Combinatorics - A Toric Algebraic Geometry Approach

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Sum, Product and Power

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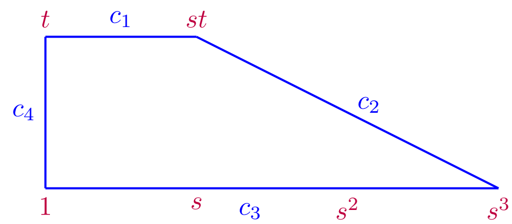
Caution. Within this notes we will only work on varieties $/\mathbb{C}$ [1].

In this talk, you will see

1. how do we construct a geometric object by combinatorial information, i.e. polytopes,
2. the correspondence between geometric information and combinatorial information,
3. how combinatorics simplifies the computation in algebraic geometry.

Example 1. Given the polytope



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we then can produce a morphism

$$\varphi : \mathbb{A}^2_{(s,t)} \to \mathbb{P}^5$$
$$(s,t) \mapsto [1, s, s^2, s^3, t, st].$$

Let $X := \overline{\varphi(\mathbb{A}^2)}$, then X is our first toric variety.

1 A Bad Definition of Toric Varieties

Definition. The n-torus is defined to be

$$(\mathbb{C}^*)^n := \text{Spec } \mathbb{C}[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}],$$

where the group structure is given component-wisely.

Proposition 1 (Good properties)**.** *1. Let T_1 and T_2 be tori and let $\varphi : T_1 \to T_2$ be a morphism that is a group homomorphism. Then the image of φ is closed in T_2.*

2. Let T be a torus and let $H \subseteq T$ be an irreducible subvariety of T that is a subgroup. Then H is a torus.

Definition. A *character* of torus $(\mathbb{C}^*)^n$ is a morphism $\chi : (\mathbb{C}^*)^n \to \mathbb{C}^*$. Dually, a *cocharacter*, or say a *one parameter subgroup* is a morphism $\lambda : \mathbb{C}^* \to (\mathbb{C}^*)^n$.

Example 2. Given an $m = (a_1, \cdots, a_n) \in \mathbb{Z}^n$, there is a character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ defined by

$$\chi^m(t_1, \cdots, t_n) = t_1^{a_1} \cdots t_n^{a_n}.$$

The amazing thing is that all characters come in this way. Also, there is a 1-PS associated with m

$$\lambda^m : \mathbb{C}^* \to (\mathbb{C}^*)^n$$
$$t \mapsto (t_1^{a_1}, \cdots, t_n^{a_n}).$$

All 1-PS's come in this way as well.

Assume that a torus T acts linearly on a finitely dimensional vector space W over $\mathbb{C}$. A basic result is that the maps $w \mapsto t \cdot w$ are simultaneously diagonalizable as follows. Given $m \in M$, define the eigenvector space

$$W_m := \{w \in W \mid t \cdot w = \chi^m(t)w \text{ for all } t \in T\}.$$

Then one can show that $W \cong \bigoplus_{m \in \mathbb{Z}^n} W_m$.

Definition (Bad Definition)**.** A *toric variety* is an irreducible variety V containing a torus $T := (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T on itself extends to an algebraic action of T on V.

2 Cones and Fourier-Motzkin

Definition. Let V be a finite dimensional vector space over $\mathbb{R}$, $S \subseteq V$ is a non-empty subset.

1. S be a *(convex) cone* if $\forall x, y \in S, \alpha, \beta \in \mathbb{R}$, if $\alpha, \beta \geq 0$ then $\alpha x + \beta y \in S$.
2. S be a *(convex) set* if $\forall x, y \in S, \alpha, \beta \in \mathbb{R}$, if $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$ then $\alpha x + \beta y \in S$.

Example 3 (Key examples).

