*. Numerical Methods :

- Numerical Methods are mathematical techniques used for solving mathematical problems that can't be solved or are difficult to solve analytically.

- A numerical solution is an approximate numerical value (a number) for the solution.

- Absolute Error, $E_A = \left| \text{true value} - \text{approximate value} \right|$

- Relative Error, $E_R = \dfrac{E_A}{\text{true value}}$

- Percentage Error, $E_p = 100 \times E_R = \dfrac{E_A}{\text{true value}} \times 100$

*. Convergence - A sequence of iterates x_k is said to be convergence to the root of α if

$$\lim_{k \to \infty} \left| x_k - \alpha \right| = 0, \quad \lim_{k \to \infty} x_k = \alpha$$

*. POLYNOMIAL EQUATIONS :

We consider the methods for determining the roots of equation $f(x) = 0$. There are two types of methods that can be used to find the roots of this equation.

i. Direct Method : exact value of roots.

ii. Iterative Method : some approximations based values.

*. ITERATIVE METHODS for SIMPLE ROOTS :

~~1. Bisection Method~~ :

- A root γ is called simple root of $f(x) = 0$ if $f(\gamma) = 0$ & $f'(\gamma) \neq 0$. Then, we can also write $\boxed{f(x) = (x-\gamma)\, g(x)}$, where $g(x)$ is bounded and $g(\gamma) \neq 0$.

- Indirect methods or Iterative methods determine one or two roots at a time.

- These indirect methods or iterative methods are further divided into 2 types :

  1. Bracketing Methods
  2. open Methods

- The Bracketing methods require the limits between which the root lies.

- Bisection Method / Bolzano Method [OR] Regular Falsi method / Method of False position are two known examples of the bracketing methods.

- The open methods require the initial estimation of the solution. Example : Secant, Fixed point Iteration, Newton - Raphson method / Netwon's Method

*. Intermediate Value Theorem :

- If α is root of equation, $f(x) = 0$, then $f(\alpha)$
- Every equation of nth degree has exactly n roots (real or imaginary)

IMP
- If $f(x)$ is cont. func. in closed interval $[a, b]$ and $f(a)$ & $f(b)$ are having opposite signs, then the equation $f(x) = 0$ has atleast one real root or odd no. of roots between a & b.

IMP
- If $f(x)$ is cont. func. in $[a, b]$ and $f(a)$ & $f(b)$ are of same sign, then $f(x) = 0$ has no roots or even no. of roots between a & b.

*. We will discuss the following indirect or iterative methods

1. Bisection Method.
2. Secant Method (Chord Method) and Method of False Position (Regular Falsi Method).
3. Iterative Method.
4. Newton Raphson Method.

* ALGORITHM OF BISECTION METHOD :

Step 1 : • Find interval (a, b) using Intermediate value theorem [Trial & error]

• choose lower $x = a$ & upper $x = b$ guesses for roots such that the function changes sign over interval.

$$\boxed{f(a) \cdot f(b) < 0}$$

Step 2 : Estimate the root, $\boxed{x_K = \frac{a+b}{2}}$

Step 3 : Make the sub root location as

• If $f(x_K) = 0 \implies x_K$ is the root.

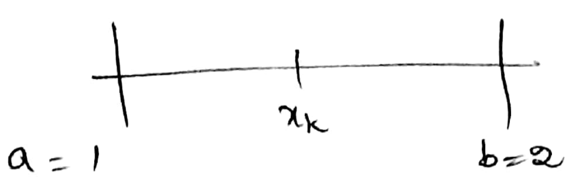
• If $f(a) \cdot f(x_K) < 0$ or $f(x_K) > 0 \implies$ root lies between $[a, x_K]$ repeat step - 2.

• If $f(b) \cdot f(x_K) < 0$ or $f(x_K) < 0 \implies$ root lies between $[x_K, b]$ repeat step - 2.

Example :

Perform 4th iteration of bisection method to obtain the root of the equation $f(x) = x^3 - x - 1 = 0$

$$f(x) = x^3 - x - 1 = 0$$



$f(a) = 1^3 - 1 - 1 = -1 < 0$

$f(b) = 2^3 - 2 - 1 = 5 > 0$

$f(a) \cdot f(b) < 0$

root x_1 lies between $[1, 2]$

$$x_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

$$f(x_1) = f(1.5) = (1.5)^3 - (1.5) - 1 = 0.875 > 0$$

So, $f(a) = -1$, $f(b) = 5$

$$f(a) \cdot f(x_1) = -1 \times 0.875 = -0.875 < 0$$

$\Rightarrow$ root lies between $[a, x_1] = [1, 1.5]$

Now, $a = 1 \Rightarrow f(a) = -1$
$b = 1.5 \Rightarrow f(b) = 0.875$

root x_2 lies between $[1, 1.5]$

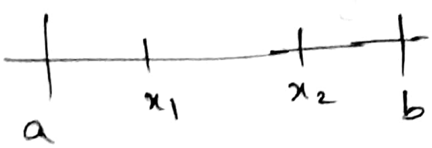
$$x_2 = \frac{1 + 1.5}{2} = 1.25 \Rightarrow f(x_2) = -0.296875$$

$$f(a) \cdot f(x_2) = -1 \times -0.296875 = 0.296875 > 0 \quad \times$$

$$f(b) \cdot f(x_2) = 0.875 \times -0.296875 < 0 \quad \checkmark$$

$\Downarrow$

root lies between $[x_2, b] = [1.25, 1.5]$



Now here $a = 1.25$
$\Rightarrow f(a) = f(1.25) = -0.29687$
& $b = 1.5 \Rightarrow f(b) = 0.875$

root x_3 lies between $[1.25, 1.5]$

$$x_3 = \frac{1.25 + 1.5}{2} = 1.375$$

$$f(x_3) = f(1.375) = 0.224609375 > 0$$

$f(a) \cdot f(x_3) < 0 \rightarrow$ root lies between $[a, x_3]$

$[a, x_3] = [1.25, 1.375]$

Here, $a = 1.25 \rightarrow f(a) = -0.296875$

$b = 1.375 \rightarrow f(b) = 0.224609375$

root x_4 lies between $[1.25, 1.375]$

$$x_4 = \frac{1.25 + 1.375}{2} = 1.3125$$

$f(x_4) = f(1.3125) = -0.05151367$

$f(b) \cdot f(x_4) = f(1.375) \cdot f(1.3125) < 0 \Rightarrow$

root x_5 lies between $[1.3125, 1.375]$

$$x_5 = \frac{1.3125 + 1.375}{2} = 1.34375$$

No. of iterations = 4 (given in Question)

x_k is answer or root of $f(x) = 0$.

i.e. k = iterations + 1 = 4+1 = 5

$\rightarrow x_5$ is root of $f(x)$

$\therefore$ Root of $f(x) = 0$ is $\boxed{x_5 = 1.34375}$

It lies in the interval $[1.3125, 1.375]$

- The convergence of the bisection method is slow as it is simply based on halving the interval.
- This methods fails if there is a discontinuous interval. / discontinuity.
- Bisection method cannot be applied over an interval where $f(x)$ alway takes same sign.
- This method fails for complex roots.
- If one of the initial guesses a_0 or b_0 is closer to the exact solution, it will take larger number of iterations to reach the root.

*. SECANT METHOD : (Better than Regula Falsi) Convergence = 1.61

- If x_{K+1} and x_K are two approximation to the root than the next approximation is given as

$$\boxed{x_{K+1} = x_K - \left[\frac{x_K - x_{K-1}}{f_K - f_{K-1}}\right] \cdot f_K} \quad , \quad K = 0, 1, 2, \ldots$$

*

- This method is also called secant Method or chord Method.
- Convergence of this method, $= 1.618$

Example :

A real root of the equation $x^3 - 5x + 1 = 0$ lies in the interval $(0, 1)$. Perform Four iteration at the secant Method.

Sol : Given, $f(x) = x^3 - 5x + 1$

$x_0 = 0 \Rightarrow f(x_0) = 0^3 - 5(0) + 1 = 1$

$x_1 = 1 \Rightarrow f(x_1) = 1^3 - 5(1) + 1 = -3$

$$\boxed{x_{n+1} = \frac{x_{n-1} \cdot f(x_n) - x_n \cdot f(x_{n-1})}{f(x_n) - f(x_{n-1})}} \quad n = 1, 2, \ldots$$

Put $n = 1 \Rightarrow x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$x_2 = \dfrac{0.(3) - 1.(1)}{-3 - 1} = \dfrac{-1}{-4} = \dfrac{1}{4} \Rightarrow f(x_2) = \dfrac{15}{64}$

Put $n = 2 \Rightarrow x_3 = \dfrac{x_1 \cdot f(x_2) - x_2 \cdot f(x_1)}{f(x_2) - f(x_1)}$

$$\Rightarrow x_3 = \frac{1 \cdot \frac{15}{64} - \frac{1}{4} \cdot (-3)}{\frac{15}{64} - (-3)} = 0.18644$$

$$f(x_3) = f(0.18644) = 0.07428$$

Put $n=3 \Rightarrow$ $$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$x_4 = \frac{15}{64}\left[0.7428\right] - (0.18644)\left[\frac{15}{64}\right] = 0.20174$$

$$f(x_4) = f(0.20174) = -0.00048$$

Put $n=4 \Rightarrow$ $$x_5 = \frac{x_3 \cdot f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

$$x_5 = \frac{(0.18644)\left[-0.00048\right] - 0.20174\left[0.07428\right]}{-0.00048 - 0.07428}$$

$$\boxed{x_5 = 0.20081}$$

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*. REGULA FALSI METHOD : (False Position Method)

- Convergence $\simeq 1$

Algorithm :

① Find the interval $[a,b]$ such that $f(a) \cdot f(b) < 0$.

② Find $$\boxed{c = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)}}$$ *

③ $f(a) \cdot f(c) < 0$ roots, the $f(b) \cdot f(c) < 0$ roots lie in (c,b)

④ Repeat 1, 2 (Same to Bisection Method)

Example :

Find a real root of $x^3 - 2x - 5 = 0$ using the False position method upto Four iterations.

Given, $f(x) = x^3 - 2x - 5 = 0$

(I)

$f(2) = 8 - 4 - 5 = -1$ $\quad a = 2$

$f(3) = 27 - 6 - 5 = 16$ $\quad b = 3$

$$C = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)} = \frac{2(16) - 3(-1)}{16 - (-1)} = \frac{32 + 3}{17}$$

$$C = \frac{35}{17} = 2.0588$$

$f(c) = f(2.0588) = -0.3908 < 0$

So, root lies between $[c, b]$.

(II)

$a = 2.0588$ $\quad b = 3$

$f(a) = -0.3908$ $\quad f(b) = 16$

$$C = \frac{16(2.0588) - 3(-0.3908)}{16 - (-0.3908)} = 2.0812$$

$f(c) = f(2.0812) = -0.1479 < 0$

So, root lies between $[c, b]$

(III)

$a = 2.0812$ $\quad b = 3$

$f(a) = -0.1479$ $\quad f(b) = 16$

$$c = \frac{16(2.0812) - 3(-0.1479)}{16 - (-0.1479)} = 2.0896$$

$f(c) = f(2.0896) = -0.0551 < 0$

So, roots lies between $[c, b]$

(IV) $a = 2.0896$ $\quad$ $b = 3$

$f(a) = -0.0551$ $\quad$ $f(b) = 16$

$$c = \frac{2.0896(16) - 3(-0.0551)}{16 - (-0.0551)} = 2.0927$$

$f(c) = f(2.0927) =$ ____

So, root lies between ____

Stop since upto 4th iteration (Given in Question) ∴

∴ Required root $= 2.0927$

✱. NEWTON RAPHSON METHOD :

- In this method we approach drawing Tangent (method) to the curve of $f(x)$.
- Generally Tangent equation ⇒ $\boxed{y - y_1 = \frac{dy}{dx}(x - x_1)}$
- When the Tangent of that $f(x)$ cuts the x-axis (root) then that is the solution / root of $f(x)$.
- Convergence ≈ 2. (Best Method & Fast than all)

- This is a open Method. So no need of estimating between intervals.

- The approach is same as Secant Method but just change in Formula.

$$* \quad \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \qquad f'(x_n) \neq 0.$$

Example : Find the solution of equation $x^3 - 3x - 5 = 0$ using Newton Raphson method.

Given, $f(x) = x^3 - 3x - 5 = 0$

$f(2) = 8 - 6 - 5 = -3$

$f(3) = 27 - 9 - 5 = 16$

Take $f(x_0) = -3 \Rightarrow \boxed{x_0 = 2}$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

So, $f(x) = x^3 - 3x - 5 \Rightarrow f(x_0) = f(2) = -3$

$f'(x) = 3x^2 - 3 \Rightarrow f'(x_0) = f'(2) = 9$

$n=0$ $\quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{(-3)}{9} = 2 + \frac{1}{3} = \frac{7}{3}$

$n=1$ $\quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.33 - \frac{(2.33)^3 - 3(2.33) - 5}{3(2.33)^2 - 3}$

$x_2 = 2.2805 \Rightarrow f(x_2) = (2.2805)^3 - 3(2.2805) - 5$

(n=2) $x_3 = x_2 - \dfrac{x_2^3 - 3x_2 - 5}{3x_2^2 - 3} = 2.2805 - \dfrac{\cdots}{\cdots}$

$x_3 = 2.2805 - \dfrac{(2.2805)^3 - 3(2.2805) - 5}{3(2.2805) - 3} = 2.2790$

(n=3) $x_4 = x_3 - \dfrac{x_3^3 - 3x_3 - 5}{3x_3^2 - 3} = 2.2790 - \left(\dfrac{(2.2790)^3 \cdots}{3(2.2790)^2 - 3}\right)$

$x_4 = 2.2788$

Here, $|x_4 - x_3| < 10^{-3}$ (very less error)

$\Rightarrow$ At $n=3 \Rightarrow \underline{\underline{x_4 = 2.2790}}$ is the root.

- In this method, stopping criteria is $|x_{n+1} - x_n| <$ less error

then x_{n+1} = root of $f(x)$.

<u>Example</u> : Use N.R.M. to find real root of $\cos x - xe^x = 0$ corrected to four decimal places.

<u>sol</u>: Given, $f(x) = \cos x - xe^x$

$f'(x) = -\sin x - e^x - xe^x$

$f(0) = 1$

$= -\sin x - e^x(x+1)$

$f(1) = -2.1779$

Assume x_0 between 0 & 1 $\Rightarrow x_0 = 0.5$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(n=0) $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 0.5 - \dfrac{f(0.5)}{f'(0.5)} = 0.51802$

$n=1 \Rightarrow x_2 = x_1 + \dfrac{\cos x_1 - x_1 e^{x_1}}{\sin x_1 + e^{x_1}(x_1+1)} = 0.5180$

Here $|x_2 - x_1| = |0.5180 - 0.51802| < 10^{-4}$.

So, this is stopping criteria. $\Rightarrow x_2$ is root.

$\therefore$ Root of $\cos x - xe^x = 0$ is $\boxed{x = 0.5180}$

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*. Example : Apply Newton Raphson method to solve the equation $2(x-3) = \log_{10} x$

Sol :

$f(x) = 2x - 6 - \log_{10} x$

$f(3) = 6 - 6 - \log_{10} 3 = -0.4772$

$f(4) = 8 - 6 - \log_{10} 4 = 1.39794$

So, root lies between 3 & 4 $\Rightarrow x_0 = \dfrac{3+4}{2} = 3.5$

$f(x) = 2x - 6 - \log_{10} x$ $\left\{ \because \log_{10} x = 0.4343 \log_e x \right\}$

$f'(x) = 2 - \dfrac{0.4343}{x}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\boxed{x_{n+1} = x_n - \left[\frac{2x_n - 6 - \log_{10} x_n}{2 - \frac{0.4343}{x_n}} \right]}$$

Put $n=0 \Rightarrow x_1 = x_0 - \left[\dfrac{2x_0^2 - 6x_0 - 0.4343x_0 \log_e x_0}{2x_0 - 0.4343}\right]$

$x_1 = 3.25696$

Put $n=1 \Rightarrow x_2 = 3.256366$

Put $n=2 \Rightarrow x_3 = 3.256310$

Here $|x_3 - x_2| = |3.256310 - 3.256366| < 10^{-4}$. So, the stopping criteria is satisfied

$\Rightarrow \boxed{x_3 = 3.256}$ is the root of $f(x) = 0$

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*. ITERATION METHOD : (Fixed Point Iteration Method)

$\rightarrow$ convergence value ≈ 2

- Suppose we have equation $f(x) = 0$. The equation can be expressed as $x = \phi(x)$ where,

  $\rightarrow$ Choose the minimum at $x = x_0$

  $|\phi'(x)|_{at\ x = x_0} < 1$

- Then iterative method applied, The successive approximation is given by,

  $x_n = \phi(x_{n-1})$, $x_1 = \phi(x_0)$ ...
  $x_2 = \phi(x_1)$ ...

