On Different Construction of Blow-up and Its Geometric Properties

1 Introduction

In algebraic geometry, we have to deal with singularities of varieties. The problem of **resolution of singularities** asks whether every algebraic variety X has a resolution, a non-singular variety Y with a proper birational map $f : Y \to X$. We already have different ways to resolve a singularity, and blowing up is one of the most important methods.

Usually $\mathbb{C}^n$ is a denotation geometrically, while $\mathbb{A}^n_{\mathbb{C}}$ is more algebraically. However we will regard them equivalently through this paper. $\mathbb{A}^n, \mathbb{P}^n$ are $\mathbb{A}^n_{\mathbb{C}}$ and $\mathbb{CP}^n$ respectively except indicated explicitly. All of the construction can be transplanted over another (algebraic closed) field.

In order to get a full understand of this paper, the readers are supposed to know the results from commutative algebra, properties of projective spaces, some results from geometry and topology, basic construction of algebraic geometry, complex geometry and sheaves. All of the definition will be mentioned later in the text however it may be extremely confusing if it is the first time to encounter these terminologies.

1.1 Acknowledgements

2 Construction: Geometry

2.1 Blow-up of $\mathbb{C}^n$ at Origin

We will start from the simplest case, i.e. blowing up the origin of $\mathbb{C}^n$.

Consider a subset of $\mathbb{C}^n \times \mathbb{P}^{n-1}$

$$\tilde{\mathbb{C}^n} := \{((a_1, \cdots, a_n), [b_1, \cdots, b_n]) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid a_i b_j = a_j b_i, 1 \leq i, j \leq n\}.$$

It is easy to see that we have a natural projection from $\tilde{\mathbb{C}^n}$ to $\mathbb{C}^n$:

$$\pi : ((a_1, \cdots, a_n), [b_1, \cdots, b_n]) \mapsto (a_1, \cdots, a_n).$$

Definition. The set $\tilde{\mathbb{C}^n}$ along with the map $\pi : \tilde{\mathbb{C}^n} \to \mathbb{C}^n$ is called the *blow-up* of $\mathbb{C}^n$ at the origin.

One can see that the blow-up of $\mathbb{C}^n$ is exactly $\mathcal{O}(-1)$ of $\mathbb{P}^{n-1}$.

Immediately we have these properties of $\tilde{\mathbb{C}^n}$:

For any point $O \neq z = (a_1, \cdots, a_n) \in \mathbb{C}^n$, there is at least a $1 \leq i_0 \leq n$ s.t. $a_{i_0} \neq 0$ hence $b_{i_0} \neq 0$. Thus the equations can be written as $\frac{b_j}{b_{i_0}} = \frac{a_j}{a_{i_0}}, 1 \leq j \leq n$. Therefore there is only **ONE** point x in $\tilde{\mathbb{C}^n}$ with $\pi(x) = z$, i.e. $x = ((a_1, \cdots, a_n), [a_1, \cdots, a_n])$.

$\pi^{-1}(O) \cong \mathbb{P}^{n-1}$. Indeed, any point $[b_1, \cdots, b_n] \in \mathbb{P}^{n-1}$ satisfies the equations if $(a_1, \cdots, a_n) = 0$.

The points of $\pi^{-1}(O)$ are in 1-1 corresponding to the lines through the origin in $\mathbb{C}^n$. A line through the origin l can be given by the parametric equation $x_i = a_i t$ where $1 \leq i \leq n$ and $a_i \in \mathbb{C}$ are not all zero, thus $[a_1, \cdots, a_n]$ is a point in $\mathbb{P}^{n-1}$. Now consider the line $\tilde{l} = \pi^{-1}(l - O)$ in $\tilde{\mathbb{C}^n} - \pi^{-1}(O)$, it is $\{((a_1 t, \cdots, a_n t), [a_1, \cdots, a_n]) \mid t \in \mathbb{C}^*\}$. These equations also make sense for $t = 0$ and give the closure $\bar{l}$ of $\tilde{l}$ in $\tilde{\mathbb{C}^n}$. $\tilde{l}$ meets $\pi^{-1}(O)$ in the point $[a_1, \cdots, a_n] \in \mathbb{P}^{n-1}$, so we see that sending l to Q gives us the correspondence between lines through origin in $\mathbb{C}$ and points of $\pi^{-1}(O)$.

