Anisotropic Math

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Preliminaries

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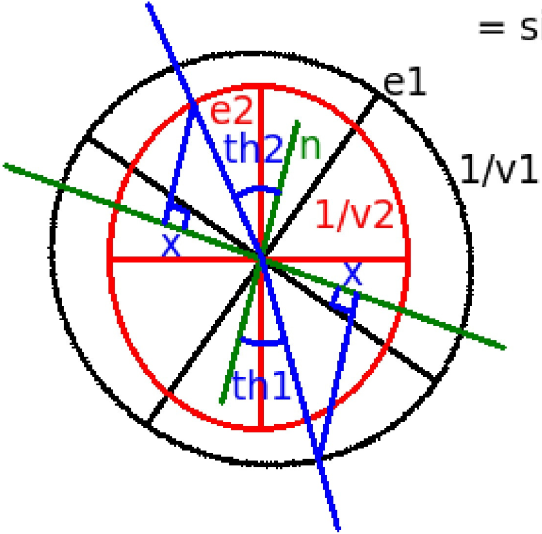
Snell's Law Intuition

$$x = \sin(th1)/v1 = \sin(th2)/v2$$



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- Phase Velocities (eigenvectors of G):

$$(G_{ik} - \rho v^2 \delta_{ik}) \bar{U}_k = 0$$

Stress Tensor Simplification

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- The 81 element tensor can be represented by a 6x6 matrix

$$
\begin{array}{ccc}
c_{ijkl} & \rightarrow & C_{IJ} \\
11 & \rightarrow & 1 \\
22 & \rightarrow & 2 \\
33 & \rightarrow & 3 \\
23 & \rightarrow & 4 \\
13 & \rightarrow & 5 \\
12 & \rightarrow & 6
\end{array}
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- Furthermore, due to symmetry, we can further simplify this matrix

$$
C = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\
C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2}
\end{pmatrix}
$$

Exact Phase Velocities (transverse anisotropy)

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- $$\begin{aligned}
V_{qP}(\xi) &= \sqrt{\frac{C_{11}\,sin^2(\xi) + C_{33}\,cos^2(\xi) + C_{44} + \sqrt{M(\xi)}}{2\rho}} \\
V_{qS}(\xi) &= \sqrt{\frac{C_{11}\,sin^2(\xi) + C_{33}\,cos^2(\xi) + C_{44} - \sqrt{M(\xi)}}{2\rho}} \\
V_S(\xi) &= \sqrt{\frac{C_{66}\,sin^2(\xi) + C_{44}\,cos^2(\xi)}{\rho}} \\
M(\xi) &= \left[(C_{11} - C_{44})sin^2(\xi) - (C_{33} - C_{44})cos^2(\xi)\right]^2 + (C_{13} + C_{44})^2 sin^2(2\xi)
\end{aligned}$$

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- $$\begin{aligned} V_{qP}(\xi) &\approx V_{P0}(1 + \delta sin^2(\xi)cos^2(\xi) + \epsilon sin^4(\xi)) \\ V_{qS}(\xi) &\approx V_{S0}\left[1 + \left(\frac{V_{P0}}{V_{S0}}\right)^2 (\epsilon - \delta)sin^2(\xi)cos^2(\xi)\right] \\ V_S(\xi) &\approx V_{S0}(1 + \gamma sin^2(\xi)) \end{aligned}$$

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- $$\begin{aligned} \epsilon &= \frac{C_{11} - C_{33}}{2C_{33}} \\ \delta &= \frac{(C_{13} + C_{44})^2 - (C_{33} - C_{44})^2}{2C_{33}(C_{33} - C_{44})} \\ \gamma &= \frac{C_{66} - C_{44}}{2C_{44}} \\ V_{P0} &= \sqrt{\frac{C_{33}}{\rho}} \\ V_{S0} &= \sqrt{\frac{C_{44}}{\rho}} \end{aligned}$$

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- $$\frac{dT}{ds} = |\vec{p} \frac{\partial H}{\partial \vec{p}}|$$

