Representation Homology

and some computations with unipotent coefficients

Guanyu Li

BUGCAT, November 11 2023

Representations

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Let V be a k-vector space. When we say "a representation V", there are generally three settings:

1. For a discrete group π (later we shall use G to denote an algebraic group, and here we use Greek letters to denote a discrete group), a representation of π is a group homomorphism
$$\pi \to \mathrm{GL}(V),$$
2. for a Lie algebra $\mathfrak{g}$, a representation is a Lie-algebra homomorphism
$$\mathfrak{g} \to \mathfrak{gl}(V),$$
and

3. for a k-algebra A, a representation is an algebra homomorphism
$$A \to \mathrm{End}_k(V).$$

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For any case, we can construct a space (or one could call it a variety), universally parameterizing all representations.

Groups and Lie algebras

Definition

An affine group scheme G ovr k is a functor

$$G : k - \mathbf{CommAlg} \to \mathbf{Gp}$$

that is representable.

Examples

GL_2 is the affine group scheme

$$\begin{aligned} k - \mathbf{CommAlg} &\to \mathbf{Gp} \\ A &\mapsto GL_2(A) \end{aligned}$$

where $GL_2(A)$ is the group of 2×2 matrices with determinant invertible. The representative is $k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, (\det X)^{-1}]$, i.e.

$$GL_2 \cong \mathrm{Hom}(k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, (\det X)^{-1}], -).$$

Given a Lie algebra $\mathfrak{g}$, it could be viewed as a functor

$$\begin{aligned} k-\mathbf{CommAlg} &\to \mathbf{Lie}_k \\ A &\mapsto \mathfrak{g}(A), \end{aligned}$$

where $\mathfrak{g}(A)$ is the Lie algebra with vector space $g \otimes A$ and bracket $[\alpha \otimes a, \beta \otimes b] := [\alpha, \beta] \otimes (ab)$. This can be viewed as the Lie algebra $\mathfrak{g}$ with coefficients in A.

Examples

$\mathfrak{gl}_2$ is the functor

$$A \mapsto \mathfrak{gl}_2(A)$$

where $\mathfrak{gl}_2(A)$ is the Lie algebra of 2×2 matrices with coefficients in A.

Space parameterizing representations

Theorem

Given a (discrete) group π and an affine group scheme G over k, the functor

$$\begin{aligned} \mathrm{Rep}_G(\pi) : k-\mathbf{CommAlg} &\to \mathbf{Set} \\ A &\mapsto \mathrm{Hom}_{\mathbf{Gp}}(\pi, G(A)) \end{aligned}$$

is representable. The representative is denoted by $(\pi)_G$. In other words, there is a natural isomorphism

$$\hom_{k-\mathbf{CommAlg}}((\pi)_G, A) \cong \hom_{\mathbf{Gp}}(\pi, G(A)) \quad (1)$$

for any commutative k-algebra A.

Theorem (cont.)

The affine scheme $(\mathrm{Spec}\ (\pi)_G)$ associated to the ring $(\pi)_G$ is called the **representation scheme**. Under mild conditions this scheme is a variety and it is called the **representation variety**.

This gives a functor

$$(-)_G : \mathbf{Gp} \to k - \mathbf{CommAlg},$$

which is the left adjunction of $G : k - \mathbf{CommAlg} \to \mathbf{Gp}$.

Examples

Let $\pi = \pi_1(T) = \mathbb{Z} = \langle x, y \mid xy = yx \rangle$, and let $G = GL_2 = \mathrm{Spec}\ k[a, b, c, d, (ad - bc)^{-1}]$. Then

$$(\pi)_{GL_2} \cong \frac{k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}, (\det X)^{-1}, (\det Y)^{-1}]}{XYX^{-1}Y^{-1} = I}.$$

This can be interpreted as all pairs of matrices $(X, Y) \in GL_2$ s.t. $XY = YX$.

Space parameterizing Lie representations

Space parameterizing Lie representations

Theorem

Given a finite dimensional Lie algebra $\mathfrak{g}$ and a Lie algebra $\mathfrak{a}$, the functor $\mathrm{Rep}_{\mathfrak{g}}(\mathfrak{a}) : k - \mathbf{CommAlg} \to \mathbf{Set}$

$$A \mapsto \mathrm{Hom}_{\mathbf{Lie}}(\mathfrak{a}, \mathfrak{g}(A))$$

is representable, where $\mathfrak{g}(A)$ is "the Lie algebra $\mathfrak{g}$ with coefficient A". The representative is denoted by $(\mathfrak{a})_{\mathfrak{g}}$. In other words, there is a natural isomorphism

$$\hom_{k-\mathbf{CommAlg}}((\mathfrak{a})_{\mathfrak{g}}, A) \cong \hom_{\mathbf{Lie}}(\mathfrak{a}, \mathfrak{g}(A)) \tag{2}$$

for any commutative k-algebra A. This gives a functor

$$(-)_{\mathfrak{g}} : \mathbf{Lie} \to k - \mathbf{CommAlg},$$

which is the left adjunction of $\mathfrak{g} : k - \mathbf{CommAlg} \to \mathbf{Lie}$.

