Realization

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Index

→ Introduction

→ Direct realization structures
  → Direct type I
  → Direct type II

→ Cascade and parallel implementation
  → Cascade realization
  → Parallel implementation

→ Summary

Discrete time system: Transfer function

A *N*th order difference equation describing a causal system can be written as

$$y(n) + b_1 y(n-1) + b_2 y(n-2) + \ \ldots + b_N y(n-N) = a_0 x(n) + a_1 x(n-1) + \ \ldots + a_N x(n-N)$$

where $x(n-i)$ is the input sequence, $y(n-i)$ is the output sequence,
a_i, b_i are real coefficients.

The above difference equation can be written as

$$y(n) = \sum_{i=0}^{N} a_i x(n-i) - \sum_{i=1}^{N} b_i y(n-i)$$

It corresponds to the following transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} a_i\, z^{-i}}{1 + \sum_{i=1}^{N} b_i\, z^{-i}}$$

Here, you may notice that we are using z^{-i}

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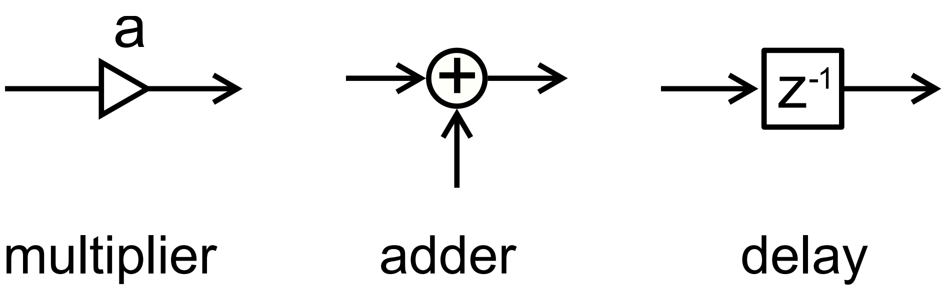
Structure realization

For a digital filter such as bellow, how can we express it using a **block diagram**.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} a_i \, z^{-i}}{1 + \sum_{i=1}^{N} b_i \, z^{-i}}$$

Basic elements for block diagram



Realization of a digital filter

→**Direct realization:**
- →**Type I structure**
- →**Type II structure**

→Cascade realization
→Parallel realization

Direct type I structure

The transfer function $H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} a_i z^{-i}}{1+\sum_{i=1}^{N} b_i z^{-i}}$

corresponds to the following difference equation

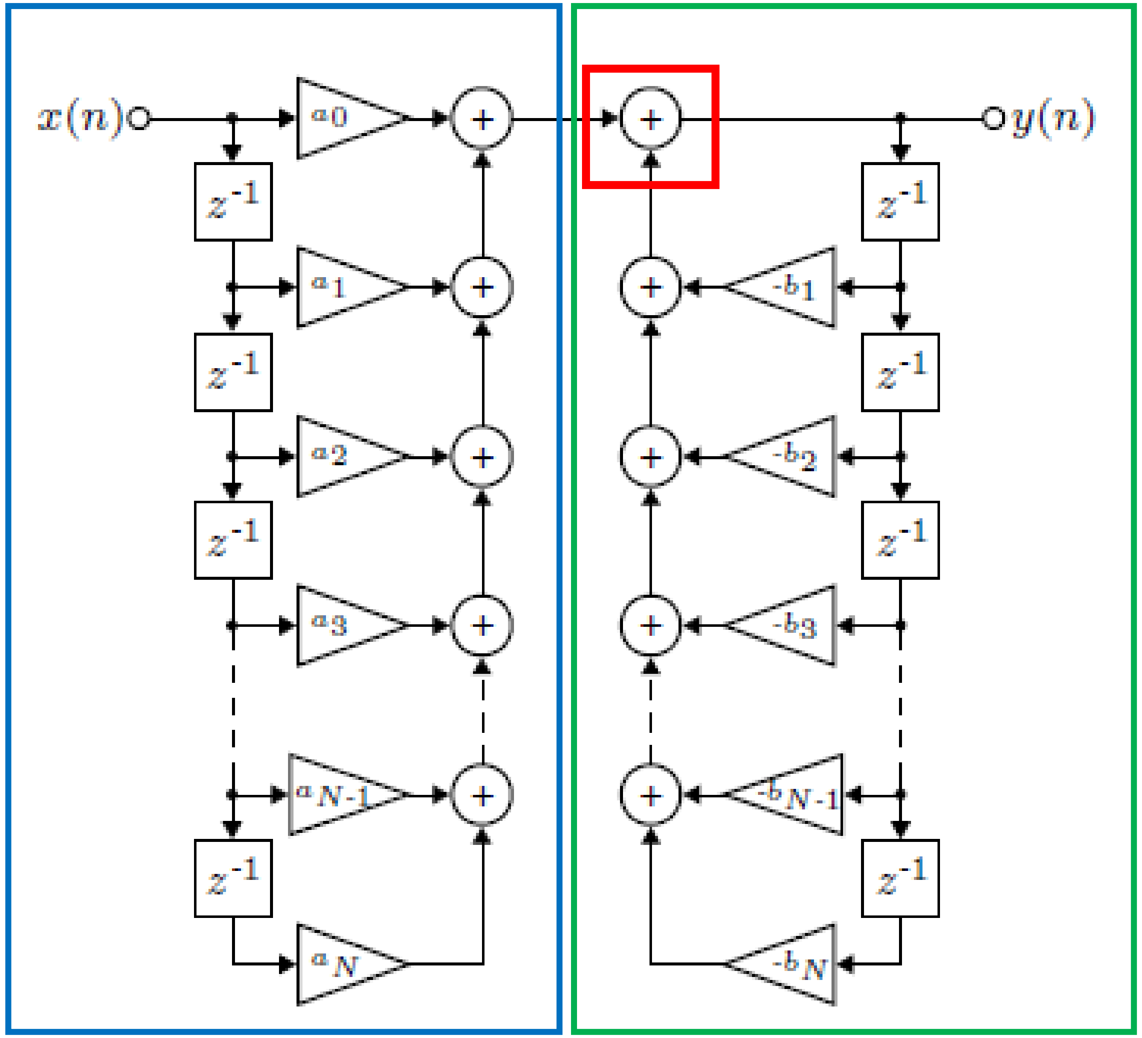
$$y(n) = \sum_{i=0}^{N} a_i x(n-i) - \sum_{i=1}^{N} b_i y(n-i)$$