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$$
\begin{aligned}
H &= \left(\frac{|\nabla T|^2}{U(\xi)^2} - 1 \right) \\
&= \left(\frac{|\nabla T|^2}{U_0^2} (1 + \delta cos^2(\xi) sin^2(\xi) + \epsilon sin^4(\xi))^2 - 1 \right) \\
&= \left(\frac{|\nabla T|^2}{U_0^2} f(\xi) - 1 \right)
\end{aligned}
$$

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- $$\frac{\partial f(\xi)}{\partial \vec{x}} = \frac{\partial}{\partial \vec{x}}(1 + \delta sin^2(\xi) cos^2(\xi) + \epsilon sin^4(\xi))^2$$
  $$= 2(1 + \delta sin^2(\xi) cos^2(\xi) + \epsilon sin^4(\xi))$$
  $$[(\delta(sin^2(\xi) - cos^2(\xi)) - 2\epsilon sin^2(\xi))\frac{\partial cos^2(\xi)}{\partial \vec{x}}$$
  $$+ sin^2(\xi) cos^2(\xi)\frac{\partial \delta}{\partial \vec{x}}$$
  $$+ sin^4(\xi)\frac{\partial \epsilon}{\partial \vec{x}}$$
  $$]$$

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- $$\frac{\partial cos^2(\xi)}{\partial \vec{x}} = 2 \frac{|\vec{\eta}|^2 |\vec{p}|^2 (\vec{\eta} \cdot \vec{p})(\frac{\partial \vec{\eta}}{\partial \vec{x}} \cdot \vec{p}) - (\vec{\eta} \cdot \vec{p})^2 |\vec{p}|^2 (\frac{\partial \vec{\eta}}{\partial \vec{x}} \cdot \vec{\eta})}{|\vec{\eta}|^4 |\vec{p}|^4}$$

Equations of Motion (anisotropic, last but not least)

