# TSIA202A – Booklet of Exercises

Marco Cagnazzo, François Roueff

# Contents



# 1 General reminders and notation

## 1.1 Gaussian r.v.'s, vectors, processes

Except for the zero-variance case, a real valued **Gaussian random variable** $X$ has the following probability density function (pdf):

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

A random vector $\mathbf{X} = [X_1, X_2, \dots, X_n]^T$ is a **Gaussian random vector** if and only if, $\forall u \in \mathbb{R}^n, Y = u^T\mathbf{X} = \sum_{i=1}^n u_i X_i$ is a Gaussian r.v. The pdf of a Gaussian vector is completely defined by the mean vector $\mu = \mathbb{E}[\mathbf{X}]$ and the covariance matrix $\Gamma = \mathbb{E}\left[\mathbf{X}^c \mathbf{X}^{cT}\right]$

A random process $\{X_t, t \in \mathbb{Z}\}$ is a **Gaussian random process** if and only if for all finite set of indexes $I \subset \mathbb{Z}, I = \{t_1, t_2, \dots, t_n\}$, the random vector $[X_{t_1}, X_{t_2}, \dots, X_{t_n}]^T$ is a Gaussian vector.

## 1.2 Functions of r.v.'s and of random processes

Let $X$ be a real-valued r.v. and let $g$ a real function. Let us suppose that $g$ is derivable over $\mathbb{R}$, except for a set whose measure is zero, *e.g.*, a numerable set of points. If we define a new r.v. $Y = g(X)$, the pdf of $Y$ is related to that of $X$ as follows:

$$p_Y(y) = \begin{cases} 0 & \text{if the equation in the variable } x,\ g(x) = y, \text{ has no solution} \\ \sum_{i=1}^{N_y} \frac{p_X(x_i(y))}{|g'(x_i(y))|} & \text{if } g(x) = y \text{ has } N_y \geq 1 \text{ solutions, referred to as } \{x_i(y)\}_{i=1,\dots,N_y} \end{cases}$$

We can also consider function of multiple r.v.'s. A particularly interesting case is when a random process is obtained by applying a function to another random process:

$$X_t = g_t(\{Z_s, s \in \mathbb{Z}\})$$

| Shortcut | Meaning |
|---|---|
| $\overline{X}$ | Conjugate of $X$ |
| $A^T$ | Transpose pf $A$ |
| $A^H$ | Hermitian of $A$, *i.e.* $\overline{A^T}$ |
| $\mathbb{N}_0$ | Natural numbers including zero |
| $\mathbb{R}^+$ | Positive real numbers: $\{x \in \mathbb{R} \| x > 0\}$ |
| $\mathbb{R}_0^+$ | Non-negative real numbers: $\{x \in \mathbb{R} \| x \geq 0\}$ |
| $\mathbf{1}_A(x)$ | Indicator function of set $A$: $\mathbf{1}_A(x) = 1$ if and only if $x \in A$; otherwise, $\mathbf{1}_A(x) = 0$ |
| r.v. | random variable |
| pdf | probability density function |
| $X \sim P$ | $X$ is a r.v. distributed with law $P$ |
| $\mathcal{N}\left(\mu, \sigma^2\right)$ | Gaussian r.v. with mean $\mu$ and variance $\sigma^2$ |
| $\mathbb{E}\left[X\right]$ | Expectation of the r.v. $X$ |
| $X^c$ | Centered version of $X$: $X^c = X - \mathbb{E}\left[X\right]$ |
| $\mathsf{Var}\left(X\right)$ | Variance of the r.v. $X$: $\mathsf{Var}\left(X\right) = \mathbb{E}\left[\lvert X^c\rvert^2\right]$ |
| $\mathsf{Cov}\left(X, Y\right)$ | $\mathbb{E}\left[X^c \overline{Y^c}\right]$ |
| $\{X_t, t \in \mathbb{Z}\}$ | Discrete random process |
| s.o.1 | A process $\{X_t, t \in \mathbb{Z}\}$ is *stationary at order 1* if and only if $\mathbb{E}\left[X_t\right]$ does not depend on $t$ |
| s.o.2 | A process $\{X_t, t \in \mathbb{Z}\}$ is *stationary at order 2*, if and only if $\forall t \in \mathbb{Z}$, $\mathbb{E}\left[\lvert X_t\rvert^2\right] < +\infty$ and $\forall t, h \in \mathbb{Z}, \mathsf{Cov}\left(X_t, X_{t+h}\right)$ does not depend on $t$ |
| w.s. | weakly stationary, *i.e.*, s.o.1 and s.o.2 |
| $\gamma_X(h)$ | For $\{X_t, t \in \mathbb{Z}\}$ s.o.2, $\gamma_X(h) = \mathsf{Cov}\left(X_{t+h}, X_t\right) = \mathsf{Cov}\left(X_h, X_0\right)$ |
| $\delta_h$ | The Kronecker's delta: $\delta : h \in \mathbb{Z} \to \delta_h$; if $h = 0$, $\delta_h = 1$; otherwise, $\delta_h = 0$ |

Table 1: Shortcuts and notation used throughout this document.

A special case is when the transformation is the same at each time (*i.e.* $g$ does not depend on $t$) and it has a finite number of inputs. Apart from a time shift, this can be written as:

$$X_t = g(Z_t, Z_{t-1}, \ldots, Z_{t-k+1})$$

This is called a *moving transformation*. It can be shown that, for a moving transformation, if $g$ is measurable and $\{Z_t, t \in \mathbb{Z}\}$ i.i.d., then $\{X_t, t \in \mathbb{Z}\}$ is strictly stationary.

A particularly interesting case of moving transformation is a linear filter:

$$Y_t = \sum_{n \in \mathbb{Z}} \alpha_n X_{t-n}$$

If the support of $\alpha$ is finite, this filter is called Finite Impulse Response (FIR); otherwise it is an Infinite Impulse Response (IIR).

### 1.2.1 Example: inversion of a FIR

Let us remember a particularly simple case of invertible filter. Let $\theta \in \mathbb{C}$ and $|\theta| < 1$. We introduce the following $L^1$ sequences:

$$\begin{aligned} a &: n \in \mathbb{Z} \to \delta_n - \theta\delta_{n-1} \\ b &: n \in \mathbb{Z} \to \begin{cases} \theta^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \\ c = (a * b) &: n \in \mathbb{Z} \to \sum_{k \in \mathbb{Z}} a_k b_{n-k} \end{aligned}$$

It is easy to find that $(a * b) = \delta$. In that case, we say that a FIR having $a$ as impulse response, can be inverted by an IIR having $b$ as impulse response, since the cascade of $a$ and $b$ will not change an input signal.

