Expandindo algumas Séries de Taylor

Para aproveitar ao máximo as analogias entre os exercícios 1. e 2., realizamos a expansão da Série de Taylor em relação as duas variáveis realizando um passo à frente ou para trás ao mesmo tempo, a partir do operador $\pm$:

$$
\begin{aligned}
u(x_i \pm h, y_j) = u_i(x_i, y_j) + (\pm h)\frac{\partial u}{\partial x}(x_i, y_j) + \frac{(\pm h)^2}{2!}\frac{\partial u^2}{\partial^2 x}(x_i, y_j) \\
+\frac{(\pm h)^3}{3!}\frac{\partial u^3}{\partial^3 x}(x_i, y_j) + \frac{(\pm h)^4}{4!}\frac{\partial u^4}{\partial^4 x}(x_i, y_j)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
u(x_i, y_j \pm k) = u_i(x_i, y_j) + (\pm k)\frac{\partial u}{\partial y}(x_i, y_j) + \frac{(\pm k)^2}{2!}\frac{\partial u^2}{\partial^2 y}(x_i, y_j) \\
+\frac{(\pm k)^3}{3!}\frac{\partial u^3}{\partial^3 y}(x_i, y_j) + \frac{(\pm k)^4}{4!}\frac{\partial u^4}{\partial^4 y}(x_i, y_j)
\end{aligned} \tag{2}
$$

Direção x

Avançada A partir de Equação 1, tomando o caso da adição, truncando a série até a segunda derivada e depois isolando o termo que descreve a primeira derivada:

$$u(x_i + h, y_j) = u_i(x_i, y_j) + h\frac{\partial u}{\partial x}(x_i, y_j) + \frac{h^2}{2!}\frac{\partial u^2}{\partial^2 x}(x_i, y_j)$$

$$h\frac{\partial u}{\partial x}(x_i, y_j) = u(x_i + h, y_j) - u_i(x_i, y_j) - \frac{h^2}{2!}\frac{\partial u^2}{\partial^2 x}(x_i, y_j)$$

$$\frac{\partial u}{\partial x}(x_i, y_j) = \frac{u(x_i + h, y_j) - u_i(x_i, y_j) - \frac{h^2}{2!}\frac{\partial u^2}{\partial^2 x}(x_i, y_j)}{h}$$

$$\frac{\partial u}{\partial x}(x_i, y_j) = \frac{u(x_i + h, y_j) - u_i(x_i, y_j)}{h} - \frac{h}{2}\frac{\partial u^2}{\partial^2 x}(x_i, y_j)$$

O ETL pode ser representado pelo último termo ao escolher $\xi_i \in [x_i, x_i + h]$, pelo Teorema do Valor Intermediário

$$
\begin{aligned}
\frac{\partial u}{\partial x}(x_i, y_j) = \frac{u(x_i + h, y_j) - u_i(x_i, y_j)}{h} \\
\text{ETL: } -\frac{h}{2}\frac{\partial u^2}{\partial^2 x}(\xi_i, y_j)\blacksquare
\end{aligned} \tag{3}
$$

Atrasada A mesma lógica pode ser aplicada, mas utilizando a subtração no lugar da adição para Equação 1:

$$u(x_i - h, y_j) = u_i(x_i, y_j) - h\frac{\partial u}{\partial x}(x_i, y_j) + \frac{h^2}{2!}\frac{\partial u^2}{\partial^2 x}(x_i, y_j)$$

$$h\frac{\partial u}{\partial x}(x_i, y_j) = -u(x_i - h, y_j) + u_i(x_i, y_j) + \frac{h^2}{2!}\frac{\partial u^2}{\partial^2 x}(x_i, y_j)$$

$$\frac{\partial u}{\partial x}(x_i, y_j) = \frac{u_i(x_i, y_j) - u(x_i - h, y_j)}{h} + \frac{h}{2}\frac{\partial u^2}{\partial^2 x}(x_i, y_j)$$

$$\begin{gathered} \frac{\partial u}{\partial x}(x_i, y_j) = \frac{u_i(x_i, y_j) - u(x_i - h, y_j)}{h} \\ \text{ETL:} \;\; \frac{h}{2}\frac{\partial u^2}{\partial^2 x}(\xi_i, y_j)\blacksquare \end{gathered} \tag{4}$$

