From Tor to DAG
Deriving the representation variety

Guanyu Li

Abstract

Homological Algebra explores derived functors. They come from left/right exact functors (between abelian categories) that are not exact. Their constructions, which involve projective/injective resolutions, can be challenging to grasp, but they hold significance across various mathematical disciplines.

We will use the example of Tor functor, to explain why the derived functor are special, in the sense that it is the functor closest to the original one among all the functors preserving quasi-isomorphisms. This property allows us to extend the notion of derived functors to a wide range of settings, including non-abelian categories. For instance, the singular homology functor defined on the category of topological spaces is a derived functor. With such an enhancement, we can establish the foundation of derived algebraic geometry.

Representations play a crucial role in modern algebra. Surprisingly, all representations can be parameterized by an affine scheme known as the representation variety. Despite being useful, representation varieties have certain inherent limitations. Firstly, these varieties tend to have singularities that make it challenging to understand their geometry. Secondly, when we apply them to take the fundamental group of a topological space, we lose higher-order information about the space. To address these issues, we introduce Derived Algebraic Geometry (DAG).

DAG offers several reasons for its usefulness. Many classical objects, such as the cotangent complex and the intersection formula, find a more natural representation in the derived setting. Additionally, DAG can be seen as approximating the underlying space through a sequence of smooth spaces, providing additional tools for studying scheme geometry.

In this talk, I will delve into the foundations of DAG, explaining how they can be used to derive the representation variety. The discussion will cover intriguing properties, selected computations, and potential applications if time permits.

Contents

It is well-known that tensor product is right exact, which means for any S.E.S. of mudules

$$0 \to M \to N \to P \to 0,$$

there is an induced exact sequence

$$M \otimes_R Q \to N \otimes_R Q \to P \otimes_R Q \to 0.$$

This makes tensor product a bad-behavioured functor in terms of homological algebra. The classical way of dealing this problem is to extend this exact sequence on the left, which gives the so-called derived functor.

1 Several definitions of Tor

Definition. Given a unital ring R, let $R-\mathbf{Mod}$ be the category of left R-modules, and let $\mathbf{Mod}-R$ be the category of right R-modules. Given $M \in \mathbf{Mod}-R$ and $N \in R-\mathbf{Mod}$, there exist projective resolutions

$$P_\bullet \to M, \quad Q_\bullet \to N.$$

Then there are isomorphisms

$$H_i(P_\bullet \otimes_R N) \cong H_i(M \otimes_R Q_\bullet) \cong H_i(\operatorname{Tot}(P_\bullet \otimes Q_\bullet)).$$

We call this the i-th torsion group $\operatorname{Tor}_i^R(M,N)$, and call $\operatorname{Tor}_i^R(-,-)$ the derived functor of $-\otimes_R -$.

Definition. A (covariant) homological δ-functor $\mathcal{A} \to \mathcal{B}$ is a collection of functors $\{T_n : \mathcal{A} \to \mathcal{B}\}_{n\in\mathbb{N}}$ and natural transformations $\{\delta_i : T_n \Rightarrow T_{n-1}\}_{n\in\mathbb{N}}$ s.t. for any S.E.S.

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{A}$, there is an associated long exact sequence

$$\cdots \to T_i(X) \to T_i(Y) \to T_i(Z) \xrightarrow{\delta_{X,Z}} T_{i-1}(X) \to \cdots,$$

natural with S.E.S..

Theorem 1. *The set of functors $\{\operatorname{Tor}_i^R(-,N)\}_{i\in\mathbb{N}}$ is the unique set of functors satisfying*

1. $\operatorname{Tor}_0^R(M,N) = M \otimes_R N$.
2. $\operatorname{Tor}_i^R(P,N) = 0$ *if $i > 0$ and P is projective.*
3. $\{\operatorname{Tor}_i^R(-,N)\}_{i\in\mathbb{N}}$ *is a homological δ-functor.*

Slogan 1. The derived functor is the unique functor that is special to some certain objects.

