MATH 171 - WIM

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The Basel Problem

1. Introduction and Historical Context

Fourier analysis is a field that, though exceedingly important for the study of physical and mathematical systems alike, often eludes students about what it represents. To shed some light on how ubiquitous the possible applications of Fourier analysis are, one can find special cases of it dating back to ancient Babylonian astronomy [**Prestini**]. The rigorous development of the theory happened mostly in the 18th and early 19th centuries through contributions from many of the greatest mathematical minds, such as Gauss, Euler, Lagrange, and, as the name of the theory indicates, Jean-Baptiste Fourier.

For the most part, these mathematicians didn't collaborate. Instead, in their individual work they stumbled upon special cases of what was later called Fourier Analysis following the significant contributions of its namesake in generalizing the approach. Though the previous mathematicians solved the mechanics behind a vibrating string, the heat equation, and orbits, Fourier argued that these solutions came from the same principle: that all functions can be modeled as sums of trigonometric series.

In this paper, we will walk through the derivation of Fourier analysis up until the point at which we can solve the Basel Problem, that is, show that:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

The earliest proof of this was one of Euler's first breakthroughs, which later inspired minds such as Riemann and Weierstrass in their work in analysis.

2. Convergence of Fourier Series

Theorem 2.1 (Generalized Fourier Series). *For S a complete set of orthogonal functions in class of functions A, any $f \in A$ can be written as a linear combination of $f_n \in S$:*

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

Any coefficient c_k can be calculated by applying the dot product with the respective f_k and using the orthogonality of the f_n's:

$$\langle f, f_k\rangle = \sum_{n=1}^{\infty} c_n \langle f_n, f_k\rangle = c_k \langle f_k, f_k\rangle$$

$$c_k = \frac{\langle f, f_k\rangle}{\langle f_k, f_k\rangle} = \frac{\int_a^b f(x)\overline{f_k(x)}dx}{\int_a^b f_k(x)\overline{f_k(x)}dx} = \frac{\int_a^b f(x)\overline{f_k(x)}dx}{\int_a^b |f_k(x)|^2 dx}$$

*These coefficients c_k are called **Fourier Coefficients** and the associated series is called the **Generalized Fourier Series**:*

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{\langle f, f_k\rangle}{\langle f_k, f_k\rangle} f_k(x).$$

The first thing to note from this theorem is since the Fourier coefficients are found via an integral, the function f to be written as the Fourier series must have some integrability condition. There are numerous equivalent conditions for this purpose, but a relatively simple, general one is that of Riemann Integrability:

Definition 2.2 (**Riemann Integrability**)**.** A function $f : [0, L] \to \mathbb{R}$ is said to be **Riemann integrable** if it is bounded and for all $\epsilon > 0$ there exists a subdivision of the interval $[0, L]$, $0 = x_0 < x_1 < ... < x_{n-1} < x_n = L$, such that the upper and lower sums:

$$\mathcal{U} = \sum_{i=1}^{n} [\sup_{x_{i-1}\leq x\leq x_i} f(x)](x_i - x_{i-1})$$

$$\mathcal{L} = \sum_{i=1}^{n} [\inf_{x_{i-1}\leq x\leq x_i} f(x)](x_i - x_{i-1})$$

satisfy $\mathcal{U} - \mathcal{L} < \epsilon$

The next thing to note is the orthogonality requirement for the functions f_k. This is to guarantee that there is no overlap between different indices in the series. The most common choices for such functions are $\sin(\frac{2\pi kx}{L})$ and $\cos(\frac{2\pi kx}{L})$, equivalent to the complex exponential $e^{2\pi ikx/L}$, which one can easily verify form an orthogonal set for different values of k. Then, the generalized Fourier series in an interval $[0, L]$ can be specialized to:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k f_k(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) f_k(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi ikx/L}$$

Where $\hat{f}(k) = \frac{1}{L}\int_0^L f(x)e^{-2\pi ikx/L}$ is complex-valued and essentially a way to write c_k for sines and cosines without separating them into two sums (and a 0^{th} coefficient term). From now on, any reference to Fourier series is specifically regarding this special form.

