

Structural Properties of Decomposable Digraphs

by

Gabriella Juhl Jensen

supervised by

Prof. Jørgen Bang-Jensen



UNIVERSITY OF SOUTHERN DENMARK

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

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Part I

Primitive graphs and gabbigubbi

0.1 Introduction

Why we need graphs.

Chapter 1

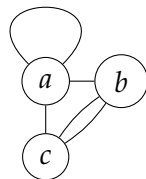
Intro

1.1 Graphs and Digraphs

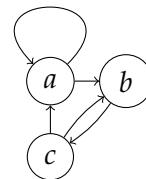
Before going deep into structural properties of decomposable digraphs we first need to establish what a graph is. For some graph $G(V, E)$ where V and E are two sets containing the **vertices** (also commonly called nodes) and **edges** of the graph respectively. We define the **size** of the graph to be the number of vertices $|V|$ this is also known as **cardinality** of V . An **edge** $e \in E$ where $e \equiv (a, b)$ and $\{a, b\} \subseteq V$ then e is an edge in G , e is said to be **incident** to a and b . We call $a, b \in V$ **adjacent** if there is an edge (a, b) or (b, a) (two given vertices connected by an edge is said to be adjacent). If an edge goes from and to the same vertex (a, a) it is called a **loop**. The set of edges e_1, \dots, e_k is usually describe with the letter E where each edge contains a pair of vertices that are adjacent.

In a graph we have something called a **walk** which is a alternately ordering of vertices and edges in the graph G where the edge in between the two vertices in the ordering is an edge between the vertices in G (for (a, e_1, b) to be a walk the edge e_1 has to be between a and b). We call a walk closed if the first vertex in the walk is the same as the last.

Every vertex $v \in V$ of $G(V, E)$ have a **degree** denoted $d(v)$ which is the number of incident edges to v . A **path** in a graph is a walk where each vertex in the ordering can only appear one time. A cycle is a closed walk where the only vertex present more than one time is the first vertex (for the walk to be closed the first vertex has to appear last to also called a closed path). Let X be a subset of the vertices $X \subseteq V$ then we say that $V \setminus X$ is the set of vertices without the vertices in X , i.e. $V \setminus X \equiv V - X$. A subgraph H of G can contain any of the vertices and the arcs connected to the chosen vertices in H . you can not have an edge connecting no vertices in H but you do not have to choose all the arcs in G between the chosen vertices in H for H being a subgraph.



(a) graph $G(V, E)$ is an example of a graphs, the red edge is a loop, and all pair of vertices in this graph is adjacent.



(b) This is an orientation of the edges in the graph which makes this a digraph

Before delving more specific into graphs and digraphs we must establish some important prerequisite and properties. A graph is called **simple** if there is no loops and no multiple edges. With multiple edges it means multiple edges between the same pair of vertices like in Figure 1.1a between b and c .

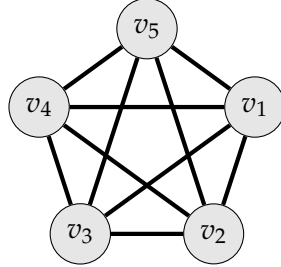


Figure 1.2: Complete graph with 5 vertices.

A graph is **connected** if there exists a path between all pair of vertices in the graph and **disconnected** otherwise. A graph is called **complete** if there for all pair of vertices in the graph is an edge between them see Figure 1.2.

If we instead of edges have **arcs** between the vertices we call it a **digraph**. An arc is describe just like an egde with two adjacent vertices (a, b) the first vertex mentioned in an arc is the vertex **from** where the arc starts also called the **tail**, the second vertex is where the arc is pointing **to** also called **head**. The set of arcs is normally denoted A like the set of edges is denoted E . So the arc (a, b) goes from a to b , if you wanted it the other way around the arc is (b, a) . These graph containig only arcs and no edges is called a digraph $G(V, A)$ which is what we in this project are focusing on see Figure 1.1b.

For two vertices x and y in $D(V, A)$ then if we have an arc from x to y we say that x **dominates** y this is denoted like this $x \rightarrow y$. If we talk about subgraphs A and B , then A **dominates** B if for all $a \in A$ and $b \in B$, $a \rightarrow b$. If there is no arcs from B to A we denote it $A \mapsto B$ and if both $A \rightarrow B$ and $A \mapsto B$ we say that A **completely dominates** B and this is denoted $A \Rightarrow B$.

