Formelsammlung FEL-3 VO

basierend auf dem Skript von Prof. Reiter

16. Dezember 2022

1. Spannungstensor

Tensor: $[\sigma_{ij}] = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$

Spannungsvektor: $\vec{\sigma}_n = [\sigma_{ij}] \cdot \vec{n} \quad \sigma_{ni} = \sigma_{ij} n_j \qquad \vec{n}$ ... muss Einheitsvektor sein

Normalspannung: $\sigma_{nn} = \vec{\sigma}_n \cdot \vec{n} = \{[\sigma_{ij}] \cdot \vec{n}\} \cdot \vec{n}$

Schubspannung: $\tau = \sqrt{\sigma^2 - \sigma_n^2}$ mit $\sigma = ||\vec{\sigma}_{nn}|| \qquad \vec{\tau} = \sigma_n - \sigma_{nn} \cdot \vec{n} \quad \rightarrow \quad \tau = ||\vec{\tau}||$

Hauptnormalspannungen: $-\sigma^3 + I_1 \cdot \sigma^2 - I_2 \cdot \sigma + I_3 = 0$ (Eigenwertproblem, lösen mit Rechenknecht)

Koeffizienten (Invarianten) des Spannungstensors:

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \mathrm{spur}[\sigma_{ij}]$$

$$I_3 = \det[\sigma_{ij}] = \sigma_1 \cdot \sigma_2 \cdot \sigma_3$$

$$I_2 = \det \begin{vmatrix} \sigma_{xx} & \sigma_{yx} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} + \det \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \det \begin{vmatrix} \sigma_{xx} & \sigma_{zx} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} = \sigma_1 \cdot \sigma_2 + \sigma_2 \cdot \sigma_3 + \sigma_3 \cdot \sigma_1$$

Normalrichtungen zu den Hauptnormalspannungen:

$\{[\sigma_{ij}] - \sigma_i \cdot [I]\} \cdot \vec{n}_i = \vec{0} \quad [I]$ ... Einheitsmatrix $\quad \sigma_i$ ... Einsetzten von σ_1, σ_2 und $\sigma_3 \rightarrow \vec{n}_1,\ \vec{n}_2$ und $\vec{n}_3$

Kesselformeln: $\sigma_{xx} = \frac{p_{ü}}{2\,t} R \qquad \sigma_{yy} = \sigma_{\varphi\varphi} = 2\ \sigma_{xx}$ (ESZ) $\quad t$ ... Wandstärke

Krümmungsradius: (hier?) $R = \frac{E}{\sigma_F} \sqrt{\frac{1}{4} \left[3h^2 - M^{EP} \frac{12}{b\sigma_F} \right]}$

2. Verzerrungstensor

Tensor: $[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{zx} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \frac{\gamma_{xy}}{2} & \frac{\gamma_{zx}}{2} \\ \frac{\gamma_{xy}}{2} & \varepsilon_{yy} & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{yz}}{2} & \varepsilon_{zz} \end{bmatrix}$ $\qquad \varepsilon_i = \vec{n}_i^\top \cdot [\varepsilon_{ij}] \cdot \vec{n}_i$

Materialgesetz: (isentrope Materiale)

$$\varepsilon_{xx} = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 - \sigma_3)] + \alpha_T \cdot \Delta T \qquad \varepsilon_{yy} = \frac{1}{E}[\sigma_2 - \nu(\sigma_3 - \sigma_1)] + \alpha_T \cdot \Delta T$$

$$\varepsilon_{zz} = \frac{1}{E}[\sigma_3 - \nu(\sigma_1 - \sigma_2)] + \alpha_T \cdot \Delta T \qquad \gamma_{xy} = \frac{1}{G} \cdot \sigma_{xy} = \frac{2 \cdot (1+\nu)}{E} \cdot \sigma_{xy}$$

Materialgesetz 2??? $G = \frac{E}{2(1+\nu)}$ $\quad K = \frac{E}{3(1-2\nu)}$ $\quad \mu = \frac{E}{2(1+\nu)}$ kann nicht stimmen

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \rightarrow \quad \sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