Example 1. *Given a (commutative) ring R with an element $r \in R$, s.t. r is not a nonzerodivisor. Given any S.E.S. of R-modules*

$$0 \to M \to N \to P \to 0,$$

there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0, \end{array}$$

where the snake lemma induces a long exact sequence

$$0 \to_r M \to_r N \to_r P \to M/rM \to N/rN \to P/rP \to 0.$$

This means that

$$T_n(M) := \begin{cases} rM & n = 1 \\ M/rM & n = 0 \\ 0 & \textit{otherwise} \end{cases}$$

is a δ-functor.

Definition. A morphism between δ-functors $\varphi : S \Rightarrow T$ is a set of natural transformations $\{\varphi_n : S_n \Rightarrow T_n\}_{n\in\mathbb{N}}$ cpmmuting with the connecting morphisms.

A δ-functor extending a given functor $F : \mathcal{A} \to \mathcal{B}$ is a δ-functor $\{T_n : \mathcal{A} \to \mathcal{B}\}_{n\in\mathbb{N}}$ s.t. $T_0 = F$.

Definition. A homological δ-functor $\{T_n : \mathcal{A} \to \mathcal{B}\}_{n\in\mathbb{N}}$ is called universal if it is terminal among all functor extending a given functor $F : \mathcal{A} \to \mathcal{B}$.

Theorem 2. *The set of functors $\{\operatorname{Tor}_i^R(-,N)\}_{i\in\mathbb{N}}$ is universal.*

This might give us some reasons why we care about Tor among all δ-functors. However, why do we care about δ-functors?

2 Deriving a functor between abelian categories

One of the reasons that we care about Tor is that we care more about complexes up to quasi-isomorphisms than complexes, but tensor product does not preserve quasi-isomorphisms. One the one hand, Tor (and other δ-functors) measures how much a tensor product is away from being exact, and on the other hand, Tor (and other δ-functors) turns S.E.S. to long exact sequence, which is of a form of being exact.

The next problem is to make the discussion above mathematically precise.

Definition. The derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ is defined to be the (Gabriel-Zisman) localisation

$$K(\mathcal{A})[Q-iso^{-1}],$$

where $K(\mathcal{A})$ is the homotopy category of $\mathcal{A}$, defined to be the category with object $\mathrm{ob}\,\mathbf{Com}_\bullet(\mathcal{A})$, and hom-set

$$\hom_{K(\mathcal{A})}(X,Y) := \frac{\hom_{\mathbf{Com}_\bullet(\mathcal{A})}(X,Y)}{\text{chain homotopies}}.$$

Proposition 3. *There is a natural equivalence of categories*

$$D(\mathcal{A}) \simeq \mathbf{Com}_\bullet(\mathcal{A})[Q-iso^{-1}].$$

Definition. Given functors $F : \mathcal{C} \Rightarrow \mathcal{D}F : \mathcal{C} \Rightarrow \mathcal{D}$, the right Kan extension of F along G is a functor $\mathcal{R}_G(F) : \mathcal{E} \to \mathcal{D}$ together with a natural transformation $\epsilon : \mathcal{R}_G(F) \circ G \Rightarrow F$ satisfying the commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "G"'] & {} \arrow[d, phantom, "\epsilon\Uparrow"] & \mathcal{D} \\
& \mathcal{E} \arrow[ur, "\mathcal{R}_G(F)"'] &
\end{tikzcd}$$

s.t. for any functor $H : \mathcal{E} \to \mathcal{D}$ and natural transformation $\xi : H \circ G \Rightarrow F$, there is a unique natural transformation $\delta : H \Rightarrow \mathcal{R}_G(F)$ making the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "G"'] & & \mathcal{D} \\
& \mathcal{E} \arrow[ur, bend left, "\mathcal{R}_G(F)"] \arrow[ur, bend right, "H"'] \arrow[ur, phantom, "\Leftarrow\delta"] &
\end{tikzcd}$$

commute, i.e. $\epsilon \circ (\delta G) = \xi$. In other words, $(\mathcal{R}_G(F), \eta)$ is terminal among all the extensions.

Back to our setting, the categories are categories of complexes, with their localisations. To make things actually work, we need a bit more definitions.