1. Let $A \in \mathbb{R}^{m \times n}$, $A = [\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n]$ where $\boldsymbol{a}_i \in \mathbb{R}^m$. Define
$$\mathrm{vcone}(A) := \{x_1 \boldsymbol{a}_1 + \cdots + x_n \boldsymbol{a}_n \mid x_i \geq 0\} = \{A\boldsymbol{x} \mid \boldsymbol{x} \geq \boldsymbol{0}\},$$
where a cone of this form is called finitely generated.
2. Notations as before, define
$$\mathrm{conv}(A) = \{x_1 \boldsymbol{a}_1 + \cdots + x_n \boldsymbol{a}_n \mid x_i \geq 0, \sum_{i=1}^{n} x_i = 1\},$$
where a set of this form is called a polytope.
3. Let $B = \begin{bmatrix} \boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{b}_n \end{bmatrix}$ where $\boldsymbol{b}_i \in (\mathbb{R}^m)^*$. Define
$$\mathrm{hcone}(B) := \{\boldsymbol{y} \in (\mathbb{R}^m)^* \mid \langle \boldsymbol{y}, \boldsymbol{b}_i \rangle \leq 0\} = \{\boldsymbol{y} \in (\mathbb{R}^m)^* \mid \; B\boldsymbol{y} \leq 0\}.$$
4. $A \in \mathbb{R}^{m \times n}, \boldsymbol{b} \in \mathbb{R}^m$, define
$$P(A, \boldsymbol{b}) := \{\boldsymbol{x} \in \mathbb{R}^n \mid \; A\boldsymbol{x} \leq \boldsymbol{b}\}.$$
This is called to be polyhedral.

Definition. For a cone $\sigma \subseteq V$, its dual cone is
$$\sigma^\vee := \{\boldsymbol{y} \in V^* \mid \langle \boldsymbol{y}, \boldsymbol{x} \rangle \leq 0, \forall \boldsymbol{x} \in V\}.$$

Lemma 1.

1. *If $\sigma = \mathrm{vcone}(A)$, then $\sigma^\vee = \mathrm{hcone}(A^T)$.*
2. *σ is a vcone if and only if it is an hcone.*
3. *$\sigma^{\vee\vee} = \sigma$.*

Theorem 2 (Fourier-Motzkin Elimination)**.**

Theorem 3 (Weyl-Minkowski)**.** *Let $\sigma \subseteq \mathbb{R}^n$ be a cone. Then σ is finitely generated if and only if σ is polyhedral.*

3 Polytope Geometry

Definition. Let $\sigma \subseteq \mathbb{R}^n$ be a polyhedral cone. The *linear space* of σ is the largest subspace contained in σ. The cone σ is said to be *pointed* or *strongly convex* if its linear space is 0.

Proposition 4.

1. *The following are equivalent:*
   - *(a)* $\sigma \cap -\sigma = 0$.
   - *(b) σ is pointed.*
   - *(c) There is a $u \in \sigma^\vee$ with $\sigma \cap U^\perp = 0$.*
   - *(d) $\sigma^\vee$ spans V.*
   - *(e)* $\dim \sigma = \dim \sigma^\vee = \dim V$.

2. *Any cone $\sigma \subseteq V$ can be written as the sum of a linear space and a pointed cone. In face*

$$\sigma = L + \tau,$$

where $L := \sigma \cap -\sigma$ and $\tau := \sigma \cap L^{\perp}$ is pointed.

Definition. A *face* of a polyhedral cone $\sigma \subseteq \mathbb{R}^n$ is a subset $\tau \subseteq \sigma$ of the form

$$\tau := \sigma \cap u^{\perp}$$

for some $u \in \sigma^{\vee}$. A 1-dimensional face is called an *edge* or an *extremal ray*. A 1-codimensional face is called a facet.

Definition. Given any $u \in \mathbb{R}^n$, we define

$$H_m := \{f \in (\mathbb{R}^n)^* \mid \langle f, u \rangle = 0\}$$

and

$$H_m^+ := \{f \in (\mathbb{R}^n)^* \mid \langle f, u \rangle \geq 0\}.$$

Lemma 2. 1. *σ it self is a face.*

2. *The smallest face is $\sigma \cap -\sigma$.*

3. *A face τ of σ is also a polyhedral cone.*

4. *A face of a face is also a face.*

5. *If $\sigma = \mathrm{vcone}(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n)$, $\boldsymbol{u} \in \sigma^{\vee}$, then $\tau = \mathrm{vcone}(\boldsymbol{a}_i \mid \langle \boldsymbol{u}, \boldsymbol{a}_i \rangle = 0)$.*

Definition. A cone σ in $V = \mathbb{R}^n$ is said to be *rational* if it is generated by vectors (i.e. vcone) in $\mathbb{Q}^n$ (or equivalently $\mathbb{Z}^n$).

