# Analysis

# Assignments for 22 (Group 1)/23 April (Groups 2, 3).

## Solutions

32. (a) For $n = 1$ the inequality holds with equality.

For $n = 2$ the inequality is the defining property of a convex function.

Let us assume that the inequality holds for some $n \in \mathbb{N}$ (induction assumption), and let $\lambda_1 \geq 0, \ldots, \lambda_{n+1} \geq 0$ be such that $\lambda_1 + \ldots + \lambda_{n+1} = 1$. We can assume that $\lambda_{n+1} \neq 1$.

$$\varphi\left(\underbrace{\lambda_1 \cdot x_1 + \ldots + \lambda_n \cdot x_n}_{(1-\lambda_{n+1})\cdot y} + \lambda_{n+1} \cdot x_{n+1}\right) \overset{\substack{\text{by the def. of} \\ \text{convexity}}}{\leq} (1-\lambda_{n+1}) \cdot \varphi(y) + \lambda_{n+1} \cdot \varphi(x_{n+1}), \tag{1}$$

where $y = \frac{\lambda_1}{1-\lambda_{n+1}} \cdot x_1 + \ldots + \frac{\lambda_n}{1-\lambda_{n+1}} \cdot x_n$. Observe that $\frac{\lambda_1}{1-\lambda_{n+1}} + \ldots + \frac{\lambda_n}{1-\lambda_{n+1}} = 1$ and all the terms in the sum are non-neagtive, so $y$ is a convex combination of $x_1, \ldots, x_n$. Therefore, using the induction assumption,

$$\varphi(y) = \varphi\left(\frac{\lambda_1}{1-\lambda_{n+1}} \cdot x_1 + \ldots + \frac{\lambda_n}{1-\lambda_{n+1}} \cdot x_n\right) \leq \frac{\lambda_1}{1-\lambda_{n+1}} \cdot \varphi(x_1) + \ldots + \frac{\lambda_n}{1-\lambda_{n+1}} \cdot \varphi(x_n) \tag{2}$$

Combining (1) and (2), the statement follows.

(b) Let $x_1, \ldots, x_n > 0$. First we set the $G \leq A$ inequality, then we use equivalent operations in each step.

$$\sqrt[n]{x_1 \cdot \ldots \cdot x_n} \leq \frac{x_1 + \ldots + x_n}{n} \tag{3}$$

$$\Updownarrow$$

$$x_1 \cdot \ldots \cdot x_n \leq \left(\frac{x_1 + \ldots + x_n}{n}\right)^n \tag{4}$$

$$\Updownarrow$$

$$\log(x_1) + \ldots + \log(x_n) \le n \cdot \log\left(\frac{x_1 + \ldots + x_n}{n}\right) \tag{5}$$

$$\Updownarrow$$

$$\frac{\log(x_1) + \ldots + \log(x_n)}{n} \le \log\left(\frac{x_1 + \ldots + x_n}{n}\right) \tag{6}$$

Inequality (6) is the definition of concavity applied to the function $\log(x)$. $\square$

33. We will apply the definition of convexity with the arguments $x_1 = 0,\ x_2 = 1$, and with the weights $\lambda_1 = 1 - x,\ \lambda_2 = 1 - \lambda_1 = x$ :

$$f\left((1-x)\cdot 0 + x \cdot 1\right) \le (1-x) \cdot \underbrace{f(0)}_{0} + x \cdot \underbrace{f(1)}_{1} = x\,,$$

which proves the statement.

34. First derivative:

$$f'(x) = \frac{2}{1+x^2} - \frac{1}{2x^2} > 0 \iff 1 + x^2 > 4x^2 \iff 1 > 3x^2 \iff \frac{1}{\sqrt{3}} > |x|\,.$$

Further

$$f''(x) = \frac{1}{x^3} - \frac{4x}{(1+x^2)^2} > 0 \iff (1+u)^2 > 4u^2 \text{ (with } u = x^2 \ge 0) \iff u > 1 \iff |x| > 1\,.$$

The implications were already posted.

