List of Papers

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1 Grothendieck Duality Made Simple [6]

2 Stable and unitary vector bundles on a compact Riemann surface [3]

3 Space of Unitary Vector Bundles on a Compact Riemann Surface [4]

We will assume the following:

Theorem 3.1. *Let $\mathcal{A}$ be an abelian category. If $M \in \mathrm{ob}\ \mathcal{A}$ has a Jordan-Hölder series, then its cycle of simple components is determined uniquely up to isomorphism. If $\mathcal{A}$ is both Artinian and Noetherian, then every object in $\mathcal{A}$ has a Jordan-Hölder series.*

Let X be a Riemann surface with genus $g \geq 2$, where $\mathscr{O}_X(1)$ is a fixed very ample invertible sheaf on X. For any coherent sheaf $\mathscr{F}$ over X, we denote by $\mathscr{F}(m)$ the coherent sheaf $\mathscr{F} \otimes \mathscr{O}_X(m)$. Similarly for F a holomorphic bundle over X.

For a coherent sheaf $\mathscr{E}$ on X, we denote by $Q(\mathscr{E}/p(x))$ the family of all coherent sheaves $\mathscr{F}$ on X such that $\mathscr{F}$ is a quotient of $\mathscr{F}$ and the Hilbert polynomial of $\mathscr{F}$ is $p(x)$, where $p(x)$ is a given linear polynomial, i.e. $p(x) = \dim H^0(\mathscr{F}(m)) - \dim H^1(\mathscr{F}(m))$ for m sufficiently large.

We now have the following theorem due to Grothendieck:

Theorem 3.2. *There is a (unique) structure of a projective algebraic scheme on $Q = Q(\mathscr{E}/p(x))$ and a surjective homomorphism $\theta : p_1^*(\mathscr{E}) \to \mathscr{F}$ of coherent sheaves on $X \times Q$ such that*

1. *$\mathscr{F}$ is flat over Q.*
2. *the restriction of homomorphism θ to $X \times \{q\} \cong X$ corresponds to the element of $Q(\mathscr{E}/p(x))$ represented by q.*
3. *given a surjective homomorphism $\varphi : p_1^*(\mathscr{E}) \to \mathscr{G}$ of coherent sheaves on $X \times T$, where T is an algebraic (resp. analytic) scheme such that $\mathscr{G}$ is flat over T, and the Hilbert polynomial of $\mathscr{G}_t$-restriction of $\mathscr{G}$ to $X \times \{t\}$ is $p(x)$, there exists a (unique) morphism $f : T \to Q$ such that $\varphi : p_1^*(\mathscr{E}) \to \mathscr{G}$ is the inverse image of $\theta : p_1^*(\mathscr{E}) \to \mathscr{F}$ by the morphism f.*

Proposition 3.3. *Let $R_1 = R_1(\mathscr{E}/p(x))$ be the subset of $Q = Q(\mathscr{E}/p(x))$ consisting of points $q \in Q$ such that $\mathscr{F}_q$ is locally free on X. Then R_1 is an open subset of Q, and the restriction $\mathscr{F}_1$ of $\mathscr{F}$ to $X \times R_1$ is locally free of constant rank.*

To prove this we need a lemma from commutative algebra:

Lemma 3.1. *Let $\varphi : A \to B$ be a homomorphism of commutative rings such that A is local with the maximal ideal $\mathfrak{m}$, and B is Noetherian. Suppose that for every maximal ideal $\mathfrak{n}$ of B, $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$, i.e. $\varphi(\mathfrak{m})B$ is contained in the radical of B. For a B-module M of finite type such that it is flat over A, then M is free over B if $M/\mathfrak{m}M$ is free over $B/\mathfrak{m}B$.*

For G the automorphism group of the coherent sheaf $\mathscr{E}$, we have an action of G on $Q = Q(\mathscr{E}/p(x))$, and $R_1(\mathscr{E}/p(x))$ is a G-invariant subset of Q. Let d be the degree and r be the rank of any F_q, $q \in R_1(E/p)$. We note that

$$\dim H^0(F_q) \geq d - r(g-1)$$

by Riemann-Roch.