Definition. Given a complex $X_\bullet$ in an abelian category $\mathcal{A}$, its projective Cartan-Eilenberg resolution is an exact sequence $P_{\bullet,\bullet}$ in $\mathbf{Com}_\bullet(\mathcal{A})$

$$\cdots \to P_{\bullet,q} \to \cdots \to P_{\bullet,1} \to \cdots \to P_{\bullet,0} \to \cdots \to X_\bullet \to 0$$

s.t. all of the following sequences

$$\begin{aligned}
\cdots \to P_{p,q} \to \cdots \to P_{p,1} \to P_{p,0} \to X_p \to 0 \\
\cdots \to Z(P)_{p,q} \to \cdots \to Z(P)_{p,1} \to Z(P)_{p,0} \to X_p \to 0 \\
\cdots \to B(P)_{p,q} \to \cdots \to B(P)_{p,1} \to B(P)_{p,0} \to X_p \to 0 \\
\cdots \to H(P)_{p,q} \to \cdots \to H(P)_{p,1} \to H(P)_{p,0} \to X_p \to 0
\end{aligned}$$

are projective resolutions for all p.

Proposition 4. *If $\mathcal{A}$ is an abelian category with enough projectives (or injectives), then every complex $X_\bullet$ in $\mathbf{Com}_\bullet(\mathcal{A})$ has a Cartan-Eilenberg projective (or injective) resolution.*

Thus given any complex of R-modules $M_\bullet, N_\bullet \in \mathbf{Com}_{\geqslant 0}(R-\mathbf{Mod})$, the hyper-tor $\mathbb{T}\mathrm{or}(M,N)$ is defined to be

$$\mathbb{T}\mathrm{or}_i(M,N) := H_i(\mathrm{Tot}(P_{\bullet,\bullet} \otimes_R Q_{\bullet,\bullet})),$$

where $P_{\bullet,\bullet} \to M_\bullet$ and $Q_{\bullet,\bullet} \to N_\bullet$ are Cartan-Eilenberg resolutions.

Corollary 4.1. *1. The functor* $\mathbb{T}\mathrm{or}$ *is a universal* δ*-functor.*

2. When $M_\bullet, N_\bullet$ *are concentrated at* 0*,* $\mathbb{T}\mathrm{or}$ *goes back to* Tor.

3. The functor $\mathbb{T}\mathrm{or}$ *is a right Kan extension.*

Slogan 2. The derived functor is the functor closest to the original functor among all preserving quasi-isomorphisms.

Now we have new problems - All the results work over abelian categories. However there are other functors that behave like δ-functors, which is not defined over abelian categories.

Example 2. *Consider in the category* $\mathbf{Top}$*, for any CW pair* (X, A)*, there is an induced long exact sequence*

$$\cdots \to \tilde{H}_i(A) \to \tilde{H}_i(X) \to \tilde{H}_i(X, A) \to \tilde{H}_{i-1}(A) \to \cdots .$$

Despite the proof being almost the same as Tor *being a* δ*-functor, (reduced) singular homology functor is never a derived functor, simply because* $\mathbf{Top}$ *is not linear.*

3 Deriving a functor in a more general setting

Definition. A model category is a category together with collections of morphisms $\mathrm{WE}, \mathrm{Fib}$ and Cof, called weak equivalences, fibrations, and cofibrations, satisfying

1. The category $\mathcal{M}$ is (finitely) complete and cocomplete.
2. 2-out-of-3 property: Given any morphism f, g and $g \circ f$ s.t. two of them are WE, then so is the third.
3. If f is the retract of g, and if g belongs to any of W, Fib or Cof, then so is f.
4. Given any commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, hook, "i"'] & E \arrow[d, two heads, "p"] \\ X \arrow[r, "f"] \arrow[ur, dashed, "h"] & B, \end{tikzcd}$$

   where i is a cofibration and p is a fibration, with either i or p is a WE., then there exists such a lift $h : C \to B$ making the diagram commutative.

5. Any morphism $f : A \to B$ in $\mathcal{M}$ has decompositions $f = q \circ i = p \circ j$, where p, q are fibrations, i, j are cofibrations, and i, p are weak equivalences.

