Example :

$$\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$$

Determine if the given set is linearly dependent or not ?

Consider, $S = \left\{ \begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix} \right\}$ of vectors in the vector space R^4, determining whether S is linearly independent or linearly dependent.

Step 1 : Make a homogeneous system of equations.

The set $S = \{v_1, v_2\}$ of vectors in R^4 is linearly independent if the only solution of $c_1 v_1 + c_2 v_2 = 0$ is $c_1, c_2 = 0$.

otherwise (i.e. for a solution with atleast some non-zero values exists), S is linearly dependent.

$$c_1 \begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Rearranging the left hand side,

$$-2c_1 + 3c_2 = 0$$
$$4c_1 - 6c_2 = 0$$
$$6c_1 - 9c_2 = 0$$
$$10c_1 + 15c_2 = 0$$

The above matrix is equivalent to following homogeneous system of equations.

Step - 2 : Convert into row reduced echelon form.

Trivial solution $\rightarrow$ S is linearly independent

Both trivial & non - trival solution $\rightarrow$ S is linearly dependent

$$\Rightarrow \begin{bmatrix} -2 & 3 & 0 \\ 4 & -6 & 0 \\ 6 & -9 & 0 \\ 10 & 15 & 0 \end{bmatrix}$$

Apply

$R_1 \leftarrow -\frac{1}{2} R_1$

$$\Rightarrow = \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 4 & -6 & 0 \\ 6 & -9 & 0 \\ 10 & 15 & 0 \end{bmatrix}$$

Apply

$R_2 \leftarrow \frac{1}{4} \cdot R_2$

$$= \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 1 & -\frac{3}{2} & 0 \\ 6 & -9 & 0 \\ 10 & 15 & 0 \end{bmatrix}$$

Apply

$R_3 \leftarrow \frac{1}{6} \cdot R_3$

$$= \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 1 & -\frac{3}{2} & 0 \\ 1 & -\frac{3}{2} & 0 \\ 10 & 15 & 0 \end{bmatrix}$$

Apply $R_4 \leftarrow \frac{1}{10} \cdot R_4$

$$= \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 1 & -\frac{3}{2} & 0 \\ 1 & -\frac{3}{2} & 0 \\ 1 & -\frac{3}{2} & 0 \end{bmatrix}$$

Apply $R_2 \leftarrow R_2 - R_1$

$$= \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \\ 1 & -\frac{3}{2} & 0 \\ 1 & \frac{3}{2} & 0 \end{bmatrix}$$

Apply $R_3 \leftarrow R_3 - R_1$

$$= \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{3}{2} & 0 \end{bmatrix}$$

Apply $R_4 \leftarrow R_4 - R_1$

$$= \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

Apply $R_4 \leftrightarrow R_2$

$$= \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Apply $R_2 \leftarrow \frac{1}{3} R_2$

$$= \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Apply $R_1 \leftarrow R_1 + \frac{3}{2} R_2$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is the row reduced echoleon form obtained.

Step-3 : Converting back to system of linear equation.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$$

this corresponds to the system

$$1 \cdot c_1 = 0$$

$$1 \cdot c_2 = 0$$

$\Rightarrow \boxed{c_1 = 0}$, $\boxed{c_2 = 0}$

$\therefore \{v_1, v_2\}$ are linearly independent.

$c_1 v_1 + c_2 v_2 = 0$ & $(c_1, c_2) = (0, 0)$

*. GAUSS ELIMINATION METHOD :

Ex : Solve the following equation by Gauss Elimination Method

$$x - y + 2z = 3$$

$$x + 2y + 3z = 5$$

$$3x - 4y - 5z = -13$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -13 \end{bmatrix} \Rightarrow AX = B$$

$$\Downarrow$$

$\{A|B\}$ Augum.

$$[A|B] = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 3 & -4 & -5 & -13 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & -1 & -11 & -22 \end{bmatrix} \quad R_3 \to 3R_3 + R_2$$

$$= \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -32 & -64 \end{bmatrix}$$

Convert back to Linear equations $\Rightarrow$

$$x - y + 2z = 3$$
$$3y + z = 2$$
$$-32z = -64$$

$\Rightarrow \boxed{z = 2}$ $3y + 2 = 2 \Rightarrow \boxed{y = 0}$

$\Rightarrow 3 = x - 0 + 2(2) \Rightarrow \boxed{x = -1}$

Solution $\Rightarrow$ $\left.\begin{matrix} x = -1 \\ y = 0 \\ z = 2 \end{matrix}\right\}$ ✓

* Solve the system of equations :

$$\left.\begin{aligned} 2x + y + z &= 10 \\ 3x + 2y + 3z &= 18 \\ x + 4y + 9z &= 16 \end{aligned}\right\}$$ By using Gauss Jordan and Gauss Elimination.

Sol :

Given, $x + 4y + 9z = 16$

$2x + y + z = 10$

$3x + 2y + 3z = 18$

{ GAUSS ELIMINATION }

$$\begin{bmatrix} 1 & 4 & 9 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \\ 18 \end{bmatrix}$$ $\Rightarrow AX = B$

$\{A|B\}$ Augumented Matrix

$$[A|B] = \begin{bmatrix} 1 & 4 & 9 & 16 \\ 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \end{bmatrix}$$ $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$= \begin{bmatrix} 1 & 4 & 9 & 16 \\ 0 & -7 & -17 & -22 \\ 0 & -10 & -24 & -30 \end{bmatrix}$$ $R_3 \to 7R_3 - 10R_2$

$$= \begin{bmatrix} 1 & 4 & 9 & 16 \\ 0 & -7 & -17 & -22 \\ 0 & 0 & 2 & 10 \end{bmatrix}$$ Convert back to linear system

$x + 4y + 9z = 16$

$-7y - 17z = -22$

$2z = 10$

$\Rightarrow \boxed{z = 5}$, $\boxed{y = -9}$, $\boxed{x = 7}$

Solving the same : $x + 4y + 9z = 16$

using {GAUSS JORDAN ELIMINATION} $2x + y + z = 10$

$3x + 2y + 3z = 18$

$$\begin{bmatrix} 1 & 4 & 9 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \\ 18 \end{bmatrix} \Rightarrow AX = B$$

$\{A|B\}$: Augumented Matrix

$$[A|B] = \begin{bmatrix} 1 & 4 & 9 & 16 \\ 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

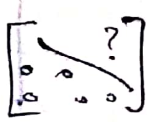
$$= \begin{bmatrix} 1 & 4 & 9 & 16 \\ 0 & -7 & -17 & -22 \\ 0 & -10 & -24 & -30 \end{bmatrix} \quad \begin{array}{l} R_3 \to 7R_3 - 10R_2 \\ R_1 \to 7R_1 + 4R_2 \end{array}$$

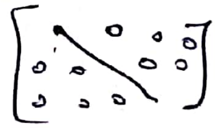
$$= \begin{bmatrix} 7 & 0 & -5 & 24 \\ 0 & -7 & -17 & -22 \\ 0 & 0 & 2 & 10 \end{bmatrix} \quad \begin{array}{l} R_1 \to 2R_1 + 5R_3 \\ R_2 \to 2R_2 + 17R_3 \end{array}$$

$$= \begin{bmatrix} 14 & 0 & 0 & 98 \\ 0 & -14 & 0 & 126 \\ 0 & 0 & 2 & 10 \end{bmatrix}$$

Convert back to linear system $\Rightarrow$ $14x = 98$, $-14y = 126$, $2z = 10$

$\Rightarrow$ $\boxed{x = 7}$ $\boxed{y = -9}$ $\boxed{z = 5}$

Gauss Elimination Method : Convert into Upper Triangular Matrix 

Gauss Jordan Elimination : Diagonal Matrix 

*. GAUSS JORDON METHOD :

- This method is modification of "Gauss Elimination" Method.
- Convert into Diagonal Matrix form, for direct solution.

Example :

Apply Gauss Jordon method to solve the equation :

$$x + y + z = 9$$

$$2x - 3y + 4z = 13$$

$$3x + 4y + 5z = 40$$

Solution :

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix} \rightarrow AX = B$$

$\{A|B\}$ Augumented Matrix

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$= \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{bmatrix}$$

$R_1 \rightarrow 5R_1 + R_2$

$R_3 \rightarrow 5R_3 + R_2$

$$= \begin{bmatrix} 5 & 0 & 7 & 40 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 12 & 60 \end{bmatrix} \qquad \begin{array}{l} R_2 \to 6R_2 - R_3 \\ R_1 \to 12R_1 - 7R_3 \end{array}$$

$$= \begin{bmatrix} 60 & 0 & 0 & 60 \\ 0 & -30 & 0 & -90 \\ 0 & 0 & 12 & 60 \end{bmatrix}$$

Convert back to linear system of equations

$$\Rightarrow \quad 60x = 60 \quad \Rightarrow \quad \boxed{x = 1}$$

$$-30y = -90 \quad \Rightarrow \quad \boxed{y = +3}$$

$$12z = 60 \quad \Rightarrow \quad \boxed{z = 5}$$

solutions.

---

*. Algorithm to find A^{-1} (Inverse of Matrix A) :

Row reduced the augumented matrix [A I]. If A is row equivalent to I, then [A I] is row equivalent to $[I \ A^{-1}]$. otherwise, A doesn't have an inverse.

Example :

Find inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

Sol : Row reduced augumented Matrix [A I]. If A is row equivalent to I, then [A I] is row equivalent to $[I \ A^{-1}]$. otherwise, A doesnot have an inverse.

$$A \cdot A^{-1} = I \quad \Rightarrow \quad A^{-1} = \underset{\text{reduced}}{\text{row}} \underbrace{[A|I]}_{\text{Aug}}$$

$$[A\ I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\overset{\text{Interchange}}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R_3 \to R_3 - 4R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \underset{R_3 \to R_3 + 3R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-9}{2} & 7 & \frac{-3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$A \sim I \Rightarrow A$ is invertible $\qquad [A\ I] \approx [I\ A^{-1}]$

$$\therefore A^{-1} = \begin{bmatrix} \frac{-9}{2} & 7 & \frac{-3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

---

*. DETERMINANT :

det A = row reduced echeleon form of A $\longrightarrow$ product of diagonal entries

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow[\text{rref}]{\text{After}} \begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix} \Rightarrow \det A = pqr$$

*. LU DECOMPOSITION : (Factorization Method)

- It is also called as Cholesky's Method.

Consider the system of linear equations as:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$AX = B \qquad \{A|B\} \text{ Augumented Matrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = L \cdot U$$

L = Lower Triangular

U = Upper Triangular

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$AX = B \Rightarrow (LU)\cdot X = B \Rightarrow L\cdot(UX) = B$$

let $UX = y \Rightarrow L \cdot y = B$ (Conversion)

- In computing element of L & U :

① first row of U : U_{11} U_{12} U_{13}

② first column of L : L_{21} L_{31}

③ Second row of U : U_{21} U_{23}

④ Second Column of L : L_{32}

⑤ Third row of U : U_{33}

Example : (IMP)

Solve the following system of equation by L U. Decomposition method.

$$x + 5y + z = 14$$

$$2x + y + 3z = 13$$

$$3x + y + 4z = 17$$

Converting into Matrix form :

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

$$A \quad X = B$$

So, $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = L \cdot U$ (Method 2)

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix}$$

$\boxed{U_{11} = 1}$

$\boxed{U_{12} = 5}$

$\boxed{U_{13} = 1}$

$L_{21} \cdot U_{11} = 2$

$\boxed{L_{21} = 2}$

$L_{31}\, U_{11} = 3$

$\boxed{L_{31} = 3}$

$L_{21}\, U_{12} + U_{22} = 1$

$(2)(5) + U_{22} = 1$

$\boxed{U_{22} = -9}$

$L_{31} U_{12} + L_{32} U_{22} = 1$

$(3)(5) + L_{32}(-9) = 1$

$\boxed{L_{32} = \frac{14}{9}}$

$L_{21} U_{13} + U_{23} = 3$

$(2)(1) + U_{23} = 3 \Rightarrow \boxed{U_{23} = 1}$

$\boxed{U_{23} = 1}$

$L_{31} U_{13} + L_{32} U_{23} + U_{33} = 4$

$(3)(1) + \left(\frac{14}{9}\right)(1) + U_{33} = 4 \Rightarrow \boxed{U_{33} = \frac{-5}{9}}$

So, substitute all these values in L, U

$$A = L \cdot U = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{bmatrix}}_{U}$$

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

Convert into linear system of equations

$y_1 = 14$

$2y_1 + y_2 = 13 \Rightarrow \boxed{y_2 = -15}$

$3y_1 + \frac{14}{9} y_2 + y_3 = 17 \Rightarrow \boxed{y_3 = -\frac{5}{3}}$

Now, $UX = Y$

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -15 \\ -5/3 \end{bmatrix}$$

Convert into system of equations again,

$-\frac{5}{9} z = -\frac{5}{3} \Rightarrow \boxed{z = 3}$

$-9y + z = -15 \Rightarrow \boxed{y = 2}$

$x + 5y + z = 14 \Rightarrow \boxed{x = 1}$

Final solution of the system.

MODULE - 2

*. VECTOR SPACE :

- A vector is an arrow with a length (called magnitude) and have direction.

- operations on vectors :

① Addition - internal composition

② Scalar Multiplication - external composition

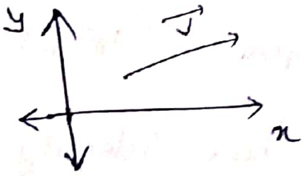
↓

This scalar is taken from Field of Scalar (Ring Theory) - $\{Q, R, C, \text{etc} \dots\}$

- 2D plane :

$\vec{v} = x\hat{i} + y\hat{j}$

$\vec{v} = (x, y)$



- 3D plane :

$\vec{v} = (x, y, z)$

$\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$

- Generalizing this to nD space :

→ All the collection of vectors (n dimensional vectors) is called n dimensional vector space.

$$\vec{v} = \{v_1, v_2, v_3, \dots\dots, v_{n-1}, v_n\}$$

→ Vector space : An algebraic structure with two operations addition, scalar multiplication.

- Set as vector space over a field $F = (R)$ if it satisfies certain properties.

* Vector Space holds the following properties :

For every u, v, w in V, $\forall\ a, b \in R$

i. $u + v \in V$

ii. $a.u \in V$

iii. $u + v = v + u$

iv. $u + (v + w) = (u + v) + w$

v. $u + (-u) = 0 = (-u) + u$

vi. $0 + y = y + 0 = y$, $y \in V$

vii. $a(u + v) = au + av$

viii. $(a + b)u = au + bu$

ix. $(ab)u = a(bu)$

x. $1.u = u.1 = u$

- R^n is a vector space of $V = \{n_1, n_2, n_3 \dots n\}$
- $V = \{0\}$ is a vector space (Trivial vector space)
- every vector space must have identity element under addition. So, $V = \{0\}$ is the smallest vector space.

* SUB SPACES : (from a big Vector space)

- A sub space of a vector V is a subset H that satisfies three conditions :

(1) Zero vector is in H, $\{0\} \in H$

(2) H is closed under addition, $u + v \in H$ where $u \in H$, $v \in H$

(3) H is closed under multiplication of scalars. $c \in R$ – Field; $c.u \in H$ – Vector Space

- $\{0\}$ is a subspace of R^n

Q) Let $v_1, v_2 \in V$. Show $H = \text{span}\{v_1\ v_2\}$ is a subspace of V.

Proof : 1) $0 = 0v_1 + 0v_2 \in H$

2) $u = s_1 v_1 + s_2 v_2 \in H$

$v = t_1 v_1 + t_2 v_2 \in H$

$u + v = (s_1 + t_1) v_1 + (s_2 + t_2) v_2 \in H$

3) $c \cdot u = (cs_1) v_1 + (cs_2) v_2 \in H$

$\therefore$ $H = \text{span}\{v_1\ v_2\}$ is a subspace of V.

- Theorem : If $\{v_1\ v_2\ v_3\ \ldots\ v_p\} \in V$, then H is $H = \text{span}\{v_1\ v_2\ v_3 \ldots v_p\}$ is a subspace of V.

- A subspace is also a vector space AND A vector space is ~~not~~ a subspace. (but of itself)

IMP
- R^2 is ~~a~~ not a subspace of R^3.

$$\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \xrightarrow[\text{Subspace of } R^3]{\text{is a}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

IMP
- If $cu = 0 \Rightarrow \boxed{u = 0}$, $c \neq 0$, $u \in V$

*. NULL SPACE :

- The null space of $m \times n$ Matrix A, is the set of all solutions to $Ax = 0$.

$$\boxed{\text{Nul } A = \{ x : x \in \mathbb{R}^n \ \& \ A \cdot x = 0 \}}$$

Ex : Is $u = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ in Nul A where $A = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}$

We know that null space is the set of vectors such that $A \cdot u = 0$

$$\begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 25 \\ 65 \\ 40 \end{bmatrix} + \begin{bmatrix} -63 \\ -69 \\ -42 \end{bmatrix} + \begin{bmatrix} 38 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $u = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$ is a null space of A.

---

*. Theorem : If A is $m \times n$, Nul A is subspace of $\mathbb{R}^n$

Ex : Find Explicit Solutions to Nul A. Find the spanning set of $Ax = 0$.

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \quad R_1 \rightarrow R_1 - 3R_2$$

$$A = \begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{bmatrix}_{2\times 4} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{4\times 1}$$

$$x_1 - 7x_3 + 6x_4 = 0 \Rightarrow x_1 = 7x_3 - 6x_4$$

$$x_2 + 4x_3 - 2x_4 = 0 \qquad x_2 = -4x_3 + 2x_4$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7x_3 - 6x_4 \\ -4x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Spanning sets = $\left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. So, this is

also called "SPAN"

IMP the explicit solutions to the null space so if we want to find any vector u such that $A \cdot u = 0$. then we can pick any linear combination of these two vectors from the spanning set and products the $\{0\}$.

