Analysis

Assignment for 25(Group 1)/26 March (Groups 2, 3)

Solutions

10. (a) $b_5 = \left(\frac{6}{5}\right)^6 \approx 2.986 < 3$.

    (b) $b_n \leq b_{n-1} \iff 1 + \frac{1}{n} \leq \sqrt[n+1]{b_{n-1}} = G$ for n copies of $1 + \frac{1}{n-1}$ and the number 1 once. For these quantities the harmonic mean is $H = \frac{n+1}{1+n(1+\frac{1}{n-1})^{-1}} = \frac{n+1}{n} = 1 + \frac{1}{n}$, which shows the assertion.

    (c) $a_n < (1+\frac{1}{n})^n(1+\frac{1}{n}) = b_n \leq 3$ for $n \geq 5$ because of (a), and $a_n < 3$ for $n \in \{1, 2, 3, 4\}$ by direct verification. Therefore the limit of a_n exists; it is usually denoted by Euler's number e.

    (d) $0 \leq b_n - a_n = \frac{1}{n}a_n \leq \frac{3}{n} \to 0$ as $n \to \infty$, so $b_n = b_n - a_n + a_n \to 0 + e = e = \lim\limits_{n\to\infty} a_n$.

11. We know that $\frac{1}{n} \to 0$ converges but $(-1)^n$ diverges (does not converge). Therefore by the addition theorem, the sum of these two sequences cannot converge.

12. Case distinction: case $\ell = 0$, then $|(-1)^n d_n| = |d_n| \to \ell = 0$ shows $(-1)^n d_n \to 0$. Case $\ell \neq 0$: then $(-1)^n \ell$ diverges (multiplication or division theorem), but $|(-1)^n d_n - (-1)^n \ell| = |d_n - \ell| \to 0$, and hence by the addition theorem $(-1)^n d_n$ must diverge. Actually, this sequence has two accumulation points $-\ell$ and $+\ell$.

13. First of all, we have $1 \leq \sqrt[n]{n} \leq 1 + \frac{2}{\sqrt{n}} \to 1$ by Exercise 7. Hence the comparison criterion shows $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. Now, since $1 \leq \log n \leq n$ for all $n \geq 3$, we have

$$\sqrt[n]{n} \leq \sqrt[n]{n \log n} \leq \sqrt[n]{n^2} = \left(\sqrt[n]{n}\right)^2 .$$

As the leftmost and the rightmost sequence both tend to one (the latter to $1^2 = 1$ by the multiplication theorem) as $n \to \infty$, the comparison criterion yields convergence of $\sqrt[n]{n \log n}$ to 1.

14. Two ways to solution: either invoking theory of series and the knowledge that the exponential series $\sum_{n=0}^{\infty} \frac{a^n}{n!}$ converges, so the expression $\frac{a^n}{n!}$ necessarily tends to zero as $n \to \infty$. Or we do it directly via the observation that for all $n \geq N = \lceil 2a \rceil$, we have

$$0 \leq \frac{a^{n+1}}{(n+1)!} \leq \frac{1}{2} \frac{a^n}{n!} \leq \frac{1}{2} \frac{1}{2} \frac{a^{n-1}}{(n-1)!} \leq \cdots \leq \left(\frac{1}{2}\right)^{n-N} \frac{a^N}{N!} = \frac{(2a)^N}{N!} \left(\frac{1}{2}\right)^n \to 0 \quad \text{as } n \to \infty\,,$$

which by the way establishes convergence of the exponential series by the quotient criterion (using the majorization criterion and convergence of the geometric series).

15. We show $\frac{n!}{n^n} \to 0$ as $n \to \infty$ by the comparison criterion:

$$0 \leq \frac{n!}{n^n} = \frac{1 \cdot 2 \cdots (n-1) \cdot n}{n \cdot n \cdots n \cdot n} = \frac{1}{n} \frac{2}{n} \cdots \frac{n}{n} < \frac{1}{n} \cdot 1 \cdots 1 = \frac{1}{n} \to 0 \quad \text{as } n \to \infty\,.$$

16. Use the addition theorem for series and the geometric series formula $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$:

$$\sum_{n=0}^{\infty} \left[\left(\frac{2}{5}\right)^n + \left(\frac{3}{5}\right)^n\right] = \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n = \frac{5}{3} + \frac{5}{2} = \frac{25}{6}\,.$$

17. Since $\log n \to \infty$, we know $\frac{1}{\log n} \to 0$ as $n \to \infty$, so by Leibniz' theorem, the first series converges. But as $\frac{1}{\sin n}$ does not converge to zero because of $|\frac{1}{\sin n}| \geq \frac{1}{1} = 1$ for all n, again by Leibniz' theorem, the second series diverges.

