RELATIVE REGULAR SEQUENCES AND GENERALIZED COHOMOLOGY OF INFINITE REAL GRASSMANNIANS

IGOR KRIZ, GUCHUAN LI

Abstract. We introduce a generalization of the algebraic notion of a regular sequence, which we call a relative regular sequence, and prove some structural properties of this concept. Applications of this concept arise in algebraic topology, where they help clarify the structure of generalized cohomology rings of infinite real Grassmannians.

1. Introduction

In commutative algebra, many properties of a quotient of a commutative ring $R/(a_1, \dots, a_n)$ are understood when the elements $a_1, \dots, a_n$ form a *regular sequence*, meaning that

$$0 \longrightarrow R/(a_1, \dots, a_i) \xrightarrow{a_{i+1}} R/(a_1, \dots, a_i) \longrightarrow R/(a_1, \dots, a_i, a_{i+1}) \longrightarrow 0$$

is a short exact sequence. In that case, we have a "cube-like" free R-resolution of the quotient ring which gives a bound on its cohomological dimension as an R-module. The construction, (under additional conditions) also gives information about other invariants, such as local cohomology or dimension. On the other hand, if $a_1, \dots, a_n$ do not form a regular sequence, algebraic information about the ring $R/(a_1, \dots, a_n)$ becomes much more scarce. That is, of course, perfectly understandable, since from the geometric point of view, we are dealing with an arbitrary closed subscheme of $Spec(R)$.

There are examples in algebraic topology where this discussion comes into play. One striking example is Wilson's calculation [7] of the complex cobordism cohomology of real Grassmannians. One has

$$MU^*BO(n) = MU_*[[c_1, \dots, c_n]]/(c_1 - \widetilde{c}_1, \dots, c_n - \widetilde{c}_n) \tag{1}$$

where c_k are the Conner-Floyd Chern classes of the complexification of the universal real bundle γ_k on $BO(k)$ and $\widetilde{c}_k$ are their complex conjugates. This result was further generalized by Kitchloo and Wilson [4], proving similar formulas where the complex cobordism MU is

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replaced by the generalized cohomology theories $E(k)$ or $E\mathbb{R}(k)$. The formula (1) is extremely elegant, but since the elements $c_i - \widetilde{c}_i$ do not form a regular sequence, our information about the ring it describes is limited, unless we obtain more information about these elements. One result that, which was proved by Kono and Yagita [3], says that $BP^*BO(n)$ is Landweber flat (see also the results of [6, 1], which apply to more general situations). More complete information proves to be both non-trivial and useful (cf. [2] for $n = 2$).

In this paper, we introduce the following generalization of the concept of a regular sequence, which is relevant to this example.

Definition 1. *Let R be a commutative ring. A* relative regular sequence *consists of elements a_i, b_i, c_i where*

$$b_i c_i \equiv a_i \mod (a_1, \dots, a_{i-1}),$$

$i = 1, \dots, n$ which satisfy the condition

$$Ann_{R/(a_1,\dots,a_{i-1})}(b_i) = Ann_{R/(a_1,\dots,a_{i-1})}(a_i) = b_{i-1} \tag{2}$$

for $i = 1, \dots, n$. (For the sake of the case $i = 1$, we put $a_0 = b_0 = 0$, $c_0 = 1$.)

This reduces to the ordinary concept of a regular sequence in the case when the elements c_i are units. The relevance of this concept to the formula (1) and its generalizations is expressed by the following:

Theorem 2. *We have*

$$MU^*BO(n) = MU_*[[c_1, \dots, c_n]]/(a_1, \dots, a_n) \tag{3}$$

for some relative regular sequence $a_i = b_i c_i$, $i = 1, \dots, n$, where, moreover,

$$a_i \equiv c_i - \widetilde{c}_i \mod (c_{i+1}, \dots, c_n). \tag{4}$$

This is, in fact, a consequence of the following result:

Theorem 3. *Let $R = A[[c_1, \dots, c_n]]$ and suppose $a_1, \dots, a_n, \beta_1, \dots, \beta_n \in R$ are elements such that, letting*

$$A_m = A[[c_1, \dots, c_n]]/(a_1, \dots, a_m, c_{m+1}, \dots, c_n),$$

we have short exact sequences

$$0 \longrightarrow A_m/(\beta_m) \xrightarrow{c_m} A_m \longrightarrow A_{m-1} \longrightarrow 0 \tag{5}$$

