Anisotropic Ray Tracing
TTI Tilted Transverse Isotropy
Eikonal Equation:

$$G(x, \nabla\tau(x)) = 1 \tag{1}$$

Eigenvalues of Christoffel matrix:

$$\begin{aligned}
G(x,p) &= \frac{1}{2}\left((a+b) + \sqrt{(a-b)^2 + 4c^2}\right) \text{pressure} && (2)\\
G(x,p) &= \frac{1}{2}\left((a+b) - \sqrt{(a-b)^2 + 4c^2}\right) \text{shear} && (3)\\
a &= (1+2\epsilon)v_p^2||p||^2 + \left[v_s^2 - (1+2\epsilon)v_p^2\right]\zeta^2 && (4)\\
b &= v_s^2||p||^2 + (v_p^2 - v_s^2)\zeta^2 && (5)\\
c^2 &= h(||p||^2 - \zeta^2)\zeta^2 && (6)\\
h &= \left[(1+2\delta)v_p^2 - v_s^2\right]\left[v_p^2 - v_s^2\right] && (7)\\
\zeta &= p\cdot\vec{\eta} && (8)\\
\vec{\eta} &= (\cos\beta\sin\alpha, \sin\beta\sin\alpha, \cos\alpha) && (9)
\end{aligned}$$

Hamiltonian Equation(s):

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}\nabla_p G(x,p) && (10)\\
\frac{dp}{dt} &= -\frac{1}{2}\nabla_x G(x,p) && (11)
\end{aligned}$$

Alkilifah approximation ($v_s = 0$), solve only for pressure waves:

$$\begin{aligned}
a &= (1+2\epsilon)v_p^2||p||^2 - (1+2\epsilon)v_p^2\zeta^2 && (12)\\
b &= v_p^2\zeta^2 && (13)\\
h &= (1+2\delta)v_p^4 && (14)\\
A &= \frac{1}{v_p^2}(a+b) = \frac{1}{v_p^2}\left((1+2\epsilon)v_p^2||p||^2 - (1+2\epsilon)v_p^2\zeta^2 + v_p^2\zeta^2\right) && (15)\\
&= (1+2\epsilon)||p||^2 - 2\epsilon\zeta^2 && (16)\\
c^2 &= h(||p||^2 - \zeta^2)\zeta^2 && (17)
\end{aligned}$$

$$
\begin{aligned}
A_1 &= \frac{4c^2-4ab}{v_p^4} = \frac{4h(||p||^2-\zeta^2)\zeta^2 - 4(1+2\epsilon)v_p^2(||p||^2-\zeta^2)v_p^2\zeta^2}{v_p^4} && (18)\\
&= \frac{4(1+2\delta)v_p^4(||p||^2-\zeta^2)\zeta^2 - 4(1+2\epsilon)v_p^2(||p||^2-\zeta^2)v_p^2\zeta^2}{v_p^4} && (19)\\
&= 8(\delta-\epsilon)\zeta^2(||p||^2-\zeta^2) && (20)\\
D &= A^2 + A_1 = \left(\frac{a+b}{v_p^2}\right)^2 + \left(\frac{4c^2-4ab}{v_p^4}\right) = \left(\frac{a^2+2ab+b^2+4c^2-4ab}{v_p^4}\right)\\
&= \left(\frac{a^2-2ab+b^2+4c^2}{v_p^4}\right) = \left(\frac{(a-b)^2+4c^2}{v_p^4}\right) && (21)\\
B &= A+\sqrt{D} = \left(\frac{(a+b)+\sqrt{(a-b)^2+4c^2}}{v_p^2}\right) && (22)\\
G &= \frac{1}{2}v_p^2 B && (23)
\end{aligned}
$$

So, the Hamiltonian system becomes:

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}\nabla_p G(x,p) = \frac{1}{4}v_p^2\nabla_p B && (24)\\
\frac{dp}{dt} &= -\frac{1}{2}\nabla_x G(x,p) = -\frac{1}{2}v_p B\nabla_x v_p - \frac{1}{4}v_p^2\nabla_x B && (25)\\
\nabla_p B &= \nabla_p A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_p A + \nabla_p A_1\right] && (26)\\
\nabla_x B &= \nabla_x A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_x A + \nabla_x A_1\right] && (27)\\
A &= (1+2\epsilon)||p||^2 - 2\epsilon\zeta^2 && (28)\\
\nabla_p A &= 2(1+2\epsilon)\vec{p} - 4\epsilon\zeta\vec{\eta} && (29)\\
\nabla_x A &= 2(||p||^2-\zeta^2)\nabla_x\epsilon - 4\epsilon\zeta\nabla_x\zeta && (30)\\
A_1 &= 8(\delta-\epsilon)\zeta^2(||p||^2-\zeta^2) && (31)\\
\nabla_p A_1 &= 16(\delta-\epsilon)\left[\zeta^2\vec{p} + \zeta(||p||^2-2\zeta^2)\vec{\eta}\right] && (32)\\
\nabla_x A_1 &= 8\zeta^2(||p||^2-\zeta^2)(\nabla_x\delta - \nabla_x\epsilon) + 16\zeta(||p||^2-2\zeta^2)(\delta-\epsilon)\nabla_x\zeta\\
\nabla_x\zeta &= \left[(p_1\cos\beta + p_2\sin\beta)\cos\alpha - p_3\sin\alpha\right]\nabla_x\alpha + \sin\alpha(-p_1\sin\beta + p_2\cos\beta)\nabla_x\beta
\end{aligned}
$$

So, everything is in terms of cubes $v_p, \alpha, \beta, \epsilon, \delta$ and their gradients. Since the angles α, β are periodic, we need to worry about phase unwrapping in order to perform $\nabla_x\alpha, \nabla_x\beta$. Alternatively, we can estimate $\nabla_x\zeta$ directly.

$$\begin{aligned}\nabla_x\zeta &= \nabla_x\vec{p}\cdot\vec{\eta} = (\vec{p}\cdot\nabla_x)\vec{\eta}\text{this is probabaly wrong, but needs verification}\\ \nabla_x\zeta &= \nabla_x\vec{p}\cdot\vec{\eta} = \vec{p}\times(\nabla_x\times\vec{\eta}) + (\vec{p}\cdot\nabla_x)\vec{\eta}\text{this version is probabaly correct}\end{aligned}$$

VTI Vertical Transverse Isotropy

VTI is a special case of TTI, where the tilted axis $\vec{\eta}$ is not tilted, but is parallel to the z axis. So, we substitute $\zeta = \vec{p}\cdot\vec{\eta} = p_3$. Furthermore, $\nabla_x\zeta = 0$.

Let us denote $\tilde{p}^2 = p_1^2 + p_2^2 = ||p^2|| - p_3^2$ for convenience.

Our equations become: (I realize there is probably an easier way to derive these, then through back substitution, but this way is more useful as a sanity check)

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}\nabla_p G(x,p) = \frac{1}{4}v_p^2\nabla_p B && (33)\\
\frac{dp}{dt} &= -\frac{1}{2}\nabla_x G(x,p) = -\frac{1}{2}v_p B\nabla_x v_p - \frac{1}{4}v_p^2\nabla_x B && (34)\\
\nabla_p B &= \nabla_p A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_p A + \nabla_p A_1\right] && (35)\\
\nabla_x B &= \nabla_x A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_x A + \nabla_x A_1\right] && (36)\\
A &= (1+2\epsilon)||p||^2 - 2\epsilon\zeta^2 && (37)\\
&= ||p||^2 + 2\epsilon||\tilde{p}||^2 && (38)\\
\nabla_p A &= 2(1+2\epsilon)\vec{p} - 4\epsilon\zeta\vec{\eta} && (39)\\
&= 2(1+2\epsilon)\vec{p} - 4\epsilon(0,0,p_3) && (40)\\
\frac{\partial A}{\partial p_3} &= 2p_3 && (41)\\
(i\neq 3)\frac{\partial A}{\partial p_i} &= 2(1+2\epsilon)p_i && (42)\\
\nabla_x A &= 2(||p||^2 - \zeta^2)\nabla_x\epsilon - 4\epsilon\zeta\nabla_x\zeta && (43)\\
&= 2||\tilde{p}||^2\nabla_x\epsilon && (44)\\
A_1 &= 8(\delta-\epsilon)\zeta^2(||p||^2 - \zeta^2) && (45)\\
&= 8(\delta-\epsilon)p_3^2||\tilde{p}||^2 && (46)\\
\nabla_p A_1 &= 16(\delta-\epsilon)\left[\zeta^2\vec{p} + \zeta(||p||^2 - 2\zeta^2)\vec{\eta}\right] && (47)\\
&= 16(\delta-\epsilon)\left[p_3^2\vec{p} + (||\tilde{p}||^2 - p_3^2)(0,0,p_3)\right] && (48)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial A_1}{\partial p_3} &= 16(\delta-\epsilon)||\tilde{p}||^2 p_3 && (49)\\
(i \neq 3)\frac{\partial A_1}{\partial p_i} &= 16(\delta-\epsilon)p_3^2 p_i && (50)\\
\nabla_x A_1 &= 8\zeta^2(||p||^2-\zeta^2)(\nabla_x\delta-\nabla_x\epsilon)+16\zeta(||p||^2-2\zeta^2)(\delta-\epsilon)\nabla_x\zeta\\
&= 8p_3^2||\tilde{p}||^2(\nabla_x\delta-\nabla_x\epsilon)
\end{aligned}$$