Another useful concept to keep in mind for later is that of functions on the circle.

Definition 2.3 (**Function on the Circle**)**.** On the unit circle, points can be written as $e^{i\theta}$ for an angle θ with the x-axis. The real and imaginary components are the x and y coordinates respectively. A function F is said to be ***on the circle*** if $\forall\theta \in \mathbb{R}$ we can define:

$$f(\theta) = F(e^{i\theta}),$$

where one can verify that the properties of F (integrability, continuity, differentiability, ...) are determined by those of f. Furthermore, f is 2π-periodic (can be modified to an arbitrary period L via change of variables), so any interval of size 2π fully captures the behavior of F.

With these preliminary definitions and notation, we can work towards the convergence of Fourier Series to the function at hand. Precisely, what we want to show is:

$$\lim_{n\to\infty} \Big(\sum_{k=-n}^{n} \hat{f}(k) e^{2\pi ikx/L} \Big) = f(x) \qquad \forall x \in [0, L].$$

Unfortunately, math isn't kind. To satisfy this "for all x", additional conditions regarding differentiability are needed. One can intuitively see this by noting that Fourier series are continuous by construction but can describe discontinuous functions (Riemann integrability is weaker than continuity), so there are points in which the limit won't match the discontinuity.

To tread through this issue carefully, it's best to define other properties of sums. In particular, the notions of Cesàro Summability and convolutions will prove useful:

Definition 2.4 (**Cesàro Mean and Cesàro Summability**)**.** Consider a partial sum of complex numbers $s_n = \sum_{k=-n}^{n} c_k$. The **Cesàro Mean** of s_n is

$$\sigma_n = \frac{s_0 + s_1 + ... + s_{n-1} + s_n}{n}.$$

An infinite series $\sum_{k=-\infty}^{\infty} c_k$ is said to be **Cesàro Summable** if

$$\lim_{n\to\infty} (\sigma_n) = \sigma$$

for some σ.

Definition 2.5 (**Convolution**)**.** Given two 2π-periodic functions f, g on $\mathbb{R}$, the convolution $f * g$ on $[0, L]$ is defined as:

$$(f * g)(x) = \frac{1}{L}\int_0^L f(y)g(x-y)dy = \frac{1}{L}\int_0^L f(x-y)g(y)dy.$$

The reason these notions are useful is because of how Fourier coefficients are defined: $\hat{f}(k) = \frac{1}{L}\int_0^L f(x)e^{-2\pi ikx/L}$, meaning that, for the **Dirichlet Kernel** $D_n = \sum_{k=-n}^{n} e^{ikx}$:

$$s_n = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi ikx/L} = (f * D_n)(x)$$

Now, it's clear that the convergence of s_n is related to the behavior of D_n.

Theorem 2.6. *If f is integrable on the circle and $K_n(x)$ satisfies the following properties (in which case, the family $\{K_n(x)\}_{n=1}^{\infty}$ is said to be a family of **good kernels**):*

(1) $\forall n \geq 1 : \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1$;
(2) $\exists M > 0$ *such that,* $\forall n \geq 1 : \quad \int_{-\pi}^{\pi} |K_n(x)|dx < M$;
(3) $\forall \delta > 0, \int_{-\delta<|x|<\pi} |K_n(x)|dx \to 0$ *as* $n \to \infty$;

then:

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

wherever f is continuous on the circle. If f is continuous everywhere, the limit is uniform.

