System Analysis in Z domain

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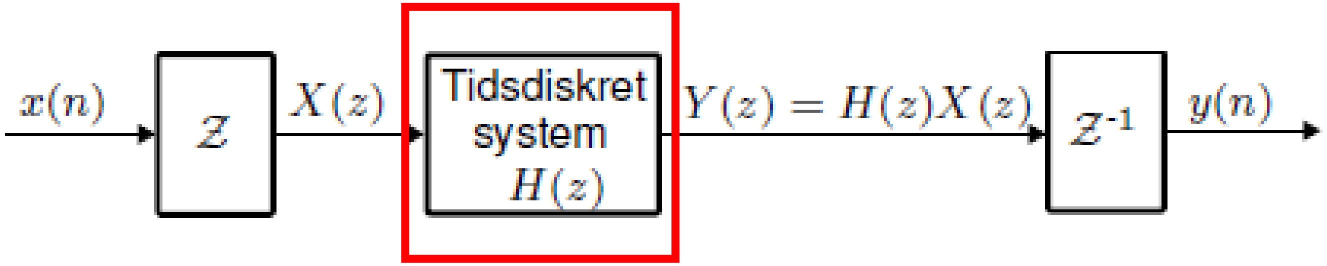
Z domain in discrete signal processing

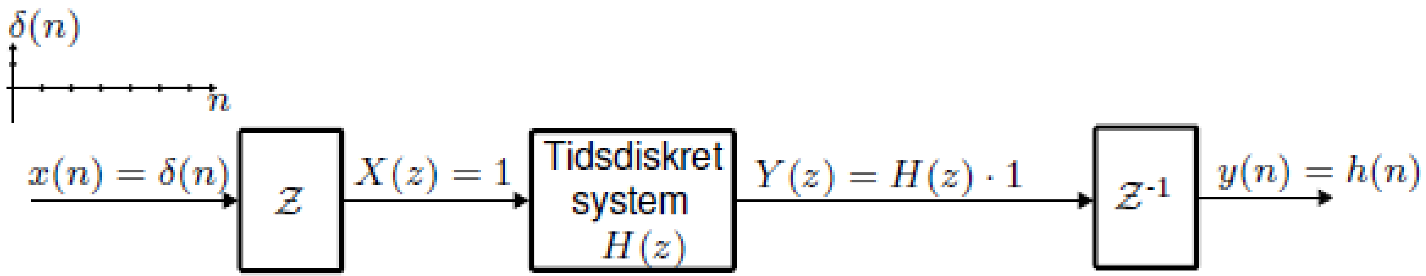
→ Perform z transform on input signal x(n)
→ Describe the discrete time system in z domain H(z)
→ Obtain output Y(z) = H(z)X(z)
→ Perform inverse z transform on Y(z) to obtain y(n)



Impulse response

The impulse response of a time-discrete system, denoted as $h(n)$, is the system's output when the input is an impulse $\delta(n)$. It provides a fundamental description of the system.

The principle is illustrated in the following figure.



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Impulse response

Given that $Z\big(\delta(n)\big) = 1$, the output response in z-domain is

$$Y(z) = H(z)X(z) = H(z)$$

The impulse response sequence can be calculated by inverse z-transform of $H(z)$, i.e.

$$h(n) = Z^{-1}\big(H(z)\big)$$

A system's impulse response sequence $h(n)$ is found by inverse z-transform of the system's transfer function $H(z)$.

Example

Considering the following transfer function, and determine the impulse response sequence of the system $h(n)$.

$$H(z) = \frac{1 + 0.4z^{-1}}{1 - 0.7z^{-1} + 0.1z^{-2}}$$

Firstly, we need to rewrite the transfer function with position powers

$$H(z) = \frac{z^2 + 0.4z}{z^2 - 0.7z + 0.1}$$

The transfer function can be factorized as

$$H(z) = \frac{z(z + 0.4)}{(z - 0.5)(z - 0.2)}$$

It can be seen that it has two zeros $z_1 = 0$ and $z_2 = -0.4$
And two poles $p_1 = 0.5$ and $p_2 = 0.2$.

Example

Through partial fraction solution

$$\frac{H(z)}{z} = \frac{k_1}{z - p_1} + \frac{k_2}{z - p_2}$$

Where

$$k_1 = (z - p_1)\frac{H(z)}{z}|_{z=p_1} = \frac{z + 0,4}{z - 0,2}|_{z=0,5} = 3$$
$$k_2 = (z - p_2)\frac{H(z)}{z}|_{z=p_2} = \frac{z + 0,4}{z - 0,5}|_{z=0,2} = -2$$

Thus, the transfer function becomes

$$H(z) = 3\frac{z}{z - 0,5} - 2\frac{z}{z - 0,2}$$

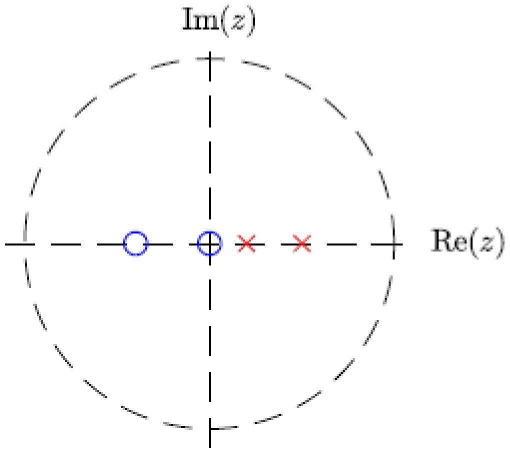
Inverse z-transform

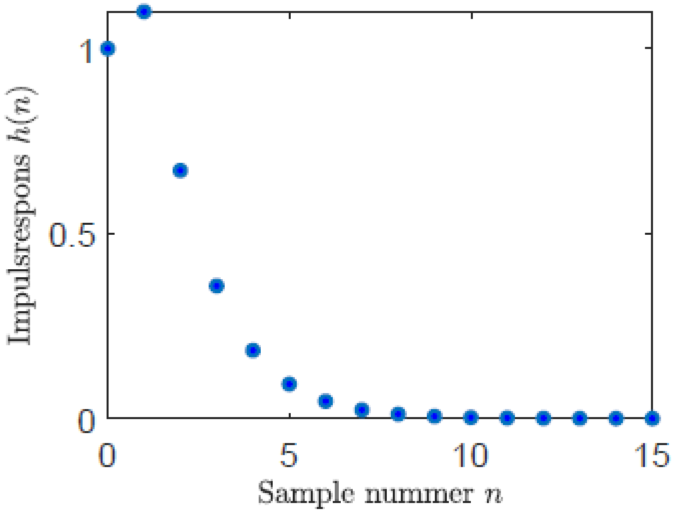
$$h(n) = \mathcal{Z}^{-1}[H(z)] = 3 \cdot 0,5^n - 2 \cdot 0,2^n = 3e^{-0,693n} - 2e^{-1,61n}$$

We can draw it

$$H(z) = \frac{z(z+0,4)}{(z-0,5)(z-0,2)}$$

$$h(n) = \mathcal{Z}^{-1}[H(z)] = 3 \cdot 0,5^n - 2 \cdot 0,2^n = 3e^{-0,693n} - 2e^{-1,61n}$$





Matlab function

‘tf()’ is used to create a transfer function, either for **continuous-time system**, or **discrete-time system**.

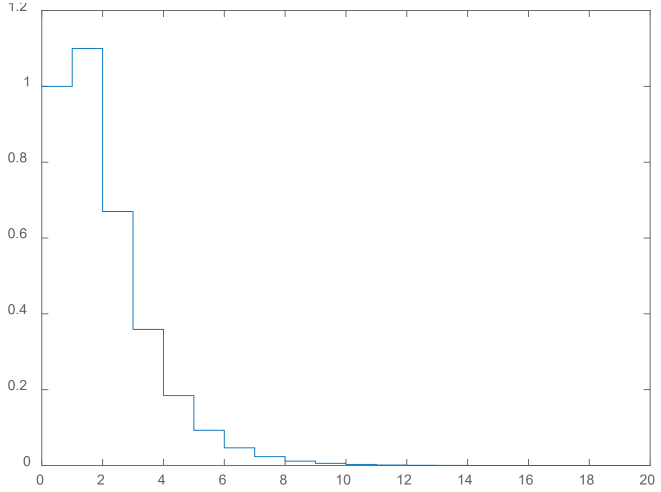
→ **continuous-time system:** sys = tf(numerator,denominator)

→ **discrete-time system:** sys = tf(numerator,denominator, ts)

To leave the sample time unspecified, set ‘ts’ to -1.