3.1 Categories of Vector Bundles on a Riemann Surface

Let $\mathcal{V}$ be the additive category of vector bundles on a compact Riemann surface X, and let $\mathcal{V}^0$ be the full subcategory of vector bundles of degree 0. (Here the degree of a line bundle is defined to be the degree of its determinant bundle.)

Definition. A vector bundle $V \in \mathcal{V}$ is said to be *semi-stable* (resp. *stable*) if for every proper holomorphic subbundle W of V, we have

$$\frac{d(W)}{r(W)} \leq \frac{d(V)}{r(V)}$$

where $\frac{d(V)}{r(V)}$ is called the *slope* of V.

Let $\mathcal{S}$ be the full subcategory of $\mathcal{V}^0$ consisting of semi-stable vector bundles of degree 0.

Proposition 3.4. *The category $\mathcal{S}$ is abelian, Artinian, and Noetherian. Furthermore, if $\alpha \in \mathrm{Hom}(V, W)$, then α is of constant rank on the fibres of V.*

Proof. It suffices to show that ker α, coker α, and coim α are all of degree 0. By semi-stability, all degrees are ≤ 0. If $d(\ker\,\alpha) < 0$, then by $0 = d(V) = d(\ker\,\alpha) + d(\mathrm{coker}\,\alpha)$ we get a contradiction. Similarly for others. □

By GAGA [1], the compact Riemann surface X is uniquely determined by its underlying structure of a non-singular algebraic variety, and a holomorphic vector bundle V on X has a unique underlying structure of an algebraic vector bundle.

Definition. A subcategory $\mathcal{B}$ of $\mathcal{V}$ is said to be *bounded* if there is an algebraic family of vector bundles $\{V_t\}_{t\in T}$ parametrized by an algebraic scheme T such that given $V \in \mathcal{B}$, there is a $t \in T$ for $V \cong V_t$.

Proposition 3.5. *The subcategory $\mathcal{S}_n$ of $\mathcal{S}$ consisting of semi-stable vector bundles of degree 0 and rank $\leq n$, n being a fixed positive integer, is bounded.*

Proof. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bounded subcategories of $\mathcal{V}$, then the subcategory $\mathcal{B}$ consisting of vector bundles which are extensions of an object in $\mathcal{B}_1$ by an object in $\mathcal{B}_2$ is again bounded. Hence it suffices to prove that the stable bundles are bounded. But a stable bundle is indecomposable [3], whence we can use a result by Atiyah [2]. □

Let $R(E/p)$ denote the subset of $R_1(E/p)$ (as in Theorem 3.2) consisting of points $q \in R_1(E/p)$ such that the canonical mapping $H^0(E) \to H^0(F_q)$ is an isomorphism and $\dim H^0(F_q) = d - r(g-1)$. It can be proved that $R(E/p)$ is invariant under the action of Aut E, and further it is open (in which the semi-continuity theorem will be used).

Proposition 3.6. *Let $\mathcal{B}$ be a bounded subcategory of $\mathcal{V}$ such that every object of $\mathcal{B}$ has the same rank r and degree d. Then there is a positive integer m_0 such that, if $m \geq m_0$ and p the Hilbert polynomial of $V(m)$, $V \in \mathcal{B}$, we have*

1. *For every $V \in \mathcal{B}$, $H^1(V(m)) = 0$ and $H^0(V(m))$ generates $V(m)$ (i.e., $H^0(V(m))$generates the fibre of $V(m)$ at every $x \in X$). In particular, the rank of $H^0(V(m))$ is independent of $V \in \mathcal{B}$. Let this be p.*
2. *If E is the trivial vector bundle of rank p, the set of all $q \in R(E/p)$ such that $F_q \cong V(m)$ for a given $V \in \mathcal{B}$ is non-empty and is precisely an orbit for the operation of the group $G = \mathrm{Aut}\ E$.*
3. *If $\{V_i\}$ is an algebraic (resp. analytic) family of vector bundles on X parametrized by an algebraic (resp. analytic) scheme T such that for every $t \in T$, $V_t(m) \in \mathcal{B}$, then given $t_0 \in T$, there is a neighbourhood T_0 of t_0 and a morphism $f : T_0 \to R(E/p)$ such that if $q = f(t)$, $t \in T_0$, $F_q \cong V_t(m)$.*
4. *There is an open, irreducible, non-singular subvariety U of $R(E/p)$ invariant under the action of G such that, given $V \in \mathcal{B}$, there is a $q \in U$ with $F_q \cong V(m)$.*