$\tilde{\mathbb{C}^n}$ is irreducible. Indeed $\tilde{\mathbb{C}^n} = \tilde{\mathbb{C}^n} - \pi^{-1}(O) \cup \pi^{-1}(O)$, where the first piece is isomorphic to $\mathbb{C}^n - O$ which is obviously irreducible, and every point of the second part $\pi^{-1}(O)$ is in the closure of some line of $\mathbb{C}^n - \tilde{\pi}^{-1}(O)$. Hence $\mathbb{C}^n - \tilde{\pi}^{-1}(O)$ is dense in $\tilde{\mathbb{C}^n}$ and $\tilde{\mathbb{C}^n}$ is irreducible.

Here we give an example to illustrate how this process works.

The blow-up of $\mathbb{R}^2$ at the origin can be visualized as follows: we have known that points not the origin are in a 1-1 correspondence by π, so we leave them stable. However we need to replace the origin by $\mathbb{RP}^1$, i.e. a circle. For any line l parameterized by $x = a_1 t, y = a_2 t$, $\pi^{-1}(l - O)$ lies in $\tilde{\mathbb{R}^2} - \mathbb{RP}^1$, and the closure of $\pi^{-1}(l - O)$ consist of the point $[a_1, a_2]$ in $\mathbb{RP}^1$. So the disconnected part of $\pi^{-1}(l - O)$ are glued by the point. This operation works like we glue the antipodal points together. Topologically, this is regarding the missing point as a disk, gluing the boundary of the disk with the boundary of a Möbius band, since both of the boundary are circles.

Using the technique above, it is easy to blow up an algebraic set at the origin.

Definition. If V is a closed algebraic set of $\mathbb{C}$ passing through the origin, the *blow-up* of V at the origin is $\tilde{V} = \overline{\pi^{-1}(V - O)}$, where $\pi : \tilde{\mathbb{C}^n} \to \mathbb{C}^n$ is the blow-up of $\mathbb{C}^n$ as above. We denote also by $\pi : \tilde{V} \to V$ the morphism by the restriction of $\pi : \tilde{\mathbb{C}^n} \to \mathbb{C}^n$ to $\tilde{V}$.

2.2 A Little Further: Surgery

Consider every point on a complex manifold have a local Euclidean coordinate, so to blow up a (complex) manifold at a point, it suffices to "cut off" a small neighborhood of the point, blow-up the small piece at the origin and then glue it back up to the manifold. Yet we need some extra technique to do this surgery-like modification.

Let M be a complex manifold, and $S \subset M$ a compact submanifold of M. We construct a new manifold $\tilde{M} = (M - S) \cup \tilde{S}$ replacing S by another complex manifold $\tilde{S}$ as follows:

Take domains W, W_1 s.t. $S \subset W_1 \subset \overline{W_1} \subset W \subset M$ where $\overline{W}$ is compact. Let $\tilde{S}$ be another compact manifold s.t. $\tilde{S} \subset \tilde{W_1} \subset \overline{(\tilde{W_1})} \subset \tilde{W}$, and a biholomorphic surjective map $\psi : \tilde{W} - \tilde{S} \to W - S$ s.t. $\underline{\psi} : (\tilde{W} - \tilde{S}) = W - S$. Let $\tilde{M}$ be the manifold obtained by gluing $M - \overline{W_1}$ identifying $P \in W - \overline{W_1}$ with $\tilde{P} = \psi^{-1}(P) \in \tilde{W} - \overline{\tilde{W_1}}$ via ψ:

$$\tilde{M} = \left(M - \overline{W_1}\right) \cup \tilde{W}.$$

Since ψ is biholomorphic, $\tilde{M}$ becomes a manifold. Thus (see Figure 1)

$$\tilde{M} = (M - W) \cup \tilde{W}.$$

We would like to use this to construct the blow-up of a manifold at a point.

For example, let $M = \mathbb{P}^2$, and let $[z_0, z_1, z_2]$ be its homogeneous coordinate and $P = [1, 0, 0]$. We denote by $\mathbb{P}^1_\infty$ the projective line $z_0 = 0$. Then

$$\mathbb{P}^2 = \mathbb{C}^2 \cup \mathbb{P}^1_\infty$$

where

$$U_0 = \{[z_0, z_1, z_2] \mid z_0 \neq 0\} = \{[1, \frac{z_1}{z_0}, \frac{z_2}{z_0}] \mid z_0 \neq 0\} = \{(w_1, w_2)\} = \mathbb{C}^2.$$

We call the line $\mathbb{P}^1_\infty$ the line at infinity. Since we have known how to blow up $\mathbb{C}^2$ at the origin, i.e. blow up U_0 at the point $[1, 0, 0]$, and $\pi|_{\tilde{U_0} - \pi^{-1}(O)} : \tilde{U_0} - \pi^{-1}(O) \to U_0 - O$ is biholoporphic, we can replace U_0 by $\tilde{U_0}$. That is blowing up $\mathbb{P}^2$ at the point P.

