Design of IIR filter (2)

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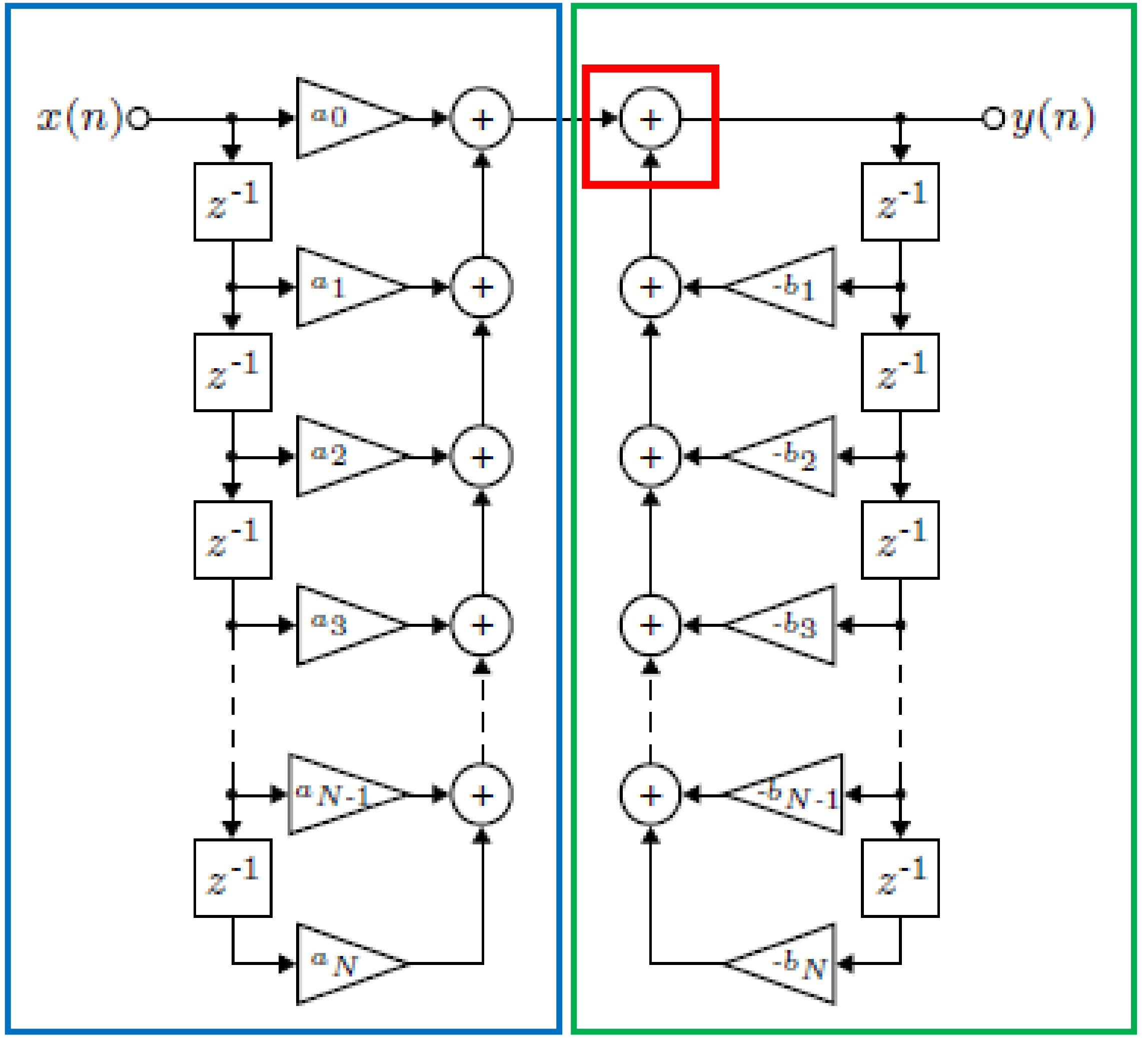
Recap: realization

Transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} a_i\, z^{-i}}{1 + \sum_{i=1}^{N} b_i\, z^{-i}}$$

Difference equation

$$y(n) = \sum_{i=0}^{N} a_i x(n-i) - \sum_{i=1}^{N} b_i y(n-i)$$



Recap: Filter designs

The goal is to make it satisfy the desired frequency response $H_d(e^{j\omega})$

IIR

$$H(z) = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{M} a_k\, e^{-jk\omega}}$$

FIR

$$H(z) = \sum_{k=0}^{M} b_k e^{-jk\omega}$$

→ An FIR filter has 5 to 10 times larger realization structure than a corresponding IIR filter.
→ An FIR filter is always stable as it only has zero points.
→ An FIR filter is called a **non-recursive structure**, while an IIR filter is called a **recursive structure**.
→ An FIR filter is less sensitive to coefficient changes and rounding errors than an IIR filter.

