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Abstract

We study the effects of warm water on the local penguin population. The major finding is that it is extremely difficult to induce penguins to drink warm water. The success factor is approximately $-e^{-i\pi} - 1$.

1 Introduction

Systems with uncertain model

$$\dot{x}(t) = f(x(t), u(t)) + \xi(x(t), t), \quad x(0) = x_0$$
$$y(t) = Cx(t) + \eta(t)$$

$x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$; $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\xi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$, $C \in \mathbb{R}^{p\times n}$, $\eta \in \mathbb{R}^p$, x_0 is given

$$\mathbb{R}^+ = \{s \in \mathbb{R}, s > 0\} \oplus \{0\}$$

2 Problem statement

Consider the reference trajectory

$$x^*(t) \in C^1$$

that is governed by

$$\dot{x}^*(t) = h(x^*), \quad h : \mathbb{R}^n \to \mathbb{R}^n$$

The problem is to design u such that the tracking error $\delta \in \mathbb{R}^n$, $\delta = x - x^*$ is driven to the origin:

1. asymptotically

$$a)\limsup_{t\to\infty} \delta(t) = 0$$
$$b)\limsup_{t\to\infty} \|\delta(t)\| = 0$$

2. exponentially:$\exists\alpha > 0$ such that

$$\|\delta(t)\| \leq \|\delta(0)\|^{+} e^{-\alpha t}$$

$$\|\delta(0)\|^{+} = \max_{\delta(0)\in\Xi_0} \|\delta(0)\|$$

3. in finite-time: $\exists T > 0$ such that

$$\|\delta(t)\| \leq \beta_f,\ \forall t \geq T > 0$$

Note: ideally, $\beta_f = 0$

3 Difficulties in the problem statement

There are several issues that must be considered before the problem can be solved:

1. The mathematical structure of f is uncertain
2. The effect of control uncertain, there is not an affine representation

$$\begin{aligned} \dot{x}(t) &= f_a(x(t)) + g_a(x(t))u(t) + \xi(x(t), t) \\ f(x, u) &= f_a(x) + g_a(x)u \end{aligned}$$

   with $f_a : \mathbb{R}^n \to \mathbb{R}^n$, $g_a : \mathbb{R}^n \to \mathbb{R}^{n\times m}$

3. Controllability condition can be assumed?
4. Is ξ integrable?

4 The neural network approximation

There are several problem statement for the approximation based on NN

$$\begin{gathered} f(x, u) \cong f_N(x, u) + \tilde{f}(x, u) \text{ mild approximation} \\ f(x, u) \cong f_M(x, u) + f_N(x, u) + \tilde{f}(x, u) \\ f(x, u) \cong f_M(x, u) + f_N(x) + g_N(x)u + \tilde{f}(x, u) \end{gathered}$$

where $f_M(x, u)$ is a partial section of the model $f(x, u)$

What is the rationally of the NN approximation: Kolmogorov approximation theory, Stone-Weierstrass consistent approximation of continuous functions by polynomials and Cybenko theorems about using sigmoidal functions:

Sigmoidal function: a monotonically continuous function $S(x)$ is called sigmoidal if

$$\begin{gathered} S'(x) > 0 \\ \lim_{x\to\infty} S'(x) = 0 \\ \lim_{x\to-\infty} S'(x) = 0 \end{gathered}$$

Single layer approximation

$$\begin{aligned} d(x) &\cong W_1\sigma_1(x) + \tilde{f}(x) \\ f_N(x, n_1) &= W_1\sigma_{1,n_1}(x) \end{aligned}$$

$d : \mathbb{R}^n \to \mathbb{R}^n$, $W_1 \in \mathbb{R}^{n\times n_1}$, $\sigma_1 : \mathbb{R}^n \to \mathbb{R}^{n_1}$,

$$\left\| \tilde{f}(x) \right\|$$

is a mononically decreasing function with respect to n_1.

$$\lim_{n_1 \to +\infty} \left\| \tilde{f}(x) \right\| = 0$$

Multi-layer approximation

$$\begin{aligned} d(x) &\cong W_1\sigma_1(W_2\sigma_2(W_3\sigma_3(x)) + \tilde{f}(x) \\ f_N(x, n_1) &= W_1\sigma_1(W_2\sigma_2(W_3\sigma_3(x)) \end{aligned}$$

$W_1 \in \mathbb{R}^{n\times n_1}$, $\sigma_1 : \mathbb{R}^n \to \mathbb{R}^{n_1}$, $W_2 \in \mathbb{R}^{n\times n_2}$, $\sigma_2 : \mathbb{R}^n \to \mathbb{R}^{n_2}, W_3 \in \mathbb{R}^{n\times n_3}$, $\sigma_3 : \mathbb{R}^n \to \mathbb{R}^{n_3}$.