In a digraph we have something called the **underlying graph**. An underlying graph of a digraph is where all arcs are replaced by edges (edge is used every time we talk about undirected edges between vertices, when using directions it is called an arc). Let $X \subseteq V$ Then we can make the subdigraph $D \langle X \rangle$ which is the subgraph D induced by the set X meaning that all the vertices is from X and the arcs is from $A \in G$ but where both head and tail is incedent to the vertices in X . We will denote the graph $D \langle V \setminus X \rangle$ for some $X \subseteq V$ as $D - X$. A digraph is **connected** if the underlying graph is connected, (also called weakly connected), a digraph can be **strongly connected** and **semi connected** too. A digraph is called **semi connected** if there for each pair u and v exists a path from either u to v or v . It is said to be **strongly connected** if for each pair of vertices u and v there exists a path from both u to v and v to u . A strongly connected digraph is also called a **strong** digraph. A strong digraph have a subset S called a **seperator** if $D - S$ is not strong, we also say that S **seperates** D . A seperator S is called **minimal seperator** of D if there exists no proper subset $X \subset S$ that separates D . Now we can introduce a **k -strong** digraph D which is a strong digraph with $|V| \geq k$ and a minimal seperator S on $|S| = k$.

In a digraph $D(V, A)$ we mostly use the **degree** as two different degrees namely **out degree**, $d^+(v)$, and **in degree**, $d^-(v)$, that is the arcs from v and to v respectively. In a

digraph D we can talk about the over all **minimum out degree**, $\delta^+(D) = \min\{d^+(v)|v \in V\}$ and **minimum in degree**, $\delta^-(D) = \min\{d^-(v)|v \in V\}$ sometime we is going to need the minimum of these to $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ and called the **minimum degree**. For every vertex v the vertices that is **adjacent** with v we denote $N^+(v)$ and $N^-(v)$ as the set of vertices that is dominated by (**out-nieghbours**) v and dominates (**in-nieghbours**) v , respectively. This means that $d^+(v) = |N^+(v)|$ and $d^-(v) = |N^-(v)|$.

For simplicity when mentioning paths and cycles in digraphs it will be **directed** paths and cycles if not anything else is mensioned. By **directed** means that we go from tail to head on every arc on the path or cycle. When mentioning paths in a digraph it sometimes makes more sence specifying the head an tail of the path, so a path from s to t is denoted (s, t) -**path**. In some digraphs there is more than one path between the same two vertices these paths can use the same arcs or same vertices or be totally distinct from eachother, the maximum number of disjoint path between two vertices in a digraph is denoted $\lambda_D(s, t)$

1.2 somthing else

The reason for grouping the digraphs into smaller collections of digraphs (like tournaments is a smaller collection of semicomplete digraphs) is because of problems that is hard to solve on general graph but is easy/polynomial solvable on specific graphs.

A group of these problems is called NP-hard problems which sometimes sound easy solvable for graphs but only for some specific graphs we know how to solve it in polynomial time.

Definition 1.2.1. *define NP-hard problems*

In this paper we focusing on the specific digraphs that are **decomposable**. A **decomposable** digraph is a digraph $D = H[G_1, G_2, \dots, G_H]$ where each G_i is sconnected graphs replacing each vertex of the digraph H

1.3 Computational complexities

1.4 Classes of Digraphs

We can use specific collection of graphs to solve various problems faster or easier. Like finding paths in digraphs or cycles or more specific things, but in general des more we know about a digraph we can use to solve hard problems which in general would be time consuming like the problems that are NP-hard. By some quick fast algorithm you can checks wheter a digraph belongs to a certion **class** of digraphs. A class of digraph is a collection of digraph with certain properties incommen like **tournaments**.

1.4.1 introduction to some digraph classes

Tournaments is a digraph where the underlying graph is complete. So a complete graph of order 5 any orientation of the edges concludes in a tournament. Strong digraphs is also

in it self a classification of digraphs. Classes of digraphs can be overlapping each other or be fully contain in each other like tournaments is fully contain in the class called semicomplete digraph. A **semicomplete** digraph is where the underlying graph is complete multigraph, there can be some multiple edges in between the same pair of vertices in the underlying graph. Since the class called semicomplete digraphs contains all digraphs where the underlying graph is a complete multigraph it clearly also contains the graph with only one arc between every pair of vertices (Tournaments). A **complete** digraph is where every pair of vertices $a, b \in V$ the arc (a, b) and (b, a) is present in the graph.

