Backtesting of Value at Risk (VaR)

Shu Dong

1 Introduction

Backtesting of value at risk, the concept introduced in the Quantitative Risk Management section 9.3 (McNeil, Frey, and Embrechts 2015), is the method used to evaluate how the risk measurement work. More specifically, since we know that $\mathrm{VaR}_\alpha(L)$ represents the loss with the probability (at least) α will not be exceeded, if we estimate the value at risk at the α percentage level, there will exist $(1-\alpha)\%$ of the data from the theoretical data set will exceed the $\alpha\%$ the threshold of loss. Backtesting is used to test whether the performance of estimation of the value at risk performs effectively.

- There were three major steps for doing the backtesting evaluation:
  1. We should have the estimated value for $\mathrm{VaR}_\alpha(X)$ for the given α significance level.
  2. By using the backtesting method for our estimated value $\mathrm{VaR}_\alpha(X)$, we should test statistics and get the p-value and use it to determine whether our result rejects the null hypothesis.
  3. We can make the conclusion whether our performance of estimation of the value at risk performs effectively according to the result of testing hypothesis.

2 Getting the Estimated Value for $\mathrm{VaR}_\alpha(X)$ Given $\alpha = 0.95$

Since we want to choose a suitable estimated value for $\mathrm{VaR}_\alpha(X)$, we can find a suitable financial data set and use the historical daily stock prices from the last ten years selected of the stock for the Yahoo Finance https://ca.finance.yahoo.com/. However, in order to minimize the influence of the existence of the stock split and the stock dividend, we pick the daily adjusted close price to reduce effects which may cause experimental errors as much as possible.

```
# Import the selected data set
apple = read.csv("AAPL.csv", header = TRUE)
```

Since we know that the negative daily log return for the t-th day is $(\log(S_{t-1}/S_t))$ where S_t denotes the t-th daily adjusted close price and we only focus on the loss part of the value at risk, we apply the negative daily log return for the adjusted close price of the selected data set.

```r
# Apply the negative daily log return for the selected data
negative_log_return = -diff(log(apple$Adj.Close))
```

In order to get the estimated value for $\mathrm{VaR}_\alpha(X)$ for the selected data, we can fit normal distribution for the sequence of negative daily log returns. Since we know that $\mathrm{VaR}_\alpha(X) = \mu + \sigma\Phi^{-1}(\alpha)$, we can get the mean ($\mu$) and the standard derivation (σ) for the sequence of negative log returns by using maximum-likelihood method.

```r
# Fit normal distribution
library(MASS)
fitdistr(negative_log_return, "normal")
```

```
##        mean             sd
##   -0.0008607853    0.0183325403
##  ( 0.0003653379) ( 0.0002583329)
```

Therefore, we can get

$$parameter = \begin{cases} -0.0008607853 & \mu \\ 0.0183325403 & \sigma \end{cases}$$

Then, since we assume that $\alpha = 0.95$, we can use the parameters which we found and put them in the equation $\mathrm{VaR}_\alpha(X) = \mu + \sigma\Phi^{-1}(\alpha)$ and we can get

$$\begin{aligned} \mathrm{VaR}_{0.95}(X) &= \mu + \sigma\Phi^{-1}(0.95) \\ &= -0.0008607853 + 0.0183325403 \cdot 1.645 \\ &= 0.02929624 \end{aligned}$$

```r
# Check the calculation of the value at risk
Var_estimate = qnorm(0.95, mean = mean(negative_log_return),
                     sd = sd(negative_log_return))
```

By using R to check the result of the calculation above, we can know that the estimated value for $\mathrm{VaR}_{0.95}(X)$ is 0.02929955 based on the data which we selected.

3 Method 1: Using Unconditional Method to do Backtesting

Under the unconditional method, we need to count the number of exceptions (the losses larger than estimated for $\mathrm{VaR}_\alpha(X)$) and compare them with the confidence interval in order to determine whether the exceptions are in the statistical limits. If the number of exceptions are in the statistical limits, we reject the null hypothesis.

