MODULE - I

* What is Information Theory ?

- The movements of and transformations of information, just like those of a fluid, are constrained by mathematical and physical laws. These laws have deep connections with :

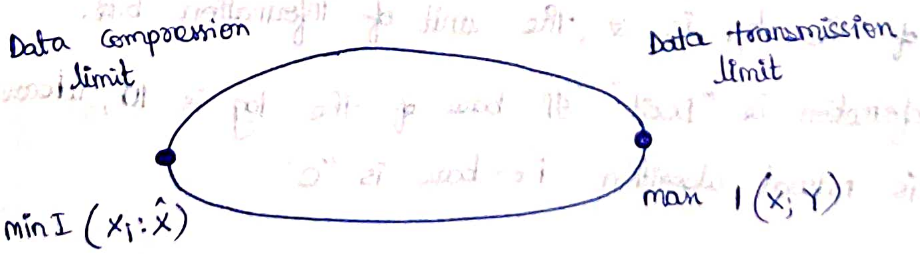
1) Probability theory, statistics, and combinatorics.
2) Thermodynamics (statistical physics).
3) Spectral Analysis, Fourier (and other) transforms.
4) Sampling theory, prediction, estimation theory.
5) Electrical engineering (bandwidth ; signal to noise ratio).
6) complexity theory (minimal description length).
7) signal processing, representation, compressibility.

* Scope of Information Theory :

- Information theory is concerned with fundamental limitations of communication.

- Entropy of the data, (H) is its compression limit of data compress. Ex : How many bits are required to represent music source.

- The ultimate limit of reliable communication over a noisy channel is the channel capacity (C), its rate limit. Ex : Bits sent in one second over telephone line.



Information theory as the extreme points of communication theory.

- The data compression minimum $I(X;\hat{X})$ lies at one extreme of the set of communication ideas. All data compression schemes require description rates at least equal to this minimum.

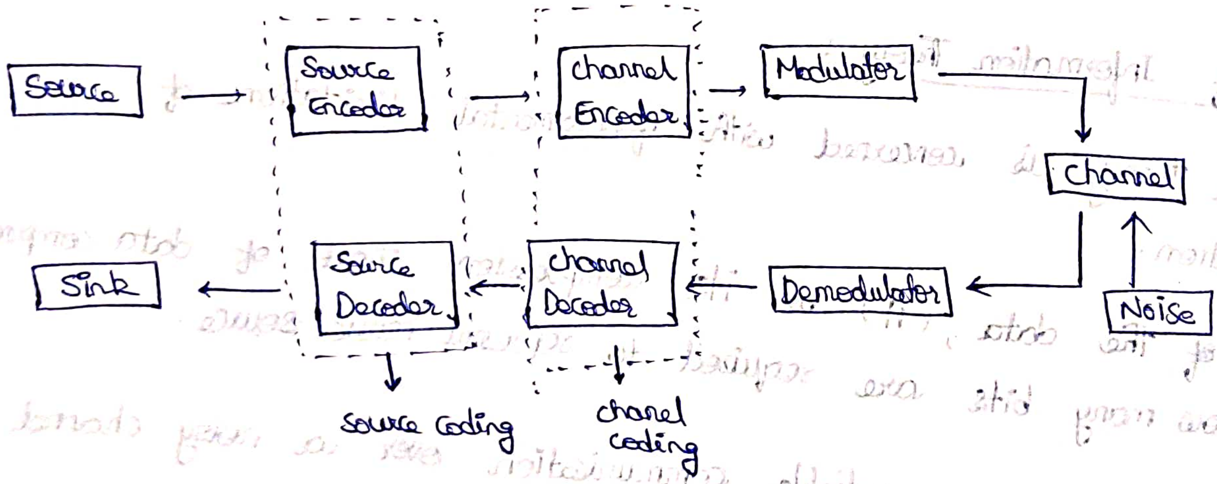
- At the other extreme is the data transmission maximum $I(X;Y)$, known as the channel capacity. Thus all modulation schemes and data compression schemes lie between these limits.

*. SCOPE OF CODING THEORY :

- Concerned with practical techniques to realize the limits specified by information theory.

1) Source Coding : converts source output to bits.

2) Channel Coding : adds extra bits to data transmitted over the channel. This redundancy helps combat the errors introduced in transmitted bits due to channel noise.



- Information is measured in terms of its probability.

$$\boxed{I(x) = -\log_2 P(x)}$$

- If the base of the log is 2, the unit of information bits.

- The unit of information is "Decit" If base of the log is 10, whereas it is "nat" if it is natural algorithm i.e. base is "e"

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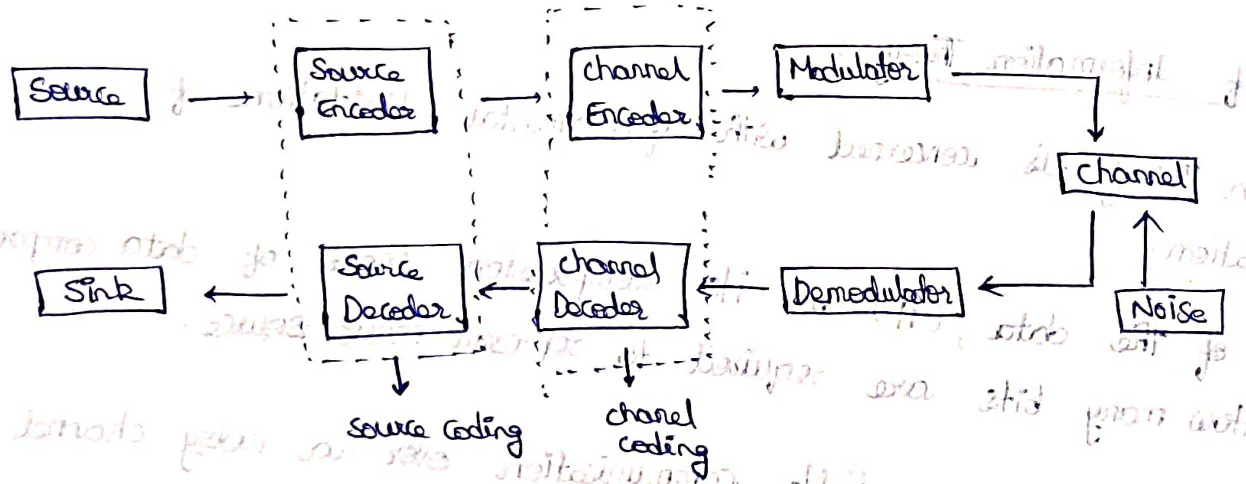

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*. RANDOM VARIABLE :

- Random Variables are variables that take on values determined by probability distributions. They may be discrete or continuous, in either their domain or their range.

*. ENTROPY :

- Measure of self Information.
- The entropy is a measure of the average uncertainty in the random variable.
- It is the no. of bits on average required to describe the random variable.

$$\boxed{I(x) = \log_2\left(\frac{1}{P_x}\right) = -\log_2 P_x} \; *$$

- Let X be a discrete random variable (DRV) with alphabet $\mathcal{X}$ and probability mass function $p(x) = P_r\{X = x\}$, $x \in X$.
- The entropy $H(x)$ of a discrete random variable X is defined by:

$$\boxed{H(x) = -\sum_{x=i} P(i) \log_2 P(x)} \; *$$

EXAMPLES :

* Find out the entropy of throwing a dice.

Sol:
$$X = \{1, 2, 3, 4, 5, 6\}$$

$$P(x) = \left\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right\}$$

$$H(x) = -\sum_{i=1}^{6} P_i \log_2(P_i) = -\frac{1}{6}\log\left(\frac{1}{6}\right) - \frac{1}{6}\log\left(\frac{1}{6}\right) - \frac{1}{6}\log\left(\frac{1}{6}\right)$$
$$-\frac{1}{6}\log\left(\frac{1}{6}\right) - \frac{1}{6}\log\left(\frac{1}{6}\right) - \frac{1}{6}\log\left(\frac{1}{6}\right)$$

$$= -\left[\frac{1}{6}\log\left(\frac{1}{6}\right)\right] \times 6 = 2.584 \text{ bits}$$

* If source generates a symbol with their associated probabilities as $\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32} \right\}$. Calculate the entropy.

Sol:
$$H(X) = -\sum_{i=1}^{5} P(x) \log P(x)$$
$$= \sum_{i=1}^{5} P(x) \log\left(\frac{1}{P(x)}\right)$$
$$= \frac{1}{2}\log_2\left(\frac{1}{\frac{1}{2}}\right) + \frac{1}{4}\log_2\left(\frac{1}{\frac{1}{4}}\right) + \frac{1}{8}\log_2\left(\frac{1}{\frac{1}{8}}\right) + \frac{1}{16}\log_2\left(\frac{1}{\frac{1}{16}}\right) + \frac{1}{32}\log_2\left(\frac{1}{\frac{1}{32}}\right)$$
$$= \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 + \frac{1}{16}\log_2 16 + \frac{1}{32}\log_2 32$$
$$= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} = \underline{\underline{1.7825 \text{ bits}}}$$

* CONCLUSION:

- More Probable – More Certain / less uncertain / less entropy
- Less Probability – Less chance / More uncertain / More entropy
- More uncertainity – More Entropy
- Less uncertainity – less Entropy
- Entropy (Source sure event) = 0
- Entropy (Impossible) = 1

* Properties of Entropy:

$$H(X) = -\sum P_i \log P_i$$

- $H(X) \geq 0$; $0 \leq P(x) \leq 1 \Rightarrow \log \frac{1}{P(x)} \geq 0$
- $H_b(X) = (\log_b a) H_a(X)$

*. Consider a random variable that has a uniform distribution over 32 outcomes. Determine its entropy?