---

*. COLUMN SPACE :

- If A is $m \times n$ and $A = [a_1, a_2, a_3, \dots, a_n]$

$$\text{Col } A = \text{span}\{a_1, a_2, \dots, a_n\} :-$$

$$= \{ b \mid b = A \cdot x \text{ for some } x \in R^n \}$$

Example : Find a matrix A such that $W = \text{Col } A$.

$$W = \left\{ \begin{bmatrix} -a + 2b \\ a - 2b \\ 3a - 6b \end{bmatrix} : a, b \in R \right\}.$$

So, $W = \text{span}(A)$.

$$W = \begin{bmatrix} -a + 2b \\ a - 2b \\ 3a - 6b \end{bmatrix} = a\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix}$$

$S_1 \quad v_1 \qquad S_2 \quad v_2$

So, span of $A = [v_1, v_2] = \left\{ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix} \right\}$

Span set $= A = \underbrace{[v_1 \;\; v_2]}_{\text{written together}} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 3 & -6 \end{bmatrix}$

IMP

- If $Ax = b$ has a solution for every b in R^m

then $\underline{\text{Col } A = R^m}$ & $b = a_1v_1 + a_2v_2 + \cdots a_nv_n$

<u>Example</u> : Find A so that $\text{Col } A = W$. If $\text{Col } A$ is a subspace of R^k, what is value of K ?

Given,

$$W = \begin{bmatrix} b-c \\ 2b+c+d \\ 5c-4d \\ d \end{bmatrix} = b\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix}$$

So, $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}_{4\times 3}$

$m = 4$

$n = 3$

Col space of $R^m = R^4$

$\therefore R^K = R^4 \Rightarrow \boxed{K = 4}$

IMP If the Matrix is of order $m \times n$ then the Column space is subspace of R^m.

Null space is subspace of R^n. IMP

* ~~COMPLEX VECTOR SPACE~~:

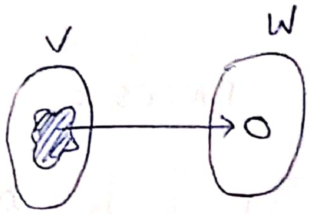
~~V is a vector space over a scalar field~~

*. A linear transformation from vector space V to vector space W assigns each $x \in V$ to a unique $T(x) \in W$ such that :

i. $T(u+v) = T(u) + T(v) \quad \forall \; u, v \in V$

ii. $T(c \cdot u) = c \cdot T(u) \quad \forall \; u \in V, c \in R$

*. KERNEL : $\{ u \in V \mid T(u) = 0 \}$. It is same as finding the Null space of Vector space



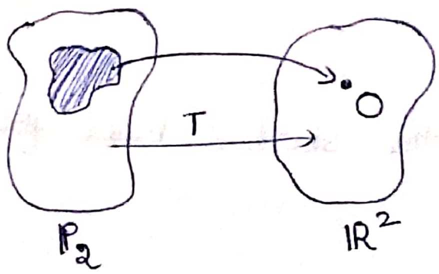
*. RANGE : $\{ T(x) \in W \mid x \in V \}$

The range of T is subspace of W.

*. Define $T = \mathbb{P}_2 \rightarrow \mathbb{R}^2$ by $T(p) = \begin{bmatrix} P(0) \\ P(0) \end{bmatrix}$. Find the polynomi $P_1, P_2 \in \mathbb{P}_2$ that span the Kernel of T and describe the range of T.

$P(t) = a_0 + a_1 t + a_2 t^2$

$P(0) = a_0$



range $= \left\{ \begin{bmatrix} a_0 \\ a_0 \end{bmatrix}, a_0 \in \mathbb{R} \right\}$

As $P(0)$ = Kernal of $T = 0$

$P(0) = a_0 = 0 \Rightarrow a_0 = 0$

$P(t) = 0 + a_1 t + a_2 t^2 = a_1 t + a_2 t^2$

$P(t) = a_1 \cdot t + a_2 \cdot t^2$ is similarly as,

Linearly independent $s_1 V_1 + s_2 V_2$ where span $\{v_1, v_2\}$

So, $P_1(t) = t$, $P_2(t) = t^2$, span $\{t, t^2\}$

---

*. A Linearly independant set is an indexed set of vectors $\{v_1, v_2, \ldots v_p\}$ such that $c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$ has only the trivial solution.

$$c_1 = c_2 = \cdots = c_p = 0$$

*. BASES :

- Let H be a subspace of vector space V. $B = \{b_1, \ldots b_p\}$ in V is a basis for H if :

  i. B is linearly independent.

  ii. $H = \text{span}\{b_1, b_2, \ldots b_p\}$

Ex : $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is the standard bases for $\mathbb{R}^3$.

$\{1, t, t^2, t^3, t^4\}$ is the standard bases for $\mathbb{P}_4$.

Ex : Determine if $\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$ is basis of $\mathbb{R}^3$.

Sol :

$$= \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ -3 & -4 & 1 \end{bmatrix} \underset{R_3 \to R_3 + 3R_1}{\sim} \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 5 & -5 \end{bmatrix}$$

$$\underset{R_3 \to R_3 - 5R_2}{\sim} \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

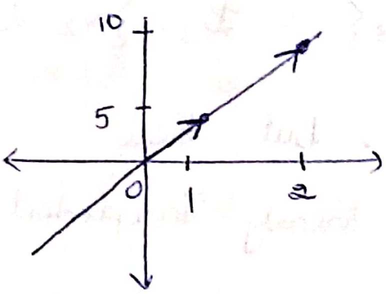
So, now, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} +3 \\ +1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$

we can express, $v_3 = v_1 - v_2$, $v_1 = v_2 + v_3$,

$v_2 = v_1 - v_3$

So, the given span $\{ \ldots \}$ are not bases of $\mathbb{R}^3$.
because they are linearly dependent. (not span).

Ex : Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = 5x$.



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 5x \end{bmatrix} = x \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

s. v_1

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\} = \text{Basis} = B$$

*- THE SPANNING SET THEOREM :

- Let $S = \{v_1, \ldots v_p\}$ be a set in V, and $H = \text{span}\{v_1 \ldots v_p\}$ then

(i) If a vector $v_K \in S$ is a linear combination of the remaining vectors in S, then $S - \{v_K\}$ still spans H.

(ii) If $H \neq \{0\}$, some subset of S is a basis for H

Ex : Let $v_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 9 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$ and $H = \text{span}\{v_1\ v_2\ v_3\}$. Also, $4v_1 + 5v_2 - 3v_3 = 0$. Find the three distinct bases for H.

Sol : Let B_1, B_2, B_3 be bases for H.

We know that, these vectors can be re written as linear combination of one another. So, this means we can pick any one of these vectors and we can remove it. So,

$B_1 = \{v_1\ v_2\}$ $B_2 = \{v_1\ v_3\}$ $B_3 = \{v_2\ v_3\}$

These are 3 distinct bases for H. But these $\{v_1 v_2\}$, $\{v_1 v_3\}$, $\{v_2 v_3\}$ must be linearly independent.

Ex : Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and

$H = \left\{ \begin{bmatrix} a \\ b \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. We can write every vector $\begin{bmatrix} a \\ b \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (b-a)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + a\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Is,

$\{v_1\ v_2\ v_3\}$ a basis for H ?

Sol :

$$\begin{bmatrix} a \\ b \\ b \end{bmatrix} = a v_1 + (b-a) v_2 + a v_3$$

$$= \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b-a \\ b-a \end{bmatrix} + \begin{bmatrix} 0 \\ a \\ a \end{bmatrix}.$$

But, Here both v_1, v_3 doesnot belong to H. Though v_1, v_2 and v_3 are linearly dependent sets, you cannot form the Basis of H. As clearly states that every vector that belongs to H, must be written in the form [a b b] and clearly v1 and v3 violate the condition

*. NOTE : If $B = \{b_1 \ldots b_n\}$ be a basis for a vector space V. Then for each $x \in V$, there exists unique scalars $c_1, \ldots, c_n$ such that,

$$\boxed{x = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n}$$

Ex : Given, $b_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $b_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ and $[x]_B = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

where $x \in \mathbb{R}^2$. Find x.

$x_B = \{c_1, c_2\}$

$$x = c_1 b_1 + c_2 b_2$$

$$= -2\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 5\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -26 \\ 1 \end{bmatrix}$$

*. DIMENSION :

- If V is spanned by finite set, then V is finite - dimensional. The dimension of V, dim V, is the no. of vectors in a basis for V.

  - $\dim 0 = 0$
  - $\dim \mathbb{R}^n = n$ (standard basis)
  - $\dim \mathbb{P}_n = n+1$

    $\{1, t, t^2, \ldots t^n\}$

Theorem : Let H be a subspace of finite dimensional vector space V. Any linearly independent set in H can be expanded to a basis for H, and

$$\boxed{\dim H \leq \dim V}$$

IMP
*. The dimension of Nul A is the no. of free variables in $Ax = 0$.

IMP
* The dimension of Col A is the no. of pivot columns in A.

Ex :

$$\hat{A} = \left[\begin{array}{cccc:c} 1 & 3 & 6 & 2 & 0 \\ 0 & 4 & 1 & 9 & 0 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & 3 & 6 & 2 \\ 0 & 4 & 1 & 9 \end{bmatrix}_{2\times4} \qquad b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \text{Zero Space}$$

2 free variables $\Rightarrow$ $\dim(\text{Nul } A) = 2$

2 pivot columns $\Rightarrow$ $\dim(\text{Col } A) = 2$

Ex: Find the dimension of the subspace spanned by

$$\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}-3\\-1\\-1\end{bmatrix}, \begin{bmatrix}5\\2\\2\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix}.$$

$\text{span}\{v_1\ v_2\ v_3\ v_4\} \sim [v_1\ v_2\ v_3\ v_4] = A$

$$\Rightarrow A = \left[\begin{array}{ccc:c}1 & -3 & 5 & 2\\0 & -1 & 2 & 1\\2 & -1 & 2 & -1\end{array}\right]$$

After performing reduced echelon form

$$\sim \begin{array}{cccc} \begin{bmatrix}1 & 0 & 0 & -1\\0 & 1 & 0 & -1\\0 & 0 & 1 & 0\end{bmatrix} \\ v_1 \quad v_2 \quad v_3 \quad v_4 \end{array}.$$

Here, $v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $v_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$v_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $v_4 = \begin{bmatrix}-1\\-1\\0\end{bmatrix}$

$\boxed{v_4 = -v_1 - v_2}$

dim of $\mathbb{B} = 3$

dim of $\mathbb{R}^3 = 3$

---

*. ROW SPACE :

- The set of all linear combinations of row vectors is called the row space.

$$A = \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 3\\0 & 0 & 1 & 2\end{bmatrix}$$

$r_1 = (1, 2, 3, 4)$

$r_2 = (0, 1, 2, 3)$

$r_3 = (0, 0, 1, 2)$

- Row A is subspace of $\mathbb{R}^n$.
- We could write Row A as Col A^T.

Ex : Find row space of A (Row A) given,

$$A = \begin{bmatrix} 1 & -4 & 9 & 7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$$

Reducing this into Echeloen row

$$\sim A = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times4}$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 5 & 0 \\ 5 & -6 & 0 \end{bmatrix}_{4\times3}$$

$$\text{Row } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 5 \end{bmatrix}^T, \begin{bmatrix} 0 \\ -2 \\ 5 \\ -6 \end{bmatrix}^T \right\} = \text{Col } A^T$$

- The rank of A is dim (Col A)
- The rank of A^T is dim (Col A^T) which is dim (Row A)

Ex : If $A \sim B$ are dim (Row B) = 8,

What is rank of A^T = ?

$\Rightarrow$ Rank of A^T = dim (Row B) = dim (Row A) = 8.

IMP

*. RANK THEOREM :

IMP

- Given, matrix $A_{m\times n}$, $\boxed{\text{rank } A + \dim(\text{Nul } A) = n}$ *

IMP

m – rows

n – columns

implies, dim (Col A) = rank A

$$\Rightarrow \boxed{\underbrace{\dim(\text{Col } A)}_{\text{pivot columns}} + \underbrace{\dim(\text{Nul } A)}_{\text{free variables / no. of non pivot columns}} = n}$$

Total columns = n

Ex : If the null space of a 7×6 matrix is 5-dimensional, What is rank of A?

$$\text{Rank } A + \dim(\text{nul } A) = n$$

$$\text{Rank } A + 5 = 6 \Rightarrow \underline{\text{Rank } A = 1}$$

Ex : If A is 6×4, what is the smallest possible dimension of Nul A ?

n - Columns of A

Given, $n = 4$

Rank of $A \leq n$ (dimension of Matrix)

To be least Nul A, Rank A $= n$

$\Rightarrow$ Rank A + dim (nul A) $= n$

n + dim (nul A) $= n$ $\Rightarrow$ dim (nul A) $= 0$

$\therefore$ A matrix can have smallest possible dim (nul A) $= 0$ of order $m \times n$.

Ex : Given an $m \times n$ matrix A, which subspaces are $\mathbb{R}^m$? $\mathbb{R}^n$?

	Col A^T / Row A	Nul A	Col A
$\mathbb{R}^m$			✓ are in
$\mathbb{R}^n$	✓ are in	✓ are in	

Ex : Let $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find v such that $u \cdot v^T =$

$$= \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix}$$

$$\underbrace{u}_{2\times 1} \cdot \underbrace{v^T}_{1\times 3} = \underbrace{\begin{bmatrix} & & \end{bmatrix}}_{2\times 3}$$

$v^T \rightarrow 1\times 3 \Rightarrow v$ is 3×1 dimension

Let $v^T = \begin{bmatrix} x & y & z \end{bmatrix}_{1\times 3}$, $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1}$

$$\cancel{\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times 1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix}}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2\times 1} \begin{bmatrix} x & y & z \end{bmatrix}_{1\times 3} = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix}$$

$$\begin{bmatrix} x & y & z \\ 2x & 2y & 2z \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \end{bmatrix} = v^T$$

$$\Rightarrow v = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

$\boxed{\text{NOTE}}$ If A be an $m\times n$ matrix, then $\text{Rank}(AB) \le \text{Rank}(A)$ and B be an $n\times p$ matrix, $\text{Rank}(AB) \le \text{Rank}(B)$.

IMP Example : Let $A = \begin{bmatrix} 2 & -1 & -3 \\ -4 & 2 & 6 \end{bmatrix}$

i. Null space of A :

$$\begin{bmatrix} 2 & -1 & -3 \\ -4 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Big\downarrow$ row reduced Echelon form

$$\begin{bmatrix} 1 & -1/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 - \frac{1}{2}x_2 - \frac{3}{2}x_3 = 0$$

$$x_1 = \frac{1}{2}x_2 + \frac{3}{2}x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \qquad x_2, x_3 \in \mathbb{R}$$

No. of free variables $= 2$ (x_2, x_3)

$\dim(\text{Nul } A) = 2$

$\text{Nul } A = N(A) = \text{Span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

ii. Column space of A :

$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$

Column space after simplifying since they are linearly dependent

$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} +2 \\ -4 \end{bmatrix} \right\}$ Rank $(A) = 1$

- Rank (A) = 1
- Rank theorem Rank (A) + dim (Nul A) = n

$\therefore$ 1 + 2 = 3 (proved)

dim (Col A) = rank (A) = 1

~~dim (Col A^T) = dim (row A) = 1 (verify same)~~

ii. <u>TRANSPOSE</u> :

$$A = \begin{bmatrix} 2 & -1 & -3 \\ -4 & 2 & 6 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & -4 \\ -1 & 2 \\ -3 & 6 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & -4 \\ -1 & 2 \\ -3 & 6 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{3\times 1}$$

$\downarrow$ reduced form

$$\begin{bmatrix} +1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} +x_1 - 2x_2 = 0 \\ x_1 = 2x_2 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad x_2 \in \mathbb{R}$$

$$\text{Nul } A^T = N(A^T) = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}, \quad \dim(\text{nul } A^T) = 1$$

$$\text{Col } A^T = \text{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\} \qquad \dim(\text{Col } A^T) = 1$$

$$\text{Rank}(A^T) = \dim(\text{Col } A^T) = 1$$

$$\dim(\text{Col } A^T) = \dim(\text{row } A) = 1$$

row space of A = Col space of A^T

$$\text{row } A = \text{Span}\left\{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right\}$$

row space of A^T = Col space of A

$$\text{Col } A = \text{row } A^T = \text{Span}\left\{\begin{bmatrix} 2 \\ -4 \end{bmatrix}\right\}.$$

iii. LEFT NULL SPACE :

$$A^T \cdot X = 0$$

Take transpose both sides,

$$(A \cdot X)^T = (0)^T$$

$$\boxed{\underset{1 \times n}{(X)^T} \cdot A = 0^T}$$

$$\text{Nul } A^T = \{ x \mid A^T \cdot x = 0 \} = \{ x \mid x \cdot A = 0^T \}$$

$$\boxed{\text{Left Nullspace of } A = \text{Null space } (A^T)}$$ * IMP

$$\text{Left Nul } A = \text{Nul}(A^T) = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

