Rearrangements of Functions

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Outline

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Distribution Function and Rearrangement

Definition

Let $n \in \mathbb{N}$*. We let* $\mathcal{L}_n : \mathcal{L}(\mathbb{R}^n) \to [0, \infty]$ *be the Lebesgue measure on* R^n*, where* $\mathcal{L}(\mathbb{R}^n)$ *is the set of Lebesgue measurable subsets of* $\mathbb{R}^n$*.*

Definition

Let $f : \mathbb{R} \to [0, \infty)$ *be Lebesgue measurable. We define the distribution function of* f*,* $\mu_f : [0, \infty) \to [0, \infty]$ *via*

$$\mu_f(t) = \mathcal{L}_1(\{x \in \mathbb{R} : f(x) > t\}).$$

Definition

Let $f : \mathbb{R} \to [0, \infty)$ *be Lebesgue measurable. We say that* f^* *is a rearrangement of* f *if* $\mu_f = \mu_{f^*}$*.*

A core motivation for this construction is the equality between the integrals of any two rearrangements f and f^*:

$$\int_{\mathbb{R}^n} f(x)dx = \int_0^\infty \mu_f(t)dt = \int_0^\infty \mu_{f^*}(t)dt = \int_{\mathbb{R}^n} f^*(x)dx.$$

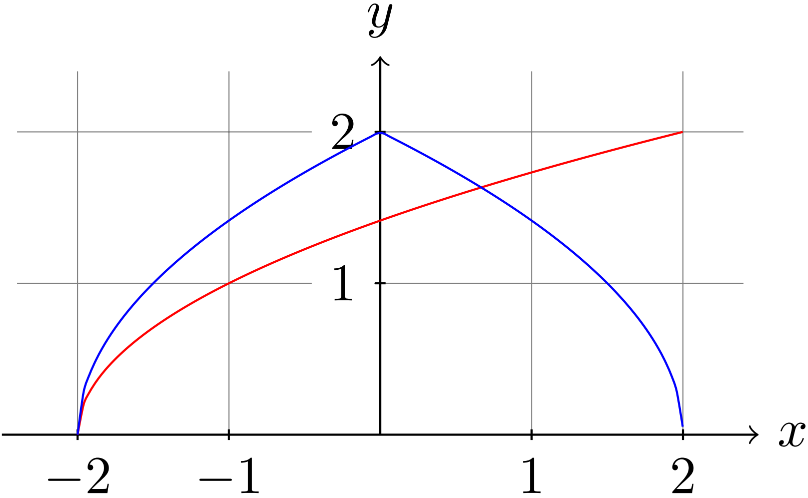
Our project focuses on the preservation of other function characteristics (variation and absolute continuity) for some specific rearrangements (symmetric decreasing and two-point).

Symmetric Decreasing Rearrangement

Definition

Let $f : \mathbb{R} \to [0, \infty)$ *be Lebesgue measurable. We define the symmetric decreasing rearrangement of* f, $f^{\#} : \mathbb{R} \to [0, \infty)$, *via*

$$\begin{aligned} f^{\#}(x) &= \int_0^{\infty} \chi_{U_t}(x)\, dt, \ \text{where } U_t = \left(-\frac{\mu_f(t)}{2}, \frac{\mu_f(t)}{2}\right) \\ &= \sup\{t \in \mathbb{R} : |x| < \mu_f(t)/2\}. \end{aligned}$$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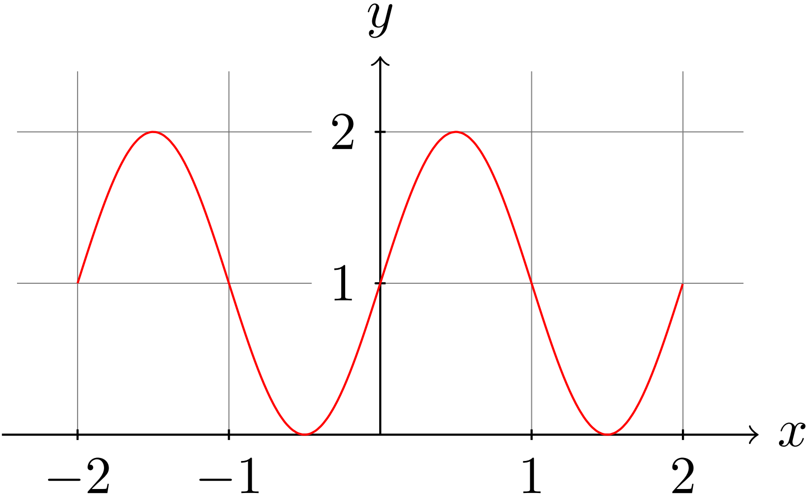
Definition

