Convolution

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Topics to be covered in this course

→ Sampling and reconstruction
→ Aliasing
→ Quantization and dynamic range
→ Implementation
→ Conversion time-frequency domain
→ Z transform
→ Linear Time Invariant system
→ System analysis
→ Window functions
→ Filter design (FIR and IIR)

Discrete time signal – basic operation

$x(n)$ is the input signal.

→ Multiplier

$$y(n) = \alpha x(n)$$

→ Adder

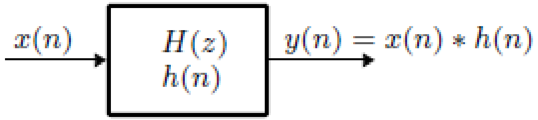
$$y(n) = x_1(n) + x_2(n)$$

→ Delay

$$y(n) = x(n-1)$$

Time-discrete convolution

Time-discrete convolution can be used to calculate the output sequence for a time-discrete system using the input sequence $x(n)$ and the impulse response $h(n)$.



The convolution sum is defined as

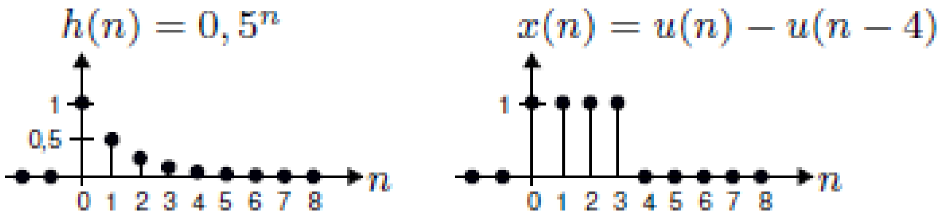
$$y(n) = x(n) * h(n) = \sum_{m=0}^{n} x(m)h(n-m)$$

The output sequence $y(n)$ can also be calculated by

$$y(n) = h(n) * x(n) = \sum_{m=0}^{n} h(m)x(n-m)$$

Example

We consider a system with the following impulse response and input sequence.



It is desired to determine the output sequence $y(n)$ using discrete-time convolution

$$y(n) = h(n) * x(n) = 0.5^n * [u(n) - u(n-4)]$$

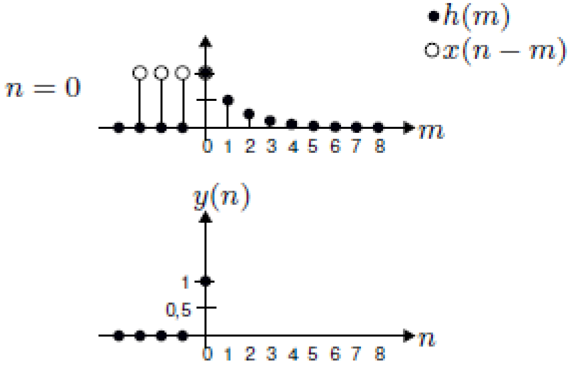
Where $u(n)$ is the unit sequence

Example

The output sequence is calculated using

$$y(n) = \sum_{m=0}^{n} h(m)x(n-m)$$

For $n = 0$, we have $y(0) = h(0)x(0)$



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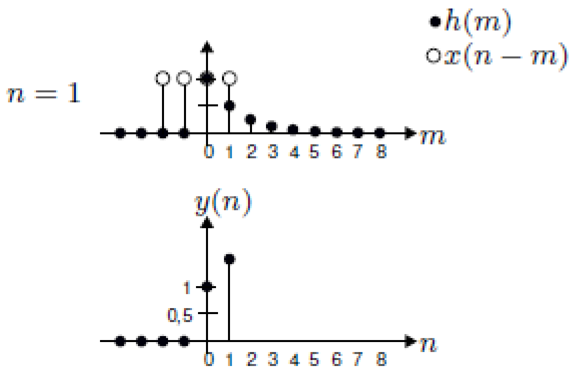

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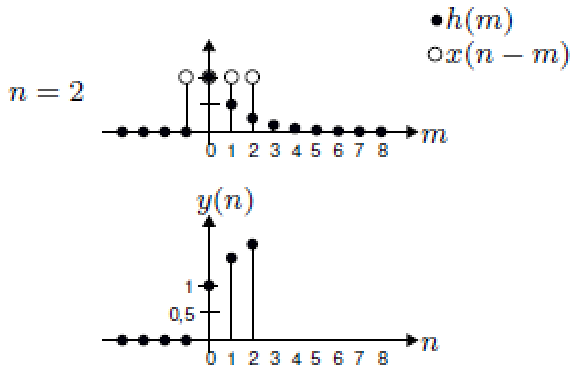