(where the second arrow is the projection) and

$$0 \longrightarrow A_{m-1}/(\beta_{m-1}) \xrightarrow{\beta_m} A_m \longrightarrow A_m/(\beta_m) \longrightarrow 0 \tag{6}$$

(where the second arrow is the projection). Suppose further that the compositions

$$A_{m-1}/(\beta_{m-1}) \xrightarrow{\beta_m} A_m \longrightarrow A_{m-1} \tag{7}$$

(where the second map is the projection) are injective. Then there exist s a relative regular sequence $a_i, b_i, \overline{c}_i$, $i = 1, \ldots, n$ such that

$$A_m = R/(a_1, \ldots, a_m, \overline{c}_{m+1}, \ldots, \overline{c}_n),$$

$$B_m = A_m/(\beta_m) = R/(a_1, \ldots, a_{m-1}, b_m, \overline{c}_{m+1}, \ldots, \overline{c}_n),$$

and b_m, $\overline{c}_m$ coincide with β_m, c_m modulo

$$(c_{m+1}, \ldots, c_n) = (\overline{c}_{m+1}, \ldots, \overline{c}_n).$$

Now being the quotient by a relative regular sequence gives some insight to the structure of a ring. For example, we also have a "cube-like" resolution:

Proposition 4. *Let a_1, b_i, c_i be a relative regular sequence, $i = 1, \ldots, n$ in a ring R. Then the R-modules $\overline{A}_n = R/(a_1, \ldots, a_n)$, $\overline{A}_n/(b_n)$ have free R-resolutions of the form*

$$R \longrightarrow \ldots \longrightarrow \bigoplus_{\binom{n}{k}} R \longrightarrow \ldots \longrightarrow R. \tag{8}$$

Proof. This follows immediately from the short exact sequences

$$0 \longrightarrow \overline{A}_{n-1}/(b_{n-1}) \xrightarrow{a_n} \overline{A}_{n-1} \longrightarrow \overline{A}_n \longrightarrow 0, \tag{9}$$

$$0 \longrightarrow \overline{A}_{n-1}/(b_{n-1}) \xrightarrow{b_n} \overline{A}_{n-1} \longrightarrow \overline{A}_n/(b_n) \longrightarrow 0 \tag{10}$$

and the functoriality of resolutions. □

However, one might wish for a more explicit statement in the case of the Wilson ring (1). To this end, the concept of a relative regular sequence must be strengthened. Let (a_i, b_i, c_i) be a relative regular sequence. We shall define an *admissible sequence* recursively as follows:

(1) $(b_1), (c_1)$ are admissible sequences.

(2) If $(\ldots, \widehat{b_i})$ is an admissible sequence, where $\widehat{b_i}$ stands either for b_i or for b_i/c_{i-1}, then the sequences

$$(\ldots, \widehat{b_i}, b_{i+1}/c_i),\ (\ldots, \widehat{b_i}, c_{i+1})$$

are admissible.

(3) If a sequence $(\dots, c_i)$ is admissible, then the sequences

$$(\dots, c_i, c_{i+2}),\ (\dots, c_i, b_{i+2})$$

are admissible.

Definition 5. *A relative regular sequence is called* strictly relative regular *if every admissible sequence is regular.*

One may wonder if more needs to be said about why the elements b_{i+1}/c_i make sense, but this is readily clarified by the following easy result:

Lemma 6. *For a relatively regular sequence (a_i, b_i, c_i), b_{i+1} is uniquely divisible by c_i in $R/(a_1, \dots, a_{n-2}, b_{n-1})$. (In particular, c_i is regular.)*

Proof. Putting $\overline{A}_i = R/(a_1, \dots, a_i)$, the statement follows from the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\overline{A}_{i-1}/(b_{i-1}) \arrow[r, dotted, "\frac{b_{i+1}}{c_i}"] \arrow[d, "b_i"] & \overline{A}_{i-1}/(b_{i-1}) \arrow[d, "a_i"] \\
\overline{A}_{i-1} \arrow[r, "b_{i+1}"] \arrow[d] & \overline{A}_{i-1} \arrow[d] \\
\overline{A}_i/(b_i) \arrow[r, "b_{i+1}"] \arrow[d] & \overline{A}_i \arrow[d] \\
0 & 0
\end{tikzcd}$$

