

Departamento de Eletrónica, Telecomunicações e
Informática

LECTURE 2: FML - LINEAR REGRESSION

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LINEAR REGRESSION - outline

1. Univariate linear regression

- Cost (loss) function - Mean Squared Error (MSE)
- Cost function convergence
- Gradient descent algorithm

2. Multivariate linear regression

- Overfitting problem

3. Regularization => way to deal with overfitting

Supervised Learning - CLASSIFICATION vs REGRESSION

Classification- the label is an integer number.
(e.g. 0, 1 for binary classification)

Regression - the label is a real number.

Examples of regression problems:

- Weather forecast
- Predicting wind velocity from temperature, humidity, air pressure
- Time series prediction of stock market indices
- Predicting sales amounts of new product based on advertising expenditure

Standard Notations in this course

x – input vector of features, attributes

y – output vector of labels, ground truth, target

m - number of training examples

n – number of features

$h_{\theta}(x)$ - model (hypothesis)

θ - vector of model parameters

Training set: data matrix X (m rows, n columns)

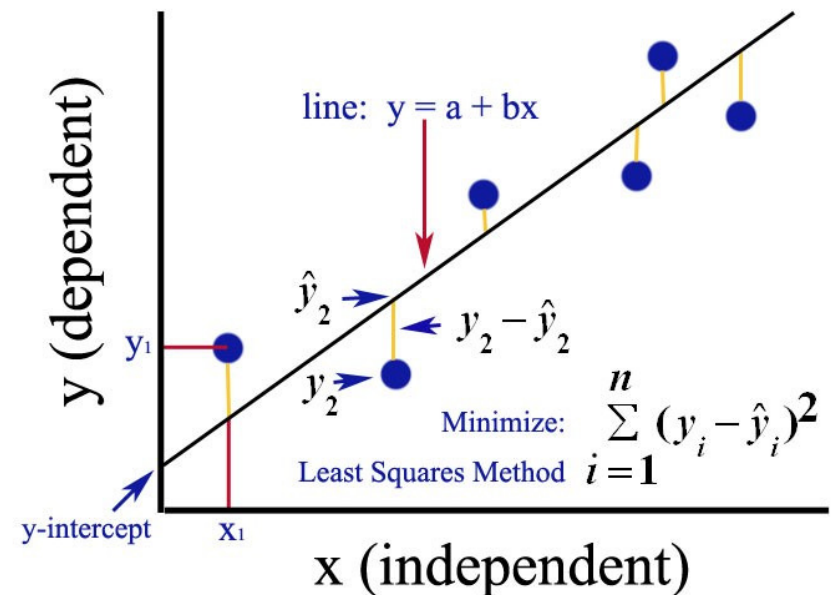
	feature x_1	feature x_2	feature x_n	output(label) y
Example 1	$x_1^{(1)}$			$x_n^{(1)}$	$y^{(1)}$
Example 2	$x_1^{(2)}$			$x_n^{(2)}$	$y^{(2)}$
...					
Example i	$x_1^{(i)}$			$x_n^{(i)}$	$y^{(i)}$
...					
...					
Example m	$x_1^{(m)}$			$x_n^{(m)}$	$y^{(m)}$

Supervised Learning – univariate regression

Problem: Learning to predict the housing prices (output, predicted variable) as a function of the living area (input, feature, predictor)

observation – prediction = Error (“residual”)

Living area (feet ²)	Price (1000\$s)
2104	400
1600	330
2400	369
1416	232
3000	540
⋮	⋮



Supervised Learning – univariate regression

$$h_{\theta}(x) = \theta_0 + \theta_1 x_1 = \begin{bmatrix} 1 & x_1 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} = \vec{x}^T \vec{\theta}$$

=> in Python => np.dot(X, Theta)

$$\vec{x} = \begin{bmatrix} x_0 = 1 \\ x_1 \end{bmatrix}$$

	X_0 (extra column)	feature x_1 (living area)	output(label) y (price)
Example 1=>	1	$x^{(1)}$	$y^{(1)}$
Example 2=>	1	$x^{(2)}$	$y^{(2)}$
	1		
Example m=>	1	$x^{(m)}$	$y^{(m)}$

Mean Square Error (MSE)

Linear Model (hypothesis) =>

$$h_{\theta}(x) = \theta^T x = \theta_0 + \theta_1 x_1$$

Cost (loss) function =>
(Mean Square Error)

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2$$

m – number of training examples

Goal =>

$$\min_{\theta} J(\theta)$$

Gradient descent algorithm =>
iterative algorithm; at each
iteration all parameters (theta)
are updated simultaneously

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

alpha – learning rate > 0

Linear Regression

(computing the gradient)

Cost function =>

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2$$

Cost function gradients =>

vector with partial derivatives of J with respect to each parameter for one example ($m=1$)

$$\begin{aligned} \frac{\partial}{\partial \theta_j} J(\theta) &= \frac{\partial}{\partial \theta_j} \frac{1}{2} (h_{\theta}(x) - y)^2 \\ &= 2 \cdot \frac{1}{2} (h_{\theta}(x) - y) \cdot \frac{\partial}{\partial \theta_j} (h_{\theta}(x) - y) \\ &= (h_{\theta}(x) - y) \cdot \frac{\partial}{\partial \theta_j} \left(\sum_{i=0}^n \theta_i x_i - y \right) \\ &= (h_{\theta}(x) - y) x_j \end{aligned}$$

$$h_{\theta}(x) = \theta^T x = \theta_0 + \theta_1 x_1$$

Cost function gradients =>

for m examples

$$\frac{\partial J(\theta)}{\partial \theta_j} = \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

Linear Regression – iterative gradient descent algorithm (summary)

