MAT2011 Graph Theory and Applications

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Introduction to Graphs

- A graph G is a finite nonempty set V of objects called vertices together with a set E of 2-element subsets of V called edges.
- Vertices are sometimes called points or nodes, while edges are sometimes referred to as lines or links.
- Each edge $\{u, v\}$ of V is commonly denoted by uv or vu.
- If e = uv, then the edge e is said to join u and v.
- The number of vertices in a graph G is the order of G and the number of edges is the size of G. We often use n for the order of a graph and m for its size.
- To indicate that a graph G has vertex set V and edge set E, we sometimes write $G = (V, E)$. To emphasize that V is the vertex set of a graph G, we often write V as V(G). For the same reason, we also write E as E(G).
- A graph of order 1 is called a trivial graph and so a nontrivial graph has two or more vertices. A graph of size 0 is an empty graph and so a nonempty graph has one or more edges.

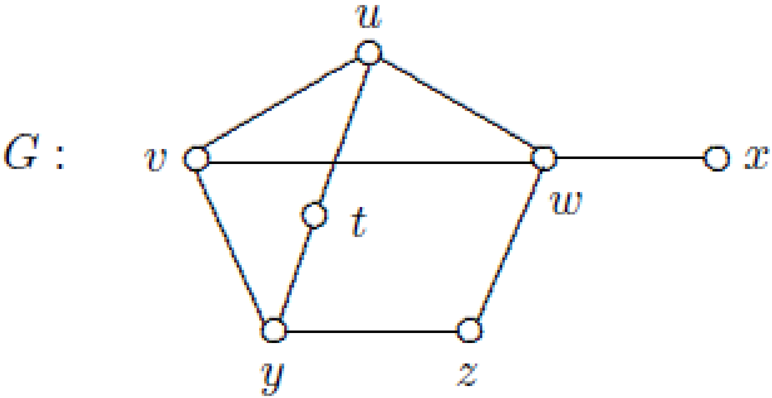

Figure 1.1: A graph

- If uv is an edge of G, then u and v are adjacent vertices. Two adjacent vertices are referred to as neighbors of each other.
- The vertex u and the edge uv are said to be incident with each other. Similarly, v and uv are incident.
- The degree of a vertex v in a graph G is the number of vertices in G that are adjacent to v. Thus the degree of a vertex v is the number of the vertices in its neighborhood $N(v)$
- A vertex of degree 0 is referred to as an isolated vertex and a vertex of degree 1 is an end-vertex or a leaf. An edge incident with an end-vertex is called a pendant edge.
- The largest degree among the vertices of G is called the maximum degree of G is denoted by $\Delta(G)$. The minimum degree of G is denoted by $\delta(G)$.
- If v is a vertex of a graph G of order n, then $0 \leq \delta(G) \leq deg(G) \leq \Delta(G) \leq n-1$

Theorem 1.1 **(The First Theorem of Graph Theory)** *If G is a graph of size m, then*

$$\sum_{v \in V(G)} \deg v = 2m.$$

Proof. When summing the degrees of the vertices of G, each edge of G is counted twice, once for each of its two incident vertices. ■

Corollary 1.2 *Every graph has an even number of odd vertices.*

Proof. Suppose that G is a graph of size m. By Theorem 1.1,

$$\sum_{v \in V(G)} \deg v = 2m,$$

which is, of course, an even number. Since the sum of the degrees of the even vertices of G is even, the sum of the degrees of the odd vertices of G must be even as well, implying that G has an even number of odd vertices. ■

Introduction to Graph Coloring

- A proper vertex coloring of a graph G is an assignment of colors to the vertices of G, one color to each vertex, so that adjacent vertices are colored differently. The positive integers (typically 1,2,...,k, for some positive integer k) are commonly used for the colors.
- A proper coloring can be considered as a function $c : V(G) \longrightarrow N$ (where N is the set of positive integers) such that $c(u) \neq c(v)$ if u and v are adjacent in G. If each color used is one of k given colors, then we refer to the coloring as a k-coloring.
- Suppose that c is a k-coloring of a graph G, where each color is one of the integers 1,2,...,k, as mentioned above. If $V_i (1 \leq i \leq k)$ is the set of vertices in G colored i (where one or more of these sets may be empty), then each nonempty set V_i is called a color class and the nonempty elements of $\{V_1, V_2, ..., V_k\}$ produce a partition of V(G).

Chromatic number of a graph G

- A graph G is k-colorable if there exists a coloring of G from a set of k colors. In other words, G is k-colorable if there exists a k-coloring of G. The minimum positive integer k for which G is k-colorable is the chromatic number of G and is denoted by $\chi(G)$. (The symbol χ is the Greek letter chi.)

Three different colorings of a graph H are shown in Figure 6.1. The coloring in Figure 6.1(a) is a 5-coloring, the coloring in Figure 6.1(b) is a 4-coloring, and the coloring in Figure 6.1(c) is a 3-coloring. Because the order of G is 9, the graph H is k-colorable for every integer k with $3 \leq k \leq 9$. Since H is 3-colorable, $\chi(H) \leq 3$. There is, however, no 2-coloring of H because H contains triangles and the three vertices of each triangle must be colored differently. Therefore, $\chi(H) \geq 3$ and so $\chi(H) = 3$.

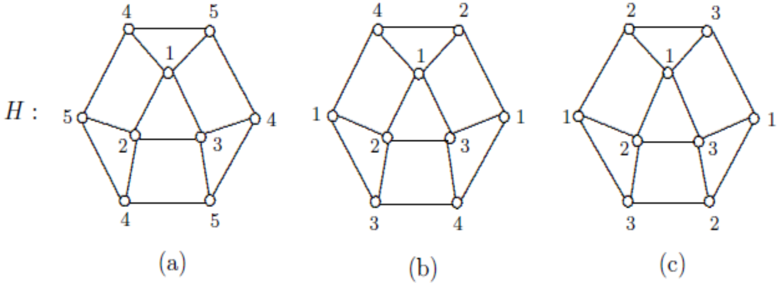

Figure 6.1: Colorings of a graph H

For any graph G with n vertices, $1 \leq \chi(G) \leq n$

Theorem 6.1 *If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.*

Proof. Suppose that $\chi(G) = k$. Then there exists a k-coloring c of G. Since c assigns distinct colors to every two adjacent vertices of G, the coloring c also assigns distinct colors to every two adjacent vertices of H. Therefore, H is k-colorable and so $\chi(H) \leq k = \chi(G)$. ∎

Recall that the clique number $\omega(G)$ of a graph G is the order of the largest clique (complete subgraph) of G. The following result is an immediate consequence of Theorem 6.1.

Corollary 6.2 *For every graph G, $\chi(G) \geq \omega(G)$.*

Proposition 6.3 *For graphs $G_1, G_2, \ldots, G_k$ and $G = G_1 \cup G_2 \cup \cdots \cup G_k$,*

$$\chi(G) = \max\{\chi(G_i) : 1 \leq i \leq k\}.$$

Corollary 6.4 *If G is a graph with components $G_1, G_2, \ldots, G_k$, then*

$$\chi(G) = \max\{\chi(G_i) : 1 \leq i \leq k\}.$$

Some bounds for the chromatic number

Every complete k-partite graph has chromatic number k.

A graph G of order n has chromatic number n if and only if $G = K_n$.

A graph G of order n has chromatic number 1 if and only if $G = \overline{K}_n$.

A nonempty graph G has chromatic number 2 if and only if G is bipartite.

Proposition 6.7 *A nontrivial graph G is 2-colorable if and only if G is bipartite.*

Theorem 6.8 *If every vertex of a graph G lies on at most k odd cycles for some nonnegative integer k, then*

$$\chi(G) \leq \left\lceil \frac{1 + \sqrt{8k + 9}}{2} \right\rceil.$$

A graph G has chromatic number at least 3 if and only if G contains an odd cycle.

Proposition 6.9 *For every integer $n \geq 3$,*

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Many bounds (both upper and lower bounds) have been developed for the chromatic number of a graph. Two of the most elementary bounds for the chromatic number of a graph G involve the independence number $\alpha(G)$, which, recall, is the maximum cardinality of an independent set of vertices of G. The lower bound is especially useful.

Theorem 6.10 *If G is a graph of order n, then*

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n - \alpha(G) + 1.$$

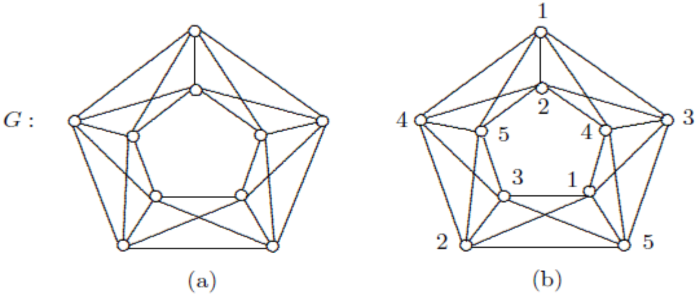

Figure 6.2: A 5-chromatic graph G with $\alpha(G) = 2$ and $\omega(G) = 4$

Applications of Coloring

Example 6.11 *At a gathering of eight employees of a company, which we denote by $A = \{a_1, a_2, \ldots, a_8\}$, it is decided that it would be useful to have these individuals meet in committees of three to discuss seven issues of importance to the company. The seven committees selected for this purpose are*

$$A_1 = \{a_1, a_2, a_3\},\ A_2 = \{a_2, a_3, a_4\},\ A_3 = \{a_4, a_5, a_6\},\ A_4 = \{a_5, a_6, a_7\},$$
$$A_5 = \{a_1, a_7, a_8\},\ A_6 = \{a_1, a_4, a_7\},\ A_7 = \{a_2, a_6, a_8\}.$$

If each committee is to meet during one of the time periods

1-2 *pm*, 2-3 *pm*, 3-4 *pm*, 4-5 *pm*, 5-6 *pm*,

then what is the minimum number of time periods needed for all seven committees to meet?

Solution. No two committees can meet during the same period if some employee belongs to both committees. Define a graph G whose vertex set is

$$V(G) = \{A_1, A_2, \ldots, A_7\}$$

where two vertices A_i and A_j are adjacent if $A_i \cap A_j \neq \emptyset$ (and so A_i and A_j must meet during different time periods). The graph G is shown in Figure 6.3. The answer to the question posed in the example is therefore $\chi(G)$. Since each committee consists of three members and there are only eight employees in all, it follows that the independence number of G is $\alpha(G) = 2$. By Theorem 6.10, $\chi(G) \geq n/\alpha(G) = 7/2$ and so $\chi(G) \geq 4$. Since there is a 4-coloring of G, as shown in Figure 6.3, it follows that $\chi(G) = 4$. Hence the minimum number of time periods needed for all seven committees to meet is 4. According to the resulting color classes, one possibility for these meetings is

1-2 pm: A_1, A_4; 2-3 pm: A_2, A_5; 3-4 pm: A_3; 4-5 pm: A_6, A_7. ♦

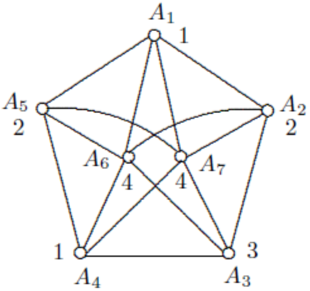

Figure 6.3: The graph G in Example 6.11

Example 6.12 *In a rural community, there are ten children (denoted by c_1, c_2, ..., c_{10}) living in ten different homes who require physical therapy sessions during the week. Ten physical therapists in a neighboring city have volunteered to visit some of these children one day during the week but no child is to be visited twice on the same day. The set of children visited by a physical therapist on any one day is referred to as a tour. It is decided that an optimal number of children to visit on a tour is 4. The following ten tours are agreed upon:*

$$\begin{array}{lll} T_1 = \{c_1, c_2, c_3, c_4\}, & T_2 = \{c_3, c_5, c_7, c_9\}, & T_3 = \{c_1, c_2, c_9, c_{10}\}, \\ T_4 = \{c_4, c_6, c_7, c_8\}, & T_5 = \{c_2, c_5, c_9, c_{10}\}, & T_6 = \{c_1, c_4, c_6, c_8\}, \\ T_7 = \{c_3, c_4, c_8, c_9\}, & T_8 = \{c_2, c_5, c_7, c_{10}\}, & T_9 = \{c_5, c_6, c_8, c_{10}\}, \\ T_{10} = \{c_6, c_7, c_8, c_9\}. & & \end{array}$$

It would be preferred if all ten tours can take place during Monday through Friday but the physical therapists are willing to work on the weekend if necessary. Is it necessary for someone to work on the weekend?

Solution. A graph G is constructed with vertex set $\{T_1, T_2, \ldots, T_{10}\}$, where T_i is adjacent to T_j $(i \neq j)$ if $T_i \cap T_j \neq \emptyset$. (See Figure 6.4.) The minimum number of days needed for these tours is $\chi(G)$. Since $\{T_2, T_3, T_5, T_7, T_9, T_{10}\}$ induces a maximum clique in G, it follows that $\omega(G) = 6$. By Theorem 6.2, $\chi(G) \geq 6$. There is a 6-coloring of G (see Figure 6.4) and so $\chi(G) = 6$. Thus, visiting all ten children requires six days and it is necessary for some physical therapist to work on the weekend. ♦

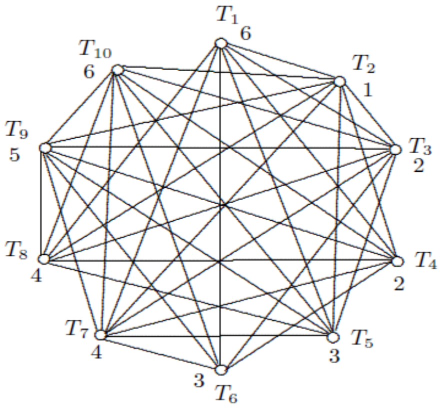

Figure 6.4: The graph G in Example 6.12

Example 6.13 *At a regional airport there is a facility that is used for minor routine maintenance of airplanes. This facility has four locations available for this purpose and so four airplanes can conceivably be serviced at the same time. This facility is open on certain days from 7 am to 7 pm. Performing this maintenance requires $2\frac{1}{2}$ hours per airplane; however, three hours are scheduled for each plane. A certain location may be scheduled for two different planes if the exit time for one plane is the same as the entrance time for the other. On a particular day, twelves airplanes, denoted by $P_1, P_2, \ldots, P_{12}$, are scheduled for maintenance during the indicated time periods:*

P_1 : 11 *am* - 2 *pm*;	P_2 : 3 *pm* - 6 *pm*;	P_3 : 8 *am* - 11 *am*;
P_4 : 1 :30 *pm* - 4 :30 *pm*;	P_5 : 1 *pm* - 4 *pm*;	P_6 : 2 *pm* - 5 *pm*;
P_7 : 9 :30 *am* - 12 :30 *pm*;	P_8 : 7 *am* - 10 *am*;	P_9 : *noon* - 3 *pm*;
P_{10} : 4 *pm* - 7 *pm*;	P_{11} : 10 *am* - 1 *pm*;	P_{12} : 9 *am* - *noon*.

Can a maintenance schedule be constructed for all twelve airplanes?

Solution. A graph G is constructed whose vertex set is the set of airplanes, that is, $V(G) = \{P_1, P_2, \ldots, P_{12}\}$. Two vertices P_i and P_j $(i \neq j)$ are adjacent if their scheduled maintenance overlaps (see Figure 6.5). Since there are only four locations available for maintenance, the question is whether the graph G is 4-colorable. In fact, $\chi(G) = \omega(G) = 4$, where $\{P_1, P_7, P_{11}, P_{12}\}$ induces a maximum clique in G. Ideally, it would be good if each color class has the same number of vertices (namely three) so that each maintenance crew services the same number of planes during the day. The 4-coloring of G shown in Figure 6.5 has this desired property. ♦

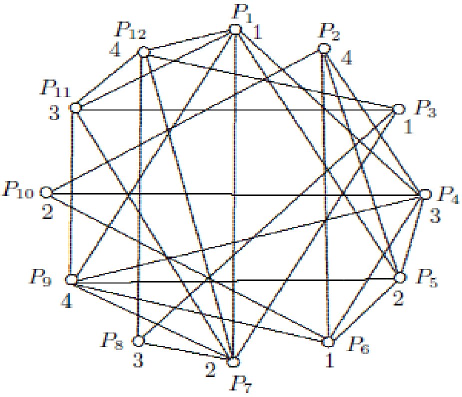

Figure 6.5: The graph G in Example 6.13

Example 6.14 *Two dentists are having new offices designed for themselves. In the common waiting room for their patients, they have decided to have an aquatic area containing fish tanks. Because some fish require a coldwater environment while others are more tropical and because some fish are aggressive with other types of fish, not all fish can be placed in a single tank. It is decided to have nine exotic fish, denoted by* $F_1, F_2, \ldots, F_9$*, where the fish that cannot be placed in the same tank as* F_i $(1 \le i \le 9)$ *are indicated below.*

$$\begin{array}{lll} F_1: \ F_2, F_3, F_4, F_5, F_6, F_8, & F_2: \ F_1, F_3, F_6, F_7, & F_3: \ F_1, F_2, F_6, F_7, \\ F_4: \ F_1, F_5, F_8, F_9, & F_5: \ F_1, F_4, F_8, F_9, & F_6: \ F_1, F_2, F_3, F_7, \\ F_7: \ F_2, F_3, F_6, F_9, & F_8: \ F_1, F_4, F_5, F_9, & F_9: \ F_4, F_5, F_7, F_8. \end{array}$$

What is the minimum number of tanks required?

Solution. A graph G is constructed with vertex set $V(G) = \{F_1, F_2, \ldots, F_9\}$, where F_i is adjacent to F_j $(i \ne j)$ if F_i and F_j cannot be placed in the same tank (see Figure 6.6). Then the minimum number of tanks required to house all fish is $\chi(G)$. In this case, $\omega(G) = 4$, so $\chi(G) \ge 4$. However, $n = 9$ and $\alpha(G) = 2$ and so $\chi(G) \ge 9/2$. Thus $\chi(G) \ge 5$. A 5-coloring of G is given in Figure 6.6, implying that $\chi(G) \le 5$ and so $\chi(G) = 5$. ♦

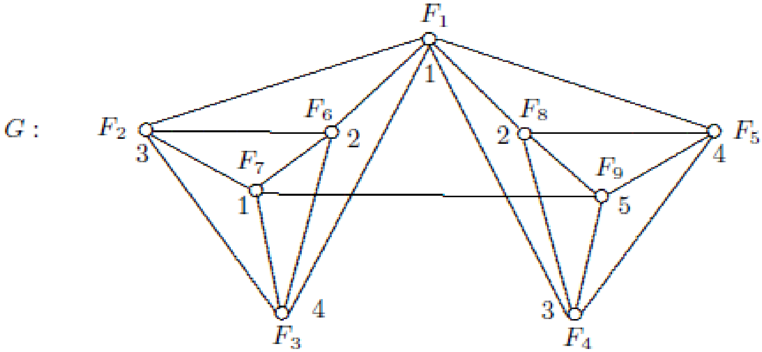

Figure 6.6: The graph of Example 6.14

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Complete coloring and Achromatic number

- By a complete coloring of a graph G, we mean a proper vertex coloring of G having the property that for every two distinct colors i and j used in the coloring, there exist adjacent vertices of G colored i and j.
- A complete coloring in which k colors are used is a complete k-coloring.
- If a graph G has a complete k-coloring, then G must contain at least ${}^{k}C_2$ edges. Consequently, if the size of a graph G is less than ${}^{k}C_2$ for some positive integer k, then G cannot have a complete k-coloring.
- If G is a k-chromatic graph, then every k-coloring of G is a complete coloring of G.
- If there is a complete k-coloring of a graph G for some positive integer k, then it need not be the case that $\chi(G) = k$.

For example, the 3-coloring of the path P_4 given in Figure is a complete 3-coloring and yet $\chi(P_4) = 2$. Indeed, the 4-coloring of the path P_8 is a complete 4-coloring.