Let $f : \mathbb{R} \to [0, \infty)$ *and* $\alpha \in \mathbb{R}$. *We set*

$$I_+ = \begin{cases} (-\infty, \alpha) & \alpha \geq 0 \\ (\alpha, \infty) & \alpha < 0 \end{cases} \text{ and } I_- = \begin{cases} (\alpha, \infty) & \alpha \geq 0 \\ (-\infty, \alpha) & \alpha < 0 \end{cases}.$$

We let $\sigma : \mathbb{R} \to \mathbb{R}$ *be the reflection map about* α, *i.e.* $\sigma(x) = 2\alpha - x$. *We then define the two-point rearrangement across* α, $f^\sigma : \mathbb{R} \to \mathbb{R}$, *via*

$$f^\sigma(x) = \begin{cases} \max\{f(x), f(\sigma(x))\} & x \in I_+ \\ \min\{f(x), f(\sigma(x))\} & x \in \mathbb{R} \setminus I_+ \end{cases}$$



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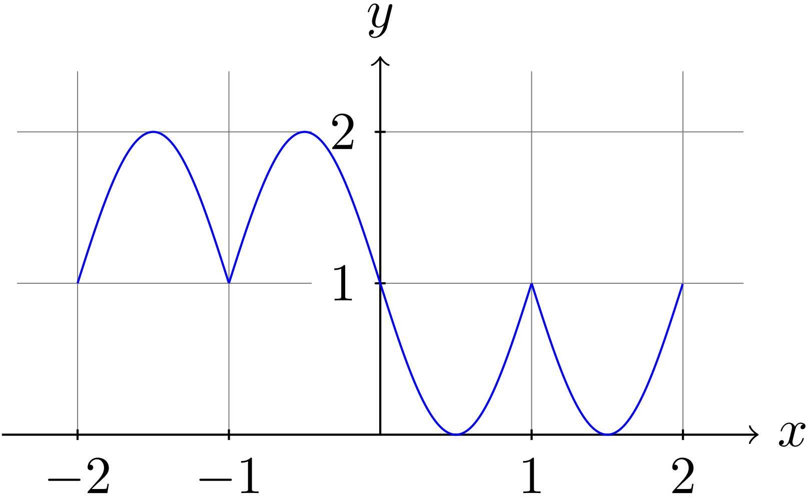
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Definition

Let $E \subseteq \mathbb{R}^n$ and $f : \mathbb{R} \to E$. We define the variation of f as

$$Var(f) = \sup \left\{ \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| \right\}$$

with the supremum taken over all increasing finite sequences $\{x_0, \ldots, x_N\} \subseteq \mathbb{R}$. We can also write $Var_I(f)$ given an interval I to represent the same expression with the restriction that $\{x_0, \ldots, x_N\} \subseteq I$. We say that $f \in BV(I; E)$ (or simply $BV(I)$ when $E = \mathbb{R}$) if $Var_I(f) < \infty$.

Definition

Let $f : [a, b] \to \mathbb{R}^n$ define a curve in $\mathbb{R}^n$. We define the length of such curve to be the variation of f.

To find the length of a curve, we can approximate it with progressively more accurate polygonal curves. Using pointwise variation also allows us to deal with some "messier" functions.

Definition

Let $f \in AC(\mathbb{R})$. We define $AC_f : (0, \infty) \to (0, \infty)$ via

$$AC_f(\delta) = \sup \left\{ \sum_{k=1}^{\ell} |f(b_k) - f(a_k)| \right\},$$

with the supremum taken over all finite collections of nonoverlapping open intervals $\{(a_k, b_k)\}_{k=0}^{\ell}$ with combined length less than δ.

Remark

For a general $f : \mathbb{R} \to \mathbb{R}$, f is absolutely continuous if and only if $\lim_{\delta \to 0} AC_f(\delta) = 0$.