Initialize model parameters (e.g. $\theta = 0$)

Repeat until J converge {

Compute Linear Regression Model =>

$$h_{\theta}(x) = \theta^T x = \theta_0 + \theta_1 x_1$$

Compute cost function =>

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2$$

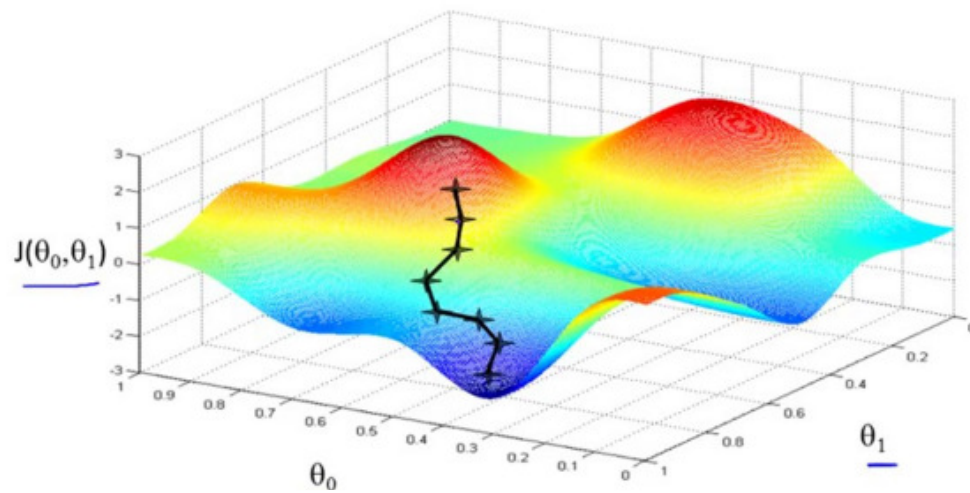
Compute cost function gradients =>

$$\frac{\partial J(\theta)}{\partial \theta_j} = \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

Update parameters =>

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

}



Batch/mini batch/stochastic gradient descent for parameter update

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

Batch learning (classical approach):

update parameters after all training examples have been processed, repeat several iterations until convergence

Mini batch learning (if big training data):

divide training data into small batches update parameters after each mini batch has been processed, repeat until convergence

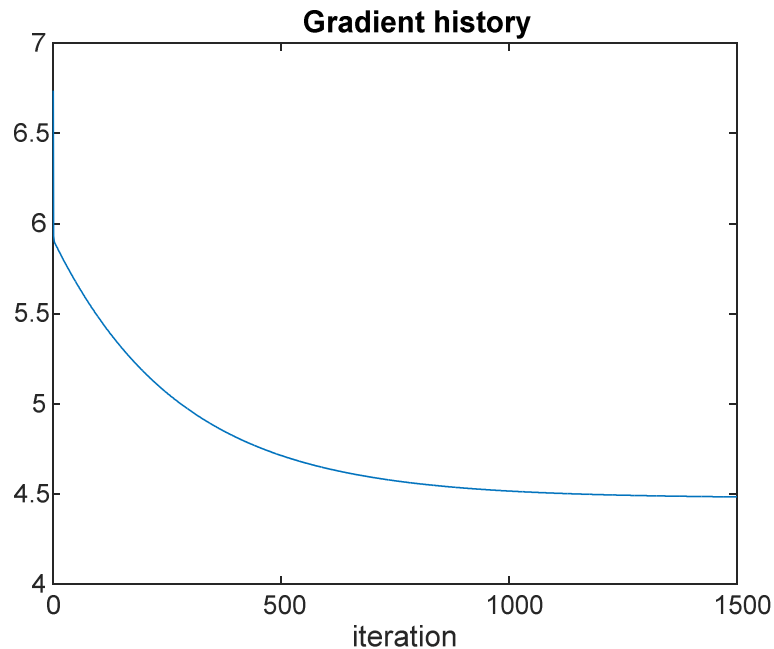
Stochastic (incremental) learning (large-scale ML problems):

update parameters after every single training example has been processed.

Stochastic Gradient Descent (SGD): the true gradient is approximated by a gradient at a single example. Adaptive learning rate.