□

It is worth noticing that if we denote by $B(n)$ resp. $C(n)$ the number of admissible sequences ending with $\widehat{b}_n$ resp. c_n, we have

$$B(1) = C(1) = B(2) = C(2) = 1,$$

$$B(i) = C(i) = C(i-2) + B(i-1),\ i \geq 2,$$

so we see that

$$B(i) = C(i) = \Phi_i$$

is the ith Fibonacci number.

Let (a_n, b_n, c_n) be a relative regular sequence of length n in a ring R. We find it useful to order admissible sequences lexicographically according to the key

$$\widehat{b}_1 < c_1 < \widehat{b}_2 < c_2 < \cdots < \widehat{b}_n < c_n.$$

Let R_i, $i = 1, \dots \Phi_{n+2}$ denote the quotient of R by the ith admissible sequence with respect to the above lexicogrphical ordering of all admissible sequences ending with c_{n-1}, $\widehat{b}_n$ or c_n. Note that R_i, $i = 1, \dots, \Phi_{n+1}$ will be precisely the quotients by the admissible sequences ending with c_{n-1} or $\widehat{b}_n$.

Theorem 7. *For a strongly relative regular sequence (a_i, b_i, c_i) of length n in a ring R, there is an increasing filtration of R-modules $F_i\overline{A}_n$ on $\overline{A}_n = R/(a_1, \dots, a_n)$, resp. $G_i(\overline{A}_n/(b_n))$ on $\overline{A}_n/(b_n)$ such that $F_0\overline{A}_n = 0$, $G_0(\overline{A}_n/(b_n)) = 0$, $F_{\Phi_{n+2}}\overline{A}_n = \overline{A}_n$ resp. $G_{\Phi_{n+1}}(\overline{A}_n/(b_n)) = \overline{A}_n/(b_n)$, and*

$$R_i \cong F_i/F_{i-1} \cong G_i/G_{i-1}. \tag{11}$$

Additionally, the isomorphism (11) is induced on R_i by multiplication by

$$c_{I_i} = \prod_{j \in I_i} c_j$$

where I_i is the set of all indices j where the ith admissible sequence contains $\widehat{b}_j$.

Comment: The result of Theorem 7 can be interpreted as a stratification on the affine schemes $Spec(\overline{A}_n)$, $Spec(\overline{A}_n/(b_n))$. The pure strata are ordered by our lexicographic ordering, where the ith stratum is

$$Spec(R_i[c_{I_i}^{-1}]).$$

Theorem 8. *The relative regular sequence on $R = MU^*BU(n)$ induced by Theorem 3 from $\overline{A}_n = MU^*BO(n)$ is strongly relative regular.*

2. Proof of Theorem 3

Proof of Theorem 3. Put $B_m = A_m/(\beta_m)$. The injectivity of (7) implies a diagram of the following form:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & B_n \arrow[r, "Id"] \arrow[d, "c_n"] & B_n \arrow[d, "c_n"] & \\
0 \arrow[r] & B_{n-1} \arrow[r, "\beta_n"] \arrow[d, "Id"] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_{n-1} \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
\tag{12}
$$

We put $b_n = \beta_n$. We shall, by convention, denote the elements of a factor ring by the same symbol as their lifts when the lift is arbitrary. We shall, in fact, construct, by downward induction on k, rings of the form

$$\overline{A}_k = R/(a_1, \dots, a_k),$$

$$\widehat{A}_k = R/(a_1, \dots, a_k, \overline{c}_{k+1}),$$

and elements

$$\overline{c}_k, b_{k-1} \in \overline{A}_k$$

such that

$$b_k \overline{c}_k = b_k b_{k-1} = 0$$

and putting $\overline{B}_k = \overline{A}_k/(b_k)$,

$$\widehat{B}_{k-1} = \overline{A}_k/(b_{k-1}, \overline{c}_k),$$

we have a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \overline{B}_k \arrow[r, "Id"] \arrow[d, "\overline{c}_k"] & \overline{B}_k \arrow[d, "\overline{c}_k"] & \\
0 \arrow[r] & \widehat{B}_{k-1} \arrow[r, "b_k"] \arrow[d, "Id"] & \overline{A}_k \arrow[r] \arrow[d] & \overline{B}_k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{B}_{k-1} \arrow[r] & \widehat{A}_{k-1} \arrow[r] \arrow[d] & \overline{C}_k \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{13}$$