First part of the structure is

$$\sum_{i=0}^{N} a_i x(n-i)$$

Second part of the structure is

$$-\sum_{i=1}^{N} b_i y(n-i)$$



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Example

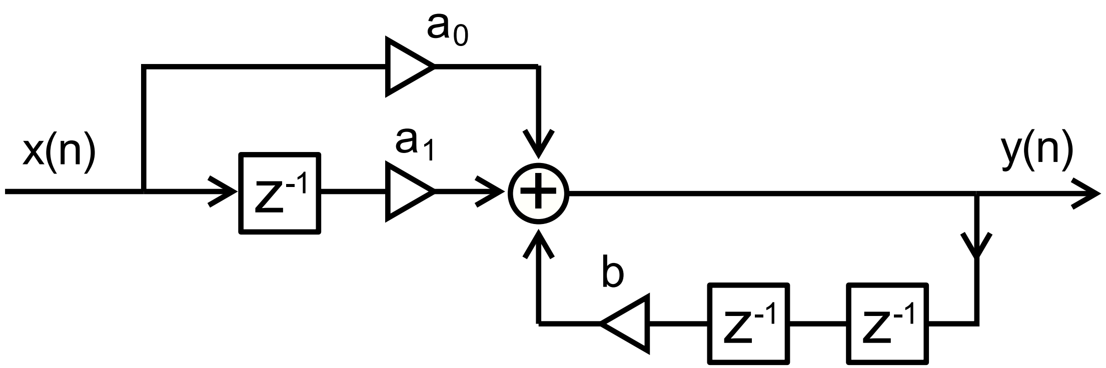
$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_0 + a_1 z^{-1}}{1 + b \cdot z^{-2}}$$

We can rewrite the above to

$$y(n) + 0 \cdot y(n-1) + b \cdot y(n-2) = a_0 x(n) + a_1 x(n-1) + 0 \cdot x(n-2)$$

$$y(n) = a_0 x(n) + a_1 x(n-1) - b \cdot y(n-2)$$

Let's draw the block diagram



Exercise

Let's draw the block diagram of the following 4th order high pass filter

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0,60 - 2,39z^{-1} + 3,60z^{-2} - 2,39z^{-3} + 0,60z^{-4}}{1 - 2,98z^{-1} + 3,43z^{-2} - 1,78z^{-3} + 0,36z^{-4}}$$

Direct type II

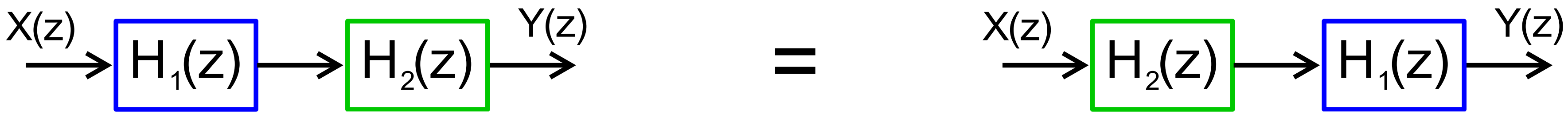
Given the following transfer function

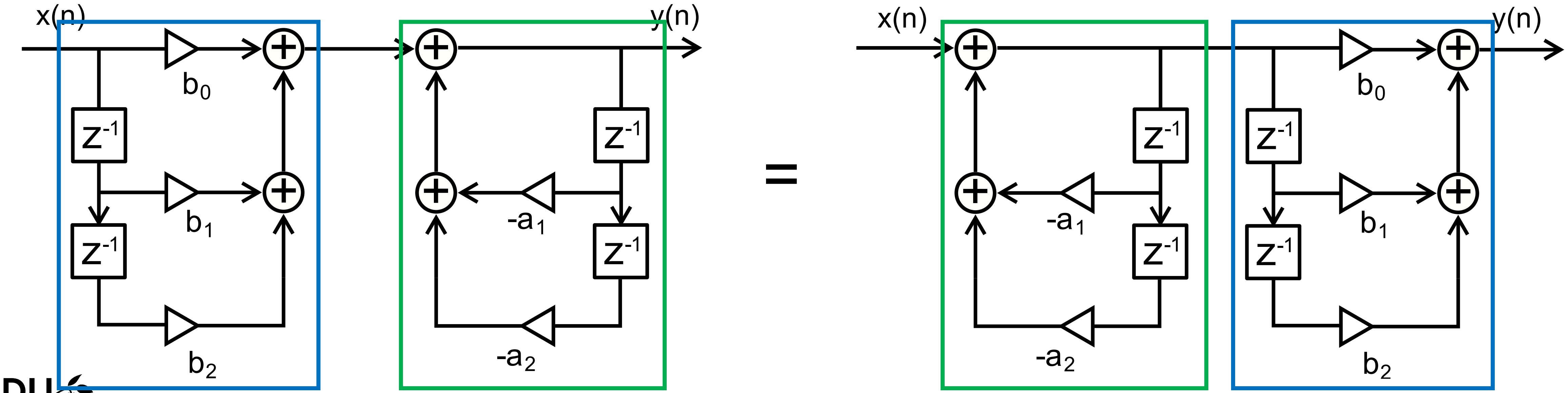
$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} a_i\, z^{-i}}{1 + \sum_{i=1}^{N} b_i\, z^{-i}}$$

It can be rewritten as

$$H(z) = \frac{W(z)}{X(z)}\frac{Y(z)}{W(z)} = \frac{1}{1 + \sum_{i=1}^{N} b_i\, z^{-i}} \sum_{i=0}^{N} a_i\, z^{-i} = H_1(z)\, H_2(z)$$

