An Algebraic Introduction to Representation Homology

Notes for A-Exam Presentation

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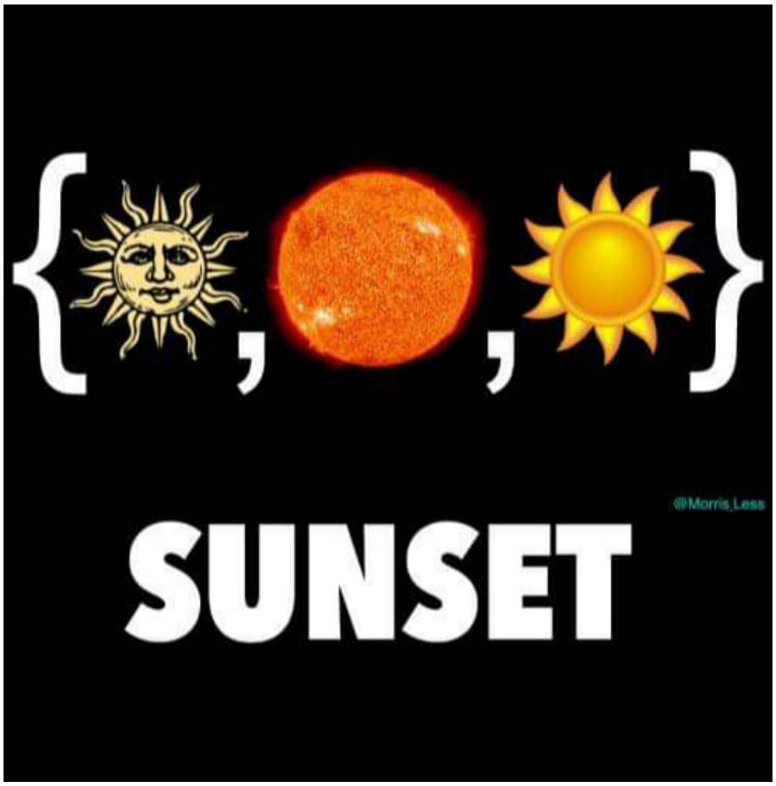

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Contents

1 Hochschild (Co)homology

Through out the talk, I shall use k to denote the ground commutative ring with unit. k-Algebras are unital, but not necessarily commutative. Non-unital k-algebras will be explicitly pointed out.

Definition. Given a k-algebra A and (A, A)-bimodule M, define

$$C_n(A, M) := M \otimes_k A^{\otimes n},$$

where $A^{\otimes n} := A \otimes_k \cdots \otimes_k A$ with the boundary maps

$$\begin{aligned} \partial_n : C_n(A, M) &\to C_{n-1}(A, M) \\ m \otimes a_1 \otimes \cdots \otimes a_n &\mapsto ma_1 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ &\quad + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}, \end{aligned}$$

then $(C_\bullet(A, M), \partial_\bullet)$ is called the Hochschild complex, whose homology group is called the Hochschild homology group of A with coefficients in M, denoted by $HH_\bullet(A, M)$. In particular, if $M = A$, we denote by $HH_\bullet(A)$ the Hochschild homology group.

Definition. Given k-algebra A with opposite algebra A°, let $A^e := A \otimes_k A^{\mathrm{op}}$ and define an action

$$(a \otimes b)m := amb$$

for any (A, A)-bimodule. Then the following is called the Bar complex:

$$C_\bullet^{\mathrm{bar}} : \cdots \xrightarrow{\partial_{n+1}^{\mathrm{bar}}} A^{\otimes n+1} \xrightarrow{\partial_n^{\mathrm{bar}}} A^{\otimes n} \xrightarrow{\partial_{n-1}^{\mathrm{bar}}} \cdots \xrightarrow{\partial_1^{\mathrm{bar}}} A^{\otimes 2} \to 0,$$

where $A^{\otimes 2}$ is of degree 0, and $\partial_n^{\mathrm{bar}} := \sum_{i=0}^{n-1} (-1)^i d_i$. The multiplication

$$\mu : A \otimes_k A \to A$$

gives an augmentation of $C_\bullet^{\mathrm{bar}}$.