NOTE Always Rank of A = Rank of A^T

Conclusion : (IMP)

i. Null space of A = Nul A = Span$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Dimensions of Null space of A = dim (Nul A) = 2

ii. Null space of A^T = Nul A^T = Span$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Dimensions of Null space of A^T = dim (Nul A^T) = 1

iii. Left Null Space of A = Left Nul A = span$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Dimensions of Left Null space of A = dim(Left Nul A) = 1

iv. Left Null space of A^T = Left Nul A^T = Span$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Dimensions of Left Null space of A^T = dim (Left Nul A^T) = 2

v. Column space of A = Col A = Span$\left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\}$

Dimensions of Column space of A = dim (Col A) = 1

vi. Column space of A^T = Col A^T = Span$\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$

Dimensions of Column space of A^T = dim (Col A^T) = 1

vii. Row space of A^T = Row A^T = Span$\left\{ \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\}$

Dimensions of Row space of A^T = dim (Row A^T) = 1

viii. Row space of A = Row A = Span$\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$

Dimensions of Row space of A = dim(Row A) = 1

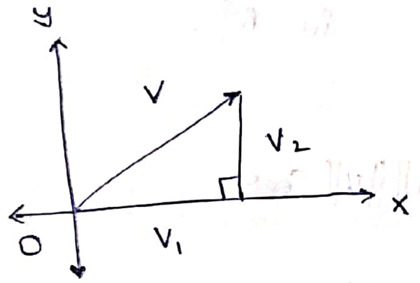
ix. Rank (A) = 1 x. Rank (AT) = 1

MODULE - 3 : INNER PRODUCT SPACES

LENGTH :

- The length or magnitude, of the vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in R^2 denoted by $\|v\|$, is by the Pythagorean theorem.

$$\boxed{\|v\| = \sqrt{v_1^2 + v_2^2}}$$



Ex : If $v = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

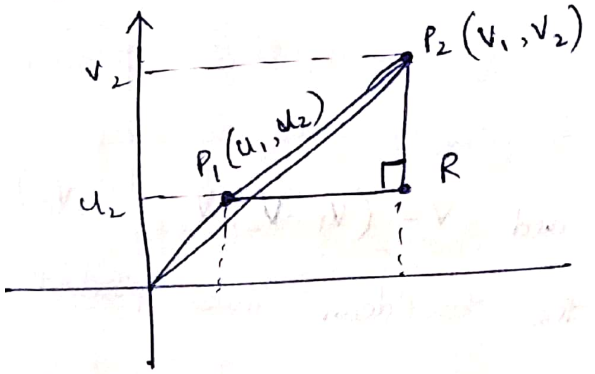
then $\|v\| = \sqrt{(2)^2 + (-5)^2}$
$= \sqrt{29}$

- For $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in R^3 $\Rightarrow$ $\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

- If $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in R^2,

The distance between u and v is,

$$\sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2} = \|v - u\|$$



Distance between the points $P_1(u_1, u_2)$ and $P_2(v_1, v_2)$.

*. DIRECTION :

The direction of vector in R^2 is given by specifying its angle of inclination, or slope. The direction of vector v in R^3 is specified by giving the cosines of the angles that the vector v makes with positive x-, y-, z-axes, these are called direction cosines.

- Suppose $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be two vectors in R^3.

$$\|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta$$

Thus,

$$\boxed{\cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{\|u\| \cdot \|v\|}}, \quad 0 \le \theta \le \pi$$

- The length of vector and the cosine of angle between two non zero vectors in R^2 or R^3 can be expressed in terms of the dot product.

standard inner product or dot product $= u_1v_1 + u_2v_2 + u_3v_3$. in R^3.

(or) $u_1v_1 + u_2v_2$ in R^2,

*. EUCLIDEAN INNER PRODUCT :

- If $u = (u_1, u_2, u_3, \ldots, u_n)$ and $v = (v_1, v_2, v_3, \ldots, v_n)$ are vectors in R^n then the Euclidean inner product $u \cdot v$ is,

$$u \cdot v = u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n.$$

* PROPERTIES :

- If u, v, w are vectors in R^n and K is any scalar,

$\oplus$ $u \cdot v = v \cdot u$ $\qquad$ $\oplus$ $(u+v) \cdot w = u \cdot (w) + (v \cdot w) = uw + vw$

$\oplus$ $(k \cdot u) \cdot v = k (u \cdot v)$ $\qquad$ $\oplus$ $v \cdot v \geq 0$ $\qquad$ $\oplus$ $v \cdot v = 0 \Leftrightarrow v = 0$

* NORM & DISTANCE :

- We define the Euclidean norm (or Euclidean length) of a vector $u = (u_1, u_2, u_3, \ldots, u_n)$ in R^n by

$$\|u\| = (u \cdot u)^{1/2} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

- The Euclidean distance between the points $u = (u_1, u_2, \ldots u_n)$ and $v = (v_1, v_2, \ldots v_n)$ in R^n is defined by

$$d(u,v) = \|u - v\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

* THEOREM : (CAUCHY - SCHWARZ INEQUALITY in R^n)

- If $u = (u_1, u_2, u_3 \ldots u_n)$ and $v = (v_1, v_2, v_3 \ldots v_n)$ are vectors in R^n, then $|u \cdot v| \leq \|u\| \cdot \|v\|$.

Properties :

- $\|u\| \geq 0$
- $\|u\| = 0$ if & only if $u = 0$
- $\|k \cdot u\| = |k| \, \|u\|$
- $\|u + v\| \leq \|u\| + \|v\|$. (Triangle inequality)
- $d(u,v) \geq 0$ $\qquad$ • $d(u,v) = d(v,u)$
- $d(u,v) \leq d(u,w) + d(w,v) \longrightarrow$ Triangle inequality

- If u, v, w are vectors in R^n with the Euclidean inner product, then $u \cdot v = \frac{1}{4} \|u+v\|^2 - \frac{1}{4} \|u-v\|^2$.

*. MATRIX FORMULA for DOT PRODUCT:

- If $u = [u_1\ u_2 \ldots u_n]^T$ and $v = [v_1\ v_2\ v_3 \ldots v_n]^T$

then,

$$\left. \begin{aligned} u \cdot v &= v^T . u \\ Au \cdot v &= u . A^T v \\ u \cdot Av &= A^T u . v \end{aligned} \right\}$$

- If $A = [a_{ij}]$ is an $m \times r$ matrix and $B = [b_{ij}]$ is an $r \times n$ matrix, then the ij entry of AB is

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j} + \ldots a_{ir} b_{rj}$$

which is the dot product of i th row vector of A and the j th column vector of B.

- Thus, if row vectors of A are $r_1, r_2 \ldots r_m$ and column vectors of B are $c_1, c_2, \ldots c_n$:

$$AB = \begin{bmatrix} r_1 \cdot c_1 & r_1 \cdot c_2 & \ldots & r_1 \cdot c_n \\ r_2 \cdot c_1 & r_2 \cdot c_2 & \ldots & r_2 \cdot c_n \\ \vdots & \vdots & & \vdots \\ r_m \cdot c_1 & r_m \cdot c_2 & \ldots & r_m \cdot c_n \end{bmatrix}$$

Example :

A Linear system written in dot product form is, system

$$3x_1 - 4x_2 + x_3 = 1$$
$$2x_1 - 7x_2 - 4x_3 = 5$$
$$x_1 + 5x_2 - 8x_3 = 0$$

dot product form

$$\begin{bmatrix} (3,-4,1)\cdot(x_1,x_2,x_3) \\ (2,-7,-4)\cdot(x_1,x_2,x_3) \\ (1,5,-8)\cdot(x_1,x_2,x_3) \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$

*- INNER PRODUCT SPACES :

- Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be vectors in R^2.

We define $(u,v) = u_1 v_1 - u_2 v_1 - u_1 v_2 + 3 u_2 v_2$

- If V is inner product space, we define the distance between two vectors u and v in V as,

$$d(u,v) = \|u - v\|$$

- Let V be an inner product space. Two vectors u and v in V are orthogonal if

$$(u,v) = u \cdot v = u^T \cdot v = 0.$$

- Let V be inner space. A set S of vectors in V is called orthogonal if any two distinct vectors in S are orthogonal.

- If, in addition, each vector in S is of unit length then S is called orthonormal.

*. Definition :

- Two subspaces V and W of vector space are orthogonal if every vector v in V is perpendicular to every vector w in W.

orthogonal subspaces	$v^T w = 0$	for all v in V and for all w in W.

Ex : If $x_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, and $x_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

then $\{x_1, x_2, x_3\}$ is an orthogonal set. The vectors

$$u_1 = \begin{bmatrix} \frac{1}{\sqrt5} \\ 0 \\ \frac{2}{\sqrt5} \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} -2/\sqrt5 \\ 0 \\ 1/\sqrt5 \end{bmatrix}$$ are unit vectors orthonormal

in the directions of x_1 and x_2, respectively. Since x_3 is also a unit vector, we conclude that $\{u_1, u_2, x_3\}$ is an orthogonal set.

Theorem : let $S = \{u_1, u_2 \dots u_n\}$ be an finite orthogonal set of non zero vectors in an inner product space V. Then S is linearly independent.

*. ORTHOGONALITY OF FOUR SUBSPACES :

- The row space is perpendicular to null space.
- Every vector x in the null space is perpendicular to every row of A, because $Ax = 0$.
- The null-space $N(A)$ and the row-space $C(A^T)$ are orthogonal subspaces of R^n.
- The column space is perpendicular to null-space of A^T.
- Every vector y in the null space of A^T is perpendicular to every column of A.
- The left null space $N(A^T)$ and the column space $C(A)$ are orthogonal in R^m.

Ex. : Basis for an orthogonal Complement

- Let W be the subspace for R^6 spanned by vectors

$w_1 = (1, 3, -2, 0, 2, 0)$ $\quad$ $w_2 = (2, 6, -5, -2, 4, -3)$

$w_3 = (0, 0, 5, 10, 0, 15)$ $\quad$ $w_4 = (2, 6, 0, 8, 4, 18)$

Find basis of orthogonal complement of W.

Sol : The subspace W is the same as the row space of the matrix.

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

Since the row space and null space of A are orthogonal complements, our problem reduces to finding a basis for the null space of this matrix.

$$V_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for this null space. Expressing these vectors in comma-delimited form (to match that of w_1, w_2, w_3, w_4), we obtain the basis vectors.

$V_1 = (-3, 1, 0, 0, 0, 0)$; $V_2 = (-4, 0, -2, 1, 0, 0)$

$V_3 = (-2, 0, 0, 0, 1, 0)$.

You may want to check that these vectors are orthogonal to w_1, w_2, w_3, w_4 by computing the necessary dot products.

*. ORTHOGONAL COMPLEMENTS :

- The orthogonal complement of subspace V contains every vector that is perpendicular to V. This orthogonal subspace is denoted by $V^{\perp}$ ('V perp')

- The Null space is orthogonal complement of the row space. every x that is perpendicular to the rows satisfies $Ax = 0$, and lies in the null space.

- If v is orthogonal to the null space, it must be in the row space.

- The left null space & column space are orthogonal in R^m and they are orthogonal complements.

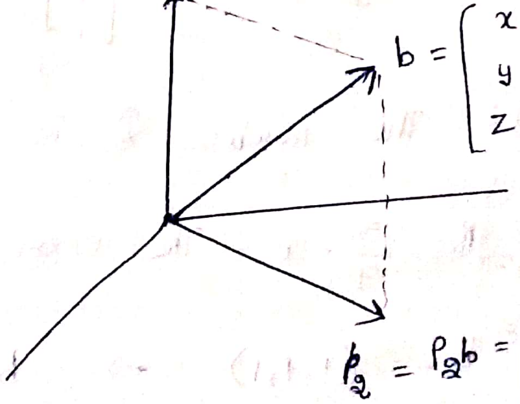
*. <u>Theorem</u> :

- The column space & row space both have dimension r.
- The null space have dimensions $n-r$ and $m-r$.
- $N(A)$ is the orthogonal complement of row space $C(A^T)$ in R^n.
- $N(A^T)$ is the orthogonal complement of column space $C(A)$ in R^m.

*. <u>PROJECTIONS</u> :

- When b is projected onto a line, its projection p is the part of b along that line. If b is projected onto a plane p is the part in the plane. The projection p is Pb. The projection matrix P multiplies b to give p.

Projection $P_1 = P_1 \cdot b = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$



$b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Projections, $p_1 = P_1 \cdot b$ — z axis

$p_2 = P_2 \cdot b$ — xy plane

$p_2 = P_2 b = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

- Projection Matrix onto z axis : $P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- onto the xy plane : $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- The vectors $p_1 + p_2 = b$ & Matrices $P_1 + P_2 = I$

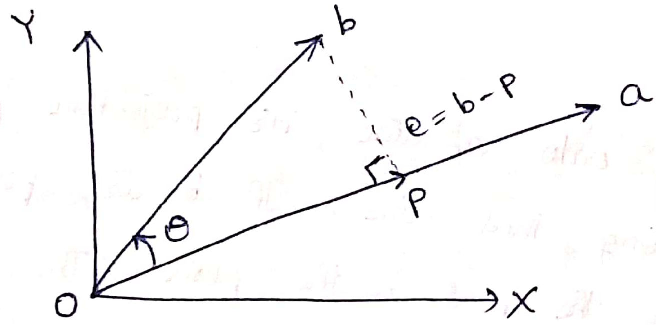
*. PROJECTION ONTO A LINE :

- The projection of b onto a line a is the vector

$$\boxed{\hat{p} = \hat{x} \cdot a = \frac{a^T \cdot b}{a^T \cdot a} \cdot a}$$

Special Case :

1. If $b = a$ then $\hat{x} = 1$. The projection of a onto a is itself
2. If b is perpendicular to a then $a^T \cdot b = 0$. Projection $p = 0$.



$$p = \hat{x} a = \frac{a^T b}{a^T a} \cdot a$$

The projection p of b onto a line

EX : Projection $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto $a = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ to find $p = \hat{x} a$

Sol : The number $\hat{x}$ is ratio of $a^T b = 5$ to $a^T a = 9$

$p = \frac{5}{9} \cdot a$. The error vector between b & p is $e = b - p$

$b = (1, 1, 1) \Rightarrow b = p + e$

$p = \frac{5}{9} a = \left(\frac{5}{9}, \frac{10}{9}, \frac{10}{9}\right)$ & $e = b - p = \left(\frac{4}{9}, \frac{-1}{9}, \frac{-1}{9}\right)$

- The error e should be perpendicular to $a = (1, 2, 2)$ and it is : $e^T a = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$

- $p = \frac{a^T b}{a^T \cdot a} \cdot a$ has length $\boxed{\|p\| = \|b\| \cos\theta}$

- Projection Matrix , $P = \frac{a \cdot a^T}{a^T \cdot a}$ & $\boxed{p = P \cdot b}$

- The projection matrix P is $m \times m$ & rank $= 1$.

Ex : Find the projection matrix $P = \frac{a \cdot a^T}{a^T \cdot a}$ onto the line through $a = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Sol : Multiply column a times row a^T & divide by $a^T \cdot a$

here $a^T \cdot a = 9$

Projection matrix, $P = \frac{a \cdot a^T}{a^T \cdot a} = \frac{1}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$

$$P = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$$

This matrix projects any vector b onto a .

$p = P \cdot b$ for $b = (1, 1, 1)$ (from previous example)

$$p = P \cdot b = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix}$$

* Note : If P projects onto one subspace , $I - P$ projects onto the perpendicular subspace . Here $I - P$ projects onto the plane perpendicular to a .

*. PROJECTION ONTO A SUBSPACE :

- The combination $p = \hat{x}_1 a_1 + \cdots + \hat{x}_n a_n = A \cdot \hat{x}$ that is closest to b comes from $\hat{x}$.

Find $\hat{x}$ $(n \times 1)$ $\Rightarrow$ $\boxed{A^T (b - A\hat{x}) = 0}$ * [OR] $\boxed{A^T \cdot A \hat{x} = A^T \cdot b}$ *

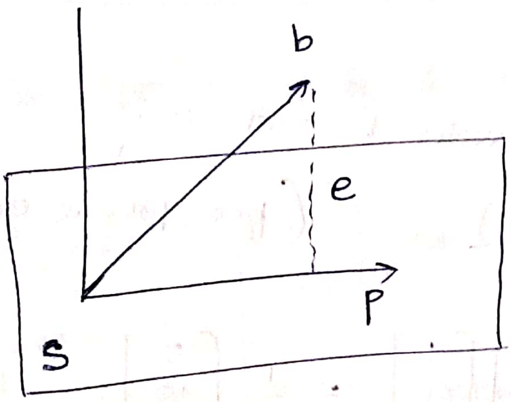
- This symmetric matrix $A^T \cdot A$ is n by n. It is invertible if the a's are independent. The solution is

* $\boxed{\hat{x} = (A^T \cdot A)^{-1} \cdot A^T \cdot b}$ Find $p\,(m \times 1)$:

$$\boxed{p = A \cdot \hat{x} = A (A^T A)^{-1} \cdot A^T \cdot b} \; *$$

- Now is the projection matrix,

$P\,(m \times m)$ $\Rightarrow$ $\boxed{P = A (A^T \cdot A)^{-1} \cdot A^T}$ *



$$p = A \cdot \hat{x}$$
$$= A (A^T \cdot A)^{-1} \cdot A^T \cdot b$$
$$= P \cdot b$$

IMP * $\boxed{\text{The projection } p \text{ of } b \text{ onto } S = \text{Column space of } A.}$

Ex: If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$. Find $\hat{x}$ & p also P.

Sol: $A^T \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$

$A^T \cdot b = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

Normal equation, $A^T \cdot A \cdot \hat{x} = A^T \cdot b$, $\hat{x} = ?$

$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \Rightarrow \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

The combination $p = A \cdot \hat{x}$ is projection of b onto the column space of A.