Sol: $\frac{1}{32}, \frac{1}{32}, \frac{1}{32}, \ldots\ldots, \frac{1}{32}$

$$= \left[\frac{1}{32} \times \log_2\left(\frac{1}{1/32}\right)\right] \times 32 = \frac{32}{32} \log_2 2^5 = 5 \text{ bits/message}$$

*. If 2^M equal probable messages then its entropy will be maximum $\approx M$ bits/message.

Ex: 16 messages $\approx$ are equiprobable

$$\approx \left[\frac{1}{16} \log_2\left(\frac{1}{\frac{1}{16}}\right)\right] \times 16 = \log_2 2^4 = 4 \log_2 2 \approx 4 \text{ bits/message}.$$

*. The probability of winning for the eight horses participating in horse race are $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}, \frac{1}{64}\right)$. Calculate the entropy of the horse race.

Sol: $= \frac{1}{2} \log_2 2 + \frac{1}{2} \log_2 4 + \frac{1}{2} \log_2 8 + \log_2 16 + \left(4 \times \frac{1}{64} \log_2 64\right)$

$$= \frac{1}{2} + \frac{2}{2} + \frac{3}{2} + \frac{4}{2} + \frac{4 \times 6}{64} = $$

*. Information Rate:

M ($M_1\ M_2\ M_3$) $\rightarrow$ Source $\xrightarrow{r}$ messages/sec

$\boxed{R = rH}$ bits/sec

Ex: Let $X = \begin{cases} a & \text{with probability } 1/2 \\ b & \text{with probability } 1/4 \\ c & \text{with probability } 1/8 \\ d & \text{with probability } 1/8 \end{cases}$. The entropy of X is:

Sol: $H(x) = -\frac{1}{2} \log \frac{1}{2} - \frac{1}{4} \log \frac{1}{4} - \frac{1}{8} \log \frac{1}{8} - \frac{1}{8} \log \frac{1}{8} = \frac{7}{4}$ bits

IMP

*. A discrete memory less source has six symbols with probabilities $P_A = \frac{1}{2}$, $P_B = \frac{1}{4}$, $P_C = \frac{1}{8}$, $P_D = P_E = \frac{1}{20}$, $P_F = \frac{1}{40}$. Determine :

(i) The entropy of source :

(ii) The amount of information contained in message ABABBA.

(iii) The amount of information contained in the message FDDFDF

Sol :

$$H = -\sum_{i=1}^{6} P_i \log P_i$$

$$= \frac{1}{2}\log_2\left(\frac{1}{1/2}\right) + \frac{1}{4}\log_2\left(\frac{1}{1/4}\right) + \frac{1}{8}\log_2\left(\frac{1}{1/8}\right) + \frac{2}{20}\log_2\left(\frac{1}{1/20}\right) + \frac{1}{40}\log_2\left(\frac{1}{1/40}\right)$$

$$= \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 + \frac{2}{20}\log_2 20 + \frac{1}{40}\log_2 40$$

$$= 1.94024 \text{ bits / symbols}$$

ii. $I(x) = \log_2\left(\frac{1}{P(x)}\right)$

$$P(ABABBA) = P_A \times P_B \times P_A \times P_B \times P_B \times P_A$$

$$= \frac{1}{2} \times \frac{1}{4} \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{2} = \left(\frac{1}{2}\right)^9$$

$$\therefore I(ABABBA) = \log_2\left(\frac{1}{(\frac{1}{2})^9}\right) = 9 \text{ bits}$$

iii. $P_F = \frac{1}{40}$; $P_D = \frac{1}{20}$

$$I(FDDFDF) = 28.93 \text{ bits}$$

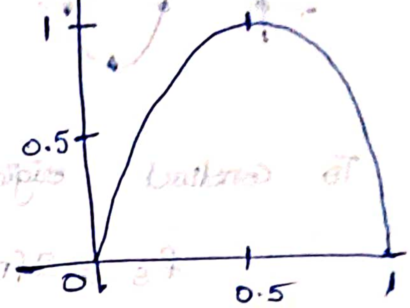
* Entropy of Binary Memory less Source :

Let $X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$

Sol: The Entropy will be :

$$H(X) = -p \log p - (1-p) \log (1-p) = H(P)$$

$H(x) = 1$ when $p = \frac{1}{2}$.



*. An event has 6 possible outcomes with probabilities $P_1 = \frac{1}{2}$, $P_2 = \frac{1}{4}$, $P_3 = \frac{1}{8}$, $P_4 = \frac{1}{16}$, $P_5 = \frac{1}{32}$, $P_6 = \frac{1}{32}$. Find the entropy of the system and the rate of information if there are 16 outcomes per second.

Sol:
$$H = -\sum_{i=1}^{6} P_i \log_2 P_i$$

Given $r = 16$ outcomes / second. Need to find, $R = ?$

$$H = \frac{1}{2} \log_2 2 + \frac{1}{4} \log_2 4 + \frac{1}{8} \log_2 8 + \frac{1}{16} \log_2 16 + \frac{2}{32} \log_2 32$$

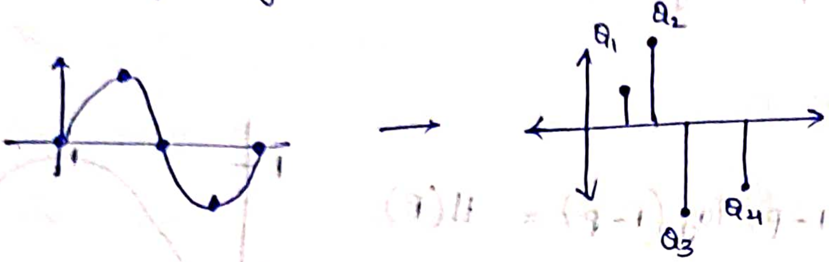
$$= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \left(\frac{5}{32} + \frac{5}{32}\right) = \frac{16 + 16 + 12 + 8 + 10}{32} = \frac{62}{32} \ \frac{\text{bits}}{\text{outcome}}$$

$\boxed{R = r \cdot H} = \frac{16 \text{ outcomes}}{\text{Seconds}} \times \frac{162}{32} \frac{\text{bits}}{\text{outcomes}} = 3.1$ bits / seconds.

*. A analog signal is bandlimited to "B" Hz sampled at the Nyquist rate and the samples are quantized into 4 levels. The quantization levels $Q_1 Q_2 Q_3 Q_4$ (messages) are assumed independent and occur with probabilities $P_1 = P_4 = \frac{1}{8}$ and $P_2 = P_3 = \frac{3}{8}$.

(a) Find the information rate per second

(b) Find the information rate per second if all the messages are equally likely.



$\boxed{N \geq 2 f_m} = 2B$

To construct original signal, the sample rate,

$f_s \geq 2f_m$ (Nyquist Rate)

$\boxed{f_s = 2B}$

(a)

i. $P_1 = P_4 = \frac{1}{8}$; $P_2 = P_3 = \frac{3}{8}$

$$H = P_1 \log_2\left(\frac{1}{P_1}\right) + P_2 \log_2\left(\frac{1}{P_2}\right) + P_3 \log_2\left(\frac{1}{P_3}\right) + P_4 \log_2\left(\frac{1}{P_4}\right)$$

$$= \sum_{i=1}^{4} P_i \log_2\left(\frac{1}{P_i}\right) = \left(\frac{1}{8}\log_2 8\right) * 2 + 2 * \left(\frac{3}{8}\log_2\left(\frac{8}{3}\right)\right)$$

$H = 1.81127$ Bits/msg

$r = 2B \Rightarrow \boxed{R = r \cdot H} = 2B(1.81128) \approx 3.622B$ bits/sec

(b) $P_1 = P_2 = P_3 = P_4 = \frac{1}{2}$

$$H = \sum_{i=1}^{4} P_i \log_2\left(\frac{1}{P_i}\right) = 4 \times \frac{1}{2}\log_2 2 = 2 \text{ Bits/msg}$$

$R = r \cdot H = (2B)(2) = 4B$ bits/sec

Analog signal → Sampling (DT) → Quantization → Encoding

Q_1	00	Q_3	10
Q_2	01	Q_4	11

IMP *

Consider a telegraph source having two symbols dot and dash. The dot duration is 0.2 sec and the dash duration is 3 times of the dot duration. The probability of occurance of dot is twice than that of dash, and time between symbol is 0.2 seconds. Calculate information rate of the telegraph source. (Assume the string consists of 1200 symbols).