3.2 Category of Points of N-folded Grassmannians

Through out this section, we shall use $Gr_{p,r}(E)$ denoting the Grassmannian of p dimensional sub-spaces of E which is a $\mathbb{C}$-vector space of rank r, and use $Gr^{p,r}(E)$ denoting the Grassmannian of p dimensional quotient spaces of E which is a $\mathbb{C}$-vector space of rank r. Hence there is a canonical isomorphism $Gr_{p,r}(E) \cong Gr^{r-p,r}(E)$. Let $Gr^N_{p,r}(E)$ denote the N-fold product of $Gr_{p,r}(E)$.

Definition. Let N be a fixed positive integer. We denote $\mathcal{G}^N$ the category whose objects are points of $Gr^N_{p,r}(E)$, where E is any vector space of rank $r \geq 0$ and $0 \leq p \leq r$.

A morphism $\alpha : Y \to X$, for $Y = \{F_i\}_{1\leq i\leq N} \in Gr^N_{q,s}(F)$ and $X = \{E_i\}_{1\leq i\leq N}$ in this category is a linear map $\bar{\alpha} : F \to E$ (called the underlying linear map) such that $\bar{\alpha}(F_i) \subseteq E_i$.

It is not hard to see $\mathcal{G}^N$ is an additive category, and satisfies these properties:

1. α is a monomorphism (resp. epi) if and only if $\bar{\alpha}$ is injective (resp. surjective).
2. α has a kernel if and only if the rank of $K_i = \ker\,\alpha \cap F_i$ is independent for i, and then $\{K_i\}_{1\leq i\leq N}$ is the kernel of α. If α has kernel, then its coimage exists.

3. α has a cokernel if and only if the rank of $M_i = \pi(E_i)$ is independent for i where $\pi : E \to \text{coker } \alpha$ is the canonical projection, and then $\{M_i\}_{1\le i\le N}$ is the cokernel of α. If α has kernel, then its image exists.

4. If α has both kernel and cokernel, then the image and coimage of α exist and the canonical morphism from the coimage to the image is an isomorphism if and only if $r(F_i) - r(K_i) = r(E_i) - r(M_i)$ for all $1 \le i \le N$.

If α is a monomorphism (resp. epi) and has a cokernel (resp. kernel), then we say that $0 \to Y \xrightarrow{\alpha} X$ (resp. $Y \xrightarrow{\alpha} X \to 0$) is exact. In this case, let Z be the cokernel (resp. kernel) of α and $\beta : X \to Z$ (resp. $\beta : Z \to Y$) be the canonical morphism, we see by the previous comments that α is the kernel of β. Thus we write that $0 \to Y \to X \to Z \to 0$ (resp. $0 \to Z \to Y \to X \to 0$) is exact.

Let n be a integer ≥ 2, then we denote by $\mathcal{G}^{N,n}$ the full subcategory of $\mathcal{G}^N$ consisting of objects which are points of $Gr^N_{r(n-1),rn}(E)$, where E is any vector space of rank $r \ge 0$ and $0 \le p \le r$. It is not hard to show that a morphism in $\mathcal{G}^{N,n}$ is a monomorphism (resp. epimorphism) if and only if it is so in $\mathcal{G}^N$.