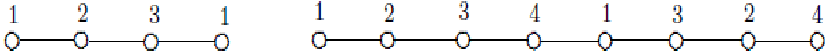


Figure 12.1: A complete 3-coloring of P_4 and a complete 4-coloring of P_8

Achromatic Number

- The largest positive integer k for which G has a complete k-coloring is the achromatic number of G, which is denoted by $\psi(G)$.
- $\psi(G) \geq \chi(G)$ for every graph G.
- if G is a graph of order n, then $\psi(G) \leq n$
- For a complete graph K_n, $\psi(K_n) = \chi(K_n) = n$
- The paths P_4 and P_8 illustrate the fact that a bipartite graph need not have achromatic number 2. Every complete bipartite graph, however, does have achromatic number 2.

Theorem

Every complete bipartite graph has achromatic number 2.

Proof.

Suppose, to the contrary, that there is some complete bipartite graph G such that $\psi(G) \neq 2$. Since $\chi(G) = 2$, it follows that $\psi(G) = k$ for some integer $k \geq 3$. Let there be given a complete k-coloring of G. Then two vertices in some partite set of G must be assigned distinct colors, say i and j. Since this coloring is a proper coloring, no vertex in the other partite set of G is colored i or j. Therefore, G does not contain adjacent vertices colored i and j, producing a contradiction. $\square$

Proposition *If G is a graph of size m, then*

$$\psi(G) \leq \frac{1+\sqrt{1+8m}}{2}.$$

Proof. Suppose that $\psi(G) = k$. Then

$$m \geq \binom{k}{2} = \frac{k(k-1)}{2}.$$

Solving this inequality for k, we obtain $\psi(G) = k \leq \frac{1+\sqrt{1+8m}}{2}$.

Theorem 12.9 *For each $n \geq 2$, $\psi(P_n) = \max\left\{k : \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right)(k-2) + 2 \leq n\right\}$.*

According to Theorem 12.9, $\psi(P_7) = 3$ and $\psi(P_{11}) = 5$. A complete 3-coloring of P_7 and a complete 5-coloring of P_{11} are given in Figure 12.4 (see Exercise 9).

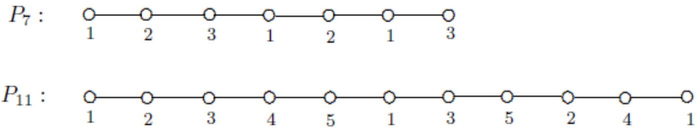


Figure 12.4: The graphs P_8 and P_{11} with $\psi(P_7) = 3$ and $\psi(P_{11}) = 5$

The following is then a consequence of Theorem 12.9 (see Exercise 10).

Corollary 12.10 *For every positive integer k, there exists a positive integer n such that $\psi(P_n) = k$.*

Theorem 12.11 *For each $n \geq 3$, $\psi(C_n) = \max\left\{k : k\left\lfloor \frac{k}{2} \right\rfloor \leq n\right\} - s(n)$, where $s(n)$ is the number of positive integer solutions of $n = 2x^2 + x + 1$.*

According to Theorem 12.11, $\psi(C_{10}) = 5$ and $\psi(C_{11}) = 4$. A complete 5-coloring of C_{10} and a complete 4-coloring of C_{11} are given in Figure 12.5 (see Exercise 11).

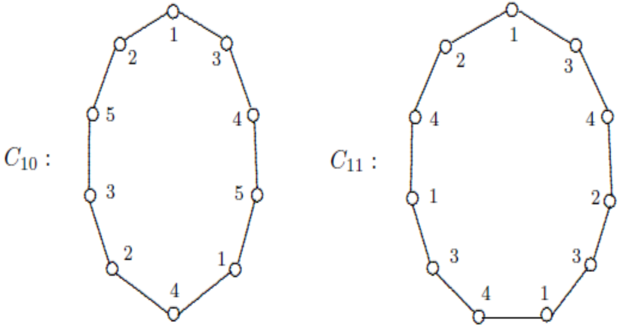


Figure 12.5: The graphs C_{10} and C_{11} with $\psi(C_{10}) = 5$ and $\psi(C_{11}) = 4$

Complete Coloring and Achromatic Number

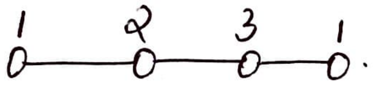
Complete coloring

A proper vertex coloring in which for every two distinct colors i and j used in the coloring, there exist adjacent vertices of G colored i and j.

* If there is a complete coloring on k colors, then there will be at least kC_2 edges.

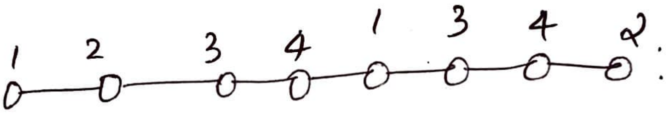
Eg.

Path P_4



Here $3C_2 = 3$ and here we have exactly 3 edges and all the combinations $(1,2), (1,3)$ and $(2,3)$ appeared on adjacent vertices

Path P_8:



~~Here~~ $4C_2 = \frac{4 \cdot 3}{1 \cdot 2} = 6$ and we have a total of 7 edges here. Also all the combinations $(1,2), (1,3), (1,4), (2,3), (2,4)$ and $(3,4)$ is appeared on adjacent vertices.

Note

1) $\psi(G) \geq \chi(G)$.

2) $\psi(G) \leq n$, where n is the total number of vertices in the graph G

3) For a complete graph, the achromatic number is n.

4) For a bipartite graph $\psi(G) \geq 2$.

5) For a complete bipartite graph $\psi(G) = 2$.

Proof.

First we will prove that the achromatic number of a bipartite graph is at least 2 and after that we will prove that the achromatic number of complete bipartite graph is exactly 2.

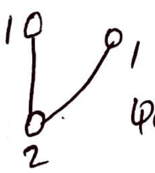
1) Bipartite graph

We know that all paths are bipartite graphs and we already seen that the achromatic number of P_4 is 3 and that of P_8 is 4. Also we can prove that the achromatic number of P_2 is 2. It is illustrated below.

P_2 or $K_{1,1}$

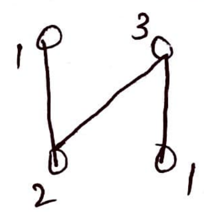
$\psi(G)=2$.

P_3 or $K_{2,1}$



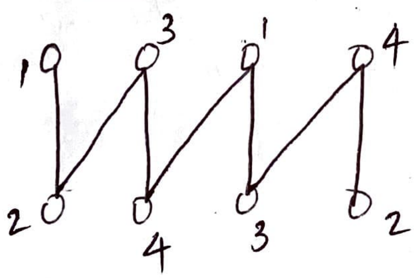
$\psi(G)=2$

P_4 or $K_{2,2}$.



$\psi(G)=3$.

P_8 or $K_{4,4}$.



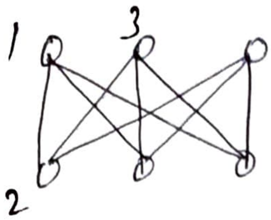
$\psi(G)=4$.

Thus for $\psi(K_{1,1})=2$, $\psi(K_{2,1})=2$, $\psi(K_{2,2})=3$, $\psi(K_{4,4})=4$, ...

Thus the achromatic number of a bipartite graph is at least 2.

2. Complete bipartite graph.

Here we have to prove that the achromatic number is 2. On the contrary assume that it is more than 2 (let's say 3). Consider an example $K_{3,3}$.



* If there is a complete coloring on 3 colors, then all the combinations (1,2) ~~(1,3)~~ (1,3) and (2,3) should appear on adjacent vertices. So assign color 1 to first vertex in the first partite set and color 2 to the first vertex in the second partite set. Color 3 can assign to the second vertex in the first partition. Thus we will get the combinations (1,2) and (3,2) which are appeared on two adjacent vertices. But we ~~will not~~ cannot assign the pair of colors (1,3) to two adjacent vertices. Thus our assumption is wrong and thus for a complete bipartite graph, the achromatic number is 2.

* If G is a graph of size m, then $\psi(G) \leq \frac{1+\sqrt{1+8m}}{2}$.

Ans suppose $\psi(G) = k$. Then there will be at least kC_2 edges.

Thus $m \geq kC_2 = \frac{k\cdot(k-1)}{1\cdot 2}$ ie $m \geq \frac{k(k-1)}{2}$.

$\Rightarrow 2m \geq k^2 - k$ or $k^2 - k - 2m \leq 0$. (quadratic in k)

Roots of $k^2 - k - 2m$ is $\frac{1 \pm \sqrt{1+8m}}{2}$ exclude the root $\frac{1-\sqrt{1+8m}}{2}$.

$\therefore k \leq \frac{1+\sqrt{1+8m}}{2}$ //

Theorem.

For each $n \geq 2$, $\varphi(P_n) = \max\{k : (\lfloor \frac{k}{2} \rfloor + 1)(k-2) + 2 \leq n\}$

Example.

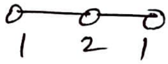
$n = 3$. ie P_3.

If $k = 2$, $(\lfloor \frac{k}{2} \rfloor + 1)(k-2) + 2 = (\lfloor \frac{2}{2} \rfloor + 1)(2-2) + 2 = 2$. Also $2 \leq 3$.

If $k = 3$, $(\lfloor \frac{k}{2} \rfloor + 1)(k-2) + 2 = (\lfloor \frac{3}{2} \rfloor + 1)(3-2) + 2 = 2 \cdot 1 + 2 = 4$.

But $4 \not\leq 3$ #.

$\therefore$ the maximum value of k in which the inequality holds is 2. $\therefore \varphi(P_3) = 2$.



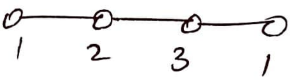
$n = 4$ ie P_4.

If $k = 2$, $(\lfloor \frac{k}{2} \rfloor + 1)(k-2) + 2 = 2$ and $2 \leq 4$ inequality holds.

If $k = 3$, $(\lfloor \frac{k}{2} \rfloor + 1)(k-2) + 2 = 4$ and $4 \leq 4$ inequality holds.

If $k = 4$, $(\lfloor \frac{k}{2} \rfloor + 1)(k-2) + 2 = (2+1)2 + 2 = 8 \not\leq 4$ inequality not holds.

$\therefore$ the maximum value of k in which inequality holds is 3. $\therefore \varphi(P_4) = 3$.



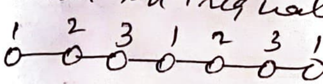
$n = 7$ ie P_7.

If $k = 2$; $(\lfloor \frac{k}{2} \rfloor + 1)(k-2) + 2 = 2$ and $2 \leq 7$ inequality holds.

If $k = 3$ $(\lfloor \frac{k}{2} \rfloor + 1)(k-2) + 2 = 4$ and $4 \leq 7$ inequality holds.

If $k = 4$ $(\lfloor \frac{k}{2} \rfloor + 1)(k-2) + 2 = 8$ and $8 \not\leq 7$ inequality not holds

$\therefore$ the maximum value of k in which the inequality holds is 3 and thus $\varphi(P_7) = 3$.



$n = 11$, i.e P_{11}.

If $k=2$, $(\lfloor \frac{k}{2} \rfloor + 1)(k-2)+2 = 2$ and $2 \le 11$ inequality holds

If $k=3$, " " $= 4$ and $4 \le 11$ " "

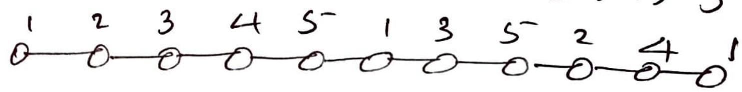
If $k=4$, " " $= 8$ and $8 \le 11$ " "

If $k=5$, $(\lfloor \frac{k}{2} \rfloor + 1)(k-2)+2 = 3 \times 3 + 2 = 11$ and $11 \le 11$ and so inequality holds.

If $k=6$, $(\lfloor \frac{k}{2} \rfloor + 1)(k-2)+2 = 4 \times 4 + 2 = 18$ and $18 \not\le 11$

$\therefore$ inequality does not holds.

$\therefore$ the maximum value of k in which ineq holds is 5 and so $\varphi(P_{11}) = 5$,

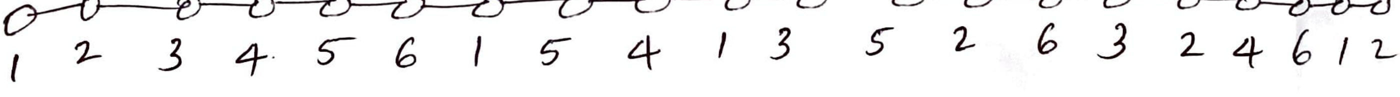


$n = 20$, i.e P_{20}.

For $k = 2, 3, 4, 5, 6$ the inequality will holds.

If $k = 7$, $(\lfloor \frac{k}{2} \rfloor + 1)(k-2)+2 = 4 \times 5 + 2 = 22 \not\le 20$.

$\therefore$ the maximum value of k for which the inequality holds is 6. $\therefore$ $\varphi(P_{20}) = 6$.



(1,2) (1,3) (1,4) (1,5) (1,6) (2,3) (2,4) (2,5) (2,6),
(3,4) (3,5) (3,6) (4,5) (4,6) (5,6)

$6C_2 = \frac{6 \times 5}{1 \cdot 2} = 3 \times 5 = 15$ and we have a total of 19 edges.

Theorem

For each $n \geq 3$, $\varphi(C_n) = \max\{k : k\lfloor \frac{k}{2} \rfloor \leq n\} - s(n)$
where $s(n)$ is the number of integer solutions of
$n = 2x^2 + x + 1$

Examples

$n = 3$ ie C_3.

$s(n)$ = integer solutions of $3 = 2x^2 + x + 1 \Rightarrow 2x^2 + x - 2 = 0$.

solutions of $2x^2 + x - 2 = 0 \Rightarrow \frac{-1 \pm \sqrt{1+16}}{4} = \frac{-1 \pm \sqrt{17}}{4}$. no integer solns

$\therefore s(n) = 0$.

$\therefore$ when $k = 2$, $k\lfloor \frac{k}{2} \rfloor = 2\lfloor \frac{2}{2} \rfloor = 2 \leq 3$ holds the inequality

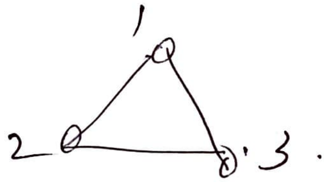
and $k - s(n) = 2 - 0 = 2$.

when $k = 3$, $k\lfloor \frac{k}{2} \rfloor = 3 \cdot \lfloor \frac{3}{2} \rfloor = 3 \times 1 = 3$ and $3 \leq 3$, inequality holds.

$k - s(n) = 3 - 0 = 3$.

when $k = 4$, $k\lfloor \frac{k}{2} \rfloor = 4 \cdot \lfloor \frac{4}{2} \rfloor = 4 \times 2 = 8 \nleq 3$ inequality not holds.

$\therefore$ ~~$\varphi(C_3) = 3$~~ $\varphi(C_3) = 3$



$n = 4$ ie C_4.

solutions of $2x^2+x+1 = n \Rightarrow 2x^2+x+1 = 4$.

ie $2x^2+x-3 = 0$.

$$\frac{-1 \pm \sqrt{1+24}}{4} = \frac{-1 \pm \sqrt{25}}{4} = \frac{-1 \pm 5}{4}.$$

ie $\frac{-6}{4} = \frac{-3}{2}$ and $\frac{-1+5}{4} = \frac{4}{4} = 1$.

ie $-3/2$ and 1 are the solutions and the integer solution is 1 $\therefore$ there is only one integer solution and thus $s(n) = 1$.

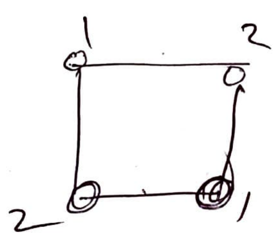
If $k = 2$, $k \cdot \lfloor \frac{k}{2} \rfloor = 2 \cdot \lfloor \frac{2}{2} \rfloor = 2 \times 1 = 2 \leq 4$. (holds)

$k = 3$, $3 \cdot \lfloor \frac{3}{2} \rfloor = 3 \cdot 1 = 3 \leq 4$ holds.

$k = 4$, $4 \cdot \lfloor \frac{4}{2} \rfloor = 4 \times 2 = 8 \nleq 4$ not holds.

$\therefore$ the maximum value of k which holds the inequality is 3. $\therefore$ $\varphi(C_4) = k - s(n)$

$= 3 - 1 = \underline{\underline{2}}$.



Q10. $n=10$ ie C_{10}

solutions of $n=2x^2+x+1 \Rightarrow 2x^2+x+1=10$

ie $2x^2+x-9=0$

$$\frac{-1\pm\sqrt{1+72}}{4} = \frac{-1\pm\sqrt{73}}{4}$$ no integer solutions

$\therefore s(n)=0$.

If $k=2$, $k\cdot\left\lfloor\frac{k}{2}\right\rfloor = 2\cdot\left\lfloor\frac{2}{2}\right\rfloor = 2 \le 10$.

$k=3$, $3\cdot\left\lfloor\frac{3}{2}\right\rfloor = 3\times1 = 3\le 10$.

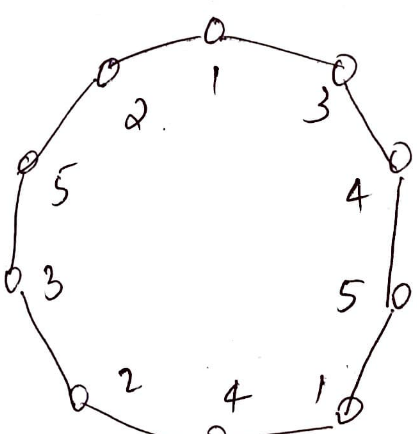
$k=4$, $4\cdot\left\lfloor\frac{4}{2}\right\rfloor = 4\times2 = 8\le 10$.

$k=5$, $5\times\left\lfloor\frac{5}{2}\right\rfloor = 5\times2 = 10\le 10$

$\Big\}$ inequality holds.

$k=6$, $6\cdot\left\lfloor\frac{6}{2}\right\rfloor = 6\times3 = 18 \not\le 10$.

$\therefore$ the maximum value of k in which the inequality holds is 5. $\therefore$ the achromatic number of C_{10} is $\psi(C_{10}) = k - s(n) = 5 - 0 = \underline{\underline{5}}$.



(1,2) (1,3) (1,4)
(1,5) (2,3) (2,4)
(2,5) (3,4) (3,5)
(4,5).

$^5C_2 = \frac{5\cdot4}{1\cdot2} = 10$ and we have 10 edges here.

$n = 11$ ie C_{11}.

solutions of $n = 2x^2 + x + 1$. $\Rightarrow 2x^2 + x + 1 = 11$.

ie $2x^2 + x - 10 = 0$.

$$\frac{-1 \pm \sqrt{1 + 80}}{4} = \frac{-1 \pm \sqrt{81}}{4} = \frac{-1 \pm 9}{4}.$$

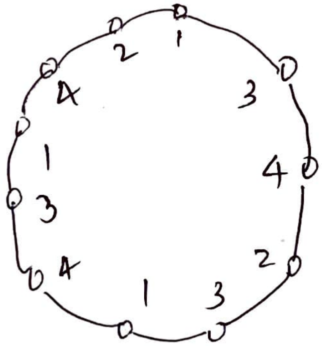
$-\frac{10}{4}$ and $\frac{8}{4}$ ie. -2.5 and 2 $\therefore$ we have only one integer solutions $\therefore$ $s(n) = 1$.

if $k = 2$, $k\left\lfloor\frac{k}{2}\right\rfloor = 2 \cdot \left\lfloor\frac{2}{2}\right\rfloor = 2 \leq 11$.
$k = 3$, $k \cdot \left\lfloor\frac{k}{2}\right\rfloor = 3 \cdot \left\lfloor\frac{3}{2}\right\rfloor = 3 \leq 11$.
$k = 4$ $k\left\lfloor\frac{k}{2}\right\rfloor = 4 \cdot \left\lfloor\frac{4}{2}\right\rfloor = 4 \times 2 = 8 \leq 11$
$k = 5$ $k\left\lfloor\frac{k}{2}\right\rfloor = 5 \cdot \left\lfloor\frac{5}{2}\right\rfloor = 10 \leq 11$.

inequality holds.