A modified type of NN approximation

$$\begin{aligned} d(x) &\cong Ax + W_1\sigma_1(x) + \tilde{f}(x) \\ &\cong \begin{bmatrix} A & W_1 \end{bmatrix} \begin{bmatrix} x \\ \sigma_1(x) \end{bmatrix} + \tilde{f}(x) \\ &\cong W_1\sigma_1(x) + Ax - Ax + \tilde{f}(x) = Ax + W_1\sigma_1(x) + \tilde{f}_A(x) \\ \tilde{f}_A(x) &= \tilde{f}(x) - Ax \end{aligned}$$

$$\begin{aligned} d(z) &\cong \begin{bmatrix} A & W_1 & W_2 \end{bmatrix} \begin{bmatrix} x \\ \sigma_1(x) \\ \sigma_2(x)u \end{bmatrix} + \tilde{f}(x) \\ &\cong \begin{bmatrix} A & W_1 & W_2 \end{bmatrix} \begin{bmatrix} x \\ \sigma_1(x) \\ \sigma_2(u) \end{bmatrix} + \tilde{f}(x) \\ &\cong \begin{bmatrix} A & W_1 & W_2 \end{bmatrix} \begin{bmatrix} x \\ \sigma_1(x) \\ \sigma_2(x, u) \end{bmatrix} + \tilde{f}(x) \end{aligned}$$

$z = \begin{bmatrix} x \\ u \end{bmatrix}$

5 Identifier design

$$\dot{x}\,(t) = f_a(x\,(t)) + g_a(x\,(t))u\,(t) + \xi\,(x\,(t)\,, t)\,, \qquad x\,(0) = x_0$$

NN-representation of the nonlinear system

$$\dot{x}\,(t) \cong \begin{bmatrix} A & W_1 & W_2 \end{bmatrix} \begin{bmatrix} x\,(t) \\ \sigma_1(x\,(t)) \\ \sigma_2(x\,(t))u\,(t) \end{bmatrix} + \tilde{f}(x) + \xi\,(x\,(t)\,, t)\,, \qquad x\,(0) = x_0$$

1. $x \in int\,\{X\} \subset \mathbb{R}^n$
2. Image of g is known
3. Exists a Hurwitz matrix A such that the approximation is valid
4. $\tilde{f}(x) \in \tilde{F}$

$$\tilde{F} = \left\{ \tilde{f}(x) \mid \left\| \tilde{f}(x) \right\|^2 \leq \tilde{f}_0 + \tilde{f}_1 \left\| x \right\|^2, \ \ \forall x \in int\,\{X\} \right\}$$

5. $\xi\,(x\,(t)\,, t) \in \Psi$

$$\Psi = \left\{ \xi(x) \mid \left\| \xi(x,t) \right\|^2 \leq \tilde{f}_0 + \tilde{f}_1 \left\| x \right\|^2, \ \ \forall x \in int\,\{X\}\,, \forall t \geq 0, \ \ \lim_{T \to \infty} \int\limits_{t=0}^{T} \left\| e^{A(t-\tau)} \xi \right\| d\tau < +\infty \right\}$$

6. $W_1^0 \quad W_2^0$ such that

$$\begin{bmatrix} W_1^0 & W_2^0 \end{bmatrix} = \underset{W_1 \in \mathbb{R}^{n \times n_1} \quad W_2 \in \mathbb{R}^{n \times n_2}}{\arg\min} \left\| \tilde{f}(x) \right\|^2$$

7. The admissible set of controls is

$$U_{adm} = \left\{ u \mid \left\| u \right\|^2 \leq u_0 + u_1 \left\| x \right\|^2, \ \ \forall x \in int\,\{X\}\,, \forall t \geq 0, \ u_0 \in \mathbb{R}^+, u_1 \in \mathbb{R}^+ \right\}$$

Design of the identifier

$$\dot{x}\,(t) \cong \begin{bmatrix} A & W_1 & W_2 \end{bmatrix} \begin{bmatrix} x\,(t) \\ \sigma_1(x\,(t)) \\ \sigma_2(x\,(t))u\,(t) \end{bmatrix} + \tilde{f}(x) + \xi\,(x\,(t)\,, t)\,, \qquad x\,(0) = x_0$$

$$\frac{d}{dt}\hat{x}\,(t) = \hat{f}\left(\hat{x}\,(t)\,, x\,(t)\,, u(t) \backslash \ \hat{W}_1 \quad \hat{W}_2 \ \right)$$

Lets try to find $\hat{f}$ and $\hat{W}_1 \quad \hat{W}_2$ such that

$$\limsup_{t \to \infty} \left\| x - \hat{x} \right\| = \beta$$

$$\hat{f}\left(\hat{x}\,(t)\,, x\,(t)\,, u(t) \backslash \ \hat{W}_1 \quad \hat{W}_2 \ \right) = \begin{bmatrix} A & \hat{W}_1 & \hat{W}_2 \end{bmatrix} \begin{bmatrix} \hat{x}\,(t) \\ \sigma_1(\hat{x}\,(t)) \\ \sigma_2(\hat{x}\,(t))u\,(t) \end{bmatrix}$$

$$\frac{d}{dt}\hat{x}\left(t\right) = \left[\begin{array}{ccc} A & \hat{W}_1 & \hat{W}_2 \end{array}\right] \left[\begin{array}{c} \hat{x}\left(t\right) \\ \sigma_1(\hat{x}\left(t\right)) \\ \sigma_2(\hat{x}\left(t\right))u\left(t\right) \end{array}\right]$$

