K-Theory in Algebraic Geometry and Motivic Cohomology

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Inverse of a function

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There is a famous and beautiful in topology

Theorem 1 (Atiyah-Hirzebruch)**.** *For any finite dimensional CW complex X there is a spectral sequence*

$$E_2^{p,q} = H^p(X, \pi_q(BU)) \Rightarrow K^{p+q}(X)$$

where H^i takes the singular cohomology of a space.

We also have K-theories in algebraic geometry, so it is reasonable to have an analogy of Atiyah-Hirzebruch. Then the problem is find the correct notion of singular cohomology in algebraic geometry.

Definition. Let X be a variety. The **group of cycles** on X is the free abelian group

$$Z^*(X) := \mathbb{Z}\langle \text{closed subvarieties } Y \subset X \rangle,$$

which is naturally graded codimensionally. Elements in $Z^*(X)$ are called cycles.

The reason we wanted the group to be codimensionally graded is that we want a well-defined mutiplication over $Z^*(X)$, where the multiplication $[C][C']$ of two class represents the intersection $C \cap C'$. But closed subvarieties can intersect in the wrong dimension, for instance $C = C' = \operatorname{Spec} \mathbb{C}[x, y, z]/(x)$ in $\mathbb{CP}^2$. Classically the solution is the following

Definition. Let X be a k-variety. A **rational equivalence** between cycles $A, B \in Z^*(X)$ is a cycle $\gamma \in Z^*(X \times_k \mathbb{A}_k^1)$ s.t. $\gamma \cap X \times \{0\} = A$ and $\gamma \cap X \times \{1\} = B$.

Definition. The **Chow group** of X is the quotient

$$\mathrm{CH}^*(X) := Z^*(X)/\sim$$

where $\sim$ is the rational equivalence.

Notice that the Chow Group is naturally graded, by dimension. But here we write it codimensionally, where the reason was that we want to endow it with a multiplication.

Definition. We say that subvarieties A, B of a variety X **intersect transversely** at a point P if A, B and X are all smooth at P and the tangent spaces to A and B at P together span the tangent space to X, i.e.

$$T_P A + T_P B = T_P X,$$

or equivalent

$$\operatorname{codim}(T_P A \cap T_P B) = \operatorname{codim} T_P A + \operatorname{codim} T_P B.$$

Also, we say that two subvarieties $A, B \subseteq X$ are **generically transverse**, or that they **intersect generically transversely**, if they meet transversely at a general point of each component C of $A \cap B$.

The important thing is that we can perturb the subschemes a little bit so that they intersect properly.

Lemma 1 (Moving lemma)**.** *Let X be a smooth quasi-projective variety, then*

1. *For every $\alpha, \beta \in \mathrm{CH}^*(X)$, there are generically transverse cycles $A, B \in Z^*(X)$ with $[A] = \alpha$ and $[B] = \beta$.*
2. *The class $[A \cap B]$ is independent of the choice of such cycles A and B.*

Theorem 2. *For a smooth quasi-projective variety X, there is a unique product structure on $\mathrm{CH}^*(X)$ satisfying the condition that if two subvarieties A, B of X are generically transverse, then*

$$[A \cap B] = [A][B].$$

*The structure makes $\mathrm{CH}^*(X)$ a associative, commutative ring graded by codimension, called the **Chow ring** of X.*

Intuitively rational equivalence looks like homotopy equivalence between algebraic cycles. And Bloch [1] found that the Chow groups are actually determined by these homotopy information.

Theorem 3. *Let X be a variety, then there is a CW complex $Z^*(X,-)$ s.t. $\mathrm{CH}^*(X) = \pi_0(Z^*(X,-))$.*

Proof. Let
$$\Delta^n := \mathrm{Spec}\ k[x_0, \cdots, x_n]/(x_0 + \cdots + x_n - 1) \cong \mathbb{A}_k^n$$
be the standard algebraic n-simplex. By the contravariant functor Spec, we can form this a simplicial set. Take

$$z^*(X,n) \subseteq Z^*(X \times_k \Delta^n)$$

to be the abelian group generated by all closed subschemes meeting all faces $X \times_k \Delta^m \subset X \times_k \Delta^n$ properly. These groups are stable under pullback by the degeneracy maps between the standard algebraic simplices. We obtain in this way a simplicial complex of graded abelian groups
$$z^*(X,-) : \cdots z^*(X,1) \rightrightarrows z^*(X,0)$$
where $Z^*(X,-)$ is taken to be the geometric realization of $z^*(X,-)$. □

And it is naturally to see the higher homotopy groups of this space, where they were called the higher Chow groups by Bloch.