Matlab function: impulse()

```
H_z = tf([1, 0.4, 0],[1, -0.7, 0.1], -1)
n=0:20;
impulse(H_z,n)
```



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Let’s take a closer look

A transfer function can be factorized as

$$H(z) = a_0 \frac{(z - z_1)(z - z_2) \dots (z - z_N)}{(z - p_1)(z - p_2) \dots (z - p_N)}$$

where z_i for $i = 1, \dots, N$ are zeros, and p_i for $i = 1, \dots, N$ are poles

Using partial fraction splitting, the transfer function can be written

$$\frac{H(z)}{z} = \frac{k_1}{z - p_1} + \frac{k_2}{z - p_2} + \cdots + \frac{k_N}{z - p_N}$$

where $k_1, k_2, \dots, k_N$ are coefficients

Thus, the impulse response sequence can be written as

$$h(n) = h_1(n) + h_2(n) + \cdots + h_N(n)$$

Given the pole p_i the following relationship:

$$z = p_i = e^{s_i T} = e^{\sigma_i T} e^{j\omega_i T}$$

We have

$$\boxed{h_i(n) = \mathcal{Z}^{-1}\left[\frac{z}{z - p_i}\right] = e^{s_i n T} = e^{\sigma_i n T} e^{j\omega_i n T}}$$

Assuming it has ‘Simple poles’, meaning that all poles have multiplicity of 1.

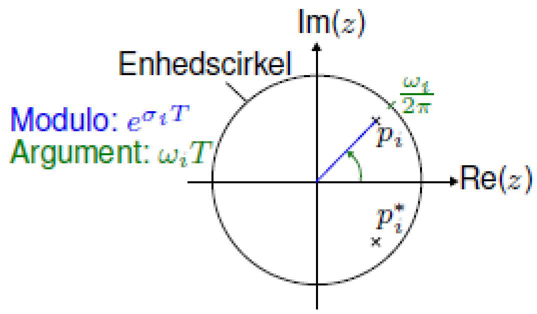
Relation between pole placement and impulse response

Let's consider the following impulse response

$$h_i(n) = e^{\sigma_i nT} e^{j\omega_i nT} = e^{\sigma_i nT} \angle \omega_i nT$$

Euler's formula:
$e^{ix} = \cos(x) + i\sin(x)$

Therefore, the pole p_i can be written in polar form as $p_i = e^{\sigma_i T} \angle \omega_i T$



From the above impulse response sequence $h_i(n)$:

- **Modulus** of $h_i(n)$ changes by a factor $e^{\sigma_i T}$ **between two consecutive samples**.
- **Argument** of $h_i(n)$ is changed by $\omega_i T$ **between two consecutive samples**.

Example: Impulse response of a 2nd order system

Let's consider the following 2nd order system

$$H(z) = \frac{z}{z^2 + 1.697z + 1.44}$$

The transfer function $H(z)$ has 1 zero $z = 0$, and 2 poles $p_1 = 1.2 \cdot e^{j3\pi/4}$ and $p_2 = 1.2 \cdot e^{-j3\pi/4}$

The impulse response for the system is calculated via partial fraction

$$\frac{H(z)}{z} = \frac{k_1}{z - p_1} + \frac{k_2}{z - p_2}$$

where

$$k_1 = (z - p_1)\frac{H(z)}{z}\bigg|_{z = p_1} = \frac{1}{p_1 - p_2} = \frac{1}{1.2(e^{j3\pi/4} - e^{-j3\pi/4})} = \frac{1}{2.4j \sin\left(\frac{3\pi}{4}\right)} = \frac{1}{2.4 \sin\left(\frac{3\pi}{4}\right)} e^{-j\pi/2}$$

k_1 and k_2 are conjugate pair

Euler's formula:
- $e^{ix} = \cos(x) + i\sin(x)$
- $j = e^{j\pi/2}$

Example: Impulse response of a 2nd order system

$x(n)$	$X(z)$
a^n	$\frac{z}{z-a}$

The impulse response can be calculated via inverse z-transformation

$$h(n) = \frac{1}{2.4\sin\left(\frac{3\pi}{4}\right)}\left(e^{-\frac{j\pi}{2}}1.2^{\mathrm{n}}\mathrm{e}^{\mathrm{jn}\frac{3\pi}{4}} + e^{\frac{j\pi}{2}}1.2^{\mathrm{n}}\mathrm{e}^{-\mathrm{jn}\frac{3\pi}{4}}\right)$$

$$= \frac{1}{2.4\sin\left(\frac{3\pi}{4}\right)}1.2^{\mathrm{n}}\left(e^{j\left(\mathrm{n}\frac{3\pi}{4}-\frac{\pi}{2}\right)} + e^{-j\left(\mathrm{n}\frac{3\pi}{4}-\frac{\pi}{2}\right)}\right)$$

Then it can split to

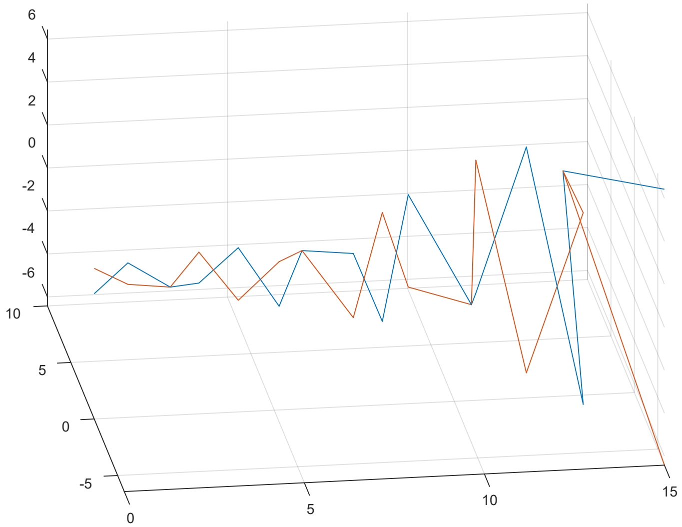
$$h_1(n) = \frac{1}{2.4\sin\left(\frac{3\pi}{4}\right)}\mathrm{e}^{0.1823\mathrm{n}}\mathrm{e}^{\mathrm{j}(\frac{3\pi}{4}n-\frac{\pi}{2})}$$

$$h_2(n) = \frac{1}{2.4\sin\left(\frac{3\pi}{4}\right)}\mathrm{e}^{0.1823\mathrm{n}}\mathrm{e}^{-\mathrm{j}(\frac{3\pi}{4}n-\frac{\pi}{2})}$$

$$\boxed{h_i(n) = e^{\sigma_i nT}e^{j\omega_i nT} = e^{\sigma_i nT}\angle\omega_i nT}$$

Example: Impulse response of a 2nd order system

```
% plot it out
n = 0:15;
h1_n = 1/(2.4*sin(3*pi/4)) * 1.2.^n .* exp(1j*(3*pi/4*n - pi/2));
h2_n = 1/(2.4*sin(3*pi/4)) * 1.2.^n .* exp(-1j*(3*pi/4*n - pi/2));
figure()
plot3(n,real(h1_n),imag(h1_n))
hold on
plot3(n,real(h2_n),imag(h2_n))
grid on
```

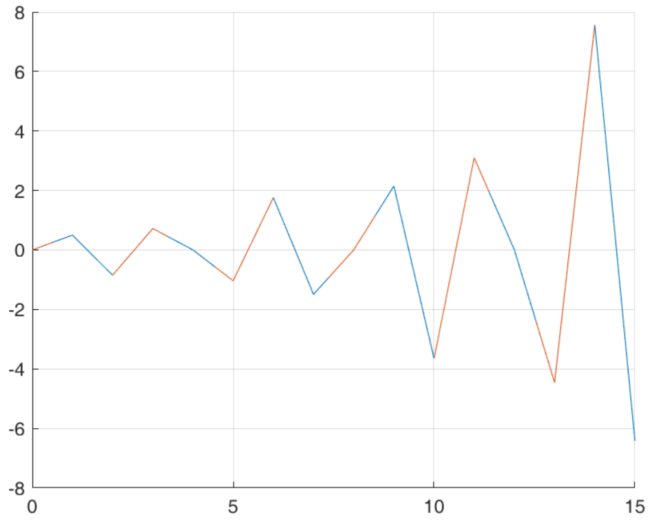


$$h_1(n) = \frac{1}{2.4\sin\left(\frac{3\pi}{4}\right)} \mathrm{e}^{0.1823\mathrm{n}} \mathrm{e}^{\mathrm{j}\left(\frac{3\pi}{4}n - \frac{\pi}{2}\right)}$$

