# Higher Hochschild homology and representation homology

Guanyu Li

Yuri Berest, chair
Daniel Halpern-Leistner, minor member
Michael Stillman, minor member

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# Outline

# Classical Hochschild homology

## Definition

Given a $k$-algebra, define

$$C_n(A) := A^{\otimes n+1},$$

where $A^{\otimes n+1} := A \otimes_k \cdots \otimes_k A$ with the boundary maps

$$\begin{aligned} \partial_n : C_n(A) &\to C_{n-1}(A) \\ a_0 \otimes a_1 \otimes \cdots \otimes a_n &\mapsto \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ &\quad + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, \end{aligned}$$

then $(C_\bullet(A), \partial_\bullet)$ is called the Hochschild complex, whose homology group is called the Hochschild homology group of $A$, denoted by $HH_\bullet(A)$.

# Construction of higher Hochschild homology

Let $\mathbf{FinSet}$ be the category of finite sets $[n] := \{0, 1, \cdots, n\}$. Let $A$ be a commutative $k$-algebra with unit. Following Loday, we define a functor $\mathcal{L}(A) : \mathbf{FinSet} \to k - \mathbf{Mod}$ by

$$[n] \mapsto A^{\otimes n+1}.$$

For a pointed map $f : [n] \to [m]$, the action of $f_*$ on $\mathcal{L}(A)$ is

$$f_*(a_0 \otimes \cdots \otimes a_n) := b_0 \otimes \cdots \otimes b_m \tag{1}$$

where

$$b_j := \prod_{f(i)=j} a_i$$

for $j = 0, \cdots, m$.

Furthermore one has the canonical embedding $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$, so one can prolong the functor $\mathcal{L}(A)$ via the Kan extension

$$\begin{tikzcd}
\mathbf{FinSet} \arrow[r, "\mathcal{L}(A)"] \arrow[d, hook] & k - \mathbf{Vect} \\
\mathbf{Set}, \arrow[ur, dashed] &
\end{tikzcd}$$

more precisely,

$$\widetilde{\mathcal{L}(A)}(X) := \operatorname{colim} \mathcal{L}(A)([n])$$

where the colimit is taken over all pointed sets inclusions $[n] \hookrightarrow X$.

Definition

In general, for any simplicial set $X : \mathbf{\Delta}^{\circ} \to \mathbf{Set}$, one can define a simplicial $k$-vector space extending $\mathcal{L}(A)$ level-wisely

$$\mathbf{\Delta}^{\circ} \xrightarrow{X} \mathbf{Set} \xrightarrow{\widetilde{\mathcal{L}(A)}} s(k - \mathbf{Vect}).$$

Then one can define *$X$-homology of $A$* by

$$HH_*(X, A) := \pi_*(\mathcal{L}(A))(X).$$

# Example

## Proposition

For the simplicial set $S^1$, $HH_*(S^1, A)$ is exactly the Hochschild homology.

## Proof

Let's take the simplicial model $S^1$ to be $\mathbf{\Delta}[1]/d^0(\mathbf{\Delta}[0]) \cup d^1(\mathbf{\Delta}[0])$. Then

$$(S^1)_k = \{(0, \cdots, 0, 1, \cdots, 1)\}/(0, \cdots, 0) \sim (1, \cdots, 1)$$

with face maps $d_i^{[k]} : (S^1)_k \to (S^1)_{k-1}$ given by

$$(c_0, \cdots, c_k) \mapsto (c_0, \cdots, \hat{c}_i, \cdots, c_k).$$

Apply the functor $\mathcal{L}(A)$, we find exactly $\mathcal{L}(A)(d_i)$ gives

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

and the last term is guaranteed by the quotient.

### Remark

The homology depends only on the homotopy type of $X$.

### Example*

We take the standard simplicial model for $S^n = \mathbf{\Delta}[n]/d^0(\mathbf{\Delta}[n-1]) \cup \cdots \cup d^n(\mathbf{\Delta}[n-1])$, where in dimension $0 < i < n$, there is no non-degenerate simplices, so

$$HH_0(S^n, A) \cong A$$

and

$$HH_i(S^n, A) = 0$$

for all $0 < i < n$.