35. Let $f(x) = x\ln^2(x)$. The function is defined only if the input of the logarithm is larger than zero, i.e. $D_f = \mathbb{R}^+ \setminus \{0\}$. Furthermore $f(x) + \infty$ as $x \to +\infty$ and by de L'Hospital's rule

$$\lim_{x \searrow 0} f(x) = \lim_{x \searrow 0} \frac{\ln^2(x)}{x^{-1}} = \lim_{x \searrow 0} \frac{[\ln^2(x)]'}{[x^{-1}]'} = \lim_{x \searrow 0} \frac{2(\ln x)/x}{-x^{-2}} = -2\lim_{x \searrow 0}(x \ln x) = 0\,.$$

The first derivative is

$$f'(x) = \ln^2(x) + 2\ln(x)\,,\ x > 0\,.$$

Then $f'(x) = 0$ if and only if $x = 1$ or $x = e^{-2}$.

| $x$ | $(0, e^{-2})$ | $e^{-2}$ | $(e^{-2}, 1)$ | $1$ | $(1, +\infty)$ |
|---|---|---|---|---|---|
| $f'(x)$ | $+$ | $0$ | $-$ | $0$ | $+$ |
| $f(x)$ | $\nearrow$ | local max | $\searrow$ | local min | $\nearrow$ |

We proceed with the second derivative:

$$f''(x) = 2\frac{\ln(x) + 1}{x} \quad x > 0\,,$$

which is zero if and only if $x = \frac{1}{e}$.

| $x$ | $(0, \frac{1}{e})$ | $\frac{1}{e}$ | $(\frac{1}{e}, +\infty)$ |
|---|---|---|---|
| $f''(x)$ | $-$ | $0$ | $+$ |
| $f(x)$ | $\smile$ | wp | $\frown$ |

Moreover

$$\lim_{x \to +\infty} f(x) = +\infty, \qquad \lim_{x \searrow 0} \frac{\ln^2(x)}{x^{-1}} \overset{(*)}{=} \lim_{x \searrow 0} \frac{2x^{-1}}{-x^{-2}} = 0\,,$$

where in $(*)$ we apply de L'Hospital.

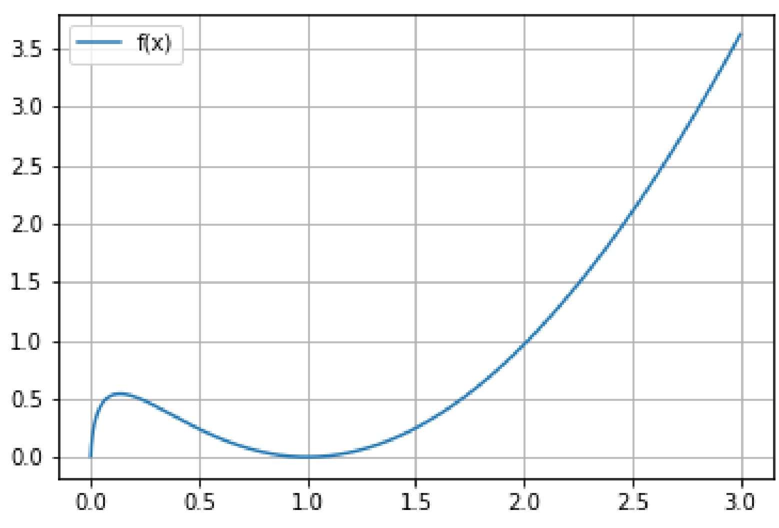


**Figure 1:** Ex.35; plot made with Python

36. For $f(x) = e^{-1/x^2}$ we have $D_f = \mathbb{R} \setminus \{0\}$. $f$ is an even function and has no zeroes.

$f'(x) = \dfrac{2}{x^3} \cdot e^{-1/x^2}$.

$D_{f'} = \mathbb{R} \setminus \{0\}$ and $f'$ has no zeroes, so the only point which is used for splitting in the first table is $x = 0$.

| $x$ | $(-\infty, 0)$ | $0$ | $(0, \infty)$ |
|---|---|---|---|
| $f'(x)$ | $-$ | | $+$ |
| $f(x)$ | $\searrow$ | | $\nearrow$ |

At $x = 0$ there is NO local minimum, since $x = 0 \notin D_f$.

$f''(x) = \left(\dfrac{4}{x^6} - \dfrac{6}{x^4}\right) \cdot e^{-1/x^2}, \quad D_{f''} = \mathbb{R} \setminus \{0\}$.