Sol: $P_{dot} = 2 P_{dash}$; $P_{dot} + P_{dash} = 1$

$$P_{dash} = \frac{1}{3}$$

$$P_{dot} = 2\left(\frac{1}{3}\right) = \frac{2}{3}$$

$$H = \frac{1}{3}\log_2\left(\frac{1}{\left(\frac{1}{3}\right)}\right) + \frac{2}{3}\log_2\left(\frac{1}{2/3}\right) = 0.9183 \text{ bits/symbol}$$

$\boxed{R = r \cdot H}$ $T_{dot} = 0.2$ sec ; $T_{dash} = 0.6$ Sec

Each string consists of 1200 symbols on the average the dots and dash will appear according to the probability,

$$\text{No. of dots} = \frac{2}{3} \times 1200 = 800$$

$$\text{No. of dash} = \frac{1}{3} \times 1200 = 400$$

Joint Pro

Total time required,

$= \left[(\text{No. of dot})(\text{dot duration})\right] + \left[(\text{No. of dash})(\text{dash duration})\right] + \left[(1200)(0.2)\right]$

$= \left[(800)(0.2)\right] + \left[(400)(0.6)\right] + \left[(1200)(0.2)\right]$

$= 640$ Sec

Average symbol rate,

$$r = \frac{1200}{T} = \frac{1200}{640 \text{ sec}} = 1.875 \text{ symbols/sec}$$

Rate of Information,

$$\boxed{R = r\,H} = (1.875)(0.9183) = 1.7218 \text{ bits/second.}$$

- To gain information is to lose uncertainity by the same amount, so I and H differ only in sign $(H = -I)$

- Entropy and Information have units of bits.

$$I_{m,n} = \log_2(P_m P_n) = \log_2 P_m + \log_2 P_n = I_m + I_n.$$

MODULE-2

*. Rule of Probability : Product Rule

$P(A,B)$ = Joint Probability of both A and B.

$= P(A|B) \cdot P(B) \approx P(B|A) \cdot P(A)$

If A & B are independent events.

$P(A|B) = P(A)$ & $P(B|A) = P(B) \Rightarrow P(A,B) = P(A) \cdot P(B)$

Ex: If $A = \{1,2,3,4,5,6\}$ & $B = \{1,2,3,4,5,6\}$

$$P(A=3, B=4) = P(A=3 | B=4) \cdot P(B=4)$$
$$= P(B=4 | A=3) \cdot P(A=3)$$

*. Rule of Probability : Sum Rule

- If event A is conditionalized on a no. of other events B, then the probability of A is the events sum of its joint Probabilities with all B:

$$P(A) = \sum_{B} P(A,B) = \sum_{B} P(A|B) \cdot P(B)$$

Marginal Probability

$$P(B) = \sum_{A} P(A,B) = \sum_{A} P(B|A) \cdot P(A)$$

*. Rule of Probability : Baye's Theorem

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

- It allows to reverse the conditionalizing of events and to compute $P(B|A)$ from the Knowledge of $P(A)$ & $P(B)$.

* . BAYES THEOREM :

$$P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)} = \frac{P(A,B)}{P(A)}$$

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} = \frac{P(A,B)}{P(B)}$$

- Entropy is the uncertainity of a single random variable.
- Conditional entropy $H(X|Y)$, is the entropy of a random variable conditional on the knowledge of another random variable.
- The reduction in uncertainity due to another random variable is called the mutual information.

Ex : $X = \{$ Probability of weather condition $\}$

$Y = \{$ Probability of raining today $\}$

imp

* JOINT ENTROPY :

- The joint Entropy $H(X,Y)$ of a pair of discrete random variables (X,Y) with a joint distribution $P(x,y)$ is defined as :

$$H(X,Y) = -\sum_{x \in X} \sum_{y \in Y} P(x,y) \log P(x,y)$$

$$H(X,Y) = \sum_{x,y} P(x,y) \cdot \log \frac{1}{P(x,y)}$$

- Joint Entropy is the additive if X and Y are independent random variables :

$$H(X,Y) = H(X) + H(Y) \quad \text{iff} \quad P(X,Y) = P(X) \cdot P(Y)$$

* Conditional Entropy of an ensemble X

- Measures the uncertainity remaining about random variable X after specifying that random variable Y has taken on particular value $y = b_j$

$$H(X|y=b_j) = \sum_{X} P(X|y=b_j) \log_2 \frac{1}{P(X|y=b_j)}$$

* Conditional Entropy of an ensemble X, given an ensemble Y :

$$H(X|Y) = \sum_{y} P(y) \left[\sum_{x} P(x|Y) \log_2 \frac{1}{P(x|Y)} \right]$$

(or)

$$\boxed{H(X|Y) = \sum_{x,y} P(x,Y) \log \frac{1}{P(X|Y)}}$$

- This measures the average uncertainity that remains about X, when Y is known.

$$H(X|Y) = \sum_{x,Y} P(x,Y) \cdot \log_2 \frac{1}{P(X|Y)}$$

- Uncertainity of Y given X : mean the conditional uncertainity of Y given the event $X = x$, averaged over the possible values x of X.

$$H(Y|X) = \sum_{x \in X} P(x) \cdot H(Y|X=x)$$

$$= - \sum_{x \in X} P(x) \sum_{y \in Y} P(Y|x) \log_2 P(Y|x)$$

$$= - \sum_{x \in X} \sum_{y \in Y} P(x,Y) \cdot \log_2 P(Y|x)$$

$$= - E \log_2 P(Y|X)$$

*. Mutual Information :

- The mutual information between two random variables measures the amount of information that one conveys about the other.

- How much information does the random variable Y give about the random variable X ?
  - Shannon's answer would be the "amount by which Y reduces the uncertainty about X", namely :

$$H(X) - H(X|Y)$$

- The mutual information between the discrete random variable X & Y is the quantity :

$$I(X;Y) = \boxed{H(X) - H(X|Y)}$$

$$I(X;Y) = \sum_{X,Y} P(X,Y) \log_2 \frac{P(X,Y)}{P(X) \cdot P(Y)}$$

- IMP If X & Y are independent random variables then the mutual information equals zero.

  Also, $\boxed{I(X;Y) = I(Y;X)}$ *

- Non-negativity : Mutual Information is always greater than zero.

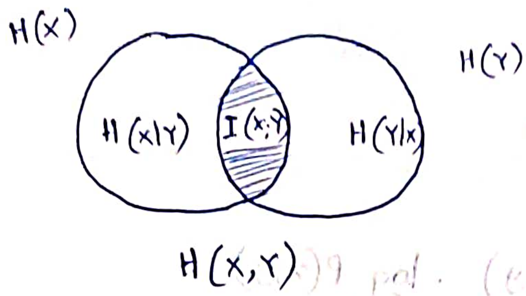
$$I(X;X) = H(X)$$

$$I(Y;Y) = H(Y)$$

$$I(X;Y) = H(X) - H(X|Y)$$

$$I(X;Y) = H(Y) - H(Y|X) = I(Y;X)$$

$$\boxed{I(X;Y) = H(X) + H(Y) - H(X,Y)} \ *$$



*. Chain Rule of Uncertainity :

- It states "the uncertainity of random variable vector equals to the uncertainity of its first component, plus the uncertainity of its second component when the first is known, ..., plus the uncertainity of its last component when all previous components are known".

$$H(X_1, X_2, X_3 \ldots X_n) = H(X_1) + H(X_2|X_1) + H(X_3|X_1, X_2) + H(X_4|X_1, X_2, X_3) + \ldots + H(X_N|X_1 \ldots X_{N-1})$$

*. Distance $D(X;Y)$ between X and Y :

- The amount by which the joint entropy of two random variable exceed their mutual information is a measure of the "distance" between them :

$$D(X;Y) = H(X;Y) - I(X;Y)$$

- Note that this quantity satisfies the standard axioms for a distance
- $D(X;Y) \geq 0$, $D(X;X) = 0$, $D(X;Y) = D(Y;X)$ and
- $D(X;Z) = D(X;Y) + D(Y;Z)$.
- Kullback – Leibler distance :

$$D_{KL}(P||Q) = \sum_{x} P(x) \log_2 \frac{P(x)}{Q(x)}$$

Summary :

1) $H(X) = -\sum_{X=x} P(x)\log_2 P(x)$

2) $H(X,Y) = -\sum_{X=x}\sum_{Y=y} P(x,y) \cdot \log_2 P(x,y)$

3) $H(X|Y) = \sum_{x,y} P(x,Y) \log \frac{1}{P(X|Y)}$

4) $I(X;Y) = \sum_{x,y} P(x,y) \cdot \log_2 \frac{P(x,y)}{P(x)P(y)}$

EXAMPLES :

Let (X,Y) have the following joint distribution

Y \ X	1	2	3	4
1	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{32}$
2	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{1}{32}$
3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
4	$\frac{1}{4}$	0	0	0

The marginal distribution of X is $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$ and the marginal distribution of Y is $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. Calculate the following :

a) $H(X)$

b) $H(Y)$

c) $H(X|Y)$

d) $H(Y|X)$

e) $H(X,Y)$

Sol:

$$H(X) = \sum_{i=1}^{4} P_i \log_2\left(\frac{1}{P_i}\right)$$

$$P(X) = \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8} \right\}$$

a) $H(X) = \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 + \frac{1}{8}\log_2 8$

$= \frac{7}{4}$ bits / symbols

b) $H(Y) = \sum_{j=1}^{4} P_j \log_2\left(\frac{1}{P_j}\right)$

$$P(Y) = \left\{ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right\}$$

$H(Y) = \frac{1}{4}\log_2 4 + \frac{1}{4}\log_2 4 + \frac{1}{4}\log_2 4 + \frac{1}{4}\log_2 4$

$= 4 \times \frac{1}{4}\log_2 4 = 2$ bits / symbols

c) $H(X,Y)$ = Joint Entropy = ?