Vergleichsspannungen

Mises: $\sigma_{v,M} = \sqrt{\frac{1}{2} \cdot \left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6 \cdot ({\tau_{xy}}^2 + {\tau_{xz}}^2 + {\tau_{yz}}^2)\right]}$

Mises einfach: $\sigma_v = \sqrt{\sigma^2 + \zeta\ \tau^2}$ $\quad \zeta = \left(\frac{\sigma_D}{\tau_D}\right)^2$ oder $= 3$ $\qquad$ Schweißnaht: $\sigma_{vs} = \sqrt{\sigma_\perp^2 + \tau_\parallel^2 + \tau_\perp^2}$

3. Biegungen, Verzerrungen und Energien (?)

Verzerrung eines Balkens $U = \int_{x_1}^{x_2} \frac{N^2}{2\ EA} \mathrm{d}x + \int_{x_1}^{x_2} \frac{M_b^2}{2\ EJ_b} \mathrm{d}x + \int_{x_1}^{x_2} \frac{M_T^2}{2\ GJ_T} \mathrm{d}x + \int_{x_1}^{x_2} \frac{Q^2}{2\ GA_s} \mathrm{d}x$

Arbeitssatz $W = U \qquad W = \frac{1}{2}\ F\ w_F$

Castigliano: $w_f = \frac{\partial U}{\partial F_i} \qquad w_H = \frac{\partial U}{\partial H}$

Menabrea: $\frac{\partial U}{\partial X_i} = 0$

$X_i \ldots$ Auflagekraft; statisch unbestimmte, innere Kraft

Äußere/Innerlich statische Bestimmtheit:

2-dimensional: $N = Z + R - 3\ K$ $\qquad$ 3-dimensional: $N = Z + R - 6\ K$

$Z \ldots$ Zwangskräfte $\quad R \ldots$ Reaktionskräfte (Lagerkräfte) $\quad K \ldots$ Körper

Ritz: siehe S.176 im Skript

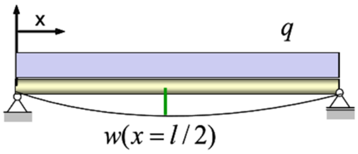
$\vec{u} \approx \tilde{u}_{(x,y,z)} = \sum_k a_k\ \vec{\varphi}_{k(x,y,z)}$ $\qquad \vec{\varphi}_{k(x,y,z)} \ldots$ Ritz'sche Ansatzfunktion $\qquad a_k \ldots$ Koeffizienten

$V \approx \tilde{V}_{(\tilde{u})} = \tilde{V}_{(a_k)} \qquad \frac{\partial \tilde{V}_{(a_k)}}{\partial a_k} = 0 \rightarrow a_k$

4. Gesamtpotentiale

4.1. Beispiel 1

$V = \pi^{(i)} + \pi^{(a)}$



Inneres Potential $\pi^{(i)}$: $V_{Biegung} = \int_0^l EJ\ w''^{\,2}\ \mathrm{d}x$ (Tilde? Dach?)

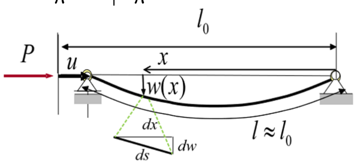
Äußeres Potential $\pi^{(a)}$: $V_{Kraft} = \pm \frac{1}{2}\ F\ w_{(f)}$

Kraft und Verschiebung in verschiedene (+) und gleiche (−) Richtung

$V_{Streckenlast} = -\int_0^l \tilde{w}_{(x)}\ q\ \mathrm{d}x$

4.2. Beispiel 2

$V = \pi^{(i)} + \pi^{(a)}$



Inneres Potential $\pi^{(i)}$ $V_{Biegung} = \frac{1}{2} \int_0^l EJ\ w''^{\,2}\ \mathrm{d}x$

Äußeres Potential $\pi^{(a)}$ $V_{Kraft} = -Pu \approx -P\ \frac{1}{2} \int_0^l w'^{\,2}\ \mathrm{d}x$

Mögliche Ansatzfunktionen:

Grundsätzlich kann man jegliche Funktion verwenden, aber diese soll der Biegelinie ähneln.