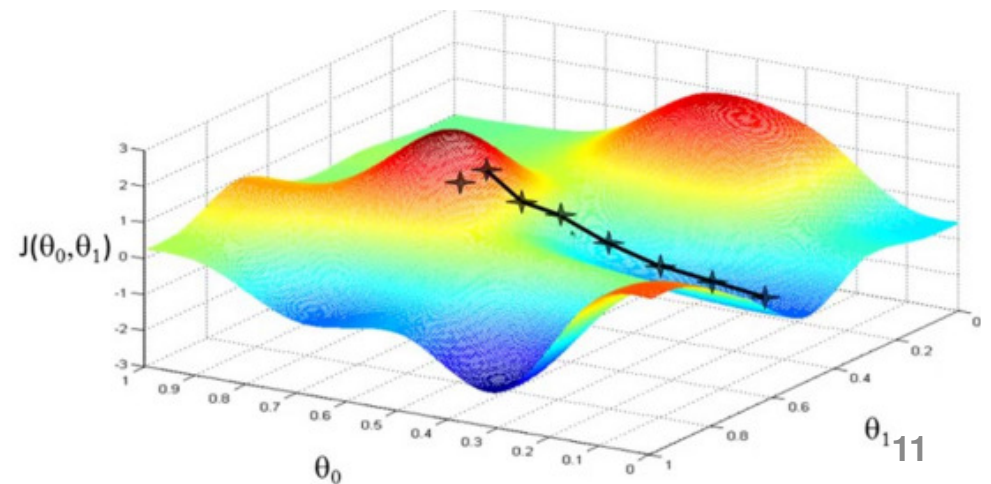
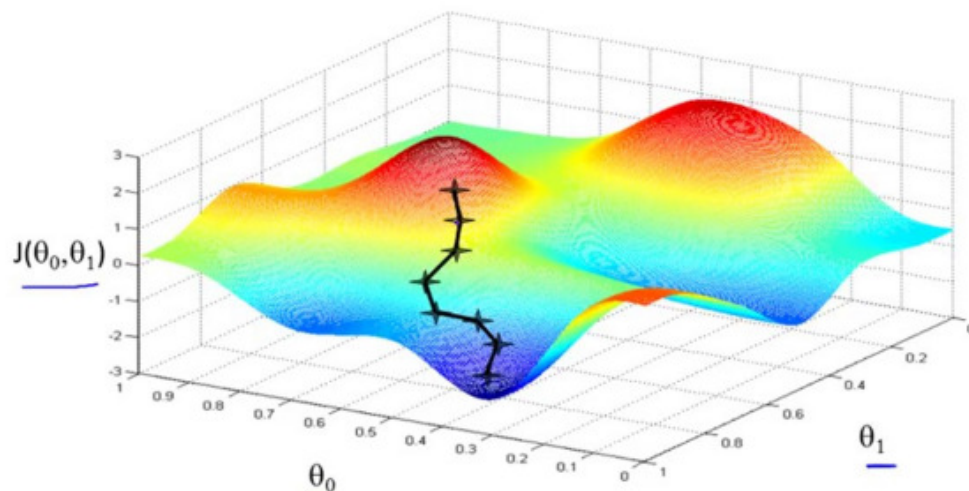
Cost function convergence

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2$$



Linear Regression (LR):

starting from different initial values of the parameters the cost function J should always converge (**maybe to a local minimum !!!**) if LR works properly.

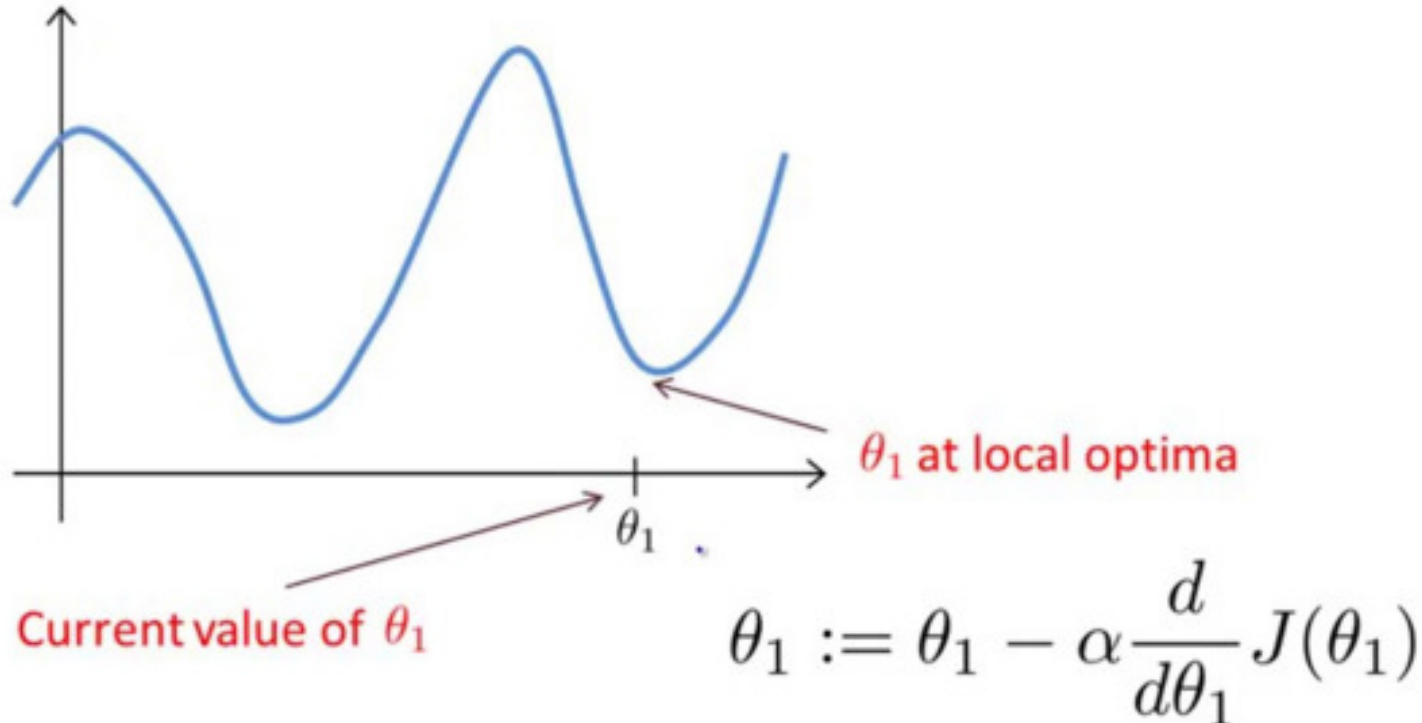


Lin Reg Cost function – local minimum

Suppose θ_1 is at a local optima as shown in the figure.