~~$H(x=1 | Y=1)$~~ $H(x=1, Y=1) = \frac{1}{8}$

$H(x=1, Y=2) = \frac{1}{16}$

$H(x=1, Y=3) = \frac{1}{32}$

$H(x=1, Y=4) = \frac{1}{32}$

$H(x=2, Y=1) = \frac{1}{16}$

$H(x=2, Y=2) = \frac{1}{8}$

$H(x=3, Y=3) = \frac{1}{32}$

$H(x=3, Y=4) = \frac{1}{32}$

$H(x=3, Y=1) = \frac{1}{16}$

$H(x=3, Y=2) = \frac{1}{16}$

$H(x=3, Y=3) = \frac{1}{16}$

$\boxed{H(x=3, Y=4) = \frac{1}{16}}$

$\boxed{H(x=4, Y=1) = \frac{1}{4}}$

$H(x=4, Y=2) = H(x=4, Y=3)$

$= H(x=4, Y=4) = 0$

$$H(X,Y) = \sum_{\forall x} \sum_{\forall y} P(x,y) \cdot \log_2\left(\frac{1}{P(x,y)}\right)$$

$$= P(x_1,y_1)\log_2\frac{1}{P(x_1,y_1)} + P(x_1,y_2)\cdot\log_2\left(\frac{1}{P(x_1,y_2)}\right) +$$

$$P(x_1,y_3)\log_2\frac{1}{P(x_1,y_3)} + P(x_1,y_4)\log_2\frac{1}{P(x_1,y_4)} + \ldots$$

$$+ \ldots\ldots + \ldots + P(x_4 y_4)\log_2\frac{1}{P(x_4 y_4)}$$

$$H(X,Y) = \left(\frac{1}{8}\log_2 8\right) + \left(\frac{1}{16}\log_2 16\right) + \left(\frac{1}{32}\log_2 32\right) + \left(\frac{1}{32}\log_2 32\right)$$

$$+\left(\frac{1}{16}\log_2 16\right) + \left(\frac{1}{8}\log_2 8\right) + \left(\frac{1}{32}\log_2 32\right) + \left(\frac{1}{32}\log_2 32\right)$$

$$+\left(\frac{1}{16}\log_2 16\right) + \left(\frac{1}{16}\log_2 16\right) + \left(\frac{1}{16}\log_2 16\right) + \left(\frac{1}{16}\log_2 16\right)$$

$$+\left(\frac{1}{4}\log_2 4\right) + 0 + 0 + 0$$

$$= \left(2\times\frac{1}{8}\log_2 8\right) + \left(6\times\frac{1}{16}\log_2 16\right) + \left(4\times\frac{1}{32}\log_2 32\right) + \left(\frac{1}{4}\log_2 4\right)$$

$$= \frac{27}{8} \text{ bits/symbols}$$

c) As $P(XY) = P(X).P(Y)$ → Independent Events

$$\boxed{H(Y|X) = H(X,Y) - H(X)}$$

$$\boxed{H(X|Y) = H(X,Y) - H(Y)}$$

Given $H(X) = \frac{7}{4}$

$H(Y) = 2$

$H(X,Y) = \frac{27}{8}$

c) $H(X|Y) = \frac{27}{8} - 2 = \frac{11}{8}$

d) $H(Y|X) = \frac{27}{8} - \frac{7}{4} = \frac{13}{8}$

*. Calculate the probability that if somebody is "tall" (meaning) taller than 6ft or whatever), that person must be male. Assume that the probability of being male is $P(M) = 0.5$ and so likewise for being female $P(F) = 0.5$. Suppose that 80% of males are T (i.e. tall), and that 6% of females are tall. Solve :

1) If you know that somebody is male, how much information do you gain (in bits) by learning that he is also tall ?

2) How much do you gain by learning that a female is tall ?

3) Finally, how much information do you gain from learning that a tall person is female ?

Sol : Given, $P(M) = 0.5$ $\quad P(F) = 0.5$

$P(T|M) = 0.2$ $\quad P(T|F) = 0.06$

1) Information Gain, $I(M|T) = ?$ [Tall Person is Male]

$I(T|M) = ?$ [Male is Tall]

We know that,

$$P(M|T) = \frac{P(M,T)}{P(T)} = \frac{P(T|M) \cdot P(M)}{P(T)}$$

Using sum & product rule :

$$P(T) = P(T|M) \cdot P(M) + P(T|F) \cdot P(F)$$

$$= (0.2)(0.5) + (0.06)(0.5) = 0.13$$

$$P(M|T) = \frac{(0.2)(0.5)}{0.13} = 0.7692$$

$$I(M|T) = -\log_2 P(M|T) = -\log_2(0.77) = 0.3785 \text{ bits}$$

$$I(T|M) = -\log_2 P(T|M)$$

$$= -\log_2(0.2)$$

$$I(T|M) = 2.319 \text{ bits}$$

2) $I(T|F) = ?$

$P(T|F) = 0.06$

$I(T|F) = -\log_2 P(T|F) = -\log_2 (0.06) = 4.05889$ bits

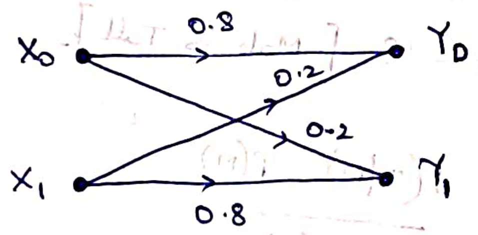
3) $I(F|T) = ?$

$$P(F|T) = \frac{P(F,T)}{P(T)} = \frac{P(T|F) \cdot P(F)}{P(T)}$$

$$= \frac{(0.06)(0.5)}{0.13} = 0.23076$$

$I(F|T) = -\log_2 P(F|T) = -\log_2 (0.23076) = 2.115477$ bits

* Calculate the resultant channel matrix for given Binary Symmetric channel ?



Sol:

$$P(Y|X) = \begin{array}{c c} & \begin{array}{cc} Y_0 & Y_1 \end{array} \\ \begin{array}{c} X_0 \\ X_1 \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

Assume, $[P(X_0) \;\; P(X_1)] = [0.4 \;\; 0.6]$

Now,

$P(Y) = P(X) \cdot P(Y|X)$

$$[P(Y_0) \;\; P(Y_1)] = [0.4 \;\; 0.6] \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} = [0.44 \;\; 0.56]$$

Here,

$$I(X;Y) = \sum_{x,y} P(x,y) \log \frac{P(x,y)}{P(x)\,P(y)}$$

$$= P(x_0\, y_0) \log \frac{P(x_0\, y_0)}{P(x_0)\,P(y_0)} + P(x_0\, y_1) \log \frac{P(x_0\, y_1)}{P(x_0)\,P(y_1)}$$

$$+ P(x_1\, y_0) \log \frac{P(x_1\, y_0)}{P(x_1)\,P(y_0)} + P(x_1\, y_1) \log \frac{P(x_1\, y_1)}{P(x_1)\,P(y_1)}$$

First we will have to calculate Joint Probability Matrix :

$$P(Y|X) = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \quad ; \quad P(X) = [P(X_0) \;\; P(X_1)] = [0.4 \;\; 0.$$

$P(Y_1|X_0) = 0.2$

$P(Y_0|X_0) = 0.8$ $\qquad$ $P(Y_0|X_1) = 0.2$

$P(X_1) = 0.6$

$P(Y_1|X_1) = 0.8$ $\qquad$ $P(X_0) = 0.4$

So, Joint Probabilities are :

$$P(X_0\, Y_0) = P(Y_0|X_0) \cdot P(X_0) = (0.8)(0.4) = 0.32$$

$$P(X_0\, Y_1) = P(Y_1|X_0) \cdot P(X_0) = (0.2)(0.4) = 0.08$$

$$P(X_1\, Y_0) = P(Y_0|X_1) \cdot P(X_1) = (0.2)(0.6) = 0.12$$

$$P(X_1\, Y_1) = P(Y_1|X_1) \cdot P(X_1) = (0.8)(0.6) = 0.48$$

Hence,

$$P(X\,Y) = \begin{array}{c|cc} & Y_0 & Y_1 \\ \hline X_0 & 0.32 & 0.08 \\ X_1 & 0.12 & 0.48 \end{array}$$

Now,

$$I(X,Y) = \sum_{x,y} P(x,y) \log \frac{P(x,y)}{P(x)\,P(y)}.$$

And $P(Y_0) = 0.44$

$P(Y_1) = 0.56$

$$= P(x_0\, Y_0) \log_2 \frac{P(x_0, y_0)}{P(x_0) \cdot P(Y_0)} + P(x_0\, Y_1) \log \frac{P(x_0\, Y_1)}{P(x_0)\, P(Y_1)}$$

$$P(x_1\, Y_0) \log \frac{P(x_1\, Y_0)}{P(x_1)\, P(Y_0)} + P(x_1\, Y_1) \log \frac{P(x_1\, Y_1)}{P(x_1)\, P(Y_1)}$$

$$= (0.32) \log_2 \frac{0.32}{(0.4)(0.44)} + (0.08) \log \frac{0.08}{(0.4)(0.56)}$$

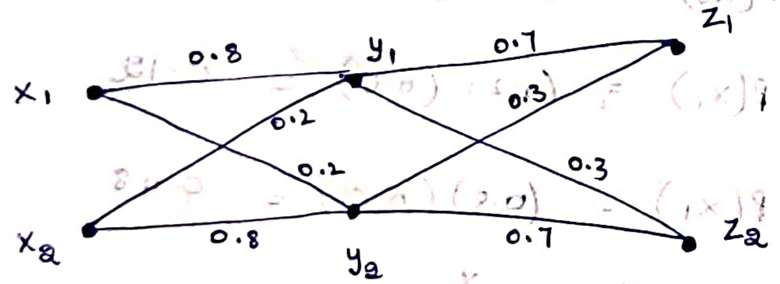
$$(0.12) \log \frac{0.12}{(0.6)(0.44)} + (0.48) \log \frac{0.48}{(0.6)(0.56)}$$

$$= 0.275998 + (-0.118834) + (-0.136500) + (0.246995)$$

$$= 0.2676595 \text{ bits / symbol}$$

---

*. Find $P(z_1)$ and $P(z_2)$ if $P(x_1) = 0.6$ and $P(x_2) = 0.4$ for the given cascaded channel.