→ Potenzfunktionen: z.B. $\varphi_{(x)} = x\ (l - x)$

→ Trigonometrische Funktionen: z.B. $\varphi_{(x)} = \sin\left(\frac{x}{l}\pi\right)$

A. Euler-Knickfälle

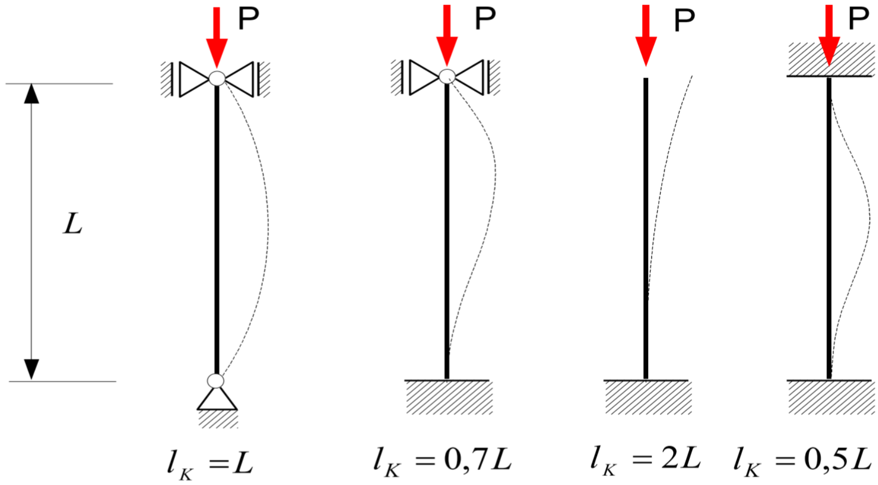
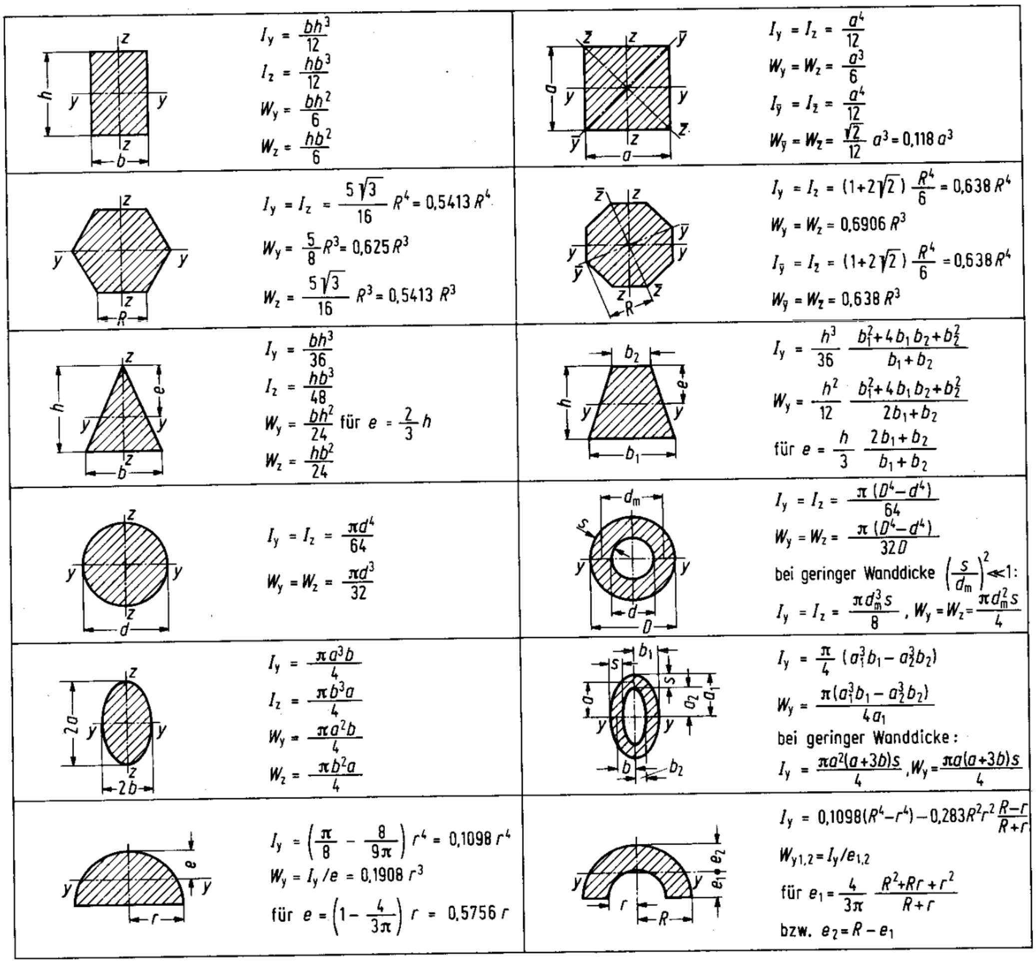