Let us show that $c = \delta$.

$$c_n = \sum_{k\in\mathbb{Z}} a_k b_{n-k} = b_n - \theta \cdot b_{n-1} = \begin{cases} 0 - \theta \cdot 0 = 0 & \text{if } n < 0 \\ 1 - \theta \cdot 0 = 1 & \text{if } n = 0 \\ \theta^n - \theta \cdot \theta^{n-1} = 0 & \text{if } n > 0 \end{cases} = \delta_n$$

## 1.3 Autocovariance

$$\boxed{\begin{aligned}
\mathsf{Cov}\,(X,Y) &= \mathbb{E}\left[X^c \overline{Y^c}\right] \\
\mathsf{Cov}\,(X,Y) &= \mathbb{E}\left[X\overline{Y}\right] - \mathbb{E}\,[X]\,\mathbb{E}\left[\overline{Y}\right] \\
\overline{\mathsf{Cov}\,(X,Y)} &= \mathsf{Cov}\,(Y,X) \\
\mathsf{Cov}\,(X+a,Y) &= \mathsf{Cov}\,(X,Y) \\
\mathsf{Cov}\,(X,Y+a) &= \mathsf{Cov}\,(X,Y) \\
\mathsf{Cov}\,(aX,Y) &= a\mathsf{Cov}\,(X,Y) \\
\mathsf{Cov}\,(X,aY) &= \overline{a}\mathsf{Cov}\,(X,Y) \\
\mathsf{Cov}\,(X_1+X_2,Y) &= \mathsf{Cov}\,(X_1,Y) + \mathsf{Cov}\,(X_2,Y) \\
\mathsf{Cov}\,(X,Y_1+Y_2) &= \mathsf{Cov}\,(X,Y_1) + \mathsf{Cov}\,(X,Y_2)
\end{aligned}}$$

Table 2: Covariance properties. $X, X_1, X_2, Y$ are complex or real r.v.'s; $a \in \mathbb{C}$.

The covariance of two r.v.'s has several interesting properties resumed in Tab. 2. Two real r.v. with null covariance are said to be *uncorrelated.* Two complex r.v. with null covariance are said to be *orthogonal*, while if also the *pseudo-covariance* $\mathbb{E}\,[XY]$ is null, they are said *uncorrelated.* Independent r.v.'s are uncorrelated while the converse is not true in general. A notable exception is when $(X, Y)$ is a Gaussian vector (but not when $X$ and $Y$ are marginally Gaussian and not jointly Gaussian): in that case, uncorrelatedness implies independence.

The covariance allows to define a scalar product between two r.v.'s: $\langle X_t, X_s\rangle = \mathsf{Cov}\,(X_t, X_s)$. The (squared) norm of a r.v.'s is then its variance. Note that this scalar product is not affected by the mean of the r.v.'s, since neither the covariance is. For example, a zero-norm r.v. has a null variance, but can have any mean.

We can also introduce the concept of linear independent r.v.'s. $(X_1, \ldots, X_k)$ is a set of linearly independent r.v.'s if and only if $\forall a \in \mathbb{R}^k - \{\mathbf{0}\}, \|\sum_{i=1}^k a_i X_i\|^2 = \mathsf{Var}\left(\sum_{i=1}^k a_i X_i\right) > 0$.

Note also that, if $(X_1, \ldots, X_k)$ are not linearly independent, this means that one of the $X_i$ can be expressed as a linear combination of the other r.v.'s, up to an additive constant, which does not affect the covariance. This constant is null in the case $\mathbb{E}\,[(X_1, \ldots, X_k)] = \mathbf{0}$.

For the sake of simplicity, let us prove that for some $i$, $X_i$ is a linear combination of the other r.v.'s *only in the case of a centered vector.* In this case it must exist $a \in \mathbb{R}^k - \{\mathbf{0}\}$ such that $\mathsf{Var}\left(\sum_{i=1}^k a_i X_i\right) = 0$. The vector $a$ must have at least one non-zero component, let it be $a_j$. Let also $Y = \sum_{i=1}^k a_i X_i$; since its variance is zero, $Y = \mathbb{E}\,[Y] = 0$. This implies:

$$\begin{aligned}
0 &= \sum_{i=1}^k a_i X_i = a_j X_j + \sum_{i\neq j} a_i X_i \\
a_j X_j &= -\sum_{i\neq j} a_i X_i \\
X_j &= -\sum_{i\neq j} \frac{a_i}{a_j} X_i
\end{aligned}$$

Then $X_j$ is a linear combination of other r.v.'s. It can be shown that, if the $X_i$ are not centered, the same result holds up to a constant: $X_j = -\sum_{i\neq j} \frac{a_i}{a_j} X_i + \sum_{i=1}^k \frac{a_i}{a_j} \mathbb{E}\,[X_i]$.

The **covariance matrix** of a complex-valued random vector $\mathbf{X} = [X_1, X_2, \ldots, X_n]^T$ is $\Gamma = \mathbb{E}\left[\mathbf{X}^c \mathbf{X}^{cH}\right]$. In other words, $\Gamma_{i,j} = \mathsf{Cov}\,(X_i, X_j)$. It is an Hermitian, non-negative matrix, since for all $u \in \mathbb{C}^n$ the random variable $Y = u^H X$ shall have a non negative variance:

$$\begin{aligned} 0 \leq \mathsf{Var}\left(Y\right) &= \mathbb{E}\left[\|u^H \mathbf{X} - \mathbb{E}\left[u^H \mathbf{X}\right]\|^2\right] = \mathbb{E}\left[\|u^H (\mathbf{X} - \mathbb{E}\left[\mathbf{X}\right]\|^2\right] \\ &= \mathbb{E}\left[\|u^H \mathbf{X}^c\|^2\right] = \mathbb{E}\left[u^H \mathbf{X}^c \mathbf{X}^{cH} u\right] \\ &= u^H \mathbb{E}\left[\mathbf{X}^c \mathbf{X}^{cH}\right] u = u^H \Gamma u \end{aligned}$$

The autocovariance function (acf) of a random process $\{X_t, t \in \mathbb{Z}\}$ is a function of two discrete variables $t$ and $s$:

$$\gamma(t,s) = \mathsf{Cov}\left(X_t, X_s\right)$$

A **weakly stationary process** is a process s.o.1 and s.o.2, therefore, all $X_t$ have finite quadratic mean, the mean of $X_t$ is the same for all $t$ and the autocovariance function only depend on the delay $t-s$:

$$\gamma(t,s) = \gamma(t-s) = \mathsf{Cov}\left(X_{t-s}, X_0\right)$$

In that case, we use a single-parameter notation for $\gamma$:

$$\gamma(h) = \mathsf{Cov}\left(X_h, X_0\right)$$

The acf of weakly stationary processes is an Hermitian and non-negative function. The maximum of $|\gamma|$ is in 0. The normalized acf, $\rho(h) = \frac{\gamma(h)}{\gamma(0)}$ is referred to as autocorrelation function.