Proof. Let $\epsilon > 0$ and suppose f is continuous at x. Then, pick $\delta > 0$ such that

$$|y| < \delta \implies |f(x-y) - f(x)| < \epsilon.$$

By (1), write:

$$\begin{aligned}(f * K_n)(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)f(x-y)dy - f(x) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)[f(x-y) - f(x)]dy,\end{aligned}$$

implying

$$|(f * K_n)(x) - f(x)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)[f(x-y) - f(x)]dy\right|.$$

Thus,

$$|(f*K_n)(x)-f(x)| \leq \frac{1}{2\pi}\left(\int_{|y|<\delta} |K_n(y)||f(x-y) - f(x)|dy \ + \ \int_{\delta\leq|y|\leq\pi} |K_n(y)||f(x-y) - f(x)|dy\right).$$

Then, for a bound M on f:

$$|(f * K_n)(x) - f(x)| \leq \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} |K_n(y)|dy + \frac{M}{\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)|dy.$$

Since (3) gives boundedness of the integral of $|K_n|$ for large n, we know $\exists C$ such that:

$$|(f * K_n)(x) - f(x)| \leq C\epsilon.$$

□

Analyzing the Dirichlet Kernels $D_n = \sum_{k=-n}^{n} e^{ikx}$, they don't satisfy all three properties for the theorem to be valid (they fail to meet (2)). In any case, note that we can write

$$D_n = \sum_{k=-n}^{n} e^{ikx} = \sum_{k=0}^{n} e^{ikx} + \sum_{k=-n}^{-1} e^{ikx} = \frac{1 - e^{i(n+1)x}}{1 - e^{ix}} + \frac{e^{-inx} - 1}{1 - e^{ix}} = \frac{\sin((n + \frac{1}{2})x)}{\sin(\frac{x}{2})}.$$

Now, define the n^{th} **Féjer Kernel** to be the n^{th} Cesàro Mean of the Dirichlet Kernels:

$$F_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(x).$$

It turns out that this does satisfy those three properties as we'll see.

Theorem 2.7. *For all n, the n^{th} **Féjer Kernel** can be written as:*

$$F_n(x) = \frac{1}{n}\cdot\frac{\sin^2((n+\frac{1}{2})x)}{\sin^2(\frac{x}{2})}.$$

*Further, the family of Féjer Kernels forms a family of **good kernels** (as in the definition from Theorem 2.6:*

Proof. We first obtain the closed form of the F_n to ease the computations. By definition:

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx} = (e^{-inx}+\cdots+e^{-ix})+(1+e^{ix}\cdots+e^{inx}) = e^{-ix}\left(\frac{e^{-inx}-1}{e^{-ix}-1}\right)+\frac{1-e^{i(n+1)x}}{1-e^{-ix}}.$$

Simplifying,

$$D_n(x) = \frac{e^{-inx}-e^{i(n+1)x}}{1-e^{ix}}.$$

Then, we can write:

$$\begin{aligned}
nF_n(x) &= \sum_{k=0}^{n-1} D_k(x) = \sum_{k=0}^{n-1}\frac{e^{-ikx}-e^{i(k+1)x}}{1-e^{ix}} = \frac{1}{1-e^{ix}}\left(\sum_{k=0}^{n-1} e^{-ikx} - \sum_{k=0}^{n-1} e^{i(k+1)x}\right)\\
&= \frac{1}{1-e^{ix}}\left(\frac{e^{-inx}-1}{e^{-ix}-1} - e^{ix}\frac{1-e^{inx}}{1-e^{ix}}\right) = \frac{1}{(1-e^{ix})^2}\left(e^{-i(n-1)x}-e^{ix}-e^{ix}(1-e^{inx})\right)\\
&= \frac{e^{ix}}{(1-e^{ix})^2}\left(e^{-inx}-2+e^{inx}\right) = \frac{(e^{\frac{inx}{2}}-e^{\frac{-inx}{2}})^2}{(e^{\frac{ix}{2}}-e^{\frac{-ix}{2}})^2} = \frac{\sin^2((n+\frac{1}{2})x)}{\sin^2(\frac{x}{2})}
\end{aligned}$$