Remark:

$$\frac{d}{dt}\hat{x}\left(t\right) = \left[\begin{array}{ccc} A & \hat{W}_1 & \hat{W}_2 \end{array}\right] \left[\begin{array}{c} \hat{x}\left(t\right) \\ \sigma_1(\hat{x}\left(t\right)) \\ \sigma_2(\hat{x}\left(t\right))u\left(t\right) \end{array}\right] + K_1\left(x-\hat{x}\right)$$

$$\frac{d}{dt}\hat{x}\left(t\right) = \left[\begin{array}{ccc} A & \hat{W}_1 & \hat{W}_2 \end{array}\right] \left[\begin{array}{c} \hat{x}\left(t\right) \\ \sigma_1(\hat{x}\left(t\right)) \\ \sigma_2(\hat{x}\left(t\right))u\left(t\right) \end{array}\right] + K_1\left(x-\hat{x}\right) + K_2\frac{\left(x-\hat{x}\right)}{\left\|x-\hat{x}\right\|}$$

$$\frac{d}{dt}\hat{x}\left(t\right) = \left[\begin{array}{ccc} A & \hat{W}_1 & \hat{W}_2 \end{array}\right] \left[\begin{array}{c} \hat{x}\left(t\right) \\ \sigma_1(\hat{x}\left(t\right)) \\ \sigma_2(\hat{x}\left(t\right))u\left(t\right) \end{array}\right] + K_1\left(x-\hat{x}\right) + K_2\left(x-\hat{x}\right)\left\|x-\hat{x}\right\| + K_3\left(x-\hat{x}\right)\left\|x-\hat{x}\right\|^2$$

$$\dot{x}\left(t\right) \cong \left[\begin{array}{ccc} A & W_1 & W_2 \end{array}\right] \left[\begin{array}{c} x\left(t\right) \\ \sigma_1(x\left(t\right)) \\ \sigma_2(x\left(t\right))u\left(t\right) \end{array}\right] + \tilde{f}(x) + \xi\left(x\left(t\right), t\right), \qquad x\left(0\right) = x_0$$

$$\frac{d}{dt}\hat{x}\left(t\right) = \left[\begin{array}{ccc} A & \hat{W}_1 & \hat{W}_2 \end{array}\right] \left[\begin{array}{c} \hat{x}\left(t\right) \\ \sigma_1(\hat{x}\left(t\right)) \\ \sigma_2(\hat{x}\left(t\right))u\left(t\right) \end{array}\right]$$

$$\begin{gathered} \Delta = x - \hat{x} \\ \tilde{W}_1 = \hat{W}_1 - W_1 \\ \tilde{W}_2 = \hat{W}_2 - W_2 \end{gathered}$$

$$\begin{gathered} \dot{\Delta} = \frac{d}{dt}\left(x-\hat{x}\right) = Ax\left(t\right) + W_1\sigma_1(x\left(t\right)) + W_2\sigma_2(x\left(t\right))u\left(t\right) + \tilde{f}(x) + \xi\left(x\left(t\right), t\right) - \\ A\hat{x}\left(t\right) - \hat{W}_1\sigma_1(\hat{x}\left(t\right)) - \hat{W}_2\sigma_2(\hat{x}\left(t\right))u\left(t\right) \end{gathered}$$

$$\begin{gathered} \dot{\Delta} = A\left(x\left(t\right)-\hat{x}\left(t\right)\right) + W_1\sigma_1(x\left(t\right)) - \hat{W}_1\sigma_1(\hat{x}\left(t\right)) \\ +W_2\sigma_2(x\left(t\right))u\left(t\right) - \hat{W}_2\sigma_2(\hat{x}\left(t\right))u\left(t\right) + \tilde{f}(x) + \xi\left(x\left(t\right), t\right) = \\ A\left(x\left(t\right)-\hat{x}\left(t\right)\right) + W_1\sigma_1(x\left(t\right)) - W_1\sigma_1(\hat{x}\left(t\right)) + W_1\sigma_1(\hat{x}\left(t\right)) - \hat{W}_1\sigma_1(\hat{x}\left(t\right)) \\ +W_2\sigma_2(x\left(t\right))u\left(t\right) - W_2\sigma_2(\hat{x}\left(t\right))u\left(t\right) + W_2\sigma_2(\hat{x}\left(t\right))u\left(t\right) - \hat{W}_2\sigma_2(\hat{x}\left(t\right))u\left(t\right) + \tilde{f}(x) + \xi\left(x\left(t\right), t\right) \end{gathered}$$