Example : Find a real root of equation,

Sol :

$f(x) = x^3 + x^2 - 1 = 0$

$f(0) = -1$, $f(1) = 1$

$x_0 = \dfrac{0+1}{2} = 0.5$ (assumption) since it lies between 0 & 1.

I. $x^3+x^2-1=0$

$x^3 = 1-x^2$

$x = (1-x^2)^{\frac{1}{3}}$

$\phi(x) = (1-x^2)^{\frac{1}{3}}$

$\phi'(x) = \frac{1}{3}\,\frac{2x}{(1-x^2)^{2/3}}$

$\phi'(x_0) = \left|\frac{1}{3}\,\frac{1}{(1-\frac{1}{4})^{\frac{2}{3}}}\right| =$

$x_0 = 0.5$

$= 0.4038 < 1$

II. $x^3+x^2-1=0$

$x^2 = 1-x^3$

$x = (1-x^3)^{1/2}$

$\phi(x) = (1-x^3)^{\frac{1}{2}}$

$\phi'(x) = \frac{1}{2}\,\frac{3x^2}{(1-x^3)^{\frac{1}{2}}}$

$|\phi'(x_0)| = \frac{1}{2}\,\frac{3(0.5)^2}{(1-(0.5)^3)^{\frac{1}{2}}}$

$|\phi(x_0)| = 0.40089$

< 1

III. $x^3+x^2-1=0$

$x^2(1+x) = 1$

$x = \frac{1}{\sqrt{1+x}}$

$\phi(x) = \frac{1}{\sqrt{1+x}}$

$\phi'(x) = \left|\frac{1}{2}\,\frac{1}{(1+x)^{\frac{3}{2}}}\right|$

$\phi'(x_0) = \frac{1}{2}\,\frac{1}{(1+0.5)^{\frac{3}{2}}}$

$|\phi'(x_0)| = 0.2721$

< 1

For $x = \frac{1}{\sqrt{1+x}}$; $\phi(x) = \frac{1}{\sqrt{1+x}}$

$\phi'(x_0 = 0.5) = \frac{1}{2} \times \frac{1}{(1+0.5)^{\frac{3}{2}}} = 0.2721 < 1 \rightarrow$ least value

So, our $\phi(x) = \frac{1}{\sqrt{1+x}} \Rightarrow \boxed{\phi(x_n) = \frac{1}{\sqrt{1+x_{n-1}}}}$ *

Put $n=1 \Rightarrow x_1 = \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1+0.5}} = 0.81649$

Put $n=2 \Rightarrow x_2 = \frac{1}{\sqrt{1+x_1}} = \frac{1}{\sqrt{1+0.8149}} = 0.74196$

Put $n=3 \Rightarrow x_3 = \frac{1}{\sqrt{1+x_2}} = \frac{1}{\sqrt{1+0.74196}} = 0.75767$

Put $n=4 \Rightarrow x_4 = \frac{1}{\sqrt{1+x_3}} = \frac{1}{\sqrt{1+0.75767}} = 0.75427$

Put $n=5 \Rightarrow x_5 = \dfrac{1}{\sqrt{1+x_4}} = \dfrac{1}{\sqrt{1+0.75427}} = 0.75500$

Put $n=6 \Rightarrow x_6 = 0.75488$

Put $n=7 \Rightarrow x_7 = 0.75487$

$|x_7 - x_6| = |0.75487 - 0.75488| < 10^{-4} \Rightarrow x_7$ is root

$\therefore$ Root of $x^3+x^2-1=0$ is $\underline{\underline{x_7 = 0.75487}}$

Example: Find the roots of $\cos x = 3x-1$, corrected upto four decimal places by using iteration method.

Sol: $f(x) = \cos x - 3x - 1 = 0$ $\qquad \frac{\pi}{2} = 1.570$

$f(0) = 1-0+1 = 2$

$f\left(\frac{\pi}{2}\right) = 0 - 3\left(\frac{\pi}{2}\right) + 1 = -3.712$

root lies between 0 & $\frac{\pi}{2}$ $\Rightarrow$ Assume $x_0 = 0$ for easy calculation

$x = \phi(x)$: form conversion $\Rightarrow$

$\cos x - 3x - 1 = 0 \Rightarrow x = \dfrac{1+\cos x}{3} = \phi(x)$

$\phi'(x) = \dfrac{-\sin x}{3} \Rightarrow |\phi'(x)| = \dfrac{\sin x}{3}$

$|\phi(x_0)|_{\text{at } x=0} = \dfrac{\sin 0}{3} = 0 < 1$ (✓) can be considered

So, $\phi(x) = \dfrac{1+\cos x}{3} = x$

Then, $\boxed{x_n = \dfrac{1+\cos x_{n-1}}{3}}$ * our function to proceed applying Iteration method

Put $n=1 \Rightarrow x_1 = \dfrac{1+\cos x_0}{3} = \dfrac{1+\cos 0}{3} = 0.66667$

Put $n=2 \Rightarrow x_2 = \dfrac{1+\cos x_1}{3} = \dfrac{1+\cos(0.66667)}{3} = 0.595296$

Put $n=3 \Rightarrow x_3 = \dfrac{1+\cos x_2}{3} = \dfrac{1+\cos(0.595296)}{3} = 0.609328$

Put $n=4 \Rightarrow x_4 = \dfrac{1+\cos x_3}{3} = \dfrac{1+\cos(0.609328)}{3} = 0.606678$

Put $n=5 \Rightarrow x_5 = \dfrac{1+\cos(0.606678)}{3} = 0.607182$

Put $n=6 \Rightarrow x_6 = \dfrac{1+\cos(0.607182)}{3} = 0.607086$

Put $n=7 \Rightarrow x_7 = \dfrac{1+\cos(0.607086)}{3} = 0.607105$

Put $n=8 \Rightarrow x_8 = \dfrac{1+\cos(0.607105)}{3} = 0.607101$

Now, $|x_8 - x_7| = |0.607101 - 0.607105| < 10^{-4}$. Therefore correct root of $f(x) = 0$ is $\underline{x_8 = 0.607101}$

*. Non-Linear System using Newton's Method :

- Newton Raphson method described in previous page is extended in terms of general case of system of n non linear equations with n unknowns,

$$\left.\begin{aligned} f_1(x_1, x_2, x_3 \dots x_n) &= 0 \\ f_2(x_1, x_2, x_3, \dots x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, x_3, \dots x_n) &= 0 \end{aligned}\right\}$$

where $f_i(x_1 \dots x_n)$ are non-linear functions

- To solve such system of equations, we follow Newton's method by :

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} - J^{-1} \cdot \begin{bmatrix} f_1(x_i\ y_i) \\ f_2(x_i\ y_i) \end{bmatrix}$$

Here J^{-1} is Jacobian inverse.

$$J = \begin{bmatrix} \left(\dfrac{\partial f_1}{\partial x}\right)_{at\ x = x_1} & \left(\dfrac{\partial f_1}{\partial y}\right)_{at\ x = x_2} \\ \left(\dfrac{\partial f_2}{\partial x}\right)_{at\ x = x_1} & \left(\dfrac{\partial f_2}{\partial y}\right)_{at\ x = x_2} \end{bmatrix}$$

*. GAUSS ELIMINATION METHOD :

Ex : Solve the following equation by Gauss Elimination Method :

$$x - y + 2z = 3$$

$$x + 2y + 3z = 5$$

$$3x - 4y - 5z = -13$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -13 \end{bmatrix} \rightarrow AX = B$$

$\Downarrow$

$\{A|B\}$ Augmented

$$[A|B\} = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 3 & -4 & -5 & -13 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & -1 & -11 & -22 \end{bmatrix} \quad R_3 \rightarrow 3R_3 + R_2$$

$$= \begin{bmatrix} 1 & -11 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -32 & -64 \end{bmatrix}$$

Convert back to linear system $\Rightarrow$

$$x - y + 2z = 3$$
$$3y + z = 2$$
$$-32z = 64$$

$\Rightarrow$ $\boxed{z = 2}$ $\quad 3y + 2 = 2 \rightarrow \boxed{y = 0}$

$3 = x - 0 + 2(2) \Rightarrow \boxed{x = -1}$

Sol : $\left.\begin{matrix} x = -1 \\ y = 0 \\ z = 2 \end{matrix}\right\}$

- Gauss Elimination Method : Convert into Upper Triangular Matrix

- Gauss Jordan Elimination : Diagonal Matrix

- Homogeneous System : $AX = B \Rightarrow B = 0$
- Non-Homogenous System : $AX = B \Rightarrow B \neq 0$

- Rank of Matrix : No. of non-zero rows in matrix
- Augmented Matrix - $[A|B] = \tilde{A}$

- Consistent System : $\text{Rank}(A) = \text{Rank}(\tilde{A})$
  - → Unique Solution - $\text{Rank}(A) = \text{Rank}(\tilde{A}) = n$
  - → infinite solution - $\text{Rank}(A) = \text{Rank}(\tilde{A}) < n$

- Inconsistent System : $\text{Rank}(A) \neq \text{Rank}(\tilde{A})$
  - → no solution - $\text{Rank}(A) < \text{Rank}(\tilde{A})$

*. Gauss Elimination Method, Gauss Jordan Method, LU decomposition Method, Thomas Method, tridiagonal Matrix Method → DIRECT METHODS.

*. Gauss - Seidal Method, Jacobi iteration Method → ITERATIVE METHODS

*. LU Decomposition : (Factorization Method)

- It is also called as cholaskey's Method.

Consider the system of linear equations as :

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$$

$$AX = B \qquad [A|B] : \text{Augumented Matrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = L \cdot U$$

L - Lower Triangular

U - upper Triangular

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, \quad U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$AX = B \Rightarrow (LU) \cdot X = B \Rightarrow L \cdot (Ux) = B$$

Let $Ux = y \Rightarrow L \cdot y = B$ (conversion)

↓ back substitution

then, $Ux = y$

$$[U|y] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = X \text{ (obtained)}$$

Example : Solve the following system of equations by LU decomposition method.

$$x + 5y + z = 14$$

$$2x + y + 3z = 13$$

$$3x + y + 4z = 17$$

Sol : Converting these into Matrix form :

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix} \Rightarrow A \quad X = B$$

So, $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = L \cdot U$ (Method)

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} U_{11} & L_{21} U_{12} + U_{22} & L_{21} U_{13} + U_{23} \\ L_{31} U_{11} & L_{31} U_{12} + L_{32} U_{22} & L_{31} U_{13} + L_{32} U_{23} + U_{33} \end{bmatrix}$$

$\boxed{U_{11} = 1}$ $\qquad L_{21} \cdot U_{11} = 2$ $\qquad L_{21} U_{12} + U_{22} = 1$

$\boxed{U_{12} = 5}$ $\qquad \boxed{L_{21} = 2}$ $\qquad (2)(5) + U_{22} = 1$

$\boxed{U_{13} = 1}$ $\qquad \boxed{U_{22} = -9}$

$L_{31} U_{11} = 3$

$\boxed{L_{31} = 3}$

$L_{21} U_{13} + U_{23} = 3$

$(2)(1) + U_{23} = 3$

$\boxed{U_{23} = 1}$

$L_{31} U_{12} + L_{32} U_{22} = 1$

$(3)(5) + L_{32}(-9) = 1$

$\boxed{L_{32} = \frac{14}{9}}$

$L_{31} U_{13} + L_{32} U_{23} + U_{33} = 4$

$(3)(1) + \left(\frac{14}{9}\right)(1) + U_{33} = 4$

$\boxed{U_{33} = -\frac{5}{9}}$

So, substitute all these values in L, U

$$A = L \cdot U = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix}}_{U}$$

$L \cdot Y = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

convert back into system of linear equation

$\boxed{y_1 = 14}$

$2y_1 + y_2 = 13 \Rightarrow \boxed{y_2 = -15}$

$3y_1 + \frac{14}{9} y_2 + y_3 = 17 \Rightarrow \boxed{y_3 = -\frac{5}{3}}$

Now, $UX = Y$

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -15 \\ -5/3 \end{bmatrix}$$

Convert into system of equations again,

$$-\frac{5}{9}z = -\frac{5}{3} \Rightarrow \boxed{z = 3}$$

$$-9y + z = -15 \Rightarrow \boxed{y = 2}$$

$$x + 5y + z = 14 \Rightarrow \boxed{x = 1}$$

Final solution of the system.

---

*. TRIDIAGONAL SYSTEM : (Thomas Algorithm)

- Tridiagonal Matrix :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4\times4}$$

Super Diagonal or upper diagonal

Main diagonal $\{ [a_{ij}] , i = j \}$

Sub diagonal or lower diagonal

- A matrix other than these elements (lower, upper, main diagonal) are 0s is called Tridiagonal Matrix.

Working Rule :

Step 1 : For the first equation, for the new elements, we have

$$a_1 = \frac{a_1}{d_1} \quad , \quad \gamma_1 = \frac{\gamma_1}{d}$$

Step 2 : For each of the equations from $i = 2, \ldots n-1$

$$a_i = \frac{a_i}{d_i - b_i a_{i-1}} \quad , \quad r_i = \frac{r_i - b_i r_{i-1}}{d_i - b_i a_{i-1}}$$

Step 3 :

$$r_n = \frac{r_n - b_n r_{n-1}}{d_n - b_n a_{n-1}}$$

Step 4 : Solve by back substitution,

$$\boxed{x_n = r_n}$$

$$x_i = r_i - a_i x_{i+1} \quad , \quad i = n-1, n-2, \ldots 1$$

* **Example :**

Solve the following tridiagonal system using Thomas Method

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + 2x_3 - x_4 &= 0 \\ -x_3 + 2x_4 &= 1 \end{aligned}$$

Sol :

$$d = \overset{d_1\ d_2\ d_3\ d_4}{(2, 2, 2, 2)} \qquad a = \overset{a_1\ a_2\ a_3\ a_4}{(-1, -1, -1, 0)}$$

$$b = \overset{b_1\ b_2\ b_3\ b_4}{(0, -1, -1, -1)} \qquad r = \overset{r_1\ r_2\ r_3\ r_4}{(1, 0, 0, 1)} \qquad \boxed{n = 4}$$

① For first equation, $a_1 = \frac{a_1}{d_1} = \frac{-1}{2}$

$$r_1 = \frac{r_1}{d_1} = \frac{1}{2}$$

(update these values in array).