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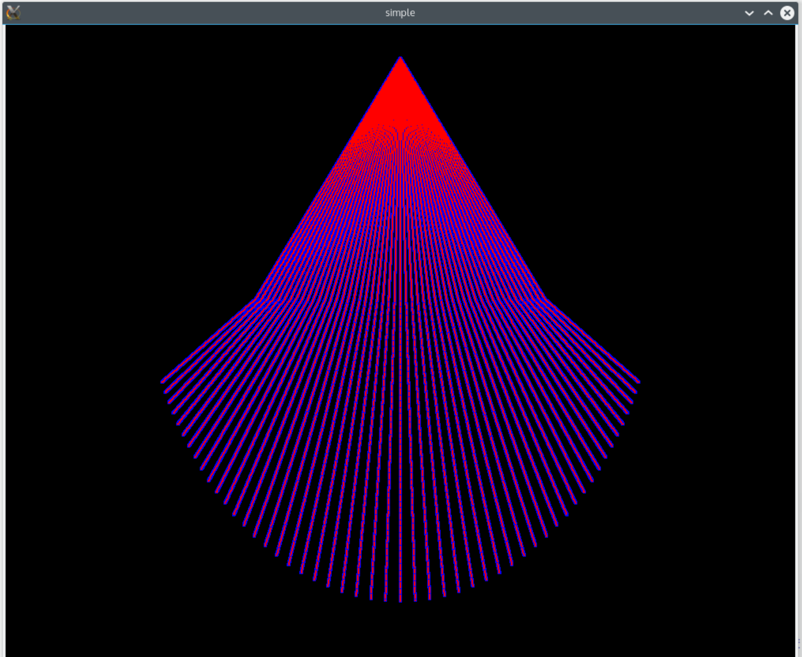
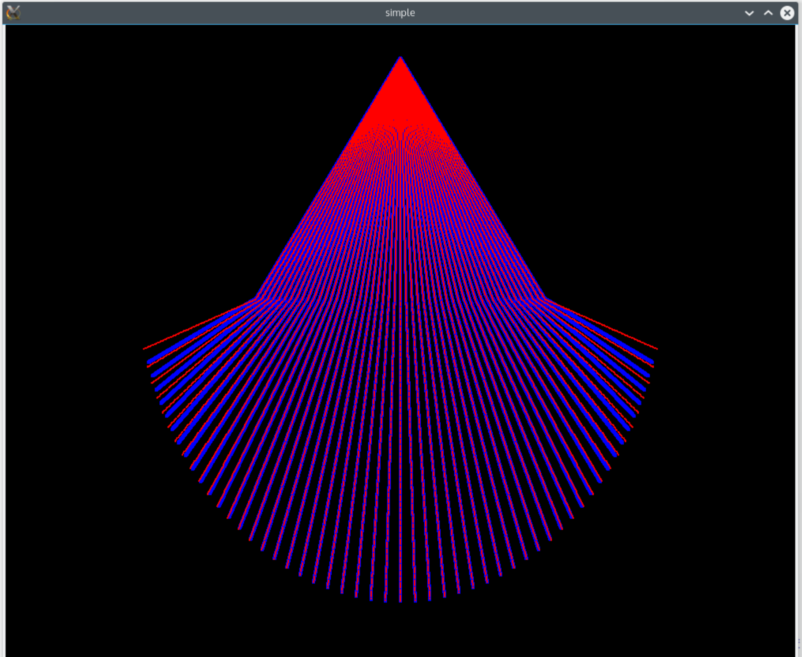
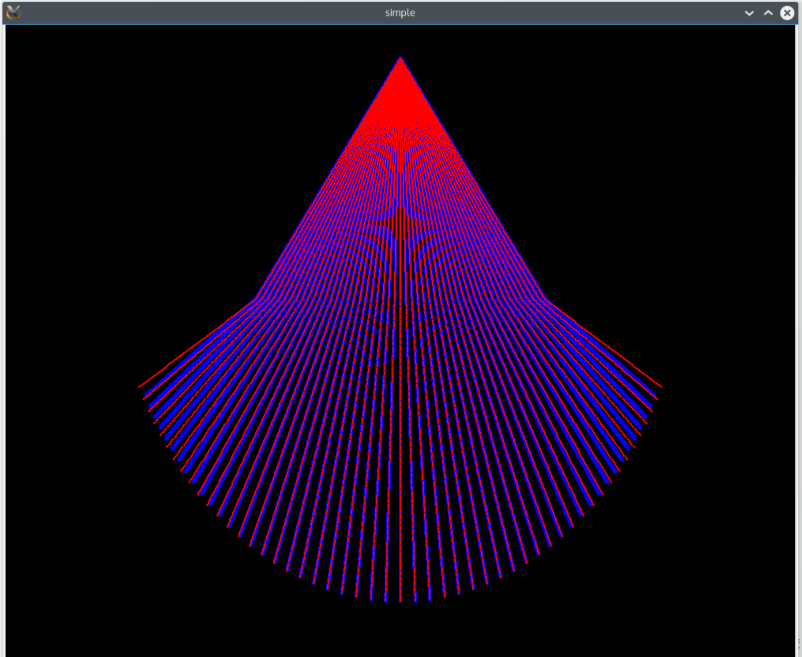
Flat Isotropic