Therefore, we get

$$F_n(x) = \frac{1}{n}\frac{\sin^2((n+\frac{1}{2})x)}{\sin^2(\frac{x}{2})}.$$

Now, for the properties, (1) amounts to solving an integral, and, since $F_n > 0$, (2) just requires multiplying both sides by 2π, meaning that any $M > 2\pi$ is an upper bound. Since sines are bounded, it's also easy to see that as n goes to infinity F_n goes to zero. Then, since the interval in which we evaluate the integral is finite, the integral is also zero, showing (3). $\square$

With these new tools, we can use Theorem 2.6 to get a result involving D_n instead of F_n:

Theorem 2.8. *If f is integrable on the circle, then the Fourier Series of f is Cesàro Summable to f at every point where f is continuous. If f is continuous everywhere, then it is uniformly Cesàro Summable.*

Proof. This comes directly from applying Theorem 2.6 to the Féjer Kernels. Since the Féjer Kernels satisfy the properties of good kernels while being the Cesàro Mean of the Dirichlet Kernels, we conclude that the Fourier Series of f is Cesàro Summable to f at every point where f is continuous. □

Corollary 2.9. *Continuous functions on the circle can be uniformly approximated by trigonometric polynomials.*

Proof. Since partial sums are trigonometric polynomials, this directly follows the theorem (Cesàro Means are partial sums). □

Theorem 2.10. ***(Best Approximation):*** *If f is integrable on the circle with Fourier coefficients c_n, then:*

$$\|f - S_n(f)\| \leq \left\| f - \sum_{|k| \leq n} z_k e_k \right\|$$

For arbitrary complex numbers z_n and partial Fourier Series S_n associated with f. Equality happens precisely when $z_k = c_k$.

Proof. Consider:

$$f - \sum_{|k| \leq n} z_k e_k = f - S_n + \sum_{|k| \leq n} b_k e_k$$

where $b_k = c_k - z_k$. Then, applying the Pythagorean Theorem:

$$\left\| f - \sum_{|k| \leq n} z_k e_k \right\| = \|f - S_n\| + \left\| \sum_{|k| \leq n} b_k e_k \right\|,$$

and, since norms are positive-definite,

$$\left\| f - \sum_{|k| \leq n} z_k e_k \right\| \geq \|f - S_n\|.$$

□

What this theorem gives is that the Fourier Series for a given function f is the **best approximation** to that function using trigonometric polynomials. With this fact in mind, we need only show one more lemma to finally prove that $||f - S_n|| \to 0$ as $n \to \infty$.

Lemma 2.11. *Suppose f is integrable on the circle and bounded by M. Then, there exists a sequence of continuous functions on the circle $\{f_k\}_{k=1}^{\infty}$ such that:*

$$\forall k \in \mathbb{R}, \sup_{x \in [-\pi, \pi]} |f_k(x)| \leq M$$

and, as k goes to infinity,

$$\int_{-\pi}^{\pi} |f(x) - f_k(x)|dx \to 0.$$

Proof. Let $\epsilon > 0$. Using the integrability of f, we can partition the interval $[-\pi, \pi]$ as $-\pi = x_0 < x_1 < ... < x_{n-1} < x_n = \pi$ such that the upper and lower sums differ by at most ϵ. Now define the step-function f^* as:

$$f^*(x) = \sup_{x_{i-1} \le y \le x_i} f(y) \text{ if } x \in [x_{i-1}, x_i[\text{ for } 1 \le i \le n.$$

Then, f^* is the least upper bound of quantities bounded by M, so $|f^*| \le M$. Furthermore, $f^*(x) - f(x) \ge 0$, meaning we can write the integral:

$$\int_{-\pi}^{\pi} |f^*(x) - f(x)| = \int_{-\pi}^{\pi} (f^*(x) - f(x)) < \epsilon.$$

This function f^* isn't continuous, but we can modify it and the intervals to get continuity and the lemma. Let $\delta > 0$ and $\tilde{f}(x) = f^*(x)$ whenever x is at a distance $d_i \ge \delta$ from any of the x_i. When $d_i < \delta$ for some i, let $\tilde{f}(x)$ be comprised of the lines connecting $\tilde{f}(x \pm \delta) = f^*(x \pm \delta)$. Further, let $\tilde{f}(-\pi) = \tilde{f}(\pi) = 0$.

