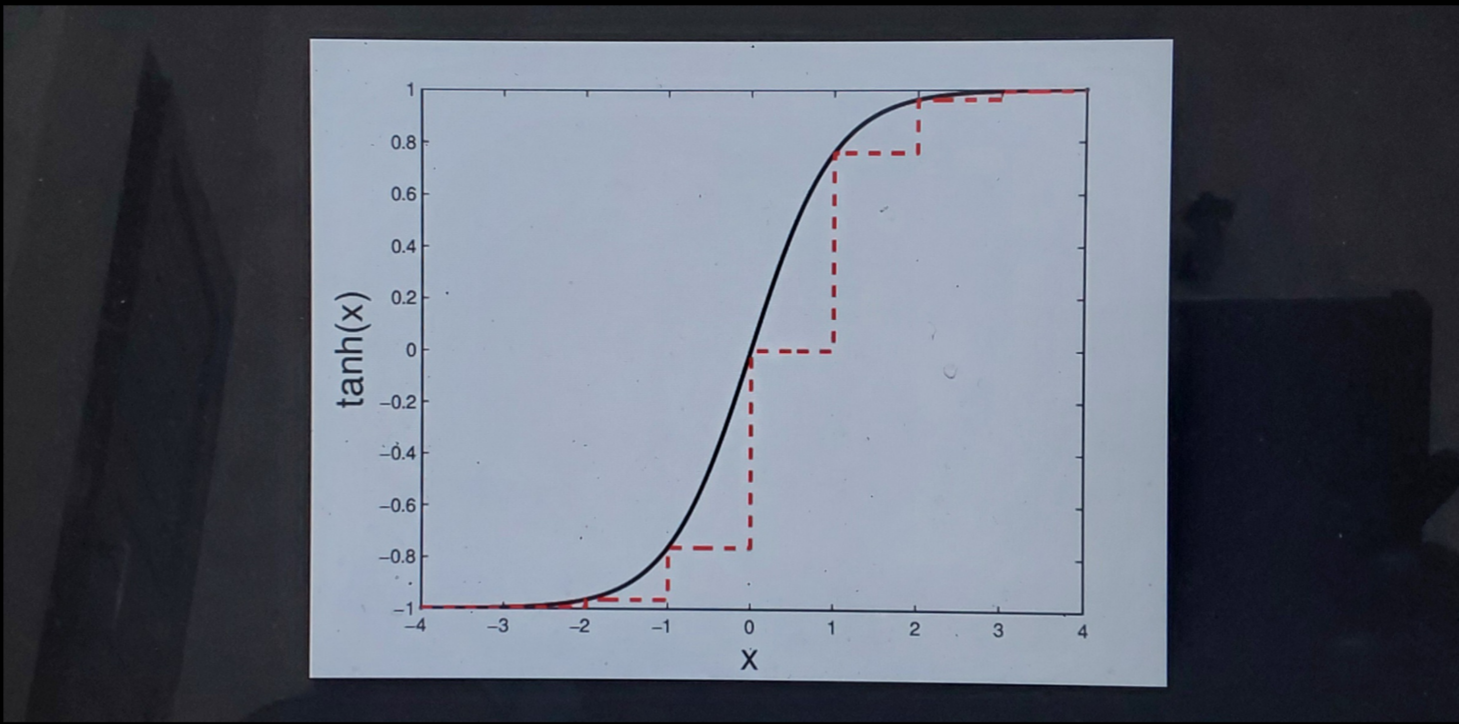
Graph of $\tanh(x)$ vs x



$$\boxed{\begin{aligned} x^{i+1} &= x^i - \mu d_i \left(2^{-i} y^i\right) \\ y^{i+1} &= y^i + d_i \left(2^{-i} x^i\right) \\ z^{i+1} &= z^i - d_i e^i \end{aligned}}$$

For hyperbolic rotations:

- $\mu = -1$
- $e^i = \tanh^{-1}\left(2^{-i}\right)$

Operating equations for hyperbolic rotations:

$$\boxed{\begin{aligned} x_{i+1} &= x_i + d_i \left(2^{-i} y^i\right) \\ y_{i+1} &= y_i + d_i \left(2^{-i} x_i\right) \\ z_{i+1} &= z_i - d_i \tanh^{-1}\left(2^{-i}\right) \end{aligned}}$$