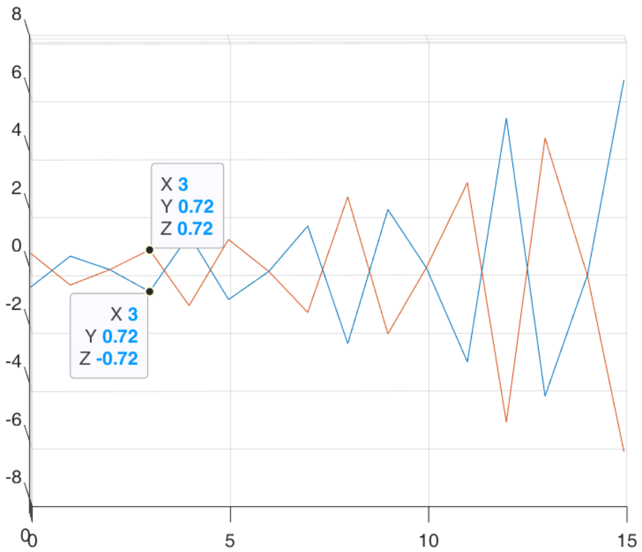
$$h_2(n) = \frac{1}{2.4\sin\left(\frac{3\pi}{4}\right)} \mathrm{e}^{0.1823\mathrm{n}} \mathrm{e}^{-\mathrm{j}\left(\frac{3\pi}{4}n - \frac{\pi}{2}\right)}$$

$$h_i(n) = e^{\sigma_i nT} e^{j\omega_i nT} = e^{\sigma_i nT} \angle \omega_i nT$$

Rotate view to real



Rotate view to imaginary



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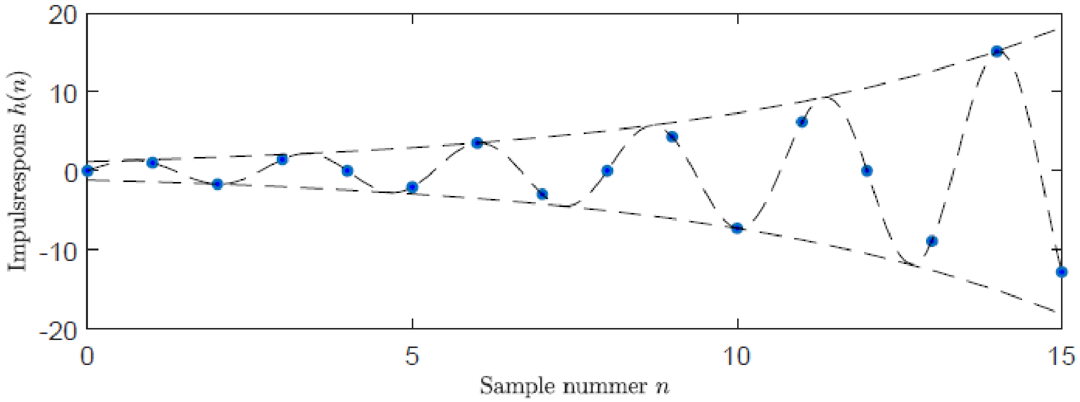
Example: Impulse response of a 2nd order system




```
n = 1:15;
H_n = h1_n + h2_n;
plot(h)
```

The output $h(n)$ keeps increasing with n.
This system is not stable!

Stability

A system's stability state can be one of the following

→ **Stable system**: A system is stable if its impulse response $h(n)$ goes to zero as n goes to infinity

$$|h(n)| \to 0 \qquad for \quad n \to \infty$$

→ **Marginally stable system**: A system is marginally stable if its impulse response $h(n)$ goes towards constant value different from zero or oscillates with constant amplitude and frequency as n goes towards infinity.

→ **Unstable system**: A system is unstable if its impulse response $h(n)$ grows without limit as n goes to infinity

$$|h(n)| \to \infty \qquad for \quad n \to \infty$$

Determination of stability

Transfer function $H(z)$ has poles $p_1, p_2, \ldots, p_N$.

→ The system is **stable** if all poles lie within the unit circle, i.e.

$$|p_i| < 1 \quad for \quad i = 1, 2, \ldots, N$$

→ The system is **marginally stable** if at least one pole (e.g. p_j) lies on the unit circle, while the other poles lie within the unit circle, i.e.

$$|p_i| \leq 1 \quad for \quad i = 1, 2, \ldots, N$$

or

$$|p_j| = 1 \quad for \quad j \in \{1, 2, \ldots, N\}$$

→ The system is unstable if a pole (e.g. p_j) lies outside the unit circle, i.e.

$$|p_j| > 1 \quad for \quad j \in \{1, 2, \ldots, N\}$$

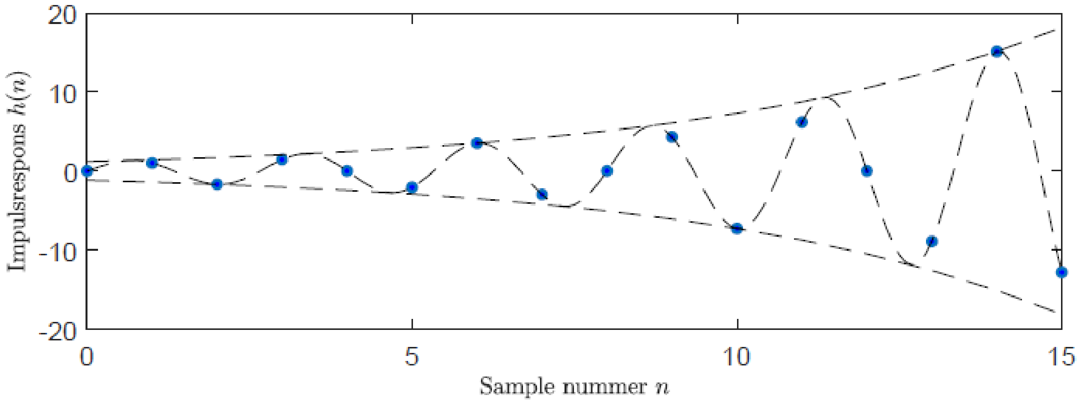
The previous example

The transfer function is $H(z) = \frac{z}{z^2+1.697z+1.44}$

Its poles are

$p_1 = 1.2 \cdot e^{j3\pi/4}$ and $p_2 = 1.2 \cdot e^{-j3\pi/4}$

And $|\boldsymbol{p}| > \mathbf{1}$ **applies**!



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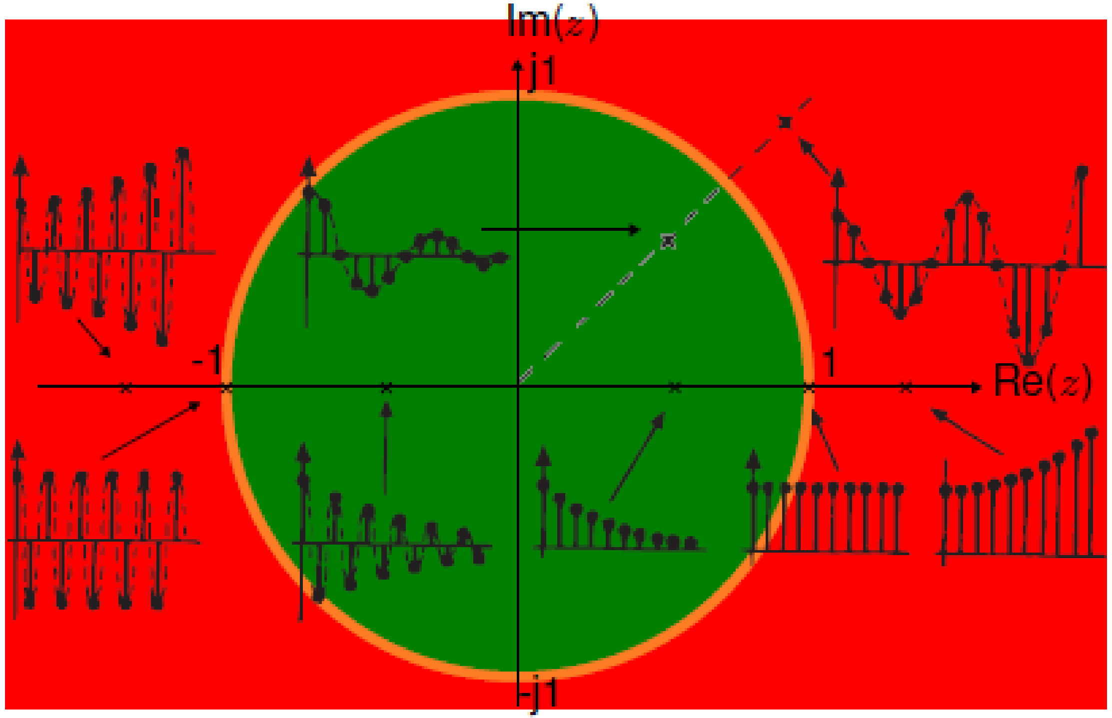
Impulse response in z-plane related to pole position

