A.3 Cruise Control Example

This section presents the modelling and control of a vehicle that drives on a road with varying inclination. The aim of the control is to regulate the speed of the car to a constant setpoint, by controlling the throttle position, which is assumed to be proportional to a force pointing in the forwards direction. There are three requirements to the control system

1. The overshoot should be smaller than 20 %.
2. The settling time should be shorter than 8 s.
3. There should be no steady state error when a constant reference is applied to the system.

The section contains modelling, linearization, and controller design for a considered cruise control example.

A.3.1 Modelling

The considered vehicle is shown in Figure A.8 that illustrates the forces acting on the vehicle, which are: the force controlled by the throttle F_t, the drag force F_d, and the gravitational force F_g.

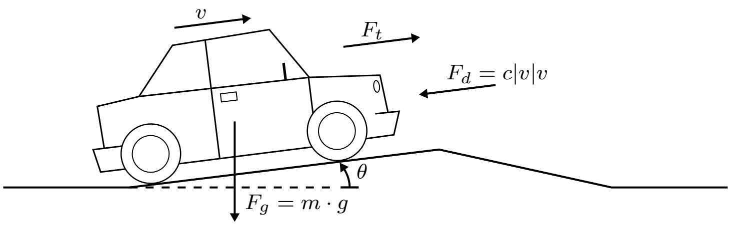

Figure A.8: Vehicle driving with speed v on a road with inclination θ.

From Figure A.8 and Newton's 2nd law, the following expression is obtained for the dynamics of the vehicle

$$m\dot{v} = F_t - F_v - F_g \cdot \sin\theta \qquad \text{[N]}$$

where v is the velocity of the vehicle [m/s], m is the mass of the vehicle [kg], and θ is the inclination of the road [rad]. By inserting expressions for F_v and F_g, the dynamics can be rewritten as

$$m\dot{v} = F_t - c|v|v - mg\sin\theta \qquad \text{[N]}$$

where g is the gravitational acceleration [m/s^2] and $|v|$ denotes the absolute value of v. It is seen that the above equation is a 1st order differential equation with a controlled input $u = F_t$ and an exogenous disturbance input $d = mg\sin\theta$. In addition the model is nonlinear, since the drag force depends on $|v|v$. In conclusion, the system model is given by the following differential equation

$$\dot{v} = f(v, u, d) = \frac{1}{m}u - \frac{c}{m}|v|v - \frac{1}{m}d \qquad [\text{m/s}^2] \tag{A.5}$$

where u is the input to the vehicle [N] and d is a disturbance acting on the vehicle [N] (The disturbance is given by $d = mg\sin\theta$). The parameter values and variables associated to the cruise control system are provided in Table A.1.

Name	**Symbol**	**Value**	**Unit**
Drag coefficient	c	10	m^2/s^2
Gravitational acceleration	g	9.82	m/s^2
Mass of vehicle	m	900	kg
Velocity of vehicle	v	-	m/s
Inclination of road	θ	-	rad

Table A.1: Variables and parameters of the cruise control system.

A.3.2 Linearization

A linearized model is derived in the following that will form the basis for the controller design. The linearization of the differential equation shown in (A.5) is accomplished for a fixed speed denoted by $\bar{v}$ and constant disturbance $\bar{d} = 0$ N, which gives a stationary value of u as (from $\dot{v} = f(\bar{v}, \bar{u}, \bar{d}) = 0$ m/s^2)

$$\bar{u} = c|\bar{v}|\bar{v} \qquad [\mathrm{N}]$$

The linearized model of the system is obtained from a first-order Taylor approximation of $\dot{v}$, which is given by

$$\dot{v} \approx f(\bar{v}, \bar{u}, \bar{d}) + \left.\frac{\partial f(v,u,d)}{\partial v}\right|_{(v,u,d)=(\bar{v},\bar{u},\bar{d})} \cdot \hat{v} + \left.\frac{\partial f(v,u,d)}{\partial u}\right|_{(v,u,d)=(\bar{v},\bar{u},\bar{d})} \cdot \hat{u} + \left.\frac{\partial f(v,u,d)}{\partial d}\right|_{(v,u,d)=(\bar{v},\bar{u},\bar{d})} \cdot \hat{d}$$

where $\hat{a} = a - \bar{a}$. The approximation is equal to

$$\dot{v} \approx \underbrace{\frac{1}{m}\bar{u} - \frac{c}{m}|\bar{v}|\bar{v} - \frac{1}{m}\bar{d}}_{f(\bar{v},\bar{u},\bar{d})} - 2\frac{c}{m}\bar{v}\hat{v} + \frac{1}{m}\hat{u} - \frac{1}{m}\hat{d}.$$