We can reorganize / switch sub-system





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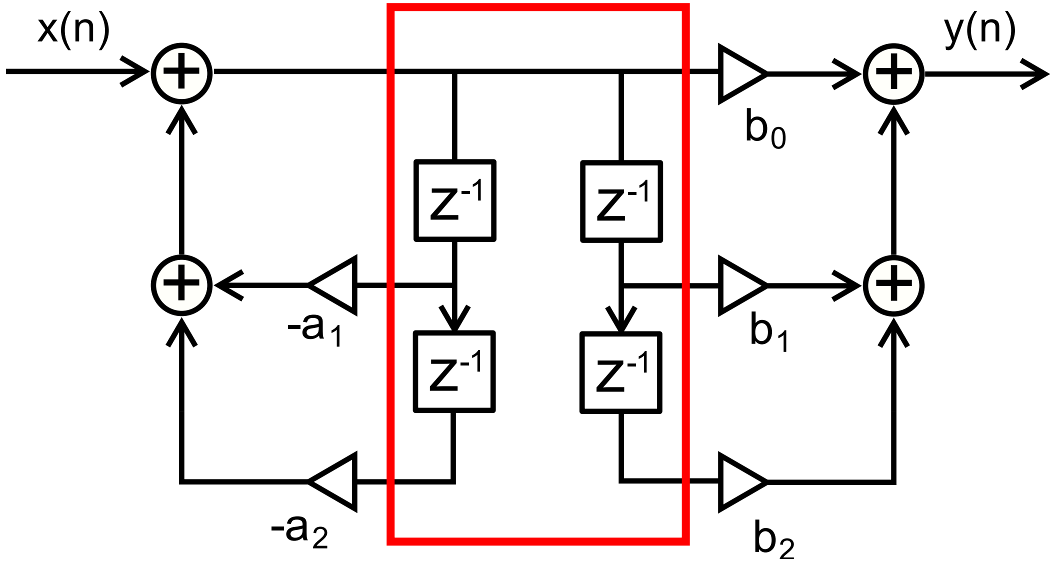
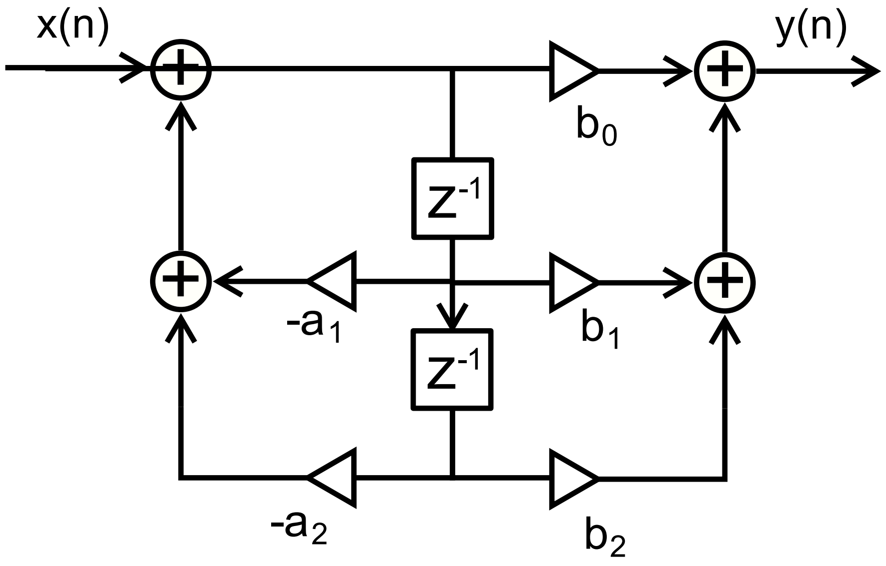
Direct type II



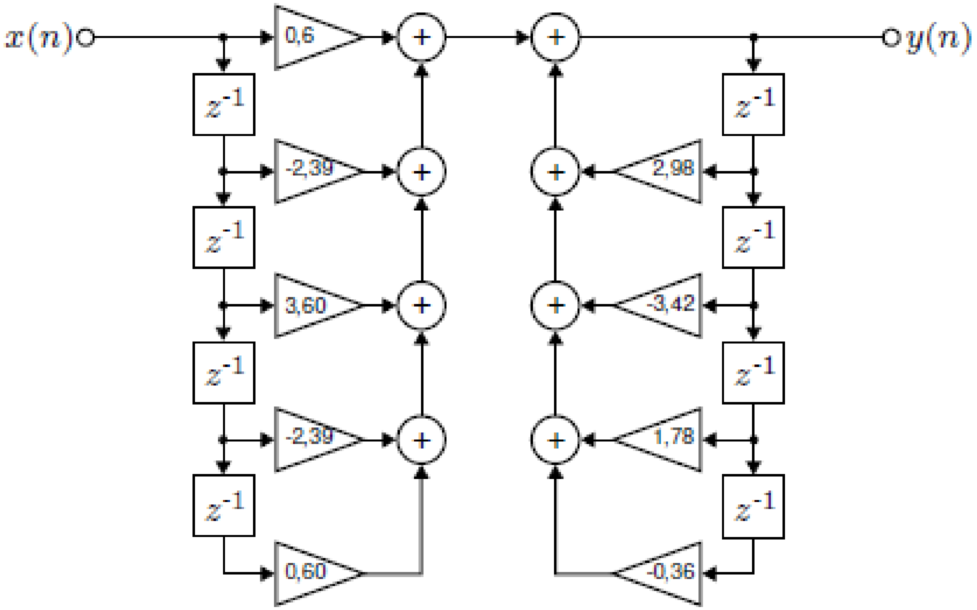


How many delay elements in this case?

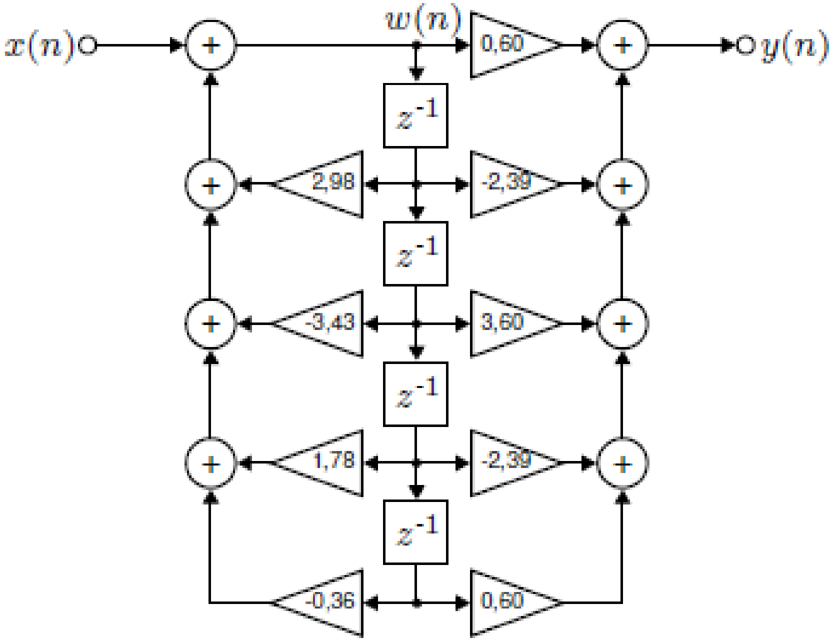
Example (Type II)

Let's consider the same 4th order transfer function in the previous example

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0,60 - 2,39z^{-1} + 3,60z^{-2} - 2,39z^{-3} + 0,60z^{-4}}{1 - 2,98z^{-1} + 3,43z^{-2} - 1,78z^{-3} + 0,36z^{-4}}$$



