Jacobi Modular Forms - 30 ans après

notes from the Coursera lectures by Valery Gritsenko

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Contents

1 Motivations

The motivations for studying Jacobi modular forms are related to the study of the following mathematical objects:

- generating functions for the number of representations of integers by quadrating forms,
- the partition function,
- the definition of modular forms,
- the elliptisation of Ramanujan Δ-function.

1.1 Lattices

Definition L is an integral even positive definite quadratic lattice iff it has the following properties:

- it is a lattice: L is a free $\mathbb{Z}$ module $\simeq \mathbb{Z}^n, n > 0$,
- it is integral: there exists a symmetric, bilinear pairing $L \times L \to \mathbb{Z}$,
- it is even: $(v, v) \in 2\mathbb{Z}, \forall v \in L$,
- it is positive definite: $(v, v) > 0, \forall v \neq 0$

1.2 The theta function

Let L be an integral even positive definite quadratic lattice, and $r_L(2n) = \#\{v \in L|(v, v) = 2n\}$. Let $\theta_L(q) = \sum_{n\geq 0} r_L(2n)q^n$ and $q = e^{2i\pi\tau}, \tau \in \mathbb{H}_1 = \{x+iy|y > 0\}$. We have $\theta(\tau + 1) = \theta(\tau)$ from the definition of q, but there is an additional hidden symmetry:

$$\vartheta(-\frac{1}{\tau}) = (\frac{\tau}{i})^{\frac{n}{2}}(\det L)^{-\frac{1}{2}}\vartheta_{L^*}(\tau)$$

where $L^* = \{u \in L \otimes \mathbb{Q}|\forall v \in L, (u, v) \in \mathbb{Z}\}$ is the dual lattice of L.
Let us assume that $L^* = L$, which is equivalent to having $\det L = 1$ (and implies $n \equiv 0[8]$). Then we have:

$$\begin{cases} \vartheta_L(-\frac{1}{\tau}) = \tau^{\frac{n}{2}}\vartheta_L(\tau) \\ \vartheta_L(\tau + 1) = \vartheta(\tau) \end{cases}$$

ϑ is then called a modular form of weight n over $SL_2(\mathbb{Z})$. Now we remark that $SL_2(\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})|\det M = 1\}$ acts on $\mathbb{H}_1$ with

$$M : \tau \to \frac{a\tau + b}{c\tau + d} \in \mathbb{H}_1$$

Note that $SL_2(\mathbb{Z})$ is generated by $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

1.3 Modular and abelian transforms

1.3.1 The partition function

Let L be an even integer positive definite lattice, $\tau \in \mathbb{H}_1$ and $z \in L \otimes \mathbb{C} \simeq \mathbb{C}^n$. We pose

$$\vartheta_L(\tau, z) = \sum_{v \in L} e^{i\pi\left((v,v)\tau + 2(v,z)\right)}$$

This is the generating function for the distributions of lattice points of given norm and fixed scalar product with z. $\vartheta_L(\tau, z)$ is holomorphic over $\mathbb{H}_1 \times (L \otimes \mathbb{C})$. Its symmetries are given by:

$$\vartheta_L\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) = \tau^{\frac{n}{2}} e^{i\pi\frac{(z,z)}{\tau}} \vartheta(\tau, z) \quad \text{if } L = L^* \qquad (1)$$

$$\vartheta_L(\tau, z + \lambda\tau + \mu) = e^{-i\pi\left(((\lambda,\lambda)\tau + 2(\lambda, z)\right)} \vartheta_L(\tau, z), \quad \lambda, \mu \in \mathbb{Z} \qquad (2)$$

Exercise 1: Prove the identities (use Poisson summation).

1.3.2 Pullbacks of the partition function

If $z = 0$ then $\theta_L(\tau, 0) = \theta_L(\tau)$.
Let $u \in L, (u,u) = 2m > 0$ and $z = u.Z, Z \in \mathbb{C}$. Then

$$\vartheta_L(\tau, u.Z) = \sum_{v \in L} e^{i\pi\left((v,v)\tau + 2(u,v)Z\right)} \triangleq \vartheta_{L,u}(\tau, Z)$$

$\vartheta_{L,u}(\tau, Z)$ is the Jacobi ϑ-series of Eichler and Zagier [5].

1.3.3 Pullbacks of the ϑ-functions

$$\vartheta_{L,u}(\tau, Z) = \sum_{n \geq 0, l \in \mathbb{Z}} r_{L,u}(n,l) q^n r^l, \quad q = e^{2i\pi\tau}, r = e^{2i\pi Z}$$

$$r_{L,u}(n,l) = \#\{v \in L | (v,v) = 2n, (u,v) = l\}$$

Let E_8 be the even integral lattice of determinant 1 such that $E_8 = E_8^*$. If u is a root of E_8 (i.e. $(u,u) = 2$), then $u^{\perp}_{E_8} = E_7$. In this case, $r_{E_8,u}(n,0) = \#\{v \in E_8 | (v,v) = 2n, (u,v) = 0\} = r_{E_7}(n)$, so the Eichler-Zagier ϑ-function provides a tool for studying E_7 via E_8, which in turn is simpler.

1.3.4 Elliptisation of the Ramanujan θ-function

Definition: a modular form of weight $k \in \mathbb{Z}$ w.r.t. the modular group $SL_2(\mathbb{Z})$ is a holomorphic function:

$$\begin{aligned} f : \mathbb{H}_1 &\to \mathbb{C} \text{ satisfies:} \\ f\left(\frac{a\tau + b}{c\tau + d}\right) &= (c\tau + d)^k f(\tau) \qquad (M) \\ f(\tau) &= \sum_{n \geq 0} a(n) e^{2i\pi n\tau} \qquad (F) \end{aligned}$$

$(M) \iff f$ is periodic.
(F): $f(\tau)$ is holomorphic at $i\infty \iff f(q)$ holomorphic at $q = 0$.

Notation: $M_k = M_k(SL_2(\mathbb{Z})$ is the space of modular forms of fixed weight k. At this stage we admit that $\dim_{\mathbb{C}} M_k < +\infty\ \forall k$.

As per the definition in [8] the Ramanujan Δ function is

$$\Delta(\tau) = q \prod_{n\geq 1} (1-q^n)^{24} \in M_{12}^{\text{cusp}}$$

We can lift this function as follows:

$$G_{4,4}(\tau, Z) = qr^{-4} \prod_{n\geq 1} (1-q^{n-1}r)^8 (1-q^n r^{-1})^8 (1-q^n)^8$$

where $G_{4,4}$ is a Jacobi form of weight 4 and index 4.

1.4 Exercises week 1: even integral lattices

2 Definition of Jacobi forms

The plan of the section is to state a first definition of the Jacobi modular forms. Then we will present the Jacobi modular group Γ^J. In particular we will see that

$$SL_2(\mathbb{Z}) \hookrightarrow \Gamma^J \hookrightarrow Sp_2(\mathbb{Z})$$

Finally we will reach the second definition of Jacobi modular forms, based on the action of the Jacobi group.

2.1 First definition

Let $\tau \in \mathbb{H} = \{\tau \in \mathbb{C}, \Im\tau > 0\}$, $z \in \mathbb{C}$ and $k \in \mathbb{Z}$ the weight of the form, $m \in \mathbb{N}$ its index.

Consider a holomorphic function $\phi(\tau, z) : \mathbb{H}_1 \times \mathbb{C} \to \mathbb{C}$. We state the following equations for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $\lambda, \mu \in \mathbb{Z}$:

$$\begin{aligned} \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) &= (c\tau+d)^k e^{2i\pi m \frac{cz^2}{c\tau+d}} \phi(\tau, z) \qquad (M) \\ \phi(\tau, z+\lambda\tau+\mu) &= e^{-2i\pi m(\lambda^2\tau+2\lambda z)} \phi(\tau, z) \qquad (E) \end{aligned}$$

(M) is the modular equation, (E) the elliptic equation. The latter is close to a double periodicity of ϕ over the elliptic curve $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$.
From these equations one obtains immediately $\phi(\tau+1, z) = \phi(\tau, z)$, and
$\phi(\tau, z+1) = \phi(\tau, z)$. So ϕ has a Fourier expansion:

$$\phi(\tau, z) = \sum_{n,l\in\mathbb{Z}} a(n,l) e^{2i\pi(n\tau+lz)}$$

Definition: ϕ is called a holomorphic (resp. cusp, resp. weak) Jacobi form of

weight k and index m if $a(n,l) = 0$ unless $4nm - l^2 \geq 0$ (resp. $4nm - l^2 > 0$, resp. $n \geq 0$).
Notation:

- $J_{k,m}$ is the space of holomorphic Jacobi forms.

- $J_{k,m}^{\text{cusp}}$ is the space of holomorphic Jacobi cusp forms

- $J_{k,m}^{w}$ is the space of weak Jacobi forms.

One has:

$$J_{k,m}^{\text{cusp}} \subset J_{k,m} \subset J_{k,m}^{w}$$

We take for granted that $\dim J_{k,m}^{w} < +\infty$

Example: Let $u \in L$, L an even positive definite quadratic positive lattice. The Fourier expansion of the Jacobi modular form is given by definition as:

$$\theta_{L,U}(\tau, z) = \sum_{v \in L} e^{i\pi\left((v,v)\tau + 2(u,v)z\right)} = \sum_{n,l \in \mathbb{Z}} r_{L,u}(n,l) e^{2i\pi(n\tau + lz)}$$

The Fourier coefficients are given by

$$r_{L,u}(n,l) = \#\{v \in L | (v,v) = 2n, (u,v) = l\}$$

So we have

$$r_{L,u}(n,l) \neq 0 \implies 4nm - l^2 > 0 \implies \left(\begin{smallmatrix} (v,v) & (u,v) \\ (u,v) & (u,u) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 4n & l \\ l & m \end{smallmatrix}\right) \succ 0$$

2.2 A remark for the modular equation

To comment about the curious transformation $z \to \frac{z}{c\tau+d}$ in the modular equation, one notes that for an elliptic curve the invariance by $z \to z + \lambda\tau + \mu$ for $\lambda, \mu \in \mathbb{Z}$ means that $z \in \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) = C_\tau$. However, $C_\tau \simeq C_{M\langle\tau\rangle}$, with $M\langle\tau\rangle = \frac{a\tau+b}{c\tau+d}$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. But $C_{M\langle\tau\rangle} = \mathbb{Z}M\langle\tau\rangle + \mathbb{Z} = \frac{(a\mathbb{Z}+b\mathbb{Z})\tau + (c\mathbb{Z}+d\mathbb{Z})}{c\tau+d}$. Since $(a,b) = (c,d) = 1$, this simplifies as $\mathbb{Z}M\langle\tau\rangle + \mathbb{Z} = (\mathbb{Z} + \mathbb{Z}\tau)/(c\tau + d)$. This hints at a relation between Jacobi modular forms and the universal elliptic curve $SL_2(\mathbb{Z}) \setminus \mathbb{H}_1 \times \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$.

2.3 Motivation for the Jacobi modular group

1) If we specialize the modular equation to $z = 0$ we obtain

$$\phi\left(\frac{a\tau + b}{c\tau + d}, 0\right) = (c\tau + d)^k \phi(\tau, 0)$$

So if $\phi \in J_{k,m}$:

$$\phi(\tau, 0) = \sum_{n \in \mathbb{Z}} \Big(\sum_{l \in \mathbb{Z}, 4nm - l^2 \geq 0} a(n,l) e^{2i\pi n\tau} \Big)$$

Since $m \geq 0$, $\phi(\tau, 0) = \sum_{n\geq 0} c(n) q^n, q = e^{2i\pi\tau}$. So $\phi(\tau, z) \in J_{k,m} \implies \phi(\tau, 0) \in M_k\left(SL_2(\mathbb{Z})\right)$.

Question 1: lifting of modular forms For $f \in M_k(SL_2(\mathbb{Z}))$, can we find $\phi \in J_{k,m}$ such that $\phi(\tau, 0) = f$? We will see later that the answer is positive.

Question 2: other specializations. Are $\phi(\tau, \frac{1}{2}), \phi(\tau, \frac{\tau+1}{2}), \phi(\tau, \frac{a\tau+b}{N})$ for $a, b \in \mathbb{Z}/N\mathbb{Z}$ modular forms ? This leads to another definition of Jacobi forms. And finally,
Question 3: What is the Jacobi modular group ?

3 The Jacobi modular group

3.1 The slash operator

Let $M \in SL_2(\mathbb{Z})$, $f : \mathbb{H}_1 \to \mathbb{C}$. Let us define $f_{|k} M(\tau) = (c\tau + d)^{-k} f(\tau)$ called the "slash k" operator. We remind that $j(M, \tau) = (c\tau + d)$ is the automorphic factor. Then one has

$$f_{|k}(M_1 M_2) = (f_{|k} M_1)_{|k} M_2$$

Proof: One needs to show first that $M_1 M_2 \langle\tau\rangle = M_1 \langle M_2 \langle\tau\rangle\rangle$, and then that $j(M_1 M_2, \tau) = j(M_1, M_2\langle\tau\rangle) j(M_2, \tau)$, the so-called "cocycle" condition. Cf exercises.

Definition: A modular form of weight $k \in \mathbb{Z}$ w.r.t. $SL_2(\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that $f_{|k}(M) = f$, $\forall M \in SL_2(\mathbb{Z})$, and such that f is holomorphic at $i\infty$.
Question: How can one give a definition of Jacobi modular forms in a similar way? To draw a parallel:

$$\begin{array}{rccl} \Gamma^J & \dashrightarrow & SL_2(\mathbb{Z}) & \\ \cdot_{|k,m} & \dashrightarrow & \cdot_{|k} & \\ \text{s.t. } \phi_{|k,m} g & = & \phi & \quad \forall g \in \Gamma^J \end{array}$$

3.2 $Sp_n(\mathbb{Z})$ as a generalization of $SL_2(\mathbb{Z})$

$$Sp_n(\mathbb{Z}) = \{M \in M_2(\mathbb{Z}) |^t M \left(\begin{smallmatrix} 0 & -E_n \\ E_n & 0 \end{smallmatrix}\right) M = \left(\begin{smallmatrix} 0 & -E_n \\ E_n & 0 \end{smallmatrix}\right)$$

Exercise 1: Prove that $Sp_1(\mathbb{Z}) = SL_2(\mathbb{Z})$.

Exercise 2:
a) Prove that $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right) \in Sp_n(\mathbb{Z}) \iff D \in GL_n(\mathbb{Z})\,, A = {}^tD^{-1}$
b) Prove that $\left(\begin{smallmatrix} E_n & B \\ 0 & E_n \end{smallmatrix}\right) \in Sp_n(\mathbb{Z}) \iff {}^tB = B \in M_n(\mathbb{Z})$.

3.3 The Siegel modular group

$Sp_n(\mathbb{R})$ acts on the Siegel upper half-plane
$\mathbb{H}_n = \{Z = X + iY | X, Y \in M_n(\mathbb{R}), X = {}^tX\,, Y = {}^tY\,, Y > 0\}$. Note that $\mathbb{H}_{n=1} = \mathbb{H}_1$ so notations are consistent with the previous lecture.

$Sp_n(\mathbb{R})$ acts on $\mathbb{H}$: let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp_n(\mathbb{R})$. Define

$$M\langle Z\rangle = (AX+B)(CZ+D)^{-1}$$

Exercises will put emphasis on this group action. We obtain for $F : \mathbb{H}_n \to \mathbb{C}, M \in Sp_n(\mathbb{R}), k \in \mathbb{Z}$:

$$\begin{aligned} (F_{|k}M)(Z) &= \det(CZ+D)^{-k} F(M\langle Z\rangle) \\ F_{|k}M_1M_2 &= (F_{|k}M_1)_{|k}M_2 \end{aligned}$$

3.4 Siegel modular forms

Definition: A Siegel modular form of weight k w.r.t. $Sp_n(\mathbb{Z}), n \geq 2$ is a holomorphic function $F : \mathbb{H}_n \to \mathbb{C}$ such that $F_{|k}M = F$, $\forall M \in Sp_n(\mathbb{Z})$.
Remark: There is no condition on the Fourier expansion for $n \geq 2$.

3.5 The Jacobi modular group

The Jacobi modular group is

$$\Gamma^J = \left\{ \left(\begin{array}{cccc} * & 1 & * & * \\ * & 1 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1 \end{array} \right) \in Sp_2(\mathbb{Z}) \right\}$$

Checks are made in the exercises. Γ^J is a (parabolic) subgroup of $Sp_2(\mathbb{Z})$. Also,

$$\begin{array}{ccc} SL_2(\mathbb{Z} & \hookrightarrow & \Gamma^J \\ \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) & \to & \left[\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right] = \left(\begin{array}{cccc} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \end{array}$$

And

$$M = \left(\begin{array}{cccc} a & 1 & b & * \\ * & 1 & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & 1 \end{array}\right) \Longrightarrow \left[\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right]^{-1} M = \left(\begin{array}{cccc} 1 & 0 & 0 & p \\ -q & 1 & p & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{array}\right) \in \Gamma^J \subset Sp_2(\mathbb{Z})$$

The Heisenberg group is a subgroup $H(\mathbb{Z}) < \Gamma^J$ described by

$$H(\mathbb{Z} = \left\{ \left[\left(\begin{array}{c} p \\ q \end{array}\right), r\right] = \left(\begin{array}{cccc} 1 & 0 & 0 & p \\ -q & 1 & p & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{array}\right) | p,q,r \in \mathbb{Z} \right\}$$

It represents the subgroup of unipotent matrices in $Sp_2(\mathbb{Z})$.

The second definition of Jacobi forms:
$\mathbb{H}_2 = \{Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in M_2(\mathbb{C} | \Im Z > 0\}$. We remark that $\Im Z > 0 \iff \tau, \omega \in$

$\mathbb{H}_1, \Im\tau.\Im\omega - (\Im z)^2 > 0$. So $\forall(\tau, z) \in \mathbb{H} \times \mathbb{C}, \exists\omega \in \mathbb{H}_1$ such that $\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathbb{H}_2$.
For $\phi \in J_{k,m}$ let us define $\tilde{\phi}_m(\tau, z) \triangleq \phi(\tau, z)e^{2i\pi m\omega}$, where m is called the index.

Then ϕ satisfies both the modular and the elliptic equations iff:

$$\tilde{\phi}_m : \mathbb{H}_2 \to \mathbb{C} : \tilde{\phi}_{m|k}g = \tilde{\phi}_m \forall g \in \Gamma^j$$

3.6 The subgroups of the Jacobi modular group

As shown above, we have $SL_2(\mathbb{Z}) < Sp_2(\mathbb{Z})$. In particular, $\forall g \in Sp_2(\mathbb{Z}), \exists M \in SL_2(\mathbb{Z})$ s.t.

$$[M]^{-1}g = \begin{pmatrix} 1 & * & 0 & * \\ * & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Using the symplectic relations, one obtains constraints between the blocks which lead to

$$[M]^{-1}g = \begin{pmatrix} 1 & 0 & 0 & p \\ -q & 1 & p & r \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} p \\ q \end{pmatrix}, r\right]$$

Thus we obtain

$$\Gamma^J = [SL_2(\mathbb{Z})]\,.H(\mathbb{Z})$$

Properties of $H(\mathbb{Z})$:
1) We have

$$\left[\begin{pmatrix} p \\ q \end{pmatrix}, r\right] \cdot \left[\begin{pmatrix} p' \\ q' \end{pmatrix}, r'\right] = \left[\begin{pmatrix} p+p' \\ q+q' \end{pmatrix}, r + r' + \begin{vmatrix} p & p' \\ q & q' \end{vmatrix}\right]$$

This immediately implies that $H(\mathbb{Z})$ is not commutative.
2) From above we deduce

$$\left[\begin{pmatrix} p \\ q \end{pmatrix}, r\right]^{-1} = \left[\begin{pmatrix} -p \\ -q \end{pmatrix}, -r\right]$$

3) The commutators are described by

$$h.h'.h^{-1}h'^{-1} = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 2\begin{vmatrix} p & p' \\ q & q' \end{vmatrix}\right]$$

4) The center of $H(\mathbb{Z})$ is

$$Z\big(H(\mathbb{Z})\big) = \left\{\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, r\right], r \in \mathbb{Z}\right\}$$

5)We have the exact sequence

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow H(\mathbb{Z}) \longrightarrow & \mathbb{Z} \times \mathbb{Z} \longrightarrow 0 \\ r & \longrightarrow \left[\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right), r\right] \longrightarrow & \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{Z}^2 \end{array}$$

The Heisenberg group is thus the central extension of $\mathbb{Z}^2$.