### 1.4 Noise

A **weak white noise** is a real-valued, weakly stationary process $\{X_t, t \in \mathbb{Z}\}$, with zero-mean and impulsive acf: $\gamma_X(h) = \sigma^2 \delta(h)$. In other words, for all $t \neq s$, $X_t$ and $X_s$ are uncorrelated variables.

A **strong white noise** is a real-valued, zero-mean, i.i.d. process. Note that a strong white noise is also a weak white noise, since i.i.d. implies weak stationarity and impulsive acf. On the contrary, a weak white noise is not necessarily a strong one, since uncorrelated r.v.'s may be dependent.

In both cases, we usually consider finite, positive variance $\sigma^2 = \mathsf{Var}\left(X_t\right)$.

## 2 Gaussian vectors

*Exercise* 2.1 (Functions of Gaussian random variables). Let $X \sim \mathcal{N}(0,1)$, $a \in \mathbb{R}^+$ and $Y^a = X\mathbf{1}_{\{|X|<a\}} - X\mathbf{1}_{\{|X|\geq a\}}$.

1. Give the law of $Y^a$
2. Compute $\mathsf{Cov}\left(X, Y^a\right)$. For which value $a_0$ of a the covariance is null? Are $X$ and $Y^{a_0}$ independent?
3. Is $(X, Y^{a_0})$ a Gaussian vector?
4. For $a \neq a_0$, is $(X, Y^a)$ a Gaussian vector?

## 3 Stationarity

*Exercise* 3.1 (Uncorrelated processes). Let $\{X_t, t \in \mathbb{Z}\}$ and $\{Y_t, t \in \mathbb{Z}\}$ be two weakly stationary (w.s.), uncorrelated random processes. Show that $\{Z_t = X_t + Y_t, t \in \mathbb{Z}\}$ is weakly stationary. Find the covariance function of $Z_t$ from those of $X_t$ and $Y_t$ and the spectral measure of $Z_t$ from those of $X_t$ and $Y_t$

*Exercise* 3.2 (Functions of strong white noise). Let $\{\epsilon_t, t \in \mathbb{Z}\}$ be a strong white noise with $\mathbb{E}\left[\epsilon_0^2\right] < \infty$. For each of the following processes (functions of the white noise), find out if they are weakly stationary or strictly stationary.

1. $W_t = a + b\epsilon_t + c\epsilon_{t-1}$, with $a, b, c$ real numbers

2. $X_t = \epsilon_t \epsilon_{t-1}$
3. $Y_t = (-1)^t \epsilon_t$
4. $Z_t = \epsilon_t + Y_t$

*Exercise* 3.3 (Structured covariance matrix). Let us consider a real number $\rho$; we define $\Sigma_2 = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$. Moreover, let $\forall t \in \mathbb{Z}$, $\Sigma_t$ be a $t \times t$ matrix with diagonal elements equal to 1, and out-of-diagonal elements equal to $\rho$:

$$\Sigma_t = \begin{bmatrix} 1 & \rho & \rho & \dots & \rho \\ \rho & 1 & \rho & \dots & \rho \\ \rho & \rho & 1 & \dots & \rho \\ \dots & \dots & \dots & \dots & \dots \\ \rho & \rho & \rho & \dots & 1 \end{bmatrix}$$

1. Which condition on $\rho$ must be imposed such that $\Sigma_t$ is a covariance matrix for all $t$? Suggestion: decompose $\Sigma_t = \alpha I + A$, where $A$ is matrix with easy-to-find eigenvalues.
2. Build a stationary process having $\Sigma_t$ as auto-covariance matrix for all $t$.

# 4 Covariance, spectral measure and spectral density

*Exercise* 4.1 (Functions of weak white noise). Let $\{Z_t, t \in \mathbb{Z}\}$ be a weak white noise, centered, with variance $\sigma^2$. Let $a, b, c \in \mathbb{R}$. Are the following processes s.o.2? If yes, compute the autocovariance function and the spectral measure.

1. $X_t = a + bZ_0$
2. $X_t = Z_0 \cos(ct)$
3. $X_t = a + bZ_t + cZ_{t-1}$
4. $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$
5. $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$

Reminders:

$$\begin{aligned} \gamma(h) &= \int_{-\pi}^{\pi} e^{ih\lambda} \nu(d\lambda) \\ \gamma(h) &= \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda && \text{if the density } f(\cdot) \text{ exists} \\ f(\lambda) &= \frac{1}{2\pi} \sum_{h \in \mathbb{Z}} \gamma(h) e^{-ih\lambda} && \text{if } \gamma \in L^1(\mathbb{Z}) \end{aligned}$$

*Exercise* 4.2 (Autocovariance function characterization). Let us introduce the following sequence on the integers:

$$\gamma : h \in \to \mathbb{Z} \gamma(h) = \begin{cases} 1 & \text{if } h = 0 \\ \rho & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases}$$

We want to show that such a function is an autocovariance function if and only if $|\rho| \leq \frac{1}{2}$.

1. Let $\Gamma_k$ be a $k \times k$ matrix such that $\forall i, j \in \{1, 2, \dots, k\}, \Gamma_k(i, j) = \gamma(i - j)$.

$$\Gamma_k = \begin{bmatrix} 1 & \rho & 0 & 0 & \dots & 0 & 0 \\ \rho & 1 & \rho & 0 & \dots & 0 & 0 \\ 0 & \rho & 1 & \rho & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \rho & 1 \end{bmatrix}$$

   Find the recurrence equation among the determinants of matrices $\Gamma_k$

2. Show that if $\rho$ is not greater than a given value, $\Gamma_k$ is positive definite for all $k$. Use or the previous point or the Herglotz theorem.

3. Build a s.o.2 process having $\gamma(h)$ as autocovariance function.

*Exercise* 4.3 (Band-limited stationary process). Let $S(f) = \mathbf{1}_{(-f_0, f_0)}(f)$, with $f_0 \in (0, \pi)$ be the spectral density of a stationary process.

1. Compute the autocovariance function.

2. Is it $L^1$?

*Exercise* 4.4 (Process generated by linear combination). Let $\gamma$ be the autocovariance function of a stationary, zero-mean process. Let us suppose that it exist a finite subset of this process such that the corresponding autocovariance matrix is not invertible, *i.e.*, it is not full rank.