Tidying things up:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}\nabla_p G(x,p) = \frac{1}{4}v_p^2\nabla_p B && (51)\\
\frac{dp}{dt} &= -\frac{1}{2}\nabla_x G(x,p) = -\frac{1}{2}v_p B\nabla_x v_p - \frac{1}{4}v_p^2\nabla_x B && (52)\\
\nabla_p B &= \nabla_p A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_p A + \nabla_p A_1\right] && (53)\\
\nabla_x B &= \nabla_x A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_x A + \nabla_x A_1\right] && (54)\\
A &= ||p||^2 + 2\epsilon||\tilde{p}||^2 && (55)\\
\frac{\partial A}{\partial p_3} &= 2p_3 && (56)\\
(i \neq 3)\frac{\partial A}{\partial p_i} &= 2(1+2\epsilon)p_i && (57)\\
\nabla_x A &= 2||\tilde{p}||^2\nabla_x\epsilon && (58)\\
A_1 &= 8(\delta-\epsilon)p_3^2||\tilde{p}||^2 && (59)\\
\frac{\partial A_1}{\partial p_3} &= 16(\delta-\epsilon)||\tilde{p}||^2 p_3 && (60)\\
(i \neq 3)\frac{\partial A_1}{\partial p_i} &= 16(\delta-\epsilon)p_3^2 p_i && (61)\\
\nabla_x A_1 &= 8p_3^2||\tilde{p}||^2(\nabla_x\delta-\nabla_x\epsilon)
\end{aligned}$$

Plugging stuff in:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}\nabla_p G(x,p) = \frac{1}{4}v_p^2\nabla_p B && (62)\\
\frac{dp}{dt} &= -\frac{1}{2}\nabla_x G(x,p) = -\frac{1}{2}v_p B\nabla_x v_p - \frac{1}{4}v_p^2\nabla_x B && (63)\\
D &= A^2 + A_1 && (64)
\end{aligned}$$

$$
\begin{aligned}
&= \left[||p||^2 + 2\epsilon||\tilde{p}||^2\right]^2 + 8(\delta - \epsilon)p_3^2||\tilde{p}||^2 && (65)\\
&= \left[(1+2\epsilon)||\tilde{p}||^2 + p_3^2\right]^2 + 8(\delta - \epsilon)p_3^2||\tilde{p}||^2 && (66)\\
\nabla_p B &= \nabla_p A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_p A + \nabla_p A_1\right] && (67)\\
\frac{\partial B}{\partial p_3} &= 2p_3 + \frac{1}{2}D^{-\frac{1}{2}}\left[2((1+2\epsilon)||\tilde{p}||^2 + p_3^2)2p_3 + 16(\delta-\epsilon)||\tilde{p}||^2 p_3\right] && (68)\\
&= 2p_3\left(1 + D^{-\frac{1}{2}}\left[((1+2\epsilon)||\tilde{p}||^2 + p_3^2) + 4(\delta-\epsilon)||\tilde{p}||^2\right]\right) && (69)\\
(i \neq 3)\frac{\partial B}{\partial p_i} &= 2(1+2\epsilon)p_i + \frac{1}{2}D^{-\frac{1}{2}}\left[2((1+2\epsilon)||\tilde{p}||^2 + p_3^2)2(1+2\epsilon)p_i + 16(\delta-\epsilon)p_3^2 p_i\right] && (70)\\
&= 2p_i\left((1+2\epsilon) + D^{-\frac{1}{2}}\left[((1+2\epsilon)||\tilde{p}||^2 + p_3^2)(1+2\epsilon) + 4(\delta-\epsilon)p_3^2\right]\right) && (71)\\
\nabla_x B &= \nabla_x A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_x A + \nabla_x A_1\right] && (72)\\
&= \nabla_x A + \frac{1}{2}D^{-\frac{1}{2}}\left[2(||p||^2 + 2\epsilon||\tilde{p}||^2)2||\tilde{p}||^2\nabla_x\epsilon + 8p_3^2||\tilde{p}||^2(\nabla_x\delta - \nabla_x\epsilon)\right]\\
&= \nabla_x A + 4\frac{1}{2}D^{-\frac{1}{2}}\left[((1+2\epsilon)||\tilde{p}||^2 + p_3^2)||\tilde{p}||^2\nabla_x\epsilon + 2p_3^2||\tilde{p}||^2(\nabla_x\delta - \nabla_x\epsilon)\right]\\
&= 2\left(||\tilde{p}||^2\nabla_x\epsilon + D^{-\frac{1}{2}}\left[((1+2\epsilon)||\tilde{p}||^2 + p_3^2)||\tilde{p}||^2\nabla_x\epsilon + 2p_3^2||\tilde{p}||^2(\nabla_x\delta - \nabla_x\epsilon)\right]\right)
\end{aligned}
$$