3

Put $i=2$

$$a_2 = \frac{a_2}{d_2 - b_2 a_1} = \frac{-1}{2-(-1)(-\frac{1}{2})} = -\frac{2}{3}$$

$$\gamma_2 = \frac{\gamma_2 - b_2 \gamma_1}{d_2 - b_2 a_1} = \frac{0-(-1)(\frac{1}{2})}{2-(-1)(-\frac{1}{2})} = \frac{1}{3}$$

Put $i=3$

$$a_3 = \frac{a_3}{d_3 - b_3 a_2} = \frac{-1}{2-(-1)(-\frac{2}{3})} = -\frac{3}{4}$$

$$\gamma_3 = \frac{\gamma_3 - b_3 \gamma_2}{d_3 - b_3 a_2} = \frac{1}{4}$$

Put $i=4$

$$a_4 = \frac{a_4}{d_4 - b_4 a_3} \qquad \gamma_4 = \frac{\gamma_4 - b_4 \gamma_3}{d_4 - b_4 a_3}$$

$$a_4 = 0 \qquad \gamma_4 = 1$$

4 Solving by back substitution, ($n=4$ stop)

$$x_4 = \gamma_4 = 1$$

$$x_3 = \gamma_3 - a_3 x_4$$

$$= \frac{1}{4} - \left(-\frac{3}{4}\right)(1) = 1$$

$$x_2 = \gamma_2 - a_2 x_3 = \frac{1}{3} - \left(-\frac{2}{3}\right)(1) = 1$$

$$x_1 = \gamma_1 - a_1 x_2 = \frac{1}{2} - \left(-\frac{1}{2}\right)(1) = 1$$

$\boxed{x = (1,1,1,1)}$ is the required solution.

ITERATIVE METHODS :

*. JACOBI ITERATION METHOD :

Example :

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

Sol :

diagonal entries, other elements

$|27| > |6| + |-1|$ ✓

diagonal entries

$|15| > |6| + |2|$ ✓

diagonal entries

$|54| > |1| + |1|$ ✓

} DIAGONAL DOMINANT MATRIX .

So, we can use Jacobi method for given system of equation ,

$$\Rightarrow \quad x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 6x - 2z)$$

$$z = \frac{1}{54}(110 - x - y)$$

$x^{(i)}$ - x value in ith iteration

$y^{(i)}$ - y value in ith iteration

$z^{(i)}$ - z value in ith iteration

Let initial value, $x_0 = 0$, $y_0 = 0$, $z_0 = 0$

i. First iteration,

$$x^1 = \frac{85}{27} = 3.148 \qquad y^1 = \frac{72}{15} = 4.8 \qquad z^1 = \frac{110}{54} = 2.037$$

(update these values & use it in next iteration)

ii. Second iteration,

$$x^{(2)} = \frac{1}{27}\left[85 - 6(4.8) + 2(0.37)\right] = 2.157$$

$$y^{(2)} = \frac{1}{15}\left[72 - 6(3.148) - 2(2.037)\right] = 3.269$$

$$z^{(2)} = \frac{1}{54}\left[110 - 3.148 - 2.037\right] = 1.89$$

iii. Third iteration,

$$x^{(3)} = \frac{1}{27}\left[85 - 6(3.269) + 1.89\right] = 2.492$$

$$y^{(3)} = \frac{1}{15}\left[72 - 6(2.157) - 2(1.89)\right] = 3.685$$

$$z^{(3)} = \frac{1}{54}\left[110 - 2.157 - 3.269\right] = 1.937$$

iv. Fourth iteration,

$$x^{(4)} = \frac{1}{27}\left[85 - 6(3.685) + 1.937\right] = 2.401$$

$$y^{(4)} = \frac{1}{15}\left[72 - 6(2.492) + 2(1.937)\right] = 3.545$$

$$z^{(4)} = \frac{1}{54}\left[10 - 2.492 - 3.685\right] = 1.925$$

v. Fifth iteration,

$$x^{(5)} = \frac{1}{27}\left[85 - 6(3.545) + 1.925\right] = 2.432$$

$$y^{(5)} = \frac{1}{15}\left[72 - 6(2.401) + 2(1.925)\right] = 3.583$$

$$z^{(5)} = \frac{1}{54}\left[10 - 2.401 - 3.545\right] = 1.927$$

vi. Sixth iteration,

$$x^{(6)} = 2.423$$

$$y^{(6)} = 3.57$$

$$z^{(6)} = 1.926$$

Seventh iteration,

$$x^{(7)} = 2.426$$

$$y^{(7)} = 3.574$$

$$z^{(7)} = 1.926$$

The iterations (6, 7) give particularly same values, therefore we can stop the iteration. Hence, the solution is (7th iteration values)

$$\Rightarrow \left.\begin{aligned} x &= 2.423 \approx 5/2 \\ y &= 3.574 \approx 7/2 \\ z &= 1.926 \approx 2 \end{aligned}\right\} \text{Approximations.}$$

---

*. GAUSS - SEIDAL ITERATIVE METHODS :

Ex : solve the two system of equations using Gauss - Seidal iterative method.

$$2x_1 - x_2 + 0 \cdot x_3 = 7$$

$$-x_1 + 2x_2 - x_3 = 1$$

$$0 \cdot x_1 - x_2 + 2x_3 = 1$$

sol :

$$\left.\begin{aligned} |2| &> |-1| + |0| \checkmark \\ |2| &\geq |-1| + |-1| \checkmark \\ |2| &> |0| + |-1| \checkmark \end{aligned}\right\} \text{DIAGONALLY DOMINENT MATRIX}$$

Now,

$$x_1 = \frac{1}{2}(7 + x_2)$$

$$x_2 = \frac{1}{2}(1 + x_1 + x_3)$$

$$x_3 = \frac{1}{2}(1 + x_2)$$

Initial values, Approximation $\Rightarrow$ $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$

i. First Iteration,

$$x_1^{(1)} = \frac{7}{2} = 3.5$$

$$x_2^{(1)} = \frac{1}{2}\left(1 + x_1^{(1)} + x_3^{(0)}\right) = \frac{1}{2}(1 + 3.5 + 0) = 2.25$$

$$x_3^{(1)} = \frac{1}{2}(1 + 2.25) = \cancel{2.25}\ 1.625$$

ii. Second iteration,

$$x_1^{(2)} = \frac{1}{2}(7 + 2.25) = 4.625$$

$$x_2^{(2)} = \frac{1}{2}(1 + 4.625 + 1.625) = 3.625$$

$$x_3^{(2)} = \frac{1}{2}(1 + 3.625) = 2.3125$$

iii. Third iteration,

$$x_1^{(3)} = \frac{1}{2}(7 + 3.625) = 5.3125$$

$$x_2^{(3)} = \frac{1}{2}(1 + 5.3125 + 2.3125) = 4.3125$$

$$x_3^{(3)} = \frac{1}{2}(1 + 4.3125) = 2.65625$$

iv. Fourth iteration,

$$x_1^{(4)} = \frac{1}{2}(7 + 4.3125) = 5.65625$$

$$x_2^{(4)} = \frac{1}{2}(1 + 5.65625 + 2.65625) = 4.65625$$

$$x_3^{(4)} = \frac{1}{2}(1 + 4.65625) = 2.828125$$

v. 5th iteration,

$$x_1^{(5)} = 5.8281$$

$$x_2^{(5)} = 4.8281$$

$$x_3^{(5)} = 2.9106$$

6th iteration,

$$x_1^{(6)} = 5.4140$$

$$x_2^{(6)} = 4.9140$$

$$x_3^{(6)} = 2.9570$$

7th iteration,

$$x_1^{(7)} = 5.9570$$

$$x_2^{(7)} = 4.9570$$

$$x_3^{(7)} = 2.9785$$

. . . . Continue upto 11 th iteration

In 11th iteration,

$$x_1^{(11)} \approx 6$$

$$x_2^{(11)} \approx 5$$

$$x_3^{(11)} \approx 3$$

∴ Solution of the system of equation $\Rightarrow$

$$x_1 = 6$$

$$x_2 = 5$$

$$x_3 = 3$$

*. EIGEN VALUES & EIGEN VECTORS :

Consider the homogeneous system of equation, $AX = \lambda X$

(or) $\boxed{(A - \lambda I) X = 0}$

- Homogeneous + Trivial (or) non - Trivial + Consistance.

$\quad$ Trivial $\downarrow$ $x = 0$; non - Trivial $\downarrow$ $x \neq 0$

Ex : Consider $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix} = A - \lambda I$$

$$|A - \lambda I| = (4-\lambda)(2-\lambda) - 3 \Rightarrow \boxed{\lambda^2 - 6\lambda + 5 = 0}$$

Characteristic equation

Solving this characteristic equation,

$$\lambda^2 - 6\lambda + 5 = 0 \Rightarrow \boxed{\lambda = 1, 5} \rightarrow \text{eigen values}$$

To check if obtained eigen values $\Rightarrow$

- Sum of diagonal entries = Sum of Eigen values.
- determinant of Matrix = product of Eigen values.

*. Eigen - everything :

- Eigenvalue λ
- Eigen vector v
- Eigen pair λv
- $A \cdot v = \lambda v$
- A is $n \times n$ matrix
- Solving polynomials, systems of equations & differential eq, with applications to physics, geology, civil engineering, mechanical engineering, machine learning, you name it.

*. POWER METHOD :

- Given by formula, $$\boxed{b_{k+1} = \frac{A \cdot b_k}{\|A b_k\|}}$$

where, b_k - n element vector

A - $n \times n$ matrix

$\|A \cdot b_k\|$ - is maximum of resulting vector

- Initial b is non-zero and usually random numbers.
- Iterate until $b_{k+1} \approx b_k$ (eigen vector) and $\|A b_k\|$ (eigen value)

<u>Example</u> : Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Find eigen values using power method.

$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ → assumption

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = Ab_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \xrightarrow{\text{Take 2 common}} = 2\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} \xrightarrow{\text{Take 1.5 common}} = 1.5\begin{bmatrix} 0.666 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.666 \\ 1 \end{bmatrix} = 1.66\begin{bmatrix} 0.6 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.6 \\ 1 \end{bmatrix} = 1.6\begin{bmatrix} 0.625 \\ 1 \end{bmatrix}$$

$$\Rightarrow 1.625\begin{bmatrix} 0.61538 \\ 1 \end{bmatrix}, \quad 1.61538\begin{bmatrix} 0.619048 \\ 1 \end{bmatrix}$$

. . . . . . Converges 8 iterations later,

$$= \underset{\substack{\downarrow \\ \text{value}}}{1.618034} \underset{\substack{\downarrow \\ \text{vector}}}{\begin{bmatrix} 0.618034 \\ 1 \end{bmatrix}} \Longrightarrow \boxed{\lambda = 1.618034} \text{ Eigen value}$$

$$\boxed{v = \begin{bmatrix} 0.618034 \\ 1 \end{bmatrix}}$$

Eigen vector

Example : Determine dominant eigen value, eigen vector using power method for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, input, $V_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (assumption)

iteration 1 :

$$y_1 = A V_0 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$V_1 = \frac{y_1}{\max(AV_0)} = \frac{1}{7}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4285 \\ 1 \end{bmatrix}$$

Iteration 2 :

$$y_2 = A \cdot V_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.4285 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.42857 \\ 5.28571 \end{bmatrix}$$

$$V_2 = \frac{y_2}{\max(AV_1)} = \frac{1}{5.2857}\begin{bmatrix} 2.42857 \\ 5.28571 \end{bmatrix} = \begin{bmatrix} 0.45946 \\ 1 \end{bmatrix}$$

iteration 3 :

$$y_3 = A \cdot V_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.45946 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.45946 \\ 5.37838 \end{bmatrix}$$

$$V_3 = \frac{y_3}{\max(AV_2)} = \frac{1}{5.37838}\begin{bmatrix} 2.45946 \\ 5.37838 \end{bmatrix} = \begin{bmatrix} 0.45729 \\ 1 \end{bmatrix}$$

iteration 4 :

$$y_4 = A \cdot V_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0.45729 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.45729 \\ 5.37187 \end{bmatrix}$$

$$V_4 = \frac{y_4}{\max(AV_3)} = \frac{1}{5.37187}\begin{bmatrix} 2.45729 \\ 5.37187 \end{bmatrix} = \begin{bmatrix} 0.45744 \\ 1 \end{bmatrix}$$

iteration 5 :

$$y_5 = A.V_4 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.45744 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.45744 \\ 5.37232 \end{bmatrix}.$$

$$V_5 = \frac{y_5}{\max(A\,V_4)} = \frac{1}{5.37232} \begin{bmatrix} 2.45744 \\ 5.37232 \end{bmatrix} = \begin{bmatrix} 0.457426 \\ 1 \end{bmatrix}$$

iteration 6 :

$$y_6 = A \cdot V_5 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0.457426 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.457426 \\ 5.372278 \end{bmatrix}$$

$$V_6 = \frac{y_6}{\max(AV_5)} = \frac{1}{5.372278} \begin{bmatrix} 2.457426 \\ 5.372278 \end{bmatrix} = \begin{bmatrix} 0.457427 \\ 1 \end{bmatrix}$$

When do we stop the iteration :

The iterations are stopped when all the magnitudes of difference of ratios are less than the given error tolerance.

$$\boxed{\text{ratio } (\gamma_K) = \frac{y_{K+1}}{V_K} \quad \downarrow \quad \text{IMP} \bigstar \ \max(AV_K) \text{ values}}$$

$|\gamma_1 - \gamma_2|$, $|\gamma_2 - \gamma_3|$, $|\gamma_3 - \gamma_4|$, $|\gamma_4 - \gamma_5|$, ... < error (small difference)

We obtain ratio as,

$$\frac{2.45743}{0.45743} = 5.37225, \ 5.37229$$

$|5.37225 - 5.37229| = 0.00004 < 0.00005$. Hence the dominant eigen value, $= \underline{5.3722}$

$\therefore$ Eigen value = 5.3722 (dominant)

$\therefore$ Eigen vector = $\begin{bmatrix} 0.4575 \\ 1 \end{bmatrix}$ (at 6th iteration)

<u>Example</u>: Determine smallest eigen value for given matrix $A = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix}$ using power method

<u>Sol</u>: The concept of inverse power method goes to be,...

- Largest eigen value of $A = \dfrac{1}{\text{smallest eigen value of } A^{-1}}$

Needed
$*$ Smallest eigen value of $A = \dfrac{1}{\text{largest eigen value of } A^{-1}}$

So, to find smallest eigen value of A, we need to find dominant eigen value of matrix A^{-1} & reciprocate it.

Given, $A = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 2 & 3 & 10 \end{bmatrix}$

Now, we take our given matrix as $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 2 & 3 & 10 \end{bmatrix}$ and find dominant eigen vector, value using Power method.

So, $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 2 & 3 & 10 \end{bmatrix}$, $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

1st iteration,

$$A x_0 = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 2 & 3 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 14 \\ 15 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_1 = \frac{1}{15} \begin{bmatrix} 5 \\ 14 \\ 15 \end{bmatrix} = \begin{bmatrix} 0.333 \\ 0.933 \\ 1 \end{bmatrix}$$

2nd iteration,

$$A x_1 = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 2 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.333 \\ 0.933 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.267 \\ 12.467 \\ 13.467 \end{bmatrix}$$

and by scaling we obtain the approximation,

$$x_2 = \frac{1}{13.467} \begin{bmatrix} 4.267 \\ 12.467 \\ 13.467 \end{bmatrix} = \begin{bmatrix} 0.317 \\ 0.926 \\ 1 \end{bmatrix}$$

3rd iteration,

$$A x_2 = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 2 & 3 & 10 \end{bmatrix} \begin{bmatrix} 0.317 \\ 0.926 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.243 \\ 12.411 \\ 12.411 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_3 = \frac{1}{13.411}\begin{bmatrix} 4.243 \\ 12.411 \\ 13.411 \end{bmatrix} = \begin{bmatrix} 0.316 \\ 0.925 \\ 1 \end{bmatrix}$$

4th iteration,

$$Ax_3 = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 9 \\ 2 & 3 & 10 \end{bmatrix}\begin{bmatrix} 0.316 \\ 0.925 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.242 \\ 12.409 \\ 13.409 \end{bmatrix}$$

and by scaling we obtain the approximation

$$x_4 = \frac{1}{13.409}\begin{bmatrix} 4.242 \\ 12.409 \\ 13.409 \end{bmatrix} = \begin{bmatrix} 0.316 \\ 0.925 \\ 1 \end{bmatrix}$$

$\therefore$ The dominant eigenvalue $\lambda = 13.409 \approx 13.41$

and the dominant eigen vector is,

$$= \begin{bmatrix} 0.316 \\ 0.925 \\ 1 \end{bmatrix} \cong \begin{bmatrix} 0.32 \\ 0.93 \\ 1 \end{bmatrix}$$

MODULE - 3

*. INTERPOLATION :

The technique or method of estimating unknown values inside range from given set of observations is Interpolation.

