Internship Report - MSc Mod4NeuCog (M1)

Spherical forward model for extended EEG sources

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Summary

Internship description

This mini-project was proposed by the Athena project-team of Inria Sophia Antipolis and the goal was to develop a spherical forward model for extended EEG sources.

One of the interests of the Athena team is to study and improve techniques of acquisition and processing data from the brain, and this internship was aligned with this topic. Using extended sources is an improvement of the model in the sense of physical realism when compared with others that assume brain sources as punctual dipoles.

To reach the proposed goal, it was necessary to study and understand the spherical model and an already existing code (in C++) of a specific model. My main activity was to rewrite that algorithm in Python, verify its accuracy and improve the code in order to make it compatible with the extended sources.

1 - Introduction

To calculate the potential resultant from the brain activity captured by electrodes during an encephalography (EEG), it is necessary to model the electrical propagation inside the human head.

This topic is known as the inverse or the forward problem and depends on diverse assumptions about the head geometry, its physical properties and on the definition of the sources from which the potential is generated. The discussion of the mathematical aspects of the problem [2] and some different types of models [1] can be found in the literature.

For this work, the analytical model utilized for the calculation of the potential was the one proposed by Hedou [3]. With this model it is possible to calculate the potential received by an electrode placed at any layer of the head (from the inner sphere to the surface) considering the following assumptions: (I) the head is represented by spherical layers [Figure 1] and (II) the brain sources are modeled as dipoles.

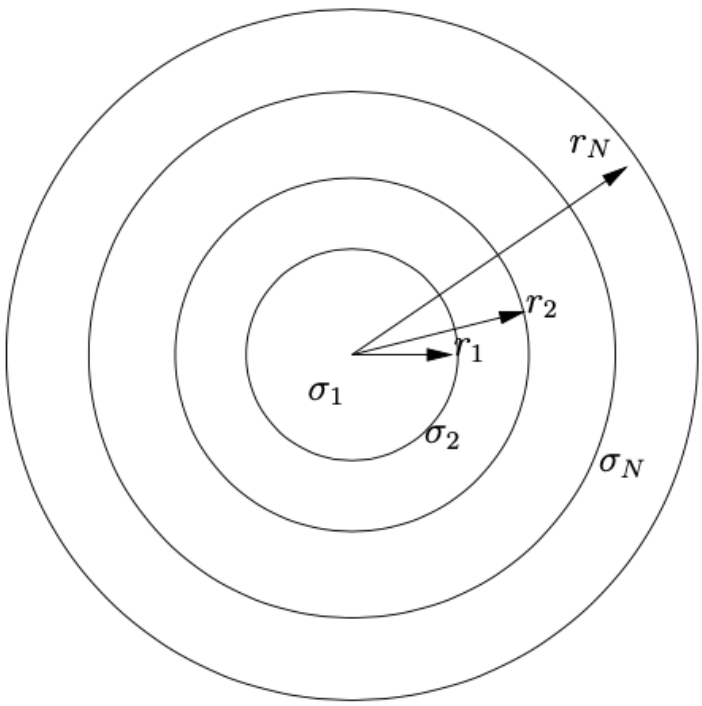

Figure 1 - Representation of a spherical multilayer model of the head. (r is the radius and σ is the conductivity of each layer)

2 - Analytical model

For the utilized model, the potential (ϕ) calculation, considering a dipole positioned in the plane xOz and with the moment oriented along the direction z, follows the equations below:

- For $0 \leq r < b$:

$$\phi_1(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=o}^{l}[C_{lm_1} + M_{lm}^{Inf}]r^l \cos(m\varphi)P_l^m(\cos\theta)$$

$$\text{with } M_{lm}^{Inf} = \frac{M}{4\pi\sigma_1 b^2}(2-\delta_m^0)(m-l-1)\frac{(l-m)!}{(l+m)!}b^{-l}P_{l+1}^m(cos\theta_0)$$

- For $b < r \leq r_1$:

$$\phi_1(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=o}^{l}[C_{lm_1}r^l + M_{lm}^{Sup}r^{-(l+1)}]\cos(m\varphi)P_l^m(\cos\theta)$$

$$\text{with } M_{lm}^{Sup} = \frac{M}{4\pi\sigma_1 b^2}(2-\delta_m^0)\frac{(l-m)!}{(l+m-1)!}b^{l+1}P_{l-1}^m(cos\theta_0)$$

- For $r_{i-1} \leq r \leq r_i$, $i = 2,...,n$:

$$\phi_i(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=o}^{l}[C_{lm_i}r^l + D_{lm_i}r^{-(l+1)}]\cos(m\varphi)P_l^m(\cos\theta)$$

Where:

(r,θ,φ): cartesian coordinates (x,y,z) of the electrode position in terms of (r,θ,φ);

r: electrode position radius;

b: dipole radius;

r_i: layers radius;

C_{lm_i}, D_{lm_i}: unknown variables determined by the interface and boundary conditions;

P_l^m: Legendre Polynomials [4];

M: moment radius;

σ_1: conductivity of inner sphere layer;

δ_m^0: variable dependent on m $[if\,(m = 0) : 1; else : 0]$;

θ_0: theta angule of the dipole position.