Example 3. *The category* $\mathbf{Top}$ *has a model structure with*

1. W *consists of weak homotopy equivalences.*

2. Fib *consists of Serre fibrations.*

3. Cof *consists of retracts of relative cell attachment.*

Example 4. *The category* $\mathbf{Com}_{\geqslant 0}(\mathbf{Ab})$ *has a projective model structure with*

1. W *consists of quasi-isomorphisms.*

2. Fib *consists of levelwise surjections.*

3. Cof *consists of levelwise injections with projective cokernels.*

Proposition 5. *For any R-module M, N,*

$$\mathrm{Ext}^i(M, N) \cong \mathrm{Hom}_{\mathrm{Ho}\ \mathbf{Com}_{\geqslant 0}(\mathbf{Ab})}(M, N[i]).$$

Example 5. *The category $\mathbf{\Delta}$ is the category consists of $[n]$, where $[n]$ is the small category $\{0 \to 1 \to \cdots \to n\}$. The category $s\mathbf{Set}$ is defined to be $\mathrm{Funct}(\mathbf{\Delta}^\circ, \mathbf{Set})$.*

There is a model structure on $s\mathbf{Set}$ satisfying

1. *W consists of Quillen weak equivalences.*
2. Cof *consists of all levelwise injections.*
3. Fib *consists of all Kan fibrations.*

Definition. Given model categories $\mathcal{M}, \mathcal{N}$ and adjunction

$$F : \mathcal{M} \leftrightarrows \mathcal{N} : G,$$

they are called a Quillen pair if any of the following (hence all) is satisfied:

1. F sends cofibrations in $\mathcal{M}$ to cofibrations in $\mathcal{N}$, and sends acyclic cofibrations in $\mathcal{M}$ to acyclic cofibrations in $\mathcal{N}$.
2. G sends fibrations in $\mathcal{N}$ to fibrations in $\mathcal{M}$, and sends acyclic fibrations in $\mathcal{N}$ to acyclic fibrations in $\mathcal{M}$.
3. F sends cofibrations in $\mathcal{M}$ to cofibrations in $\mathcal{N}$, and G sends fibrations in $\mathcal{N}$ to fibrations in $\mathcal{M}$.

Theorem 6. *Given Quillen pair $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$, then the left (total) derived functor $\mathbb{L}F$ of F and the right (total) derived functor $\mathbb{R}G$ of G exist, and*

$$\mathbf{L}F : \mathrm{Ho}\ \mathcal{M} \leftrightarrows \mathrm{Ho}\ \mathcal{N} : \mathbf{R}G$$

form an adjunction. For any $X \in \mathrm{Ho}\ \mathcal{M}$ and $f \in \mathrm{hom}_{\mathrm{Ho}\ \mathcal{M}}(X, Y)$,

$$\mathbf{L}F(X) = \gamma F(QX),\ \mathbf{L}F(f) = \gamma F(Qf),$$

where $QX$is the cofibrant replacement of X and Qf is the cofibrant replacement of f.

Example 6. *Given a commutative ring R, then the adjunction*

$$- \otimes_R N : R - \mathbf{Mod} \leftrightarrows R - \mathbf{Mod} : \mathrm{Hom}_R(N, -)$$

gives rise to an adjunction

$$- \otimes_R N : \mathbf{Com}_{\geqslant 0}(R - \mathbf{Mod}) \leftrightarrows \mathbf{Com}_{\geqslant 0}(R - \mathbf{Mod}) : \mathrm{Hom}_R(N, -),$$

which forms a Quillen pair.

Given R-modules M, N. By Theorem 6, $- \otimes^{\mathbb{L}}_R N(M) := \mathbb{L}(- \otimes_R N)(M)$ is $QM \otimes_R N$, where QM is the cofibrant replacement of M. Unravel this, $QM \to M$ is a quasi-isomorphism and $0 \to QM$ is a levelwise injection with projective cokernel. This is exactly saying Tor *is in this sense the derived functor.*

Definition. Given model categories $\mathcal{M}, \mathcal{N}$ and Quillen pair

$$F : \mathcal{M} \leftrightarrows \mathcal{N} : G,$$

they are called a Quillen equivalence if

$$f^{\#} : A \to G(X)$$

is a weak equivalence iff

$$f^{\flat} : F(A) \to X$$

is a weak equivalence, where A is a cofibrant object in $\mathcal{M}$ and X is a fibrant object in $\mathcal{N}$.