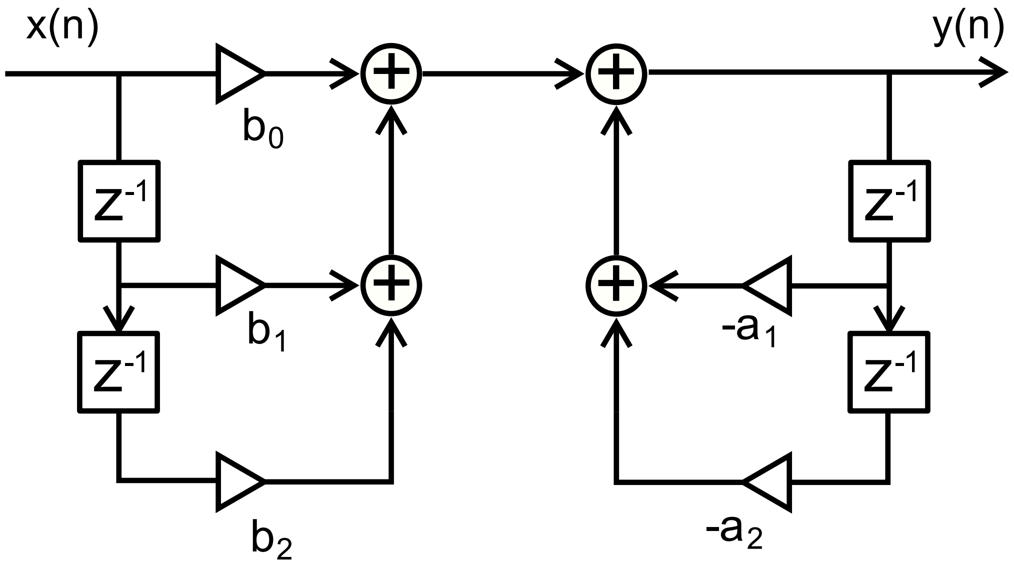
Type I



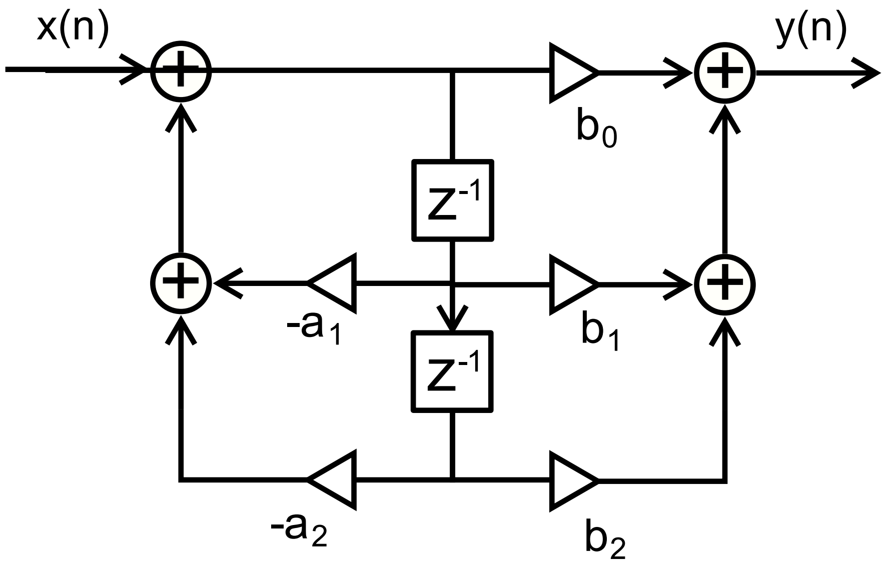
Type II

Type I v.s. Type II

→ The type I structure uses 2N delay elements, i.e. 2N values must pass through the filter.
→ **The type II structure uses only N delay elements, which requires less memory.**



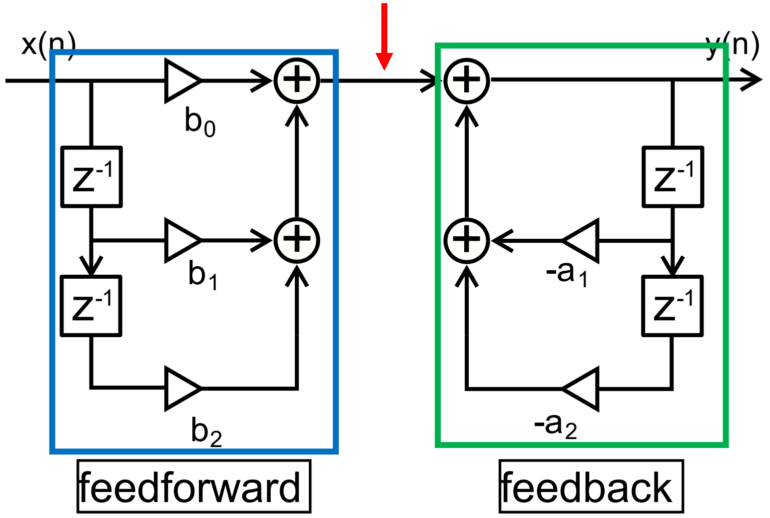
Type I



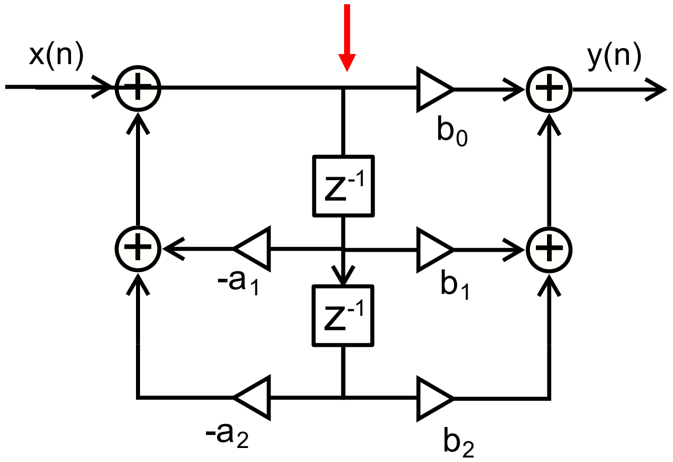
Type II

Why not always use Type II?

→ Type I uses separate feedforward and feedback loop, while Type II combines them together. The intermediate states in Type II can experience **larger signal swings**.



I will use this as an example for later Simulink tutorial.



→ In Type II, the coefficient a_i act on the internal states, which make the system **more sensitive to the rounding errors**.

In digital filter, the coefficients are presented with a finite number of bits, which means that they are rounded.