Abbildung 4.9. Kritische Knicklängen l_k für „Standard-Randbedingungen“

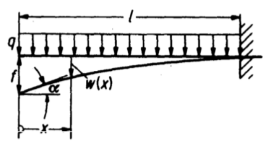
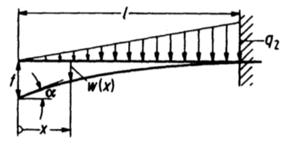
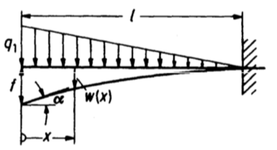
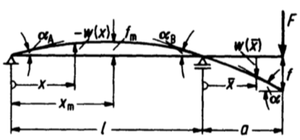
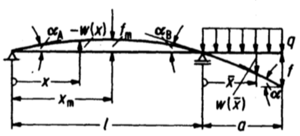
B. aus MEL2VO

B.1. Trägheitsmomente



$I_y = \frac{bh^3}{12}$, $I_z = \frac{hb^3}{12}$, $W_y = \frac{bh^2}{6}$, $W_z = \frac{hb^2}{6}$	$I_y = I_z = \frac{a^4}{12}$, $W_y = W_z = \frac{a^3}{6}$, $I_{\bar y} = I_{\bar z} = \frac{a^4}{12}$, $W_{\bar y} = W_{\bar z} = \frac{\sqrt{2}}{12} a^3 = 0{,}118\, a^3$
$I_y = I_z = \frac{5\sqrt{3}}{16} R^4 = 0{,}5413\, R^4$, $W_y = \frac{5}{8} R^3 = 0{,}625\, R^3$, $W_z = \frac{5\sqrt{3}}{16} R^3 = 0{,}5413\, R^3$	$I_y = I_z = (1+2\sqrt{2})\frac{R^4}{6} = 0{,}638\, R^4$, $W_y = W_z = 0{,}6906\, R^3$, $I_{\bar y} = I_{\bar z} = (1+2\sqrt{2})\frac{R^4}{6} = 0{,}638\, R^4$, $W_{\bar y} = W_{\bar z} = 0{,}638\, R^3$
$I_y = \frac{bh^3}{36}$, $I_z = \frac{hb^3}{48}$, $W_y = \frac{bh^2}{24}$ für $e = \frac{2}{3}h$, $W_z = \frac{hb^2}{24}$	$I_y = \frac{h^3}{36}\,\frac{b_1^2+4b_1b_2+b_2^2}{b_1+b_2}$, $W_y = \frac{h^2}{12}\,\frac{b_1^2+4b_1b_2+b_2^2}{2b_1+b_2}$, für $e = \frac{h}{3}\,\frac{2b_1+b_2}{b_1+b_2}$
$I_y = I_z = \frac{\pi d^4}{64}$, $W_y = W_z = \frac{\pi d^3}{32}$	$I_y = I_z = \frac{\pi(D^4-d^4)}{64}$, $W_y = W_z = \frac{\pi(D^4-d^4)}{32D}$, bei geringer Wanddicke $\left(\frac{s}{d_m}\right)^2 \ll 1$: $I_y = I_z = \frac{\pi d_m^3 s}{8}$, $W_y = W_z = \frac{\pi d_m^2 s}{4}$
$I_y = \frac{\pi a^3 b}{4}$, $I_z = \frac{\pi b^3 a}{4}$, $W_y = \frac{\pi a^2 b}{4}$, $W_z = \frac{\pi b^2 a}{4}$	$I_y = \frac{\pi}{4}(a_1^3 b_1 - a_2^3 b_2)$, $W_y = \frac{\pi(a_1^3 b_1 - a_2^3 b_2)}{4a_1}$, bei geringer Wanddicke: $I_y = \frac{\pi a^2(a+3b)s}{4}$, $W_y = \frac{\pi a(a+3b)s}{4}$
$I_y = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4 = 0{,}1098\, r^4$, $W_y = I_y/e = 0{,}1908\, r^3$, für $e = \left(1 - \frac{4}{3\pi}\right) r = 0{,}5756\, r$	$I_y = 0{,}1098(R^4-r^4) - 0{,}283 R^2 r^2 \frac{R-r}{R+r}$, $W_{y1,2} = I_y/e_{1,2}$, für $e_1 = \frac{4}{3\pi}\,\frac{R^2+Rr+r^2}{R+r}$, bzw. $e_2 = R - e_1$