Corollary 6.1. *If $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$ is a Quillen equivalence, then the adjunction*

$$\mathbf{L}F : \operatorname{Ho}\mathcal{M} \leftrightarrows \operatorname{Ho}\mathcal{N} : \mathbf{R}G$$

is an equivalence of categories.

Example 7. *The adjunction*

$$|-| : s\mathbf{Set} \rightleftarrows \mathbf{Top} : S$$

is a Quillen equivalence. Hence the study of spaces up to weak homotopy equivalence is the same as the study of simplicial sets.

Definition. Given a category $\mathcal{C}$, an object A is called an abelian group object if the functor

$$\hom_{\mathcal{C}^\circ}(-, A) \to \mathbf{Set}$$

factors through the category $\mathbf{Ab}$. The sub-category of $\mathcal{C}$ consisting of abelian group objects is denoted by $\mathcal{C}_{\mathrm{ab}}$.

Example 8. *For the category* $\mathbf{Set}$*,* $\mathbf{Set}_{\mathrm{ab}} = \mathbf{Ab}$ *and there is an adjunction*

$$\mathbb{Z}[-] : \mathbf{Set} \leftrightarrows \mathbf{Ab} : U$$

extending to a Quillen pair

$$\mathbb{Z}[-] : s\mathbf{Set} \leftrightarrows s\mathbf{Ab} : U.$$

Then by Theorem 6, $\mathbb{LZ}[-]$ *exists, and it is* $\mathbb{Z}[-]$ *applied onto cofibrant replacement. Since all objects in* $s\mathbf{Set}$ *are cofibrant,*

$$\mathbb{LZ}[-] = \mathbb{Z}[-]$$

and the composition

$$\mathbf{Top} \xrightarrow{S} s\mathbf{Set} \xrightarrow{\mathbb{Z}[-]} s\mathbf{Ab}$$

realises the singular homology functor as a derived functor.

4 Cotangent complexes and DAG

Example 9. *Given a commutative algebra R. For the category $R - \mathbf{Alg}$, $(R - \mathbf{Alg})_{\mathrm{ab}} = \{0\}$. Instead, one has*

$$\operatorname{Hom}_{R-\mathbf{Alg}_{/A}}(X, A \circlearrowright M) \cong \operatorname{Der}_R(X, M) \cong \operatorname{Hom}_A(A \otimes_X \Omega_{X/R}, M)$$

for any $A \in R - \mathbf{Alg}$ and A-module M, where $A \circlearrowright M$ is the R-algebra with multiplication $(a, m)(b, n) = (ab, an + bm)$.

This gives rise to an adjunction

$$A \otimes_R \Omega_{-/R} : R - \mathbf{Alg}_{/A} \leftrightarrows R - \mathbf{Mod} : A \circlearrowright -,$$

which means the category of abelian objects in $R - \mathbf{Alg}_{/A}$ is A-modules. The adjunction could be extended naturally to a Quillen pair

$$A \otimes_R \Omega_{-/R} : s(R - \mathbf{Alg}_{/A}) \leftrightarrows s(R - \mathbf{Mod}) : A \circlearrowright -,$$

where given any R-algebra X over A, the left derived functor of $A \otimes_R \Omega_{-/R}$ is called the cotangent complex.

Lemma 1. *Using the same setting as Example 9, one has*

$$\operatorname{Hom}_{R-\mathbf{Alg}_{/A}}(X, A \circlearrowright M) \cong \operatorname{Der}_R(X, M) \cong \operatorname{Hom}_A(A \otimes_R \Omega_{X/R}, M).$$

Proof. 1. The R-algebra structure on $A \circlearrowright M$ is $r \mapsto (r, 0)$, and the $/A$ structure is $(a, m) \mapsto a$. Since M is an A-module, it is automatically an $A \circlearrowright M$-module via the structure map, so

$$(a, n) \cdot m := a \cdot m.$$

2. Given any algebra map $f : X \to A \circlearrowright M, x \mapsto (a, m) = f(x)$, construct

$$\begin{aligned} D_f : X &\to M \\ x &\mapsto m = (f(x))_2, \end{aligned}$$

then we need to verify that this is a derivation. On the one hand,

$$D_f(xy) = (f(x)f(y))_2 = an + bm = (a, m) \cdot n + (b, n) \cdot m = xD_f(y) + yD_f(x),$$

and on the other hand, if x is in the image of R under the structure map $\alpha : R \to X$,

$$D_f(x) = (\alpha(r), 0)_2 = 0,$$

hence D_f is an R-derivation.