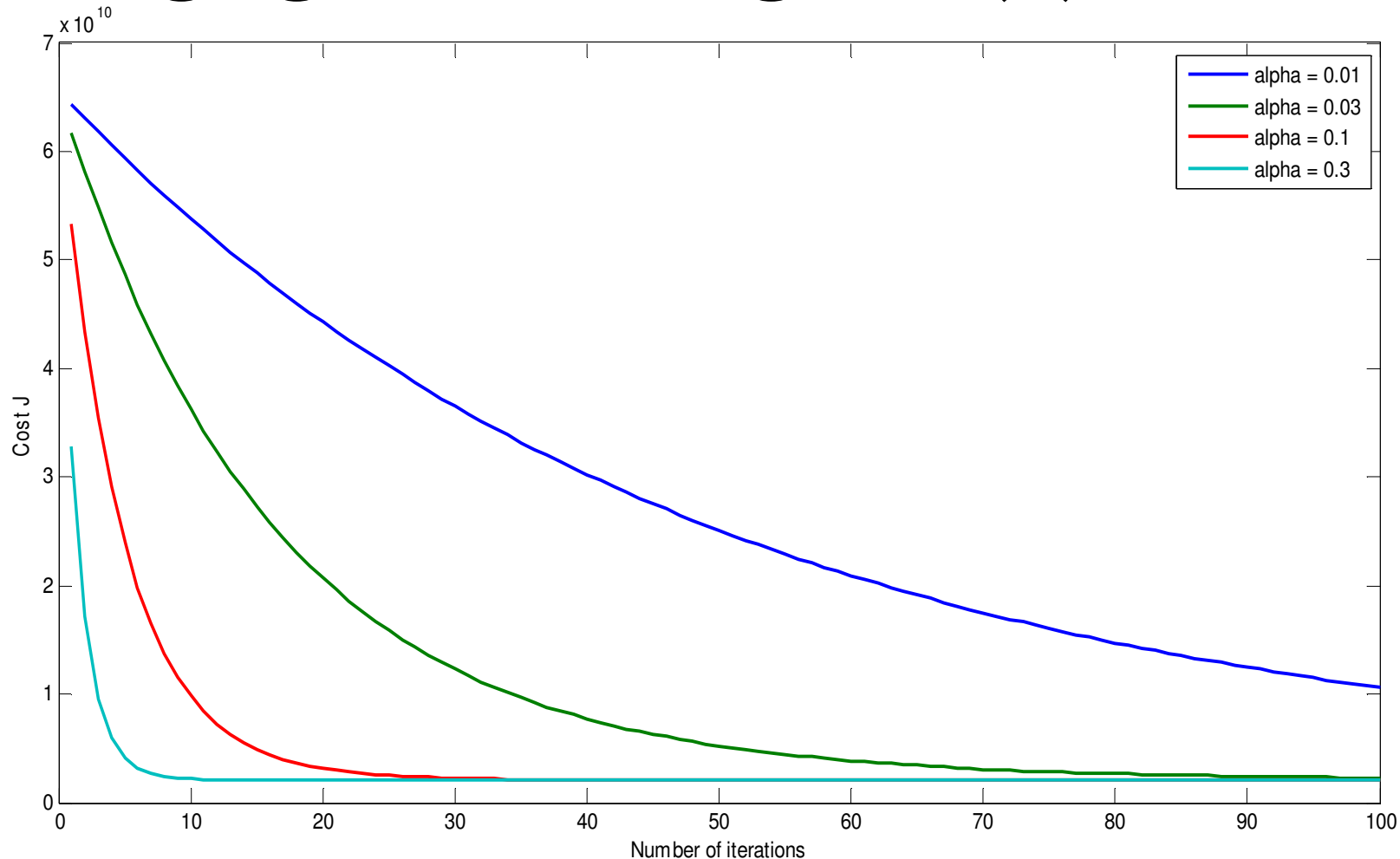
What will one step of Gradient Descent do ?

- 1) Leave θ_1 unchanged
- 2) Change θ_1 in a random direction
- 3) Decrease θ_1
- 4) Move θ_1 in direction to the global minimum of J



Cost function convergence

changing the learning rate (α) - 100 iter.

