Structurale properties of decomposable digraphs

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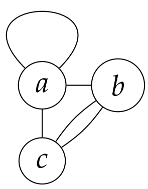
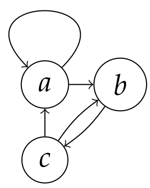
Intro

Graphs and Digraphs

Before going deep into structural properties of decomposable digraphs we first need to establish what a graph is. For some graph $G(V, E)$ where V and E are two sets contaning the **vertices** (also commonly called nodes) and **egdes** of the graph respectivlely. We define the **size** of the graph to be the number of vertices $|V|$ this is also known as **cardinality** of V. An **edge** $e \in E$ where $e \equiv (a, b)$ and $\{a, b\} \subseteq V$ then e is an edge in G, e is said to be **incedent** to a and b. We call $a, b \in V$ **adjecent** if there is an edge (a, b) or (b, a) (two given vertices connected by an edge is said to be adjecent). If an edge goes from and to the same vertex (a, a) it is called a **loop**. The set of edges $e_1, \ldots, e_k$ is usally describe whit the letter E where each edge contains a pair of vertices that are adjecent.
In a graph we have something called a **walk** which is a alternately ordering of vertices an edges in the graph G where the edge in between the two vertices in the ordering is an edge between the vertices in G (for (a, e_1, b) to be a walk the edge e_1 has to be between a and b). We call a walk closed if the first vertex in the walk is the same as the last.
Every vertex $v \in V$ of $G(V, E)$ have a **degree** denoted $d(v)$ which is the number of incident edges to v. A **path** in a graph is a walk where each vertex in the ordering can only apear one time. A cycle is a closed walk where the only vertex pressent more then one time is the first vertex(for the walk to be closed the first vertex has to apear last to also called a closed path). Let X be a subset of the vertices $X \subseteq V$ then we say that $V \backslash X$ is the set of vertices with out the vertices in X, i.e. $V/X \equiv V - X$. A subgraph H of G can contain any of the vertices and the arcs connected to the chossen vertecies in H. you can not have an edge conecting no vertices in H but you do not have to choose all the arcs in G between the chossen vertices in H for H being a subgraph.



(a) graph $G(V, E)$ is an example of a graphs, the red edge is a loop, and all pair of vertices in this graph is adjecent.



(b) This is an oriantation of the edges in the graph which makes this a digraph

Before delving more specific into graphs and digraphs we must establish some important prerequisite and properties. A graph is called **simple** if there is no loops and no multiple edges. With multiple edges it means multiple edges between the same pair of vertices like in Figure 1.1a between b and c.

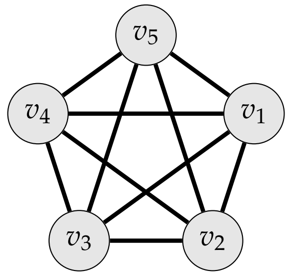


Figure 1.2: Complete graph with 5 vertices.

A graph is **connected** if there exists a path between all pair of vertices in the graph and **disconnected** otherwise. A graph is called **complete** if there for all pair of vertices in the graph is an edge between them see Figure 1.2.

If we instead of edges have **arcs** between the vertices we call it a **digraph**. An arc is describe just like an egde with two adjecent vertices (a, b) the first vertex mentioned in an arc is the vertex **from** where the arc starts also called the **tail**, the second vertex is where the arc is pointing **to** also called **head**. The set of arcs is normaly denoted A like the set of edges is denoted E. So the arc (a, b) goes from a to b, if you wanted it the other way around the arc is (b, a). These graph contaning only arcs and no edges is called a digraph $G(V, A)$ which is what we in this project are focusing on see Figure 1.1b.
For two vertices x and y in $D(V, A)$ then if we have an arc from x to y we say that x **dominates** y this is denoted like this $x \to y$. If we talk about subgraphs A and B, then A **dominates** B if for all $a \in A$ and $b, \in B$, $a \to b$. If there is no arcs from B to A we denote it $A \mapsto B$ and if both $A \to B$ and $A \mapsto B$ we say that A **completely dominates** B and this is denoted $A \Rightarrow B$.