Theorem

Let $f : [a, b] \to \mathbb{R}$. Then f is absolutely continuous in $[a, b]$ iff f is differentiable $\mathcal{L}_1$-a.e. in $[a, b]$, f' is Lebesgue integrable, and the fundamental theorem of calculus is valid, that is, for all $x, x_0 \in [a, b]$,

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\, dt.$$

Theorem (Hilden (5))

Let $f \in W^{1,p}(\mathbb{R}^n)$ be non-negative. Then $f^{\#} \in W^{1,p}(\mathbb{R}^n)$ and

$$\left\|\nabla f^{\#}\right\|_{L^p} \leq \|\nabla f\|_{L^p} \text{ for all } 1 \leq p \leq \infty.$$

(Here $W^{1,p}(\mathbb{R}^n)$ represents a Sobelev Space.)

Theorem (Spencer (1))

Let $f \in C_c^{\infty}(\mathbb{R}^n)$ be non-negative. Then

$$\left\|\nabla f^{\#}\right\|_{L^p} \leq \|\nabla f\|_{L^p} \text{ for all } 1 \leq p \leq \infty.$$

Theorem (Main Result 1)

Let $u \in BV(\mathbb{R}; [0, \infty))$. *Then* $u^{\#} \in BV(\mathbb{R}; [0, \infty))$ *and* $Var(u^{\#}) \leq Var(u)$.

Theorem (Main Result 2)

Let $u \in AC(\mathbb{R}; [0, \infty))$. *Then* $u^{\#} \in AC(\mathbb{R}; [0, \infty))$ *and* $AC_{u^{\#}} \leq AC_u$.

We will draw conclusions about the symmetric decreasing rearrangement using an approximation with a series of two-point rearrangements.

Theorem (Hardy-Littlewood Inequality for Two-Point Rearrangements)

Let $f, g \in L^1(\mathbb{R}^n; [0, \infty))$ and σ be a reflection map. Then

$$\int_{\mathbb{R}^n} f(x)g(x)\ dx \leq \int_{\mathbb{R}^n} f^{\sigma}(x)g^{\sigma}(x)\ dx,$$

with equality iff $(f(x) - f(\sigma(x)))(g(x) - g(\sigma(x))) = 0$ for $\mathcal{L}_1$-a.e. $x \in \mathbb{R}$.

Theorem (Helly's Selection Theorem)

Let $I \subseteq \mathbb{R}$ be an interval and $\{f_n\}_{n=0}^{\infty} \subseteq BV(I; \mathbb{R})$ be a sequence of functions such that $\sup\{Var(f_n) : n \in \mathbb{N}\} < \infty$ and $\{f_n(x) : n \in \mathbb{N}\}$ is bounded for some $x \in I$. Then there exists a subsequence $\{f_{n_k}\}_{k=0}^{\infty}$ and a function $f \in BV(I; \mathbb{R})$ such that $f_{n_k} \to f$ pointwise as $n \to \infty$.

Lemma

Let $(a, b, c, d) \in \mathbb{R}^4$. *Then* $|a - c| + |b - d| \geq |\max\{a, b\} - \max\{c, d\}| + |\min\{a, b\} - \min\{c, d\}|$.

Lemma

Let $f \in BV(\mathbb{R}; [0, \infty))$. *Then* $f^\sigma \in BV(\mathbb{R}; [0, \infty))$ *and* $Var(f^\sigma) \leq Var(f)$.

Lemma

Let $f \in AC(\mathbb{R}; [0, \infty))$. *Then* $f^\sigma \in AC(\mathbb{R}; [0, \infty))$ *and* $AC_{f^\sigma} \leq AC_f$.

It often proves useful to refine our collections of points/intervals to be more symmetric with respect to σ.