Recap: Design of digital IIR filter

1. The filter's specifications are drawn up
2. The z-domain transfer function of the filter is set up using

Matched z-transform:
Transfer poles and zeros from s-domain to z-domain

$$H(s) = \frac{(s-z_1)(s-z_2)\dots(s-z_N)}{(s-p_1)(s-p_2)\dots(s-p_N)}$$

$$H(z) = \frac{(z-e^{z_1T})(z-e^{z_2T})\dots(z-e^{z_NT})}{(z-e^{p_1T})(z-e^{p_2T})\dots(z-e^{p_NT})}$$

Impulse invariant z-transform:
Z transform of the sampled impulse response of the analog filter.

$$H(s) = \sum_{i=1}^{N} \frac{r_i}{s-p_i}$$

$$H(z) = T\sum_{i=1}^{N} \frac{r_i}{1-e^{p_iT}z^{-1}}$$

3. Select realization structure
   → Matched z-transform: cascade
   → Impulse invariant z-transform: parallel

Bilinear z-transform

The basic idea is to design the digital filter by converting a continuous-time transfer function to a discrete-time equivalent.

Through bilinear z-transform, s in $H(s)$ is substituted by a function of z, namely

$$H(z) = H(s)\Big|_{s=f(z)}$$

Again, the relationship between s and z is $z = e^{sT}$

We talk about this in the matched z-transform

In this case, we can have

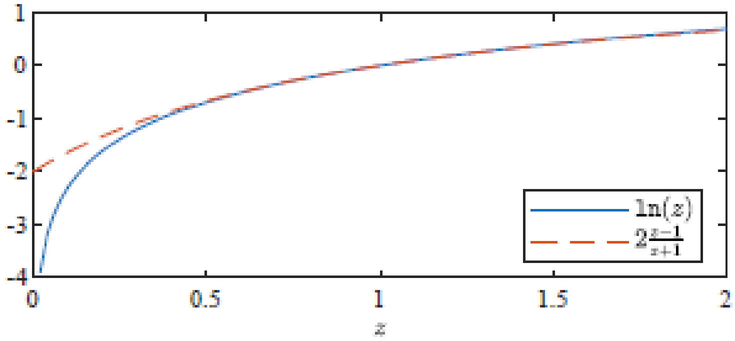
$$\ln z = sT$$

Approximation of logarithm

Please note that the natural logarithm is defined as an odd sum

$$\ln z = 2\left[\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \ \dots\right] \approx 2\,\frac{z-1}{z+1}$$

$$\boxed{\left|\frac{z-1}{z+1}\right| < 1}$$



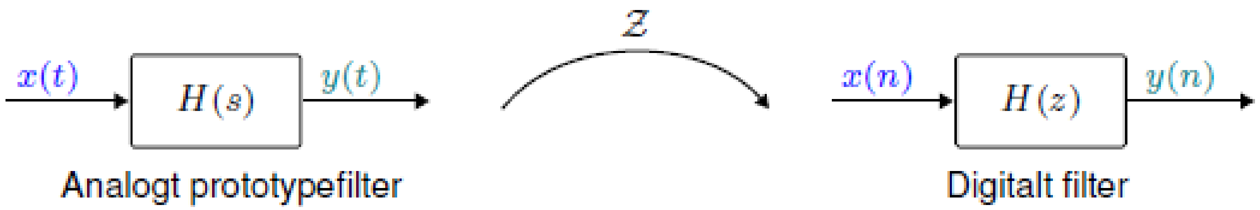
Principle

Given that

$$\ln(z) = sT \approx 2\frac{z-1}{z+1}$$

a discrete transfer function is thus obtained by bilinear z-transform as

$$\boxed{H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}}$$



Non-linear frequency mapping

The unit circle in the z-plane corresponds to the imagine axis in the s-plane.

The relationship between $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$ can be found by substituting $z = e^{j\omega T}$ to

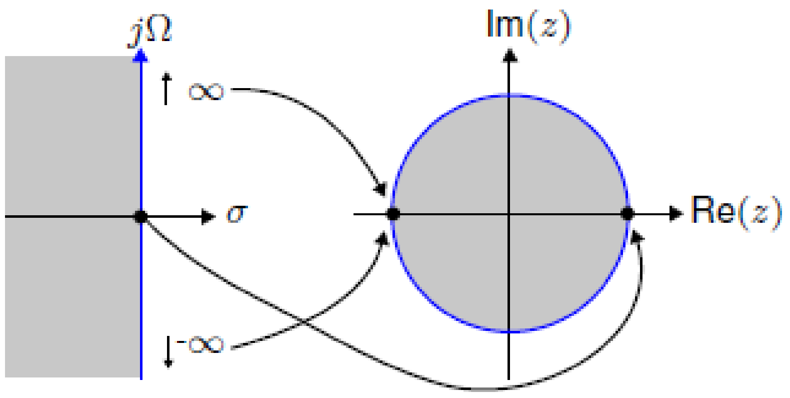
$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$