2.3 Blow-up of $\mathbb{C}^n$ along a Affine Variety

For an *affine variety*, we mean an **irreducible** algebraic subset of $\mathbb{A}^n$, denoted as

$$Z(f_1, \cdots, f_k) = \{(a_1, \cdots, a_n) \in \mathbb{A}^n \mid f_i(a_1, \cdots, a_n) = 0 \quad \forall 1 \leq i \leq k\},$$

where $(f_1, \cdots, f_k)$ is a prime ideal. We can always find finitely many generators as a result of Hilbert basis theorem.

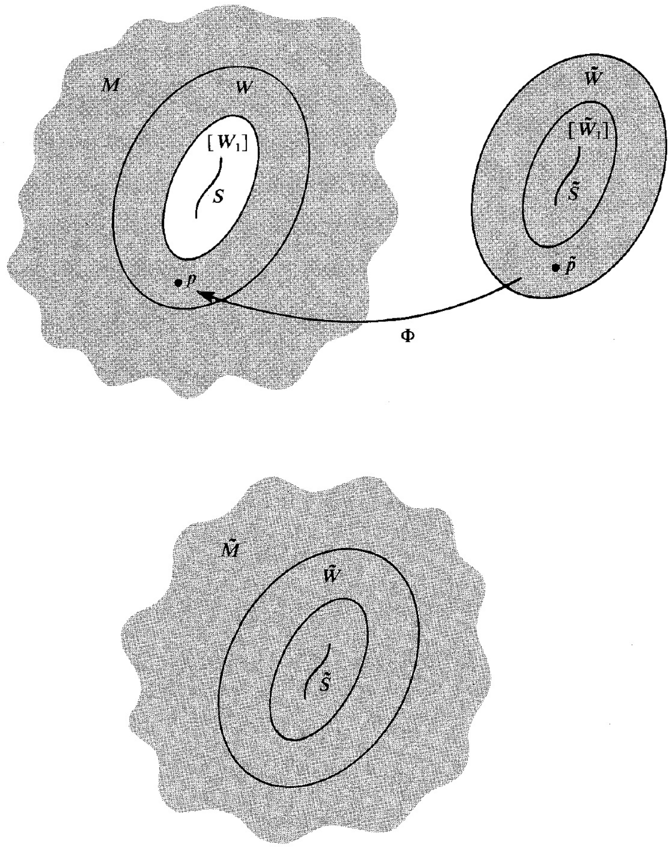

Figure 1: The Surgery of a Manifold

We define $A(V) = k[x_1, \cdots, x_n]/(f_1, \cdots, f_n)$ as the *coordinate ring* of affine variety $V = Z(f_1, \cdots, f_k)$, and let $k(V)$ be the quotient field, which is called the *field of fractional functions* on V. If P is a point of V, we define

$$\mathcal{O}_{V,P} = \Big\{ \frac{f}{g} \mid f, g \in A(V) \text{ and } g(P) \neq 0 \Big\}$$

as the *local ring* of V at point P. If $U \subset V$ is a nonempty open set, we set

$$\mathcal{O}_V(U) := \bigcap_{P \in U} \mathcal{O}_{V,P}.$$

This is a subring of $k(V)$, and we call this the *the ring of regular function* on U. Note that the local ring of V at point P is actually a local ring.

Lemma. *Let U be an open set in an affine variety X. A set theory map $\varphi : U \to k$ is a rational function at the point P if and only if there is an open neighborhood of P in U s.t. there are polynomials $f, g \in k[x_1, \cdots, x_n]$, with $g(Q) \neq 0$ and $\varphi(Q) = \frac{f(Q)}{g(Q)}$ for all $Q \in V$, and φ is a regular function on U if it is regular at every point in U.*

A **morphism** $\varphi : X \to Y$ between two affine variety is a continuous map s.t. $f^*(\mathcal{O}_Y(U)) \subseteq \mathcal{O}_X(f^{-1}(U))$, i.e. for all regular function f on an open set U of Y, $\varphi^{\#}(f) = f \circ \varphi$ is still regular. One can see that the definition of morphism is actually the definition of morphism of ringed spaces.

Notice that for the blow-up at the origin, $\pi : \tilde{\mathbb{C}}^n - O \to \mathbb{C}^n - O$ is a birational map.