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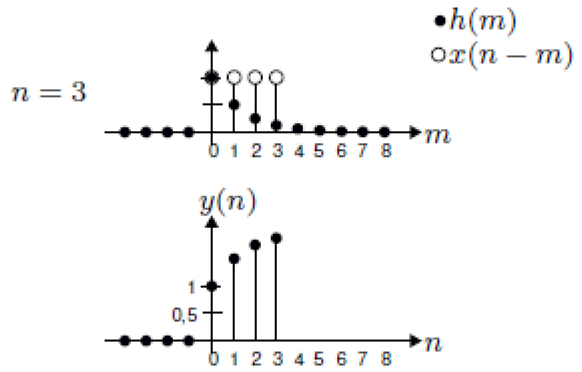

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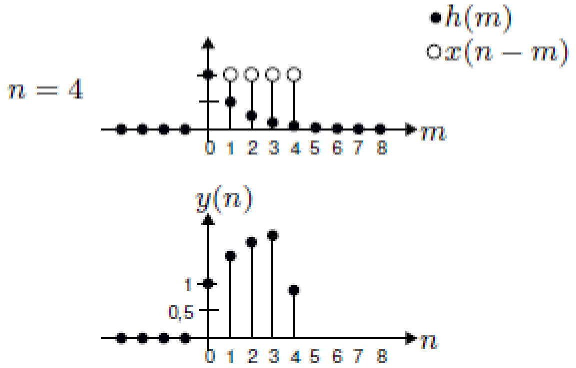

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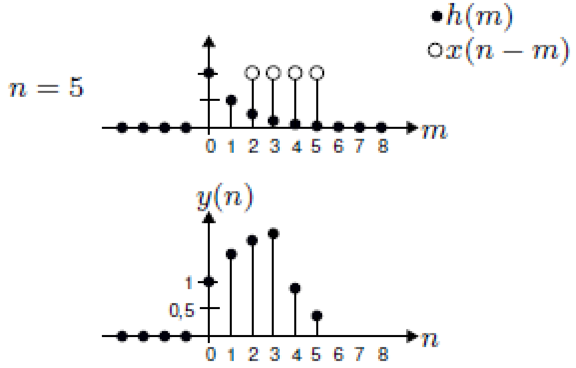

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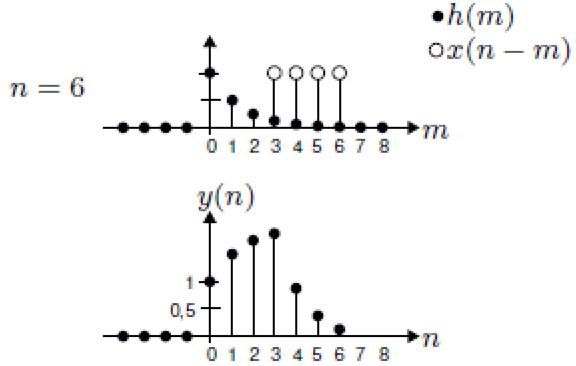

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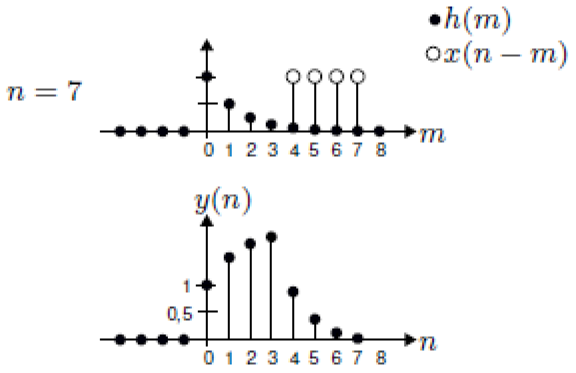

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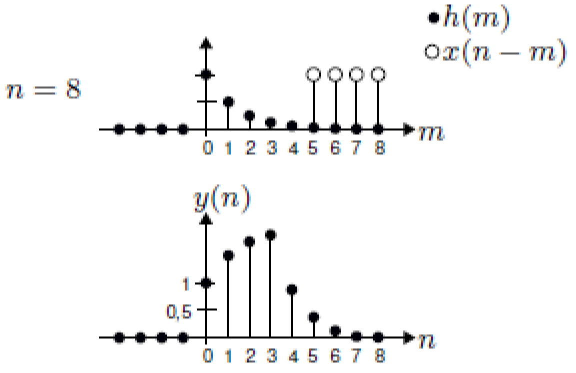

