Discrete Fourier Transform

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Topics to be covered in this course

→ Sampling and reconstruction
→ Aliasing
→ Quantization and dynamic range
→ Implementation
→ Conversion time-frequency domain
→ Z transform
→ Linear Time Invariant system (LTI)
→ System analysis
→ Window functions
→ Filter design
→ Impulse response (FIR and IIR)

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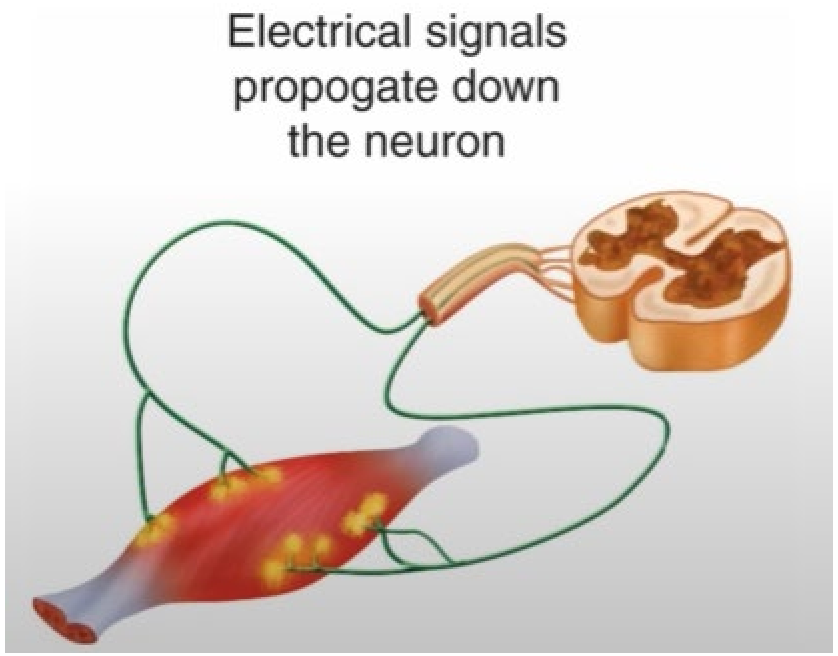
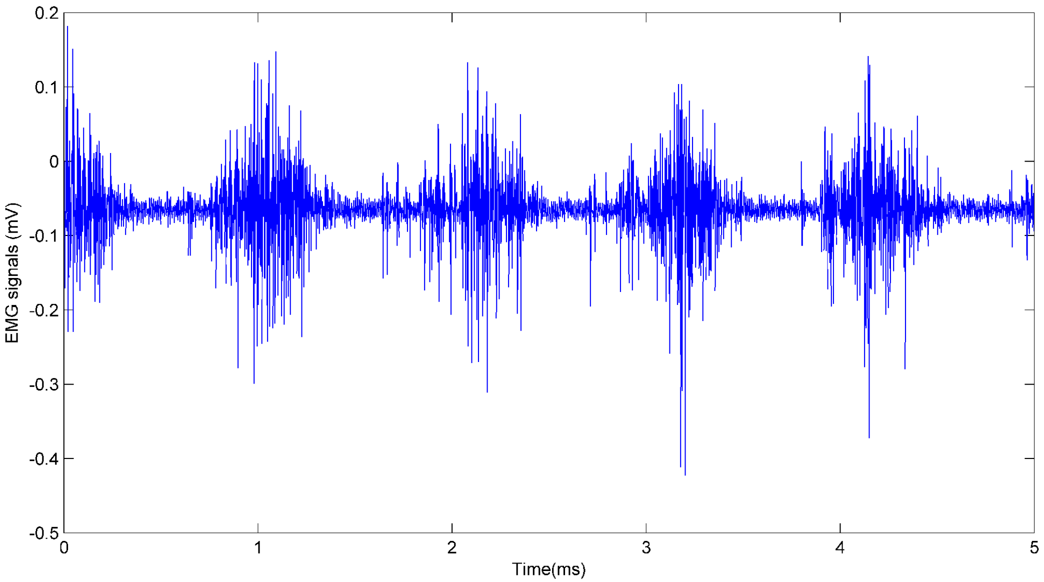
Muscle activity by Electromyography (EMG)