$$\begin{gathered} \dot{\Delta} = A\Delta + W_1\left(\sigma_1(x\left(t\right)) - \sigma_1(\hat{x}\left(t\right))\right) + \tilde{W}_1\sigma_1(\hat{x}\left(t\right)) + \\ W_2\left(\sigma_2(x\left(t\right))u\left(t\right) - \sigma_2(\hat{x}\left(t\right))u\left(t\right)\right) + \tilde{W}_2\sigma_2(\hat{x}\left(t\right))u(t) + \tilde{f}(x) + \xi\left(x\left(t\right), t\right) \end{gathered}$$

$$\begin{gathered} V\left(\Delta, \tilde{W}_1, \tilde{W}_2\right) = \left\|\Delta\right\|_P^2 + \left\|\tilde{W}_1\right\|_{K_1,M}^2 + \left\|\tilde{W}_2\right\|_{K_2,M}^2 \\ = \Delta^\intercal P\Delta + tr\left\{\tilde{W}_1^\intercal K_1\tilde{W}_1\right\} + tr\left\{\tilde{W}_2^\intercal K_2\tilde{W}_2\right\} \\ = \Delta^\intercal P\Delta + \det\left\{\tilde{W}_1^\intercal K_1\tilde{W}_1\right\} + \det\left\{\tilde{W}_2^\intercal K_2\tilde{W}_2\right\} \\ = \Delta^\intercal P\Delta + \lambda_{\max}\left\{\tilde{W}_1^\intercal K_1\tilde{W}_1\right\} + \lambda_{\max}\left\{\tilde{W}_2^\intercal K_2\tilde{W}_2\right\} \\ = \Delta^\intercal P\Delta + tr\left\{\tilde{W}_1^\intercal K_1\tilde{W}_1\right\}^{p_1} + tr\left\{\tilde{W}_2^\intercal K_2\tilde{W}_2\right\}^{p_2} \\ P > 0, P = P^\intercal, P \in \mathbb{R}^{n\times n} \end{gathered}$$

Remark:

$$\dot{\Delta} = A\Delta + W_1\left(\sigma_1(x\left(t\right)) - \sigma_1(\hat{x}\left(t\right))\right) + \tilde{W}_1\sigma_1(\hat{x}\left(t\right)) +$$
$$W_2\left(\sigma_2(x\left(t\right))u\left(t\right) - \sigma_2(\hat{x}\left(t\right))u\left(t\right)\right) + \tilde{W}_2\sigma_2(\hat{x}\left(t\right))u(t) + \tilde{f}(x) + \xi\left(x\left(t\right), t\right)$$

$$= A\Delta + W_1\left(\sigma_1(x\left(t\right)) - \sigma_1(\hat{x}\left(t\right))\right) + W_2\left(\sigma_2(x\left(t\right))u\left(t\right) - \sigma_2(\hat{x}\left(t\right))u\left(t\right)\right)$$
$$+\tilde{f}(x) + \xi\left(x\left(t\right), t\right) +$$
$$M\sigma_1(\hat{x}\left(t\right))V_{\tilde{W},1} + M\sigma_2(\hat{x}\left(t\right))u(t)V_{\tilde{W},2},$$

Kronecker product

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ae+bf \\ ce+df \end{bmatrix}$$

$$\begin{bmatrix} e & 0 & f & 0 \\ 0 & e & 0 & f \end{bmatrix}\begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix} = \begin{bmatrix} ae+bf \\ ce+df \end{bmatrix}$$

$$V\left(\Delta, \tilde{W}_1, \tilde{W}_2\right) = \|\Delta\|_P^2 + \left\|V_{\tilde{W},1}\right\|_{K_1}^2 + \left\|V_{\tilde{W},2}\right\|_{K_2}^2$$

Stability analysis version 1

$$V\left(\Delta, \tilde{W}_1, \tilde{W}_2\right) = \Delta^\intercal P\Delta + tr\left\{\tilde{W}_1^\intercal K_1\tilde{W}_1\right\} + tr\left\{\tilde{W}_2^\intercal K_2\tilde{W}_2\right\}$$

$$V\left(\Delta, \tilde{W}_1, \tilde{W}_2\right) \geq 0$$
$$V\left(0,0,0\right) = 0$$
$$\lim_{\Delta\to\infty, \tilde{W}_1\to\infty, \tilde{W}_2\to\infty} V\left(\Delta, \tilde{W}_1, \tilde{W}_2\right) \to \infty$$

$\frac{d}{dt}V\left(\Delta(t), \tilde{W}_1(t), \tilde{W}_2(t)\right) \leq 0,$ origin is an stable equilibrium point in the Lyapunov sense

$\frac{d}{dt}V\left(\Delta(t), \tilde{W}_1(t), \tilde{W}_2(t)\right) < 0$ origin is an asymtotically stable equilibrium point in the Lyapunov

$\frac{d}{dt}V\left(\Delta(t), \tilde{W}_1(t), \tilde{W}_2(t)\right) < 0$ for all $\left\{\Delta(t), \tilde{W}_1(t), \tilde{W}_2(t)\right\} \in \Omega$ not equal to the entire space of apropiate c
we have what we call practical stable equilibrium point