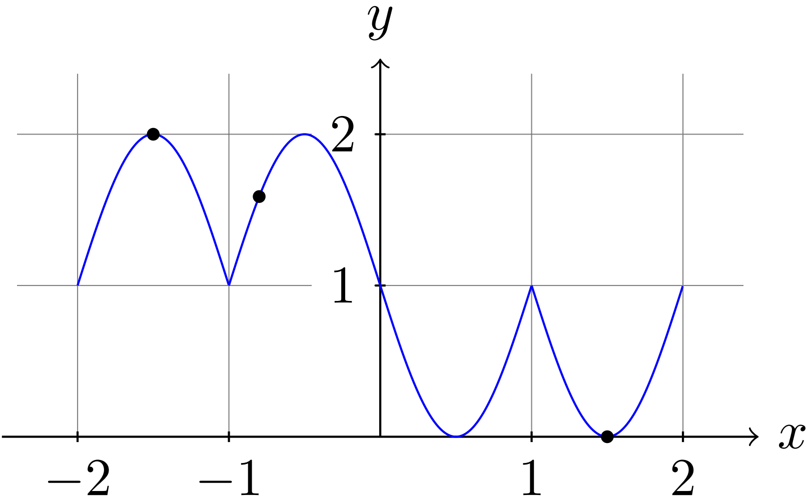
Lemma

Let $f \in BV(\mathbb{R};[0,\infty))$. Then $f^\sigma \in BV(\mathbb{R};[0,\infty))$ and $Var(f^\sigma) \leq Var(f)$.

It suffices to show that for an arbitrary increasing sequence $\{x_0, \dots, x_N\} \subseteq \mathbb{R}$ that

$$\sum_{i=1}^{N} |f^\sigma(x_i) - f^\sigma(x_{i-1})| \leq \mathrm{Var}(f).$$

Towards that end, we refine to a collection $\{y_0, \dots, y_{2\ell}\}$ which is symmetric with respect to σ and use this new sequence in conjunction with the first lemma.



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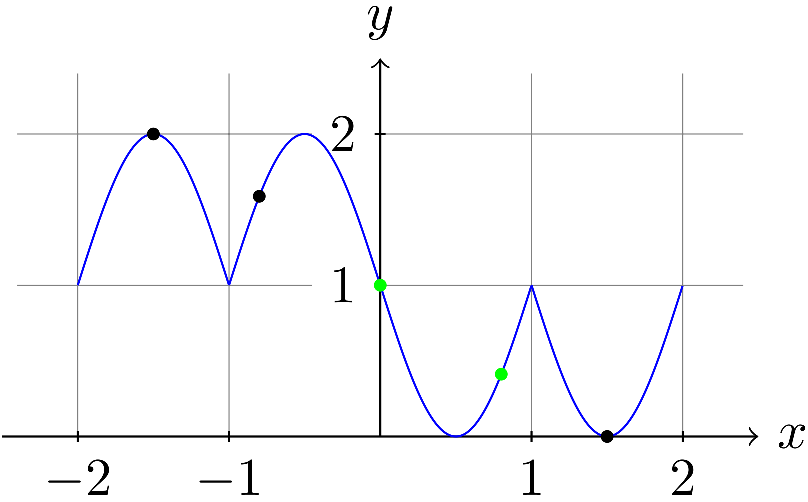
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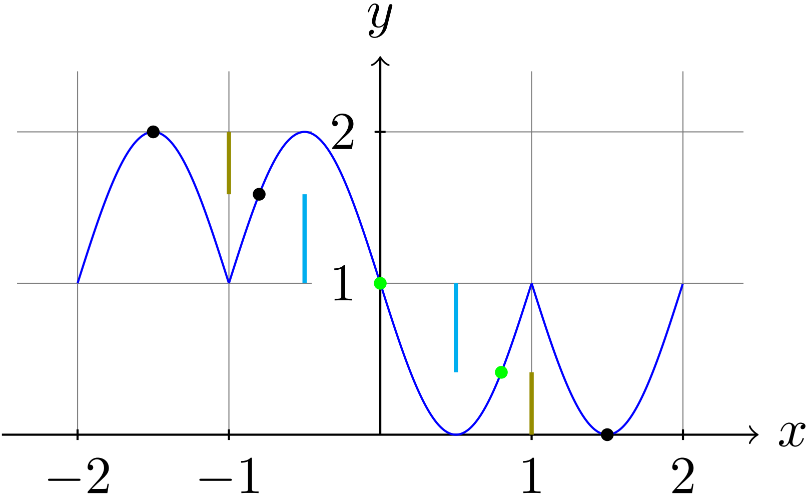