Exercise:
6) Let $V_H : H(\mathbb{Z}) \to \{\pm 1\}$, $V_H\left(\left[\binom{p}{q}, r\right]\right) = (-1)^{p+q+pq+r}$.
Prove that V_H is a character. What is its kernel ?

7) Prove that $SL_2(\mathbb{Z})$ acts on $H(\mathbb{Z})$ by conjugation with :

$$[M]\left[\begin{pmatrix} p \\ q \end{pmatrix}, r\right][M]^{-1} = \left[M\begin{pmatrix} p \\ q \end{pmatrix}, r\right]$$

8) Prove that $H(\mathbb{Z}) \triangleleft \Gamma^J$ and that $\Gamma^J = SL_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$.

3.7 The action of Γ^J on the Siegel upper half-plane $\mathbb{H}_2$

Given the decomposition $\Gamma^J = SL_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$ we can study the action of Γ^J via the action of its subgroups. Let $Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathbb{H}_2$, $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then

$$[M]\langle Z \rangle = \begin{pmatrix} \frac{a\tau+b}{c\tau+d} & \frac{z}{c\tau+d} \\ \frac{z}{c\tau+d} & \omega - \frac{cz^2}{c\tau+d} \end{pmatrix}$$

this corresponds to the transformation of the equation (M_J):

$$\left(\phi(\tau, z)e^{2i\pi m\omega}\right)_{|k}[M] = (c\tau+d)^{-k}e^{-2i\pi m\frac{cz^2}{c\tau+d}}\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)e^{2i\pi m\omega}$$

Regarding the action of $H(\mathbb{Z})$, let $h = \left[\binom{p}{-q}, r\right] \in H(\mathbb{Z})$. Then we have

$$\begin{aligned} h\langle Z \rangle &= \begin{pmatrix} 1 & 0 & 0 & p \\ q & 1 & p & r \\ 0 & 0 & 1 & -q \\ 0 & 0 & 0 & 1 \end{pmatrix}\langle\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}\rangle \\ &= \begin{pmatrix} \tau & q\tau+z+p \\ q\tau+z+p & q^2\tau+2qz+qp+\omega+r \end{pmatrix} \end{aligned}$$

We can compare the previous expression to the elliptic equation:

$$\phi(\tau, z)e^{2im\pi\omega}{}_{|k}h = e^{2i\pi m(q^2\tau+2qz+pq+r)}\phi(\tau, z+q\tau+p)e^{2im\pi\omega}$$

The Γ^J action on Jacobi forms
Definition Let $\phi : \mathbb{H}_1 \times \mathbb{C} \to \mathbb{C}$. We note that $\forall(\tau, z) \in \mathbb{H}_1 \times \mathbb{C}, \exists\omega : \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathbb{H}_2$.
Let $g \in \Gamma^J(\mathbb{R})$. Let us pose

$$\phi_{|k,m}g \triangleq \left(\left(\phi(\tau, z)e^{2i\pi m\omega}\right)_{|k}g\right)e^{-2i\pi m\omega}$$

We obtain that:
1) $\phi_{|k,m}g$ depends only on τ and z,

2) $\phi_{|k,m}g_1g_2 = (\phi_{|k,m}g_1)_{k,m}g_2$ is the right action of the real Jacobi group.
We remind the modular equation for the Jacobi forms:

$$\left(\phi(\tau,z)e^{2i\pi m\omega}\right)_{|k}[M] = (c\tau+d)^{-k}e^{-2i\pi m\frac{cz^2}{c\tau+d}}\phi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)e^{2i\pi m\omega}$$

For $h=\left[\binom{p}{-q},r\right]\in H(\mathbb{R})$, we have

$$\begin{aligned} h\langle Z\rangle &= \begin{pmatrix} 1 & 0 & 0 & p \\ q & 1 & p & r \\ 0 & 0 & 1 & -q \\ 0 & 0 & 0 & 1 \end{pmatrix}\langle\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}\rangle \\ &= \begin{pmatrix} \tau & z+p \\ q\tau+z+p & qz+\omega+r \end{pmatrix}\begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \tau & q\tau+z+p \\ q\tau+z+p & q^2\tau+2qz+qp+\omega+r \end{pmatrix} \end{aligned}$$

So

$$\phi(\tau,z)e^{2i\pi m\omega}{}_{|k}h = e^{2i\pi m(q^2\tau+2qz+pq+r)}\phi(\tau,z+q\tau+p)e^{2i\pi m\omega}$$

3.8 The second definition of Jacobi modular forms:

Definition:
A holomorphic function $\phi:\mathbb{H}_1\times\mathbb{C}$ is a Jacobi modular form of weight k and index $m\in\mathbb{Z}_{\geq 0}$ if $\forall g\in\Gamma^J$: $\phi_{|k,m}g=\phi$. Moreover, $\phi(\tau,z)$ has a "good" Fourier expansion

$$\phi(\tau,z)=\sum_{4nm-l^2\geq 0}a(n,l)e^{2i\pi(n\tau+lz)}$$

i.e. $a(n,l)=0$ unless $4nm-l^2\geq 0$.

Note that we can write the Fourier expansion as:

$$\phi(\tau,z)e^{2im\pi\omega}\sum_{N\geq 0}a\left(\left(\begin{smallmatrix} n & l/2 \\ l/2 & m \end{smallmatrix}\right)\right)e^{2i\pi\mathrm{Tr}(N.Z)},\ z=\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right)\in\mathbb{H}_2$$

A holomorphic Jacobi form is holomorphic function on $\mathbb{H}_2$ which is modular with respect to the parabolic subgroup $\Gamma^J < Sp_2(\mathbb{Z})$. We have thus the following illustration of the structure of modular forms:

$$\begin{array}{ccc} SL_2(\mathbb{Z})\hookrightarrow & \Gamma^J & \hookrightarrow Sp_2(\mathbb{Z}) \\ \text{modular forms}\rightarrow & \text{Jacobi forms} & \rightarrow\text{Siegel modular forms} \end{array}$$

4 Special values of Jacobi forms

Problem: how can we analyze the modular properties of $\phi(\tau,\frac{1}{2})$, $\phi(\tau,\frac{\tau+1}{2})$ or $\phi(\tau,q\tau+p)$ for $q,p\in\mathbb{Q}$ where $\phi\in J_{k,m}$ of for z any point of finite order in $\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$? For this we will study the ${}_{|k,m}$-action of Γ^J.
Let $X=\binom{p}{-q}\in\mathbb{Q}^2$. Then

$$\phi_{|k,m}[X,0]=e^{2i\pi m(q^2\tau+2qz+pq)}\phi(\tau,z+q\tau+p)$$

Specialising to $z = 0$ we obtain

$$\phi_{|k,m}[X, 0] = e^{2i\pi m(q^2\tau + pq)}\phi(\tau, q\tau + p)$$

Let us put $\Gamma_X = \left\{M \in SL_2(\mathbb{Z}) | MX \equiv X[\mathbb{Z}^2]\right\} < SL_2(\mathbb{Z})$. We have $\Gamma(N) < \Gamma_X$, where $\Gamma(N) = \{M \in SL_2(\mathbb{Z}) | MX \equiv E_2[N]\}$ and N is the minimum positive integer such that $N.X \in \mathbb{Z}^2$. N is called the level of X.

Exercise Check that $\Gamma(N) < \Gamma_X$.

Theorem: Let $\phi \in J_{k,m}$, $X = \binom{p}{-q} \in \mathbb{Q}^2$. Then the function $\phi_X(\tau) = e^{2i\pi m(q^2\tau+pq)}\phi(\tau, q\tau + p)$ is a modular form of weight k with respect to Γ_X with a character $\chi_X(M)$:

$$\chi_X(M) = e^{2i\pi \det(MX,X)}$$

Proof:
Let us investigate the behaviour of $\phi_{|k,m}[X, 0]$ under $_{|k,m}[M], M \in SL_2(\mathbb{Z})$. We have $[X, 0].[M] = [M] \left([M]^{-1}.[X, 0].[M]\right)$. Since $\phi \in J_{k,m}$, $\phi_{|k,m}[M] = \phi$. Then

$$\left(\phi_{k,m}[X, 0]\right)_k [M] = \phi_{k,m}[M^{-1}X, 0], \quad \forall M \in SL_2(\mathbb{Z})$$

Let $M \in \Gamma_X : MX - X \in \mathbb{Z}^2$. Then

$$\begin{aligned} [M^{-1}X, 0] &= = [M^{-1}X, 0].[X, 0]^{-1}.[X, 0] \\ &= [M^{-1}X, 0].[-X, 0].[X, 0] \\ &= [M^{-1}X - X, -\det(M^{-1}X, X)].[X, 0] \\ &= [M^{-1}X - X, -\det(X, MX)].[X, 0] \end{aligned}$$

So we obtain for ϕ the transformation equation

$$\phi_{|k,m}[M^{-1}X, 0] = e^{2i\pi m \det(MX,X)}\phi_{|k,m}[X, 0]$$

We have thus proved the modular property for the pullback $\phi_X(\tau)$:

$$(\phi_{|k,m}[X, 0])_{|k}[M] = e^{2i\pi m \det(MX,X)}\phi_{|k,m}[X, 0] \triangleq \chi_X(M)\phi_{|k,m}[X, 0]$$

, where $M \to \chi_X(M)$ is a character of Γ_X. As a result, we obtain for $z = 0$:

$$\phi_{X|k}M = e^{2i\pi m \det(MX,X)}\phi_X \quad \forall M \in \Gamma_X$$

We check the Fourier expansion for the modular form.

$$\begin{aligned} \phi_X(\tau) &= e^{2i\pi m(q^2\tau+pq)}\phi(\tau, q\tau + p) \\ &= e^{2i\pi m(q^2\tau+pq)} \sum_{4nm-l^2\geq 0} a(n, l)e^{2i\pi(n\tau+l(q\tau+p)} \\ &= e^{2i\pi mpq} \sum_{4nm-l^2\geq 0} a(n, l)e^{2i\pi lp}e^{2i\pi(mq^2+lq+n)\tau} \end{aligned}$$

$N = mq^2 + lq + n \geq 0$ since its discriminant is $\Delta = l^2 - 4nm \leq 0$. So ϕ is holomorphic at $i\infty$. To finish the proof, we have to study the Fourier expansion at all cusps: $\phi_X|M\, \forall M \in SL_2(\mathbb{Z})$.

$$\phi_{X|k,}M = \phi_{|k,m}[M^{-1}X, 0])_{z=0}$$

Note that $M^{-1}X = \binom{p'}{q'} \in \mathbb{Q}^2$. This thus leads to a similar calculation. So one concludes that $(\phi_{X|k}M)(\tau) = \sum_{N\geq 0} c(N)e^{2i\pi N\tau}$ and so that $\phi_X(\tau)$ is holomorphic on all cusps of Γ_X. $\square$

Remark: $a(n,l) = 0$ unless $4nm - l^2 \geq 0 \iff \forall X \in \mathbb{Q}^2 : \phi_X(\tau)$ is holomorphic at $i\infty$. The latter interpretation is for $\phi(\tau)$ as a pullback, while the former stands for Siegel modular forms.

5 The zeros of elliptic functions

Theorem : Let $\phi : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, $m \in \mathbb{Z}, \tau \in \mathbb{H}_1$ - considered fixed here. We assume that

$$\phi(z + \lambda\tau + \mu) = e^{-2im\pi(\lambda^2\tau + 2\lambda z)}\phi(z) \quad \forall \lambda, \mu \in \mathbb{Z} \qquad \text{(E)}$$

Then if ϕ is not identically 0, then ϕ has exactly $2m$ zeros in any fundamental domain $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$, counting multiplicities.

Proof:
Let $D_{z_0} = \{z_0 + x + iy | x, y \in [0,1[\,\}$ be a model of $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$. By varying z_0 we can make sure that $\phi(z)$ does not vanish on the boundary ∂D_{z_0}. The integral $\int_{\partial D_{z_0}} \frac{\phi'(z)}{\phi(z)} dz$ is equal to the number of zeroes of $\phi(z)$ in the fundamental domain. Let us consider the parallelepiped $z_0 \xrightarrow{A} z_0 + 1 \xrightarrow{B} z_0 + 1 + \tau \xrightarrow{C} z_0 + \tau \xrightarrow{D} z_0$. From (E) we get $\frac{\phi'(z+1)}{\phi(z+1)} = \frac{\phi'(z)}{\phi(z)}$. Moreover, $\frac{\phi'(z+\tau)}{\phi(z+\tau)} = \frac{\phi'(z)}{\phi(z)} - 4im\pi$. So we obtain, taking into account the orientations:

$$\int_C \frac{\phi'(z)}{\phi(z)} dz = -\int_A \frac{\phi'(z)}{\phi(z)} dz + 4im\pi$$

$$\int_B \frac{\phi'(z)}{\phi(z)} dz = -\int_D \frac{\phi'(z)}{\phi(z)} dz$$

So

$$\frac{1}{2i\pi} \int_{\partial D_{z_0}} \frac{\phi'(z)}{\phi(z)} dz = \frac{1}{2i\pi} \int_{ABCD} \frac{\phi'(z)}{\phi(z)} dz = 2m$$

$\square$

5.1 Applications

1) If ϕ is meromorphic, then $2m$ is the number of zeros minus the number of poles.

2) If ϕ is holomorphic, then $2m \geq 0$ - this is why Jacobi forms are defined

only for $m \geq 0$.

3) The function ϕ from the theorem is determined up to a constant by its zeroes:
Proof: Let us assume that $\exists \phi, \psi$ s.t. $\phi(z_1) = \psi(z_1) = 0, \ldots, \phi(z_{2m} = \psi(z_{2m}) = 0$. So $\forall z_0 \notin \{z_1, \ldots, z_{2m}\}$, $\phi(z_0)\psi(z_0) \neq 0$.
So, using the theorem, $\phi(z)\psi(z_0) - \phi(z_0)\psi(z)$ has at least $2m+1$ zeroes, so vanishes identically. Hence we get $\psi(z) = \frac{\psi(z_0)}{\phi(z_0)}\phi(z)$.

4) $\phi \in J^w_{k,m}$, $z \to \phi(\tau, z)$ satisfies the conditions of the theorem. So $\phi(\tau, z)$ has exactly $2m$ zeroes in any fundamental domain $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$.

5) **Case of $\phi \in J^w_{k,0}$, for** $m = 0$: $\phi(\tau, z) - \phi(\tau, 0)$ has a zero for $z = 0$ So $\phi(\tau, z) = \phi(\tau, 0) \in M_k$. So $J_{k,0} \subseteq J^w_{k,0} = M_k \subseteq J_{k,0}$ and hence $J_{k,0} = J^w_k = M_k$.

Theorem: $\dim J_{k,m} < +\infty$.

Proof: $\forall X = \binom{p}{-q} \in \mathbb{Q}^2$, $\forall \phi \in J_{k,m}$, then $\phi_X(\tau) \triangleq e^{2i\pi(q^2\tau + pq)}\phi(\tau, q\tau + p) \in M_k(\Gamma_X, \chi_X)$, with $\Gamma_X \in SL_2(\mathbb{Z})$. Then the map

$$\phi(\tau, z) \to \left(\phi_{X_i}(\tau)\right)_{i=1}^{2m}, \; X_i \neq X_j \in \mathbb{Q}^2$$

is injective (because of corollary 3 above: ϕ is determined uniquely up to $\mathbb{C}^\times$ by its zeros). So

$$\dim J_{k,m} \leq \sum_{i=1}^{2m} \dim M_{X_i}(\Gamma_{X_i}, \chi_{X_i}) < +\infty$$

$\square$

We will see below that this is a crude upper bound which can be improved upon.

5.2 Taylor expansion of Jacobi forms

Let $\phi(\tau, z) = \sum_{4nm - l^2 \geq 0} a(n, l) q^n r^l \in J_{k,m}$, with $q = e^{2i\pi\tau}$ and $r = e^{2i\pi z}$. So

$$\phi(\tau, z) = \sum_{d \geq d_0 \geq 0} f_d(\tau) z^d,$$

where $d_0 = \mathrm{ord}_{|z=0}\phi(\tau, z)$. We have

$$f_0(\tau) = \phi(\tau, 0) = \sum_{n \geq 0} \sum_{l, 4nm - l^2 \geq 0} a(n, l) q^n \in M_k\big(SL_2(\mathbb{Z})\big)$$

Also, since $\phi(\tau+1, z) = \phi(\tau, z)$, then $f_d(\tau+1) = f_d(\tau)$, so $f_d(\tau) = \sum_{n \geq 0} c(n) q^n$.

Remark: We have the following property: if $d > 0$, then $f_d(\tau) = \sum_{n > 0} c(n) q^n$: there is no constant term since $a(n, l) = 0$ for $n = 0, \forall l \neq 0$.

For a proof of this, observe that any z^d term for $d > 0$ must thus come from $n > 0$. So if $d > 0$, $f_d(\tau) = \sum_{n>0} c(n)q^n$, however f_d is not in general a modular form.

Proposition: $\phi \in J_{k,m}$ is determined uniquely by its first $(2m+1)$ Taylor coefficients $\big(f_0(\tau), \dots, f_{2m}(\tau)\big)$.

Proof: Let ϕ, ψ share the same first Taylor coefficients at order $0, \dots, 2m$. Then $\mathrm{ord}_{|z=0}(\phi - \psi) \geq 2m+1$, and since $\phi - \psi$ is of order m it must vanish identically, hence $\phi \equiv \psi$. $\square$

As a side remark, one notes that for $\phi \in J_{k,m}$, since $-E_2 \in SL_2(\mathbb{Z})$, $\phi(\tau, -z) = (-1)^k \phi(\tau, z)$. So

$$\phi(\tau, z) = \sum_{d \equiv k[2], d \geq 0} f_d(\tau) z^d$$

Proposition: For $\phi \in J_{k,m}$, $f_0(\tau) = \phi(\tau, 0) \in M_k$. If $\phi(\tau, 0) \equiv 0$, then $f_{d_0}(\tau) \in S_{k+d_0}$ - a cusp form.

Proof: $\forall M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, we have

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{2i\pi m \frac{cz^2}{c\tau + d}} \phi(\tau, z)$$

Expanding both sides, we get:

$$\sum_{d' \geq 0} f_{d'}(M\langle\tau\rangle) \frac{z^{d'}}{(c\tau + d)^{d'}} = (c\tau + d)^k e^{2i\pi m \frac{cz^2}{c\tau + d}} \sum_{d' \geq 0} f_{d'}(\tau) z^{d'}$$

Getting the full series requires the Taylor expansion of $e^{2i\pi m \frac{cz^2}{c\tau + d}}$. However, if $d_0 = \mathrm{ord}_{|z=0}\phi(\tau, z)$, then we obtain:

$$f_{d_0}(M\langle\tau\rangle) = (c\tau + d)^{k+d_0} f_{d_0}(\tau)$$

So $f_{d_0}(\tau) \in M_{k+d_0}$. Moreover, we see that $f_{d_0} \in S_{k+d_0}$ if $d_0 > 0$ since there is no constant term after the remark above. $\square$

Applications: We want to understand the structure behind dim $J_{k,m}$.
1) $J_{2,1} = \{0\}$.

2) We have: dim $J_{4,1} \leq 1$, dim $J_{6,1} \leq 1$, dim $J_{8,1} \leq 1$.
Proof:
Let $\phi(\tau, z) \in J_{2k,1}$. Then $\phi(\tau, z) = f_0(\tau) + f_2(\tau)z^2 + f_4(\tau)z^4 + \dots$
From the proposition before, $f_0(\tau) \in M_{2k}$, and after [8], we know that $M_2k = \mathbb{C}E_{2k}(\tau)$ for $2k = 4, 6, 8, 10$. If $f_0(\tau) \equiv 0$, then $f_2(\tau)$ has to be a cusp form of weight $2k+2$, but $S_{2k+2} = \{0\}$, $2k+2 \leq 10$. So for weights $2k = 2, 4, 6, 8$, $\phi(\tau, z) \in J_{2k,1}$ is determined by $f_0(\tau)$.

