MATH 120 - WIM

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Doubling the Cube and Trisecting the Angle

1. Introduction

When people think of mathematics, it's not uncommon for their view to be of a field full of numbers, symbols, and calculations. While that may represent what students learn until high school, this reflects a modern view of mathematics that would elude many of the great minds in history. The Ancient Greeks, whose mathematical prowess was among the most advanced in the world for a millennium, would have no idea how to read a modern book on algebra, even if it were translated.

The algebraic notation took hundreds of years to be fully formalized and symbolic, which raises the question: how did people study mathematics without such tools? The answer lies in geometry. The philosopher and mathematician Thales of Miletus, who brought geometry from Egypt to Greece, is credited with saying: "The greatest is space, for it holds all things," a view numerous thinkers shared for centuries to follow.

While fields such as algebra and analysis are of an abstract nature, it was often hard for mathematicians to separate geometry from the real-world tools used to represent it, especially in Ancient Greece. In particular, Western mathematics was heavily reliant on Euclid's Postulates, which, given that they only referred to lines and circles, made a compass and a straightedge the logical tools for working with geometry.

The Greeks studied what could be constructed with these tools, discovering numerous theorems and manipulations. Even those thousands of years ago, they tried to rigorously prove and formalize what they were working with, leaving to future mathematicians what they couldn't accomplish. Constructions that allowed for the bisection of any arbitrary angle and doubling of a square with a straightedge and compass were discovered early in the study of geometry, but for centuries, mathematicians weren't able to expand these notions to trisecting an angle or doubling a cube's volume, which are the two problems to be demonstrated throughout this paper.

2. Constructible Numbers

Let K be an infinite set of ordered pairs that is exactly what can be constructed using only a straightedge and compass. To represent these measurements on a plane, define an origin and what your unit is. Then, any point (a, b) in K can be represented on a coordinate plane by taking a to be the horizontal coordinate and b the vertical.

Opening the compass to the unit length and progressively marking a line, any n number of units can be drawn, so, by construction, all integers can be represented in this plane and hence belong to K (Figure 1).

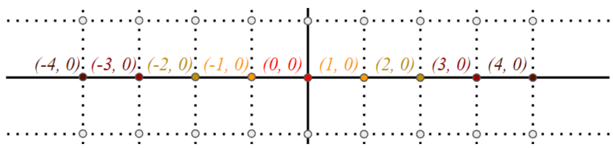


FIGURE 1. Fig 1: Integer lengths on the coordinate plane

Using similar triangles and defining some unit length, multiplication, and division and be performed on previously drawn lengths, so all rationals are in K (Figure 2) ([**Sumant**]).

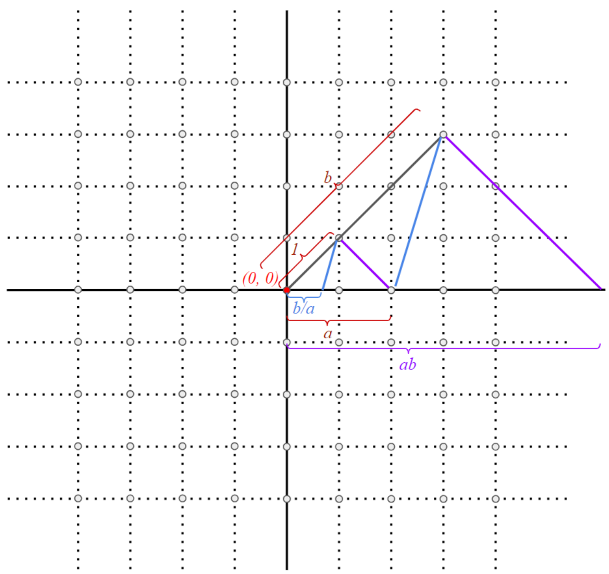
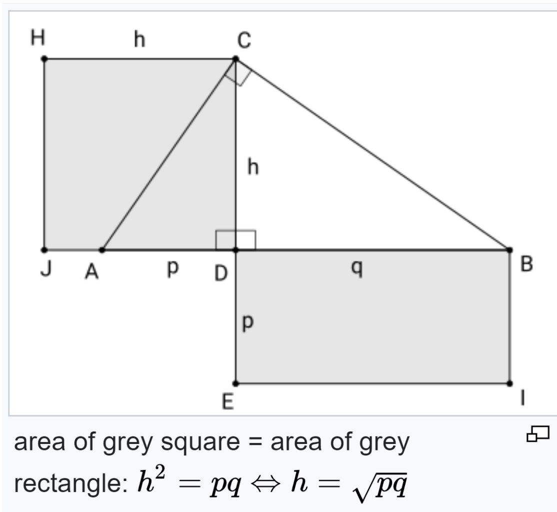