whose associated graded object with respect to the ideal

$$(c_{k+1}, \dots, c_n)$$

is a sum of copies of (12) with n replaced by k. Then the right hand column of (13) gives a short exact sequence of the form

$$0 \to R/(a_1, \dots, a_{k-2}, b_{k-1}) \to R/(a_1, \dots, a_k) \to \overline{B}_k \to 0 \tag{14}$$

(This is seen by thinking of $\overline{B}_k$ as a "twisted power series ring in $(c_k, \dots, c_n)$"; modulo $(c_k, \dots, c_n)$, this holds with hats replaced by bars. The hats indicate adding the appropriate $(c_k, \dots, c_n)$-multiple correction terms.)

Now we will construct elements $\overline{c}_{k-1}, b_{k-2} \in R/(a_1, \dots, a_{k-1})$ such that

$$\overline{c}_{k-1} b_{k-1} = b_{k-2} b_{k-1} = 0. \tag{15}$$

For the case of $\overline{c}_{k-1}$, we use (14). This gives

$$c_{k-1} b_{k-1} = c_k^{m_k} \dots c_n^{m_n} t \in R/(\widehat{\alpha}_1, \dots, \widehat{\alpha}_{k-1})$$

such that $t \notin (c_k, \dots, c_n)$. However, we also know that t is divisible by b_{k-1} in $R/(a_1, \dots, a_{k-1}, c_k \dots, c_n)$, since the associated graded sequence of (14) is also exact. In other words, we can write

$$t = b_{k-1} q \mod (c_k, \dots, c_n) \in R/(a_1, \dots, a_{k-1}). \tag{16}$$

Thus, by replacing c_{k-1} with

$$c_{k-1} - c_k^{m_k} \dots c_n^{m_n} q,$$

we can increase $\sum\limits_{i \geq k} m_i$. Continuing this procedure, we can construct $\overline{c}_{k-1}$.

For b_{k-2}, recall (6) with $m = k$. Now consider the composition

$$R/(a_1,\dots,a_{k-3}) \xrightarrow{b_{k-1}} R/(a_1,\dots,a_{k-2}) \longrightarrow R/(a_1,\dots,a_{k-2},b_{k-1}) \longrightarrow 0$$

Considering the associated graded objects with respect to $(c_k,\dots,c_n)$, and factoring the first term by β_{k-1}, we get (6). This means that

$$\beta_{k-2}b_{k-1} = c_k^{m_k}\dots c_n^{m_n}t \in R/(a_1,\dots,a_{k-1}).$$

However, again, we must have

$$b_{k-1} \mid t \in R/(a_1,\dots,a_{k-1},c_k,\dots,c_n)$$

by the exactness of (6). In other words, again, we have (16). Replacing β_{k-2} by

$$\beta_{k-2} - c_k^{m_k}\dots c_n^{m_n}q,$$

again, increases $\sum\limits_{i\geq k} m_i$. Continuing this procedure, we can construct b_{k-2}.

By (15), then, we have an analog of (13) with k replaced by $k-1$, so the induction is complete.

Now the axioms of a relative regular sequence follow from (5), (6). □

Proof that (7) is injective for $A_n = MU^*BO(n)$, $B_n = \widetilde{MU}^*BO(n)^{\gamma^n_{\mathbb{R}}}$. Expressing formally c_n as a product of "elementary factors"

$$x_1\dots x_n,$$

the composition (7) can be calculated by setting $x_n = 0$ in the expression

$$\frac{x_1\dots x_n - i(x_1)\dots i(x_n)}{x_1\dots x_n}.$$

We see that this is equal to

$$\frac{x_1\dots x_{n-1} + i(x_1)\dots i(x_{n-1})}{x_1\dots x_{n-1}}.$$

In $B_{n-1} = A_{n-1}/(\beta_{n-1})$, however, this is equal to 2 which is a regular element, since by the paper of Wilson [7], B_{n-1} has no torsion. This means that (7) in fact is injective after being composed with the projection $A_{n-1} \to B_{n-1}$, and thus is injective. □