Lemma 1.1. *If the given k-algebra A has a unit 1, then $(C_\bullet^{\mathrm{bar}}, \partial_\bullet^{\mathrm{bar}})$ is an augmentation of A as a complex of A-bimodules.*

Notice that this immediately implies that

$$HH_*(A) \cong H_*(M \otimes_{A^e} C_\bullet^{\mathrm{bar}}),$$

so one similarly defines the Hochschild cohomology by

$$HH^*(A, M) := H^*(\mathrm{Hom}_{A^e}(C_\bullet^{\mathrm{bar}}, M)),$$

Theorem 1.1. *For a unital k-algebra A, the augmented bar complex is a free A^e-module resolution of the A^e-module A.*

The idea of the proof is as follows: the Hochschild complex $(C_\bullet(A, M), \partial_\bullet)$ is pre-simplicial (with face maps satisfying $d_i^{[n]} d_j^{[n]} = d_{j-1}^{[n]} d_i^{[n]}$ for $i < j$); with the existence of unit actually gives degeneracy maps. These make $(C_\bullet(A, M), \partial_\bullet)$ a simplicial object where the simplicial identities give the desired result.

Corollary 1.1.1. *Given a unital k-algebra A, if A is a projective (flat) k-module, then for any A-bimodule M, there is a natural isomorphism*

$$HH_n(A, M) \cong \mathrm{Tor}_n^{A^e}(M, A).$$

Proof. □

So one might ask, what else could we get from the simplicial perspective point of view?

2 Higher Hochschild (Co)homology

From now on, let k be a field.

2.1 Categorical Reformulation

Let $\mathbf{FinSet}_*$ be the category of pointed finite sets $[n] := \{0, 1, \cdots, n\}$ with base point 0. Let A be a *commutative* k-algebra, with unit and let M be an A-module, considered as a symmetric (A, A)-bimodule. Following Loday, we define a functor $\mathcal{L}(A, M) : \mathbf{FinSet}_* \to k - \mathbf{Mod}$ by

$$[n] \mapsto M \otimes_k A^{\otimes n}.$$

For a pointed map $f : [n] \to [m]$, the action of f_* on $\mathcal{L}(A, M)$ is

$$f_*(a_0 \otimes \cdots \otimes a_n) := b_0 \otimes \cdots \otimes b_m \tag{1}$$

where

$$b_j := \prod_{f(i)=j} a_i$$

for $j = 0, \cdots, m$. (This is where the commutativity is used!) The reason why we want the finite set to be pointed is also here, where a_0 has to be mapped to the first position.

Furthermore one has the canonical embedding $\mathbf{FinSet}_* \hookrightarrow \mathbf{Set}_*$, so one can prolong the functor $\mathcal{L}(A, M)$ via the Kan extension

$$\begin{tikzcd} \mathbf{FinSet}_* \arrow[r, "\mathcal{L}(A,M)"] \arrow[d, hook] & k - \mathbf{Vect} \\ \mathbf{Set}_*, \arrow[ur, dashed] & \end{tikzcd}$$

more precisely,

$$\mathcal{L}(A, M)(X) := \operatorname{colim} \mathcal{L}(A, M)([n])$$

where the colimit is taken over all pointed sets inclusions $[n] \hookrightarrow X$.