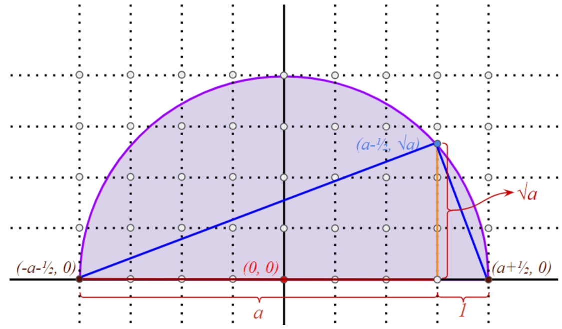
FIGURE 2. Fig 2: Rational lengths on the coordinate plane

Finally, recall the geometric mean theorem to find the square root of a given number. The theorem states that the square of the altitude h of a right triangle is equal to the product of the two segments p, and q, of the hypothenuse separated at the intersection between the hypotenuse and the altitude (Figure 3a) ([**Euclid**]). This means that if p is a unit, the altitude must be the square root of q and vice-versa. (Figure 3b)

This reasoning can be further expanded to represent all roots by powers of 2, as if q is already a square root, h will be a fourth root, and so on. Having shown these elements of K, it follows to demonstrate whether they are all such elements. To do this, recall that all that can be drawn with a straightedge are lines; and with a compass, circles with a known



(A) Fig 3a: Geometric mean theorem



(B) Fig 3b: Square roots in the coordinate plane via the geometric mean theorem

center. Therefore, the only possible constructions are centers of circles, and intersections of the forms: line-line, line-circle, and circle-circle.

Looking at these shapes algebraically, lines can be expressed as linear equations, and circles as quadratic equations, so these constructions are quadratic and only involve differences, sums, products, and ratios, so the set K of the rationals, their 2^n roots, and linear combinations of the two is the entirety of K. Then, to determine whether an arbitrary angle can be trisected or any cube can be doubled, it is imperative to show that they can't be attained just using these tools.

3. Trisecting the Angle

Simply put, the question is: knowing that for any angle θ, there is a right triangle whose hypothenuse has length 1 and sides $\cos(\theta)$, and $\sin(\theta)$, is it always possible to make a right triangle with sides $\cos(3\theta)$ and $\sin(3\theta)$ using only the operations viable in K?

The first useful observation to make when tackling this problem is that there are trisectable angles. For instance, both a 180° and a 60° angles can be trivially drawn. To show that an arbitrary θ can't be trisected, first assume it can and produce a counterexample, as that ends up significantly less laborious than a direct proof. To do this, start with a "nice" angle that doesn't have a trivial trisection, such as 30° or 60°, and see if a contradiction can be produced. This is done as follows:

Say $\theta = 60°$. Using the triple angle formula, $\cos(\theta) = 4\cos 3(3\theta) - 3\cos(3\theta)$, write $\cos(60°) = 1/2 = 4\cos 3(20°) - 3\cos(20°)$. Multiplying by 8 and defining $x = \cos(20°)$ yields the polynomial $p(x) = 8x^3 - 6x - 1$ that has $\cos(20°)$ as a root.

Now, remember the Rational Roots Theorem ([**Descartes**]): Any polynomial equation $0 = a_0 + a_1x + ... + a_nx^n$ with nonzero a_0, and a_n can only have rational roots of the form $x = bc$ for b an integer divisor of a_0, and c an integer divisor of a_n.

For the polynomial $p(x)$, $a_n = 8$ and $a_0 = 1$, which leaves the following possibilities for rational roots: 1/8, 1/4, 1/2, 1. Plugging and checking, however, shows that none of these are roots, meaning $p(x)$ has no rational roots (Table 1).

