1 Introduction

In algebraic geometry, we have to deal with singularities of varieties. The problem of **resolution of singularities** asks whether every algebraic variety X has a resolution, a non-singular variety Y with a proper birational map $f : Y \to X$. We already have different ways to resolve a singularity, and blowing up is one of the most important methods.

2 Construction: Geometry

2.1 Blow-up of $\mathbb{C}^n$ at Origin

We will start from the simplest case, i.e. blowing up the origin of $\mathbb{C}^n$.

2.2 Blow-up of $\mathbb{C}^n$ at a Affine Variety

For an *affine variety*, we mean an irreducible algebraic subset of $\mathbb{A}^n$, denoted as

$$V(f_1, \cdots, f_k) = \{(a_1, \cdots, a_n) \in \mathbb{A}^n \mid f_i(a_1, \cdots, a_n) = 0 \ \ \forall 1 \leq i \leq k\}.$$

We can always find finitely many generators as a result of Hilbert basis theorem.
The blow-up of $\mathbb{A}^n$ with respect to the subvariety $V(f_1, \cdots, f_k)$ is given by

$$\{((a_1, \cdots, a_n), [b_1, \cdots, b_k]) \mid b_i f_j(a_1, \cdots, a_n) = b_j f_i(a_1, \cdots, a_n)\}$$

3 Construction: Blow-up Algebra

3.1 Blow-up Algebra

Definition. Let R be a ring and let $I \subset R$ be an ideal of R. The *blow-up algebra* or *Rees algebra* , associated with the pair (R, I), is the graded R-algebra

$$\mathrm{Bl}_I(R) := \bigoplus_{n \geq 0} I^n = R \oplus I \oplus I^2 \oplus \cdots$$

3.2 Proj Construction

4 Comparison: Algebraic Variety and Scheme

We first give a full generalization of blow-up. Let X be a scheme, and let $\mathcal{I}$ be a coherent sheaf of ideals on X. We say the blow-up of X with respect to $\mathcal{I}$ is a scheme $\tilde{X}$ along with a morphism $\pi : \tilde{X} \to X$, such that $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\tilde{X}}$ is a invertible sheaf, with the universal property: for any scheme and morphism $f : Y \to X$ such that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ is a invertible sheaf, there is a unique factorization:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\pi"] & X \\ & Y \arrow[u, "f"'] \arrow[ul, "\tilde{f}"] \end{tikzcd}$$

Here we need to explain some

However instead of going that far, we just consider affine scheme, i.e. $X = \text{Spec } R$ for some commutative ring R.

The first example is blowing up the maximal ideal (x, y) of ring $\mathbb{C}[x, y]$. Consider the map $f : \mathbb{C} - V(x, y) \to \mathbb{C} \times \mathbb{CP}^1, (a, b) \mapsto ((a, b), [a, b])$.

Then is blowing up the ideal (x^2, y) of ring $\mathbb{C}[x, y]$.