$k = 6$ $k\left\lfloor\frac{k}{2}\right\rfloor = 6 \cdot \left\lfloor\frac{6}{2}\right\rfloor = 18 \not\leq 11$.

$\therefore$ the max. value of k in which the inequality holds is 5. $\therefore \psi(C_{11}) = k - s(n) = 5 - 1 = \underline{4}$.



$(1,2)(1,3)(1,4)\ (2,3)(2,4)(3,4)$.

$4C_2 = \frac{4 \cdot 3}{1 \cdot 2} = 6$. and we have a total of 11 edges.

$5C_2 = \frac{5 \cdot 4}{1 \cdot 2} = 10$ necessary condition. not sufficient.

MAT2011 Graph Theory and Applications

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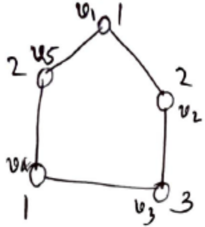
b-coloring coloring and b-chromatic number

- A b-coloring of a graph G is a proper coloring in which every color class contains at least one vertex that has a neighbor in each of the other color classes.
- The b-chromatic number , $\varphi(G)$ of a graph G is defined as the maximum k such that G admits a proper k-coloring in which every color class contains at least one vertex that has a neighbor in each of the other color classes.
- b-coloring is a refinement of complete coloring. Every b-coloring is a complete coloring but every complete coloring is not a b-coloring

b-colorings



color class 1 = $\{v_1, v_4\}$

color class 2 = $\{v_2, v_5\}$

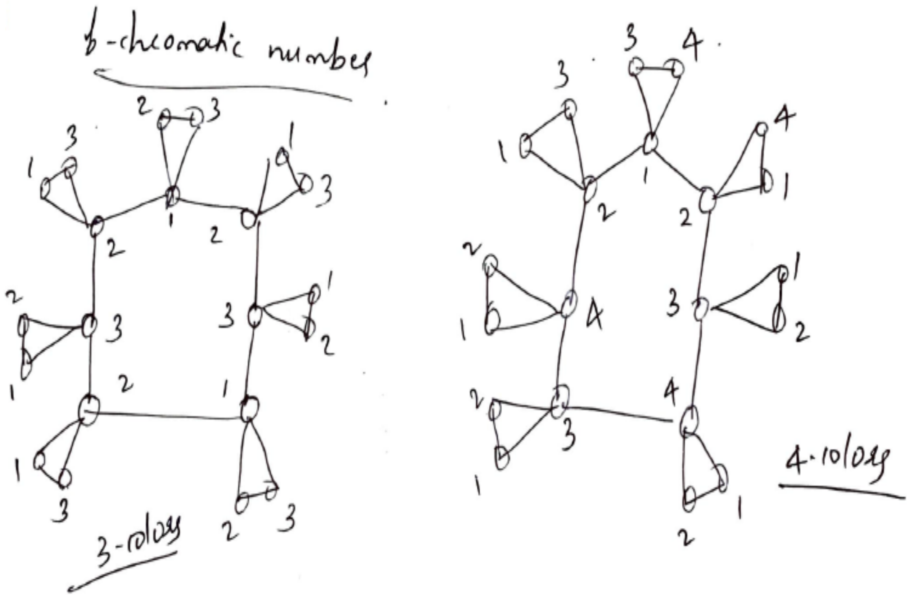
color class 3 = $\{v_3\}$.

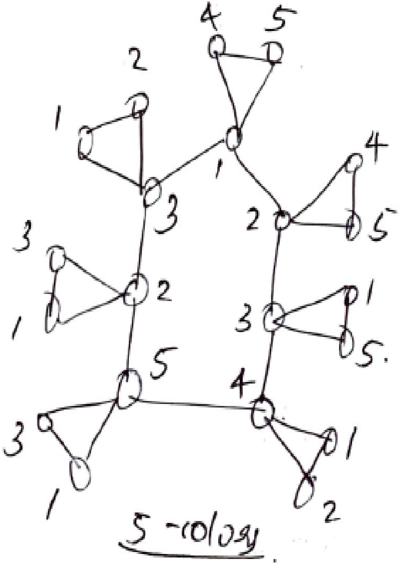
v_4 is adjacent to v_5 and v_3 with colors 2 and 3 respectively.

v_2 is adjacent to v_1 and v_3 with colors 1 and 3 respectively.

v_3 is adjacent to v_2 and v_4 with colors 2 and 1 respectively.

b-chromatic number



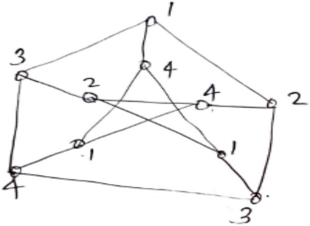


$\phi(G) = 5$

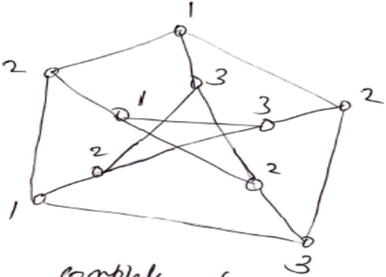
b-coloring is a refinement of complete coloring

b-coloring is a refinement of complete coloring.

Eg complete coloring of Petersen graph.



complete coloring using 4-colors.



complete coloring using 3-colors and also it is a b-coloring.

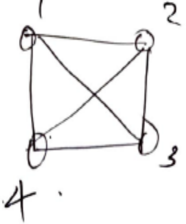
Every b-coloring is a complete coloring. But not every complete coloring is ~~not~~ a b-coloring. (Above example)

- The lower bound and upper bound of the b-chromatic number of a graph G is $\chi(G) \leq \varphi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.
- For a complete graph G, $\chi(G) = \varphi(G) = \Delta(G) + 1$
- Every proper coloring using $\chi(G)$ colors of G is a b-coloring
- If m is the b-chromatic number of a graph G, then there should be at least m vertices in G with degree at least $m - 1$.
- The above condition is not a sufficient condition because, the b-chromatic number of Petersen graph is 3 but, there are 10 vertices having degree 3.

b-chromatic number of a complete graph

complete graph.

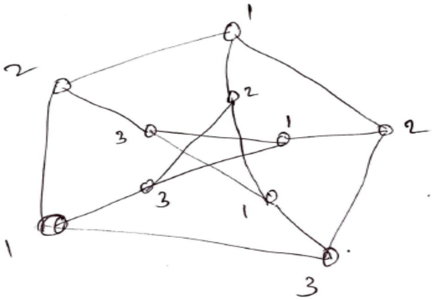
K_4



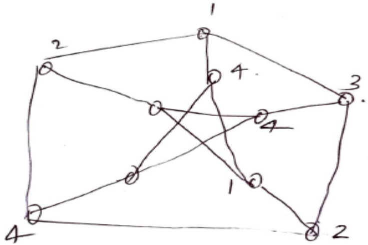
$\phi(G) = 4 = n-1 = \Delta(G)+1.$

Here $\Delta(G) = 3.$

Petersen graph.



$\varphi(G) = 3$.



No: of vertices with degree 3 = 10.
But cannot have a b-coloring using 4 colors

Applications of b-coloring in clustering of web documents

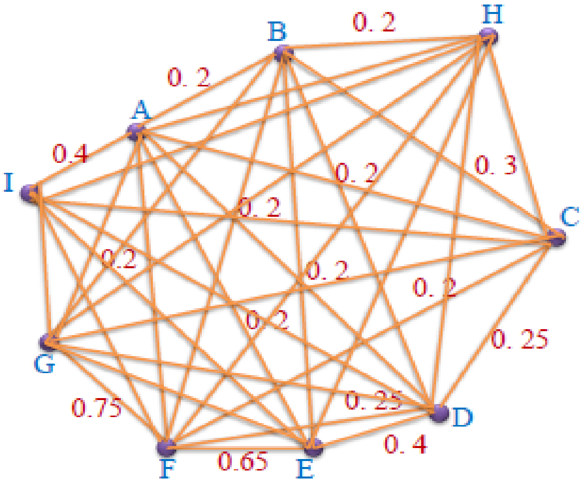
Clustering: The division of data items (objects, instances, etc.) into groups or categories, such that all objects in the same group are similar to each other, while dissimilar from objects in the other groups is called the clustering of data items.

b-coloring based clustering: Graph Construction

- The pair of data to be handled is denoted by v_t.
- The dissimilarity of a pair of data is calculated by the function $d : V \times V \longrightarrow R^+$, V is the set of data items and R^+ is the set of positive real numbers. Also this function d is symmetric.
- Map each data item to a vertex and connect each pair of vertices v_i and v_j by the edge (v_i, v_j) with label $d(v_i, v_j)$. A new parameter θ, called the threshold value which will determine the edges in the graph.
- Delete each edge (v_i, v_j) in the graph iff $d(v_i, v_j) \leq \theta$.

Example: suppose a set of data with the dissimilarities in table 1 is given. Figure 1 superior threshold θ graph for table 1 where the threshold is set to 0.15

Vertex	A	B	C	D	E	F	G	H	I
A	0								
B	0.20	0							
C	0.10	0.30	0						
D	0.10	0.20	0.25	0					
E	0.20	0.20	0.15	0.40	0				
F	0.20	0.20	0.20	0.25	0.65	0			
G	0.15	0.10	0.15	0.10	0.10	0.75	0		
H	0.10	0.20	0.10	0.10	0.05	0.05	0.05	0	
I	0.40	0.075	0.15	0.15	0.15	0.15	0.15	0.15	0



B
H
A
I
C
G
D
F
E
0. 2
0. 2
0.4
0. 2
0. 3
0. 2
0.2
0. 2
0. 2
0. 2
0. 25
0. 25
0.75
0.65
0. 4

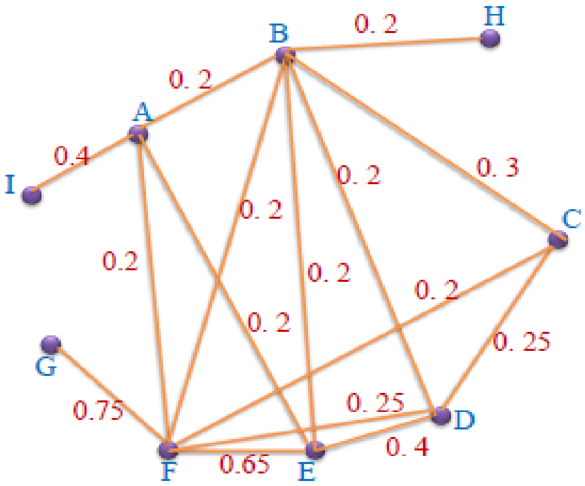


Fig. 1. A threshold graph for Table 1 (θ =0.15)

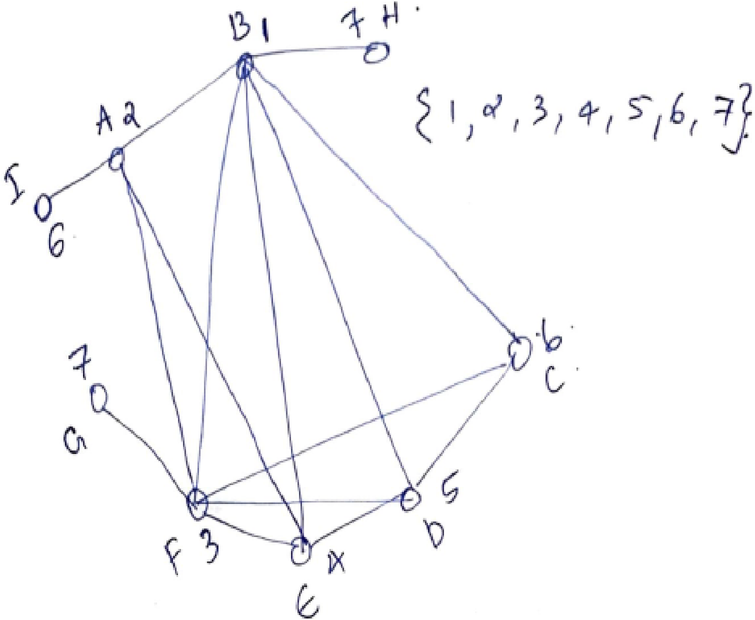
Clustering construction using b-coloring

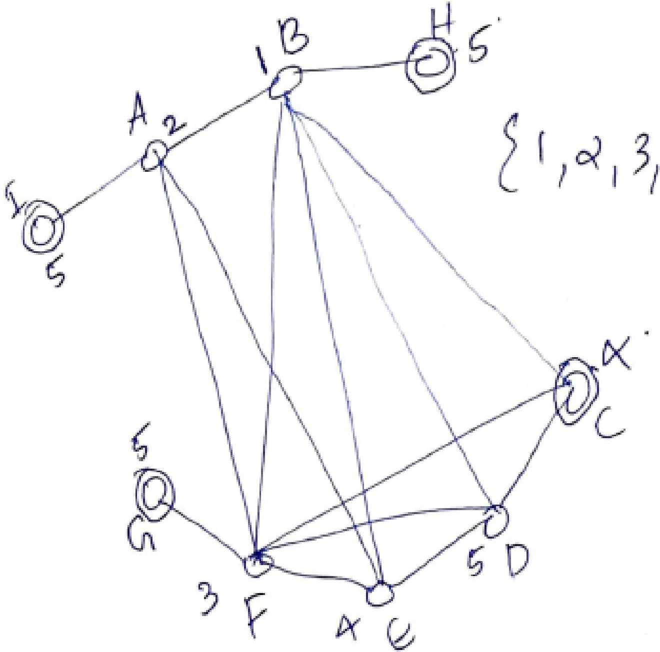
Procedure 1: The coloring initialization:

- Initialize the color of the vertices of graph by $\Delta(G) + 1$ number of colors.
- Color the vertex having maximum degree $\Delta(G)$ by the color 1 and adds it to a list S. Then, the procedure colors the remaining vertices as follows:
- The vertex v_i with the largest degree among all colored vertices belonging to S is selected
- If there are non-colored vertices v_j adjacent to v_i then a new color is assigned to every one of them and are added to S.

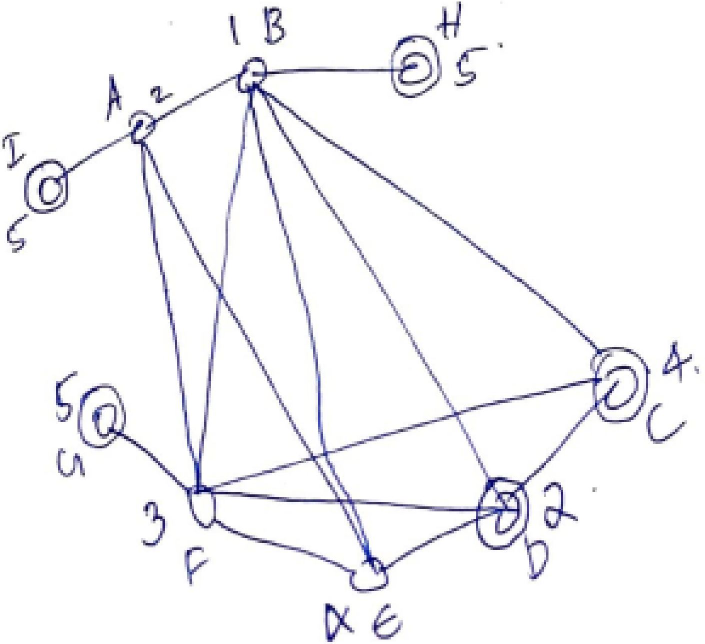
Procedure 2: find a b-coloring of G:

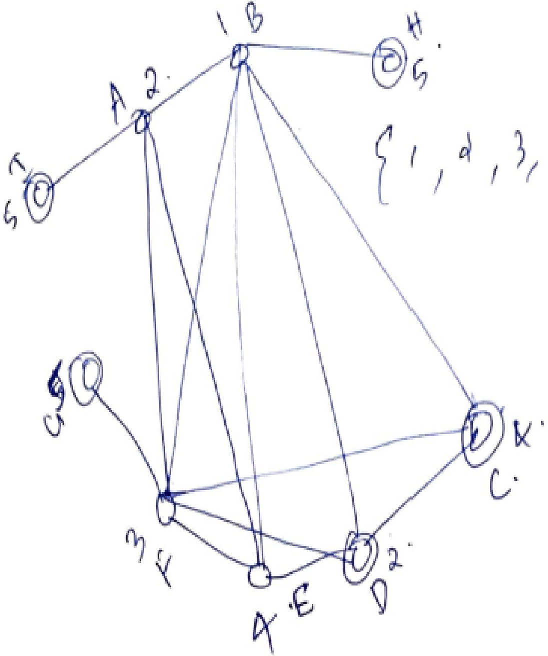
- Remove a non-dominating color p from the graph by recoloring every vertex colored with p by an already used color not appearing in its neighborhood.
- The operation is repeated until all the colors are marked as dominating.


B 1
7 H
{1, 2, 3, 4, 5, 6, 7}
A 2
I
6
6
C
7
G
5
F 3
D
E

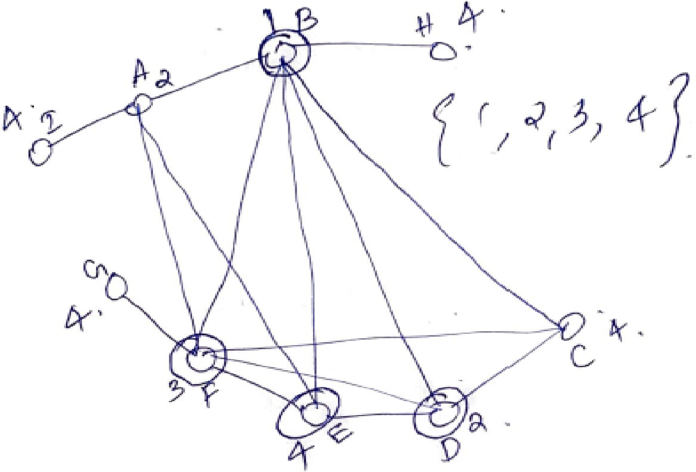


$\{1, 2, 3, 4, 5\}$.


1 B
H
5
A
2
I
5
4.
C
5
G
3
F
2
D
E



$\{1, 2, 3, 4, 5\}$.

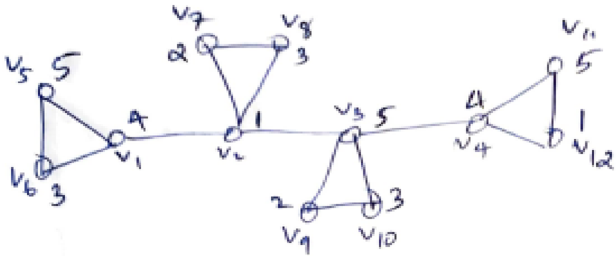


B-coloring using 4 colors

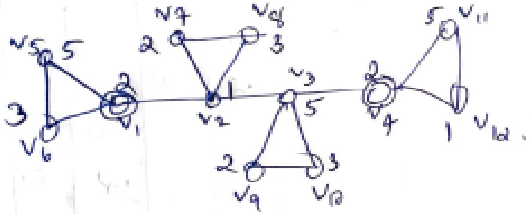
Consider the next example. Here the threshold value is 0.15 and the edge weights greater than 0.15 is given below. Using the below information, construct a grap b-coloring and hence find out the clusters.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
v_1	0											
v_2	.2	0										
v_3		.3	0									
v_4			.3	0								
v_5	.35				0							
v_6	.25					0						
v_7		.35					0					
v_8							.3	0				
v_9			.22						0			
v_{10}			.25						.2	0		
v_{11}				.24							0	
v_{12}				.35							.2	0

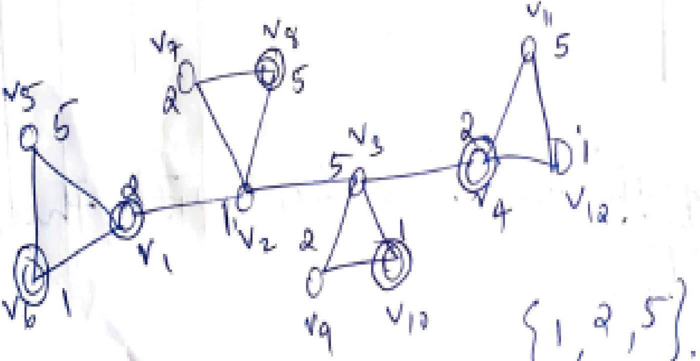
A graph on 12 vertices with threshold value · 15



$\{1, 2, 3, 4,$



$\{1, 2, 3, 5\}$



b-chromatic number 3

∴ cluster-1 = {v_2, v_6, v_{10}, v_{12}}

cluster-2 = {v_7, v_9, v_4, v_1} cluster-3 {v_5, v_8, v_3, v_1

MAT2011 Graph Theory and Applications

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Restricted Vertex Coloring

When attempting to properly color the vertices of a graph G (often with a restricted number of colors), there may be instances when

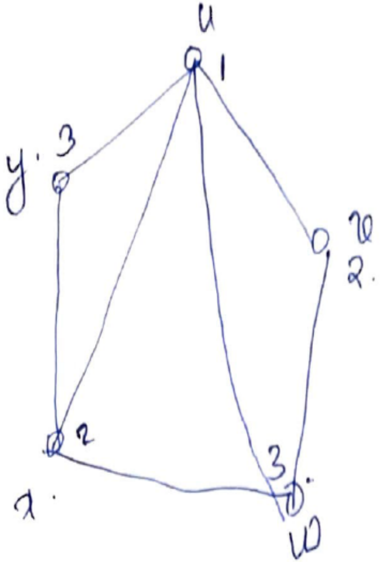
- there is only one choice for the color of each vertex of G except for the names of the colors
- every vertex of G has some preassigned restriction on the choice of a color that can be used for the vertex or
- some vertices of G have been given preassigned colors and the remaining vertices must be colored according to these restrictions.