In a digraph we have something called the **underlying graph**. An underlying graph of a digraph is where all arcs are replaced by edges (edge is used every time we talk about undirected edges between vertices, when using directions it is called an arc). Let $X \subseteq V$ Then we can make the subdigraph $D\langle X \rangle$ which is the subgraph D induced by the set X meaning that all the vertices is from X and the arcs is from $A \in G$ but where both head and tail is incedent to the vertices in X. We will denote the graph $D\langle V \backslash X \rangle$ for some $X \subseteq V$ as $D - X$. A digraph is **connected** if the underlying graph is connected, (also called weakly connected), a digraph can be **strongly connected** and **semi connected** too. A digraph is called **semi connected** if there for each pair u and v exists a path from either u to v or v. It is said to be **strongly connected** if for each pair of vertices u and v there exists a path from both u to v and v to u. A strongly connected digraph is also called a **strong** digraph. A strong digraph have a subset S called a **seperator** if $D - S$ is not strong, we also say that S **seperates** D. A seperator S is called **minimal seperator** of D if there exists no proper subset $X \subset S$ that seperates D. Now we can introduce a k**-strong** digraph D which is a strong digraph with $|V| \geq k$ and a minimal seperator S on $|S| = k$.

In a digraph $D(V, A)$ we mostly use the **dregree** as two different degrees namely **out degree**, $d^+(v)$, and **in degree**, $d^-(v)$, that is the arcs from v and to v respectively. In a

digraph D we can talk about the over all **minimum out degree**, $\delta^+(D) = \min\{d^+(v)|v \in V\}$ and **minimum in degree**, $\delta^-(D) = \min\{d^-(v)|v \in V\}$ somtime we is going to need the minimum of these to $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ and called the **minimum degree**. For every vertex v the vertices that is **adjacent** with v we denote $N^+(v)$ and $N^-(v)$ as the set of vertices that is dominated by **(out-nieghbours)**v and dominates **(in-nieghbours)**v, respectivly. This means that $d^+(v) = |N^+(v)|$ and $d^-(v) = |N^-(v)|$.
For simplicity when mentioning paths and cycles in digraphs it will be **directed** paths and cycles if not anything else is mensioned. By **directed** means that we go from tail to head on every arc on the path or cycle. When mentioning paths in a digraph it sometimes makes more sence specifing the head an tail of the path, so a path from s to t is denoted **(s, t)-path**. In some digraphs there is more than one path between the same two vertices these paths can use the same arcs or same vertices or be totally distinct from eachother, the maximum number of disjoint path between two vertices in a digraph is denoted $\lambda_D(s,t)$

1.1 somthing else

The reason for grouping the digraphs into smaller collections of digraphs (like tournaments is a smaller collection of semicomplete digraphs) is because of problems that is hard to solve on general graph but is easy/polynomial solvable on specific graphs.

A group of these problems is called NP-hard problems which sometimes sound easy solvable for graphs but only for some specific graphs we know how to solve it in polynomial time.

Definition 1.1.1. *define NP-hard problems*

In this paper we focusing on the specific digraphs that are **decomposable**. A **decomposable** digraph is a digraph $D = H[G_1, G_2, \ldots, G_|H|]$ where each G_i is sconnected graphs replacing each vertex of the digraph H. ...

1.2 Computational complexities

1.3 Classes of Digraphs

We can use specific collection of graphs to solve various problems faster or easier. Like finding paths in digraphs or cycles or more specific things, but in general des more we know about a digraph we can use to solve hard problems which in general would be time consuming like the problems that are NP-hard. By some quick fast algorithm you can checks wheter a digraph belongs to a certion **class** of digraphs. A class of digraph is a collection of digraph with certain properties incommen like **tournaments**.