## Some topological background

There is a pair of adjunction

$$\mathbb{G} : s\mathbf{Set}_0 \leftrightarrows s\mathbf{Gr} : \overline{W}$$

where $\mathbb{G}$ is called the Kan loop group construction and $\overline{W}G$ is the classfying simplicial complex.
Actually the functor $\mathbb{G}$ preserves weak equivalences and cofibrations, and the functor $\overline{W}$ preserves weak equivalences and fibrations. Thus this is a pair of Quillen equivalence, which gives an equivalence of homotopy categories

$$\mathrm{Ho}\ s\mathbf{Set}_0 \simeq \mathrm{Ho}\ s\mathbf{Gr}.$$

We will need that the set of $n$-simplicies is

$$\mathbb{G}X_n := \langle X_{n+1} \rangle / \langle s_0(x) = 1, \forall x \in X_n \rangle \cong \langle B_n \rangle,$$

where $B_n := X_{n+1} - s_0(X_n)$ and the isomorphism is induced by the inclusion $B_n \hookrightarrow X_n$.

# Definition of representation homology

Let $\mathfrak{G}$ be the full subcategory of $\mathbf{Gr}$ whose objects are the (finitely generated) free groups $\langle n \rangle = \langle x_1, \cdots, x_n \rangle$ for $n \geq 0$. Then any commutative Hopf algebra $H$ gives a $\mathfrak{G}$-module

$$\begin{aligned} \mathfrak{G} &\to k - \mathbf{Vect} \\ \langle n \rangle &\mapsto H^{\otimes n}, \end{aligned}$$

which will be denoted by $\underline{H}$. Actually, the functor $\underline{H}$ takes values in the category of commutative algebras. Then consider the inclusion of categories $\mathfrak{G} \hookrightarrow \mathbf{FreeGr}$ where $\mathbf{FreeGr}$ is the full subcategory of all free groups, there is a Kan extension of $\underline{H}$ along the inclusion

$$\begin{tikzcd} \mathfrak{G} \arrow[r, "\underline{H}"] \arrow[d, hook, "i"] & k - \mathbf{Vect} \\ \mathbf{FreeGr} \arrow[ur, dashed, "\underline{H}"'] & \end{tikzcd}$$

also denoted by $\underline{H}$.

The composition of functors

$$\mathbf{\Delta}^{\circ} \xrightarrow{\mathbb{G}X} \mathbf{FreeGr} \xrightarrow{\underline{H}} \boldsymbol{k} - \mathbf{CommAlg}$$

defines a simplicial commutative algebra $\underline{H}(\mathbb{G}X)$ for any reduced simplicial set $X$.

Definition

The representation homology of $X$ in $H$ is defined by

$$\mathrm{HR}_*(X, H) := \pi_*(\underline{H}(\mathbb{G}X)).$$

# How are they related

**Theorem**

For any commutative Hopf algebra $H$ and any simplicial set $X$, there is a natural isomorphism of graded commutative algebras

$$HR_*(\Sigma(X_+), H) \cong HH_*(X, H).$$

# Another definition*

Given a (discrete) group $\Gamma$, the functor

$$\begin{aligned}\mathrm{Rep}_G(\Gamma) : k-\mathbf{CommAlg} &\to \mathbf{Set}\\ A &\mapsto \mathrm{Hom}_{\mathbf{Gr}}(\Gamma, G(A))\end{aligned}$$

is representable. The representative is denoted by $(\Gamma)_G$.
This gives a functor

$$(-)_G : \mathbf{Gr} \to k-\mathbf{CommAlg},$$

which is the left adjunction of $G : k-\mathbf{CommAlg} \to \mathbf{Gr}$.

# Another definition*

Extend the functor to be a functor

$$\boldsymbol{s}\mathbf{Gr} \to s(k - \mathbf{CommAlg}) \tag{2}$$

level-wisely, still denoted by $(-)_G$.

## Proposition

The functor $(-)_G$ maps weak equivalences between cofibrant objects in $\boldsymbol{s}\mathbf{Gr}$ to weak equivalences in $s(k - \mathbf{CommAlg})$, and hence has a total left derived functor.

For a fixed simplicial group $\Gamma \in \boldsymbol{s}\mathbf{Gr}$, one can formally define the representation homology of $\Gamma$ in $G$

$$HR_*(\Gamma, G) := \pi_*\mathbb{L}(\Gamma)_G,$$

where $\mathrm{DRep}_G(\Gamma) := \mathrm{Spec}\ \mathbb{L}(\Gamma)_G$ is called the representation scheme.