$p = 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$. The error is

$e = b - p = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Similarly $\boxed{P = A \cdot (A^T \cdot A)^{-1} \cdot A^T}$

We get $P = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$

- It satisfies 3 conditions: (must for all) * IMP

1. error e $\perp^r$ to both the columns of A.
2. $p = P \cdot b$ (check)
3. $P^2 = P$ because 2nd projection doesn't change 1st projection.

* Summary : (Algorithm) *IMP

- To find projection p, solve $\boxed{A^T \cdot A\hat{x} = A^T \cdot b}$
- Then $p = A \cdot \hat{x}$, here $\hat{x}$ is obtained from .
- Error, $e = b - p = b - A \cdot \hat{x}$
- Projection matrix, $P = A(A^T \cdot A)^{-1} \cdot A^T$
- Verify $p = P \cdot b$ or not. & $e \perp$ both columns of A.
- This matrix P, satisfies $P^2 = P$
- The distance from b to the subspace $C(A) = \|e\|$.

*· ORTHOGONAL SETS & ORTHOGONAL BASES :

- A set of vectors $\{u_1, u_2, \dots u_n\}$ in R^n is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is if $u_i \cdot u_j = 0$ whenever $i \neq j$

Ex : Show that $\{u_1, u_2, u_3\}$ is an orthogonal set, where

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

Sol : Consider the three possible pairs of distinct vectors namely $\{u_1, u_2\}$, $\{u_1, u_3\}$, $\{u_2, u_3\}$

$$u_1 \cdot u_2 = 3(-1) + 1(2) + 1(1) = 0$$
$$u_1 \cdot u_3 = 3(-1/2) + 1(-2) + 1(7/2) = 0$$
$$u_2 \cdot u_3 = -1(-1/2) + 2(-2) + 1(7/2) = 0$$

each pair of distinct vectors is orthogonal, and so $\{u_1, u_2, u_3\}$ is an orthogonal set. See

Theorem: If $S = \{u_1, \dots u_p\}$ is an orthogonal set of non zero vectors in R^n, then S is linearly independent & hence is basis for subspace spanned by S.

* An orthogonal basis for a subspace W of R^n is a basis for W that is also an orthogonal set.

Theorem: Let $\{u_1 \dots u_p\}$ be an orthogonal basis for a subspace W of R^n. For each y in W, the weights in the linear combination $y = c_1u_1 + c_2u_2 + \dots + c_pu_p$ are:

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j} \quad (j = 1, \dots p)$$

- It is easy to visualize the case in which $W = R^2 =$ span (u_1, u_2) with u_1 & u_2 orthogonal. Any y in R^2 can be written in the form

$$\boxed{y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} \cdot u_2}$$

Ex: The set $S = \{u_1, u_2, u_3\}$ in previous example is orthogonal basis for R^3. Express the vector $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ as a linear combination of vectors in S.

Sol: Compute,

$$y \cdot u_1 = 11 \qquad y \cdot u_2 = -12 \qquad y \cdot u_3 = -33$$

$$u_1 \cdot u_1 = 11 \qquad u_2 \cdot u_2 = 6 \qquad u_3 \cdot u_3 = \frac{33}{2}$$

By theorem,

$$\boxed{y = \frac{y \cdot u_1}{u_1 \cdot u_1} \cdot u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} \cdot u_2 + \frac{y \cdot u_3}{u_3 \cdot u_3} \cdot u_3}$$

$$\Rightarrow \quad y = \frac{11}{11} u_1 + \frac{-12}{6} u_2 + \frac{-33}{\frac{33}{2}} u_3$$

$$\Rightarrow \quad y = u_1 - 2u_2 - 2u_3$$

Theorem: An $m \times n$ matrix U has orthonormal columns if and only if $U^T \cdot U = I$. $\|Ux\| = \|x\|$

*. NORMALIZING VECTOR :

- In inner product spaces, the solution of a problem can often be simplified by choosing a basis in which the vectors are orthogonal to one another.

*. THE GRAM - SCHMIDT PROCESS :

- It is a process for providing an orthogonal or orthonormal basis for any non zero subspace of R^n.

*IMP $\langle u_i, v_i \rangle$ = dot product of (u_i, v_i) = $\hat{u}_i \cdot \hat{v}_i$.

$\|v_i\|^2$ = length of vector v_i $= \left| \sqrt{v_{i_1}^2 + v_{i_2}^2 + \cdots} \right|$

Algorithm :

To convert a basis $\{u_1, u_2, \dots u_r\}$ into an orthogonal basis $\{v_1, v_2, \dots v_r\}$, perform the following computations.

step 1 : $v_1 = u_1$

step 2 : $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} \cdot v_1$

step 3 : $v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$

step 4 : $v_4 = u_4 - \frac{\langle u_4, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_4, v_2 \rangle}{\|v_2\|^2} v_2 - \frac{\langle u_4, v_3 \rangle}{\|v_3\|^2} v_3$

⋮

(continue for r steps)

optional : To convert the orthogonal basis into an orthonormal basis $\{q_1, q_2, \dots q_r\}$ normalize the orthogonal basis vectors

Additionally : $\text{span}\{v_1, v_2, \dots, v_r\} = \text{span}\{u_1, u_2, \dots, u_r\} = \text{span}\{q_1, q_2, \dots, q_r\}$.

Ex: Using Gram schmidt process, Assume that the vector space R^3 has the Euclidean inner product. And transform the basis vectors $u_1 = (1,1,1)$, $u_2 = (0,1,1)$, $u_3 = (0,0,1)$ into an orthogonal basis $\{v_1, v_2, v_3\}$ and then normalize the orthogonal basis vectors to obtain an orthonormal basis $\{q_1, q_2, q_3\}$.

Sol : Apply above algorithm,

Step 1 : $v_1 = u_1 = (1,1,1)$

Step 2 : $v_2 = u_2 - \text{proj}_{w_1} u_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

$= (0,1,1) - \frac{2}{3}(1,1,1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Step 3 : $v_3 = u_3 - \dfrac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \dfrac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$

$= (0,0,1) - \frac{1}{3}(1,1,1) - \dfrac{1/3}{2/3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$

Thus,

$v_1 = (1,1,1)$, $v_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $v_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$ form an orthogonal basis for R^3. The norms of these vectors are $\|v_1\| = \sqrt{3}$, $\|v_2\| = \frac{\sqrt{6}}{3}$, $\|v_3\| = \frac{1}{\sqrt{2}}$. So, an orthonormal basis for R^3 is,

$q_1 = \dfrac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$q_2 = \dfrac{v_2}{\|v_2\|} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$q_3 = \dfrac{v_3}{\|v_3\|} = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ are orthonormal basis for R^3.

*. LEGENDRE POLYNOMIALS :

Let the vector space P_2 have the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx .$$

Apply Gram - Schmidt process to transform the standard basis $\{1, x, x^2\}$ for P_2 into an orthonormal (or) orthogonal basis $\{ \phi_1(x), \phi_2(x), \phi_3(x) \}$.

Ex : Take $u_1 = 1$, $u_2 = x$, $u_3 = x^2$

Step 1 : $v_1 = u_1 = 1$

Step 2 : We have,

$$\langle u_2, v_1 \rangle = \int_{-1}^{1} x \cdot dx = 0$$

So, $$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = u_2 = x$$

Step 3 : We have,

$$\langle u_3, v_1 \rangle = \int_{-1}^{1} x^2\, dx = \left.\frac{x^3}{3}\right]_{-1}^{1} = 2/3$$

$$\langle u_3, v_2 \rangle = \int_{-1}^{1} x^3\, dx = \left.\frac{x^4}{4}\right]_{-1}^{1} = 0$$

$$\|v_1\|^2 = \langle v_1, v_1 \rangle = \int_{-1}^{1} 1 \cdot dx = \left. x \right]_{-1}^{1} = 2$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2 = x^2 - 1/3$$

$\phi_1(x) = 1$, $\phi_2(x) = x$, $\phi_3(x) = x^2 - 1/3$ $\rightarrow$ orthogonal basis

*. QR FACTORIZATION :

Ex : Find Q and R for QR Factorization of $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Sol : $A = Q \cdot R$

Q : orthogonal matrix (has orthonormal columns)

R : upper Triangular Matrix

Columns of $A \rightsquigarrow$ orthonormal set using Gram schmidt process

$$A = \left\{ \begin{matrix} 1 \\ -1 \\ 1 \\ 1 \end{matrix} \right\}, \left\{ \begin{matrix} 2 \\ 1 \\ 1 \\ 1 \end{matrix} \right\}, \left\{ \begin{matrix} 3 \\ 1 \\ 1 \\ 1 \end{matrix} \right\}$$

orthogonal Set, $X = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ -3 \\ -3 \end{bmatrix} \right\}$

orthonormal set, $\hat{x} = \left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 5/\sqrt{76} \\ 7/\sqrt{76} \\ 1/\sqrt{76} \\ 1/\sqrt{76} \end{bmatrix}, \begin{bmatrix} 4/\sqrt{38} \\ -2/\sqrt{38} \\ -3/\sqrt{38} \\ -3/\sqrt{38} \end{bmatrix} \right\}$

So,
$$Q = \begin{bmatrix} \frac{1}{2} & \frac{5}{\sqrt{76}} & \frac{4}{\sqrt{38}} \\ \frac{-1}{2} & \frac{7}{\sqrt{76}} & \frac{-2}{\sqrt{38}} \\ \frac{1}{2} & \frac{1}{\sqrt{76}} & \frac{-3}{\sqrt{38}} \\ \frac{1}{2} & \frac{1}{\sqrt{76}} & \frac{-3}{\sqrt{38}} \end{bmatrix}$$

$$\boxed{R = Q^T \cdot A}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{\sqrt{76}} & \frac{7}{\sqrt{76}} & \frac{1}{\sqrt{76}} & \frac{1}{\sqrt{76}} \\ \frac{4}{\sqrt{38}} & \frac{-2}{\sqrt{38}} & \frac{-3}{\sqrt{38}} & \frac{-3}{\sqrt{38}} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3/2 & 2 \\ 0 & \sqrt{19}/2 & \frac{12\sqrt{19}}{19} \\ 0 & 0 & \frac{2\sqrt{38}}{19} \end{bmatrix}$$

$$Q^T \cdot A = R$$

$$\therefore \quad Q = \begin{bmatrix} \frac{1}{2} & \frac{5}{\sqrt{76}} & \frac{4}{\sqrt{38}} \\ \frac{-1}{2} & \frac{7}{\sqrt{76}} & \frac{-2}{\sqrt{38}} \\ \frac{1}{2} & \frac{1}{\sqrt{76}} & \frac{-3}{\sqrt{38}} \\ \frac{1}{2} & \frac{1}{\sqrt{76}} & \frac{-3}{\sqrt{38}} \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 3/2 & 2 \\ 0 & \sqrt{19}/2 & \frac{12\sqrt{19}}{19} \\ 0 & 0 & \frac{2\sqrt{38}}{19} \end{bmatrix}$$

*. LEAST SQUARES APPROXIMATIONS :

1. Solving $A^T \cdot A \cdot \hat{x} = A^T \cdot b$ gives the projection $p = A \cdot \hat{x}$ of b onto the column space of A.

2. When $Ax = b$ has no solution, $\hat{x}$ is the "least - squares" solution. $\|b - A\hat{x}\|^2 = \text{minimum}$

3. Setting partial derivaties of $E = \|Ax - b\|^2$ to zero $\left(\frac{\partial E}{\partial x_i} = 0\right)$ also produces $A^T \cdot A \cdot \hat{x} = A^T \cdot b$.

4. To fit points $(t_1, b_1), \ldots, (t_m, b_m)$ by a straight line, A has columns $(1, \ldots, 1)$ & $(t_1, \ldots, t_m)$

5. In that case $A^T \cdot A$ is the 2×2 matrix $\begin{bmatrix} m & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{bmatrix}$ and $A^T \cdot b$ is the vector $\begin{bmatrix} \Sigma b_i \\ \Sigma t_i b_i \end{bmatrix}$

- When $Ax = b$ has no solution, multiply by A^T

$$\boxed{A^T \cdot A \cdot \hat{x} = A^T \cdot b}$$ and solve.

Ex: A crucial application of least squares is fitting a straight line to m points. Start with three points: Find the closest line to the points $(0,6)$, $(1,0)$, $(2,0)$

Sol: No straight line $b = C + Dt$ goes through those three points. We are asking for two numbers C and D that satisfy three equations: $n = 2$, $m = 3$. Here are the three equations at $t = 0, 1, 2$ to match the given values $b = 6, 0, 0$:

$t = 0$ The 1st point on the line $b = C + Dt$ if $C + D \cdot 0 = 6$

$t = 1$ The 2nd point on the line $b = C + Dt$ if $C + D \cdot 1 = 0$

$t = 2$ The 3rd point is on the line $b = C + Dt$ if $C + D \cdot 2 = 0$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \; ; \; x = \begin{bmatrix} C \\ D \end{bmatrix} \; ; \; b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$Ax = b$ not solvable

when $Ax = b$ has no solution, multiply by A^T and solve

$$\boxed{A^T \cdot A \cdot \hat{x} = A^T \cdot b}$$

$$A^T \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$A^T \cdot b = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

$(A^T \cdot A \cdot \hat{x} = A^T \cdot b)$ solve this

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \text{ gives } \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$C + Dt \Rightarrow \begin{matrix} C = 5 \\ D = -3 \end{matrix} \Rightarrow 5 - 3t$ will be best line for those 3 points.

*. MINIMIZING ERROR :

- The least squares solution $\hat{x}$ makes $E = \|A\hat{x} - b\|^2$ as small as possible.

Ex : Find the least-squares solution of the inconsistent system $Ax = b$ for :

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Sol : To use normal equations (3), compute :

$$A^T \cdot A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

$$A^T \cdot b = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

Then the equation $A^T \cdot Ax = A^T \cdot b$ becomes

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$(A^T \cdot A)^{-1} = \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

and then solve $A^T \cdot A\hat{x} = A^T \cdot b$ as

$$\boxed{\hat{x} = (A^T \cdot A)^{-1} \cdot A^T \cdot b}$$ * IMP

$$= \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

least square error,

$$\| b - A \cdot \hat{x} \| \approx$$

Ex : Infinitely Many Least Square Solutions :

a) Find the least squares solutions, the least squares error vector, and the least squares error of the system

$$3x_1 + 2x_2 - x_3 = 2$$

$$x_1 - 4x_2 + 3x_3 = -2$$

$$x_1 + 10x_2 - 7x_3 = 1$$

Sol : The matrix form of the system is $Ax = b$ where

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{bmatrix} \text{ and } b = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

It follows that,

$$A^T \cdot A = \begin{bmatrix} 11 & 12 & -7 \\ 12 & 120 & -84 \\ -7 & -84 & 59 \end{bmatrix} \; \& \; A^T \cdot b = \begin{bmatrix} 5 \\ 22 \\ -15 \end{bmatrix}$$

$A^T \cdot A \cdot x = A^T \cdot b$ is,

$$\left[\begin{array}{ccc|c} 11 & 12 & -7 & 5 \\ 12 & 120 & -84 & 22 \\ -7 & -84 & 59 & -15 \end{array}\right]$$

The row reduced echelon form of this Matrix is,

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 1 & -5/7 & \frac{13}{84} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From which it follows that there are infinitely many least squares solution and that they are given by parametric equations.

$$x_1 = \frac{2}{7} - \frac{1}{7}t$$

$$x_2 = \frac{13}{84} + \frac{5}{7}t$$

$$x_3 = t$$

As a check, let us verify that all least squares solutions produce the same least squares error vector & same least sq. Error

$$b - A\cdot\hat{x} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{bmatrix}\begin{bmatrix} \frac{2}{7} - \frac{1}{7}t \\ \frac{13}{84} + 5/7\, t \\ t \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{7}{6} \\ -\frac{1}{3} \\ 11/6 \end{bmatrix} = \begin{bmatrix} 5/6 \\ -5/3 \\ -5/6 \end{bmatrix}$$

Since $b - Ax$ does not depend on t, all least square solutions produce the same error vector namely.

$$\|b - A\cdot\hat{x}\| = \sqrt{\left(\frac{5}{6}\right)^2 + \left(-\frac{5}{3}\right)^2 + \left(-\frac{5}{6}\right)^2} = \frac{5}{6}\sqrt{6}$$

IMP

<u>Theorem</u>: If A is $m\times n$ matrix, with L.I.D column vectors & if $A = Q\cdot R$ (is QR decompose of A), then for each b in R^m the system $Ax = b$ has <u>UNIQUE least SQ. Solution</u> given as,

$$\boxed{x = R^{-1}\cdot Q^{T}\cdot b} \; *.$$

*. HERMITIAN AND UNITARY MATRICES :

- If A is a complex matrix, then the conjugate transpose of A, denoted by A^*, is defined by,

$$\boxed{A^* = \bar{A}^T}$$

- A square matrix A is said to be unitary if,

$$A \cdot A^* = A^* \cdot A = I$$

or equivalently if, $\boxed{A^* = A^{-1}}$

and it is said to be Hermitian if, $\boxed{A^* = A}$

Ex : Unitary :

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} i \\ -\frac{1}{\sqrt{2}} i & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Hermitian :

$$A = \begin{bmatrix} 1 & i & 1+i \\ -i & -5 & 2-i \\ 1-i & 2+i & 3 \end{bmatrix}$$

MODULE-4 : EIGEN VALUES - EIGEN VECTORS

*. Eigen Value & Eigen Vector :

- To find the eigen values of $n \times n$ matrix A, we rewrite $Ax = \lambda x$ as $Ax = \lambda I x$ or equivalently,

$$\boxed{(\lambda I - A)x = 0}$$

- For λ to be eigen value, there must be a non zero solution of this equation. However, by theorem (TB) the above equation has a non zero solution if & only iff

$$\boxed{\det(\lambda I - A) = 0}$$

- This is called the characteristic equation of A; the scalar satisfying this equation are the eigenvalues of A. When expanded, the determinant $\det(\lambda I - A)$ is a polynomial p in λ called the "characteristic polynomial" of A.