Lemma 3. *A cone $\sigma \subseteq \mathbb{R}^n$ is rational if and only if $\sigma^{\vee}$ is rational.*

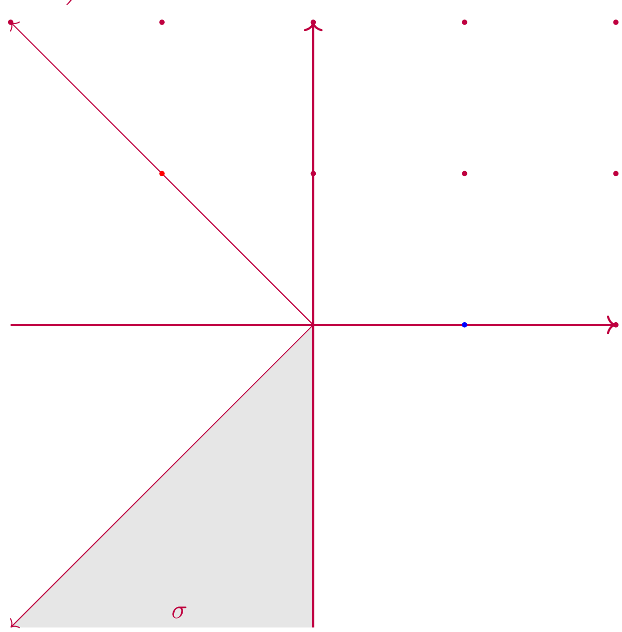
Definition. Let M be a lattice, then $M \cong \mathbb{Z}^n$ for some integer n which is the rank. We have $M \subseteq M_{\mathbb{Q}} := \mathbb{Q}^n \subseteq M_{\mathbb{R}} := \mathbb{R}^n$, and $M_k := M \otimes_{\mathbb{Z}} k$. Let N be the dual lattice $:= \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) = M^*$. We say σ is a cone in N if σ is a rational polyhedral cone in $N_{\mathbb{R}}$.

Definition. Let σ be a cone in N, we define

$$S_\sigma := \sigma^{\vee} \cap M.$$

Note that S_σ is a semi-group. Natural question: why not $S_\sigma := \sigma \cap N$?

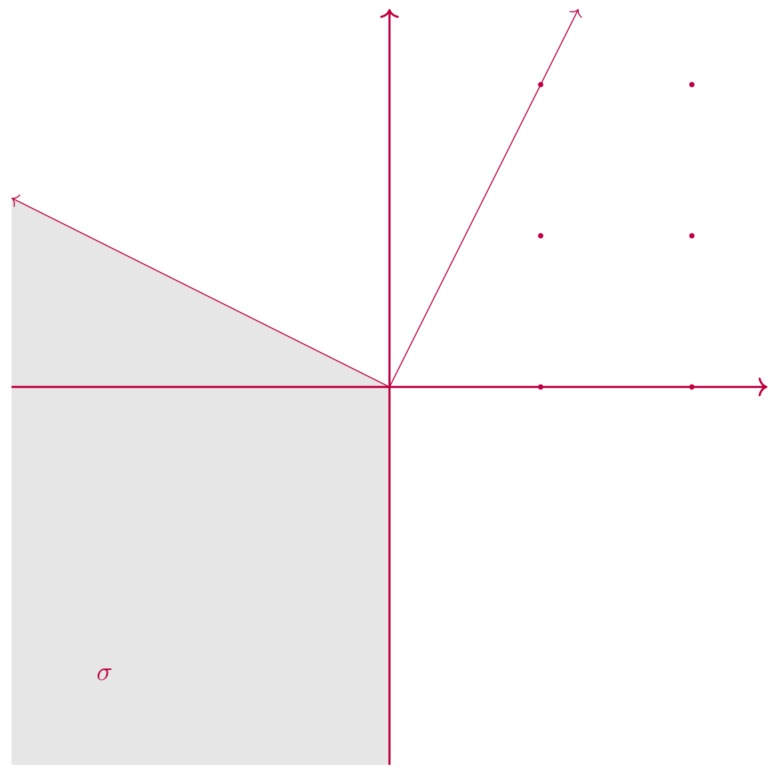
Example 4. Let $\sigma := \mathrm{vcone}\begin{pmatrix} -1 & \\ -1 & -1 \end{pmatrix}$.



One can read from the picture that

$$S_\sigma = \left\langle \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle.$$

Example 5. Let $\sigma := \operatorname{vcone}\begin{pmatrix} -2 & \\ 1 & -1 \end{pmatrix}$.