3. Given any R-derivation $D : X \to M$, define

$$\begin{aligned} f^D : X &\to A \circlearrowright M \\ x &\mapsto (g(x), D(x)) \end{aligned}$$

where $g : X \to A$ is the structure map. One could easily verify that this is an algebra map.

4. By commutative algebra,

$$\mathrm{Der}_R(X, M) \cong \mathrm{Hom}_X(\Omega_{X/R}, M),$$

hence the last isomorphism.

$\square$

Definition. The simplicial S-module

$$\mathcal{L}_{S/R} := \Omega_{P_*/R} \otimes_{P_*} S$$

is called the cotangent complex of S/R, where $P_* \to R$ is a cofibrant replacement of S over R.

Maybe one would like to ask where does the story start. Given a sequence of algebras $R \to A \to B$, there is an exact sequence

$$B \otimes_A \Omega_{A/R} \to \Omega_{B/R} \to \Omega_{B/A} \to 0.$$

This similar behaviour suggests us to think about the extended exact sequence. The left derived functor actually does this, which means there is a long exact sequence of B-modules

$$H_i(B \otimes_A L_{A/R}) \to H_i(L_{B/R}) \to H_i(L_{B/A}) \to H_{i-1}(B \otimes_A L_{A/R}).$$

Theorem 7. *Let $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$ be an adjoint pair and suppose $\mathcal{M}$ is a cofibrantly generated model category. Let I and J be chosen sets of generating cofibrations and acyclic cofibrations, respectively. Define a morphism $f : X \to Y$ in $\mathcal{N}$ to be a weak equivalence or a fibration if $G(f)$ is a weak equivalence or fibration in $\mathcal{M}$. Suppose further that*

1. *the right adjoint $G : \mathcal{N} \to \mathcal{M}$ commutes with sequential colimits; and*

2. *every cofibration in $\mathcal{N}$ with the LLP with respect to all fibrations is a weak equivalence.*

Then $\mathcal{N}$ becomes a cofibrantly generated model category. Furthermore the sets $\{F(i) \mid i \in I\}$ and $\{F(j) \mid j \in J\}$ generate the cofibrations and the acyclic cofibrations of $\mathcal{N}$ respectively.

Corollary 7.1. *Given a field k of characteristic 0, there is a (big) model category structure on cdga with weak equivalences the quasi-isomorphisms and fibrations the degree-wise surjections making the adjunction*

$$\mathrm{Sym} : \mathbf{Com}_{\geqslant 0}(k) \leftrightarrows \mathrm{CDGA}_k : U$$

a Quillen adjunction, where Sym *is the free commutative algebra functor and U is the forgetful functor.*

Furthermore there is a Quillen equivalence

$$N : s(k - \mathbf{CA}) \simeq \mathrm{CDGA}_k : \Gamma$$

where N is the normalisation functor.

This means that we have two equivalent models of derived algebraic geometry. From now on we do not see differences between DAG's and simplicial algebras.

Example 10. *Consider $A = k[x,y]/(y^2 - x^3)$, then*

$$P := k[x,y][T, dT = y^2 - x^3]$$

is a cofibrant replacement of A. Thus by the definition,

$$\mathcal{L}_{S/R} = (\Omega_{k[x,y]/k} \xrightarrow{\Omega_d} \Omega_{k[x,y]/k}) \otimes_P A.$$

Notice that $d = (y^2 - x^3)$ and $\Omega_d = 2y - 3x^2$, one has

$$\mathcal{L}_{S/R} = Adx \oplus Ady \xrightarrow{2y-3x^2} Adx \oplus Ady.$$

Thus $H_0 = \Omega_{A/k}$ and $H_1 = 0$.

Definition. A derived scheme is a pair $(X, \mathscr{O}_X)$, where X is a topological space and $\mathscr{O}_X$ is a sheaf of simplicial algebras over X s.t.