$$P(y) = P(x) \cdot P(Y|x)$$

$$P(z) = P(y) \cdot P(z|Y)$$

(or) $$P(z|x) = P(Y|x) \cdot P(z|Y).$$

$$P(z|Y) = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} \text{ and } P(Y|x) = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

Now, resultant channel matrix

$$P(Z|X) = P(Y|X) \cdot P(Z|Y)$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix}$$

$P(z_1)$ and $P(z_2)$ = ?

$$P(Z) = P(X) \cdot P(Z|X)$$

$$= \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix}$$

$$P(Z) = \begin{bmatrix} 0.524 & 0.476 \end{bmatrix} \approx \begin{bmatrix} P(z_1) & P(z_2) \end{bmatrix}$$

$\therefore$ $P(z_1) = 0.524$ & $P(z_2) = 0.476$

Note: If the symbols are statistically independent, then

$$\boxed{H(S^n) = n \cdot H(S)}$$ where, n = order of extension

Proof

* Extension of Discrete Memoryless channel:

Given source, $S = \{s_0, s_1, s_2\}$

$$P(S) = \left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\}$$

Extension of source by order '2' i.e.

$H(S^2) = ?$ $\Rightarrow$ $S^2 = \{s_0, s_1, s_2\} \cdot \{s_0, s_1, s_2\}$

$$= \begin{Bmatrix} s_0 s_0, & s_0 s_1, & s_0 s_2, \\ s_1 s_0, & s_1 s_1, & s_1 s_2, \\ s_2 s_0, & s_2 s_1, & s_2 s_2 \end{Bmatrix}.$$

$$P(S) = \left\{ \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right\}$$

$$P(S^2) = \left\{ \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right\} \cdot \left\{ \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \right\}$$

$$= \left\{ \begin{matrix} \frac{1}{4} \times \frac{1}{4}, & \frac{1}{2} \times \frac{1}{2}, & \frac{1}{4} \times \frac{1}{4} \\ \frac{1}{2} \times \frac{1}{4}, & \frac{1}{2} \times \frac{1}{2}, & \frac{1}{2} \times \frac{1}{4} \\ \frac{1}{4} \times \frac{1}{4}, & \frac{1}{4} \times \frac{1}{2}, & \frac{1}{4} \times \frac{1}{4} \end{matrix} \right\} = \left\{ \begin{matrix} \frac{1}{16}, & \frac{1}{8}, & \frac{1}{16} \\ \frac{1}{8}, & \frac{1}{4}, & \frac{1}{8} \\ \frac{1}{16}, & \frac{1}{8}, & \frac{1}{16} \end{matrix} \right\}$$

$$H(S^2) = \sum_{i=1}^{n} P_i \log_2\left(\frac{1}{P_i}\right)$$

$$= \frac{1}{16}\log_2 16 + \frac{1}{8}\log_2 8 + \frac{1}{16}\log_2 16 + \frac{1}{8}\log_2 8 + \frac{1}{4}\log_2 4$$
$$+ \frac{1}{8}\log_2 8 + \frac{1}{16}\log_2 16 + \frac{1}{8}\log_2 8 + \frac{1}{16}\log_2 16$$

$$= \frac{4}{16} + \frac{3}{8} + \frac{4}{16} + \frac{4}{16} + \frac{3}{8} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{3}{8} + \frac{4}{16}$$

$$= 4\left(\frac{4}{16}\right) + 4\left(\frac{3}{8}\right) + \frac{2}{4}$$

$$H(S^2) = 3$$

$$H(S) = \sum_{i=1}^{n} P_i \log_2\left(\frac{1}{P_i}\right)$$

$$= \frac{1}{4}\log_2 4 + \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 = \frac{2}{4} + \frac{1}{2} + \frac{2}{4} = \frac{2+2+2}{4}$$

$$H(S) = \frac{6}{4} = \frac{3}{2}$$

$$\Rightarrow \boxed{H(S^2) = 2 \cdot H(S)}$$

$H(S^2) = 3$

$H(S) = 3/2$

$3 = 2 \times \frac{3}{2}$ (True)

MODULE-3 : LOSSLESS COMPRESSION

* Rate of Information transmission over a Discrete channel :

- The entropy of the symbol gives average amount of information going into channel :

$$\boxed{H(x) = \sum_{i=1}^{n} P_i \log_2\left(\frac{1}{P_i}\right)}$$

- If the source generates 'r' symbols per second. Then the rate of information, $R = r \cdot H$ bits/sec

$$\boxed{D_{in} = r \cdot H(x)}$$

- Errors are introduced in the channel during transmissional and thus there is information loss in the channel.

Information Loss $= H(x|Y)$

$\therefore$ Transmitted Information $= H(x) - H(x|Y)$

- The average rate of information across the channel,

$$\therefore \boxed{D_t = \left[H(x) - H(x|Y)\right] \cdot r} \text{ bits/sec}$$

- Case-i : When noise is too large, then X and Y are statistically independent. $\boxed{H(x|Y) = H(x)}$

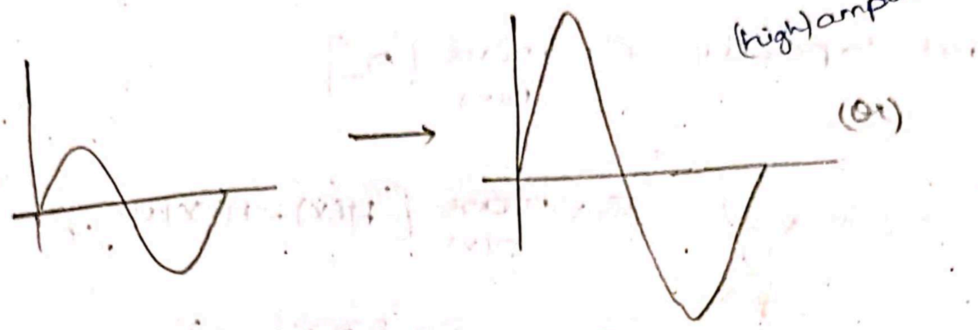
As $H(x|Y) = H(x)$; no information is transmitted over the channel

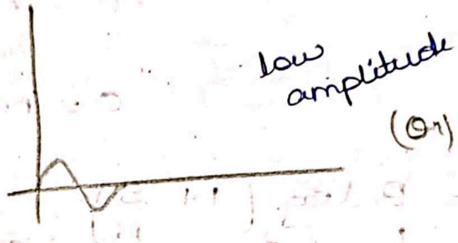
$\therefore H(x|Y) = H(x)$

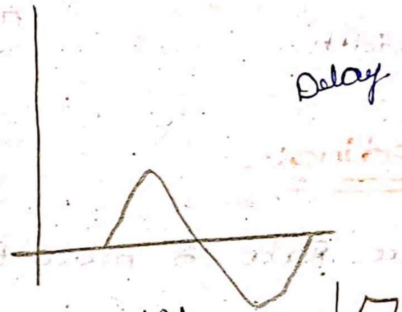
- Case-ii : Errorless Transmission, No information loss $\boxed{H(x|Y) = 0}$ and $\boxed{D_t = D_{in}}$

Channel Capacity :- (Highest Info Rate)

Reliable Transmission :- (shape of signal should be preserved)







but it should not be like

⇓

(Distortion)

* Capacity of channel = max of Mutual Info b/w i/p & o/p of channel.

* $W \xrightarrow{\text{msg}}$ [Encoder f_n] $\xrightarrow{X^n}$ (Encoded Sequence) [Channel $P(Y|X)$] $\xrightarrow{Y^n}$ (Received Seq) [Decoder g_n] $\rightarrow \hat{W}$ (Estimated msg)

Probability of error $\quad P_e = P_r(\hat{W} \neq W)$.

Types of source coding:-

(i) Lossless:- No loss of Info.

(ii) Lossy:- There is loss of Info.

Discrete memory less source:-

Source $\longrightarrow x_0, x_1, \ldots, x_n$.

$X = \{x_0, x_1, x_2, \ldots, x_n\}$

$\boxed{R = rH}$ Enthropy

$P(x_0)\ P(x_1) \ldots P(x_n)$

Easiest way of source coding:-

*

symbols	prob	codeword	Length
A	0.5	00	2
B	0.25	01	2
C	0.125	10	2
D	0.125	11	2

Equal length code: So simply write 00, 01, 10, 11.

Avg length = $\sum$ 2 bits/message

* For variable length codeword

[another way]

sym	prob	code	Length	codeword
A	0.5	0	1	0
B	0.25	1	1	10
C	0.125	10	2	110
D	0.125	11	2	111

No logic simply write

For sample sym diff codeword

$$\boxed{\text{Avg Length} = \sum P_i n_i}$$

P_i → prob

n_i → length of particular sym

Enthropy of source coding $\boxed{H = \sum P_i \log \frac{1}{P_i}}$

Efficiency = $\boxed{\frac{\bar{L}}{H} = \eta}$

Properties:-

* Non-Singular:-

If $x_i \neq x_j \Rightarrow c(x_i) \neq c(x_j)$

means if msg is diff then code must diff

* Uniquely Decodable:-

$x_1 x_2 -- x_m \neq x_1' x_2' -- x_m'$

Non Eg:- {0, 010, 01, 10}

msg = 01010.