Definition. An object $X = \{E_i\}_{1\le i\le N} \in Gr^N_{p,r}(E)$ is said to be *semi-stable* (resp. *stable*) if, for every subspace F of E (resp. proper subspace) we have

$$\frac{\frac{1}{N}\sum_{i=1}^N r(F \cap E_i)}{p} \le \frac{r(F)}{r}.$$

Also, for $X = \{E_i\}_{1\le i\le N} \in Gr^{p,r}_N(E)$, the canonical image of X in $Gr^N_{r-p,r}(E)$ is semi-stable (resp.) if and only if, for every subspace F of E,

$$\frac{\frac{1}{N}\sum_{i=1}^N r(F_i)}{p} \ge \frac{r(F)}{r}.$$

Proposition 3.7. *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence in $\mathcal{G}^N$ with Y, Z, X in $\mathcal{G}^{N,n}$. Then X is semi-stable if and only if both Y and Z are semi-stable.*

We also denote by $\mathcal{K}^{N,n}$ the full subcategory of $\mathcal{G}^{N,n}$ consisting of the semi-stable objects of $\mathcal{G}^{N,n}$. It is not hard to show that a morphism in $\mathcal{K}^{N,n}$ is a monomorphism (resp. epimorphism) if and only if it is so in $\mathcal{G}^N$.

Proposition 3.8. *Let $\alpha : Y \to X$ (resp. $\alpha : X \to Y$) be a monomorphism (resp. epi) in $\mathcal{G}^N$ with $X, Y \in \mathcal{G}^{N,n}$. Then if X is semi-stable, $0 \to Y \to X$ (resp. $X \to Y \to 0$) is exact, and Y is semi-stable.*

Proposition 3.9. *Let X be a stable object of $\mathcal{G}^{N,n}$. Then if $\alpha : X \to Y$ is a morphism in $\mathcal{K}^{N,n}$, then either α is 0, or $0 \to Y \to X$ is exact.*

Definition. An object $X \in \mathcal{G}^{N,n}$ is said to have a *stable series* S if there is an increasing sequence $S = \{X_i\}_{q\le i\le m}$

$$X_1 \subset X_2 \subset \cdots \subset X_m = X$$

of subobjects of X such that every one of the canonical monomorphisms $X_i \to X_{i+1}$ has a cokernel X_{i+1}/X_i, and $X_1, \cdots, X_m/X_{m-1}$ are all stable objects of $\mathcal{G}^{N,n}$.

By an application of Proposition 3.7, it follows that $X \in \mathcal{K}^{N,n}$ if X has a stable series S. We denote by $\mathcal{A}^{N,n}$ the full subcategory of $\mathcal{K}^{N,n}$ consisting of those objects which possess stable series.

Proposition 3.10. *The category $\mathcal{A}^{N,n}$ is abelian, artinian, and noetherian, and the simple object in it are precisely the stable objects.*

3.3 Connecting Two Categories

Let X be a Riemann surface with genus $g \ge 2$. Let $\mathcal{V}_1$ be the full subcategory of the additive category $\mathcal{V}$ of holomorphic vector bundles on X, having the property that all objects are globally generated. Let $\mathfrak{n}$ be an ordered set of N distinct points $P_1, \cdots, P_N$ on X. Then if V is an object of $\mathcal{V}_1$ of rank r, let E be the vector space $H^0(V)$ and p be its dimension.

We define a functor $\tau(\mathfrak{n})$, which to V associates the point $x \in Gr^{r,p}_N(E) \cong Gr^N_{p-r,p}(E)$ such that the i-th coordinate of x is precisely the quotient vector space of E represented by the fibre of V at P_i. Thus $\tau(\mathfrak{n})$ is an additive functor from $\mathcal{V}_1$ into $\mathcal{G}^N$.

It is not hard to verify that if $0 \to V_1 \to V_2 \to V_3 \to 0$ is exact in $\mathcal{V}$ such that V_i are in $\mathcal{V}_1$ and $H^1(V_i) = 0$, then

$$0 \to \tau(\mathfrak{n})(V_1) \to \tau(\mathfrak{n})(V_2) \to \tau(\mathfrak{n})(V_3) \to 0$$

is exact. If $\mathcal{D}$ is any subcategory of $\mathcal{V}$, we denote by $\mathcal{D}(m)$ the subcategory of $\mathcal{V}$ of all objects $V(m)$ where $V \in \mathcal{D}$.