$f''(x) = 0 \iff x_{1,2} = \pm\sqrt{\frac{2}{3}}$.

So the points used for splitting in the second table are $0$ and $\pm\sqrt{\frac{2}{3}}$.

| $x$ | $\left(-\infty, -\sqrt{\frac{2}{3}}\right)$ | $-\sqrt{\frac{2}{3}}$ | $\left(-\sqrt{\frac{2}{3}}, 0\right)$ | $0$ | $\left(0, \sqrt{\frac{2}{3}}\right)$ | $\sqrt{\frac{2}{3}}$ | $\left(\sqrt{\frac{2}{3}}, \infty\right)$ |
|---|---|---|---|---|---|---|---|
| $f''(x)$ | $-$ | $0$ | $+$ | | $+$ | $0$ | $-$ |
| $f(x)$ | $\frown$ | Wp | $\smile$ | | $\smile$ | Wp | $\frown$ |

$\lim\limits_{|x| \to +\infty} e^{-1/x^2} = 1$, so the horizontal asymptote is $y = 1$.

As $\lim\limits_{x \to 0} e^{-1/x^2} = 0$, with the extension $f(0) := 0$ the function becomes continuous on $\mathbb{R}$.

The extended function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at $x = 0$:

$f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{e^{-1/x^2}}{x} = \lim\limits_{t \to \infty} \dfrac{t}{e^{t^2}} = \lim\limits_{t \to \infty} \dfrac{1}{2te^{t^2}} = 0$, where we have put $t := \frac{1}{x}$ for $x \neq 0$. Now for the extended function we have similarly $f''(0) = \lim\limits_{x \to 0} \dfrac{f'(x) - f'(0)}{x - 0} =$

$\lim\limits_{x\to 0}\dfrac{2x^3e^{-1/x^2}}{x} = \lim\limits_{t\to\infty}\dfrac{2t^2}{e^{t^2}} = \lim\limits_{t\to\infty}\dfrac{2}{e^{t^2}} = 0$ and so on (can be made precise by induction, using the formula $f^{(k)}(x) = P_k(t)e^{-t^2}$, where $P_k(t)$ is a polynomial in $t = \frac{1}{x}$ satisfying the recursion formula $P_{k+1}(t) = -t^2P_k'(t) - 2t^3P_k(t)$ for all $t \neq 0$). All derivatives of $f$ at $x = 0$ exist and are equal to zero, despite the fact that the function is **not** constantly zero.

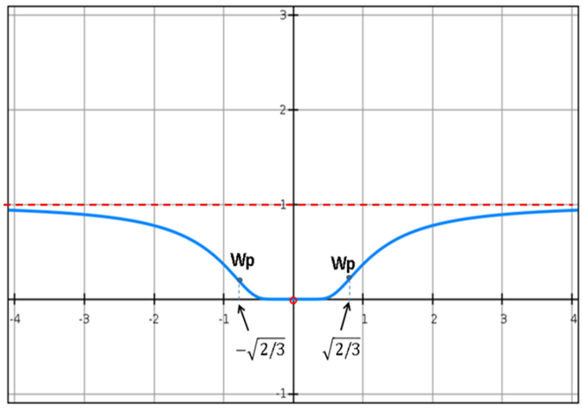


**Figure 2:** The plot to Ex. 36. is made by graphsketch.com

37. Let $f(x) = \frac{1}{\ln(x)}$, then $D_f = \mathbb{R}^+ \setminus (\{0\} \cup \{1\})$.

$$f'(x) = \frac{-1}{\ln^2(x)}\frac{1}{x} = -\frac{1}{x\ln^2(x)}.$$

We can use the results of the previous exercises to compute the sign of $f'(x)$:

| $x$ | $(0,1)$ | $(1,+\infty)$ |
|---|---|---|
| $f'(x)$ | $-$ | $-$ |
| $f(x)$ | $\searrow$ | $\searrow$ |

So the function is always decreasing and we cannot detect local minima/maxima.

$$f''(x) = \frac{\ln(x) + 2}{x^2\ln^3(x)},$$

which is zero if and only if $x = e^{-2}$.