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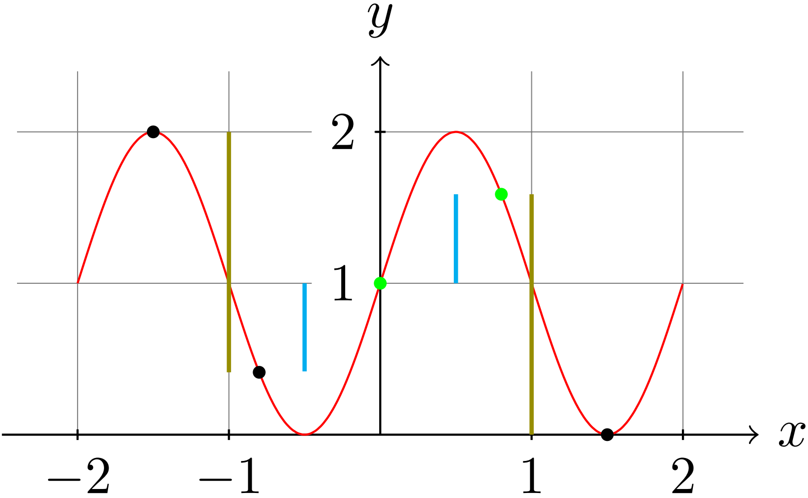

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Lemma

Let $\{f_n\}_{n=0}^{\infty}$ be such that $f_n \to f$ pointwise as $n \to \infty$. Then for any reflection σ, we have that $(f_n)^{\sigma} \to f^{\sigma}$ pointwise as $n \to \infty$.

Lemma

Let $\{f_n\}_{n=0}^{\infty} \subseteq AC(\mathbb{R})$ and $g : (0, \infty) \to (0, \infty)$ such that $AC_{f_n} \leq g$ for all $n \in \mathbb{N}$. Further suppose that $f_n \to f$ pointwise as $n \to \infty$. Then f is absolutely continuous and $AC_f \leq g$.

Lemma

Let $f \in BV(\mathbb{R}, [0, \infty))$. The following are equivalent:

1. $f^\sigma = f$ *a.e. for all reflections* σ, f *is right continuous on* $[0, \infty)$, *and* f *is left continuous on* $(-\infty, 0]$.
2. $f^\sigma = f$ *for all reflections* σ, f *is right continuous on* $[0, \infty)$, *and* f *is left continuous on* $(-\infty, 0]$.
3. f *is even and* f *is nonincreasing and right continuous on* $[0, \infty)$.
4. $f^\# = f$.

Theorem (Main Result 1)

Let $u \in BV(\mathbb{R}; [0, \infty))$. Then $u^{\#} \in BV(\mathbb{R}; [0, \infty))$ and $Var(u^{\#}) \leq Var(u)$.

For $u \in BV(\mathbb{R}; [0, \infty))$, the symmetric decreasing rearrangement satisfies

$$\mathrm{Var}(u^{\#}) = 2(u^{\#}(0) - \lim_{x \to \infty} u^{\#}(x)).$$

If $u^{\#}$ is constant, the result is trivial. Otherwise, for sufficiently small $\varepsilon > 0$ we can set

$$t_0 = u^{\#}(0) - \frac{\varepsilon}{4} \text{ and } t_1 = \lim_{x \to \infty} u^{\#}(x) + \frac{\varepsilon}{4}$$

and observe that both $\mu_u(t_0)$ and $\mu_u(t_1)$ are nonzero and finite.

Theorem (Main Result 1)

Let $u \in BV(\mathbb{R}; [0, \infty))$. Then $u^{\#} \in BV(\mathbb{R}; [0, \infty))$ and $Var(u^{\#}) \leq Var(u)$.

We therefore can find $\{x_0, x_1, x_2\}$ such that

$$u(x_0) \leq t_1, \;\; u(x_1) \geq t_0, \text{ and } u(x_2) \leq t_1.$$

We then bound

$$\begin{aligned} \mathrm{Var}(u) &\geq |u(x_1) - u(x_0)| + |u(x_2) - u(x_1)| \\ &\geq 2(t_0 - t_1) = \mathrm{Var}(u^{\#}) + \varepsilon \end{aligned}$$

and deduce that $u^{\#} \in BV(\mathbb{R}; [0, \infty))$ and $\mathrm{Var}(u^{\#}) \leq \mathrm{Var}(u)$.

