# Lecture 6 - Upper Confidence Bound Algorithms

Guiliang Liu

The Chinese University of Hong Kong, Shenzhen

DDA4230: Reinforcement Learning
Course Page: [Click]

# DDA 4230 Resources

Please join our Slack group.

https://join.slack.com/t/slack-us51977/shared_invite/zt-22g8b40v8-0qSs9o0G3~8hXHwWydlCpw

Please check our course page.

https://guiliang.github.io/courses/cuhk-dda-4230/dda_4230.html

# Motivation

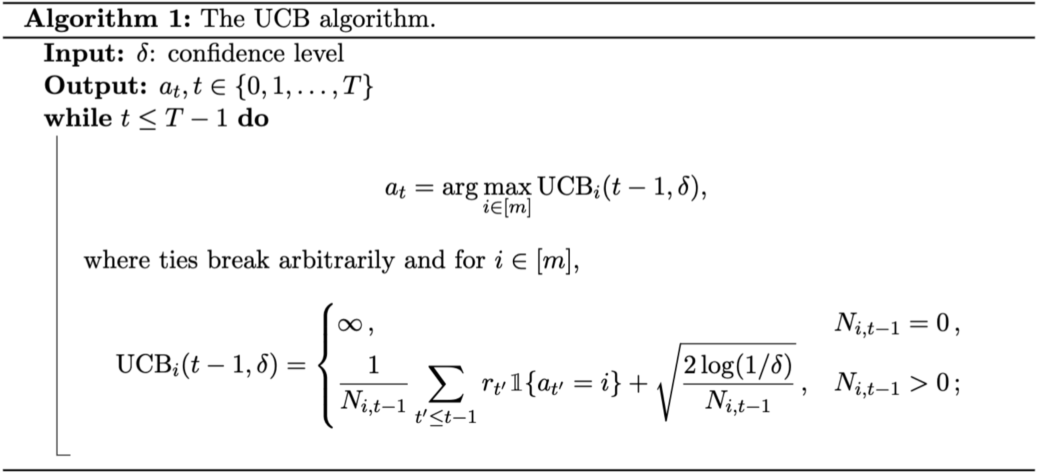
Recall the previously introduced algorithms:

- Execute $\varepsilon$-greedy by choosing $\varepsilon_t = \min\{1, Ct^{-1}\Delta_{\min}^{-2} m\}$.
- Run ETC with $k = \lceil \frac{2}{\Delta^2} W(\frac{T^2\Delta^4}{32\pi}) \rceil$

**Limitation** of $\varepsilon$-greedy and ETC.

1. Executing the algorithm requires the knowledge of $\Delta$, which is usually not available in real applications.
2. The algorithm uses $T$, but the horizon is unknown in real applications.
3. The theoretical result obtained by ETC is applied to 2-armed bandits only.

# The UCB Algorithms



**Algorithm 1:** The UCB algorithm.

```
Input: $\delta$: confidence level
Output: $a_t, t \in \{0, 1, \dots, T\}$
while $t \leq T - 1$ do
    $$a_t = \arg\max_{i \in [m]} \mathrm{UCB}_i(t-1, \delta),$$
    where ties break arbitrarily and for $i \in [m]$,
    $$\mathrm{UCB}_i(t-1, \delta) = \begin{cases} \infty\,, & N_{i,t-1} = 0\,, \\ \frac{1}{N_{i,t-1}} \sum_{t' \leq t-1} r_{t'} \mathbb{1}\{a_{t'} = i\} + \sqrt{\frac{2\log(1/\delta)}{N_{i,t-1}}}\,, & N_{i,t-1} > 0\,; \end{cases}$$
```

# The Optimism Principle

The UCB algorithm is based on the principle of optimism in the face of uncertainty, which states that

*one should act as if the environment is as nice as plausibly possible.*

In fact, this principle is applicable to other bandit algorithms as well and is beyond the finite-armed stochastic bandit problem.

# The Optimism Principle

For UCB, the optimism principle means using the data observed so far to assign to each arm a value, called the upper confidence bound. The first term,

$$\hat{\mu}_{i,t-1} = \frac{1}{N_{i,t-1}} \sum_{t' \leq t-1} r_{t'} \mathbb{1}\{a_{t'} = i\},$$

is the empirical mean of the rewards collected from arm $i$, where $N_{i,t-1} = \sum_{t' \leq t-1} \mathbb{1}\{a_{t'}\}$ is the number of times arm $i$ has been pulled up to time $t-1$.

# The Optimism Principle

Recall the Chernoff-Hoeffding bound for $n$ independent 1-sub-Gaussian random variables

$$\mathbb{P}(\overline{X} - \mathbb{E}[\overline{X}] \leq z) \geq 1 - \exp(-nz^2/2)\,.$$

The term $\sqrt{\frac{2\log(1/\delta)}{N_{i,t-1}}}$, is an at least $(1-\delta)$-order statistics of $\mu_i$. With high probability the UCB term is an overestimate of the unknown mean, if $N_{i,t-1}$ is a constant

$$\mathbb{P}(\mu_i \geq \hat{\mu}_{i,t-1} + \sqrt{\frac{2\log(1/\delta)}{N_{i,t-1}}}) \leq \delta\,.$$

While $N_{i,t-1}$ is also a random variable that is not independent of $\hat{\mu}_{i,t-1}$, the claim holds up to constant factors (Exercise 7.1 on the book).

# The Regret of UCB Algorithms

The UCB Algorithm explores all arms exactly once and then estimates each arm using the (sample-mean based) upper bound of its $\delta$-confidence interval.

Intuitively, the arm chosen in round $t$ either

- Has a large sample mean,
- Remain underexplored compared to other arms.

# The Regret of UCB Algorithms

The key ingredient lies in choosing a good confidence level $\delta$, which again balances the trade-off between exploration and exploitation.

### Theorem

*Assume the rewards of arms are* 1*-sub-Gaussian. Let* $\delta = T^{-2}$*. The regret under UCB is at most*

$$\overline{R}_T \leq 3\sum_{i=1}^{m} \Delta_i + \sum_{i:\Delta_i>0} \frac{16 \log T}{\Delta_i}.$$

UCB does not require knowledge on the suboptimality gaps.

# The Regret of UCB Algorithms

The UCB Theorem may seem loose when $\Delta_i$ are small. This can be fixed by separating the arms into two parts: those with a sub-optimality gap less than $\sqrt{16m \log T / T}$ and greater than $\sqrt{16m \log T / T}$. Bounding $\mathbb{E}[N_{i,T}]$ by $T$ in the first part and by the UCB Theorem in the second part gives

$$\overline{R}_T \leq 3 \sum_{i \in [m]} \Delta_i + 8\sqrt{mT \log T}. \tag{1}$$

# The Regret of UCB Algorithms

There are a few things we could consider for extension.

- The confidence level in UCB Theorem depends on horizon $T$. This can be removed by choosing $\delta$ in a decreasing format, say, $\delta_t = (1 + t \log^2 t)^{-1}$.
- The Hoeffding inequality used in the algorithm can be rather loose sometimes. For example, consider the Bernoulli bandits whose means are close to 0 or 1. In such situations, one could apply the Chernoff bound instead, which gives a confidence interval based on relative entropy.

# Question and Answering (Q&A)