ i = 1, 2$. Hence the closure is smooth.

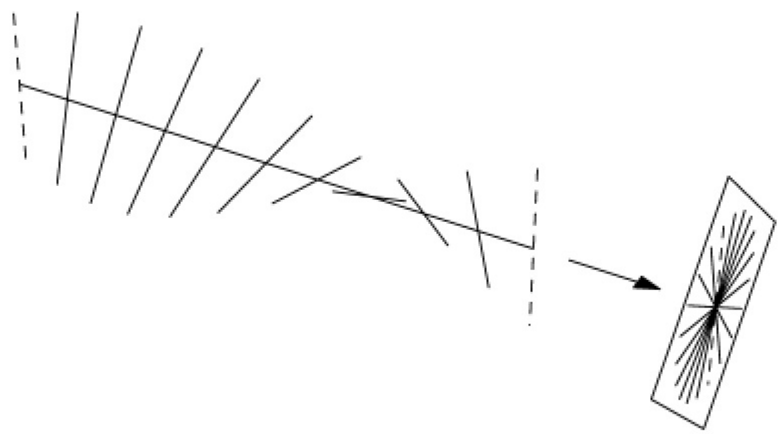

Figure 3: Desingularization of All the Lines through the Origin

Actually, from this point of view, we can say that the blow-up of $\mathbb{C}^n$ at the origin is separating all the lines through the origin. So, there is supposed to be a projective line replacing the origin, corresponding each of a point of the exceptional bundle to a line. And this is what the figure above illustrating.

The next example is more interesting. It says by blowing up the singularity of a conic, we do get a ruled surface. Consider

$$V = Z(z - x^2 - y^2)$$

is a affine algebraic set of $\mathbb{C}^3$, and it consists of all the lines $l = (\sin\theta \cdot t, \cos\theta \cdot t, t), \theta \in [0, 2\pi]$. Let $\widetilde{V}$ be its blow-up at the origin, we have seen that the lines with different direction can be separated. Since the lines described above satisfying this condition, $\widetilde{V}$ are the union of $\tilde{l}$. And we can analogize it as an elliptic hyperboloid with one sheet.

6 Conclusion

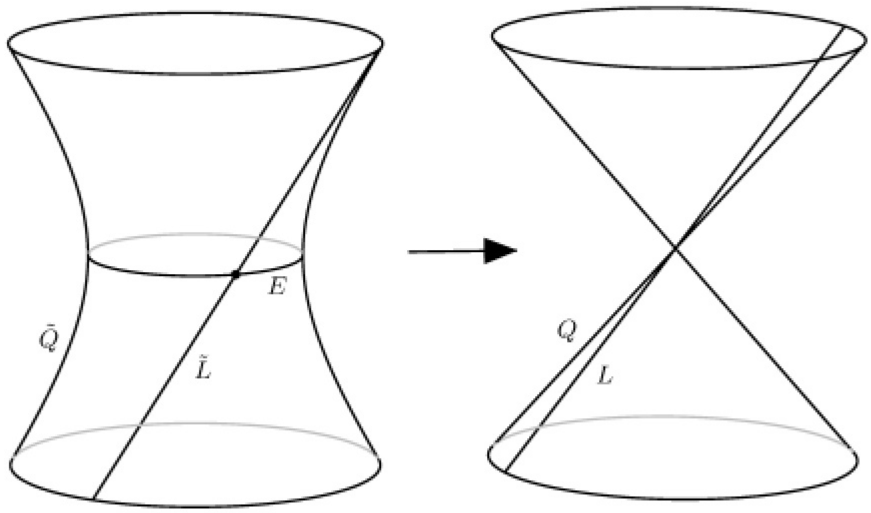

Figure 4: Desingularization of a Cone the Origin

Acknowledgement

References

[EH] D. Eisenbud, J. Harris, *The Geometry of Schemes*, Grad. Texts in Math. 197, Springer-Verlag, New York, 2000.

[GH] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[Ha] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. 52, Springer-Verlag, New York-Heidelberg, 1977.

[Hi] R. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math, 70 1964, 109-203; 79 1964, 205-326

[Huy] D. Huybrechts, *Complex Geometry: An Introduction*, Universitext, Springer-Verlag, Berlin-Heidelberg, 2005.

[Ken] U. Kenji, *Algebraic Geometry 1,2*, Mathematical Monographs 185,197 - Iwanami Series in Modern Mathematics, Amer. Math. Soc., Providence, RI, 1999.

[SKKT] K. E. Smith, L. Kahanpää, P. Kekäläinen and W. Traves *An Invitation to Algebraic Geometry*, Universitext, Springer-Verlag, New York, 2000.

[Va] R. Vakil, *The Rising Sea: Foundations of Algebraic Geometry*, preprint 2017.