Remark. $\mathcal{L}(A, M)$ can be generalized for a CDGA, where the functor $\mathcal{L}(A, M) : \mathbf{FinSet}_* \to k - \mathbf{Mod}$ on objects is

$$[n] \mapsto M \otimes_k A^{\otimes n},$$

and for a pointed map $f : [n] \to [m]$, the action of f_* on $\mathcal{L}(A, M)$ is

$$f_*(a_0 \otimes \cdots \otimes a_n) := (-1)^{\epsilon(f,a)} b_0 \otimes \cdots \otimes b_m \tag{2}$$

where $b_j := \prod_{f(i)=j} a_i$ for $j = 0, \cdots, m$ and

$$\epsilon(f, a) := \sum_{j=1}^{n-1} |a_j| \left(\sum_{k \in I_j} |a_k| \right)$$

where $I_j = \{k > j \mid 0 \le f(k) \le f(j)\}$.

Remark. The functor can be generalized to an arbitrary functor $F : \mathbf{FinSet}_* \to k - \mathbf{Vect}$, with the same construction.

In general, for any pointed simplicial set $X : \mathbf{\Delta}^\circ \to \mathbf{Set}_*$, one can define a simplicial k-vector space extending $\mathcal{L}(A, M)$ level-wisely

$$\mathbf{\Delta}^\circ \xrightarrow{X} \mathbf{Set}_* \xrightarrow{\mathcal{L}(A,M)_*} s(k - \mathbf{Vect}).$$

Then one can define *X-homology of A with coefficient in M* [?MR0339132] by

$$H_*^X(A, M) := \pi_*(\mathcal{L}(A, M))(X).$$

In particular,

Proposition 2.1. *For the pointed simplicial set S^1, $H_*^{S^1}(A,M)$ is exactly the Hochschild homology.*

Proof. Let's take the simplicial model S^1 to be $\mathbf{\Delta}^{[1]}/\{0,1\}$. Then

$$(S^1)_k = \{(0,\cdots,0,1,\cdots,1)\}/(0,\cdots,0)\sim(1,\cdots,1)$$

(we regard $(0,\cdots,0,1,\cdots,1)$ with i 0's as i) with face maps

$$\begin{aligned} d_i^{[k]} : (S^1)_k &\to (S^1)_{k-1} \\ (a_0,\cdots,a_k) &\mapsto (a_0,\cdots,\hat{a}_i,\cdots,a_k) \end{aligned}$$

and degeneracy maps

$$\begin{aligned} s_j^{[k]} : (S^1)_k &\to (S^1)_{k+1} \\ (a_0,\cdots,a_k) &\mapsto (a_0,\cdots,a_j,a_j,a_{j+1},\cdots,a_k). \end{aligned}$$

Apply the functor $\mathcal{L}(A,M)$, we find exactly $\mathcal{L}(A,M)(d_i)$ gives

$$m\otimes a_1\otimes\cdots\otimes a_n \mapsto m\otimes a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n$$

and the last term is guaranteed by the quotient. □

Remark. For another model $S^1 = B\mathbb{Z}$, huge

$$\begin{aligned} d_i : \mathbb{Z}^n &\to \mathbb{Z}^{n-1} \\ (g_1,\cdots,g_n) &\mapsto \begin{cases} (g_2,\cdots,g_n) & i=0 \\ (g_1,\cdots,g_i+g_{i+1},\cdots,g_n) & 0<i<n \\ (g_1,\cdots,g_{n-1}) & i=n \end{cases} \end{aligned}$$

Lemma 2.1. *The homology group $H_*^X(A,M)$ depends only on the homotopy type of X.*

Proof Sketch. There is a 'fundamental SS'

$$E^2_{p,q} = \mathrm{Tor}_p^{\mathbf{FinSet}_*}(\mathcal{J}_q(H_*X),F) \Rightarrow \pi_{p+q}(F(X)),$$

implying that for any map $X\to Y$ inducing an isomorphism $H_*X\to H_*Y$, there is an isomorphism $\pi_*(F(X))\to\pi_*(F(Y))$. □

2.2 Higher Hochschild Homology

Definition. The S^d-homology of A with coefficient in M

$$H_*^{S^d}(A,M) = \pi_*(\mathcal{L}(A,M))(S^d)$$

is called the d-th higher Hochschild homology, or d-th Pirashvili-Hochschild homology, denoted by $HH_*^{[d]}(A,M)$.