B.2. Biegelinien

	Belastungsfall	Gleichung der Biegelinie	Durchbiegung	Neigungswinkel
8		$w(x) = \frac{ql^4}{24EI_y}\left[3 - 4\frac{x}{l} + \left(\frac{x}{l}\right)^4\right]$	$f = \frac{ql^4}{8EI_y}$	$\alpha = \frac{ql^3}{6EI_y}$
9		$w(x) = \frac{q_2 l^4}{120EI_y}\left[4 - 5\frac{x}{l} + \left(\frac{x}{l}\right)^5\right]$	$f = \frac{q_2 l^4}{30EI_y}$	$\alpha = \frac{q_2 l^3}{24EI_y}$
10		$w(x) = \frac{q_1 l^4}{120EI_y}\left[11 - 15\frac{x}{l} + 5\left(\frac{x}{l}\right)^4 - \left(\frac{x}{l}\right)^5\right]$	$f = \frac{11}{120}\frac{q_1 l^4}{EI_y}$	$\alpha = \frac{q_1 l^3}{8EI_y}$
11		$0 \leq x \leq l$: $w(x) = -\frac{Fal^2}{6EI_y}\left[\frac{x}{l} - \left(\frac{x}{l}\right)^3\right]$ $0 \leq \bar{x} \leq a$: $w(\bar{x}) = \frac{Fa^3}{6EI_y}\left[2\frac{l}{a}\frac{\bar{x}}{a} + 3\left(\frac{\bar{x}}{a}\right)^2 - \left(\frac{\bar{x}}{a}\right)^3\right]$	$f = \frac{Fa^2(l+a)}{3EI_y}$ $f_m = \frac{Fal^2}{9\sqrt{3}\,EI_y}$ in $x_m = \frac{l}{\sqrt{3}}$	$\alpha = \frac{Fa(2l+3a)}{6EI_y}$ $\alpha_A = \frac{Fal}{6EI_y}$ $\alpha_B = \frac{Fal}{3EI_y}$
12		$0 \leq x \leq l$: $w(x) = -\frac{qa^2 l^2}{12EI_y}\left[\frac{x}{l} - \left(\frac{x}{l}\right)^3\right]$ $0 \leq \bar{x} \leq a$: $w(\bar{x}) = \frac{qa^4}{24EI_y}\left[4\frac{l}{a}\frac{\bar{x}}{a} + 6\left(\frac{\bar{x}}{a}\right)^2 - 4\left(\frac{\bar{x}}{a}\right)^3 + \left(\frac{\bar{x}}{a}\right)^4\right]$	$f = \frac{qa^3(4l+3a)}{24EI_y}$ $f_m = \frac{qa^2 l^2}{18\sqrt{3}\,EI_y}$ in $x_m = \frac{l}{\sqrt{3}}$	$\alpha = \frac{qa^2(l+a)}{6EI_y}$ $\alpha_A = \frac{qa^2 l}{12EI_y}$ $\alpha_B = \frac{qa^2 l}{6EI_y}$