Proposition 3.11. *Let $\mathcal{B}$ be a bounded subcategory of $\mathcal{V}$. Then one can find a positive integer m and an ordered set $\mathfrak{n}$ of N distinct points $P_1, \cdots, P_N$ on X such that*

1. *$\mathcal{B}(m) \subseteq \mathcal{V}_1$ and*
2. *$\tau(\mathfrak{n})(V_1)$ and $\tau(\mathfrak{n})(V_2)$ for $V_1, V_2 \in \mathcal{B}(m)$ are isomorphic if and only if $V_1 \cong V_2$.*

Definition. A vector bundle $V \in \mathcal{V}$ is said to be *generically generated* by a linear subspace F of $H^0(V)$ if there is at least one $x \in X$ such that F generates the fibre of V at x.

Let $\mathcal{W}$ be the full subcategory of v consisting of objects $V \in \mathcal{V}$ such that

1. if G is any sub-bundle of V with the property

   $$\frac{d(G)}{r(G)} \geq \frac{d(V)}{r(V)}$$

   then $H^1(G) = 0$ and $H^0(G)$ generates G, and
2. if G is a proper sub-bundle such that $H^0(G)$ generates G generically, then

   $$\frac{r(H^0(G))}{r(G)} > \frac{r(H^0(V))}{r(V)}$$

   if $\frac{d(G)}{r(G)} < \frac{d(V)}{r(V)}$.

We see that $\mathcal{W} \subseteq \mathcal{V}_1$. Let $V \in \mathcal{W}$ and G a sub-bundle of V such that

$$\frac{d(G)}{r(G)} > \frac{d(V)}{r(V)} \qquad \text{resp.} \frac{d(G)}{r(G)} = \frac{d(V)}{r(V)}.$$

Then since $H^1(V) = H^1(G) = 0$, we have by the Riemann-Roch

$$\frac{r(H^0(V))}{r(V)} = \frac{d(V)}{r(V)} - g + 1 \qquad \frac{r(H^0(G))}{r(G)} = \frac{d(G)}{r(G)} - g + 1.$$

Therefore it follows that

$$\frac{r(H^0(G))}{r(G)} > \frac{r(H^0(V))}{r(V)} \qquad \text{resp.} \frac{r(H^0(G))}{r(G)} = \frac{r(H^0(V))}{r(V)}.$$

Proposition 3.12. *Let $\mathcal{B}$ be a bounded subcategory of $\mathcal{W}$. Then given any ordered set $\mathfrak{n}$ of distinct points $P_1, \cdots, P_N$ on X such that N is sufficiently large, the functor has the following property that, for $V \in \mathcal{B}$, $\tau(\mathfrak{n})(V)$ is a stable (resp. semi-stable) object of $\mathcal{G}^N$ if and only if V is a stable (resp. semi-stable) object of $\mathcal{V}$.*

To prove this, we need this lemma:

Lemma 3.2. *Let $V \in \mathcal{V}$ and F a subspace of $H^0(V)$ which generates V generically. Let μ be the number of distinct points such that F does not generate the fibre of V ate y, then*

$$\mu \leq d(V).$$

Proposition 3.13. *Let $\mathcal{B}$ be a bounded subcategory of $\mathcal{V}$. Then we can find an integer m_0 such that if $m \geq m_0$, then $\mathcal{B}(m) \subseteq \mathcal{W}$.*

We also need this lemma:

Lemma 3.3. *Let $V \in \mathcal{V}$ and $H^0(V)$ generate V generically. Then we have*

$$r(H^0(V)) \le d(V) + r(V).$$

Theorem 3.14. *Let $\mathcal{B}$ be a bounded subcategory of $\mathcal{V}$. Then we can find an integer m and an ordered set $\mathfrak{n}$ of distinct points $P_1, \cdots, P_N$ on X such that*