We have

$$s = \frac{2}{T}\frac{1 - e^{-j\omega T}}{1 + e^{-j\omega T}} = \frac{2}{T}\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{e^{j\omega T/2} + e^{-j\omega T/2}}$$

$$s = \frac{2}{T}\frac{2j\sin\left(\frac{\omega}{2}T\right)}{2\cos\left(\frac{\omega}{2}T\right)} = \frac{2j}{T}\tan\left(\frac{\omega}{2}T\right)$$

$$\boxed{\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}T\right)}$$

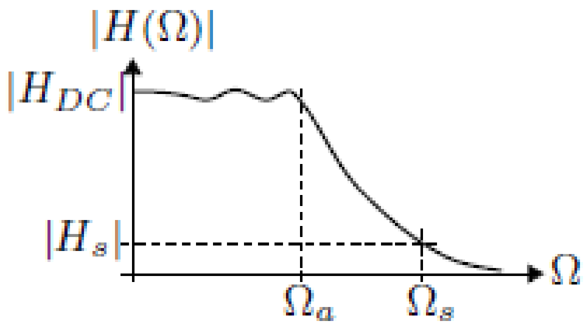


$s = j\Omega$ $\quad$ $z = e^{j\omega T}$

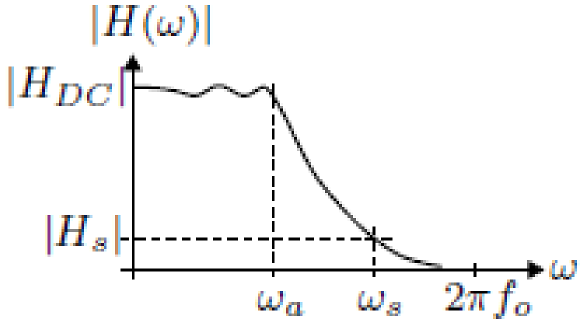
Frequency warping

Since the frequency range $0 < f < \infty$ for the **analog filter** is transformed to the frequency range $0 < f < f_o$ for the **digital filter**, the frequency axis is deformed.
This affects the cutoff frequency ω_a and the stopband frequency ω_s of the digital filter.

Analog filter



Digital filter

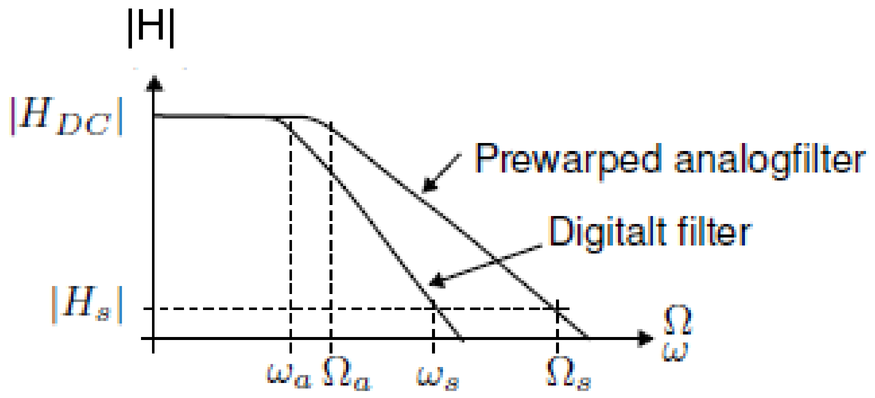


Q: How to choose specs for the analog filter (Ω_a and Ω_s), so that the desired digital filter can have (ω_a and ω_s)?

$$\Omega_a = \frac{2}{\mathrm{T}} \tan\left(\frac{\omega_a T}{2}\right)$$
$$\Omega_s = \frac{2}{\mathrm{T}} \tan\left(\frac{\omega_s T}{2}\right)$$