One can read from the picture that

$$S_\sigma = \left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle.$$

Proposition 5 (Gordan's Lemma)**.** *If σ is a cone in N, then S_σ is a finitely generated semi-group.*

Proof. Let $\sigma^\vee = \operatorname{vcone}(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_s)$, and let $K := \{x_1\boldsymbol{v}_1 + \cdots + x_s\boldsymbol{v}_s \mid 0 \le x_i < 1\}$. Then $\sigma^\vee \cap M$ is generated by $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_s\} \cap K$. $\square$

Definition. Let σ be a pointed cone. Consider $0 \neq m \in \sigma^\vee \cap M = S_\sigma$, it is called *irreducible* if for any decomposition $m = k + l$ in S_σ, either $k = 0$ or $l = 0$.

Proposition 6. *Let σ be a pointed polyhedral cone in $\mathbb{R}^n$, and let*

$$H := \{m \in S_\sigma \mid m \text{ is irreducible}\},$$

then

1. *$|H| < \infty$.*
2. *H generates S_σ.*
3. *Every generating set contains H.*

Here the set is called the Hilbert basis.

Definition. A polytope P is said to be *simplicial* if all its facets are simplices.

Definition. A lattice polytope $P \subseteq M_{\mathbb{R}}$ is said to be *normal* if

$$(kP) \cap M + (lP) \cap M = ((k+l)P) \cap M$$

for all $k, l \in \mathbb{N}$.

Theorem 7. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope of dimension $n \geq 2$, then kP is normal for all $k \geq n-1$.*

Definition. A sub-semi-group $S \subseteq M$ is said to be *saturated* if whenever $m \in M$ and $pm \in M$ for some $p \in \mathbb{N}_+$, $m \in S$.

Definition. A lattice polytope $P \subseteq M_{\mathbb{R}}$ is said to be *very ample* if for every vertex $m \in P$, the semi-group $S_{P,m} := \mathbb{N}\langle P \cap M - m \rangle$ is saturated in M.

Proposition 8. *A normal lattice polytope is very ample.*

Definition. A *fan* Σ in $N_{\mathbb{R}}$ is a finite collection of cones such that:

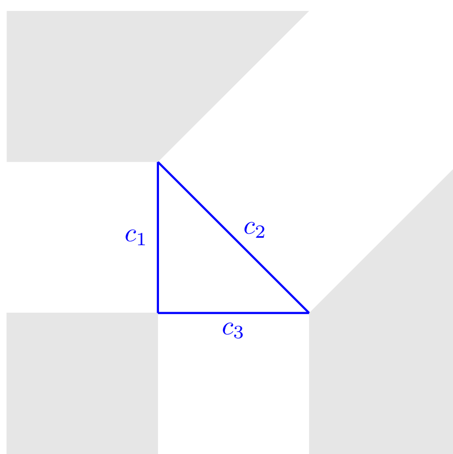
1. Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.
2. For all $\sigma \in \Sigma$, each face of σ is also in Σ.
3. For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

Definition. Given a full-dimensional lattice polytope P in $\mathbb{R}^n$, for each face F of the polytope, define

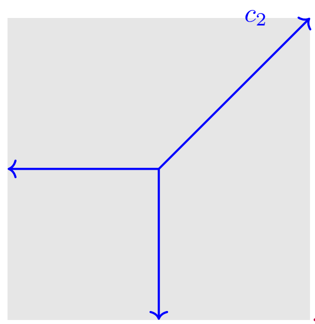
$$\sigma_F := \{\boldsymbol{u} \in N_{\mathbb{R}} \mid \langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{u} \rangle \leq 0, \forall \boldsymbol{x} \in F, \boldsymbol{y} \in P\}.$$

Let $\Sigma_P := \{\sigma_F \mid F \subseteq P \text{ is a face}\}$, then Σ_P is a fan, called the normal fan associated with the polytope P.