Example 2.1. *We take the standard simplicial model for S^n, where in dimension $0<i<d$, there is no non-degenerate simplices, so*

$$HH_0^{[d]}(A,M)\cong M$$

and

$$HH_i^{[d]}(A,M) = 0$$

for all $0<i<d$.

Example 2.2. *There is always a stable*

$$HH_d^{[d]}(A,M)\cong HH_1^{[1]}(A,M)\cong \Omega^1_A\otimes M.$$

Actually this holds for a large class of functors.

3 Generalization: Where are the Problems?

3.1 Topological Interpretation

Since the construction is defined only for commutative algebras, people have made efforts to generalize the definition.

Pirashvili himself generalized this higher Hochschild homology for non-commutative algebras, using a combinatorial construction called ordered simplicial sets. ?????

However, the good thing is that, the category $\mathbf{Set}_*$ or $\mathbf{Set}$ has good correspondence to topologies, but not for .....

Theorem 3.1. *There is a pair of adjunction*

$$\mathbb{G} : s\mathbf{Set}_0 \leftrightarrows s\mathbf{Gp} : \overline{W}$$

where $\mathbb{G}$ is called the Kan loop group construction and $\overline{W}G$ is the classfying simplicial complex.

Actually the functor $\mathbb{G}$ preserves weak equivalences and cofibrations, and the functor $\overline{W}$ preserves weak equivalences and fibrations. Thus this is a pair of Quillen equivalence, which gives an equivalence of homotopy categories

$$\text{Ho } s\mathbf{Set}_0 \simeq \text{Ho } s\mathbf{Gp}.$$

The detailed construction is as follows: Given a reduced simplicial set X, the set of n-simplicies is

$$\mathbb{G}X_n := \langle X_{n+1}\rangle / \langle s_0(x) = 1, \forall x \in X_n\rangle \cong \langle B_n\rangle,$$

where $B_n := X_{n+1} - s_0(X_n)$ and the isomorphism is induced by the inclusion $B_n \hookrightarrow X_n$. The degeneracy maps $s_j^{\mathbb{G}X} : \mathbb{G}X_n \to \mathbb{G}X_{n+1}$ are induced by $s_{j+1} : X_{n+1} \to X_{n+2}$, and the face maps $d_i^{\mathbb{G}X} : \mathbb{G}X_n \to \mathbb{G}X_{n-1}$ are given by

$$d_i^{\mathbb{G}X}(x) := \begin{cases} d_1(x) \cdot (d_)(x))^{-1} & i = 0 \\ d_{i+1}(x) & \text{otherwise.} \end{cases}$$

Corollary 3.1.1. *The Kan loop group construction $\mathbb{G}X$ is semi-free.*

For the other direction,

$$WG_n := G_n \times G_{n-1} \times \cdots \times G_0,$$

and

$$d_i(g_n, g_{n-1}, \cdots, g_0) = \begin{cases} (d_i g_n, d_{i-1}g_{n-1}, \cdots, (d_0 g_{n-1})g_{n-i-1}, g_{n-i-2}, \cdots, g_0) & i < n \\ (d_n g_n, d_{n-1}g_{n-1}, \cdots, d_1 g_1) & i = n \end{cases}$$

$$s_i(g_n, g_{n-1}, \cdots, g_0) = (s_i g_n, s_{i-1}g_{n-1}, \cdots, s_0 g_{n-i}, e, g_{n-i-1}, \cdots, g_0)$$

There is an action $G \times WG \to WG$

$$(h, (g_n, g_{n-1}, \cdots, g_0)) \mapsto (hg_n, g_{n-1}, \cdots, g_0)$$

and $\overline{W}G := WG/G$.