The full derivative with respect to time of V (Lie derivative)

$$V\left(\Delta, \tilde{W}_1, \tilde{W}_2\right) = \Delta^\intercal P \Delta + tr\left\{\tilde{W}_1^\intercal K_1 \tilde{W}_1\right\} + tr\left\{\tilde{W}_2^\intercal K_2 \tilde{W}_2\right\}$$

$$\begin{gathered}\dot{V}(t) \triangleq \frac{d}{dt} V\left(\Delta(t), \tilde{W}_1(t), \tilde{W}_2(t)\right) = \frac{\partial}{\partial \Delta} V\left(\Delta(t), \tilde{W}_1(t), \tilde{W}_2(t)\right) \frac{d}{dt}\Delta + \\ \frac{\partial}{\partial \tilde{W}_1} V\left(\Delta(t), \tilde{W}_1(t), \tilde{W}_2(t)\right) \frac{d}{dt}\tilde{W}_1 + \\ \frac{\partial}{\partial \tilde{W}_2} V\left(\Delta(t), \tilde{W}_1(t), \tilde{W}_2(t)\right) \frac{d}{dt}\tilde{W}_2 \\ = 2(P\Delta)^\intercal \frac{d}{dt}\Delta + tr\left\{2\left(K_1\tilde{W}_1\right)^\intercal \frac{d}{dt}\tilde{W}_1\right\} + tr\left\{2\left(K_2\tilde{W}_2\right)^\intercal \frac{d}{dt}\tilde{W}_2\right\} \\ = 2\Delta^\intercal P^\intercal \frac{d}{dt}\Delta + 2tr\left\{\tilde{W}_1^\intercal K_1^\intercal \frac{d}{dt}\tilde{W}_1\right\} + 2tr\left\{\tilde{W}_2^\intercal K_2^\intercal \frac{d}{dt}\tilde{W}_2\right\}\end{gathered}$$

$$\begin{gathered}2\Delta^\intercal P^\intercal \frac{d}{dt}\Delta = \\ 2\Delta^\intercal P^\intercal \left(A\Delta + W_1(\sigma_1(x(t)) - \sigma_1(\hat{x}(t))) + \tilde{W}_1\sigma_1(\hat{x}(t))\right) + \\ 2\Delta^\intercal P^\intercal \left(W_2(\sigma_2(x(t))u(t) - \sigma_2(\hat{x}(t))u(t)) + \tilde{W}_2\sigma_2(\hat{x}(t))u(t) + \tilde{f}(x) + \xi(x(t), t)\right) =\end{gathered}$$

$$\begin{gathered}2\Delta^\intercal P^\intercal A\Delta = \Delta^\intercal(P^\intercal A)\Delta + \Delta^\intercal(A^\intercal P)\Delta = \Delta^\intercal(PA + A^\intercal P)\Delta \\ 0 < P = P^\intercal\end{gathered}$$

$$2\Delta^\intercal P^\intercal W_1(\sigma_1(x(t)) - \sigma_1(\hat{x}(t)))$$

Young, inequality

$$\begin{gathered}X, Y \in \mathbb{R}^{p_1 \times p_2} \\ X^\intercal Y + Y^\intercal X \leq X^\intercal \Lambda X + Y^\intercal \Lambda^{-1} Y, \quad \Lambda > 0, \Lambda = \Lambda^\intercal, \Lambda \in \mathbb{R}^{p_1 \times p_1}, \text{ (Young's)} \\ X, Y \in \mathbb{R}^{p_1 \times 1} \\ 2X^\intercal Y \leq X^\intercal \Lambda X + Y^\intercal \Lambda^{-1} Y \text{ (Lambda inequality)}\end{gathered}$$

$$\begin{gathered}X, Y \in \mathbb{R}^{p_1 \times p_2} \\ X^\intercal Y + Y^\intercal X \leq X^\intercal (1+\varepsilon) I X + Y^\intercal (1+\varepsilon)^{-1} I Y \\ \leq (1+\varepsilon) X^\intercal X + (1+\varepsilon)^{-1} Y^\intercal Y \text{ (Peter-Paul inequality)}\end{gathered}$$

Step 1

$$2\Delta^\intercal P^\intercal A\Delta = \Delta^\intercal(P^\intercal A)\Delta + \Delta^\intercal(A^\intercal P)\Delta = \Delta^\intercal(PA + A^\intercal P)\Delta$$

Step 2

$$2\underbrace{\Delta^\intercal P W_1}_{X^\intercal}\underbrace{(\sigma_1(x(t)) - \sigma_1(\hat{x}(t)))}_{Y} \leq \Delta^\intercal P^\intercal W_1 \Lambda_1 W_1^\intercal P \Delta + (\sigma_1(x(t)) - \sigma_1(\hat{x}(t)))^\intercal \Lambda_1^{-1} (\sigma_1(x(t)) - \sigma_1(\hat{x}(t)))$$