The blow-up of $\mathbb{A}^n$ with respect to the subvariety $X = Z(f_1, \cdots, f_k)$ is given by

$$\tilde{X} := \{((a_1, \cdots, a_n), [b_1, \cdots, b_k]) \mid b_i f_j(a_1, \cdots, a_n) = b_j f_i(a_1, \cdots, a_n)\},$$

which is a subset of $\mathbb{A}^n \times \mathbb{P}^{k-1}$, along with the projection $\pi : \tilde{X} \to \mathbb{A}^n$. This can be characterized as the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathbb{A}^n \arrow[d] & (\mathbb{A}^n - X) \cup E \arrow[l] \arrow[d] \\
\mathbb{A}^n \times \{0\} \arrow[r] & \mathbb{A}^n \times \mathbb{A}^k & \mathbb{A}^n \times \tilde{\mathbb{A}^k} \arrow[l, "\pi"]
\end{tikzcd}$$

A simple example is the blow-up of $\mathbb{C}^n$ along a linear subspace $\mathbb{C}^m$ satisfying $z_{m+1} = \cdots = z_n = 0$ which will be used next. Since linear space is also a variety, the blow-up can be derived as

$$\mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) := \{((z_1, \cdots, z_n), [x_{m+1}, \cdots, z_n]) \mid z_i x_j = z_j x_i,\ i, j = m+1, \cdots, n\},$$

with a projection $\pi : \mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) \to \mathbb{C}^n$.

2.4 Ultimate Generalization: Blowing up along a Submanifold

Finally we can construct the blow-up of an n-dimensional complex manifold X along an arbitrary submanifold $Y \subset X$ of dimension m. In order to do so, we choose an atlas $X = \bigcup \varphi(U_i)$, $\varphi(U_i) \to X$ where U_i are open in $\mathbb{C}^n$ and $\varphi(U_i \bigcap \mathbb{C}^m) = \varphi(U_i) \bigcap Y$ for $\mathbb{C}^m = \{(z_1, \cdots, z_n) \mid z_{m+1} = \cdots = z_n = 0\}$.

3 Construction: Algebra

3.1 Blow-up Algebra

Definition. Let R be a ring and let $I \subset R$ be an ideal of R. The *blow-up algebra* or *Rees algebra* , associated with the pair (R, I), is the graded R-algebra

$$\mathrm{Bl}_I(R) := \bigoplus_{n \geq 0} I^n = R \oplus I \oplus I^2 \oplus \cdots$$

For a *graded ring*, we mean a ring R that is the direct sum of abelian groups R_i s.t. $R_i R_j \subseteq R_{i+j}$. We would write

$$R = \bigoplus_{n \in \mathbb{N}_0} R_n,$$

where the elements in any factor R_n of the decomposition are called the *homogeneous elements* of *degree* n. Every elements $a \in R$ can be written as a sum $a = a_{i_1} + \cdots + a_{i_k}$ where a_{i_j} lying different R_{i_j} are called the *homogeneous components* of a. A graded module M over a graded ring R can be written as

$$M = \bigoplus_{n \in \mathbb{N}_0} M_n,$$

where $R_i M_j \subseteq M_{i+j}$. A graded algebra A over a ring R is an algebra if it is graded as a ring. It is clear that the blow-up algebra is a graded R-algebra. And this is the key property we will use following.

3.2 Proj Construction

Having the corresponding ring of blow-up, we still need to realize the geometrical object by this blow-up algebra. This moment, the simple spectrum does not work since the blow-up should be 'projective'. The process can be described as Proj construction. First we give two examples to illustrate the construction. This is exactly the analogy of the construction of projective spaces.

Here we start with two examples, that how we construct the projective spaces by the coordinate ring, then the generalization.

Consider affine lines

$$\begin{aligned} U_0 &= (\text{Spec } \mathbb{C}[x], \mathcal{O}_{\text{Spec } \mathbb{C}[x]}) \\ U_1 &= (\text{Spec } \mathbb{C}[y], \mathcal{O}_{\text{Spec } \mathbb{C}[y]}), \end{aligned}$$

one can define an affine scheme structure on an open set X_x of $X = \text{Spec } \mathbb{C}[x]$ as follows:

$$U_{01} = (\text{Spec } \mathbb{C}[x, \frac{1}{x}], \mathcal{O}_{\text{Spec } \mathbb{C}[x, \frac{1}{x}]}).$$