Elliptical anisotropy

For elliptical anisotropy, we set $\delta = \epsilon$. And, of course, the gradients are also equal, $\nabla_x\delta = \nabla_x\epsilon$.

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}\nabla_p G(x,p) = \frac{1}{4}v_p^2\nabla_p B && (73)\\
\frac{dp}{dt} &= -\frac{1}{2}\nabla_x G(x,p) = -\frac{1}{2}v_p B\nabla_x v_p - \frac{1}{4}v_p^2\nabla_x B && (74)\\
\nabla_p B &= \nabla_p A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_p A + \nabla_p A_1\right] && (75)\\
\nabla_x B &= \nabla_x A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_x A + \nabla_x A_1\right] && (76)\\
A &= (1+2\epsilon)||p||^2 - 2\epsilon\zeta^2 && (77)\\
&= (1+2\delta)||p||^2 - 2\delta\zeta^2 && (78)\\
\nabla_p A &= 2(1+2\epsilon)\vec{p} - 4\epsilon\zeta\vec{\eta} && (79)\\
&= 2(1+2\delta)\vec{p} - 4\delta\zeta\vec{\eta} && (80)\\
\nabla_x A &= 2(||p||^2 - \zeta^2)\nabla_x\epsilon - 4\epsilon\zeta\nabla_x\zeta && (81)
\end{aligned}
$$

$$\begin{aligned}
&= 2(||p||^2 - \zeta^2)\nabla_x \delta - 4\epsilon\zeta\nabla_x\zeta && (82)\\
A_1 &= 8(\delta - \epsilon)\zeta^2(||p||^2 - \zeta^2) && (83)\\
&= 0 && (84)\\
\nabla_p A_1 &= 16(\delta - \epsilon)\left[\zeta^2\vec{p} + \zeta(||p||^2 - 2\zeta^2)\vec{\eta}\right] && (85)\\
&= 0 && (86)\\
\nabla_x A_1 &= 8\zeta^2(||p||^2 - \zeta^2)(\nabla_x\delta - \nabla_x\epsilon) + 16\zeta(||p||^2 - 2\zeta^2)(\delta - \epsilon)\nabla_x\zeta\\
&= 0 && (87)\\
& && (88)
\end{aligned}$$

Plugging stuff in:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}\nabla_p G(x,p) = \frac{1}{4}v_p^2\nabla_p B && (89)\\
\frac{dp}{dt} &= -\frac{1}{2}\nabla_x G(x,p) = -\frac{1}{2}v_p B\nabla_x v_p - \frac{1}{4}v_p^2\nabla_x B && (90)\\
D &= A^2 + A_1 && (91)\\
&= A^2 && (92)\\
B &= A + \sqrt{D} && (93)\\
&= 2A && (94)\\
\nabla_p B &= \nabla_p A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_p A + \nabla_p A_1\right] && (95)\\
&= \nabla_p A + \frac{1}{2}\frac{1}{A}\left[2A\nabla_p A\right] && (96)\\
&= 2\nabla_p A && (97)\\
\nabla_x B &= \nabla_x A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_x A + \nabla_x A_1\right] && (98)\\
&= \nabla_x A + \frac{1}{2}\frac{1}{A}\left[2A\nabla_x A\right] && (99)\\
&= 2\nabla_x A && (100)
\end{aligned}$$