3) We have $\dim J_{10,1}^{\mathrm{cusp}} \leq 1, \quad \dim J_{8,2}^{\mathrm{cusp}} \leq 1, \quad \dim J_{6,3}^{\mathrm{cusp}} \leq 1$

Proof for $J^{\text{cusp}}_{6,3}$: Similarly, $\phi(\tau,z) = f_0(\tau)+f_2(\tau)z^2+f_4(\tau)z^4+f_6(\tau,z)z^6+\dots$
Since ϕ is a cusp form and $S_6 = \{0\}$, then:
$f_2 \in S_8 = \{0\} \implies f_4 \in S_{10} = \{0\} \implies f_6(\tau) \in S_{12} = \mathbb{C}\Delta(\tau)$ where $\Delta(\tau)$ is the Ramanujan Δ function. We obtain that $\phi_{6,3}$ is determined uniquely by $f_6(\tau) \implies \dim J^{\text{cusp}}_{6,3} \leq 1$.
The same argument provides the estimates for $J^{\text{cusp}}_{10,1} \leq 1$ and $J^{\text{cusp}}_{8,2}$. $\square$

In the next section, we are going to prove that $\dim J^{\text{cusp}}_{6,3} = \dim J^{\text{cusp}}_{8,2} = \dim J^{\text{cusp}}_{10,1} = 1$.

6 The Jacobi ϑ-series and Dedekind η-function

The Dedekind η function is defined as:

$$\begin{aligned} \eta : \mathbb{H}_1 &\to \mathbb{C} \\ \tau &\to \eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n) \end{aligned}$$

Its properties are:

$$\eta(\tau+1) = \eta(\tau)$$
$$\eta\left(\frac{-1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\eta(\tau)$$

where the branch of the square root is taken s.t. $\sqrt{\tau} > 0$ if $\tau \in \mathbb{R}^+$. We have also:

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = v_\eta(M)(c\tau+d)^{\frac{1}{2}}\eta(\tau), \quad \forall M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$$

where $v_\eta(M)^{24} = 1$, and $v_\eta(M_1)v_\eta(M_2) = \pm v_\eta(M_1 M_2)$ is a projective character. v_η is called a multiplier system of the modular group $SL_2(\mathbb{Z})$.
$v_\eta^2 : SL_2(\mathbb{Z}) \to \mathbb{C}$ is a character of order 12. We also have

$$\eta(\tau)^{2k} \in M_k\big(SL_2(\mathbb{Z}), (v_\eta^2)^k\big) \quad ,\forall k \geq 1$$
$$\eta(\tau)^{24} = q\prod_{n\geq 1}(1-q^n)^{24} = \Delta(\tau) \in S_{12}\big(SL_2(\mathbb{Z})\big)$$

The following two identities are due to Euler:

$$\eta(\tau) = \sum_{n\geq 1}\left(\frac{-12}{n}\right)q^{n^2/24}$$

where

$$\left(\frac{12}{n}\right) = \begin{cases} +1 & \text{if } n \equiv \pm 1[12] \\ -1 & \text{if } n \equiv \pm 5[12] \\ 0 & \text{if } (n,12) \neq 1 \end{cases}$$

Also,

$$\eta(\tau)^3 = \sum_{n\geq 1}(\frac{-4}{n})nq^{n^2/8}$$

$$\left(\frac{-4}{n}\right) = \begin{cases} \pm 1 & \text{if } n \equiv \pm 1[4] \\ 0 & \text{if } n \equiv 0[2] \end{cases}$$

The Jacobi ϑ-series is defined for $\tau \in \mathbb{H}_1, z \in \mathbb{C}$:

$$\begin{aligned} \vartheta(\tau, z) &= \sum_{n \in \mathbb{Z}, n \equiv 1[2]} (-1)^{\frac{n-1}{2}} e^{i\pi(\frac{n^2}{4}\tau + nz)} \\ &= \sum \left(\frac{-4}{n}\right) q^{n^2/8} r^{n/2}, \qquad q = e^{2i\pi\tau},\ r = e^{2i\pi z} \\ &= -q^{1/8} r^{-1/2} \prod_{n \geq 1} (1 - q^{n-1} r)(1 - q^n r^{-1})(1 - q^n) \end{aligned}$$

Proposition: $\vartheta(\tau, z)$ converges absolutely and normally on any compact set of $\mathbb{H}_1 \times \mathbb{C}$.

Proof: Let $|\Im z| < C$, $\Im \tau > \epsilon$, and n_0 such that $C - \epsilon \frac{n_0}{4} > 0$. Then

$$\begin{aligned} |e^{-i\pi(\frac{n^2}{4}\tau + nz)}| < e^{\pi n C - \pi\epsilon\frac{n^2}{4}} &= e^{\pi n(C - \epsilon\frac{n_0}{4})} e^{-\pi\epsilon\frac{n(n-n_0)}{4}} \\ &< (e^{-\tau\epsilon})^{\frac{n(n-n_0)}{4}} \end{aligned}$$

which establishes the required convergence properties. $\square$

Note that $\vartheta(\tau, -z) = -\vartheta(\tau, z)$ since $\left(\frac{-4}{-n}\right) = -\left(\frac{-4}{n}\right)$. In particular we get $\vartheta(\tau, 0) = 0$.

6.1 Quasi-periodicity of ϑ

We have $\forall \lambda, \mu \in \mathbb{Z}$,

$$\vartheta(\tau, z + \lambda\tau + \mu) = (-1)^{\lambda+\mu} e^{-i\pi(\lambda^2\tau + 2\lambda z)} \vartheta(\tau, z)$$

Note in comparison that the elliptic equation for $J_{k,m}$ carried a factor $e^{2i\pi m(\lambda^2\tau + 2\lambda z)}$. So ϑ satisfies the elliptic equation with $m = \frac{1}{2}$ apart from the extra $(-1)^{\lambda+\mu}$ factor. Since

$$\vartheta(\tau, z) = \sum_{n \in \mathbb{Z}} \left(\frac{-4}{n}\right) e^{i\pi(\frac{n^2}{4}\tau + n(z + \lambda\tau + \mu))}$$

the even n terms cancel: one can sum over $n \equiv 1[2]$, so $e^{i\pi n\mu} = (-1)^\mu$. Let $(\frac{n}{2} + \lambda)^2 = (\frac{N}{2})^2$.

$$\tau(\frac{n^2}{4} + n\lambda + \lambda^2) - \lambda^2\tau + z(n + 2\lambda) - 2\lambda z$$

So

$$(*) = \left(\sum_n \left(\frac{-4}{n}\right) e^{i\pi(\frac{n^2}{4}\tau + Nz)}\right) e^{-i\pi(\lambda^2\tau + 2Nz)}$$

We have $N = n + 2\lambda \implies \left(\frac{-4}{n}\right) = (-1)^{\lambda}\left(\frac{-4}{n+2\lambda}\right)$. For $\lambda \equiv 0[2]$, this is obvious. For $\lambda \equiv 1[2]$, this comes from the fact that $\left(\frac{-4}{n}\right)$ exchanges the values 1 and 3 mod 4. Hence

$$(*) = (-1)^{\lambda+\mu)}\vartheta(\tau, z)e^{-i\pi(\lambda^2\tau+2\lambda z)}$$

So the Jacobi ϑ-series is a Jacobi form of index $m = \frac{1}{2}$. We can define its divisor:

$$\vartheta(\tau, z) = 0 \iff z \in \mathbb{Z}\tau + \mathbb{Z}$$

The order of 0 is 1. For the proof we use the theorem about the number of zeros in a fundamental domain of ϑ.

$$\frac{\vartheta_z(\tau, z+\tau)}{\vartheta(\tau, z+\tau)} = \frac{\vartheta_z(\tau, z)}{\vartheta(\tau, z)} - 2i\pi$$

So there is a single zero in a fundamental domain of ϑ, and $\vartheta(\tau, 0) = 0 \implies z \in \mathbb{Z}\tau + \mathbb{Z}$.

Let us analyse the derivative of ϑ. We have

$$\begin{aligned}
\frac{\partial\vartheta(\tau, z)}{\partial z}_{|z=0} &= \frac{\partial}{\partial z}\left(\sum_n \left(\frac{-4}{n}\right) q^{\frac{n^2}{8}} r^{\frac{n}{2}}\right)_{|z=0}, \qquad q = e^{2i\pi\tau},\ r = e^{2i\pi z} \\
&= i\pi \sum_{n\in\mathbb{Z}} n \left(\frac{-4}{n}\right) q^{\frac{n^2}{8}} \\
&= 2i\pi\eta(\tau)^3
\end{aligned}$$

where the last equation is one of Euler's identities. So we have $\vartheta(\tau, 0) = 0$, $\partial_z\vartheta(\tau, z)_{|z=0} = 2i\pi\eta(\tau)^3$, where $\eta(\tau) \in M_{\frac{1}{2}}(SL_2(\mathbb{Z}, v_\eta)$, $v_\eta^{24} \equiv 1$.

Hence The Jacobi ϑ-series is quasiperiodic of index $m = \frac{1}{2}$. The Dedekind η-function is a modular form w.r.t. $SL_2(\mathbb{Z}$. We are going to present its quasi-periodicity in automorphic format.

Quasi-periodicity of ϑ in terms of Γ^J
For $\left(\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right) \in \mathbb{H}_2$, the $Sp_2(\mathbb{Z})$ action writes:

$$(E) \iff \left(\vartheta(\tau, z)e^{i\pi\omega}\right)_{|[\left(\begin{smallmatrix}\mu\\ -\lambda\end{smallmatrix}\right),0]} = (-1)^{\lambda+\mu+\lambda\mu}\vartheta(\tau, z)e^{i\pi\omega}$$

The last factor $\lambda\mu$ comes from the action on ω. So we have:

$$\begin{aligned}
v_H : H(\mathbb{Z} &\to \{\pm 1\} \\
v_H\left(\left[\begin{pmatrix}\mu\\ -\lambda\end{pmatrix}, r\right]\right) &= (-1)^{\lambda+\mu+\lambda\mu+r} \\
v_H(ghg^{-1}) &= v_H(h), \qquad \forall h \in H(\mathbb{Z}),\ g \in SL_2(\mathbb{Z})
\end{aligned}$$

6.2 Modularity of $\vartheta(\tau, z)$

For $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ we have:

$$\begin{array}{rrcl} (M_T): & \vartheta(\tau+1, z) & = & e^{i\frac{\pi}{4}}\vartheta(\tau, z) \\ (M_S): & \vartheta(-\dfrac{1}{\tau}, z) & = & \sqrt{\dfrac{\tau}{i}}e^{i\pi\frac{z^2}{\tau}}\vartheta(\tau, z) \end{array}$$

The last equation shows that ϑ is modular of weight $\frac{1}{2}$.
Proof: We will show these identities without using the Poisson formula. Changing z into τz, we get

$$(M_{S'}): \quad \vartheta(-\frac{1}{\tau}, z\tau) = \sqrt{\frac{\tau}{i}}e^{i\pi\tau z^2}\vartheta(\tau, z\tau) \triangleq \sqrt{\frac{\tau}{i}}\xi_\tau(z)$$

By definition, we see that the l.h.s. is a quasi-periodic function for the lattice $-\frac{1}{\tau}\mathbb{Z} + \mathbb{Z}$. We want to prove that ξ is also quasi-periodic w.r.t. the same lattice. Then, according to the theorem for the number of zeros, we will get that $\vartheta(-\frac{1}{\tau}, z) = c(\tau)\xi_\tau(z)$. First,

$$\begin{array}{rcl} \xi_\tau(z+1) & = & e^{i\pi(z+1)^2}\vartheta(\tau, z\tau+\tau) \\ & = & -e^{i\pi(z^2+2z+1)\tau}e^{-i\pi(\tau+2z\tau)}\vartheta(\tau, z\tau) \\ & = & -e^{i\pi z^2\tau}\vartheta(\tau, z\tau) \\ & = & -\xi_\tau(z) \end{array}$$

Then,

$$\begin{array}{rcl} \xi_\tau(z-\dfrac{1}{\tau}) & = & e^{i\pi(z-\frac{1}{\tau})^2\tau}\vartheta(\tau, z\tau-1) \\ & = & (-1)e^{i\pi z^2\tau}\vartheta(\tau, z\tau)e^{i\pi(-\frac{2z}{\tau}+\frac{1}{\tau^2})\tau} \\ & = & (-1)\xi_\tau(z)e^{-i\pi(-\frac{1}{\tau}+2z)} \end{array}$$

This is the same equation as for the Jacobi ϑ-series. Since $\xi_\tau(z)$ satisfies the same relations and have the same zeros, they are equal up to a constant $c(\tau)$:

$$\vartheta(-\frac{1}{\tau}, z) = c(\tau)e^{i\pi z^2\tau}\vartheta(\tau, z\tau)$$

To find $c(\tau)$ we apply $\partial_{z|z=0}$: the l.h.s. gives $2i\pi\eta(-\frac{1}{\tau})^3 = 2i\pi\left(\sqrt{\frac{\tau}{i}}\right)^3\eta(\tau)$, while the r.h.s. is $(\partial_z\vartheta(\tau, z))_{|z=0} = 2i\pi\eta(\tau)^3$. Hence $c(\tau) = -\sqrt{\frac{\tau}{i}}$. Since S and T generate the full modular group, we can state the following result:

$$\vartheta\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = v_\eta(M)^3(c\tau+d)^{\frac{1}{2}}e^{i\pi\frac{cz^2}{c\tau+d}}\vartheta(\tau, z)$$

Hence $\vartheta(\tau, z)$ is a Jacobi form of weight $\frac{1}{2}$ and index $\frac{1}{2}$ w.r.t. $\Gamma^J = SL_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$, $v_H : H(\mathbb{Z}) \to \{\pm 1\}$ and multiplier system $v_\eta^3 : SL_2(\mathbb{Z} \to \{\sqrt[8]{1}\}$:

$$\vartheta(\tau, z) \in J_{\frac{1}{2},\frac{1}{2}}(v_\eta^3 \ltimes v_H)$$

Last, we need to check the holomorphicity of $\vartheta(\tau,z)$. Let us consider its Fourier expansion:

$$\vartheta(\tau,z)=\sum_{n\in\mathbb{Z}}\left(\frac{-4}{n}\right)q^{\frac{n^2}{8}}r^{\frac{n}{2}}\in J_{\frac12,\frac12}$$

The coefficients of the expansion are $a(\frac{n^2}{8},\frac{n}{2})=\left(\frac{-4}{n}\right)$. The hyperbolic norm of the index here is $4nm-l^2=4\frac{n^2}{8}\frac{1}{2}-(\frac{n}{2})^2=0$. Hence $a(n,l)\neq 0 \implies 4nm-l^2=0$. One can remark that for a modular form $f(\tau)=\sum_{n\geq 0}a_nq^n$, only $a(0)$ verifies this property. In our case, the Jacobi ϑ-series generalises it to all coefficients.

6.3 $\vartheta(\tau,z)$ and examples of Jacobi modular forms

1) $\phi_{-2,1}(\tau,z)=\frac{\vartheta(\tau,z)^2}{\eta(\tau)^6}\in J_{-2,1}$, since it has trivial multiplier system, and $v_H^2\equiv 1$.

2) $\phi_{-1,\frac12}(\tau,z)=\frac{\vartheta(\tau,z)}{\eta(\tau)^3}\in J_{-1,\frac12}(v_H)$

Exercise: check that the first coefficient of the Fourier expansion of $\phi_{-1,\frac12}$ is $(r^{1/2}-r^{-1/2})q^0$, and that for its square $\phi_{-2,1}$ it is $(r-2+r^{-1})q^0$.

Lemma: $J_{-2,1}^{weak}=\mathbb{C}.\phi_{-2,1}$

Proof: Let $\psi_{-2,1}(\tau,z)=f_{-2}(\tau)+f_0(\tau)z^2+\ldots\in J_{-2,1}^{weak}$. f_2 must be a holomorphic modular form of weight -2, so $f_2=0$. Hence $f_0(\tau)\in M_0\left(SL_2(\mathbb{Z})\right)=\mathbb{C}$. If $f_0\equiv 0$ then $\psi_{-2,1}\equiv 0$ since otherwise it would thus have a zero of order at least 4 in 0, which is impossible given its index.

Lemma: $J_{10,1}^{cusp}=\mathbb{C}.\phi_{10,1}$, where $\phi_{10,1}=\eta(\tau)^{18}\vartheta(\tau,z)^2$.

Proof: The index of $\vartheta(\tau,z)^2\in J_{1,1}(v_\eta^6)$ is 1, its weight 1 and its character is $v_\eta: SL_2(\mathbb{Z})\to\{\sqrt[4]{1}\}$. Let $\psi_{10,1}^{cusp}(\tau,z)=f_{10}^{cusp}(\tau)+f_{12}^{cusp}(\tau)z^2+\ldots\in J_{10,1}^{cusp}$. Then f_{10} must be a cusp form of weight 10 so $f_{10}\equiv 0$. Then $f_{12}(\tau)\in S_{12}=\mathbb{C}\Delta$. Finally, if $f_{12}(\tau)\equiv 0$, then $\psi_{10,1}\equiv 0$ because it would have thus a zero of order 4 in $z=0$. $\square$

We remind that $\dim J_{8,2}^{cusp}\leq 1$ and $\dim J_{6,3}^{cusp}\leq 1$.
Let us define similarly, $\phi_{8,2}=\vartheta(\tau,z)^4\eta(\tau)^{12}=\left(\eta(\tau)^3\vartheta(\tau,z)\right)^4\in J_{8,2}^{cusp}$, and $\phi_{6,3}(\tau,z)=\left(\eta(\tau)\vartheta(\tau,z)\right)^6\in J_{6,3}^{cusp}$. Then we have also:

$$\begin{aligned}J_{8,2}^{cusp}&=\mathbb{C}.\phi_{8,2}\\ J_{6,3}^{cusp}&=\mathbb{C}.\phi_{6,3}\end{aligned}$$

Also, $\vartheta(\tau,z)^8\in J_{4,4}$ - but is not a cusp form. We will study this function later on in relation with the Jacobi Eisenstein series.

Question: Now we have just constructed 3 cusp forms $\phi_{10,1},\phi_{8,2}$ and $\phi_{6,3}$ of even weight, how can we construct a Jacobi form of odd weight ?

Let us look at the following example: $\phi_{11,2}(\tau,z)=\eta(\tau)^{21}\vartheta(\tau,2z)\in J_{11,2}^{cusp}$.

Proposition: Let $\vartheta_t(\tau,z) \triangleq \vartheta(\tau,tz)$, $t\in\mathbb{N}$. Then:
1) $\vartheta_t(\tau,z)\in J_{\frac{1}{2},\frac{t^2}{2}}(v_\eta^3 \ltimes v_H^t)$
2) $\forall\phi\in J_{k,m}$, $\phi(\tau,tz)\in J_{k,t^2m}$.

Proof: The modular equation writes

$$\vartheta_t\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=v_\eta^3(M).(c\tau+d)^{\frac{1}{2}}.e^{i\pi\frac{ct^2z^2}{c\tau+d}}.\vartheta(\tau,tz)$$

The elliptic equation is

$$\vartheta(\tau,z+\lambda\tau+\mu)=(-1)^{\lambda t+\mu t}e^{i\pi(\lambda^2t^2+2\lambda t^2 z)}\vartheta(\tau,z)$$

$\square$

We see that t^2 can be used as the index in the formulae above. Hence we can construct Jacobi forms of odd weight, and also Jacobi forms of weight 4, which will turn out to be cusp forms.

7 ϑ-blocks and Θ-quarks

If one looks back at the material we have covered so far, we have looked at Jacobi forms from two different angles:

- as functions $f(\tau,z)$ satisfying both the modular and elliptic equations - which deal with respectively the variables τ and z, and which are holomorphic at ∞,
- as Γ^J-modular forms $\phi(\tau,Z)e^{2im\pi\omega}$ defined over $\mathbb{H}_2$, where

  $$\Gamma^J\sim(SL_2(\mathbb{Z})\ltimes H(\mathbb{Z}))<Sp_2(\mathbb{Z})$$

  and Γ^J is a parabolic subgroup of $Sp_2(\mathbb{Z})$ (cf [6]).

One notes that the holomorphicity at ∞ in the first definition is equivalent in the second definition to ϕ being merely holomorphic of $\mathbb{H}_2$. The latter implies in particular that Jacobi forms are a graded ring since they are holomorphic by multiplication.

We have also later defined the Jacobi ϑ-function:

$$\vartheta(\tau,z)e^{i\pi\omega}\in J_{\frac{1}{2},\frac{1}{2}}(v_\eta^3\ltimes v_H)$$

where $v_H : H(\mathbb{Z})\to\{\pm1\}$ is defined by $v_H\left(\left[\left(\begin{smallmatrix}\mu\\-\lambda\end{smallmatrix}\right),r\right]\right)=(-1)^{\mu+\lambda+\mu\lambda+r}$ - the presence of the term $\lambda\mu$ in v_H hints at the fact that $\theta(\tau,z)$ is not a mere product of two distinct functions of respectively τ and z. Finally, $\forall g\in SL_2(\mathbb{Z})$, we have

$$v_H(ghg^{-1})=v_H(h)$$

Problem: How can we construct as many Jacobi forms as possible using only two building blocks: η and ϑ ?