The points of Spec $\mathbb{C}[x, \frac{1}{x}]$ are the maximal ideals $(x - c)$ where $c \neq 0$ together with (0). It is obvious that $\mathcal{O}_{\text{Spec }\mathbb{C}[x,\frac{1}{x}]} = \mathcal{O}_X|_{X_x}$. Similarly we have another affine scheme structure as

$$U_{10} = (\text{Spec }\mathbb{C}[y, \frac{1}{y}], \mathcal{O}_{\text{Spec }\mathbb{C}[y,\frac{1}{y}]}).$$

The isomorphism

$$\begin{aligned} \varphi \quad : \quad & \mathbb{C}[y, \frac{1}{y}] \to \mathbb{C}[x, \frac{1}{x}] \\ & f(y, \frac{1}{y}) \mapsto f(x, \frac{1}{x}) \end{aligned}$$

induces an isomorphism of affine schemes $(\varphi^*, \varphi^\#) : U_{01} \to U_{10}$. Through this isomorphism, U_0 and U_1 can be glued, yielding the scheme

$$\mathbb{P}^1 = (Z, \mathcal{O}_Z)$$

where

$$Z = X \cup_{\varphi^*} Y$$

and $\mathcal{O}_Z|_X = \mathcal{O}_X$ and $\mathcal{O}_Z|_Y = \mathcal{O}_Y$. Hence $\mathcal{O}_Z$ is obtained by identifying $\mathcal{O}_X|_{X_x}$ and $\mathcal{O}_Y|_{Y_y}$ through $\varphi^\#$.

The next example show some generalization of the previous one.

4 Comparison: Algebraic Variety and Scheme

We first give a full generalization of blow-up algebraically. Let X be a scheme, and let $\mathcal{I}$ be a coherent sheaf of ideals on X. We say the blow-up of X with respect to $\mathcal{I}$ is a scheme $\tilde{X}$ along with a morphism $\pi : \tilde{X} \to X$, such that $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\tilde{X}}$ is a invertible sheaf, with the universal property: for any scheme and morphism $f : Y \to X$ such that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ is a invertible sheaf, there is a unique factorization:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\pi"] & X \\ & Y \arrow[ul, "\tilde{f}"] \arrow[u, "f"'] \end{tikzcd}$$

Here we need to explain some terminologies.

However instead of going that far, we just consider affine scheme, i.e. $X =$ Spec R for some commutative ring R.

The first example is blowing up the maximal ideal (x, y) of ring $\mathbb{C}[x, y]$. Consider the map $f : \mathbb{C} - V(x, y) \to \mathbb{C} \times \mathbb{P}^1, (a, b) \mapsto ((a, b), [a, b])$.

Then is blowing up the ideal (x^2, y) of ring $\mathbb{C}[x, y]$.

5 Geometric Properties of Blow-ups

First we claim that the blow-up at the origin is actually a manifold. It suffices to prove that $\mathcal{O}(-1) = \tilde{\mathbb{C}^n}$ is a vector bundle over $\mathbb{P}^{n-1}$. Let $p : \mathcal{O}(-1) \to \mathbb{P}^{n-1}$ be the canonical projection and let $\mathbb{P}^{n-1} = \bigcup_{i=1}^{n-1} U_i$ be the standard open covering. A canonical trivialization of $\mathcal{O}(-1)$ over U_i is given by

$$\begin{array}{rcl} \varphi : p^{-1}(U_i) & \cong & \mathbb{C} \times U_i \\ (z, l) & \mapsto & (z_i, l) \end{array}$$

and hence the transition maps $\varphi_{ij} : \mathbb{C} \to \mathbb{C}$ are $w \mapsto \frac{z_i}{z_j} w$. Therefore the blow-up at a point is naturally a manifold.

Using the same technique, the blow-up of $\mathbb{C}^n$ along a linear subspace is also a manifold.

We would conclude the thesis by some examples of applications of blow-ups.

Consider the curve $C \subset \mathbb{C}^2$ given by $z_1 \cdot z_2 = 0$. Obviously the origin is not smooth. On the other hand, the closure of C in $\tilde{\mathbb{C}^2}$ is the union of two separated lines $\overline{\pi^{-1}(l_1 - O)}$ and $\overline{\pi^{-1}(l_2 - O)}$, where $l_i = \{(z_1, z_2) | z_i = 0\},\ i = 1, 2$. Hence the closure is smooth.

References

[Ha] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. 52, Springer-Verlag, New York-Heidelberg, 1977.

[Huy] D. Huybrechts, *Complex Geometry: An Introduction*, Universitext, Springer-Verlag, Berlin-Heidelberg, 2005.

[Vak] R. Vakil, *The Rising Sea: Foundations of Algebraic Geometry*, preprint 2017.