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Dæmatími

Dæmi 2

Lýsing: Let $f_k(n)$ be the number of partitions of n into k distinct parts. Give a combinatorial proof that

$$f_k\left(n + \binom{k}{2}\right) = p_k(n)$$

Lausn:

a. We have

$$P(x) = \prod_{j\geq 1} \frac{1}{1-x^j} = \prod_{j\geq 1}(1 + x^j + x^{2j} + x^{3j} + \cdots)$$

Here, j is kind of the size of the part and

$$Q(x) = \prod_{j\geq 2} \frac{1}{1-x^j}$$

But then

$$P(x) = \frac{1}{1-x}Q(x) \quad \text{or} \quad Q(x) = (1-x)P(x)$$

b. From $Q(x) = (1-x)P(x)$ we have

$$\begin{aligned}\sum_{n\geq 0} q(n)x^n &= (1-x)\sum_{n\geq 0} p(n)x^n \\ &= \sum_{n\geq 0} p(n)x^n - x\sum_{n\geq 0} p(n)x^n\end{aligned}$$

However we have

$$x\sum_{n\geq 0} p(n)x^n = \sum_{n\geq 0} p(n)x^{n+1} = \sum_{n\geq 1} p(n-1)x^n$$

so we get

$$\sum_{n\geq 0} p(n)x^n - x\sum_{n\geq 0} p(n)x^n = 1 + \sum_{n\geq 1}(p(n) - p(n-1))x^n$$

We can now equate the coefficients, that is

$$q(n) = p(n) - p(n-1), \quad n \geq 1$$

For the combinatorial proof, we first rewrite our expression as

$$q(n) + p(n-1) = p(n)$$

We notice that $\{(\lambda_1, \dots, \lambda_k) \vdash n\}$ is the disjoint union

$$\text{(i)} \quad \{(\lambda_1, \dots, \lambda_k) \vdash n \mid \lambda_k \neq 1\} \cup \{(\lambda_1, \dots, \lambda_k) \vdash n \mid \lambda_k = 1\} \quad \text{(ii)}$$

The cardinality of (1) is $q(n)$ by definition and the cardinality of (ii) is $p(n-1)$ by removing $\lambda_k = 1$.

Dæmi 5

Lýsing Let S_n be the set of permutations of $[n]$. A *descent* of $\pi = a_1 a_2 \dots a_n \in S_n$ is an index i such that $a_i > a_{i+1}$.

Lausn:

a. We have

$$f(n) = f(n-1) - 1 + f(n-1) + n - 1 = n + 2(f(n-1) - 1)$$

b. We have

$$\sum_{n\geq 1} f(n)x^n = \sum_{n\geq 1}(2f(n-1) + n - 2)x^n$$

so

$$\begin{aligned} F(x) - 1 &= 2\sum_{n\geq 1} f(n-1)x^n + \sum_{n\geq 1} nx^n - 2\sum_{n\geq 1} x^n \\ &= 2xF(x) + x\sum_{n\geq 1} nx^{n-1} - \frac{2x}{1-x} \end{aligned}$$

Notice that

$$x\sum_{n\geq 1} nx^{n-1} = \frac{d}{dx}\left(\sum_{n\geq 0} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

Then we have

$$F(x) - 1 = 2xF(x) + \frac{1}{(1-x)^2} - \frac{2x}{1-x}$$

so

$$F(x) = \frac{1-3x-3x^2}{(1-x)^2(1-2x)}$$

c. For the closed formula we could use partial-fraction decomposition.