Appendix A

Examples of Dynamical Systems

The purpose of this section is to provide models of dynamical systems, which are analyzed in the main part of the document.

A.1 Mass-Spring-Damper System

The purpose of this section is to model the mass-spring-damper system shown in Figure A.1, which is affected by an external force F.

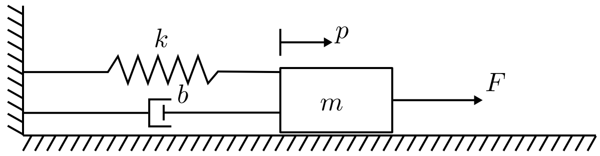

Figure A.1: Diagram of mass-spring-damper system.

To model the system, the following physical laws are exploited

- **Newton's 2nd law**: The sum of forces F_i acting on an object is equal to the mass m of the object multiplied by the acceleration a of the object, i.e.

$$ma = \sum_i F_i \qquad [\mathrm{N}]$$

- **Hooke's law**: The force needed to extend or compress a spring by some distance p, scales linearly with respect to the distance, i.e.,

$$F_{\text{spring}} = kp \qquad [\mathrm{N}]$$

  where k is the stiffness of the spring [N/m].

- **Damper**: The force needed to compress a damper scales linearly with respect to the velocity v, i.e.,

$$F_{\text{damper}} = bv \qquad [\mathrm{N}]$$

  where b is the damping coefficient [N/(m/s)].

To model a mechanical system, a freebody diagram is set up that shows all forces acting on each body in the system. Figure A.2 shows a free-body diagram for the mass-spring-damper system that consists of only one body on which the forces F, F_{spring}, and F_{damper} act. Based on Figure A.2, Newton's 2nd law can be employed to obtain (note that the arrows in Figure A.2 indicate positive direction)[1]

$$m\ddot{p} = F - F_{\text{spring}} - F_{\text{damper}}$$

[1] We define $\ddot{p} := \frac{d^2p}{dt^2}$

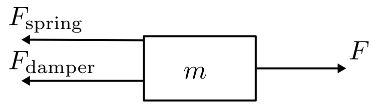

Figure A.2: Free-body diagram of mass-spring-damper system.

Subsequently, equations for the forces F_{spring} and F_{damper} can be inserted to obtain the following description of the system dynamics

$$m\ddot{p} = F - kp - b\dot{p}. \tag{A.1}$$

Remark that the reference frame used for describing the system dynamics (for the position p) must be inertial (non accelerating); thus, it should be attached to the fixed wall, not the moving mass. For convenience, the model of the system (A.1) is rewritten as

$$\ddot{p} = -\frac{k}{m}p - \frac{b}{m}\dot{p} + \frac{F}{m} \tag{A.2}$$

This is a second order ordinary differential equation. However, the system can be written as a system of first order differential equations by defining the variable $v = \dot{p}$ (the velocity), and realizing that $\dot{v} = \ddot{p}$. Then the system can be written on state space form as (the output is chosen to be the position p)

$$\begin{bmatrix}\dot{p}\\ \dot{v}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}p\\ v\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}F$$
$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}p\\ v\end{bmatrix}$$

The system can also be represented as the transfer function

$$H(s) = \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

Step responses of the system are provided in Figure A.3 with different values for parameters b, k, m, to give an intuition about the dynamics of the system.

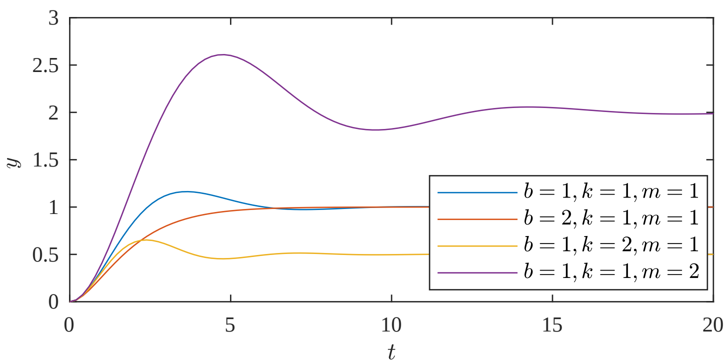

Figure A.3: Step responses of mass-spring-damper system.