1. *If $V \in \mathcal{B}$, then $H^1(V(m)) = 0$ and $H^0(V(m))$ which generates $V(m)$ so that, in particular $\mathcal{B}(m) \subseteq \mathcal{V}_1$.*
2. *$\tau(\mathfrak{n})(V_1)$ and $\tau(\mathfrak{n})(V_2)$ for $V_1, V_2 \in \mathcal{B}(m)$ are isomorphic if and only if $V_1 \cong V_2$.*
3. *If $V \in \mathcal{B}(m)$, $\tau(\mathfrak{n})(V)$ is stable (resp. semi-stable) if and only if V is stable (resp. semi-stable).*

Corollary 3.14.1. *Take for $\mathcal{B}$ the abelian sub-category $\mathcal{S}_r$ of $\mathcal{V}$ consisting of the semi-stable bundles of degree 0 and rank r. Let m and $\mathfrak{n}$ be chosen as in Theorem 3.14, and $n = d(L) - g + 1$, where L is the line bundle defined by the invertible sheaf $\mathscr{O}_X(m)$. Let $\tau_1 : \mathcal{S}_r \to \mathcal{G}^N$ be the functor $\tau_1(V) := \tau(\mathfrak{n})(V(m))$. Then we have*

1. *$\tau_1(\mathcal{S}_r)$ is contained in the abelian sub-category $\mathcal{A}^{N,n}$ of $\mathcal{G}^N$, and the functor $\tau_1 : \mathcal{S}_r \to \mathcal{A}^{N,n}$ is an exact functor.*
2. *For $V \in \mathcal{S}_r$, $\tau(\mathfrak{n})(V)$ is stable (resp. semi-stable) if and only if V is stable (resp. semi-stable).*
3. *For $V_1, V_2 \in \mathcal{S}_r$, $\tau_1(V_1)$ and $\tau_1(V_2)$ for $V_1, V_2 \in \mathcal{B}(m)$ are isomorphic if and only if $V_1 \cong V_2$. In particular,* $\mathrm{gr}\ V_1 \cong \mathrm{gr}\ V_2$ *if and only if* $\mathrm{gr}\ \tau_1(V_1) = \mathrm{gr}\ \tau_1(V_2)$.

Corollary 3.14.2. *Let $\{V_t\}_{t \in T}$ be an algebraic family of vector bundles parametrized by an algebraic scheme T. Then subset T_s (resp. T_{ss}) of points $t \in T$ such that V_t is stable (resp. semi-stable) is open in T. Similarly for an analytic family, then $T - T_s$ (resp. $T - T_{ss}$) is an analytic subset of T.*

Theorem 3.15. *Let $\{z_i\}_{1 \le i \le m}$ be a sequence of stable objects in $G^{N,n}$. Let T be the subset of $Gr^N_{r(n-1),rn}(E)_{ss}$ consisting of the points x in $\mathcal{A}^{N,n}$, i.e. those having a stable series*

$$x_1 \subset x_2 \subset \cdots \subset x_m = x$$

such that $x_1 \cong z_1, x_2/x_1 \cong z_2, \cdots, x_m/x_{m-1} = z_m$. Then T is closed and $GL(E)$ invariant.

If else, T is the subset of $Gr^N_{r(n-1),rn}(E)_{ss}$ consisting of the points having a stable series with a fixed cycle of stable components C. Then T is closed and $GL(E)$ invariant. If T_1 and T_2 are two such subsets associated to two distinct cycles of stable components, then $T_1 \cap T_2 = \emptyset$.

Corollary 3.15.1. *Let $X = Gr^N_{r(n-1),rn}(E)$. Then the categorical quotient Y of X_{ss} modulo $PGL(E)$ exists, and is a projective variety. Further if $\varphi : X_{ss} \to Y$ is the canonical morphism and x_1 and x_2 are two points of X_{ss} such that they belong to $\mathcal{A}^{N,n}$ and $\mathrm{gr}\ x_1 \neq \mathrm{gr}\ x_2$, then $\varphi(x_1) \neq \varphi(x_2)$.*

3.4 The Main Theorem and Its Proof

Let X as before be a Riemann surface with genus $g \geq 2$, with $p : \tilde{X} \to X$ the covering space of X. Let $\mathcal{S}$ be the category of semi-stable vector bundles of degree 0.