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Property of Convolution

→ Commutative

$$a(n) * b(n) = b(n) * a(n)$$

→ Associative

$$\big(a(n) * b(n)\big) * c(n) = a(n) * \big(b(n) * c(n)\big)$$

→ Distributive

$$a(n) * \big(b(n) + c(n)\big) = a(n) * b(n) + a(n) * c(n)$$

Example & Matlab code

You are given an input signal x = [1,2,3]
A filter h=[0.5, 0.5]
Please manually compute the convolution $y(n) = x(n) * h(n)$

$$y(n) = \sum_{k=0}^{\infty} h_k x(n-k)$$

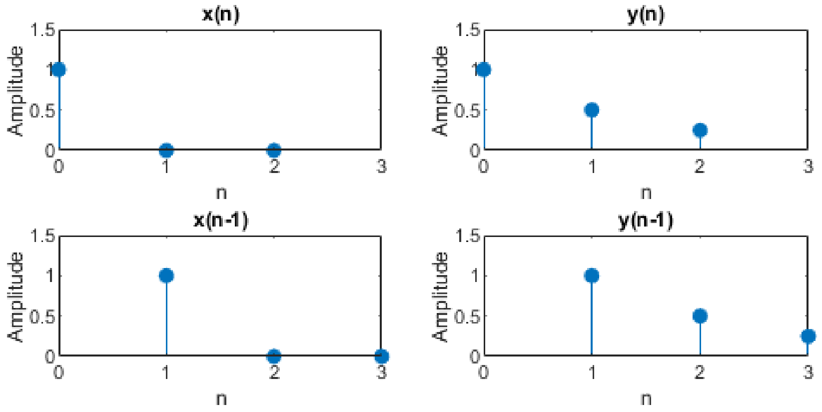
Then you try using Matlab:

```
x = [1, 2, 3]; % Input signal
h = [0.5, 0.5]; % Impulse response
y = conv(x, h); % Convolution
```

Linear Time Invariant (LTI)

Let the outputs of a system be $y_1(n)$ and $y_2(n)$ respectively, when the inputs are $x_1(n)$ and $x_2(n)$. The system is called **Linear** if the output $y(n) = ay_1(n) + by_2(n)$ in response to input $x(n) = ax_1(n) + bx_2(n)$

Let a system have output $y(n)$ when the input is $x(n)$. The system is called time invariant if $x(n - n_0)$ produces y$(n - n_0)$

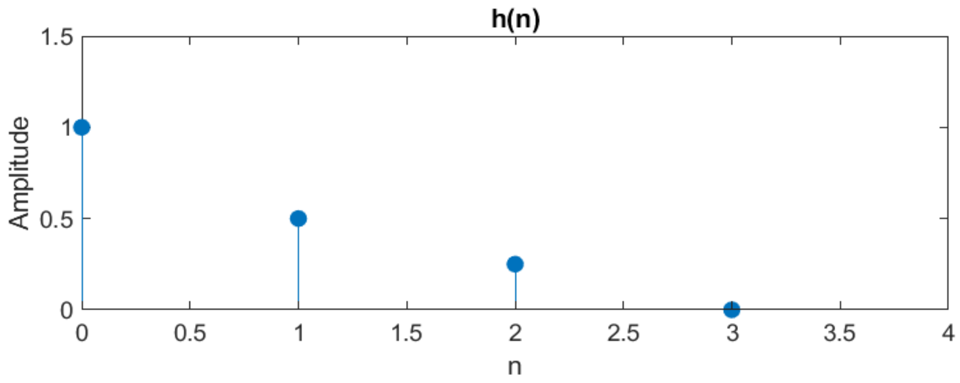


A system is **LTI (Linear and Time Invariant)** if it is both linear and time invariant

Causal

A system with input $x(n)$ and output $y(n)$ is **causal** if $y(n)$ depends only on $x(n)$, $x(n-1)$, $x(n-2)$, …, but not on $x(n+1)$, $x(n+2)$, …

The LTI system is **causal** if and only if $h(-1) = h(-2) = \cdots = 0$
i.e. $h(n) = 0$, $\forall n < 0$



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