# Understanding Distance to Default (DD) and Probability of Default (PD)

## Symbol Definitions (Used Throughout)

**Core Symbols** (no hats except in naive section): - **V** = Firm asset value (USD) - **F** = Face value of debt (USD) - **σ\_V** = Asset volatility (annual, decimal) - **σ\_E** = Equity volatility (annual, decimal) - **r\_f** = Risk-free rate (annual, decimal) - **T** = Time horizon (years, typically 1) - **Φ** = Standard normal cumulative distribution function

**Barrier Convention**: For banks, F = total liabilities (deposits are liabilities and dominate funding).

**Units Convention**: Millions converted to dollars once, not twice.

## ⏰ Time Index Conventions

**Critical**: All variables are indexed by time to prevent lookahead bias.

* **E\_t**: Market capitalization observed during year t
* **F\_t**: Barrier at t (for banks: total liabilities at t)
* **r\_{f,t}**: One-year risk-free rate observed at t
* **σ\_{E,t-1}**: Equity volatility computed only from returns up to t-1 (no future data)
* \*\*μ̂\_t\*\*: For naive approach, equals r\_{i,t-1} (previous year’s equity return)
* **V\_t, σ\_{V,t}**: Solved at time t using only information available at t
* **T**: Time horizon = 1 year (from t to t+1)

**Key Statements**: - DD\_{m,t} equals d₂ when μ = r\_{f,t} (Q-measure) - DD\_{naive,t} uses μ̂*t = r*{i,t-1} and is a P-style score unless calibrated - **No lookahead**: σ\_{E,t-1} window ends strictly at t-1

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## Introduction

This document explains how we measure the financial health and default risk of banking institutions using two complementary metrics: **Distance to Default (DD)** and **Probability of Default (PD)**.

The foundation comes from Merton (1974), who viewed equity as a call option on firm assets. This was extended by Kealhofer, McQuown, and Vasicek (KMV) for publicly traded firms.

**Key References:** - Merton, R. C. (1974). “On the Pricing of Corporate Debt: The Risk Structure of Interest Rates” - Bharath, S. T., & Shumway, T. (2008). “Forecasting Default with the Merton Distance to Default Model” - Tepe, M., Thastrom, P., and Chang, R. (2022). “How Does ESG Activities Affect Default Risk”

## What is Distance to Default?

**Distance to Default (DD)** measures how many volatility steps separate a firm’s assets from its debt level.

* **DD = 5**: Assets are 5 volatility steps above debt
* **DD = 1**: Only 1 volatility step from trouble
* **DD < 0**: Assets already below debt

## What is Probability of Default?

**Probability of Default (PD)** converts DD into a percentage chance of default within time T.

PD = Φ(-DD)

### Practical Interpretation

| DD | PD | Risk Level |
| --- | --- | --- |
| 1 | 15.87% | Very High |
| 3 | 0.135% | High |
| 4.47 | 0.000391% | Moderate |
| 5 | 0.0000287% | Low |
| 5.7 | 5.99×10⁻⁷% | Very Low |
| ≥7 | ≈0† | Very Low |

**†Note**: For DD ≥ 7, PD approaches machine precision limits. We report “effectively zero” and round to 3-4 significant digits only.

## The Merton Model Foundation

### Core Insight

Shareholders hold a **call option** on firm assets with strike price F.

### The Two Key Equations

#### 1. Equity Value Equation

E\_t = V\_t × Φ(d₁) - F\_t × e^(-r\_{f,t}×T) × Φ(d₂)

Where: - **Φ(d₂)** = Risk-neutral probability V\_T > F - **Φ(d₁)** = Option delta (equity sensitivity to assets)

#### 2. Equity Volatility Equation

σ\_{E,t-1} = (V\_t/E\_t) × Φ(d₁) × σ\_{V,t}

### The d₁ and d₂ Terms

d₁ = [ln(V\_t/F\_t) + (r\_{f,t} + 0.5×σ²\_{V,t})×T] / (σ\_{V,t}×√T)  
d₂ = d₁ - σ\_{V,t}×√T

**Derivation of d₂**:

d₂ = d₁ - σ\_{V,t}×√T  
 = [ln(V\_t/F\_t) + (r\_{f,t} + 0.5×σ²\_{V,t})×T] / (σ\_{V,t}×√T) - σ\_{V,t}×√T  
 = [ln(V\_t/F\_t) + (r\_{f,t} - 0.5×σ²\_{V,t})×T] / (σ\_{V,t}×√T)

### Critical Statement

**Distance to default equals d₂ when μ = r\_f**:

DD = [ln(V\_t/F\_t) + (μ - 0.5×σ²\_{V,t})×T] / (σ\_{V,t}×√T)  
  
When μ = r\_{f,t}:  
DD\_m = d₂  
PD\_m = Φ(-d₂)

## Market-Based Approach

### Overview

Uses current market data to estimate default risk under the **risk-neutral (Q-measure)** where μ = r\_{f,t}.

**Barrier Convention**: F\_t = total liabilities (for banks).

**Rationale**: Deposits are liabilities and dominate bank funding. This follows standard practice for financial institutions.

