List of Papers

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1 Stable and unitary vector bundles on a compact Riemann surface [3]

2 Space of Unitary Vector Bundles on a Compact Riemann Surface [4]

2.1 Basic Information

1. **Reading time:** 2019-Nov to
2. **Classification:** AG, GIT, Construction of Muduli spaces
3. **Content:**
4. **Main background:** Jordan-Hölder in an Abelian category, The Functor Qout.

We will assume the following:

Theorem 2.1. *Let $\mathcal{A}$ be an abelian category. If $M \in \text{ob } \mathcal{A}$ has a Jordan-Hölder series, then its cycle of simple components is determined uniquely up to isomorphism. If $\mathcal{A}$ is both Artinian and Noetherian, then every object in $\mathcal{A}$ has a Jordan-Hölder series.*

2.2 Categories of Vector Bundles on a Riemann Surface

Let $\mathcal{V}$ be the additive category of vector bundles on a compact Riemann surface X, and let $\mathcal{V}^0$ be the full subcategory of vector bundles of degree 0. (Here the degree of a line bundle is defined to be the degree of its determinant bundle.)

Definition. A vector bundle $V \in \mathcal{V}$ is said to be *semi-stable* (resp. *stable*) if for every proper holomorphic subbundle W of V, we have

$$\frac{d(W)}{r(W)} \leq \frac{d(V)}{r(V)}$$

where $\frac{d(V)}{r(V)}$ is called the *slope* of V.

Let $\mathcal{S}$ be the full subcategory of $\mathcal{V}^0$ consisting of semi-stable vector bundles of degree 0.

Proposition 2.2. *The category $\mathcal{S}$ is abelian, Artinian, and Noetherian. Furthermore, if $\alpha \in \text{Hom}(V, W)$, then α is of constant rank on the fibres of V.*

Proof. It suffices to show that ker α, coker α, and coim α are all of degree 0. By semi-stability, all degrees are ≤ 0. If $d(\ker \alpha) < 0$, then by $0 = d(V) = d(\ker \alpha) + d(\text{coker } \alpha)$ we get a contradiction. Similarly for others. ☐

By GAGA [2], the compact Riemann surface X is uniquely determined by its underlying structure of a nonsingular algebraic variety, and a holomorphic vector bundle V on X has a unique underlying structure of an algebraci vector bundle.

Definition. A subcategory $\mathcal{B}$ of $\mathcal{V}$ is said to be *bounded* if there is an algebraic family of vector bundles $\{V_t\}_{t \in T}$ parametrized by an algebraic scheme T such that given $V \in \mathcal{B}$, there is a $t \in T$ for $V \cong V_t$.

Proposition 2.3. *The subcategory $\mathcal{S}_n$ of $\mathcal{S}$ consisting of semi-stable vector bundles of degree 0 and rank $\leq n$, n being a fixed positive integer, is bounded.*

Proof. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bounded subcategories of $\mathcal{V}$, then the subcategory $\mathcal{B}$ consisting of vector bundles which are extensions of an object in $\mathcal{B}_1$ by an object in $\mathcal{B}_2$ is again bounded. Hence it suffices to prove that the stable bundles are bounded. But a stable bundle is indecomposable [3], whence we can use a result by Atiyah [1]. ☐

2.3 Category of Points of N-folded Grassmannians

Through out this section, we shall use $Gr_{p,r}(E)$ denoting the grassmannian of p dimensional subspaces of E which is a $\mathbb{C}$-vector space of rank r, and use $Gr^{p,r}(E)$ denoting the grassmannian of p dimensional quotient spaces of E which is a $\mathbb{C}$-vector space of rank r. Hence there is a canonical isomorphism $Gr_{p,r}(E) \cong Gr^{r-p,r}(E)$. Let $Gr^N_{p,r}(E)$ denote the N-fold product of $Gr_{p,r}(E)$.

Definition. Let N be a fixed positive integer. We denote $\mathcal{G}^N$ the category whose objects are points of $Gr^N_{p,r}(E)$, where E is any vector space of rank $r \geq 0$ and $0 \leq p \leq r$.

A morphism $\alpha : Y \to X$, for $Y = \{F_i\}_{1\leq i\leq N} \in Gr^N_{q,s}(F)$ and $X = \{E_i\}_{1\leq i\leq N}$ in this category is a linear map $\bar{\alpha} : F \to E$ (called the underlying linear map) such that $\bar{\alpha}(F_i) \subseteq E_i$.

It is not hard to see $\mathcal{G}^N$ is an additive category, and satisfies these properties:

1. α is a monomorphism (resp. epi) if and only if $\bar{\alpha}$ is injective (resp. surjective).

2. α has a kernel if and only if the rank of $K_i = \ker \alpha \cap F_i$ is independent for i, and then $\{K_i\}_{1\leq i\leq N}$ is the kernel of α. If α has kernel, then its coimage exists.