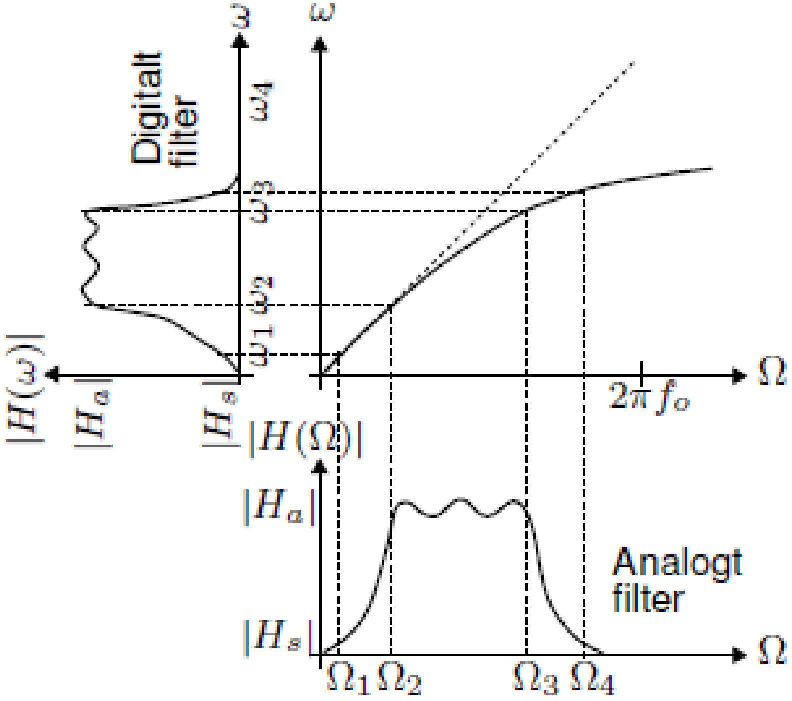
Frequency prewarping

The idea of prewarping is to adjust the analog filter's cutoff frequency Ω_a and stopband frequency Ω_s so that the converted digital filter's cutoff frequency ω_a and stopband frequency ω_s align with the desired frequencies.



Frequency prewarping

The consequence of frequency warping can be seen from the following bandpass filter.



$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}T\right)$$

Example

An analog 5th order Butterworth low pass filter $H(s)$ has -3 dB cutoff frequency $f_3 = \mathbf{3}$ **kHz** and -30 dB stopband frequency $f_{30} = \mathbf{6}$ **kHz**.

The converted digital filter by bilinear z-transform with a sampling frequency of f_s =16 kHz has

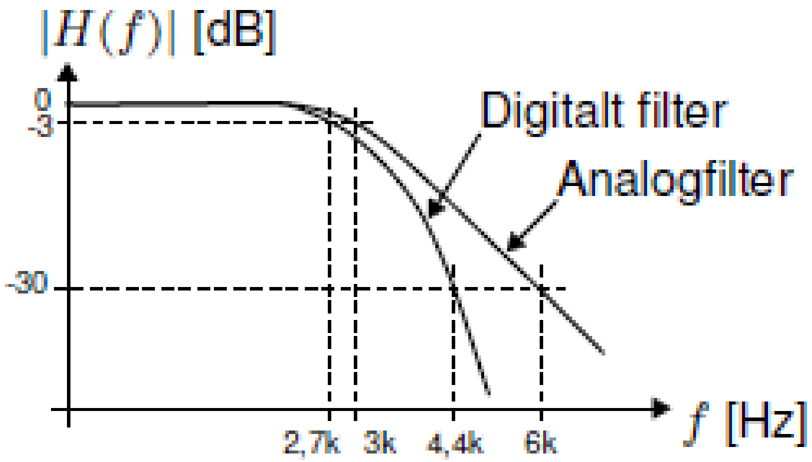
$$\omega_3 = \frac{2}{\mathrm{T}} \tan^{-1}\left(\frac{\Omega_a T}{2}\right) = 2f_s \tan^{-1}\left(\frac{2\pi f_3}{2f_s}\right) = 17.03\mathrm{e}3 \text{ rad/s} = \mathbf{2.7kHz}$$

$$\omega_{30} = \frac{2}{\mathrm{T}} \tan^{-1}\left(\frac{\Omega_s T}{2}\right) = 2f_s \tan^{-1}\left(\frac{2\pi f_{30}}{2f_s}\right) = 27.74\mathrm{e}3 \text{ rad/s} = \mathbf{4.4kHz}$$

Matlab:
atan()

Example

The consequence of frequency warping for the low-pass filter can be seen from the following amplitude characteristic.



Example

Let's try pre-warping.
Design a low pass digital filter: cutoff frequency $f_3 = 3$ kHz, stopband frequency $f_{30} = 6$ kHz, and sampling frequency f_s =16 kHz.

1. pre-warping analog filter: find the corresponding frequencies using

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega T}{2}\right) \quad \text{[rad/s]}$$

The prewrapped cutoff frequency: $\Omega_3 = 21.38$k rad/s, and stopband frequency $\Omega_{30} = 77.25$k rad/s, corresponding to 3.4 kHz and 12.3 kHz.

Example

Now we need to design a lowpass analog filter with cutoff frequency: $\Omega_3 = 21.38$k rad/s, and stopband frequency $\Omega_{30} = 77.25$k rad/s.

Let's use a Bessel filter. Try different order N in Matlab to assure stopband frequency is satisfied

```
>> [b, a] = besself(N, wa);
>> Hs = tf(b,a)
>> bode(Hs)
```

Through some trials, the required order N is found 3.