Proof Sketch of AC Result

We build ourselves up to all of $AC(\mathbb{R};[0,\infty))$ in four steps:

$$
\begin{aligned}
& u \in AC(\mathbb{R};[0,\infty)) \text{ with compact support} \\
\to\ & u \in AC(\mathbb{R};[0,\infty)) \text{ with constant } c \geq 0 \text{ such that } u \geq c \\
& \text{ and } u = c \text{ outside of a finite interval} \\
\to\ & u \in AC(\mathbb{R};[0,\infty)) \text{ with constant } c \geq 0 \text{ such that } u \geq c \\
& \text{ and } u(x) \to c \text{ as } x \to \pm\infty \\
\to\ & \text{General } u \in AC(\mathbb{R};[0,\infty))
\end{aligned}
$$

Proposition

Let $u \in AC(\mathbb{R}; [0,\infty))$ *have compact support. Then* $u^{\#} \in AC(\mathbb{R}; [0,\infty))$ *and* $AC_{u^{\#}} \leq AC_u$.

We set

$$\mathcal{P}_u = \{u^{\sigma_1 \dots \sigma_n} : \sigma_i \text{ reflections}, \; n \in \mathbb{N}\}.$$

We also let $f : [0,\infty) \to [0,\infty)$ be a strictly decreasing, bounded, integrable function such that $f(t) \to 0$ as $t \to \infty$. We then set $g : \mathbb{R} \to [0,\infty)$ via $g(x) = f(|x|)$. Then for any reflection σ we have $g^{\sigma} = g$.

Proposition

Let $u \in AC(\mathbb{R}; [0, \infty))$ have compact support. Then $u^{\#} \in AC(\mathbb{R}; [0, \infty))$ and $AC_{u^{\#}} \leq AC_u$.

We then consider

$$F(w) = \int_{\mathbb{R}} w(x) g(x) \, dx.$$

Note that

$$w(x) \leq \|u\|_\infty \implies F(w) \leq \|u\|_\infty \int_{\mathbb{R}} g(x) \, dx.$$

We therefore can define $\ell = \sup_{w \in \mathcal{P}_u} F(w)$ and find $\{w_n\}_{n=0}^{\infty} \subseteq \mathcal{P}_u$ such that $F(w_n) \to \ell$ as $n \to \infty$.

AC Step 1

Lemma

Let $f \in BV(\mathbb{R};[0,\infty))$. Then $f^\sigma \in BV(\mathbb{R};[0,\infty))$ and $Var(f^\sigma) \leq Var(f)$.

Theorem (Helly's Selection Theorem)

Let $I \subseteq \mathbb{R}$ be an interval and $\{f_n\}_{n=0}^\infty \subseteq BV(I;\mathbb{R})$ be a sequence of functions such that $\sup\{Var(f_n) : n \in \mathbb{N}\} < \infty$ and $\{f_n(x) : n \in \mathbb{N}\}$ is bounded for some $x \in I$. Then there exists a subsequence $\{f_{n_k}\}_{k=0}^\infty$ and a function $f \in BV(I;\mathbb{R})$ such that $f_{n_k} \to f$ pointwise as $n \to \infty$.

We obtain a subsequence $\{v_m\}_{m=0}^\infty \subseteq \{w_n\}_{n=0}^\infty$ and $v \in BV(\mathbb{R};[0,\infty))$ such that $v_m \to v$ pointwise as $m \to \infty$.

Proposition

Let $u \in AC(\mathbb{R};[0,\infty))$ have compact support. Then $u^{\#} \in AC(\mathbb{R};[0,\infty))$ and $AC_{u^{\#}} \leq AC_u$.

Properties of v:

- $\mathrm{Var}(v) \leq \mathrm{Var}(u)$ as $\mathrm{Var}(v_m) \leq \mathrm{Var}(u)$ for $m \in \mathbb{N}$
- $F(v) = \int_{\mathbb{R}} v(x)g(x)\, dx = \ell$
- $\mu_v = \mu_u \implies v^{\#} = u^{\#}$
- For any reflection σ, $(v_n)^{\sigma} \to v^{\sigma}$
- $F(v^{\sigma}) = \ell$ (Use of Hardy-Littlewood Inequality)
- $(v(x) - v(\sigma(x)))(g(x) - g(\sigma(x))) = 0$ for $\mathcal{L}_1$-a.e. $x \in \mathbb{R}$
- $v^{\sigma}(x) = v(x)$ for $\mathcal{L}_1$-a.e. $x \in \mathbb{R}$.