Example 6. Given the polytope



we shall have the normal fan



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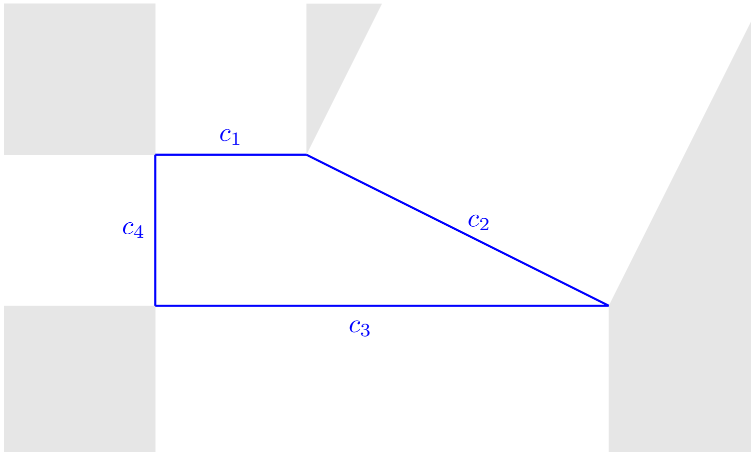
Example 7. Given the polytope



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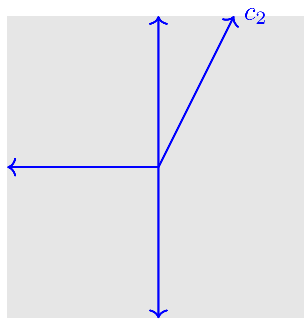

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4 Toward Toric Varieties

Here comes our main construction:

Definition. Let $A_\sigma := \mathbb{C}[S_\sigma]$ be a $\mathbb{C}$-algebra, such that

1. $\{t^m \mid m \in S_\sigma\}$ forms a $\mathbb{C}$-basis for A_σ.
2. $t^{m_1} t^{m_2} = t^{m_1+m_2}$.

We then call $U_\sigma := \operatorname{Spec} A_\sigma$ the affine toric variety associated to σ.

If $S_\sigma = \langle m_1, \cdots, m_r \rangle$, then A_σ is generated (as a ring) by $t^{m_1}, \cdots, t^{m_r}$. In particular A_σ is Noetherian. Consider

$$0 \to \operatorname{Ker} \varphi \to \mathbb{C}[x_1, \cdots, x_r] \xrightarrow{\varphi} A_\sigma \to 0$$

where φ maps x_i to t^{m_i}. $I_\sigma := \operatorname{Ker} \varphi$ is called the toric ideal.

Theorem 9. *All affine toric varieties defined at the beginning come from this way.*

Example 8. Let $\sigma := \operatorname{vcone} \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$. Then $S_\sigma = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$ and thus $\mathbb{C}[S_\sigma] = \mathbb{C}[s,t]$, $U_\sigma = \mathbb{C}^2$. Similarly, if $\sigma := \operatorname{vcone}(-I_n)$, then $U_\sigma = \mathbb{C}^n$.

Example 9. Let $\sigma := \operatorname{vcone} \begin{pmatrix} -1 & \\ -1 & -1 \end{pmatrix}$. Then $S_\sigma = \left\langle \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$ and thus $\mathbb{C}[S_\sigma] = \mathbb{C}[s^{-1}t^{-1}, t] = \mathbb{C}[x,y]$, $U_\sigma = \mathbb{C}^2$.

Example 10. Let $\sigma := \operatorname{vcone}\begin{pmatrix} -2 & \\ 1 & -1 \end{pmatrix}$. From previous computation we know that $A_\sigma = \mathbb{C}[s, st, st^2] = \mathbb{C}[x, y, z]/(y^2 - xz)$.

Question: the previous examples have 2 generators and 3 generators respectively. What are the differences?

Example 11. Let $\sigma := \operatorname{vcone}(0)$, then $S_\sigma = \mathbb{Z}^2$ and $A_\sigma = \mathbb{C}[s, s^{-1}, t, t^{-1}]$, hence U_σ is the 2-torus.

Proposition 10. *If $S_\sigma = \langle m_1, \cdots, m_r \rangle$, then I_σ is generated by*

$$\{x_1^{a_1} \cdots x_r^{a_r} - x_1^{b_1} \cdots x_r^{b_r} \mid \sum_{i=1}^{r} (a_i - b_i) m_i = 0\}.$$

Definition. Let σ be a pointed cone in $N = \mathbb{Z}^n$. We say

1. σ is *simplicial* if the number of extremal rays $= \dim \sigma$.
2. σ is *smooth* if σ is generated by a part of a $\mathbb{Z}$-basis of N.

