Laplace transform & Z transform

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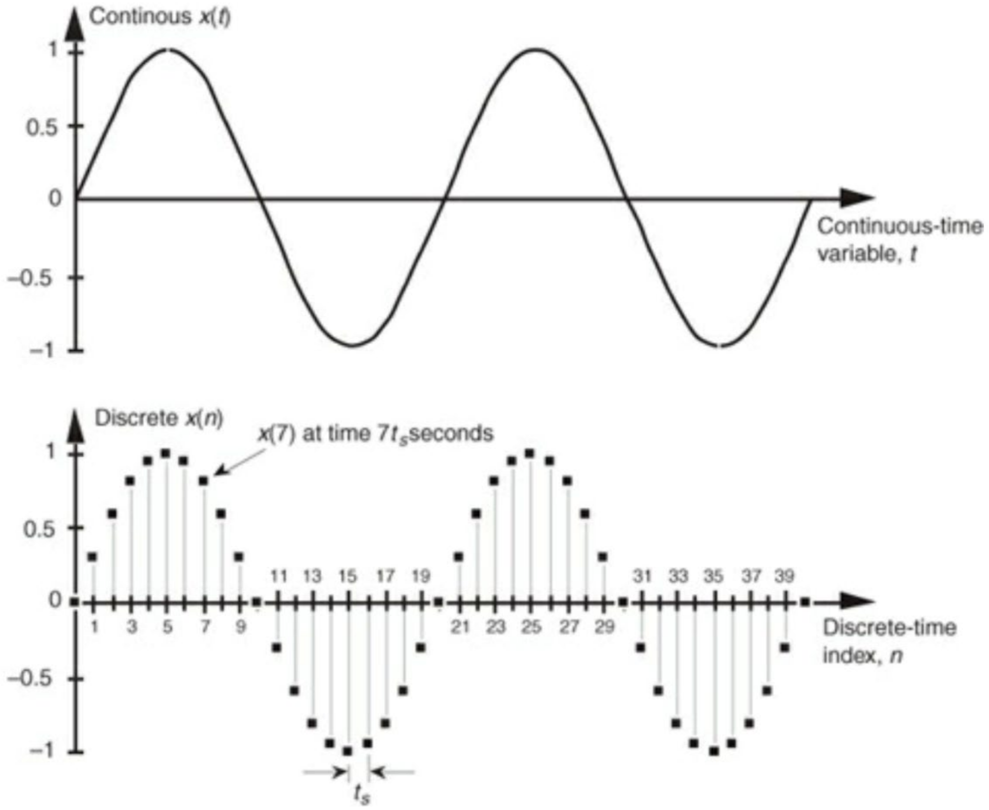
Purpose of this course

→ What is z-transform?
→ Why do we bother to use the z-transform?

Discrete time signal v.s. continuous time signal



Discrete time signal v.s. continuous time signal

For discrete time signal $x(n)$,
the **z-transform** of a causal sequence $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

For continuous time signal $x(t)$,
Laplace transform of a signal $x(t)$ is defined as

$$X(s) = \mathcal{L}\{x(t)\} = \int_{0}^{\infty} x(t) e^{-st} dt$$

Discrete time system v.s. continuous time system

Discrete time system

Transfer function of a discrete time system:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} a_i \, z^{-i}}{1 + \sum_{i=1}^{N} b_i \, z^{-i}}$$

Continuous time system

Transfer function of a continuous time system

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{i=0}^{N} a_i \, s^{i}}{1 + \sum_{i=1}^{N} b_i \, s^{i}}$$

$X()$ represents the input signal,
$Y()$ represents the output signal

Complex Fourier transform -> Laplace transform

Complex Fourier transform:

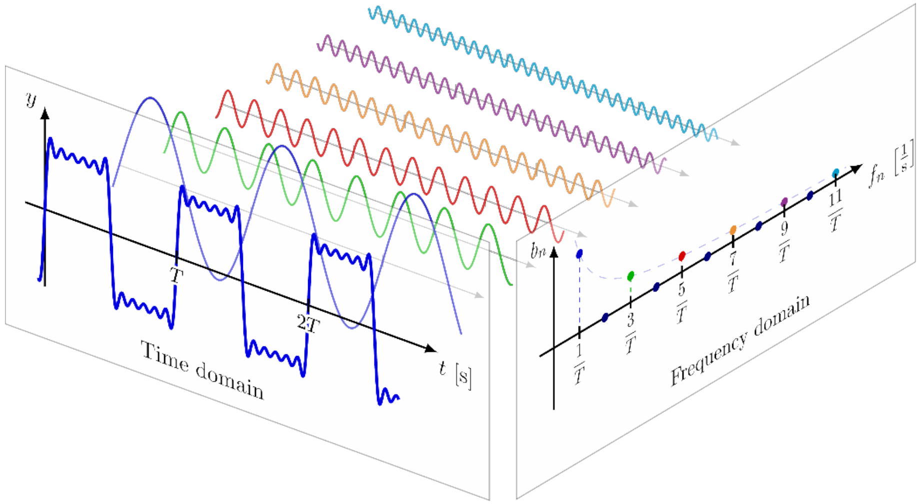
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

It can be extended to Laplace transform

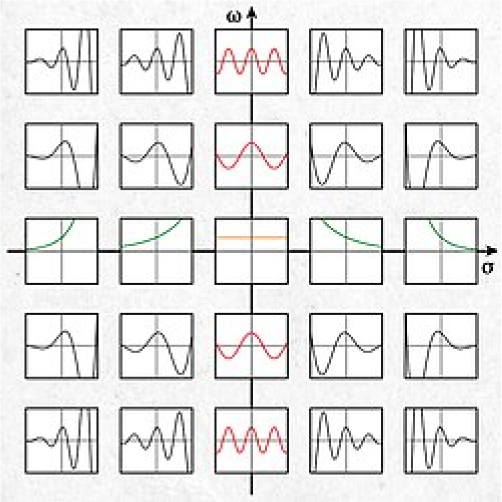
$$X(\sigma,\omega) = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt$$

What does $e^{-\sigma t}$ mean?

