Analysis

Assignments for 01 (Group 1)/02 April (Groups 2, 3).

Solutions

22. The domain of f is given by those $x \in \mathbb{R}$ such that

$$2x - \sqrt{x^2 - 1} > 0 \qquad \text{and} \qquad x^2 - 1 \geq 0\,.$$

Therefore we have to solve the following system of inequalities:

$$\begin{cases} 2x > 0 \\ 4x^2 > x^2 - 1 \\ x^2 - 1 \geq 0 \end{cases} \implies \begin{cases} x > 0 \\ x^2 > -\frac{1}{3} \\ |x| \geq 1 \end{cases} \implies \begin{cases} x > 0 \\ x^2 > -\frac{1}{3} \\ x \geq 1\,. \end{cases}$$

Since the previous system is satisfied for all $x \geq 1$, we obtain that $Dom_f = \{x \in \mathbb{R} : x \geq 1\}$ or $Dom_f = [1, +\infty)$.

23. To find the domain of g, observe that the inequality $\sin(x) \geq 1$ is satisfied only for the values $x \in \mathbb{R}$ such that $\sin(x) = 1$, i.e. $\{x \in \mathbb{R} : x = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\}$. In particular in correspondence of those values for x, we have $g(x) = 0$.
Hence $Dom_g = \{\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\}$.

24. The domain of h is given by those $x \in \mathbb{R}$ such that

$$\begin{cases} \frac{x^2+2x}{x-1} \geq 0 & (A) \quad \text{and} \\ x^2 - 1 > 0\,. & (B) \end{cases}$$

Define $h_1(x) := \frac{x^2+2x}{x-1} = \frac{N(x)}{D(x)}$ and let us check for which $x \in \mathbb{R}$ this function is nonnegative, to satisfy the first inequality of previous system. Since it is the ratio of two functions, it is positive when numerator and denominator have the same sign, i.e. either both positive or both negative.

Now

$$N(x) = x(x+2) \geq 0 \implies (x \geq 0 \wedge x \geq -2) \vee (x \leq 0 \wedge x \leq -2) \implies x \leq -2 \vee x \geq 0\,.$$

and likewise $N(x) < 0 \iff -2 < x < 0$. Further $D(x) < 0 \iff x < 1$ and

$$D(x) > 0 \iff x - 1 > 0 \iff x > 1\,.$$

By combination of the conditions on the denominator and on the numerator, (*A*) is satisfied for all $x \in \mathbb{R}$ such that

$$-2 \leq x \leq 0 \vee x > 1\,. \qquad (C)$$

Notice that the inequality in (*B*) is fulfilled for those x such that $x < -1 \vee x > 1$. Combined with condition (*C*) this yields

$$Dom_h = \{x \in \mathbb{R} : -2 \leq x < -1 \ \vee \ x > 1\} = [-2, -1) \cup (1, +\infty)\,.$$

25. Let $a(x) = \arcsin\left(\frac{2x-2}{x-2}\right)$. Observe that the arcsine is a function defined on $[-1, 1]$, hence the denominator $x - 2$ cannot be zero and

$$Dom_a = \left\{x \in \mathbb{R} \setminus \{2\} : -1 \leq \frac{2x-2}{x-2} \leq 1\right\}\,. \tag{1}$$

Isolating x in (1), it is easy to check $x > 2$ is impossible, as this would imply $2x - 2 \leq x - 2$ by the rightmost inequality, which is equivalent to $x < 0$. On the other hand, if $x < 2$, then the inequalities in (1) are equivalent to the two conditions $2 - x \geq 2x - 2 \geq x - 2$ which in turn are equivalent, isolating x again, to $x \in [0, \frac{4}{3}]$.

26. The function $f(x) = x^2 - 4x + 9 = (x-2)^2 + 5$ is not invertible in its domain, since it is not injective as $f(1) = f(3) = 6$; its graph is a parabola with symmetry axis $x = 2$ and

vertex given by the point $V = (2,5)$. Either of the parts on one side of the symmetry axes would correspond to strictly monotone functions. Hence two invertible restrictions of f are

$$f_1 : (-\infty, 2] \to [5, +\infty), \qquad f_2 : [2, +\infty) \to [5, +\infty).$$

Notice that the restrictions on the image set are considered to make f_1 and f_2 both surjective in contrast with f, and with the restriction on the domains both are bijective. Now that we have two bijective functions we are allowed to invert them. Hence we have to solve the following equations with respect to x, by fixing a level/image $y \in [5, +\infty)$:

$$x^2 - 4x + 9 - y = 0 \implies x = 2 \pm \sqrt{y-5}\,.$$

Therefore

$$f_1^{-1}(y) = 2 - \sqrt{y-5}\,, \qquad f_2^{-1}(y) = 2 + \sqrt{y-5}\,, \qquad y \in [5, +\infty)\,.$$

27. Observe that $Dom_f = \mathbb{R}$ and that to be invertible the function f has to be surjective and injective. Notice that the cosine it is not injective on its domain, since for a given $y \in [-1,1]$, there correspond infinitely many $x \in \mathbb{R}$ such that $f(x) = y$. To this extent we have to restrict the domain to fulfill injectivity.
By the graph of the cosine we can see that an example of interval where there is a one-to-one relationship (bijection) between x and y, is $I = [0, \pi]$. In our case

$$3x \in [0, \pi] \implies x \in [0, \pi/3]\,.$$

Since f it is not even surjective, to make it so we consider as the image set (Bildmenge) its own image, so since $\cos(3x) \in [-1,1]$ for all $x \in [0, \pi/3]$, we have

$$0 \leq 2\cos(3x) + 2 \leq 4, \forall x \in [0, \pi/3]\,.$$

Hence $Im(f) = [0,4]$ and the function $f : [0, \pi/3] \to [0,4]$ is now invertible with inverse $f^{-1} : [0,4] \to [0, \pi/3]$. To find the analytic expression of f^{-1}, fix $y \in [0,4]$ and consider the equation

$$y = 2\cos(3x) + 2\,.$$

If we solve this with respect to x we obtain:

$$y - 2 = 2\cos(3x) \implies \cos(3x) = \frac{y-2}{2} \implies x = \frac{1}{3}\arccos\left(\frac{y-2}{2}\right).$$

Therefore the analytic form of f^{-1} is given by

$$f^{-1}(y) = \frac{1}{3}\arccos\left(\frac{y-2}{2}\right), \qquad \text{where } y \in [0,4].$$

28. (a) $f(x)$ is not continuous at $x_0 = 0$ (and therefore also not differentiable).

We show that the function have an essential discontinuity at $x_0 = 0$.

Look at also Classification of discontinuities at

https://en.wikipedia.org/wiki/Classification_of_discontinuities .

Let $x_k = \frac{1}{k\cdot\pi}$. The sequence (x_k) is a null sequence, i.e. $\lim\limits_{k\to\infty} x_k = 0$.

Along the sequence (x_k)

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} \sin\left(\frac{1}{x_k}\right) = \lim_{k\to\infty} \sin(k\pi) = 0.$$

Let $x_m = \frac{1}{2m\pi + \frac{\pi}{2}}$. The sequence (x_m) is a null sequence, i.e. $\lim\limits_{k\to\infty} x_m = 0$.

Along the sequence (x_m)

$$\lim_{m\to\infty} f(x_m) = \lim_{m\to\infty} \sin\left(\frac{1}{x_m}\right) = \lim_{m\to\infty} \sin\left(2m\pi + \frac{\pi}{2}\right) = 1.$$

So $\lim\limits_{x\to 0} f(x) \;\nexists$. (Look at the blue curve in the figure.)

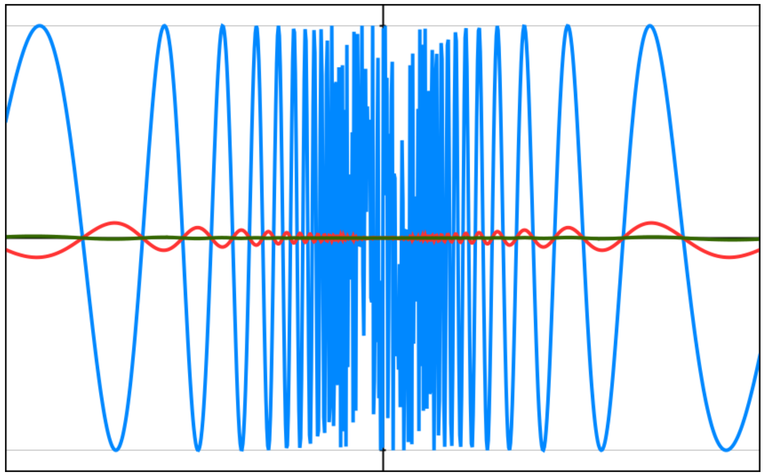
(b) $g(x)$ is continuous at $x_0 = 0$:

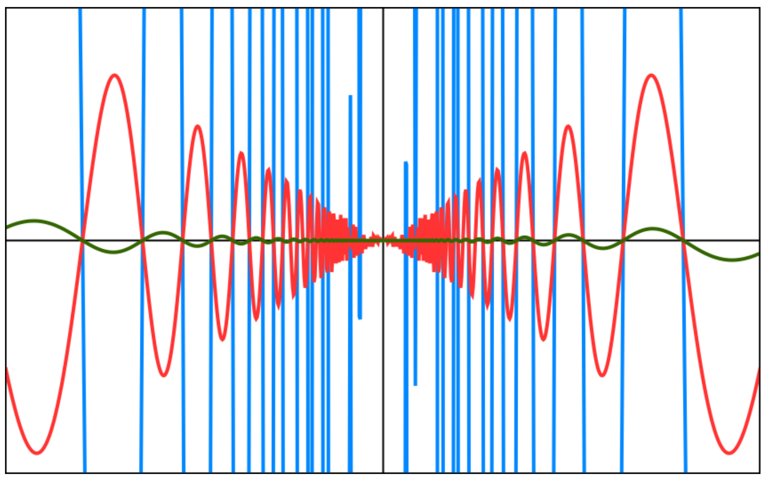
$\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0$, because $x \to 0$ and $\sin\left(\frac{1}{x}\right)$ is bounded.

(Look at the red curve in the figure.)

$g(x)$ is not differentiable at $x_0 = 0$:

$g'(0) = \lim\limits_{x\to 0} \dfrac{g(x) - g(0)}{x - 0} = \lim\limits_{x\to 0} \sin\left(\frac{1}{x}\right) \;\nexists$ (look at Exercise (a)).





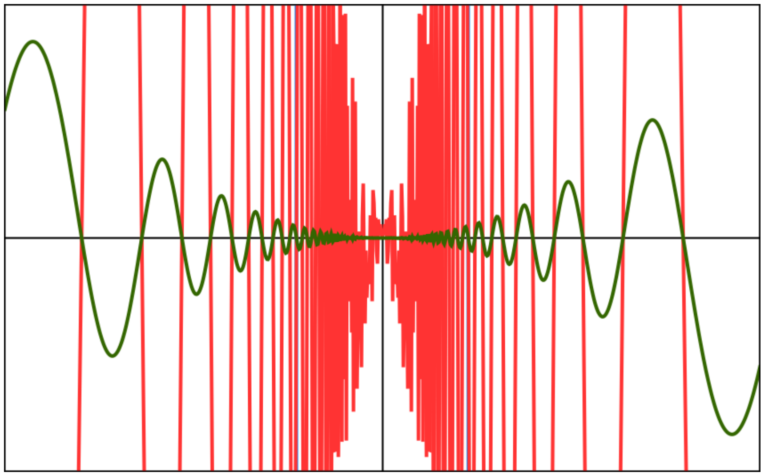
(c) $h(x)$ is continuous at $x_0 = 0$:

$\lim\limits_{x\to 0} h(x) = \lim\limits_{x\to 0} x^2 \cdot \sin\left(\frac{1}{x}\right) = 0$, because $x^2 \to 0$ and $\sin\left(\frac{1}{x}\right)$ is bounded.

$h(x)$ is differentiable at $x_0 = 0$:

$$h'(0) = \lim_{x\to 0} \frac{h(x) - h(0)}{x - 0} = \lim_{x\to 0} x \cdot \sin\left(\tfrac{1}{x}\right) = 0\,.$$

(Look at the green curve in the figure.)



(d) Yes, for instance the function h from Exercise (c). The derivative function is

$$h'(x) = \begin{cases} 2x \cdot \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

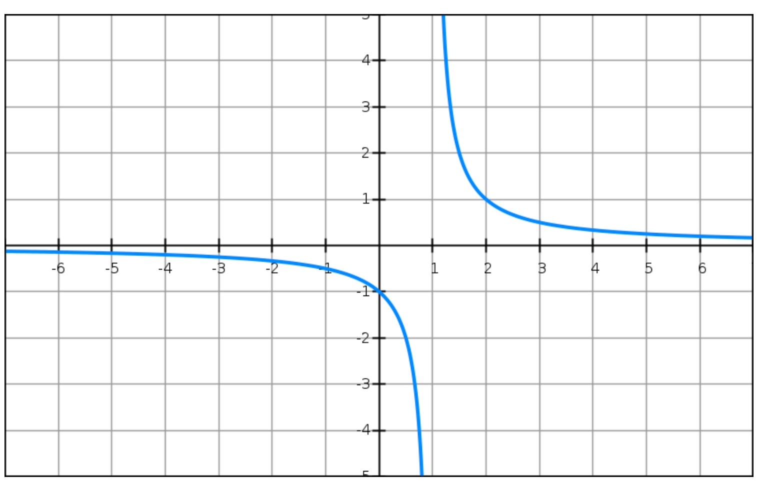
$\lim\limits_{x\to 0} h'(x) \nexists$. (One can show similarly as in Exercise (a)).

29. AN 6.1. (a)

$x = 1$ is a discontinuity point, namely a pole, which is an essential discontinuity.

Look at https://en.wikipedia.org/wiki/Classification_of_discontinuities .

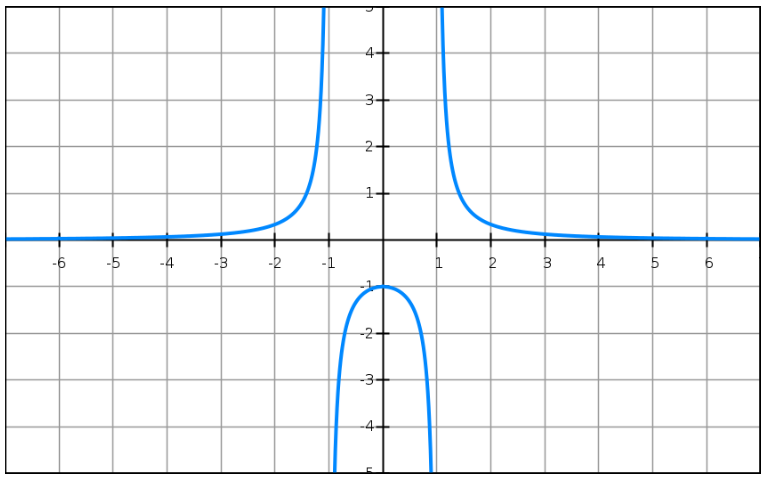
$\lim\limits_{x\to 1+} \dfrac{1}{x-1} = \infty, \quad \lim\limits_{x\to 1-} \dfrac{1}{x-1} = -\infty.$



AN 6.1. (b)

$x = \pm 1$ are points of discontinuities, namely poles.

$$\lim_{x\to 1+} \frac{1}{x^2-1} = \infty, \quad \lim_{x\to 1-} \frac{1}{x^2-1} = -\infty, \quad \lim_{x\to -1+} \frac{1}{x^2-1} = -\infty, \quad \lim_{x\to 1-} \frac{1}{x^2-1} = \infty.$$

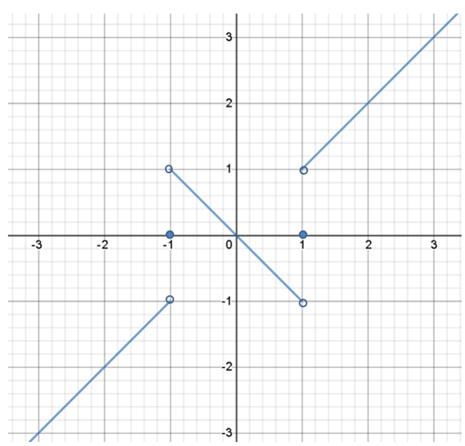


AN 6.2. (d)

$x = \pm 1$ are points of discontinuities, namely jumps.

$$\lim_{x\to 1+} x\cdot \mathrm{sgn}(x^2-1) = 1, \quad \lim_{x\to 1-} x\cdot \mathrm{sgn}(x^2-1) = -1,$$

$$\lim_{x\to -1+} x\cdot \mathrm{sgn}(x^2-1) = 1, \quad \lim_{x\to 1-} x\cdot \mathrm{sgn}(x^2-1) = -1.$$



30.(a) The Dirichlet function is discontinuous **in all points**.

The explanation is based on topological properties of the set of real numbers.

Let $x_0 \in \mathbb{Q}$. There is a sequence (y_k) of *irrational* numbers, such that $\lim\limits_{k\to\infty} y_k = x_0$. We say for this porperty that the irrational numbers are dense in the real numbers.

And then

$$\lim_{k\to\infty} f(y_k) = 0 \;\neq\; f\left(\lim_{k\to\infty} y_k\right) = f(x_0) = 1\,, \quad ,$$

(look at Übertragungsprinzip on page 130 in the textbook), therefore the function is not continuous at x_0 (i.e., in the rational points).

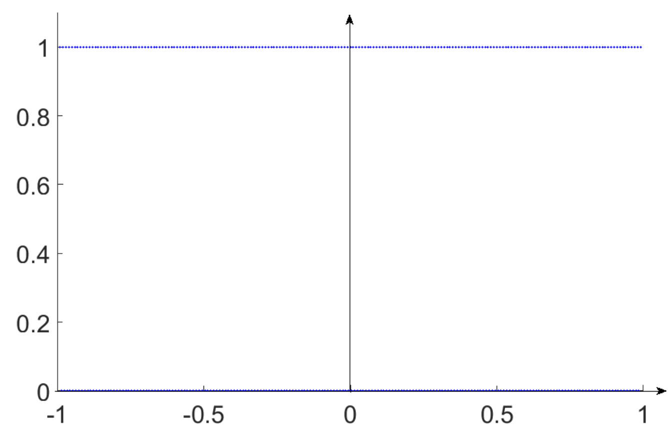
Let $y_0 \in \mathbb{Q}^*$. There is a sequence (x_k) of *rational* numbers, such that $\lim\limits_{k\to\infty} x_k = y_0$. We say for this porperty that the rational numbers are dense in the real numbers.

And then

$$\lim_{k\to\infty} f(x_k) = 1 \;\neq\; f\left(\lim_{k\to\infty} x_k\right) = f(y_0) = 0\,, \quad ,$$

(look at Übertragungsprinzip on page 130 in the textbook), therefore the function is not continuous at y_0 (i.e., in the irrational points).

As a consequence: the Dirichlet function is discontinuous in **all points**.



30.(b) The function $g(x)$ is continuous at $x_0 = 0$:

$\lim_{x \to 0} g(x) = 0 = g(0)$ according to the definition of $g(x)$.

The discontinuity of $g(x)$ at $x_0 \neq 0$ can be shown similarly as in 30.(a).

The funkcion $g(x)$ is *not* differentiable at $x_0 = 0$:

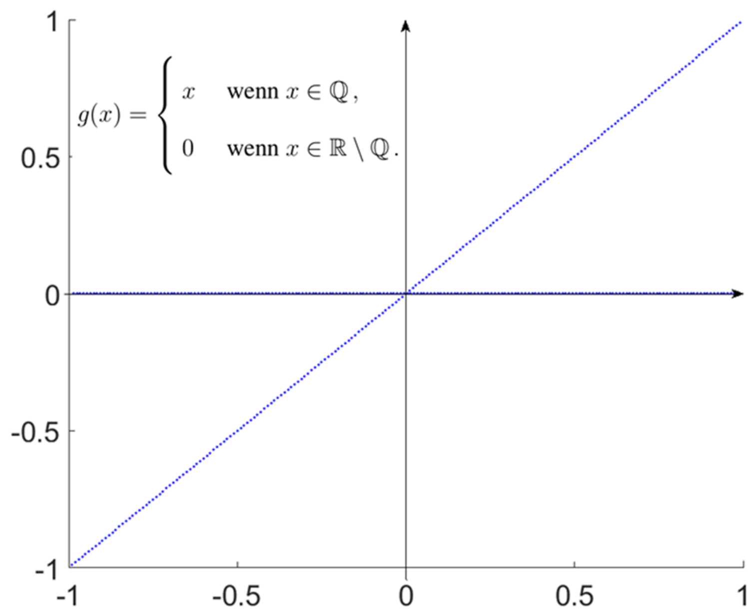
Let (y_k) be a sequence of *irrational* numbers, such that $\lim_{k \to \infty} y_k = 0$.

Then $\lim_{k \to \infty} \dfrac{g(y_k) - g(0)}{y_k - 0} = 0$, because $g(y_k) = 0$ for all k, $g(0) = 0$, and $y_k \neq 0$, since $y_k \in \mathbb{R} \setminus \mathbb{Q}$.

Let (x_k) be a sequence of *rational* numbers, such that $\lim_{k \to \infty} x_k = 0$, and let us assume, that $x_k \neq 0$ for all k.

Then $\lim_{k \to \infty} \dfrac{g(x_k) - g(0)}{x_k - 0} = 1$, because $g(x_k) = x_k$ for all k and $g(0) = 0$.

So $g'(0) = \lim_{x \to 0} \dfrac{g(x) - g(0)}{x - 0} \quad \nexists$.



30.(c) The function $h(x)$ is continuous at $x_0 = 0$:

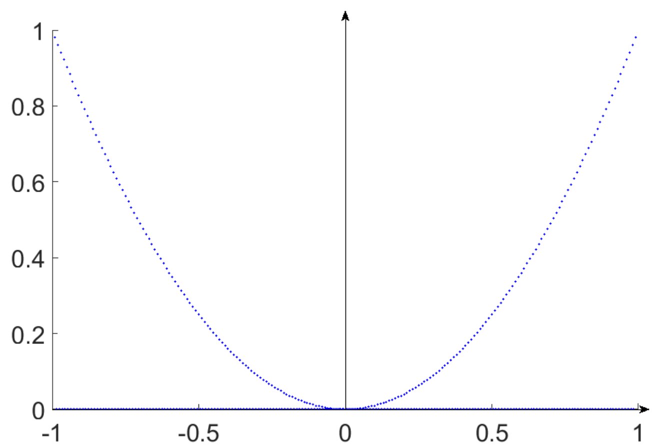
$\lim_{x \to 0} h(x) = 0 = h(0)$ according to the definition of $h(x)$.

The discontinuity of $h(x)$ at $x_0 \neq 0$ can be shown in the same way as in 30.(a) and 30.(b).

The function $h(x)$ is differentiable at $x_0 = 0$:

$h'(0) = \lim\limits_{x\to 0} \dfrac{h(x)-h(0)}{x-0} = 0,$ according to the definition of $h(x)$:

the numerator is either 0 or x^2, therefore the difference quotient $\dfrac{h(x)-h(0)}{x-0}$ takes the value 0 or x, therefore for the differential quotient (i.e. for the derivative) $\lim\limits_{x\to 0} \dfrac{h(x)-h(0)}{x-0}$ we obtain 0.



31. (a)

$$\lim_{x\to 0+} \frac{\sqrt{1+2x^2}-\sqrt{1-2x^2}}{3x} = \lim_{x\to 0+} \frac{4x}{3\cdot\left(\sqrt{1+2x^2}+\sqrt{1-2x^2}\right)} = 0 \quad \& \quad f(0) = b$$

$\Longrightarrow$ for $b = 0,\ a \in \mathbb{R}$ the function f is continuous.

(b)

$$f'_+(0) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{\sqrt{1+2x^2}-\sqrt{1-2x^2}}{3x^2} =$$

$$\lim_{x\to 0+} \frac{4}{3\left(\sqrt{1+2x^2}+\sqrt{1-2x^2}\right)} = \frac{2}{3}.$$

$f'_-(0) = a.$

$\Longrightarrow$ for $b = 0,\ a = \dfrac{2}{3}$ the function f is differentiable.