Coloring of a graph with such restrictions are known as restricted vertex coloring

Uniquely colorable graph

- Suppose that G is a k-chromatic graph. Then every k-coloring of G produces a partition of V (G) into k independent subsets (color classes). If every two k- colorings of G result in the same partition of V (G) into color classes, then G is called uniquely k-colorable or simply uniquely colorable.
- The complete graph K_n is uniquely colorable
- The complete k-partite graphs are also uniquely colorable.

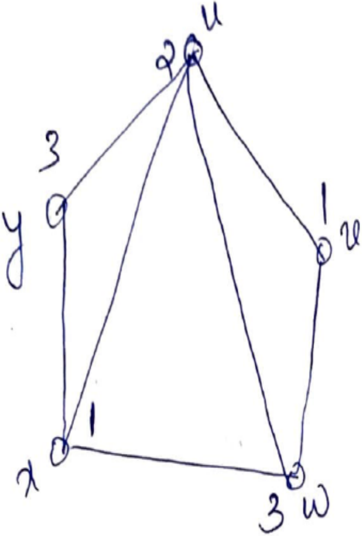
Example of uniquely colorable graphs



colour class-1 = {u}.

" 2 = {v, x}.

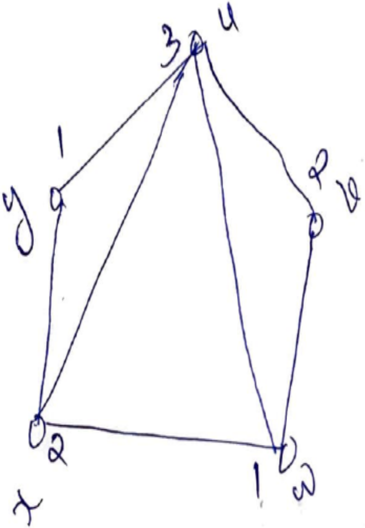
" 3 = {w, y}.



1 – {v, x}

2 – {u}

3 – {y, w}

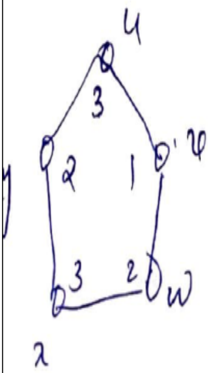


$1 - \{w, y\}$

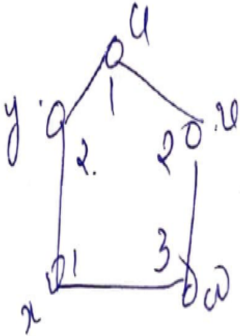
$2 - \{v, x\}$.

$3 - \{u\}$.

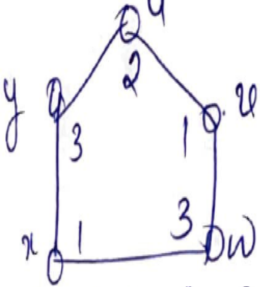
Example of a graph wich is not uniquely colorable



$\{v\}, \{y, w\}, \{x, u\}$.



$, \{v, y\}$. $\{w\}, \{u, x\}$



$\{u\}, \{v, x\}, \{w, y\}$

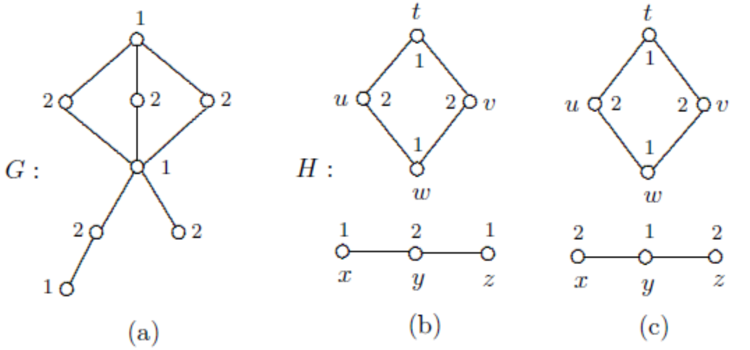


Figure A graph that is uniquely 2-colorable and another that is not

List Coloring

- Let G be a graph for which there is an associated set L(v) of permissible colors for each vertex v of G. The set L(v) is commonly called a color list for v.
- A list coloring of G is then a proper coloring c of G such that $c(v) \in L(v)$for each vertex v of G. A list coloring is also referred to as a choice function.
- If $\mathcal{L} = \{L(v) : v \in V(G)\}$ is a set of color lists for the vertices of G and there exists a list coloring for this set L of color lists, then G is said to be $\mathcal{L}$-choosable or $\mathcal{L}$-list-colorable.
- The list chromatic number $\chi_l(G)$ of G is the minimum positive integer k such that G is k-choosable. Then $\chi_l(G) \geq \chi(G)$.

Example

Consider the cycle C_4 and suppose that we are given any four color lists $L(v_i), 1 \leq i \leq 4$, with $|L(v_i)| = 2$. Let $L(v_1) = \{a, b\}$. We consider three cases.

- Case 1: $a \in L(v_2) \bigcap L(v_4)$. In this case, assign v_1 the color b and v_2 and v_4 the color a. Then there is at least one color in $L(v_3)$ that is not a. Assigning v_3 that color gives C_4 a list coloring for this collection of lists.
- Case 2: The color a belongs to exactly one of $L(v_2)$ and $L(v_4)$. If there is some color $x \in L(v_2) \bigcap L(v_4)$, then assign v_2 and v_4 the color x and v_1 the color a. There is at least one color in $L(v_3)$ different from x. Assign v_3 that color. Hence there is a list coloring of C_4 for this collection of lists. Next, suppose that there is no color belonging to both $L(v_2)$ and $L(v_4)$. If $a \in L(v_3)$, then assign a to both v_1 and v_3. There is a color available for both v_2 and v_4.

If $a \notin L(v_3)$, then assign v_1 the color a, assign v_2 the color y in $L(v_2)$ different from a, assign v_3 any color z in $L(v_3)$ different from y, and assign v_4 any color in $L(v_4)$ different from z. This is a list coloring for C_4.

- Case 3: $a \notin L(v_2) \bigcup L(v_4)$. Then assign v_1 the color a and v_3 any color from $L(v_3)$. Hence there is an available color from $L(v_2)$ and $L(v_4)$ to assign to v_2 and v_4, respectively. Therefore, there is a list coloring of C_4 for this collection of lists.

(1) $a \in L(v_2) \cap L(v_4)$

(2) $a \in L(v_2)$ or ~~to~~ $a \in L(v_4)$
but not in both.

2.(a) $x \in L(v_2) \cap L(v_4)$.

2(b) $x \notin L(v_2) \cap L(v_4)$.

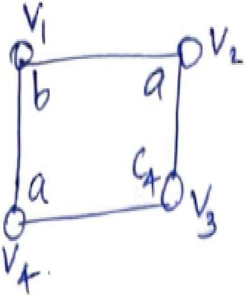
2(b)(i) $a \in L(v_3)$

2(b)(ii) $a \notin L(v_3)$.

(3) $a \notin L(v_2) \cup L(v_4)$.

Case -1.

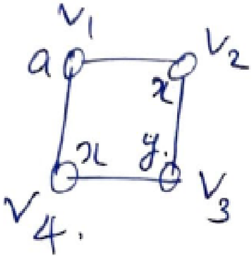
$a \in L(v_2) \cap L(v_4)$. $L(v_1) = \{a, b\}$.



Case - 2.

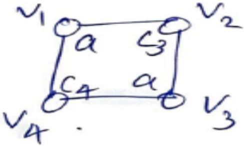
$a \in L(v_2)$ or $L(v_4)$ not in both.

If $x \in L(v_2) \cap L(v_4)$, then

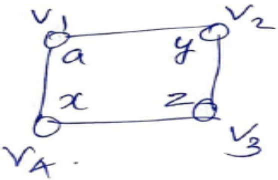


If there is no colour belonging to both $L(v_2)$ and $L(v_4)$.

If $a \in L(v_3)$

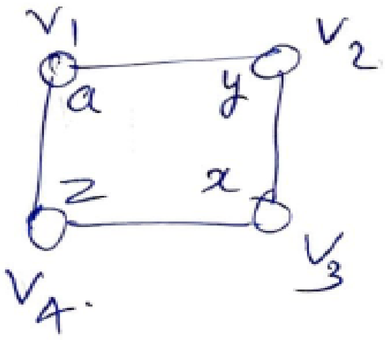


If $a \notin L(v_3)$



Case - 3

$a \notin L(v_2) \cup L(v_4)$.



MAT2011 Graph Theory and Applications

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Chromatic Polynomial

- For a graph G and a positive integer λ, the number of different proper λ-colorings of G is denoted by $P(G, \lambda)$ and is called the chromatic polynomial of G.
- Two λ-colorings c and c' of G from the same set $\{1, 2, ..., \lambda\}$ of λ colors are considered different if $c(v) \neq c'(v)$ for some vertex v of G.
- if $\lambda < \chi(G)$, then $P(G, \lambda) = 0$.
- $P(G, 0) = 0$.

Example

Proposition *Let G be a graph. Then $\chi(G) = k$ if and only if k is the smallest positive integer for which $P(G, k) > 0$.*

As an example, we determine the number of ways that the vertices of the graph G of Figure 8.3 can be colored from the set $\{1, 2, 3, 4, 5\}$. The vertex v can be assigned any of these 5 colors, while w can be assigned any color other than the color assigned to v. That is, w can be assigned any of the 4 remaining colors. Both u and t can be assigned any of the 3 colors not used for v and w. Therefore, the number $P(G, 5)$ of 5-colorings of G is $5 \cdot 4 \cdot 3^2 = 180$. More generally, $P(G, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$ for every integer λ.

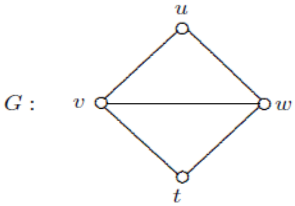


Figure 8.3: A graph G with $P(G, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$

There are some classes of graphs G for which $P(G, \lambda)$ can be easily computed.

Theorem *For every positive integer* λ,

(a) $P(K_n, \lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1) = \lambda^{(n)}$,

(b) $P(\overline{K}_n, \lambda) = \lambda^n$.

In particular, if $\lambda \geq n$ in Theorem 8.5(a), then

$$P(K_n, \lambda) = \lambda^{(n)} = \frac{\lambda!}{(\lambda-n)!}.$$

We now determine the chromatic polynomial of C_4 in Figure 8.4. There are λ choices for the color of v_1. The vertices v_2 and v_4 must be assigned colors different from the that assigned to v_1. The vertices v_2 and v_4 may be assigned the same color or may be assigned different colors. If v_2 and v_4 are assigned the same color, then there are $\lambda - 1$ choices for that color. The vertex v_3 can then be assigned any color except the color assigned to v_2 and v_4. Hence the number of distinct λ-colorings of C_4 in which v_2 and v_4 are colored the same is $\lambda(\lambda - 1)^2$.

If, on the other hand, v_2 and v_4 are colored differently, then there are $\lambda - 1$ choices for v_2 and $\lambda - 2$ choices for v_4. Since v_3 can be assigned any color except the two colors assigned to v_2 and v_4, the number of λ-colorings of C_4 in which v_2

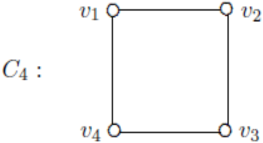


Figure 8.4: The chromatic polynomial of C_4

and v_4 are colored differently is $\lambda(\lambda-1)(\lambda-2)^2$. Hence the number of distinct λ-colorings of C_4 is

$$\begin{aligned} P(C_4,\lambda) &= \lambda(\lambda-1)^2+\lambda(\lambda-1)(\lambda-2)^2 \\ &= \lambda(\lambda-1)(\lambda^2-3\lambda+3) \\ &= \lambda^4-4\lambda^3+6\lambda^2-3\lambda \\ &= (\lambda-1)^4+(\lambda-1). \end{aligned}$$

Edge coloring and chromatic index

- An edge coloring of a graph G is an assignment of colors to the edges of G, one color to each edge.
- If adjacent edges are assigned distinct colors, then the edge coloring is a proper edge coloring.
- A proper edge coloring that uses colors from a set of k colors is a k-edge coloring. That is a k-edge coloring of a graph G can be described as a function $c : E(G) \longrightarrow \{1, 2, ..., k\}$ such that $c(e) \neq e(f)$ for every two adjacent edges e and f in G.
- The chromatic index (or edge chromatic number) $\chi'(G)$ of a graph G is the minimum positive integer k for which G is k-edge colorable.

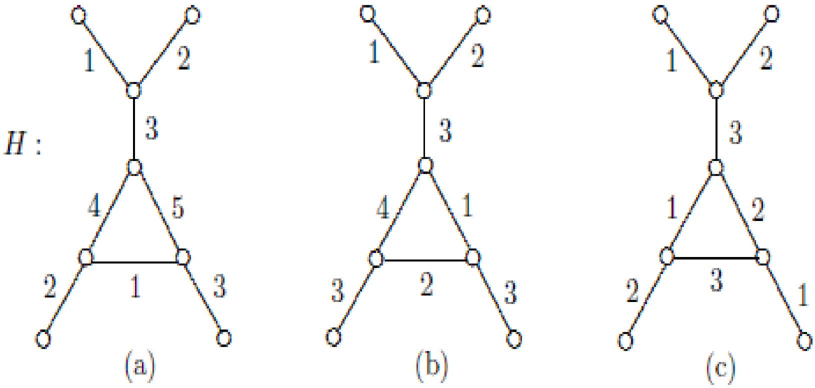


Figure   Edge colorings of a graph

Edge independence number

The edge independence number $\alpha'(G)$ of a nonempty graph G is the maximum number of edges in an independent set of edges of G. Furthermore, if the order of G is n, then $\alpha'(G) \leq n/2$.

Theorem *If G is a graph of size $m \geq 1$, then*

$$\chi'(G) \geq \frac{m}{\alpha'(G)}.$$

Proof. Suppose that $\chi'(G) = k$ and that $E_1, E_2, \ldots, E_k$ are the edge color classes in a k-edge coloring of G. Thus $|E_i| \leq \alpha'(G)$ for each i $(1 \leq i \leq k)$. Hence

$$m = |E(G)| = \sum_{i=1}^{k} |E_i| \leq k\alpha'(G)$$

and so $\chi'(G) = k \geq \frac{m}{\alpha'(G)}$. ■

Since every edge coloring of a graph G must assign distinct colors to adjacent edges, for each vertex v of G it follows that $\deg v$ colors must be used to color the edges incident with v in G. Therefore,

$$\chi'(G) \geq \Delta(G)$$

for every nonempty graph G.

In the graph G of order $n = 7$ and size $m = 10$ of Figure 10.2, $\Delta(G) = 3$. Hence $\chi'(G) \geq 3$. On the other hand, $X = \{uz, vx, wy\}$ is an independent set of three edges of G and so $\alpha'(G) \geq 3$. Because, $\alpha'(G) \leq n/2 = 7/2$, it follows that $\alpha'(G) = 3$. By Theorem , $\chi'(G) \geq m/\alpha'(G) = 10/3$ and so $\chi'(G) \geq 4$. The 4-edge coloring of G in Figure 10.2 shows that $\chi'(G) \leq 4$ and so $\chi'(G) = 4$.

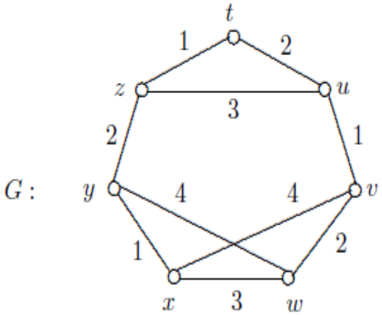


Figure A graph with chromatic index 4

Theorem (**Vizing's Theorem**) *For every nonempty graph G,*

$$\chi'(G) \leq 1 + \Delta(G).$$

By Vizing's theorem, it follows that for every nonempty graph G, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = 1+\Delta(G)$. A graph G belongs to or is of **Class one** if $\chi'(G) = \Delta(G)$ and is of **Class two** if $\chi'(G) = 1 + \Delta(G)$. Consequently, a major question in the

We now look at a few well-known graphs and classes of graphs to determine whether they are of Class one or Class two. We begin with the cycles. Since the cycle C_n $(n \geq 3)$ is 2-regular, $\chi'(C_n) = 2$ or $\chi'(C_n) = 3$. If n is even, then the edges may be alternately colored 1 and 2, producing a 2-edge coloring of C_n. If n is odd, then $\alpha'(C_n) = (n-1)/2$. Since the size of C_n is n, it follows by Theorem 10.1 that $\chi'(C_n) \geq n/\alpha'(C_n) = 2n/(n-1) > 2$ and so $\chi'(C_n) = 3$. Therefore,

$$\chi'(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Since $\Delta(C_n) = 2$, it follows that C_n is of Class one if n is even and of Class two if n is odd.

We now turn to complete graphs. Since K_n is $(n-1)$-regular, either $\chi'(K_n) = n-1$ or $\chi'(K_n) = n$. If n is even, then it follows by Theorem 4.15 that K_n is 1-factorable, that is, K_n can be factored into $n-1$ 1-factors $F_1, F_2, \ldots, F_{n-1}$. By assigning each edge of F_i $(1 \le i \le n-1)$ the color i, an $(n-1)$-edge coloring of K_n is produced. If n is odd, then $\alpha'(K_n) = (n-1)/2$. Since the size m of K_n is $n(n-1)/2$, it follows by Theorem 10.1 that $\chi'(K_n) \ge m/\alpha'(K_n) = n$. Thus $\chi'(K_n) = n$. In summary,

$$\chi'(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Consequently, the chromatic index of every nonempty complete graph is an odd integer. Since $\Delta(K_n) = n-1$, it follows, as with the cycles C_n, that K_n is of Class one if n is even and of Class two if n is odd.

Of course, both the cycles and complete graphs are regular graphs. For an r-regular graph G, either $\chi'(G) = r$ or $\chi'(G) = r+1$. If $\chi'(G) = r$, then there is an r-edge coloring of G, resulting in r color classes $E_1, E_2, \ldots, E_r$.