3 - Coding the model

In order to write the algorithm presented above and correctly compute the potential captured by an electrode positioned at any layer of the head and considering a dipole source inside the head, it is necessary to make some smaller calculations previously:

1. Rotate the determined dipole to guarantee that it meets the premise of being positioned in the plane xOz and with the moment oriented along the direction z,
2. Define the dipole and electrode position in terms of (r, θ, φ),
3. Find the unknown variables C_{lm_i}, D_{lm_i}.

3.1 - Dipole rotation

To fully rotate a dipole positioned at (x, y, z) with moment (m_x, m_y, m_z), it is necessary to make three smaller rotations [5]:

- rotation around axis Ox of angle α with:

$$\alpha = \arctan(\frac{m_y}{m_z}) \,,$$

$$position(x', y', z') = (x, y\cos\alpha - z\sin\alpha, y\sin\alpha + z\cos\alpha),$$

$$moment(m'_x, m'_y, m'_z) = (m_x, 0, \sqrt{m_y^2 + m_z^2}).$$

- rotation around axis Oy of angle β with:

$$\beta = \arctan(\frac{m'_x}{m'_z}),$$

$$position(x'', y'', z'') = (x' \cos\beta - z' \sin\beta, y', x' \sin\beta + z' \cos\beta),$$

$$moment(m''_x, m''_y, m''_z) = (0, m'_y, \sqrt{m'^2_x + m'^2_z}).$$

- rotation around axis Oz of angle γ with:

$$\gamma = \arctan(\frac{-y''}{x''}),$$

$$position(x''', y''', z''') = (x'' \cos\gamma - y'' \sin\gamma, x'' \sin\gamma + y'' \cos\gamma, z''),$$

$$moment(m'''_x, m'''_y, m'''_z) = (m''_x, m''_y, m''_z).$$

After those three rotations, the dipole positioned at (x''', y''', z''') is in the plane xOz and the moment (m'''_x, m'''_y, m'''_z) is oriented along the direction z.

It is important to also apply the rotation for the electrode position using the same constants (α, β, γ).

3.2 - Position in terms of (r, θ, φ)

The potential calculation model requires the position cartesian coordinates defined in terms of (r, θ, φ) [Figure 2]. Finding them is not a complex task, it can be made by just applying the following equations:

$$r = \sqrt{x^2 + y^2 + z^2},$$

$$\theta = \arccos(\frac{z}{r}),$$

$$\varphi = \arctan(\frac{y}{x}).$$

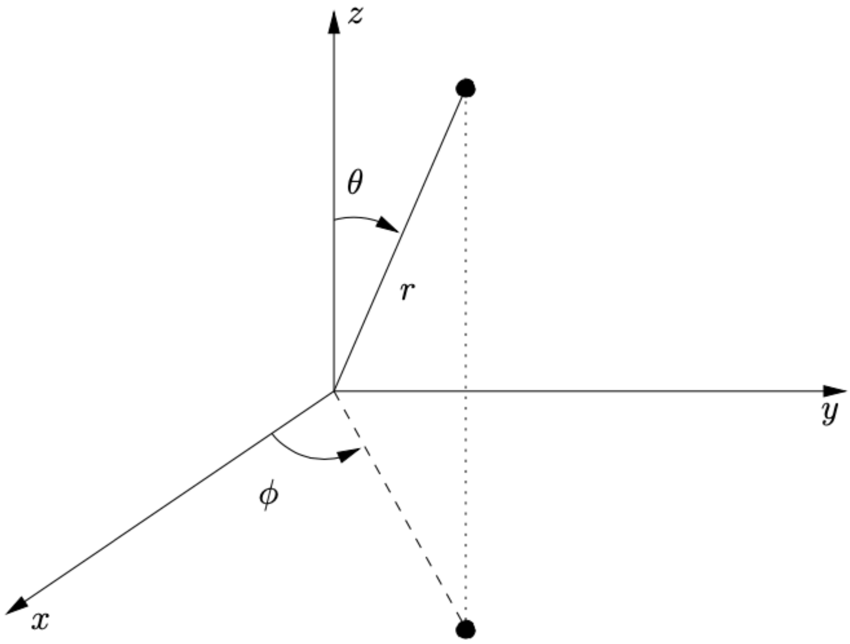

Figure 2 - Cartesian coordinates in terms of (x, y, z) and (r, θ, φ)