The approximation of $\dot{v}$ reduces to the following differential equation

$$\dot{v} \approx -2\frac{c}{m}\bar{v}\hat{v} + \frac{1}{m}\hat{u} - \frac{1}{m}\hat{d} \tag{A.6}$$

which is the linearized model of the system. Finally, a transfer function of the system is set up in the following by Laplace transforming the differential equation (A.6) and assuming zero initial conditions

$$\hat{v}(s) = \underbrace{\frac{1/(2c\bar{v})}{\frac{m}{2c\bar{v}}s + 1}}_{=G(s)} \left(\hat{u}(s) - \hat{d}(s)\right) \tag{A.7}$$

It is seen that the model is a first-order system

$$G(s) = \frac{k}{\tau s + 1}$$

where the gain k and the time constant τ are given by

$$k = \frac{1}{2c\bar{v}} \qquad \text{and} \qquad \tau = \frac{m}{2c\bar{v}}$$

The objective of the control system is to maintain a constant speed of 10 m/s; thus, the operating point for the linearization of the system is chosen to be $(\bar{v}, \bar{u}, \bar{d}) = (10\ \mathrm{m/s}, 1000\ \mathrm{N}, 0\ \mathrm{N})$. This gives a gain $k = 0.005$ and a time constant $\tau = 4.5$.

Verification of Model

Before proceeding with the controller design, the nonlinear model and the linearized models are compared to determine how well the linearized model describes the dynamics of the system. The comparison is accomplished by applying step inputs with different amplitudes to the models. The input to the linearized model is $\hat{u}$, the initial condition is $\hat{v}(0) = 0$, and the output is $\hat{v}$. The input to the nonlinear model is $u = \bar{u} + \hat{u}$, the initial condition is $v(0) = \bar{v} + \hat{v}(0) = \bar{v}$, and the output is v. To compare the two systems, $\hat{v}$ is computed from the nonlinear system model as $\hat{v} = v - \bar{v}$. The comparison of the two models is presented in Figure A.9 that shows the step responses for inputs with different amplitudes, and in Figure A.10 that shows the difference between the outputs.

From the figures, it is seen that the responses of the linearized model and the nonlinear model are similar when the system is close to the operating point of the system (i.e., when $\hat{v} \approx 0$ m/s); however, when the system is getting further away from the operating point, the discrepancy between the outputs gets larger.

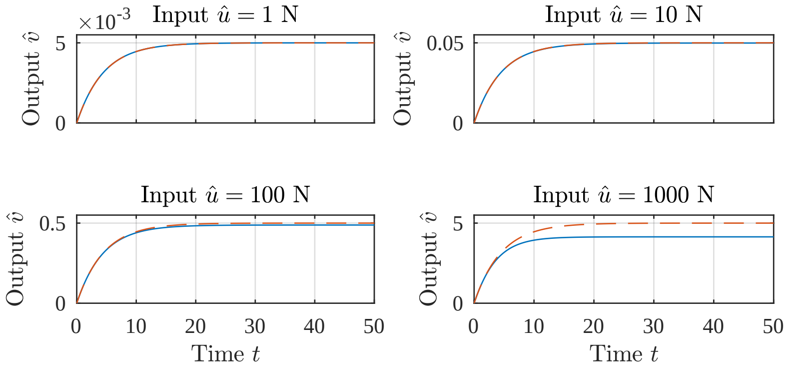

Figure A.9: Step responses of the nonlinear model (blue solid lines) and the linearized model (red dashed lines) when applying steps with magnitude $\hat{u} = 1, 10, 100, 1000$.

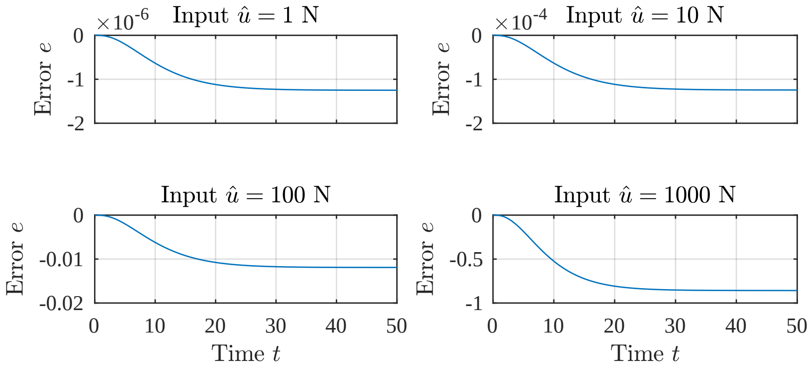

Figure A.10: Difference between the output of the nonlinear model and the output of the linearized model when applying steps with magnitude $\hat{u} = 1, 10, 100, 1000$.