pole $p_i = e^{\sigma_i T} e^{j\omega_i T}$



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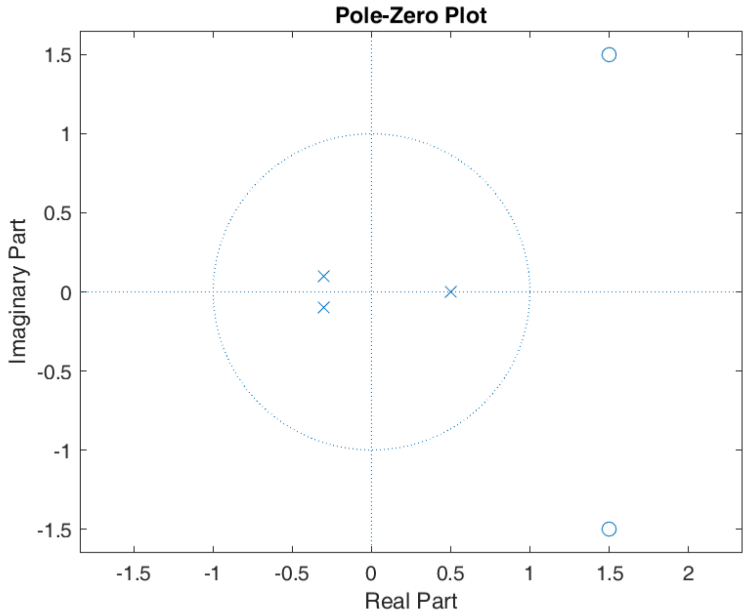
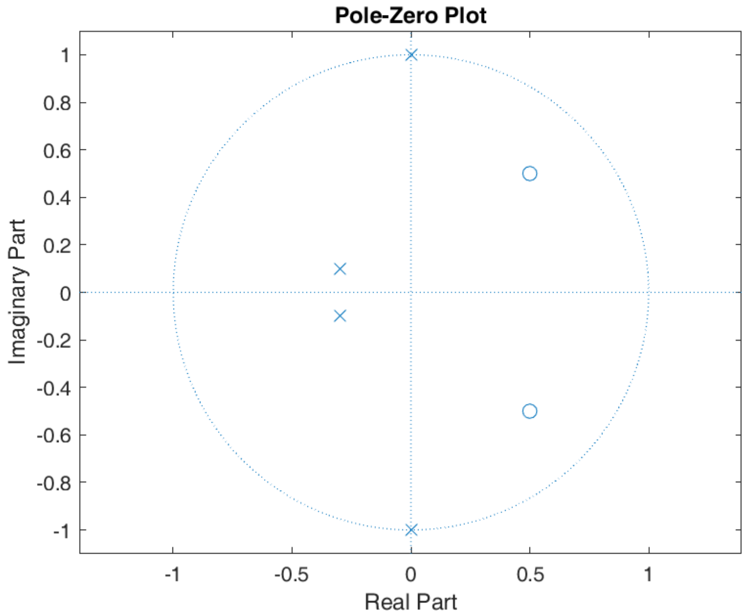
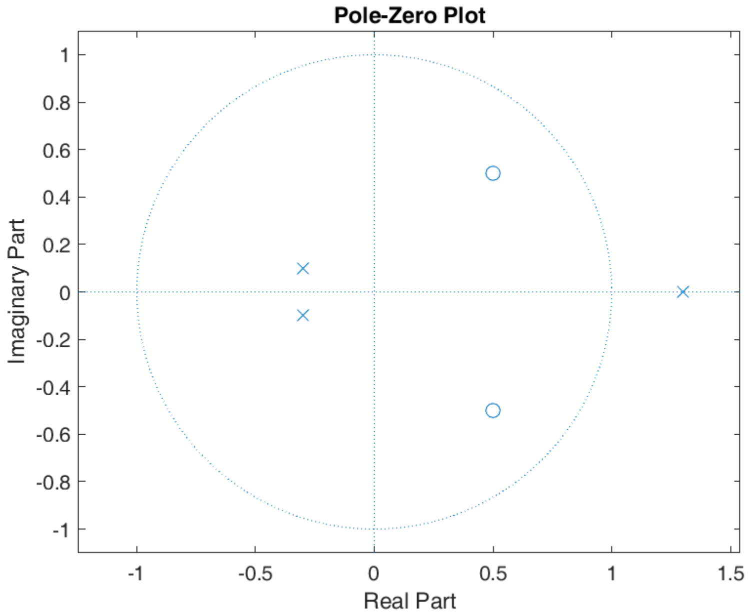
Are these systems stable?







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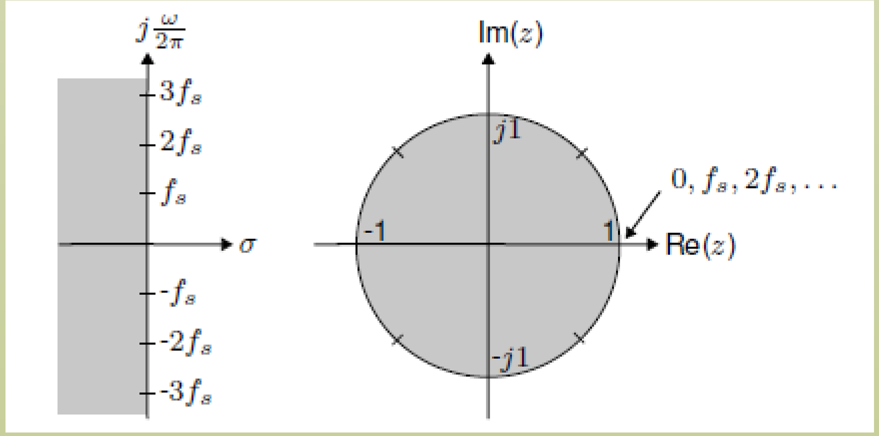
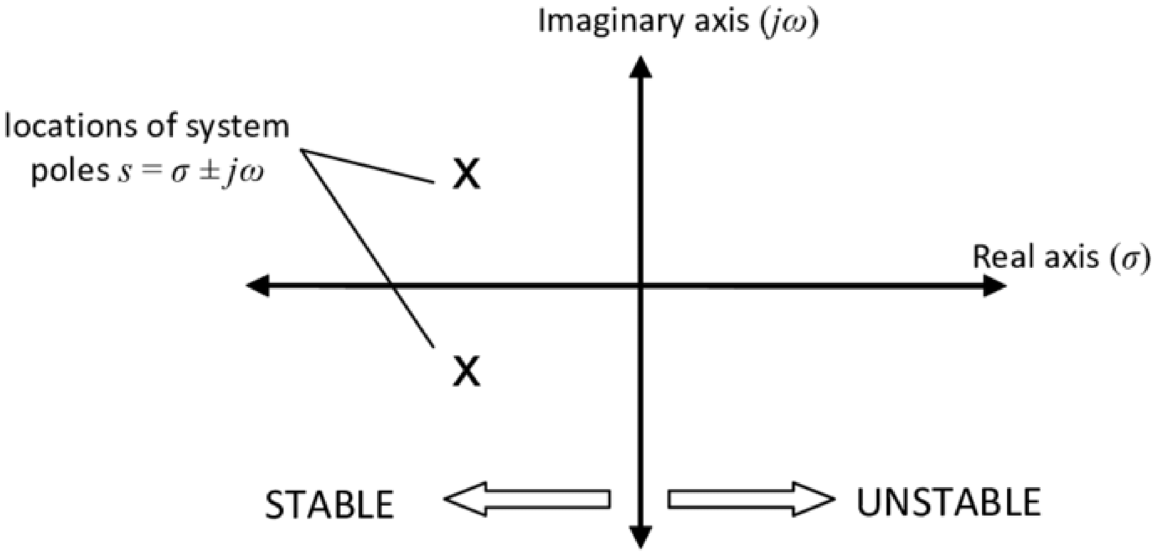
What about in continuous time system (s-domain)?