	1	1/2	1/4	1/8
$+$	$p(1) = 1$	$p(1/2) = -3$	$p(1/4) = -2.375$	$p(1/8) = -1.734375$
$-$	$p(-1) = -3$	$p(-1/2) = 1$	$p(-1/4) = 0.375$	$p(-1/8) = -0.265625$

Table 1

Knowing that there must be some irrational element to all roots, look at the irrational part of K: 2^n roots and their linear combinations with rationals (i.e., every irrational number in K is of the form $a + \sqrt{b}$ for $a \in \mathbb{Q}$, $b \in \mathbb{Q}(\sqrt{})$, and b not a perfect square). Then, assume for some a, b that $a + \sqrt{b}$ is a root of $p(x)$. It quickly follows that $a - \sqrt{b}$ would also be a root since:

If $a + \sqrt{b}$ is a root of some polynomial $q(x)$ with rational coefficients, $q(x)$ must have a degree greater than 1 (a degree 1 would be of the form $x - (a + b)$, which trivially has no rational coefficients). With some work, it can be concluded that the minimal polynomial has a degree of exactly 2. The Fundamental Theorem of Algebra yields that all polynomials with real coefficients can be factored into linear and quadratic terms with real coefficients ([**Polynomial**]), meaning that if $p(x)$ can't be divided by a linear term as shown, it must be by a quadratic term. Guessing that $a - b\sqrt{c}$ will also be a root:

$$(x - (a + \sqrt{b}))(x - (a - \sqrt{b})) = x^2 - x(a - \sqrt{b}) - x(a + \sqrt{b}) + (a - \sqrt{b})(a + \sqrt{b})$$

$$(x - (a + \sqrt{b}))(x - (a - \sqrt{b})) = x^2 - 2xa + (a^2 - b)$$

Which is monic with rational coefficients, so it is the minimal polynomial. Then, any polynomial with rational coefficients and $a + b\sqrt{c}$ as a root must also have $a - b\sqrt{c}$ as a root because the minimal polynomial will divide it. The contradiction is that factoring a cubic expression of the form $x^3 + \beta x + \gamma = 0$ to have a factor of $x^2 - 2xa + (a^2 - b)$ yields:

$$x^3 + \beta x + \gamma = (x^2 - 2xa + (a^2 - b))(x - C) \qquad \text{for some unknown C}$$

$$x^3 + \beta x + \gamma = x^3 + x^2(-C - 2a) + x(aC + (a^2 - b)) + C(b - a^2)$$

Knowing the x^2 term is 0 in the original expression, so $-C - 2a = 0$ and $C = -2a$, so C is rational and a root, a contradiction, meaning an arbitrary $a + \sqrt{b}$ can't be a root, even though it represents any element of K.

This gives the conclusion that the roots of $p(x) = 8x^3 - 6x - 1$ can't be constructed using only the operations in K, so a length equal to the $\cos(20°)$ can't be made. Since one angle can't be trisected, the general trisection must be impossible, thus ending the demonstration.

4. Doubling the Cube

This problem is: from a side with length 1, can a length of $\sqrt[3]{2}$ be constructed? Since the volume of the cube is simply the side length cubed, this is the side length that yields a cube with twice the volume of the original one.

This proof of impossibility can be structured analogously to the previous one. If $\sqrt[3]{2}$ can be constructed, then there exists a polynomial with rational coefficients such that $\sqrt[3]{2}$ is a root, such as$q(x) = x^3 - 2$. By the Rational Roots Theorem, any rational roots would need to be ± 1, ± 2, which can again be plugged into $q(x)$ to verify whether they are roots (Table 2):

	1	2
$+$	$q(1) = -1$	$q(2) = 6$
$-$	$q(-1) = -3$	$q(-2) = -10$

Table 2

The lack of such roots also means that $x^3 - 2$ can't be factored, as that would require linear terms of the form $x - k$ where k is a rational root, which doesn't exist. The last thing to check are the square roots, but the x^2 term also has a 0 coefficient here, so the roots must add to 0. This necessarily produces a contradiction when looking at the possible roots of q(x) that include a square root.

Then, it can finally be concluded that it is not possible to construct a segment of length $\sqrt[3]{2}$ with a straightedge and compass, meaning doubling the cube isn't possible with such tools.

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