1. $(X, \pi_0(\mathscr{O}_X))$ is a classical scheme.
2. $\pi_i(\mathscr{O}_X)$ is a quasi-coherent sheaf over X for all $i > 0$.

Example 11. *The cotangent complex is "naturally derived".*

Let X be a scheme over a given field k and $x \in X(k)$. Put $\mathcal{L}_{X,x} := x^* \mathcal{L}_{X/k}$, then

$$\mathrm{Ext}^0(\mathcal{L}_{X,x}, k) = \mathrm{Hom}_{\mathbf{Sch}_*}(D_0, (X,x))$$

where $D_0 := \mathrm{Spec}\ k[\epsilon] = \mathrm{Spec}\ k[t]/t^2$. However, for other $\mathrm{Ext}^i(\mathcal{L}_{X,x}, k)$ where $i > 0$, they are not representable in the category $\mathbf{Sch}_*$. But if we enlarge the category, then they are representable

$$\mathrm{Ext}^i(\mathcal{L}_{X,x}, k) \cong \mathbb{R}\mathrm{Hom}(D_i, (X,x))$$

where $\mathrm{Hom}_{\mathrm{Ho}\ s\mathbf{Sch}_*}(D_i, (X,x))$ and $D_i = \mathbb{R}\mathrm{Spec}\ k \oplus k[i]$.

This result also has a global version.

Example 12. *A scheme X/S is a functor*

$$X : \mathbf{Sch}^\circ \to \mathbf{Set}$$

that is (co)representable. Equivalently, it is a functor

$$X : \mathbf{Alg} \to \mathbf{Set}$$

with some (weaker) representability.

One direction of generalisation is to include automorphisms, to have stacks and algebraic stacks, where in this way, a stack is a functor

$$\mathbf{Sch} \to \mathbf{Gpd} \hookrightarrow \mathbf{Cat}$$

with descent.

If we consider $\mathbf{Cat}$ is the category with 2-information, we should extend it to higher level. A derived scheme in this sense is a functor

$$s\mathbf{Sch}^\circ \to s\mathbf{Set}$$

that is representable, so are derived algebraic stacks. We will not use these generalisations - which requires the language of infinity categories.

5 Derived representation variety

Let V be a k-vector space. When we say "a representation V", there are generally three settings:

1. For a discrete group π (later we shall use G to denote an algebraic group, and here we use Greek letters to denote a discrete group), a representation of π is a group homomorphism

$$\pi \to \mathrm{GL}(V),$$

2. for a k-algebra A, a representation is an algebra homomorphism

$$A \to \mathrm{End}_k(V),$$

3. and for a Lie algebra $\mathfrak{g}$, a representation is a Lie-algebra homomorphism

$$\mathfrak{g} \to \mathrm{End}(V).$$

For any case, we can construct a scheme (we shall show later that these are varieties), universally parametrizing all representations. Precisely, we have theorems

Theorem 8. *Given a (discrete) group π, the functor*

$$\begin{aligned} \mathrm{Rep}_G(\pi) : k - \mathbf{CommAlg} &\to \mathbf{Set} \\ A &\mapsto \mathrm{Hom}_{\mathbf{Gp}}(\pi, G(A)) \end{aligned}$$

is representable. The representative is denoted by $(\pi)_G$.

This gives a functor

$$(-)_G : \mathbf{Gp} \to k - \mathbf{CommAlg},$$

which is the left adjunction of $G : k - \mathbf{CommAlg} \to \mathbf{Gp}$. In other words, there is a natural isomorphism

$$\hom_{\boldsymbol{k}-\mathbf{CommAlg}}((\pi)_G, A) \cong \hom_{\mathbf{Gp}}(\pi, G(A))$$

for any commutative k-algebra A.