Proposition

Let $u \in AC(\mathbb{R}; [0, \infty))$ have compact support. Then $u^{\#} \in AC(\mathbb{R}; [0, \infty))$ and $AC_{u^{\#}} \leq AC_u$.

We now let ψ be the function obtained by making v right continuous on $[0, \infty)$ and left continuous on $(-\infty, 0]$. This manipulation ensures that $\psi^{\#} = \psi$, and we can deduce the desired bound:

$$\mathrm{Var}(\psi) \leq \mathrm{Var}(v) \leq \mathrm{Var}(u) \text{ and } \psi = \psi^{\#} = v^{\#} = u^{\#}$$
$$\implies u^{\#} \in BV(\mathbb{R}; [0, \infty)) \text{ and } \mathrm{Var}(u^{\#}) \leq \mathrm{Var}(u).$$

Corollary

Let $u \in AC(\mathbb{R}; [0, \infty))$, $c \geq 0$, and $R \geq 0$ be such that $u \geq c$ and $u = c$ outside of $(-R, R)$. Then $u^{\#} \in AC(\mathbb{R}; [0, \infty))$ and $AC_{u^{\#}} \leq AC_u$.

This follows when we consider $v \in AC(\mathbb{R}; [0, \infty))$ defined via $v(x) = u(x) - c$. We have that $v^{\#}(x) = u^{\#}(x) - c$, and as v vanishes at infinity we can apply the previous proposition and obtain the desired result for $u^{\#}$.

Proposition

Let $u \in AC(\mathbb{R};[0,\infty))$ and $c \geq 0$ be such that $u \geq c$ and $\lim_{x\to\infty} u(x) = \lim_{x\to-\infty} u(x) = c$. Then $u^{\#} \in AC(\mathbb{R};[0,\infty))$ and $AC_{u^{\#}} \leq AC_u$.

For $n \in \mathbb{N}$ we define $u_n : \mathbb{R} \to [0,\infty)$ via

$$u_n(x) = \begin{cases} u(x) & |x| \leq n \\ c & |x| > n \end{cases}$$

We let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $c \leq u(x) < c + \varepsilon/2$ outside of $(-N, N)$. Doing so ensures that $AC_{u_n} < AC_u + \varepsilon$ for $n \geq N$. We then can apply the previous corollary to find

$$AC_{u_n^{\#}} \leq AC_{u_n} < AC_u + \varepsilon \text{ for } n \geq N.$$

AC Step 3

Proposition

Let $u \in AC(\mathbb{R};[0,\infty))$ and $c \geq 0$ be such that $u \geq c$ and $\lim_{x\to\infty} u(x) = \lim_{x\to-\infty} u(x) = c$. Then $u^{\#} \in AC(\mathbb{R};[0,\infty))$ and $AC_{u^{\#}} \leq AC_u$.

We know $u_n \to u$ pointwise as $n \to \infty$, and some analysis of these functions yields that $u_n^{\#} \to u^{\#}$ pointwise as $n \to \infty$.

Lemma

Let $\{f_n\}_{n=0}^{\infty} \subseteq AC(\mathbb{R})$ and $g : (0,\infty) \to (0,\infty)$ such that $AC_{f_n} \leq g$ for all $n \in \mathbb{N}$. Further suppose that $f_n \to f$ pointwise as $n \to \infty$. Then f is absolutely continuous and $AC_f \leq g$.

We then apply this lemma to see that $AC_{u^{\#}} \leq AC_u + \varepsilon$. As this holds for all $\varepsilon > 0$, we conclude that $AC_{u^{\#}} \leq AC_u$.

Theorem

Let $u \in AC(\mathbb{R}; [0, \infty))$. Then $u^{\#} \in AC(\mathbb{R}; [0, \infty))$ and $AC_{u^{\#}} \leq AC_u$.

We set

$$c = \max\{\lim_{x \to \infty} u(x), \lim_{x \to -\infty} u(x)\}.$$

and define $v : \mathbb{R} \to [0, \infty)$ via $v(x) = \max\{u(x), c\}$. As the maximum of two functions, we can bound

$$AC_v \leq AC_u + AC_c = AC_u.$$

The previous proposition grants us that $AC_{v^{\#}} \leq AC_v$. The construction of v ensures that $v^{\#} = u^{\#}$, and as such we deduce that $u^{\#} \in AC(\mathbb{R}; [0, \infty))$ and $AC_{u^{\#}} \leq AC_u$.

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