Given the following projection relationship:





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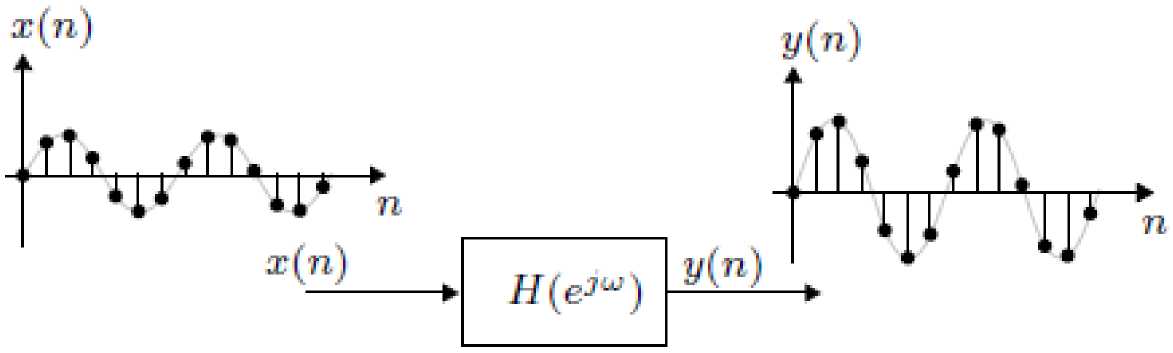
Frequence response analysis



A frequency response analysis provides the response of a system to a sinusoidal input sequence. Here it is assumed that the sinusoidal sequence has been imprinted from time $-\infty$ (the analysis disregards transient response).

Frequence response analysis

A system's frequency response is the response (output signal) when a **sinusoidal input** is applied to a system.

The output signal of a system with input signal $A\cos(\omega t)$ can be determined as

$$y(t) = H(j\omega)\, x(t) = H(j\omega)\, A\cos(\omega t)$$

Given Euler equation:

$$A\cos(\omega t) = \frac{A}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

The response can be written as

$$y(t) = \frac{A}{2}\left(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\right)$$

Frequence response analysis

$$y(t) = \frac{A}{2}\left(H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\right)$$

In polar form, $H(j\omega) = M(\omega)e^{j\varphi(\omega)}$, so

$$y(t) = \frac{A}{2}M(\omega)\left(e^{j\left(\omega t+\varphi(\omega)\right)} + e^{-\left(j\omega t+\varphi(\omega)\right)}\right) = AM(\omega)\cos(\omega t + \varphi(\omega))$$

where

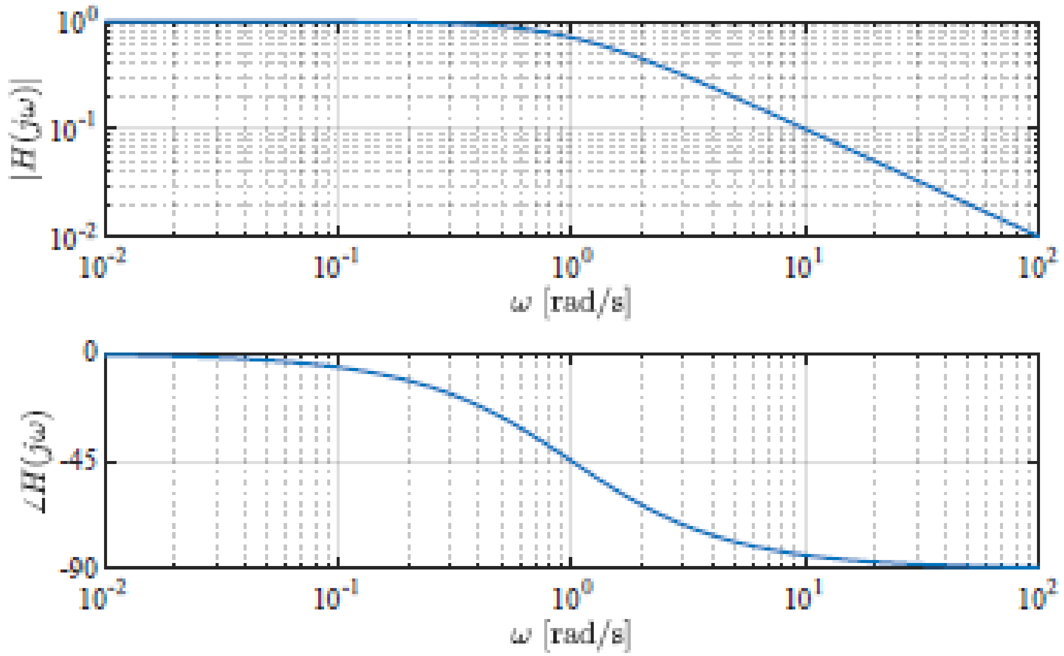
$$M(\omega) = |H(j\omega)| \quad \text{and} \quad \varphi(\omega) = \angle H(j\omega)$$

Bode plot

A **Bode plot** is used to visualize the frequency response, and is usually drawn on a logarithmic scale.

$$y(t) = AM(\omega)\cos(\omega t + \varphi(\omega))$$

where
$\boldsymbol{M(\omega)} = |H(j\omega)|$ and $\boldsymbol{\varphi(\omega)} = \angle H(j\omega)$



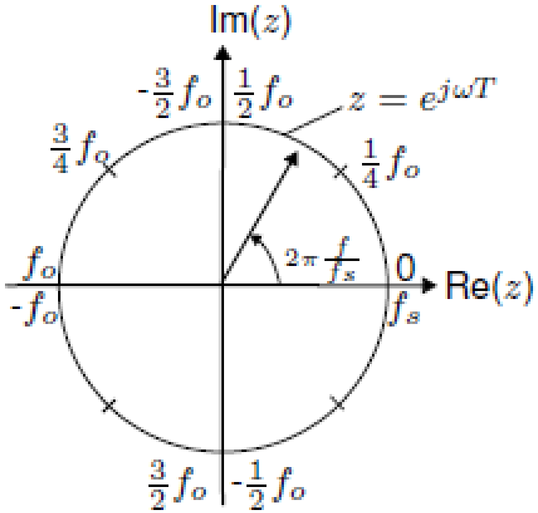
Hand calculation of frequency response

To study the frequency response of a system, we only look at values of z that lie on the unit circle, i.e.

$$z = e^{j\omega T}$$

The frequency response of a time-discrete system $H(z)$ is thus given as

$$H(z)\Big|_{z=e^{j\omega T}} = H\left(e^{j\omega T}\right) = \frac{Y(e^{j\omega T})}{X(e^{j\omega T})}$$



In polar form, the frequency response is

$$H(j\omega) = |H(\omega)|\,\angle\varphi(\omega)$$

The **amplitude** is normally given in dB, i.e.

$$|H(\omega)| = 20\log\frac{|Y(j\omega)|}{|X(j\omega)|} \qquad \text{[dB]}$$

Ultra cautious!
The ‘log()’ function in Matlab means $y = \ln(x)$.
You should use the ‘log10()’ function

Example

Let’s consider the following transfer function

$$H(z) = \frac{z^2 + 0.4z}{z^2 - 0.7z + 0.1}$$

To determine the frequency response, we calculate $H(j\omega)$, i.e.

$$H(z)\Big|_{z\,=\,e^{j\omega T}} = H\left(e^{j\omega T}\right) = \frac{e^{j2\omega T} + 0.4e^{j\omega T}}{e^{j2\omega T} - 0.7e^{j\omega T} + 0.1}$$

The frequency response of input signal $x(n) = \sin(\omega nT)$, and $\omega T = 1$

$$H(j\omega) = \frac{e^{j2} + 0.4e^{j}}{e^{j2} - 0.7e^{j} + 0.1} = 0.92 - j1.37 = 1.65\angle - 56°$$

Use Matlab for calculation:
(exp(2j) + 0.4*exp(1j))/(exp(2j)-0.7*exp(1j)+0.1)