Ex :

Find eigenvalues of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$

Sol:

The characteristic polynomial of A is,

$$\det(\lambda I - A) = \det\left[\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}\right]$$

$$= \det\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{bmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

- The eigen values of A must therefore satisfying the cubic equation,

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

*- Theorem :

If A is $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal) then the eigenvalues of A are entries on the main diagonal of A

Ex :

The eigen values of A = $\frac{1}{2}, \frac{2}{3}, \frac{-1}{4}$ for A

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{2}{3} & 0 \\ 5 & -8 & \frac{-1}{4} \end{bmatrix}$$ (lower triangular matrix)

*- Theorem :

- If A is an $n \times n$ matrix and λ is a real number, then the following are equivalent :

1. λ is an eigen value of A.
2. The system of equations $(\lambda I - A)x = 0$ has non-trivial solutions.
3. There is a nonzero vector x in R^n such that $Ax = \lambda x$.
4. λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.

• **Note :** The spectral radius $\rho(A)$ of a matrix A is the modulus of its dominant eigen value ; that is, $\rho(A) = \max\{|\lambda_i|\}$ $(i = 1, 2, 3, \dots n)$.

*· FINDING BASES FOR EIGENSPACES :

- The eigenvectors of A corresponding to an eigenvalue λ are the nonzero x that satisfy $Ax = \lambda x$.
- Equivalently, the eigenvectors corresponding to λ are the non-zero vectors in the solution space of $(\lambda I - A)x = 0$.
- We call this solution space the eigen space of A corresponding to λ.

Ex : Find the bases for the eigenspaces of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

Sol : The characteristic equation of matrix A is $\lambda^3 - 5\lambda^2 + 8\lambda + (-4) = 0$, or in factored form, $(\lambda-1)(\lambda-2)^2 = 0$. Thus eigen values of $A = (\lambda = 1, \lambda = 2)$. So there are two eigenspaces of A.

$$(\lambda I - A)x = 0 \Rightarrow \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $\lambda = 2$, then

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Solving this yield $x_1 = -s$, $x_2 = t$, $x_3 = s$
- Thus, the eigen vectors of A corresponding to $\lambda = 2$ are the non-zero vectors of the form

$$x = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- The vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent and form a basis for the eigenspace corresponding to $\lambda = 2$.

- Similarly, the eigen vectors of A corresponding to $\lambda = 1$ are the non-zero vectors of the form

$$x = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

$\therefore$ Thus, $[-2 \ 1 \ 1]^T$ is the basis for eigen space corresponding to $\lambda = 1$.

*. GEOMETRIC AND ALGEBRAIC MULTIPLICITY :

- If λ_0 is an eigenvalue of an $n \times n$ matrix A then the dimensions of the eigenspace corresponding to λ_0 is called GM of λ_0 and

- The no. of times $(\lambda - \lambda_0)$ appears as factor in the characteristic polynomial of A is called AM of A.

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*. Theorem :

- If K is +ve integer, λ is eigen value of a matrix A, and x is corresponding eigenvector then,

$\Rightarrow \lambda^K$ is eigen value of A^K, x is corresponding E.Vector

Ex: Given, $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. Apply above theorem ↑.

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*. Theorem :

- A square matrix A is invertible if and only if $\lambda = 0$ is not an eigen value of A.

V.IMP

- If A is $m \times n$ matrix and if $T_A : R^n \rightarrow R^n$ is multiplication by A, then the following are equivalent :

1. A is invertible.
2. $Ax = 0$ has only the trivial solution
3. The reduced row - echelon form of A is I_n.

4. A is expressible as a product of elementary matrices

5. $Ax = b$ is consistent with every $n \times 1$ matrix b.

6. $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.

7. $\det(A) \neq 0$.

8. The range of T_A is R^n.

9. T_A is one to one.

10. The column vectors of A are linearly independent.

11. The row vectors of A are linearly independent.

12. The column vectors of A span R^n.

13. The row vectors of A span R^n.

14. The column vectors of A form basis for R^n.

15. The row vectors of A form basis for R^n.

16. A has rank n.

17. A has nullity 0.

18. The orthogonal complement of the nullspace of A is R^n.

19. The orthogonal complement of the rowspace of A is $\{0\}$.

20. $A^T \cdot A$ is invertible.

21. $\lambda = 0$ is not eigen value of A.

*. SOME USEFUL PROPERTIES OF EIGENVALUES :

- Let A be an $n \times n$ matrix, with eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$ then :

1. The sum of eigen values of A = Trace (A).
2. The product of eigen values of A = det (A).
3. Eigen values of A^{-1} are $\lambda_1^{-1}, \lambda_2^{-1}, \ldots \lambda_n^{-1}$.
4. Eigen values of A^T are $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$
5. If K is scalar then eigen values of KA are $K\lambda_1, K\lambda_2, K\lambda_3, \ldots K\lambda_n$.
6. If K is scalar and I is $n \times n$ identity (unit) matrix respectively, then the eigen values of $A \pm KI$ are

   $\lambda_1 \pm K, \lambda_2 \pm K, \ldots \lambda_n \pm K$
7. If K is positive integer, then eigen values of A^K are $\lambda_1^K, \lambda_2^K, \ldots \lambda_n^K$.

*. The characteristic and Minimal Polynomial of a Matrix :

- Let A be an $n \times n$ matrix. We associate two polynomials to A.

1. The characteristic polynomial of A is defined as $f(x) = \det(x \cdot 1 - A)$, where x is the variable of the polynomial and 1 represents the identity matrix. $f(x)$ is a monic polynomial of degree n.

2- The minimal polynomial of A, which we will denote by $\mu(x)$, is defined by the following properties :

- $\mu(x)$ is monic (i.e., its leading coefficient is 1)
- $\mu(A) = 0$
- $\mu(x)$ is monic polynomial of the smallest possible degree such that $\mu(A) = 0$.

They also satisfy the following properties.

- If $g(x)$ is another polynomial, then $g(A) = 0$ if and only if $\mu(x)$ divides $g(x)$.
- $f(x)$ is multiple of $\mu(x)$.

Example : Find the eigen values & eigen vectors of Matrix

$$A = \begin{bmatrix} -2 & 1 & 1 \\ -11 & 4 & 5 \\ -1 & 1 & 0 \end{bmatrix}.$$

Sol : $|A - \lambda I| = 0 \Rightarrow \left| \begin{bmatrix} -2 & 1 & 1 \\ -11 & 4 & 5 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 1 & 1 \\ -11 & 4-\lambda & 5 \\ -1 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow (-2-\lambda)[-\lambda(4-\lambda)-5] + -1(11\lambda+5) + 1(-11+4-\lambda)$$

$\Rightarrow (d+2)\left[d(4-d)\right] + 5(d+2) - 11d - 5 - 11 + 4 - d = 0$

$\Rightarrow -d(d+2)(d-4) + 5d + 10 - 11d - 5 - 11 + 4 - d = 0$

$\Rightarrow -\cancel{d^2 2d} \; -d\left(d^2 + 2d - 4d - 8\right) + (-7d) - 2 = 0$

$\Rightarrow -d^3 + 2d^2 + 4d^2 + 8d - 7d - 2 = 0$

$\Rightarrow -d^3 + 2d^2 + d - 2 = 0$ (or) $\boxed{d^3 - 2d^2 - d + 2 = 0}$

is characteristic equation.

Solving characteristic equation results in Eigen values.

So, $d^3 - 2d^2 - d + 2 = 0$

$1 - 2 - 1 + 2$

$d=1$	1	-2	-1	2
	0	1	-1	-2
	1	-1	-2	0

$(d-1)$ is the factor

$\Rightarrow d_1 = 1$

$d^2 - d - 2 = 0 \Rightarrow d^2 - 2d + d - 2 = 0$

$\Rightarrow d(d-2) + 1(d-2) = 0 \Rightarrow d_2 = 2, \; d_3 = -1$

$\therefore$ Eigen values are : $\boxed{d_1 = 2}$, $\boxed{d_3 = -1}$, $\boxed{d_2 = 1}$

Now, for Eigen vectors,

- For $\underline{d_1 = 2} \Rightarrow$ $\begin{bmatrix} -4 & 1 & 1 \\ -11 & 2 & 5 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

It is null space of Matrix $\Rightarrow$ Solve to find $x_1 \; x_2 \; x_3$ which are eigen vector values of $d_1 = 2$

RREF of $\begin{bmatrix} -4 & 1 & 1 \\ -11 & 2 & 5 \\ -1 & 1 & -2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_3 = t \Rightarrow x_1 = t$, $x_2 = 3t$

Eigen vector for $\lambda_1 = 2$, $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 3t \\ t \end{bmatrix}$

- $\boxed{X_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} t \quad \text{for } \lambda_1 = 2}$

- For $\lambda_2 = 1 \Rightarrow A_2 - 1 \cdot I = \begin{bmatrix} -3 & 1 & 1 \\ -11 & 3 & 5 \\ -1 & 1 & -1 \end{bmatrix}$

Row reduced Echelon form of this matrix is

$$\begin{bmatrix} -3 & 1 & 1 \\ -11 & 3 & 5 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the null space of this matrix, we get Eigen vector for corresponding eigen value, $\lambda_2 = 1$.

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} t \Rightarrow \text{Eigen Vector}, \boxed{X_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ for } \lambda_2 = 1}$$

For $\lambda_3 = -1$ $\Rightarrow$ $A_2 + I = \begin{bmatrix} -1 & 1 & 1 \\ -11 & 5 & 5 \\ -1 & 1 & 1 \end{bmatrix}$

Row reduced Echelon form of this matrix is,

$$\begin{bmatrix} -1 & 1 & 1 \\ -11 & 5 & 5 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $X_3 = \begin{bmatrix} 0 \\ -t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t$ $\Rightarrow$ $\boxed{\text{Eigen vector } X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ for } \lambda_3 = -1}$

$\therefore$ Eigen values $= \{2, 1, -1\}$

Eigen vectors $= \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ respectively.

Ex: Find Eigen values, Eigen vectors & Multiplicities of this matrix, $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ Characteristic equation, $-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$

roots of characteristic equation are Eigen values $\rightarrow$ $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 4$

For every λ (eigen value) we need its corresponding eigen vector x.

- For $\lambda_1 = 1$:

$$A - \lambda_1 \cdot I = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Row reduced echelon form of this matrix is,

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Null space of this Matrix is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So, As $\lambda = 1$ is the repeating Eigen value.

$\Rightarrow$ We obtain two Eigen vectors by substituting 0 & 1 in free variables. like

$$\begin{matrix} t = 1 \\ s = 0 \end{matrix} \Rightarrow \boxed{X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \quad \text{And} \quad \begin{matrix} t = 0 \\ s = 1 \end{matrix} \Rightarrow \boxed{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = X_2}$$

are the two Eigen vectors for $\lambda_1 = 1$

- For $\lambda_3 = 4$:

$$A - \lambda_3 I = \begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix}$$

Row reduced Echoleon form of this matrix is,

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ +1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Null space of this matrix is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \boxed{X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}$$

$\therefore$ Eigen Values $= \{1, 1, 4\}$ and

Eigen vectors $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Algebraic Multiplicity = ~~Degree~~ Highest exponent of the eigen value λ in C-Eq

characteristic equation : $(\lambda-1)^2(\lambda-4) = 0$

For $\lambda = 1 \Rightarrow AM = 2$ $GM = 2$

For $\lambda = 4 \Rightarrow AM = 1$ $GM = 1$

IMP

- Geometric Mullplicity = No. of variables - Rank

$\Rightarrow \boxed{GM = n - R(A)}$

* MATRIX SIMILARITY :

- Two $n \times n$ matrices A and B are called similar if there exists an invertible $n \times n$ matrix P such that

$$\boxed{B = P^{-1} \cdot A \cdot P}$$

- A square Matrix A is said to be diagonalizable if A is similar to a diagonal Matrix, i.e. $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

* THE DIAGONALIZATION THEOREM :

- If $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigen vectors

- $A = PDP^{-1}$, with D a diagonal Matrix, if and only if the columns of P are n linearly independent eigen-vectors of A.

- D entries are eigen values diagonally placed to the eigen vectors in P correspondingly.

- Eigen Vector Basis : A is diagonalizable if and only if there are enough eigen vectors to form a basis of $\mathbb{R}^n$. It is called Eigen Vector Basis of $\mathbb{R}^n$.

* Algorithm to Diagonalize Matrix :

1. Find Eigen values of A.
2. Find linearly independent (n) eigen vectors of A.
3. Construct Matrix P from eigen vectors.

4. Construct Matrix D from Eigen values obtained.

5. Verify $AP = P \cdot D$ (equivalent to $A = PDP^{-1}$).

6. Then the given Matrix A is diagonalizable.

* THEOREM : (any)

- An $n \times n$ matrix with n distinct eigen values is Diagonalizable.
- An $n \times n$ matrix with n linearly independent Eigen vectors.
- Sum of Geometric multiplicities $= n$.
- If for every eigenvalue of $n \times n$ matrix, the Geometric Multiplicity equals the Algebraic multiplicity then A is said to be diagonalizable.

* DIAGONALIZATION OF SYMMETRIC MATRICES :

- Every symmetric matrix can be Diagonalized.
- $A = A^T \Rightarrow$ A is symmetric with some non singular eigen matrix P such that $\boxed{P^{-1} \cdot A \cdot P = D}$.
- Note : All roots of characteristic polynomial of a symmetric matrix are real numbers.
- Note : If A is symmetric matrix, then eigenvectors that belong to distinct eigenvalues of A are orthogonal.
- $A^{-1} = A^T$ or $A \cdot A^T = I \Rightarrow$ A is orthogonal.

Theorem :

- $\boxed{P^{-1}AP = P^{T} \cdot A \cdot P}$ $\Rightarrow$ orthogonal matrix A

- The $n \times n$ matrix A is orthogonal if & only if the columns (rows) of A form an orthonormal set.

*- Algorithm for Diagonalization of Symmetric Matrix :

1. Find a basis for each eigen space of A.
2. Apply the Gram - Schmidt process to each of these bases, to obtain an orthonormal basis for each eigenspaces.
3. Form the matrix P whose columns are the basis vectors constructed in step 2 ; this matrix orthogonally diagonalizes A.

Ex : Find an orthogonal matrix P that diagonalizes

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Sol : The characteristic equation of A is

$$\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{bmatrix} = (\lambda - 2)^2 (\lambda - 8)$$

$$(\lambda - 2)^2 (\lambda - 8) = 0 \Rightarrow \lambda = 2, \quad \lambda = 8$$

$\Rightarrow$ The basis of the eigenspace corresponding to $\lambda = 2$ is

$$u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Applying Gram - Schmidt process to $\{u_1, u_2\}$ yields

$$v_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

$\Rightarrow$ The basis of the eigenspace corresponding to $\lambda = 8$ is

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Applying Gram - Schmidt process to $\{u_3\}$ yields

$$v_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Thus, $P = [v_1 \ v_2 \ v_3] = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

orthogonally diagonalizes A.

*. CAYLEY - HAMILTON THEOREM :

- A square matrix A satisfies its own characteristic equation ; that is if $\lambda^n + C_{n-1}\lambda^{n-1} + C_{n-2}\lambda^{n-2} + \dots + C_1\lambda_1 + C_0 = 0$ is the characteristic equation of $n \times n$ matrix A then,

$$\boxed{A^n + C_{n-1}A^{n-1} + C_{n-2}A^{n-2} + \dots + C_1 A + C_0 I = 0}$$

Ex : Calculate e^{At} and $\sin(At)$ when $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \lambda_1 = \lambda_2 = 1$ (Eigen values)

$\boxed{e^{At} = \alpha_0 I + \alpha_1 A}$ with $e^t = \alpha_0 + \alpha_1$

$t \cdot e^t = \alpha_1$ leading to $e^{At} = \begin{bmatrix} e^t & -te^t \\ 0 & e^t \end{bmatrix}$

Similarly, $\sin At = \alpha_0 I + \alpha_1 A$ with

$\sin t = \alpha_0 + \alpha_1$, $t\cos t = \alpha_1$ leading to $\sin At = \begin{bmatrix} s & -tc \\ 0 & s \end{bmatrix}$

$$= \begin{bmatrix} \sin t & -t\cos t \\ 0 & \sin t \end{bmatrix}$$

---

*. Inverse using Cayley Hamilton Theorem,

$$\boxed{A^{-1} = \frac{-1}{C_0}\left(A^{n-1} + C_{n-1}A^{n-2} + \dots + C_1 I\right)}$$

* Theorem :

- If k is a positive integer, λ is an eigen value of a matrix A, and x is a corresponding eigen vector, then λ^k is an eigenvalue of A^k and x is a corresponding eigenvector.

* POSITIVE DEFINITE MATRICES :

- A symmetric Matrix A is positive definite if and only if all the eigenvalues of A are positive.

- S is positive definite if $x^T.S.x > 0$ for every nonzero vector x : 2×2 $x^T.S.x = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$= ax^2 + 2bxy + cy^2 > 0.$

- If S and T are symmetric positive definite, So is S + T.

