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Dæmatími, PS2

Problem 1.14

Description:

a. Give a combinotorical proof of the identity

$$\sum_{n=k}^{m} \binom{n}{k} = \binom{m+1}{k+1}$$

b. Calculate the sum

$$\sum_{k=0}^{n} (1 + 3k + 3k^2) \quad \text{when} \quad n = 100^{100} - 1$$

c. Deveise a strategy to compute a closed form of any given sum $\sum_{k=0}^{n} p(k)$ in which $p(k)$ is a polynomial in k

Solution:

b & c. We want to calculate

$$\sum_{k=0}^{n} (1 + 3k + 3k^2) \quad \text{where} \quad n = 100^{100} - 1$$

Notice that

$$\begin{aligned}
\binom{k}{0} &= 1 \\
\binom{k}{1} &= k \\
\binom{k}{2} &= \frac{k(k-1)}{2} = \frac{1}{2}k + \frac{1}{2}k^2 \\
\binom{k}{3} &= \frac{k(k-1)(k-2)}{6} = \frac{1}{3}k - \frac{1}{2}k^2 + \frac{1}{6}k^3
\end{aligned}$$

and in general $\binom{k}{d}$ is a polynomial of degree d

Let $P_d = \{p(k) \in \mathbb{Q}[k] \mid \deg p(k) \leq d\}$. The standard basis for P_d is

$$1, k, k^2, \ldots, k^d$$

Another basis is

$$\binom{k}{0}, \binom{k}{1}, \binom{k}{2}, \dots, \binom{k}{d}$$

Let $T : \mathbb{Q}^{d+1} \to \mathbb{Q}^{d+1}$ be defined by

$$T(a_0, \dots, a_d) = (b_0, \dots, b_d) \quad \text{where} \quad \sum_{m=0}^{d} a_m k^m = \sum_{m=0}^{d} b_m \binom{k}{m}$$

The matrix representation for this change of basis would be

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Which is the inverse matrix of

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

Which is simply the matrix of the coefficients we calculated for $\binom{k}{m}$ above.

If we multiply our matrix by the vector we want to calculate, that is $\mathbf{v} = (1, 3, 3, 0)$ we have that $M\mathbf{v} = (1, 6, 6, 0)$

$$\sum_{k=0}^{n}(1+3k+3k^2) = \sum_{k=0}^{n}\left(\binom{k}{0} + 6\binom{k}{1} + 6\binom{k}{2}\right) = \binom{n+1}{0} + 6\binom{n+1}{1} + 6\binom{n+1}{3}$$

Problem 1.15

Description: How many ways c_n are there to tile (cover) a $1 \times n$ rectangle with bricks of type $\square$, $\boxed{}$ and $\blacksquare$?

Solution: Here we have

$$G = \varepsilon + (\square + \boxed{} + \blacksquare)G$$

so

$$\begin{aligned} F &= 1 + (x + 2x^2)F \\ &= \frac{1}{1 - x - 2x^2} \\ &= \frac{1}{3}\left(\frac{2}{1-2x} + \frac{1}{1-(-x)}\right) \\ &= \frac{1}{3}\left(\sum_{n\geq 0} 2^{n+1}x^n + \sum_{n\geq 0}(-1)^n x^n\right) \\ &= \sum_{n\geq 0}\frac{1}{3}(2^{n+1} + (-1)^n)x^n \end{aligned}$$

Problem 2.3

Description:

a. Let $f(n)$ be the number of integer partitions of n into distinct parts. Find an expression for the ordinary generating function $F(x) = \sum_{n \geq 0} f(n)x^n$

b. Let $g(n)$ be the number of integer partitions of n into odd parts. Find and expression for the ordinary generating function $G(x) = \sum_{n \geq 0} g(n)x^n$

c. Prove that $f(n) = g(n)$ for all $n \in \mathbb{N}$.

Solution: The generating function for partitions into distinct parts is

$$F(x) = (1 + x)(1 + x^2)(1 + x^3) \cdots$$

The generating function for partitions into odd parts is

$$G(x) = \frac{1}{1 - x} \cdot \frac{1}{1 - x^3} \cdot \frac{1}{1 - x^5} \cdots$$

Notice that

$$\begin{aligned} G(x) &= \frac{1 - x^2}{1 - x} \cdot \frac{1 - x^4}{1 - x^2} \cdot \frac{1 - x^6}{1 - x^3} \cdots \\ &= \prod_{i \geq 1} \frac{1 - (x^i)^2}{1 - x^i} \\ &= \prod_{i \geq 1} (1 + x^i) \\ &= (1 + x)(1 + x^2)(1 + x^3) \cdots = F(x) \end{aligned}$$

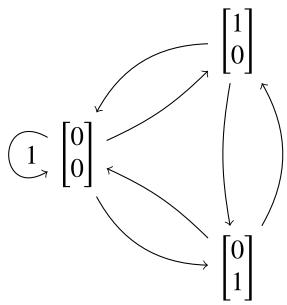
Problem 2.6

Description Let $a(m, n)$ be the number of $m \times n$ matricies with entries in $\{0, 1\}$ with no two adjacent ones in rows or columns. Find generating functions for:

a. The number $a(1, n)$ b. The number $a(2, n)$ c. The number $a(3, n)$

d. For a fixed, but general, m, can you say anything about the generating function for the numbers $a(n, m)$?

Solution: We will use the transfer matrix method. The graph representing the possible combinations for the $2 \times n$ matricies is



The adjacency matrix for the graph is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The generating function for the number of combinations is then the sum of all entries in

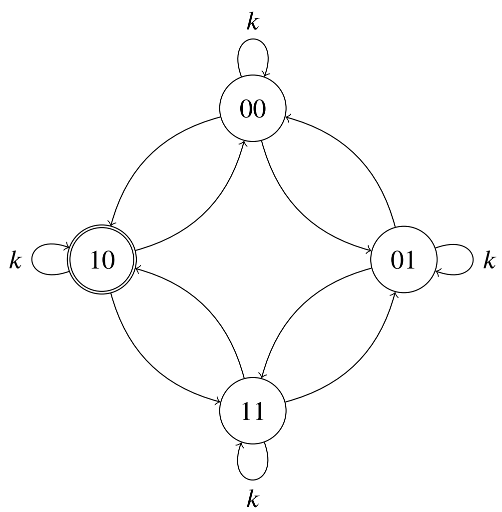
$$B = (I - xA)^{-1}$$

however we need to account for the fact that the walks of length 0 are the combinations counted in $a(2, 1)$.

Problem 2.7

Description: Each of n (distinguishable) telephone poles is painted red, blue, or any of an additional k colours. Determine the number of ways this can be done so that an odd number are painted red and an even number blue.

Solution: If we represent whether the red and blue poles are odd or even with a 1 or 0, we can represent their state with a 2-bit binary string where the first bit represents red and the second one blue (read from left to right). Thus we get the DFA



The adjacency matrix representing this digraph is

$$A = \begin{bmatrix} k & 1 & 1 & 0 \\ 1 & k & 0 & 1 \\ 1 & 0 & k & 1 \\ 0 & 1 & 1 & k \end{bmatrix}$$

The generating function for walks of length n that start and end at the accepted state would be the sum of the entries in row 1 of $B = (I - xA)$, that is

$$F(x) = \frac{1 - 2kx + (k^2 - 2)x^2}{1 - 3kx + (3k^2 - 4)x^2 - (k^3 - 4k)x^3}$$