Fourier Transform



Laplace Transform



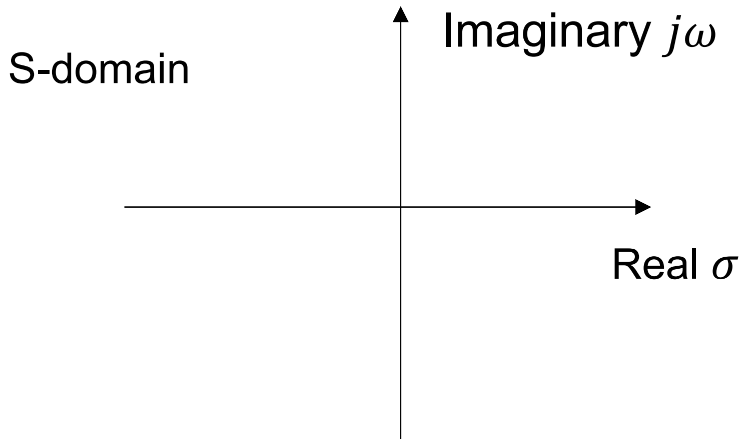
Laplace transform and S domain

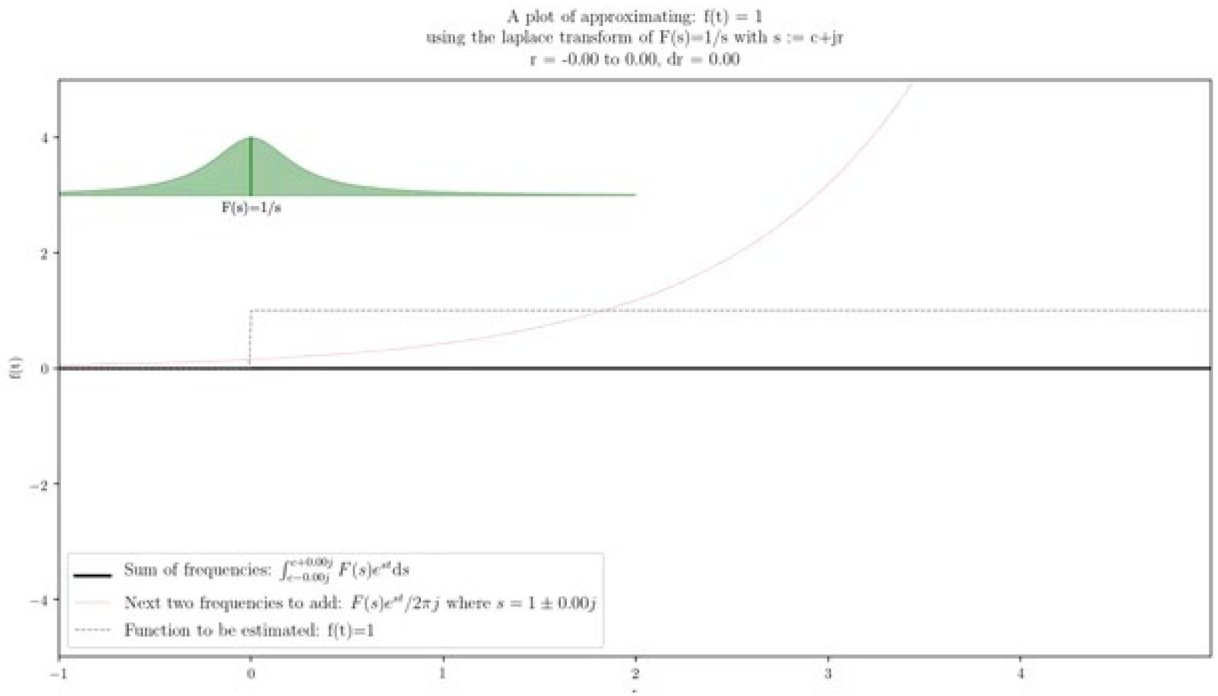
The **Laplace transform** changes a signal in the time domain into a signal in the **s-domain**.

$$X(\sigma, \omega) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt}$$

where $s = \sigma + j\omega$





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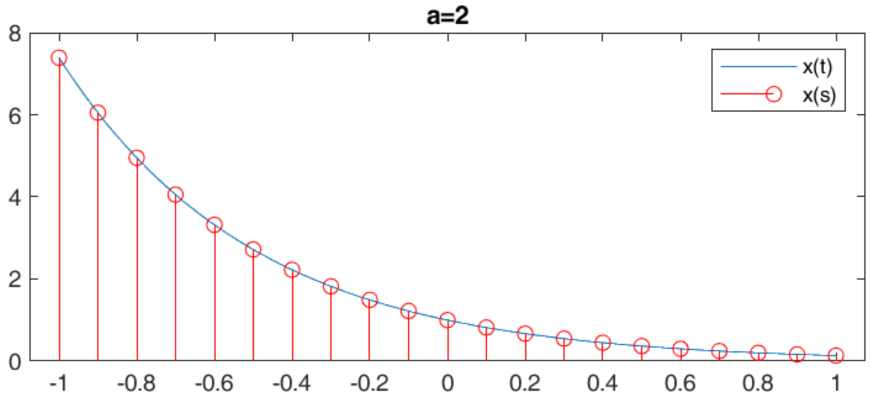
Example



Let's consider a **continuous** signal

$$x(t) = e^{-at}$$

Its Laplace transform is

$$X(s) = \frac{1}{s+a}$$

After impulse sampling, we receive the following sequence

$$x(nT) = e^{-anT}, \qquad \text{a} > 0$$

Laplacian transform

$$X_s(s) = \sum_{n=0}^{\infty} x(nT)e^{-snT}$$

$$= \sum_{n=0}^{\infty} e^{-anT}e^{-snT} = \sum_{n=0}^{\infty} e^{-(s+a)nT}$$

For $|x| < 1$, infinite quotient series

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

It can be rewritten as

$$X_s(s) = \frac{1}{1-e^{-(s+a)T}}$$

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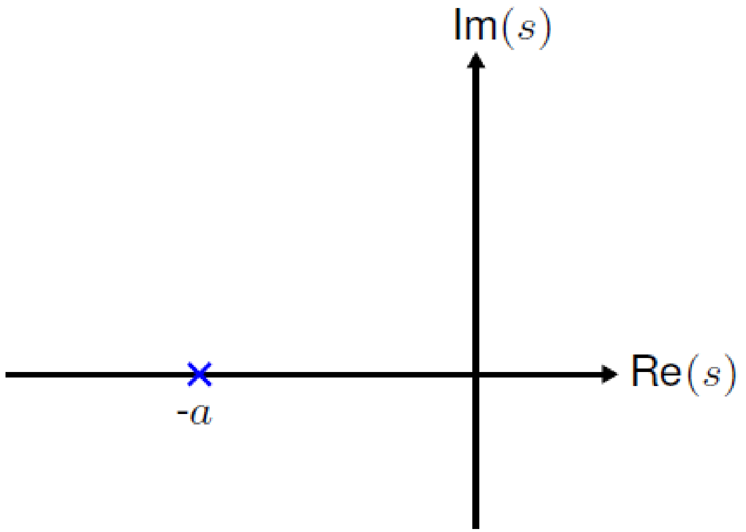
Continuous signal

$$X(s) = \frac{1}{s+a}$$

This is a transfer function with one pole in $s = -a$



Impulse sampled sequence

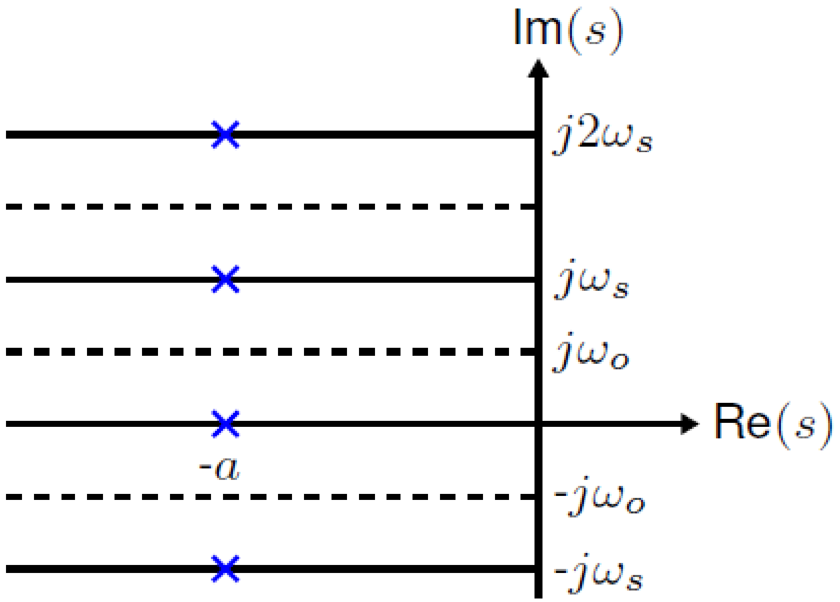
$$X_s(s) = \frac{1}{1 - e^{-(s+a)T}}$$

It has poles

$$s = -a \pm jm\frac{2\pi}{T} = -a \pm jm2\pi f_s$$

$e^{-(s+a)T} = 1$

Euler's identity
$e^{2m\pi} = 1$



When sampling, the pole-zero diagram along the imaginary axis is repeated periodically with the sample frequency.