$$\begin{gathered}\text{Rayleigh's inequality } z^\intercal H z,\ H > 0,\ \lambda_{\min}\{H\}\|z\|^2 \leq z^\intercal H z \leq \lambda_{\max}\{H\}\|z\|^2 \\ \leq \Delta^\intercal P^\intercal W_1 \Lambda_1 W_1^\intercal P \Delta + \lambda_{\max}\left\{\Lambda_1^{-1}\right\}\|\sigma_1(x(t)) - \sigma_1(\hat{x}(t))\|^2 \\ \leq \Delta^\intercal P^\intercal W_1 \Lambda_1 W_1^\intercal P \Delta + \lambda_{\max}\left\{\Lambda_1^{-1}\right\} L_\sigma \|x(t) - \hat{x}(t)\|^2 \\ \leq \Delta^\intercal P^\intercal W_1 \Lambda_1 W_1^\intercal P \Delta + \lambda_{\max}\left\{\Lambda_1^{-1}\right\} L_\sigma \|\Delta\|^2\end{gathered}$$

Step 3

$$2\Delta^{\intercal} P\tilde{W}_1\sigma_1(\hat{x}(t)) = 2\sigma_1^{\intercal}(\hat{x}(t))\tilde{W}_1^{\intercal} P\Delta$$
$$= tr\left\{2\tilde{W}_1^{\intercal} P\Delta\sigma_1^{\intercal}(\hat{x}(t))\right\}$$

$$x, y \in \mathbb{R}^{p_1}, \quad (x, y) = x^{\intercal} y = tr\left\{yx^{\intercal}\right\}$$

Step 4

$$2\Delta^{\intercal} PW_2\left(\sigma_2(x(t))u(t) - \sigma_2(\hat{x}(t))u(t)\right) \leq \Delta^{\intercal} P^{\intercal} W_2\Lambda_2 W_2^{\intercal} P\Delta + \lambda_{\max}\left\{\Lambda_2^{-1}\right\} L_\sigma \left\|\Delta u\right\|^2$$
$$\leq \Delta^{\intercal} P^{\intercal} W_2\Lambda_2 W_2^{\intercal} P\Delta + \lambda_{\max}\left\{\Lambda_2^{-1}\right\} L_\sigma \left\|\Delta\right\|^2 \left\|u\right\|^2$$

Step 5

$$2\Delta^{\intercal} P\tilde{W}_2\sigma_2(\hat{x}(t))u(t) = tr\left\{2\tilde{W}_2^{\intercal} P\Delta\left[\sigma_2(\hat{x}(t))u(t)\right]^{\intercal})\right\}$$

Step 6

$$2\Delta^{\intercal} P\tilde{f}(x) \leq \Delta^{\intercal} P\Lambda_3 P\Delta + \tilde{f}^{\intercal}(x)\Lambda_3^{-1}\tilde{f}(x)$$

Step 7

$$2\Delta^{\intercal} P\xi\left(x(t), t\right) \leq \Delta^{\intercal} P\Lambda_4 P\Delta + \xi^{\intercal}\left(x(t), t\right)\Lambda_4^{-1}\xi\left(x(t), t\right)$$

gathering all the previous results yields

$$\dot{V}(t) \leq$$
$$2\Delta^{\intercal} P^{\intercal}\frac{d}{dt}\Delta + 2tr\left\{\tilde{W}_1^{\intercal} K_1^{\intercal}\frac{d}{dt}\tilde{W}_1\right\} + 2tr\left\{\tilde{W}_2^{\intercal} K_2^{\intercal}\frac{d}{dt}\tilde{W}_2\right\} \leq$$

$$\Delta^{\intercal}\left(PA + A^{\intercal} P\right)\Delta +$$
$$\Delta^{\intercal} P^{\intercal} W_1\Lambda_1 W_1^{\intercal} P\Delta + \lambda_{\max}\left\{\Lambda_1^{-1}\right\} L_\sigma\left\|\Delta\right\|^2 +$$
$$-> tr\left\{2\tilde{W}_1^{\intercal} P\Delta\sigma_1^{\intercal}(\hat{x}(t))\right\} +$$
$$\Delta^{\intercal} P^{\intercal} W_2\Lambda_2 W_2^{\intercal} P\Delta + \lambda_{\max}\left\{\Lambda_2^{-1}\right\} L_\sigma\left\|\Delta\right\|^2\left\|u\right\|^2 +$$
$$-> tr\left\{2\tilde{W}_2^{\intercal} P\Delta\left[\sigma_2(\hat{x}(t))u(t)\right]^{\intercal})\right\} +$$
$$\Delta^{\intercal} P\Lambda_3 P\Delta + \tilde{f}^{\intercal}(x)\Lambda_3^{-1}\tilde{f}(x) +$$
$$\Delta^{\intercal} P\Lambda_4 P\Delta + \xi^{\intercal}\left(x(t), t\right)\Lambda_4^{-1}\xi\left(x(t), t\right) +$$
$$2tr\left\{\tilde{W}_1^{\intercal} K_1^{\intercal}\frac{d}{dt}\tilde{W}_1\right\} + 2tr\left\{\tilde{W}_2^{\intercal} K_2^{\intercal}\frac{d}{dt}\tilde{W}_2\right\} \leq$$