Plugging some more stuff in:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}\nabla_p G(x,p) = \frac{1}{4}v_p^2\nabla_p B && (101)\\
&= \frac{1}{2}v_p^2\nabla_p A && (102)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}v_p^2(2(1+2\delta)\vec{p} - 4\delta\zeta\vec{\eta}) && (103)\\
&= v_p^2((1+2\delta)\vec{p} - 2\delta\zeta\vec{\eta}) && (104)\\
&= v_p^2((1+2\delta)\vec{p} - 2\delta(\vec{p}\cdot\vec{\eta})\vec{\eta}) && (105)\\
&= v_p^2\vec{p_1}((1+2\delta) - 2\delta\cos^2\theta) && (106)\\
&= v_p^2\vec{p_1}(1+2\delta\sin^2\theta) && (107)\\
\frac{dp}{dt} &= -\frac{1}{2}\nabla_x G(x,p) = -\frac{1}{2}v_p B\nabla_x v_p - \frac{1}{4}v_p^2\nabla_x B && (108)\\
&= -v_p A\nabla_x v_p - \frac{1}{2}v_p^2\nabla_x A && (109)\\
&= -v_p((1+2\delta)||p||^2 - 2\delta\zeta^2)\nabla_x v_p - \frac{1}{2}v_p^2(2(||p||^2-\zeta^2)\nabla_x\delta - 4\epsilon\zeta\nabla_x\zeta)\\
&= -v_p((1+2\delta)||p||^2 - 2\delta\zeta^2)\nabla_x v_p - v_p^2((||p||^2-\zeta^2)\nabla_x\delta - 2\epsilon\zeta\nabla_x\zeta)
\end{aligned}$$

If we perform some analysis on this result, we can easily see that when the angle between the symmetry axis, and p is zero, $\frac{dx}{dt} = v_p^2\vec{p}$. When the angle is 90^o, $\frac{dx}{dt} = v_p^2\vec{p}(1+2\delta)$. Furthermore, we can generalize this to $\frac{dx}{dt}(\theta) = v_p^2\vec{p_1}(1+2\delta\sin^2(\theta))$, where θ is the angle between $\vec{p_1}$ and the symmetry axis $\vec{\eta}$.

Weak Anisotropy

For weak anisotropy, we make the assumption that $\epsilon, \delta << 1$.

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}\nabla_p G(x,p) = \frac{1}{4}v_p^2\nabla_p B && (110)\\
\frac{dp}{dt} &= -\frac{1}{2}\nabla_x G(x,p) = -\frac{1}{2}v_p B\nabla_x v_p - \frac{1}{4}v_p^2\nabla_x B && (111)\\
\nabla_p B &= \nabla_p A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_p A + \nabla_p A_1\right] && (112)\\
\nabla_x B &= \nabla_x A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_x A + \nabla_x A_1\right] && (113)\\
A &= (1+2\epsilon)||p||^2 - 2\epsilon\zeta^2 && (114)\\
\nabla_p A &= 2(1+2\epsilon)\vec{p} - 4\epsilon\zeta\vec{\eta} && (115)\\
\nabla_x A &= 2(||p||^2-\zeta^2)\nabla_x\epsilon - 4\epsilon\zeta\nabla_x\zeta && (116)\\
A_1 &= 8(\delta-\epsilon)\zeta^2(||p||^2-\zeta^2) && (117)\\
\nabla_p A_1 &= 16(\delta-\epsilon)\left[\zeta^2\vec{p} + \zeta(||p||^2-2\zeta^2)\vec{\eta}\right] && (118)\\
\nabla_x A_1 &= 8\zeta^2(||p||^2-\zeta^2)(\nabla_x\delta - \nabla_x\epsilon) + 16\zeta(||p||^2-2\zeta^2)(\delta-\epsilon)\nabla_x\zeta
\end{aligned}$$

$$(119)$$

Isotropic Medium (yet another sanity check)

For isotropic media, the quantities ϵ, δ are zero, and obviously their gradients also become zero.