Example – through z transform

Let’s consider the following transfer function:

$$H(z) = \frac{z^2 + 0.4z}{z^2 - 0.7z + 0.1}$$

The z-transform of the input signal $x(n) = \sin(\omega nT)$ is

$$X(z) = \frac{\sin(\omega T)\, z}{z^2 - 2\cos(\omega T)\, z + 1}$$

ZT6	$\sin \omega_0 nT$	$\frac{(\sin \omega_0 T)z}{z^2 - 2(\cos \omega_0 T)z + 1}$

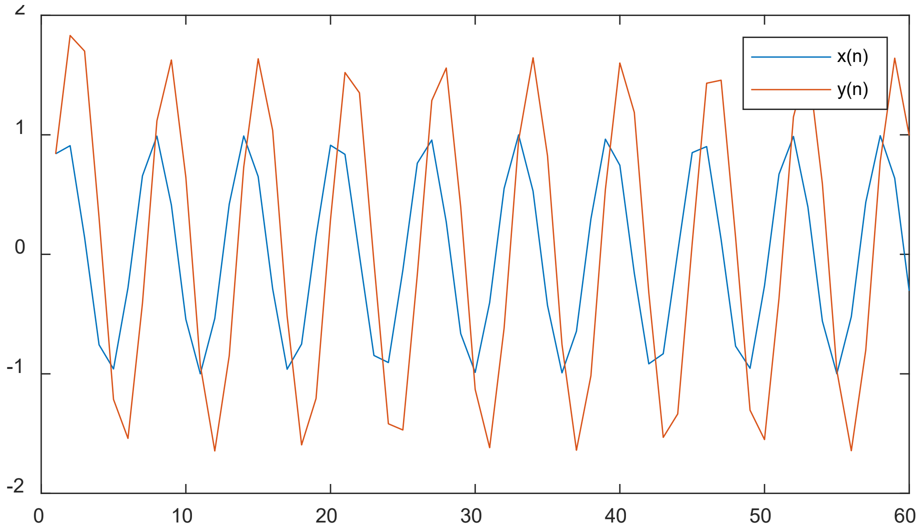
We can calculate $Y(z)$, and let’s study an example frequency response $\boldsymbol{\omega T} = \mathbf{1}$

$$Y(z) = H(z)X(z) = \frac{z^2 + 0.4z}{z^2 - 0.7z + 0.1} \cdot \frac{0.84z}{z^2 - 1.08z + 1}$$

Implement inverse z-transform, and we have $y(n)$. Then we can plot the input and output.

Example

Let's compare the input and output signal $x(n) = \sin(n)$



$$H(j\omega) = 1.65\angle -56°$$

```
syms z
Y = 0.84*z*z*(z+0.4)/((z^2-
0.7*z+0.1)*(z^2-1.08*z+1));
y= iztrans(Y);

syms n
y_res = subs(y, n, {1:60});

n=1:60;
x=sin(n);

plot(n,x);
hold on
plot(n, y_res);
```

Matlab function: subs()
Symbolic substitution. Check this function by yourself.

Matlab function – bode()

Function 'bode()' is used to **obtain results** and **visualize** the frequency response of the system, either **continuous time system** or **discrete time system**.

Important trick!!

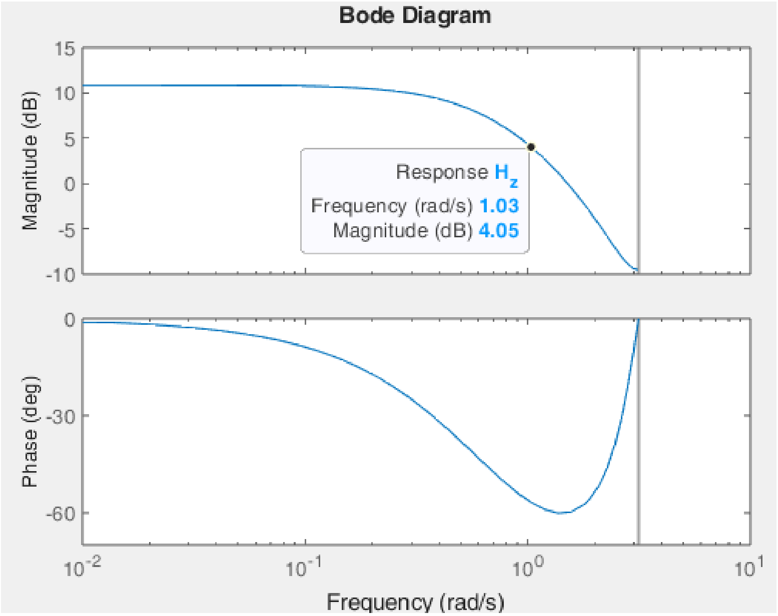
The 'bode()' function by default plots in 'rad/s'. Add the following commands to plot in 'Hz':

```
options.FreqUnits = 'Hz';
body(H_z, options);
```

The transfer function in the exmaple: $H(z) = \frac{z^2+0.4z}{z^2-0.7z+0.1}$

```
%% Matlab function
H_z = tf([1, 0.4, 0], [1, -0.7, 0.1], -1)
bode(H_z)
```

Be aware that we are checking input frequency $\omega T = 1$



The result just now:

$$H(j\omega) = 1.65\angle - 56°$$

And

$$20\log(1.65) = 4.3497$$

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Question?

What about $\omega T = 10$?

Graphical determination of frequency response

A transfer function can be factorized as

$$H(z) = a_0 \frac{(z - z_1)(z - z_2) \cdots (z - z_N)}{(z - p_1)(z - p_2) \cdots (z - p_N)}$$

where z_i for $i = 1, \ldots, N$ are the zeros, and p_i for $i = 1, \ldots, N$ are the poles.

The output amplitude of $H(z)$ depends on all these partial factors.

$$|H(z)| = a_0 \frac{|z - z_1||z - z_2| \cdots |z - z_N|}{|z - p_1||z - p_2| \cdots |z - p_N|}$$

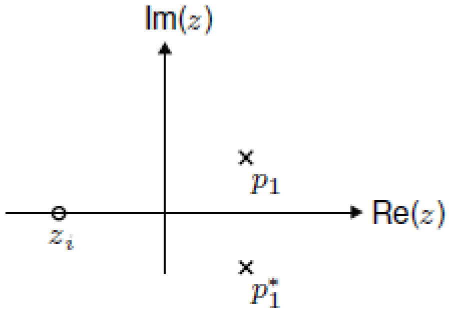
Graphical determination: Amplitude

Let’s consider the following transfer function

$$H(z) = a_0 \frac{(z - z_1)}{(z - p_1)(z - p_1^*)}$$

Whose amplitude is

$$|H(z)| = a_0 \frac{|z - z_1|}{|z - p_1||z - p_1^*|}$$



Here in the z plane
is the system

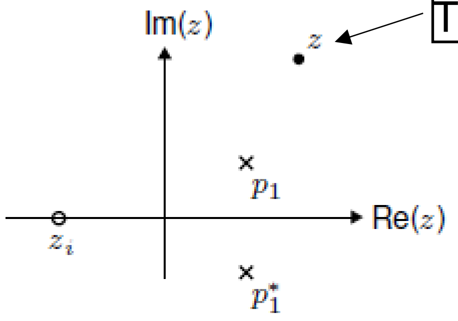
Graphical determination: Amplitude

Let's consider the following transfer function

$$H(z) = a_0 \frac{(z - z_1)}{(z - p_1)(z - p_1^*)}$$

Whose amplitude is

$$|H(z)| = a_0 \frac{|z - z_1|}{|z - p_1||z - p_1^*|}$$



This 'z' represents the input signal

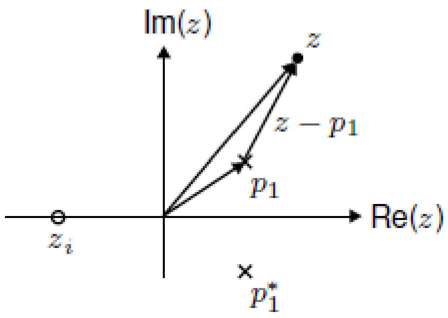
Graphical determination: Amplitude

Let’s consider the following transfer function

$$H(z) = a_0 \frac{(z - z_1)}{(z - p_1)(z - p_1^*)}$$

Whose amplitude is

$$|H(z)| = a_0 \frac{|z - z_1|}{|z - p_1||z - p_1^*|}$$



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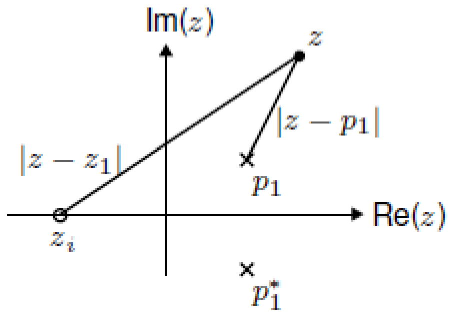


Graphical determination of frequency response

A transfer function can be factored as

$$H(z) = a_0 \frac{(z-z_1)(z-z_2)\cdots(z-z_N)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

where z_i for $i = 1, \ldots, N$ are the zeros, and p_i for $i = 1, \ldots, N$ are the poles.