$$\Delta^{\intercal}\Big(PA + A^{\intercal} P + PW_1\Lambda_1 W_1^{\intercal} P + \lambda_{\max}\left\{\Lambda_1^{-1}\right\} L_\sigma I_n + PW_2\Lambda_2 W_2^{\intercal} P + \lambda_{\max}\left\{\Lambda_2^{-1}\right\} L_\sigma\left\|u\right\|^2 I_n + P\Lambda_3 P + P\Lambda_4$$
$$\tilde{f}^{\intercal}(x)\Lambda_3^{-1}\tilde{f}(x) + \xi^{\intercal}\left(x(t), t\right)\Lambda_4^{-1}\xi\left(x(t), t\right) +$$
$$tr\left\{2\tilde{W}_1^{\intercal} P\Delta\sigma_1^{\intercal}(\hat{x}(t))\right\} + tr\left\{2\tilde{W}_2^{\intercal} P\Delta\left[\sigma_2(\hat{x}(t))u(t)\right]^{\intercal})\right\} + 2tr\left\{\tilde{W}_1^{\intercal} K_1^{\intercal}\frac{d}{dt}\tilde{W}_1\right\} + 2tr\left\{\tilde{W}_2^{\intercal} K_2^{\intercal}\frac{d}{dt}\tilde{W}_2\right\}$$

$$\Delta^{\intercal}\left(PA + A^{\intercal} P + P\left(W_1\Lambda_1 W_1^{\intercal} + W_2\Lambda_2 W_2^{\intercal} + \Lambda_3 + \Lambda_4\right)P + \lambda_{\max}\left\{\Lambda_1^{-1}\right\} L_\sigma I_n + \lambda_{\max}\left\{\Lambda_2^{-1}\right\} L_\sigma\left\|u\right\|^2 I_n\right)$$
$$\tilde{f}^{\intercal}(x)\Lambda_3^{-1}\tilde{f}(x) + \xi^{\intercal}\left(x(t), t\right)\Lambda_4^{-1}\xi\left(x(t), t\right) +$$
$$tr\left\{2\tilde{W}_1^{\intercal} P\Delta\sigma_1^{\intercal}(\hat{x}(t)) + 2\tilde{W}_1^{\intercal} K_1^{\intercal}\frac{d}{dt}\tilde{W}_1\right\} + tr\left\{2\tilde{W}_2^{\intercal} P\Delta\left[\sigma_2(\hat{x}(t))u(t)\right]^{\intercal}) + 2\tilde{W}_2^{\intercal} K_2^{\intercal}\frac{d}{dt}\tilde{W}_2\right\}$$

$$\dot{V}\left(t\right) \leq$$
$$\Delta^{\intercal} \Big(PA + A^{\intercal} P + PW_1 \Lambda_1 W_1^{\intercal} P + \lambda_{\max}\left\{\Lambda_1^{-1}\right\} L_{\sigma} I_n + PW_2 \Lambda_2 W_2^{\intercal} P + \lambda_{\max}\left\{\Lambda_2^{-1}\right\} L_{\sigma} \left\|u\right\|^2 I_n + P\Lambda_3 P +$$
$$\tilde{f}^{\intercal}(x)\Lambda_3^{-1}\tilde{f}(x) + \xi^{\intercal}\left(x\left(t\right),t\right)\Lambda_4^{-1}\xi\left(x\left(t\right),t\right) +$$
$$tr\left\{2\tilde{W}_1^{\intercal} P\Delta\sigma_1^{\intercal}(\hat{x}\left(t\right))\right\} + tr\left\{2\tilde{W}_2^{\intercal} P\Delta\left[\sigma_2(\hat{x}\left(t\right))u(t)\right]^{\intercal})\right\} + 2tr\left\{\tilde{W}_1^{\intercal} K_1^{\intercal} \frac{d}{dt}\tilde{W}_1\right\} + 2tr\left\{\tilde{W}_2^{\intercal} K_2^{\intercal} \frac{d}{dt}\tilde{W}$$

$$\Delta^{\intercal} \Big(PA + A^{\intercal} P + PW_1 \Lambda_1 W_1^{\intercal} P + \lambda_{\max}\left\{\Lambda_1^{-1}\right\} L_{\sigma} I_n + PW_2 \Lambda_2 W_2^{\intercal} P + \lambda_{\max}\left\{\Lambda_2^{-1}\right\} L_{\sigma} \left(u_0 + u_1 \left\|x\right\|^2\right) I_n + P$$
$$\lambda_{\max}\left\{\Lambda_3^{-1}\right\}\left(\tilde{f}_0 + \tilde{f}_1 \left\|x\right\|^2\right) + \lambda_{\max}\left\{\Lambda_4^{-1}\right\}\left(\xi_0 + \xi_1 \left\|x\right\|^2\right) +$$
$$tr\left\{2\tilde{W}_1^{\intercal} P\Delta\sigma_1^{\intercal}(\hat{x}\left(t\right)) + 2\tilde{W}_1^{\intercal} K_1^{\intercal} \frac{d}{dt}\tilde{W}_1\right\} + tr\left\{2\tilde{W}_2^{\intercal} P\Delta\left[\sigma_2(\hat{x}\left(t\right))u(t)\right]^{\intercal}) + 2\tilde{W}_2^{\intercal} K_2^{\intercal} \frac{d}{dt}\tilde{W}_2\right\}$$