Therefore, we can get

$$\begin{cases} X & \text{loss random variable for a portfolio} \\ R & \text{number of exceptions} \\ T & \text{total number of observations} \\ \hat{c} = \frac{R}{T} & \text{failure rate} \end{cases}$$

We can assume that the estimated value for $\mathrm{VaR}_\alpha(X)$ is in the confidence interval $100(1-c)\%$, and then we can expect that $\mathbb{P}(X > \widehat{VaR}_\alpha(X)) = c$. It means that the estimated value $\hat{c}$ is an unbiased estimator for the failure rate c.

Since we know that if our estimated value for $\mathrm{VaR}_\alpha(X)$ is accurate, then the number of exceptions R follows a binomial distribution:

$$P(R = r) = \binom{T}{r} c^r (1-c)^{T-r} \qquad , r = 0, 1, \dots, T$$

Also, if the number of observations T is large, the binomial distribution can be approximated by the normal distribution by using the central limiting theorem:

$$\frac{R - cT}{\sqrt{c(1-c)T}} \approx N(0,1)$$

Therefore, by using the estimated value for value at risk for $\alpha = 0.95$ which we calculated above, we can know the values for the parameters R, T, $\hat{c}$, respectively.

```
# Find the value for R, T, c
Var_estimate
```

```
## [1] 0.02929955
```

```
R = sum(negative_log_return > Var_estimate)
total_obs = length(negative_log_return)
c = 1 - 0.95

R
```

```
## [1] 112
```

```r
total_obs
```

```
## [1] 2518
```

```r
c
```

```
## [1] 0.05
```

Thus, we can get

$$\begin{cases} R = 112 \\ T = 2518 \\ c = 0.05 \end{cases}$$

Since the number of observations which we get is 2518 which is large, we use the central limiting theorem to test the statistics.

```r
# Find the p-value for the estimated value for the value at risk using method 1
test_stat = (R - c * total_obs) / sqrt(c * (1 - c) * total_obs)
p_value_1 = 2 * (1 - pnorm(abs(test_stat)))

p_value_1
```

```
## [1] 0.2037343
```

Therefore, we can get the p-value for the estimated value of $\mathrm{VaR}_{0.95}(X)$ is 0.2037343 by using the unconditional method to do backtesting.

4 Method 2: Using Conditional Method to do Backtesting

Under the conditional method, for $t = 1, \dots, T$, we can have

$$\begin{cases} I_t = 1 & \text{exception occurs with the t-th observation} \\ I_t = 0 & \text{exception does not occur with the t-th observation} \end{cases}$$

Also, for $t = 1, \dots, T-1$, we can have $X_t = I_t$ and $Y_t = I_{t+1}$. In order to test whether our estimated value for $\mathrm{VaR}_\alpha(X)$ is accurate, it means that we need to determine whether the exceptions ($X_t = I_t$ and $Y_t = I_{t+1}$) are independent to each other. If the exceptions are dependent to each other, we reject the null hypothesis.

Therefore, we can use the likelihood ratio test to test statistics which is

$$D = -2\ln(\frac{\text{maximum likelihood for null model}}{\text{maximum likelhood for alternative model}})$$

So, we can firstly find the exceptions ($X_t = I_t$ and $Y_t = I_{t+1}$).

```
# Find exceptions X_t and Y_t
exception = as.numeric(negative_log_return > Var_estimate)
X_t = exception[-total_obs]
Y_t = exception[-1]
```

Then, we can let n_{ij} be the number of t such that $X_t = i$ and $Y_t = j$, and $n_{00}+n_{01}+n_{10}+n_{11} = T-1$. So, after knowing the exceptions, we can generate the contingency table.

```
# Generate the contingency table
count = table(Y_t, X_t)
n_00 = count[1, 1]
n_10 = count[1, 2]
n_01 = count[2, 1]
n_11 = count[2, 2]

col_1 = c(n_00, n_01)
col_2 = c(n_10, n_11)
name_col = c("X = 0", "X = 1")
name_row = c("Y = 0", "Y = 1")
con_table = setNames(data.frame(col1 = col_1, col2 = col_2,
                                row.names = name_row), name_col)

library(knitr)
knitr::kable(con_table, "pipe")
```