| $x$ | $(0, e^{-2})$ | $e^{-2}$ | $(e^{-2}, 1)$ | $(1, +\infty)$ |
|---|---|---|---|---|
| $f''(x)$ | $-$ | 0 | $-$ | $+$ |
| $f(x)$ | $\frown$ | wp | $\frown$ | $\smile$ |

$$\lim_{x\to 0^+} f(x) = 0, \quad \lim_{x\to 1^-} f(x) = -\infty, \quad \lim_{x\to 1^+} f(x) = +\infty, \quad \lim_{x\to +\infty} f(x) = 0.$$

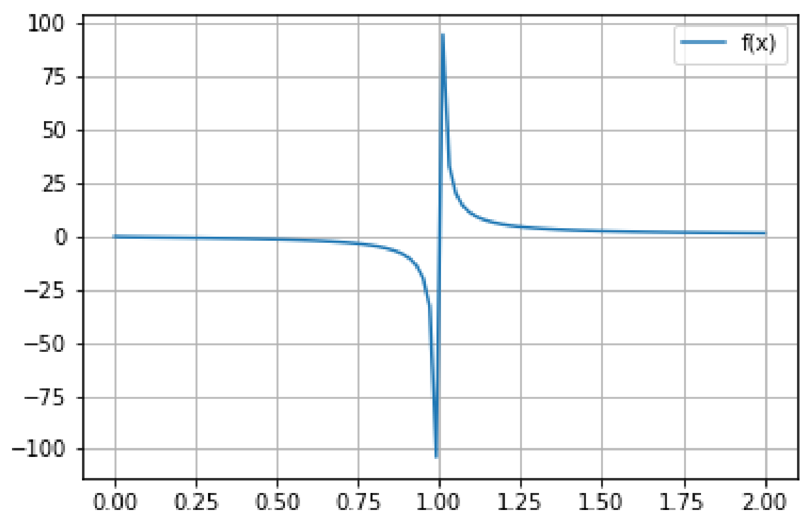


**Figure 3:** Ex.37; plot made with Python

38. Let $f(x) = x^2 e^{-x^2}$, then $D_f = \mathbb{R}$ and $f(x) = 0$ if and only if $x = 0$ and $f(x) \searrow 0$ as $|x| \to +\infty$. For the first derivative we have

$$f'(x) = 2xe^{-x^2}(1 - x^2) = 0 \iff x = 0 \lor x = \pm 1.$$

Finally for the second derivative

$$f''(x) = 2e^{-x^2}(1-4x^2+x^4) = 0 \iff x^4-4x^2+1 = 0 \iff (x^2-2)^2 = 4-1 = 3 \iff x^2 = 2\pm\sqrt{3}.$$

Therefore $f''(x) = 0$ for

$$x = -\sqrt{2+\sqrt{3}}, \quad x = -\sqrt{2-\sqrt{3}}, \quad x = \sqrt{2-\sqrt{3}}, \quad x = \sqrt{2+\sqrt{3}}.$$

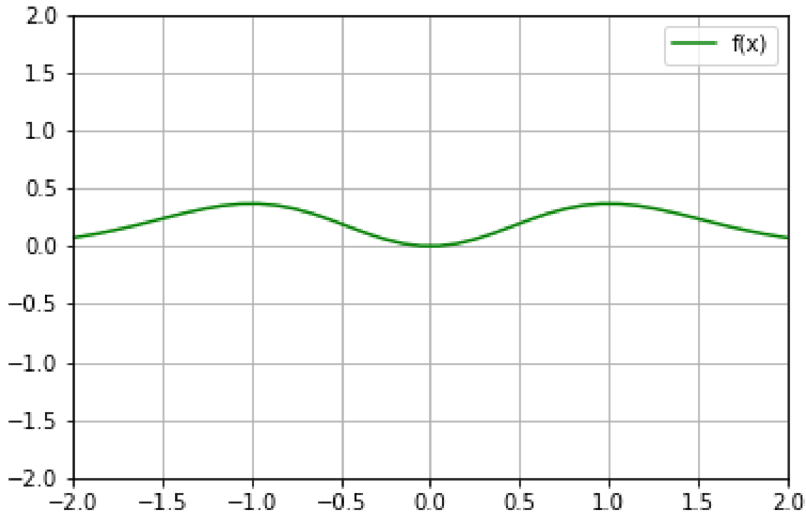


**Figure 4:** Ex.38; plot made with Python

39. Let $f(x) = \frac{\ln(x)}{x}$, then $D_f = \mathbb{R}_+ \setminus \{0\}$ and $f(x) = 0$ if and only if $x = 1$. Furthermore $f(x) \to -\infty$ as $x \searrow 0$ and by de L'Hospital's rule $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1/x}{1} = 0$.

