1 Introduction

In algebraic geometry, we have to deal with singularities of varieties. The problem of **resolution of singularities** asks whether every algebraic variety X has a resolution, a non-singular variety Y with a proper birational map $f : Y \to X$. We already have different ways to resolve a singularity, and blowing up is one of the most important methods.

Usually $\mathbb{C}^n$ is a denotation geometrically, while $\mathbb{A}^n_{\mathbb{C}}$ is more algebraically. However we will regard them equivalently through this paper. $\mathbb{A}^n, \mathbb{P}^n$ are $\mathbb{A}^n_{\mathbb{C}}$ and $\mathbb{CP}^n$ respectively except indicated explicitly. All of the construction can be transplanted over another (algebraic closed) field.

In order to get a full understand of this paper, the readers are supposed to know the results from commutative algebra, properties of projective spaces, some results from geometry and topology, basic construction of algebraic geometry and sheaves. All of the definition will be mentioned later in the text however it may be extremely confusing if it is the first time to encounter these terminologies.

2 Construction: Geometry

2.1 Blow-up of $\mathbb{C}^n$ at Origin

We will start from the simplest case, i.e. blowing up the origin of $\mathbb{C}^n$.

Consider a subset of $\mathbb{C}^n \times \mathbb{P}^{n-1}$

$$\tilde{\mathbb{C}^n} := \{((a_1, \cdots, a_n), [b_1, \cdots, b_n]) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid a_i b_j = a_j b_i, 1 \leq i, j \leq n\}.$$

It is easy to see that we have a natural projection from $\tilde{\mathbb{C}^n}$ to $\mathbb{C}^n$:

$$\pi : ((a_1, \cdots, a_n), [b_1, \cdots, b_n]) \mapsto (a_1, \cdots, a_n).$$

Definition. The set $\tilde{\mathbb{C}^n}$ along with the map $\pi : \tilde{\mathbb{C}^n} \to \mathbb{C}^n$ is called the *blow-up* of $\mathbb{C}^n$ at the origin.

Immediately we have these properties of $\tilde{\mathbb{C}^n}$:

For any point $O \neq z = (a_1, \cdots, a_n) \in \mathbb{C}^n$, there is at least a $1 \leq i_0 \leq n$ s.t. $a_{i_0} \neq 0$ hence $b_{i_0} \neq 0$. Thus the equations can be written as $\frac{b_j}{b_{i_0}} = \frac{a_j}{a_{i_0}}, 1 \leq j \leq n$. Therefore there is only **ONE** point x in $\tilde{\mathbb{C}^n}$ with $\pi(x) = z$, i.e. $x = ((a_1, \cdots, a_n), [a_1, \cdots, a_n])$.

$\pi^{-1}(O) \cong \mathbb{P}^{n-1}$. Indeed, any point $[b_1, \cdots, b_n] \in \mathbb{P}^{n-1}$ satisfies the equations if $(a_1, \cdots, a_n) = 0$.

The points of $\pi^{-1}(O)$ are in 1-1 corresponding to the lines through the origin in $\mathbb{C}^n$. A line through the origin l can be given by the parametric equation

$x_i = a_i t$ where $1 \leq i \leq n$ and $a_i \in \mathbb{C}$ are not all zero, thus $[a_1, \cdots, a_n]$ is a point in $\mathbb{P}^{n-1}$. Now consider the line $\tilde{l} = \pi^{-1}(l - O)$ in $\tilde{\mathbb{C}^n} - \pi^{-1}(O)$, it is $\{((a_1 t, \cdots, a_n t), [a_1, \cdots, a_n]) \mid t \in \mathbb{C}^*\}$. These equations also make sense for $t = 0$ and give the closure $\bar{l}$ of $\tilde{l}$ in $\tilde{\mathbb{C}^n}$. $\tilde{l}$ meets $\pi^{-1}(O)$ in the point $[a_1, \cdots, a_n] \in \mathbb{P}^{n-1}$, so we see that sending l to Q gives us the correspondence between lines through origin in $\mathbb{C}$ and points of $\pi^{-1}(O)$.

$\tilde{\mathbb{C}^n}$ is irreducible. Indeed $\tilde{\mathbb{C}^n} = \tilde{\mathbb{C}^n} - \pi^{-1}(O) \cup \pi^{-1}(O)$, where the first piece is isomorphic to $\mathbb{C}^n - O$ which is obviously irreducible, and every point of the second part $\pi^{-1}(O)$ is in the closure of some line of $\mathbb{C}^n - \tilde{\pi}^{-1}(O)$. Hence $\mathbb{C}^n - \tilde{\pi}^{-1}(O)$ is dense in $\tilde{\mathbb{C}^n}$ and $\tilde{\mathbb{C}^n}$ is irreducible.

Here we give an example to illustrate how this process works.