3.3 - Unknown variables

The last elements missing in order to finally be able to calculate the potential are the coefficients C_{lm_i}, D_{lm_i}. As commented before, those are unknown variables determined by the interface and boundary conditions. To find them, it is necessary to solve a system presented by Hedou [3]. To do that, the matrix equation ($Ax = b$) has to be utilized (with n being the number of layers of the head) where:

$$A_{[2n-1],[2n-1]} = \begin{bmatrix} r_1^l & -r_1^l & \cdots & 0 & 0 & -\frac{1}{r_1^{(l+1)}} & \cdots & 0 & 0 \\ 0 & r_2^l & \ddots & 0 & 0 & \frac{1}{r_2^{(l+1)}} & \ddots & 0 & 0 \\ & & \ddots & \ddots & & & \ddots & \ddots & \\ 0 & 0 & \cdots & r_{n-1}^l & -r_{n-1}^l & 0 & \cdots & \frac{1}{r_{n-1}^{(l+1)}} & -\frac{1}{r_{n-1}^{(l+1)}} \\ \sigma_1 l r_1^{l-1} & -\sigma_2 l r_1^{l-1} & \cdots & 0 & 0 & \frac{\sigma_2(l+1)}{r_1^{(l+2)}} & \cdots & 0 & 0 \\ 0 & \sigma_2 l r_2^{l-1} & \ddots & 0 & 0 & -\frac{\sigma_2(l+1)}{r_2^{(l+2)}} & \ddots & 0 & 0 \\ & & \ddots & \ddots & & & \ddots & \ddots & \\ 0 & 0 & \cdots & \sigma_{n-1} l r_{n-1}^{l-1} & -\sigma_n l r_{n-1}^{l-1} & 0 & \cdots & -\frac{\sigma_{n-1}(l+1)}{r_{n-1}^{(l+2)}} & \frac{\sigma_n(l+1)}{r_{n-1}^{(l+2)}} \\ 0 & 0 & \cdots & 0 & l r_n^{l-1} & 0 & \cdots & 0 & -\frac{(l+1)}{r_n^{(l+2)}} \end{bmatrix}$$

$$x_{[2n-1],[1]} = \begin{bmatrix} C_{lm_1} \\ C_{lm_2} \\ \vdots \\ C_{lm_{n-1}} \\ C_{lm_n} \\ D_{lm_2} \\ \vdots \\ D_{lm_{n-1}} \\ D_{lm_n} \end{bmatrix} \quad \text{and} \quad b_{[2n-1],[1]} = \begin{bmatrix} -r_1^{-(l+1)} M_{lm}^{Sup} \\ 0 \\ \vdots \\ 0 \\ \sigma_1(l+1) r_1^{-(l+2)} M_{lm}^{Sup} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

4 - Adapting the model to extended sources

After finishing coding the model in python and checking the results with the older C++ code, the next step was to adapt the model to utilize extended sources.

The rationale behind it is that the EEG electrodes do not capture a brain activity generated in a specific point of the brain (what is suggested by the models that use a punctual dipole source). In fact, the activity registered comes from the activation of a mesh of neurons in some area of the brain. Thinking of it, a more realistic model could be constructed by assuming that the source is not generated as one dipole, but as a bunch of them concentrated in a delimited space.

One possibility to calculate the potential for extended sources is to assume that the mesh can be modeled by a tetrahedron and that there can exist multiple dipole sources coming from all of the area of this three dimensional structure.

The document in the annex is a mathematical demonstration made by Theodore on how to write an equation to calculate the potential using the tetrahedron assumption as the extended source. It can be simplified as:

$$\phi' = \frac{\sum_{\lambda_1 \in I_1} \sum_{\lambda_2 \in I_2} \sum_{\lambda_3 \in I_3} \phi}{(n+1)^3}$$

Where:

I_1: $[0,h,2h,\ldots,(n-1)h,1]$, where $h = \dfrac{1}{n}$;

I_2: $[0,h,2h,\ldots,(n-1)h,1-\lambda_1]$, where $h = \dfrac{1-\lambda_1}{n}$;

I_3: $[0,h,2h,\ldots,(n-1)h,1-\lambda_1-\lambda_2]$, where $h = \dfrac{1-\lambda_1-\lambda_2}{n}$;

$n \in \mathbb{R} \,|\, n \geq 1$;

ϕ: potential calculated using the traditional model with the dipole source:

- positioned at: $(x, y, z) = v_1\lambda_1 + v_2\lambda_2 + v_3\lambda_3 + v_4(1 - \lambda_1 - \lambda_2 - \lambda_3)$,

- with moment: $(m_x, m_y, m_z) = m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 + m_4(1 - \lambda_1 - \lambda_2 - \lambda_3)$.