$$f'(x) = \frac{1-\ln(x)}{x^2} = 0 \iff x = e.$$

$$f''(x) = \frac{2\ln(x)-3}{x^3} = 0 \iff x = e^{\frac{3}{2}}.$$

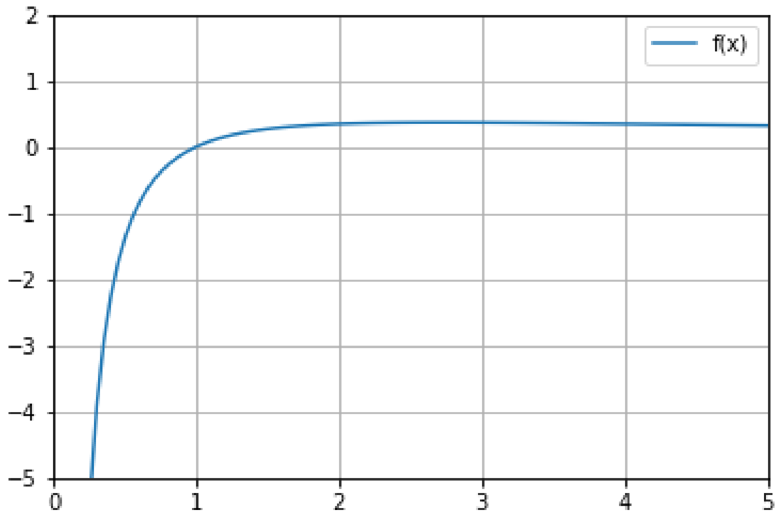


**Figure 5:** Ex.39; plot made with Python

40. Let $f(x) = \sqrt[3]{(x^2-1)}$, then $D_f = \mathbb{R}$ and $f(x) \to +\infty$ as $|x| \to +\infty$.

$$f'(x) = \frac{2x}{3\sqrt[3]{(x^2-1)^2}}\,, |x| \neq 1\,.$$

| $x$ | $(-\infty, 0)$ | $0$ | $(0, +\infty)$ |
|---|---|---|---|
| $f'(x)$ | $-$ | $0$ | $+$ |
| $f(x)$ | $\searrow$ | local min | $\nearrow$ |

$$f''(x) = -2\frac{(x^2+3)}{9\sqrt[3]{(x^2-1)^5}}\,, |x| \neq 1\,,$$

which is never zero and it is positive if and only if $x \in (-1, 1)$; in particular $f''(0) = \frac{2}{3}$.

| $x$ | $(-\infty, -1)$ | $(-1, 1)$ | $(1, +\infty)$ |
|---|---|---|---|
| $f''(x)$ | $-$ | $+$ | $-$ |
| $f(x)$ | $\frown$ | $\smile$ | $\frown$ |

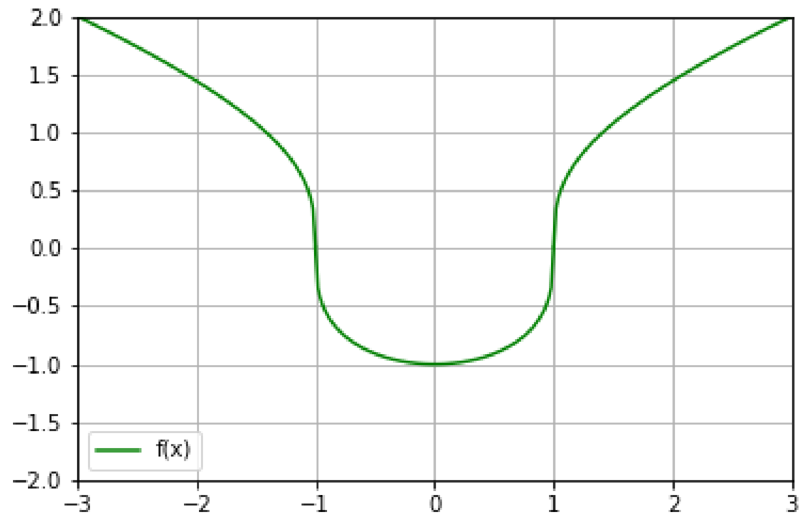


**Figure 6:** Ex.40; plot made with Python

41. Let $f(x) = \frac{4x}{4-x^2}$, then $D_f = \mathbb{R} \setminus (\{-2\} \cup \{2\})$ and $f(x) \to 0$ as $|x| \to +\infty$.