Ex :

year x	$f(x)$ population
1971	1000
1981	1025
1991	1080
2001	1120
2011	1200

range = [1971, 2011]

Suppose asked what is the population in year 1977 ?

(or) population in year 1993 ?

Equal intervals

- → difference in value is same
- → step value is equal

Unequal intervals

- → step value is not equal

1. For Equal intervals :

- Newton Forward.
- Gauss Forward, Gauss Backward, Striling method, Bessels.
- Newton Backward.

2. For Unequal intervals :

- Lagranges interpolation Formula
- Newton divided difference.

*. Extrapolation : The technique or method of estimating unknown values from given set of observations beyond, outside the range is called Extrapolation.

Ex : what is population in year 2015 ? 2020 ?

*. INTERPOLATION FOR EQUAL INTERVAL :

Ex: Estimate the population in 1895, 1925 from the following statistics ?

year	1891	1901	1911	1921	1931
population	46	66	81	93	101

Sol :

$$\boxed{\Delta f(x) = f(x+h) - f(x)}$$ (delta - Δ)

Asked is 1895 $\rightarrow$ apply Newton forward formula.

Newton Forward Formula :

$$f(a+hu) = f(a) + \frac{u}{1!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + \cdots + \frac{u(u-1)(u-2)\cdots(u-(n-1))}{n!}\Delta^n f(a)$$

a – previous value near to given | after value near to gives as base

h – interval difference

We need to find $f(1895)$ $\Rightarrow$ $a+hu = 1895$

$1891 + 10u = 1895$

Here, $a = 1891$

$h = 10$

$10u = 1895 - 1891$

$$\boxed{u = 0.4}$$

$$f(1895) = 46 + \frac{0.4}{1}(20) + \frac{(0.4)(0.4-1)}{2}(-5) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(2) + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}(-3)$$

$= 54.8528$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1891	46				
		= 20			
1901	66		= -5	→ Take	these for 1895
		= 15		= 2	
1911	81		= -3		= -3
		= 12		= -1	
1921	93		= -4		
		= 8			
1931	101				

$\Delta f(a) = 20$ $\qquad a = 1891$

$f(a) = 46$

$\Delta^2 f(a) = -5$

$h = 10 = |(1891 - 1901)|$
$= |(1901 - 1911)|$ → equal interval

$\Delta^3 f(a) = 2$

$\Delta^4 f(a) = 3$

NEWTON'S BACKWARD :

Asked is 1925 → Apply Newton Backward Formula

$$\boxed{\nabla f(x) = f(x) - f(x-h)}$$ (nabla - ∇)

$$f(a+hu) = f(a) + \frac{u}{1!}\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a)$$

$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a) + \cdots$$

So, $f(1925) = ?$ $\Rightarrow$ $a + hu = 1925$

So, $a = 1931$ $\qquad 1931 + hu = 1925$, $h = 10$ ↓ interval

$h = 10$ $\qquad \boxed{u = -0.6}$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1891	46				
		= 20			
			= -5		
1901	66			= 2	
		= 15			
			= -3		= -3
1911	81				
		= 12		= -1	
1921	93		= -4		
		= 8			
1931	101				

Take these for 1925

$a = 1931$ $\qquad \nabla f(a) = 8$ $\qquad \nabla^3 f(a) = -1$

$f(a) = 101$ $\qquad \nabla^2 f(a) = -4$ $\qquad \nabla^4 f(a) = -3$

$$f(1925) = 101 + \frac{(-0.6)}{1!} \times 8 + \frac{(-0.6)(-0.6+1)}{2}(-4)$$

$$+ \frac{(-0.6)(-0.6+2)}{6}(-1) + \frac{(-0.6)(-0.6+2)(-0.6+3)}{24} \cdot (-3)$$

$$f(1925) = 96.8368$$

Ex: Find the no. of Men getting wages between Rs. 10 & Rs. 15 from the following data.

wages	0 - 10	10 - 20	20 - 30	30 - 40
Frequency	9	30	35	42

86) :

wages x	men $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
below 10	9			
		= 30		
			= 5	
below 20	30+9 = 39			
		= 35		= 2
below 30	39+35 = 74		= 7	
		= 42		
below 40	74+42 = 116			

Newton Forward :

$$f(a+hu) = f(a) + \frac{u}{1!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + \cdots$$

We need to find $f(15) = ?$ $\Rightarrow$ $a+hu = 15$

$10 + 10 \cdot u = 15$

$a = 10$

$\boxed{u = 0.5}$

$h = 10$

$f(a) = 9$ $\quad \Delta^2 f(a) = 5$

$\Delta f(a) = 30$ $\quad \Delta^3 f(a) = 2$

$$f(15) = 9 + \frac{0.5}{1}(30) + \frac{0.5(0.5-1)}{2}(5) + \frac{(0.5)(0.5-1)(0.5-2)}{6}(2)$$

$f(15) = 23.5 \approx 24$ frequency of men

below 15

Hence No. of men getting wages between

10 & 15 $=$ $\begin{bmatrix} \text{Men getting} \\ \text{below } 15 \end{bmatrix} - \begin{bmatrix} \text{Men getting} \\ \text{below } 10 \end{bmatrix}$

$$= 24 - 9 = \underline{\underline{15}}$$

<u>Ex</u>: Find the lowest degree polynomial $y(x)$ that fit the data $y(5) = ?$

x	0	2	4	6	8
y	5	9	61	209	501

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	5				
		$= 4$			
2	9		$= 48$		
		$= 52$		$= 48$	
4	61		$= 96$		$= 0$
		$= 148$		$= 48$	
6	209		$= 144$		
		$= 292$			
8	501				

Apply Newton Forward Method,

$$f(a+hu) = f(a) + \frac{u}{1!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + \cdots$$

Here, $f(\cancel{x}) = ?$ $\Rightarrow$ $a + hu = X$

we need to find polynomial

$a = \text{base} = 0$

$h = \text{difference} = 2$

$a + hu = x \Rightarrow u = x/2$

$$f\left(\frac{x}{2}\right) = 5 + \frac{x}{2}(4) + \frac{\left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)}{2}(48) + \frac{\left(\frac{x}{2}\right)\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)}{6}(48)$$

$\boxed{f(x) = x^3 - 2x + 5}$ we obtain the polynomial on simplifying this.

$y(5) = f(5) = (5)^3 - 2(5) + 5 = \underline{\underline{120}}$

*. MOTIVATION OF CENTRAL DIFFERENCE :

- The central difference formulae which are most suited for interpolation near the middle of tabulated set.
- The coefficients in below Central difference formulae are smaller and converge faster than those in Newton's Formulae

*. STIRLING CENTRAL DIFFERENCE :

Ex : Find U_9 if $U_0 = 14$, $U_4 = 24$, $U_8 = 32$, $U_{12} = 35$, $U_{16} = 40$.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	14 y_{-2}				
		10 Δy_{-2}			
4	24 y_{-1}		-2 $\Delta^2 y_{-2}$		
		8 Δy_{-1}		-3 $\Delta^3 y_{-2}$	
8	32 y_0		-5 $\Delta^2 y_{-1}$		10 $\Delta^4 y_{-2}$
		3 Δy_0		7 $\Delta^3 y_{-1}$	
12	35 y_1		2 $\Delta^2 y_0$		
		5 Δy_1			
16	40 y_2				

GAUSS FORWARD :

$$f(a+hu) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2}$$

$a+hu = 9$, $a = 8$, $h = 4$ $\Rightarrow$ $u = 0.25$

$$f(9) = 32 + \frac{0.25}{1!}(3) + \frac{(0.25-1)}{2}(-5) + \frac{(0.25+1)0.25(0.25+1)}{6}(7) + \frac{(0.25)(0.25+1)(0.25-1)(0.25-2)}{24}(10) = 33.1162$$

GAUSS BACKWARD :

$$f(a+hu) = y_0 + \frac{u}{1!}\Delta y_{-1} + \frac{u(u+1)}{2}\Delta^2 y_{-1} + \frac{(u-1)u(u+1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(u-1)u(u+1)(u+2)}{4!}\Delta^4 y_{-2}$$

$$f(9) = 32 + \frac{0.25}{1}(8) + \frac{(0.25)(0.25+1)}{2}(-5) + \frac{(0.25-1)0.25(0.25+1)}{6}(-3)$$

$$+ \frac{(0.25-1)0.25(0.25+1)(0.25+2)}{24}(10) = 33.1162$$

STIRLING FORMULAE :

$$f(a+hu) = y_0 + \frac{u}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right)$$

$$+ \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2}$$

$$f(9) = 32 + 0.25\left(\frac{3+8}{2}\right) + \frac{(0.25)^2}{2}(-5) + \frac{(0.25)((0.25)^2-1)}{6}$$

$$\times\left(\frac{-3+7}{2}\right) + \frac{(0.25)^2((0.25)^2-1)}{24}(10) = 33.1162$$

Also, stirling formula $= \dfrac{\text{Gauss F. Answer} + \text{Gauss Backward Ans}}{2}$

$$= \frac{33.1162 + 33.1162}{2} = \underline{\underline{33.1162}}$$

INTERPOLATION FOR UNEQUAL INTERVALS :

LAGRANGE'S INTERPOLATION for Unequal Interval :

Q : Find the value of y when $x = 10$ by Lagrange's Interpolation formulae

x	5	6	9	11
$f(x) = y$	12	13	14	16

Formula : Let $y = f(x)$ take values $y_0, y_1, y_2, \ldots y_n$ for the arguement x taking values $x_0, x_1, x_2, \ldots x_n$ then the polynomial by Lagrange's interpolation formula is given by

$$f(x) = \sum_{i=0}^{n} L_i y_i = L_0 y_0 + L_1 y_1 + L_2 y_2 + \cdots + L_{n-1} y_{n-1} + L_n y_n$$

where,

$$L_0 = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}$$

$$L_1 = \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}$$

$$L_2 = \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)}$$

$\vdots$

$$L_n = \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

$$f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}[12] + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}[14] + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$

Asked $f(10) = ?$

$$f(10) = \frac{4 \times 1 \times (-2)}{(-1)(-4)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(13) + \frac{(5)(4)(-1)}{(4)(3)(-2)}(14)$$

for

the

$$+ \frac{(5)(4)(1)}{(6)(5)(2)}(16) = 4 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3}$$

$$= 4 + \frac{38}{3} = \frac{50}{3} = \underline{\underline{14.666}}$$

$L_n y_n$

Same using <u>Newton Divided difference</u> :

x_i	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	5	12			
			$\frac{13-12}{6-5} = 1$		
x_1	6	13		$\frac{1/3 - 1}{9-5} = -1/6$	
			$\frac{14-13}{9-6} = 1/3$		$\frac{\frac{2}{15} + 1/6}{11-5} = \frac{1}{20}$
x_2	9	14		$\frac{1 - 1/3}{11-6} = 2/15$	
			$\frac{16-14}{11-9} = 1$		
x_3	11	16			

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) +$$
$$(x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0)$$

$$f(x) = 12 + (x-5)\,1 + (x-5)(x-6)(-1/6) + \frac{(x-5)(x-6)(x-9)}{}\left(\tfrac{1}{20}\right)$$

We need $f(10) = ?$

$$f(10) = 12 + (10-5) + (10-5)(10-6)\left(-\tfrac{1}{6}\right) + (10-5)(10-6)(10-9)\left(\tfrac{1}{20}\right)$$

$$= 12 + 5 + (5)(4)\left(-\tfrac{1}{6}\right) + (5)(4)\left(\tfrac{1}{20}\right) = 18 - \frac{10}{3}$$

$$= \underline{\underline{14.6667}}$$

Q: Use Lagranges formula to fit polynomial to the following data. Hence $y(-2)$, $y(1)$, $y(4)$

x	-1	0	2	3
y	-8	3	1	2

$$f(x) = \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3) +$$

$$\frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1) + \frac{(x-0)(x+1)(x-2)}{(3-0)(3+1)(3-2)}(2)$$

$$= (x-2)(x-3)\left[\frac{-8x}{(-1)(-3)(-4)} + \frac{3(x+1)}{(1)(-2)(-3)}\right] + (x)(x+1)\left[\frac{x-3}{3\cdot2\cdot(-1)} + \frac{2(x-2)}{4\cdot3\cdot1}\right]$$

$$= (x^2-5x+6)\left[\frac{2x}{3} + \frac{x+1}{2}\right] + (x^2+x)\left[\frac{x-2}{6} - \frac{x-3}{6}\right]$$

$$f(x) = \frac{7x^3 - 31x^2 + 28x + 18}{6}$$

$f(1) = 3.666$

$f(-2) = -36.33$

$f(4) = 13.666$ - - - - - .

Q: Apply Lagranges Formulas inversly to find values of x when $y = 19$. Given following:

x	0	1	2
y	0	1	20

→ this as $f(x)$ } continue

→ take this as x

$$x = \frac{(y-1)(y-20)}{(0-1)(0-20)}(0) + \frac{(y-0)(y-20)}{(1-0)(1-20)}(1) + \frac{(y-0)(y-1)}{(20-0)(20-1)}(2)$$

$$x_{\text{at } y=19} = 0 + \frac{(19-0)(19-20)}{1 \cdot (-19)}(1) + \frac{(19-0)(19-1)}{(20)(19)}(2)$$

$$x_{\substack{At \\ y=19}} = 0 + 1 + \frac{18}{10} = \underline{\underline{2.8}}$$

MODULE - 4 : NUMERICAL DIFFERENTIATION

Example : Given a polynomial with following data points:

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
$f(x)$	7.989	8.403	8.781	9.129	9.451	9.75	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.1$ and $x = 1.5$

Finding polynomial at $x = 1.1$ applying Newton's Forward Interpolation formula :

$$f(a+hu) = f(a) + \frac{u}{1!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)$$
$$+ \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 f(a) + \frac{u(u-1)(u-2)(u-3)(u-4)(u-5)}{6!}\Delta^6 f(a)$$

Forward & Backward Table :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
1	7.989						
		0.414					
1.1	8.403		-0.036				
		0.378		0.006			
1.2	8.781		-0.03		-0.002		
		0.348		0.004		0.001	
1.3	9.129		-0.026		-0.001		+0.004 ~~0.002~~ $= +4\times10^{-3}$
		0.322		0.003		+0.002	
1.4	9.451		-0.023		+0.002		
		0.299		0.005			
1.5	9.750		-0.018				
		0.281					
1.6	10.031						

$a = 1$, $f(a) = 7.989$, let u be variable

to find polynomial equation. $h = 0.1$ (step size)

$\Delta f(a) = 0.414$ $\qquad \Delta^3 f(a) = 0.006$

$\Delta^2 f(a) = -0.036$ $\qquad \Delta^4 f(a) = -0.002$

$\Delta^5 f(a) = 0.001$ $\qquad \Delta^6 f(a) = +0.002$

$\cancel{f(x) = f(a) + p\,\Delta y_0}$

$$\boxed{y = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{6!}\Delta^6 y_0 + \cdots}$$ *

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \cdots\right]}$$ *

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6p-6}{6}\Delta^3 y_0 + \frac{12p^2-36p+22}{24}\Delta^4 y_0 + \cdots\right]}$$ *

$$\boxed{p = \frac{x - x_0}{h} = \frac{x - \text{initial value of base}}{h}}$$

x_0 = base near to given value $= a$

IDEA ★ The other way to find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ is, Finding the polynomial first & simplifying it 'y' & then applying 1st derivative, 2nd derivative, etc... [EASY].