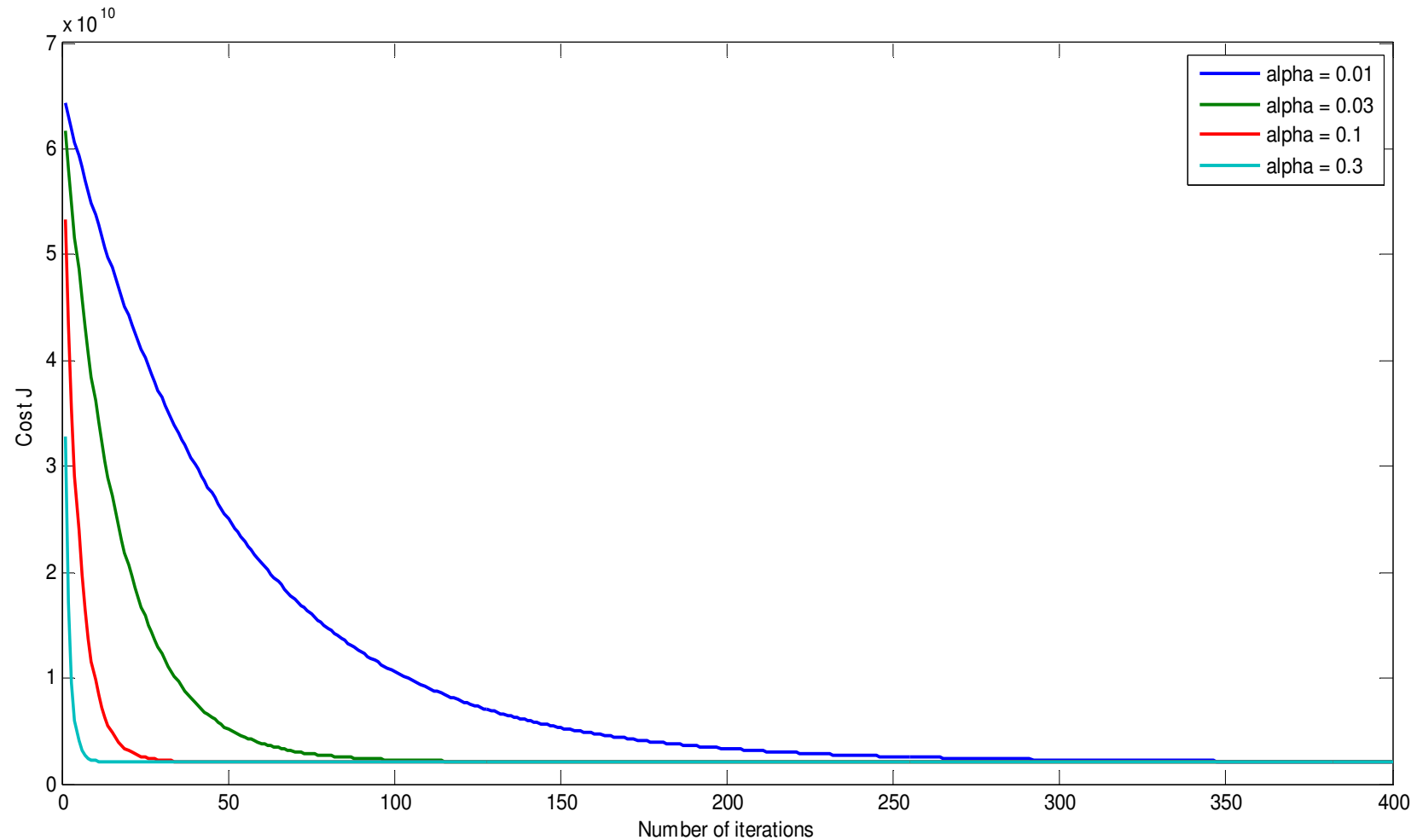


$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

If α too small : slow convergence of the cost function J (the Gradient Descent optimization can be slow)

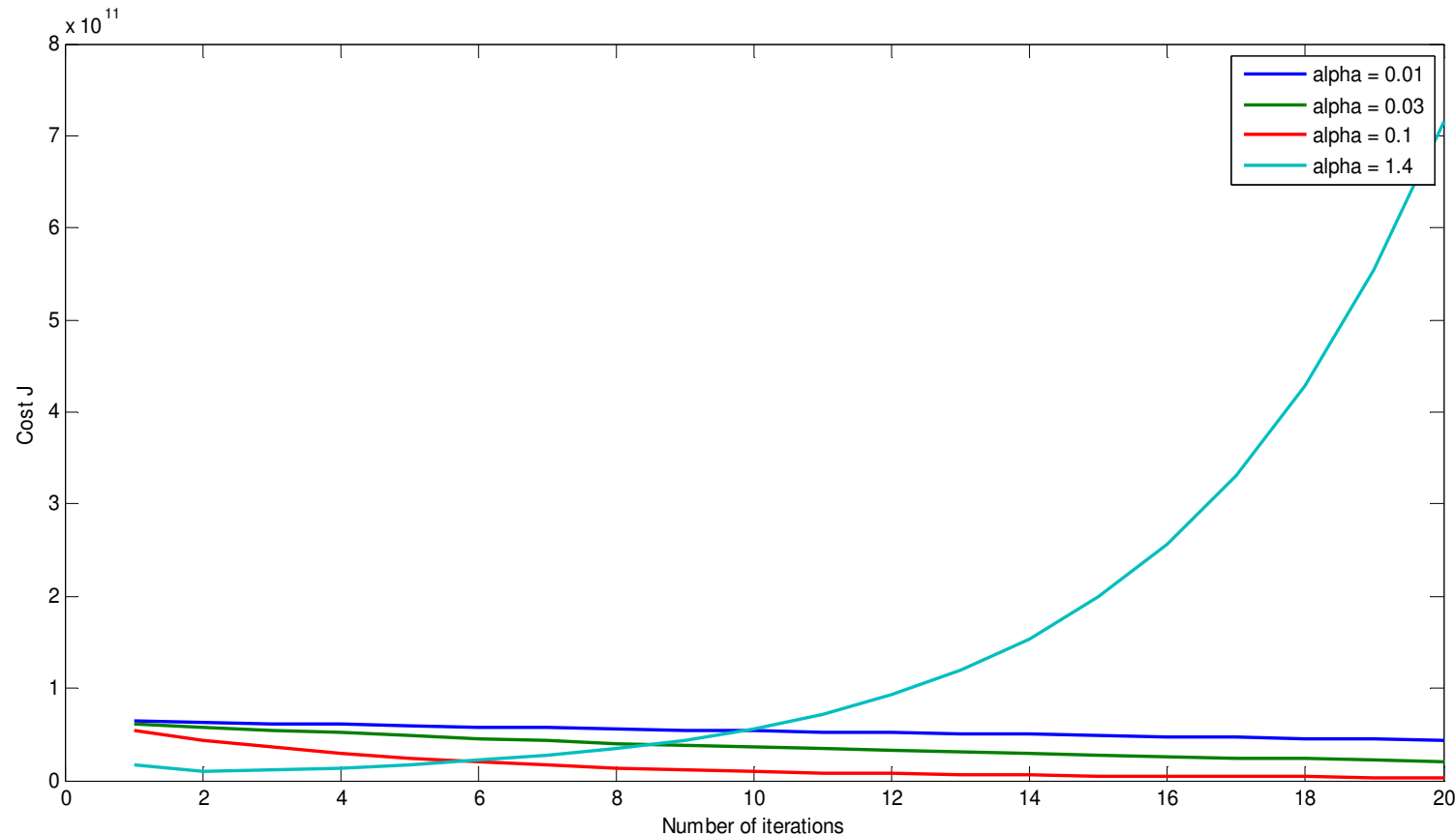
Cost function convergence

changing the learning rate (α) -400 iter.