Rounding errors

The differential function of a higher order filter:

$$y(n) + b_1 y(n-1) + b_2 y(n-2) + \ \ldots + b_N y(n-N) = a_0 x(n) + a_1 x(n-1) + \ \ldots + a_N x(n-N)$$

Let's calculate $w(n)$:

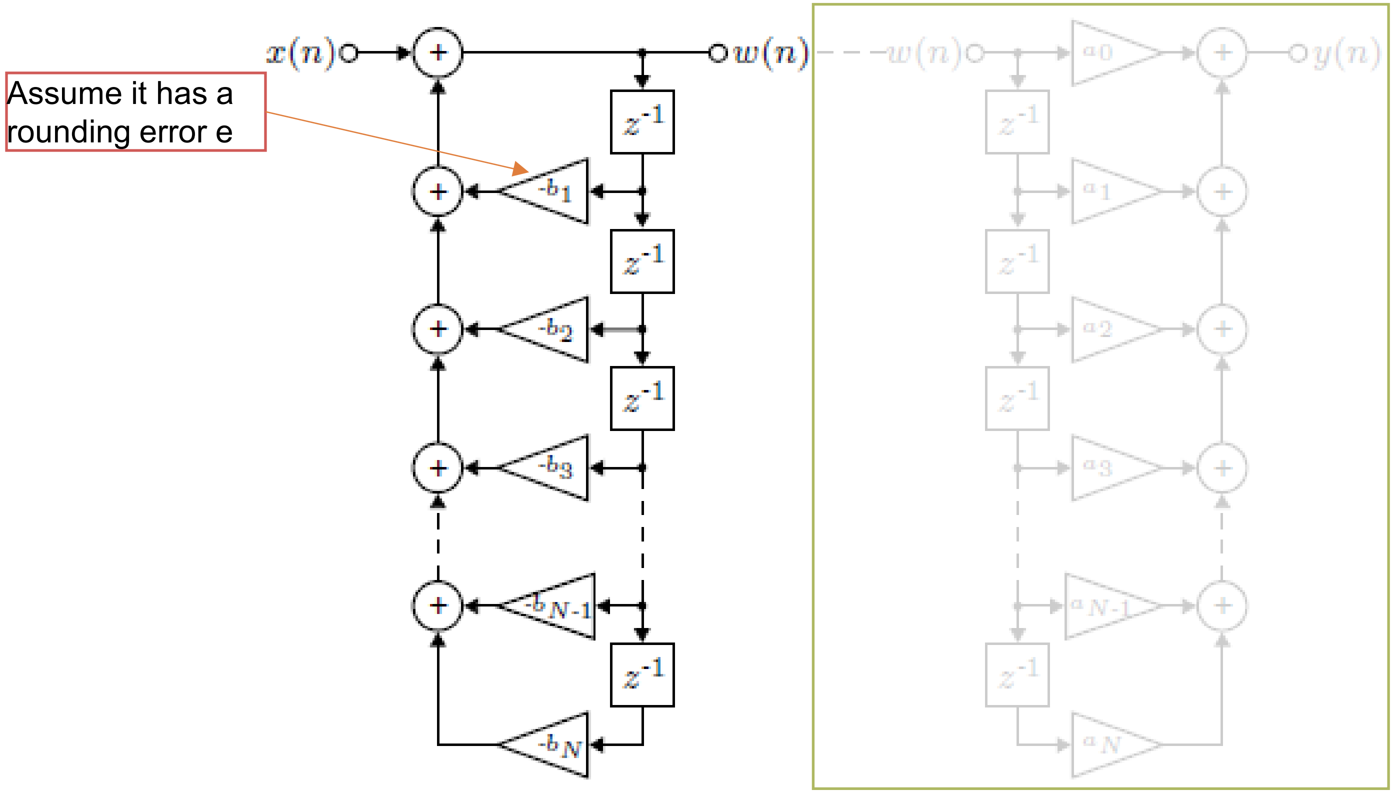
$w(0) = x(0)$

$w(1) = -(b_1 + e)w(0) + x(1)$

$w(2) = -b_2 w(0) - (b_1 + e)w(1) + x(2)$

$w(3) = -b_3 w(0) - b_2 w(1) - (b_1 + e)w(2) + x(3)$

The higher order it is, more error is amplified!



What can be the problems for direct realization?

To realize a high order digital filter, we rarely use Direct type I or Direct type II structures.

Normally, **we divide it to sections with low order (1st or 2nd order)**, and then realize the whole system via
→ Cascade structure or
→ Parallel structure

Reasons:
→ More stable and numerical precise (rounding error / quantization error)
→ Easier scaling and **dynamical range control**
→ Facilitate tuning or replacement of sections

(signal swing) Explain later in Simulink tutorial.

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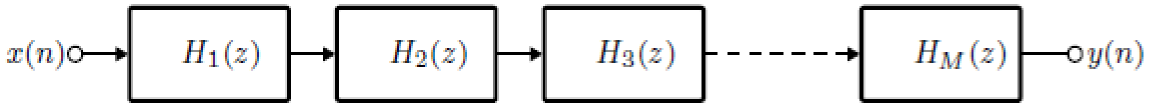
Cascade realization

A **higher-order system** is often realised as a cascade form

$$H(z) = H_1(z) \cdot H_2(z) \cdot H_3(z) \cdot \cdots \cdot H_M(z)$$



The M sections of the sub-system are 1st order or 2nd order transfer functions

$$H_k(z) = \frac{a_{0k} + a_{1k}z^{-1}}{1 + b_{1k}z^{-1}}$$

or

$$H_k(z) = \frac{a_{0k} + a_{1k}z^{-1} + a_{2k}z^{-2}}{1 + b_{1k}z^{-1} + b_{2k}z^{-2}}$$

where $k = 1, 2, \dots, M$ indicates the section number

Example

Let's consider the following 3rd order high pass filter

$$H(z) = \frac{0,3947 - 1,1841z^{-1} + 1,1841z^{-2} - 0,3947z^{-3}}{1 - 1,289z^{-1} + 0,8070z^{-2} - 0,06173z^{-3}}$$

Realize it using a cascade structure using 2 direct Type II structure.

The transfer function is written in standard form as

$$H(z) = 0,3947\frac{z^3 - 3z^2 + 3z - 1}{z^3 - 1,289z^2 + 0,8070z - 0,06173}$$

The transfer function H(z) has three zero points: $z_1 = z_2 = z_3 = 1$, and three poles $p_1 = 0.08802$, and $p_2 = p_3^* = 0.6005 + j0.5837$

The system $H(z)$ is written as a cascade of two transfer functions $H_1(z)$ and $H_2(z)$.

Matlab function:

- roots()
- [p,z] = pzmap(H_z);

It helps you calculate the zeros and poles.

Example

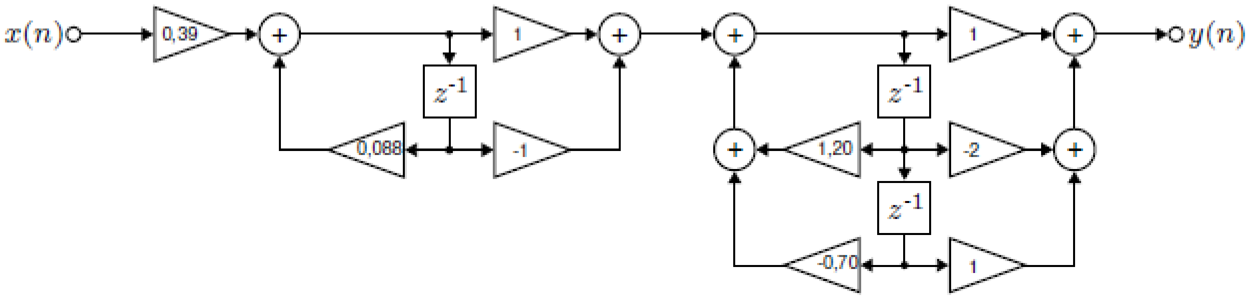
The transfer function can be factorized as

$$H(z) = a_0 H_1(z) H_2(z) = a_0 \frac{z - z_1}{z - p_1} \cdot \frac{(z - z_2)(z - z_3)}{(z - p_2)(z - p_3)}$$

The above factorization leads to

$$H(z) = 0,3947 \cdot \frac{z - 1}{z - 0,08802} \cdot \frac{z^2 - 2z + 1}{z^2 - 1,201z + 0,7013}$$

The realization of $H(z)$ with a cascade of two direct type II structures is shown below.