Theorem 3.2. *For any reduced simplicial set X, there is a weak equivalence*

$$|\mathbb{G}X| \simeq \Omega|X|.$$

Proposition 3.3. *Given any pointed simplicial set X, the Eilenberg subcomplex*

$$\overline{S}_n(X) := \{f : \Delta^n \to X \mid f(v_i) = * \textit{ for all vertices } v_i \textit{ of } \Delta^n\}$$

gives rise to a pair of Quillen equivalence

$$|-| : \mathbf{sSet}_0 \leftrightarrows \mathbf{Top}_{0,*} : \overline{S}.$$

3.2 Main Definition

Let G be an affine group scheme over k.

Lemma 3.1. *Given a (discrete) group Γ, the functor*

$$\begin{aligned}\mathrm{Rep}_G(\Gamma) : k - \mathbf{CommAlg} &\to \mathbf{Set}\\ A &\mapsto \mathrm{Hom}_{\mathbf{Gp}}(\Gamma, G(A))\end{aligned}$$

is representable. The representative is denoted by $(\Gamma)_G$.

This gives a functor

$$(-)_G : \mathbf{Gp} \to k - \mathbf{CommAlg},$$

which is the left adjunction of $G : k - \mathbf{CommAlg} \to \mathbf{Gp}$.

Now that we have a functor $(-)_G : \mathbf{Gp} \to k - \mathbf{CommAlg}$, we can extend the functor to be a functor

$$s\mathbf{Gp} \to s(k - \mathbf{CommAlg}) \tag{3}$$

level-wisely, still denoted by $(-)_G$.

Lemma 3.2. *The functor $(-)_G$ maps weak equivalences between cofibrant objects in $s\mathbf{Gp}$ to weak equivalences in $s(k - \mathbf{CommAlg})$, and hence has a total left derived functor.*

Proof. All objects in $s\mathbf{Gp}$ are fibrant, so for any weak equivalence $f : G \to H$ between cofibrant objects, there is a homotopy inverse $g : H \to G$ by Whitehead theorem. $s\mathbf{Gp}$ is a simplicial model category, the (left) homotopy can be realized via a good cylinder object which can be taken naturally via $\otimes I$. The simplicial relations are preserved by $(-)_G$, so $(f)_G$ and $(g)_G$ are mutually inverse in Ho $s(k - \mathbf{CommAlg})$. $\square$

Remark.

For a fixed simplicial group $\Gamma \in s\mathbf{Gp}$, one can formally define the representation homology of Γ in G

$$HR_*(\Gamma, G) := \pi_* \mathbb{L}(\Gamma)_G,$$

where $\mathrm{DRep}_G(\Gamma) := \mathrm{Spec}\, \mathbb{L}(\Gamma)_G$ is called the representation scheme.

Definition. For a space $X \in \mathbf{Top}_{0,*}$, the *derived representation scheme* $\mathrm{DRep}_G(X)$ is $\mathrm{Spec}\ \mathrm{DRep}_G(\Gamma X)$, where ΓX is a(ny) simplicial group model of X. The *representation homology of X in G* is then

$$HR_*(X, G) := \pi_* \mathbb{L}(\Gamma X)_G. \tag{4}$$

Example 3.1. *Let $G = \mathbb{G}_a$ be the additive group. Then for any group $\Gamma \in \mathbf{Gr}$, one has*

$$\mathrm{Hom}_{\mathbf{Gr}}(\Gamma, \mathbb{G}_a(A)) = \mathrm{Hom}_{\mathbf{Gr}}(\Gamma_{\mathrm{ab}}, \mathbb{G}_a(A)) = \mathrm{Hom}_{k-\mathbf{CommAlg}}(\mathrm{Sym}(\Gamma_{\mathrm{ab}} \otimes_{\mathbb{Z}} k), A).$$