1. Show that either $\gamma(0) = 0$, or it exists $k \geq 1$ such that:

   - $X_{k+1} \in \mathsf{Vect}(X_1, \dots, X_k)$; and
   - $(X_1, \dots, X_k)$ is a set of linearly independent vectors: $\forall a \in \mathbb{R}^k - \{\mathbf{0}\}, \mathsf{Var}\left(\sum_{i=1}^k a_i X_i\right) > 0$.

2. Let $\Gamma_k$ be the autocovariance matrix of $X_1, \dots, X_k$. Find a property of its minimum eigenvalue.

3. Show that the process $\{X_t, t \in \mathbb{Z}\}$ is linearly predictable, *i.e.*, for all $p \geq 1$, there exists a set of $k$ scalars $\phi_{p,1}, \phi_{p,2}, \dots, \phi_{p,k}$ such that:

$$X_{k+p} = \sum_{\ell=1}^{k} \phi_{p,\ell} X_\ell. \tag{1}$$

4. Show that $\sup_{p \geq 1} \sum_{\ell=1}^k |\phi_{p,\ell}|^2 < \infty$.

5. Deduce that, if in addition $\lim_{|t| \to \infty} \gamma(t) = 0$, then $\gamma(0) = 0$.

# 5 Linear filtering, ARMA processes

*Exercise* 5.1 (Linear filtering and stationarity). Let $\beta \in \mathbb{R}$, $\{S_t, t \in \mathbb{Z}\}$ a w.s., periodical (period = 4) real process, and $\{X_t, t \in \mathbb{Z}\}$ a w.s. real process, uncorrelated with $S_t$.

Let us consider the process $\{Y_t = \beta t + S_t + X_t, t \in \mathbb{Z}\}$.

1. Is $\{Y_t, t \in \mathbb{Z}\}$ w.s.?

2. Let us refer to the back-shift operator as $B$, and let us consider the process $\{\bar{S}_t = \left(1 + B + B^2 + B^3\right) \circ S_t, t \in \mathbb{Z}\}$. Show that $\gamma$ is periodic and that $\bar{S}_t = S_0 + S_1 + S_2 + S_3$

3. Let us consider the process $\{Z_t = (1 - B) \circ \left(1 + B + B^2 + B^3\right) \circ S_t, t \in \mathbb{Z}\}$. Show that $\{Z_t, t \in \mathbb{Z}\}$ is w.s. and compute $\gamma_Z$ as a function of $\gamma_X$ (autocovariance functions).

4. Find the shape of the spectral measure $\mu$ of $\{S_t, t \in \mathbb{Z}\}$.

5. Find the spectral measure of $(1 - B^4) \circ Y_t$ as a function of the spectral measure of $\{X_t, t \in \mathbb{Z}\}$.

*Exercise* 5.2 (Characterization of MA($q$)). Let $q \in \mathbb{Z}$ and $q > 0$. Let $\{X_t, t \in \mathbb{Z}\}$ be a centered w.s. real process and let $\gamma$ be its autocovariance function. Let us suppose that $\gamma$ has a compact support, *i.e.* $\forall t > q, \gamma(t) = 0$.

We also introduce

$$\mathcal{H}_t = \mathsf{Vect}(X_s, s \leq t)$$
$$\widetilde{X}_t = \mathsf{Proj}(X_t | \mathcal{H}_{t-1})$$

1. Recall why $Z_t = X_t - \widetilde{X}_t$ is a white noise.
2. Show that $X_t \perp \mathcal{H}_{t-q-1}$.
3. Deduce that $X_t \in \mathsf{Vect}(Z_s, s \in \{t, t-1, \dots t-q\})$.
4. Show that $\{X_t, t \in \mathbb{Z}\}$ is a MA($q$) process.

*Exercise* 5.3 (Sum of MA processes). Let $\{X_t, t \in \mathbb{Z}\}$ and $\{Y_t, t \in \mathbb{Z}\}$ be two real uncorrelated MA processes of order $q$ and $p$ respectively:

$$X_t = \epsilon_t + \sum_{n=1}^{q} \theta_n \epsilon_{t-n} \qquad\qquad Y_t = \eta_t + \sum_{n=1}^{p} \rho_n \eta_{t-n}$$

where $\forall n \in \{1, \dots, q\}, \theta_n \in \mathbb{R}$, $\forall n \in \{1, \dots, p\}, \rho_n \in \mathbb{R}$, $\{\epsilon_t, t \in \mathbb{Z}\}$ and $\{\eta_t, t \in \mathbb{Z}\}$ are white noises whose variances are respectively noted as $\sigma_\epsilon^2$ and $\sigma_\eta^2$. Let us also introduce $\{Z_t = X_t + Y_t, t \in \mathbb{Z}\}$.

1. Which kind of process is $\{Z_t, t \in \mathbb{Z}\}$?
2. Let us consider the case $p$=1 , $q$=1, $0<\theta_1<1$ and $0<\rho_1<1$. Show that $\{\epsilon_t, t \in \mathbb{Z}\}$ and $\{\eta_t, t \in \mathbb{Z}\}$ are uncorrelated.
3. For $p$=1 , $q$=1, $\theta_1=\rho_1=\theta$ and $0<\theta<1$, what is the innovation process for $\{Z_t, t \in \mathbb{Z}\}$?
4. For $p$=1 , $q$=1, $0<\theta_1<1$ and $0<\rho_1<1$, compute the variance of the innovation of $\{Z_t, t \in \mathbb{Z}\}$.

*Exercise* 5.4 (Sum of AR processes). Let $\{X_t, t \in \mathbb{Z}\}$ and $\{Y_t, t \in \mathbb{Z}\}$ be two real uncorrelated AR(1) processes such that

$$X_t = aX_{t-1} + \epsilon_t$$
$$Y_t = bY_{t-1} + \eta_t$$

where $a \in ]0, 1[$, $b \in ]0, 1[$. Moreover, $\{\epsilon_t, t \in \mathbb{Z}\}$ and $\{\eta_t, t \in \mathbb{Z}\}$ are white noises whose variances are respectively noted as $\sigma_\epsilon^2$ and $\sigma_\eta^2$. Let us also introduce $\{Z_t = X_t + Y_t, t \in \mathbb{Z}\}$.

1. Show that there exists a white noise $\{\xi_t, t \in \mathbb{Z}\}$ and a real number $\theta \in ]-1, 1[$ such that:
$$Z_t - (a+b)Z_{t-1} + abZ_{t-2} = \xi_t - \theta\xi_{t-1}.$$
2. Show that:
$$\xi_t = \epsilon_t + (\theta - b)\sum_{k=0}^{\infty} \theta^k \epsilon_{t-1-k} + \eta_t + (\theta - a)\sum_{h=0}^{\infty} \theta^h \eta_{t-1-h}.$$
3. Compute the prediction of $Z_{t+1}$ when $(X_s, s \leq t)$ and $(Y_s, s \leq t)$ are all known.
4. Compute the prediction of $Z_{t+1}$ when $(Z_s, s \leq t)$ are all known.