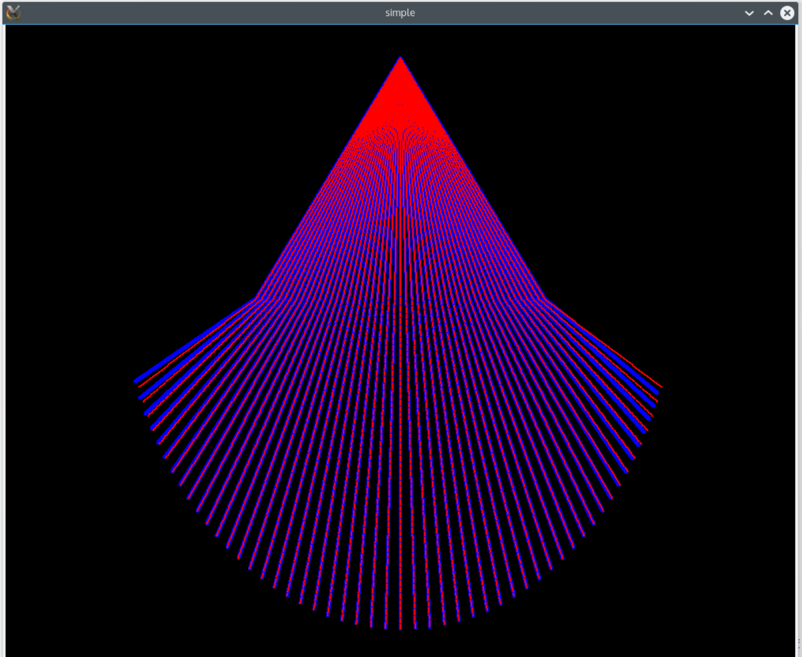
Anisotropic Bottom



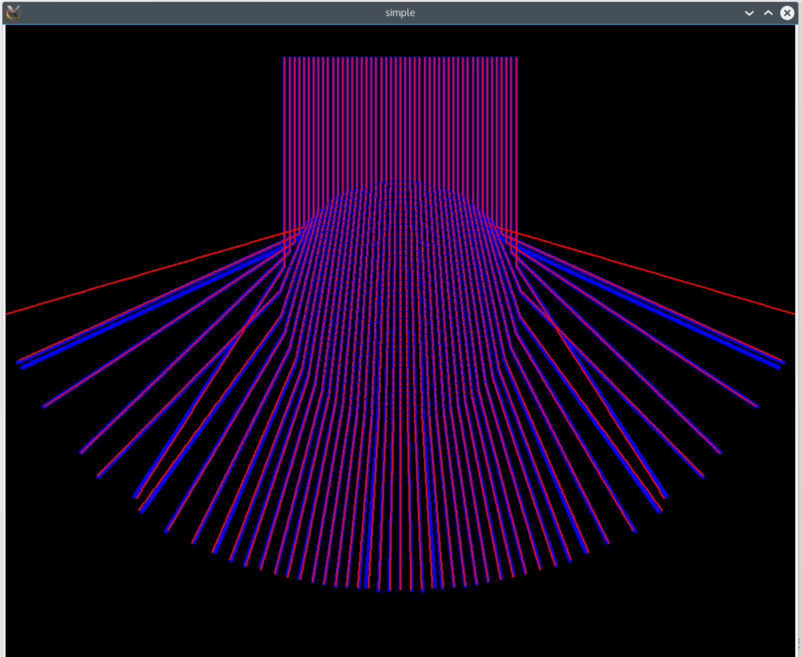
Anisotropic Parallel



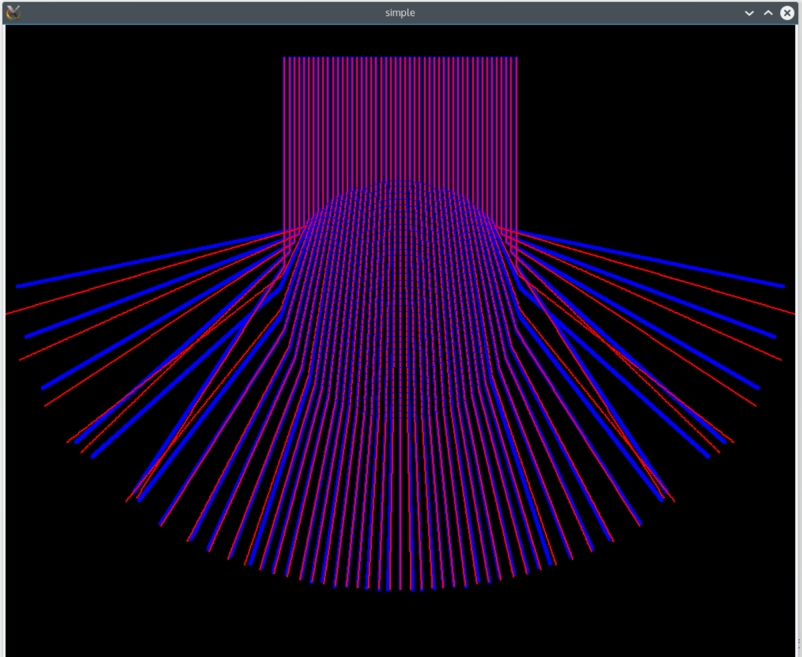
Anisotropic Perpendicular



Spherical Isotropic



Spherical Anisotropic



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    - At the end of the day, we may consider just using the ground truth solver for extreme transitions.