Now note that $\tilde{f}(x)$ coincides with f^* everywhere except at n intervals of "width" 2δ. This, combined with the boundedness by M, yields:

$$\int_{-\pi}^{\pi} |f^*(x) - \tilde{f}(x)|dx \le 2Mn(2\delta)$$

. So if we pick $\delta = \frac{\epsilon}{4Mn}$,

$$\int_{-\pi}^{\pi} |f^*(x) - \tilde{f}(x)|dx \le \epsilon$$

. Now, using the triangle inequality:

$$|f(x) - \tilde{f}(x)| = |f(x) - f^*(x) + f^*(x) - \tilde{f}(x)| \le |f(x) - f^*(x)| + |f^*(x) - \tilde{f}(x)|$$

, and thus,

$$\int_{-\pi}^{\pi} |f(x) - \tilde{f}(x)|dx \le \int_{-\pi}^{\pi} |f(x) - f^*(x)|dx + \int_{-\pi}^{\pi} |f^*(x) - \tilde{f}(x)|dx \le 2\epsilon$$

. Then, for $f_k = \tilde{f}$, so that $2\epsilon = \frac{1}{k}$, we have the lemma. □

Theorem 2.12 (Mean Square Convergence of Fourier Series). *Suppose f is integrable on the circle and its associated partial Fourier series is S_n. Then:*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - S_n(f(\theta))|^2 d\theta \to 0 \text{ as } n \to \infty$$

Proof. Suppose that f is continuous on the circle and let $\epsilon > 0$. Then, by Corollary 2.9, there is a trigonometric polynomial P of degree N such that:

$$\forall\theta, |f(\theta) - P(\theta)| < \epsilon.$$

Furthermore, by Theorem 2.10,

$$||f - S_n|| < \epsilon, \forall n \geq N.$$

This is sufficient to solve the Basel Problem, but since everything so far assumed integrable f rather than continuous, we may as well generalize further. If f is integrable, can use Lemma 2.11 for a bound M. First, choose a continuous g on the circle such that equations (1) and (2) below are satisfied:

$$\sup_{\theta\in[0,2\pi]} |g(\theta)| \leq \sup_{\theta\in[0,2\pi]} |f(\theta)| \leq M \tag{1}$$

$$\int_0^{2\pi} |f(\theta) - g(\theta)| d\theta < \epsilon^2. \tag{2}$$

Then, we obtain:

$$\begin{aligned} ||f - g||^2 &= \frac{1}{2\pi} \int_0^{2\pi} |f(\theta) - g(\theta)|^2 d\theta \\ &\leq \frac{M}{\pi} \int_0^{2\pi} |f(\theta) - g(\theta)| d\theta \\ &\leq C_1\epsilon^2. \end{aligned}$$

Now, since g is continuous, we can do as before: approximate it using some polynomial Q of degree N such that $||g - Q|| < \epsilon$ and $||f - Q|| < C_2\epsilon$. Then, applying Theorem 2.10, we get precisely

$$||f - S_n|| < \epsilon,$$

for all $n \geq N$. □

3. Uniqueness of Fourier Series

The first thing to say regarding uniqueness of Fourier Series is that Fourier Series aren't unique. While this may appear to make this section pointless, it's quite the contrary, as it means that we must tread carefully to find how un-unique they are. Since we've been working with the rather weak condition of integrability, that's what I'll maintain here, but do note that Fourier Series are indeed unique under some additional constraints (such as twice differentiability of f).

