1 Introduction

In algebraic geometry, we have to deal with singularities of varieties. The problem of **resolution of singularities** asks whether every algebraic variety X has a resolution, a non-singular variety Y with a proper birational map $f : Y \to X$. We already have different ways to resolve a singularity, and blowing up is one of the most important methods.

Usually $\mathbb{C}^n$ is a denotation geometrically, while $\mathbb{A}^n_{\mathbb{C}}$ is more algebraically. However we will regard them equivalently through this paper. $\mathbb{A}^n, \mathbb{P}^n$ are $\mathbb{A}^n_{\mathbb{C}}$ and $\mathbb{CP}^n$ respectively except indicated explicitly.

2 Construction: Geometry

2.1 Blow-up of $\mathbb{C}^n$ at Origin

We will start from the simplest case, i.e. blowing up the origin of $\mathbb{C}^n$.

Consider a subset of $\mathbb{C}^n \times \mathbb{P}^{n-1}$

$$\tilde{\mathbb{C}^n} := \{((a_1, \cdots, a_n), [b_1, \cdots, b_n]) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid a_i b_j = a_j b_i, 1 \leq i, j \leq n\}.$$

It is easy to see that we have a natural projection from $\tilde{\mathbb{C}^n}$ to $\mathbb{C}^n$:

$$\pi : ((a_1, \cdots, a_n), [b_1, \cdots, b_n]) \mapsto (a_1, \cdots, a_n).$$

Immediately we have these properties of $\tilde{\mathbb{C}^n}$:

For any point $0 \neq P \in \mathbb{C}^n$, there is at least a $1 \leq i_0 \leq n$ s.t. $a_{i_0} \neq 0$ hence $b_{i_0} \neq 0$. Thus the equations can be written as $\frac{b_j}{b_{i_0}} = \frac{a_j}{a_{i_0}}, 1 \leq j \leq n$. Therefore there is only **ONE** point x in $\tilde{\mathbb{C}^n}$ with $\pi(x) = P$, i.e. $x = ((a_1, \cdots, a_n), [a_1, \cdots, a_n])$.

$\pi^{-1}(0) \cong \mathbb{P}^{n-1}$. Indeed, any point $[b_1, \cdots, b_n] \in \mathbb{P}^n$ satisfies the equations if $(a_1, \cdots, a_n) = 0$.

The points of $\pi^{-1}(0)$ are in 1-1 corresponding to the lines through the origin in $\mathbb{C}^n$. This is just the explanation of points in projective spaces.

$\tilde{\mathbb{C}^n}$ is irreducible.

Definition. The set $\tilde{\mathbb{C}^n}$ along with the map $\pi : \tilde{\mathbb{C}^n} \to \mathbb{C}^n$ is called the *blow-up* of $\mathbb{C}^n$ at origin.

2.2 A Little Further

2.3 Blow-up of $\mathbb{C}^n$ at a Affine Variety

For an *affine variety*, we mean an **irreducible** algebraic subset of $\mathbb{A}^n$, denoted as

$$V(f_1, \cdots, f_k) = \{(a_1, \cdots, a_n) \in \mathbb{A}^n \mid f_i(a_1, \cdots, a_n) = 0 \ \ \forall 1 \leq i \leq k\}.$$

We can always find finitely many generators as a result of Hilbert basis theorem.

The blow-up of $\mathbb{A}^n$ with respect to the subvariety $V(f_1, \cdots, f_k)$ is given by

$$\{((a_1, \cdots, a_n), [b_1, \cdots, b_k]) \mid b_i f_j(a_1, \cdots, a_n) = b_j f_i(a_1, \cdots, a_n)\}$$

which is a subset of $\mathbb{A}^n \times \mathbb{P}^{k-1}$.

3 Construction: Algebra

3.1 Blow-up Algebra

Definition. Let R be a ring and let $I \subset R$ be an ideal of R. The *blow-up algebra* or *Rees algebra* , associated with the pair (R, I), is the graded R-algebra

$$\mathrm{Bl}_I(R) := \bigoplus_{n \geq 0} I^n = R \oplus I \oplus I^2 \oplus \cdots$$

3.2 Proj Construction

Having the corresponding ring of blow-up, we still need to realize the geometrical object by this blow-up algebra. This moment, the spectrum does not work since the blow-up should be 'projective'. The process can be described as Proj construction.

4 Comparison: Algebraic Variety and Scheme

We first give a full generalization of blow-up. Let X be a scheme, and let $\mathcal{I}$ be a coherent sheaf of ideals on X. We say the blow-up of X with respect to $\mathcal{I}$ is a scheme $\tilde{X}$ along with a morphism $\pi : \tilde{X} \to X$, such that $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\tilde{X}}$ is a invertible sheaf, with the universal property: for any scheme and morphism $f : Y \to X$ such that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ is a invertible sheaf, there is a unique factorization:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\pi"] & X \\ & Y \arrow[u, "f"'] \arrow[ul, "\tilde{f}"] \end{tikzcd}$$

Here we need to explain some

However instead of going that far, we just consider affine scheme, i.e. $X = \text{Spec } R$ for some commutative ring R.

The first example is blowing up the maximal ideal (x, y) of ring $\mathbb{C}[x, y]$. Consider the map $f : \mathbb{C} - V(x, y) \to \mathbb{C} \times \mathbb{P}^1, (a, b) \mapsto ((a, b), [a, b])$.

Then is blowing up the ideal (x^2, y) of ring $\mathbb{C}[x, y]$.