We have our original equations:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}\nabla_p G(x,p) = \frac{1}{4}v_p^2 \nabla_p B && (120)\\
\frac{dp}{dt} &= -\frac{1}{2}\nabla_x G(x,p) = -\frac{1}{2}v_p B \nabla_x v_p - \frac{1}{4}v_p^2 \nabla_x B && (121)\\
\nabla_p B &= \nabla_p A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_p A + \nabla_p A_1\right] && (122)\\
\nabla_x B &= \nabla_x A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_x A + \nabla_x A_1\right] && (123)\\
A &= (1+2\epsilon)||p||^2 - 2\epsilon\zeta^2 && (124)\\
&= ||p||^2 && (125)\\
\nabla_p A &= 2(1+2\epsilon)\vec{p} - 4\epsilon\zeta\vec{\eta} && (126)\\
&= 2\vec{p} && (127)\\
\nabla_x A &= 2(||p||^2 - \zeta^2)\nabla_x \epsilon - 4\epsilon\zeta\nabla_x \zeta && (128)\\
&= 0 && (129)\\
A_1 &= 8(\delta - \epsilon)\zeta^2(||p||^2 - \zeta^2) && (130)\\
&= 0 && (131)\\
\nabla_p A_1 &= 16(\delta - \epsilon)\left[\zeta^2\vec{p} + \zeta(||p||^2 - 2\zeta^2)\vec{\eta}\right] && (132)\\
&= 0 && (133)\\
\nabla_x A_1 &= 8\zeta^2(||p||^2 - \zeta^2)(\nabla_x\delta - \nabla_x\epsilon) + 16\zeta(||p||^2 - 2\zeta^2)(\delta - \epsilon)\nabla_x\zeta\\
&= 0
\end{aligned}$$

Plugging stuff in:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}\nabla_p G(x,p) = \frac{1}{4}v_p^2 \nabla_p B && (134)\\
\frac{dp}{dt} &= -\frac{1}{2}\nabla_x G(x,p) = -\frac{1}{2}v_p B \nabla_x v_p - \frac{1}{4}v_p^2 \nabla_x B && (135)\\
D &= = A^2 + A_1 = ||p||^4 && (136)\\
\nabla_p B &= \nabla_p A + \frac{1}{2}D^{-\frac{1}{2}}\left[2A\nabla_p A + \nabla_p A_1\right] && (137)
\end{aligned}$$

$$
\begin{aligned}
&= 2\vec{p} + \frac{1}{2} D^{-\frac{1}{2}} \left[2||p||^2 2\vec{p} \right] && (138) \\
&= 2\vec{p}(1 + D^{-\frac{1}{2}}||p||^2) && (139) \\
&= 4\vec{p} && (140) \\
\nabla_x B &= \nabla_x A + \frac{1}{2} D^{-\frac{1}{2}} \left[2A\nabla_x A + \nabla_x A_1 \right] && (141) \\
&= 0 && (142)
\end{aligned}
$$

Plugging some more stuff in:

$$
\begin{aligned}
B &= A + \sqrt{D} && (143) \\
&= ||p||^2 + \sqrt{||p||^4} && (144) \\
&= 2||p||^2 && (145) \\
\frac{dx}{dt} &= \frac{1}{2}\nabla_p G(x,p) = \frac{1}{4} v_p^2 \nabla_p B && (146) \\
&= \frac{1}{4} v_p^2 4\vec{p} && (147) \\
&= v_p^2 \vec{p} && (148) \\
\frac{dp}{dt} &= -\frac{1}{2}\nabla_x G(x,p) = -\frac{1}{2} v_p B \nabla_x v_p - \frac{1}{4} v_p^2 \nabla_x B && (149) \\
&= -\frac{1}{2} v_p 2||p||^2 \nabla_x v_p && (150) \\
&= -v_p ||p||^2 \nabla_x v_p && (151)
\end{aligned}
$$

If we convert to slowness, $v_p = \frac{1}{U}$, and plug that in:

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{1}{U^2}\vec{p} && (152) \\
\frac{\partial v_p}{\partial x_i} &= \frac{\partial}{\partial x_i}\frac{1}{U} = -\frac{1}{U^2}\frac{\partial U}{\partial x_i} && (153) \\
\frac{dp}{dt} &= -v_p ||p||^2 \left(\frac{\partial v_p}{\partial x}, \frac{\partial v_p}{\partial y}, \frac{\partial v_p}{\partial z} \right) && (154) \\
&= \frac{||p||^2}{U^3} \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right) && (155) \\
&= \frac{||p||^2}{U^3} \nabla U && (156)
\end{aligned}
$$

If we perform unit analysis, with the assumption that p has units of $\left[\frac{s}{m}\right]$, and of course slowness U has units $\left[\frac{s}{m}\right]$, $\left[\frac{dx}{dt}\right] = \left[\frac{m}{s}\right]$ and $\left[\frac{\vec{p}}{U^2}\right] = \left[\frac{m}{s}\right]$. Furthermore, $\left[\frac{dp}{dt}\right] = \left[\frac{1}{m}\right]$ and $\left[\frac{||p||^2}{U^3}\nabla U\right] = \left[\frac{1}{m}\right]$.