Theorem 3.1. *Suppose that f is an integrable function on the circle with $\hat{f}(n) = 0$, $\forall n \in \mathbb{Z}$. Then, $f(\theta_0) = 0$, $\forall \theta_0$ at which f is continuous.*

Proof. Assume f is real-valued (if complex, we can just separate it into real and imaginary components and follow the same argument for each one). For the sake of contradiction, take $\theta \in [-\pi, \pi]$, $\theta_0 = 0$, and $f(0) > 0$ (we can modify interval, point, and sign by change of variables, so we may make these assumptions without losing generality).

Since f is continuous at 0, choose $0 < \delta \leq \frac{\pi}{2}$ such that $f(\theta) > \frac{f(0)}{2}$ for all θ with $|\theta| < \delta$. Let $\epsilon > 0$ be sufficiently small so that, for all $\delta \leq |\theta| \leq \pi$, $|p(\theta)| = |\epsilon + \cos(\theta)| < 1 - \frac{\epsilon}{2}$. Now we choose $0 < \eta < \delta$ so that $p(\theta) \geq 1 + \frac{\epsilon}{2}$ for $|\theta| < \eta$. Now define the $p_k(\theta) = p(\theta)^k$. Lastly, let M be an upper bound on f for all θ (since integrability is stronger than boundedness). By construction, the p_k are trigonometric polynomials, so $\hat{f}(n) = 0$ for all n implies:

$$\int_{-\pi}^{\pi} f(\theta)p_k(\theta)d\theta = 0, \forall k.$$

However:

$$\int_{|\theta|<\eta} f(\theta)p_k(\theta)d\theta \geq 2\eta\frac{f(0)}{2}(1+\frac{\epsilon}{2})^k.$$

This means that, as k increases, so will the value of this integral. Further:

$$\int_{\eta\leq|\theta|<\delta} f(\theta)p_k(\theta)d\theta \geq 0 \tag{3}$$

and also

$$\left|\int_{\delta\leq|\theta|} f(\theta)p_k(\theta)d\theta\right| \leq 2\pi M(1-\frac{\epsilon}{2})^k, \tag{4}$$

meaning that the other chunks of the integral are either positive or bounded. Then, we conclude that $\int p_k(\theta)f(\theta)d(\theta) \to \infty$ as $k \to \infty$, so the integral that should be 0 by assumption blows up, which is a contradiction. □

4. Parseval's Identity and the Basel Problem

Theorem 4.1 (Parseval's Identity). *For a function $f(x)$ where $x \in [-\pi, \pi]$, we have*

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi \sum_{\infty}^{\infty} |c_n|^2,$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Proof. The first thing to note here is that, as mentioned early in the text, the functions e^{ikx} are orthogonal for different values of k. One can check that they are orthonormal, so:

$$||\sum_{k=-n}^{n} c_k e^{ikx}||^2 = \sum_{k=-n}^{n} |c_k|^2.$$

Now we also note that the following is true via the Pythagorean Theorem:

$$||f||^2 = \left\| f - \sum_{k=-n}^{n} c_k e^{ikx} \right\|^2 + \left\| \sum_{k=-n}^{n} c_k e^{ikx} \right\|^2,$$

allowing us to conclude

$$||f||^2 = ||f - S_n||^2 + \sum_{k=-n}^{n} |c_k|^2.$$

Then, since we found that $||f - S_n|| < \epsilon, \forall n \geq N$, we can take N to infinity and get:

$$||f||^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} |c_k|^2$$

which is exactly Parseval's identity. □

With this identity, it takes a few simple steps to solve the Basel problem.

Theorem 4.2 (Basel Problem).

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Proof. Let $f(x) = x$. Then,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx \implies |c_n|^2 = \begin{cases} 0, & \text{if } n = 0, \\ \frac{1}{n^2}, & \text{if } n \neq 0. \end{cases}$$

Thus,

$$||f||^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi} \frac{2\pi^3}{3} = \frac{\pi^2}{3}$$

Finally, since

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{n^2},$$

we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2}||f||^2 = \frac{\pi^2}{6},$$

as we wanted. □

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