5. Compare the variances of the prediction errors in the two previous cases.

*Exercise* 5.5 (Forward/backward prediction of a MA(1) process). Let $\{X_t = Z_t + \theta Z_{t-1}, t \in \mathbb{Z}\}$ be a real w.s. process, with $\{Z_t, t \in \mathbb{Z}\}$ centered white noise and $\theta \in ]-1,1[$.

1. Find the best (in terms of MSE) linear prediction of $X_3$ as a function of $X_1$ and $X_2$.
2. Find the best linear prediction of $X_3$ as a function of $X_4$ and $X_5$.
3. Find the best linear prediction of $X_3$ as a function of $X_1$, $X_2$, $X_4$ and $X_5$.

*Exercise* 5.6 (Canonical representation of an ARMA process). Let $\{X_t, t \in \mathbb{Z}\}$ be a centered, s.o.2 process satisfying the recurrence equation

$$X_t - 2X_{t-1} = \epsilon_t + 4\epsilon_{t-1}$$

where $\{\epsilon_t, t \in \mathbb{Z}\}$ is a white noise with variance $\sigma^2$.

1. Compute the spectral density of $\{X_t, t \in \mathbb{Z}\}$.
2. Compute the canonical representation of $\{X_t, t \in \mathbb{Z}\}$.
3. What is the variance of the innovation of $\{X_t, t \in \mathbb{Z}\}$?
4. Find a representation of $X_t$ as a function of $(\epsilon_s, s \leq t)$.

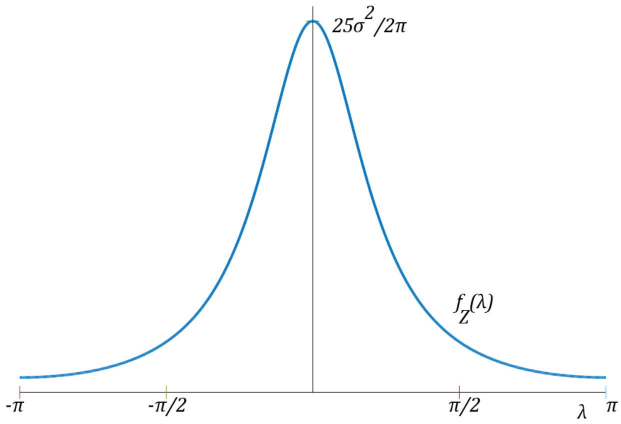


Figure 1: $f_Z(\lambda) = \frac{\sigma^2}{2\pi}\frac{8\cos\lambda+17}{5-4\cos\lambda}$

*Exercise* 5.7 (ACF of an AR(1) process). Let $\{X_t, t \in \mathbb{Z}\}$ be a w.s. process defined by:

$$X_t - \phi X_{t-1} = \epsilon_t$$

where $\phi \in ]-1,1[$ and $\{\epsilon_t, t \in \mathbb{Z}\}$ is a centered WN with variance $\sigma_\epsilon^2$.

1. Compute the weights $\psi_i$ of the representation

$$X_t = \sum_{k \in \mathbb{Z}} \psi_k \epsilon_{t-k}$$

2. Deduce the autocovariance function of $\{X_t, t \in \mathbb{Z}\}$.

## 6 Solutions

**Solution of Exercise 2.1** 1. The r.v. $Y$ satisfies the following equation: $Y = \begin{cases} X & \text{if } |X| < a \\ -X & \text{if } |X| > a \end{cases}$

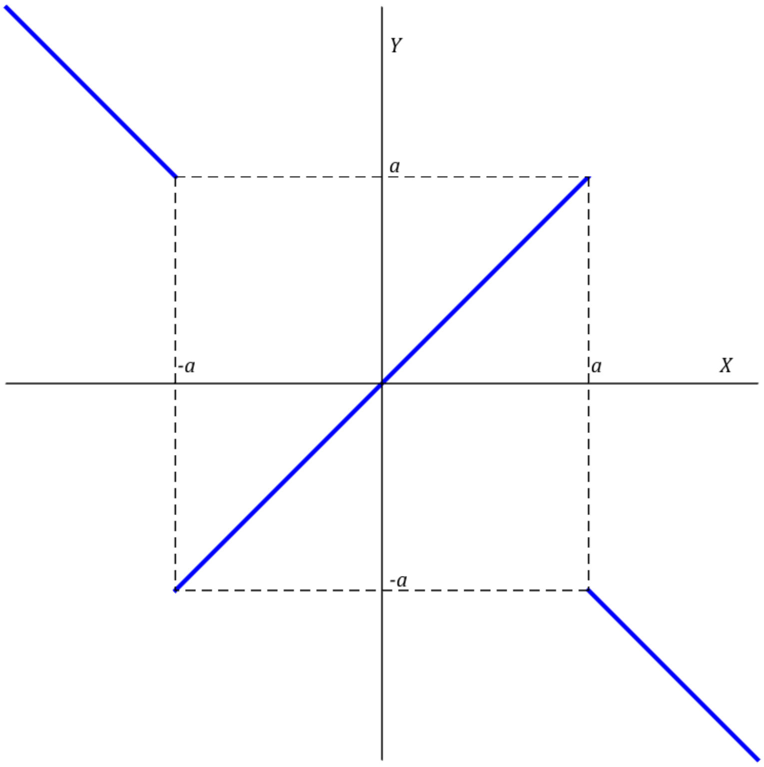


Figure 2: $Y = g(X)$

If $|y| < a$ $\quad p_Y(y) = p_X(y) = \dfrac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$

If $|y| > a$ $\quad p_Y(y) = p_X(-y) = \dfrac{e^{-\frac{(-y)^2}{2}}}{\sqrt{2\pi}}$

Thus, $Y \sim \mathcal{N}(0,1)$

2. Let us compute the covariance of $X$ and $Y^a$:

$$\begin{aligned}\mathsf{Cov}(X, Y^a) &= \mathbb{E}[XY^a] = \mathbb{E}\left[X^2\mathbf{1}_{\{|X|<a\}} - X^2\mathbf{1}_{\{|X|\geq a\}}\right] \\ &= \mathbb{E}\left[X^2\left(\mathbf{1}_{\{|X|<a\}} - \mathbf{1}_{\{|X|\geq a\}}\right)\right] = \mathbb{E}\left[X^2\left(2\mathbf{1}_{\{|X|<a\}} - 1\right)\right] \\ &= 2\mathbb{E}\left[X^2\mathbf{1}_{\{|X|<a\}}\right] - \mathbb{E}\left[X^2\right] = \sqrt{\frac{2}{\pi}}\int_{-a}^{a} x^2 e^{-\frac{x^2}{2}}\,dx - 1 = h(a)\end{aligned}$$