Organize poles and zeros into different sections

$$H(z) = H_1(z) \cdot H_2(z) \cdot H_3(z) \cdot \cdots \cdot H_M(z) = \underline{\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}} \cdot \underline{\frac{(z-z_3)(z-z_4)}{(z-p_3)(z-p_4)}} \cdot \cdots \cdot \underline{\frac{z-z_M}{z-p_M}}$$

→ For each section, it is preferred to have the same number of order for numerator and denominator
→ Conjugate pairs (zeros and poles) are put together (plot the zeros and poles to determine)
→ Zeros and poles whose positions are close should be put in the same section.

Example

Matlab code:

```
fc = 30; % cut off frequency 30 Hz
fs = 100; % sampling frequency 100 Hz
[b,a] = cheby1(5,0.5,fc/(fs/2));
H_z = tf(b,a,1/fs)
pzplot(H_z)
```

Let's consider the following 5th order 0.5 dB Chebyshev lowpass filter.

$$H(z) = \frac{0.06654\, z^5 + \ 0.3327\, z^4 + \ 0.6654\, z^3 + \ 0.6654\, z^2 + \ 0.3327\, z + 0.06654}{z^5 + \ 0.1825\, z^4 + \ 1.001\, z^3 - 0.2126\, z^2 + 0.268\, z \ - \ 0.1098}$$

Matlab code to find poles and zeros

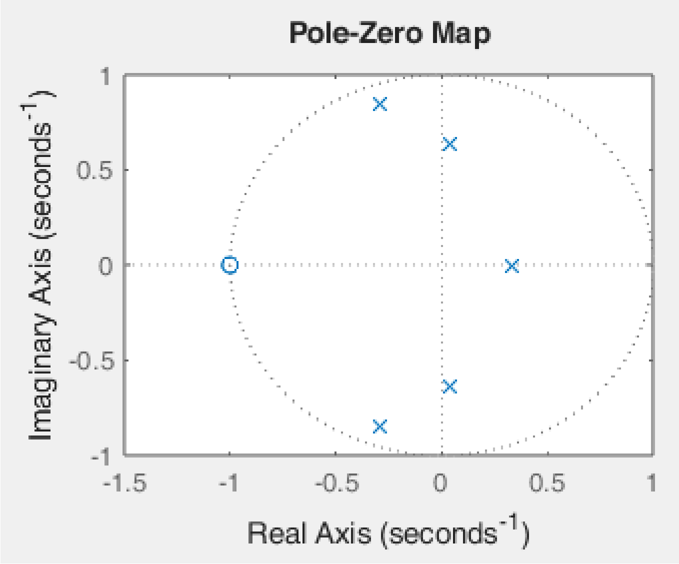
```
[p,z] = pzmap(H_z);
```

```
p =

  -0.2942 + 0.8514i
  -0.2942 - 0.8514i
   0.0357 + 0.6350i
   0.0357 - 0.6350i
   0.3345 + 0.0000i
```

```
z =

  -1.0008 + 0.0006i
  -1.0008 - 0.0006i
  -0.9997 + 0.0009i
  -0.9997 - 0.0009i
  -0.9990 + 0.0000i
```



This filter is realized as a cascade of two 2nd order filters and one 1st order filter.

$$H_1 = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}$$

$$H_2 = \frac{(z - z_3)(z - z_4)}{(z - p_3)(z - p_4)}$$

$$H_3 = \frac{z - z_5}{z - p_5}$$

Q: How to organize these sections?

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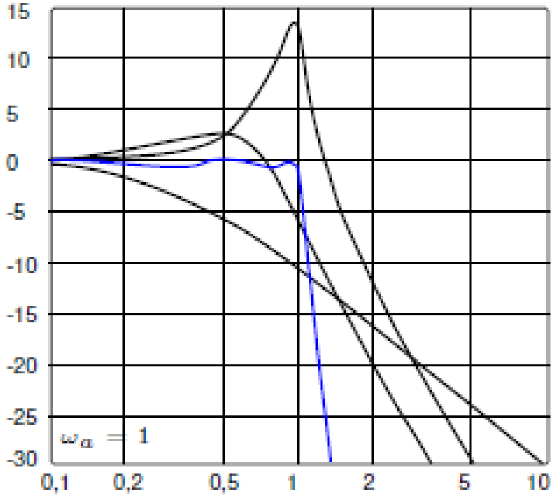
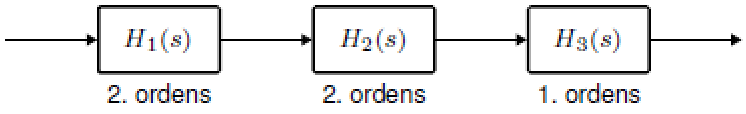
Sequence of sections

By cascading the three filters (two 2nd order filters and one 1st order filter), the desired Chebyshev filter can be constructed. The figure below shows the amplitude characteristics of the used three filters (black curves), and the Chebyshev filter (blue).



You can see that
→ one of the filters has a maximum gain of 13 dB, which can cause problems due to limited dynamic range.
→ Another filter has -10dB gain, which can lose signal close to $\omega_a = 1$

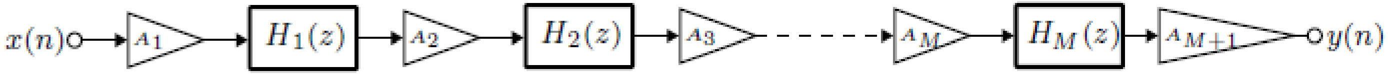
So, we can organize the filters as



Keynote:
Avoiding passing very large or small values between sections.