Theorem (König's Theorem) *If G is a nonempty bipartite graph, then*

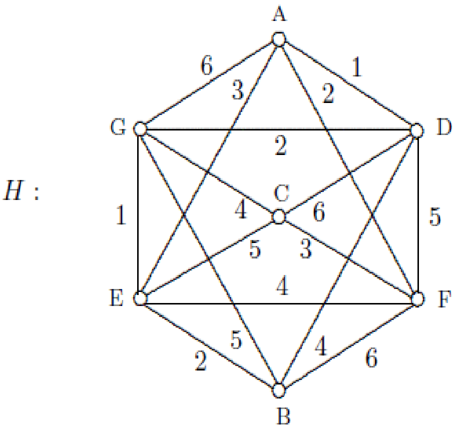
$$\chi'(G) = \Delta(G).$$

Example *A community, well known for having several professional tennis players train there, holds a charity tennis tournament each year, which alternates between men and women tennis players. During the coming year, women tennis players will be featured and the professional players Alice, Barbara, and Carrie will be in charge. Two tennis players from each of two local tennis clubs have been invited to participate as well. Debbie and Elizabeth will participate from Woodland Hills Tennis Club and Frances and Gina will participate from Mountain Meadows Tennis Club. No two professionals will play each other in the tournament and no two players from the same tennis club will play each other; otherwise, every two of the seven players will play each other. If no player is to play two matches on the same day, what is the minimum number of days needed to schedule this tournament?*

Solution. We construct a graph H with $V(H) = \{\text{A, B}, \cdots, \text{G}\}$ whoses vertices correspond to the seven tennis players. Two vertices x and y are adjacent in H if x and y are to play a tennis match against each other. The graph H is shown in Figure 10.7. The answer to the question posed is the chromatic index of H. The order of H is $n = 7$ and the degrees of its vertices are $5, 5, 5, 5, 4, 4, 4$. Thus $\Delta(H) = 5$ and the size of H is $m = 16$. Since

$$16 = m > \Delta(H) \cdot \left\lfloor \frac{n}{2} \right\rfloor = 15,$$

the graph H is overfull. By Corollary 10.10, H is of Class two and so $\chi'(H) = 1 + \Delta(H) = 6$. A 6-edge coloring of H is also shown in Figure 10.7. This provides us with a schedule for the tennis tournament taking place over a minimum of six days. ♦

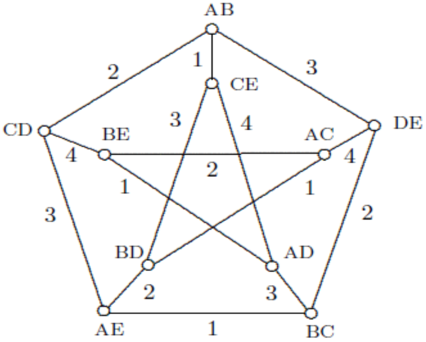


Day 1: AD, EG
Day 2: AF, BE, DG
Day 3: AE, CF
Day 4: BD, CG, EF
Day 5: BG, CE, DF
Day 6: AG, BF, CD

Figure 10.7: The graph H in Example and a 6-edge coloring of H

Example 10.12 *One year it is decided to have a charity tennis tournament consisting entirely of double matches. Five tennis players (denoted by A, B, C, D, E) have agreed to participate. Each pair $\{W, X\}$ of tennis players will play a match against every other pair $\{Y, Z\}$ of tennis players, where then $\{W, X\} \cap \{Y, Z\} = \emptyset$, but no 2-person team is to play two matches on the same day. What is the minimum number of days needed to schedule such a tournament? Give an example of such a tournament using a minimum number of days.*

Solution. We construct a graph G whose vertex set is the set of 2-element subsets of $\{A, B, C, D, E\}$. Thus the order of G is $\binom{5}{2} = 10$. Two vertices $\{W, X\}$ and $\{Y, Z\}$ are adjacent if these sets are disjoint. The graph G is shown in Figure 10.8. Thus G is the Petersen graph, or equivalently the Kneser graph $\mathrm{KG}_{5,2}$ (see Section 6.2). To answer the question, we determine the chromatic index of G. Since the Petersen graph is known to be of Class two, it follows that $\chi'(G) = 1 + \Delta(G) = 4$. A 4-edge coloring of G is given in Figure 10.8 together with a possible schedule of tennis matches over a period of four days. ♦



Day 1: AB-CE, AC-BD, AE-BC, AD-BE
Day 2: AB-CD, AC-BE, AE-BD, BC-DE
Day 3: AB-DE, AD-BC, AE-CD, BD-CE
Day 4: AC-DE, AD-CE, BE-CD

Figure 10.8: The Petersen graph G in Example 10.12 and a 4-edge coloring of G

MAT2011 Graph Theory and Applications

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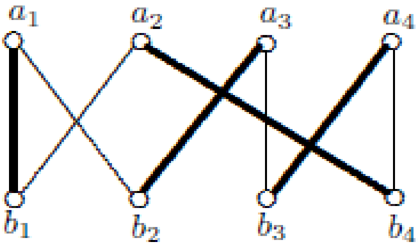
Matching

- In a graph G, a set M of edges, no two edges of which are adjacent, is called a matching.

$G:$



- Let G be a bipartite graph with partite sets U and W, where $|U| \leq |W|$. A matching in G is therefore a set $M = \{e_1, e_2, ..., e_k\}$ of edges, where $e_i = u_i w_i$ for $1 \leq i \leq k$ such that $u_1, u_2, ..., u_k$ are k distinct vertices of U and $w_1, w_2, ..., w_k$ are k distinct vertices of W.
- In this case, M matches the set $\{u_1, u_2, ..., u_k\}$ to the set $\{w_1, w_2, ..., w_k\}$.
- Necessarily, for any matching of k edges, $k \leq |U|$.
- If $|U| = k$, then U is said to be **matched** to a subset of W.
- **Hall's condition**
  For a bipartite graph G with partite sets U and W and for $S \subseteq U$, let $N(S)$ be the set of all vertices in W having a neighbor in S. Then the Hall's condition is $|N(S)| \geq |S|$ for all $S \subseteq U$

Theorem

Let G be a bipartite graph with partite sets U and W, where $|U| \leq |W|$. Then U can be matched to a subset of W if and only if Hall's condition is satisfied.

A matching M in a graph G is a

1. **maximum matching** of G if G contains no matching with more than $|M|$ edges;
2. **maximal matching** of G if M is not a proper subset of any other matching in G;
3. **perfect matching** of G if every vertex of G is incident with some edge in M.

- If M is a perfect matching in G, then G has order $n = 2k$ for some positive integer k and $|M| = k$.
- Thus only a graph of even order can have a perfect matching. Furthermore, every perfect matching is a maximum matching and every maximum matching is a maximal matching, but neither converse is true.
- For the graph $G = P_6$, the matching $M = \{v_1v_2, v_3v_4, v_5v_6\}$ is both a perfect and maximum matching, while $M' = \{v_2v_3, v_5v_6\}$ is maximal matching that is not a maximum matching.

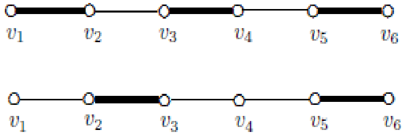


Figure : Maximum and maximal matchings in a graph

Theorem

Every r-regular bipartite graph $(r \geq 1)$ has a perfect matching.

Theorem

The Marriage Theorem *Let there be given a collection of women and men such that each woman knows exactly r of the men and each man knows exactly r of the women. Then every woman can marry a man she knows.*

Idependance in graphs

- A set M of edges in a graph G is independent if no two edges in M are adjacent.Therefore, M is an independent set of edges of G if and only if M is a matching in G.
- The maximum number of edges in an independent set of edges of G is called the edge independence number of G and is denoted by $\alpha'(G)$.
- If M is an independent set of edges in G such that $|M| = \alpha'(G)$, then M is a maximum matching in G.
- If G has order n, then $\alpha'(G) \leq n/2$ and $\alpha'(G) = n/2$ if and only if G has a perfect matching.
- An independent set M of edges of G is a maximal independent set if M is a maximal matching in G. Thus M is not a proper subset of any independent set of edges of G.
- The lower edge independence number $\alpha'_o(G)$ of G is the minimum cardinality of a maximal independent set of edges (or maximal matching) in G.

Problems

Example *As a result of doing well on an exam, six students Ashley (A), Bruce (B), Charles (C), Duane (D), Elke (E), and Faith (F) have earned the right to receive a complimentary text in either algebra (a), calculus (c), differential equations (d), geometry (g), history of mathematics (h), programming (p), or topology (t). There is only one book on each of these subjects. The preferences of the students are*

A: d, h, t; B: g, p, t; C: a, g, h; D: h, p, t; E: a, c, d; F: c, d, p.

Can each of the students receive a book he or she likes?

Solution. This situation can be modeled by the bipartite graph G of Figure 8.2(a) having partite sets $U = \{A, B, C, D, E, F\}$ and $W = \{a, c, d, g, h, p, t\}$. We are asking if G contains a matching with six edges. Such a matching does exist, as shown in Figure 8.2(b). From the matching shown in Figure 8.2(b), we see how six of the seven books can be paired off with the six students. ◇

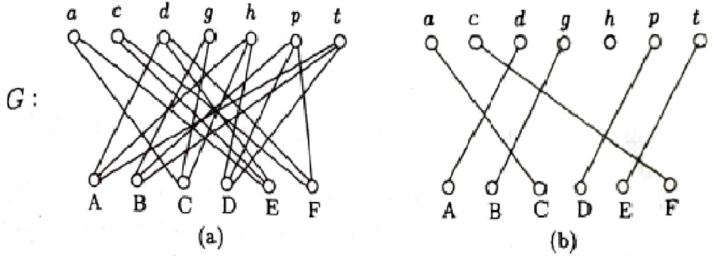


Figure 8.2: A matching in a bipartite graph

Example 8.2 *Seven seniors Ben (B), Don (D), Felix (F), June (J), Kim (K), Lyle (L), and Maria (M) are looking for positions after they graduate. The University Placement Office has posted open positions for an accountant (a), consultant (c), editor (e), programmer (p), reporter (r), secretary (s), and teacher (t). Each of the seven students has applied for some of these positions:*

B: c, e;	D: a, c, p, s, t;	F: c, r;	J: c, e, r;
K: a, e, p, s;	L: e, r;	M: p, r, s, t.	

Is it possible for each student to be hired for a job for which he or she has applied?

Solution. This situation can be modeled by the bipartite graph G of Figure 8.3, where one partite set $U = \{B, D, F, J, K, L, M\}$ is the set of students and the other partite set $W = \{a, c, e, p, r, s, t\}$ is the set of positions. A vertex $u \in U$ is joined to a vertex $w \in W$ if u has applied for position w.

The answer to this question is *no* as Ben, Felix, June, and Lyle have only applied for some or all the positions of consultant, editor, and reporter. So not all of these four students can be hired for the jobs for which they have applied. Consequently, not all seven students can be hired for the seven positions. What we have observed for the bipartite graph G of Figure 8.3 is that there is no matching with seven edges. What we gave for an explanation is that there is a subset $X = \{B, F, J, L\}$ of U containing four vertices whose neighbors belong to a set $\{c, e, r\}$ of only three vertices. As we are about to see, this is the key reason why this or any bipartite graph with partite sets U and W such that $r = |U| \leq |W|$ does not contain a matching with r edges.

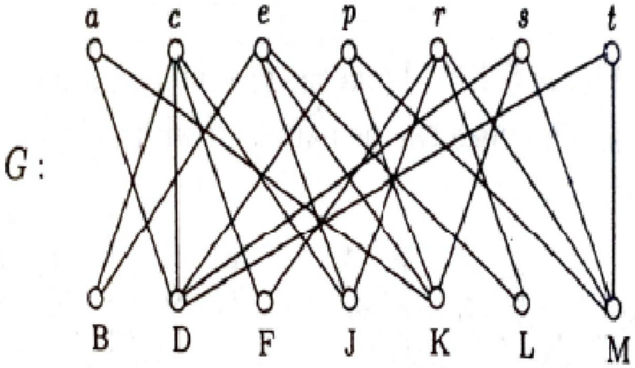


Figure A graph modeling the situation in Example

MAT2011 Graph Theory and Applications

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April 15, 2021

By a factor of a graph G, we mean a spanning subgraph of G. A k-regular factor is called a k-factor. Thus the edge set of a 1-factor in a graph G is a perfect matching in G. So a graph G has a 1-factor if and only if G has a perfect matching.

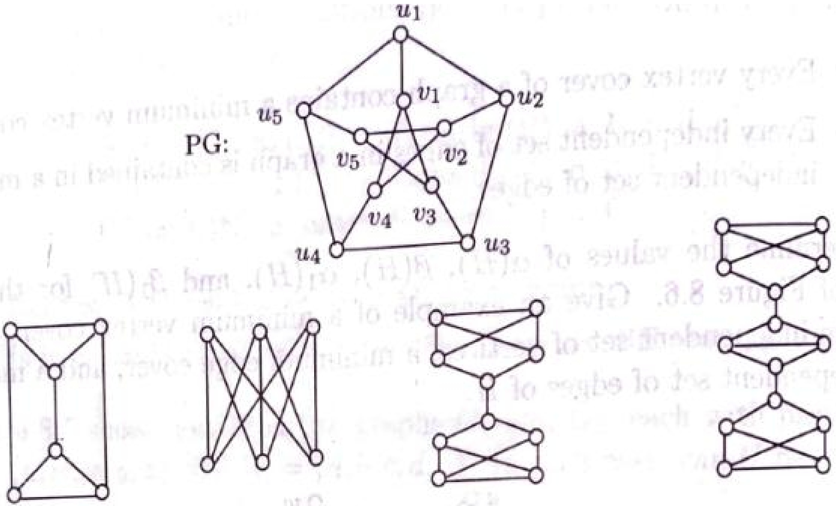


Figure 8.7: 3-regular graphs containing 1-factors

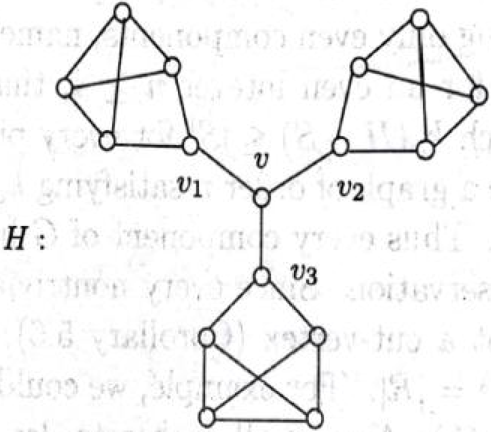


Figure 8.8: A 3-regular graph containing no 1-factor

Theorem

(Petersen's Theorem) Every three regular bridgeless graph contains a 1-factor

Theorem

Every three regular graph with atmost two bridges contains a 1-factor.

A graph is said to be 1-factorable if there exists 1-factors $F_1, F_2, ..., F_r$ of G such that $\{E(F_1), E(F_2), ..., E(F_r)\}$ is a partition of $E(G)$. Consequently, every edge of G belongs to exactly one of these 1-factor.

Theorem

The Petersen graph is not 1-factorable

Cyclic Factorization

A factorization of a graph G into k copies of a graph H is a cyclic factorization if H is drawn in an appropriate manner so that rotating H through an appropriate angle $k-1$ times produces this factorization.

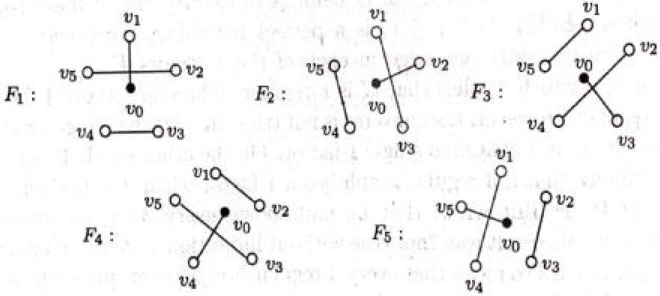


Figure 8.10: A cyclic 1-factorization of K_6

A **2-factor** in a graph G is a spanning 2-regular subgraph of G. Every component of a 2-factor is therefore a cycle. A graph G is said to be **2-factorable** if there exist 2-factors $F_1, F_2, \ldots, F_k$ such that $\{E(F_1), E(F_2), \ldots, E(F_k)\}$ is a partition of $E(G)$. The graph G is consequently $2k$-regular. That is, if G is a 2-factorable graph, then G is r-regular for some positive even integer r. In what might be considered an unexpected result, Petersen showed that the converse of this statement is true as well.

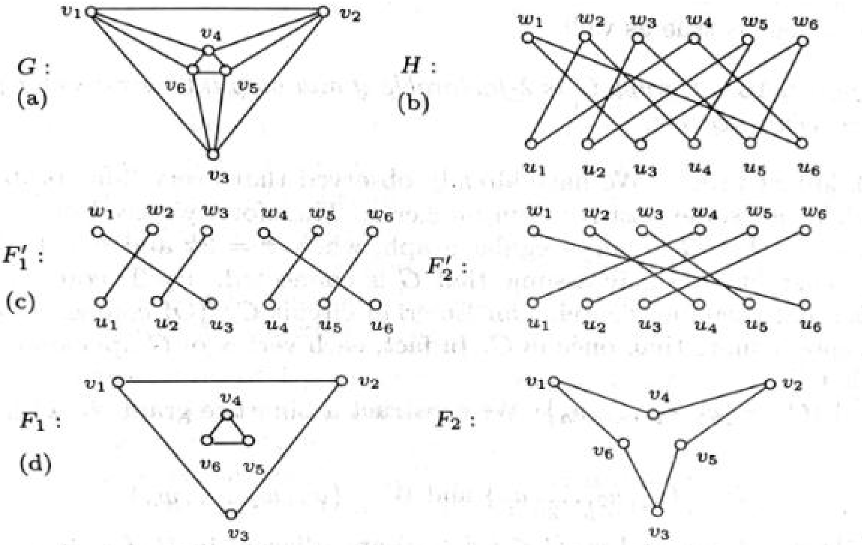


Figure 8.11: Constructing a 2-factorization of a 4-regular graph G

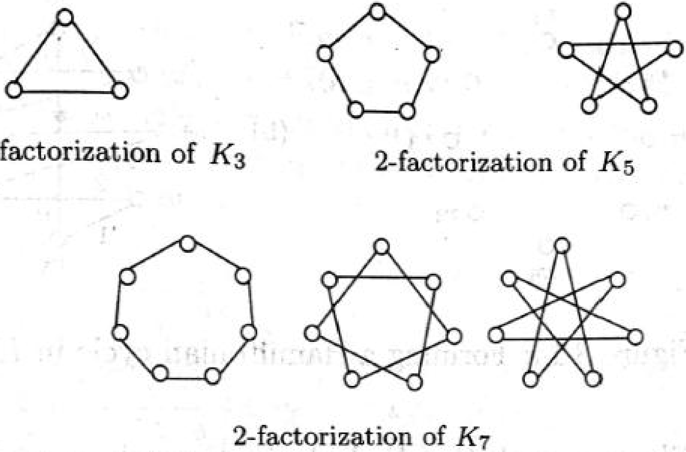
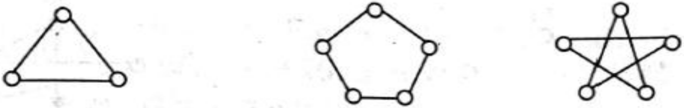


Figure 8.12: Some 2-factorizations of K_3, K_5, and K_7

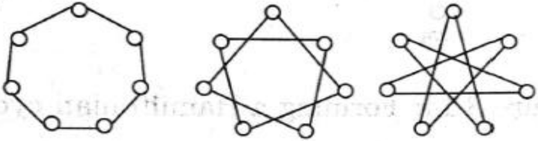
Hamiltonian Factorization

A graph G is Hamiltonian factorable if there exists a factorization F of G such that each factor in F is a Hamiltonian cycle of G.