$$p = \frac{x - x_0}{h} = \frac{x-1}{0.1}, \quad y_0 = 7.989, \quad \Delta y_0 = 0.414$$

$$\Delta^2 y_0 = -0.036 \qquad \Delta^3 y_0 = 0.006 \qquad \Delta^4 y_0 = \mp 0.002$$

$$\Delta^5 y_0 = 0.001 \qquad \Delta^6 y_0 = -0.004.$$

$$y = 7.989 + \left(\frac{x-1}{0.1}\right)(0.414) + \frac{\left(\frac{x-1}{0.1}\right)\left(\frac{x-1}{0.1}-1\right)}{2}(-0.036)$$

$$+ \frac{\left(\frac{x-1}{0.1}\right)\left(\frac{x-1}{0.1}-1\right)\left(\frac{x-1}{0.1}-2\right)}{6}(0.006) + \frac{\left(\frac{x-1}{0.1}\right)\left(\frac{x-1}{0.1}-1\right)\left(\frac{x-1}{0.1}-2\right)\left(\frac{x-1}{0.1}-3\right)(+0.002)}{24}$$

$$+ \frac{\left(\frac{x-1}{0.1}\right)\left(\frac{x-1}{0.1}-1\right)\left(\frac{x-1}{0.1}-2\right)\left(\frac{x-1}{0.1}-3\right)\left(\frac{x-1}{0.1}-4\right)}{120}(0.001)$$

$$+ \frac{\left(\frac{x-1}{0.1}\right)\left(\frac{x-1}{0.1}-1\right)\left(\frac{x-1}{0.1}-2\right)\left(\frac{x-1}{0.1}-3\right)\left(\frac{x-1}{0.1}-4\right)\left(\frac{x-1}{0.1}-5\right)}{720}(+0.004)$$

$$\boxed{\begin{aligned} y = {} & 2.77778\,x^6 - 20x^5 + 59.0278\,x^4 - 90.5\,x^3 \\ & + 73.5094\,x^2 - 23.953x + 7.127 \end{aligned}}$$ required polynomial

$$y_{\text{at } x = 1.1} = 8.40298 \sim 8.403$$

$y = 2.77778\, x^6 - 20 x^5 + 59.0278\, x^4 - 90.5\, x^3$

$+ 73.5094\, x^2 - 23.953\, x + 7.127$

$$\boxed{y' = 16.66668\, x^5 - 100 x^4 + 236.1112\, x^3 - 271.5\, x^2 + 147.0188\, x - 23.953}$$

$y'_{\text{at } x=1.1} = 3.94854$

$$\boxed{y'' = 83.3334\, x^4 - 400 x^3 + 708.3336\, x^2 - 543x + 147.0188.}$$

y'' at $x = 1.1 = -3.58911$

Now for, Finding polynomial at $x = 1.5$

$B = \frac{\cancel{x - x_0}}{\cancel{h}}$

The polynomial y remains same So y', y'' remains same. We just need to find $y_{\text{at } x=1.5}$ $y'_{\text{at } x=1.5}$ & $y''_{\text{at } x=1.5}$ by substituting.

$y = 9.75008$ at $x = 1.5$

$y' = 2.888101$ at $x = 1.5$

$y'' = -1.85526$ at $x = 1.5$

*. NUMERICAL INTEGRATION :

- The area bounded by the curve $f(x)$ and x-axis between limit a & b, is denoted by,

$$I = \int_a^b f(x)\, dx$$

- Divide the interval (a,b) into n-equal intervals with length h (step size)

i.e. $(a,b) = (a = x_0, x_1, x_2, x_3, \ldots \ldots x_{n-1}, x_n = b)$

$a = x_0$

$x_1 = x_0 + h$

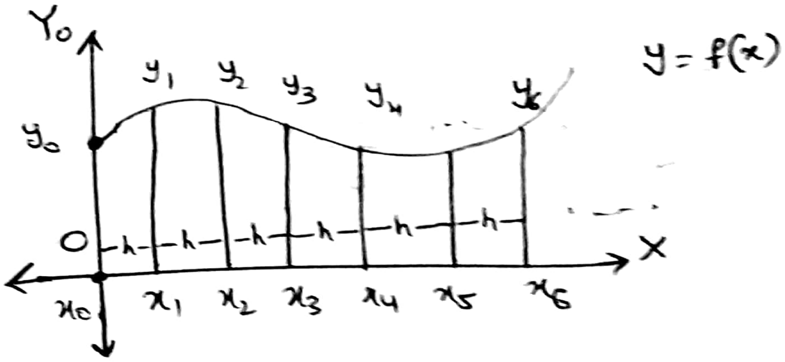
$x_2 = x_1 + h$

$\vdots$

$x_n = x_{n-1} + h$

$$\boxed{x_n = x_0 + n \cdot h}$$

[OR] $\boxed{n = \dfrac{b-a}{h}}$, $\boxed{h = \dfrac{b-a}{n}}$



$I = \int_a^b f(x)\, dx$ can be evaluated using :

(I). TRAPEZOIDAL RULE :

$$\int_a^b f(x)\, dx = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + y_3 + \cdots) \right]$$

Note : It is applicable on any no. of interval (either even, odd).

II. SIMPSONS 1/3 RD RULE :

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots)+2(y_2+y_4+y_6+\cdots)\right]$$

Note : It is applicable if the total no. of intervals is even.

III. SIMPSONS 3/8 th RULE :

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots)+2(y_3+y_6+y_9+y_{12}+\cdots)\right]$$

Note : It is applicable if the total No. of interval is multiple of 3.

Default : If the intervals is not given, take $n=6$ (or) if h is given $\left[n=\frac{b-a}{h}\right]$.

PROBLEMS :

NOTE :

- Numerical integration has numerous practical application in the field of Calculus.

- imp Simpson's 1/3 rule due to its ease in application and higher accuracy is a preferred method in various application areas as given below.

*. Area bounded by curve $y = f(x)$ between the ordinates $x = a$ and $x = b$, above x-axis is given by

$$\boxed{A = \int_a^b y \cdot dy} \; *$$

*. Volume of solid formed by revolving the curve $y = f(x)$ along x-axis between the ordinates $x = a$, $x = b$ is given by

$$\boxed{V = \int_a^b \pi y^2 \, dx} \; *$$

*. Length of an arc of $y = f(x)$ between the ordinates $x = a$; $x = b$ & x-axis is given by,

$$\boxed{\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx} \; *$$

- To find velocity when acceleration at different times is given in Tabular form.

- To find displacement when velocity is given as function of time in discrete form.

* . REMARKS :

- IMP Simphon's rules ideally returns more accurate results compared to Trapezoidal rule provided h is small, (less than one essentially).

- Simphons' $\frac{1}{3}$ rule requires odd number of points (even no. of sub-intervals) for application.

- Simphon's $3/8$ rule requires no. of sub-intervals to be multiple of 3.

PROBLEMS :

① Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule, Simphson's $1/3$ rule & Simphson's $3/8$ rule. Also find the value of π in each case.

Default case : Taking $n = 6 \Rightarrow h = \frac{b-a}{n} = \frac{1-0}{6}$

$$\boxed{h = 1/6}$$

$x_1 = 1/6$

$x_2 = 2 \times 1/6$

$x_3 = 3 \times 1/6$

$x_4 = 4 \times 1/6$

$x_5 = 5 \times 1/6$

$x_6 = 6 \times 1/6$

$y_0 = \frac{1}{1+x_0^2} = 1$

$y_1 = 1/1+x_1^2 = \frac{36}{37}$

$y_2 = 1/1+x_2^2 = 0.9$

$y_3 = 1/1+x_3^2 = 0.8$

$y_4 = 1/1+x_4^2 = 9/13$

$y_5 = 1/1+x_5^2 = \frac{36}{61}$

$y_6 = 1/1+x_6^2 = 0.5$

By Trapezoidal Rule,

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + y_3 + \cdots)\right]$$

Here $n = 6$ $\quad a = 0$ $\quad b = 1$ $\quad h = 1/6$ $\quad f(x) = \frac{1}{1+x^2}$

So, $$\int_0^1 f(x)\,dx = \int_0^1 \frac{1}{1+x^2}\,dx = \frac{\left(\frac{1}{6}\right)}{2}\left[y_0 + y_6 + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{12}\left[1 + 0.5 + 2\left(\frac{36}{37} + 0.9 + 0.8 + \frac{9}{13} + \frac{36}{61}\right)\right]$$

$$= \cancel{0.785396} \quad 0.78423$$

By Simpson's 1/3 Rule:

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{(1/6)}{3}\left[1 + 0.5 + 4\left(\frac{36}{37} + 0.8 + \frac{36}{61}\right) + 2\left(0.9 + \frac{9}{13}\right)\right]$$

$$= 0.785396$$

By Simphson's 3/8 rule:

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)\right]$$

$$= \frac{3\left(1/6\right)}{8}\left[1 + 0.5 + 3\left(\frac{36}{37} + 0.9 + \frac{9}{13} + \frac{36}{61}\right) + 2(0.8)\right]$$

$$= 0.785394$$

Also, By direct integration.

$$\int_0^1 \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_0^1 = \pi/4 - 0 = \pi/4$$

Trapezoidal $\Rightarrow$ $\pi/4 = 0.78423$ $\Rightarrow$ $\pi = 3.1369$

S. 1/3 rule $\Rightarrow$ $\pi/4 = 0.785396$ $\Rightarrow$ $\pi = 3.14158$

S. 3/8 rule $\Rightarrow$ $\pi/4 = 0.785394$ $\Rightarrow$ $\pi = 3.14157$

---

(8) Use Trapezoidal rule to compare $\int_1^2 \frac{dx}{x}$ using three intervals, Compare it with exact. $(n = 3)$

$h = \frac{2-1}{3} = 1/3$, $y = 1/x$

$x_0 = 1$ $\quad$ $y_0 = 1/x_0 = 1$

$x_1 = 1 + 1/3 = 4/3$ $\quad$ $y_1 = 1/x_1 = 3/4$

$x_2 = \frac{4}{3} + \frac{1}{3} = 5/3$ $\quad$ $y_2 = 1/x_2 = 3/5$

$x_3 = \frac{5}{3} + \frac{1}{3} = 2$ $\quad$ $y_3 = 1/x_3 = 1/2$

Applying Trapezoidal rule,

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + y_3 + 2(y_1 + y_2)\right]$$

$$= \frac{(1/3)}{2}\left[1 + 1/2 + 2\left(\frac{3}{4} + 3/5\right)\right] = 0.700$$

By actual Integration,

$$\int_1^2 \frac{1}{x}\,dx = \log x\Big]_1^2 = \log_e 2 - \log_e 1 = \log_e 2$$

$$= 0.693$$

Conclusion :

By applying these methods, we obtain the approximate values whereas by analytical methods, we get exact value. There is some small error in each of these methods which can be neglected.

③ The velocity v of airplane which starts from rest is given at fixed intervals of time t as shown :

t (minutes)	2	4	6	8	10	12	14	16	18	20
$v = f(t)$ km / minutes	8	17	24	28	30	20	12	6	2	0

Estimate the approximate distance covered in 20 minutes.

Sol : Let s be the distance covered at any instant of time t, then :

$$V = \frac{ds}{dt} \quad (\text{or}) \quad ds = v \cdot dt \Rightarrow s = \int v \cdot dt$$

$$S = \int_0^{20} ds = \int_0^{20} v \cdot dt = \frac{h}{3}\Big[(v_0 + v_{10}) + 4(v_1 + v_3 + v_5 + v_7 + v_9) + 2(v_2 + v_4 + v_6 + v_8)\Big]$$

$h = 2$, $n = 10$

$$S = \frac{2}{3}\left[(0+0) + 4\,(8+24+30+12+2) + 2\,(17+28+20+6) \right]$$

$$\therefore \quad \underline{S = 297.33} \quad \text{Km}$$

---

(4) A solid of revolution is formed by rotating about x-axis, the area between x-axis, the line $x = 0$ and a curve through the points with the following coordinates :

x	0	0.25	0.50	0.75	1
$y = f(x)$	1	0.5846	0.5586	0.5085	0.7328

Estimate the volume of solid formed, giving the answer upto 3 decimal places.

Sol :

$$\boxed{V = \int_a^b \pi y^2 \, dx} \; *$$

Applying Simpson's 1/3 rule with $h = 0.25$ $n = 4$

$$V = \pi \int_0^1 y^2 \, dx = \frac{\pi h}{3}\left[y_0^2 + y_4^2 + 4\,(y_1^2 + y_3^2) + 2y_2^2 \right]$$

$$V = \frac{\pi}{12}\left\{ \left[1^2 + (0.7328)^2\right] + 4\left[(0.5846)^2 + (0.5085)^2\right] + 2\left[(0.5586)^2\right] \right\}$$

$$= \frac{\pi}{12}\left[1.5370 + 4\,(0.60033) + 2\,(0.31203) \right]$$

$$= \underline{1.944}$$

* GAUSSIAN QUADRATURE :

- The term "Quadrature" refers to methods in which the points where the function is evaluated are chosen, so that the formula is Exact for polynomials of as high as a degree as possible.
- The most basic of these methods is "GAUSSIAN QUADRATURE"
- The general form of Gaussian Quadrature formula is,

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} c_i f(x_i)$$

choice of n

Note: This integration formula is exact for polynomial of degree upto "$2n-1$"

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2) + \dots + c_n f(x_n)$$

For $n=1$: $\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1)$ is exact for polynomial of degree 1

$$\boxed{\int_{-1}^{1} f(x)\,dx = 2 \cdot f(0) \quad \rightarrow \text{for } n=1}$$

$$\boxed{\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \quad \rightarrow \text{for } n=2}$$

$$\boxed{\int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\frac{\sqrt{3}}{\sqrt{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\frac{\sqrt{3}}{\sqrt{5}}\right) \quad \rightarrow \text{for } n=3}$$

$n=1 \rightarrow$ one point Gaussian Quadrature

$n=2 \rightarrow$ two point Gaussian Quadrature

$n=3 \rightarrow$ three point Gaussian Quadrature

Ex : Solve $\int_{-1}^{1} e^{-x^2}\,dx$

For n = 1 :

$$\int_{-1}^{1} f(x)\,dx = 2\cdot f(0)$$

$f(x) = e^{-x^2}$

$f(0) = e^{-(0)^2} = 1$

$$\int_{-1}^{1} e^{-x^2}\,dx = 2(1) = \underline{\underline{2}}$$

For n = 2 :

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$f(x) = e^{-x^2}$

$f\left(\frac{-1}{\sqrt{3}}\right) = e^{-\frac{1}{3}}$

$f\left(\frac{1}{3}\right) = e^{-\frac{1}{3}}$

$$\int_{-1}^{1} e^{-x^2}\,dx = e^{-\frac{1}{3}} + e^{-\frac{1}{3}} = \underline{1.433}$$

For n = 3 :

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9}\, f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}\, f(0) + \frac{5}{9}\, f\left(\sqrt{\frac{3}{5}}\right)$$

$$\int_{-1}^{1} e^{-x^2}\,dx = \frac{5}{9}\, e^{-3/5} + \frac{8}{9}\cdot 1 + \frac{5}{9}\cdot e^{-3/5} = \underline{1.49868}$$

The above is valid / restricted only if $-1 \le x \le 1$.

If $a \le x \le b$ then :

$$\int_{a}^{b} f(t)\,dt = \int_{-1}^{1} f\left[\frac{(b-a)x + (b+a)}{2}\right] dx$$

solve like previous.

Put $t = \dfrac{(b-a)x + (b+a)}{2}$

If $t = a \Rightarrow x = -1$

If $t = b \Rightarrow x = +1$

Ex : Evaluate $\int_0^2 e^{-t^2} dt = ?$

Here $a = 0$, $b = 2$

$$t = \frac{(b-a)x + (b+a)}{2} = \frac{(2-0)x + (2+0)}{2} \Rightarrow t = x+1 .$$

$t = x+1 \Rightarrow dt = dx$

$$\int_0^2 e^{-t^2} dt = \int_{-1}^{1} e^{-(x+1)^2} dx$$

Now. $\int_{-1}^{1} f(x)\, dx$ with $f(x) = e^{-(x+1)^2}$. Apply

Gaussian Quadrature one point $(n=1)$, two point $(n=2)$, three point $(n=3)$ & solve.