We proved previously that $\vartheta_a = \vartheta(\tau, az) \in J_{\frac{1}{2},\frac{a^2}{2}}(v_\eta^3 \ltimes v_H^a)$. The most general function can be written as $\eta^d \vartheta_{a_1} \dots \vartheta_{a_k}$, with $d, a_1, \dots, a_k \in \mathbb{Z}_{>0}$. For this to be a Jacobi modular form, since the multiplier system has the form $v_\eta^{d+3k} \ltimes v_H^{a_1+\dots+a_k}$ one must have $d + 3k \equiv 0[24]$ and $a_1 + \dots + a_k \in 2\mathbb{Z}$. Note that $d + 3k \equiv 0[24] \implies d = 3e$. We are going to consider products satisfying these condition: $e + k \equiv 0[8]$ and $a_1 + \dots + a_k \in 2\mathbb{Z}$, looking for the Jacobi modular forms of minimum weight and index: we thus restrict to $e + k = 8$.

e	k	ϕ	Space $J_{k,m}$
8	0	$\eta^{24} = \Delta$	S_{12}
7	1	$\eta^{21}\vartheta_2 = \phi_{11,2}$	$J_{11,2}^{cusp} = \mathbb{C}.\phi_{11,2}$
6	2	$\eta^{18}\vartheta^2 = \phi_{10,1}$	$J_{10,1}^{cusp} = \mathbb{C}.\phi_{10,1}$
5	3	$\eta^{15}\vartheta_2\vartheta^2 = \phi_{9,3}$	$J_{9,3}^{cusp} =?$ **(exercise)**
4	4	$\eta^{12}\vartheta^4 = \phi_{8,2}$	$J_{8,2}^{cusp} = \mathbb{C}.\phi_{8,2}$
3	5	$\eta^{9}\vartheta_2\vartheta^4 = \phi_{7,4}$	$J_{7,4}^{cusp} =?$ **(exercise)**
2	6	$\eta^{6}\vartheta^6 = \phi_{6,3}$	$J_{6,3}^{cusp} = \mathbb{C}.\phi_{6,3}$
1	7	$\eta^{3}\vartheta_2\vartheta^6 = \phi_{5,5}$	$J_{5,5}^{cusp} =?$ **(exercise)**
0	8	$\vartheta^8 = \phi_{4,4}$	non-cusp, dim$J_{4,4}$=2 **(exercise)**

Question: How can one construct a cusp form of weight 4 ? How can we construct cusp forms as products $\vartheta_{a_1} \dots \vartheta_{a_k}$?

Proposition: Let $a, b \in \mathbb{Z}_{>0}$. Then

$$\vartheta_a \vartheta_b \in J^{\mathbf{cusp}}_{1,\frac{a^2+b^2}{2}}(v_\eta^6 \ltimes v_H^{a+b}) \iff \frac{ab}{(a,b)} \text{ is even.}$$

Proof:

$$\vartheta_a(\tau,z)\vartheta_b(\tau,z) = \sum_{n_1,n_2\in\mathbb{Z}} \left(\frac{-4}{n_1 n_2}\right) q^{\frac{n_1^2+n_2^2}{8}} r^{\frac{an_1+bn_2}{2}}, \text{ with } q = e^{2i\pi\tau},\ r = e^{2i\pi z}$$

Let $N = n_1 + n_2$, $l = an_1 + bn_2$. For $m = \frac{a^2+b^2}{2}$ we have

$$\begin{aligned} \mathrm{Norm}_m(\frac{N}{8}, \frac{l}{2}) &\triangleq 4.\frac{N}{8}\frac{a^2+b^2}{2} - \left(\frac{l}{2}\right)^2 \\ &= \frac{1}{4}((n_1^2+n_2^2)(a^2+b^2) - (an_1+bn_2)^2) \\ &= \frac{1}{4}(n_1 b - n_2 a)^2 \geq 0 \end{aligned}$$

Let us assume $(a, b) = 1$ and $ab \in 2\mathbb{Z}$. Since n_1, n_2 come from ϑ expansions with coefficients $\left(\frac{-4}{n}\right)$, non-zero coefficients come from $n_1 \equiv n_2 \equiv 1[2]$, hence $(n_1 b - n_2 a)^2 > 0$ and so the product above is a cusp form. $\square$

Corollary: Let $\overrightarrow{a} = (a_1, \dots, a_k)$, $a_i \in \mathbb{Z}\forall i$ and $\vartheta_{\overrightarrow{a}} = \vartheta_{a_1} \dots \vartheta_{a_k}$. Then

$\vartheta_{\overrightarrow{a}}$ is a Jacobi cusp form if at least one of the a_i is even. Moreover,

$$\begin{array}{lcl} \vartheta_{\overrightarrow{a}} \in J_{\frac{k}{2},\frac{a_1^2+\ldots+a_k^2}{2}} & \Longleftrightarrow & k \equiv 0[8], a_1, \ldots, a_k \in 2\mathbb{Z} \\ \vartheta_{\overrightarrow{a}} \in J^{\text{cusp}}_{\frac{k}{2},\frac{a_1^2+\ldots+a_k^2}{2}} & \Longleftrightarrow & k \equiv 0[8], a_1, \ldots, a_k \in 2\mathbb{Z} \text{ and } \exists i : a_i \in 2\mathbb{Z} \end{array}$$

$\square$

Example: Let $\overrightarrow{a} = (1,1,1,1,1,1,2,2) \in \mathbb{Z}^8$. Then $\vartheta_{\overrightarrow{a}} = \vartheta^6\vartheta_2^2 \in J_{4,7}^{cusp}$. This is our first cusp form coming from ϑ products.

Exercise: Show that $J_{4,6}^{cusp} = \{0\}$ (difficult).

Theorem: We have $\dim J_{2k+1,2} = \dim S_{2k+2}$. Furthermore the two spaces are isomorphic: $J_{2k+1,2} \simeq S_{2k+2}$.

Proof: We have

$$\operatorname{div}(\phi_{2k+1,m}) \supseteq \left\{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\right\} = \frac{1}{2}(\mathbb{Z}\tau + \mathbb{Z})$$

and $\phi_{2k+1,m}(\tau, 0) \in M_{2k+1}(SL_2(\mathbb{Z})) = \{0\}$.
The elliptic equation gives

$$\phi_{2k+1,m}(\tau, \tau - \frac{\tau}{2}) = e^{2i\pi m(\tau - 2\frac{\tau}{2})}\phi_{2k+1,m}(\tau, \frac{\tau}{2}) = \phi(\tau, \frac{\tau}{2})$$

However, $\phi_{2k+1,m}(\tau, -\frac{\tau}{2}) = (-1)^{2k+1}\phi_{2k+1,m}(\tau, \frac{\tau}{2})$, so $\phi(\tau, \frac{\tau}{2}) = 0$. The same argument leads to $\phi(\frac{1}{2}) = \phi(\frac{1+\tau}{2}) = 0$.

As a side result we get this **corollary:** $J_{2k+1,1}^{weak} = \{0\}$, since members of that space have exactly 2 zeros.

Let $\phi_{-1,2} = \frac{\vartheta(\tau,2z)}{\eta(\tau)^3}$. We have $\operatorname{div}\phi_{-1,2} = \frac{1}{2}(\mathbb{Z}\tau + \mathbb{Z})$. Moreover $\frac{\phi_{2k+1,2}}{\phi_{-1,2}} \in J_{2k+2,0}^{weak}$ -valid only for J^{weak} since $\phi_{-1,2} = (r + r^{-1}) + q(\ldots)$- and hence the quotient contains only positive powers of q. But we know that $J_{2k+2,0} = M_{2k+2}(SL_2(\mathbb{Z}))$, so $\phi_{2k+1,2}(\tau, z) = \phi_{-1,2}(\tau, z).f_{2k+2}(\tau)$, where $f_{2k+2} = E_{2k+2}(\tau) = 1 + q(\ldots)$ is the Eisenstein series.

So $\phi_{-1,2}E_{2k+2}(\tau) = (r - r^{-1}) + q(\ldots)$. Since $N(q^0, r^1) = 4.0.2 - 1^2 = -1$, this product is not holomorphic. However, since $\phi_{11,2} = \phi_{-1,2}\Delta$ is holomorphic,

$$\begin{aligned} \phi_{-1,2}g_{2k+2}^{cusp} &= (\phi_{-1,2}\Delta(\tau)).g_{2k-10} \\ &= \phi_{11,2}.g_{2k-10} \in J_{2k+1,2} \end{aligned}$$

$\square$

We have thus constructed an isomorphism

$$J_{2k+1,2} \xrightarrow{\partial_z\phi(\tau,z)_{|z=0}} S_{2k+2}(SL_2(\mathbb{Z}))$$

$$J_{2k+1,2} \xleftarrow{\phi_{-1,2}\cdot g_{2k+2} = \frac{\vartheta(\tau,z)}{\eta(\tau)^6}} S_{2k+2}(SL_2(\mathbb{Z}))$$

Question: We see that $\vartheta_{a_1}\vartheta_{a_2}\dots\vartheta_{a_8} \in J^{cusp}_{4,*}$. Then using products of η and ϑ functions, can we construct Jacobi forms of weight 3? of weight 2? We will see later that the answer is positive.

Θ-quarks theorem: (Gritsenko, 2002) Let $a, b \in \mathbb{Z}_{>0}$. Then

$$\Theta_{a,b} \triangleq \frac{\vartheta_a\vartheta_b\vartheta_{a+b}}{\eta} \in \ J_{1,a^2+ab+b^2}(v_\eta^8)$$

Moreover, $\Theta_{a,b} \in J^{cusp} \iff a \neq b[3]$.

Corollary:

$$\Theta_{a_1,b_1}\Theta_{a_2,b_2}\Theta_{a_3,b_3} \in J_{3,m}$$

where $m = \sum_{i=1}^3(a_i^2 + a_ib_i + b_i^2)$. It is a cusp form if $\exists i, a_i \neq b_i[3]$.

So for instance $\Theta_{1,1} = \phi_{3,9}^{noncusp}$, and $\Theta_{1,1}^2.\Theta_{1,2} = \frac{\vartheta_3.\vartheta_2^3.\vartheta^5}{\eta^3} = \phi_{3,13}$. One can prove these are respectively the first Jacobi form of weight 3 (the first cusp form of weight 3).
Proof:

$$\vartheta_{a_1}\dots\vartheta_{a_k}(\tau,z) = \sum_{\overrightarrow{n}=(n_1,\dots,n_k)\in\mathbb{Z}^k} \left(\frac{-4}{n_1\dots n_k}\right) q^{\frac{(\overrightarrow{n},\overrightarrow{n})}{8}} r^{\frac{(\overrightarrow{n},\overrightarrow{a})}{2}}$$

The index of this Jacobi form is $m = \frac{a_1^2+\dots+a_k^2}{2}$. Let $N = \frac{(\overrightarrow{n},\overrightarrow{n})}{8}$ and $\frac{l}{2} = \frac{(\overrightarrow{n},\overrightarrow{a})}{2}$. Then

$$\begin{aligned}
\mathrm{Norm}(\frac{N}{8},\frac{l}{2}) &= 4\frac{(\overrightarrow{n},\overrightarrow{n})}{8}.\frac{(\overrightarrow{a},\overrightarrow{a})}{2} - \frac{(\overrightarrow{a},\overrightarrow{n})^2}{4} \\
&= \frac{1}{4}\sum_{1\le i<j\le k}(n_ia_j - n_ja_i)^2 \geq 0
\end{aligned}$$

Let us look for the minimum value one can obtain for odd $n_1, \dots, n_k$ for $\vartheta_{\overrightarrow{a}} = \vartheta_a\vartheta_b\vartheta_{a+b}$, with $k = 3$:

$$\begin{aligned}
12\mathrm{Norm}(\frac{N}{8},\frac{l}{2}) &= ((a+2b)n_1 - (2a+b)n_2 - (a-b)n_3)^2 \\
&\qquad +2(a^2+ab+b^2)(n_1+n_2-n_3)^2 \ \textbf{(check)} \\
&\geq 2(a^2+ab+b^2) \text{ since } (n_1+n_2-n_3) \equiv 1[2]
\end{aligned}$$

To prove that $\frac{\vartheta_a\vartheta_b\vartheta_{a+b}}{\eta}$ is holomorphic, we remind that $\eta(\tau) = q^{\frac{1}{24}}\prod(1-q^n)$ so $\eta(\tau)^{-1} = q^{-\frac{1}{24}} + \dots$ We thus have to check the following Fourier coefficients:

$$\frac{\vartheta_a\vartheta_b\vartheta_{a+b}}{\eta} = \sum f(\frac{N}{8},\frac{l}{2})q^{\frac{N}{8}-\frac{1}{24}}r^{\frac{l}{2}}$$

$$\begin{aligned}
\mathrm{Norm}(\frac{N}{8}-\frac{1}{24},\frac{l}{2}) &= \mathrm{Norm}(\frac{N}{8},\frac{l}{2}) - 4.\frac{1}{24}.\frac{a^2+b^2+(a+b)^2}{2} \\
&\geq \frac{a^2+b^2+(a+b)^2}{12} - \frac{a^2+b^2+(a+b)^2}{12} \\
&\geq 0
\end{aligned}$$

If $n_3 = n_1 + n_2 - \pm 1$, then **(to check)** an equality is possible iff $a - b \equiv 0[3]$.$\square$

As an example taken from [5], $\phi_{2,37}$ is the first cusp form of weight 2:

$$\phi_{2,37}^{cusp} = \frac{\vartheta_5 \vartheta_4 \vartheta_3^2 \vartheta_2^3 \vartheta^3}{\eta^6} = \frac{\vartheta^{(10)}}{\eta^6} \in J_{2,*}$$

Another block is of type $\frac{\vartheta^{34}}{\eta^{(30)}} \in J_{2,*}$ (Gritsenko, Yumen, 2013-2014). Nils Skoruppa proved in 1995 in his PhD thesis that there are no Jacobi forms of weight 1.

One can develop the subject using not only $\frac{\vartheta}{\eta}$-quotients, but also using $\vartheta_{\frac{3}{2}}(\tau, z) = \eta(\tau) \frac{\vartheta(\tau,2z)}{\vartheta(\tau,z)} \in J_{\frac{1}{2},\frac{3}{2}}(v_\eta \ltimes v_H)$. We have in this case

$$\vartheta_{\frac{3}{2}}(\tau, z) = \sum_n \left(\frac{12}{n}\right) q^{\frac{n^2}{24}} r^{\frac{n}{2}}$$

that can be proved using ϑ and η Fourier expansions.

Problem: Generalise the table above to $\vartheta_{\frac{3}{2}}$ while working in modulo 24. This yields many interesting equations!

8 Jacobi ϑ-series and Jacobi modular forms in several variables

Remember the Fourier expansion for ϑ:

$$\begin{aligned} \vartheta(\tau, z) &= \sum_{n\in\mathbb{Z}} \left(\frac{-4}{n}\right) q^{\frac{n^2}{8}} r^{\frac{n}{2}} \\ &= \sum_{n\in\mathbb{Z}} \left(\frac{-4}{n}\right) e^{i\pi(\frac{n^2}{4}\tau + nz)}, \qquad \tau \in \mathbb{H}, z \in \mathbb{C} \end{aligned}$$

We also constructed earlier $\phi_{-2,1}(\tau, z) = \frac{\vartheta(\tau,z)^2}{\eta(\tau)^6}$ that generates $J_{-2,1}^{weak}$, and $\phi_{10,2}(\tau, z) = \eta^{18}\vartheta^2$ which generates $J_{10,2}^{cusp}$. Moreover, $\phi_{11,2}(\tau, z) = \eta(\tau)^{21}\vartheta(\tau, 2z)$ generates $J_{11,2}$. Also, $\vartheta(\tau, z)^4 \in J_{4,4}$. We will now note $\vartheta(z) = \vartheta(\tau, z)$.

Question: What can we say about $\vartheta(\tau, z) = \vartheta(z_1) \ldots \vartheta(z_8)$ with $z = (z_1, \ldots, z_8)$? We have for the modular equation:

$$\vartheta\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^4 e^{i\pi \frac{c(z_1^2 + \ldots + z_8^2}{c\tau + d}} \vartheta(\tau, z)$$

Let us note $(z, z) = z_1^2 + \ldots + z_8^2$. For $\lambda, \mu \in \mathbb{Z}^8$, we have

$$\vartheta(\tau, z + \lambda\tau + \mu) = (-1)^{\sum_i \lambda_i + \mu_i} e^{-i\pi((\lambda,\lambda)\tau + 2(\lambda,z))} \vartheta(\tau, z)$$

To obtain a trivial character, one should have $\lambda, \mu \in D_8$, where

$$D_8 = \{(x_1, \ldots, x_8) \in \mathbb{Z}^8 | x_1 + \ldots + x_8 \in 2\mathbb{Z}\} \overset{2:1}{<} \mathbb{Z}^8$$

This is the first example of a Jacobi form for the lattice D_8.

Definition: We fix an integral positive definite lattice L of rank $n_0 > 0$ ($L \simeq \mathbb{Z}^{n_0}$). We have the lattice quadratic form $(.,.) : L \times L \to \mathbb{Z}$, and $\forall l \in L$, $(l,l) = l^2 \in 2\mathbb{Z}$, and $l \neq 0 \implies (l,l) > 0$.
Let $\tau \in \mathbb{H}_1$, $z \in L \otimes \mathbb{C}$, $\dim L \otimes \mathbb{C} = n_0$. A Jacobi form of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{Z}_{>0}$ for the lattice L is a holomorphic function $\phi : \mathbb{H}_1 \times (\mathbb{C} \otimes L)$ that satisfies:

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{i\pi\frac{c(z,z)}{c\tau+d}}\phi(\tau,z), \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \qquad (M)$$

and

$$\phi(\tau, z+\lambda\tau+\mu) = e^{-i\pi((\lambda,\lambda)\tau+2(\lambda,z))}\phi(\tau,z), \quad \lambda,\mu \in L \qquad (A)$$

(A) stands for the abelian variables equation.

We remind the definition for the dual lattice:

$$L^* = \{v \in L \otimes \mathbb{Q} | \forall v \in L, (v,l) \in \mathbb{Z}\}$$

Since $\phi(\tau+1,z) = \phi(\tau,z) = \phi(\tau,z+\mu)$ $\forall \mu \in L$, we can write the Fourier expansion of $\phi(\tau,z)$:

$$\phi(\tau,z) = \sum_{n\in\mathbb{Z}, l\in L^*} a(n,l)e^{2i\pi(n\tau+(l,z))}$$

Then
$\phi(\tau,z)$ is called a weak Jacobi form if $a(n,l) \neq 0 \implies n \geq 0$,
$\phi(\tau,z)$ is called a holomorphic Jacobi form if $a(n,l) \neq 0 \implies 2nm-(l,l) \geq 0$,
$\phi(\tau,z)$ is called a parabolic Jacobi form if $a(n,l) \neq 0 \implies 2nm-(l,l) > 0$.

$$J_{k,m}^{weak} \supset J_{k,m} \supset J_{k,m}^{cusp}$$

One can prove that all these spaces are finite dimensional.

Remark: The previous definition does not make use of the index m. We recall the classical modular equation brings a factor $e^{im\pi\frac{c(z,z)}{c\tau+d}}$, and the classical elliptic equation has a factor $e^{-im\pi((\lambda,\lambda)\tau+2(\lambda,z))}$: so m can be put inside the scalar products, and thus plays only a marginal role here: we have

$$J_{k,L,m} = J_{k,L(m),1}$$

where $L(m) = L$ and $(v,v)_{L(m)} = m(v,v)_L$ consists in a simple renormalisation operation. One can check that it corresponds to holomorphicity at ∞. We will use the notation $J_{k,L(m)} \triangleq J_{k,L(m),1}$.

Hyperbolic reformulation: Let $\phi \in J_{k,L,m}$.
Then $a(n,l) \neq 0 \implies 2nm-(l,l) \geq 0$. We represent $(n,l,m) = ne+l+mf \in S = U \oplus L(-1)$, where $U = \langle e,f\rangle \simeq \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $e^2 = f^2 = 0$, $(e,f) = 1$.
The signature of S is $(1, n_0+1)$. So $2nm-(l,l) \geq 0 \iff w \in V^+(S)$, where

$V^+(S) = \{v \in S \otimes \mathbb{R} | (v, y) \geq 0\}$ is the light cone of the future (there are 2 connected components in the positive cone: **to check**). So the index of the Fourier coefficients belongs to the positive cone of the hyperbolic lattice.