Scaling

To avoid overflow, scaling factors can be introduced between the sections



Parallel realization

A **higher-order system** can also be realised as a parallel form

$$H(z) = H_1(z) + H_2(z) + H_3(z) + \cdots + H_M(z)$$

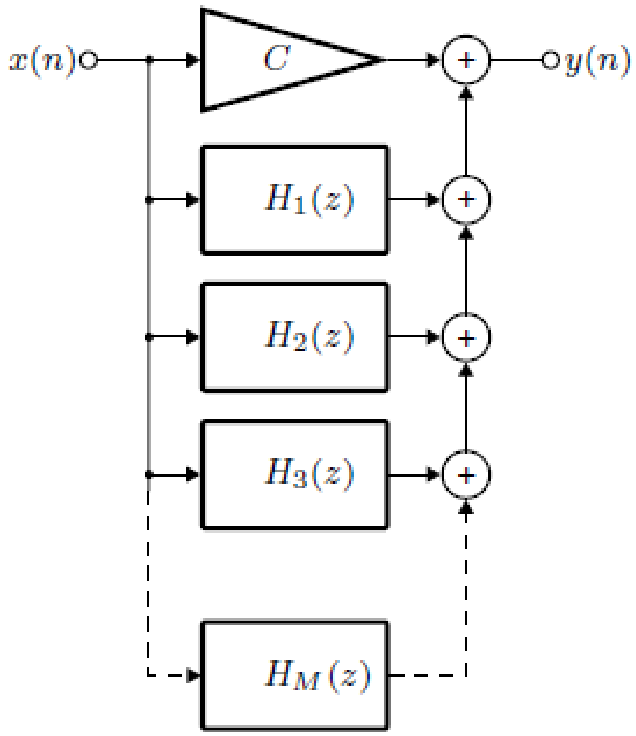
The M sections of the sub-system are 1st order or 2nd order transfer functions

$$H_k(z) = \frac{a_{0k} + a_{1k}z^{-1}}{1 + b_{1k}z^{-1}}$$

or

$$H_k(z) = \frac{a_{0k} + a_{1k}z^{-1} + a_{2k}z^{-2}}{1 + b_{1k}z^{-1} + b_{2k}z^{-2}}$$

where $k = 1, 2, \ldots, M$ indicates the section number



Example

Let's make a parallel realization of the filter

$$H(z) = \frac{0,3947 - 1,1841z^{-1} + 1,1841z^{-2} - 0,3947z^{-3}}{1 - 1,289z^{-1} + 0,8070z^{-2} - 0,06173z^{-3}}$$
$$= \frac{0,3947(z-1)(z^2-2z+1)}{(z-0,08802)(z^2-1,201z+0,7013)}$$

Using partial fraction decomposition, we can convert the above to the following

$$H(z) = C + H_1(z) + H_2(z)$$

Example

In the partial fraction decomposition, we firstly find all the poles, and then we construct

$$\frac{H(z)}{z} = \frac{C}{z} + \frac{a_{01}}{z - 0,08802} + \frac{a_{02}z + a_{12}}{z^2 - 1,201z + 0,7013}$$

where the coefficients C and a_{01} can be calculated as shown in previous lessons.

$$C = z\frac{H(z)}{z}|_{z=0} = 6,394$$

$$a_{01} = (z - 0,08802)\frac{H(z)}{z}|_{z=0,08802} = -5,637$$

In this case,

$$\frac{H(z)}{z} = \frac{6,394}{z} - \frac{5,637}{z - 0,08802} + \frac{a_{02}z + a_{12}}{z^2 - 1,201z + 0,7013}$$

Example

To find the coefficients a_{02} and a_{12}, two values for z are inserted into the equation.

We first use $z = 1$

$$\frac{H(z)}{z}|_{z=1} = \frac{6,394}{z}|_{z=1} - \frac{5,637}{z-0,08802}|_{z=1} + \frac{a_{02}z + a_{12}}{z^2 - 1,201z + 0,7013}|_{z=1}$$
$$0 = 6,394 - \frac{5,637}{1-0,08802} + \frac{a_{02} + a_{12}}{1 - 1,201 + 0,7013}$$

This gives that

$$a_{02} + a_{12} = -0,1066$$

Example

To find the coefficients a_{02} and a_{12}, two values for z are inserted into the equation.

Then we use $z = -1$

$$\frac{H(z)}{z}|_{z=-1} = \frac{6,394}{z}|_{z=-1} - \frac{5,637}{z-0,08802}|_{z=-1} + \frac{a_{02}z + a_{12}}{z^2 - 1,201z + 0,7013}|_{z=-1}$$
$$-1 = -6,394 - \frac{5,637}{-1-0,08802} + \frac{-a_{02} + a_{12}}{1+1,201+0,7013}$$

This gives that

$$-a_{02} + a_{12} = 0,6181$$

Thus, the coefficients are found from

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_{02} \\ a_{12} \end{bmatrix} = \begin{bmatrix} -0,1066 \\ 0,6181 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} a_{02} \\ a_{12} \end{bmatrix} = \begin{bmatrix} -0,3624 \\ 0,2558 \end{bmatrix}$$

Example

Eventually, the transfer function

$$H(z) = \frac{0,3947 - 1,1841z^{-1} + 1,1841z^{-2} - 0,3947z^{-3}}{1 - 1,289z^{-1} + 0,8070z^{-2} - 0,06173z^{-3}} = \frac{0,3947(z-1)(z^2-2z+1)}{(z-0,08802)(z^2-1,201z+0,7013)}$$

can be rewritten as

$$H(z) = 6,394 - \frac{5,637z}{z-0,08802} + \frac{-0,3624z^2+0,2558z}{z^2-1,201z+0,7013}$$

Matlab function: residuez() or residue()

```
% H_z = (0.3947*z^3-1.1841*z^2+1.1841*z-0.3947)/(z^3-1.289*z^2+0.8070*z-0.06173);
b_z = [0.3947, -1.1841, 1.1841, -0.3947]; % numerator coeffs for z^3 ... z^0
a_z = [1.0000, -1.2890, 0.8070, -0.06173]; % denominator coeffs for z^3 ... z^0
% Partial fraction decomposition
% r = residues, p = poles, k = direct terms (if any)
[r, p, k] = residuez(b, a);
```

```
% you will receive output
Residues (r):
 -0.1810 - 0.0328i
 -0.1810 + 0.0328i
 -5.6373 + 0.0000i

Poles (p):
  0.6005 + 0.5837i
  0.6005 - 0.5837i
  0.0880 + 0.0000i

Direct term (k):
   6.3940
```