- If columns of A are independent, then $S = A^T.A$ is positive definite.

- When a symmetric matrix S has one these 5 properties, it has them all :

1. All n pivots of S are positive.
2. All n upper left determinants are positive.
3. All n eigenvalues of S are positive.
4. $x^T.S.x$ is positive except at $x = 0$. This is the energy-based definition.
5. S equals $A^T.A$ for matrix A with independent columns.

* CONIC SECTIONS :

- A Quadratic Equation in the variables x and y has the form $ax^2 + 2bxy + cy^2 + dx + ey + f = 0$

is expressed as $[x \ \ y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + [d \ \ e]\begin{bmatrix} x \\ y \end{bmatrix} + f$

Ex : Identify & sketch the conic equation,

$$5x^2 - 6xy + 5y^2 - 24\sqrt{2}\,x + 8\sqrt{2}\,y + 56 = 0$$

Write the equation in standard form.

Sol : Rewriting the equation as,

$$[x \ \ y]\begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + [-24\sqrt{2} \ \ 8\sqrt{2}]\begin{bmatrix} x \\ y \end{bmatrix} + 56 = 0$$

We now find the eigen values of the matrix.

$A = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$. Then $|\lambda I_2 - A| = ?$

$$\Rightarrow = \begin{bmatrix} \lambda-5 & 3 \\ 3 & \lambda-5 \end{bmatrix} = (\lambda-5)(\lambda-5) - 9 = \lambda^2 - 10\lambda + 16$$

$= (\lambda-2)(\lambda-8)$. So, the Eigenvalues of A are,

$\lambda_1 = 2, \quad \lambda_2 = 8.$

Associated eigenvectors are obtained by solving the homogeneous system

$$(\lambda I_2 - A)X = 0$$

$\Rightarrow$ Thus for $\lambda_1 = 2$, we have

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} X = 0$$. So, the eigen vector of A associated with $\lambda_1 = 2$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\Rightarrow$ For $\lambda_2 = 8$, we have

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} X = 0$$. So, the eigen value vector of A associated with $\lambda_2 = 8$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Normalizing the eigen vectors, we obtain the orthogonal matrix.

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$ Then

$$P^T . A . P = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}.$$

Letting $x = P.y$ we write the transformed equation for the given conic section as

$$2x'^2 + 8y'^2 - 16x' + 32y' + 56 = 0$$

(or) $$x'^2 + 4y'^2 - 8x' + 16y' + 28 = 0$$

Translating the axes,

$$(x' - 4)^2 + 4(y' + 2)^2 + 28 = 16 + 16$$

$$(x'-4)^2 + 4(y'+2)^2 = 4$$

$$\frac{(x'-4)^2}{4} + \frac{(y'+2)^2}{1} = 1$$

Put $x'' = x'-4$, $y'' = y'+2$

$$\Rightarrow \boxed{\frac{(x'')^2}{4} + \frac{(y'')^2}{1} = 1}$$ Ellipse equation with $(4,-2)$ origin.

Note : Eigen values $\lambda_1 \lambda_2 > 0$: Ellipse
$\lambda_1 \lambda_2 < 0$: Hyperbola
$\lambda_1 \lambda_2 = 0$: Parabola

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*. If A is a Complex matrix, then the conjugate transpose of A, denoted by A^* is defined by $\boxed{A^* = \bar{A}^T}$.

*. A square Matrix A is said to be unitary if $A \cdot A^* = A^* \cdot A = I$ or equivalently, if $A^* = A^{-1}$ and is said to be Hermitian if $A^* = A$

*. A square complex matrix A is said to be unitarily diagonalizable if there is a unitary matrix P such that $\boxed{P^* A P = D}$ is a complex diagonal matrix. Any such matrix P is said to unitarily diagonalize A.

*. UNITARILY DIAGONALIZING HERMITIAN MATRIX :

Step 1 : Find a basis for each eigen space of A.

Step 2 : Apply the Gram-Schmidt process to each of these bases to obtain orthonormal bases for the eign spaces.

Step 3 : Form the Matrix P whose column vectors are the basis vectors obtained in step 2. This will be unitary matrix and will unitarily diagonalize A.

*. NORMAL MATRICES :

- A square complex matrix A is unitarily diagonalizable if and only if $\boxed{A \cdot A^* = A^* \cdot A}$. Matrices with this property are "normal".

*. SPECTRAL DECOMPOSITION :

Ex : Find a spectral decomposition of the matrix A

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Steps : ① Find an orthogonal diagonalization.

② $A = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$.

Sol: $A = Q D Q^{-1}$, $Q^{-1} = Q^T$

Q = orthonormal Matrix obtained for eigen vector Matrix P

Eigen values of $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ are

$\lambda_1 = 4$, $\lambda_2 = 1$, $\lambda_3 = 1$

So, Diagonal Matrix, $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and its

Corresponding Eigen Matrix $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ where the individual columns are Eigen vectors of A.

So, orthogonal Matrix,

$$Q = \begin{array}{c} \begin{array}{ccc} Q_1 & Q_2 & Q_3 \end{array} \\ \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{bmatrix} \end{array}$$

Now, $A = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$

where $P_i = (q_i \cdot q_i^T)$ $\rightarrow$ q_i is columns of Q Matrix

$$P_1 = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}}_{Q_1} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{Q_1^T} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$P_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$Q_2 \qquad Q_2$

$$P_3 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ 2/\sqrt{6} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

spectral decomposition :

$$A = 4 \cdot P_1 + 1 \cdot P_2 + 1 \cdot P_3$$

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* **Theorem** :

- A and $A^T.A$ have the same null space.
- A and $A^T.A$ have the same row space.
- A^T. and $A^T.A$ have the same column space.
- A and $A^T.A$ have same rank.
- A^TA is orthogonally Diagonalizable.
- The eigen values of A^TA are non negative.

Note : If A is $m \times n$ Matrix, and if $d_1, d_2, \ldots d_n$ are the eigenvalues of $A^T.A$ then the numbers

$\sigma_1 = \sqrt{d_1}$, $\sigma_2 = \sqrt{d_2}$, ... $\sigma_n = \sqrt{d_n}$ are called the singular values of A.

Ex : Find the singular values of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Sol : Given $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$A^T \cdot A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic polynomial is, $\begin{vmatrix} 2-d & 1 \\ 1 & 2-d \end{vmatrix} = 0$

$\Rightarrow d^2 - 4d + 3 = 0 \Rightarrow (d-3)(d-1) = 0$

So, the eigen values of $A^T \cdot A$ are $d_1 = 3$, $d_2 = 1$

$\Rightarrow$ singular values $\sigma_1 = \sqrt{d_1} = \sqrt{3}$ of A.

$\sigma_2 = \sqrt{d_2} = \sqrt{1} = 1$

$\therefore$ singular values of $A = \{\sqrt{3}, 1\}$.

---

*. SINGULAR VALUE DECOMPOSITION :

$$\boxed{A = U \Sigma V^T}$$ where

$$U = \left[\frac{1}{\sigma_1} AV_1 \quad \frac{1}{\sigma_2} AV_2 \quad \frac{\overset{\text{Null space}}{NS(A^T)}}{|NS(A^T)|} \right] \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

$$V^T = \left[\text{eigenvector Matrix } (A^T \cdot A) \right]^T.$$

Ex : Find the singular value Decomposition of the Matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

Sol: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$$A^T \cdot A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Eigen values of $A^T \cdot A \Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda_2 = 2, \ \lambda_1 = 3$

So, Eigen vectors of A are :

For $\lambda_1 = 3$:

$$\begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} = \begin{bmatrix} 2-3 & 0 \\ 0 & 3-3 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow$ Eigen vector for $\lambda_1 = 3$ = $\boxed{X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

For $\lambda_2 = 2$:

$$\begin{bmatrix} 2-2 & 0 \\ 0 & 3-2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$ Eigen vectors for $\lambda_2 = 2$ = $\boxed{X_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

So, $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\sigma_1 = \sqrt{d_1} = \sqrt{3}$

$\sigma_2 = \sqrt{d_2} = \sqrt{2}$

$$U = \left[\frac{1}{\sigma_1} A V_1 \quad \frac{1}{\sigma_2} A V_2 \quad \frac{\text{NullSpace}(A^T)}{|\text{NullSpace}(A^T)|} \right]$$

$$= \left[\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \frac{NS(AT)}{|NS(AT)|} \right]$$

$$NS(A^T) \Rightarrow A^T \cdot x = 0 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$\Rightarrow x = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, let $x_3 = 1$

$\Rightarrow u_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\quad |u_3| = \sqrt{1+4+1} = \sqrt{6}$

$$U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & 1/\sqrt{6} \end{bmatrix},$$

$$\boxed{A = U \Sigma V^T}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Ex : Given, $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$. Find the singular value decomposition of Given matrix A.

Sol : Step 1 : Find an orthogonal diagonalization of $A^T.A$

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}$$

$$A^T.A = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}_{3\times 2} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}_{3\times 3}$$

$$\begin{vmatrix} 80-\lambda & 100 & 40 \\ 100 & 170-\lambda & 140 \\ 40 & 140 & 200-\lambda \end{vmatrix} = 0 \Rightarrow -\lambda^3 + 450\lambda^2 - 32400\lambda = 0$$

$\Rightarrow \lambda_1 = 0, \ \lambda_2 = 90$

$\lambda_3 = 360$

Eigen vectors for $\lambda_1 = 0$: $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$\lambda_2 = 90$: $\begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix}$

$\lambda_3 = 360$: $\begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$

$$V = \begin{bmatrix} 2 & -1 & 1/2 \\ -2 & -1/2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$V = [v_1 \quad v_2 \quad v_3] = \begin{array}{ccc} d_1=0 & d_2=90 & d_3=360 \\ \end{array}$$
$$\begin{bmatrix} 2 & -1 & 1/2 \\ -2 & 1/2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Step 2 : Setup V^T & Σ Matrices

$$V^T = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 1/2 & 1 \\ 1/2 & 1 & 1 \end{bmatrix}$$

Descending order only

$$\sigma_3 = \sqrt{d_1} = 0$$
$$\sigma_2 = \sqrt{90} = \sqrt{90} = 3\sqrt{10}$$
$$\sigma_1 = \sqrt{360} = 6\sqrt{10}$$

Σ = Same as size of A $\Rightarrow$ $\Sigma_{2\times3}$

* Write eigen vector in ~~decreasing~~ (descending) order of eigen values

So, $V = [v_1 \quad v_2 \quad v_3] =$

$$\begin{array}{ccc} d_1=360 & d_2=90 & d_3=0 \end{array}$$
$$\begin{bmatrix} \frac{1}{2} & -1 & 2 \\ 1 & 1/2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$V^T = \begin{bmatrix} \frac{1}{2} & 1 & 1 \\ -1 & 1/2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

So, $\Sigma = [D \quad 0]$

$$\Sigma = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

$$\boxed{u_i = \frac{1}{\sigma_i} A v_i}$$

$$u_1 = \frac{1}{\sigma_1} A \cdot V_1 = \frac{1}{6\sqrt{10}} \begin{bmatrix} 18 \\ 6 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A \cdot V_2 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$$

So, singular value decomposition $A =$

$$A_{2\times3} = \underbrace{\begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}}_{U_{2\times2}} \underbrace{\begin{bmatrix} 6/\sqrt{10} & 0 & 0 \\ 0 & 3/\sqrt{10} & 0 \end{bmatrix}}_{\Sigma_{2\times3}} \underbrace{\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}}_{V^T_{3\times3}}$$

*. POLAR DECOMPOSITION :

- Every real square matrix can be factored into $\boxed{A = Q \cdot S}$ where Q is orthogonal and S is symmetric positive definite.

Ex : Polar Decomposition :

$$A = Q \cdot S \Rightarrow \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

- Reverse Polar Decomposition :

$$A = S^{1} \cdot Q \Rightarrow \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\boxed{Q = U V^T}$ & $\boxed{S = V \cdot \Sigma V^T}$

---

*. PSEUDOINVERSE :

Ex : Given $A = (1 \;\; 2)$. what is pseudoinverse of A

Sol : Using concept of SVD, $\boxed{A = U \cdot \Sigma \cdot V^T}$

$$A_{1\times 2} = U_{1\times 1} \, \Sigma_{1\times 2} \, V^T_{2\times 2}$$

$$A = [1 \;\; 2] \Rightarrow A^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} [1 \;\; 2] = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Eigen values of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \left| \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} \right| = 0$

$(1-\lambda)(4-\lambda) - 4 = 0 \Rightarrow \lambda = 5$ or $\lambda = 0$

Eigen vectors of $\lambda = 5$: $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Eigen vector of $\lambda = 0$: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$V = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ $\quad V^T = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\Sigma = \begin{bmatrix} \sqrt{5} & 0 \end{bmatrix}$

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{5}$

$\sigma_2 = \sqrt{\lambda_2} = \sqrt{0} = 0$

$$A = \underbrace{[1]}_{U} \underbrace{\begin{bmatrix} \sqrt{5} & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}^T}_{V}$$

$$\boxed{A^{+} \text{ (pseudoinverse)} = V \, \Sigma^{-1} . U^T}$$

$\Sigma^{-1} = \begin{bmatrix} \sqrt{5} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$ (not inverse its pseudoinverse)

$$= \underbrace{\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}}_{V} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}}_{\Sigma^{-1}} [1] = \underline{\underline{\frac{1}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}}$$

MODULE 5 : LINEAR TRANSFORMATION

*. Definition :

- If $T : V \to W$ is a function from a vector space V into a vector space W, then T is called a linear transformation from V to W if for all vectors u & v in V and all scalars c

$$T(u+v) = T(u) + T(v)$$
$$T(c \cdot u) = c \cdot T(u)$$

- In the special case where $V = W$, the linear transformati $T : V \to V$ is called a <u>linear operator</u> on V.

- Example (zero Transformation)
  → Mapping $T : V \to W$ such that $T(v) = 0$ for every v in V is a linear transformation called the zero transformation.

- Example (Identity Operator)
  → The Mapping $I : V \to I$ defined by $I(v) = v$ is called the identity operator on V.

*. ORTHOGONAL PROJECTION :

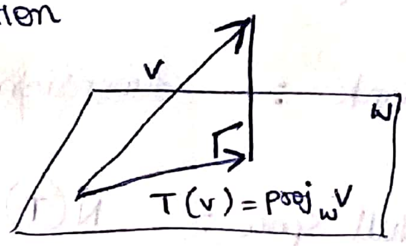
- Suppose that W is a finite - dimensional subspace of an innerproduct space V ; then the orthogonal projection of V onto W is the transformation defined by :

$$\boxed{T(v) = \text{proj}_W V}$$

If $S = \{w_1, w_2, \ldots w_r\}$ is any orthogonal basis for W, then $T(v)$ is given by formula

$$T(v) = \text{proj}_W V = \langle v, w_1 \rangle w_1 + \langle v, w_2 \rangle w_2 + \cdots + \langle v, w_r \rangle w_r$$

- This projection a linear Transformation



1. $T(u+v) = T(u) + T(v)$
2. $T(c \cdot u) = c \cdot T(u)$

Ex: A Linear Transformation from P_n to P_{n+1}

- Let $p = p(x) = c_0 + c_1 x + \cdots + c_n x^n$ be a polynomial in P_n, and define the function $T : P_n \to P_{n+1}$ by

$$T(p) = T(p(x)) = x \cdot p(x) = c_0 x + c_1 x^2 + \cdots + c_n x^{n+1}$$

- The function T is a linear Transformation :

→ For any scalar k and any polynomials p_1 and p_2 in P_n we have :

- $T(p_1 + p_2) = T(p_1(x) + p_2(x)) = x(p_1(x) + p_2(x))$

$$= x\, p_1(x) + x \cdot p_2(x) = T(p_1) + T(p_2)$$

- $T(k \cdot p) = T(k \cdot p(x)) = x(k \cdot p(x)) = k(x\, p(x))$

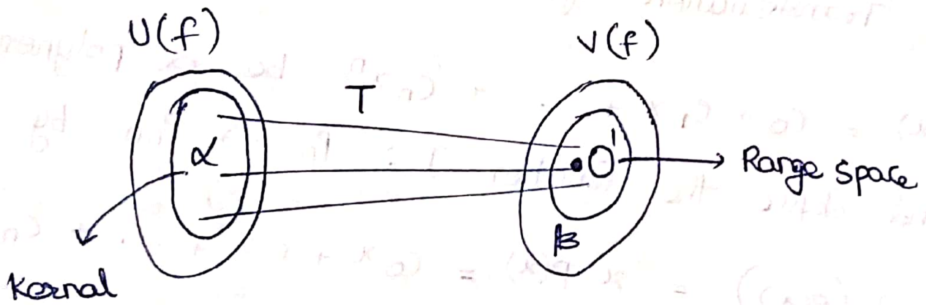
$$= k \cdot T(p)$$

- $f(a\alpha + b\beta) = a \cdot f(\alpha) + b \cdot f(\beta)$, $\forall\ a, b \in F$

$\alpha, \beta \in U$. V is called Homomorphic image of U.

*- Important Terms :-

- Range : $T(\alpha) = \beta$ $(R(T))$
- Rank : dimension of Range space of T $(\rho(T))$ denoted as
- Null space : $N(T) = \{\alpha \in U : T(\alpha) = 0 \in V\} \doteq Ker(T)$
- Nullity : dimension of Null space of T $(n(T))$ denoted as



- Kernal : Null space of T is also called Kernal of T.

$$\rho(T) = \text{rank of } T = \dim(R(T))$$

$$n(T) = \text{nullity of } T = \dim(N(T))$$

*- Standard Transformations :

- Reflection with respect to the x-axis :

$L : R^2 \to R^2$ is defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix}$$

- Projection into the xy-plane :

$L: R^3 \rightarrow R^2$ is defined by

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

- Dilation : $L: R^3 \rightarrow R^3$ is defined by $\boxed{L(u) = r \cdot u}$ $\forall\ r > 1$.