The output phase of $H(z)$ also depends on all the partial factors:

$$\angle H(z) = \psi_1 + \psi_2 + \cdots + \psi_N - (\theta_1 + \theta_2 + \cdots + \theta_N)$$

Where $\psi_i = \angle(z - z_i)$ and $\theta_i = \angle(z - p_i)$

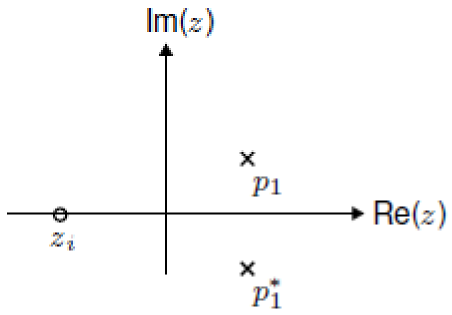
Graphical determination: Phase

Let's consider the transfer function

$$H(z) = a_0 \frac{(z - z_1)}{(z - p_1)(z - p_1^*)}$$

Whose phase

$$\angle H(z) = \psi_1 - \theta_1 - \theta_2$$



Here in the z plane
is the system

Graphical determination: Phase

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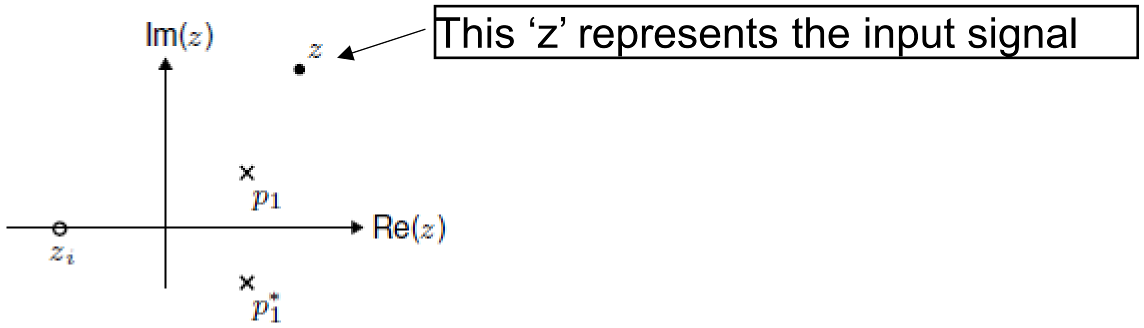

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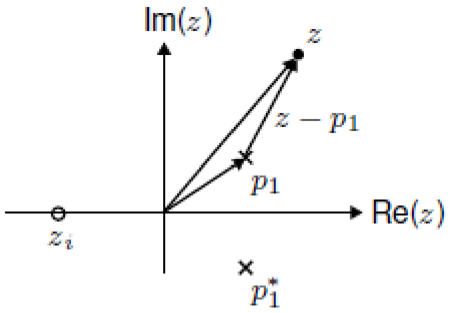

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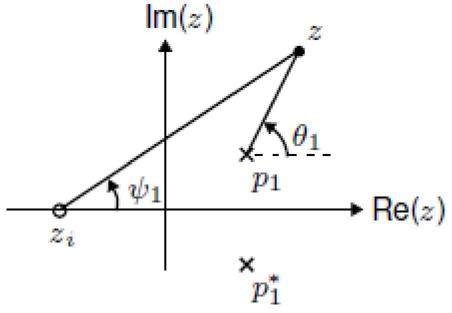
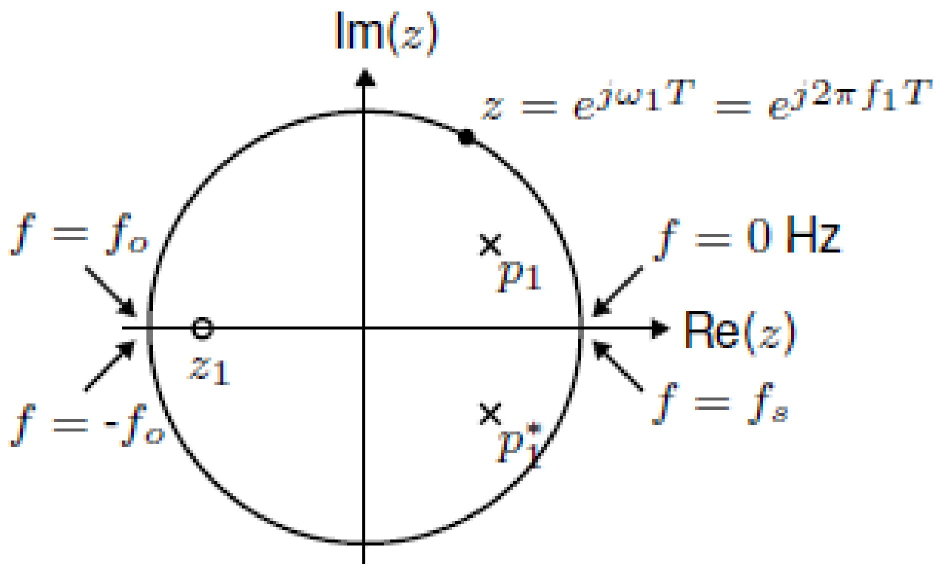


Graphical determination of frequency response

The frequency response is found by finding the amplitude and phase of $H(z)$ lies on the **unit circle, i.e.** $\boldsymbol{z = e^{j\omega T}}$



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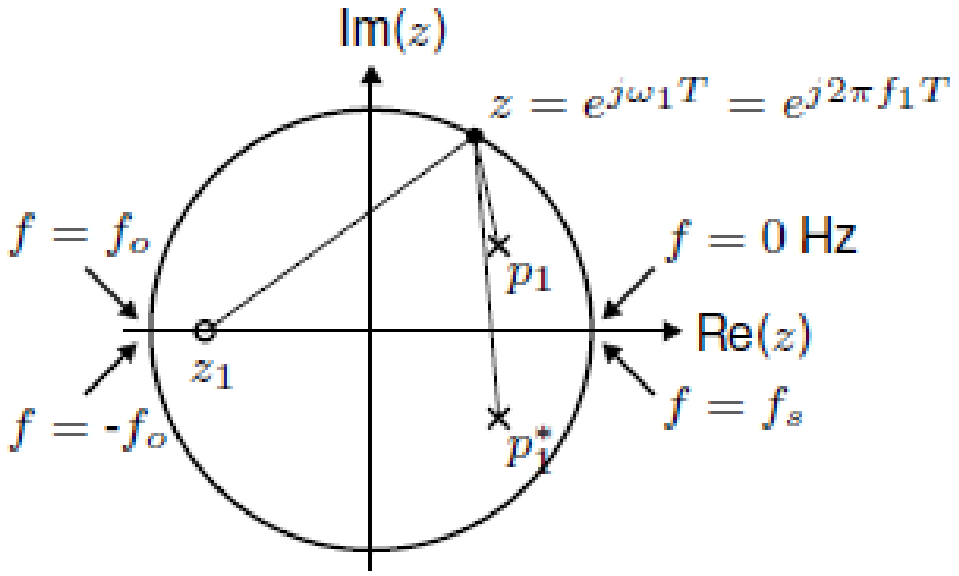

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$$|H(z)| = a_0 \frac{|z - z_1||z - z_2| \cdots |z - z_N|}{|z - p_1||z - p_2| \cdots |z - p_N|}$$