2-factorization of K_3

2-factorization of K_5



2-factorization of K_7

Figure 8.12: Some 2-factorizations of K_3, K_5, and K_7

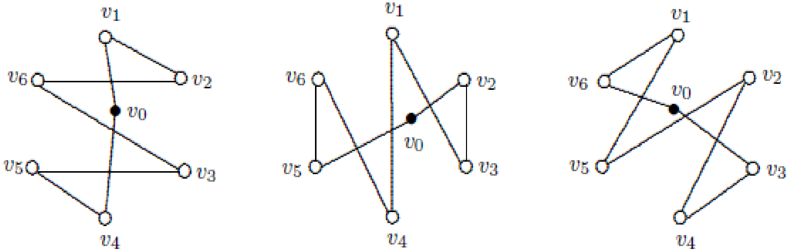


Figure : A Hamiltonian factorization of K_7

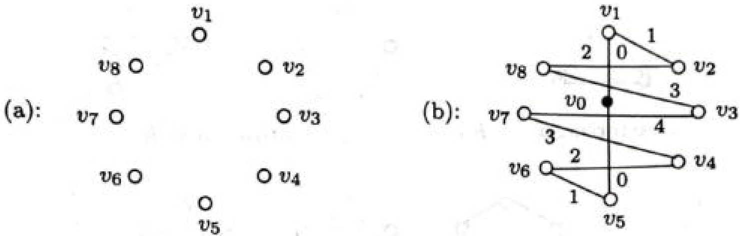


Figure 8.14: Forming a Hamiltonian cycle in K_9

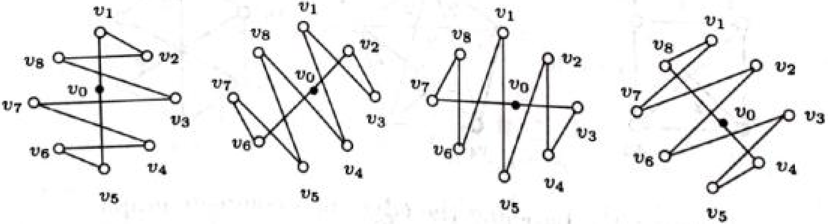


Figure 8.15: A Hamiltonian factorization of K_9

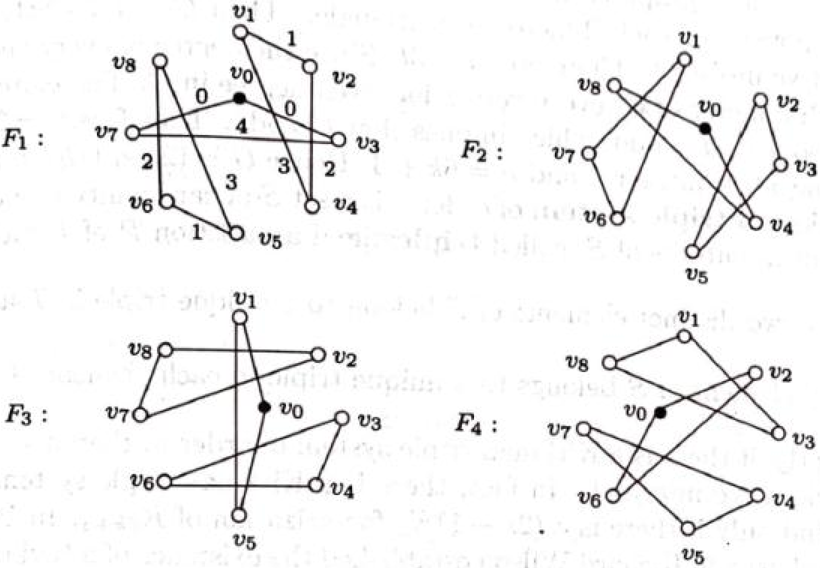


Figure 8.16: A $3K_3$-factorization of K_9

Kirkman's Schoolgirl Problem

A school mistress has fifteen schoolgirls whom she wishes to take on a daily walk. The girls are to walk in five rows of three girls each. It is required that no two girls should walk in the same row more than once a week. Can this be done?

If we think about this a bit, we see that the question can be rephrased as follows: Is there a $5K_3$-factorization of K_{15}? If we label the vertices of K_{15} by the schoolgirls, numbered $1, 2, \ldots, 15$ say, then we see that a solution is given below.

Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7
{1, 2, 3}	{1, 4, 5}	{1, 6, 7}	{1, 8, 9}	{1, 10, 11}	{1, 12, 13}	{1, 14, 15}
{4, 8, 12}	{2, 9, 11}	{2, 8, 10}	{2, 5, 7}	{2, 12, 15}	{2, 5, 6}	{2, 4, 7}
{5, 10, 14}	{3, 13, 15}	{3, 12, 14}	{3, 13, 14}	{3, 4, 6}	{3, 9, 10}	{3, 8, 11}
{6, 9, 15}	{6, 8, 14}	{4, 9, 13}	{4, 10, 15}	{5, 8, 13}	{4, 11, 14}	{5, 9, 12}
{7, 11, 13}	{7, 10, 12}	{5, 11, 15}	{6, 11, 12}	{7, 9, 14}	{7, 8, 15}	{6, 10, 13}

Although there is a $5K_3$-factorization of K_{15}, it turns out that there is no cyclic $5K_3$-factorization of K_{15} (which makes it more difficult to construct such a factorization).

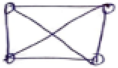
Since we saw in Figure 8.16 that K_9 is $3K_3-$factorable, the following is true:

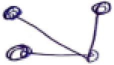
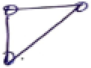
Nine schoolgirls can take four daily walks in three rows of three girls each so that no two girls walk in the same row twice.

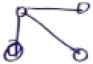
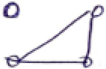
Graph Decomposition

A graph G is said to be **decomposable** into the subgraphs H_1, H_2, …, H_k if each subgraph H_i, $1 \leq i \leq k$, has no isolated vertices and $\{E(H_1), E(H_2), \ldots, E(H_k)\}$ is a partition of $E(G)$. In other words, the subgraphs H_i are not required to be spanning subgraphs of G. If, on the other hand, each subgraph H_i is a spanning subgraph of G, then the decomposition is a factorization of G. If each $H_i \cong H$ for some graph H, then the graph G is H-**decomposable** and the decomposition is an H-**decomposition**.

Decomposition of graphs

K_4 

Decomposition : H_1 :  H_2 : 

Not a decomposition H_1 :  H_2 : 

Not a decomposition H_1 :  H_2 : 

Steiner Triple System

A **Steiner triple system** of order n is a set S of cardinality n and a collection T of 3-element subsets, called **triples**, such that every two distinct elements of S belong to a unique triple in T. Therefore, there is a Steiner triple system of order n if and only if K_n is K_3-decomposable. Consequently, in order for a Steiner triple system of order n to exist, either $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$. In 1846 Kirkman showed that the converse holds as well.

Theorem 8.21 *A Steiner triple system of order $n \geq 3$ exists if and only if $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$.*

a Steiner triple system of order 7 from the set $\{1, 2, \ldots, 7\}$, namely:

$$\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}.$$

The K_3-decomposition is a **cyclic decomposition** of K_7.

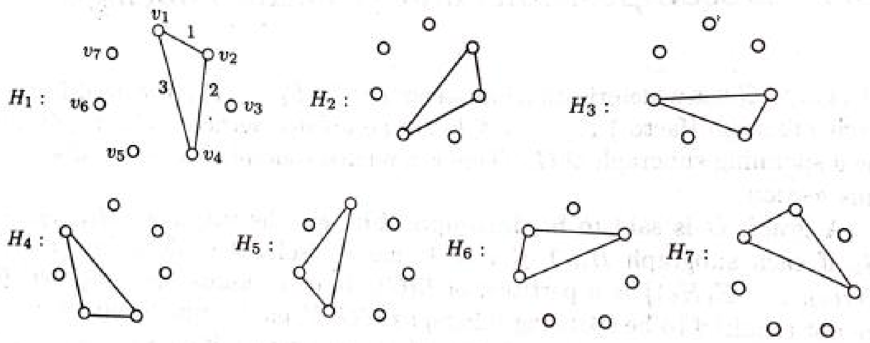


Figure 8.20: A cyclic K_3-decomposition of K_7

Let G be a graph of order n and size m. A one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ is called a **graceful labeling** of G if the induced edge labeling f': $E(G) \to \{1, 2, \ldots, m\}$ defined by

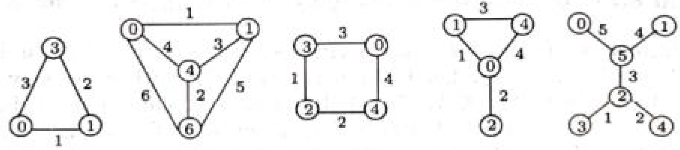
$$f'(e) = |f(u) - f(v)| \text{ for each edge } e = uv \text{ of } G$$

is also one-to-one. If f is a graceful labeling of a graph G of order n, then so too is the **complementary labeling** g: $V(G) \to \{0, 1, 2, \ldots, m\}$ of f defined by $g(v) = m - f(v)$ for all $v \in V(G)$ since, for $e = uv$,

$$g'(e) = |g(u) - g(v)| = |(n - f(u)) - (n - f(v))| = |f(u) - f(v)| = f'(e).$$

A graph G possessing a graceful labeling is called a **graceful graph.**

Figure 8.21 shows five graceful graphs, including the complete graphs K_3 and K_4 and the cycle C_4, along with a graceful labeling of each of these graphs.



Example 8.22 *The cycle C_5 is not a graceful graph.*

Solution. Let $H \cong C_5$ (see Figure 8.22(a)). Assume, to the contrary, that H is graceful. Then there exists a graceful labeling $f : V(H) \to \{0, 1, 2, 3, 4, 5\}$. Since some edge of H is labeled 5 by the induced edge labeling, there are two adjacent vertices of H labeled 0 and 5.

The only way for an edge of H to be labeled 4 is for its incident vertices to be labeled 0 and 4, or 1 and 5. Since either f or its complementary labeling assigns adjacent vertices the labels 0 and 4, we may assume that three of the five vertices of H are labeled as in Figure 8.22(b). The vertex w cannot be labeled 1 as there is already an edge labeled 4. If x is labeled 1, then w must be labeled 2 or 3, neither of which results in a graceful labeling of H. Hence one of x and w is labeled 2 and the other is labeled 3. However, neither produces a graceful labeling of H. ◊

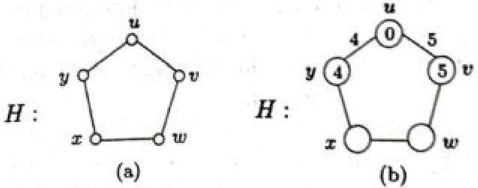
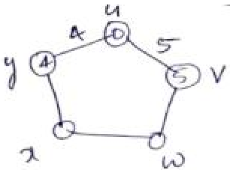


Figure 8.22: The graph H in Example 8.22

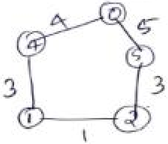
The rightmost graph shown in Figure 8.21 is, of course, a tree; in fact, it is a double star. The labeling given there shows that this tree is a graceful graph. In fact, there is a well-known conjecture due to Gerhard Ringel and Anton Kotzig.

Conjecture 8.23 *Every tree is graceful.*



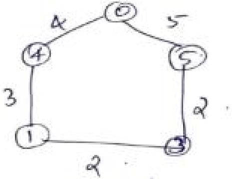
Free x and w, the remaining choices are (1,2), (1,3) and (2,3)

If it is (1,2)

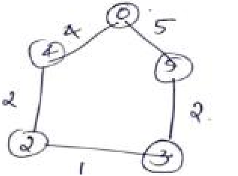


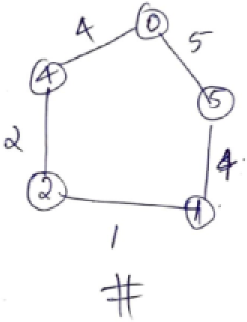
contradiction

If it is (1,3)



If it is (2,3)

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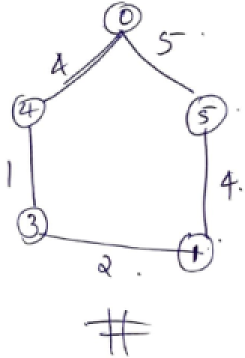
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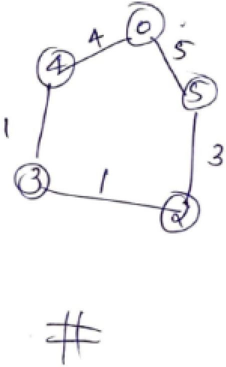
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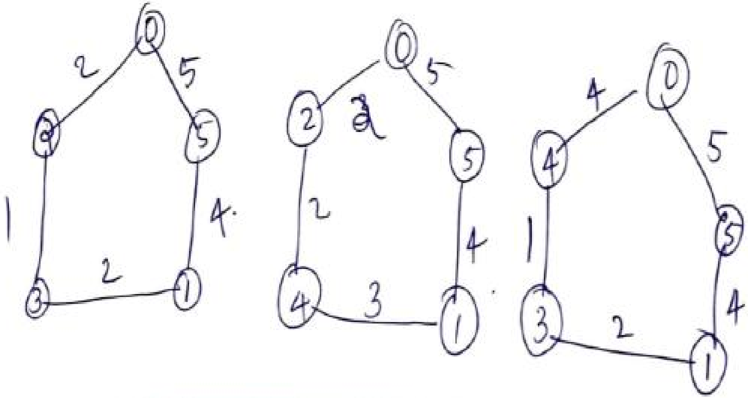

MAT2011 Graph Theory and Applications

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May 25, 2021

Instant Insanity Puzzle

Open the package. Notice that there are four different colors showing on each side of this stack of blocks. You may NEVER, EVER see them this way again. Now mix them up and then restack them so that there are again four colors, all different, showing on each side.

What is written above appears on an insert within packaging that contains four multi-colored cubes that make up a puzzle called **Instant Insanity**, which is manufactured by Hasbro Inc. (makers of toys and games). Each of the six faces of each cube is colored with one of the four colors red (R), blue (B), green (G), and yellow (Y). The object of the puzzle is to stack the cubes as in Figure 8.26, one on top of another, so that all four colors appear on each of the four sides.

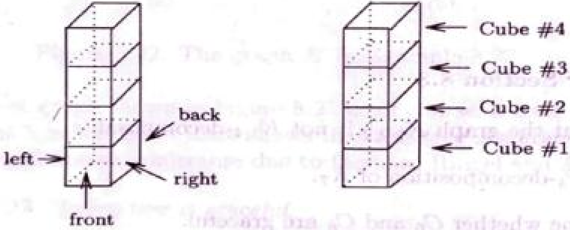


Figure 8.26: The stacking of four cubes

On the reverse side of the insert is written: *Give up?* An address is supplied where a solution of the puzzle can be obtained. Reading all of this can be quite intimidating. Indeed, even before we attempt to solve the puzzle, we are being informed that it is very unlikely that we will be successful. Let's compute the number of ways in which four cubes can be stacked.

Select one of the cubes (which we'll call the first cube) and place it on a table, say. There are three ways this can be done, according to which pair of opposite faces will be the top and bottom of the cube. These are the "buried" faces. Select one of the other four faces as the front face. Now place the second cube on top of the first cube. Any of the six faces of the second cube can be chosen to appear directly above the front face of the first cube, and each of these six faces can be positioned (rotated) in one of four ways. That is, there are $6 \cdot 4 = 24$ ways to place the second cube on top of the first cube. Consequently, the number of ways to stack all four cubes on the top of one another is $3 \cdot (24)^3 = 41,472$. Now if there is only one way to stack the cubes so that all four colors appear on all four sides, then using a trial-and-error method to solve the puzzle seems like a frustrating task and is likely to result in ... *instant insanity*.

Graph theory can help us to solve this tantalizing puzzle. Let's see how this can be done. For this purpose, it is convenient to have a way of representing a cube and the locations of the colors on its faces. See Figure 8.27.

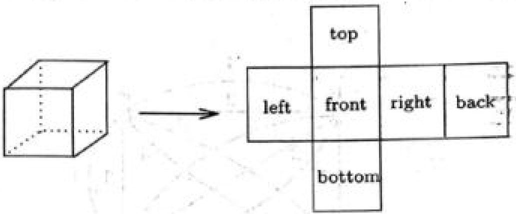


Figure 8.27: The six faces of a cube

We are now prepared to present an example.

Example 8.25 *Consider the four multi-colored cubes given in Figure 8.28.*

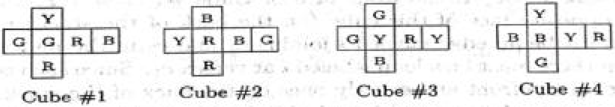


Figure 8.28: The four cubes in an Instant Insanity puzzle

Solution. With each of the four cubes of Figure 8.28, we associate a multigraph (allowing loops) of order 4 and size 3. The vertex set of each multigraph is the set {R, B, G, Y} of four colors and there is an edge joining color c_1 and color c_2 (possibly $c_1 = c_2$) whenever there is a pair of opposite faces colored c_1 and c_2. If there are two (or three) opposite faces colored c_1 and c_2, then the multigraph has two (or three) parallel edges joining the vertices c_1 and c_2. If $c_1 = c_2$, then there is a loop at c_1. The multigraphs corresponding to the cubes of Figure 8.28 are shown in Figure 8.29.

A composite multigraph M of order 4 (with vertex set {R, B, G, Y}) and size 12 and whose edge set is the union of the edge sets of these four multigraphs is shown in Figure 8.29. In order to distinguish which edges of M came from Cube #i ($i = 1, 2, 3, 4$), those three edges of M are labeled by i. The multigraph M constructed from the multigraphs of Figure 8.29 is shown in Figure 8.30.

Let's pause for a moment while we review what we are seeking. Since our goal to stack the four cubes on top of one another so that all four colors appear

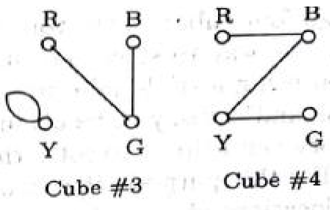


Figure 8.29: The four multigraphs in Example 8.25

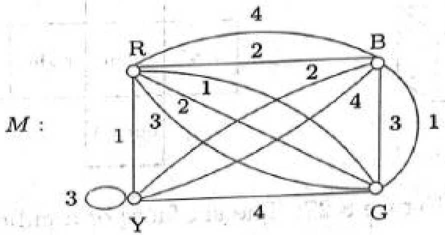


Figure 8.30: The composite multigraph of Example 8.25

on all four sides, of course, all four colors must appear on both the front and the back of the stack. If the front face of Cube #i ($i = 1, 2, 3, 4$) is colored c_1 and the opposite face of this cube (on the back of the stack) is colored c_2, then there must be an edge labeled i joining c_1 and c_2 in the multigraph M. If $c_1 = c_2$, then there must be a loop labeled i at vertex c_1. Since each color appears exactly once on the front and exactly once on the back of the stack, there must be a 2-regular spanning submultigraph M' (a 2-factor) of M (where a loop is considered to have degree 2) such that there is exactly one edge labeled 1, 2, 3, and 4. Similarly, corresponding to the right and left sides of the stack, there is a 2-regular spanning submultigraph M'' of M whose edge set is disjoint from that of M'. On the basis of these observations, we seek two edge-disjoint spanning 2-regular submultigraphs, where there is one edge labeled 1, 2, 3, and 4 in each of these two submultigraphs. If such a pair of multigraphs does not exist, then the puzzle can have no solution. If such a pair M', M'' of multigraphs exists, then they can be used to solve the puzzle, that is, to stack the cubes appropriately. Any 2-regular spanning submultigraph must be one of the seventeen multigraphs shown in Figure 8.31.