At $n = 2 \Rightarrow e^{-\left(\frac{-1}{\sqrt{3}}+1\right)^2} + e^{-\left(\frac{1}{\sqrt{3}}+1\right)^2} \approx 0.9195$

Similarly do at $n=1$, $n=3$.

SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

*. OF FIRST ORDER & FIRST DEGREE BY

NUMERICAL ANALYSIS

*. PICARD METHOD : (Successive Approximation)

$$\boxed{y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\, dx}$$

* Formula
$n = 0, 1, 2, \ldots$

Ex: Solve by Picard Method (upto 3rd Approximation)

given $\frac{dy}{dx} = x + y^2$, $y(0) = 0$. Also find $y(0.1)$

Sol: $f(x,y) = x + y^2$

~~$x_0 = y(0) = 0$~~

$x_0 = 0$, $y_0 = 0$

$$\boxed{y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\, dx}$$

Put $n=0$

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$= 0 + \int_{0}^{x} (x + 0)\, dx = \left.\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}$$

Put $n = 1$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$= 0 + \int_{0}^{x} \left(x + \frac{x^4}{4}\right) dx$$

$$= \frac{x^2}{2} + \frac{x^5}{20}$$

Put $n=2$ $\quad y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$

$$= 0 + \int_0^x \left(x + \left(\frac{x^2}{2} + \frac{x^5}{20}\right)^2\right) dx$$

$$= \int_0^x x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}\, dx = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}$$

$$y(0.1) = y_3(0.1) = \frac{(0.1)^2}{2} + \frac{(0.1)^5}{20} + \frac{(0.1)^8}{160} + \frac{(0.1)^{11}}{4400}$$

$y(0.1)$ at $n=3$ $=$

Ex: Solve by Picard method, find successive approximation solution upto 4th order of Initial value problem

$$y' + y = e^x, \quad y(0) = 0$$

Sol: $\quad y' + y = e^x \Rightarrow \dfrac{dy}{dx} = e^x - y$ $\quad$ with $x_0 = 0$, $y_0 = 0$

$$f(x, y) = e^x - y$$

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\, dx$$

$$y_{n+1} = 0 + \int_0^x (e^x - y_n)\, dx \approx \int_0^x e^x - y_n\, dx$$

Put $n=0$: $\quad y_1 = \int_0^x e^x - y_0 = e^x - 1$

Put $n=1$: $\quad y_2 = \int_0^x (e^x - (e^x - 1))\, dx = x$

Put $n=2$: $y_3 = \int_0^x (e^x - x)\, dx = e^x - \frac{x^2}{2} - 1$

Put $n=3$: $y_4 = \int_0^x \left(e^x - e^x + \frac{x^2}{2} + 1\right) dx = \frac{x^3}{6} + x$

$\therefore$ upto 4th order is, $n=3 \Rightarrow \frac{x^3}{6} + x$

---

*. TAYLOR'S SERIES METHOD :

- Consider given differential equation, $\frac{dy}{dx} = f(x,y)$ with initial conditions $y(x_0) = y_0$
- Then, the Taylor's series for $y(x)$ around $x = x_n$. with $h = x_{n+1} - x_n$ is given by,

$$y_{n+1} = y_n + h \cdot y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$$

Ex :

Solve $\frac{dy}{dx} = x+y$ by Taylor's Series method, start from $x=1$, $y=0$ and carry to $x = 1.2$ with $h = 0.1$

Sol : Given, $\frac{dy}{dx} = x+y = f(x,y)$

Also, $x_0 = 1$, $y_0 = 0$, $h = 0.1$

$x_1 = x_0 + h = 1 + 0.1 = 1.1$

$y' = \frac{dy}{dx} = x+y$

$y_0 = 0$

$y' = x + y$ $\quad\quad y_0' = x_0 + y_0 = 1 + 0 = 1$

$y'' = 1 + y'$ $\quad\quad y_0'' = 1 + y_0' = 1 + 1 = 2$

$y''' = y''$ $\quad\quad y_0''' = y_0'' = 2$

$y^{IV} = y'''$ $\quad\quad y_0^{IV} = y_0''' = 2$

and so on

By Taylor's Series,

$$y_1 = y(x_1) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV} + \cdots$$

$$y_1 = y(1.1) = 0 + (0.1)(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2) + \cdots$$

$$y_1 = 0.11034 \approx 0.1103 \quad \text{(upto 4 decimals)}$$

Now, $x_1 = 1.1$, $y_1 = 0.1103$, $h = 0.1$

$\therefore x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

Asked in question $y_2 = ?$ at $x_2 = 1.2$

$y_1' = x_1 + y_1 = 1.1 + 0.1103 = 1.2103$

$y_1'' = 1 + y_1' = 1 + 1.2103 = 2.2103$

$y_1''' = y_1'' = 2.2103$; $\quad y_1^{IV} = y_1''' = 2.2103$

By Taylor's series,

$$y_2 = y(x_2) = y_1 + h\cdot y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y^{IV} + \cdots$$

$$y(1.2) = (0.1103) + (0.1)(1.2103) + \frac{(0.1)^2}{2!}(2.2103) + \frac{(0.1)^3}{3!}(2.2103) + \frac{(0.1)^4}{4!}(2.2103)$$

$y(1.2) = y_2 = 0.24276 \approx 0.2428$ (upto 4 decimals)

$\therefore$ At $x = 1.2 \Rightarrow y_2 = 0.2428$

Ex: Use Taylor's series method to solve $\frac{dy}{dx} = 2y + 3e^x$ with initial conditions, $x_0 = 0$, $y_0 = 1$.

Find approximate value of y for $x = 0.1$ & $x = 0.2$

Sol:
$$\frac{dy}{dx} = 2y + 3e^x = f(x, y)$$

$x_0 = 0 \quad y_0 = 1$

$x_1 = 0.1 \Rightarrow h = x_1 - x_0 = 0.1 - 0 = 0.1$

$y_0 = 1$

$y' = \frac{dy}{dx} = 2y + 3e^x \qquad y_0' = 2y_0 + 3e^{x_0} = 2(1) + 3e^{(0)} = 5$

$y'' = 2y' + 3e^x \qquad y_0'' = 2y_0' + 3e^{x_0} = 13$

$y''' = 2y'' + 3e^x \qquad y_0''' = 2y_0'' + 3e^{x_0} = 29$

$y^{IV} = 2y''' + 3e^x \qquad y_0^{IV} = 2y_0''' + 3e^{x_0} = 61$

By Taylor's series,

$$y_1 = y(x_1) = y_0 + h \cdot y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV} + \cdots$$

$$y(0.1) = 1 + (0.1)(5) + \frac{(0.1)^2}{2!}(13) + \frac{(0.1)^3}{3!}(29) + \frac{(0.1)^4}{4!}(61) + \cdots$$

$y(0.1) = y_1$ at $x = 0.1 = 1.5701$

≈ 1.57 (upto 2 decimals)

Now, $x_1 = 0.1$, $y_1 = 1.570$

$x_2 = 0.2 \Rightarrow h = x_2 - x_1 = 0.2 - 0.1 = 0.1$

Now, $y_1' = 2y_1 + 3e^{x_1} = 2(1.570) + 3e^{(0.1)} = 6.4555$

$y_1'' = 2y_1' + 3e^{x_1} = 2(6.4555) + 3e^{(0.1)} = 16.2265$

$y_1''' = 2y_1'' + 3e^{x_1} = 2(16.2265) + 3e^{(0.1)} = 35.7685$

$y_1^{IV} = 2y_1''' + 3e^{x_1} = 2(35.7685) + 3e^{(0.1)} = 74.8525$

By Taylor's Series,

$$y_2 = y(x_2) = y_1 + h \cdot y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{IV} + \cdots$$

$$= 1.570 + (0.1)(6.4555) + \frac{(0.1)^2}{2!}(16.2265) + \frac{(0.1)^3}{3!}(35.7685)$$

$$+ \frac{(0.1)^4}{4!}(74.8525) + \cdots$$

y_2 at $x = 0.2 = y(0.2) = y(x_2) = 2.3029$

y_1 at $x = 0.1 = 1.570$ (upto 3 decimal places)

y_2 at $x = 0.2 = 2.303$

---

*. EULER METHOD : [RUNGE - KUTTA METHOD OF I ORDER]

- Formula :

$$\boxed{y_{n+1} = y_n + h \cdot f(x_n, y_n)}$$

Euler's Formula Method

Ex: Solve $\frac{dy}{dx} = x+y$ with boundary condition $y=1$ at ,

Find approximate value at y for $x = 0.1$ with

stepsize of 5. (Use Euler's Formula to evaluate)

Sol: $h = \frac{0.1}{5} = 0.02$

$x_0 = 0$ $\quad$ $x_2 = 0.04$ $\quad$ $x_4 = 0.08$

$x_1 = 0.02$ $\quad$ $x_3 = 0.06$ $\quad$ $x_5 = 0.1$

$f(x,y) = x+y$ $\qquad$ $f(x_n, y_n) = x_n + y_n$

By Euler's Formula:

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

$$y_{n+1} = y_n + h\,(x_n + y_n)$$

Put $n=0$: $\quad y_1 = y_0 + h(x_0 + y_0) = 1 + 0.02(0+1)$

$y_1 = 1.02$

Put $n=1$: $\quad y_2 = y_1 + h\,(x_1 + y_1) = 1.02 + 0.02\,(0.02 + 1.02)$

$y_2 = 1.0408$

Put $n=2$: $\quad y_3 = y_2 + h(x_2 + y_2) = 1.0408 + (0.02)\begin{pmatrix} 0.04 + \\ 1.0408 \end{pmatrix}$

$y_3 = 1.0624$

Put $n=3$ $\Rightarrow$ $y_4 = y_3 + h\,(x_3 + y_3) = 1.0848$

Put $n=4$ $\Rightarrow$ $y_5 = y_4 + h(x_4 + y_4) = 1.1081$

*. EULER'S MODIFIED METHOD :

- It is also called " Runge Kutta Method of II order ".

Formula *

$$\boxed{\begin{aligned} y^*_{n+1} &= y_n + h\, f(x_n, y_n) \\ y_{n+1} &= y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1}) \right] \end{aligned}}$$

* Modify Euler

Ex :

Given $\frac{dy}{dx} = x^2 + y$ with $y(0) = 1$, find $y(0.02)$ & $y(0.04)$ by Euler's modified method.

Sol : Given, $f(x, y) = x^2 + y$

$y_0 = 1$

$x_0 = 0$] $h = 0.02$ (difference) $y_1^* = ?$

$x_1 = 0.02$] $h = 0.02$

$x_2 = 0.04$ $y_2^* = ?$

Put n=0 .

$$y_1^* = y_0 + h\, f(x_0, y_0)$$

$$y_1^* = y_0 + h\left(x_0^2 + y_0^2\right) = 1 + 0.02\left(0^2 + 1^2\right)$$

$$y_1^* = 1.02$$

Put n=0 in Eulers' Modified

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1}) \right]$$

$$y_1 = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^*) \right)$$

$$y_1 = y_0 + \frac{h}{2}\left((x_0^2 + y_0) + (x_1^2 + y_1^*) \right)$$

$$y_1 = 1 + \frac{0.02}{2}\left((0^2+1) + (0.02)^2 + 1.02\right) = 1.0202$$

Again put $n = 1$ in Euler Formula:

$$y_2^* = y_1 + h \cdot f(x_1, y_1)$$

$$y_2^* = y_1 + h(x_1^2 + y_1) = 1.0202 + 0.02\left((0.02)^2 + 1.0202\right)$$

$$y_2^* = 1.0406$$

Put $n = 1$ in Euler's Modified formula:

$$y_2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^*)\right]$$

$$y_2 = y_1 + \frac{h}{2}\left[(x_1^2 + y_1) + (x_2^2 + y_2^*)\right]$$

$$y_2 = 1.0202 + \frac{0.02}{2}\left[(0.02)^2 + 1.0202 + (0.04)^2 + 1.0406\right]$$

$$\boxed{y_2 = 1.0408} \quad \& \quad \boxed{y_1 = 1.0202}$$

(*) RUNGE - KUTTA METHOD OF FOURTH ORDER :

Very Very IMportant

- Consider initial value problem $\frac{dy}{dx} = f(x, y)$

where $y(x_0) = y_0$

$$K_1 = h\, f(x_n, y_n)$$

$$K_1 = h \cdot f(x_n, y_n)$$

$$K_2 = h \cdot f\left(x_n + \frac{h}{2},\ y_n + \frac{K_1}{2}\right)$$

$$K_3 = h \cdot f\left(x_n + \frac{h}{2},\ y_n + \frac{K_2}{2}\right)$$

$$K_4 = h \cdot f\left(x_n + h,\ y_n + K_3\right)$$

Slopes

$$K = \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

$$\boxed{y_{n+1} = y_n + K} \rightarrow \text{solution.}$$

Ex: Given: $\frac{dy}{dx} = x + y^2$ with $y(0) = 1$. Find $y(0.2)$ where $h = 0.1$ using R.K method of IV order.

Sol: $f(x, y) = x + y^2$

$x_0 = 0$ $\quad y_0 = 1$

$x_1 = 0.1$

$x_2 = x_1 + h = 0.2$

$x_3 = x_2 + h = 0.3 \ldots$

Put $n = 0$:

$$K_1 = h \cdot f(x_0, y_0)$$
$$= h \cdot (x_0 + y_0^2) = 0.1$$

$$K_2 = h \cdot f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right)$$

$$= 0.1\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)^2\right] = 0.11525$$

$$K_3 = h \cdot f\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right)$$

$$= 0.1\left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.11525}{2}\right)^2\right] = 0.1169$$

$$K_4 = h \cdot f\left(x_0 + h\ ,\ y_0 + K_3\right)$$

$$= h \cdot \left[(x_0 + h) + (y_0 + K_3)^2 \right] = 0.1 \left[(0 + 0.1) + (1 + 0.1 \right.$$

$$= 0.1347$$

$$K = \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

$$= \frac{1}{6}\left(0.1 + 2(0.11525) + 2(0.1169) + 0.1347\right)$$

$$K = 0.1165$$

Put $n=0$

$$y_{n+1} = y_n + K \Rightarrow y_1 = y_0 + K$$

$$y_1 = 1 + 0.1165 = \underline{\underline{1.1165}}$$

<u>Put $n = 1$:</u>

$$K_1 = h \cdot f(x_1, y_1) = h(x_1 + y_1^2) = 0.1\left[0.1 + (1.1165)^2\right]$$

$$= 0.1347$$

$$K_2 = h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = h\left(x_1 + \frac{h}{2} + \left(y_1 + \frac{K_1}{2}\right)^2\right) = 0.1552$$

$$K_3 = h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = h\left[x_1 + \frac{h}{2} + \left(y_1 + \frac{K_2}{2}\right)^2\right] = 0.1576$$

$$K_4 = h \cdot f\left(x_1 + h, y_1 + K_3\right) = h\left[x_1 + h + (y_1 + K_3)^2\right] = 0.1823$$

$$K = \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right) = 0.1572$$

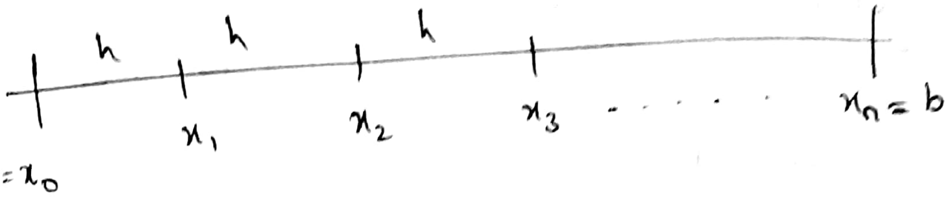
$$y_2 = y_1 + K = 1.1165 + 0.1572 = \underline{\underline{1.2737}}$$

$$\therefore \boxed{y_1 = 1.1165} \ \& \ \boxed{y_2 = 1.2737}$$

*. FINITE DIFFERENCE METHOD :

- This method is used to solve ODEs - Bounded value problems (BVP).