The function $h : a \to h(a)$ is continuous and strictly increasing. Moreover $h(0) = -1$ and $\lim_{a\to+\infty} h(a) = \mathbb{E}\left[X^2\right] = 1$. Therefore, $\exists a_0 \in ]0, +\infty[$: $h(a_0) = 0$. For such a value $a_0$, $X$ and $Y^{a_0}$ are uncorrelated but they are not independent, since $Y|X$ is deterministic. Another way to show that $X$ and $Y^{a_0}$ are not independent is the following. Since they are both Gaussian, if they were independent, the vector $(X, Y^{a_0})$ would be a Gaussian Vector, therefore $X + Y^{a_0}$ would be Gaussian. But this is impossible, since $X + Y^{a_0} = 2X\mathbf{1}_{|X|<a_0}$ cannot be larger than $2a_0$. This also answers to points 3. As for point 4, the since $X + Y^a$ is not a Gaussian r.v. for any real positive $a$, the vector $(X, Y^a)$ cannot be a Gaussian vector.

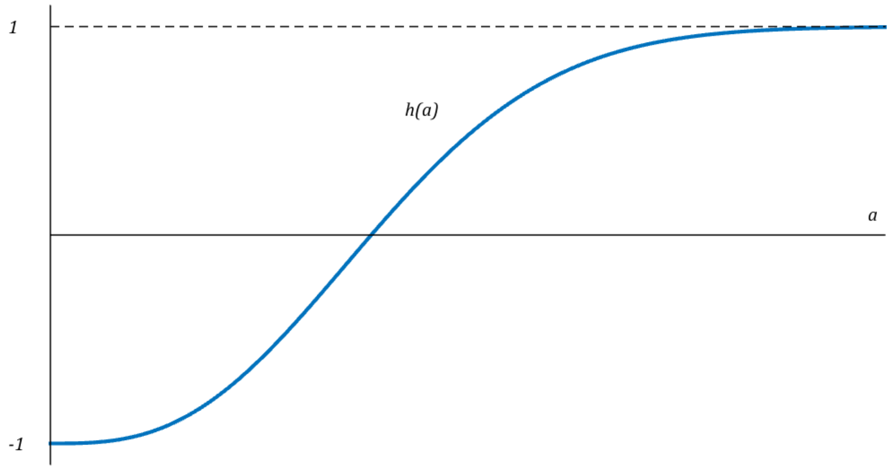


Figure 3: Function $h(a) = \mathsf{Cov}\left(X, Y^a\right)$

**Solution of Exercise 3.1** First, since $\{X_t, t \in \mathbb{Z}\}$ and $\{Y_t, t \in \mathbb{Z}\}$ are w.s.,

$$\mathbb{E}\left[X_t\right] = \mu_X \qquad \mathbb{E}\left[Y_t\right] = \mu_Y$$
$$\mathsf{Cov}\left(X_t, X_s\right) = \gamma_X(t-s) \qquad \mathsf{Cov}\left(Y_t, Y_s\right) = \gamma_Y(t-s)$$

Moreover, $\{X_t, t \in \mathbb{Z}\}$ and $\{Y_t, t \in \mathbb{Z}\}$ are uncorrelated, meaning that $\forall t, s, \mathsf{Cov}\left(X_t, Y_s\right) = 0$, therefore we find:

$$\begin{aligned}
\mathbb{E}\left[Z_t\right] &= \mathbb{E}\left[X_t + Y_t\right] = \mu_X + \mu_Y \\
\mathsf{Cov}\left(Z_t, Z_s\right) &= \mathsf{Cov}\left(X_t + Y_t, X_s + Y_s\right) = \mathsf{Cov}\left(X_t, X_s\right) + \mathsf{Cov}\left(X_t, Y_s\right) + \mathsf{Cov}\left(Y_t, X_s\right) + \mathsf{Cov}\left(Y_t, Y_s\right) \\
&= \gamma_X(t-s) + \gamma_Y(t-s)
\end{aligned}$$

Therefore $\{Z_t, t \in \mathbb{Z}\}$ is w.s. with $\mathbb{E}\left[Z_t\right] = \mu_X + \mu_Y$ and $\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h)$. From the previous point we deduce that the spectral measure of $\{Z_t, t \in \mathbb{Z}\}$ is the sum of those of $\{X_t, t \in \mathbb{Z}\}$ and $\{Y_t, t \in \mathbb{Z}\}$.

**Solution of Exercise 3.2** We remind that if $g$ is a measurable moving transformation, it preserves the strict stationarity, meaning that, since $\{\epsilon_t, t \in \mathbb{Z}\}$ is strictly stationary, so $g(\epsilon)$ is.

1. and 2. We are in the case of a moving transformation. In both cases $g$ is measurable, so $\{W_t, t \in \mathbb{Z}\}$ and $\{X_t, t \in \mathbb{Z}\}$ are strictly stationary.

3. This is not a moving transformation. Actually, $Y_t$ is alternatively equal to $\epsilon_t$ and $-\epsilon_t$. Since $\{\epsilon_t, t \in \mathbb{Z}\}$ is iid, the pdf of $Y_t$ is

$$p_Y(y) = \begin{cases} p_\epsilon(y) & \text{if } t \text{ is even} \\ p_\epsilon(-y) & \text{if } t \text{ is odd} \end{cases}$$

Therefore, if the pdf of $\epsilon_t$ is symmetric, $\{Y_t, t \in \mathbb{Z}\}$ is iid; otherwise, it is not strictly stationary.

As for weak stationarity, it is achieved if $\mathbb{E}\left[\epsilon_t\right] = 0$. This actually implies that $\mathbb{E}\left[Y_t\right] = 0$. Moreover,

$$\mathsf{Cov}\left(Y_t, Y_s\right) = \begin{cases} \mathbb{E}\left[\epsilon_0^2\right] & \text{if } t = s \\ \mathsf{Cov}\left(\pm\epsilon_t, \pm\epsilon_s\right) = 0 & \text{otherwise} \end{cases}$$

Thus, $Y_t$ is w.s. if $\mathbb{E}\left[\epsilon_t\right] = 0$.

4. In that case,$Z_t = 2\epsilon_t$ if $t$ is even, and $Z_t = 0$ if $t$ is odd ,implying that:

$$\mathbb{E}\left[Z_t\right] = \begin{cases} 0 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases} \qquad \mathsf{Var}\left(Z_t\right) = \begin{cases} 4\sigma_\epsilon^2 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}$$

Therefore $\{Z_t, t \in \mathbb{Z}\}$ is s.o.1, but it is s.o.2 if and only if $\sigma_\epsilon^2 = 0$: in that case, $\epsilon_t = Z_t = 0$ for all $t$.