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Relation between s-domain and z-domain

Laplace transform of **sequence** $\boldsymbol{x(n)}$

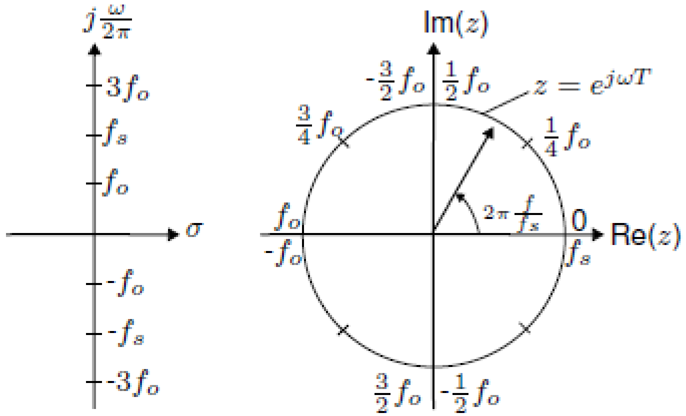
$$X_s(s) = \sum_{n=0}^{\infty} x(n)e^{-snT}$$

z transform of **sequence** $\boldsymbol{x(n)}$

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

$X_s(s) = X(z)$ when $z = e^{sT}$

Where $s = \sigma + j\omega$



s domain

z domain

Q: What f_0 here mean?

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Z transform

For discrete signal process, we use z transform

Definition

The z transform of a sequence $x(n)$ is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Notation $X(z) = \mathcal{Z}[x(n)]$

The inverse z transform

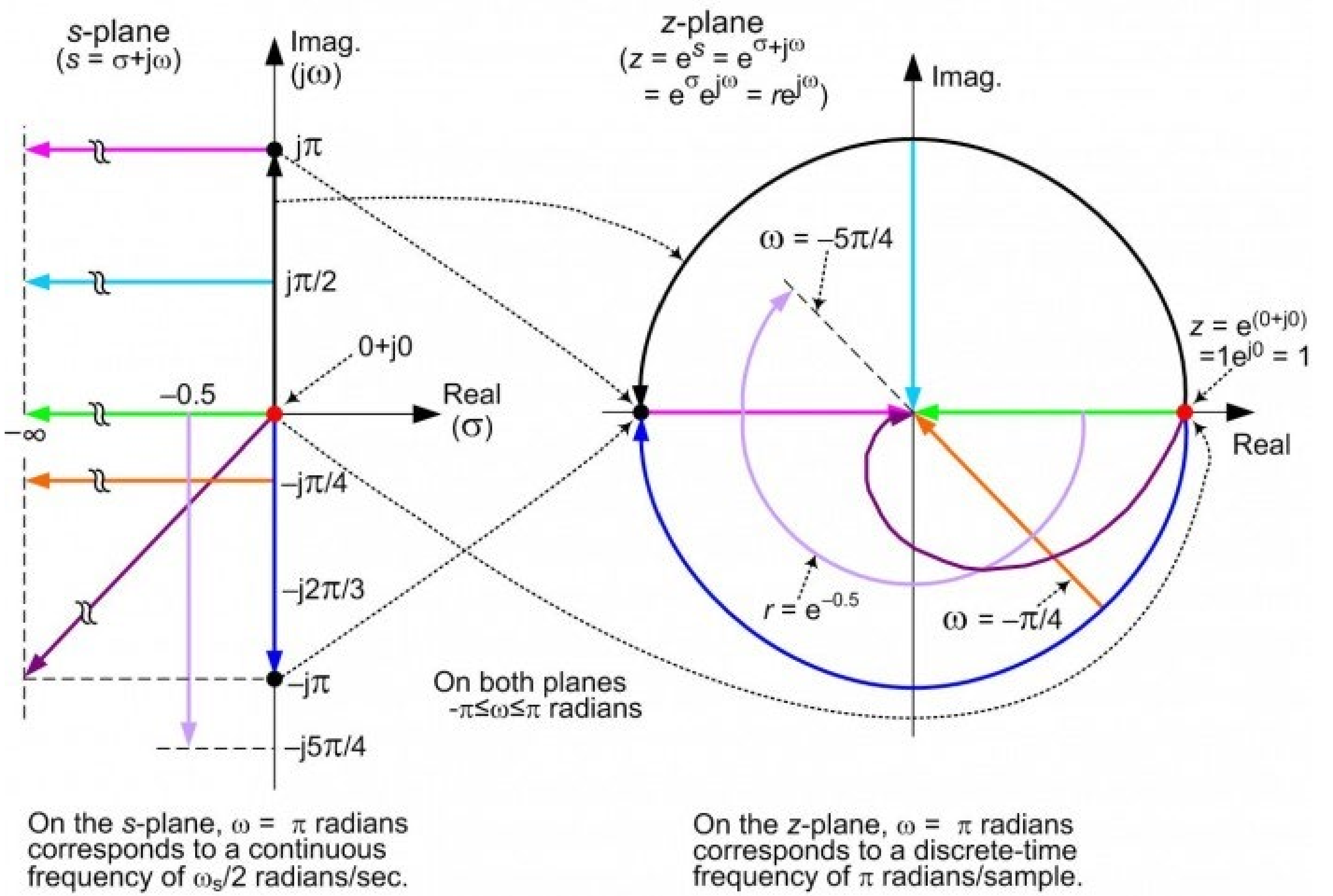
$$x(n) = \mathcal{Z}^{-1}[X(z)]$$

Relation between s plane and z plane

The relation between s plane and z plane as follows

$$z = e^{sT} = e^{\sigma / f_s} \angle 2\pi \frac{f}{f_s}$$

z-plane is a polar coordinate!



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Imagine axis of s-plane

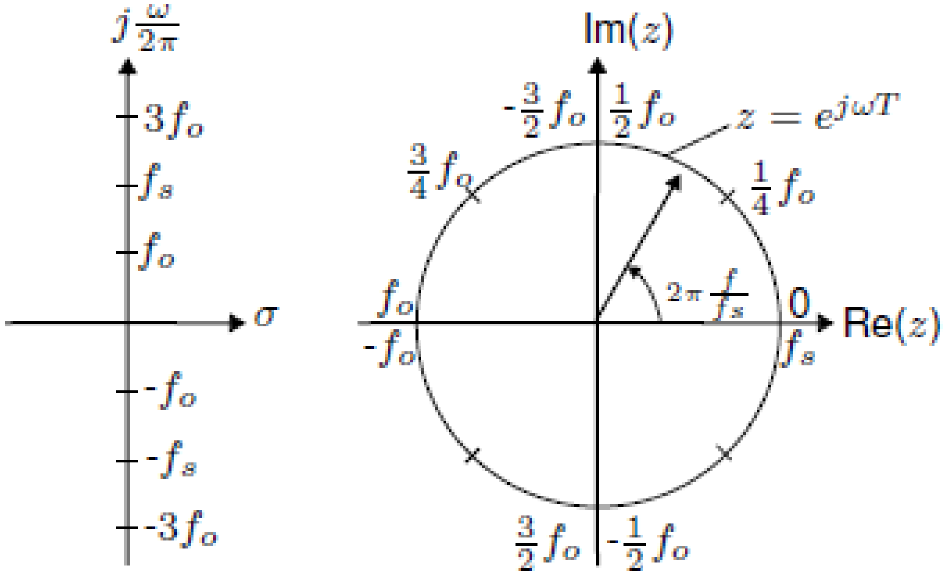
The Imaginary axis of the s-plane can be understood as a set where $\sigma = 0$
In this case,