**⏰ Timing Rule**: σ\_{E,t-1} uses t-1 only, E\_t, F\_t, r\_{f,t} are at t, unknowns V\_t and σ\_{V,t} are solved at t, horizon is t to t+1.

### Observable Inputs (with Timing)

| Variable | Symbol | Description | Units | Timing |
| --- | --- | --- | --- | --- |
| Market Cap | E\_t | Total share value | USD | t |
| Equity Vol | σ\_{E,t-1} | Stock volatility | Decimal | t-1 |
| Total Liabilities | F\_t | Total liabilities | USD | t |
| Risk-Free Rate | r\_{f,t} | Safe return | Decimal | t |
| Time Horizon | T | Analysis period | Years | 1 |

**Data Source Note**: Millions converted to dollars once, not twice.

### Solver Process

**Initial Guesses**:

V₀ = E\_t + F\_t  
σ\_{V,0} = σ\_{E,t-1}

**Convergence Criteria**: - |E\_model - E\_t| < 10⁻⁶ - |σ\_{E,model} - σ\_{E,t-1}| < 10⁻⁴

**Solver Gates**: - Only converged solutions proceed to DD\_m calculation - Non-converged cases dropped from summaries - Convergence rate reported in output

### Market DD and PD Formulas

DD\_m = [ln(V\_t/F\_t) + (r\_{f,t} - 0.5×σ²\_{V,t})×T] / (σ\_{V,t}×√T) = d₂  
  
PD\_m = Φ(-d₂) [Q-measure probability]

### Market Variables Dictionary (with Timing)

| Variable | Symbol | Units | Timing | Role |
| --- | --- | --- | --- | --- |
| market\_cap | E\_t | USD | t | Solver input |
| equity\_vol | σ\_{E,t-1} | Decimal | t-1 | Solver input |
| total\_liabilities\_usd | F\_t | USD | t | Barrier |
| rf | r\_{f,t} | Decimal | t | Drift |
| asset\_value | V\_t | USD | t | Solver output |
| asset\_vol | σ\_{V,t} | Decimal | t | Solver output |
| DD\_m | DD\_m | Std devs | t | Final metric |
| PD\_m | PD\_m | Probability | t | Q-measure |

## Accounting-Based Approach

### Overview

Uses balance sheet data with simplified proxies. Provides a **P-style score** when using μ̂ = r\_{i,t-1}.

**Barrier Convention**: F\_t = total liabilities (for banks).

**Rationale**: Deposits are liabilities and dominate bank funding. This follows standard practice for financial institutions.

**⏰ Timing Rule**: σ\_{E,t-1} uses t-1 only, E\_t, F\_t are at t, μ̂*t = r*{i,t-1}, unknowns V̂*t and σ̂*{V,t} computed at t, horizon is t to t+1.

### Key Simplifications

1. **No solver**: Direct formulas
2. **Book values**: From financial statements
3. **Drift**: μ̂ = r\_{i,t-1} (previous year’s equity return)

### Naive Proxies (Bharath-Shumway)

**Asset Value Proxy**:

V̂\_t = Ê\_t + F\_t

**Asset Volatility Proxy**:

σ̂\_{V,t} = (Ê\_t/(Ê\_t+F\_t)) × σ\_{E,t-1} + (F\_t/(Ê\_t+F\_t)) × σ̂\_{D,t}  
  
where σ̂\_{D,t} = 0.05 + 0.25 × σ\_{E,t-1}

**Drift Proxy**:

μ̂ = r\_{i,t-1} (previous year's equity return)

### Equity Proxy for Banks

**Critical**: Book equity means total equity from balance sheet, not assets minus deposits or debt securities.

Ê\_t = (Price-to-Book)\_t × (Total Equity)\_t

**Units**: Ensure consistent scaling (millions to dollars once).

### Accounting DD and PD Formulas

DD\_a = [ln(V̂\_t/F\_t) + (μ̂ - 0.5×σ̂²\_{V,t})×T] / (σ̂\_{V,t}×√T)  
  
PD\_a = Φ(-DD\_a) [P-style score]

**Important**: This is a P-style score. **Calibrate to real PD via logit or hazard model if publishing PD.**

### Accounting Variables Dictionary (with Timing)

| Variable | Symbol | Units | Timing | Role |
| --- | --- | --- | --- | --- |
| total\_equity | - | Millions USD | t | Book equity source |
| total\_liabilities | F\_t | USD | t | Barrier |
| E\_proxy | Ê\_t | USD | t | P/B × equity |
| sigma\_E | σ\_{E,t-1} | Decimal | t-1 | Volatility input |
| V\_hat | V̂\_t | USD | t | Ê\_t + F\_t |
| sigma\_V\_hat | σ̂\_{V,t} | Decimal | t | Weighted avg |
| mu\_hat | μ̂ | Decimal | t-1 | r\_{i,t-1} |
| DD\_a | DD\_a | Std devs | t | Final metric |
| PD\_a | PD\_a | Probability | t | P-style score |