1.3.1 introduction to some digraph classes

Tournaments is a digraph where the underlying graph is complete. So a complete graph of order 5 any orientation of the edges concludes in a tournament. Strong digraphs is also

in it self a classification of digraphs. Classes of digraphs can be overlapping each other or be fully containt in each other like tournaments is fully containt in the class called semicomplete digraph. A **semicomplete** digraph is where the underlying graph is complete multigraph, there can be some multiple edges in between the same pair of vertices in the underlying graph. Since the class called semicomplete digraphs contains all digraphs where the underlying graph is a complete multigraph it clearly also contains the graph with only one arc between every pair of vertecies (Tournaments). A **complete** digraph is where every pair of vertices $a, b \in V$ the arc (a, b) and (b, a) is present in the graph.
If you can split the graph into two sets of vertices A and B such that $A \cup B = V$ and there is no arcs inside these sets, then we classify this as an **bipartite** digraph This means all arcs in the graph is in the form (a, b) or (b, a) for all $a \in A$ and $b \in B$. The sets A and B are called the partites of $D(V, A)$. The underlying graph of a bipartite digraph is also called bipartite since there is no edges inside A or B. If there exists more then two of these partite sets we call the digraphs **multipartite**, since there is multiple partite sets in the graph, bipartite sets $\subset$ multipartite.
A much used type of digraph is an **acyclic** digraph. It is a digraph where for an specific ordering of the vertices $V = v_1, v_2, \ldots, v_n$ the arcs in the digraph is (v_i, v_j) where $i < j$ for all $(v_i, v_j) \in A$. This ordering is called an **acyclic ordering** and can also be used to order strong components in an non-strong digraph such that the ordering of the componentent $C_1, C_2, \ldots C_k$ is an acyclic digraph when contracting the components into k vertices. When classifying digraphs there is several ways of doing this, like **transitive** digraphs which are digraphs where for all vertices $a, d, c \in V$ where the arc (a, b) and b, c is present in the digraph ($\in A$), the arc (a, c) has to be a part of A too. using the same kind of classification there is digraphs which are **Quasi-transitive** which is forall vertices $a, d, c \in V$ where the arc (a, b) and b, c is present in the digraph ($\in A$), a and c has to be adecent by at *least* one (more arcs in between are also allowed) arc in either direction ((a, c) or (c, a)). These graphs are going to be mentioned a lot in this thises since the graph is also what we call **decomposable**.
Decomposable digraphs is also a clasification of graphs which are decomposable, for a graph D to be decomposable we have $H_1, H_2, \ldots, H_k$ **houses** and S where $V(S) = s_1, s_2, \ldots, s_k$ which are all digraphs by them self but if each s_i is replaced by the digraph H_i $i = 1, 2, \ldots, k$ we have the graph D, where $H_i \to H_j \in D$ if $s_i \to s_j \in S$ denote this decomposition like $D = S[H_1, H_2, \ldots H_k]$. This is the class of digraphs we are focusing on in this thises. If all the houses are independent sets we call $D = S[H_1, H_2, \ldots, H_k]$ the extension of S. If S is a semicomplete digraph we call the extensin of these **extended semicomplete** digraph. Like we already mentioned Quasi-transitive digraphs are decomposable but we have several classes that are decomposable, and another class of digraphs that is giong to be used a lot in this is **locally semicomplete** digraphs.
First we are going to introduce **in-locally semicomplete** digraphs and **out-locally semicomplete** digraphs which is for every in-nieghboor of a vertex $x \in V$ they have to be adjecent ($x \cup N^-(x)$ induces a semicomplete digraph) is the in-locally semicomplete digraph if it is true for all $x \in V$. Respectively it is called an out-locally semicomplete digraph if $\forall x \in V$ the out-nieghboors, $N^+(x)$, has to be adjecent. If a digraph is both in-locally semicomplete and out-locally semicomplete, it is called a **locally semicomplete** digraph. Why both Quasi transitive digraphs and some locally semicomplete digraphs are decomposeble will be described in section **??**.
The last class of digraph that are important for this thises is the round digrphs. A digraph is called a **round** digraph if there exists an ordering of the vertices $v_1, v_2, \ldots, v_n$ such that for all v_i, $N^+(v_i) = v_{i+1}, v_{i+2}, \ldots, v_{i+d^+(v_i)}$ and $N^-(v_i) = v_{i-d^-(v_i)}, v_{i-(d^-(v_i)-1)}, \ldots, v_{i-1}$.

1.3.2 Semicomplete Digraphs

1.3.3 muligvis Transitive Digraphs

1.3.4 Strong Digraphs

1.3.5 Round Digraphs

Bibliography