$\boxed{01|010} = \boxed{010|10} = \boxed{01|0|10}$

Eg:- {00, 010, 11, 110}.

* prefix code / Instantaneous code:-

→ No code word should be prefix of another code word.

Non Eg:- {10, 00, 11, 1111}

Huffman Coding

① Huffman Code:-

Eg:-

sym	prob
S_0	0.1
S_1	0.2
S_2	0.4
S_3	0.2
S_4	0.1

Find the codeword using Huffman coding

$$H(S) = \sum_{\forall i} P_i \log \frac{1}{P_i}$$

$$= 0.4 \log \frac{1}{0.4} + 2\left(0.2 \log \frac{1}{0.2}\right) + 2\left(0.1 \log \frac{1}{10}\right)$$

$$= 2.12193$$

∴ Arrange all symbols in Decreasing order of prob

Sol:- As high as possible $\boxed{\text{(Low variance method)}}$

S_K	Stage I (Decreasing)	Stage II	Stage III	Stage IV
S_2	0.4	0.4	0.4	0.6
S_1	0.2	0.2	0.4	0.4
S_3	0.2	0.2	0.2	
S_0	0.1	0.2		
S_4	0.1			

(process:- add least 2 prob & Keep as high as u can)

∴ Start coding

	I	II	III	IV
S_2	0.4	0.4	0.4	0.6 0
S_1	0.2	0.2	0.4 0	0.4 1
S_3	0.2	0.2 0	0.2 1	
S_0	0.1 0	0.2 1		
S_4	0.1 1			

By tracing

codeword	length
00	2
01	2
11	2
010	3
110	3

(Not Final) reverse all

by Tracing path.

∴ Final → [00, 10, 11, 010, 011]

~~Av.~~ length of codeword :- ~~2+0.2+0.2+0.3+0.3~~

$$\boxed{\sum P_i L_i}$$

$\Rightarrow (0.4)(2) + (0.2)(2) + (0.2)(2) + (0.1)(3) + (0.1)3$

$\Rightarrow \underline{\underline{2.2.}}$

Efficiency :- $\boxed{\eta = \frac{H(x)}{\bar{L}}}$

$= \frac{2.12}{2.2} = \underline{\underline{96\%}}$

redundancy :- $\boxed{r = 1 - \eta}$ $= 1 - 0.96 = \underline{\underline{0.04.}}$

codevariance : $\boxed{\sigma^2 = \sum_{\forall k} P_k (L_k - \bar{L})^2}$

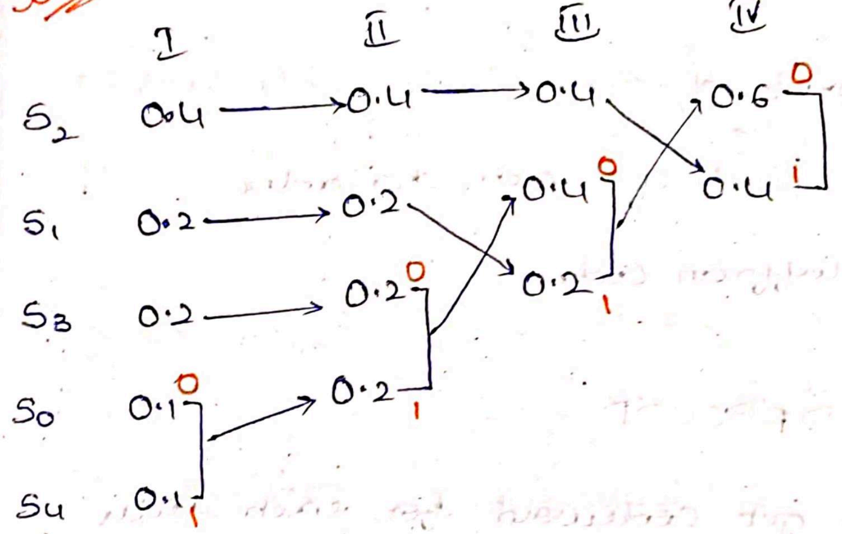
$\Rightarrow (0.4)(2-2.2)^2 + (0.2)(2-2.2)^2 + (0.2)(2-2.2)^2 + (0.1)(3-2.2)^2 + (0.1)(3-2.2)^2$

$\Rightarrow (0.4)(0.2)^2 + (0.2)(0.2)^2 + (0.2)(0.2)^2 + (0.1)(0.8)^2 + (0.1)(0.8)^2$

$\Rightarrow 0.016 + 0.008 + 0.008 + 0.064 + 0.064$

$\Rightarrow \boxed{0.16}$

Sol:-B As low as possible



Tracing Code	Final
1	1
10	01
000	000
0100	0010
1100	0011

* Find efficiency, reduntancy, Avg length for B.

⇒ $\boxed{\text{More efficiency \& low variance}}$ ⇒ performs well.

* If they didn't mention any method ⇒

use method-1 [As high as possible]

b/c we get low variance.

2] Two-pass Huffman Code:-

* Used when prob of symbol is unknown.

Step-1:- Measure prob of each character

Step-2:- Make Huffman code.

message: ABAACDABCCDD----

ques:-1] Ask to get codeword for each letter

ques:-2] They may ask to write seq of letter for the code like 11001001 0 ⇒ ABDA.

ques:-3] Ask to Decode 0010111 0011 ⇒ BCAAD—

ques:-4]

sym	prob
m_1	0.4
m_2	0.6

Find out the huffman code for <u>Second-order Extension</u>

⇒

m_1m_1	0.4×0.4
m_1m_2	0.4×0.6
m_2m_1	0.6×0.4
m_2m_2	0.6×0.4

perform Huffman for this table

↓

arrange the prob in decrease order & perform Huff

Eg:- msg = ABABABABABABACADABACADABACADABACAD

apply Huffman code (min variance method) to find out codeword for "ABCDDCBA".

*. The parity check bits of (8,4) linear block code are :

$$c_1 = m_2 + m_3 + m_4$$

$$c_2 = m_1 + m_3 + m_4$$

$$c_3 = m_2 + m_3 + m_4$$

$$c_4 = m_1 + m_2 + m_3$$

where m_1, m_2, m_3, m_4 are the message bits ?

i. Determine the Generator & Parity check Matrix for systematic code.

ii. Comment on error detection & error correction capabilities of code.

iii. If the received sequence is 10110101, determine the transmitted word.

Sol:

(i)

$$c_1 = m_2 + m_3 + m_4$$

$$c_2 = m_1 + m_3 + m_4$$

$$c_3 = m_2 + m_3 + m_4$$

$$c_4 = m_1 + m_2 + m_3$$

$$P' = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$n = 8$

$k = 4$

$q = n - k = 8 - 4 = 4$

Generator matrix, $G = [I_K \mid P_{K \times q}]_{K \times n}$

$$G = [I_4 \mid P_{4\times4}]_{4\times8} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

(ii)

(iii) Parity check Matrix, $H = [P^T \mid I_q]$

$$P = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \Rightarrow P^T = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow H^T = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S = Y \cdot H^T$$

received codeword, $Y = 10110101$

$$S = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}_{1\times 8} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{8\times 4} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}_{1\times 4}$$

error in the S $\Rightarrow$ $\boxed{X = E \oplus Y}$

received codeword has 8th row error, The transmitted codeword is the received codeword $\therefore \Rightarrow E = 0\,0\,0\,0\,0\,0\,0\,1$

(ii.) Error detecting capability :

w_i = no. of 1's in codeword

No. of errors that can be detected

$d_{min} \geq s+1$

$d_{min} = w_{min} = 3$

$\therefore$ It can detect $\underline{2}$ errors at each codeword.

Error Correcting Capability :

No. of errors that can be corrected :

$$\boxed{d_{min} \geq 2t + 1}$$

$d_{min} = \omega_{min} = 3$

$\therefore$ It can correct 1 error at each codeword.