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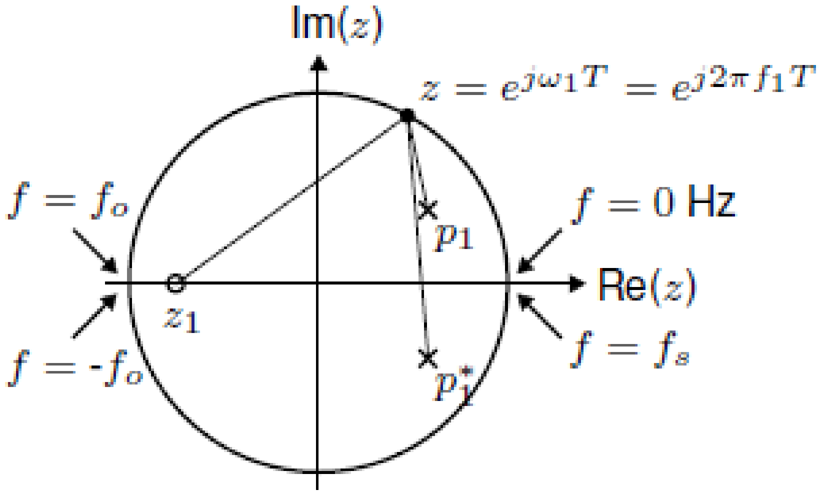
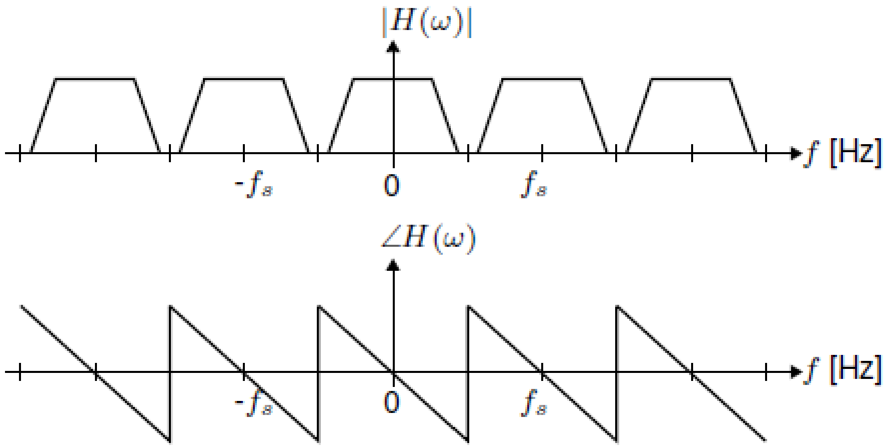
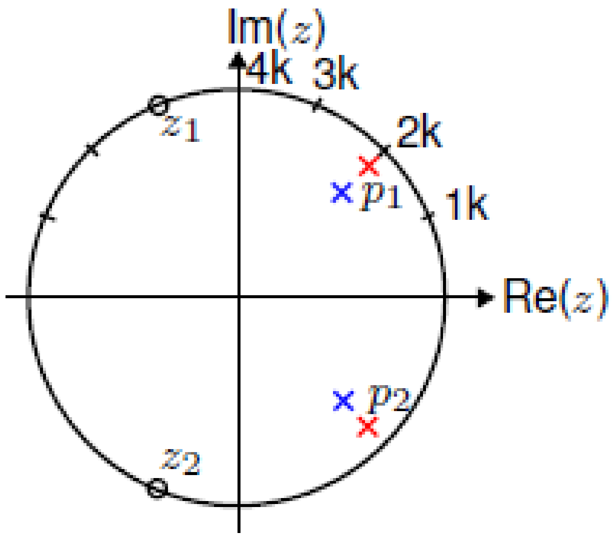
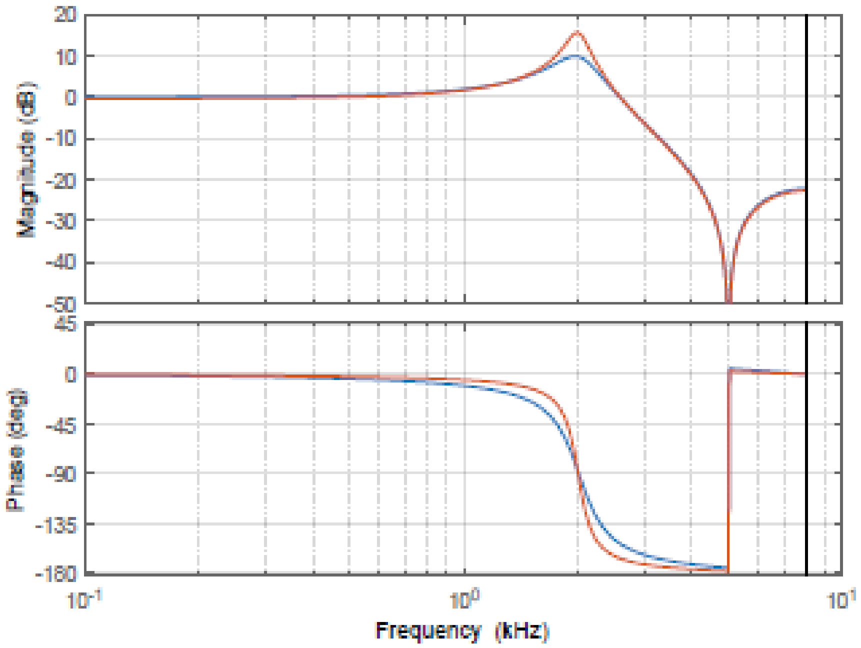




Example 1

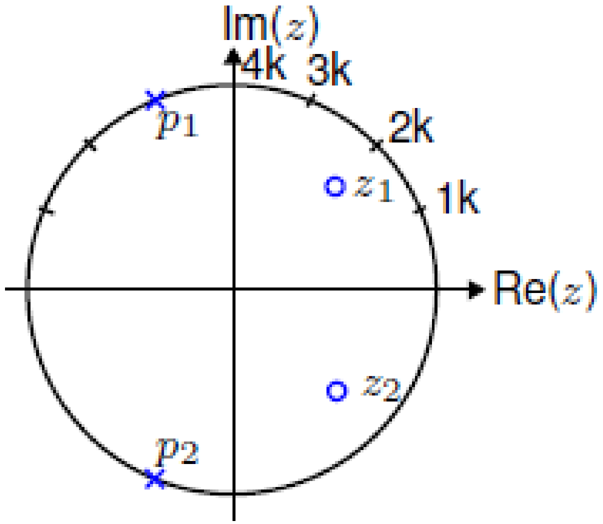
What does the frequency response look like for a transfer function with the following pole-zero diagram?

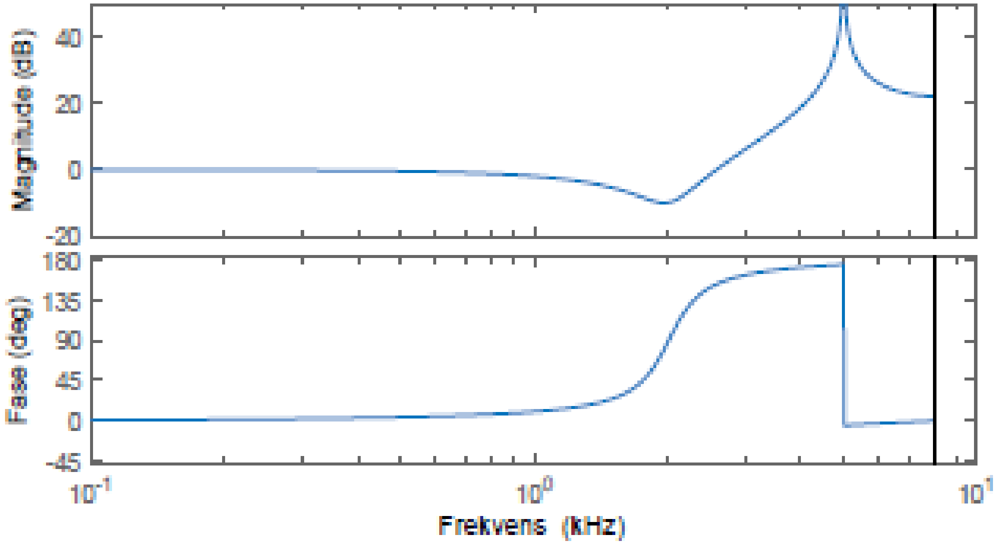




Example 2

What does the frequency response look like for a transfer function with the following pole-zero diagram?





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Sample rate

The importance of the sample frequency for the frequency response is only to move the graph along the frequency axis.

$$\boxed{z = e^{j\omega T}}$$