

Algorithm for Finding linkage in a Totally ϕ -decomposable Digraph

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Decomposable Digraph

A digraph $D = S[H_1, \dots, H_s]$ is called Decomposable where $|S| = s$ and each H_i are disjoint digraphs replacing every vertex in S . The digraphs H_i will in this presentation be called **hauses**. For a set of digraphs ϕ we can talk about ϕ -decomposable digraphs. For a digraph $D = S[H_1, \dots, H_s]$ to be ϕ -decomposable either $D \in \phi$ or $S \in \phi$.

For a digraph $D = S[H_1, \dots, H_s]$ to be totally ϕ -decomposable it has to be ϕ -decomposable and if $D \notin \phi$, then $S \in \phi$ and each H_i has to be totally ϕ -decomposable for $i = 1, \dots, s$.

The weak linkage Problem

Given two pair of vertices s_1, t_1 and s_2, t_2 finding arc-disjoint paths between each pair is 2-weak linkage problem. Then the k -weak linkage problem is finding k arc-disjoint paths between k pair of terminals. where a terminal pair is a source and a sink in the paths of the solution of the linkage problem. When talking about linkage problem for decomposable digraph, we can have houses with terminals in and some without any terminals. The houses with no terminals in is called **clean houses**. Then a terminal pair can either be inside the same house or in different houses. If a terminal pair is contained inside the same house it is called an internal pair otherwise we call the pair external. The same with a path if it is fully contained inside a house it is called an internal path otherwise it is external. you can have an external path for a internal pair if the path goes out of the house and in again.

Notation for the weak linkage problem

Every thing on this list is denoted like this unless it is specified noot to be like that.

- ▶ a natrue number k is going to repricent the number of terminal pairs there in the linkage problem.
- ▶ Π denote the set of terminal pairs $(s_1, t_1), \dots, (s_k, t_k)$.
- ▶ D is the totally ϕ -decomposable digraph where each hause is denoted H_i for $i = 1, \dots, s$.

Bombproof sets

Definition

[1] We say that a class of digraphs ϕ is Bombproof if there exists a polynomial algorithm \mathcal{A}_ϕ to find a totally ϕ -decomposition of every totally ϕ -decomposable digraph and, for every integer c , there exists a polynomial algorithm \mathcal{B}_ϕ to decide the weak k -linkage problem for the class

$$\phi(c) := \bigcup_{D \in \phi} D(c) \quad (1)$$

for an integer c , the class denoted $D(c)$ is the class of digraphs D for which you can first add as many parallel arcs to arcs that already exist in D and then blow up b vertices where $0 \leq b \leq c$ to obtain a digraph that has a size $\leq c$.

Contraction clean vertices

Lemma

[1] Let D be a digraph, Π a list of k terminal pairs and $H \subset D$ a clean module with respect to Π . Let D' be the contraction of H into a single vertex h . Then D has a weak Π -linkage if and only if D' has a weak Π -linkage.

Max inside Hause External Path

Lemma

Let $D = S[H_1, \dots, H_s]$ be a decomposable digraph, let Π' be a list of h terminal pair and let F be a set of arcs in D satisfying that $d_F^-(v), d_F^+(v) \leq r$ for all $v \in V(D)$. If $(D \setminus F, \Pi')$ has a weak linkage, then it has a weak linkage P_1, \dots, P_h such that, we have $|V(\bigcup_{i \in P_j} P_i \cap H_j)| \leq 2h(h+r)$, for every $j \in [1, \dots, s]$, where denotes the set of indices i for which P_i is external.

Algorithm for Digraph with Deleted Arcs

Lemma

Let \mathcal{C} be a class of digraphs for which there exists an algorithm \mathcal{A} to decide the weak k -linkage problem, whose running time is bounded by $f(n, k)$. Let $D = (V, A)$ be a digraph, Π a list of k pairs of terminals and $F \subseteq V \times V$ such that $D' := (V, A \cup F)$ is a member of \mathcal{C} . There exists an algorithm \mathcal{A}^- , whose running time is bounded by $f(n, k + |F|)$, to decide whether D has a weak Π -linkage.

The theorem of the main Algorithm

Theorem

Let ϕ be a bombproof class of digraph. There is a polynomial algorithm \mathcal{M} that takes as input a 5tuple $[D, k, k', \Pi, F]$ where D is a totally ϕ -decomposable digraph, k, k' are natural numbers with $k' \leq k$, Π is a list of k' terminal pairs and $F \subseteq A(D)$ is a set of arcs satiesfying

$$d_F^-(v), d_F^+(v) \leq k - k' \text{ for all } v \in V(D) \quad (2)$$

$$|F| \leq (k - k')2k$$

and decides wheter $D \setminus F$ contains a weak Π -linkage.



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Disjoint paths in decomposable digraphs.

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