Hs =

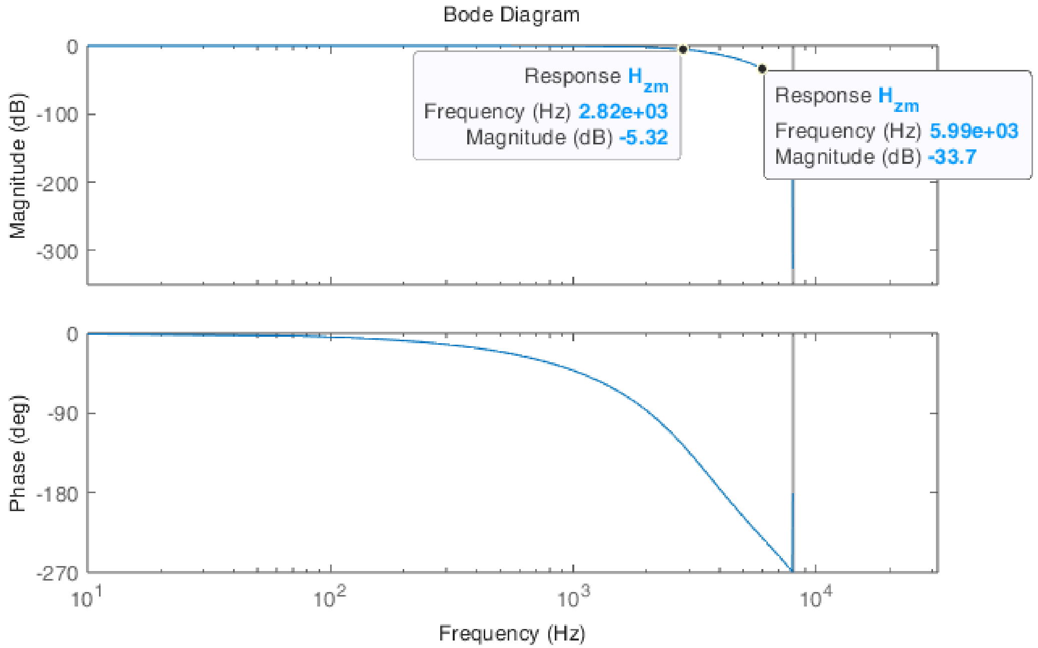
$$\frac{9.773e12}{s^3 + 5.201e04\ s^2 + 1.127e09\ s + 9.773e12}$$

Example

Then we convert the transfer function of analog filter to digital filter using

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

$$H(z) = \frac{2.443 + 7.33 * z^{-1} + 7.33 * z^{-2} + 2.443 * z^{-3}}{32.97 - 21.54 * z^{-1} + 9.575 * z^{-2} - 1.45 * z^{-3}}$$



Matlab:

```
fs = 16000;
H_z = c2d(Hs, 1/fs, 'tustin');
bode(H_z)
```

Matlab function: bilinear()

[zd,pd,kd] = bilinear(z,p,k,fs)
[numd,dend] = bilinear(num,den,fs)

Check its use by yourself.

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Design procedure

1. From the given design requirements of the digital filter, find the analog filter's cutoff frequency Ω_a and stopband frequency Ω_s (pre-wrap).

2. Determine the transfer function of the analog filter $H(s)$

3. Convert the analog filter to the digital filter by bilinear z-transform

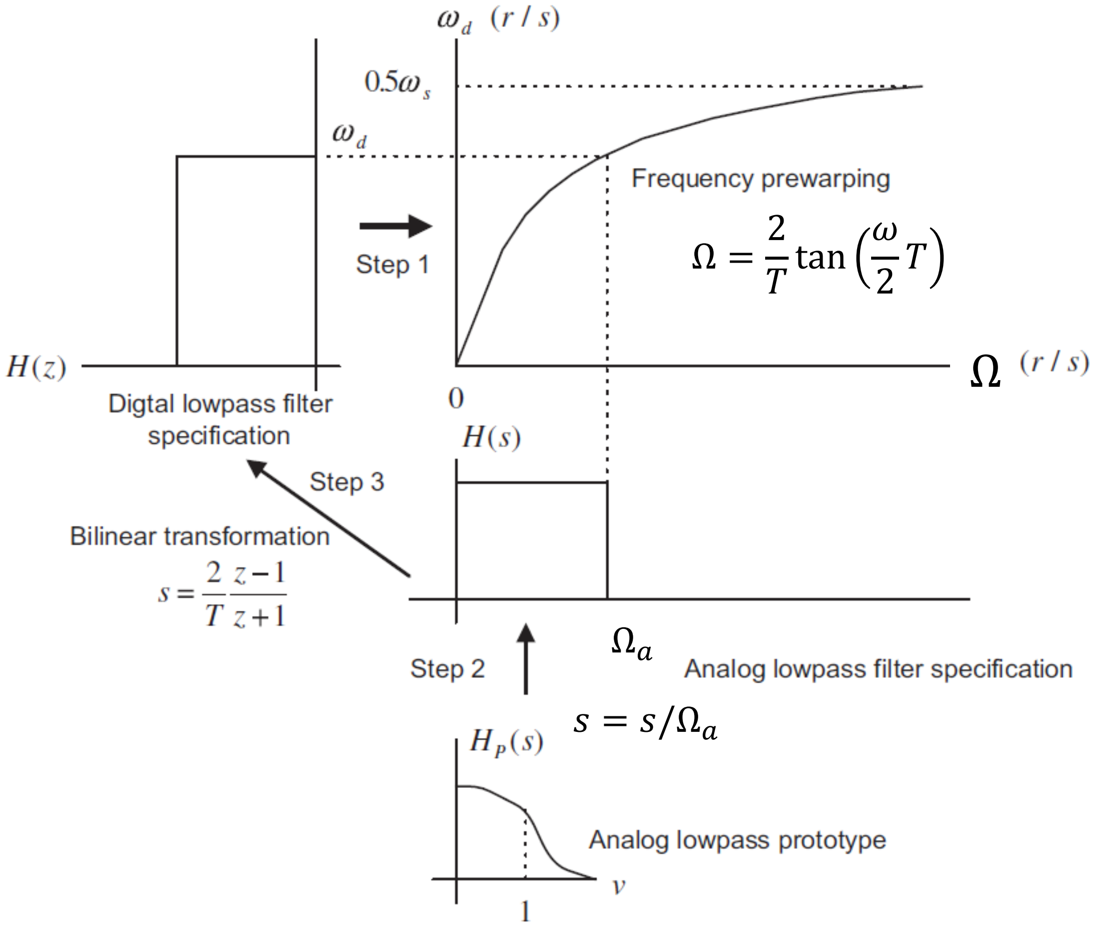
$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

