Anisotropic Wave Propagation

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Isotropic Elastic Modulus Matrix

$$C = \begin{pmatrix} C_{33} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{33} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{pmatrix}$$

$$C_{12} = C_{33} - 2C_{55}$$

VTI Elastic Modulus Matrix

$$C = \begin{pmatrix} C_{11} & C_{11} - 2C_{66} & C_{13} & 0 & 0 & 0 \\ C_{11} - 2C_{66} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}$$

TTI Elastic Modulus Matrix

$C_{TTI} = MC_{VTI}M^T$
M - Bond rotation matrix

Exact Phase Velocities

$$
\begin{aligned}
V_{qP}(\xi) &= \sqrt{\frac{C_{11} sin^2(\xi) + C_{33} cos^2(\xi) + C_{44} + \sqrt{M(\xi)}}{2\rho}} \\
V_{qS}(\xi) &= \sqrt{\frac{C_{11} sin^2(\xi) + C_{33} cos^2(\xi) + C_{44} - \sqrt{M(\xi)}}{2\rho}} \\
V_{S}(\xi) &= \sqrt{\frac{C_{66} sin^2(\xi) + C_{44} cos^2(\xi)}{\rho}} \\
M(\xi) &= \left[(C_{11} - C_{44}) sin^2(\xi) - (C_{33} - C_{44}) cos^2(\xi)\right]^2 + (C_{13} + C_{44})^2 sin^2(2\xi)
\end{aligned}
$$

Thomson Approximation - Weak Anisotropy $\delta, \gamma, \epsilon << 1$

$$
\begin{aligned}
V_{qP}(\xi) &\approx V_{P0}(1 + \delta sin^2(\xi)cos^2(\xi) + \epsilon sin^4(\xi)) \\
V_{qS}(\xi) &\approx V_{S0}\left[1 + \left(\frac{V_{P0}}{V_{S0}}\right)^2 (\epsilon - \delta) sin^2(\xi)cos^2(\xi)\right] \\
V_S(\xi) &\approx V_{S0}(1 + \gamma sin^2(\xi))
\end{aligned}
$$

Thomson Parameers from Elastic Moduli

$$
\begin{aligned}
\epsilon &= \frac{C_{11} - C_{33}}{2C_{33}} \\
\delta &= \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2C_{33}(C_{33} - C_{44})} \\
\gamma &= \frac{C_{66} - C_{44}}{2C_{44}} \\
V_{P0} &= \sqrt{\frac{C_{33}}{\rho}} \\
V_{S0} &= \sqrt{\frac{C_{44}}{\rho}}
\end{aligned}
$$

Elastic Moduli from Thomson Parameters

$$
\begin{aligned}
C_{33} &= V_{P0}^2 \rho \\
C_{44} &= V_{S0}^2 \rho \\
C_{11} &= 2\epsilon C_{33} + C_{33} \\
C_{66} &= 2\gamma C_{44} + C_{44} \\
C_{13} &= \sqrt{2\delta C_{33}(C_{33} - C_{44}) + (C_{33} - C_{44})^2} - C_{44}
\end{aligned}
$$

Isotropic Wave Propagation

$$\rho \frac{\partial^2 U}{\partial t^2} = \nabla^2 U + S$$

$$\rho_{i,j} \left(\frac{U_{i,j}^{n+1} - 2U_{i,j}^{n} + U_{i,j}^{n-1}}{\Delta t^2} \right) = \left(\frac{U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n}}{\Delta x^2} \right) + \left(\frac{U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}}{\Delta z^2} \right) + S_{i,j}^{n}$$

$$U_{i,j}^{n+1} = 2U_{i,j}^{n} - U_{i,j}^{n-1} + \frac{\Delta t^2}{\Delta x^2 \rho_{i,j}} \left(U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n} \right) + \frac{\Delta t^2}{\Delta z^2 \rho_{i,j}} \left(U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n} \right) + \frac{\Delta t^2}{\rho_{i,j}} S_{i,j}^{n}$$

Anisotropic Wave Propagation

$$
\begin{aligned}
\rho \frac{\partial^2 U}{\partial t^2} &= \frac{\partial}{\partial x}\left(A\frac{\partial}{\partial x} + C\frac{\partial}{\partial z}\right)U + \frac{\partial}{\partial z}\left(G\frac{\partial}{\partial x} + Q\frac{\partial}{\partial z}\right)U + S \\
\rho_{i,j}\left(\frac{U_{i,j}^{n+1} - 2U_{i,j}^{n} + U_{i,j}^{n-1}}{\Delta t^2}\right) &= \left(\frac{A_{i+1/2,j}(U_{i+1,j}^{n} - U_{i,j}^{n}) - A_{i-1/2,j}(U_{i,j}^{n} - U_{i-1,j}^{n})}{\Delta x^2}\right) \\
&+ \left(\frac{Q_{i,j+1/2}(U_{i,j+1}^{n} - U_{i,j}^{n}) - Q_{i,j-1/2}(U_{i,j}^{n} - U_{i,j-1}^{n})}{\Delta z^2}\right) \\
&+ \left(\frac{C_{i+1,j}(U_{i+1,j+1}^{n} - U_{i+1,j-1}^{n}) - C_{i-1,j}(U_{i-1,j+1}^{n} - U_{i-1,j-1}^{n})}{4\Delta x\Delta z}\right) \\
&+ \left(\frac{G_{i,j+1}(U_{i+1,j+1}^{n} - U_{i-1,j+1}^{n}) - G_{i,j-1}(U_{i+1,j-1}^{n} - U_{i-1,j-1}^{n})}{4\Delta x\Delta z}\right) \\
&+ S_{i,j}^{n} \\
A_{i+1/2,j} &= \frac{1}{2}(A_{i+1,j} + A_{i,j}) \\
Q_{i,j+1/2} &= \frac{1}{2}(Q_{i,j+1} + Q_{i,j})
\end{aligned}
$$