Also, $\mathbb{G}X$ is a canonical simplicial model for $|X|$, so

$$HR_*(X, G) \cong \pi_*(\mathbb{G}X_G).$$

Applying this we have

$$\begin{aligned}HR_*(X, \mathbb{G}_a) &\cong \pi_* \mathrm{Sym}((\mathbb{G}X)_{\mathrm{ab}} \otimes_{\mathbb{Z}} k)\\ &\cong \pi_* \mathrm{Sym}(\pi_*(\mathbb{G}X)_{\mathrm{ab}} \otimes_{\mathbb{Z}} k)\\ &\cong \pi_* \mathrm{Sym}(\pi_*(\mathbb{G}X)_{\mathrm{ab}} \cong H_{*+1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} k)\\ &\cong \pi_* \mathrm{Sym}(\pi_*(\mathbb{G}X)_{\mathrm{ab}} \cong H_{*+1}(X, k))\end{aligned}$$

where Sym *is the graded symmetric product and* $\pi_*(\mathbb{G}X)_{\mathrm{ab}} \cong H_{*+1}(X, \mathbb{Z})$*.*

Theorem 3.4. *The derived representation functor $\mathbf{L}(-)_G$ preverses all (small) homotopy colimits.*

4 What are Their Relations?

Let's bring up another definition, which gives us a new point of view of the representation homology, leading to the relation of representation homology and higher Hochschild homology.

Let $\mathfrak{G}$ be the full subcategory of $\mathbf{Gp}$ whose objects are the (finitely generated) free groups $\langle n\rangle = \langle x_1, \cdots, x_n\rangle$ for $n \geq 0$. Then any commutative Hopf algebra H gives a $\mathfrak{G}$-module

$$\begin{aligned} \mathfrak{G} &\to k - \mathbf{Vect} \\ \langle n\rangle &\mapsto H^{\otimes n}, \end{aligned}$$

which will be denoted by $\underline{H}$. Actually, the functor $\underline{H}$ takes values in the category of commutative algebras. Then consider the inclusion of categories $\mathfrak{G} \hookrightarrow \mathbf{FreeGp}$ where $\mathbf{FreeGp}$ is the full subcategory of all free groups, there is a Kan extension of $\underline{H}$ along the inclusion

$$\begin{tikzcd} \mathfrak{G} \arrow[r, "\underline{H}"] \arrow[d, hook, "i"] & k - \mathbf{Vect} \\ \mathbf{FreeGp} \arrow[ur, dashed, "\underline{H}"'] & \end{tikzcd}$$

also denoted by $\underline{H}$. Thus the composition of functors

$$\mathbf{\Delta}^{\circ} \xrightarrow{\mathbb{G}X} \mathbf{FreeGp} \xrightarrow{\underline{H}} \boldsymbol{k} - \mathbf{CommAlg}$$

defines a simplicial commutative algebra $\underline{H}(\mathbb{G}X)$ for any reduced simplicial set X.

Lemma 4.1. *The assignment $H \mapsto \underline{H}$ is an equivalence of the category of commutative Hopf algebras over k and the category $\mathfrak{G} - \mathbf{Mod}$.*

Proof. $\square$

Lemma 4.2. *The category $\mathfrak{G}$ is a strict monoidal category with $\otimes$ being free product s.t. $\langle n\rangle \otimes \langle m\rangle = \langle n+m\rangle$.*

Definition. The representation homology of X in H is defined by

$$\mathrm{HR}_*(X, H) := \pi_*(\underline{H}(\mathbb{G}X)).$$

Proposition 4.1. *Let G be an affine group scheme over k with coordinate ring $H = \mathscr{O}(G)$. Then for any $X \in \mathbf{Set}_0$, there is a natural isomorphism of graded commutative algebras*

$$\mathrm{HR}_*(X, H) \cong \mathrm{HR}_*(X, G).$$

In particular, $\mathrm{HR}_0(X, \mathscr{O}(G)) = \pi_1(X)_G$.