$$z = e^{j\omega T}$$

When $(\omega = 2\pi f)$

$$z = 1\angle 2\pi \frac{f}{f_s}$$

$s = \sigma + j\omega$



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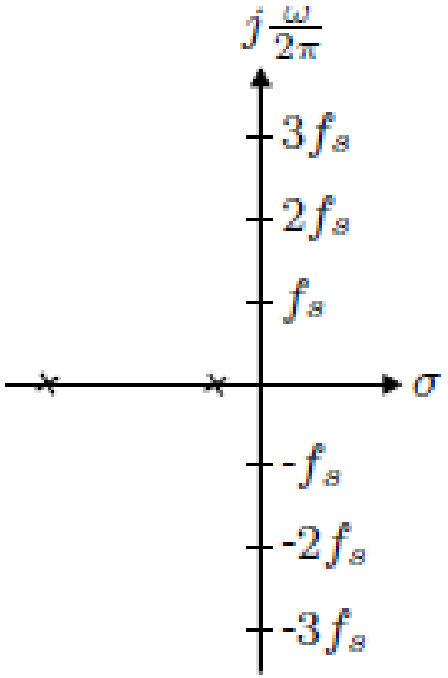
Real axis of s-plane

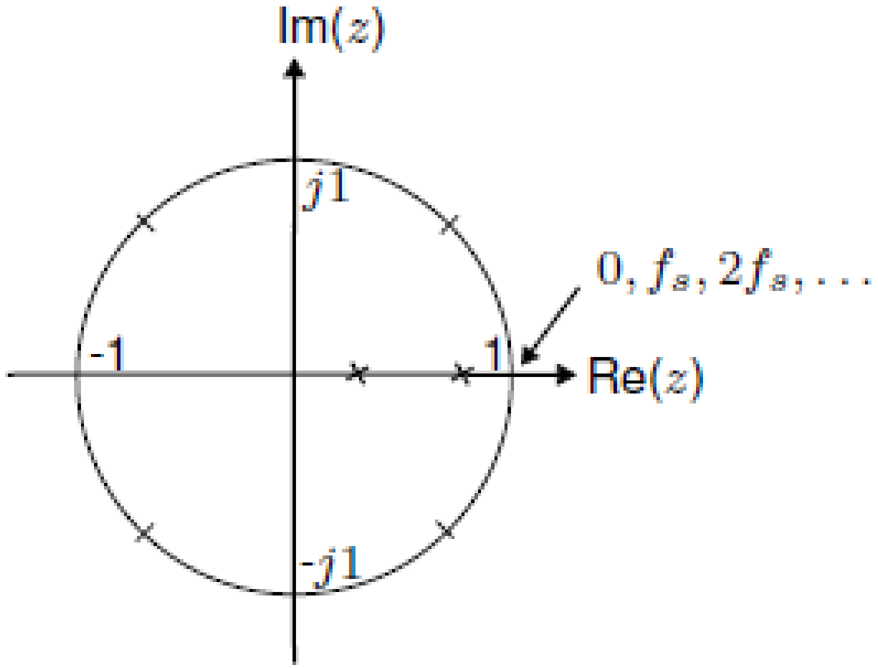
The real axis of the s-plane can be understood as a set where $\omega = 0$

$$z = e^{\sigma/f_s} \angle 0^{\circ}$$

$s = \sigma + j\omega$

Q: only the positive Re(z)? Or the whole axis of Re(z)?





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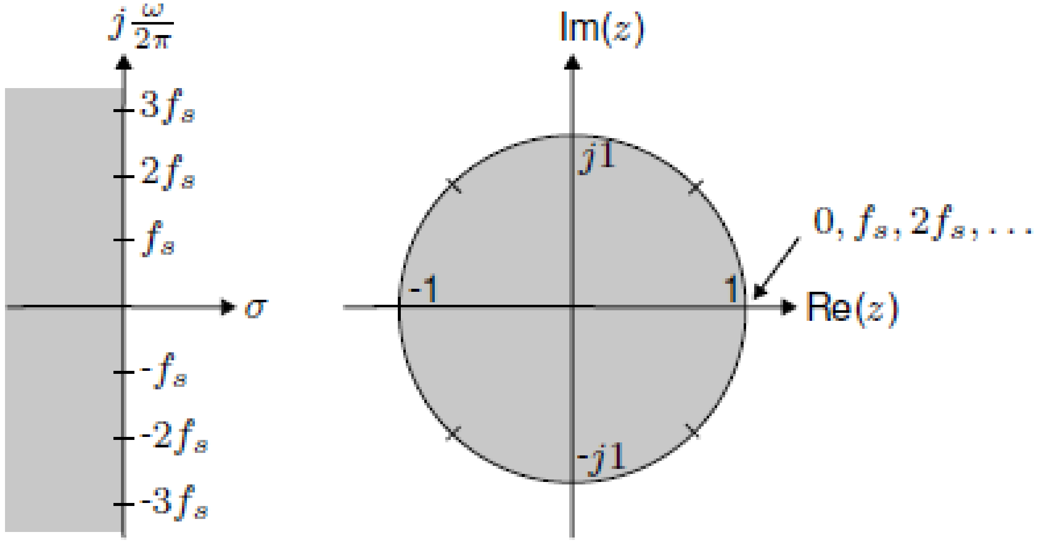
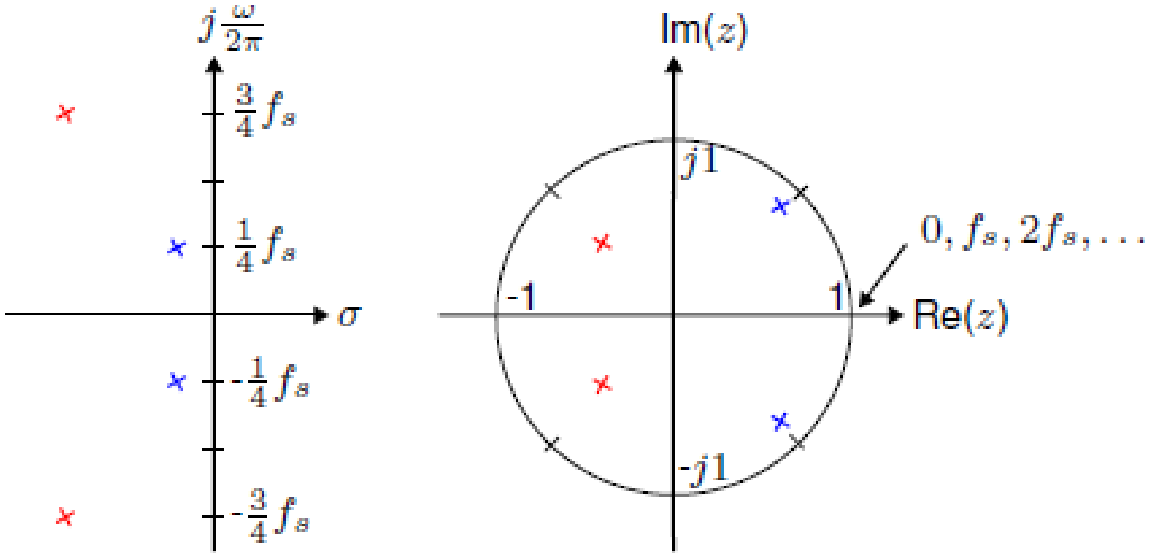
Left half-plane of s-plane

The left half-plane of s-plane → $\sigma < 0$
It means that $|z| = e^{\sigma/f_s} < 1$.
Thus, the left half-plane becomes the interior of the unit circle.