Corollary: Let $\phi_i \in J_{k_i,L,m_i}$ for $i = 1, 2$. Then $\phi_1\phi_2 \in J_{k_1+k_2,L,m_1+m_2}$ since after taking the product, the sum still belongs to the same cone.

One can thus define the graded ring

$$J_{*,L,*} = \bigoplus_{k,m} J_{k,L,m}$$

Question: What is the structure of $J^{weak}_{*,L,*}$? This intervenes in particular in the theory of Frobenius varities.

Remark: In Eichler-Zagier ([5]), a Jacobi form $\phi \in J^{EZ}_{k,m}$ is defined by a scalar abelian variable and no lattice. We can cast these Jacobi forms into the lattice context by using $< 2 >= A_1 = \mathbb{Z}u$, where $(u, u) = 2$. Then

$$J^{EZ}_{k,m} = J_{k,\langle 2\rangle,m} = J_{k,\langle 2m\rangle}$$

An index m corresponds to the lattice $\langle 2m\rangle = A_1(m)$. The graded ring of Eichler and Zagier is defined as $J_{*,A_1,*} = M_*(SL_2(\mathbb{Z}).[\phi_{-2,1}, \phi_{0,1}, \phi(-1, 2)]$, where $\phi_{-1,2} = \frac{\vartheta(\tau,2z)}{\eta(\tau)^3}$. We would now like to construct new examples of Jacobi forms in several variables.

8.1 Examples

Let $D_m = \{(x_1, \ldots, x_m) \in \mathbb{Z}^m | x_1 + \ldots + x_m \equiv 0[2]\} \overset{2:1}{<} \mathbb{Z}^m$. The root system of D_m is defined as

$$R(D_m) = \{v \in D_m | (v, v) = 2\}$$

One can check that $|R(D_m)| = 2m(m - 1)$. Then

$$\phi_{-m,D_m} = \frac{\vartheta(z_1)}{\eta^3(\tau)} \cdots \frac{\vartheta(z_m)}{\eta(\tau)^3} \in J^{weak}_{-m,D_m}$$

Exercise: Analyse the dimension of this space J^{weak}_{-m,D_m}.

The function $\vartheta_{D_8}(\tau, z) = \vartheta(z_1) \ldots \vartheta(z_8) \in J_{4,D_8}$ is called the singular weight for D_8 - it is the function achieving the minimum weight for this lattice. As other examples, we have:

$$\phi_{D_7}(\tau, z) = \eta(\tau)^3\vartheta(z_1) \ldots \vartheta(z_7) \in J^{cusp}_{5,D_7}$$
$$\phi_{D_6}(\tau, z) = \eta(\tau)^6\vartheta(z_1) \ldots \vartheta(z_6) \in J^{cusp}_{6,D_6}$$

Let $A_n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{Z}^{n+1} | \sum_i x_i = 0\} \subset D_n$, and $R(A_n)$ be its root system.

Exercise: Find $|R(A_n)|$.
Then

$$\phi_{-(n+1),A_n} = \frac{\vartheta(z_1)}{\eta(\tau)^3} \cdots \frac{\vartheta(z_n)}{\eta(\tau)^3} \frac{\vartheta(-z_1 - \ldots - z_n)}{\eta(\tau)^3} \in J^{weak}_{-(n+1),A_n}$$

This is the main generator of the graded ring of weak Jacobi forms. We can also construct for the lattice A_7:

$$\phi_{4,A_7} = \vartheta(z_1) \ldots \vartheta(z_7)\vartheta(z_1 + \ldots + z_7) \in J_{4,A_7}$$

which is related to ϑ_{D_8}.

Exercise: Is ϕ_{4,A_7} a cusp form ?

Example:
E_8 is the even unimodular lattice (one has $E_8^* = E_8$). In order to prove that $\vartheta_{E_8}(\tau, z) = \sum_{v \in E_8} e^{i\pi((v,v)\tau + 2(v,z))} \in J_{4,E_8}$, we need to prove the abelian and the modular equations. The abelian equation is a simple exercise. The modular equation can be proved using the Poisson summation formula for E_8.
For a comparison with the classical theory of modular forms, we note that

$$\begin{aligned} \vartheta_{E_8}(\tau) &= \sum_{v \in E_8} e^{i\pi(v,v)\tau} \in M_4(SL_2(\mathbb{Z})) \\ &= E_4(\tau) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n \end{aligned}$$

For the Jacobi form, $\vartheta_{E_8}(\tau, z)$ gives many examples in the sense of Eichler and Zagier. Let $u \in E_8$, $(u,u) = 2m > 0$. Let $z = u.Z$, $Z \in \mathbb{C}$. Let $\vartheta_{E_8,u} \triangleq \vartheta_{E_8}(\tau, z)_{|z=u.Z}$. So $\vartheta_{E_8,u} \in J_{4,m}$, where $m = \frac{(u,u)}{2}$.

$$\begin{aligned} \vartheta_{E_8,u}(\tau, Z) &= \sum_{v \in E_8} e^{i\pi((v,v)\tau + 2(u,v)Z)} \\ &= 1 + \sum_{n \in \mathbb{Z}_{>0}} a(n,l) e^{2i\pi(n\tau + lZ)} \end{aligned}$$

where $a(n,l) = \#\{v \in E_8 | (v,v) = n, (v,u) = l\}$.

Let $r \in R(E_8)$, $(r,r) = 2$ (note that $|R(E_8)| = 240$). Then $\vartheta_{E_8,r}(\tau, Z) = 1 + q(\ldots) \in J^{EZ}_{4,1}$. Since the space $M_4(SL_2(\mathbb{Z}))$ is of dimension 1, and as the constant terms are equal, we have $\vartheta_{E_8}(\tau, 0) = E_4(\tau)$.

Proposition: $J^{EZ}_{4,1} = \mathbb{C}.\vartheta_{E_8,r}$.
Proof:
Let $\phi(\tau, z) \in J^{EZ}_{4,1}$. We have $\phi(\tau, 0) = c.E_4(\tau) \implies \phi'(\tau, z) = \phi(\tau, z) - c\phi_{E_8,r}(\tau, z)_{|z=0} \equiv 0$. But $\phi'(\tau, z) = f_6 z^2 + f_8 z^4 + \ldots$, and $f_6 \in S_6(SL_2(\mathbb{Z})) = \{0\}$. So $\mathrm{ord}_{z=0}\phi'_{4,1} \geq 4$, and using the theorem about the number of zeros of Jacobi forms, we get $\phi'_{4,1} \equiv 0$. So $\vartheta_{E_8}(\tau, z) = E_{4,1}(\tau, z)$ and its Fourier expansion is in the space of Eichler-Zagier Jacobi forms. $\square$

Fact: Let $\phi(\tau, z) \in J_{k,l}$. Then $u \in L$, $u \neq 0$ gives that $\phi_u(\tau, Z) \triangleq \phi(\tau, z)_{|z=u.Z} \in J^{EZ}_{k,\frac{(u,u)}{2}}$, and this is a holomorphic Jacobi form.

$$\phi(\tau, z) = \sum_{2n-(l,l)\geq 0, l\in L^*} e^{2i\pi(n\tau+(l,u)Z)}$$

From Eichler and Zagier, we have $4n\frac{(u,u)}{2} - (l,u)^2 \geq 0$. Since $L > 0$, we have $(l,u)^2 \leq (l,l)(u,u)$, and hence

$$2n(u,u) - (l,u)^2 \geq 2n(u,u) - (l,l)(u,u) = u^2(2n - (l,l)) \geq 0$$

So we proved that $\phi_u(\tau, z)$ is holomorphic at ∞ in the sense of Eichler and Zagier ([5]). If $(l,u)^2 < (l,l)(u,u)$ then we can get a cusp form: we will see that this can lead to a more explicit way of building the ϑ-quarks.

9 Jacobi forms in many variables and the splitting principle

9.1 Θ-quarks as pullbacks of Jacobi modular forms

We remind the previous material related to Θ-quarks. We have $\Theta_{a,b} \in J_{1,a^2+ab+b^2}(v_\eta^8)$. Using such quarks we constructed for instance $\phi_{3,13} \in J^{cusp}_{3,13}$. Also, for $\phi \in J_{k,L}$, $z \in L \otimes \mathbb{C}$, $\mathrm{rank}(L) = n_0$, then

$$\phi(\tau, z) = \sum_{n\geq 0, l\in L^*} a(n,l) e^{2i\pi(n\tau+(l,z))} \in J_{k,L}$$

ϕ is holomorphic at ∞ if $a(n,l) \neq 0 \implies 2m - (l,l) \geq 0$. Also, for $f = (n, l, 1) \in U \oplus L(-1)$, $2n - (l,l) \geq 0$ means that $f \in V^+(U \oplus L(-1))$.
In the case of $L = E_8$, we have

$$\vartheta_{E_8}(\tau, z) = \sum_{v\in E_8} e^{i\pi((v,v)\tau+2(v,z))} \in J_{4,E_8}$$

We will now consider more general functions

$$\forall u \in L : \begin{cases} J_{k,L} \longrightarrow J_{k,\frac{(u,u)}{2}}, & (u,u) = u^2 = 2m \\ \phi(\tau, z) \longrightarrow \phi(\tau, z)_{|z=Z.u} = \phi(\tau, u.Z), & Z \in \mathbb{C} \end{cases}$$

If $u = 0$, then we get $\phi(\tau, 0) \in M_k(SL_2(\mathbb{Z})$.

Let $u \in L$, $u \neq 0$, $u^\perp = M < L$. Moreover,

$$M \oplus \langle u \rangle \;<\; L \;<\; L^* \;<\; M^* \oplus \langle u^* \rangle$$

and $\langle u^* \rangle = \mathbb{Z}\frac{u}{(u,u)}$. Let us take $z \in L \otimes \mathbb{C}$, then $z = z_M + Z.u$, $Z \in \mathbb{C}$. The pullback we want to analyse is $\phi(\tau, z_M + Z.u)_{|z=0} \triangleq \phi_u(\tau, z_M) \in J_{k,M}$, where $M = u_L^\perp$.
To analyse the Fourier expansion of $z\phi_u(\tau, z_M)$ we write $l \in L^*$ as $l = l_m + l'$, $l_m \in M^*$, $l' \in \langle u^* \rangle$:

$$l \in L^* < M^* \oplus \mathbb{Z}\frac{u}{(u,u)} \implies l = l_m + (l,u)\frac{u}{(u,u)} \triangleq l_m + l_u$$

We now consider the Fourier expansion at the point $z = 0$.

$$\begin{aligned}\phi_u(\tau, z_M) &= \sum_{\substack{n\in\mathbb{Z}_{\geq 0}\\ l_n\in M^*}} \sum_{\substack{l_u\in<u>^*\\ 2n=(l_m,l_m)\geq(l_u,l_u)}} a(n, l_M \dot{+} l_u) e^{2i\pi(n\tau+(l_n,z_M))}\\ &= \sum a_u(n, l_M) e^{2i\pi(n\tau+(l_n,z_M))}\end{aligned}$$

Where we emphasized the direct sum behind the indices using the sign $\dot{+}$. It might happen that $a(n, l_M + l_u) \neq 0 \implies l_u \neq 0$. In this case, $\phi_u(\tau, z_M) \in J^{cusp}_{k,m}$.

Problem: Let us define $\mathrm{ord}_\infty \phi(\tau, z) = \min_{a(n,l)\neq 0}\{2n - (l, l)\} \geq 0$ since $\phi \in J_{k,l}$, and $\mathrm{ord}_\infty \phi > 0$ if it is a Jacobi cusp form. How can we estimate $\mathrm{ord}_\infty \vartheta_{D_m}(\tau, z)_{|u^\perp}$ for $u \in D_m$?
Let $u = 2(b_1, \ldots, b_m)$, $(b_1, \ldots, b_m) = 1$ and $b_1 + \ldots + b_m \equiv 1[2]$ (in this case we call u a primitive vector of D_m). By definition,

$$\begin{aligned}\vartheta_{D_m}(\tau, z) &= \vartheta(z_1)\ldots\vartheta(z_m)\\ &= \sum_{n_1,\ldots,n_m} \left(\frac{-4}{n_1\ldots n_m}\right) e^{2i\pi(\frac{n_1^2+\ldots+n_m^2}{8}\tau+\frac{n_1}{2}z_1+\ldots+\frac{n_m}{2}z_m)}\end{aligned}$$

Let $l_i = \frac{n_i}{2}$. Then

$$\begin{aligned}\vartheta_{D_m}(\tau, z) &= \sum_{n_1,\ldots,n_m} \left(\frac{-4}{n_1\ldots n_m}\right) e^{i\pi((l_1^2+\ldots+l_m^2+2\sum_i z_i l_i)}\\ &= \sum_{l_1,\ldots,l_m\in D_m^*\subset\frac{1}{2}\mathbb{Z}^m} \left(\frac{-4}{(2l_1)\ldots(2l_m)}\right) e^{i\pi((l,l)+2(l,z))}\end{aligned}$$

We can compare this equation with the series corresponding to $\vartheta_{E_8}(\tau, z)$: we find the same terms, with the additional presence in the case of D_m of the character. For $\vartheta_{D_m|u^\perp}$ we have $2(l, z) = 2(l_M + (l, u)\frac{u}{(u,u)}, z_M + Z.u)$, and $Z = 0$. We have $(l, u) = (2l_1b_1 + \ldots + 2l_mb_m)$. But $2l_i \equiv 1[2]$ because of the presence of the character in the expansion. So $(l, u) = 0$.

So we see that for ϑ_{D_m}, $a_u(n, l_M) \neq 0 \implies 2n - (l_M, l_M) \geq (l_u, l_u)$. Since $l_u = (l, u)\frac{u}{(u,u)}$, $(l_u, l_u) \geq (\frac{u}{(u,u)})^2 = \frac{1}{(u,u)}$. We have thus proved the following theorem:

Theorem: Let $u = 2(b_1, \ldots, b_m) \in D_m$, u a primitive vector, then

$$\mathrm{ord}_\infty \vartheta_{D_m|u^\perp} \geq \frac{1}{(u, u)}$$

Example:
Let $u = 2(1, 1, 1) \in D_3$, $u^\perp_{D_3} = A_2$. Then $\vartheta_{D_m|u^\perp} = \vartheta(z_1)\vartheta(z_2)\vartheta(-(z_1 + z_2))$. Then $\mathrm{ord}_\infty \vartheta_{D_m|u^\perp} \geq \frac{1}{12}$. Since we also have $\mathrm{ord}_\infty \eta(\tau) = 2.\frac{1}{24} = \frac{1}{12}$, then

$$\frac{\vartheta(z_1)\vartheta(z_2)\vartheta(-(z_1 + z_2))}{\eta(\tau)} = \Theta_{A_2}(\tau, z_1, z_2) \in J_{1,A_2}(v_\eta^8)$$

Note: this is the denominator function of the affine Lie algebra A_2. We remind the definition of the Θ-quark for $(a,b,-(a+b)) \in A_2$:

$$\Theta_{a,b}(\tau,z) = \frac{\vartheta(\tau,az)\vartheta(\tau,bz)\vartheta(\tau,-(a+b)z)}{\eta(\tau)}$$

This is a conceptual explanation of the reason why Θ-quarks are holomorphic Jacobi forms.

Exercise: Find (a,b) such that $\Theta_{a,b}(\tau,z) \in J^{cusp}_{1,a^2+ab+b^2}(v_\eta^8)$.

To get a trivial character from the formula above, one can consider the product of 3 Θ-quarks:

$$\Theta_{3A_2} \triangleq \Theta_{A_2}(\tau,z_1,z_2)\Theta_{A_2}(\tau,z_3,z_4)\Theta_{A_2}(z_5,z_6) \in J_{3,3A_2}$$

, where the lattice is defined as $3A_2 = A_2 \oplus A_2 \oplus A_2$.

We want now to develop the general theory of Jacobi ϑ-functions in many variables. We will see that some constructions are simpler and clearer in the general case than for forms in a single variable.

9.2 The splitting principle

We want to represent $\phi(\tau,z) \in J^{weak}_{k,L}$ as

$$\phi(\tau,z) = \sum_{h\in L^*/L} f_h(\tau).\vartheta_{L,h}(\tau,z) = \vec{f}(\tau).\vec{\vartheta}_{L,h}$$

Remarks:

- $|L^*/L| = \det(L)$,
- $\vartheta_{L,h} = \sum_{v\in L} e^{i\pi((v+h)^2\tau+2(v+h,z))}$. If $L^* = L$, then $L^*/L = \{0\}$, and $\phi_h(\tau,z) = \sum_{v\in L} e^{i\pi((v,v)+2(v,z))}$

To prove the splitting principle, we have to reorganize the Fourier expansion of the Jacobi ϑ-series: $\forall\lambda,\mu \in L$,

$$\begin{aligned} \phi(\tau,z) &= \phi(\tau,z+\lambda\tau+\mu)e^{2i\pi(\frac{1}{2}\lambda^2\tau+(\lambda,z))} \\ &= \sum_{n,l} a(n,l)e^{2i\pi\left(n\tau+(l,z+\lambda\tau+\mu)+\frac{1}{2}\lambda^2\tau+(\lambda,z)\right)} \\ &= \sum_{n,l} a(n,l)e^{2i\pi\left(\tau(n+(l,\tau)+\frac{1}{2}(\lambda,\lambda))+(z,l+\lambda)\right)} \end{aligned}$$

So $a(n,l) = a(n+(l,\lambda)+\frac{1}{2}\lambda^2, l+\lambda)$.
Note that the hyperbolic norm of $(n+(l,\lambda)+\frac{1}{2}\lambda^2, l+\lambda)$ is $2n-(l,l)$.

Proposition: The Fourier coefficient $a(n,l)$ depends only on the hyperbolic norm $2n-(l,l)$ of the indices and on the class of l mod L. This is true for weak, holomorphic or cusp Jacobi forms.

Let $N = \frac{1}{2}(2n - (l,l)) = n - \frac{l^2}{2}$. Then $a(n,l) = a(N + \frac{l^2}{2}, l)$, where l is considered modulo L. Let h be defined modulo L. Then $\forall v \in L$, $a(N + \frac{h^2}{2}, h) = a(N + \frac{(h+v)^2}{2}, h+v)$. So

$$\begin{aligned}
\phi(\tau, z) &= \sum_{h \in L^*/L} \sum_{\substack{N \in \mathbb{Q} \\ N \equiv -\frac{h^2}{2}[1]}} a(N + \frac{h^2}{2}, h) \sum_{v \in L} e^{2i\pi(N\tau + \frac{(h+v)^2}{2}\tau + (h+v,z))} \\
&= \sum_{h \in L^*/L} \sum_{\substack{N \in \mathbb{Q} \\ N \equiv -\frac{h^2}{2}[1]}} a(N + \frac{h^2}{2}, h) e^{2i\pi N\tau} \sum_{v \in L} e^{2i\pi(\frac{(h+v)^2}{2}\tau + (h+v,z))} \\
&= \sum_{h \in L^*/L} \sum_{\substack{N \in \mathbb{Q} \\ N \equiv -\frac{h^2}{2}[1]}} a(N + \frac{h^2}{2}, h) e^{2i\pi N\tau} \vartheta_{L,h}(\tau, z) \\
&= \sum_{h \in L^*/L} f_h(\tau) \vartheta_{L,h}(\tau, z)
\end{aligned}$$

And as announced we get a sum of ϑ-functions, defined over the lattice L with characteristic h with

$$\vartheta_{L,h}(\tau, z) = \sum_{v \in L} e^{2i\pi(\frac{(h+v)^2}{2}\tau + (h+v,z))}$$

and

$$f_h(\tau) = \sum_{\substack{N \in \mathbb{Q} \\ N \equiv -\frac{h^2}{2}[1]}} a(N + \frac{h^2}{2}, h) e^{2i\pi N\tau}$$

We have thus just proved the following result:

Theorem: For any $\phi \in J_{k,L}^{weak}$,

$$\phi(\tau, z) = \sum_{h \in L^*/L} f_h(\tau) \vartheta_{L,h}(\tau, z)$$

Let us now consider the non-zero coefficients Fourier expansion of f_h: for $\phi \in J_{k,L}$ (resp. $\phi \in J_{k,L}^{cusp}$), $a(n,l) \neq 0 \implies 2n - l^2 = 2N \geq 0$, and so ϕ is holomorphic at $i\infty$ (resp. $\lim_{\tau \to i\infty} f_h(\tau) = 0$).