Theorem 11. *Let σ be a pointed cone in N and $\dim \operatorname{Span} \sigma = k$, then the following are equivalent:*

1. *U_σ is smooth.*
2. *$U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$.*
3. *σ is smooth.*

Theorem 12. *For an affine toric variety V, the following are equivalent:*

1. *V is normal.*
2. *$V := \operatorname{Spec} \mathbb{C}[S]$ where S is a saturated affine semi-group.*
3. *$V \cong U_\sigma$ for some rational polyhedral cone.*

Proposition 13. *Let $V = \operatorname{Spec} \mathbb{C}[S]$ be the affine toric variety of the affine semi-group S. Then there is a correspondence between the following:*

1. *$\mathbb{C}$-points in V.*
2. *Maximal ideals $\mathfrak{m}$ in $\mathbb{C}[S]$.*
3. *Semi-group homomorphisms $S \to \mathbb{C}$, where $\mathbb{C}$ is considered as a semi-group under multiplication.*

We can actually use this proposition to get the group action.

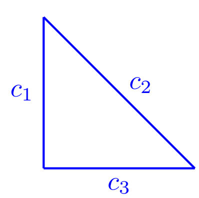
Proposition 14. *Let $\tau = \sigma \cap u^\perp$ be a face of σ where $u \in \sigma^\vee$. Then the semi-group algebra $\mathbb{C}[S_\tau]$ is the localization of $\mathbb{C}[S_\sigma]$ at the point $t^m \in \mathbb{C}[S_\sigma]$.*

Actually Proposition 14 gives us the information which is called the gluing data. Suppose τ is a common face of σ_1, σ_2, then we have

$$U_{\sigma_1} \hookleftarrow U_\tau \hookrightarrow U_{\sigma_2},$$

and we can glue U_{σ_1} and U_{σ_2} along U_τ. But where can we find the structures including common faces? The definition of a fan and here is also where polytopes can be related.

Example 12. Consider the normal fan of the polytope $P =$



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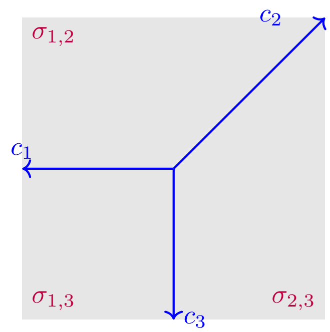
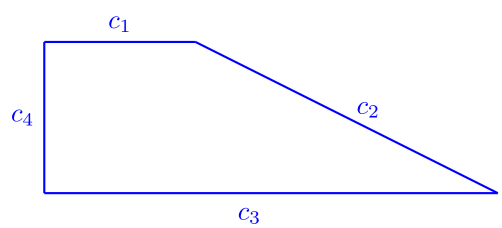
then by previous computation, the associated fan



has affine pieces $U_{1,2} = \text{Spec } \mathbb{C}[s^{-1}, st] = \mathbb{C}^2, U_{1,3} = \text{Spec } \mathbb{C}[s^{-1}, t^{-1}] = \mathbb{C}^2, U_{1,3} = \text{Spec } \mathbb{C}[st, t^{-1}] = \mathbb{C}^2$, and gluing affine open subsets $U_1 = \mathbb{C}[s^{-1}, st]_{s^{-1}} = \text{Spec } \mathbb{C}[s^{-1}, t^{-1}]_{s^{-1}}$, $U_2 = \text{Spec } \mathbb{C}[s^{-1}, st]_{st} = \mathbb{C}[st, t^{-1}]_{st}$, and $U_3 = \mathbb{C}[s^{-1}, t^{-1}]_{t^{-1}} = \mathbb{C}[st, t^{-1}]_{t^{-1}}$. This gives the variety $\mathbb{CP}^2$.

Example 13. Consider the normal fan of the polytope $P =$



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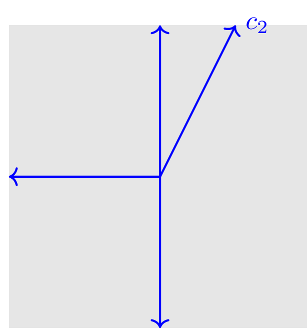