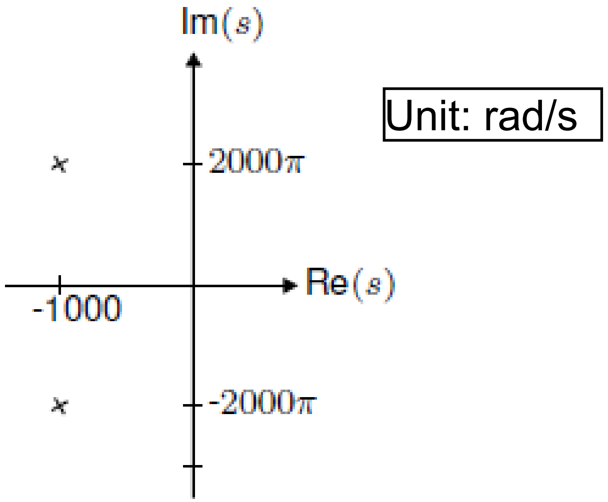
Example

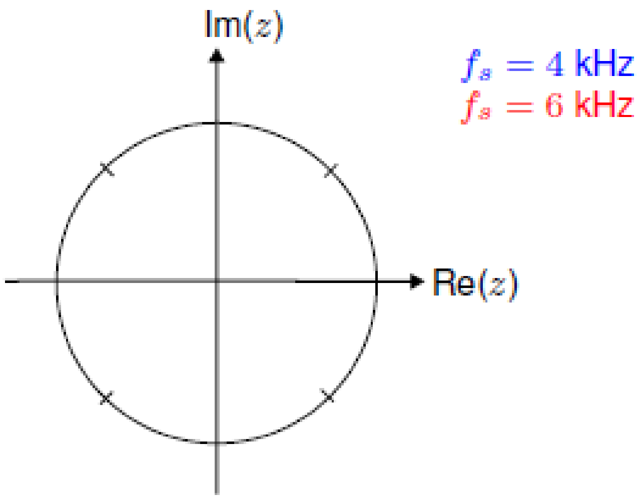
$$z = e^{sT} = e^{\sigma/f_s} \angle 2\pi \frac{f}{f_s}$$



Exercise

Where in the z plane are the poles when fs = 4 kHz and fs = 6 kHz?





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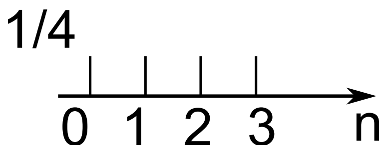
Converge

Region of convergence $(ROC) = \{z | \sum_n x(n) z^{-n} \ converge\}$

Where z is complex number.

Example 1

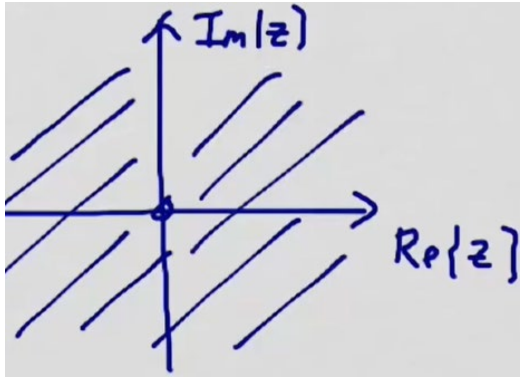


4 points rectangular window: $x(n) = \frac{1}{4}\sum_{k=0}^{3} \delta(n-k)$

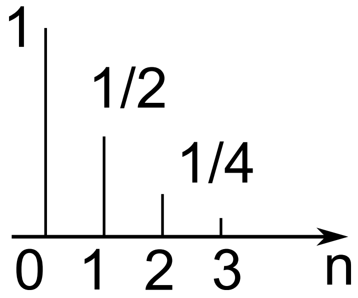
$$X(z) = \frac{1}{4}(1 + z^{-1} + z^{-2} + z^{-3})$$

ROC: the range of z that makes $X(z)$ converge?

$$ROC = \{z|z \neq 0\}$$



Example 2

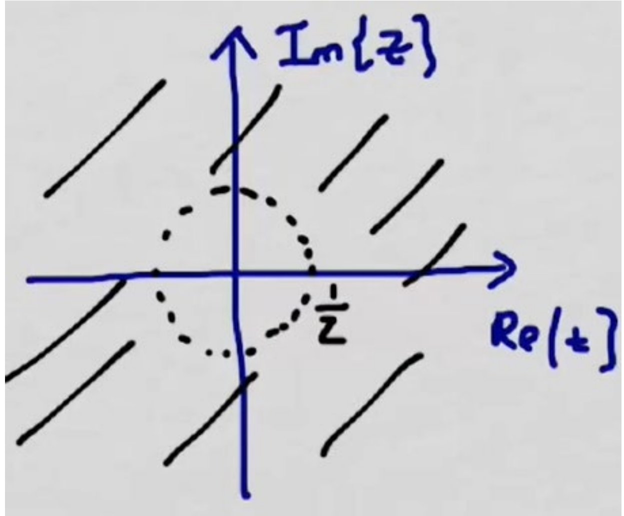


$$x(n) = \left(\frac{1}{2}\right)^n$$

$$X(Z) = \sum_{n=0}^{\infty} (\frac{1}{2})^n z^{-n} = \sum_{n=0}^{\infty} (\frac{1}{2z})^n$$

$X(Z)$ converge if $\left|\frac{1}{2z}\right| < 1$,

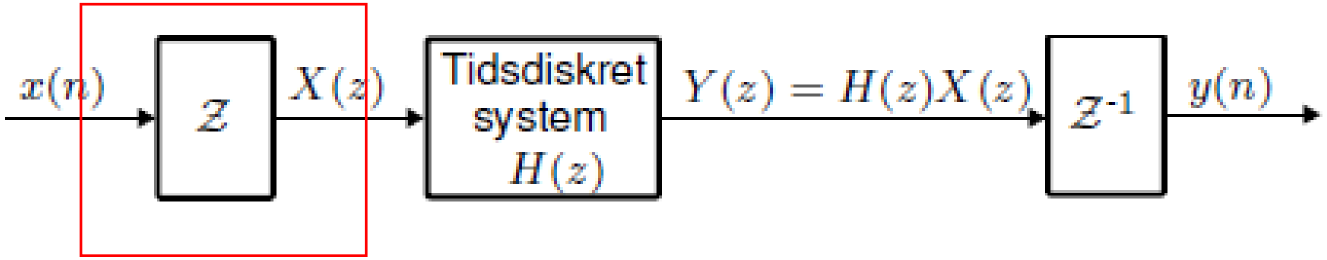
$$ROC = \left\{z \middle| \, |z| > \frac{1}{2}\right\}$$



Z domain in discrete signal processing

Process

→ Perform z transform on input signal x(n)
→ Describe the discrete time system in z domain H(z)
→ Obtain output Y(z) = H(z)X(z)
→ Perform inverse z transform on Y(z) to obtain y(n)



Z-transformation rules

A number of transformation rules can facilitate the calculation of the z-transformation
Especially the rules Z1 and Z2 are often used in this course.