If you can split the graph into two sets of vertices A and B such that $A \cup B = V$ and there is no arcs inside these sets, then we classify this as an **bipartite** digraph. This means all arcs in the graph is in the form (a, b) or (b, a) for all $a \in A$ and $b \in B$. The sets A and B are called the partite sets of $D(V, A)$. The underlying graph of a bipartite digraph is also called bipartite since there is no edges inside A or B . If there exists more than two of these partite sets we call the digraphs **multipartite**, since there is multiple partite sets in the graph, bipartite sets \subset multipartite.

A much used type of digraph is an **acyclic** digraph. It is a digraph where for an specific ordering of the vertices $V = v_1, v_2, \dots, v_n$ the arcs in the digraph is (v_i, v_j) where $i < j$ for all $(v_i, v_j) \in A$. This ordering is called an **acyclic ordering** and can also be used to order strong components in a non-strong digraph such that the ordering of the component C_1, C_2, \dots, C_k is an acyclic digraph when contracting the components into k vertices. When classifying digraphs there is several ways of doing this, like **transitive** digraphs which are digraphs where for all vertices $a, b, c \in V$ where the arc (a, b) and b, c is present in the digraph ($\in A$), the arc (a, c) has to be a part of A too. using the same kind of classification there is digraphs which are **Quasi-transitive** which is for all vertices $a, b, c \in V$ where the arc (a, b) and b, c is present in the digraph ($\in A$), a and c has to be adjacent by at least one (more arcs in between are also allowed) arc in either direction ((a, c) or (c, a)). These graphs are going to be mentioned a lot in this thesis since the graph is also what we call **decomposable**.

Decomposable digraphs is also a classification of graphs which are decomposable, for a graph D to be decomposable we have H_1, H_2, \dots, H_k **houses** and S where $V(S) = s_1, s_2, \dots, s_k$ which are all digraphs by them self but if each s_i is replaced by the digraph H_i $i = 1, 2, \dots, k$ we have the graph D , where $H_i \rightarrow H_j \in D$ if $s_i \rightarrow s_j \in S$ denote this decomposition like $D = S[H_1, H_2, \dots, H_k]$. This is the class of digraphs we are focusing on in this thesis. If all the houses are independent sets we call $D = S[H_1, H_2, \dots, H_k]$ the extension of S . If S is a semicomplete digraph we call the extension of these **extended semicomplete** digraph. Like we already mentioned Quasi-transitive digraphs are decomposable but we have several classes that are decomposable, and another class of digraphs that is going to be used a lot in this is **locally semicomplete** digraphs.

First we are going to introduce **in-locally semicomplete** digraphs and **out-locally semicomplete** digraphs which is for every in-neighbor of a vertex $x \in V$ they have to be adjacent ($x \cup N^-(x)$ induces a semicomplete digraph) is the in-locally semicomplete digraph if it is true for all $x \in V$. Respectively it is called an out-locally semicomplete digraph if $\forall x \in V$ the out-neighbors, $N^+(x)$, has to be adjacent. If a digraph is both in-locally semicomplete and out-locally semicomplete, it is called a **locally semicomplete** digraph. Why both Quasi transitive digraphs and some locally semicomplete digraphs are decomposable will be described in section ??.

The last class of digraph that are important for this thesis is the round digraphs. A digraph is called a **round** digraph if there exists an ordering of the vertices v_1, v_2, \dots, v_n such that for all v_i , $N^+(v_i) = v_{i+1}, v_{i+2}, \dots, v_{i+d^+(v_i)}$ and $N^-(v_i) = v_{i-d^-(v_i)}, v_{i-d^-(v_i)-1}, \dots, v_{i-1}$.

1.4.2 Semicomplete Digraphs

1.4.3 muligvis Transitive Digraphs

1.4.4 Strong Digraphs

1.4.5 Round Digraphs

Chapter 2

Decomposable Digraphs

Decomposable digraphs is what we in this thises is focusing on. We have introduced short what a decomposable digraph is but there is subclasses to focus on and a lot of other crucial definitions and theroems to cover about these digraphs before delving into the NP-hard problems.

2.1 Genral about Decomposable digraphs

Recall that a decomposable digraph D can be decomposed into a main graph S where $|S| = k$ and k houses H_1, H_2, \dots, H_k , where each vertex in $S = \{v_1, v_2, \dots, v_k\}$ is replaced by the house H_i replace v_i and the arcs between the houses is as follows $H_i \rightarrow H_j$ in D if $v_i \rightarrow v_j$ in S remember that for a set X to dominate an other set Y (meaning every vertex in the dominating set dominates every vertex in the dominated set) we denoted it $X \rightarrow Y$. If no arc between v_a and v_b in S then there is no arc between the sets H_a and H_b in D . The thing about decomposable digraphs is that if there is an arc between H_i and H_j either one of the houses totally dominates the other (ex. $H_i \Rightarrow H_j$) or they dominate each other (ex. $H_i \rightarrow H_j$ and $H_j \rightarrow H_i$).