(*) $X = E \oplus Y$

$$X = \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ \hline \end{array}$$

$\rightarrow$

$\therefore$ Correctly transmitted codeword = 1 0 1 1 0 1 0 0

---

m_1	m_2	m_3	m_4	c_1	c_2	c_3	c_4	weight
0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	0	4
0	0	1	0	1	1	1	1	5
0	0	1	1	0	0	0	1	3
0	1	0	0	1	0	1	1	4
0	1	0	1	0	1	0	1	4
0	1	1	0	0	1	0	0	3
0	1	1	1	0	0	1	0	3
1	0	0	0	0	1	0	1	5
1	0	0	1	1	0	1	1	4
1	0	1	0	1	0	1	0	4
1	0	1	1	0	1	0	0	5
1	1	0	0	1	1	1	0	3
1	1	0	1	0	0	0	0	4
1	1	1	0	0	0	0	1	

*. Construct a Systematic (8,4) cyclic code with generator polynomial $G(P) = P^3 + P + 1$.

i) Determine the Generator Matrix

ii) Decode the received codeword is 11011001 using Syndrome decod

Sol: ① $G(P) = P^3 + P + 1$

tth row of Generator Matrix is given by,

$$\boxed{P^{n-t} + R_t(P) = Q_t(P) \cdot [G(P)]}$$

Here $n = 8$

$K = 4$

$q = n - k = 8 - 4 = 4$

For $t = 1$:

$$P^{8-1} + R_1(P) = Q_1(P) \cdot G(P) \Rightarrow P^7 + R_1(P) \text{ is polynomial for 1st row of Generator Matrix}$$

$$R_1(P) = \text{rem}\left(\frac{P^{8-1}}{G(P)}\right) = \text{rem}\left(\frac{P^7}{P^3+P+1}\right)$$

$R_1(P) = 1$ $\Rightarrow P^7 + 1$

$Q_1(P) = P^4 + P^2 + P + 1$

$$\begin{array}{l} P^3+P+1 \,)\, P^7 \,(\, P^4+P^2+P+1 \\ \quad P^7 + P^5 + P^4 \\ \hline \quad P^5 + P^4 \\ \quad P^5 + P^3 + P^2 \\ \hline \quad P^4 + P^3 + P^2 \\ \quad P^4 + P^2 + P \\ \hline \quad P^3 + P \\ \quad P^3 + P + 1 \\ \hline \quad 1 \end{array}$$

1st row of Generator Matrix is $P^7 + 1$

= 1 0 0 0 0 0 0 1

For $t = 2$:

$$P^{8-2} + R_2(P) = Q_2(P) \cdot G(P)$$

$$R_2(P) = \text{rem}\left(\frac{P^{8-2}}{G(P)}\right) = \text{rem}\left(\frac{P^6}{P^3+P+1}\right)$$

$R_2(P) = P^2 + 1$ $\Rightarrow$ 2nd row polynomial is $P^6 + P^2 + 1$

$Q_2(P) = P^3 + P + 1$

$$\begin{array}{l} P^3+P+1 \,)\, P^6 \,(\, P^3+P+1 \\ \quad P^6 + P^4 + P^3 \\ \hline \quad P^4 + P^3 \\ \quad P^4 + P^2 + P \\ \hline \quad P^3 + P^2 + P \\ \quad P^3 + P + 1 \\ \hline \quad P^2 + 1 \end{array}$$

2nd row of Generator Matrix = 0 1 0 0 0 1 0 1

For $t = 3$:

$p^{8-3} + R_3(P) = Q_3(P) \cdot G(P)$

$R_3(P) = \text{rem}\left(\frac{p^{8-3}}{G(P)}\right) = \text{rem}\left(\frac{p^5}{p^3+p+1}\right)$

$$\begin{array}{r} p^3+p+1 \,) \; p^5 \; (p^2+1 \\ p^5+p^3+p^2 \\ \hline p^3+p^2 \\ p^3+p+1 \\ \hline p^2+p+1 \end{array}$$

$R_3(P) = p^2+p+1$ $\Rightarrow$ 3rd row Generator polynomial is, p^5+p^2+p+1

$Q_3(P) = p^2+1$

3rd row of Generator Matrix = 0 0 1 0 0 1 1 1

For $t = 4$:

$p^{8-4} + R_4(P) = Q_4(P) \cdot G(P)$

$R_4(P) = \text{rem}\left(\frac{p^{8-4}}{G(P)}\right) = \text{rem}\left(\frac{p^4}{p^3+p+1}\right)$

$$\begin{array}{r} p^3+p+1 \,) \; p^4 \; (p \\ p^4+p^2+p \\ \hline p^2+p \end{array}$$

$R_4(P) = p^2+p$ $\Rightarrow$ 4th row polynomial is p^4+p^2+p

$Q_4(P) = p$

4th row of Generator Matrix = 0 0 0 1 0 1 1 0

$\therefore$ Generator Matrix, $G(P) = \begin{bmatrix} 1 & 0 & 0 & 0 & : & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & : & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & : & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & : & 0 & 1 & 1 & 0 \end{bmatrix}$

(ii) Now, the received codeword is 1 1 0 1 1 0 0 1

Given, $Y = 1\,1\,0\,1\,1\,0\,0\,1$

$Y(P) = p^7 + p^6 + p^4 + p^3 + 1$

$G(P) = p^3 + p + 1$ (Given in question)

Syndrome Polynomial, $S(P) = \text{rem}\left[\frac{Y(P)}{G(P)}\right]$

$$\rightarrow \quad S(P) = \text{rem}\left[\frac{P^7+P^6+P^4+P^3+1}{P^3+P+1}\right]$$

$$
\begin{array}{r l}
P^3+P+1 \,\big) & P^7+P^6+P^4+P^3+1 \,\big(\, P^4+P^3+P^2+P+1 \\
 & \underline{P^7+P^5+P^4} \\
 & P^6+P^5+P^3+1 \\
 & \underline{P^6+P^4+P^3} \\
 & P^5+P^4+1 \\
 & \underline{P^5+P^3+P^2} \\
 & P^4+P^3+P^2+1 \\
 & \underline{P^4+P^2+P} \\
 & P^3+P+1 \\
 & \underline{P^3+P+1} \\
 & 0
\end{array}
$$

remainder, rem = 0

$= (0\,0\,0\,0)$

$S = 0\,0\,0\,0$

$S(P) = 0$

Error decoding Table : (8,4) cyclic code

E =	0	0	0	0	0	0	0	0	– No error
E =	1	0	0	0	0	0	0	0	– error in 1st bit
E =	0	1	0	0	0	0	0	0	– error in 2nd bit
E =	0	0	1	0	0	0	0	0	– error in 3rd bit
E =	0	0	0	1	0	0	0	0	– error in 4th bit
E =	0	0	0	0	1	0	0	0	– error in 5th bit
E =	0	0	0	0	0	1	0	0	– error in 6th bit
E =	0	0	0	0	0	0	1	0	– error in 7th bit
E =	0	0	0	0	0	0	0	1	– error in 8th bit

Also, we got Generator Matrix :

$$G(P) = \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \quad P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \Rightarrow P^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$H = \left[P^T \,\middle|\, I_q\right] = \left[\begin{array}{cccc:cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$H^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here we got

$S = [0\ 0\ 0\ 0]$

$\Rightarrow S = [0]$

$$\boxed{X = Y}$$

$\therefore$ No error $\Rightarrow$ Received codeword = Transmitted codeword.

* Determine the encoded message for the following (12,8) code using the generator polynomial $G(P) = P^4 + P^3 + 1$ in non-systematic form in cyclic code

(i) 1 1 0 0 1 1 0 0

(ii) 0 1 0 1 1 1 1 1

Sol :

Given $n = 12$, $K = 8$

$q = n - K = 12 - 8 = 4$

$G(P) = P^4 + P^3 + 1$

(12,8) is the cyclic block code

1st row of $G(P)$:

as $K = 8$

$K - 1 = 7$

$P^7 . G(P) = P^7 (P^4 + P^3 + 1) = P^{11} + P^{10} + P^7$

1st row is : [1 1 0 0 1 0 0 0 0 0 0 0]

2nd row of $G(P)$:

as $K = 8$

$K - 2 = 6$

$P^6 . G(P) = P^6 (P^4 + P^3 + 1) = P^{10} + P^9 + P^6$

2nd row is : [0 1 1 0 0 1 0 0 0 0 0 0]

3rd row of $G(P)$:

as $K = 8$

$K - 3 = 5$

$P^5 . G(P) = P^5 (P^4 + P^3 + 1) = P^9 + P^8 + P^5$

3rd row is : [0 0 1 1 0 0 1 0 0 0 0 0]

4th row of $G(P)$:

as $K = 8$

$K - 4 = 4$

$$P^4 \cdot G(P) = P^4 (P^4 + P^3 + 1) = P^8 + P^7 + P^4$$

4th row : $[0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0]$

5th row of $G(P)$:

as $K = 8$

$K - 5 = 3$

$$P^3 \cdot G(P) = P^3 (P^4 + P^3 + 1) = P^7 + P^6 + P^3$$

5th row is : $[0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0]$

6th row of $G(P)$:

as $K = 8$

$K - 6 = 2$

$$P^2 \cdot G(P) = P^2 (P^4 + P^3 + 1) = P^6 + P^5 + P^2$$

6th row is : $[0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0]$

7th row of $G(P)$:

as $K = 8$

$K - 7 = 1$

$$P \cdot G(P) = P (P^4 + P^3 + 1) = P^5 + P^4 + P$$

7th row is : $[0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0]$

8th row of $G(P)$:

as $K = 8$

$K - 8 = 0$

$$P^0 \cdot G(P) = G(P) = P^4 + P^3 + 1$$

8th row is : $[0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1]$

So, $G(P) = \begin{bmatrix} 1&1&0&0&1&0&0&0&0&0&0&0 \\ 0&1&1&0&0&1&0&0&0&0&0&0 \\ 0&0&1&1&0&0&1&0&0&0&0&0 \\ 0&0&0&1&1&0&0&1&0&0&0&0 \\ 0&0&0&0&1&1&0&0&1&0&0&0 \\ 0&0&0&0&0&1&1&0&0&1&0&0 \\ 0&0&0&0&0&0&1&1&0&0&1&0 \\ 0&0&0&0&0&0&0&1&1&0&0&1 \end{bmatrix}$

Now,

$$\boxed{X(P) = M(P) \cdot G(P)}$$

i. For $m(P) = 11001100$

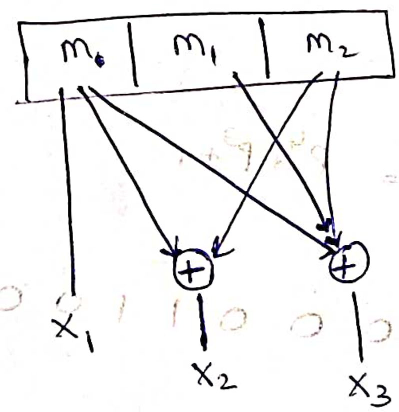
$X(P) = [1\,1\,0\,0\,1\,1\,0\,0] \cdot G(P) = [1\,0\,1\,0\,0\,1\,1\,0\,1\,1\,0\,0]$

ii. For $m(P) = 01011111$

$X(P) = [0\,1\,0\,1\,1\,1\,1\,1] \cdot G(P) = [0\,1\,1\,1\,0\,1\,0\,1\,0\,1\,1\,1]$

---

*. For the given convolutional encoder configuration



i. Draw the state diagram and code trellis diagram.

ii. Encode the sequence 1110 using code tree.