Regel	$x(n)$	$X(z)$
Z1	$ax_1(n) + bx_2(n)$	$aX_1(z) + bX_2(z)$
Z2	$x(n-m)$	$z^{-m}X(z)$
Z3	$x(n)a^{-n}$	$X(az)$
Z4	$x(n)s^{-bn}$	$X(e^{bT}z)$
Z5	$\sum_{m=0}^{n} x(m)h(n-m)$	$X(z)H(z)$

Prove

The z-transform of $x(n-m)$ is

$$\mathcal{Z}[x(n-m)] = \sum_{n=0}^{\infty} x(n-m)z^{-n}$$

where $x(n)$ is a **causal sequence**.

Therefore, the above can be written

$$\mathcal{Z}[x(n-m)] = \sum_{n=m}^{\infty} x(n-m)z^{-n}$$

By defining $k \coloneqq n-m$, we have

$$\mathcal{Z}[x(n-m)] = \sum_{k=0}^{\infty} x(k)z^{-(k+m)} = z^{-m}\sum_{k=0}^{\infty} x(k)z^{-k}$$

$$\mathcal{Z}[x(n-m)] = z^{-m}X(z)$$

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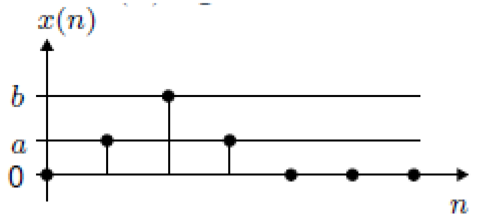
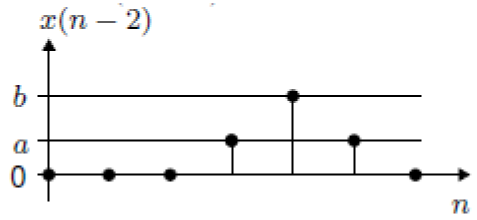
Example

Let's compare the sequence $x(n)$ and the delayed sequence $x(n-2)$ in the z-domain.





You can determine the z-transform of $x(n)$ as

$$X(z) = \mathcal{Z}[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n}$$

From the graph it can be seen that

$$X(z) = az^{-1} + bz^{-2} + az^{-3}$$

$$\mathcal{Z}[x(n-2)] = z^{-2} \sum_{n=0}^{\infty} x(n) z^{-n} = az^{-3} + bz^{-4} + az^{-5}$$

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Common z-transform for signal

Par	$x(n)$	$X(z)$
ZT1	$\delta(n)$	1
ZT2	$u(n)$	$\frac{z}{z-1}$
ZT3	n	$\frac{z}{(z-1)^2}$
ZT4	a^n	$\frac{z}{z-a}$
ZT5	$e^{s_0 nT}$	$\frac{z}{z-e^{s_0 T}}$
ZT6	$\sin \omega_0 nT$	$\frac{(\sin \omega_0 T)z}{z^2-2(\cos \omega_0 T)z+1}$
ZT7	$\cos \omega_0 nT$	$\frac{z^2-(\cos \omega_0 T)z}{z^2-2(\cos \omega_0 T)z+1}$

Example

Calculate the z-transform of **unit sequence** $\boldsymbol{u(n)}$

$$U(z) = Z[u(n)] = \sum_{n=0}^{\infty} u(n) z^{-n}$$

Since $u(n) = 1$ is for $n \geq 0$, so we have

$$U(z) = \sum_{n=0}^{\infty} z^{-n}$$

Geometric series sum:

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

Therefore,

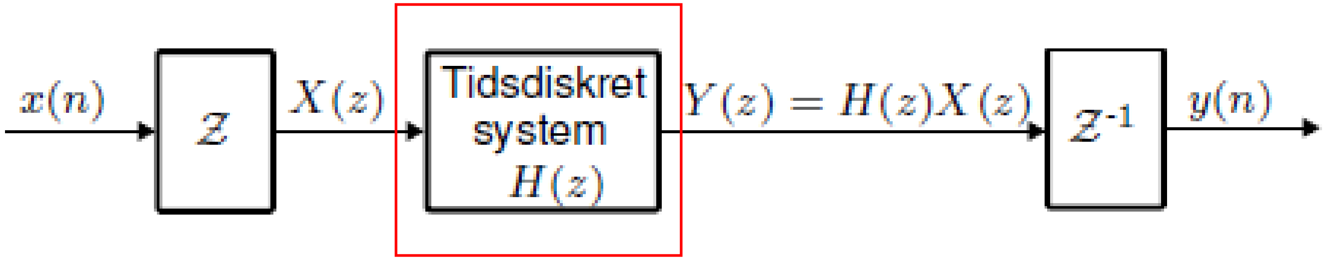
$$U(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

Transfer function in z domain

Discrete systems can be described by a transfer function as

$$H(z) = \frac{Y(z)}{X(z)}$$

$H(z)$ is a transfer function, and $X(z)$, $Y(z)$ is the input and output sequence



Discrete time system: Transfer function

A *N*th order difference equation describing a causal system can be written as

$$y(n) + b_1 y(n-1) + b_2 y(n-2) + \ldots + b_N y(n-N) = a_0 x(n) + a_1 x(n-1) + \ldots + a_N x(n-N)$$

where $x(n-i)$ is the input sequence, $y(n-i)$ is the output sequence,
a_i, b_i are real coefficients.

The above difference equation can be written as

$$y(n) = \sum_{i=0}^{N} a_i x(n-i) - \sum_{i=1}^{N} b_i y(n-i)$$

If a b-coefficient is different from zero, then the differential equation is called a *recursive algorithm*.

We will talk more about this in FIR and IIR

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Transfer function in z domain

A transfer function is found by z-transformation of a differential equation of the form

$$y(n) + \sum_{i=1}^{N} b_i y(n-i) = \sum_{i=0}^{N} a_i x(n-i)$$

Recall: rule is used for the z-transformation

$$z^{-m} X(z) = \mathcal{Z}\big(x(n-m)\big)$$

Therefore, we have

$$Y(z)\left(1 + \sum_{i=1}^{N} b_i z^{-i}\right) = X(z) \sum_{i=0}^{N} a_i z^{-i}$$

The transfer function thus becomes

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} a_i z^{-i}}{1 + \sum_{i=1}^{N} b_i z^{-i}}$$

It can be rewritten as

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} a_i z^{N-i}}{1 + \sum_{i=1}^{N} b_i z^{N-i}}}$$

Example: 1st and 2nd order transfer function

A first order system with difference equation

$$y(n) = a_0 x(n) + a_1 x(n-1) - b_1 y(n-1)$$

Its transfer function after z-transform

$$Y(z)(1 + b_1 z^{-1}) = X(z)(a_0 + a_1 z^{-1})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_0 + a_1 z^{-1}}{1 + b_1 z^{-1}} = \frac{a_0 z + a_1}{z + b_1}$$

A second order system with difference equation

$$y(n) = a_0 x(n) + a_1 x(n-1) + a_2 x(n-1) - b_1 y(n-1) \; - b_2 y(n-2)$$

Its transfer function after z-transform

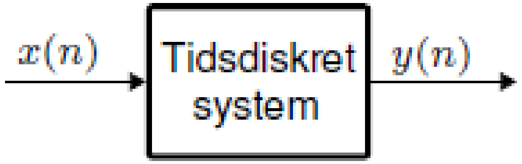
$$Y(z)(1 + b_1 z^{-1} + b_2 z^{-2}) = X(z)(a_0 + a_1 z^{-1} + a_2 z^{-2})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}} = \frac{a_0 z^2 + a_1 z + a_2}{z^2 + b_1 z + b_2}$$

Example: First order differential equation

Let’s consider the following differential equation (transfer function):

$$y(n) = x(n) + 0.5y(n-1)$$



What will be the output sequence $y(n)$ when the input sequence $\boldsymbol{x(n)}$ **is a unit sequence** $\boldsymbol{u(n)}$ and $\boldsymbol{y(n) = 0}$ **for** $\boldsymbol{n < 0}$?