Where:

v_1, v_2, v_3, v_4: tetrahedron vertices;

m_1, m_2, m_3, m_4: moment of each tetrahedron vertice.

5 - Discussion

One way of verifying the results of the extended source model is to calculate the potential assuming a tiny tetrahedron around a specific dipole. With those conditions, for the same head geometry, with equal physical properties and assuming an electrode placed at the same position, the results of the both models presented in this work should converge ($\phi' \approx \phi$).

This type of test was not extensively made during this mini-project, just a few examples were checked in order to verify the extended source model consistency. On the other hand, making those tests and studying the behavior of the differences while increasing the size of the tetrahedrons and changing the concentration of the sources could be an interesting sequel for this work. This type of study could reveal an estimated impact of the error created by the dipole source assumption in the traditional models.

The python algorithm code is available at github.

References

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[2] SARVAS, J. ***Basic mathematical and electromagnetic concepts of the biomagnetic inverse problem***. Phys. Med. Biol. 32:11–22. doi: 10.1088/0031-9155/32/1/004. 1987.

[3] HEDOU, V. ***Méthodes Numériques pour la Modélisation Electro-anatomique du Cerveau.*** (Unpublished doctoral dissertation), Université de Rennes I, pp. 25-33, 1997.

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Annex

The purpose of this document is to describe how to describe the electric potential generated by an extended source in terms of electric potentials computed with various dipolar sources.

1 Notation

We assume that we know how to compute the potential at $\mathbf{r}$ $V_\delta(\mathbf{J}_p, \mathbf{r})$ in a given domain Ω for a dipole placed $\mathbf{r}'$ and of moment $\mathbf{J}_p(\mathbf{r}')$.

We know that $V_\delta(\mathbf{J}_p, \mathbf{r})$ satisfies:

$$\nabla \cdot (\sigma \nabla V_\delta(\mathbf{J}_p, \mathbf{r})) = \nabla \cdot (\mathbf{J}_p(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')) \tag{1}$$

We also assume that $V(\mathbf{J}_p, \mathbf{r})$ is the potential in the same domain Ω for a distributed source $\mathbf{J}_p(\mathbf{r})$.

$V(\mathbf{J}_p, \mathbf{r})$ satisfies:

$$\nabla \cdot (\sigma \nabla V(\mathbf{J}_p, \mathbf{r})) = \nabla \cdot \mathbf{J}_p(\mathbf{r}) \tag{2}$$

We also assume that in both cases, we have the same boundary condition.

As an example, you can imagine that we are looking at the potential in a nested sphere geometry, with the boundary condition that there is no normal current exiting the outer sphere (because the outer medium is non-conductive).

2 Expression of $V(\mathbf{J}_p, \mathbf{r})$ using $V_\delta(\mathbf{J}_p, \mathbf{r})$

Let us start with Eq. 1. We have:

$$\begin{aligned}
\nabla \cdot (\sigma \nabla V_\delta(\mathbf{J}_p, \mathbf{r})) &= \nabla \cdot (\mathbf{J}_p(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')) && (3)\\
\int \nabla \cdot (\sigma \nabla V_\delta(\mathbf{J}_p, \mathbf{r}))\, d\mathbf{r}' &= \int \nabla \cdot (\mathbf{J}_p(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}'))\, d\mathbf{r}' && (4)\\
\nabla \cdot \left(\sigma \nabla \int V_\delta(\mathbf{J}_p, \mathbf{r}) d\mathbf{r}'\right) &= \nabla \cdot \int \mathbf{J}_p(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}') d\mathbf{r}' && (5)\\
\nabla \cdot \left(\sigma \nabla \int V_\delta(\mathbf{J}_p, \mathbf{r}) d\mathbf{r}'\right) &= \nabla \cdot \mathbf{J}_p(\mathbf{r}) && (6)
\end{aligned}$$

By identification with Eq. 2, we have:

$$V(\mathbf{J}_p, \mathbf{r}) = \int V_\delta(\mathbf{J}_p, \mathbf{r}) d\mathbf{r}' \tag{7}$$

The domain of integration is not described but typically is the support of the source $\mathbf{J}_p(\mathbf{r})$. This shows that $V(\mathbf{J}_p, \mathbf{r})$ can be computed as the integral of the various solutions $V_\delta(\mathbf{J}_p, \mathbf{r})$ which correspond of a superposition of dipoles located at $\mathbf{r}'$ with moments $\mathbf{J}_p(\mathbf{r}')$.