$$f'(x) = \frac{4(x^2+4)}{(4-x^2)^2} > 0, \forall x \in D_f.$$

Hence $f(x)$ is increasing in its whole domain.

$$f''(x) = \frac{8x(x^2+12)}{(4-x^2)^3},$$

which vanishes in zero and is positive if and only if $x \in (-\infty, -2)$ or $x \in (0, 2)$:

| $x$ | $(-\infty,-2)$ | $(-2,0)$ | $0$ | $(0,2)$ | $(2,+\infty)$ |
|---|---|---|---|---|---|
| $f''(x)$ | $+$ | $-$ | $0$ | $+$ | $-$ |
| $f(x)$ | $\smile$ | $\frown$ | wp | $\smile$ | $\frown$ |

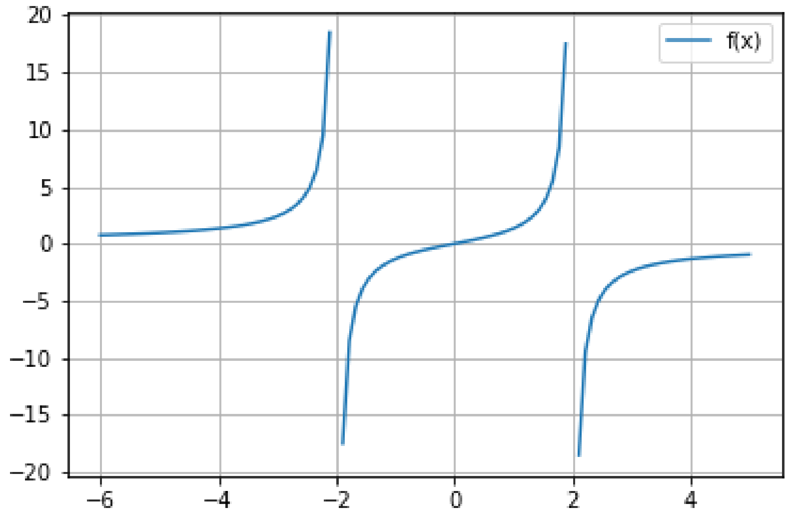


**Figure 7:** Ex.41; plot made with Python

42. Let $f(x) = \sqrt[3]{x+1}$, then $D_f = \mathbb{R}$ with $f(x) \to \pm\infty$ as $x \to \pm\infty$.

$$f'(x) = \frac{1}{3\sqrt[3]{(x+1)^2}} > 0, \qquad \forall x \in \mathbb{R} \setminus \{-1\}.$$

Therefore $f(x)$ is always increasing in its domain. Likewise for all $x \neq -1$

$$f''(x) = -\frac{2}{9\sqrt[3]{(x+1)^5}},$$

which never vanishes but it's positive if and only if $x < -1$, hence

| $x$ | $(-\infty, -1)$ | $(-1, +\infty)$ |
|---|---|---|
| $f''(x)$ | $+$ | $-$ |
| $f(x)$ | $\smile$ | $\frown$ |

$$\lim_{x \to +\infty} f(x) = +\infty, \quad \lim_{x \to -\infty} f(x) = -\infty, \quad \lim_{x \to -1^{\pm}} f(x) = 0.$$