We can calculate when $n = 0, 1, 2, \ldots$

$$
\begin{aligned}
y(0) &= x(0) + 0,5y(-1) = 1 \\
y(1) &= x(1) + 0,5y(0) = 1,5 \\
y(2) &= x(2) + 0,5y(1) = 1,75 \\
y(3) &= x(3) + 0,5y(2) = 1,875
\end{aligned}
$$

Poles and zeros of a transfer function

A function is called a transfer function if it can be written in the form as

$$H(z) = \frac{P(z)}{Q(z)}$$

Both $P(z)$ and $\mathrm{Q}(z)$ are polynomials in z

The roots of $P(z)$ are called **zeros** of $X(z)$
The roots of $\mathrm{Q}(z)$ are called **poles** of $X(z)$

Example

Considering the following

$$H(z) = \frac{0.58\ - 0.58z^{-1}}{1 - 0.16z^{-1}}$$

The transfer function can be rewritten as

$$H(z) = 0.58\frac{z - 1}{z - 0.16}$$

zero for $\mathrm{H}(z)$ is $z = 1$,
pole for $\mathrm{H}(z)$ is $z = 0.16$

Pole-Zero diagram

Let's consider the following transfer function

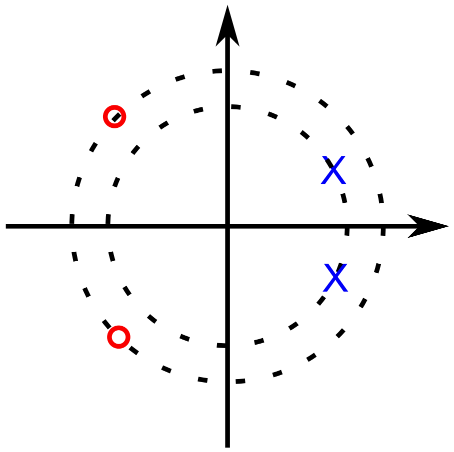
$$H(z) = \frac{0.5 + 0.707z^{-1} + 0.5z^{-2}}{1 - 1.386z^{-1} + 0.64z^{-2}} = 0.5\,\frac{z^2 + 1.414z + 1}{z^2 - 1.386z + 0.64}$$

The zeros are found from the roots of the numerator polynomial

$$z^2 + 1,414z + 1 = 0 \qquad \Rightarrow \qquad z = -0,707 \pm j0,707$$

The poles are found from the roots of the denominator polynomial

$$z^2 - 1,386z + 0,64 = 0 \qquad \Rightarrow \qquad z = 0,693 \pm j0,4$$



Factorization

From the fundamental theorem of algebra, a transfer function can be written as

$$H(z) = a_0 \frac{(z - z_1)(z - z_2) \dots (z - z_N)}{(z - p_1)(z - p_2) \dots (z - p_N)}$$

Where z_i for $i = 1, \dots, N$ are the zeros of the transfer function
p_i for $i = 1, \dots, N$ are the poles of the transfer function

Matlab function – roots()

returns the roots of the polynomial

For example, $3x^2 - 2x - 4 = 0$

```
p = [3, -2, -4];
r = roots(p)
```

```
r = 2×1

    1.5352
   -0.8685
```

x1
x2

Matlab function: tf2zp()

Find the zeros, poles, and gain of the system.

Let's consider the transfer function in the previous example:

$$H(z) = \frac{0.5 + 0.707z^{-1} + 0.5z^{-2}}{1 - 1.386z^{-1} + 0.64z^{-2}} = 0.5\frac{z^2 + 1.414z + 1}{z^2 - 1.386z + 0.64}$$

Z should be in positive power!!

```
b=[0.5, 0.5*1.414, 0.5*1]; % numerators
a=[1, -1.386, 0.64]; % denominators
[z,p,k] = tf2zp(b,a);
```

```
>> z

z =

  -0.7070 + 0.7072i
  -0.7070 - 0.7072i

>> p

p =

   0.6930 + 0.3997i
   0.6930 - 0.3997i

>> k

k =

    0.5000
```

$$H(z) = 0.5\frac{\big(z - (-0.707 + 0.7072i)\big)\big(z - (-0.707 - 0.7072i)\big)}{\big(z - (0.693 + 0.3997i)\big)\big(z - (0.693 - 0.3997i)\big)}$$

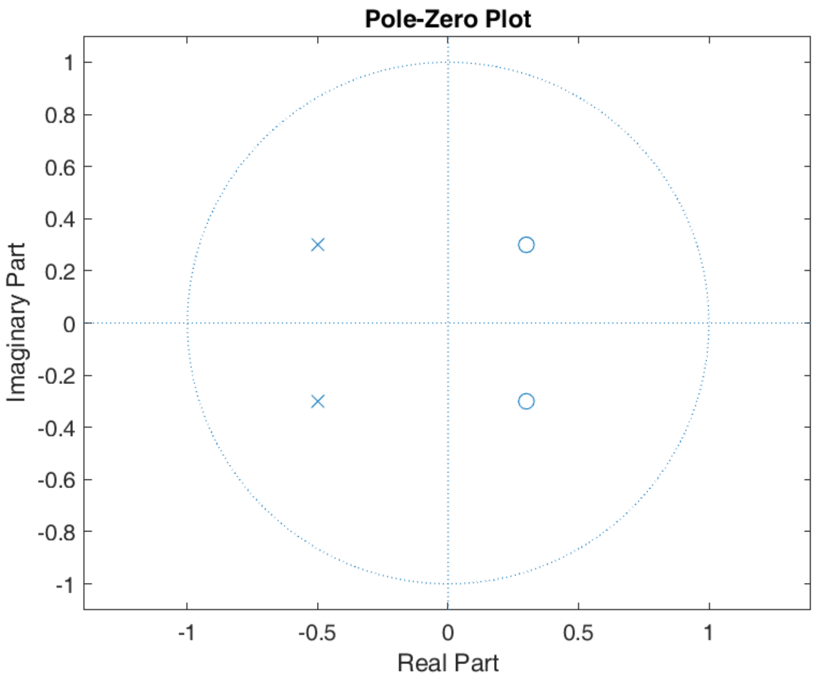
Matlab – zplane()