Definition. A holomorphic vector bundle is said to be *unitary* if it is the vector bundle associated to a unitary representation of $\pi_1(X)$.

It was proved in [3] that a holomorphic vector bundle V on X is isomorphic to a unitary vector bundle on X if and only if V is a direct sum of stable vector bundles of degree 0. Therefore for $V \in \mathcal{S}$, $\mathrm{gr}\ V$ represents an isomorphic class of unitary vector bundle.

Definition. For two elements in $\mathcal{S}$, we say they are *strongly equivalent* if $\mathrm{gr}\ V_1 = \mathrm{gr}\ V_2$.

Theorem 3.16. *Let U_r denote the set of isomorphic classes of unitary vector bundles of rank r, or equivalently the set of equivalence classes of semi-stable vector bundles of rank r and degree 0.*

Then there is a unique structure of a normal projective variety on U_r such that, if $\{V_t\}_{t\in T}$ is an algebraic (resp. analytic) family of semi-stable vector bundles of rank r and degree 0, then the mapping $T \to U_r$ defined by $t \to \mathrm{gr}\ V_t$ is a morphism.

Proof. We shall use the notations mentioned in Theorem 3.14. Let $p(x)$ be the Hilbert polynomial of $V(m)$ where $V \in \mathcal{S}_r$, $r = r(V)$. Let $R = R(E/p)$ be the scheme as in Proposition 3.6, E being the trivial bundle of rank p, $p = \dim H^0(V(m))$. Consider the canonical morphism $\tau = \tau(\mathfrak{n}) : R \to Gr^N_{p,p-r}(E) \cong Gr^{p,r}_N(E)$ where $E = H^0(E)$ which to $q \in R$ associates the point x of $Gr^{p,r}_N(E)$ such that $p_i(x)$, the i-th canonical projection on to $Gr^{p,r}(E)$, is precisely the fibre of the vector bundle at P_i.

Let R_{ss} be the subset of R consisting of points $q \in R$ such that F_q is semi-stable. Then R_{ss} is an open, non-singular and irreducible subset of R invariant under $G = \mathrm{Aut}\ E$. Further given $V \in \mathcal{S}_r$, $r = r(V)$, there is a $q \in R$ such that $F_q \cong V$, and the set of such q constitutes precisely one orbit under G.

Let $Z = Gr^N_{r(n-1),rn}(E)_{ss}$ ($n := d(L) - g + 1$, where L is the line bundle associated to the invertible sheaf $\mathscr{O}_X(m)$). Then $\tau(R_{ss}) \subseteq Z$. In fact, if $q \in R$, $\tau(q) \in \mathcal{A}^{N,n}$; further if $q_1, q_2 \in R$ such that $\mathrm{gr}\ F_{q_1} \neq \mathrm{gr}\ F_{q_2}$, then $\mathrm{gr}\ \tau(q_1) \neq \mathrm{gr}\ \tau(q_2)$.

Let G be the automorphism group of $H^0(E)$, then we see that τ is a G-morphism. Let

$$\varphi : Z \to Y$$

be the categorical quotient of Z by G, then Y is projective and if $q_1, q_2 \in R$ such that $\mathrm{gr}\ F_{q_1} \neq \mathrm{gr}\ F_{q_2}$, we have $\varphi \circ \tau(q_1) \neq \varphi \circ \tau(q_2)$ by Corollary 3.15.1.

Let Y_1' be the closure of $\varphi \circ \tau(R_{ss})$. Then the canonical morphism $\psi' : R_{ss} \to Y_1'$ is dominant and G-invariant, i.e. two points in the same orbit are mapped onto the same point by ψ'. Let Y_1 be the normalization of Y_1' and $p : Y_1 \to Y_1'$ be the canonical morphism.