Proof of Theorem 7. Induction on n. Suppose the statement is true with n replaced by $n-1$. We use the short exact sequence

$$0 \longrightarrow \overline{A}_{n-1}/(b_{n-1}) \xrightarrow{b_n} \overline{A}_{n-1} \longrightarrow \overline{A}_n/(b_n) \longrightarrow 0.$$

In our definition, the maps preserve the filtration, and by the assumption of being strongly relative regular, the sequence exact on the associated graded object. This implies the induction statement for $\overline{A}_n/(b_n)$.

For the case of $\overline{A}_n$, one additionally uses the short exact sequence

$$0 \longrightarrow \overline{A}_n/(b_n) \xrightarrow{c_n} \overline{A}_n \longrightarrow \overline{A}_{n-1} \longrightarrow 0$$

along with the induction hypothesis on the last term. □

Proof of Theorem 8. Let

$$\iota(x) = 1 + \frac{i(x)}{x}.$$

Writing, in $MU_*[[c_1, \dots, c_n]]$, $c_i = \sigma_i(x_1, \dots, x_n)$ where σ_i is the ith elementary symmetric polynomial, put

$$\beta_n = 1 - \frac{i(x_1) \dots i(x_n)}{x_1 \dots x_n}. \tag{17}$$

We can also define $\beta_n \in MU_*[[c_1, \dots, c_N]]$ for $N > n$ as the symmetrization of (17) (i.e. the same polynomial of elementary symmetric polynomials). Explicitly, writing

$$\sigma_k(\iota) = \sigma_k(\iota(x_1), \dots, \iota(x_n)),$$

we then have

$$\beta_n = 1 - (-1)^n - (-1)^{n-1}\sigma_1(\iota) - \dots - \sigma_n(\iota).$$

Now we claim that $(\beta_1, \dots, \beta_n)$ form a regular sequence in $MU_*[[c_1, \dots, c_n]]$ and that moreover, 2 is a non-zero divisor in $MU_*[[c_1, \dots, c_n]]/(\beta_1, \dots, \beta_n)$. To show this, write

$$\begin{aligned} &\beta_1 = 2 - \sigma_1(\iota), \\ &\beta_2 = \sigma_1(\iota) - \sigma_2(\iota), \\ &\beta_3 = 2 - \sigma_1(\iota) + \sigma_2(\iota) - \sigma_3(\iota), \\ &\dots \end{aligned} \tag{18}$$

We see that by a simple linear transformation, we obtain the list

$$\begin{aligned} &2 - \sigma_1(\iota), \\ &2 - \sigma_2(\iota) \\ &2 - \sigma_3(\iota), \\ &\dots \\ &2 - \sigma_n(\iota). \end{aligned} \tag{19}$$

Thus, our claim can be deduced by studying β_n in the ring

$$MU_*[[c_1, \dots, c_n]]/(c_1, \dots, c_{n-1}). \tag{20}$$

Now thinking of the c_i as the elementary symmetric polynomials of "elementary factors" x_j, reducing modulo $c_1, \dots, c_{n-1}$ is (after extending scalars) equivalent to setting

$$x_j = x\zeta_n^j$$

where ζ_n is the primitive nth root of unity. The element b_n is then unit multiple of

$$\frac{x^n - i(x)i(\zeta_n x)\dots i(\zeta_n^{n-1}x)}{x^n}. \tag{21}$$

We see however that in the element (21), none of the terms of the form

$$v_k^n x^{(2^k-1)n},$$

for $k \geq 1$, can cancel out against other terms. Note that these terms translate to

$$v_k^n c_n^{2^k-1}. \tag{22}$$

Now the same argument shows that every admissible sequence is regular, noting that β_{n+1} reduces to 2 in $MU_*[[c_1, \dots, c_n]]$, noting that the statement is only non-trivial at the terms $\widehat{b}_k$, where the terms (22) cannot cancel against any other terms.

□

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