4. The filter is realized as a cascaded realization structure.

Example

A lowpass digital filter with a cutoff frequency of 300 Hz is to be designed with a sampling frequency of 16 kHz. Please use bilinear z-transform method, and the **frequency-normalized** analog prototype filter should be

$$H_p(s) = \frac{1}{s+1}$$



Example

A lowpass digital filter with a cutoff frequency of 300 Hz is to be designed with a sampling frequency of 16 kHz. Please use bilinear z-transform method, and the **frequency-normalized** analog prototype filter should be

$$H_p(s) = \frac{1}{s+1}$$

Step 1: find the pre-wrapped frequency

$$\Omega_a = \frac{2}{\mathrm{T}} \tan\left(\frac{\omega_a T}{2}\right)$$

Step 2: denormalization

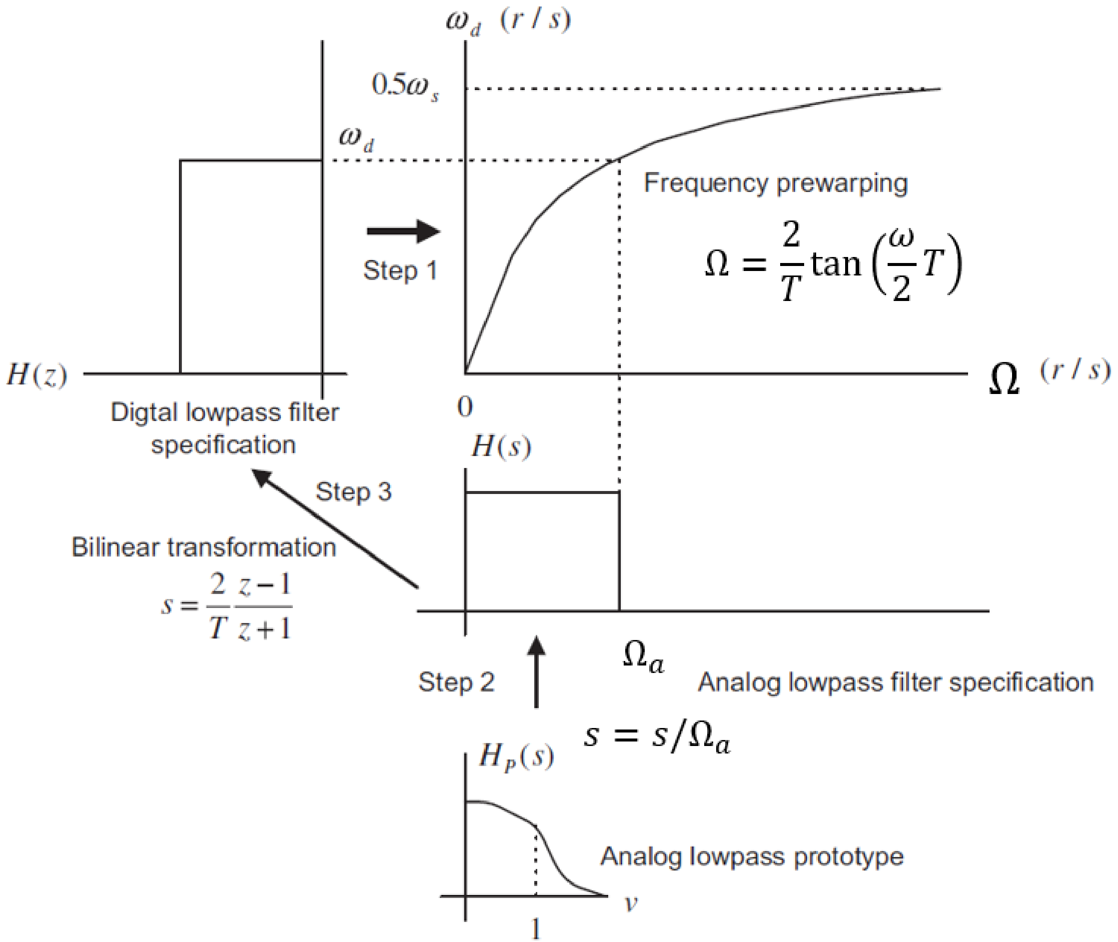
$$H(s) = H_p(s)|_{s=\frac{s}{\Omega_a}} = \frac{1}{s/\Omega_a + 1} = \frac{\Omega_a}{s+\Omega_a}$$