A.3.3 Controller Design

The aim of this section is to present the design of a control system that complies with the specification given in the beginning of the section, which is an overshoot smaller than 20 %, a settling time shorter than 8 s, and no steady state error when the reference velocity is constant. Therefore, the system should be of type 1, and the closed loop poles should be located in the gray shaded area shown in Figure A.11.

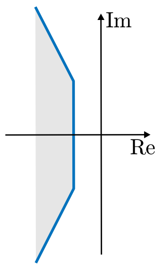

Figure A.11: Illustration of pole placement requirements for the closed-loop transfer function. A second order system (with no zeros) has an overshoot smaller than 20 % and a settling time shorter than 8 s if its poles are located in the gray shaded area.

Figure A.11 is obtained by assuming that the closed-loop transfer function is a second-order system. Then the time-domain requirements are transformed into frequency domain requirements as follows

$$\zeta \geq \sqrt{\frac{(\frac{\log(M_p)}{-\pi})^2}{1 + (\frac{\log(M_p)}{-\pi})^2}} \qquad \text{and} \qquad \sigma \geq \frac{4.6}{t_s}$$

where M_p is the overshoot ($M_p = 0.2$) and t_s is the settling time ($t_s = 8$ s). From these numbers, the frequency domain requirements are $\zeta \geq 0.46$ and $\sigma \geq 0.58$.

The control system can be designed using different methods, including the root locus design method; however, since the considered system is quite simple, the controller is designed by pole placement. To eliminate the steady state error in case of a constant reference, the closed-loop system should be of type 1. Therefore, it is chosen to design a PI controller for the cruise control system. The transfer function for the controller is

$$K(s) = K_p \frac{s + \frac{1}{T_i}}{s}$$

where K_p is the proportional gain and T_i is the integral time. A block diagram of the closed-loop system is shown in Figure A.12.

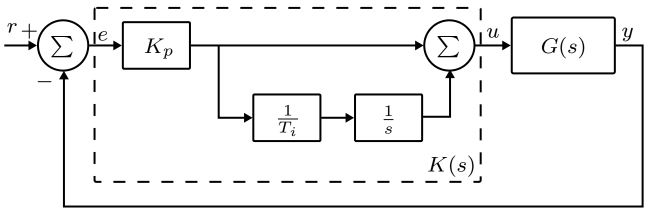

Figure A.12: Block diagram of the closed-loop system that consists of a plant model $G(s)$ and a PI controller $K(s)$.

From the two transfer functions $G(s)$ and $K(s)$, the closed loop system is described by the transfer function

$$H(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} = \frac{\frac{kK_p\left(s+\frac{1}{T_i}\right)}{s(\tau s+1)}}{1 + \frac{kK_p\left(s+\frac{1}{T_i}\right)}{s(\tau s+1)}} = \frac{kK_p\left(s + \frac{1}{T_i}\right)}{\tau s^2 + (1 + kK_p)s + \frac{kK_p}{T_i}}$$

To find an expression for the poles of the system, the characteristic equation given next is studied

$$1 + G(s)K(s) = 0$$

This implies that

$$\tau s^2 + (1 + kK_p)s + \frac{kK_p}{T_i} = 0 \tag{A.8}$$

which implies that the closed-loop system poles are located at

$$s = \frac{-(1 + kK_p) \pm \sqrt{(1 + kK_p)^2 - 4\frac{\tau k K_p}{T_i}}}{2\tau} \tag{A.9}$$

It is seen that the two closed-loop poles can be placed anywhere in the complex plane, by appropriate selection of K_p and T_i. It is however also seen from the following expression that the closed-loop transfer function has a zero

$$G(s)K(s) = \frac{kK_p}{\tau s + 1} \frac{s + \frac{1}{T_i}}{s} = 0$$

and that it is located at

$$s = -\frac{1}{T_i}.$$

To select the controller gains, one could find the worst performing controller that complies with the specification, as this often results in a controller with small gains. A second order system with $\zeta = 0.46$ and $\sigma = 0.58$ has poles at $s = -0.58 \pm 1.12$ and can be obtained by selecting gains $K_p = 835$ and $T_i = 0.58$. The closed-loop system has a zero at $s = -1.71$; hence, this design will not work, as the presence of a zero affects the system dynamics by increasing the overshoot. This statement is confirmed by the step response shown in Figure A.13 from which it is seen that the overshoot is 28 % and the settling time is approximately 8 s.