Theorem 9. *Let A be a finitely generated unital associative algebra over k. The functor*

$$\begin{aligned} \mathrm{Rep}_V^A : k - \mathbf{CommAlg} &\to \mathbf{Set} \\ B &\mapsto \mathrm{Hom}_{k-\mathbf{Alg}}(A, \mathrm{End}(V) \otimes B) \end{aligned}$$

is representable. The representative, denoted by A_V, is called the representation scheme *of A in V.*

Similarly this means the functor $\mathrm{End}(V) \otimes -$ admits a left adjunction

$$(-)_V : k - \mathbf{Alg} \to k - \mathbf{CommAlg}.$$

Example 13. *Let $\pi = \langle x, y \mid x^2 = y^3 \rangle$, and let $G = SL_2 = \mathrm{Spec}\ k[a, b, c, d]/(ad - bc - 1)$. Then*

$$(\pi)_{SL_2} \cong \frac{\mathrm{Spec}\ k[a, b, c, d, e, f, g, h]}{ad - bc - 1, eh - fg - 1, X^2 Y^{-3}}.$$

All the results could be extended to higher levels:

Theorem 10. *The extended adjunction*

$$(-)_V : s(k - \mathbf{Alg}) \to s(k - \mathbf{CommAlg}) : \mathrm{End}(V) \otimes -$$

is a Quillen adjunction so $(-)_V$ admits a left derived functor $\mathbb{L}(-)_V$, which is called the derived representation variety.

Theorem 11. *The extended adjunction*

$$(-)_G : s\mathbf{Gp} \to s(k - \mathbf{CommAlg}) : G$$

is not a Quillen pair, but $(-)_G$ admits a left derived functor $\mathbb{L}(-)_G$, which commutes with homotopy colimits.

There is a Quillen adjunction

$$\mathbb{G} : s\mathbf{Set}_0 \leftrightarrows s\mathbf{Gp} : \bar{W}$$

where $s\mathbf{Set}_0$ is the (sub)category consisting of all reduced simplicial sets. The functor $\mathbb{G}$ is called Kan loop group construction, associating a simplicial group $\mathbb{G}(X)$ to a reduced simplicial set X_*.

The Quillen equivalence in Example 7 restricted on to $s\mathbf{Set}_0$ gives rise to another Quillen equivalence

$$|-| : s\mathbf{Set}_0 \rightleftarrows \mathbf{Top}_{*,0} : S$$

where $\mathbf{Top}_{*,0}$ denotes the (sub)category of all pointed, connected topological spaces. With all the equivalences, one have the derived representation variety defined as

$$\mathbb{L}(-)_G : \text{Ho } s\mathbf{Top}_{*,0} \to \text{Ho } s(k - \mathbf{CommAlg})$$

and given a (pointed connected) topological space X and a group scheme over k,

$$HR_i(X, G) := H_i(\mathbb{L}(X)_G)$$

is called the representation homology of X with coefficient G.

Theorem 12. *The derived representation functor $\mathbf{L}(-)_G$ preverses all (small) homotopy colimits.*

Example 14. *Let X be a link. Under the Artin representation, it could be presented by an element in the braid group*

$$B_n = \langle \sigma_1, \cdots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1 \rangle$$

of order n. On generators the action is

$$\begin{gathered}(\sigma_i)_* : \mathscr{O}(G^n) \to \mathscr{O}(G^n)\\ f(g_1, \cdots, g_i, g_{i+1}, \cdots, g_n) \mapsto f(g_1, \cdots, g_i g_{i+1} g_i^{-1}, g_i, \cdots, g_n).\end{gathered}$$

For a braid $\beta \in B_n$, let $\mathscr{O}(G^n)_\beta$ be the $\mathscr{O}(G^n)$-bimodule whose underlying vector space is $\mathscr{O}(G^n)$ itself, the left action is given by multiplication while the right action is twisted by the automorphism β_*. By Theorem 12,

$$\begin{aligned}\mathscr{O}[\mathrm{DRep}_G(\mathbb{R}^3 - L)] &\cong \mathrm{hocolim}[\mathscr{O}(G^n) \xleftarrow{(\beta_*, \mathrm{id})} \mathscr{O}(G^n) \otimes_k \mathscr{O}(G^n) \xrightarrow{(\mathrm{id}, \mathrm{id})} \mathscr{O}(G^n)]\\ &= \mathscr{O}(G^n) \otimes^{\mathbb{L}}_{\mathscr{O}(G^{2n})} \mathscr{O}(G^n)_\beta.\end{aligned}$$