Math 4500 HW #08 Solutions

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This solution set is not error-free. Please email me (gl479@cornell.edu) if you spot any errors or typos!

Problem 1 (Exercise 5.6.4 (10 pts))**.** When $X^2 = -\det(X)I$, show that

$$e^X = \cos(\sqrt{\det(X)})I + \frac{\sin(\sqrt{\det(X)})}{\sqrt{\det(X)}}X.$$

Solution. By definition

$$\begin{aligned}
e^X &= \sum_{n=0}^{\infty} X^n \\
&= \sum_{n=0}^{\infty} \left(\frac{X^{2n}}{(2n)!} + \frac{X^{2n+1}}{(2n+1)!} \right) \\
&= \sum_{n=0}^{\infty} \left(\frac{X^{2n}}{(2n)!} + \frac{X^{2n}X}{(2n+1)!} \right) \\
&= \sum_{n=0}^{\infty} \left(\frac{(-\det(X))^n}{(2n)!} I + \frac{(-\det(X))^n}{(2n+1)!} X \right) \\
&= \sum_{n=0}^{\infty} \frac{(-\det(X))^n}{(2n)!} I + \sum_{n=0}^{\infty} \frac{(-\det(X))^n}{(2n+1)!} X \\
&= \cos(\sqrt{\det(X)})I + \frac{\sin(\sqrt{\det(X)})}{\sqrt{\det(X)}}X.
\end{aligned}$$

$\square$

Problem 2 (Exercise 5.6.5 (10 pts))**.** Using Exercise 5.6.4 and the fact that $\mathrm{Trace}(X) = 0$, show that if

$$e^X = \begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix},$$

then $\cos(\sqrt{\det(X)}) = -1$, in which case $\sin(\sqrt{\det(X)}) = 0$, and there is a contradiction.

Solution. By the previous problem

$$\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix} = e^X = \cos(\sqrt{\det(X)})I + \frac{\sin(\sqrt{\det(X)})}{\sqrt{\det(X)}}X.$$

Take trace of both sides, then

$$-2 = \mathrm{Trace}\left(\cos(\sqrt{\det(X)})I + \frac{\sin(\sqrt{\det(X)})}{\sqrt{\det(X)}}X \right).$$

But Trace is linear so

$$-2 = \cos(\sqrt{\det(X)})\mathrm{Trace}(I) + \frac{\sin(\sqrt{\det(X)})}{\sqrt{\det(X)}}\mathrm{Trace}(X) = 2\cos(\sqrt{\det(X)}),$$

hence $\cos(\sqrt{\det(X)}) = -1$. Therefore $\sin(\sqrt{\det(X)})^2 = 1 - \cos^2(\sqrt{\det(X)}) = 0$, thus $\sin(\sqrt{\det(X)}) = 0$. But this means

$$e^X = \cos(\sqrt{\det(X)})I + \frac{\sin(\sqrt{\det(X)})}{\sqrt{\det(X)}}X = \begin{pmatrix} -1 & 0 \\ & -1 \end{pmatrix},$$

a contradiction. □

Problem 3 (Exercise 6.1.3 (5 pts))**.** Show that $SU(n)$ is a normal subgroup of $U(n)$ by describing it as the kernel of some homomorphism.

Solution. We know that

$$\det : U(n) \to \mathbb{C}$$

is a group homomorphism, because $\det(AB) = \det(A)\det(B)$. Hence the kernel of $\det$, which is $SU(n)$, is a normal subgroup. □

Problem 4 (Exercise 6.1.4 (5 pts))**.** Show that $T_I(SU(n))$ is an ideal of $T_I(U(n))$ by checking that it has the required closure properties.

Solution. We know that

$$T_I(SU(n)) = \{A \in M_n(\mathbb{C}) \mid A + \bar{A}^T = 0, \mathrm{Trace}(A) = 0\}$$

and

$$T_I(U(n)) = \{A \in M_n(\mathbb{C}) \mid A + \bar{A}^T = 0\}.$$

For any $A \in T_I(U(n))$ and $B \in T_I(SU(n))$,

$$\mathrm{Trace}\,[A, B] = \mathrm{Trace}(AB - BA) = 0$$

since $\mathrm{Trace}\,AB = \mathrm{Trace}\,BA$. On the other hand

$$AB - BA + \overline{AB - BA}^T = AB - BA - \bar{A}^T\bar{B}^T - \bar{B}^T\bar{A}^T = 0.$$

Hence $T_I(SU(n))$ is an ideal of $T_I(U(n))$. □

Problem 5 (Exercise 6.3.4 (10 pts))**.** Find a 1-dimensional ideal J of $\mathfrak{u}(n)$, and show that J is the tangent space of $Z(U(n))$.

Solution. Put $J := \{\theta i I \mid \theta \in \mathbb{R}\}$. We have already know that $Z(U(n)) = \{e^{i\theta}I \mid \theta \in \mathbb{R}\}$. Since $\alpha(t) := e^{i\theta t}I$ is a path in $Z(U(n))$ and $\alpha'(0) = \theta i I$, we know that $J \subseteq T_I(Z(U(n)))$. Conversely, for any path $\alpha(t)$ s.t. $\alpha(0) = I \in Z(U(n))$, it has the form $\alpha(t) = e^{i\theta(t)}I$ where $\theta(t)$ is a continuous map from $\mathbb{R}$ to $\mathbb{R}$ s.t. $\theta(0) = 0$. Thus $\alpha'(0) = \theta'(0)iI \in J$. This proves that J is an ideal. Apparently J is of dimension 1, and it is an ideal since it is the tangent space of some normal subgroup. □

Problem 6 (Exercise 6.3.5 (5 pts))**.** Also show that the $Z(U(n))$ is the image, under the exponential map, of the ideal J in Exercise 6.3.4.

Solution. For any $e^{i\theta}I \in Z(U(n))$, we have $\theta i I \in J$ s.t. $e^{\theta i I} = e^{i\theta}I \in Z(U(n))$. □

Problem 7 (Exercise 5.5.1 (0 pts))**.**

Solution. □

Problem 8 (Exercise 5.5.2 (0 pts))**.**

Solution. □

Problem 9 (Exercise 5.5.4 (0 pts))**.**

Solution. □

Problem 10 (Exercise 6.4.1 (0 pts))**.**

Solution. □

Problem 11 (Exercise 6.4.2 (0 pts))**.**

Solution. □