Examples

Let $\mathfrak{a}$ be the two dimensional abelian Lie algebra, and let $\mathfrak{g} = \mathfrak{gl}_2$. Then

$$(\mathfrak{a})_{\mathfrak{g}} \cong \frac{k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}]}{[X, Y] = 0}. \tag{3}$$

This can be interpreted as all pairs of matrices $(X, Y) \in \mathfrak{gl}_2$ s.t. $[X, Y] = 0$.

This example is the famous (additive) commuting variety.

Classical results, and problems

Theorem, [1]

Let $\mathfrak{g}$ be a reductive Lie algebra over an algebraic closed field k of characteristic 0 and let $C(\mathfrak{g}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid [x, y] = 0\}$. Then $C(\mathfrak{g})$ is an irreducible algebraic variety.

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Theorem, [2]

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{gl}_n$, then $C_{\mathfrak{h}}(\mathfrak{gl}_n) = \{(x, y) \in \mathfrak{gl}_n \times \mathfrak{gl}_n \mid [x, y] \in \mathfrak{h}\}$ is a complete intersection, one component of which is exactly $C(\mathfrak{gl}_n)$.

However,

1. The parameterizing spaces lose all higher information of the topological space (in the example mentioned above, the space was T^2).
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Luckily, one could introduce derived tools, to resolve those problems in some sense.

Deriving the representation schemes

Deriving the representation schemes

One could extend the adjunction

$$(-)_G : \mathbf{Gp} \rightarrow k - \mathbf{CommAlg},$$

levelwise to simplicial setting, then one has adjunction

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between two model categories, and

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Theorem, [3]

For the extended adjunction

$$(-)_G : s\mathbf{Gp} \to s(k - \mathbf{CommAlg}) : G$$

$(-)_G$ admits a (total) left derived functor $\mathbb{L}(-)_G$, which commutes with homotopy colimits.

Derived functors

For algebraists who are not familiar with homotopically deriving a functor, you could think $\mathbb{L}(-)_G$ is something behaves like $- \otimes^{\mathbb{L}} -$, the derived tensor product, whose homology is Tor.

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In fact, $\mathbb{L}(-)_G$ is a derived tensor product.

There is a chain of Quillen equivalences

$$\mathbf{Top}_{*,0} \simeq s\mathbf{Set}_0 \simeq s\mathbf{Gp}$$

where $s\mathbf{Set}_0$ is the (sub)category consisting of all reduced simplicial sets and $\mathbf{Top}_{*,0}$ is the (sub)category of all pointed, connected topological spaces.

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So what we actually need are

1. an affine group scheme G over k, and
2. a pointed, connected topological space X.

Then the homology of

$$\mathbb{L}(X)_G \in \mathrm{Ho}\ s(k - \mathbf{CommAlg})$$

is called the *representation homology* of X with coefficient G, denoted by $HR_*(X, G)$.

Also for the Lie case, one has

Theorem, [4]

Given a finite dimensional Lie algebra $\mathfrak{g}$, the adjunction

$$(-)_{\mathfrak{g}} : s\mathbf{Lie} \rightarrow s(k - \mathbf{CommAlg}) : \mathfrak{g}(-)$$

is an Quillen equivalence, so $(-)_{\mathfrak{g}}$ admits a (total) left derived functor $\mathbb{L}(-)_{\mathfrak{g}}$.

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Then the homology of

$$\mathbb{L}(\mathfrak{a})_{\mathfrak{g}} \in \mathrm{Ho}\ s(k - \mathbf{CommAlg})$$

is called the *representation homology* of $\mathfrak{a}$ with coefficient $\mathfrak{g}$, denoted by $HR_*(\mathfrak{a}), \mathfrak{g})$.