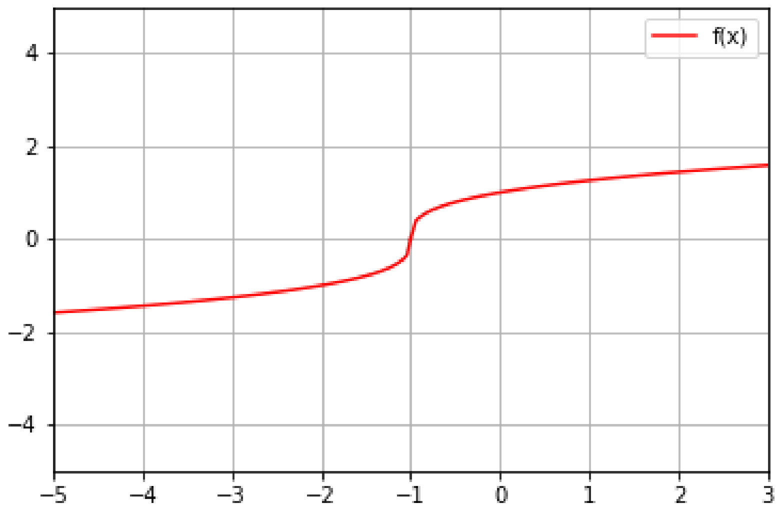


**Figure 8:** Ex.42; plot made with Python

43. Let $f(x) = \frac{x^2-3}{x+2} = (x-2) + \frac{1}{(x+2)}$, with $D_f = \mathbb{R} \setminus \{-2\}$.

$$f'(x) = 1 - \frac{1}{(x+2)^2} = \frac{x^2+4x+3}{(x+2)^2},$$

which is zero if $x = -3$ or if $x = -1$.

| $x$ | $(-\infty, -3)$ | $-3$ | $(-3, -2)$ | $(-2, -1)$ | $-1$ | $(-1, +\infty)$ |
|---|---|---|---|---|---|---|
| $f'(x)$ | $+$ | $0$ | $-$ | $-$ | $0$ | $+$ |
| $f(x)$ | $\nearrow$ | local max | $\searrow$ | $\searrow$ | local min | $\nearrow$ |

$$f''(x) = \frac{2}{(x+2)^3},$$

which is positive for $x > -2$ and never zero.

| $x$ | $(-\infty, -2)$ | $(-2, +\infty)$ |
|---|---|---|
| $f''(x)$ | $-$ | $+$ |
| $f(x)$ | $\frown$ | $\smile$ |

$$\lim_{x\to+\infty} f(x) = +\infty, \quad \lim_{x\to-\infty} f(x) = -\infty, \quad \lim_{x\to-2^+} f(x) = +\infty, \quad \lim_{x\to-2^-} f(x) = -\infty.$$

In particular the vertical asymptote is given by $x = -2$. To find the inclined one, observe $f(x) = (x-2) + \frac{1}{x+2}$ which implies

$$\lim_{|x|\to+\infty} |f(x) - (x-2)| = \lim_{|x|\to+\infty} \frac{1}{|x+2|} = 0\,,$$

which implies that the equation of the inclined asymptote is given by $y = x - 2$.

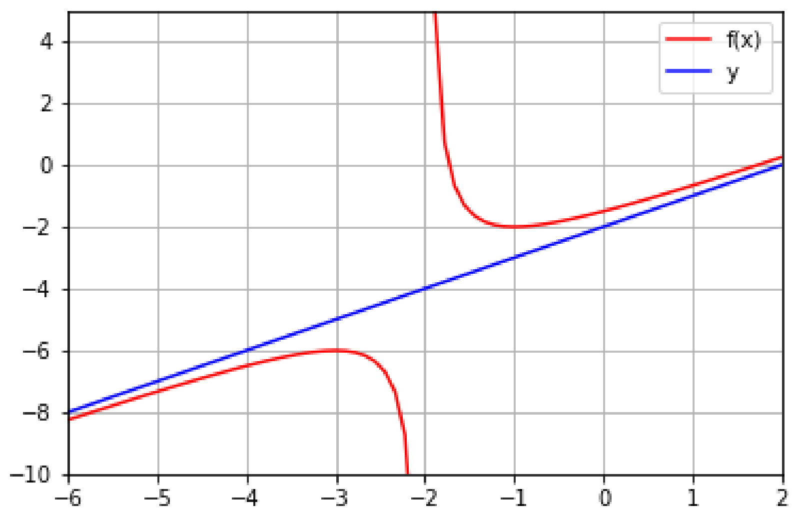


**Figure 9:** Ex.43; plot made with Python