$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

LinReg Cost function convergence - learning rate variation (α)

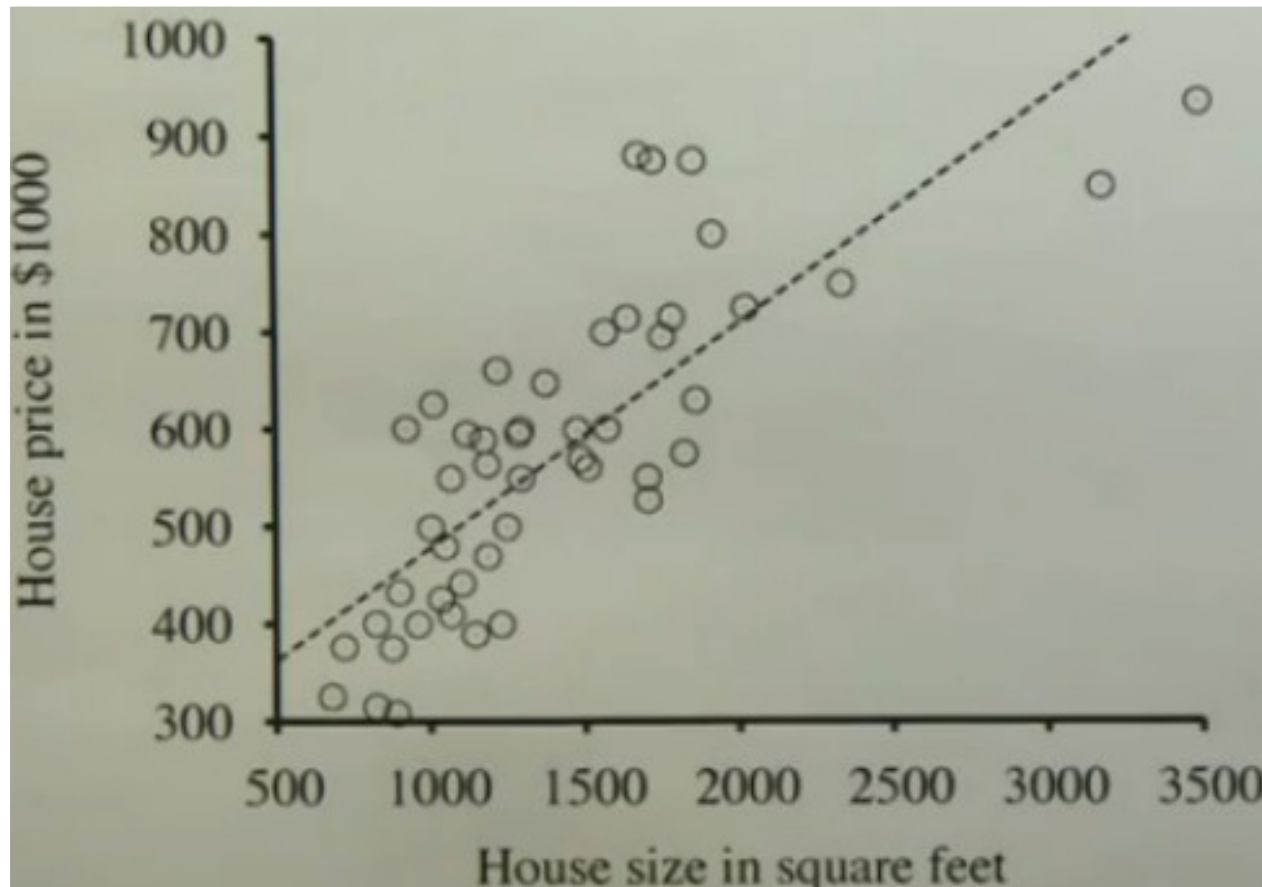


If α too large: the cost function J may no converge (decrease at each iteration). It may diverge !

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

Univariate Regression

$$h_{\theta}(x) = \theta^T x = \theta_0 + \theta_1 x_1$$



Given the house area, what is the most likely house price?
If univariate linear regression model is not sufficiently good model,
add more data (ex. # bedrooms).

Multivariate Regression

Problem: Learning to predict the housing price as a function of living area & number of bedrooms.