Definition. Let X be a variety, the **higher Chow group** is defined to be

$$\mathrm{CH}^*(X,i) := \pi_i(Z^*(X,-)).$$

3 The Motivic Cohomology

For the higher Chow group we defined, there are many good properties we can expect:

Theorem 4. *For a quasi-projective scheme X over a field k, the properties of $\mathrm{CH}^*(X,i)$ could have are:*

1. ***Functoriality**: Covariant for proper maps, contravariant for flat maps. Contravariant for arbitrary maps when X is smooth.*

2. ***Homotopy**: The pullback map*
$$\pi^* : Z^*(X,-) \to Z^*(X \times_k \mathbb{A}_k^1, -)$$
is a quasi-isomorphism.

3. ***Localization**: For Y a closed subscheme of X with pure codimension d, there is a long exact sequence*
$$\cdots \to \mathrm{CH}^*(X-Y, n+1) \to \mathrm{CH}^{*-d}(Y,n) \to \mathrm{CH}^*(X,n) \to \mathrm{CH}^*(X-Y,n) \to \cdots.$$

4. ***Local-to-global Spectral Sequence**: If $\boldsymbol{z}_X^*(-)$ is the complex of Zariski sheaves concentrated in negative degrees given by $U \mapsto z^*(U,-)$, then*
$$\mathrm{CH}^*(X,i) \cong H^{-n}(X, \boldsymbol{z}_X^*(-)).$$
In particular, given $r \geq 0$ there is a spectral sequence
$$E_2^{p,q} = H^p(X, \mathbf{CH}^r(-q)) \Rightarrow \mathrm{CH}^r(X, -p-q)$$
where $\mathbf{CH}^r(-q)$ is the Zariski sheaf associated to the presheaf $U \mapsto \mathrm{CH}^r(q)$.

5. ***Multiplicativity**: There is a*
$$\mathrm{CH}^p(X,q) \otimes \mathrm{CH}^r(Y,s) \to \mathrm{CH}^{p+r}(X \times Y, q+s).$$
Pulling back along the diagonal induces a product structure
$$\mathrm{CH}^p(X,q) \otimes \mathrm{CH}^r(X,s) \to \mathrm{CH}^{p+r}(X, q+s).$$
when X is smooth.

6. ***Chern Classes**: For E a rank n vector bundle over X, there are well-defined operators $c_i(E) : \mathrm{CH}^p(X, q) \to \mathrm{CH}^{p+i}(X, q)$ for $1 \le i \le n$ with correct functoriality. In mparticular, writing ξ the first Chern class of $\mathscr{O}(1)$ on $\mathbb{P}(E) \xrightarrow{\pi} X$, one has the projective bundle theorem*

$$\left(\bigoplus_{i=0}^{n-1} \xi^i\right) \circ \pi^* : \mathrm{CH}^*(X, m) \simeq \mathrm{CH}^*(\mathbb{P}(E), m)$$

as well as the usual Chern class identity

$$\xi^n + c_1 \xi^{n-1} + \cdots + c_n = 0.$$

7. ***K-Theory**:* $\mathrm{CH}^p(X, q) \otimes \mathbb{Q} \cong \mathrm{gr}^p_\gamma G_q(X) \otimes \mathbb{Q}$.

8. ***Riemann-Roch**: For a quasi-projective scheme X over field k, the Riemann-Roch map τ*

$$\tau : G_m(X)_{\mathbb{Q}} \to \bigoplus_d \mathrm{CH}^d(X, m)_{\mathbb{Q}}$$

is an isomorphism.

The higher Chow groups behave really like a cohomology theory, but it is not. However, we can modify it to be a cohomology theory:

Definition. Let X be a smooth variety, then the **motivic cohomology groups** $H^p(X; \mathbb{Z}(q))$ for $p, q \in \mathbb{Z}$ are given by

$$H^p(X; \mathbb{Z}(q)) := \begin{cases} \mathrm{CH}^q(X, 2q - p), & q \ge 0 \text{ and } 2q - p \ge 0 \\ 0, & \text{otherwise.} \end{cases}$$

The (naive) definition of motivic cohomology fulfills our expectations a lot:

Theorem 5. *When k is a field,*

$$H^j(\mathrm{Spec}\, k, \mathbb{Z}(j)) = K^M_j(k)$$

where $K^M_j(k)$ is the j-th Milnor K-theory of a field.

Theorem 6. *For every smooth scheme X over a field, there is a spectral sequence analogous to the Atiyah-Hirzebruch spectral sequence in topology:*

$$E_2^{p,q} = H^p(X, \mathbb{Z}(q/2)) \Rightarrow K_{-p-q}(X).$$

And for an arbitrary scheme over a field, there is a spectral sequence from motivic homology to G-theory.

4 $\mathbb{A}^1$ Homotopy Theory????

5 The Category of Motives????

I'm disabled to talk about why motives are what we want to study in algebraic geometry. But the motivic cohomology turns out to be related with the theory of motives,

Suppose DM is the derived category of mixed motives.

6 Maps to Étale Cohomology

References

[1] Spencer Bloch. Algebraic cycles and higher K-theory. *Adv. in Math.*, 61(3):267–304, 1986.