$x_1 = m$

$x_2 = m \oplus m_2$

$x_3 = m \oplus m_1 \oplus m_2$

Code rate, $r = \frac{k}{n} = \frac{1}{3}$

no. of message bit $= k = 1$

no. of encoded output, $n = 3$

- First shift message will enter M.
- second shift message will enter M_1.
- third shift message will enter M_2.
- forth shift message will not enter and will be discarded.

dimension of Code $= (n,k) = (3,1)$

- Constraint length - no. of shift over which a single message bit can influence the encoder output. $K =$

m_2	m_1	States
0	0	a
0	1	b
1	0	c
1	1	d

m = Current message bit

$m_2 m_1$ = current state

$m_1 m$ = next state

00 - a
01 - b
10 - c
11 - d

State table :

m_2	m_1	m	Current state	Next state	x_1	x_2	x_3	decimal
0	0	0	a	a	0	0	0	0
0	0	1	a	b	1	1	1	7
0	0	1	b	d	1	1	0	6
1	1	1	d	d	1	0	1	5
1	1	0	d	c	0	1	0	2

output encoded : 111 110 101 010

Trick :

a $\xrightarrow{0}$ a

a $\xrightarrow{1}$ b

b $\xrightarrow{1}$ ~~d~~ c

b $\xrightarrow{1}$ d

d $\xrightarrow{0}$ c

c $\xrightarrow{0}$ a

c $\xrightarrow{1}$ b

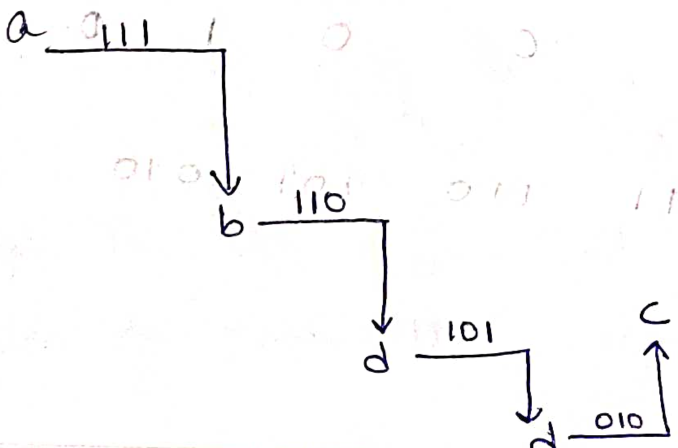
d $\xrightarrow{1}$ d

m_2	m_1	m	Current State	next State	x_1	x_2	x_3	decimal
0	0	0	a	a	0	0	0	0
0	0	1	a	b	1	1	1	7
0	1	0	b	c	0	0	1	1
0	1	1	b	d	1	1	0	6
1	0	0	c	a	0	1	1	3
1	0	1	c	b	1	0	0	4
1	1	0	d	c	0	1	0	2
1	1	1	d	d	1	0	1	5

(ii) Code tree :

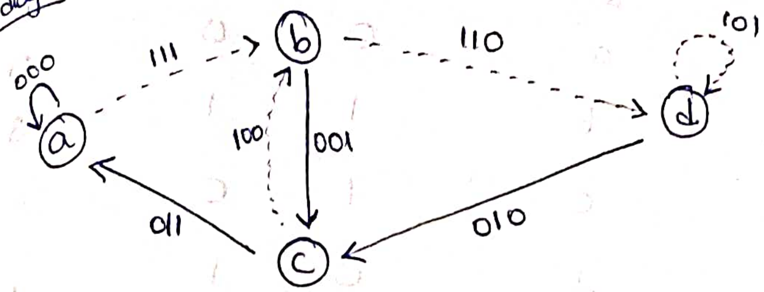
Given sequence, 1110

1 ↓ ↑ 0

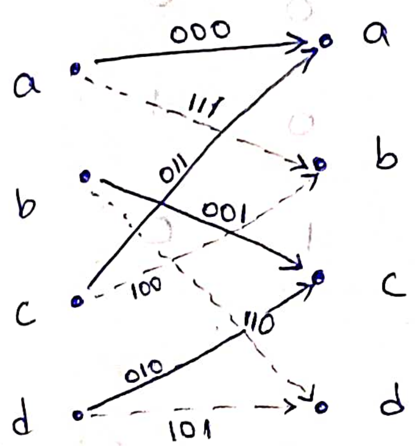


encoded sequence 111 110 101 010

State diagram :



Trelli's diagram :



Conclusion :

encoded output : 111 110 101 010

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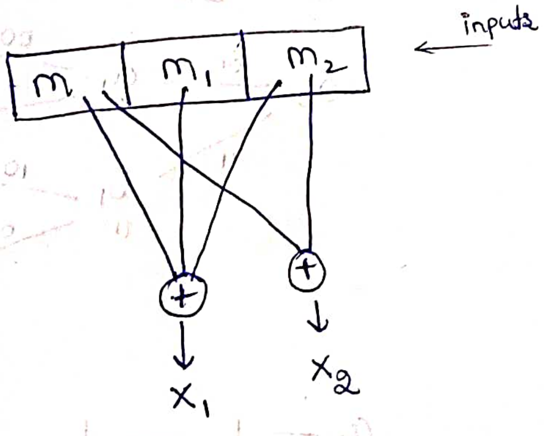
*. For the Convolutional encoder configuration with Code rate $\frac{1}{2}$, Constraint length 3 and if the outputs are expressed as $x_1 = m \oplus m_1 \oplus m_2$ and as $x_2 = m \oplus m_2$

i. Draw the diagram of Convolutional encoder

ii. Decode the sequence 01011110 using Code trellis and veterbi.

1:

(i)



$x_1 = m \oplus m_1 + m_2$

$x_2 = m \oplus m_2$

(ii) Given sequence, 01011110

Code rate, $r = \frac{k}{n} = \frac{1}{2}$

$k = 1$
$n = 2$

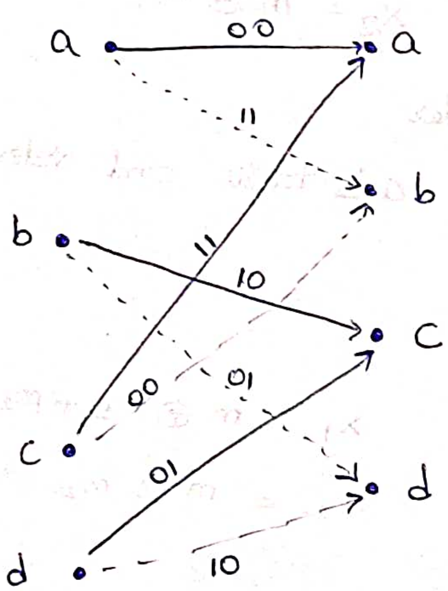
Constraint length = 3

dimension = $(n, k) = (2, 1)$

m_2	m_1	m	Current state	next state	output x_1	x_2	decimal
0	0	0	a	a	0	0	0
0	0	1	a	b	1	1	3
0	1	0	b	c	1	0	2
0	1	1	b	d	0	1	1
1	0	0	c	a	1	1	3
1	0	1	c	b	0	0	0
1	1	0	d	c	0	1	1
1	1	1	d	d	1	0	2

decode the Sequence : 01 01 11 10

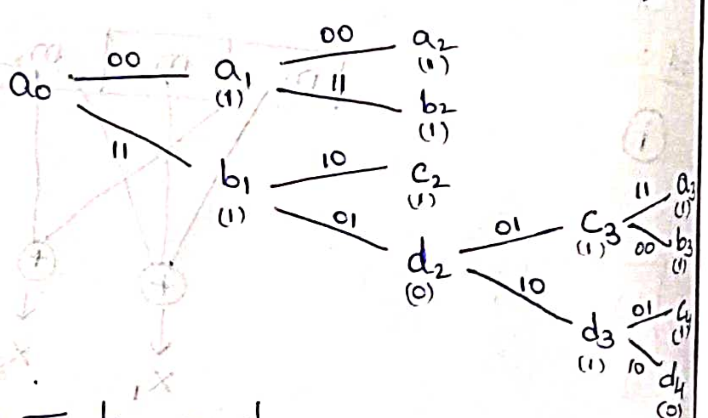
Trelli's diagram :



- veterbi decoding (also called Maximum Likelihood decoding)
- Hamming distance :

01 & 10 $\}\rightarrow \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \}$ ②

output : $\boxed{01}$ $\boxed{01}$ $\boxed{11}$ $\boxed{10}$



$a_0 \overset{}{\underset{11}{—}} b_1 \underset{01}{—} d_2 \underset{10}{—} d_3 \underset{10}{—} d_4$

1 1 1 1

encoded actual : 1111