Centrada A diferença centrada pode ser obtida a partir de uma combinação das Séries de Taylor expandidas na Equação 1. Tomando ambas as séries até a terceira derivada, e as subtraindo:

$$a: u(x_i + h, y_j) = u_i(x_i, y_j) + h\frac{\partial u}{\partial x}(x_i, y_j) + \frac{h^2}{2!}\frac{\partial u^2}{\partial^2 x}(x_i, y_j)$$
$$+\frac{h^3}{3!}\frac{\partial u^3}{\partial^3 x}(x_i, y_j)$$

$$b: u(x_i - h, y_j) = u_i(x_i, y_j) - h\frac{\partial u}{\partial x}(x_i, y_j) + \frac{h^2}{2!}\frac{\partial u^2}{\partial^2 x}(x_i, y_j)$$
$$-\frac{h^3}{3!}\frac{\partial u^3}{\partial^3 x}(x_i, y_j)$$

$$a - b: u(x_i + h, y_j) - u(x_i - h, y_j) = 2h\frac{\partial u}{\partial x}(x_i, y_j) + 2\frac{h^3}{3!}\frac{\partial u^3}{\partial^3 x}(x_i, y_j)$$

$$2h\frac{\partial u}{\partial x}(x_i, y_j) = u(x_i + h, y_j) - u(x_i - h, y_j) - 2\frac{h^3}{3!}\frac{\partial u^3}{\partial^3 x}(x_i, y_j)$$

$$\frac{\partial u}{\partial x}(x_i, y_j) = \frac{u(x_i + h, y_j) - u(x_i - h, y_j)}{2h} - \frac{h^2}{6}\frac{\partial u^3}{\partial^3 x}(x_i, y_j)$$

$$\begin{gathered} \frac{\partial u}{\partial x}(x_i, y_j) = \frac{u(x_i + h, y_j) - u(x_i - h, y_j)}{2h} \\ \text{ETL:} \; -\frac{h^2}{6}\frac{\partial u^3}{\partial^3 x}(x_i, y_j)\blacksquare \end{gathered} \tag{5}$$

Centrada da segunda derivada No item anterior, é possível somar as duas equações ao invés de subtrair, isso nos proporciona:

$$u(x_i + h, y_j) + u(x_i - h, y_j) = 2u_i(x_i, y_j) + 2\frac{h^2}{2!}\frac{\partial u^2}{\partial^2 x}(x_i, y_j)$$

Não consta nenhum termo após a segunda derivada, por isso, adicionamos a soma dos termos da quarta derivada, proveniente da Equação 1

$$u(x_i + h, y_j) + u(x_i - h, y_j) = 2u_i(x_i, y_j) +$$
$$2\frac{h^2}{2!}\frac{\partial u^2}{\partial^2 x}(x_i, y_j) + 2\frac{h^4}{4!}\frac{\partial u^2}{\partial^2 x}(x_i, y_j)$$

Isolando a segunda derivada

$$
\begin{aligned}
h^2 \frac{\partial u^2}{\partial^2 x}(x_i, y_j) = u(x_i + h, y_j) - 2u_i(x_i, y_j) + u(x_i - h, y_j) \\
-2\frac{h^4}{24}\frac{\partial u^2}{\partial^2 x}(x_i, y_j) \\
\frac{\partial u^2}{\partial^2 x}(x_i, y_j) = \frac{u(x_i + h, y_j) - 2u_i(x_i, y_j) + u(x_i - h, y_j)}{h^2} \\
-\frac{h^4}{12}\frac{\partial u^2}{\partial^2 x}(x_i, y_j) \\
\frac{\partial u^2}{\partial^2 x}(x_i, y_j) = \frac{u(x_i + h, y_j) - 2u_i(x_i, y_j) + u(x_i - h, y_j)}{h^2} \\
\text{ETL: } -\frac{h^4}{12}\frac{\partial u^2}{\partial^2 x}(x_i, y_j)\blacksquare
\end{aligned}
\tag{6}
$$

Direção y

Só inverte as ordem, porra.