Solutions to Problem Set 1 (6.042J)

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September 18, 2017

Problem 1

Assume that $\log_4 6 = \frac{m}{n}$ is rational. We have:

$$\begin{aligned} 4^{\frac{m}{n}} &= 6 && (1)\\ (4^{\frac{m}{n}})^n &= 6^n && (2)\\ 4^m &= 6^n && (3)\\ (2 \times 2)^m &= (2 \times 3)^n && (4) \end{aligned}$$

Divide both sides by 2^n:

$$\frac{2^{2m}}{2^n} = 3^n \tag{5}$$

$$2^{2m-n} = 3^n \tag{6}$$

The LHS in (6) is even while the RHS is odd. This means $2^{2m-n} \neq 3^n$. Thus, $\log_4 6$ is irrational by contradiction.

Problem 2

$P(n) ::= \{\forall n \in \mathbb{N} | n \leq 3^{\frac{n}{3}} \equiv n^3 \leq 3^n\}$. Counterexample set: $C(n) ::= \{n \in \mathbb{N} | n \neq 3^{\frac{n}{3}}\}$. Calculation of $P(n)$ with $n \leq 4$ gives:

$$P(0) ::= 0 \leq 1$$

$$P(1) ::= 1 \leq 3$$

$$P(2) ::= 8 \leq 9$$

$$P(3) ::= 27 \leq 27$$

$$P(4) ::= 64 \leq 81$$

By the well-ordering principle, C has a minimum element c. This means $c - 1$ is a non-negative integer ≥ 4 and $P(c-1)$ is true

$$\begin{aligned} P(c-1) &::= (c-1)^3 \leq 3^{c-1} \equiv 3(c-1)^3 \leq 3^c && (7)\\ P(c) &::= c^3 \leq 3^c \equiv c^3 \leq 3(c-1)^3 && (8)\\ &\equiv c \leq \sqrt[3]{3}(c-1) && (9) \end{aligned}$$

(9) is always true for $c \geq 4$ and $P(n)$ holds for $n \leq 4$. This means $C(n)$ is an empty set by contradiction and $P(n)$ holds for every non-negative integer n.

Problem 3

a)

Truth table for (P IMPLIES Q) OR (Q IMPLIES P):

P	Q	P IMPLIES Q	Q IMPLIES P	(P IMPLIES Q) OR (Q IMPLIES P)
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

The OR operation always gives True value no matter what truth values its variables may have.

b)

$R ::= (P$ AND $Q)$ OR (NOT(P) AND NOT(Q))

c)

Case 1: P is valid iff NOT(P) is not satisfiable
P is valid which means NOT(P) is always false. Thus, NOT(P) is not satisfiable since it can only produce false in this case.
Case 2: NOT(P) is not satisfiable iff P is valid
NOT(P) is not satisfiable means it can never be true, so NOT(P) is always false which means P is always true.

d)

$P_1, ..., P_k$ is consistent iff $P_1, ..., P_k$ is satisfiable. We can write S as the following:
$S ::= P_1$ AND P_2 AND ... AND P_k