# Another definition*

## Definition

For a space $X \in \mathbf{Top}_{0,*}$, the *derived representation scheme* $\mathrm{DRep}_G(X)$ is $\mathrm{Spec}\ \mathrm{DRep}_G(\Gamma X)$, where $\Gamma X$ is a(ny) simplicial group model of $X$. The *representation homology of $X$ in $G$* is then

$$HR_*(X, G) := \pi_* \mathbb{L}(\Gamma X)_G. \tag{3}$$

## Proposition

Let $G$ be an affine group scheme over $k$ with coordinate ring $H = \mathscr{O}(G)$. Then for any $X \in \mathbf{Set}_0$, there is a natural isomorphism of graded commutative algebras

$$\mathrm{HR}_*(X, H) \cong \mathrm{HR}_*(X, G).$$

# Example*

Let $G = \mathbb{G}_a$ be the additive group. Then for any group $\Gamma \in \mathbf{Gr}$, one has

$$\mathrm{Hom}_{\mathbf{Gr}}(\Gamma, \mathbb{G}_a(A)) = \mathrm{Hom}_{k-\mathbf{CommAlg}}(\mathrm{Sym}(\Gamma_{\mathrm{ab}} \otimes_{\mathbb{Z}} k), A).$$

Also, $\mathbb{G}X$ is a canonical simplicial model for $|X|$, so

$$HR_*(X, G) \cong \pi_*(\mathbb{G}X_G).$$

Applying this we have

$$\begin{aligned} HR_*(X, \mathbb{G}_a) &\cong \pi_* \mathrm{Sym}((\mathbb{G}X)_{\mathrm{ab}} \otimes_{\mathbb{Z}} k) \\ &\cong \mathrm{Sym}(\pi_*(\mathbb{G}X)_{\mathrm{ab}} \otimes_{\mathbb{Z}} k) \\ &\cong \mathrm{Sym}(\pi_*(\mathbb{G}X)_{\mathrm{ab}} \cong H_{*+1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} k) \\ &\cong \mathrm{Sym}(\pi_*(\mathbb{G}X)_{\mathrm{ab}} \cong H_{*+1}(X, k)) \end{aligned}$$

where $\mathrm{Sym}$ is the graded symmetric product and $\pi_*(\mathbb{G}X)_{\mathrm{ab}} \cong H_{*+1}(X, \mathbb{Z})$.

# Example

Let's consider when $X = T^2$ be the 2-torus. Notice that $T^2 = \mathrm{hocolim}(\{*\} \leftarrow S^1_c \xrightarrow{\alpha} S^1_a \vee S^1_b)$, then by applying the Kan loop group construction we have a simplicial group model for $T^2$

$$\mathbb{G}(T^2) = \mathrm{hocolim}(\{*\} \leftarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} * \mathbb{Z}).$$

Take the functor $(-)_G$ and by a fact that the derived representation functor commutes with (small) colimits,

$$\begin{aligned} \mathscr{O}(\mathrm{DRep}_G(T^2)) &= \mathrm{hocolim}(k \leftarrow \mathscr{O}(G) \xrightarrow{\alpha_*} \mathscr{O}(G \times G)) \\ &\cong \mathscr{O}(G \times G) \otimes^{\boldsymbol{L}}_{\mathscr{O}(G)} k. \end{aligned}$$

Therefore

$$\mathrm{HR}_*(T^2, G) \cong \mathrm{Tor}_*^{\mathscr{O}(G)}(\mathscr{O}(G \times G), k).$$

We consider the case where $G = \mathbb{G}_m = \mathrm{Spec}\ k[x, x^{-1}]$, then the map

$$\begin{aligned} \alpha_* : \mathscr{O}(G) &\to \mathscr{O}(G \times G) \\ f(x) &\mapsto f([y, z]) = f(1). \end{aligned}$$

The resolution $P_\bullet$ of $k$ over $k[x, x^{-1}]$ satisfies $P_0 = k[x, x^{-1}]$, then the kernel of

$$k[x, x^{-1}] \to P_0 \twoheadrightarrow k$$

is $k[x, x^{-1}] \cdot (x - 1)$, therefore $P_1 = k[x, x^{-1}] \cdot w$ where the differential $d : w \mapsto x - 1$. This is exactly the Kozsul complex.

# Conclusion

Thank you for listening!