The weak forms case is the following: let $\bar{h} \in L^*/L$ and $|\bar{h}|^2 = \min_{v \in \bar{h}}(v,v)$. Let $\mu(L) \triangleq \max_{\bar{h} \in L^*/L} |\bar{h}|^2 = \max_{\bar{h} \in L^*/L} (\min_{v \in \bar{h}}(v,v))$. We have $a(n,l) \neq 0 \implies n \geq 0$. But $n = N + \frac{l^2}{2} \geq 0$. So $a(N + \frac{h^2}{2}, h) \neq 0 \implies N \geq -\frac{h^2}{2} = -\frac{1}{2}\mu(N)$.

Proposition: Let $\phi \in J_{k,L}^{weak}$. Then $a(n,l) \neq 0 \implies 2n - (l,l) \geq -\mu(L)$.

Example: Let $\phi \in J_{k,m}^{EZ,\,weak} = J_{k,\langle 2m \rangle}^{weak}$ where $\langle 2m \rangle = \mathbb{Z}u$, $u^2 = 2m$. Then $\mu(\langle 2m \rangle) = \left\{ \frac{x^2}{2m} \right\} \implies \mu(\langle 2m \rangle) \geq -\frac{m}{2}$.

So $a(u, x\frac{u}{(u,u)}) \neq 0 \implies 2n - \frac{x^2}{2m} \geq -\frac{m}{2} \implies 4nm - x^2 \geq -m^2$: this important result is proved in the book of Eichler and Zagier [5] page 105. Some applications can be found for instance in the theory of automorphic Borcherds products.

Now, we know that $J_{k,m}^{EZ} \subset J_{k,m}^{EZ,\,weak}$. For the space $J_{k,m}^{EZ}$, we have $a(n,l) \neq 0 \implies 4nm - l^2 \geq 0$, and for $J_{k,m}^{weak}$, $a(n,l) \neq 0 \implies 4nm - l^2 \geq -m^2$.

Let us consider now the case of unimodular lattices.

Example:
Let $L^* = L$, $L^*/L = \{0\}$. The splitting principle then gives for $\phi \in J_{k,l}^{weak}$:

$$\phi(\tau, z) = f_0(\tau)\vartheta_L(\tau, z)$$

where $f_0(\tau) = \sum_{n \in \mathbb{Z}_{\geq 0}} a(n, 0)e^{2i\pi n\tau}$. In this case, there are no differences between weak and holomorphic forms: $\mu(L) = 0 \implies J_{k,l}^{weak} \equiv J_{k,l}$ if L is unimodular.
Moreover, $\phi(\tau, 0) = f_0(\tau)\vartheta_L(\tau, 0)$, where $\vartheta_L(\tau, 0) = \sum_{v \in L} e^{i\pi(v,v)\tau} \in M_{\frac{n_0}{2}}(SL_2(\mathbb{Z}))$, $L \simeq \mathbb{Z}^{n_0}$. So we have proved the following proposition:

Proposition: Let $L = L^*$. Then

$$J_{k,L}^{weak} = J_{k,L} \simeq M_{k-\frac{n_0}{2}}(SL_2(\mathbb{Z}))$$

Also, we have

$$J_{k,L}^{cusp} \simeq S_{k-\frac{n_0}{2}}(SL_2(\mathbb{Z}))$$

Since in the last case, the Fourier expansion of ϑ_L starts as $1 + q(\ldots)$.

Problem: Define the algebraic structure of the bigraded ring

$$J_{*,E_8,*}^{weak,\,W(E_8)} \bigoplus_{k,m \in \mathbb{Z}} J_{k,E_8,m}^{weak}$$

where $W(E_8)$ is the Weyl group generated by the reflections. This problem is much simpler in the case where the lattice is D_8 using the arguments above than for E_8.

In the next section, we will look more in depth at the splitting we proved:

$$\phi(\tau, z) = (\vec{f}_h(\tau))_{h \in L^*/L} . (\vec{\vartheta}_{L,h})_{h \in L^*/L}$$

The vector of ϑ-series has a very well-defined behaviour w.r.t. the modular transformations: it is called the Weil representation. This means in turn that $\vec{f}(\tau)$ is a vector-valued modular form defined by this Weil representation.

10 The Weil representation

As discussed above, this representation is closely related to the splitting principle. Let $\phi \in J_{k,L}$. We have

$$\phi(\tau, z) = \sum_{h \in L^*/L} f)h(\tau)\vartheta_{k,h}(\tau, z) = \Phi_L(\tau).\Theta(\tau, z)$$

where $\Phi_L(\tau) = (f_h)_{h\in L^*/L}$ and $\Theta_L(\tau,z) = (\vartheta_{L,h}(\tau,z))_{h\in L^*/L}$. Moreover,

$$f_h(\tau) = \sum_{\substack{N\in\mathbb{Q}_{\geq 0}\\ N\equiv -\frac{h^2}{2}[\mathbb{Z}]}} a(N+\frac{(h,h)}{2},h)e^{2i\pi N\tau}$$

N is non-negative since $2N = 2(N+\frac{h^2}{2}) - h^2 \geq 0$ and ϕ is holomorphic. Note that h^2 is well defined modulo $2\mathbb{Z}$

10.1 The discriminant group of L

We define $D(L) = L^*/L$. We have $|D(L)| = \det L$.
For $h \in L^*$, $(h+v,h+v) = h^2 + 2(h,v) + (v,v)$, and since the lattice is even, the class modulo 2 is well-defined. Moreover, $\forall h, g \in L^*, u, v \in L$ we have

$$\begin{aligned}(h+v,g+u) &= (h,g)+(v,g)+(h,u)+(v,u)\\ &= (h,g)[\mathbb{Z}]\end{aligned}$$

So on the discriminant group, we have a symmetric bilinear pairing:

$$(.,.):\quad D(L)\times D(L)\to\mathbb{Q}/\mathbb{Z}$$

This pairing is induced by the quadratic form over the lattice. Moreove,r $\forall\bar{h}\in D(L)$, $(\bar{h},\bar{h})\in\mathbb{Q}/2\mathbb{Z}$. So $N\equiv -\frac{h^2}{2}[\mathbb{Z}]$ is well defined: the sum from the splitting principle takes place over the classes of the discriminant group.

10.2 Modular properties of $\Theta_L(\tau,z)$

One has

$$\begin{aligned}\vartheta_{L,h}(\tau+1,z) &= \sum_{v\in L} e^{i\pi((v+h,v+h)(\tau+1)+2(v+h,z))}\\ &= e^{i\pi(h,h)}\vartheta_{L,h}(\tau,z)\end{aligned}$$

So $\vartheta_{L,h}(\tau+1,z) = U(T)\vartheta_{L,h}(\tau,z)$, where $U(T) = \mathrm{diag}(e^{i\pi(h,h)})_{h\in L^*/L}$, and $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. We also have (the proof relies on Poisson summation):

$$\Theta_L\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = \tau^{\frac{n_0}{2}}(-i)^{\frac{n_0}{2}}(\det L)^{-\frac{1}{2}}\left(e^{2i\pi(g,h)}\right)_{g,h\in L^*/L} e^{i\pi\frac{(z,z)}{\tau}}\Theta_L(\tau,z)$$

So Θ_L transforms as a Jacobi form of weight $\frac{n_0}{2}$, but is vector-valued. with

$$U(S) = (-i)^{\frac{n_0}{2}}(\det L)^{-\frac{1}{2}}\left(e^{2i\pi(g,h)}\right)_{g,h\in L^*/L}\qquad S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$$

One can check that both $U(S)$ and $U(T)$ are unitary matrices. Since $\langle S,T\rangle = SL_2(\mathbb{Z})$, we have:

$$\forall M\in SL_2(\mathbb{Z}),\quad \Theta_L(M\langle\tau\rangle,\frac{z}{c\tau+d}) = (c\tau+d)^{\frac{n_0}{2}}U(M)\Theta_L(\tau,z)$$

where $U(M)$ is a unitary matrix.
In the case where n_0 is even, $U: SL_2(\mathbb{Z})\to U(\mathbb{C}^{\det L})$. If $n_0\equiv 1[2]$, the result

remains true, but U is defined from $\widetilde{SL_2(\mathbb{Z})}$ which is the double cover of $SL_2(\mathbb{Z})$. In both cases, the representation is called the Weil representation of the lattice L. For odd n_0, we need to consider the double cover since the choice of the square root will determine a projective representation. We summarize our results:

Theorem: Let $\phi \in J_{k,L}$. Then $\phi(\tau, z) = \Phi_L(\tau).\Theta_L(\tau, z)$ and $\Phi(\tau)$ is a vector-valued modular form with respect to the full modular group $SL_2(\mathbb{Z})$:

$$\Phi_L\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{k-\frac{n_0}{2}}.\overline{U(M)}.\Phi_L(\tau), \quad \forall M \in SL_2(\mathbb{Z})$$

This comes from the fact that $U(M)$ is unitary. Moreover, the corresponding vector-valued modular form is holomorphic at $i\infty$:

$$f_h(\tau) = \sum_{\substack{N\in\mathbb{Q}_{\geq 0}\\ h\in L^*/L}} a(N + \frac{h^2}{2}, h)e^{2i\pi N\tau}$$

and is parabolic (with $\sum_{N\in\mathbb{Q}_{>0}}$ if $\phi \in J^{cusp}_{k,L}$, which results from the splitting principle and the fact that the vector-valued ϑ-series of all Jacobi theta-characteristics is a vector-valued modular form.

Corollary 1: $J_{k,L}$ depends only on the weight k and the discriminant group $D(L) = L^*/L$:

$$J_{k_1,L_1} \simeq J_{k_1+\frac{\mathrm{rank}L_2-\mathrm{rank}L_2}{2},L_2} \quad \text{if } D(L_1) \simeq D(L_2)$$

Corollary 2: Let $L^* = L$. Then $J^{(cusp)}_{k,L} \simeq M^{(cusp)}_{k-\frac{\mathrm{rank}L}{2}}(SL_2(\mathbb{Z})$.

We now want to analyse the minimum weight of Jacobi forms. We have:

Corollary 3: Let $J_{k,L} \neq \{0\}$. Then $k \geq \frac{\mathrm{rank}L}{2} = \frac{n_0}{2}$.
$k = \frac{n_0}{2}$ is called the singular weight.

Examples:

$$\vartheta_{D_8}(\tau, z) = \vartheta(z_1)\ldots\vartheta(z_8) \in J_{4,D_8}$$
$$\vartheta_{E_8}(\tau, z) = 1 + \sum_{v\in E_8^*} e^{i\pi((v,v)\tau+2(v,z))} \in J_{4,E_8}$$

Since $D_8 \subset E_8$, $\vartheta_{E_8}(\tau, z) \in J_{4,D_8}$. So $\dim J_{4,D_8} \geq 2$ (in fact, it is equal to 2). How can one find all the Jacobi forms of singular weight for a lattice L? We know that $\phi \in J_{k,L} \implies \phi(\tau, z) = \Phi_L(\tau)\Theta_L(\tau, z)$, and $\Phi_L(\tau)$ is a holomorphic modular form. If Φ_L has a positive weight, then $k \geq \frac{n_0}{2}$. If $k = \frac{n_0}{2}$, then $\Phi_L(\tau) \equiv C$ is a constant. To find $\Phi_L(\tau)$ of weight 0, we need to analyse $U(T)$ and $U(S)$. One can for instance go to proving that dimension of the Jacobi forms of singular weight for D_8 is 2.

10.3 Jacobi forms of singular weight for D_8 (or D_m)

We remind that $D_m = \{(x_1, \ldots, x_m) \in \mathbb{Z}^m | x_1 + \ldots + x_m \in 2\mathbb{Z}\}$
Exercise: Show that $D_m^*/D_m = \{0, e_1, \frac{\pm e_1 + e_2 + \ldots + e_m}{2}\} \triangleq \{h_0, h_1, h_2^+, h_3^-\}$

We have

$$U(T) = \operatorname{diag}(e^{i\pi(h,h)})_{h \in D(L)} = \operatorname{diag}(1, -1, 1, 1) \quad \text{in the case of } D_8$$

$$U(S) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

where $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ are common eigenvectors of both $U(S)$ and $U(T)$, so there are 2 singular weight Jacobi forms:

$$\begin{aligned} \phi_{D_8}(\tau, z) &= \vartheta_{D_8, h_0} + \vartheta_{D_8, h_2} \\ \psi_{D_8}(\tau, z) &= \vartheta_{D_8, h_2} - \vartheta_{D_8, h_3} \end{aligned}$$

By looking at the first Fourier coefficient, we can see that $\phi_{D_8}(\tau, z) = \vartheta_{E_8}(\tau, z)$ and that $\phi_{D_8}(\tau, z) = \pm^? \vartheta_{D_8}(\tau, z)$.

We can analyse any lattice with the same method: one first needs to find the matrices of the Weil representation for T and S, and then find common eigenvectors. However, common eigenvectors can be associated to potentially different eigenvalues: this means that in some cases we get Jacobi modular forms of singular weight with a character.

We have encountered this situation in the case of $A_1 = \langle 2 \rangle$ which led to the classical Eichler-Zagier Jacobi forms:

$$\vartheta(\tau, z) \in J_{\frac{1}{2}, \frac{1}{2}}(v_\eta^3 \ltimes v_H)$$

and

$$\vartheta_{\frac{3}{2}}(\tau, z) \in J_{\frac{1}{2}, \frac{3}{2}}(v_\eta \ltimes v_H), \quad \vartheta_{\frac{3}{2}}(\tau, z) = \eta(\tau) \frac{\vartheta(\tau, 2z)}{\vartheta(\tau, z)}$$

where $\vartheta_{\frac{3}{2}}$ is called the "quintuple product". These are the only Jacobi forms of singular weight for the full modular group.

We now consider the next lowest possible weight. We fix L of rank n_0. $k = \frac{n_0}{2}$ is called singular, the next possible weight is $k = \frac{n_0}{2} + \frac{1}{2}$ is called the critical weight. It is in particular the first possible weight for parabolic Jacobi forms. For $\phi_{\frac{n_0}{2}, L}(\tau, z) = \sum h \in L^*/L c_n \vartheta_{L,h}(\tau, z)$, we have $a(n, l) \neq 0 \implies 2n - (l, l) = 0$, which is the main property of Jacobi forms of singular weights: this function in particular is never parabolic. The critical weight is the first where one can hope to find them.
A theorem by Nils Skoruppa (1986) in his PhD thesis shows however that $J_{1,m}^{EZ} = \{0\}$.

10.4 Jacobi modular (cusp) forms of critical weight

Example:
Let $v = (1, \ldots, 1) \in D_8$. Then

$$\vartheta_{D_8|v^\perp} = \vartheta(z_1)\ldots\vartheta(z_7)\vartheta(-(z_1 + \ldots + z_7)) \in J_{4,A7}$$

The critical weight for A_7 is 4.

Exercise: is $\vartheta_{D_8|v^\perp}$ a cusp form ?

Example:
Let $u = 2(1, \ldots, 1, 2) \in D_8$, u is primitive. We remind that $\mathrm{ord}_\infty\phi(\tau, z) = \min_{a(n,l)\neq 0}\{2n - (l, l)\}$. Then $\mathrm{ord}_\infty(\vartheta_{D_8|u^\perp}) \geq \frac{1}{(u,u)} = \frac{1}{4(7+4)} = \frac{1}{44} > 0$. So $\vartheta_{D_8|u^\perp} \in J^{cusp}_{4,u^\perp}$, $\mathrm{rank} u^\perp = 7$ and $\vartheta_{D_8|u^\perp} \neq 0$.

10.5 Dimension formula for $J_{k,L}$

From the splitting principle we have $\dim J_{k,L}$ $=\dim \Phi_{k-\frac{n_0}{2}}(\tau), \overline{U}\}$, where $\overline{U}$ is the conjugate of the Weil representation. The reader is referred to Freitag's paper for the formula, which is derived using the Riemann-Roch formula. The list of small weights for lattices of rank n_0 is

- $k = \frac{n_0}{2}$: the singular weight,
- $k = \frac{n_0+1}{2}$: the critical weight,
- $k = n_0 + 2$: the canonical weight.

The ideas behind are that if $n_0 = 1$, one can lift $\phi_{k,m}$ into $\mathrm{Lift}(\phi_{k,m})$ on $\mathbb{H}_2$. This lifting construction gives Siegel modular forms, and the existence of cusp forms in $J^{cusp}_{3,m}$ is shown to lead to the non-vanishing of the cohomology groups $H_3(\Gamma_m, \mathbb{C})$.

Problem: How can we analyse $f_h(\tau)$ as modular forms with respect to congruence subgroups ?

Definition: The level of the lattice L is $\min\{q > 0 | q.L^* \subset L\}$.
Since $|L^*/L| = \det L$, then $q | 2\det L$.

We remind $\Gamma_0(q) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) |\, c \equiv 0[q]\right\}$.
$\forall h \in L^*/L$ and $\forall M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ we have

$$f_h\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{k-\frac{n_0}{2}}\chi_L^{(8)}(M)e^{i\pi ab(h,h)}f_{ah}(\tau)$$

if $n_0 \equiv 0[2]$, then $\chi(M)$ is not of order 8, but then

$$\chi_L(M) = (\mathrm{sign}(d))^{\frac{d_0}{2}}\left(\frac{(-1)^{\frac{n_0}{2}}.\det L}{|d|}\right)$$

If $L^*/L = C_p$ is a constant, p a prime the orbit is a cyclic group. We then have a chance to write all the $f_h(\tau)$ in terms of $f_0(\tau)$. If this is true, then we can prove that

$$J_{k,L} \simeq \vec{\Phi}_{k-\frac{n_0}{2},L}(\tau) \simeq \mathcal{M} \subset M_{k-\frac{n_0}{2}}(\Gamma_0(q))$$

This is described in [5] in relation with the Jacobi forms of index 1, $p = 2$ in this case. In a theorem by [5] we can see that the main objects for $J_{k,L}$ are the discriminant group $D(L)$ and the orthogonal group $O(D(L))$. So Jacobi modular forms and vector-valued Jacobi forms are equivalent objects, but sometimes the Jacobi modular forms have nice features available such as their ring structure etc.

11 Automorphic correction of Jacobi forms

This construction works well in the case of Eichler-Zagier, single variable Jacobi forms, but also in the case of several variables.

Theorem: (from Eichler-Zagier [5] -3)
Let $\phi(\tau, z) \in J^{weak}_{k,m}$ have Taylor expansion $\phi(\tau, z) = \sum_{n\geq 0} f_n(\tau)z^n$. Then the $f_n(\tau)$ are quasi-modular forms of weight $n + k$.

We will prove the theorem and provide definitions in the rest of the section.

11.1 Modular differential operators

We define the operator D as:

$$D = \frac{1}{2i\pi}\frac{d}{d\tau} = q\frac{d}{dq} \quad : \quad \sum_n a(n)q^n \to \sum_n na(n)q^n$$

This operator keeps holomorphicity, but in general does not preserve modularity. It does however in one special case: let $f_0 \in M^{weak}_0(SL_2(\mathbb{Z}))$, $f_0(\tau) = \sum_{n\geq -N} a(n)q^n$. The modularity of f_0 writes

$$f_0\left(\frac{a\tau + b}{c\tau + d}\right) = f_0(\tau), \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$$

We apply D to this equation:

$$(c\tau + d)^{-2}(Df_0)\left(\frac{a\tau + b}{c\tau + d}\right) = (Df_0)(\tau)$$

this implies that D sends $M^{weak}_0(SL_2(\mathbb{Z}))$ into $M^{weak}_2(SL_2(\mathbb{Z}))$. For the case of weight k, we fix $f(\tau) \in M_k(SL_2(\mathbb{Z}))$. Then

$$\begin{aligned} D_k f(\tau) &\triangleq \eta^{2k} D\left(\frac{f(\tau)}{\eta^{2k}(\tau)}\right) \\ &= \eta^{2k}\frac{(Df)\eta^{2k} - 2kf\eta^{2k-1}D(\eta)}{\eta^{4k}} \\ &= (Df)(\tau) - 2k\frac{D(\eta)}{\eta}f(\tau) \end{aligned}$$

We have $D\left(\frac{f}{\eta^{2k}}\right) \in M_2^{weak}\left(SL_2(\mathbb{Z}), v_\eta^{-2k}\right)$. The multiplication by η^{2k} cancels the character and brings the result into $M_{2k+2}^{weak}\left(SL_2(\mathbb{Z})\right)$.