Since R_{ss} is non-singular in particular normal, we have a morphism $\psi : R_{ss} \to Y_1$ such that $\psi' = p \circ \psi$. Since p is an isomorphism on a non-empty open subset of R_{ss}, it follows that ψ is G-invariant on a non-empty G-invariant open subset of R_{ss}, which implies that ψ is G-invariant on the entire R_{ss}. We note also that, if $q_1, q_2 \in R$ such that $\mathrm{gr}\ F_{q_1} \neq \mathrm{gr}\ F_{q_2}$, then $\psi(q_1) \neq \psi(q_2)$.

It remains to prove that Y_1 is actually what we want. So we will define a set-theoretical bijection $U_r \to Y_1$. Let now U denote the space of all unitary representations of $\pi_1(X)$ of rank r. Then we know U is a compact subspace of the analytic space $R(r)$ of all representations of π of rank r, and we know that the canonical family of vector bundles on X parametrized by $R(r)$ (namely the one which assigns to each point θ of $R(r)$ the holomorphic vector bundle on X associated to the representation θ of π) is an analytic family. By the property (3) of the Proposition 3.6, we see that given a point θ of $R(r)_{ss}$ there is a neighbourhood K of θ in $R(r)_{ss}$ and an analytic morphism $f : K \to R_{ss}$ such that if $k \in K$ and $q = f(k)$, F_q is isomorphic to the vector bundle associated to the representation of π given by k.

Now if $f_1 : K_1 \to R_{ss}$ and $f_2 : K_2 \to R_{ss}$ are two such morphisms, K_1, K_2 being open in R_{ss}, then if $k \in K_1 \cap K_2$, we see that $f_1(k)$ and $f_2(k)$ lie in the same orbit under G. Hence we conclude that $\psi \circ f_1$ and $\psi \circ f_1$ coincide in $K_1 \cap K_2$. From these considerations, we get an analytic morphism from $R(r)_{ss}$ to R_{ss}, and the restriction of this morphism defines a *continuous* mapping $g : U \to Y_1$.

By the compactness of U, we know $g(U)$ is closed (in the usual Hausdorff sense) in Y. Further if U_0 is the subspace of U consisting of the irreducible unitary representations of π, we have $g(U_0) = \psi(R)$, since a vector bundle of degree 0 is stable if and only if it is isomorphic to a vector bundle associated to an irreducible unitary of π. Since ψ is dominant, $\psi(R_s)$ contains a non-empty (Zariski) open subset of Y by a theorem of Chevalley. This implies that $g : U \to Y_1$ is surjective, and we note that if θ_1 and θ_2 are in U, $g(\theta_1) = g(\theta_2)$ if and only if the unitary bundles on X defined by the representations θ_1 and θ_2 of π are isomorphic □

Proof Idea. For paratramizing semi-stable vector bundles, we take any $V \in \mathcal{S}_r$. If V is globally generated, then V is a quotient of $E \otimes \mathscr{O}_X$, where $E = H^0(V)$. By doing this, we can use the Quot scheme (3.2) to parametrize the vector bundles. However, there are immediately several issues:

1. The quotients may only be coherent sheaves, instead of vector bundles.
2. The parametrizing map may not be surjective.
3. The parametrizing map may not be injective.

For 1, we first take $R_1 = R_1(E/p)$ the subset of $Q = Q(\mathscr{E}/p(x))$ consisting of points $q \in Q$ such that $\mathscr{F}_q$ is locally free on X (Proposition 3.3).

But before getting into the bijectiveness, there is still some issue: what if V is not globally generated? By proposition 3.6, we should twist it sufficiently. And also the first part of this property tells us we sort of get all vector bundles of X.

There is also a natural way to study the quotient scheme: we map (a subscheme of) it to the Grassmannian, or N-folded (dual) Grassmannian. Loosely, this process is taking N points on the Riemann surface X, get N quotient space of E which turns out to be a point in Gr^N. □

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