- ~~The general linear two point BVP is~~



$x_1 = x_0 + h$, $x_2 = x_1 + h$, $x_3 = x_2 + h$

$\boxed{x_n = x_0 + n \cdot h}$; $n = 1, 2, 3, \ldots$

*. Approximation of First order derivative :

① FORWARD DIFFERENCE

From Taylor's series,

$$y(x_{i+1}) = y_{i+1} = y_i + h \cdot y_i' + \frac{h^2}{2!} y_i'' + \ldots$$

If we truncate from first order,

$$y_{i+1} = y_i + h \cdot y_i' \Rightarrow \boxed{y_i' = \frac{y_{i+1} - y_i}{h}}$$ * Forward

Here $y(x_i) = y_i$, $y(x_{i+1}) = y_{i+1}$

② BACKWARD DIFFERENCE

From Taylor's series, $y_{i-1} = y_i - h \cdot y_i' + \frac{h^2}{2!} y_i'' + \ldots$

$$y_{i-1} = y_i - h \cdot y_i'$$

$$\boxed{y_i' = \frac{y_i - y_{i-1}}{h}}$$ * Backward

③ CENTRAL DIFFERENCE for II order

$$\boxed{y'(x_i) = \frac{1}{2h}\left[y_{i+1} - y_{i-1}\right]}$$ * central

*. Approximation of second order derivative :

at x_i

$y''(x_i)$, y_i''

- Central difference approximation :

$$\boxed{y''(x_i) = \frac{1}{h^2}\left[y_{i+1} - 2y_i + y_{i-1}\right]}$$ *

- equal space : Forward, Backward, Central, Taylor, Runge Kutta
- Unequal space : Picard's Method.

Ex : Solve the boundary value problem

$xy'' + y = 0$, $y(1) = 1$, $y(2) = 2$ by second order finite difference method with $h = 0.25$

Sol : Given second order finite difference method $\Rightarrow$ Apply Central difference for II order

$$\boxed{y''(x_i) = \frac{1}{h^2}\left[y_{i+1} - 2y_i + y_{i-1}\right]}$$ *

$$n = \frac{b-a}{h} = \frac{2-1}{0.25} = 4$$

$x_0 = 1.0$, $x_1 = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$ $x_4 = 2$

x_0 ↓ boundary point

x_1, x_2, x_3: INTERNAL POINTS

x_4 ↓ boundary point

Find for $i = 1, 2, 3$

$y(1) = y(x_0) = y_0 = 1$

$y(2) = y(x_4) = y_4 = 2$

By using central difference :

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$x \cdot y_i'' + y_i = 0 \Rightarrow x\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] + y_i = 0$$

$$16 \cdot x_i y_{i+1} - 32 x_i y_i + 16 x_i y_{i-1} + y_i = 0$$

$$\boxed{16 x_i y_{i-1} + (1 - 32 x_i) y_i + 16 x_i y_{i+1} = 0}$$

Put $i = 1$: $x_1 = 1.25$ $y_0 = 1.0$

$$16 x_1 y_0 + (1 - 32 x_1) y_1 + 16 x_1 y_2 = 0$$

$$20 y_0 - 39 y_1 + 20 y_2 = 0$$

★ $-39 y_1 + 20 y_2 = -20$ — (1) ✓

Put $i = 2$: $x_2 = 1.5$

$$16 x_2 y_1 + (1 - 32 x_2) y_2 + 16 x_2 y_3 = 0$$

★ $24 y_1 - 47 y_2 + 24 y_3 = 0$ — (2) ✓

Put $i = 3$: $x_3 = 1.75$, $y_4 = 2.0$

$$16 x_3 y_2 + (1 - 32 x_3) y_3 + 16 x_3 y_4 = 0$$

$$28 y_2 - 55 y_3 + 28 y_4 = 0$$

★ $28 y_2 - 55 y_3 = -56$ — (3) ✓

We got the system of equations as,

$$-39y_1 + 20y_2 = -20$$

$$24y_1 - 47y_2 + 24y_3 = 0$$

$$28y_2 - 55y_3 = -56$$

(Matrix form)

So, converting them into linear system & solve for

$y_1, y_2, y_3 = ?$

$$\begin{bmatrix} -39 & 20 & 0 \\ 24 & -47 & 24 \\ 0 & 28 & -55 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -20 \\ 0 \\ -56 \end{bmatrix}$$

Solving by Gauss Elimination,

$$[A|b] = \begin{bmatrix} -39 & 20 & 0 & -20 \\ 24 & -47 & 24 & 0 \\ 0 & 28 & -55 & -56 \end{bmatrix}$$

$$R_1 \to \frac{R_1}{-39} \Rightarrow \begin{bmatrix} 1 & \frac{-20}{39} & 0 & \frac{20}{39} \\ 24 & -47 & 24 & 0 \\ 0 & 28 & -55 & -56 \end{bmatrix}$$

$$R_2 - 24R_1 \to R_2 \Rightarrow \begin{bmatrix} 1 & \frac{-20}{39} & 0 & \frac{20}{39} \\ 0 & \frac{-1353}{39} & 24 & \frac{-480}{39} \\ 0 & 28 & -55 & -56 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{\frac{-135}{39}} \quad \begin{bmatrix} 1 & -20/39 & 0 & 20/39 \\ 0 & 1 & -936/1353 & 480/1353 \\ 0 & 28 & -55 & -56 \end{bmatrix}$$

$$R_3 - 28R_2 \to R_3 \Rightarrow \begin{bmatrix} 1 & \frac{-20}{39} & 0 & \frac{20}{39} \\ 0 & 1 & \frac{-936}{1353} & \frac{480}{1353} \\ 0 & 0 & \frac{-48207}{1353} & \frac{-89208}{1353} \end{bmatrix}$$

Back substitution,

$$\frac{-48207}{1353} y_3 = \frac{-89208}{1353} \Rightarrow y_3 = 1.85052$$

$$y_2 - \frac{936}{1353} y_3 = \frac{480}{1353} \rightarrow y_2 = 1.63495$$

$$y_1 - \frac{20}{39} y_2 = \frac{20}{39} \Rightarrow y_1 = 1.35126$$

$\therefore$ Internal point values are y_1, y_2, y_3.

Ex: Using second order finite difference method, find $y(0.25)$, $y(0.5)$, $y(0.75)$ satisfying the differential equation $y'' - y = x$ and subject to the conditions $y(0) = 0$, $y(1) = 2$.

Sol: $y'' - y = x$, $y(0) = 0$, $y(1) = 2$

$h = 0.25$, $n = \frac{1-0}{0.25} = 4$

So, $y_0 = 0$ $\quad y_4 = y(1) = 2$

$$\boxed{\frac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right) - y_i = x_i}$$

Substitute it in diff. eq.

$\Rightarrow \quad 16y_{i-1} - 33y_i + 16y_{i+1} = x_i$

For $i=1$ $\Rightarrow$ $x_1 = 0.25$ $\quad y_0 = 0$

$$16y_0 - 33y_1 + 16y_2 = 0.25$$

★ $-33y_1 + 16y_2 = 0.25$ —— (1) ✓

For $i=2$ $\Rightarrow$ $x_2 = 0.5$

★ $16y_1 - 33y_2 + 16y_3 = 0.5$ —— (2) ✓

For $i=3$ $\Rightarrow$ $x_3 = 0.75$, $y_4 = 2$

$$16y_2 - 33y_3 + 16y_4 = 0.75$$

★ $16y_2 - 33y_3 = -31.25$ —— (3) ✓

$$\begin{pmatrix} -33 & 16 & 0 \\ 16 & -33 & 16 \\ 0 & 16 & -33 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.5 \\ -31.25 \end{pmatrix}$$

Solving using Gauss Elimination,

$$\begin{bmatrix} 1 & -0.48485 & 0 \\ 0 & 1 & -0.63385 \\ 0 & 0 & -22.8584 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -0.007576 \\ -0.02461 \\ -30.85624 \end{bmatrix}$$

$$\left.\begin{aligned} \Rightarrow \quad y_1 &= 0.39534 \\ y_2 &= 0.83102 \\ y_3 &= 1.34989 \end{aligned}\right\} \text{Solution of Internal points}$$

MODULE-6 : Numerical Methods for PDE

*. FINITE DIFFERENCE METHOD for PDE :

- The general linear partial differential equation of the second order in two independent variables is of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2}$$

$$+ F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

- Such a PDE is said to be

$$\left.\begin{array}{ll} \text{i. elliptical, if} & B^2-4AC<0 \\ \text{ii. parabolic, if} & B^2-4AC=0, \\ \text{iii. Hyperbolic, if} & B^2-4AC>0. \end{array}\right\} \text{nature of PDE}$$

*. Classification of PDE :

① $U_{xx} + 4U_{xy} + 4U_{yy} - U_x + 2U_y = 0$

$A = 1, \quad B = 4, \quad C = 4$

$B^2-4AC = 16-16 = 0$ (parabolic)

② $x^2\frac{\partial^2 u}{\partial x^2} + (1-y^2)\frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty$
$\quad -1 < y < 1$

$x^2 U_{xx} + (1-y^2) U_{yy} = 0$

$A = x^2, \quad C = 1-y^2, \quad B = 0$

$$B^2 - 4AC = -4x^2(1-y^2) = 4x^2(y^2-1)$$

if $x \in (-\infty, \infty) \Rightarrow x^2$ is $+$ve

$y \in (-1, 1) \Rightarrow y^2$ is $+$ve, $y^2 - 1 < 0$

$\therefore\ 4x^2(y^2-1) \Rightarrow 4x^2(y^2-1) < 0$

$B^2 - 4AC < 0 \Rightarrow$ elliptical.

③ $$(1+x^2)\frac{\partial^2 u}{\partial x^2} + (5+2x^2)\frac{\partial^2 u}{\partial x \partial t} + (4+x^2)\frac{\partial^2 u}{\partial t^2} = 0$$

$A = 1+x^2$, $B = 5+2x^2$, $C = 4+x^2$

$$B^2 - 4AC = (5+2x^2)^2 - 4(1+x^2)(4+x^2)$$

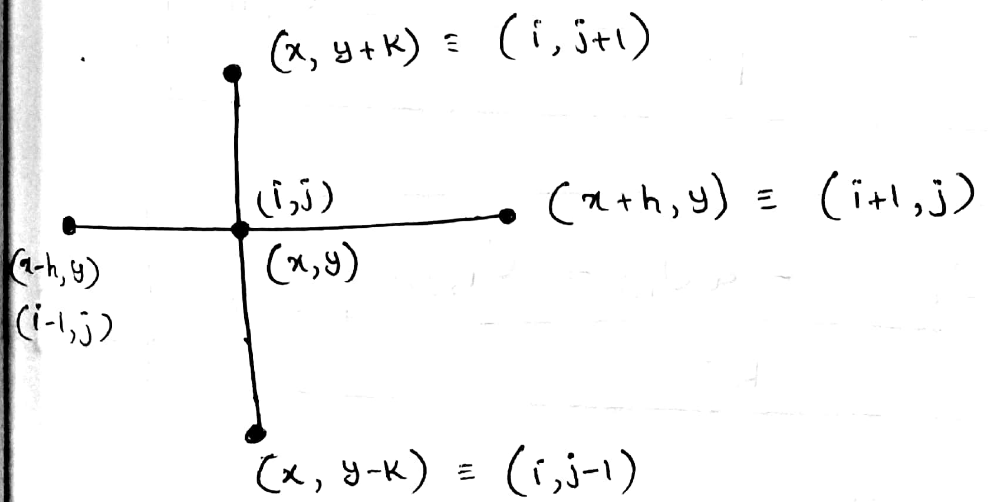
$$= 25 + 4x^2 + 20x^2 - 4x^2 - 16 - 20x^2 = 9 > 0$$

$\therefore$ Hyperbolic is the nature of PDE.

---

* Forward PDE :

$\Delta x = h$
$\Delta y = K$



* $$\boxed{\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h)}$$ Forward w.r.t x
order I

* $$\boxed{\frac{\partial u}{\partial x} = \frac{U_{i,j} - U_{i-1,j}}{h} + O(h)}$$ Backward w.r.t x order I

* $$\boxed{\frac{\partial u}{\partial x} = \frac{U_{i+1,j} - U_{i-1,j}}{2h} + O(h^2)}$$ Central w.r.t. x order I

* $$\boxed{\frac{\partial u}{\partial y} = \frac{U_{i,j+1} - U_{i,j}}{k} + O(k)}$$ Forward w.r.t y order I

* $$\boxed{\frac{\partial u}{\partial y} = \frac{U_{i,j} - U_{i,j-1}}{k} + O(k)}$$ Backward w.r.t y order I

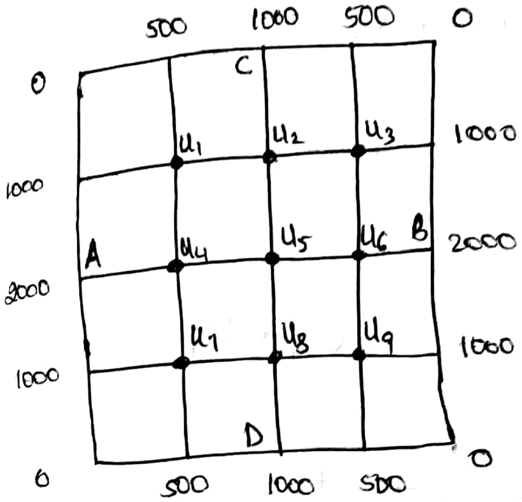
* $$\boxed{\frac{\partial u}{\partial y} = \frac{U_{i,j+1} - U_{i,j-1}}{2k} + O(k^2)}$$ Central w.r.t x order I

* $$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2} + O(h^2)}$$ Central w.r.t x order II

* $$\boxed{\frac{\partial^2 \upsilon}{\partial y^2} = \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{k^2} + O(k^2)}$$ Central II order

FINITE DIFFERENCE SOLUTION OF LAPLACE EQUATION :

*. Ex: Solve the elliptical equation $U_{xx} + U_{yy} = 0$ for the square mesh of the following figure with boundary values as shown.