Conjugate pair can be made as a single section

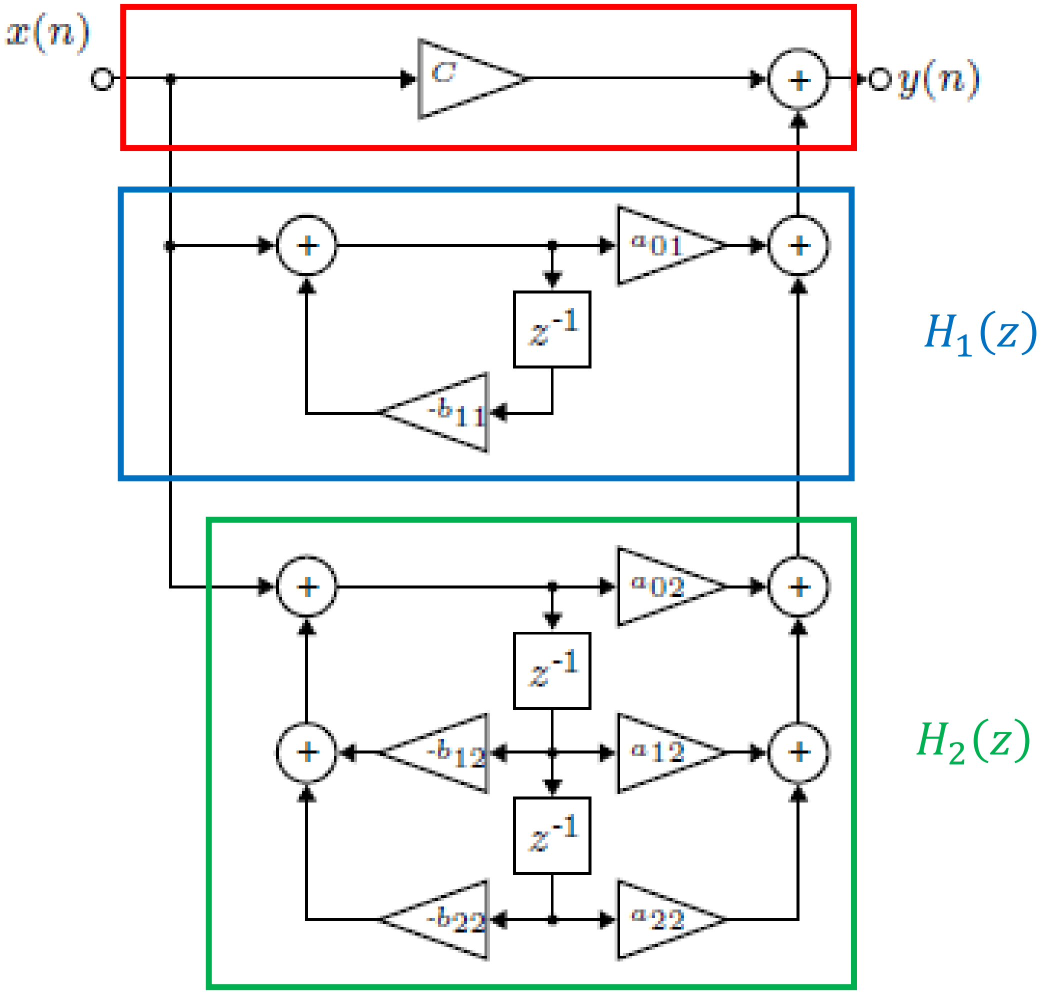
It corresponds to

$$H(z) = 6.394 + \frac{-0.1810 - 0.0328i}{z-(0.6005 + 0.5837i)} + \frac{-0.1810 + 0.0328i}{z-(0.6005 - 0.5837i)} + \frac{-5.6373}{z-0.088}$$

Example

Based on the decomposed transfer function, the block diagram of parallel form can be drawn.

$$H(z) = \underbrace{6,394}_{C} \underbrace{- \frac{5,637z}{z - 0,08802}}_{H_1(z)} + \underbrace{\frac{-0,3624z^2 + 0,2558z}{z^2 - 1,201z + 0,7013}}_{H_2(z)}$$



Cascade v.s. parallel structure

Whether the filter should be realized using
→ **cascade structure:** $H(z) = H_1(z) \cdot H_2(z) \cdot H_3(z) \cdot \cdots \cdot H_M(z)$
→ **parallel structure:** $H(z) = H_1(z) + H_2(z) + H_3(z) + \cdots + H_M(z)$

Rules:

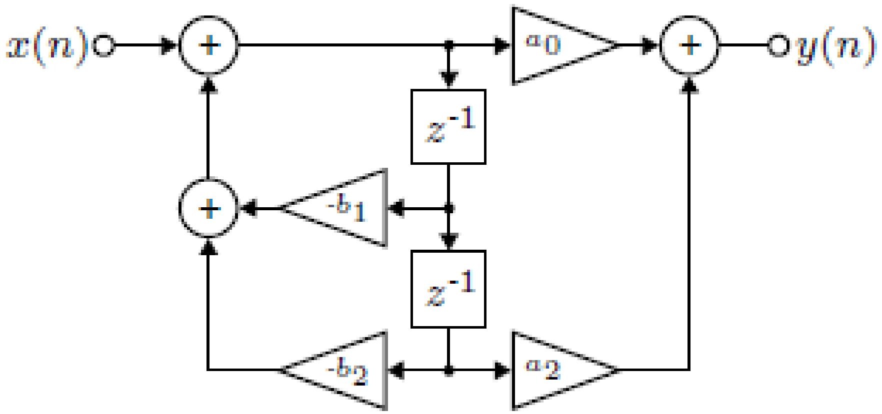
→ **Choose parallel structure for FIR (**$H(z) = \sum_{i=0}^{N} a_i z^{-i}$**), and cascade structure for IIR (**$H(z) = \frac{\sum_{i=0}^{N} a_i z^{-i}}{1+\sum_{i=1}^{N} b_i z^{-i}}$**).**

→ **Cascade structure is harder for tuning than parallel structure.**
→ **Parallel structure can lead to sum of large signals, which can have higher risk of overflow.**

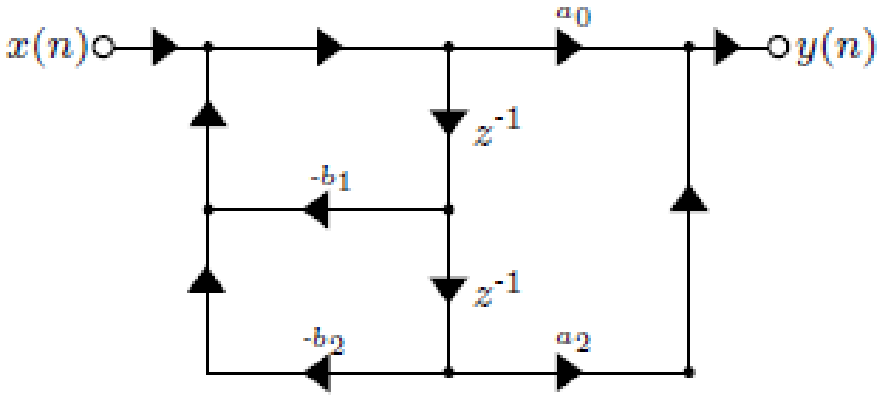
Block diagrams and signal flow graphs

Both **block diagram** and **signal flow graph** can be used to illustrate the implementation of **discrete-time systems**.

Block diagram



Signal flow graph



Transpose of the system

Given a signal flow graph of a system H(z), if the directions of all branches are ‘reversed’, the system function of the ‘transpose form’ remains the same.