18. We look at the quotient of two subsequent terms:

$$\frac{(2n+2)!}{[(n+1)!]^2} \Big/ \frac{(2n)!}{[n!]^2} = \frac{(2n+2)(2n+1)}{(n+1)^2} = \frac{4n^2+5n+2}{n^2+2n+1} \to 4 > 1\,,$$

and hence the first series diverges by the quotient criterion. Similarly for the second quotient:

$$\frac{(n+1)!(2n+2)}{(n+1)^{n+1}} \Big/ \frac{n!(2n)}{n^n} = \frac{n+1}{n} \left(\frac{n}{n+1}\right)^n \to \frac{1}{e}\,,$$

and hence the second series converges by the quotient criterion. Additional question after the hint: we know $2 < (1+\frac{1}{n})^n \leq e \leq (1+\frac{1}{n})^{n+1} < 3$ for any $n \geq 5$ by Exercises 8 and 10.

19. (a) Let us consider the function $f(x) = ||x| - 1| - |x|$. First of all f is defined for any real number, i.e. $Dom_f = \mathbb{R}$. When facing a function that is a composition of absolute values one may try to expand them using the definition of absolute value:

$$|x| = \begin{cases} x & \text{for } x \geq 0\,, \\ -x & \text{for } x < 0\,. \end{cases}$$

Clearly if $x = 0$ then $|x| = 0$. Therefore the considered function can be rewritten as a function defined by cases

$$f(x) = \begin{cases} |x - 1| - x & \text{for } x \geq 0\,, \\ |-x - 1| + x & \text{for } x < 0\,. \end{cases}$$

In particular we have $f(0) = 1$. A further expansion can be applied:

$$f(x) = \begin{cases} -1 & \text{for } x \geq 1\,, \\ -2x + 1 & \text{for } 0 < x < 1\,, \\ 2x + 1 & \text{for } -1 < x \leq 0\,, \\ -1 & \text{for } x \leq -1\,. \end{cases}$$

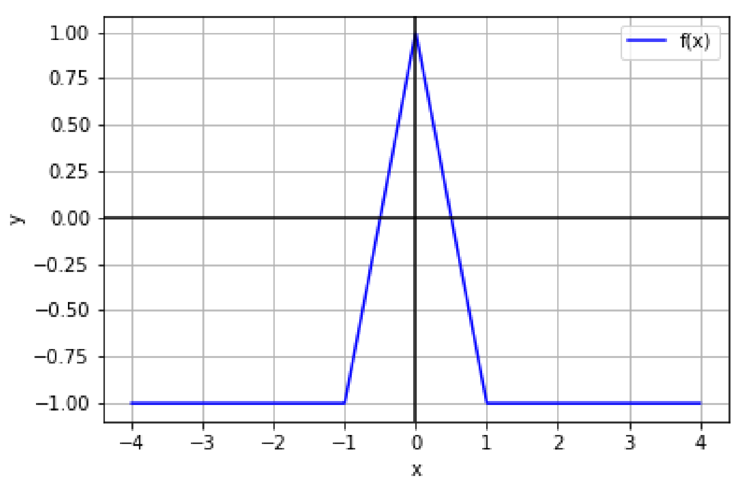

Figure 1: Plot of the function $f(x)$ in Ex.19(a) with Python.

(b) Let us now consider the function $f(x) = |1 - |x||$. Observe that the same reasoning as explained above, i.e. a stepwise cases specification, can be also be applied here. As follows we propose a simpler solution. Observe that $Dom_f = \mathbb{R}$ and in particular $Img_f = \mathbb{R}^+$, since it is equal to an absolute value that returns always a positive real number. Define the function $g(x) = 1 - |x|$ and observe that $f(x) = |g(x)|$. Therefore if we draw the function $g(x)$, $f(x)$ will coincide with $g(x)$ when $g(x)$ is positive and with $-g(x)$ when $g(x)$ is negative:

$$f(x) = \begin{cases} g(x) & \text{for } \; g(x) \geq 0\,, \\ -g(x) & \text{for } \; g(x) < 0\,. \end{cases}$$

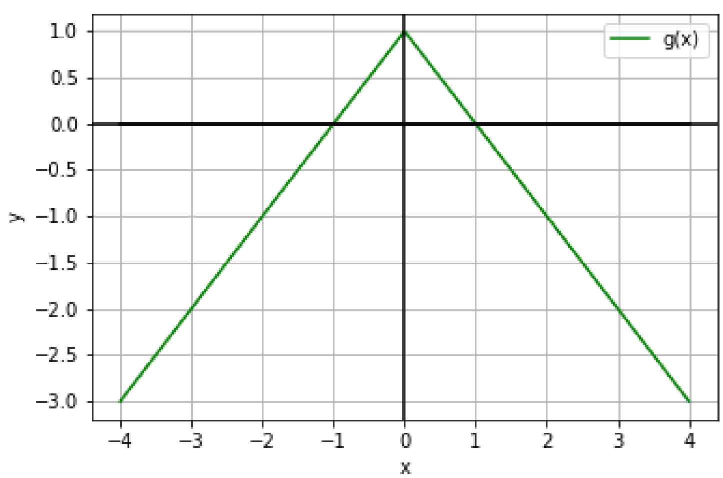

Figure 2: Plot of the function $g(x)$ in Ex.19(b) with Python.

Hence applying to $g(x)$ the absolute value is like mirroring it with respect to the x-axis, i.e. we can apply symmetry with respect to the x-axis to obtain $f(x)$:

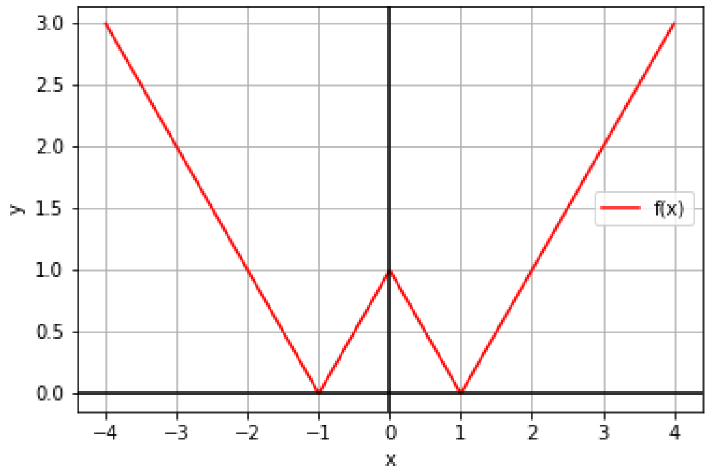

Figure 3: Plot of the function $f(x)$ in Ex.19(b) with Python.

20. (a) Let us consider f and g to be even (*gerade*), then

$$(f+g)(x) = f(x) + g(x) \overset{(*)}{=} f(-x) + g(-x) = (f+g)(-x)\,,$$
$$(f \cdot g)(x) = f(x) \cdot g(x) \overset{(*)}{=} f(-x) \cdot g(-x) = (f \cdot g)(-x)\,,$$

where in $(*)$ we applied the definition of an even function.

(b) Let f be an even function and g an odd (*ungerade*) function, i.e.

$$f(x) = f(-x), \quad \forall x \in Dom_f, \qquad g(-x) = -g(x), \quad \forall x \in Dom_g\,.$$

Then

$$(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = (f \circ g)(x)\,.$$

21. (a) Let $f(x) = \log(1+x^2)$ be the considered function and observe that it is always nonnegative, since the argument of the logarithm is always greater or equal than 1. Therefore the lower bound constant of the function is $y = 0$ and is reached if and only if $x = 0$. Thus

$$f(x) \geq f(0) = 0\,, \qquad \forall x \in \mathbb{R}\,.$$

Moreover observe that since the logarithm grows to infinity as the argument goes to infinity, then there exists no constant $M \in \mathbb{R}^+$ such that

$$f(x) \leq M, \qquad \forall x \in \mathbb{R}\,.$$

(b) Let $f(x) = e^{-|x-\mu|}$, with $\mu \in \mathbb{R}$. Notice that the exponential function is always nonnegative, therefore $f(x) \geq 0$ for all $x \in \mathbb{R}$. On the other hand observe that the argument of the exponential is always negative or zero. Hence since the exponential is a strictly monotone function its maximum and thus upper bound is reached when $x = \mu$, i.e.

$$f(x) = e^{-|x-\mu|} \leq e^0 = 1, \qquad \forall x \in \mathbb{R}\,.$$