Returning to our example, we see that the multigraph M of Figure 8.30 contains the two edge-disjoint 2-regular spanning submultigraphs M' and M'', where the edges of these two multigraphs are labeled 1, 2, 3, and 4 (shown in Figure 8.32(a)). The multigraph M' will correspond to the front and back of the stack to be produced and M'' will correspond to the right and left sides. (We could reverse M' and M'' if we desire.) For the purpose and convenience of stacking the cubes, we direct the edges of each component of M' and M'' so

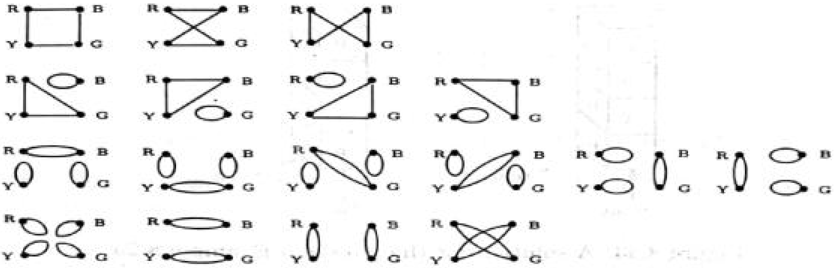


Figure 8.31: The seventeen 2-regular spanning multigraphs

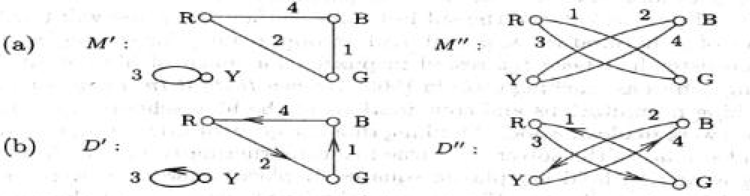
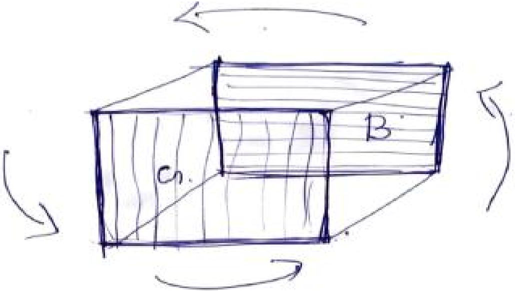


Figure 8.32: Two 2-regular spanning submultigraphs for Example 8.25

that a directed cycle results. Thus two (directed) multigraphs D' and D'' are produced, as shown in Figure 8.32(b)

With the aid of the (directed) multigraphs D' and D'' of Figure 8.32(b), we now stack the cubes. Since the arc (G, B) is labeled 1 in M', we place Cube #1 so that a green face appears in the front and a blue face on the back. Since the arc (G, R) is labeled 1 in M'', we rotate this cube (keeping a green face in the front and a blue face on the back) until we have a green face on the right and a red face on the left. We now proceed in the same way with the other three cubes and ... Voila! The puzzle has been successfully solved (see Figure 8.33). ◇



Problems

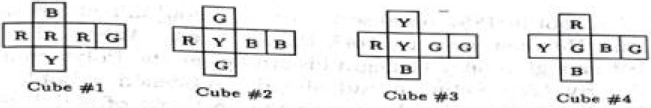


Figure 8.34: Instant Insanity puzzle for Exercise 8.39

8.40 Solve the Instant Insanity puzzle in Figure 8.35 by providing (a)–(d) as in Exercise 8.39.

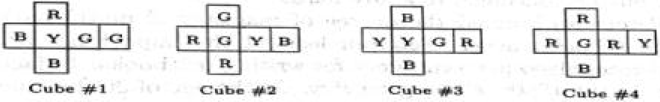


Figure 8.35: Instant Insanity puzzle for Exercise 8.40

8.41 Solve the Instant Insanity puzzle in Figure 8.36 by providing (a)–(d) as in Exercise 8.39.

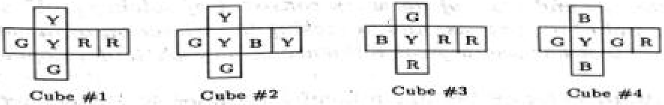
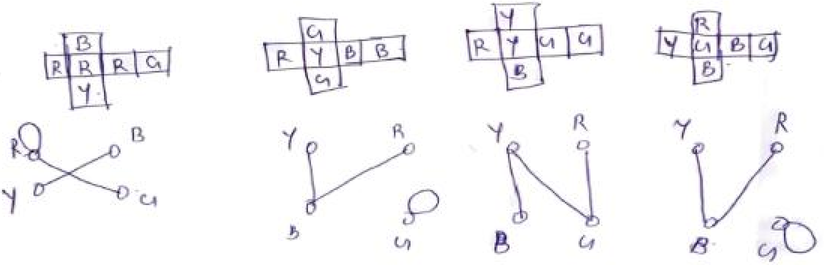
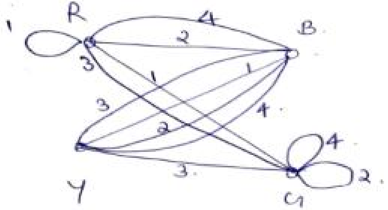


Figure 8.36: Instant Insanity puzzle for Exercise 8.41

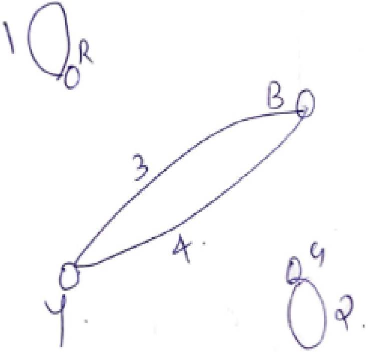
Answer for the first question

Q-1

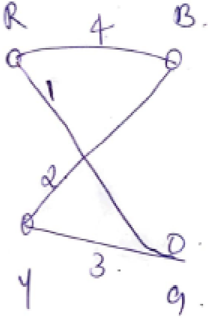




Edge distinct 1-factors are;



2 Regular spanning subgr



2 Regular spanning subgraph.

MAT2011 Graph Theory and Applications

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May 26, 2021

Distance in graphs

Let's review the definition of distance in a graph. For two vertices u and v in a graph G, the **distance** $d(u, v)$ from u to v is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ **geodesic.** In order for $d(u, v)$ to be defined for all pairs u, v of vertices in G, the graph G must be connected. We therefore assume that G is a connected graph. The term distance that we just defined satisfies all four of the following properties in any connected graph G.

1. $d(u, v) \geq 0$ for all $u, v \in V(G)$.
2. $d(u, v) = 0$ if and only if $u = v$.
3. $d(u, v) = d(v, u)$ for all $u, v \in V(G)$ [**the symmetric property**].
4. $d(u, w) \leq d(u, v) + d(v, w)$ for all $u, v, w \in V(G)$ [**the triangle inequality**].

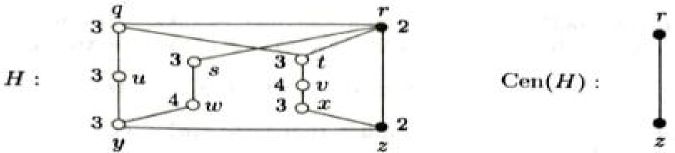
That a connected graph satisfies all four of these properties should be clear, with the possible exception of property 4 (the triangle inequality), which we now verify. Let P_1 be a $u - v$ geodesic and P_2 a $v - w$ geodesic in the graph G. The path P_1 followed by P_2 produces a $u - w$ walk of length $d(u, v) + d(v, w)$. By Theorem 1.6, G contains a $u - w$ path whose length is at most $d(u, v) + d(v, w)$. Therefore, $d(u, w) \leq d(u, v) + d(v, w)$. Since the distance d satisfies property 2 (the symmetric property), we can refer to the distance *between* two vertices rather than the distance *from* one vertex *to* another.

The fact that the distance d satisfies properties 1–4 means that d is a **metric** and $(V(G), d)$ is a **metric space**. It is ordinarily very useful when a distance is a metric as this concept has been studied widely. There are many concepts involving connected graphs that are defined in terms of distance and which are valuable in providing information about these graphs.

For a vertex v in a connected graph G, the **eccentricity** $e(v)$ of v is the distance between v and a vertex farthest from v in G. The minimum eccentricity among the vertices of G is its **radius** and the maximum eccentricity is its **diameter**, which are denoted by $\text{rad}(G)$ and $\text{diam}(G)$, respectively. A vertex v in G is a **central vertex** if $e(v) = \text{rad}(G)$ and the subgraph induced by the central vertices of G is the **center** $\text{Cen}(G)$ of G. If every vertex of G is a central vertex, then $\text{Cen}(G) = G$ and G is called **self-centered**. For example, if $G \cong C_n$ for $n \geq 3$, then G is self-centered.

To illustrate the concepts we have just presented, consider the graph H of Figure 12.2, where each vertex is labeled by its eccentricity. Since the smallest eccentricity is 2, $\text{rad}(H) = 2$. Because the largest eccentricity is 4, $\text{diam}(H) = 4$. The center of H is also shown in Figure 12.2.

There are a number of observations that can be made about the graph H of Figure 12.2. We have already mentioned that $\text{rad}(H) = 2$ and $\text{diam}(H) = 4$. The



$\text{rad}(H) = 2$ and $\text{diam}(H) = 4$

Figure 12.2: The eccentricities of the vertices of a graph

terms "radius" and "diameter" are familiar because of circles, where, of course, the diameter is always twice the radius. This fact together with the knowledge that $\text{diam}(H) = 2\,\text{rad}(H)$ for the graph H of Figure 12.2 might reasonably suggest that $\text{diam}(G) = 2\,\text{rad}(G)$ for *every* connected graph G. Such is not the case, however. Figure 12.3 shows three graphs G_2, G_3, and G_4, each of which has radius 2, where $\text{diam}(G_k) = k$ for $k = 2, 3, 4$. There is, therefore, no identity that relates the radius and the diameter of a graph. As we now show, Figure 12.3 illustrates the only possible diameters for a graph having radius 2.

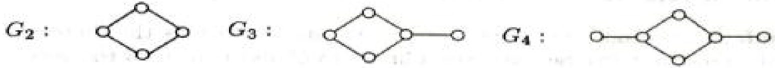


Figure 12.3: Three graphs having radius 2

Theorem 12.1 *For every nontrivial connected graph G,*

$$\operatorname{rad}(G) \leq \operatorname{diam}(G) \leq 2\operatorname{rad}(G).$$

Proof. [direct proof] The inequality $\operatorname{rad}(G) \leq \operatorname{diam}(G)$ is immediate since the smallest eccentricity cannot exceed the largest eccentricity. Let u and v be two vertices such that $d(u,v) = \operatorname{diam}(G)$ and let w be a central vertex of G. Therefore, $e(w) = \operatorname{rad}(G)$. Hence the distance between w and any other vertex of G is at most $\operatorname{rad}(G)$. By the triangle inequality,

$$\operatorname{diam}(G) = d(u,v) \leq d(u,w) + d(w,v) \leq \operatorname{rad}(G) + \operatorname{rad}(G) = 2\operatorname{rad}(G). \quad \blacksquare$$

Another observation about the graph H in Figure 12.2 is that the eccentricities of every two adjacent vertices differ by at most 1. This statement too is true for all connected graphs.

Theorem 12.2 *For every two adjacent vertices u and v in a connected graph,*

$$|e(u) - e(v)| \leq 1.$$

Proof. [direct proof] Assume, without loss of generality, that $e(u) \geq e(v)$. Let x be a vertex that is farthest from u. So $d(u, x) = e(u)$. By the triangle inequality,

$$e(u) = d(u, x) \leq d(u, v) + d(v, x) \leq 1 + e(v).$$

Hence $e(u) \leq 1 + e(v)$, which implies that $0 \leq e(u) - e(v) \leq 1$. Therefore, $|e(u) - e(v)| \leq 1$. ∎

In much the same way, the following can be proved (see Exercise 12.10).

Theorem 12.3 *Let u and v be adjacent vertices in a connected graph G. Then*

$$|d(u, x) - d(v, x)| \leq 1$$

for every vertex x of G.

Returning once again to the graph H of Figure 12.2, we see that $\mathrm{Cen}(H) \cong K_2$. This brings up a natural question. Which graphs can be the center of some graph? Stephen Hedetniemi showed that "every graph" is the answer to this question.

Theorem 12.4 *Every graph is the center of some graph.*

Proof. [direct proof] Let G be a graph. We show that G is the center of some graph. First, add two new vertices u and v to G and join them to every vertex of G but not to each other. Next, we add two other vertices u_1 and v_1, where we join u_1 to u and join v_1 to v. The resulting graph is denoted by F (see Figure 12.4).

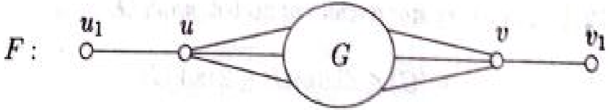


Figure 12.4: The graph H in the proof of Theorem 12.4

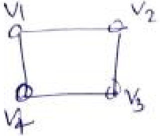
Since $e(u_1) = e(v_1) = 4$, $e(u) = e(v) = 3$, and $e(x) = 2$ for every vertex x in G, it follows that $V(G)$ is the set of central vertices of F and so $\text{Cen}(F) = G$. ∎

Self centred graphs

If the centre of a graph G is G itself, then it is called self centred graph.

Eg:- Cycle C_n

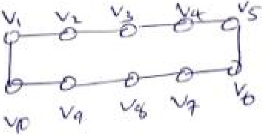
C_4 =



$e(v_1) = 2$
$e(v_2) = 2$
$e(v_3) = 2$
$e(v_4) = 2$.

minimum eccentricity is 2 and here all vertices are central vertices. The graph induced by the central vertices is C_4 itself. $\therefore$ C_4 is self centred.

C_{10}:



$e(v_i) = 5 \quad \forall i$

mini eccentricity $= 5$.

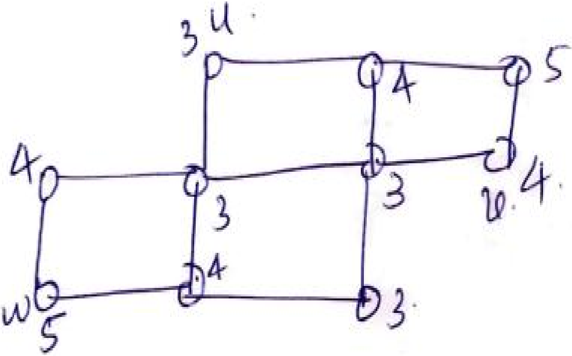
$\therefore$ Radius of $C_{10} = 5$ and here all vertices are central vertices. The graph induced by the central vertices is C_{10} itself. $\therefore$ C_{10} is self centered.

All cycles are self centered.

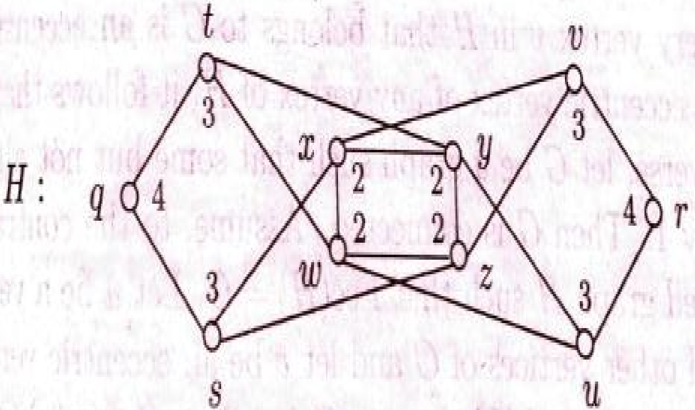
<u>Eccentric vertex</u>.

A vertex v is an eccentric vertex of a graph ~~of~~ G if v is an eccentric vertex of some vertex of G.
In other words, a vertex v is an eccentric vertex of G if v is farthest from some vertex of G.

Problems



u is an eccentric vertex in G.



gure 12.10: A graph each of whose vertices is an eccentric vertex

Let G be a connected graph. The **eccentric subgraph** $\mathrm{Ecc}(G)$ of G is the subgraph of G induced by the set of eccentric vertices of G. For example, a connected graph F and its eccentric subgraph are shown in Figure 12.11. If every vertex of a graph G is an eccentric vertex, then $\mathrm{Ecc}(G) = G$.

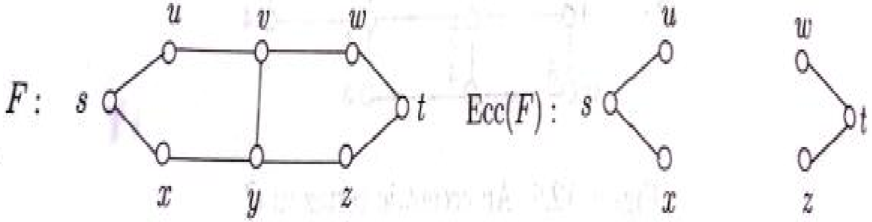


Figure 12.11: A graph and its eccentric subgraph

A vertex v in a connected graph G is called a **peripheral vertex** if $e(v) = \text{diam}(G)$. Thus, in certain sense, a peripheral vertex is opposite to a central vertex. The subgraph of G induced by its peripheral vertices is the **periphery** $\text{Per}(G)$. For the graph H of Figure 12.2, which is redrawn in Figure 12.7, the periphery of H is shown in Figure 12.7.

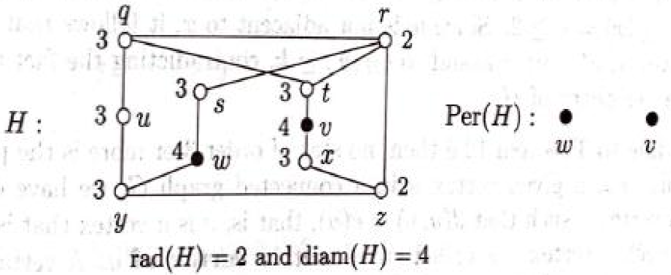


Figure 12.7: The eccentricities of the vertices of a graph

The periphery of the graph H of Figure 12.7 is isomorphic to $2K_1$ (that is, it consists of two isolated vertices) and so it is disconnected. Is the periphery of every graph disconnected? The answer is no, as the graph F of Figure 12.8 shows. Each vertex of F is labeled with its eccentricity. Since $\text{diam}(F) = 3$, it follows that $\text{Per}(F) \cong C_6$, which is connected. In fact, if $G \cong C_n$, then $\text{Per}(G) \cong C_n$ for $n \geq 3$. Could it be then that *every* graph is the periphery of some graph? Halina Bielak and Maciej Syslo showed that the answer to this question is no.

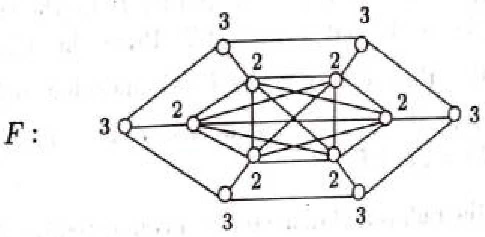


Figure 12.8: A graph F with $\text{Per}(F) = C_6$

Theorem 12.6 *A nontrivial graph G is the periphery of some graph if and only if every vertex of G has eccentricity 1 or no vertex of G has eccentricity 1.*

According to Theorem 12.6 then, no star of order 3 or more is the periphery of a graph. For a given vertex u in a connected graph G, we have discussed seeking a vertex v such that $d(u, v) = e(u)$, that is, v is a vertex that is farthest from u. Such a vertex v is called an **eccentric vertex of** u. A vertex v is an **eccentric vertex of the graph** G if v is an eccentric vertex of some vertex of G. In other words, a vertex v is an eccentric vertex of G if v is farthest from *some* vertex of G.

Consider the graph G of Figure 12.9, where each vertex is labeled with its eccentricity. For example, $e(u) = 3$. Since $d(u, v) = 3$, it follows that v is an eccentric vertex of u. Because there is a $u - v$ path of length 3 in G, there is certainly a $v - u$ path of length 3 in G. This does not mean, however, that u is an eccentric vertex of v as there may be a vertex farther from v than u is. This only implies therefore that $e(v) \geq 3$. In fact, $e(v) = 4$ and so u is *not* an eccentric vertex of v, although w *is* an eccentric vertex of v. More generally, if a vertex y is an eccentric vertex of a vertex x in a connected graph, then $e(y) \geq e(x)$.

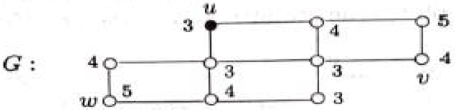


Figure 12.9: An eccentric vertex in G

If a vertex x in a connected graph G is a peripheral vertex of G, then, as we have seen, $e(x) = \text{diam}(G)$. Necessarily then, there exists a vertex y such that $d(x, y) = e(x) = \text{diam}(G)$. This also implies, however, that $d(x, y) = e(y) = \text{diam}(G)$ and that y is a peripheral vertex of G as well. Therefore, every peripheral vertex of G is an eccentric vertex. The converse is not true, however. We saw that the vertex v in the graph G of Figure 12.9 is an eccentric vertex of G but that v is not a peripheral vertex of G.