- This is $\tanh^{-1}(2^{-i})$ only.
- Not for any $-1 \le x \le 1$
- CORDIC helps to find any $\tanh^{-1}(x)$ where $-1 \le x \le 1$

Look up table:

i	0	1	2	3	4	5	6	7	8
2^{-i}	1	0.5	0.25	0.125	0.0625	0.03125	0.0156	0.0078	0.0039
$\tanh^{-1}(2^{-i})$	∞	0.549	0.2554	0.1256	0.0625	0.0312	0.0156	0.0078	0.0039

Q). Compute $\tanh^{-1}(0.4)$; $\mu = -1$

↳ For $\tanh^{-1}(y)$, set $\boxed{x=1,\ z=0}$ ⇒ Initial Values

→ For $i=0$, $\tanh^{-1}(2^{-0}) = \tanh^{-1}(1) = \infty$

⇒ $\boxed{\text{Hence we begin computation from } i=1}$

operating equations in hyperbolic mode

$$\boxed{\begin{aligned} x_{i+1} &= x_i + d_i\left(2^{-i} y_i\right) \\ y_{i+1} &= y_i + d_i\left(2^{-i} x_i\right) \\ z_{i+1} &= z_i - d_i \tanh^{-1}\left(2^{-i}\right) \end{aligned}}$$

$\tanh^{-1}(x)$ computation is done in vectoring mode. In vectoring mode, aim is to keep rotating till $y_n \to 0$

$$d_i = -\underbrace{\text{sign}}\,(x_i y_i) = -\underbrace{\text{sgn}}\,(x_i y_i)$$

↳ Signum function

$\boxed{\text{We do NOT do step 0 in } \tanh^{-1}(y) \text{ computation as } \tanh^{-1}(1) = \infty. \text{ Start from step 1 (i.e., } i=1)}$

Step1 :

$\boxed{x_1 = 1; \quad z_1 = 0}$ → Need to start from here to compute $\tanh^{-1}(y)$

$\boxed{y_1 = 0.4}$ → I/p given by user

$d_1 = -sgn(x_1 y_1) = -sgn(1 \times 0.4) = -1$

This '-1' is encoded as 0 in the program.

Step2 :

$x_2 = x_1 + d_1(2^{-1} \times y_1) = 1 + \left(-1 \times \frac{y_1}{2}\right) = 1 - \frac{0.4}{2} = 1 - 0.2 = 0.8$

$y_2 = y_1 + d_1(2^{-1} \cdot x_1) = 0.4 + \left(-1 \times \frac{1}{2} \times 1\right) = 0.4 - 0.5 = -0.1$

$z_2 = z_1 - \left[d_1 \tanh^{-1}\left(\frac{1}{2}\right)\right] = 0 - (-1 \times 0.549) = 0.549$

$d_2 = -sgn(0.8 \times -0.1) = 1$

Step3 :

$x_3 = x_2 + (d_2 \times 2^{-2} \times y_2) = 0.8 + \left(\frac{-0.1}{4}\right) = 0.775$

$y_3 = y_2 + (d_2 \times 2^{-2} \times x_2) = -0.1 + \left(1 \times \frac{x_2}{4}\right) = -0.1 + \left(\frac{0.8}{4}\right) = 0.1$

$z_3 = z_2 - \left[d_2 \tanh^{-1}\left(\frac{1}{2^2}\right)\right] = 0.549 - 0.2554 = 0.2936$

$d_3 = -sgn(0.775 \times 0.1) = -1$

Step4 :

$x_4 = x_3 + (d_3 \cdot 2^{-3} y_3) = 0.775 + \left(\frac{-1 \times 0.1}{8}\right) = 0.7625$

$y_4 = y_3 + (d_3 2^{-3} x_3) = 0.1 - \left(\frac{0.775}{8}\right) = 0.003125$

$z_4 = z_3 - \left[d_3 \tanh^{-1}\left(\frac{1}{8}\right)\right] = 0.2936 - (-0.1256) = 0.4192$

$\rightarrow \quad d_4 = -\operatorname{sgn}(0.7625 \times 0.003125) = -1$

→ For hyperbolic computations, we need to repeat iteration 4.