Example : when $X = T^2$

Examples

For the topological 2-torus T^2, it has a homotopy colimit interpretation

$$T^2 \simeq \mathrm{hocofib}\ S^1 \xrightarrow{\alpha} S^1 \vee S^1 = \mathrm{hocolim}[* \leftarrow S^1 \xrightarrow{\alpha} S^1 \vee S^1] \quad (4)$$

where $\alpha : S^1 \to S^1 \vee S^1$ is the continuous map inducing a map $\pi_1(S^1) \to \pi_1(S^1 \vee S^1), 1 \mapsto aba^{-1}b^{-1}$, thus

$$\begin{aligned}(T^2)_G &\cong \mathrm{hocolim}[(*)_G \leftarrow (S^1)_G \xrightarrow{\alpha} (S^1 \vee S^1)_G] \\ &\cong k \otimes^{\mathbb{L}}_{\mathcal{O}(G)} \mathcal{O}(G)^{\otimes 2}\end{aligned}$$

where $\mathcal{O}(G)$ is the coordinate ring of G and $\mathcal{O}(G)^{\otimes 2}$ is an $\mathcal{O}(G)$-module via α_*.

Example cont.

Examples

When G is a matrix group (for instance GL_2 or U_3), $\mathcal{O}(G)$ is generated by a matrix of variables, and the structure map is

$$\alpha_* : X \mapsto YZY^{-1}Z^{-1} \tag{5}$$

Thus the complex could be represented by the Koszul complex

$$\mathrm{Kos}_*^{\times} := k[Y, Z, T; dT = \alpha_*(X) = YZY^{-1}Z^{-1} - I]. \tag{6}$$

Apparently $H_0(\mathrm{Kos}_*^{\times}) = \frac{k[Y,Z]}{YZY^{-1}Z^{-1}=I}$, which is the multiplicative commuting variety.
In this sense, the representation homology $HR_*(T^2, G)$ is the derived multiplicative commuting variety.

Example : Lie case

Examples

For the 2-dimensional abelian Lie algebra $\mathfrak{a}$, it has a semi-free resolution

$$0 \leftarrow L(2) \xleftarrow{[\alpha,\beta]\leftarrow\!\shortmid\gamma} L(1) \rightarrow 0$$

where $L(n)$ is the free Lie algebra generated by n-elements.
When $\mathfrak{g}$ is a matrix Lie algebra (for instance $\mathfrak{gl}_2$ or $\mathfrak{u}_3$), it also has a Koszul complex representative

$$\mathrm{Kos}_*^+ := k[A, B, S; dS = [A, B]]. \tag{7}$$

Result

Let U_n be the affine group scheme consisting of all unipotent matrices, $\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}$.

Theorem (Li)

The commuting variety $C(U_n)$ is a complete intersection if and only if

$$HR_i(T^2, U_n) = 0 \quad \forall i \geq n.$$

Result

Let U_n be the affine group scheme consisting of all unipotent matrices, $\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}$.

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Corollary

- The commuting variety $C(U_n)$ is a complete intersection if $n = 2, 3, 4, 5$.
- The commuting variety $C(U_n)$ is NOT a complete intersection if $n = 6$.

Theorem (Li)

There is an isomorphism of graded vector spaces

$$HR_i^{\mathrm{Gp}}(T^2, U_n) \cong HR_i^{\mathrm{Lie}}(\mathfrak{a}, \mathfrak{u}_n).$$

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Thank you!

Why U_n?

Dyson Conjecture

The Laruant polynomial

$$\prod_{1 \le i \ne j \le n} (1 - t_i/t_j)^{a_i}$$

has constant $\frac{(a_1+\cdots+a_n)!}{a_1!\cdots a_n!}$.

q-Dyson Conjecture

The Laruant polynomial

$$\prod_{1 \le i \ne j \le n} (\frac{t_i}{t_j}; q)_{a_i} (\frac{qt_i}{t_j}; q)_{a_j}$$

has constant $\frac{(q;q)_{a_1+\cdots,a_n}}{(q;q)_{a_1}\cdots(q;q)_{a_n}}$. $[(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k).]$

(Strong) Mcdonald Conjecture [5]

If we know

$$H^*_{\mathrm{CE}}(\mathfrak{g}[x]/(x^n)),$$

then q-Dyson conjecture can be derived, where $\mathfrak{g}$ is reductive.

Roughly, [4]

If we know

$$\mathrm{DRep}_{\mathfrak{g}}(k[i] \oplus k[j])^{\mathfrak{g}},$$

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Roughly, [4]

If we know

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then q-Dyson conjecture can be derived, where $\mathfrak{g}$ is reductive and i, j are natural number of different oddity.

So what if $\mathfrak{g}$ is not reductive?