$$\left\|x\right\|^2 < x^+$$

$$\dot{V}\left(t\right) \leq$$
$$\Delta^{\intercal} \left(PA + A^{\intercal} P + PW_1 \Lambda_1 W_1^{\intercal} P + \lambda_{\max}\left\{\Lambda_1^{-1}\right\} L_{\sigma} I_n + PW_2 \Lambda_2 W_2^{\intercal} P + \lambda_{\max}\left\{\Lambda_2^{-1}\right\} L_{\sigma} \left(u_0 + u_1 x^+\right) I_n + P\Lambda_3 P\right.$$
$$\lambda_{\max}\left\{\Lambda_3^{-1}\right\}\left(\tilde{f}_0 + \tilde{f}_1 x^+\right) + \lambda_{\max}\left\{\Lambda_4^{-1}\right\}\left(\xi_0 + \xi_1 x^+\right) +$$
$$2tr\left\{\tilde{W}_1^{\intercal}\left(P\Delta\sigma_1^{\intercal}(\hat{x}\left(t\right)) + K_1^{\intercal}\frac{d}{dt}\tilde{W}_1\right)\right\} + 2tr\left\{\tilde{W}_2^{\intercal}\left(P\Delta\left[\sigma_2(\hat{x}\left(t\right))u(t)\right]^{\intercal}) + K_2^{\intercal}\frac{d}{dt}\tilde{W}_2\right)\right\}$$

$$\dot{V}\left(t\right) \leq$$
$$\Delta^{\intercal}\left(PA + A^{\intercal}P + PRP + Q\right)\Delta + \beta_0 +$$
$$2tr\left\{\tilde{W}_1^{\intercal}\left(P\Delta\sigma_1^{\intercal}(\hat{x}\left(t\right)) + K_1\frac{d}{dt}\tilde{W}_1\right)\right\} + 2tr\left\{\tilde{W}_2^{\intercal}\left(P\Delta\left[\sigma_2(\hat{x}\left(t\right))u(t)\right]^{\intercal}) + K_2\frac{d}{dt}\tilde{W}_2\right)\right\}$$

$$R = W_1\Lambda_1 W_1^{\intercal} + W_2\Lambda_2 W_2^{\intercal} + \Lambda_3 + \Lambda_4$$
$$Q = \lambda_{\max}\left\{\Lambda_1^{-1}\right\} L_{\sigma} I_n + \lambda_{\max}\left\{\Lambda_2^{-1}\right\} L_{\sigma}\left(u_0 + u_1 x^+\right) I_n$$
$$\beta_0 = \lambda_{\max}\left\{\Lambda_3^{-1}\right\}\left(\tilde{f}_0 + \tilde{f}_1 x^+\right) + \lambda_{\max}\left\{\Lambda_4^{-1}\right\}\left(\xi_0 + \xi_1 x^+\right)$$

$$\frac{d}{dt}\tilde{W}_1 = -K_1^{-1}P\Delta\sigma_1^{\intercal}(\hat{x}\left(t\right)); \quad \frac{d}{dt}\tilde{W}_2 = -K_2^{-1}P\Delta\left[\sigma_2(\hat{x}\left(t\right))u(t)\right]^{\intercal}$$

$$\dot{V}\left(t\right) \leq \Delta^{\intercal}\left(PA + A^{\intercal}P + PRP + Q\right)\Delta + \beta_0$$

Assumption: Exists $\alpha > 0$ such that

$$PA + A^{\intercal}P + PRP + Q \leq -\alpha P$$

$$\dot{V}\left(t\right) \leq -\alpha\Delta^{\intercal}P\Delta + \beta_0 < 0$$

$$\alpha\Delta^{\intercal}P\Delta > \beta_0$$

a) On the boundedness of Δ

$$\dot{V}(t) \leq -\alpha \Delta^{\intercal} P \Delta + \beta_0$$

$$\int_0^T \dot{V}(t)\, dt = V(T) - V(0) \leq -\int_0^T \alpha \Delta^{\intercal} P \Delta dt + \beta_0 T$$

$$\sup_{T\to\infty} \lim \frac{1}{T} : \alpha \int_0^T \Delta^{\intercal} P \Delta dt \leq \beta_0 T + V(0) - V(T)$$

$$\alpha \sup_{T\to\infty} \lim \frac{1}{T} \int_0^T \Delta^{\intercal} P \Delta dt \leq \beta_0 + \sup_{T\to\infty} \lim \left(\frac{V(0) - V(T)}{T} \right)$$

$$\sup_{T\to\infty} \lim \frac{1}{T} \int_0^T \Delta^{\intercal} P \Delta dt \leq \frac{\beta_0}{\alpha} => \Delta^{\intercal} P \Delta \text{ is also bounded}$$