Zero-pole plot for discrete-time systems

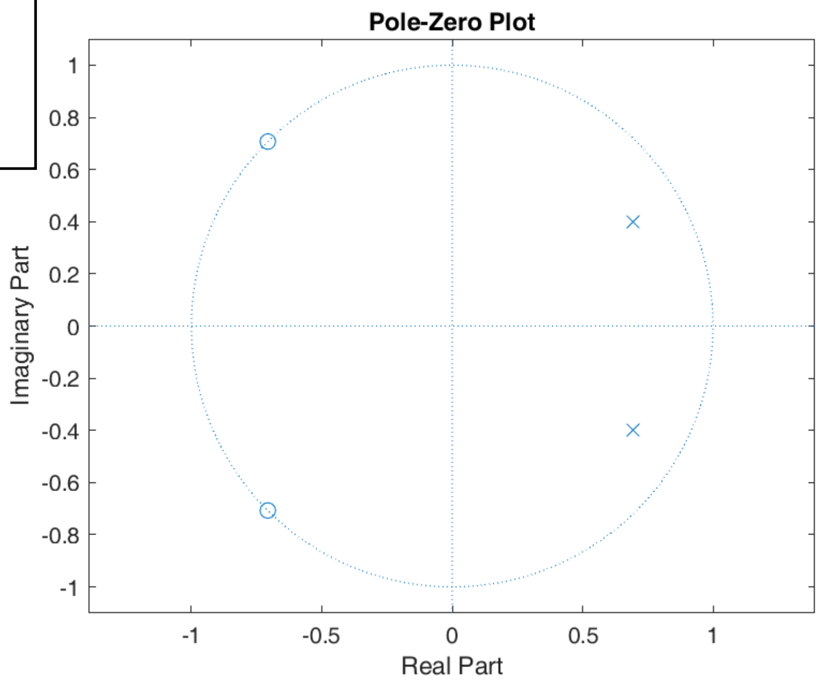
zplane(z, p) – plots the zeros and poles defined by z and p, which should be as **column vectors.**

zplane(b, a) – plots the zeros and poles defined by a transfer function whose numerator and denominator are b and a. Note that b and a should be as **row vectors.**

```
z=[0.3+0.3i; 0.3-0.3i];
p=[-0.5+0.3i; -0.5-0.3i];
zplane(z,p);
```




```
b=[0.5, 0.5*1.414, 0.5*1];
a=[1, -1.386, 0.64];
zplane(b,a)
% it is the previous
example
```

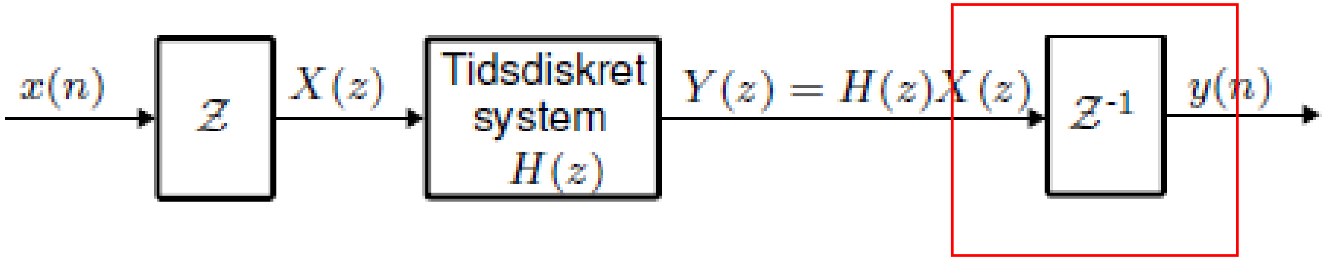


Inverse z-transform

Inverse z-transformation is used to determine the output response $y(n)$ of a discrete-time system for a given input $x(n)$.

The processing of discrete system (**RECAP**):

1. The input sequence $x(n)$ is z-transformed.
2. The system's transfer function $H(z)$ is set up with positive powers of z
3. The output response in z-domain is calculated $Y(z) = H(z)X(z)$
4. The output sequence $y(n)$ is calculated by inverse z-transform of $Y(z)$.



Example

Considering the following transfer function

$$H(z) = \frac{1}{1 - 0.5z^{-1}}$$

We use the inverse z-transform to determine the output response $y(n)$ when the **input $\boldsymbol{x(n)}$ is impulse $\boldsymbol{\delta(n)}$**.

1. The input sequence $x(n) = \delta(n)$ is z-transformed using lookup table

$$X(z) = \mathcal{Z}[x(n)] = \mathcal{Z}[\delta(n)] = 1$$

2. The system's transfer function $H(z)$ is set up with positive powers of z

$$H(z) = \frac{z}{z - 0.5}$$

3. The output response $Y(z)$ is calculated

$$Y(z) = H(z)X(z) = \frac{z}{z - 0.5} \cdot 1$$

4. According to the lookup table (ZT4)

$$y(n) = \mathcal{Z}^{-1}\left[\frac{z}{z - 0.5}\right] = 0.5^n$$

Partial fraction

The procedure for partial fraction solution is

1. Set up expressions for $Y(z)$ with positive power of z in factorization form

$$Y(z) = \frac{T(z)}{N(z)} = \frac{T(z)}{(z-p_1)(z-p_2)\dots(z-p_N)}$$

2. Divide $Y(z)$ by z so that the order number of the denominator is greater than the order number of the numerator. This expression resolves into fractions

$$\frac{Y(z)}{z} = \frac{T(z)}{zN(z)} = \frac{k_1}{z-p_1} + \frac{k_2}{z-p_2} + \cdots + \frac{k_N}{z-p_N}$$

3. Numerator coefficients k_i are calculated as

$$k_i = (z-p_i)\frac{Y(z)}{z}\bigg|_{z=p_i}$$

4. Write down $\frac{Y(z)}{z}$ in fractional form and multiply by z.
5. Inverse z-transform all the fractions. (Table)

Example

Considering transfer function

$$H(z) = \frac{z}{z - 0.5}$$

We use inverse z-transformation to determine the output response $y(n)$ when the input stimulus $x(n)$ is an **unit sequence** $u(n)$, i.e.

$$Y(z) = H(z)X(z) = \frac{z}{z - 0.5}\frac{\mathrm{z}}{z - 1} = \frac{z^2}{(z - 0.5)(z - 1)}$$

We follow the procedure for partial fraction solution
1. Set up expressions for Y(z) with positive powers of z in factored form

$$Y(z) = \frac{T(z)}{N(z)} = \frac{z^2}{(z - 0.5)(z - 1)}$$

where z = 0.5 and z = 1 are roots of the denominator polynomial of Y (z)

Example continue…

2. Divide Y(z) by z so that the order of the denominator is greater than the order of the numerator

$$\frac{Y(z)}{z} = \frac{z}{(z-0.5)(z-1)} = \frac{k_1}{z-0.5} + \frac{k_2}{z-1}$$

3. The numerator coefficients are calculated as

$$k_1 = (z-p_1)\frac{Y(z)}{z}\bigg|_{z=p_1} = \frac{z}{z-1}\bigg|_{z=0.5} = -1$$

$$k_2 = (z-p_2)\frac{Y(z)}{z}\bigg|_{z=p_2} = \frac{z}{z-0.5}\bigg|_{z=1} = 2$$

4. Write Y (z) z in fractional form and multiply by z

$$Y(z) = -\frac{1}{z-0.5}z + \frac{2}{z-1}\mathrm{z}$$

5. Inverse z-transform all the fractions. (Table ZT4)

$$y(n) = \mathcal{Z}^{-1}[Y(z)] = -\mathcal{Z}^{-1}\left[\frac{z}{z-0.5}\right] + 2\mathcal{Z}^{-1}\left[\frac{z}{z-1}\right] = -0.5^n + 2\cdot 1^n = 2-0.5^n$$

Matlab

Check functions:
  ztrans()
  iztrans()

```
syms n
f = sin(n);
fz = ztrans(f)
```

```
syms z
F = 2*z/(z-2)^2;
iztrans(F)
```