# Algebraic Topology is Inevitable

A friendly introduction to the applications of AT in other mathematics

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# I'm not a topologist!

# What is Algebraic Topology

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**Question**

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**Almost certainly wrong definition**

Algebraic topology is the study of (maybe generalised) homology groups, cohomology groups, and homotopy groups of spaces, and the connections among these algebraic objects.

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2. Hurewicz: Suppose $X$ is a connected topological space. Then the map

$$\pi_1(X) \to H_1(X), [\gamma] \mapsto (\sigma : \Delta^1 \to X, t \mapsto \gamma(t))$$

is a surjective group homomorphism, whose kernel is exactly the commutator subgroup $[\pi_1(X), \pi_1(X)]$. Hence $H_1 = (\pi_1)_{\mathrm{ab}}$.

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4. In fact, $\pi_n$'s are quite powerful. Whitehead: Suppose $f : X \to Y$ is a map between "good" topological spaces, if the induced maps

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

   are isomorphisms for all $x_0 \in X$ and all $n \geq 0$, then $f$ is a homotopy equivalence.

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   We turned topological information into algebraic information!

## Examples

5. The homotopy groups are powerful, however it is hard to compute. Even the question

$$\pi_i(S^n) = ?$$

is not fully answered. This question is still somewhat one of the central problems in AT.

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What do we mean by the dimension of a space?

1. One might say $\dim \mathbb{R}^n$ is the number of elements in a basis, which is $n$. However this is by linear algebra, not by topology.
2. One might say in geometry, we *define* a space locally looking like $\mathbb{R}^n$ of dimension $n$. Then there is another question arise immediately: How do we know

$$\mathbb{R}^0, \mathbb{R}^1, \cdots, \mathbb{R}^n, \cdots$$

are different?

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   2. Both sides deformation retracts to spheres, so we have $S^{m-1} \cong S^{n-1}$.
   3. By homology comparison, $m = n$.

## Examples

Imagine that one is tieing one's shoes, using such a way.

In reality one knows that this works - it is different from being untied.

## Examples

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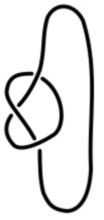
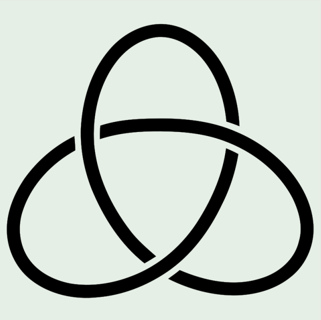


We need to know that this is different from a circle.

## Examples

After doing a little thing, we get this beautiful picture:



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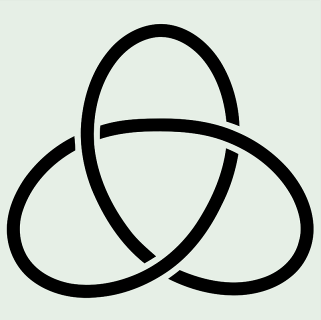

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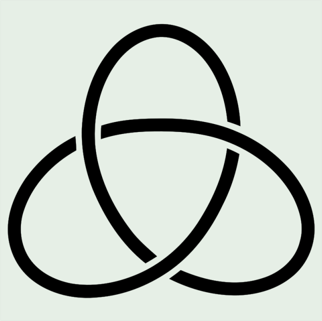

However if you have some knowledge of topology: No matter how we tangle the (closed) shoelace, it is still a circle itself!

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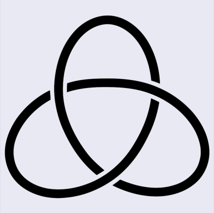

- We call such a knot trefoil and we should have that trefoil is different from an untwisted circle.

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But how? We have algebraic topology!

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   where the last isomorphism is by the substitution $y = x_2x_1, x = x_1y$ so $x_1 = xy^{-1}$ and $x_2 = y^2x^{-1}$.

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If this two groups are different, then mathematically we are done!
But again, why they are different?
Is it possible to see $\langle x, y \mid x^3 = y^2\rangle$ is nonabelian, for instance $xy \neq yx$?

# Applications in Combinatorics

## Examples

- A *finite presentation* of a group $G$ is a quotient

$$G \cong F/N,$$

where $F = \langle x_1, \cdots, x_m \rangle$ is the free group generated by $x_1, \cdots, x_m$, and $N$ is the smallest normal subgroup generated by elements $R_1, \cdots, R_n$ which are called relations. Clearly an element $w \in F$ is in the normal subgroup $N$ if and only if it can be written in the form

$$w = \prod_{i=1}^{N} T_i R_{j_i}^{e_i} T_i^{-1} \tag{1}$$

where $T_i \in F$ for all $i$ and $e_i = \pm 1$.

## Examples

- The *Word Problem* is to determine when two elements of $F$ (or two 'words') represent the same element of $G$. Equivalently, given an element $w \in F$, when does it lie in $N$ so that it has a form of 1. In general, this is always a hard problem in combinatorics. **Magnus** solved this question for $G = \langle x_1, \cdots, x_m | R_1, \cdots, R_n \rangle$ when $n = 1$. We call such groups *1-relator groups*.