To get a better formula we need to analyze the logarithmic derivative of the Dedekind η-function $\frac{D(\eta)}{\eta}$. We remind $\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n) \in M_{\frac{1}{2}}\left(SL_2(\mathbb{Z}, v_\eta)\right)$, with $v_\eta^{24} \equiv 1$. We have

$$\begin{aligned}
\frac{D(\eta)}{\eta} &= \frac{1}{24} + \sum_{n\geq 1} \frac{-nq^n}{1-q^n} \\
&= \frac{1}{24} - \sum_{n\geq 1} nq^n \left(\sum_{m\geq 0} q^{mn}\right) \\
&= \frac{1}{24} - \sum_{m,n\geq 1} nq^{mn} \\
&= \frac{1}{24} - \sum_{N\geq 1} \left(\sum_{d|N} d\right) q^N \\
&= \frac{1}{24} - \sum_{N\geq 1} \sigma_1(N) q^N, \text{ where } \sigma_k(n) = \sum_{d|n} d^k
\end{aligned}$$

The result looks very similar to the Eisenstein series we have already encountered, for instance we recall the Fourier expansion for the Eisenstein series E_4:

$$E_4(\tau) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n \in M_4(SL_2(\mathbb{Z}))$$

Let $G_2(\tau) \triangleq -\frac{D(\eta)(\tau)}{\eta(\tau)} = -\frac{1}{24} + \sum_{n\geq 1} \sigma_1(n) q^n$. Then G_2 is quasi-modular (see later for the proof). We have

$$\begin{aligned}
D_k(f) &= D(f) + 2kG_2(\tau) f \\
&= \sum_{n\geq 1} na(n) q^n + 2k \left(-\frac{1}{24} + \sum_{n\geq 1} \sigma_1(n) q^n\right) \left(\sum_{n\geq 0} a(n) q^n\right)
\end{aligned}$$

and $D_k(f)$ is holomorphic at $i\infty$, since there are no negative powers present in its Fourier expansion.

Theorem: D_k sends modular forms into modular forms:

$$D_k : M_k(SL_2(\mathbb{Z})) \longrightarrow M_{k+2}(SL_2(\mathbb{Z}))$$

Moreover, $D_k(S_k) \subset S_{k+2}$, since if $a(0) = 0$ then there are no constant terms present in the result.

Examples:
$D_4(E_4(\tau)) = D(E_4) + 8G_2E_4 \in M_6 = \mathbb{C}E_6$. By computing the constant term, one gets that $D_4(E_4(\tau)) = -\frac{1}{3}E_6$.

Exercise: Caculate $D_6(E_6)$, $D_8(E_8)$ and $D_{10}(E_{10})$. In case of $D_{10}(E_{10})$, one needs to be careful since $M_{12} = \mathbb{C}[E_{12}, \Delta]$ is 2-dimensional.

As another example, $D_{12}(\Delta) = D_{12}(\eta^{24}) \in M_{14}$. Since Δ is a cusp form, $D_{12}(\Delta) \in S_{14} = \{0\}$. So $\Delta \in \mathrm{Ker}(D_{12})$ and $D(\Delta) + 24G_2\Delta \equiv 0$.

Exercise:
1) Compute $\mathrm{Ker}D_k$
2) Show (cf the Borcherds paper [2]) that $D : M_0(SL_2(\mathbb{Z})) \longrightarrow M_2^{weak}(SL_2(\mathbb{Z}))$ is surjective, using the definition of the modular differential operator.
3) As a corollary of 2): show that if $f(\tau) = \sum_{n\geq -N} a(n)q^n \in M_2^{weak}(SL_2(\mathbb{Z}))$, then $a(0) = 0$.

Now we analyse the modular equation for the Eisenstein series $G_2(\tau) = -\frac{1}{2i\pi}\frac{\eta'(\tau)}{\eta(\tau)}$. We have

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = v_\eta(M)(c\tau+d)^{\frac{1}{2}}\eta(\tau), \quad \forall M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$$

$$(c\tau+d)^{-2}\eta'\left(\frac{a\tau+b}{c\tau+d}\right) = v_\eta(M)(c\tau+d)^{\frac{1}{2}}\eta'(\tau) + v_\eta(M)\frac{c}{2}(c\tau+d)^{-\frac{1}{2}}\eta(\tau)$$

By taking the quotient of the 2 previous expressions, we get:

$$(c\tau+d)^{-2}G_2\left(\frac{a\tau+b}{c\tau+d}\right) = G_2(\tau) - \frac{1}{4i\pi}\frac{c}{c\tau+d}$$

which shows that G_2 is not modular, but is a quasi-modular form of weight 2.

Exercise: calculation of $D(G_2)$:
Show that $D(G_2) + 2G_2^2 \in M_4$, and that the constant term $2G_2^2$ is proportional to E_4.

Theorem: (exercise)
Let $M_* = \bigoplus_{k\geq 0} M_k(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$. Then $M_*[E_2]$ is called the ring of quasi-modular forms, and we have:
1) $D : M_*[E_2] \longrightarrow M_*[E_2]$,
2) E_2, E_4 and E_6 are algebraically independent .

11.2 The automorphic correction

This methods provides a new explanation for the automorphic factor

$$\phi\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k e^{2i\pi m\frac{cz^2}{c\tau+d}}\phi(\tau,z), \quad \phi \in J_{k,m}^{weak,EZ}$$

The first explanation came from considering

$$\begin{array}{rcl} \mathbb{H}_1 \times \mathbb{C} & \longrightarrow & \mathbb{H}_2 = \{Z = \left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right), \tau, \omega \in \mathbb{H}_1, \Im Z \succ 0\} \\ \phi(\tau, Z) & \longrightarrow & \phi(\tau, Z)e^{2i\pi m\omega} \triangleq \widetilde{\phi}_m(Z) \end{array}$$

and $\widetilde{\phi}_m(Z)$ is a Γ^J-modular form on $\mathbb{H}_2$, with $\Gamma^J < Sp_2(\mathbb{Z})$.
For the second explanation, let $\Phi_m(\tau,z) = e^{-8\pi^2 mG_2(\tau)z^2}\phi(\tau,z)$. Then

$$\Phi_m\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k\Phi_m(\tau,z)$$

which shows that the extra term cancels the automorphic factor. The proof consists in writing the functional equation for the automorphic relation (**exercise**, easy) by using the quasi-modular equation.

We now use the Taylor expansion for Φ_m:

$$\Phi_m(\tau, z) = \sum_{n \geq 0} g_n(\tau) z^n$$

From the automorphic equation of Φ_m, we obtain $g_n(\tau) \in M_{k+n}(SL_2(\mathbb{Z}))$. This implies that

$$\phi(\tau, z) = e^{8\pi^2 m G_2(\tau) z^2} \sum_{n \geq 0} g_n(\tau) z^n = \sum_{n \geq 0} f_n(\tau) z^n$$

So we have proved:
Theorem: $\forall n \geq 0$, the coefficient $f_n(\tau)$ of the original Jacobi form is a quasi-modular form.

Corollary 1: Let $\phi \in J_{k,m}^{weak}$. We know that $\mathrm{ord}_{|z=0}\phi \leq 2m$ by using the theorem about the number of zeros. So

$$\begin{aligned} \dim J_{k,m}^{weak} &\leq \sum_{i=0}^{2m} \dim M_{k+i} \\ &\leq \sum_{i=0}^{m} \dim M_{k+2i} \text{ if } k \equiv 0[2] \end{aligned}$$

Let $\phi \in J_{k,m}$,

$$\begin{aligned} \phi(\tau, z) &= \sum_{\substack{n \geq 0 \\ 4mn - l^2 \geq 0}} a(n, l) e^{i\pi(n\tau + lz)} \\ &= \sum_{n \geq 0} f_n(\tau) z^n \end{aligned}$$

If we consider the Fourier expansion of $f_n(\tau)$, we see that $a(n,l)e^{2i\pi(n\tau+lz)}$ yields a strictly positive power of z only if $l \neq 0$, so we see that there are no constant terms in the Fourier expansion of $f_n(\tau) = \sum_{n>0} b_n q^n$. Now, we have

$$\Phi_m(\tau, z) = e^{-8\pi^2 G_2(\tau) z^2} \phi(\tau, z) = \sum_{n \geq 0} g_n(\tau) z^n$$

So if $\phi \in J_{k,m}^{cusp}$, then $\forall n \geq 0$, $g_n \in S_{k+n}$.

Corollary 2:
If $\phi \in J_{k,m}^{cusp}$, then $g_n(\tau) \in S_{k+n}(SL_2(\mathbb{Z}))$ if the constant term $f_0(\tau) = \phi(\tau, 0) \in S_k$, or if $f_0 \equiv 0$.

Corollary 3: For $\phi \in J_{k,m}^{cusp}$, $k \equiv 0[2]$,

$$\dim J_{k,m} < \dim M_k + \sum_{i=0}^{m} S_{k+2i}$$

11.3 Automorphic corrections of Jacobi modular forms in many variables

Let $\phi(\tau, z) \in J_{k,L}^{(weak)}$, $v \in L$ and $v \neq 0$, and let $L_1 = v_L^{\perp} < L$. For $z \in \mathbb{C} \otimes L$, we can decompose z as $z = z_1 + Z.v$ where $z_1 \in L_1 \otimes \mathbb{C}$ and $Z \in \mathbb{C}$. We take the following formula as definition of the automorphic correction for many variables:

$$\Phi_v(\tau, z_1, Z) \triangleq e^{-4\pi^2 (v,v) G_2(\tau) Z^2} \phi(\tau, z_1 + v.Z)$$

If $\Phi_v(\tau, z_1, Z) = \sum_{n \geq 0} f_n(\tau, z_1) Z^n$, then $f_n(\tau, z_1) \in J_{k+n, L_1}$ if $\phi \in J_{k,L}$, or $f_n(\tau, z_1) \in J_{k_n, L_1}^{weak}$ if $\phi \in J_{k,L}^{weak}$.
The proof is very similar to the case in one variable (**exercise**):
1) Calculate $\phi_v \left(\frac{a\tau+b}{c\tau+d}, \frac{z_1}{c\tau+d}, \frac{Z}{c\tau+d} \right)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{Z})$.
2) Calculate $\phi_v(\tau, z_1 + \lambda\tau + \mu, Z)$ for $\lambda, \mu \in L_1 = v_{L_1}^{\perp}$.

So we see that in the case of modular forms in many variables, the automorphic correction gives Jacobi forms for $L_1 < L$.

11.4 Automorphic correction for $\vartheta(\tau, z)$

Fact 1:

$$\vartheta(\tau, z) = (2i\pi z)\eta(\tau)^3 \exp\left(- \sum_{n \geq 1} \frac{2}{(2k)!} G_{2k}(\tau)(2i\pi z)^{2k} \right)$$

The proof will be shown later.
Fact 2: One can show that

$$\frac{\partial^2}{\partial z^2} \log \vartheta(\tau, z) = -\wp(\tau, z) + 8\pi^2 G_2(\tau)$$

where $\wp \in J_{2,0}^{\text{meromorphic}}$ and

$$\wp(\tau, z) = \frac{1}{z^2} + \sum_{k \geq 2} (2i\pi)^{2k} G_{2k}(\tau) \frac{z^{2k-2}}{(2k-2)!}$$

We remind that $G_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{n \geq 1} \sigma_{2k-1}(n) q^n$.

Exercise: Try to prove the Taylor expansion and the logarithmic derivative for ϑ - after which it becomes your best friend...

12 Modular differential operators for Jacobi modular forms

12.1 The heat operator of a lattice

The previous section discussed differential operators for the usual modular forms. We now want to define them for Jacobi modular forms.
Let $M = \sum_{i=1}^{n} \mathbb{Z} e_i$ be a quadratic lattice, $z \in \mathbb{C} \otimes M$, $z = \sum_{i=1}^{n} z_i e_i$, $z_i \in \mathbb{C}$

$\forall 1 \leq i \leq n$.
Let $\frac{\partial}{\partial z} = \sum_{i=1}^{n} e_i^* \frac{\partial}{\partial z_i}$, with $(e_i, e_j^*) = \delta_{i,j}$. We pose

$$\Delta \triangleq \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right)$$

Using these definitions we have (**check**):

$$\Delta e^{2i\pi(l,z)} = (2i\pi)^2.(l,l)e^{2i\pi(l,z)}$$

Let L be a positive definite lattice of rank n_0. We construct the hyperbolic lattice $L_1 = U \oplus L(-1)$ where $U \simeq \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The signature of L_1 is $(1, n_0 + 1)$.
Let $Z \in \mathbb{C} \otimes L_1$, $Z = (\tau, \omega) \dot{+} z$, $(\tau, \omega) \in U$, $z \in \mathbb{C} \otimes L$ and

$$\Delta_{L_1} = 2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \omega} - \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right)$$

We want to understand the action of this operator on Jacobi forms in analogy to the symplectic presentation of Jacobi forms:

$$\begin{array}{ccc} \phi \in J_{k,m} \dashrightarrow & \tilde{\phi}(Z) = \phi(\tau, z) e^{2im\pi\omega} & \dashrightarrow \mathbb{H}_2,\ \Gamma^J < Sp_2(\mathbb{Z}) \\ \phi(\tau, z) \in J_{k,L} \dashrightarrow & \tilde{\phi}(\tau, z, \omega) & \dashrightarrow D^{IV}, \Gamma^J < SO(2, n_0 + 2) \end{array}$$

The construction in the lattice case is explained in Gritsenko's papers with the operator

$$\begin{aligned} \Delta_{L_1} \tilde{\phi}(\tau, z, \omega) &= \left(2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \omega} - \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) \right) \phi(\tau, z) e^{2i\pi\omega} \\ &= e^{2i\pi\omega} \left(4i\pi \frac{\partial}{\partial \tau} - \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) \right) \phi(\tau, z) \end{aligned}$$

We can define H that acts on Jacobi forms of weight k and index with respect to the lattice L introduced above:

$$\begin{aligned} H\phi &= \frac{-1}{8\pi^2} \left(4i\pi \frac{\partial}{\partial \tau} - \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) \right) \phi(\tau, z) \\ &= \left(\frac{1}{2i\pi} \frac{\partial}{\partial \tau} + \frac{1}{8\pi^2} \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) \right) \phi(\tau, z) \end{aligned}$$

The operator $\left(4i\pi \frac{\partial}{\partial \tau} - \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right)\right)$ is the heat operator related to the lattice L. Note that the first part of H corresponds to $D = q\frac{d}{dq}$, so H is a generalisation of D.
Let $\phi \in J_{k,L}$, with Fourier expansion

$$\phi(\tau, z) = \sum_{\substack{n \geq 0,\, l \in L^*, \\ 2n - (l,l) \geq 0}} a(n, l) e^{2i\pi(n\tau + (l,z))}$$

We obtain

$$(H\phi)(\tau, z) = \frac{1}{2} \sum_{2n - (l,l) > 0} (2n - (l,l))\, a(n, l) e^{2i\pi(n\tau + (l,z)}$$

where only the strictly positive norm coefficients subsist. For $\phi \in J_{\frac{n_0}{2},L}$, $a(n,l) \neq 0 \implies 2n-(l,l)=0$. So $H\phi \equiv 0$ for ϕ of singular weight: this space behaves for H as constants do for D.
The modular correction for D is $D_k = D + 2kG_2$. and transforms modular forms of weight k into modular forms of weight $k+2$. To extend this to the lattice case we need to provide some formulas.

For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, let $\tilde{\tau} = \frac{a\tau+b}{c\tau+d}$ and $\tilde{z} = \frac{z}{c\tau+d}$. Using the classical formulas $\frac{\partial}{\partial X} = \frac{\partial Y}{\partial X}\frac{\partial}{\partial Y} \implies \frac{\partial}{\partial Y} = \left(\frac{\partial Y}{\partial X}\right)^{-1}\frac{\partial}{\partial X}$ one obtains (**check**)

$$\left(\frac{\partial}{\partial \tilde{\tau}}, \frac{\partial}{\partial \tilde{z}}\right) = \left((c\tau+d)^2\frac{\partial}{\partial \tau} + c(c\tau+d)\sum_{i=1}^{n_0} z_i \frac{\partial}{\partial z_i}, (c\tau+d)\frac{\partial}{\partial z}\right)$$

$$H(\tilde{\tau}, \tilde{z}) = (c\tau+d)^2\left(H(\tau,z) + \frac{1}{2i\pi}\frac{c}{c\tau+d}\sum_{i=1}^{n_0} z_i \frac{\partial}{\partial z_i}\right)$$

Let $\phi|_k M \triangleq (c\tau+d)^{-k} e^{-i\pi\frac{c(z,z)}{c\tau+d}} \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)$. Then we have the following result:

Proposition: $\forall M \in SL_2(\mathbb{Z})$, we have:

$$H(\phi|_k M) = (H\phi)|_{k+2}M + \frac{1}{4i\pi}(n_0 - 2k)\frac{c}{c\tau+d}\phi|_k M$$

In particular, if $n_0 - 2k = 0$, $k = \frac{n_0}{2}$, ϕ is of singular weight and thus $H(\phi|_k M) = (H\phi)|_{k+2}M$. The heat operator is then a modular operator since it commutes with the modular group action.

Theorem: Let $\phi \in J_{k,L}$, $H_k = H + (2k - n_0)G_2(\tau)$. Then we have

$$H_k : J_{k,L} \longrightarrow J_{k+2,L}$$

$$H_k : J_{k,L}^{cusp} \longrightarrow J_{k+2,L}^{cusp}$$

$$H_k : J_{k,L}^{weak} \longrightarrow J_{k+2,L}^{weak}$$

(in comparison, for $D_k = D + 2kG_2(\tau)$, D_k maps M_k onto M_{k+2}).

Proof:
ϕ satisfies two functional equations for the transformations under the action of $SL_2(\mathbb{Z})$ and $z \to z + \lambda\tau + \mu$).
We have proved that the heat operator is the modular operator in the case of the singular weight. So $\phi \to \eta^{2k-n_0} H\left(\frac{\phi(\tau,z)}{\eta^{2k-n_0}}\right)$ is modular. By direct computation, we can also obtain that

$$\phi(\tau,z) \to \eta^{2k-n_0} H\left(\frac{\phi(\tau,z)}{\eta^{2k-n_0}}\right) = H + (2k-n_0)G_2(\tau) = H_k$$

So $H_k(\phi)$ satisfies $SL_2(\mathbb{Z})$-functional equations with weight $k+2$.

Regarding the lattice transformations, we defined the heat operator as a function of Δ_{L_1}. For any lattice M this operator is invariant with respect to the full orthogonal group of the real space $O(\mathbb{R}\otimes M)$ (or its complex version). But the action by the differential operator on the functions is defined by the action of the hyperbolic differential operator Δ_{L_1} on $\tilde{\phi}(\tau,z,\omega)=\phi(\tau,z)e^{2i\pi\omega}$. And so the transformation $z\to z+\lambda\tau$, $\lambda\in L_1$ can be written in terms of the orthogonal group $O(U\oplus L(-1))$. This means that the definition of the differential operator in terms of the hyperbolic lattice gives the invariance in terms of the lattice transformation automatically. The invariance with respect to $z\to z+\mu$ is more or less evident.

We proved that $H_k(\phi)$ transforms like a Jacobi form of weight $k+2$, we now want to prove that $H_k\phi$ is holomorphic at $i\infty$. Let $\phi=\sum_{2n-(l,l)\geq 0}a(n,l)e^{2i\pi(n\tau+(l,z))}$. Then

$$(H_k\phi)(\tau,z)=\frac{1}{2}\sum_{2n-(l,l)>0}(2n-(l,l))\,a(n,l)e^{2i\pi(n\tau+(l,z))}+(2k-n_0)G_2(\tau)\phi(\tau,z)$$

Since $G_2(\tau)=-\frac{1}{24}+\sum_{n\geq 1}\sigma_1(n)q^n$, after multiplication by G_2 we only have standard coefficients for the hyperbolic norm. Moreover, if ϕ is a cusp form, then $H\phi$ is also a cusp form. $\square$

12.2 Applications and examples

12.2.1 Eichler-Zagier Jacobi forms

We remind that $J^{EZ}_{k,m}=J_{k,\langle 2\rangle}$. In the general case of a lattice L we note $S(L)$ its Gram matrix, $\frac{\partial}{\partial z}=\begin{pmatrix}\frac{\partial}{\partial z_1}\\ \vdots\\ \frac{\partial}{\partial z_{n_0}}\end{pmatrix}$.

We have $(\frac{\partial}{\partial z},\frac{\partial}{\partial z})=(\sum_{i=1}^{n_0}e_i^*\frac{\partial}{\partial z_i})^2=S^{-1}[\frac{\partial}{\partial z}]\overset{\langle 2m\rangle}{=}\frac{1}{2m}\frac{d^2}{dz^2}$. In our case we have

$$H_k:J_{k,m}\longrightarrow J_{k+2,m}$$

$$H_k=\frac{1}{2i\pi}\frac{d}{d\tau}+\frac{1}{8\pi^2}(\frac{\partial}{\partial z},\frac{\partial}{\partial z})=\frac{1}{2i\pi}\frac{d}{d\tau}+\frac{1}{16m\pi^2}\frac{d^2}{dz^2}$$

.