	X = 0	X = 1
Y = 0	2303	102
Y = 1	102	10

So, we can let

$$\begin{cases} \pi_0 = \frac{n_{01}}{n_{00}+n_{01}} \\ \pi_1 = \frac{n_{11}}{n_{10}+n_{11}} \\ \pi = \frac{n_{01}+n_{11}}{T-1} \end{cases}$$

By using R, we can know the value for π_0, π_1, π which are

```
# Find the value for pi
pi_hat = (n_01 + n_11) / (total_obs - 1)
pi.0_hat = n_01 / (n_00 + n_01)
pi.1_hat = n_11 / (n_10 + n_11)

pi.0_hat
```

```
## [1] 0.04241164
```

```
pi.1_hat
```

```
## [1] 0.08928571
```

```
pi_hat
```

```
## [1] 0.04449742
```

$$\begin{cases} \pi_0 = 0.04241164 \\ \pi_1 = 0.08928571 \\ \pi = 0.04449742 \end{cases}$$

Then, under the null hypothesis, we can have

$$\begin{cases} (1-\pi)^{n_{00}+n_{10}} \cdot \pi^{n_{01}+n_{11}} & \text{maximum likelihood for null model} \\ (1-\pi_0)^{n_{00}} \cdot \pi_0^{n_{01}} \cdot (1-\pi)^{n_{10}} \cdot \pi_1^{n_{11}} & \text{maximum likelhood for alternative model} \end{cases}$$

According to the likelihood ratio test, it means that

$$LR_{ind} = -2\ln(\frac{(1-\pi)^{n_{00}+n_{10}} \cdot \pi^{n_{01}+n_{11}}}{(1-\pi_0)^{n_{00}} \cdot \pi_0^{n_{01}} \cdot (1-\pi)^{n_{10}} \cdot \pi_1^{n_{11}}})$$

So, we can put the value from the contingency table to test statistics.

```
# Find the p-value for the estimated value for the value at risk using method 2
LR_stat = -2 * log((1 - pi_hat)^(n_00 + n_10) * pi_hat^(n_01 + n_11) /
                     ((1 - pi.0_hat)^n_00 * pi.0_hat^n_01 *
                        (1 - pi.1_hat)^n_01 * pi.1_hat^n_11))
p_value_2 = 1 - pchisq(LR_stat, df = 2)

p_value_2
```

```
## [1] 0.1116728
```

Therefore, we can get the p-value for the estimated value of $\mathrm{VaR}_{0.95}(X)$ is 0.1116728 by using the conditional method to do backtesting.

5 Conclusion

By using the two backtesting methods which discussed above, we can know that:

- From process of the unconditional method to do backtesting, we can get the p-value for the estimated value $\mathrm{VaR}_{0.95}(X)$ is $0.2037343 \leq$ level of significance α $(1 - 0.95 = 0.05)$.
- From process of the conditional method to do backtesting, we can get the p-value for the estimated value $\mathrm{VaR}_{0.95}(X)$ is $0.1116728 \leq$ level of significance α $(1 - 0.95 = 0.05)$.

Therefore, it means that both p-values for the estimated value $\mathrm{VaR}_{0.95}(X)$ by getting from the backtesting process all ≤ 0.05. According to the principle of the hypothesis test, we can know that if p-value $\leq$ level of significance α, the null hypothesis should be rejected at the level of significance. Thus, we can know that we should not reject the null hypothesis for both testing methods which means that our estimated value for $\mathrm{VaR}_{\alpha}(X)$ is accurate.

However, there still exist limitations to do backtesting which are:

1. The backtesting method requires sufficient detailed data from the past in order to simulate the past conditions which is not very possible or expensive to get in the real working environment.
2. The backtesting method may experience the overfitting issue which means that the model for evaluating $\mathrm{VaR}_{\alpha}(X)$ may work in the past but may not able to work in the future.

Nevertheless, the limitations of doing the backtesting cannot conceal the importance of the contribution for forming the value at risk reporting system in modern risk management. We are not able to evaluate whether the estimated value for the value at risk is accurate without using the backtesting method.

Reference

McNeil, Alexander J, Rüdiger Frey, and Paul Embrechts. 2015. *Quantitative Risk Management: Concepts, Techniques and Tools-Revised Edition.* Princeton university press.