Decomposable digraphs can be classed by a set of digraphs ϕ , we say that $D = S[H_1, H_2, \dots, H_k]$ is **ϕ -decomposable** if $D \in \phi$ or if $S \in \phi$. The chioces of H_i for $i = 1, 2, \dots, k$ does not determine anything about the digraph being ϕ -decomposable but the class of **totally ϕ -decomposable** digraphs is where D is ϕ -decomposable and each H_i is totally ϕ -decomposable. We are going to make two shuch sets of digraphs ϕ_1 which is the union of semicomplete digraph and acyclic digraph both classes deskribed in section 1.4 and ϕ_2 which is the union of semicomplete and round digraphs also deskribed in section 1.4.

$$\phi_1 = \text{Semicomplete digraphs} \cup \text{Acyclic digraphs} \quad (2.1)$$

$$\phi_2 = \text{Semicomplete digraphs} \cup \text{Round Digraphs} \quad (2.2)$$

2.2 Quasi-transitive

First we need to recall what a quasi transitive digraph is. For every triplet x, y, z in a quasi-transitive digraph if $x \rightarrow y$ (x dominates y) and $y \rightarrow z$ (y domitaes z), then there has to

be at least one arc in either direction between x and z . Quasi-transitive digraphs can be decomposed no matter if there are strong or nonstrong digraphs.

Theorem 2.2.1. [1] *Let D be a quasi-transitive digraph.*

1. *If D is not strong, then there exists a transitive acyclic digraph T on t vertices and strong quasitransitive digraphs H_1, \dots, H_t such that $D = T[H_1, \dots, H_t]$.*
2. *If D is strong, then there exists a strong semicomplete digraph S on s vertices and quasitransitive digraphs Q_1, \dots, Q_s such that each Q_i is either a single vertex or is nonstrong and $D = S[Q_1, \dots, Q_s]$.*

Proof. blablabla □

From this theorem we can see that quasi-transitive digraphs is totally ϕ_1 -decomposable. Since the transitive digraph for the nonstrong quasi-transitive digraphs is acyclic $T \in \phi_1$ and each Q_i is in itself strong quasi transitive digraphs and you can therefore use item 2.2.1 again. For the strong quasi-transitive digraphs D , S is semicomplete so $S \in \phi_1$ and each $Q_i \in \phi_1$ because it is either one vertex which is a digraph that is both acyclic and semicomplete or it is non-strong and must be quasi-transitive and therefore item 2.2.1 can be used again. So every nonstrong and strong quasi-transitive digraphs is totally ϕ_1 -decomposable.

Theorem 2.2.2. *quasi decomposition can be found in poly time*

2.3 Locally semicomplete

blablabla

Theorem 2.3.1. *round decompose locally semicomplete digraph*

Every locally semicomplete digraph can be classified into some other groups of digraphs namely semicomplete digraphs and round decomposable digraphs and the last one which is neither of the two is call evil. Round decomposable digraph $D = R[D_1, \dots, D_r]$ is where R is a round digraph of the strong componentents D_i and $|R| = r$.

Theorem 2.3.2. [1] *Let D be a locally semicomplete digraph. Then exactly one of the following possibilities holds. Furthermore, there is a polynomial algorithm that decides which of the properties hold and gives a certificate for this.*

- (a) *D is round decomposable with a unique round decomposition $R[D_1, \dots, D_r]$, where R is a round local tournament on $r \geq 2$ vertices and D_i is strong semicomplete digraph for $i = 1, 2, \dots, r$.*
- (b) *D is evil*
- (c) *D is a semicomplete digraph thet is not round decomposable.*

If the locally semicomplete digraph is nonstrong it turns out that it is decomposable this is called a semicomplete decomposition.