Proof. $\square$

Theorem 4.2. *For any commutative Hopf algebra H and any pointed simplicial set X, there is a natural isomorphism of graded commutative algebras*

$$HR_*(\Sigma X, H) \cong HH_*(X, H; k).$$

Theorem 4.3. *For any commutative Hopf algebra H and any simplicial set X, there is a natural isomorphism of graded commutative algebras*

$$HR_*(\Sigma(X_+), H) \cong HH_*(X, H).$$

There is a suspension functor defined by

$$\begin{aligned} \Sigma : \mathbf{sSet}_* &\to \mathbf{sSet}_0 \\ X &\mapsto C_*(X)/X \end{aligned}$$

where $C_*(X)$ is the reduced cone of X

$$C_*(X)_n := \{(x,m) \mid x \in X_{n-m}, 0 \le m \le n\}$$

with $(*,m) \sim *$. The structure maps are

$$\begin{aligned} d_i^{C_*[n]} &: C_*(X)_n \to C_*(X)_{n-1} \\ (x,m) &\mapsto \begin{cases} (x, m-1) & 0 \le i < m \\ (d_{i-m}^{X[n]}(x), m) & m \le i \le n \end{cases} \end{aligned}$$

and

$$\begin{aligned} s_j^{C_*[n]} &: C_*(X)_n \to C_*(X)_{n+1} \\ (x,m) &\mapsto \begin{cases} (x, m+1) & 0 \le j < m \\ (s_{j-m}^{X[n]}(x), m) & m \le j \le n \end{cases} \end{aligned}$$

where $d_1(x,1) = *$ holds for all $x \in X_0$.

Sketch proof. There are natural isomorphisms of groups $[\mathbb{G}\Sigma(X)_+]_n \cong \langle X_n \rangle$, with structure maps are compatible with those of X. Apply the functor one has

$$\underline{H}([\mathbb{G}\Sigma(X)_+]_*) \cong \underline{H}(\langle X \rangle) = X \otimes H.$$

$\square$

Example 4.1. *Let's consider when $X = T^2$ be the 2-torus. Notice that $T^2 = \mathrm{hocolim}(\{*\} \leftarrow S^1_c \xrightarrow{\alpha} S^1_a \vee S^1_b)$, then by applying the Kan loop group construction we have a simplicial group model for T^2*

$$\mathbb{G}(T^2) = \mathrm{hocolim}(\{*\} \leftarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} * \mathbb{Z}).$$

Take the functor $(-)_G$ and by Theorem 3.4,

$$\mathscr{O}(\mathrm{DRep}_G(T^2)) = \mathrm{hocolim}(k \leftarrow \mathscr{O}(G) \xrightarrow{\alpha_*} \mathscr{O}(G \times G)) \cong \mathscr{O}(G \times G) \otimes^{\boldsymbol{L}}_{\mathscr{O}(G)} k.$$

Therefore

$$\mathrm{HR}_*(T^2, G) \cong \mathrm{Tor}_*^{\mathscr{O}(G)}(\mathscr{O}(G \times G), k).$$

We consider the case where $G = \mathbb{G}_m = \mathrm{Spec}\, k[x, x^{-1}]$, then the map

$$\begin{aligned} \alpha_* &: \mathscr{O}(G) \to \mathscr{O}(G \times G) \\ f(x) &\mapsto f([y,z]) = f(1). \end{aligned}$$

The resolution $P_\bullet$ of k over $k[x, x^{-1}]$ satisfies $P_0 = k[x, x^{-1}]$, then the kernel of

$$k[x, x^{-1}] \to P_0 \twoheadrightarrow k$$

is $k[x, x^{-1}] \cdot (x-1)$, therefore $P_1 = k[x, x^{-1}] \cdot w$ where the differential $d : w \mapsto x - 1$. This is exactly the Kozsul complex.