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Word problem is generally very difficult - for instance, it is not easy to see $xy \neq yx$ in $\langle x, y \mid x^3 = y^2 \rangle$.

## Examples

- There is also complementary problem, asking the uniqueness of the expression of relations, i.e., determining all relations of $R_1, \cdots, R_n$ of the form

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Also in the case when $n = 1$, the problem is solved.

**Theorem (Magnus)**

**Theorem (Simple Identity Theorem, Lyndon)**

Let $F$ be the free group on generators $x_1, \cdots, x_m$ and $G = F/N$ where $N$ is the smallest normal subgroup containing the relation $R$. Then

$$\prod_{i=1}^{N} T_i R^{e_i} T_i^{-1} = 1$$

implies that the indices $1, \cdots, N$ can be grouped into pairs $(i, j)$ such that $t_i = t_j$, $e_i = -e_j$, and for certain integers $c_i$, $T_i \equiv T_j Q^{e_i} \pmod{N}$.

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But what does it mean?

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Then we can we say more if we have the simple identity theorem?

**Theorem**

Given a group of finite presentation $G = \langle x_1, \cdots, x_m | R \rangle$, if $R$ is not a proper power (i.e. if $R = Q^q$ then $q = 1$), its Cayley complex is *aspherical*, namely the universal covering space is contractible.

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This theorem is actually equivalent to the simple identity theorem! We again pause for a moment to see what can we derive from this theorem.

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  1. $\pi_1(BG) = G$, and
  2. the universal covering space is contractible.

  We call such a space **the** classfying space of $G$.
- For a (discrete) group $G$, its *group homology* is defined as

$$H_n(G; \mathbb{Z}) := H_n(BG; \mathbb{Z}).$$

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However, do you feel good?

## Examples

1. Given a group $G$, let

$$F_n := \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G] \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}[G]$$

where there are $n+1$ terms in total. Let $G$ acting on $F_n$ freely by

$$g \cdot (g_0 \otimes g_1 \otimes \cdots \otimes g_n) := (g \cdot g_0) \otimes g_1 \otimes \cdots \otimes g_n$$

and $F_n$ is a free $\mathbb{Z}[G]$-module with a basis of $\{1 \otimes g_1 \otimes \cdots \otimes g_n\}_{g_i \in G}$.

## Examples

2. One could define

$$d_i^{[n]} : F_n \to F_{n-1}, \;\; 0 \leq i \leq n$$

linearly extended by

$$d_i^{[n]}(1 \otimes g_1 \otimes \cdots \otimes g_n) := \begin{cases} g_1 \cdot (1 \otimes g_2 \otimes \cdots \otimes g_n) \\ (1 \otimes g_1 \otimes \cdots \otimes g_{i-1} \otimes g_i g_{i+1} \otimes \cdots \otimes g_n) \\ 1 \otimes g_1 \otimes \cdots \otimes g_{n-1} \end{cases}$$

where $\{d_i^{[n]}\}_{0 \leq i \leq n}$ is a $\mathbb{Z}[G]$-module homomorphism satisfying $d_i^{[n]} d_j^{[n]} = d_{j-1}^{[n]} d_i^{[n]}$, so $\partial_n := \sum_{i=0}^{n} (-1)^i d_i^{[n]}$ is the boundary map of the chain complex

$$0 \leftarrow F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_n} F_n \xleftarrow{\partial_{n+1}} \cdots .$$

## Examples

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## Theorem

Two definitions of group homology $H_*(G;\mathbb{Z})$ agree.

## Examples

Computation time!

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- $B\mathbb{Z} = S^1$, and $H_i(S^1) = \mathbb{Z} \oplus \mathbb{Z}$, so

$$H_n(\mathbb{Z};\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

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- Simple identity theorem says its Cayley complex has the property that the universal covering space is contractible, so **the Cayley complex is already (a choice of)** $BG$!

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- The cellular complex of the Cayley complex is easy to compute:

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## Remark

The difficulty relies on the presentation of the group / the choice of $BG$.

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But before answering these questions

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Algebraic topology is more of a way of understanding mathematics. It is a machinery, eating objects we care about, spitting out algebraic information such as numbers, polynomials, matrices, groups, modules etc. to classify the objects.

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In the famous textbook written by a (retired) professor here at Cornell, AT is divided into several parts, homologies, cohomologies, homotopies, etc. But algebraic topology is far more influencing than these studies. It gives birth to a lot of subjects like category theory, homological algebra, homotopical algebra, and so on, rooted unexpectedly in almost all mathematics.

# Applications in Geometry

### Examples

In the examples before, we see that different choices of things (the name is resolutions) give us the same homology. The general setting of this phenomenon happens is called *homological algebra*, born by algebraic topology. With AT itself, there are many big names related:

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# Thank you!