To eliminate the overshoot, it is possible to place both poles of the system on the real axis, e.g., at $s = -0.58$ by selecting the gains to $K_p = 835$ and $T_i = 2.81$; however, this would significantly increase

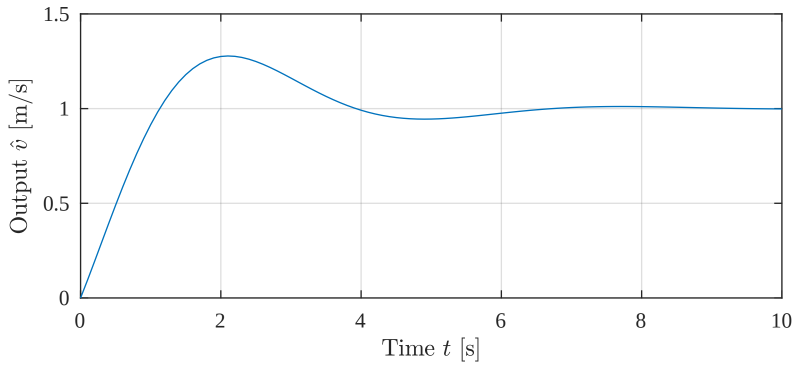

Figure A.13: Step response $\hat{r} = 1$ m/s of closed-loop system with controller having gains $K_p = 835$ and $T_i = 0.58$.

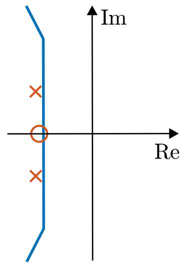

Figure A.14: Illustration of poles (crosses) and zero (circle) of the closed-loop system in relation with the performance specification indicated with the blue line.

the integral time. Therefore, the poles are chosen to be in the interior of the allowed pole region, by using gains $K_p = 1000$ and $T_i = 1.6$. The closed-loop poles and zero are illustrated in Figure A.14.

A step response of the system is shown in Figure A.15 that confirms that the linearized model complies with the specification. The nonlinear model also satisfies the specification with the chosen controller; this is concluded from a step response of the nonlinear model shown in Figure A.16.

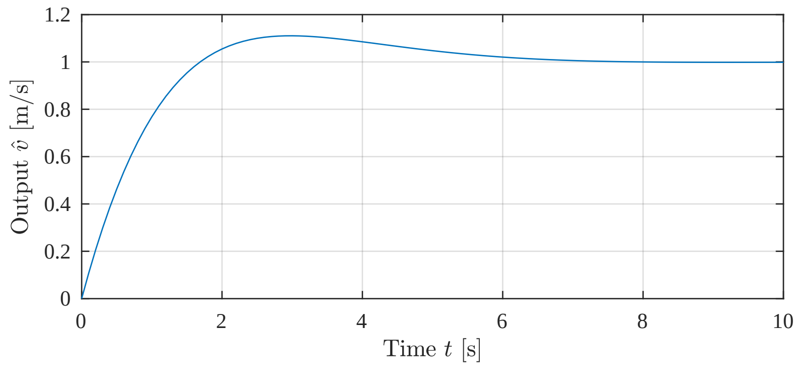

Figure A.15: Step response $\hat{r} = 1$ m/s of closed-loop system (with initial condition $v = 10$ m/s) with controller having gains $K_p = 1000$ and $T_i = 1.6$.

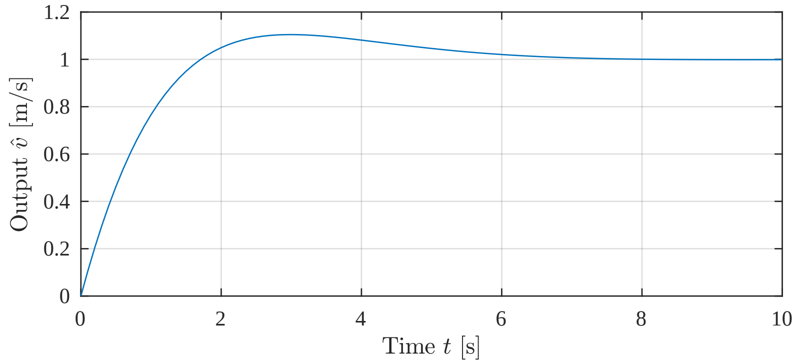

Figure A.16: Step response $r = 11$ m/s of the nonlinear system model with a PI controller having gains $K_p = 1000$ and $T_i = 1.6$.