- Contraction : $L: R^3 \rightarrow R^3$ is defined by $\boxed{L(u) = r \cdot u}$ $0 < r < 1$

- Rotation counter clockwise through angle ϕ :

$L: R^2 \rightarrow R^2$ is defined by

$$L(u) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \cdot u$$

- linear transformations preserve linear combinations.

*· Theorem

- If $T: V \rightarrow W$ is a linear transformation, then :

→ $T(0) = 0$.

→ $T(-v) = -T(v)$ $\forall$ v in V.

→ $T(v-w) = T(v) - T(w)$ $\forall$ v & w in V.

- A linear transformation is completely determined by its images of any basis vectors.

Ex : Let $L : R_4 \to R_2$ be a linear transformation and let $S = \{v_1, v_2, v_3, v_4\}$ be a basis for R_4, where

$v_1 = [1 \;\; 0 \;\; 1 \;\; 0]$, $v_2 = [0 \;\; 1 \;\; -1 \;\; 2]$, $v_3 = [0 \;\; 2 \;\; 2 \;\; 1]$

and $v_4 = [1 \;\; 0 \;\; 0 \;\; 1]$. Suppose that

$L(v_1) = [1 \;\; 2]$, $L(v_2) = [0 \;\; 3]$

$L(v_3) = [0 \;\; 0]$, $L(v_4) = [2 \;\; 0]$

Let $v = [3 \;\; -5 \;\; -5 \;\; 0]$. Find $L(v)$.

Sol :

$$v = [3 \;\; -5 \;\; -5 \;\; 0] = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4$$

$$= c_1 [1 \;\; 0 \;\; 1 \;\; 0] + c_2 [0 \;\; 1 \;\; -1 \;\; 2] + c_3 [0 \;\; 2 \;\; 2 \;\; 1]$$
$$+ c_4 [1 \;\; 0 \;\; 0 \;\; 1]$$

$$v = [c_1 + c_4 \quad c_2 + 2c_3 \quad c_3 - c_2 + 2c_3 \quad 2c_2 + c_3 + c_4]$$

$$[3 \;\; -5 \;\; -5 \;\; 0] = [c_1 + c_4 \quad c_2 + 2c_3 \quad c_1 - c_2 + 2c_3 \quad 2c_2 + c_3 + c_4]$$

$$\Rightarrow \left.\begin{aligned} c_1 + c_4 &= 3 \\ c_2 + 2c_3 &= -5 \\ c_1 - c_2 + 2c_3 &= -5 \\ 2c_2 + c_3 + c_4 &= 0 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 2 \\ c_2 &= 1 \\ c_3 &= -3 \\ c_4 &= 1 \end{aligned}$$

So, $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 2v_1 + v_2 - 3v_3 + v_4$

$L(v) = L(2v_1 + v_2 - 3v_3 + v_4)$

$\Rightarrow \; = 2L(v_1) + L(v_2) - 3L(v_3) + L(v_4)$

$= 2[1 \;\; 2] + [0 \;\; 3] - 3[0 \;\; 0] + [2 \;\; 0]$

$= [2 \;\; 4] + [0 \;\; 3] - [0 \;\; 0] + [2 \;\; 0]$

$\Rightarrow [4 \;\; 7] = L(v)$

---

*. Let $L: P_2 \to P_3$ be a linear transformation for which we know that $L(1) = 1$, $L(t) = t^2$, $L(t^2) = t^3 + t$. Find $L(2t^2 - 5t + 3)$

Sol: Basis of P_2: $\{1, t, t^2\}$ — $e_1 \;\; e_2 \;\; e_3$

$L(e_1) = 1$

$L(e_2) = t^2$

$L(e_3) = t^3 + t$

$2t^2 - 5t + 3 = 2e_3 - 5e_2 + 3e_1$

Apply L operator both sides,

$L(2t^2 - 5t + 3) = L(2e_3 - 5e_2 + 3e_1)$

$= 2L(e_3) - 5L(e_2) + 3L(e_1) = 2(t^3 + t) - 5t^2 + 3(1)$

$= 2t^3 + 2t - 5t^2 + 3 = 2t^3 - 5t^2 + 2t + 3$

$\therefore L(2t^2 - 5t + 3) = 2t^3 - 5t^2 + 2t + 3$

So, $L(at^2 + bt + c) = at^3 + bt^2 + at + c$

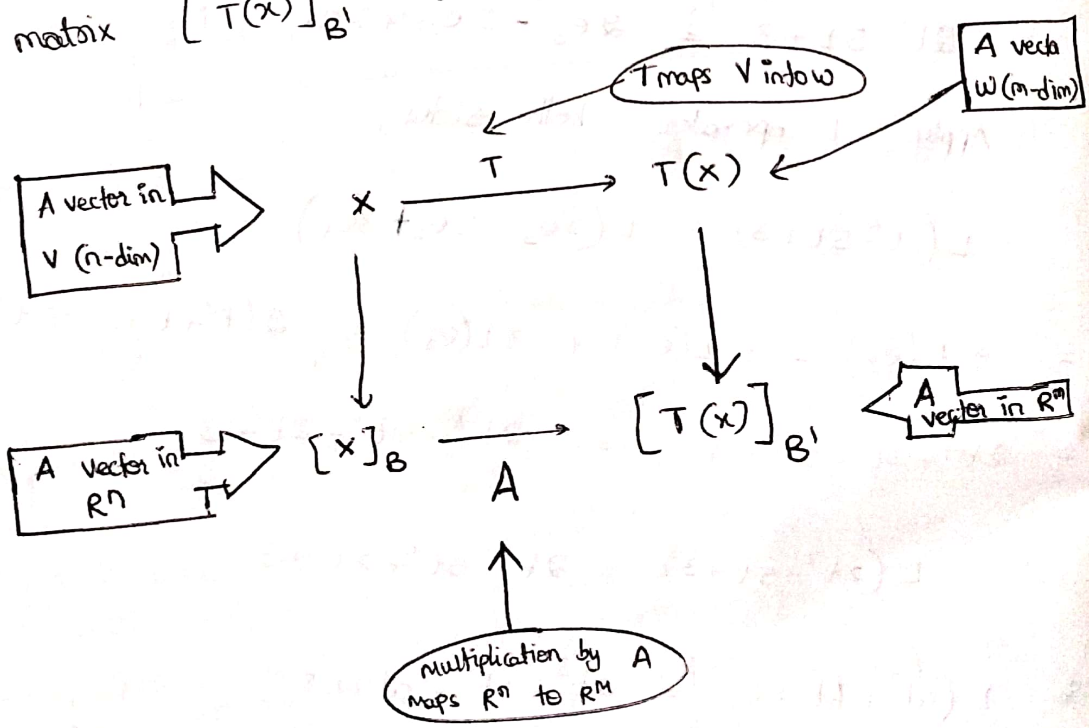
*. MATRICES OF GENERAL LINEAR TRANSFORMATIONS

- If V and W are finite-dimensional vector spaces (not necessarily R^n and R^m), then any transformation $T: V \rightarrow W$ can be regarded as matrix transformation.

- The basic idea is to work with coordinate matrices of the vectors rather than with the vectors themselves.

*. MATRICES OF LINEAR TRANSFORMATIONS :

- suppose V and W are n and m dimensional vector space and B and B' are bases for V and W; then for x in V, the coordinate matrix $[x]_B$ will be a vector in R^n, and coordinate matrix $[T(x)]_{B'}$ will be vector in R^m.



- If we let A be the standard matrix for this transformation, then $\boxed{A[x]_B = [T(x)]_{B'}}$

- The matrix A is called the matrix for T with respect to the bases B and B'

- Let $B = \{u_1, u_2, u_3, \ldots u_n\}$ (basis for V)

  Let $B' = \{u_1, u_2, u_3, \ldots u_m\}$ (basis for W

- consider an $m \times n$ matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

such that, $A[x]_B = [T(x)]_{B'}$ holds for all vectors x

i.e. $A[x]_B = [T(x)]_{B'}$ has to hold for the basis vectors $u_1, \ldots, u_n$

Thus we need,

$A[u_1]_B = [T(u_1)]_{B'}$, $A[u_2]_B = [T(u_2)]_{B'}$

$A[u_n]_B = [T(u_n)]_{B''}$. since:

$[u_1]_B = e_1$, $[u_2]_B = e_2$, ... $[u_n]_B = e_n$

We have,

$$\Rightarrow [T(u_1)]_{B'} = A[u_1]_B = Ae_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}_{m\times 1}$$

$\vdots$

$$\Rightarrow [T(u_n)]_{B'} = A[u_n]_B = Ae_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}_{m\times 1}$$

$\Rightarrow$ Thus, $\boxed{A = \left[[T(u_1)]_{B'} \ [T(u_2)]_{B'} \ \dots \ [T(u_n)]_{B'} \right]}$ which

is the matrix for T w.r.t the bases B and B'

and denoted by the symbol $[T]_{B',B}$

i.e. $\boxed{[T]_{B',B} = \left[[T(u_1)]_{B'} \ [T(u_2)]_{B'} \ \dots \ [T(u_n)]_{B'} \right]}$ *

And, $\boxed{[T]_{B',B} [x]_B = [T(x)]_{B'}}$ *

Basis for image space

Basis for domain

If $V = W$ (special case) resulting matrix is called the matrix for T w.r.t the basis B and denoted by $[T]_B$ rather than $[T]_{B,B}$.

* $\boxed{[T]_B [x]_B = [T(x)]_B}$

Ex: Let $T: P_1 \to P_2$ be the transformations defined by $T(p(x)) = x \cdot p(x)$. Find the matrix for T with respect to the standard bases,

$B = \{u_1, u_2\}$ and $B' = \{v_1, v_2, v_3\}$,

where $u_1 = 1$, $u_2 = x$; $\underbrace{v_1 = 1, v_2 = x, v_3 = x^2}$

Sol: Given, $T(p(x)) = x \cdot p(x)$

$T(u_1) = T(1) = (x)\cdot(1) = x \longrightarrow$ because $0\cdot 1 + 1\cdot x + 0\cdot x^2$

$T(u_2) = T(x) = (x)\cdot(x) = x^2 \longrightarrow$ because $0\cdot 1 + 0\cdot x + 1 x^2$

$[T(u_1)]_{B'} = [0 \ 1 \ 0]^T \sim \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$[T(u_2)]_{B'} = [0 \ 0 \ 1]^T \sim \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So, the Matrix, $T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}_{3\times 2}$

$\dim(P_1) = 2$

$\dim(P_2) = 3$

$\Downarrow$

result will be 3×2 matrix

Ex: Let $T: R^2 \to R^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix}$$

Find the matrix for the transformation T w.r.t bases $B = \{u_1, u_2\}$ for R^2 & $B' = \{v_1, v_2, v_3\}$ for R^3

where, $u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$v_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Sol:

$$T(u_1) = T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix}$$

$$T(u_1) = T\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$$

$$T(u_2) = T\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ +1 \\ -3 \end{bmatrix}$$

& $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = T(u_1)$$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$$

$$\left.\begin{aligned} c_1 - c_2 + c_3 &= 1 \\ 0 + 2c_2 + c_3 &= -2 \\ -c_1 + 2c_2 + 2c_3 &= 5 \end{aligned}\right\}$$

Solving

$c_1 = 1$ $\quad c_3 = -2$

$c_2 = 2$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = T(u_2)$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

$$\left.\begin{aligned} c_1 - c_2 + 0c_3 &= 2 \\ 0 + 2c_2 + c_3 &= 1 \\ -c_1 + 2c_2 + 2c_3 &= -3 \end{aligned}\right\}$$

solving,

$c_1 = 3$ $\quad c_3 = -1$

$c_2 = 1$

c_1, c_2, c_3 Values

So, $T(u_1) = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} = v_1 - 2v_3 \Rightarrow [T(u_1)]_{B'} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

$T(u_2) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = 3v_1 + v_2 - v_3 \Rightarrow [T(u_2)]_{B'} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

$$\Rightarrow \quad [T]_{B',B} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

---

*. **Theorems :**

- If $T : R^n \to R^m$ is linear transformation & if B and B' are the standard Bases for R^n & R^m respectively, then $\boxed{[T]_{B',B} = [T]}$ *

- If $T_1 : U \to V$ & $T_2 : V \to W$ are linear transformations & if B, B', B'' are bases for U, V, W respectively

$$\boxed{[T_2 \, T_1]_{B,B''} = [T_2]_{B'',B'} \, [T_1]_{B',B}} \quad *$$

- $[T^{-1}]_B = \left[[T]_B\right]^{-1}$ $\longrightarrow$ T is one to one $\longrightarrow$ $[T]_B$ is invertible

* Indirect Computation of Linear Transformation :

$$\begin{tikzcd} x \arrow[r, "\text{Direct}"] \arrow[d, "(1)"'] & T(x) \\ {[x]_B} \arrow[r, "\text{Multiply by}", "{[T]_{B',B}}"'] & {[T(x)]_{B'}} \arrow[u, "(3)"'] \end{tikzcd}$$

(2)

FAT

Ex : Let $T: R^2 \to R^3$ be the Linear transformation defined by $T(x) = Ax$. Find the transformation matrix B relative to the basis $U = \{u_1, u_2\}$ and $U' = \{u_1', u_2', u_3'\}$, where

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ -5 & -3 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u_1' = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad u_2' = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}, \quad u_3' = \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix}$$

$$T(u_1) = A \cdot u_1 = \underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ -2 & 1 \\ -5 & -3 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \begin{bmatrix} 5 \\ 0 \\ -11 \end{bmatrix}$$

$$T(u_2) = A \cdot u_2 = \underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ -2 & 1 \\ -5 & -3 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 3 \\ -1 \\ -8 \end{bmatrix}$$

$$c_1 u_1' + c_2 u_2' + c_3 u_3' = T(u_1)$$

$$c_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -11 \end{bmatrix}$$

$$\left.\begin{aligned} c_1 + 2c_2 + 3c_3 &= 5 \\ 3c_1 + 4c_2 + 5c_3 &= 0 \\ c_1 + 4c_2 + 5c_3 &= -11 \end{aligned}\right\}$$

Solving,

$c_1 = 11/2$ $\quad$ $c_3 = 31/2$

$c_2 = -47/2$

And also,

$$k_1 u_1' + k_2 u_2' + k_3 u_3' = T(u_2)$$

$$k_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} + k_3 \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -8 \end{bmatrix}$$

$$\left.\begin{aligned} k_1 + 2k_2 + 3k_3 &= 3 \\ 3k_1 + 4k_2 + 5k_3 &= -1 \\ k_1 + 4k_2 + 5k_3 &= -8 \end{aligned}\right\}$$

Solving,

$k_1 = 7/2$ $\quad$ $k_3 = 21/2$

$k_2 = -16$

So, $[T(u_1)]_{B'} = \begin{bmatrix} \frac{11}{2} \\ -\frac{47}{2} \\ \frac{31}{2} \end{bmatrix}$, $[T(u_2)]_{B'} = \begin{bmatrix} 7/2 \\ -16 \\ 21/2 \end{bmatrix}$

$$\therefore \quad [T]_{B',B} = \begin{bmatrix} \frac{11}{2} & \frac{7}{2} \\ \frac{-47}{2} & -16 \\ \frac{31}{2} & \frac{21}{2} \end{bmatrix}$$

IMP

* : Kernal of linear tranformation, T :

$\boxed{T_{B',B} \cdot X = 0}$, here $T_{B',B}$ is obtained from previous process

belong to $Ker(T)$ = Null space of T

IMP

* Range of linear transformation, T :

span $\{ c_1, c_2 \}$ of $T_{B',B}$ = $R(T)$

IMP

* dim (Kernel) = Nullity (T) = dim of Null space of T

dim (Range of T) = Rank(T) = dim of col-space of T.

IMP

* $\boxed{\text{Rank}(T) + \text{Nullity}(T) = \dim(\text{col of } T) = n}$

n → no. of columns of $T_{B',B}$

* If V is finite-dimensional vector space & $T: V \to V$ is a linear operator, then the following are equivalent :

- T is one to one
- $Ker(T) = 0$
- $Nullity(T) = 0$
- The range of T is V ; $R(T) = V$

* Inverse Linear Transformation :

$$T^{-1}(T(v)) = T^{-1}(w) = v$$

$$T^{-1}(T(w)) = T^{-1}(v) = w$$

Ex: Let $T: R^3 \to R^3$ be the linear operator defined by the formula $T(x_1, x_2, x_3) = (3x_1 + x_2,\ -2x_1 - 4x_2 + 3x_3,\ 5x_1 + 4x_2 - 2x_3)$.

Determine whether T is one to one; if so, find $T^{-1}(x_1, x_2, x_3)$.

Sol:

coefficients of given $T(x_1, x_2, x_3)$

$$[T] = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

$$[T]^{-1} = \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix}$$

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = [T^{-1}]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 4x_1 - 2x_2 + 3x_3 \\ -11x_1 + 6x_2 + 9x_3 \\ -12x_1 + 7x_2 + 10x_3 \end{bmatrix}.$$

$$\therefore\ T^{-1}(x_1, x_2, x_3) = [4x_1 - 2x_2 + 3x_3,\ -11x_1 + 6x_2 + 9x_3,\ -12x_1 + 7x_2 + 10x_3]$$

- $(T_1 \wedge T_2)^{-1} = (T_1)^{-1} \wedge (T_2)^{-1}$
- $(T_2 \wedge T_1)$ is one to one

} for T_1, T_2 are one to one Linear Transf.