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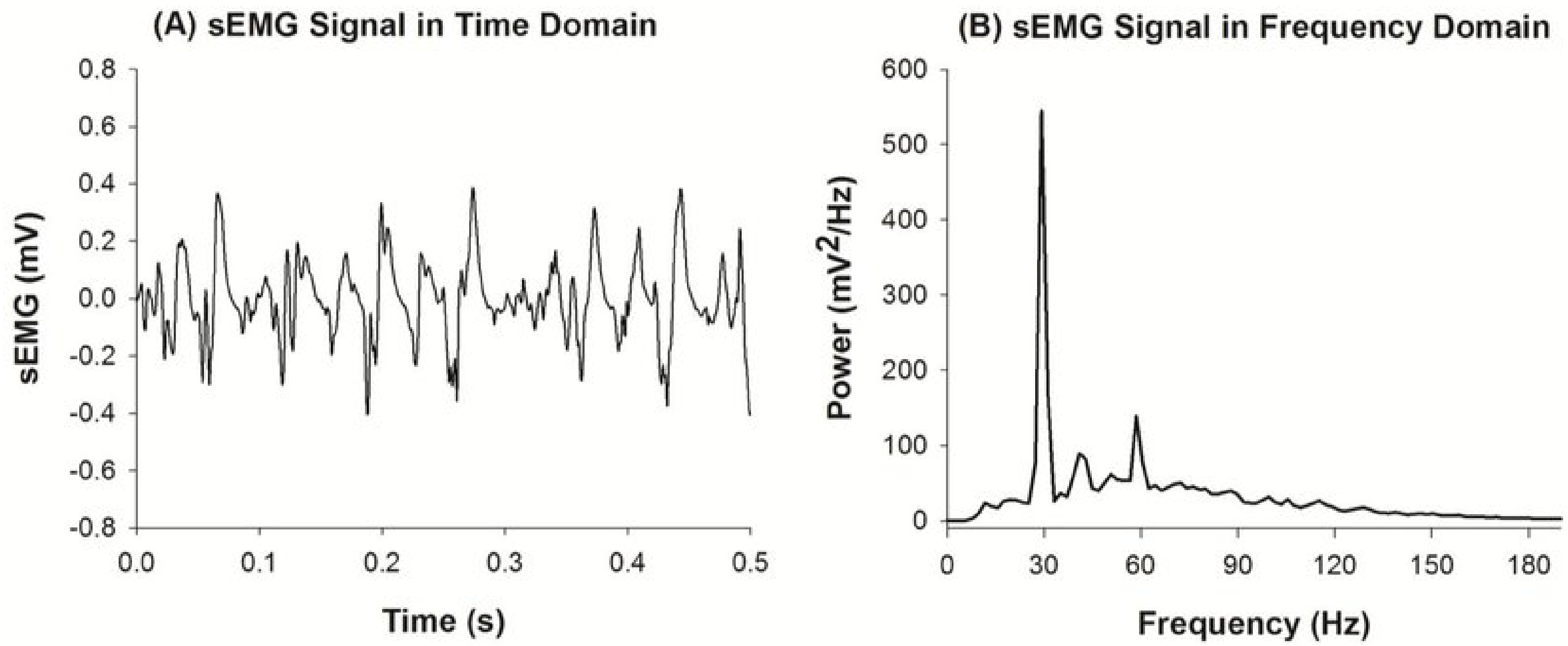
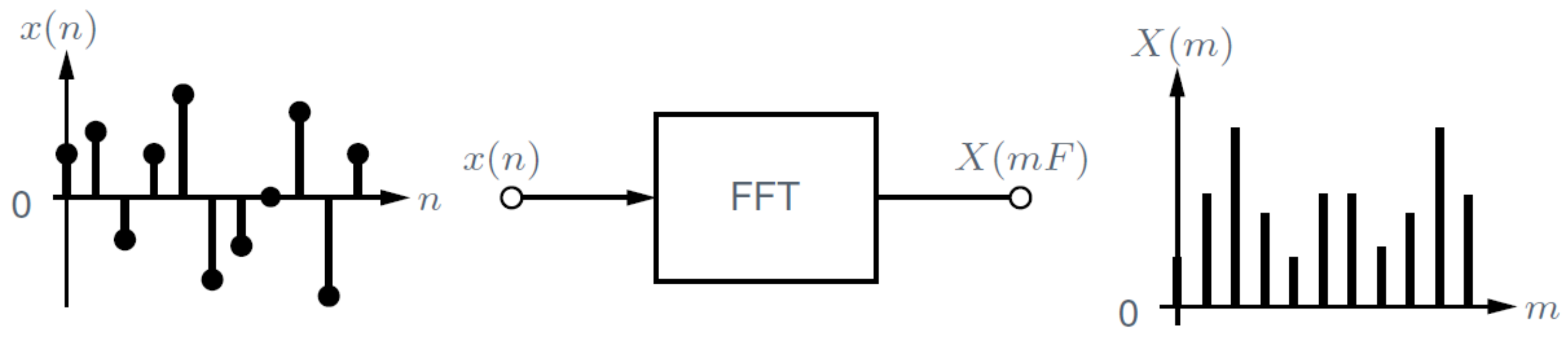
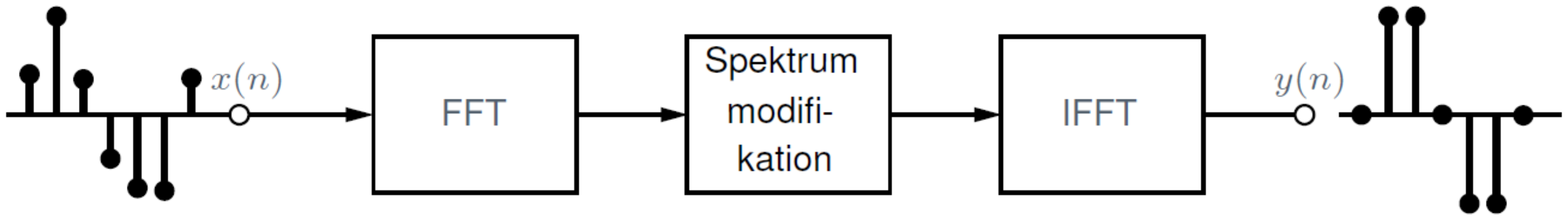
Convert time domain signal to frequency domain for analysis







