Structurale properties of decomposable digraphs

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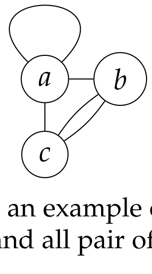
Why we need graphs.

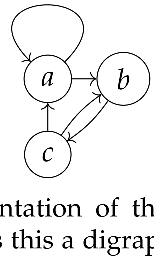
Chapter 1

Graphs and digraphs

Before going deep into structural properties of decomposable digraphs we first need to establish what a graph is. For some graph $G(V, E)$ where V and E are two sets contaning the vertices (also commonly called nodes) and egdes of the graph respectivlely. We define the **size** of the graph to be the number of vertices $|V|$ this is also known as **cardinality** of V. An **edge** $e \in E$ where $e \equiv (a, b)$ and $\{a, b\} \subseteq V$ then e is an edge in G, e is said to be **incedent** to a and b. We call $a, b \in V$ **adjecent** if there is an edge (a, b) or (b, a) (an edge between the two given vertices is said to be adjecent). If an edge goes from and to the same vertex (a, a) it is called a **loop**. The set of edges $e_1, \ldots, e_k$ is usally describe whit the letter E where each edge contains a pair of vertices that are adjecent.
In a graph we have something called a **walk** which is a alternately ordering of vertices an edges in the graph G where the edge in between the two vertices in the ordering is an edge between the vertices in G (for (a, e_1, b) to be a walk the edge e_1 has to be between a and b). We call a walk closed if the first vertex in the walk is the same as the last.
Every vertex $v \in V$ of $G(V, E)$ have a **degree** denoted $d(v)$ which is the number of incident edges to v. A **path** in a graph is a walk where each vertex in the ordering can only apear one time. A cycle is a closed walk where the only vertex pressent more then one time is the first vertex(for the walk to be closed the first vertex has to apear last to also called a closed path). Let X be a subset of the vertices $X \subseteq V$ then we say that $V \backslash X$ is the set of vertices with out the vertices in X, i.e. $V/X \equiv V - X$. A subgraph H of G can contain any of the vertices and the arcs connected to the chossen vertecies in H. you can not have an edge conecting no vertices in H but you do not have to choose all the arcs in G between the chossen vertices in H for H being a subgraph.



(a) graph $G(V, E)$ is an example of a graphs, the red edge is a loop, and all pair of vertices in this graph is adjecent.



(b) This is an oriantation of the edges in the graph which makes this a digraph

Before delving more specific into graphs and digraphs we must establish some important prerequisite and properties. A graph is called **simple** if there is no loops and no multiple edges. With multiple edges it means multiple edges between the same pair of vertices like in Figure 1.1a between b and c.

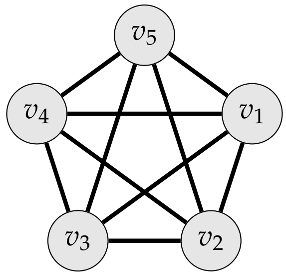


Figure 1.2: Complete graph with 5 vertices.

A graph is **connected** if there exists a path between all pair of vertices in the graph and **disconnected** otherwise. A graph is called **complete** if there for all pair of vertices in the graph is an edge between them see Figure 1.2.

If we instead of edges have **arcs** between the vertices we call it a **digraph**. An arc is describe just like an egde with two adjecent vertices (a, b) the first vertex mentioned in an arc is the vertex **from** where the arc starts also called the **tail**, the second vertex is where the arc is pointing **to** also called **head**. The set of arcs is normaly denoted A like the set of edges is denoted E. So the arc (a, b) goes from a to b, if you wanted it the other way around the arc is (b, a). These graph contaning only arcs and no edges is called a digraph $G(V, A)$ which is what we in this project are focusing on see Figure 1.1b.
For two vertices x and y in $D(V, A)$ then if we have an arc from x to y we say that x **dominates** y this is denoted like this $x \to y$. If we talk about subgraphs A and B, then A **dominates** B if for all $a \in A$ and $b, \in B$, $a \to b$. If there is no arcs from B to A we denote it $A \mapsto B$ and if both $A \to B$ and $A \mapsto B$ we say that A **completely dominates** B and this is denoted $A \Rightarrow B$.

In a digraph we have something called the **underlying graph**. An underlying graph of a digraph is where all arcs are replaced by edges (edge is used every time we talk about undirected edges between vertices, when using directions it is called an arc). Let $X \subseteq V$ Then we can make the subdigraph $D\langle X \rangle$ which is the subgraph D induced by the set X meaning that all the vertices is from X and the arcs is from $A \in G$ but where both head and tail is incedent to the vertices in X. We will denote the graph $D\langle V \backslash X \rangle$ for some $X \subseteq V$ as $D - X$. A digraph is **connected** if the underlying graph is connected, (also called weakly connected), a digraph can be **strongly connected** and **semi connected** too. A digraph is called **semi connected** if there for each pair u and v exists a path from either u to v or v. It is said to be **strongly connected** if for each pair of vertices u and v there exists a path from both u to v and v to u. A strongly connected digraph is also called a **strong** digraph. A strong digraph have a subset S called a **seperator** if $D - S$ is not strong, we also say that S **seperates** D. A seperator S is called **minimal seperator** of D if there exists no proper subset $X \subset S$ that seperates D. Now we can introduce a k**-strong** digraph D which is a strong digraph with $|V| \geq k + 1$ and a minimal seperator S on $|S| \geq k$.

In a digraph $D(V, A)$ we mostly use the **dregree** as two different degrees namely **out degree**, $d^+(v)$,and **in degree**, $d^-(v)$, that is the arcs from v and to v respectively.

1.1 somthing else

We can use these to describe som specific collection of graphs as the graph **tournaments**. **Tournaments** is a digraph where the underlying graph is complete. So a complete graph of order 5 any orientation of the edges concludes in a tournament. If instead of replacing the one edge by one arc in either direction, but instead replace it by two arcs the digraph is called **semicomplete**.

The reason for grouping the digraphs into smaller collections of digraphs (like tournaments is a smaller collection of semicomplete digraphs) is because of problems that is hard to solve on general graph but is easy/polynomial solvable on specific graphs.

A group of these problems is called NP-hard problems which sometimes sound easy solvable for graphs but only for some specific graphs we know how to solve it in polynomial time.

Definition 1.1.1. *define NP-hard problems*

In this paper we focusing on the specific digraphs that are **decomposable**. A **decomposable** digraph is a digraph $D = H[G_1, G_2, \ldots, G_|H|]$ where each G_i is sconnected graphs replacing each vertex of the digraph H. ...