* $T : V \to V$ (linear operator)

Basis $\to B$ $\quad B$

$x \in V$ (LHS), $A(x) \in V$ (RHS)

$T(x) = A x$

Task to be done: we try to search a basis B of V such that matrix A is diagonal. So that our calculation complexity becomes easy in terms of finding image, rank, Kernal, range, nullity, etc ...

* $\vec{v} \in V$ → Basis B → Coordinate vector $[v]_B$
  → Basis B' → Coordinate vector $[v]_{B'}$

change of Basis

Search of conversion matrix to get this conversion done

Such of type matrix is called "Transition Matrix"

* CHANGE OF BASIS :

- Let $S = \{v_1, v_2, \dots v_n\}$ and $T = \{w_1, w_2, \dots w_n\}$ be two ordered bases for the n-dimensional vector spaces V. If v is any vector V, then :

$$\Rightarrow v = c_1 w_1 + c_2 w_2 + \dots + c_n w_n$$

$$[v]_T = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Coordinate vector of $v \in V$ in basis, T

then $[w_j]_S = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$

- Let $[w_j]_S$ be coordinate vector of w_j w.r.t S.

- $[v]_S = c_1 [w_1]_S + c_2 \cdot [w_2]_S + c_3 [w_3]_S + \cdots + c_n [w_n]_S$

- The $n \times n$ matrix whose jth column is $[w_j]_S$ is called the "Transition Matrix" from the T-basis to S-basis & is denoted by $P_{S \leftarrow T}$.

$$\boxed{[v]_S = P_{S \leftarrow T} [v]_T} \quad *$$

(or) $$[v]_{B'} = P_{B' \leftarrow B} [v]_B$$ ✓ IMP

Ex: Let V be R^3 and let $S = \{v_1, v_2, v_3\}$ and $T = \{w_1, w_2, w_3\}$ be ordered bases for R^3, where

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and $$w_1 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$$

a. Compute the transition matrix $P_{S \leftarrow T}$ from the T-basis to the S-basis.

b. Verify Equation (3) for $v = \begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix}$

Text Book

Sol: To find $P_{S \leftarrow T}$, we need to find a_1, a_2, a_3 such that:

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = w_1$$

$$[w_1]_S = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Similarly, $b_1 V_1 + b_2 V_2 + b_3 V_3 = W_2$

$$[W_2]_S = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Find & write

And $c_1 V_1 + c_2 V_2 + c_3 V_3 = W_3$

$$[W_3]_S = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Find & re write

IMP

★ $$P_{S \leftarrow T} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$: Transposition Matrix

---

<u>Solution</u> :

(a) $$V_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$a_1 V_1 + a_2 V_2 + a_3 V_3 = W_1$$

$$a_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$$

$$b_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$$

Common for all 3

$$\left[\begin{array}{ccc|c|c|c} 2 & 1 & 1 & 6 & 4 & 5 \\ 0 & 2 & 1 & 3 & -1 & 5 \\ 1 & 0 & 1 & 3 & 3 & 2 \end{array}\right]$$

~ Augumented system of pair of Linear system of eq.

$$\text{Augmented Matrix} = \begin{bmatrix} 2 & 1 & 1 & 6 & 4 & 5 \\ 0 & 2 & 1 & 3 & -1 & 5 \\ 1 & 0 & 1 & 3 & 3 & 2 \end{bmatrix}$$

Reducing this into RREF we get

$$\sim \text{Augmented Matrix} = \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

Converting back to linear system

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

So, $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$,

$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Transition Matrix, $P_{S \leftarrow T} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

(b) eq-3 : $\boxed{[v]_S = P_{S \leftarrow T} [v]_T}$ ← from Text Book

Given $v = \begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix}$, then expressing v in terms of T-basis we have

$$v = \begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix} = 1\begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} + (-2)\begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$$

$$V = c_1 W_1 + c_2 W_2 + c_3 W_3$$ ← basis

$$c_1 \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix}$$ ← Given

$$\left.\begin{aligned} 6c_1 + 4c_2 + 5c_3 &= 4 \\ 3c_1 - c_2 + 5c_3 &= -9 \\ 3c_1 + 3c_2 + 2c_3 &= 5 \end{aligned}\right\}$$

Solving :

$c_1 = 1$

$c_2 = 2$

$c_3 = -2$

So,

$$\begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix} = 1 \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} + (-2) \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$$

$$[V]_T = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

So,

$$P_{S \leftarrow T} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

<u>Verifying RHS</u> :

$$(P_{S \leftarrow T}) \cdot ([V]_T) = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$RHS = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$$

<u>Verifying LHS</u> :

$$S = \{v_1, v_2, v_3\}$$ ← basis

$$d_1 v_1 + d_2 v_2 + d_3 v_3 = V$$

$$d_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + d_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix}$$

$$\left.\begin{aligned} 2d_1 + d_2 + d_3 &= 4 \\ 0d_1 + 2d_2 + d_3 &= -9 \\ d_1 + 0d_2 + d_3 &= 5 \end{aligned}\right\}$$

Solving :

$d_1 = 4$ $\quad$ $d_3 = 1$

$d_2 = -5$

$$[v]_S = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$$

we got LHS $= \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$, RHS $= \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$

Hence verified, $\boxed{[v]_S = P_{S \leftarrow T} [v]_T}$ is true.

---

*. Transition Matrix $P_{S \leftarrow T}$ from the T-basis to the S-basis is non-singular.

IMP *. Let S and T be the ordered bases for R^3 defined

$$\boxed{Q_{T \leftarrow S} = P^{-1}_{S \leftarrow T}}$$ *IMP

*. SIMILARITY :

- The matrix of a linear operator $T: V \to V$ depends on the basis selected for V that makes the matrix T as simple as possible — a diagonal or triangular matrix
- This search of matrix is called "Search of Good Basis."

Ex : Define - $T : R^2 \to R^2$ by $T(x) = A \cdot x$. Find a basis B for R^2 with the property that $[T]_B$ is diagonal. Given,

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$$

Sol : Calculating Eigen values of $A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -3 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda(\lambda - 4) + 3 = 0 \Rightarrow \lambda_1 = 3, \ \lambda_2 = 1$$

For $\lambda_1 = 3 \Rightarrow$ eigen vector is, $v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

For $\lambda_2 = 1 \Rightarrow$ eigen vector is, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So, $P = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \Rightarrow D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

Good Basis, $G = \{u_1, u_2\}$

$$u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \ \& \ u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[T]_G = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex: Let $T_1 : M_{22} \to P_1$ & $T_2 : P_1 \to R^3$ be the linear transformation given by $T_1\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a+c) + (b+d)x$

and $T_2(a+bx) = (a, b, b)$. Find the formula for $(T_1 \circ T_2) / (T_1 \uparrow T_2) / T_2(T_1(x))$

Sol: $(T_1 \circ T_2)(x) = T_2(T_1(x))$

$= T_2\left(T_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right)$

$= T_2\left((a+c) + (b+d)x\right) = (a+c, \; b+d, \; b+d)$

---

*- Similarity Invariants :

#	Property	Description
1.	Determinant	A & $(P^{-1}A \cdot P)$ have same determinant.
2.	Invertibility	A is invertible if & only if $P^{-1}AP$ is invertible.
3.	Rank	A and $P^{-1}A \cdot P$ have same rank.
4.	Nullity	A and $P^{-1}A \cdot P$ have same nullity.
5.	Trace	A and $P^{-1}A \cdot P$ have same trace.
6.	Characteristic polynomial	A and $P^{-1}A \cdot P$ have same char. poly.
7.	Eigen values	A & $P^{-1}A \cdot P$ have same eigen values.
8.	Eigenspace dimension	$\dim(A) = \dim(P^{-1}AP)$ eigen space.

* Find QR Factorization of $A = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$

Sol: Given, $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. We know that

$\boxed{A = Q \cdot R}$ where Q is orthogonal matrix having orthonormal columns & R is a upper Triangular Matrix

For orthogonal Matrix, Q we need orthonormal set using column vectors of given matrix A.

$A = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$. Apply Gram-Schmidt process for orthonormal set on A.

of vectors $\vec{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

According to Gram-Schmidt process,

$$\boxed{\vec{u_k} = \vec{v_k} - \sum_{j=1}^{k-1} proj_{\vec{u_k}}(\vec{v_k})}$$

where

$$proj_{\vec{u_k}}(\vec{v_k}) = \frac{\vec{u_j} \cdot \vec{v_k}}{|\vec{u_j}|} \cdot \vec{u_j}$$

is a vector projection

& the normalized vector is, $\boxed{\vec{e_k} = \frac{\vec{u_k}}{|\vec{u_k}|}}$

So, here : $\vec{u_1} = \vec{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\vec{e_1} = \frac{\vec{u_1}}{|\vec{u_1}|} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

And, $\vec{u_2} = \vec{V_2} - proj_{\vec{u_1}}(\vec{V_2}) = \begin{bmatrix} \frac{5}{2} \\ 5/2 \end{bmatrix}$

$$\vec{e_2} = \frac{\vec{u_2}}{|\vec{u_2}|} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\hat{x} = \left\{ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \right\}$$

So, $Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

Now, we need to find R, upper triangular Matrix using,

$$\boxed{R = Q^T \cdot A}$$

$\Rightarrow Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \Rightarrow Q^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

And, $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \Rightarrow Q^T \cdot A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

$$\Rightarrow = \frac{\sqrt{2}}{2} \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{5\sqrt{2}}{2} \end{bmatrix} = R_{2\times 2}$$

$$\therefore \underbrace{A_{2\times 2} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}}_{A_{2\times 2}} = \underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}}_{Q_{2\times 2}} \underbrace{\begin{bmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{5\sqrt{2}}{2} \end{bmatrix}}_{R_{2\times 2}}$$

* Find the singular value decomposition of matrix,

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}_{2\times 3}$$

Sol :

Given $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$

$$A^T \cdot A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}_{2\times 2}$$

Finding Eigen values of $A^T \cdot A$

$$\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} - d\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9-d & -9 \\ -9 & 9-d \end{bmatrix}$$

$$\begin{vmatrix} 9-d & -9 \\ -9 & 9-d \end{vmatrix} = 0 \Rightarrow \begin{aligned} d_1 &= 18 \\ d_2 &= 0 \end{aligned}$$

Corresponding Eigen vectors are :

$$d_1 = 18 \Rightarrow A - 18I = 0 \Rightarrow v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$d_2 = 0 \Rightarrow A - 0\cdot I = 0 \Rightarrow v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$V = [v_1 \ \ v_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\sigma_1 = \sqrt{d_1} = \sqrt{18} = 3\sqrt{2}$ & $\sigma_2 = \sqrt{d_2} = \sqrt{0} = 0$

Since there is only one non-zero singular value, the matrix D may be written as single entry

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$U = [u_1\ u_2\ u_3]$. We need to construct AV_1 & AV_2

$$AV_1 = \begin{bmatrix} 2/\sqrt{2} \\ -4/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix}, \quad AV_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u_1 = \frac{1}{3\sqrt{2}} AV_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

The other columns of U are found by extending the set $\{u_1\}$ to an orthonormal basis for R^3. In this case, we need two orthogonal unit vectors u_2 & u_3 that are orthogonal to u_1. Each vector must satisfy $u_1^T \cdot x = 0$, which is equivalent to the equation $x_1 - 2x_2 + 2x_3 = 0$. A basis for the solution set of this equation is,

$$w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(check w_1, w_2 are each orthogonal to u). Apply Gram-Schmidt process to $\{w_1, w_2\}$ obtained

$$u_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$

Finally, set $U = [u_1 \quad u_2 \quad u_3]$

$$U = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{-2}{\sqrt{45}} \\ \frac{-2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \\ \frac{2}{3} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix}_{3\times 3} \qquad \Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times 2}$$

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}_{2\times 2} \quad \Rightarrow \quad A_{3\times 2} \text{ (given)}$$

$$\therefore A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}_{3\times 2} = \underbrace{\begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{-2}{\sqrt{45}} \\ \frac{-2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \\ \frac{2}{3} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix}}_{U_{3\times 3}} \underbrace{\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma_{3\times 2}} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{V^T_{2\times 2}}$$

* The following data shows the atmospheric pollutants y_i (relative to an EPA standard) at half hour interval x_i. Find the equation $y = a + bx$ of the least square line that best fits the data points given by $(2,1)$, $(5,2)$, $(7,3)$, $(8,3)$. Hence predict the atmospheric pollutant at $x = 6$ half hour.

Sol: $y = a + bx$

Given data points are substituted in the system

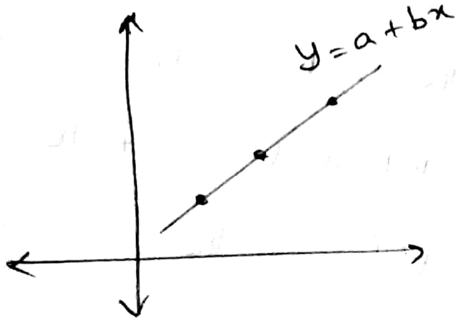
$1 = a + b(2) \Rightarrow 1a + 2b = 1$

$2 = a + b(5) \Rightarrow 1a + 5b = 2$

$3 = a + b(7) \Rightarrow 1a + 7b = 3$

$3 = a + b(8) \Rightarrow 1a + 8b = 3$

No. of data points $(m) = 4$



obtained system of eqs

$$a + 2b = 1$$
$$a + 5b = 2$$
$$a + 7b = 3$$
$$a + 8b = 3$$

Converting into matrix form,

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix}} \underset{X}{\begin{bmatrix} a \\ b \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}}$$

Cannot be Solved directly by Elimination methods.

So, we apply Least squares Approx method,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix}$$

$$A^T . A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix}_{2\times 4} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix}_{4\times 2} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}_{2\times 2}$$

$$A^T . B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix}_{2\times 4} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}_{4\times 1} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}_{2\times 1}$$

So, solution is obtained on solving $\Rightarrow$ $\boxed{(A^T . A)\,\hat{x} = (A^T . B)}$

$$\Rightarrow \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

(or) $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

$$\Rightarrow = \begin{bmatrix} \frac{71}{42} & \frac{-11}{42} \\ \frac{-11}{42} & \frac{1}{21} \end{bmatrix} \begin{bmatrix} 9 \\ 57 \end{bmatrix} = \begin{bmatrix} 9\left(\frac{71}{42}\right) + 57\left(\frac{-11}{42}\right) \\ 9\left(\frac{-11}{42}\right) + 57\left(\frac{1}{21}\right) \end{bmatrix}$$

$$\Rightarrow \cancel{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{639}{42} & -\frac{627}{42} \\ \frac{-99}{42} & \frac{57}{21} \end{bmatrix}} \approx [illegible]$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{12}{42} \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ -2 \end{bmatrix} \approx \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\therefore \boxed{y = \frac{2}{7} + (-2)\hat{x}}$$

And $\hat{x} = 6$ half hour $\Rightarrow$ $y = \frac{2}{7} + (-2)(6)$

$$y = \frac{2}{7} - 12 = \frac{-82}{7} = \underline{\underline{-11.714}}$$

*- Let $A = \begin{pmatrix} 9 & 8 \\ 8 & 9 \end{pmatrix}$. Using Cayley – Hamilton theorem find the expression $A^4 - 19A^3 + 18A^2 + A + 2I$

Sol: Let $A = \begin{pmatrix} 9 & 8 \\ 8 & 9 \end{pmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 9-\lambda & 8 \\ 8 & 9-\lambda \end{vmatrix} = 0$

$\Rightarrow (9-\lambda)^2 - 64 = 0 \Rightarrow \lambda^2 - 18\lambda + 17 = 0$

$\lambda^2 + 17 = 18\lambda \Rightarrow (\lambda^2 + 17)^2 = (18\lambda)^2$

$\lambda^4 + 17\lambda + 289 = 324\lambda^2$

(or) $\lambda^2 = 18\lambda - 17$

The matrix A can be replaced by the eigen value λ in the characteristic equation obtained using A. "Any Matrix satisfies its own characteristic equation."

$A^2 = 18A - 17I$ —(1)

Multiply A on both sides,

$A^3 = 18A^2 - 17A$

$A^3 = 18(18A - 17I) - 17A$

$A^3 = 324A - 306I - 17A$

$A^3 = 307A - 306I$

$A^4 = 307A^2 - 306A$

$= 307(18A - 17I) - 306A$

$$A^4 = 5526A - 5219I - 306A$$

$$A^4 = 5220A - 5219I$$

So, $A^4 - 19A^3 + 18A^2 + A + 2I =$

$$= [5220A - 5219I] - 19[307A - 306I] + 18[18A - 17I]$$
$$+ A + 2I$$

$$= 5220A - 5219I - 5833A + 5814I + 324A - 306I + A + 2I$$

$$= (5220A - 5833A + 324A + A) + I(-5219 + 5814 - 306 + 2)$$

$$= -288A + 291I$$

$$= -288\begin{bmatrix} 9 & 8 \\ 8 & 9 \end{bmatrix} + 291\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2592 & -2304 \\ -2304 & -2592 \end{bmatrix} + \begin{bmatrix} 291 & 0 \\ 0 & 291 \end{bmatrix}$$

$$= \begin{bmatrix} -2301 & -2304 \\ -2304 & -2301 \end{bmatrix}$$