The blow-up of $\mathbb{R}^2$ at the origin can be visualized as follows: we have known that points not the origin are in a 1-1 correspondence by π, so we leave them stable. However we need to replace the origin by $\mathbb{RP}^1$, i.e. a circle. For any line l parameterized by $x = a_1 t, y = a_2 t$, $\pi^{-1}(l - O)$ lies in $\tilde{\mathbb{R}^2} - \mathbb{RP}^1$, and the closure of $\pi^{-1}(l - O)$ consist of the point $[a_1, a_2]$ in $\mathbb{RP}^1$. So the disconnected part of $\pi^{-1}(l - O)$ are glued by the point. This operation works like we glue the antipodal points together. Topologically, this is regarding the missing point as a disk, gluing the boundary of the disk with the boundary of a Möbius band, since both of the boundary are circles.

Using the technique above, it is easy to blow up an algebraic set at the origin.

Definition. If V is a closed algebraic set of $\mathbb{C}$ passing through the origin, the *blow-up* of V at the origin is $\tilde{V} = \overline{\pi^{-1}(V - O)}$, where $\pi : \tilde{\mathbb{C}^n} \to \mathbb{C}^n$ is the blow-up of $\mathbb{C}^n$ as above. We denote also by $\pi : \tilde{V} \to V$ the morphism by the restriction of $\pi : \tilde{\mathbb{C}^n} \to \mathbb{C}^n$ to $\tilde{V}$.

2.2 A Little Further: Surgery

To blow up a (complex) manifold

2.3 Blow-up of $\mathbb{C}^n$ at a Affine Variety

For an *affine variety*, we mean an **irreducible** algebraic subset of $\mathbb{A}^n$, denoted as

$$Z(f_1, \cdots, f_k) = \{(a_1, \cdots, a_n) \in \mathbb{A}^n \mid f_i(a_1, \cdots, a_n) = 0 \ \ \forall 1 \leq i \leq k\},$$

where $(f_1, \cdots, f_k)$ is a prime ideal. We can always find finitely many generators as a result of Hilbert basis theorem.

We define $A(V) = k[x_1, \cdots, x_n]/(f_1, \cdots, f_n)$ as the *coordinate ring* of affine variety $V = Z(f_1, \cdots, f_k)$, and let $k(V)$ be the quotient field, which is called the *field of fractional functions* on V. If P is a point of V, we define

$$\mathcal{O}_{V,P} = \{\frac{f}{g} \mid f, g \in A(V) \text{ and } g(P) \neq 0\}$$

as the *local ring* of V at point P. If $U \subset V$ is a nonempty open set, we set

$$\mathcal{O}_V(U) := \bigcap_{P \in U} \mathcal{O}_{V,P}.$$

This is a subring of $k(V)$, and we call this the *the ring of regular function* on U.

The blow-up of $\mathbb{A}^n$ with respect to the subvariety $V(f_1, \cdots, f_k)$ is given by

$$\{((a_1, \cdots, a_n), [b_1, \cdots, b_k]) \mid b_i f_j(a_1, \cdots, a_n) = b_j f_i(a_1, \cdots, a_n)\}$$

which is a subset of $\mathbb{A}^n \times \mathbb{P}^{k-1}$.

3 Construction: Algebra

3.1 Blow-up Algebra

Definition. Let R be a ring and let $I \subset R$ be an ideal of R. The *blow-up algebra* or *Rees algebra* , associated with the pair (R, I), is the graded R-algebra

$$\mathrm{Bl}_I(R) := \bigoplus_{n \geq 0} I^n = R \oplus I \oplus I^2 \oplus \cdots$$

For a *graded algebra*, we mean

3.2 Proj Construction

Having the corresponding ring of blow-up, we still need to realize the geometrical object by this blow-up algebra. This moment, the simple spectrum does not work since the blow-up should be 'projective'. The process can be described as Proj construction. First we give two examples to illustrate the construction. This is exactly the analogy of the construction of projective spaces.

Here we start with two examples, that how we construct the projective spaces by the coordinate ring, then the generalization.

4 Comparison: Algebraic Variety and Scheme

We first give a full generalization of blow-up. Let X be a scheme, and let $\mathcal{I}$ be a coherent sheaf of ideals on X. We say the blow-up of X with respect to $\mathcal{I}$ is a scheme $\tilde{X}$ along with a morphism $\pi : \tilde{X} \to X$, such that $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\tilde{X}}$ is a invertible sheaf, with the universal property: for any scheme and morphism $f : Y \to X$ such that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ is a invertible sheaf, there is a unique factorization:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\pi"] & X \\ & Y \arrow[u, "f"'] \arrow[ul, "\tilde{f}"] \end{tikzcd}$$

Here we need to explain some

However instead of going that far, we just consider affine scheme, i.e. $X = \text{Spec } R$ for some commutative ring R.

The first example is blowing up the maximal ideal (x, y) of ring $\mathbb{C}[x, y]$. Consider the map $f : \mathbb{C} - V(x, y) \to \mathbb{C} \times \mathbb{P}^1, (a, b) \mapsto ((a, b), [a, b])$.

Then is blowing up the ideal (x^2, y) of ring $\mathbb{C}[x, y]$.