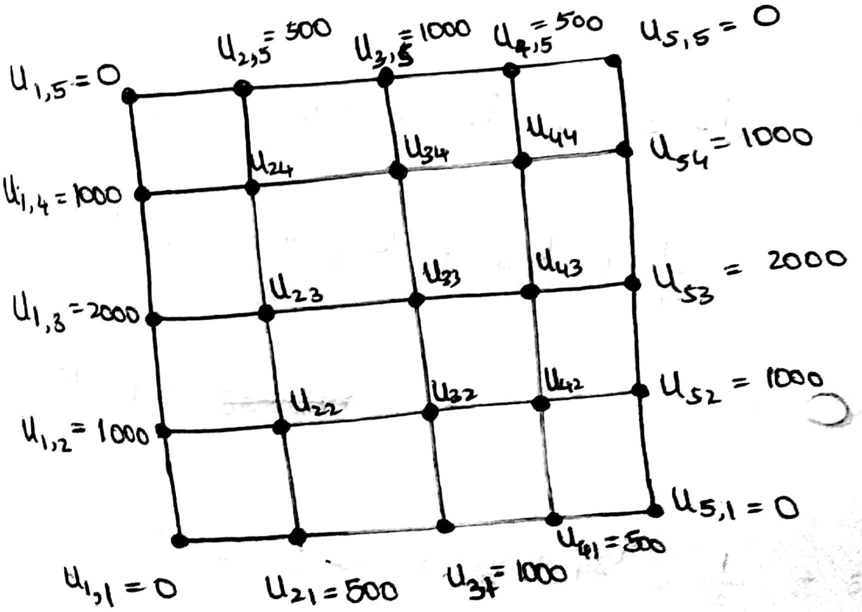
$$U_{xx} + U_{yy} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0 \qquad (h = k)$$

$\downarrow$

Applying standard 5-point formula :

$$\boxed{u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]} \; *$$



$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

$2 \le i, j \le 4$ take all $u_{ij} = 0$, $2 \le i, j \le 4$

First iteration : $j = 2$ & $i = 2, 3, 4$

$i = 2 \Rightarrow U_{22} = \frac{1}{4}\left(U_{12} + U_{32} + U_{21} + U_{23}\right)$

$= \frac{1}{4}(1000 + 0 + 500 + 0) = 375$

$i = 3 \Rightarrow U_{32} = \frac{1}{4}\left(U_{22} + U_{33} + U_{31} + U_{42}\right)$

$= \frac{1}{4}(375 + 0 + 1000 + 0) = 343.75$

$i = 4 \Rightarrow U_{42} = \frac{1}{4}\left(U_{32} + U_{41} + U_{43} + U_{52}\right)$

$= \frac{1}{4}(343.75 + 500 + 0 + 1000) = 460.94$

we obtained : $U_{22} = 375$ $U_{32} = 343.75$,

$U_{42} = 460.94$

~~Second iteration~~ : $j = 3$ & $i = 2, 3, 4$

$i = 2 \Rightarrow U_{23} = \frac{1}{4}\left(U_{13} + U_{33} + U_{22} + U_{24}\right)$

$= \frac{1}{4}(2000 + 375 + 375 + 0) = \cancel{234.38}\ 593.75$

$i = 3 \Rightarrow U_{33} = \frac{1}{4}\left(U_{34} + U_{32} + U_{23} + U_{43}\right)$

$= \frac{1}{4}(0 + 343.75 + 593.75 + 0) = 234.38$

$i=4 \Rightarrow \quad U_{43} = \frac{1}{4}(U_{44} + U_{53} + U_{42} + U_{33})$

$= \frac{1}{4}(0 + 2000 + 460.94 + 234.38)$

$= 673.83$

: $j = 4$ & $i = 2, 3, 4$

$i=2 \Rightarrow \quad U_{24} = \frac{1}{4}(U_{25} + U_{34} + U_{23} + U_{14})$

$= \frac{1}{4}(500 + 0 + 593.75 + 1000) = 523.44$

$i=3 \Rightarrow \quad U_{34} = \frac{1}{4}(U_{35} + U_{44} + U_{33} + U_{24})$

$= \frac{1}{4}(1000 + 0 + 234.38 + 523.44) = 439.55$

$i=4 \Rightarrow \quad U_{44} = \frac{1}{4}(U_{45} + U_{54} + U_{43} + U_{34})$

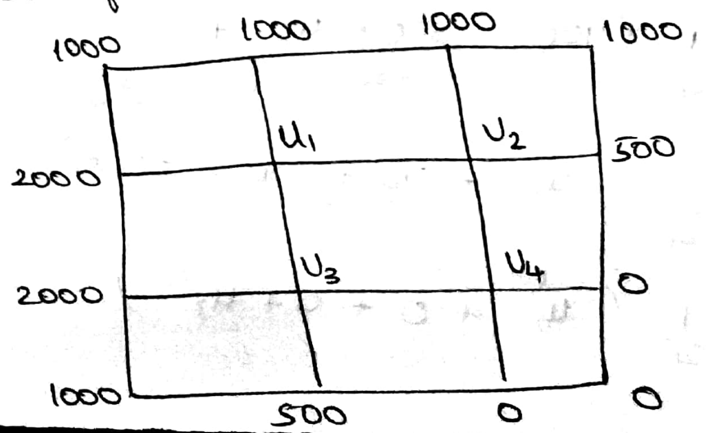
$= \frac{1}{4}(500 + 1000 + 673.83 + 439.55) = 653.345$

Second iteration : (Repeat the same from start)

Stoping criteria is $|\text{previous values} - \text{current value}| < \text{error}$.

---

*. Solve the elliptical equation $U_{xx} + U_{yy} = 0$ for the square mesh of the following figure with boundary values:



Sol : For initial approximation

$$\boxed{u_{ij} = \frac{1}{4}\left(u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}\right)}$$

is known as diagonal 5-point rule / formula.

IMP
- For initial approximation, choose either 5-point rule or diagonal 5-point rule according to more accurate value.

$u_1 = ?$ $u_2 = ?$ $u_3 = ?$ $u_4 = ?$ (Aim to calculate)

$$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000 \quad \text{(Diag.)}$$

$$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625 \quad \text{(Std.)}$$

$$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875 \quad \text{(Std.)}$$

$$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375 \quad \text{(Std.)}$$

So, initial point approx are $u_1 = 1000$, $u_2 = 625$, $u_3 = 875$, $u_4 = 375$

For iteration use standard 5-point formula :

$$u_{ij} = \frac{1}{4}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1}, u_{i,j+1}\right)$$

So,

$$u_1^{(n+1)} = \frac{1}{4}\left(1000 + u_2^{(n)} + u_3^{(n)} + 2000\right)$$

$$u_2^{(n+1)} = \frac{1}{4}\left(1000 + 500 + u_4^{(n)} + u_1^{(n+1)}\right)$$

$$u_3^{(n+1)} = \frac{1}{4}\left(u_1^{(n+1)} + u_4^{(n)} + 500 + 2000\right)$$

$$u_4^{(n+1)} = \frac{1}{4}\left(u_2^{(n+1)} + 0 + 0 + u_3^{(n+1)}\right)$$

First Iteration $\quad i = 0 = n \quad u_1^0 = 1000,\ u_2^0 = 625,\ u_3^0 = 875,$
$u_4^0 = 375$

$$u_1^{(1)} = \frac{1}{4}\left(1000 + u_2^{(0)} + u_3^{(0)} + 2000\right)$$

$$\frac{1}{4}\left(1000 + 625 + 875 + 2000\right) = 1125$$

$$u_2^{(1)} = \frac{1}{4}\left(1000 + 500 + u_4^{(0)} + u_1^{(1)}\right)$$

$$\frac{1}{4}\left(1000 + 500 + 375 + 1125\right) = 750$$

$$u_3^{(1)} = \frac{1}{4}\left(u_1^{(1)} + u_4^{(0)} + 500 + 2000\right)$$

$$= \frac{1}{4}\left(1125 + 375 + 500 + 2000\right) = 1000$$

$$u_4^{(1)} = \frac{1}{4}\left(u_2^{(1)} + 0 + 0 + u_3^{(1)}\right)$$

$$= \frac{1}{4}\left(750 + 0 + 0 + 1000\right) = 438$$

Second iteration $\quad i = 1 = n$

$u_1^{(1)} = 1125, \quad u_2^{(1)} = 750, \quad u_3^{(1)} = 1000, \quad u_4^{(1)} = 438$

$$u_1^{(2)} = \frac{1}{4}\left(1000 + 750 + 1000 + 2000\right) = 1187.5$$

$$u_2^{(2)} = \frac{1}{4}\left(1000 + 500 + 438 + 1187.5\right) = 781.375$$

$$u_3^{(2)} = \frac{1}{4}\left(1187.5 + 438 + 500 + 2000\right) = 1031.375$$

$$u_4^{(2)} = \frac{1}{4}\left(\cancel{781.375}\ 0 + 0 + 1031.375\right) = 453.1875$$

$u_1^{(2)} = 1188, \quad u_2^{(3)} = 782, \quad u_3^{(2)} = 1032, \quad u_4^{(2)} = 454$

3rd iteration : $i = 2 = n$

$u_1^{(3)} = 1204$, $u_2^{(3)} = 789$, $u_3^{(3)} = 1040$, $u_4^{(4)} = 458$

4th iteration : $i = 3 = n$

$u_1^{(4)} = 1207$, $u_2^{(4)} = 791$, $u_3^{(4)} = 1041$, $u_4^{(4)} = 458$

5th iteration : $i = 4 = n$

$u_1^{(5)} = 1208$, $u_2^{(5)} = 791.5$, $u_3^{(5)} = 1041.5$, $u_4^{(5)} = 458.25$

Thus there is no significant difference between the fourth & fifth iteration values. Hence,

$u_1 = 1208$, $u_2 = 792$, $u_3 = 1042$, $u_4 = 458$

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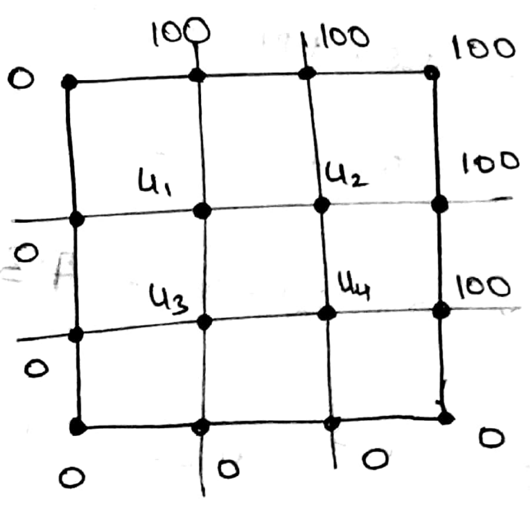
*. Solve the Poisson equation $u_{xx} + u_{yy} = -81xy$ for the square mesh $x = 0$, $y = 0$, $x = 1$, $y = 1$ given that $u(0,y) = u(x,0) = 0$ & $u(1,y) = 100$, $u(x,1) = 100$ and $h = 1/3$



$u_{xx} + u_{yy} = -81xy \quad , \quad 0 < x, y < 1$

$u(0, y) = 0 \quad , \quad u(x, 0) = 0 \quad , \quad u(1, y) = u(x, 1) = 100$

$h = k = \frac{1}{3}$

$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^4(-81)ij$

u_1 : $0 + u_2 + u_3 + 100 - 4u_1 = -2$

$-4u_1 + u_2 + u_3 = -102$ — ①

u_2 : $u_1 + 100 + u_4 + 100 - 4u_2 = -4$

$u_1 - 4u_2 + u_4 = -204$ — ②

u_3 : $0 + u_4 + 0 + u_1 - 4u_3 = -1$

$u_1 - 4u_3 + u_4 = -1$ — ③

u_4 : $u_3 + 100 + u_2 - 4u_4 = -2 \Rightarrow u_2 + u_3 - 4u_4 = -102$ — ④

So, we obtained system of equations :

$-4u_1 + 1u_2 + 1u_3 + 0u_4 = -102$

$1u_1 - 4u_2 + 0u_3 + 1u_4 = -204$

$1u_1 + 0u_2 - 4u_3 + 1u_4 = -1$

$0u_1 + 1u_2 + 1u_3 - 4u_4 = -102$

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -102 \\ -204 \\ -1 \\ -102 \end{bmatrix}$$

Apply Gauss Elimination & solve this system of Linear equation ⟹

$u_1 = 51.0833$

$u_2 = 76.5477$

$u_3 = 25.7916$

$u_4 = 51.0833$

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*. FINITE DIFFERENCE SOLUTION OF PARABOLIC EQUATION :

- The one-dimensional heat conduction equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ is a well known example of parabolic partial differential equation.

- The solution of this equation is a temperature function $u(x,t)$ which is defined for the values of x from 0 to l and for time t from 0 to ∞.

*. SOLUTION OF HEAT EQUATION : (SCHMIDT EXPLICIT)

$$\boxed{u_{i,j+1} = \alpha \cdot u_{i-1,j} + (1-2\alpha)\, u_{i,j} + \alpha \cdot u_{i+1,j}}$$

where $\alpha = \frac{kc^2}{h^2}$ is the mesh ratio parameter

It is the relation between the function values at the two time levels $j+1$ and j & is therefore called a 2-level formula.

If $0 < \alpha \leq \frac{1}{2}$ $\Longrightarrow$ Schmidt emplict formula

If other $\Rightarrow$ Crank Nicolson formula

Ex : Solve the BVP under $U_t = U_{xx}$ under the conditions $u(0,t) = u(1,t)$ and $u(x,0) = \sin(\pi x)$ $0 \leq x \leq 1$, using schmidt method with $h = 0.2$ & $\alpha = 1/2$.

Sol : $U_t = U_{xx}$, here $c^2 = 1$

$h = 0.2$, $\alpha = \frac{1}{2} = \frac{ck}{h^2}$ $\Rightarrow$ $k = 0.02$

$$u_{i,j+1} = \alpha . u_{i-1,j} + (1-2\alpha) u_{i,j} + \alpha . u_{i+1,j}$$

$$\Rightarrow u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j})$$

$$\Rightarrow u_{i,j+1} = \frac{1}{2}(\underbrace{\text{previous state} + \text{next state}}_{\sim \text{ average } \sim})$$

As $h = 0.2$ $\Rightarrow$ $x = 0, 0.2, 0.4, 0.6, 0.8, 1$

$k = 0.02$ $\Rightarrow$ $y = 0, 0.02, 0.04, 0.06, 0.08, 0.1$ $\rightarrow$ Assm

$\Rightarrow$ $u(0,0) = 0$

$u(0.2,0) = \sin\left(\frac{\pi}{5}\right) = 0.5878$

$u(0.4,0) = \sin\left(\frac{2\pi}{5}\right) = 0.95111$

$u(0.6,0) = \sin\left(\frac{3\pi}{5}\right) = 0.9511$

$u(0.8,0) = \sin\left(\frac{4\pi}{5}\right) = 0.5878$

$u(1,0) = \sin(\pi) = 0$

Since $\boxed{u(x,0) = \sin(\pi x)}$ Given

t \ x	0	0.2	0.4	0.6	0.8	1
0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	0	0.4756	0.7695	0.7695	0.4756	0
0.04	0	0.3848	0.6225	0.6225	0.3848	0
0.06	0	0.3113	0.5036	0.5036	0.3113	0
0.08	0	0.2518	0.4074	0.4074	0.2518	0
0.1	0	0.2037	0.3296	0.3296	0.2037	0

Assuming & k=0.02 interval

$u(0,t)=0$ Given

average of these two values ---

average of these two values - - - -

$u(1,t)=0$ given

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*. CRANK – NICOLSON METHOD : (Implicit)

Given, $\dfrac{\partial u}{\partial t} = c^2 \cdot \dfrac{\partial^2 u}{\partial x^2}$

Iteration formula:

$$* \quad \boxed{4u_{i,j+1} = u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}}$$

Ex : Solve $u_t = u_{xx}$, $0 \le x \le 5$, $t \geq 0$ given that $u(x,0) = 20$, $u(0,t) = 0$, $u(5,t) = 100$. Compute u for the time-step with $h=1$ by Crank – Nicolson method.

Sol : Here $c^2 = 1$, $h = 1$

Taking $\alpha = 1 = \dfrac{c^2 k}{h}$ $\Rightarrow k = 1$

Boundary condition:

j \ i	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

Crank - Nikolson Formula:

$$\boxed{4u_{i,j+1} = u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}} \quad *$$

$i=1, j=0 \Rightarrow 4u_{11} = u_{01} + u_{21} + u_{00} + u_{20}$

$\Rightarrow 4u_1 = 0 + u_2 + 0 + 20 \Rightarrow \boxed{4u_1 - u_2 = 20}$ — ①

$i=2, j=0 \Rightarrow 4u_{21} = u_{11} + u_{31} + u_{10} + u_{30}$

$\Rightarrow 4u_2 = u_1 + u_3 + 20 + 20 \Rightarrow \boxed{u_1 - 4u_2 + u_3 = 40}$ — ②

$i=3, j=0 \Rightarrow 4u_{31} = u_{21} + u_{41} + u_{20} + u_{40}$

$\Rightarrow 4u_3 = u_2 + u_4 + 20 + 20 \Rightarrow \boxed{u_2 - 4u_3 + u_4 = -40}$ — ③

$i=4, j=0 \Rightarrow 4u_{41} = u_{31} + u_{51} + u_{30} + u_{50}$

$\Rightarrow 4u_4 = u_3 + 100 + 20 + 100 \Rightarrow \boxed{u_3 - 4u_4 = -220}$ — ④

Solving system of equations ①, ②, ③, ④ by Gauss elimination direct method,

$\Rightarrow u_1 = 10.5$

$u_2 = 30.75$

$u_3 = 30.75$

$u_4 = 62.68$