**Solution of Exercise 3.3** 1. A covariance matrix is an Hermitian, non-negative matrix. Since $\rho$ is real, matrices $\Sigma_t$ are Hermitian. As for non-negativity, it is equivalent to the fact that the eigenvalues of $\Sigma_t$, let them be $\{\lambda_1, \lambda_2, \dots, \lambda_t\}$, are all non-negative.

Let us define $A$ as a $t \times t$ matrix such that $A_{i,j} = \rho$ for all $i$ and $j$. Then we have $\Sigma_t = (1-\rho)I_t + A$. Now, $\lambda_i = (1-\rho) + \omega_i$, where $\omega_i$ is the $i$-th eigenvalue of $A$. Since the rank of $A$ is 1, $t-1$ of its eigenvalues are equal to 0. Let us say that $\omega_t$ is the remaining, non null eigenvalue. Moreover, $\mathsf{Tr}(A) = \sum_{i=1}^t \omega_i = \omega_t$, but also $\mathsf{Tr}(A) = t\rho$, thus $\omega_t = t\rho$. In conclusions we have

$$\begin{aligned} \forall i \in \{1, 2, \dots, t-1\}, \lambda_i &= 1-\rho \\ \lambda_t &= 1 - \rho + t\rho = 1 + (t-1)\rho \end{aligned}$$

The non-negativity conditions are:

$$\begin{aligned} 1 - \rho &\geq 0 & 1 + (t-1)\rho &\geq 0 \\ \rho &\leq 1 & \rho &\geq -\frac{1}{t-1} \to_{t\to+\infty} 0^- \end{aligned}$$

In conclusion, $0 \leq \rho \leq 1$.

2. Let us consider a process $\{X_t = \alpha\epsilon_t + \beta Z, t \in \mathbb{Z}\}$, with $\{\epsilon_t, t \in \mathbb{Z}\}$ being a real-valued, zero-mean, unitary-variance strong white noise, $Z$ a real-valued, zero-mean, unitary-variance r.v. independent from any $\epsilon_t$, and $\alpha, \beta \in \mathbb{R}$. We would have:

$$\mathsf{Cov}\left(X_t, X_{t+h}\right) = \mathbb{E}\left[(\alpha\epsilon_t + \beta Z)(\alpha\epsilon_{t+h} + \beta Z)\right] = \alpha^2\mathbb{E}\left[\epsilon_t\epsilon_{t+h}\right] + \beta^2\mathbb{E}\left[Z^2\right] = \alpha^2\delta_h + \beta_2$$

$$\Sigma_t = \begin{bmatrix} \alpha^2+\beta^2 & \beta^2 & \beta^2 & \dots & \beta^2 \\ \beta^2 & \alpha^2+\beta^2 & \beta^2 & \dots & \beta^2 \\ \beta^2 & \beta^2 & \alpha^2+\beta^2 & \dots & \beta^2 \\ \dots & \dots & \dots & \dots & \dots \\ \beta^2 & \beta^2 & \beta^2 & \dots & \alpha^2+\beta^2 \end{bmatrix}$$

$$\begin{aligned} \alpha^2 + \beta^2 &= 1 & \beta^2 &= \rho \\ \alpha^2 &= 1-\rho & \beta^2 &= \rho \\ \alpha &= \sqrt{1-\rho} & \beta &= \sqrt{\rho} \end{aligned}$$

Since $\rho \in [0,1]$, then also $\alpha, \beta \in [0,1]$.

**Solution of Exercise 4.1** 1. $X_t = a + bZ_0$ is a constant with respect to $t$, thus strictly stationary.

$$\mathbb{E}\left[X_t\right] = a \qquad \mathsf{Cov}\left(X_t, X_{t+h}\right) = \mathsf{Cov}\left(a + bZ_0, a + bZ_0\right) = b^2\sigma^2 < +\infty$$

Since the acf is a constant, the spectral measure is $\nu(d\lambda) = b^2\sigma^2\delta(d\lambda)$.

2. $X_t = Z_0\cos(ct)$

$$\begin{aligned} \mathbb{E}\left[X_t\right] = 0 \qquad \mathsf{Cov}\left(X_t, X_{t+h}\right) &= \mathbb{E}\left[|Z_0|^2\cos(ct)\cos(ch+ct)\right] \\ &= \frac{\sigma^2}{2}\left[\cos(ch) + \cos(c(2t+h))\right] \end{aligned}$$

The covariance of $X_t$ and $X_{t+h}$ depends on $t$, thus the process is not s.o.2.

3. $X_t = a + bZ_t + cZ_{t-1}$

$$\mathbb{E}[X_t] = a \qquad \mathsf{Cov}(X_t, X_{t+h}) = \mathsf{Cov}(bZ_t + cZ_{t-1}, bZ_{t+h} + cZ_{t+h-1})$$
$$= (c^2 + b^2)\gamma_Z(h) + bc\gamma_Z(h-1) + bc\gamma_Z(h+1)$$
$$= (c^2 + b^2)\delta_h + bc\delta_{h-1} + bc\delta_{h+1}$$

Thus, $\mathsf{Cov}(X_t, X_{t+h})$ does not depend on $t$ and $\mathsf{Var}(X_t) = \gamma_X(0) = c^2 + b^2 < +\infty$. Therefore, it is a w.s. process. Finally,

$$f(\lambda) = \frac{1}{2\pi}\sum_{h\in\mathbb{Z}}\gamma(h)e^{-ih\lambda} = \frac{1}{2\pi}\left(b^2 + c^2 + 2bc\cos\lambda\right)$$

4. $X_t = Z_1\cos(ct) + Z_2\sin(ct)$

$$\mathbb{E}[X_t] = 0 \qquad \mathsf{Cov}(X_t, X_{t+h}) = \mathsf{Cov}(Z_1\cos(ct) + Z_2\sin(ct), Z_1\cos(c(t+h)) + Z_2\sin(c(t+h)))$$
$$= \sigma^2\left[\cos(ct)\cos(c(t+h)) + \sin(ct)\sin(c(t+h))\right]$$
$$= \frac{1}{2}\sigma^2\left[\cos(2ct + 2ch) + \cos(ch) + \cos(ch) - \cos(2ct + 2ch)\right] = \sigma^2\cos(ch)$$