3 P1 extended sources

In the case of a mesh constituted of a family of $\mathcal{T}$ tetrahedra, an extended source can be written as:

$$\mathbf{J}_p\mathbf{r} = \sum_{T_i \in \mathcal{T}} \sum_{j | \mathbf{r}_j \in vertices(T_i)} \mathbf{J}_p(\mathbf{r}_j) w_{T_i}^j(\mathbf{r})\ , \tag{8}$$

where $\mathbf{J}_p(\mathbf{r}_j)$ is a constant and corresponds to a moment associated with node $\mathbf{r}_j$ and $w^j_{T_i}(\mathbf{r})$ is the restriction of the basis function $w^j(\mathbf{r})$ associated with node j to the tetrahedron T_i. $w^j_{T_i}(\mathbf{r})$ is a linear function which value is 1 at node $\mathbf{r}_j$ and 0 at the other nodes of the tetrahedron T_i.

Because the potential is linear in the sources, we have:

$$V(\mathbf{J}_p, \mathbf{r}) = \sum_{T_i \in \mathcal{T}} \sum_{j | \mathbf{r}_j \in vertices(T_i)} V(\mathbf{J}_p(\mathbf{r}_j) w^j_{T_i}(\mathbf{r}), \mathbf{r}) \; . \tag{9}$$

So we just have to compute the elementary potential functions $V(\mathbf{J}_p(\mathbf{r}_j) w^j_{T_i}(\mathbf{r}), \mathbf{r})$. To do that, we can apply Eq. 7 to the extended source $\mathbf{J}_p(\mathbf{r}_j) w^j_{T_i}(\mathbf{r})$,

$$V(\mathbf{J}_p(\mathbf{r}_j) w^j_{T_i}(\mathbf{r}), \mathbf{r}) = \int_{T_i} V_\delta(\mathbf{J}_p, \mathbf{r}) d\mathbf{r}' \; . \tag{10}$$

The integration domain has been restricted to T_i because $w^j_{T_i}(\mathbf{r})$ is null outside of T_i.

Let us introduce the homogeneous coordinate $\mathbf{v}$ corresponding to $\mathbf{r}$ as:

$$\mathbf{v} = homog(\mathbf{r}) = \begin{bmatrix} \mathbf{r} \\ 1 \end{bmatrix} \; . \tag{11}$$

Without loss of generality, we can define $\mathbf{v}_0 = homog(\mathbf{r}_j)$ and similarly associate homogeneous coordinate vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ to the other 3 vertices of tetrahedron T_i. $w^j_{T_i}(\mathbf{r})$ can then be written as:

$$w^j_{T_i}(\mathbf{r}) = \begin{cases} \frac{|\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3|}{|\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3|} \text{ for } \mathbf{r} \in T_i \\ 0 \text{ otherwise} \end{cases} \tag{12}$$

Indeed, this function is linear in $\mathbf{v}$, so linear in $\mathbf{r}$ and it has all the other properties required by $w^j_{T_i}(\mathbf{r})$.

To integrate over the tetrahedron T_i, we can use the parameterization:

$$\mathbf{r} \in T_i \Leftrightarrow \mathbf{v} = \lambda_0 \mathbf{v}_0 + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + (1 - \lambda_0 - \lambda_1 - \lambda_2) \mathbf{v}_3 \; , \tag{13}$$

with $\lambda_0 \in [0 \dots 1]$, $\lambda_1 \in [0 \dots 1 - \lambda_0]$ and $\lambda_2 \in [0 \dots 1 - \lambda_0 - \lambda_1]$.

Using this change of variables, we have:

$$\begin{aligned} V(\mathbf{J}_p(\mathbf{r}_j) w^j_{T_i}(\mathbf{r}), \mathbf{r}) &= \int_{T_i} V_\delta(\mathbf{J}_p(\mathbf{r}_j) w^j_{T_i}(\mathbf{r}'), \mathbf{r}) d\mathbf{r}' & (14) \\ &= |\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3| \int_0^1 \int_0^{1-\lambda_0} \int_0^{1-\lambda_0-\lambda_1} V_\delta(\lambda_0 \mathbf{J}_p(\mathbf{r}_j), \mathbf{r}) d\lambda_2 d\lambda_1 d\lambda_0 & (15) \end{aligned}$$