has affine pieces $U_{1,2} = \text{Spec } \mathbb{C}[st^2, t] = \mathbb{C}^2, U_{1,4} = \text{Spec } \mathbb{C}[s^{-1}, t] = \mathbb{C}^2, U_{3,4} = \text{Spec } \mathbb{C}[s^{-1}, t^{-1}] = \mathbb{C}^2, U_{2,3} = \text{Spec } \mathbb{C}[st^2, t^{-1}] = \mathbb{C}^2$, and gluing affine open subsets $U_1 = \mathbb{C}[t, st^2]_t = \text{Spec } \mathbb{C}[s^{-1}, t]_t$, $U_2 = \text{Spec } \mathbb{C}[st^2, t]_{st^2} = \text{Spec } \mathbb{C}[st^2, t^{-1}]_{st^2}, U_3 = \text{Spec } \mathbb{C}[st^2, t^{-1}]_{t^{-1}} = \text{Spec } \mathbb{C}[s^{-1}, t^{-1}]_{t^{-1}}$, and $U_4 = \mathbb{C}[s^{-1}, t^{-1}]_{s^{-1}} = \mathbb{C}[t, s^{-1}]_{s^{-1}}$. This gives the variety at the very beginning. The surface is called the Hirzebruch surface $\mathbb{F}_2$.

In conclusion, we have the following

Theorem 15. *Given a fan Σ, there is a toric variety X_Σ associated with the fan. Furthermore,*

1. *X_Σ is smooth if and only if Σ is smooth, i.e. all the cones are smooth.*
2. *X_Σ is simplicial (i.e. X_Σ is an orbifold, having only finitely many quotient singularities) if and only if Σ is simplicial. When the fan Σ comes from a polytope, this is equivalent to that the polytope is simplicial.*
3. *X_Σ is complete if and only if the support $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$ is all of $N_\mathbb{R}$.*

5 Slogan: In Toric World, Geometries are Combinatorics

Theorem 16. *A normal toric variety is Cohen-Macaulay.*

When the polytope is very ample, we not only get a projective toric variety, but also the morphism how it is embedded into a projective space. This information is called a very ample line bundle.

Theorem 17 (Ehrhart)**.** *We have seen that the Ehrhart series for a full dimensional polytope $P \subseteq \mathbb{R}^n$*

$$\begin{aligned} \mathrm{Ehr}(P, -) : \mathbb{N} &\to \mathbb{N} \\ t &\mapsto \#(tP \cap \mathbb{Z}^n) \end{aligned}$$

is a polynomial with a reciprocity

$$\mathrm{Ehr}(P, -t) = (-1)^d \mathrm{Ehr}(P, t)$$

where d is the degree of the polynomial.

This theorem can be proved using toric varieties. Let V be the (projective) toric variety associated with the polytope P, then (some enlarged) the polytope gives an ample line bundle L over V. Then the Ehrhart series coincides with the Hilbert series of this line bundle almost by definition. Since the Hilbert series is a polynomial, we are done.

Theorem 18 (Stanley, 86', [2])**.** *Let P be a n-dimensional simplicial polytope, and let the f-vector $f = (f_0, \cdots, f_{n-1})$ be a sequence of numbers where f_j is the number of j-faces of P. Let $f_{-1} = 1$. Define*

$$h_i := \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_{j-1}$$

then we have the so called h-vector. The Dehn-Sommerville equations say that

$$h_i = h_{n-i}, \forall 1 \leq i \leq n,$$

which hold for any simplicial convex polytope. A sequence of integers $(k_0, \cdots, k_n)$ is said to be an M-vector (after F.S.Macauley) if

$$k_0 = 1 \text{ and } k_{i+1} \leq k_i^{\langle i \rangle} \text{ for all } 1 \leq i \leq n-1,$$

where $k_i^{\langle i \rangle}$ is defined to be

$$k_i^{\langle i \rangle} := \binom{n_i + 1}{i + 1} + \cdots \binom{n_j + 1}{j + 1},$$

where $n_i \geq n_{i-1} \geq n_j \geq j \geq 1$ are those (unique) numbers such that

$$k_i = \binom{n_i}{i} + \cdots \binom{n_j}{j}.$$

A sequence of integers $(h_0, \cdots, h_n)$ is the h-vector of a simplicial convex n-polytope if and only if $h_0 = 1$, $h_i = h_{n-i}$ and the sequence $(h_0, h_1 - h_0, \cdots, h_{[n/2]} - h_{[n/2]-1})$ is an M-vector.

This was originally conjectured by McMullen [3], and proved by Stanley using some great toric tools (Hard Lefschetz).

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[1] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[2] Richard P. Stanley. The number of faces of a simplicial convex polytope. *Adv. in Math.*, 35(3):236–238, 1980.

[3] P. McMullen. The numbers of faces of simplicial polytopes. *Israel J. Math.*, 9:559–570, 1971.