3. α has a cokernel if and only if the rank of $M_i = \pi(E_i)$ is independent for i where $\pi : E \to \text{coker } \alpha$ is the canonical projection, and then $\{M_i\}_{1\leq i\leq N}$ is the cokernel of α. If α has kernel, then its image exists.

4. If α has both kernel and cokernel, then the image and coimage of α exist and the canonical morphism from the coimage to the image is an isomorphism if and only if $r(F_i) - r(K_i) = r(E_i) - r(M_i)$ for all $1 \leq i \leq N$.

If α is a monomorphism (resp. epi) and has a cokernel (resp. kernel), then we say that $0 \to Y \xrightarrow{\alpha} X$ (resp. $Y \xrightarrow{\alpha} X \to 0$) is exact. In this case, let Z be the cokernel (resp. kernel) of α and $\beta : X \to Z$ (resp. $\beta : Z \to Y$) be the canonical morphism, we see by the previous comments that α is the kernel of β. Thus we write that $0 \to Y \to X \to Z \to 0$ (resp. $0 \to Z \to Y \to X \to 0$) is exact.

Let n be a integer ≥ 2, then we denote by $\mathcal{G}^{N,n}$ the full subcategory of $\mathcal{G}^N$ consisting of objects which are points of $Gr^N_{r(n-1),rn}(E)$, where E is any vector space of rank $r \geq 0$ and $0 \leq p \leq r$. It is not hard to show that a morphism in $\mathcal{G}^{N,n}$ is a monomorphism (resp. epimorphism) if and only if it is so in $\mathcal{G}^N$.

Definition. An object $X = \{E_i\}_{1\leq i\leq N} \in Gr^N_{p,r}(E)$ is said to be *semi-stable* (resp. *stable*) if, for every subspace F of E (resp. proper subspace) we have

$$\frac{\frac{1}{N}\sum_{i=1}^N r(F \cap E_i)}{p} \leq \frac{r(F)}{r}.$$

Also, for $X = \{E_i\}_{1\leq i\leq N} \in Gr^{p,r}_N(E)$, the canonical image of X in $Gr^N_{r-p,r}(E)$ is semi-stable (resp.) if and only if, for every subspace F of E,

$$\frac{\frac{1}{N}\sum_{i=1}^N r(F_i)}{p} \geq \frac{r(F)}{r}.$$

Proposition 2.4. *Let $0 \to Y \to X \to Z \to 0$ be an exact sequence in $\mathcal{G}^N$ with Y, Z, X in $\mathcal{G}^{N,n}$. Then X is semi-stable if and only if both Y and Z are semi-stable.*

We also denote by $\mathcal{K}^{N,n}$ the full subcategory of $\mathcal{G}^{N,n}$ consisting of the semi-stable objects of $\mathcal{G}^{N,n}$. It is not hard to show that a morphism in $\mathcal{K}^{N,n}$ is a monomorphism (resp. epimorphism) if and only if it is so in $\mathcal{G}^N$.

Proposition 2.5. *Let $\alpha : Y \to X$ (resp. $\alpha : X \to Y$) be a monomorphism (resp. epi) in $\mathcal{G}^N$ with $X, Y \in \mathcal{G}^{N,n}$. Then if X is semi-stable, $0 \to Y \to X$ (resp. $X \to Y \to 0$) is exact, and Y is semi-stable.*

Proposition 2.6. *Let X be a stable object of $\mathcal{G}^{N,n}$. Then if $\alpha : X \to Y$ is a morphism in $\mathcal{K}^{N,n}$, then either α is 0, or $0 \to Y \to X$ is exact.*

Definition. An object $X \in \mathcal{G}^{N,n}$ is said to have a *stable series* S if there is an increasing sequence $S = \{X_i\}_{q\leq i\leq m}$

$$X_1 \subset X_2 \subset \cdots \subset X_m = X$$

of subobjects of X such that every one of the canonical monomorphisms $X_i \to X_{i+1}$ has a cokernel X_{i+1}/X_i, and $X_1, \cdots, X_m/X_{m-1}$ are all stable objects of $\mathcal{G}^{N,n}$.

By an application of Proposition 2.4, it follows that $X \in \mathcal{K}^{N,n}$ if X has a stable series S. We denote by $\mathcal{A}^{N,n}$ the full subcategory of $\mathcal{K}^{N,n}$ consisting of those objects which possess stable series.

Proposition 2.7. *The category $\mathcal{A}^{N,n}$ is abelian, artinian, and noetherian, and the simple object in it are precisely the stable objects.*

2.4 Connecting Two Categories

2.5 The Main Theorem and Its Proof

References

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[2] Serre, Jean-Pierre (1956), ”Géométrie algébrique et géométrie analytique”, *Annales de l'Institut Fourier*, **6**: 1–42, DOI: 10.5802/aif.59, ISSN 0373-0956, MR 0082175

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