Figure 2.1: (a)(b) and (c)

Theorem 2.3.3. [1, 2, 3] Let D be a nonstrong locally semicomplete digraph and let D_1, D_2, \dots, D_p be the acyclic order of the strong components of D . Then D can be decomposed into $r \geq 2$ disjoint subdigraphs D'_1, D'_2, \dots, D'_r as follows:

$$D'_1 = D_p, \lambda_1 = p, \\ \lambda_{i+1} = \min\{j | N^+(D_j) \cap V(D'_i) \neq \emptyset\},$$

and

$$D'_{i+1} = D \langle V(D_{\lambda_{i+1}}) \cup V(D_{\lambda_{i+1}+1}) \cup \dots \cup V(D_{\lambda_{i+1}-1}) \rangle$$

The subdigraphs D'_1, D'_2, \dots, D'_r satisfy the properties below:

- (a) D'_i consists of some strong components that are consecutive in the acyclic ordering of the strong components of D and is semicomplete for $i = 1, 2, \dots, r$;
- (b) D'_{i+1} dominates the initial component of D'_i and there exists no arc from D'_i to D'_{i+1} for $i = 1, 2, \dots, r-1$;
- (c) if $r \geq 3$ then there exists no arc between D'_i and D'_j for i, j satisfying $|j - i| \geq 2$

For simplification of 2.3.3 the properties is drawn out in Figure 2.1 Now we focus more on the structure of the evil locally semicomplete digraph which we have not covered yet, there is a fine understanding of the structure of round decomposable and the semicomplete digraphs, even the semicomplete decomposition which is a part of the evil structure too. First we have to recall what a minimal separator from section 1.1, then use this to construct what we call a **good** separator.

Lemma 2.3.4. [1] Let S be a minimal separator of the locally semicomplete digraph D . Then either $D \langle S \rangle$ is semicomplete or $D \langle V - S \rangle$ is semicomplete.

Then a **good** separator of a locally semicomplete digraph is minimal and $D \langle V - S \rangle$ is not semicomplete. When finding a good separator in a evil locally semicomplete digraph, then the part that is left $D - S$ a semicomplete decomposition can be found it turns out that there is a lot to say about this decomposition.

Theorem 2.3.5. [1, 3] Let D be an evil locally semicomplete digraph then D is strong and satisfies the following properties.

- (a) There is a good separator S such that the semicomplete decomposition of $D - S$ has exactly three components D'_1, D'_2, D'_3 (and $D \langle S \rangle$ is semicomplete by 2.3.4);
- (b) Furthermore, for each such S , there are integers α, β, μ, ν with $\lambda_2 \alpha \leq \beta \leq p-1$ and $p+1 \leq \mu \leq \nu \leq p+q$ such that

$$N^-(D_\alpha) \cap V(D_\mu) \neq \emptyset \text{ and } N^+(D_\alpha) \cap V(D_\nu) \neq \emptyset, \quad (2.3)$$

$$\text{or } N^-(D_\mu) \cap V(D_\alpha) \neq \emptyset \text{ and } N^+(D_\mu) \cap V(D_\beta) \neq \emptyset, \quad (2.4)$$

where D_1, D_2, \dots, D_p and D_{p+1}, \dots, D_{p+q} are the strong decomposition of $D - S$ and $D \langle S \rangle$, respectively, and D_{λ_2} is the initial component of D'_2

Even though this is a structure we can work with, we can actually go deeper into the structure of this evil locally semicomplete digraph. Namly trying to group the components inside the semicomplete decomposition D'_1, D'_2, D'_3 and the good seperator S . This structer is menation in [1] but also in [4]. First we can establish this lemma which is a big part of the structure of evil locally semicomplete digraphs.

Lemma 2.3.6. [4] *Let D be an evil locally semicomplete digraph and let S be a good seperator od D . Then the following holds:*

(i) $D_p \Rightarrow S \Rightarrow D_1$.

(ii) *If sv is an arc from S to D'_2 with $s \in V(D_i)$ and $v \in V(D_j)$, then*

$$D_i \cup D_{i+1} \cup \dots D_{p+q} \Rightarrow D_1 \cup \dots \cup D_{\lambda_2-1} \Rightarrow D_{\lambda_2} \cup \dots \cup D_j$$

.

(iii) $D_{p+q} \Rightarrow D'_3$ and $D_f \Rightarrow D_{f+1}$ for $f \in [p+q]$, where $p+q+1=1$.

(iv) *If there is any arc from D_i to D_j with $i \in [\lambda_2-1]$ and $j \in [\lambda_2, p-1]$, then $D_a \Rightarrow D_b$ for all $a \in [i, \lambda_2-1]$ and $b \in [\lambda_2, j]$.*

(v) *If there is any arc from D_k to D_l with $k \in [p+1, p+q]$ and $l \in [\lambda_2-1]$, then $D_a \Rightarrow D_b$ for all $a \in [k, p+q]$ and $b \in [l]$.*

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