Step 3: convert analog filter to digital filter

$$H(z) = H(s)|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

The result should be

$$H(z) = \frac{0.056+0.056z^{-1}}{1-0.889z^{-1}}$$



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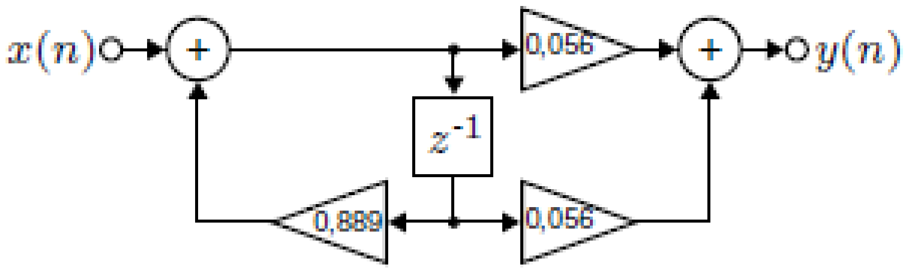
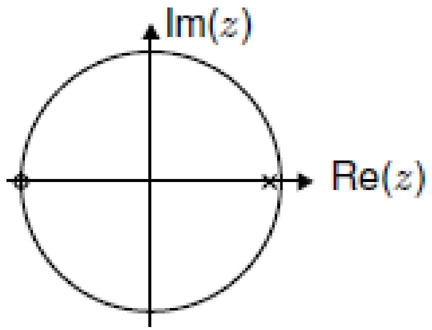
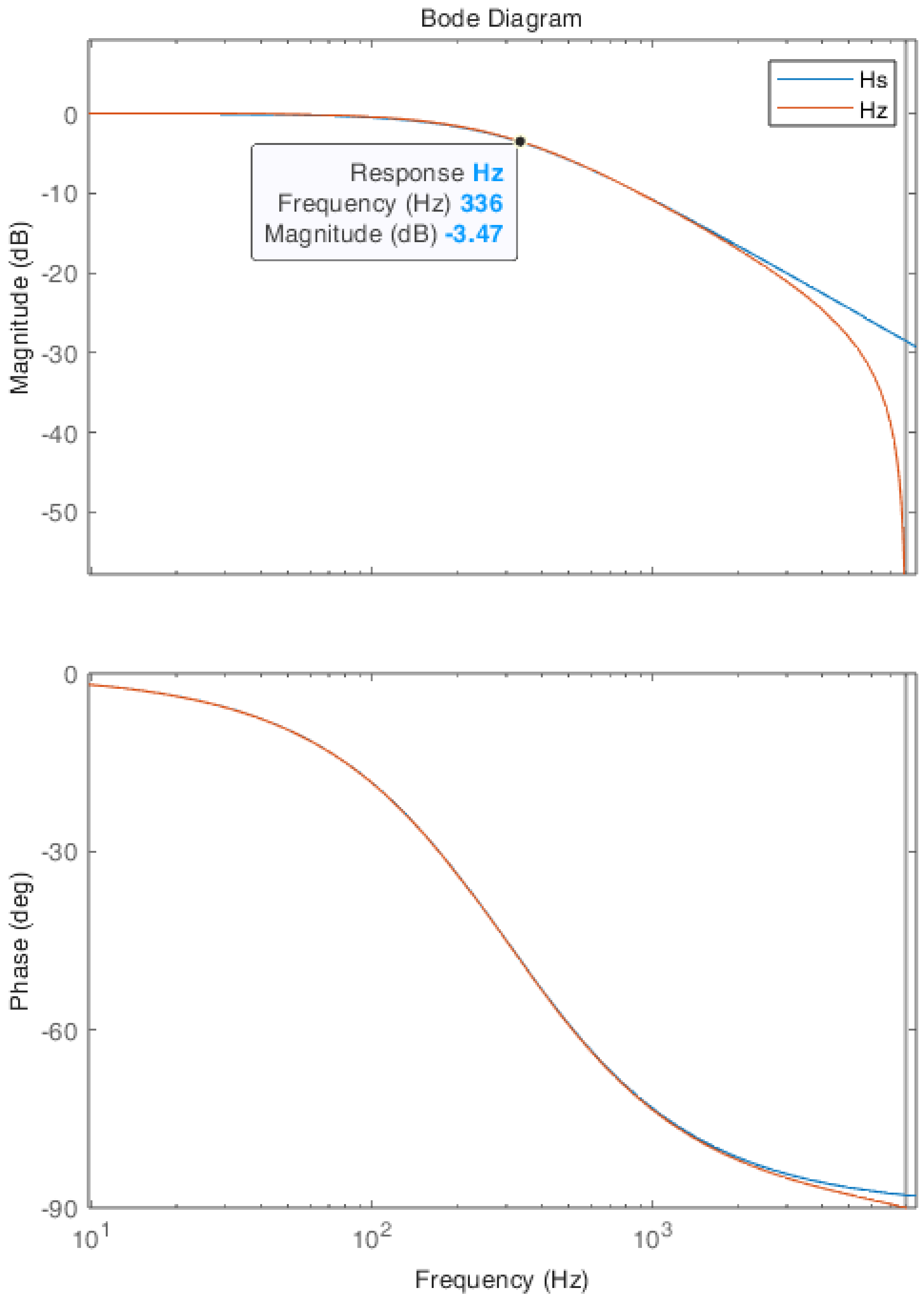
Example