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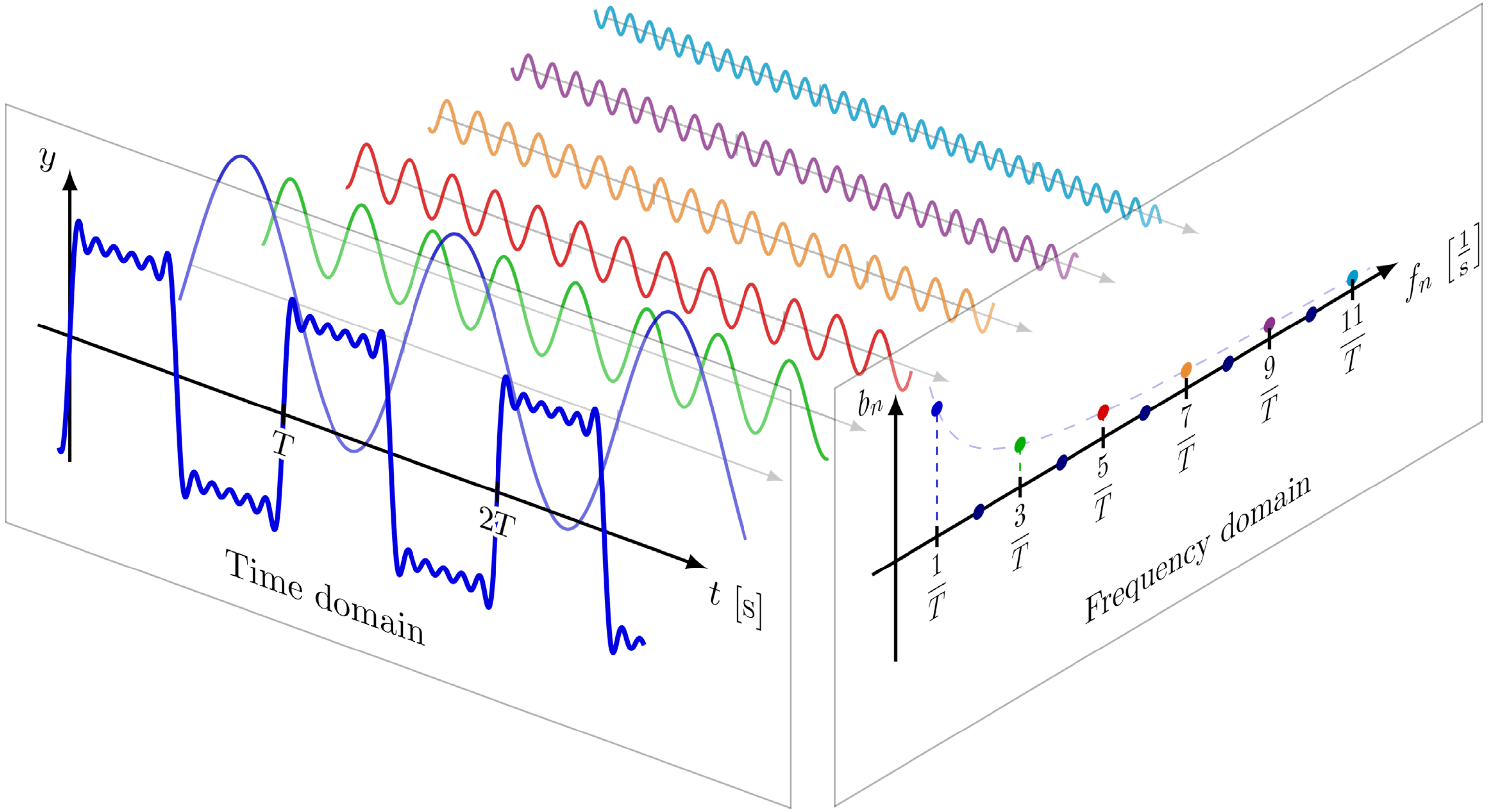
Time – Frequency convert



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Fourier Transform

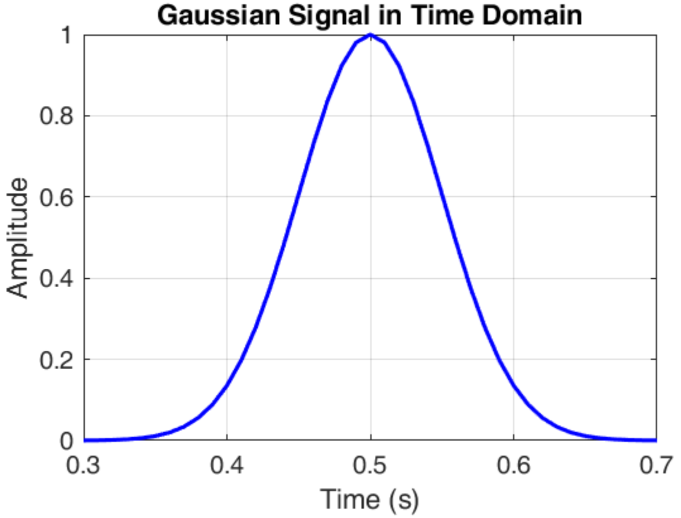
The Fourier transform is a mathematical function that provides frequency spectral analysis.

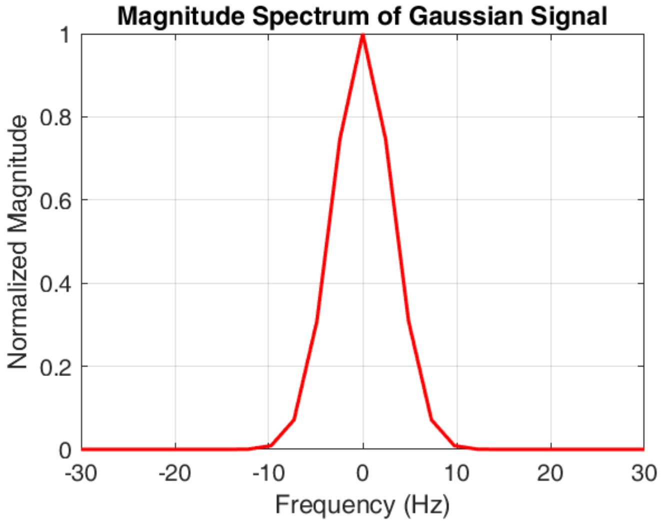
Given a signal $x(t)$, its **Fourier Transform** is defined as

$$X(\omega) = \mathcal{F}\big(x(t)\big) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

The signal $x(t)$ can also be found from the spectrum $X(\omega)$ via **Inverse Fourier Transform**

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$





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Discrete Fourier Transform

$$\boxed{\omega = 2\pi f}$$

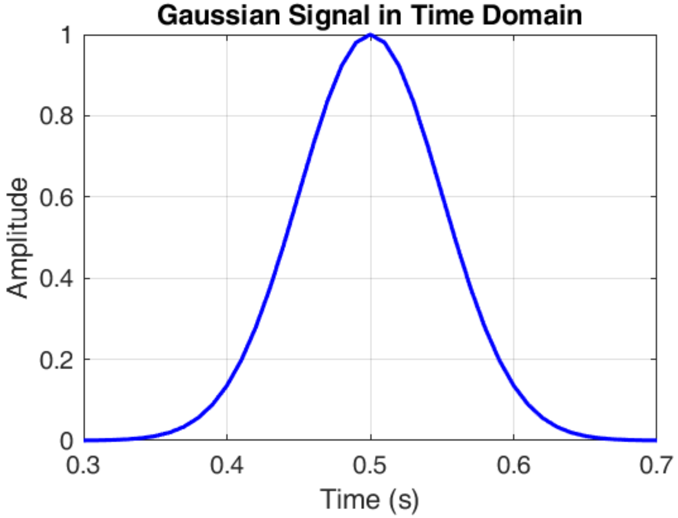
$$X(f) = \mathcal{F}\big(x(t)\big) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$

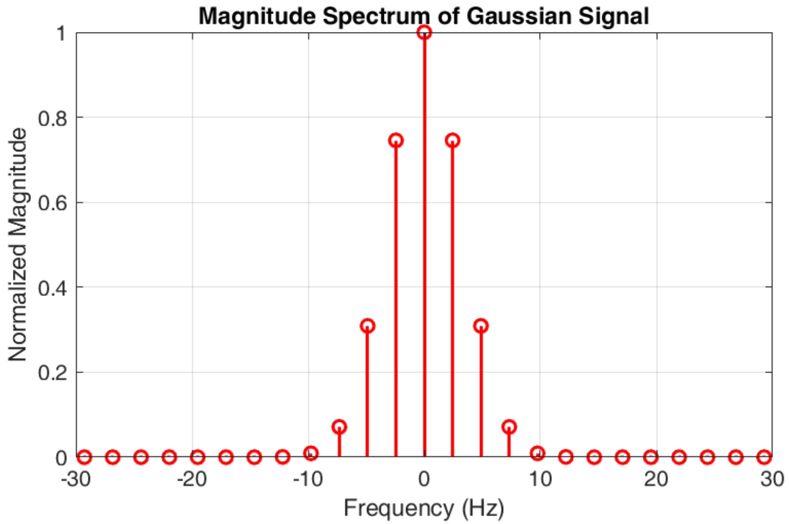
We must calculate the integral over time t from $-\infty$ to ∞

For digital signal, we can calculate the **spectrum** function $X(f)$ by discretizing the frequency range:

$$X(mF) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi mFt}\,dt$$

$$\boxed{f = mF}$$





$$X(mF) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi mFt}\,dt$$

Given the sequence $x(nT)$ which has N samples,

$$X(mF) = T\sum_{n=0}^{N-1} x(nT)e^{-j2\pi mFnT}$$

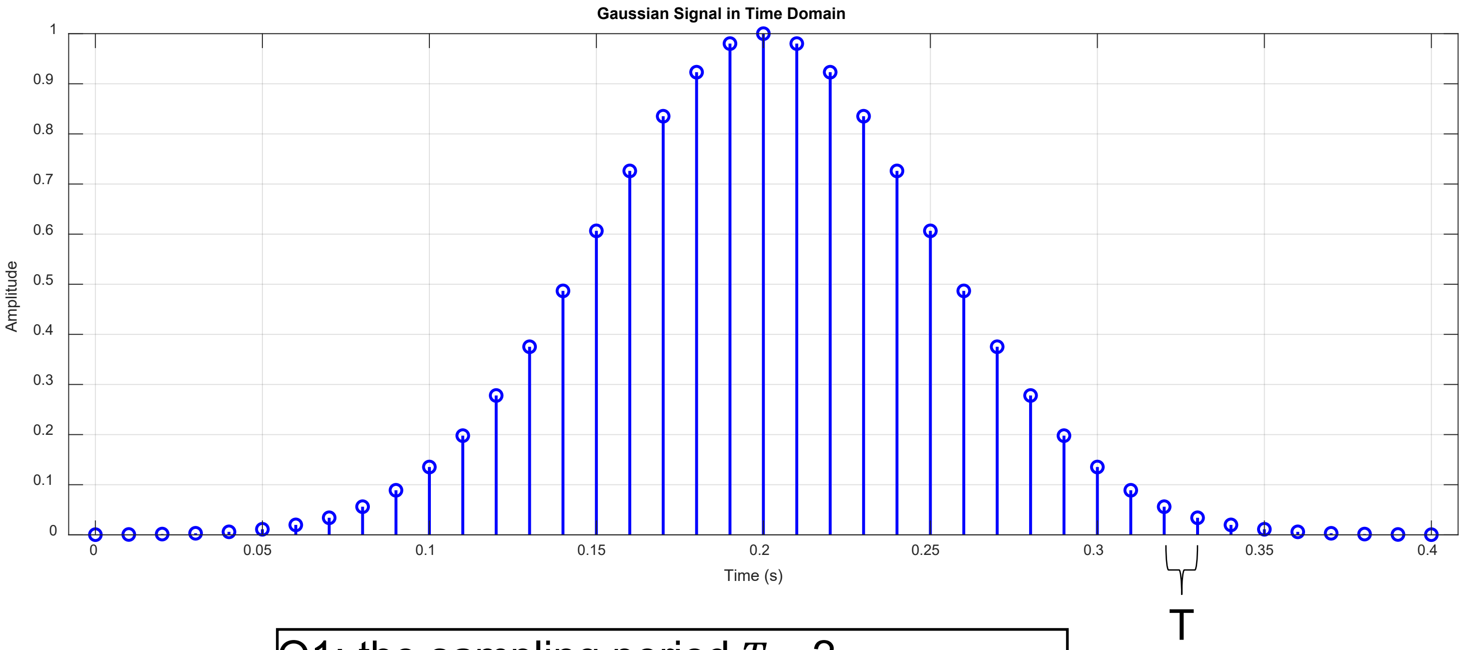
Given $f_s = NF$

$FT = 1/N$

The final DFT can be written as

$$X(m) := \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi mn}{N}}$$

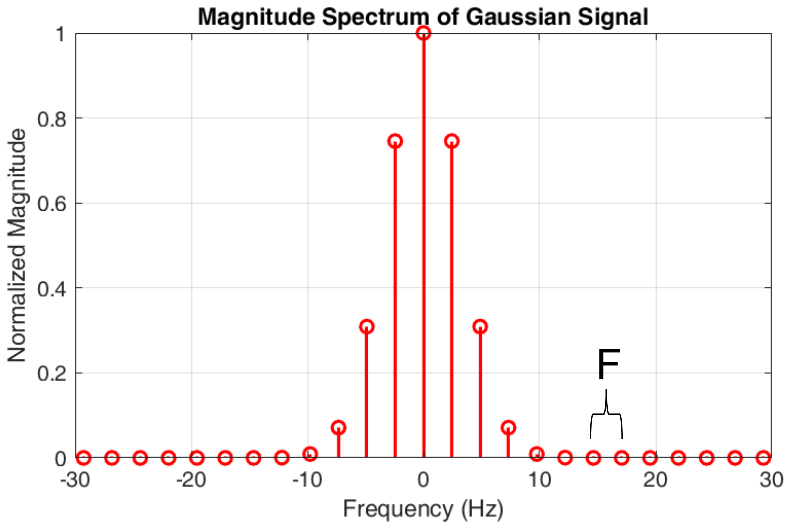
The frequency spacing between the consecutive spectral lines, called the frequency resolution, F. It also represents the m[th] harmonic.



Q1: the sampling period T =?

Then the sampling frequency $f_s = \frac{1}{T}$ =?

In this example, N=41



Inverse discrete Fourier transformation

Inverse Discrete Fourier Transform (IDFT) is used to calculate the sequence $x(n)$ giving the spectrum function $X(f)$.
For n = 0, 1, . . . , N−1

$$x(n) = \frac{1}{N}\sum_{m=0}^{N-1} X(m)\, e^{j2\pi\frac{mn}{N}}$$

It should be noted that both $X(f)$ and $x(n)$ are periodic, i.e.

$$x(n) = x(n + kN)$$
$$X(m) = X(m + kN)$$

Because $x(n)$ is a complex number, it can be rewritten as

$$x(n) = \frac{1}{N}\left(\sum_{n=0}^{N-1} X(m)\cos\left(\frac{2\pi mn}{N}\right) + j\sum_{n=0}^{N-1} X(m)\sin\left(\frac{2\pi mn}{N}\right)\right)$$

We will talk about this again in '**ADC & DAC**'!

Summary

For N-point DFT, the following transformation can be used for calculation:

$x(n)$ is a sequence sampled with interval T
The N-points DFT of $x(n)$ is given as

$$X(m) = \sum_{n=0}^{N-1} x(n) W_N^{mn}$$

for m = 0, 1, …, N-1
Where $W_N = e^{-j2\pi/N}$

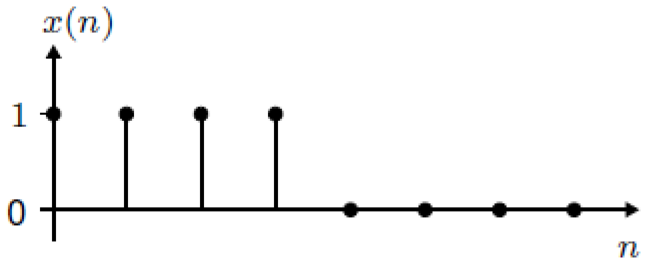
The sequence $x(n)$ can be found from the spectrum $X(m)$ as

$$x(n) = \frac{1}{N} \sum_{m=0}^{N-1} X(m) W_N^{-mn}$$

for n = 0, 1, …, N-1

Example - DFT

Considering the following square sequence $x(n)$



Its spectrum function can be calculated as follows

$$X(m) = \sum_{n=0}^{N-1} x(n) W_N^{mn}$$

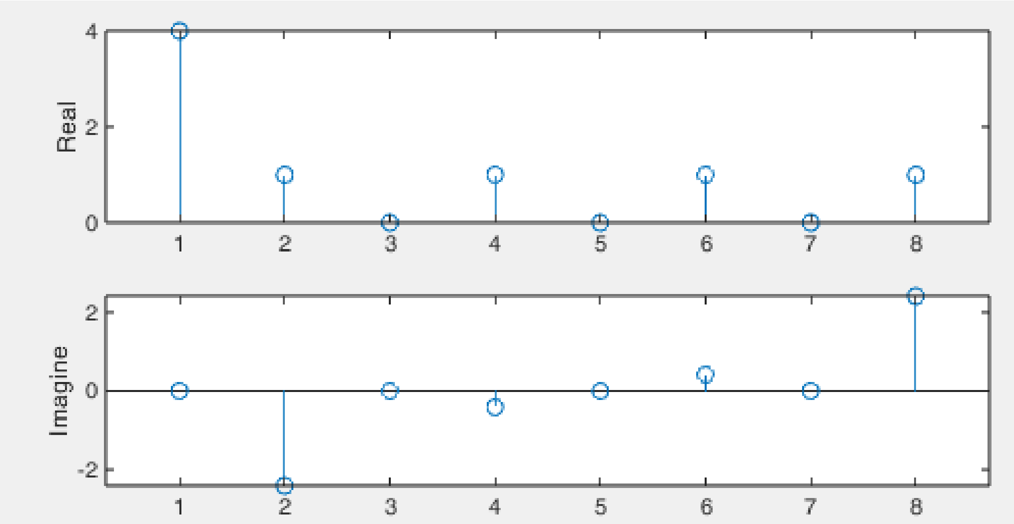
Since N=8, we have

$$X(m) = \sum_{n=0}^{7} x(n) W_8^{mn}$$
$$W_8 = e^{-j2\pi/8}$$

Thus, the spectrum function becomes

$$X(m) = \sum_{n=0}^{7} x(n) \left(\cos\left(\frac{mn\pi}{4}\right) - \mathrm{j}\sin\left(\frac{mn\pi}{4}\right)\right)$$

for m = 0, 1, … , 7




```
%% run Matlab the following

x = [1 1 1 1 0 0 0 0]; % input signal
X = zeros(1,8); % initialize

for m = 0:7
for n = 0:7
X(m+1) = X(m+1) + x(n+1)*(cos(m*n*pi/4) -
i*sin(m*n*pi/4));
end
end
figure();
subplot(2,1,1)
stem(real(X))
subplot(2,1,2)
stem(imag(X))
```

```
% try Matlab inherent function
Y= fft(x);
% plot Y and check the results
```

Example – IDFT

Considering the following sequence in frequency domain

$$X(m) = [10, \quad 2+j, \quad 0, \quad 2-j]$$

The iDFT equation is

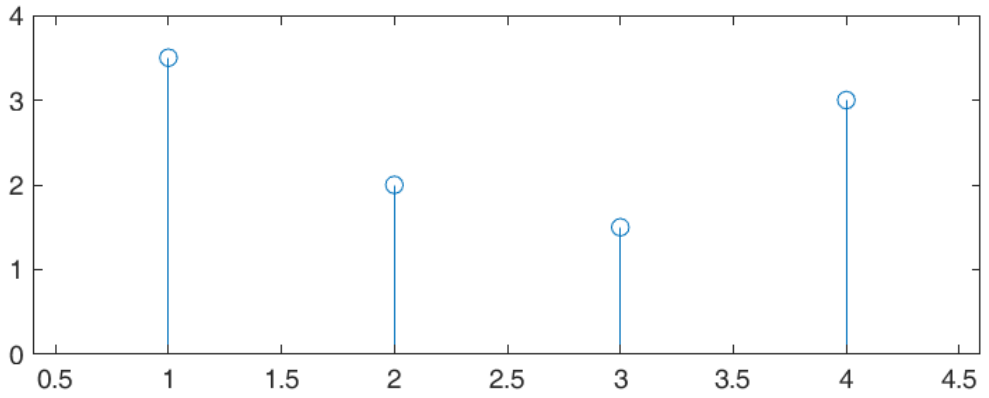
$$x(n) = \frac{1}{N}\sum_{m=0}^{N-1} X(m)\, e^{j2\pi mn/N}$$

Calculate the Time-domain signal:

N = ?

```
%%
X = [10 2+i 0 2-i]; % input signal
x = zeros(1,4);

for n = 0:3
for m = 0:3
x(n+1) = x(n+1) + X(m+1)*exp(i*2*pi*m*n/4);
end
end
x = x / 4;
stem(x)
```




```
% again, try Matlab inherent function
xx = ifft(X);
% plot xx and check the results
```

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Amplitude & phase & power spectrum

The amplitude and phase of $X(m)$ is

$$\mathrm{A_m} = \frac{1}{\mathrm{N}}|X(m)| = \frac{1}{N}\sqrt{\left[Re\left(X(m)\right)\right]^2 + \left[Im\left(X(m)\right)\right]^2} \quad \text{and} \quad \angle X(m) = \tan^{-1}\left(\frac{Im\left(X(m)\right)}{Re\left(X(m)\right)}\right)$$

Where $k = 0, 1, 2, \ldots, N-1$

Power spectrum is defined:

$$P(m) = \frac{1}{\mathrm{N}^2}|X(m)|^2 = \frac{1}{N^2}\left(\left[Re\left(X(m)\right)\right]^2 + \left[Im\left(X(m)\right)\right]^2\right)$$

What we talking about here is in the frequency domain!

Exercise