## Comparing Both Approaches

| Aspect | Market (DD\_m) | Accounting (DD\_a) |
| --- | --- | --- |
| **Measure** | Q-measure (risk-neutral) | P-style score |
| **Drift** | μ = r\_{f,t} | μ̂ = r\_{i,t-1} |
| **Asset Value** | Solved: V\_t | Proxy: V̂\_t = Ê\_t + F\_t |
| **Asset Vol** | Solved: σ\_{V,t} | Proxy: σ̂\_{V,t} |
| **Complexity** | High (solver) | Low (formulas) |
| **Use Case** | Trading, market risk | Credit, regulation |

## Results Interpretation

### Market DD Distribution (1,424 observations, converged only)

| Statistic | DD\_m | Interpretation |
| --- | --- | --- |
| 10th pct | 3.41 | Moderate buffer |
| Median | 7.05 | Typical: effectively zero PD |
| 90th pct | 12.91 | Very safe |

**Convergence Rate**: Report percentage of converged vs. total observations.

## Walkthrough Examples

### Example 1: JPMorgan Chase (JPM) - 2019

#### Market Approach

**Step 1: Inputs (with timing)**

| Input | Value | Timing |
| --- | --- | --- |
| E\_t | $387.4B | t |
| σ\_{E,t-1} | 0.227 | t-1 |
| F\_t | $516.1B | t |
| r\_{f,t} | 0.0214 | t |
| T | 1 year | - |

**Step 2: Solver Output**

* V\_t = $892.6B
* σ\_{V,t} = 0.099

**Step 3: Calculate DD\_m (showing intermediate steps)**

Numerator:

ln(V\_t/F\_t) + (r\_{f,t} - 0.5×σ²\_{V,t})×T  
= ln(892.6/516.1) + (0.0214 - 0.5×0.099²)×1  
= 0.548 + 0.0165  
= 0.564

Denominator:

σ\_{V,t}×√T = 0.099×1 = 0.099

DD\_m:

DD\_m = 0.564 / 0.099 = 5.70

**Step 4: Calculate PD\_m**

PD\_m = Φ(-5.70) = 5.99×10⁻⁷% (effectively zero)

### Example 2: Bank of America (BAC) - 2019

#### Market Approach

**Step 1: Inputs**

| Input | Value | Timing |
| --- | --- | --- |
| E\_t | $265.3B | t |
| σ\_{E,t-1} | 0.279 | t-1 |
| F\_t | $430.2B | t |
| r\_{f,t} | 0.0214 | t |

**Step 2: Solver Output**

* V\_t = $686.3B
* σ\_{V,t} = 0.108

**Step 3: Calculate DD\_m (showing intermediate steps)**

Numerator:

ln(686.3/430.2) + (0.0214 - 0.5×0.108²)×1  
= 0.467 + 0.0156  
= 0.483

Denominator:

σ\_{V,t}×√T = 0.108×1 = 0.108

DD\_m:

DD\_m = 0.483 / 0.108 = 4.47

**Step 4: Calculate PD\_m**

PD\_m = Φ(-4.47) = 0.000391% (3.91×10⁻⁴%)

### Comparison

| Bank | DD\_m | PD\_m | V\_t | σ\_{V,t} |
| --- | --- | --- | --- | --- |
| JPM | 5.70 | 5.99×10⁻⁷% | $892.6B | 9.9% |
| BAC | 4.47 | 3.91×10⁻⁴% | $686.3B | 10.8% |

## Appendix: Why d₂ is the z-score of ln V\_T vs ln F

### The Question

Under the Merton model, asset value V\_t evolves as:

V\_T = V\_t × exp[(μ - 0.5×σ²\_V)×T + σ\_V×√T×Z]

where Z ~ N(0,1). Taking logs:

ln V\_T = ln V\_t + (μ - 0.5×σ²\_V)×T + σ\_V×√T×Z

Default occurs when V\_T < F, or equivalently ln V\_T < ln F.

### The z-score

The z-score measures how many standard deviations ln V\_T is above ln F:

z = [E[ln V\_T] - ln F] / SD[ln V\_T]  
 = [ln V\_t + (μ - 0.5×σ²\_V)×T - ln F] / (σ\_V×√T)  
 = [ln(V\_t/F) + (μ - 0.5×σ²\_V)×T] / (σ\_V×√T)  
 = DD

When μ = r\_f (risk-neutral drift):

z = d₂

Therefore, **d₂ is the standardized distance between expected log asset value and log debt level**, measured in volatility units. The probability of default is:

P(ln V\_T < ln F) = P(Z < -d₂) = Φ(-d₂)

This is why DD = d₂ and PD = Φ(-d₂) under the risk-neutral measure.

## References

1. Merton, R. C. (1974). “On the Pricing of Corporate Debt: The Risk Structure of Interest Rates.” *Journal of Finance*, 29(2), 449-470.
2. Bharath, S. T., & Shumway, T. (2008). “Forecasting Default with the Merton Distance to Default Model.” *Review of Financial Studies*, 21(3), 1339-1369.
3. Tepe, M., Thastrom, P., and Chang, R. (2022). “How Does ESG Activities Affect Default Risk.” FI Consulting White Paper.
4. Crosbie, P., & Bohn, J. (2003). “Modeling Default Risk.” Moody’s KMV Technical Document.