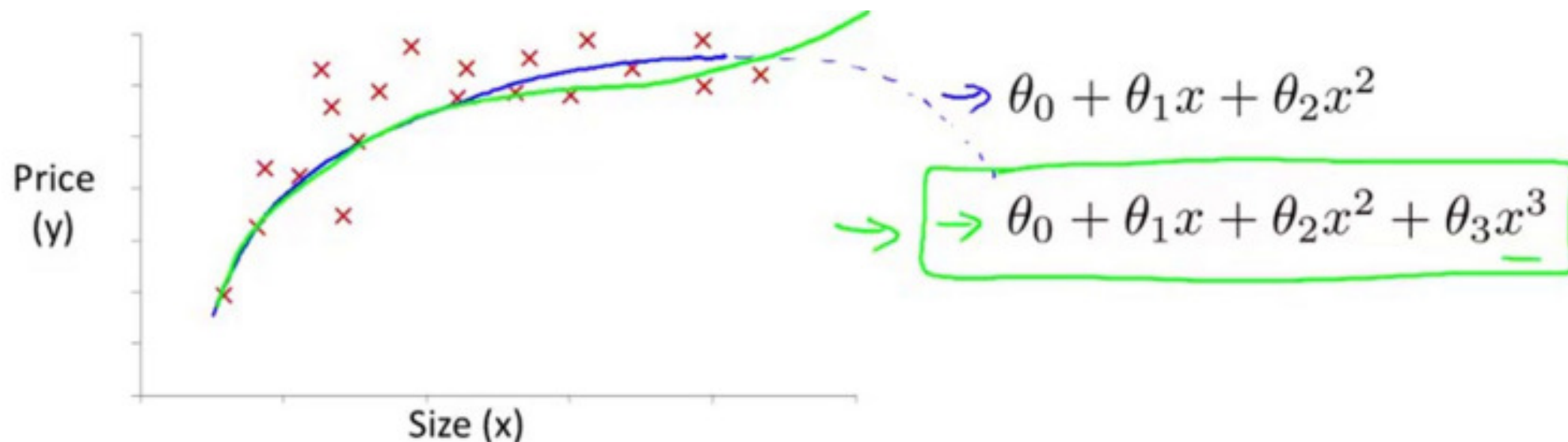
Living area (feet ²)	#bedrooms	Price (1000\$s)
2104	3	400
1600	3	330
2400	3	369
1416	2	232
3000	4	540
\vdots	\vdots	\vdots

$$h_{\theta}(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 = [\theta_0 \quad \theta_1 \quad \theta_2] \begin{bmatrix} x_0 = 1 \\ x_1 \\ x_2 \end{bmatrix} = \vec{\theta}^T \vec{x}$$

Polynomial Regression

If univariate linear regression model is not a good model, try polynomial model.

Univariate ($x_1 = \text{size}$) housing price problem transformed into multivariate (still linear !!!) regression model $x = [x_1 = \text{size}, x_2 = \text{size}^2, x_3 = \text{size}^3]$



$$\begin{aligned}h_{\theta}(x) &= \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \\&= \theta_0 + \theta_1(\text{size}) + \theta_2(\text{size})^2 + \theta_3(\text{size})^3\end{aligned}$$

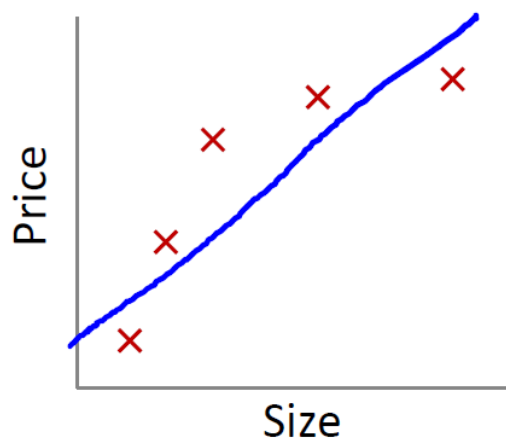
$$x_1 = (\text{size})$$

$$x_2 = (\text{size})^2$$

$$x_3 = (\text{size})^3$$

Overfitting problem

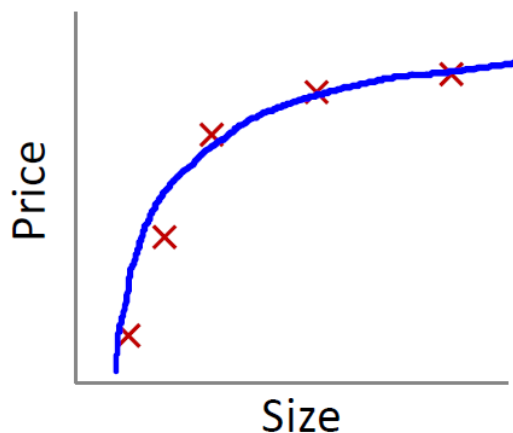
Overfitting: If we have too many features (e.g. high order polynomial model), the learned hypothesis may fit the training set very well but fail to generalize to new examples (predict prices on new examples).



underfit

(1st order polin. model)

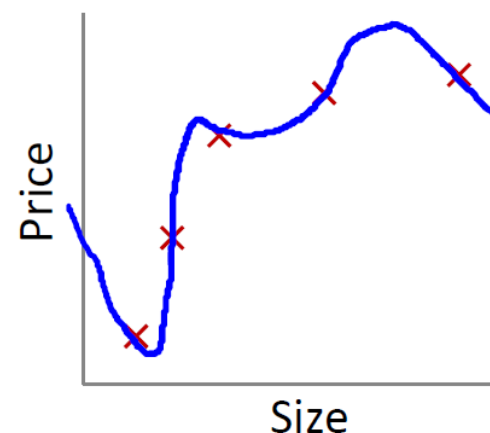
$$h_{\theta}(x) = \theta_0 + \theta_1 x$$



just right

(3rd order polinom. model)

$$h_{\theta}(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3$$



overfit

(higher ord. polinom. Model)

$$h_{\theta}(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \dots + \theta_{16} x^n$$

Overfitting problem

Overfitting: If we have too many features (x_1, \dots, x_{100}) the learned model may fit the training data very well but fails to generalize to new examples.

x_1 = size of house

x_2 = no. of bedrooms

x_3 = no. of floors

x_4 = age of house

x_5 = average income in neighborhood

x_6 = kitchen size

\vdots

x_{100}

$$h_{\theta}(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_n x_n = \vec{\theta}^T \vec{x}$$

How to deal with overfitting problem ?

1. Reduce number of features.

- Manually select which features to keep.
- Algorithm to select the best model complexity.