Step 4' [Repetition of iteration 4]

$\rightarrow x_4' = x_4 + (d_4 2^{-3} y_4) = 0.7625 + \left(\frac{-y_4}{8}\right) = 0.7625 - 0.00039 = 0.7621$

$\rightarrow y_4' = y_4 + (d_4 2^{-3} x_4) = 0.003125 + \left(\frac{-x_4}{8}\right) = 0.003125 - \frac{0.7625}{8} = -0.092$

$\rightarrow z_4' = z_4 - \left[d_4 \tanh^{-1}\left(\frac{1}{8}\right)\right] = 0.4192 - \left(-\tanh^{-1}\left(\frac{1}{8}\right)\right) = 0.5448$

$\rightarrow d_4' = -\operatorname{sgn}(0.7621 \times (-0.092)) = 1$

Step 5 :

$\rightarrow x_5 = x_4' + (d_4' 2^{-4} y_4') = 0.7621 + \left(\frac{-0.092}{16}\right) = 0.75635$

$\rightarrow y_5 = y_4' + (d_4' 2^{-4} x_4') = -0.092 + \left(\frac{0.7621}{16}\right) = -0.044$

$\rightarrow z_5 = z_4' - \left[d_4' \tanh^{-1}\left(\frac{1}{16}\right)\right] = 0.5448 - 0.0625 = 0.4823$

$\rightarrow d_5 = -\operatorname{sgn}(0.75635 \times -0.044) = 1$

Step 6

$\rightarrow x_6 = x_5 + (d_5 2^{-5} y_5) = 0.75635 + \left(\frac{-0.044}{32}\right) = 0.7549$

$\rightarrow y_6 = y_5 + (d_5 2^{-5} x_5) = -0.044 + \left(\frac{0.75635}{5}\right) = -0.0137$

$\rightarrow z_6 = z_5 - \left[d_5 \tanh^{-1}\left(\frac{1}{32}\right)\right] = 0.4823 - 0.0312 = 0.4511$

$\rightarrow d_6 = -\operatorname{sgn}(0.7549 \times -0.0137) = 1$

Step 7 :

$\rightarrow x_7 = x_6 + \left(d_6 2^{-6} y_6\right) = 0.7549 + \left(\frac{y_6}{64}\right) = 0.7549 - \frac{0.0137}{64} = 0.75468$

$\rightarrow y_7 = y_6 + \left(d_6 2^{-6} x_6\right) = -0.0137 + \left(\frac{0.7549}{64}\right) = -0.0019$

$\rightarrow z_7 = z_6 - \left(d_6 \tanh^{-1} \frac{1}{2^6}\right) = 0.4511 - \tanh^{-1}\left(\frac{1}{64}\right) = 0.4511 - 0.0052 = 0.4459$

$\rightarrow d_7 = -\operatorname{sgn}\left(0.75468 \times -0.0019\right) = 1$

Step 8 :

$\rightarrow x_8 = x_7 + \left(d_7 2^{-7} y_7\right) = 0.75468 + \left(\frac{-0.0019}{128}\right) = 0.754665$

$\rightarrow y_8 = y_7 + \left(d_7 2^{-7} x_7\right) = -0.0019 + \left(\frac{0.75468}{128}\right) = 0.00399$

$\rightarrow z_8 = z_7 - \left(d_7 \tanh^{-1}\left(\frac{1}{128}\right)\right) = 0.4459 - 0.00264 = 0.44326$

$\rightarrow d_8 = -\operatorname{sgn}\left(x_8 \cdot y_8\right) = -\operatorname{sgn}\left(0.754665 \cdot 0.00399\right) = -1$

Step 9

$\rightarrow x_9 = x_8 + \left(d_8 2^{-8} y_8\right) = 0.754665 + \left(\frac{-1 \times 0.00399}{256}\right) = 0.75468$

$\rightarrow y_9 = y_8 + \left(d_8 2^{-8} x_8\right) = 0.00399 + \left(\frac{-1}{256} \times 0.754665\right) = 0.00693$

$\rightarrow z_9 = z_8 - \left(d_8 \tanh^{-1}\left(\frac{1}{256}\right)\right) = 0.44326 - \left(-\tanh^{-1}\left(\frac{1}{256}\right)\right)$

$= 0.44326 + 0.003906$

$= 0.447166$

$\rightarrow d_9 = -\operatorname{sgn}\left(x_9 \cdot y_9\right) = -1$

$\rightarrow$ Stopping with step 9

Final Obtained Answer = $z_9 = 0.447166$

Actual Answer = $\tanh^{-1}(0.4) = 0.42364$