Filter which has a zero $z = -1$ and a pole $z = 0.889$ can be realized as







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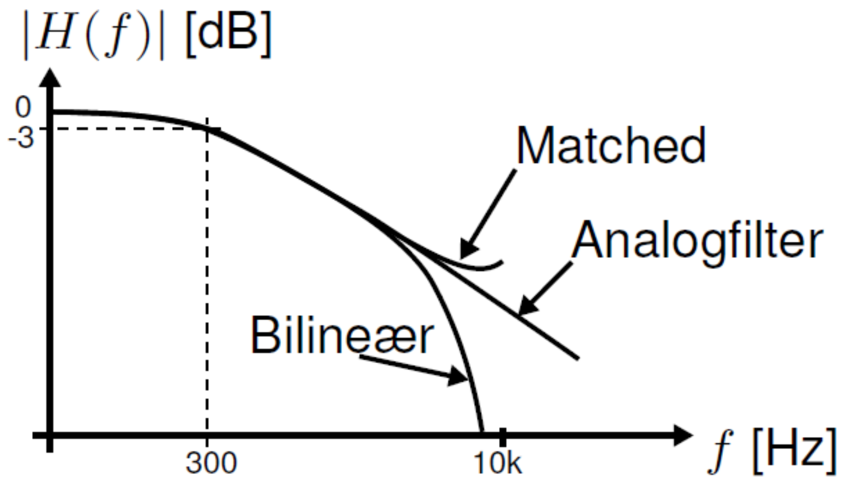
Advantage of using bilinear z transform

You can see from the amplitude characteristic of the filter that **the aliasing error is eliminated** by bilinear z-transform.



Convert directly from an analog prototype filter

Step 1: find the pre-wrapped frequency

$$\Omega_a = \frac{2}{\mathrm{T}} \tan\left(\frac{\omega_a T}{2}\right)$$

Step 2: denormalization

$$H(s) = H_p(s)\big|_{s=\frac{s}{\Omega_a}}$$

Step 3: convert analog filter to digital filter

$$H(z) = H(s)\big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

These two steps can be merged to one:

$$H(z) = H_p(s)\,\Big|_{s=C\frac{1-z^{-1}}{1+z^{-1}}}$$

where

$$\boldsymbol{C} = \frac{\mathbf{1}}{\mathbf{tan}\left(\frac{\boldsymbol{\omega_a T}}{\mathbf{2}}\right)}$$

Example

Design a digital lowpass filter with cutoff frequency 800 Hz and sampling frequency 8 kHz. Use bilinear z transform method and the **frequency-normalized prototype analog filter** is

$$H_p(s) = \frac{1}{s^2 + 1.414s + 1}$$

It is a 2nd-order Butterworth low-pass filter.

Firstly, find the prewarping constant

$$C = \frac{1}{\tan\left(\frac{\omega_a T}{2}\right)} = \frac{1}{tan\left(\frac{2\pi f_a}{2 f_s}\right)} = 3.078$$

Second, the digital filter can be calculated by the following

$$H(z) = H_p(s)\Big|_{s=C\frac{1-z^{-1}}{1+z^{-1}}}$$

Finally becomes the transfer function of the filter

$$H(z) = \frac{0.06 + 0.13z^{-1} + 0.07z^{-2}}{1 - 1.14z^{-1} + 0.41z^{-2}}$$

Example

The realization of the filter with two zeros at 1 and poles $0.57 \pm j0.29$, as shown below: