Math 833 Homework #3

Due date: Monday, Nov 15th, 23:59pm.

(This is a mandatory problem with 100 points.) 1. Consider a two-dimensional linear model:

$$\frac{\mathrm{d}u_1}{\mathrm{d}t} = F_{11}u_1 + F_{12}u_2 + \sigma_1 \dot{W}_1$$
$$\frac{\mathrm{d}u_2}{\mathrm{d}t} = F_{21}u_1 + F_{22}u_2 + \sigma_2 \dot{W}_2$$

where u_1 and u_2 are the scalar state variables, and $\dot{W}_1$ and $\dot{W}_2$ are independent white noise sources. The coefficients F_{11}, F_{12}, F_{21}, F_{22}, σ_1 and σ_2 are all constants.

a) Which condition F_{11}, F_{12}, F_{21} and F_{22} should be satisfied such that the equilibrium distribution of the system exists?

b) Let $F_{11} = F_{22} = -1$, $F_{12} = 0.5$, $F_{21} = -0.5$, $\sigma_1 = 0.2$ and $\sigma_2 = 0.5$. Start from the initial condition $x_1(0) = x_2(0) = 0.5$. Generate a time series that has in total 100 time units.

c) The observational equation is given by

$$v_1 = g_1 u_1 + \sigma_1^o,$$
$$v_2 = g_2 u_2 + \sigma_2^o,$$

where σ_1^o and σ_2^o are independent zero mean Gaussian random numbers at each observational time instant. If $g_1 = g_2 = 2$ and $\mathrm{std}(\sigma_1^o) = \mathrm{std}(\sigma_2^o) = 0.2$ and the observational time step is $\Delta t = 0.25$, then write a code for the Kalman filter, where the true signal comes from the result in Part b). Plot the posterior mean and the posterior variance of both v_1 and v_2, comparing them with the prior mean and prior variance. You may compute the root-mean-square error between the true signal and the posterior/prior mean time series. You may also compute the pattern correlation between the true signal and the posterior/prior mean time series.

d) Similar to Part c) but if $g_2 = 0$ is used while other setups remain the same, then what's the result of the Kalman filter?

e) Use the same setup as Part d) except changing $F_{12} = 0.1$ and $F_{21} = -0.1$. Then what's the result of the Kalman filter? Compare the prior distribution and posterior distribution in this case and the case in Part d) to explain the role of F_{12} and F_{21} in the Kalman filter.

2. (This is an optional problem. If you do this, you will get extra credits with up to 50 points.) Consider an one-dimensional linear model:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -au + f + \sigma \dot{W}_t,$$

with $a = 1$, $f = 1$ and $\sigma = 0.5$. Let the observational equation be

$$v = gu + \sigma^o$$

where σ^o is an independent zero mean Gaussian random number at each observational time instant. Let $g = 2$ and $\mathrm{std}(\sigma^o) = 0.2$. Generate a time series of length 100 time unit with observational time step being $\Delta t = 0.25$.
a) Write down the standard Kalman filter to solve the posterior mean and posterior covariance.
b) Write down a particle filter algorithm to solve the problem. Play with the number of particle in your simulation and explain if you need the resampling.

Attach your codes together with your answers.