2. Regularization (add extra term in cost function)

Regularization methods shrink model parameters θ towards zero to prevent overfitting by reducing the variance of the model.

2.1 Ridge Regression (L2 norm)

- Reduce magnitude of θ (but never make them =0) => keep all features
- Works well when all features contributes a bit to the output y .

2.2 Lasso Regression (L1 norm)

- May shrink some of the elements of vector θ to become 0.
- Eliminate some of the features => Serve as feature selection

Regularized Linear Regression (cost function)

Unregularized cost function =>

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2$$

Regularized cost function
(add extra regularization term
don't regularize θ_0)

Ridge Regression

$$J(\theta) = \frac{1}{2m} \left[\sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)})^2 + \lambda \sum_{j=1}^n \theta_j^2 \right]$$

$$\min_{\theta} J(\theta)$$

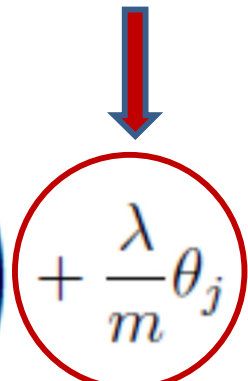
Regularized Linear Regression (cost function gradient)

**Unregularized cost
function gradients =>**

$$\frac{\partial J(\theta)}{\partial \theta_j} = \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

**Regularized cost
function gradients =>**

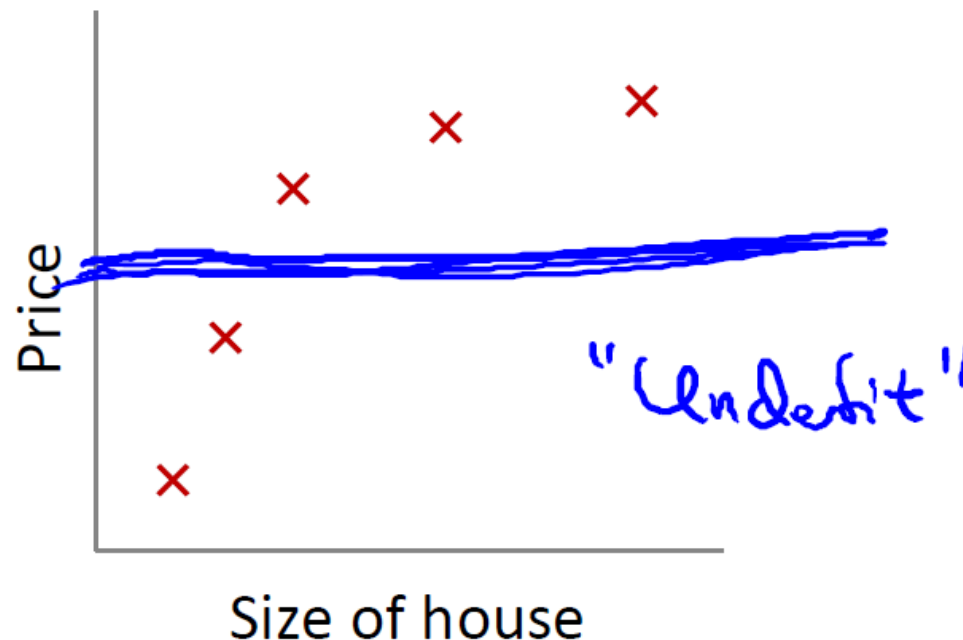
$$\frac{\partial J(\theta)}{\partial \theta_0} = \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)} \quad \text{for } j = 0$$

$$\frac{\partial J(\theta)}{\partial \theta_j} = \left(\frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)} \right) + \frac{\lambda}{m} \theta_j \quad \text{for } j \geq 1$$


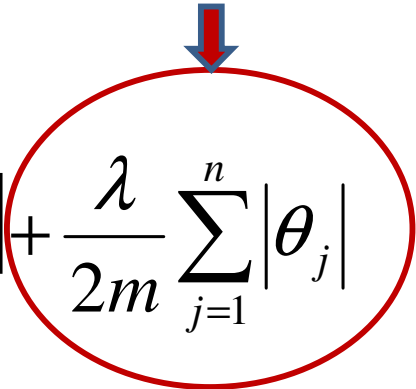
Regularized Linear Regression

What if lambda is set to an extremely large value ?

- Algorithm fails to eliminate overfitting.
- Algorithm results in under-fitting. (Fails to fit even training data well).
- Gradient descent will fail to converge.



Regularization: Lasso Regression

$$J(\theta) = \frac{1}{m} \sum_{i=1}^m \left[-y^{(i)} \log(h_{\theta}(x^{(i)})) - (1 - y^{(i)}) \log(1 - h_{\theta}(x^{(i)})) \right] + \frac{\lambda}{2m} \sum_{j=1}^n |\theta_j|$$


Ridge Regression shrinks θ towards zero, but never equal to zero => all features are included in the model no matter how small are the coefficients.

Lasso Regression is able to shrink coefficients to exactly zero => reduces the number of features. This makes Lasso Regression useful in cases with high dimension.

Lasso Regression involves absolute values (not differentiable) => computing is difficult => relevant algorithms available in sklearn Python library.

JUPYTER NOTEBOOK

Project Jupyter - 2014 by [Fernando Pérez](#)

Reference to the 3 core programming languages supported
[Julia](#), [Python](#), [R](#)

[open-source software](#), open-standards, language agnostic
[web-based interactive](#) computational environment for creating
notebook documents.

Ordered list of input/output cells which can contain code, text
(using Markup language – html, latex,), maths, plots

MATLAB-BASED IPYTHON NOTEBOOKS:

<https://anneur.ai.net/2015/11/12/matlab-based-ipython-notebooks/>.