Remember that $\phi^{weak}_{-2,1}=\mathbb{C}\phi_{-2,1}$, $\phi_{-2,1}(\tau,z)=\frac{\vartheta(\tau,z)^2}{\eta(\tau)^6}=(r-2+r^{-1})+q(\ldots)$ for $r=e^{2i\pi z}$ so the action of H_{-2} writes

$$H_{-2}\phi_{-2,1}=\left(\frac{1}{2i\pi}\frac{d}{d\tau}+\frac{1}{16\pi^2}\frac{d^2}{dz^2}\right)\phi_{-2,1}+(2k-n_0)G_2(\tau)\phi_{-2,1}(\tau,z)$$

where $2k-n_0=-5$. The q^0 term is

$$\frac{(2i\pi)^2}{16\pi^2}e^{2i\pi z}+(r-2+r^{-1})\frac{5}{24}+\frac{(-2i\pi)^2}{16\pi^2}e^{-2i\pi z}=-\frac{1}{24}(r+10+r^{-1})$$

So

$$(-24)H_{-2}\phi_{-2,1}=\phi_{0,1}=(r+10+r^{-1})+q(\ldots)\in J^{weak}_{0,1}$$

Exercise:
$J^{cusp}_{10,1} = \mathbb{C}\phi_{10,1}$, $\phi_{10,1} = \eta(\tau)^{18}\vartheta^2(\tau,z)$. So $H_{10}\phi_{10,1} = \tilde{\phi}_{12,1} \in J^{cusp}_{12,1}$ where $\tilde{\phi}(\tau,0) = c\Delta(\tau)$. So in particular one has $\frac{1}{c}\tilde{\phi}_{12,1} = \phi_{12,1}$.
Find c.

Exercise: $E_{4,1}(\tau,z) = \vartheta_{E_8}(\tau,z)|_{z=Z.v}$ for $v \in E_8$, $v^2 = 2$. Calculate $H_4 E_{4,1} \in J_{6,1}$. Calculate its specialisation $(H_4E_{4,1})(\tau,0)$.

Theorem: From Eichler-Zagier [5]. We have:

$$J^{weak}_{2*,*} = \bigoplus_{\substack{k\in\mathbb{Z}\\ m\in\mathbb{Z}_{\geq 0}}} J^{weak}_{2k,m} = M_*\left(SL_2(\mathbb{Z})\right)[\phi_{-2,1},\phi_{0,1}]$$

Proof:
We proceed by induction. Let $\phi_{2k,m} \in J^{weak}_{2k,m}$. Then $\phi_{2k,m}(\tau,0) = f_{2k}(\tau) \in M_{2k}(SL_2(\mathbb{Z}))$. Let

$$\tilde{\phi} = \phi - f_{2k}(\tau)\phi_{0,1}(\tau,z)^m \frac{1}{(12)^m}$$

Then $\tilde{\phi}_{|z=0} \equiv 0$ since $\phi_{0,1}(\tau,0) = 12$ (consider the expansion $\phi_{0,1}(\tau,z) = (r + 10 + r^{-1}) + q(\ldots))$. So $\tilde{\phi}(\tau,z) = \sum_{n\geq 2}\tilde{f}_n z^n$. Moreover, $\phi_{-2,1} = (\tau,z)\frac{\vartheta(\tau,z)^2}{\eta(\tau)^6} \implies \mathrm{ord}_{z=0}\phi_{-2,1} = 2$. Because a Jacobi form of index 1 has exactly 2 zeros of weight 1 over a fundamental domain, we obtain that

$$\frac{\tilde{\phi}}{\phi_{-2,1}} \in J^{weak}_{2k+2,m-1}$$

The proof concludes by induction over m. □

12.2.2 Generalisation

We want to consider the case of the graded ring of **weak** Jacobi forms over the lattice D_n: recall that

$$\phi_{-n,D_n}(\tau,z) = \frac{\vartheta(z_1)}{\eta(\tau)^3}\cdots\frac{\vartheta(\tau,z_n)}{\eta(\tau)^3} \in J^{weak,Sym}_{-n,D_n}$$

which are symmetric with respect to permutations of the coordinates $(z_1,\ldots,z_n)$. Then $H^{(D_n)}_{-n}\phi_{-n,D_n} = \phi_{-n+2,D_n}$.

Exercise: Find its q^0 Fourier coefficients.

Using this argument, one can try to analyse the graded ring of weak Jacobi forms of arbitrary weight for the lattice D_n and index m

$$J^{weak}_{*,D_n;*} = \bigoplus_{k\in\mathbb{Z}} J^{weak,Sym}_{k,D_n;*}$$

Conversely, if instead of weak forms one considers holomorphic Jacobi forms we obtain a graded ring with infinitely many generators. Also, the resolution of the structure of the graded ring for the lattice E_8 is still an open question.

Exercise: Consider the graded ring

$$J^{weak,EZ}_{*,*} = J^{weak}_{2*,*}[\phi_{-1,2}]$$

with $\phi_{-1,2}(\tau,z) = \frac{\vartheta(\tau,z)}{\eta(\tau)^3}$. Find the quadratic equation expressing $\phi^2_{-1,2}$ as a function of $\phi_{0,1}$ and $\phi_{-2,1}$.

The theorem by Eichler and Zagier is about the lattice $A_1 = \langle 2\rangle$. Considering A_n, we can construct

$$\phi_{-(n+1),A_n}(\tau,z) = \frac{\vartheta(\tau,z_1)}{\eta^3(\tau)}\cdots\frac{\vartheta(\tau,z_n)}{\eta(\tau)^3}\frac{\vartheta(\tau,z_1+\ldots+z_n)}{\eta(\tau)^3} \in J^{weak,Sym}_{-(n+1),A_n}$$

(note that in the case of A_1 we obtain $\frac{\vartheta(\tau,z_1)^2}{\eta(\tau)^6}$). Using the differential operator we obtain

$$H^{(A_n)}_{-(n+1)}\phi_{-(n+1)} \in J^{weak,Sym}_{-n+1,A_n}$$

Exercise:
Calculate the q^0 Fourier coefficient of $\phi_{-(n+1),A_n}$ and ϕ_{-n+1,A_n}.

12.2.3 Jacobi type forms in many variables

We introduce new functions here. Let $\tau \in \mathbb{H}_1$, $\mathfrak{z} \in L\otimes\mathbb{C}$ where L is a lattice of rank $n_0 \geq 0$, $Z \in \mathbb{C}$, $m \in \mathbb{Z}_{\geq 0}$ ($m = 0$ is possible), and $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Let ϕ be such that

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}; \frac{Z}{c\tau+d}\right) = (c\tau+d)^k e^{i\pi\frac{c(\mathfrak{z},\mathfrak{z})}{c\tau+d}} e^{2im\pi\frac{Z^2}{c\tau+d}}\phi(\tau,\mathfrak{z};Z) \quad (M)$$

and

$$\phi(\tau,\mathfrak{z}+\lambda\tau+\mu;Z) = e^{-i\pi((\lambda,\lambda)\tau+2(\lambda,\mathfrak{z}))}\phi(\tau,\mathfrak{z};Z) \quad \forall\lambda,\mu\in L \qquad (A)$$

The space of such functions is noted $TJ_{k,L;m}$. We remark that m is purely formal since $\phi \in TJ_{k,L;m} \implies \phi(\tau,\mathfrak{z};\sqrt{m^{-1}}Z) \in TJ_{k,L;1}$. In fact there are only 2 spaces of such functions: $TJ_{k,L;1}$ and $TJ_{k,L;0}$ where members of the latter have strict invariance in Z.

Let $\phi(\tau,\mathfrak{z};Z) \in TJ_{k,L;0} = \sum_{n\geq 0}\phi_{k+n}(\tau,\mathfrak{z})Z^n$, $\phi_{k+n}(\tau,\mathfrak{z}) \in J_{k+n,L}$. In the next section, we are going to consider relations of this type, and the application

$$e^{-8\pi^2 G_2(\tau)Z^2} : TJ_{k,L;1} \longrightarrow TJ_{k,L;0}$$

13 Generalisation of the Cohen-Kuznetsov-Zagier operator

13.1 The Cohen-Kuznetsov-Zagier operator

This operator was formulated by Cohen, Kuznetsov and Zagier in different contexts: we present it her for the modular forms. Let $f \in M_k(SL_2(\mathbb{Z}))$. Then the CKZ operator maps modular forms into Jacobi type forms:

$$\nabla_{CKZ} : f \longrightarrow F_f(T,Z) = (k-1)! \sum_{n \geq 0} \frac{(2i\pi)^n}{(k-1+n)!n!} f^{(n)} Z^{2n} \in TJ_{k,1}$$

Then

$$F_f\left(\frac{a\tau+b}{c\tau+d}, \frac{Z}{c\tau+d}\right) = (c\tau+d)^k e^{2i\pi \frac{cZ^2}{c\tau+d}} F_f(\tau, Z), \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$$

Application: Let $f \in M_k$, $g \in M_l$. Thanks to the cancellation of the factors $e^{2i\pi \frac{Z^2}{c\tau+d}}$ in the product $F_f(\tau,Z)F_g(\tau,iZ)$, one has

$$F_f(\tau,Z)F_g(\tau,iZ) \in TJ_{k+l,0}$$

Let us denote this series as $\sum_{n\geq 0}[f,g]_n Z^{2n}$, where $[f,g]_n \in M_{k+l+2n}$. We have

$$[f,g]_n = \sum_{i=0}^{n} (-1)^i \binom{k-1+n}{i} \binom{l-1+n}{n-i} f^{(i)} g^{(n-i)} \in S_{k+l+2n}$$

$[f,g]_n$ is called the Rankin-Cohen bracket.

We would like to generalise this construction to Jacobi forms in several variables, and provide another demonstration that $\nabla_{CKZ}(f) \in TJ_{k,1}$. To give a new, algebraic proof of the Cohen-Kuznetsov-Zagier theorem which works without any changes in the generalisation to $J_{k,L}$, our idea is the following:
1) Use the automorphic correction:

$$TJ_{k,1} \xrightarrow{\times e^{-8\pi^2 G_2(\tau) Z^2}} TJ_{k,0}$$

$$TJ_{k,1} \xleftarrow{\times e^{8\pi^2 G_2(\tau) Z^2}} TJ_{k,0}$$

2) Construct e^D, $D = q\frac{d}{dq} = \frac{1}{2i\pi}\frac{d}{d\tau}$ (in the case of trivial $L = \{0\}$), or e^H where H is the heat operator $H = -\frac{1}{8\pi^2}\left(4i\pi\frac{\partial}{\partial\tau} - (\frac{\partial}{\partial z}, \frac{\partial}{\partial z})\right)$ for a non trivial lattice. One cannot naively define e^D in a modular way since D^k is not a modular operator. So we consider instead $D_k = D + 2kG_2. : M_k \to M_{k+2}$. The operator $D_{k,n}$ defined as the composition

$$D_{k,n} : M_k \xrightarrow{D_k} M_{k+2} \xrightarrow{D_{k+2}} \dots \xrightarrow{D_{k+2n-2}} M_{k+2n}$$

has a somewhat complicated expression, we will limit ourselves to calculating its value modulo a modular form, or in fact its major term of order n.

When we analysed the action of differential operators on the quasi-modular Eisenstein series G_2, we found that $D_2 : M_2^{quasi-mod.} \longrightarrow M_4$ and $D_2(G_2) = D(G_2) + 2G_2^2 \in M_4$. So in fact

$$D(G_2) \equiv -2G_2^2 \mod M_4$$

If one calculates $D_{k,n} \mod M_*$ we obtain the following result:

Theorem:
The major term of the operator $D_{k,n}$ is

$$E_{k,n} = \sum_{\nu=0}^{n} \frac{n!}{\nu!(n-\nu)!} \frac{\Gamma(k+n)}{\Gamma(\nu+n)} (2G_2)^{n-\nu} D^{\nu}$$

And we have the following application between modular spaces:

$$M_k \stackrel{E_{k,n}}{\longrightarrow} M_{k+2n}$$

Proof:
We have to prove this is a modular operator by induction on n. We will only make use of the following relation:

$$D(G_2 \bullet) \equiv -2G_2^2 \bullet + G_2.D \mod M_*$$

Using this congruence, we can prove using elementary calculations that

$$E_{k,n+1} = D_{k+2n}(E_{k,n}) - \frac{5}{3} G_4 E_{k,n-1}$$

where E_4 is the Eisenstein series, because $D_{k+2n} = D + 2(k+2n)G_2$. One has to act by D on all Eisenstein series $(2G_2)^{n-\nu}$ in our function. Only the congruence above is needed for the identity, and this leads to the theorem. □

In order to simplify this operator, we notice the presence in the expression above of the convolution of the differential operator D^k and the powers of the quasi-modular Eistenstein series $(G_2)^{n-\nu}$. So we define the following formal power series which will play the role of e^D we asked for earlier:

$$E_k(X) = \sum_{n \geq 0} \frac{1}{n!\Gamma(k+n)} E_{k,n} X^n$$

and we can prove the following identity:

$$E_k(X) = e^{2G_2.X} \sum_{\nu \geq 0} \frac{D^{\nu}}{\nu!\Gamma(k+\nu)} X^{\nu}$$

We have thus obtained a formal decomposition of the formal power series of the quasi-exponential (or rather Bessel) function of the differential operator D. We would like to represent E in a more linear way by noting that the expression

$$E_k(X) = \sum_{n \geq 0} \frac{E_{k,n}}{n!\Gamma(k+n)} X^n$$

splits into a product of two functions:

$$E_k(X) = e^{2G_2.X} \sum_{\nu \geq 0} \frac{D^\nu}{\nu!\Gamma(k+\nu)} X^\nu = e^{2G_2.X}.\nabla_{CKZ}(X)$$

Which we can represent as the following diagram with $X = (2i\pi Z)^2 = -4\pi^2 Z^2$:

$$\begin{tikzcd}
M_k \arrow[r, "E_k(X)"] \arrow[dr, "\nabla_{CKZ}(X)"'] & TJ_{k,0} \arrow[d, "\substack{e^{-2G_2.X} = e^{8\pi^2 G_2.Z^2} \\ \text{-(automorphic} \\ \text{correction)}}"] \\
& TJ_{k,1}
\end{tikzcd}$$

This proves the Cohen-Kuznetsov-Zagier theorem which states that ∇_{CKZ} maps M_k into $TJ_{k,1}$.

Exercises:
1) Consider ∇_{CKZ} applied to G_2. For this, consider
$D_2 = D + 2G_2. : M_2^{quasi-mod} \longrightarrow M_4$, $D_2(G_2) = D(G) + 2G_2^2$. Prove that using the modification of the CKZ operator defined as

$$\tilde{\nabla} G_2 = 1 - 2 \sum_{\nu \geq 1} \frac{D^{\nu-1}(G_2)}{\nu!(\nu-1)!} (2i\pi Z)^{2\nu}$$

we have $\tilde{\nabla} G_2 \in TJ_{0,1}$.

2) Using this formula, we can generalise the Rankin-Cohen bracket. Calculate $[f, G_2]_n$, then calculate $[G_2, G_2]_n$.

3) The interest in using the Γ function in the formulae above lies in generalizing to a half-integral weight k. Modify the formulas above for k half-integral, and in particular obtain the Rankin-Cohen brackets in this case.

4) Let $k \leq 0$. In this case we need to take care about the poles of Γ. Generalise the formulae to obtain

$$\tilde{\nabla}_{CKZ}^{(k \leq 0)} \triangleq \sum_{n \geq |k|+1} \frac{E_{k,n}}{n!\Gamma(k+n)} X^n = e^{2G_2 X} \sum_{\nu \geq |k|+1} \frac{D^\nu}{\nu!(k+\nu-1)!} X^\nu$$

In particular, prove that if $f \in M_k^{meromorphic}$, $k \leq 0$, then $D^{|k|+1}(f) \in M_{|k|+2}^{meromorphic}$. This is the classical Bol identity for modular forms.

13.2 Generalisation to the case of Jacobi forms in many variables

We will now show how to amend the former section without any change to the proof, by generalising from the lattice $\{0\}$ to L, where L is a positive definite lattice of rank n_0. Let $\mathfrak{z} \in L \otimes \mathbb{C}$. We have seen the following correspondence:

$L = \{0\}$	$\mathrm{rank} L = n_0 > 0$
$D = \frac{1}{2i\pi}\frac{d}{d\tau}$	$H = -\frac{1}{8\pi^2}\left(4i\pi\frac{\partial}{\partial\tau} - (\frac{\partial}{\partial z}, \frac{\partial}{\partial z})\right)$
$D_k = D + 2kG_2.$	$H_k = H + (2k - n_0)G_2.$
k	$k - \frac{n_0}{2}$, singular weight
E^D	$E^H_{k,n} = \sum_{\nu=0}^{n} \frac{n!\Gamma(k-\frac{n_0}{2}+n)}{\nu!(n-\nu)!\Gamma(k-\frac{n_0}{2}+\nu)}(2G_2)^{n-\nu}.H^\nu$

In the proof of the theorem, we only needed the congruence

$$D(G_2\bullet) \equiv -2G_2^2 \bullet + G_2.D \mod M_*$$

which is still true in the case of the heat operator H because its main modular part $\frac{1}{2i\pi}\frac{\partial}{\partial\tau}$ is the same. So $E_{k,n}$ acts on the space of Jacobi modular forms of weight k, this operator changes the weight from k to $k+2n$. Like in the previous case we can construct the power series of this operator. We still have to cancel the numerator, then we have the convolution of the two power series: $e^{2G_2.X}$ and $\nabla_H(X) = \sum_{\nu\geq 0} \frac{H^\nu}{\nu!\Gamma(k-\frac{n_0}{2}+\nu)} X^\nu$ which is a variant of the Cohen-Kuznetsov-Zagier operator.
We need to make sure that $k \geq \frac{n_0}{2}$, which is true in particular for all holomorphic forms (cf the section about the Weil representation). We can then describe the nature of our operator using the following diagram:

$$\begin{tikzcd} J_{k,L} \arrow[r, "E^{(H)}(X)"] \arrow[dr, "\nabla_H(X)"'] & TJ_{k,L;0} \arrow[d, "e^{-2G_2.X}"] \\ & TJ_{k,L;1} \end{tikzcd}$$

Theorem:
Let $\phi \in J_{k,L}$, $H = -\frac{1}{8\pi^2}\left(4i\pi\frac{\partial}{\partial\tau} - (\frac{\partial}{\partial z}, \frac{\partial}{\partial z})\right)$ be the heat operator for the lattice L of rank n_0. Then for $\mathfrak{z} \in L \otimes \mathbb{C}$ and $Z \in \mathbb{C}$, we have

$$F_\phi(\tau, \mathfrak{z}; Z) = \sum_{\nu\geq 0} \frac{H^\nu(\phi)}{\nu!\Gamma(k - \frac{n_0}{2} + 1)}(2i\pi Z)^{2\nu} \in TJ_{k,L;1}$$

The proof is identical to the one in the case of the Cohen-Kuznetsov-Zagier modular forms operator.

Moreover, $\forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, this function satisfies the following equation:

$$F_\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}}{c\tau+d}; \frac{Z}{c\tau+d}\right) = (c\tau+d)^k e^{i\pi\frac{c(\mathfrak{z},\mathfrak{z})}{c\tau+d}} e^{2i\pi\frac{cZ^2}{c\tau+d}} F_\phi(\tau,\mathfrak{z};Z)$$

Remarks:
1) If $L = \{0\}$, the theorem is equivalent to the Cohen-Kuznetsov-Zagier theorem. We can also formulate the same series of **exercises**:

2) Calculate $[\phi_{k,L}, \psi_{l,M}]$
Like for trivial lattices, we need to analyse the products $F_\phi(\tau,\mathfrak{z};Z).F_\psi(\tau,\mathfrak{z};Z)$ $(TJ_{k+l,L\oplus M;0})$. Taking the Taylor expansion in Z one can get the generalisation of the Rankin-Cohen brackets.

3) Analyse $[\phi, G_2]$.

4) Consider the case of half-integral weight k. In particular, consider the case of the Jacobi ϑ-series from the Jacobi function $\vartheta(\tau, z)$, which leads to many beautiful theorems.

5) For $k < \frac{n_0}{2}$, we could have a pole in the Γ function. Try to consider this case, which leads to many interesting research questions.

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