Consider next the graph H shown in Figure 12.10, where $\text{rad}(H) = 2$ and $\text{diam}(H) = 4$. Since q and r are peripheral vertices (the *only* peripheral vertices of H), they are also eccentric vertices of H. The vertices x and z are also eccentric vertices of each other; while t and u are both eccentric vertices of x and z. Furthermore, w and y are eccentric vertices of each other; while s and v are both eccentric vertices of w and y. That is, every vertex of H is an eccentric vertex.

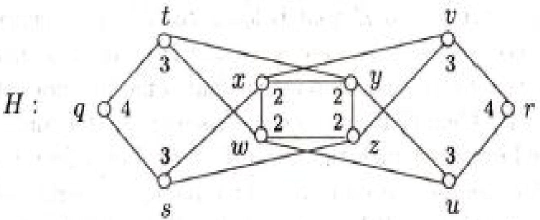


Figure 12.10: A graph each of whose vertices is an eccentric vertex

If every vertex of some graph G has the same eccentricity (and is therefore a peripheral vertex), then certainly every vertex of G is an eccentric vertex. However, the graph H of Figure 12.10 shows that every vertex of a graph can be an eccentric vertex without all the eccentricities being the same.

A connected graph G is an **eccentric graph** if every vertex of G is an eccentric vertex. Therefore, the graph H of Figure 12.10 is an eccentric graph,

MAT2011 Graph Theory and Applications

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Domination in graphs

For a vertex v of a graph G, recall that a **neighbor** of v is a vertex adjacent to v in G. Also, the **neighborhood** (or **open neighborhood**) $N(v)$ of v is the set of neighbors of v. The **closed neighborhood** $N[v]$ is defined as $N[v] = N(v) \cup \{v\}$. A vertex v in a graph G is said to **dominate** itself and each of its neighbors, that is, v dominates the vertices in its closed neighborhood $N[v]$. Therefore, v dominates $1 + \deg v$ vertices of G.

A set S of vertices of G is a **dominating set** of G if every vertex of G is dominated by some vertex in S. Equivalently, a set S of vertices of G is a dominating set of G if every vertex in $V(G) - S$ is adjacent to some vertex in S. Consider the graph G of Figure 13.1. The sets $S_1 = \{u, v, w\}$ and $S_2 = \{u_1, u_4, v_1, v_4\}$ are both dominating sets in G, indicated by solid vertices.

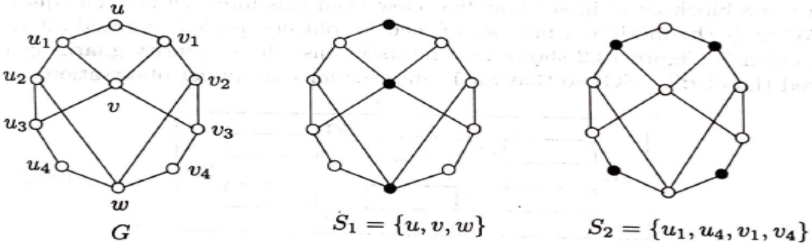


Figure 13.1: Two dominating sets in a graph G

A **minimum dominating set** in a graph G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the **domination number** of G and is denoted by $\gamma(G)$.

Since the vertex set of a graph is always a dominating set, the domination number is defined for every graph. If G is a graph of order n, then $1 \leq \gamma(G) \leq n$. A graph G of order n has domination number 1 if and only if G contains a vertex v of degree $n-1$, in which case $\{v\}$ is a minimum dominating set; while $\gamma(G) = n$ if and only if $G \cong \overline{K}_n$, in which case $V(G)$ is the unique minimum dominating set.

Let's return to the graph G of Figure 13.1. We saw that the set $S_1 = \{u, v, w\}$ is a dominating set for G. Therefore, $\gamma(G) \leq 3$. To show that the domination number of G is actually 3, it is required to show that there is no dominating set with two vertices. Notice that the order of G is 11 and that the degree of every vertex of G is at most 4. This means that no vertex can dominate more than 5 vertices and that every two vertices dominate at most 10 vertices. That is, $\gamma(G) > 2$ and so $\gamma(G) = 3$.

Applications of Domination

Let's look at a practical example involving domination. Figure 13.2 shows a portion of a city, consisting of six city blocks, determined by three horizontal streets and four vertical streets. A security protection agency has been retained to watch over the street intersections. A security guard stationed at an intersection can observe the intersection where he is located as well as all intersections up to one block away in straight line view from this intersection. The question is: What is the minimum number of security officers needed to guard all 12 intersections? Figure 13.2 shows four intersections where security guards can be placed (labeled by SG) so that all 12 intersections are under observation.

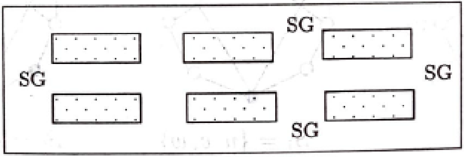


Figure 13.2: A city map

This situation can be modeled by the graph G of Figure 13.3. The graph G is actually the Cartesian product $P_3 \times P_4$, which is a bipartite graph. The street intersections are the vertices of G and two vertices are adjacent if the vertices represent intersections on the same street at opposite ends of a city block. Looking for the smallest number of security guards in the city of Figure 13.2 is the same problem as seeking the domination number of the graph G in Figure 13.3. The solid vertices in Figure 13.3 correspond to the placement of security officers in Figure 13.2.

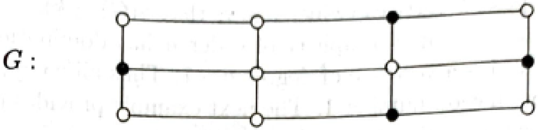


Figure 13.3: A graph modeling a city map

Example 13.1 *For the graph G in Figure* 13.3, $\gamma(G) = 4$.

Solution. Since the four solid vertices in Figure 13.3 form a dominating set of G, it follows that $\gamma(G) \leq 4$. To verify that $\gamma(G) \geq 4$, it is necessary to show that there is no dominating set with three vertices in G.

The graph G has 12 vertices, two of which have degree 4 and six have degree 3. The remaining four vertices have degree 2. Therefore, there are two vertices that dominate five vertices each and six vertices that dominate four vertices each. Conceivably, then, there is some set of three vertices that together dominate all 12 vertices of G. However, we have already noticed that G is bipartite and so its vertices can be colored with two colors, say red (R) and blue (B). Without loss of generality, we can assume that the vertices of G are colored as in Figure 13.4. Notice that the neighbors of each vertex have a color that is different from the color assigned to this vertex.

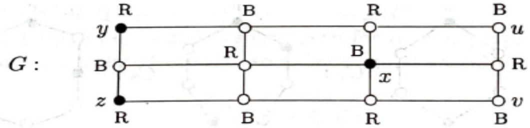


Figure 13.4: The graph $P_3 \times P_4$

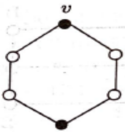
Assume, to the contrary, that G has a dominating set S containing three vertices. At least two vertices of S are colored the same. If all three vertices of S are colored the same, say red, then only three of the six red vertices will be dominated. Therefore, exactly two vertices of S are colored the same, say red, with the third vertex colored blue. If the blue vertex of S has degree at most 3, then it can dominate at most three red vertices and S dominates at most five red vertices of G, which is impossible. Hence S must contain x (see Figure 13.4) as its only blue vertex. Since y and z are the only two red vertices not dominated by x, it follows that $S = \{x, y, z\}$. However, u and v are not dominated by any vertex of S, which cannot occur. Therefore, $\gamma(G) = 4$. ◇

Example 13.2 $\gamma(C_n) = \lceil n/3 \rceil$ *for* $n \geq 3$.

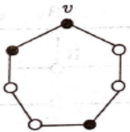
Solution. First, we write $n = 3q + r$, where $0 \leq r \leq 2$. Since C_n is 2-regular, every vertex of C_n dominates exactly three vertices. Therefore, any q vertices of C_n dominate at most $3q$ vertices of C_n. If $r = 0$, then this says that $\gamma(C_n) \geq q$. If $r = 1$ or $r = 2$, then $\gamma(C_n) \geq q + 1$.

Suppose first that $r = 0$. Let S be the set consisting of any vertex v of C_n and every third vertex of C_n begining with v as we proceed cyclically about C_n in some direction. Then every vertex of C_n is dominated by exactly one vertex of S. Since S contains exactly q vertices, $\gamma(C_n) \leq q$. Next suppose that $r = 1$ or $r = 2$. Now let S be the set consisting of any vertex v of C_n and every third vertex of C_n begining with v as we proceed cyclically about C_n in some direction until we have a total of $q + 1$ vertices (see Figure 13.5). Then, every vertex of C_n is dominated by at least one vertex of S. So S is a dominating set of C_n and $\gamma(C_n) \leq q + 1$. In both cases, $\gamma(C_n) = \lceil n/3 \rceil$. ◇

C_6 :



C_7 :



C_8 :

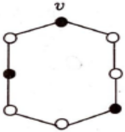
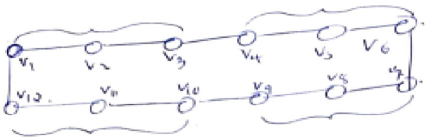


Figure 13.5: A minimum dominating set in C_n for $6 \leq n \leq 8$

In the following result both a lower bound and an upper bound are established for the domination number of a graph, each in terms of the order and the maximum degree of the graph.

Example



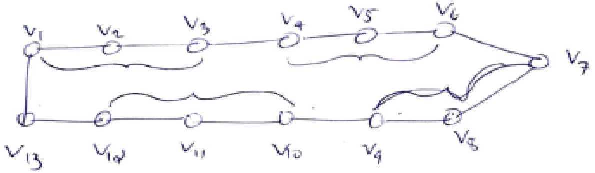
$n = 3q + r$.

$12 = 3 \times 4 + 0$.

$\therefore q = 4$ and $r = 0$.

The dominating set is $\{v_2 \; v_5 \; v_8 \; v_{11}\}$, $\gamma(G) = 4$

$(\because r = 0)$

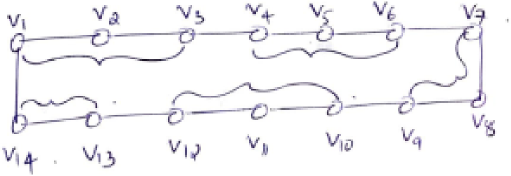


$13 = 3 \times 4 + 1$

$\therefore q = 4$, $r = 1$.

the dominating set is $\{v_2 \; v_5 \; v_8 \; v_{11} \; v_{13}\}$.

$\gamma(G) = 5 \quad (\because r > 0)$



The dominating set is $\{v_2, v_5, v_8, v_{11}, v_{13}\}$.

$\gamma(G) = 5$, $(\therefore \gamma > 0$

MAT2011 Graph Theory and Applications

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Locating number

Suppose that a certain facility consists of five rooms R_1, R_2, R_3, R_4, R_5 (shown in Figure 12.16). The distance between rooms R_1 and R_3 is 2 and the distance between R_2 and R_4 is also 2. The distance between all other pairs of distinct rooms is 1. The distance between a room and itself is 0. A certain (red) sensor is placed in one of the rooms. If a fire should take place in a room, then the sensor is able to detect the distance from the room with the red sensor to the room containing the fire. Suppose, for example, that the sensor is placed in R_1. If a fire occurs in R_3, then the sensor alerts us that a fire has occurred in a room at distance 2 from R_1; that is, the fire is in R_3 since R_3 is the only room at distance 2 from R_1. If the fire is in R_1, then the sensor indicates that the fire has occurred in a room at distance 0 from R_1; that is, the fire is in R_1. However, if the fire is in any of the other three rooms, then the sensor tells us that there is a fire in a room at distance 1 from R_1. But with this information, we cannot determine the precise room in which the fire has occurred. In fact, there is *no* room in which the (red) sensor can be placed to identify the exact location of a fire in every instance.

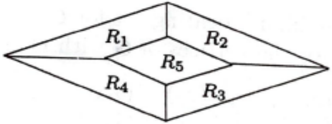

Figure 12.16: A facility consisting of five rooms

On the other hand, if we place the red sensor in R_1 and a blue sensor in R_2, and a fire occurs in R_5, say, then the red sensor in R_1 tells us that there is a fire in a room at distance 1 from R_1, while the blue sensor tells us that the fire is in a room at distance 1 from R_2, that is, the ordered pair $(1, 1)$ is produced for R_4. Since these ordered pairs are distinct for all rooms, the minimum number of sensors required to detect the exact location of any fire is 2. Care must be taken, however, as to where the two sensors are placed. For example, we cannot place sensors in R_1 and R_3 since, in this case, the ordered pairs of R_2, R_4, and R_5 are all $(1, 1)$, and we cannot determine the precise location of the fire.

The facility that we have just described can be modeled by the graph of Figure 12.17, whose vertices are the rooms and such that two vertices in this graph are adjacent if the corresponding two rooms are adjacent. This gives rise to a problem involving graphs.

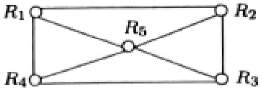

Figure 12.17: A graph modeling a facility with five rooms

Let G be a connected graph. For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices of G and a vertex v of G, the **locating code** (or simply the **code**) of v with respect to W is the k-vector

$$c_W(v) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k)).$$

The set W is a **locating set** for G if distinct vertices have distinct codes. A locating set containing a minimum number of vertices is a **minimum locating set** for G. The **location number** $\mathrm{loc}(G)$ is the number of vertices in a minimum locating set for G. For example, consider the graph G shown in Figure 12.18, which you will notice is isomorphic to the graph of Figure 12.17. The ordered set $W_1 = \{v_1, v_3\}$ is not a locating set for G since $c_{W_1}(v_2) = (1, 1) = c_{W_1}(v_4)$, that is, G contains two vertices with the same code with respect to W_1. On the other hand, $W_2 = \{v_1, v_2, v_5\}$ is a locating set for G since the codes for the vertices of G with respect to W_2 are

$$c_{W_2}(v_1) = (0,1,1), \quad c_{W_2}(v_2) = (1,0,1), \quad c_{W_2}(v_3) = (2,1,1),$$
$$c_{W_2}(v_4) = (1,2,1), \quad c_{W_2}(v_5) = (1,1,0).$$

However, W_2 is not a minimum locating set for G since $W_3 = \{v_1, v_2\}$ is also a locating set. The codes for the vertices of G with respect to W_3 are

$$c_{W_3}(v_1) = (0,1), \quad c_{W_3}(v_2) = (1,0), \quad c_{W_3}(v_3) = (2,1),$$
$$c_{W_3}(v_4) = (1,2), \quad c_{W_3}(v_5) = (1,1).$$

Since no single vertex constitutes a locating set for G, it follows that W_3 is a minimum locating set for this graph G and so $\text{loc}(G) = 2$.

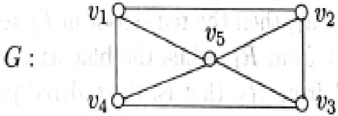

Figure 12.18: Resolving sets in a graph G

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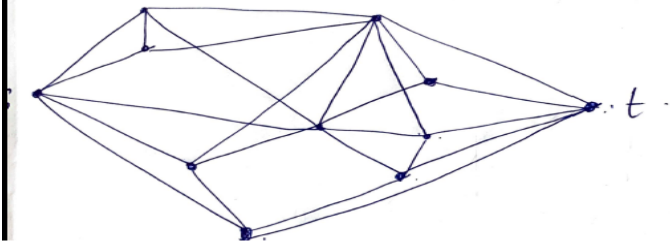
Menger's theorem.

The minimum number of ~~points~~ vertices seperating two non adjacent vertices s and t is the maximum number of disjoint s-t path.

Example.

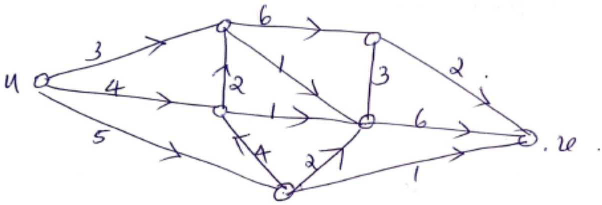


Menger's theorem is max-flow min-cut ~~in a netw~~ concept in a network

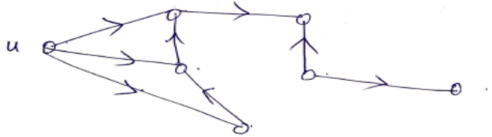
Theorem

In any network N, in which there is a path from u to v, the maximum flow from u to v is equal to the minimum cut capacity.

example.



The removal of a set of edges (there can be more than one set of such edges) will seperates the two vertices u and v. Find out the sum of capacities (edge weights) of edges in that set. Similarly find out the sum of capacities in all such sets of edges. The set of edges which gives the minimum capacity is known as the minimum cut.

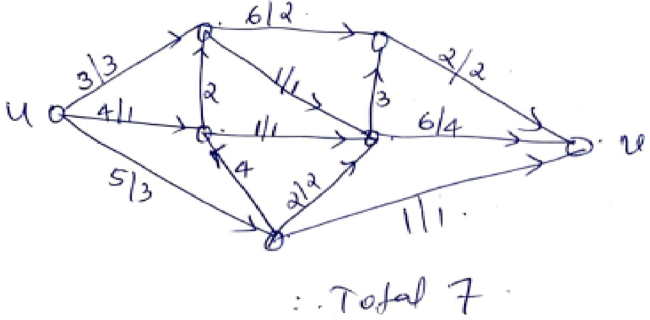


The capacities of the removed edges is

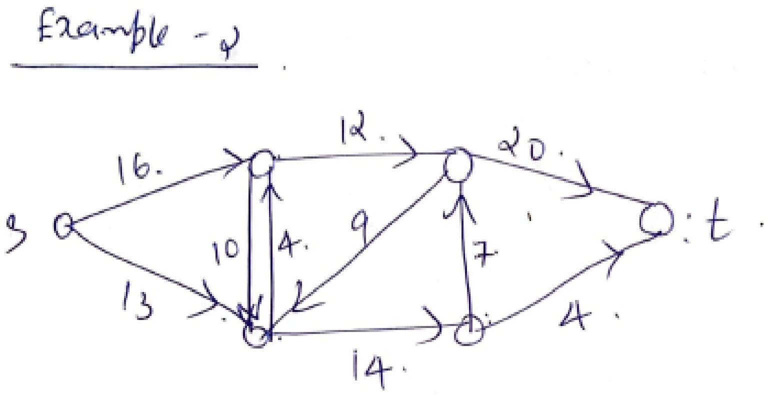
$1+1+2+1+2 = \underline{\underline{7}}$.

$\therefore$ the minimum cut capacity is 7.

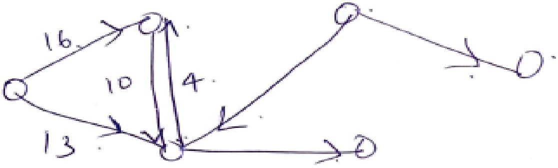
The minimum flow possible here is also 7. Thus the max-flow min cut theorem is verified here. Note that by an edge weight a/b we means that the capacity of that edge is a but we are sending only b units of items through that edge.



Verify max flow min cut theorem in the below example. The edge weight represent its capacity.



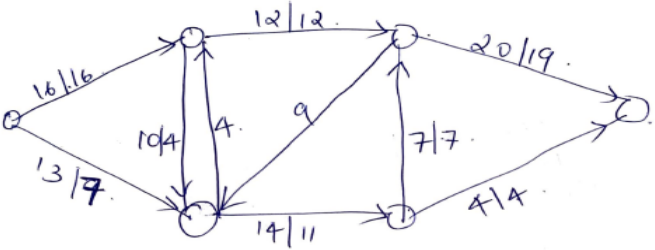
The minimum cut possible here is 23.



The capacities of the removed edges is.

$12 + 7 + 4 = \underline{\underline{23}}$.

The maximum flow possible here is also 23. Hence the theorem verified.



The maximum flow possible here is $19+4=23$ un

$\therefore$ max flow = mini · cut