Thus, $\mathsf{Cov}(X_t, X_{t+h})$ does not depend on $t$ and $\mathsf{Var}(X_t) = \sigma^2 < +\infty$. Therefore, it is a w.s. process. Finally,

$$\nu(d\lambda) = \frac{\sigma^2}{2}\left[\delta(d\lambda - c) + \delta(d\lambda + c)\right]$$

5. $X_t = Z_t\cos(ct) + Z_{t-1}\sin(ct) \Rightarrow \mathbb{E}[X_t] = 0$

$$\begin{aligned}\mathsf{Cov}(X_t, X_{t+h}) &= \mathsf{Cov}(Z_t\cos(ct) + Z_{t-1}\sin(ct), Z_{t+h}\cos(c(t+h)) + Z_{t+h-1}\sin(c(t+h)))\\
&= \sigma^2\left[\delta_h\cos(ct)\cos(c(t+h)) + \delta_{h-1}\cos(ct)\sin(c(t+h)) + \delta_{h+1}\sin(ct)\cos(c(t+h)) +\right.\\
&\left.+\ \delta_h\sin(ct)\sin(c(t+h))\right]\\
&= \sigma^2\left[\delta_h\cos(ch) + \delta_{h-1}\frac{1}{2}(\sin(c(2t+h)) + \sin(ch)) + \delta_{h+1}\frac{1}{2}(\sin(c(2t+h)) - \sin(ch))\right]\end{aligned}$$

Thus, $\mathsf{Cov}(X_t, X_{t+h})$ depends on $t$, the process is not s.o.2.

**Solution of Exercise 4.2** Let us define the sequence $d : k \in \mathbb{N}_0 \to \mathsf{det}(\Gamma_{k+1})$. We have the following:

$$k = 0 \qquad \Gamma_1 = [1] \qquad d_0 = \mathsf{det}(\Gamma_1) = 1$$

$$k = 1 \qquad \Gamma_2 = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \qquad d_1 = \mathsf{det}(\Gamma_2) = 1 - \rho^2$$

For $k \geq 2$, we can write $\Gamma_{k+1}$ as a block matrix:

$$\Gamma_{k+1} = \left[\begin{array}{c|cccccc} 1 & \rho & 0 & 0 & \dots & 0 & 0 \\ \hline \rho & & & & & & \\ 0 & & & & \Gamma_k & & \\ \dots & & & & & & \\ 0 & & & & & & \end{array}\right] = \left[\begin{array}{c|c|cccccc} 1 & \rho & 0 & 0 & \dots & 0 & 0 \\ \hline \rho & 1 & \rho & 0 & \dots & 0 & 0 \\ \cline{2-8} 0 & \rho & & & & & \\ \dots & \dots & & & \Gamma_{k-1} & & \\ 0 & 0 & & & & & \end{array}\right]$$

Therefore we have:

$$d_k = \mathsf{det}(\Gamma_{k+1}) = \mathsf{det}(\Gamma_k) - \rho\mathsf{det}\left[\begin{array}{c|cccccc} \rho & \rho & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & & & & & & \\ 0 & & & & \Gamma_{k-1} & & \\ \dots & & & & & & \\ 0 & & & & & & \end{array}\right] = d_{k-1} - \rho^2 d_{k-2}$$

Thus we have that the sequence $d$ is the solution of the following recurrent equation:

$$\begin{cases} d_k = d_{k-1} - \rho^2 d_{k-2} \\ d_0 = 0 \\ d_1 = 1 - \rho^2 \end{cases} \tag{2}$$

The characteristic equation is $x^2 - x + \rho^2 = 0$, with solutions

$$x_0 = \frac{1 - \sqrt{1 - 4\rho^2}}{2} \qquad\qquad x_1 = \frac{1 + \sqrt{1 - 4\rho^2}}{2}$$

Therefore, the sequence $d_k$ has the following form:

$$d_k = \begin{cases} \alpha x_0^k + \beta x_1^k & \text{if } x_0 \neq x_1 \Leftrightarrow |\rho| \neq \frac{1}{2} \\ (\alpha + \beta k) x_0^k & \text{if } x_0 = x_1 \Leftrightarrow |\rho| = \frac{1}{2} \Rightarrow x_0 = x_1 = \frac{1}{2} \end{cases}$$

where $\alpha$ and $\beta$ are defined by the initial conditions.

2. We have now to show that the matrices $\Gamma_k$ are positive definite given some condition on $\rho$. Using the expression Eq. (2) for the sequence of determinants, we have to find under which conditions on $\rho$, the determinants are all positive: $d_k > 0 \forall k \in \mathbb{N}_0$.

We have to consider three cases, with respect to the discriminant of the characteristic equation $x^2 - x + \rho^2 = 0$: positive, null and negative discriminant. Since $\Delta = 1 - 4\rho^2$, these conditions correspond respectively to $|\rho| < \frac{1}{2}$, $|\rho| = \frac{1}{2}$, and $|\rho| > \frac{1}{2}$.

If $\rho = |1/2|$, by applying the initial condition, one can easily find that $\alpha = 1$ and $\beta = 1/2$. In that case $d_k = (1 + \frac{k}{2}) \left(\frac{1}{2}\right)^k > 0 \forall k$. Then the $\Gamma_k$ matrices are all definite positive, thus they can be autocovariance matrices.

If $|\rho| \neq \frac{1}{2}$, one can find that $\alpha = \frac{\rho^2 - x_0}{\sqrt{\Delta}} = \frac{1}{2} - \sqrt{\Delta}\left(\frac{1}{2} + \frac{\rho^2}{\Delta}\right)$ and $\beta = \frac{x_1 - \rho^2}{\sqrt{\Delta}} = \frac{1}{2} + \sqrt{\Delta}\left(\frac{1}{2} + \frac{\rho^2}{\Delta}\right)$. Now, if $|\rho| < \frac{1}{2}$ then $\Delta > 0$ and both $\alpha$ and $\beta$ are real. It can also be proven that $\beta > 1$ , $\alpha < 0$ and $|\beta| - |\alpha > |1$. Since $0 < x_0 < x_1$, $|\beta||x_1|^n > |\alpha||x_0|^n$, proving that $\forall k \in \mathbb{N}_0, d_k > 0$, *q.d.e.*.

Finally, if $|\rho| > \frac{1}{2}$, it can be shown that $d_k$ has sinusoidal terms, hence it can be negative, which prevents $\Gamma_k$ from being an autocovariance matrix.

As alternative method, we can use the **Herglotz theorem**, stating that $\gamma(h)$ is positive if and only if it exists a positive measure $\nu$ such that $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} \nu(d\lambda)$. Here we can use the density: $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda$ where

$$f(\lambda) = \frac{1}{2\pi} \sum_{h \in \mathbb{Z}} \gamma(h) e^{-ih\lambda} \frac{1}{2\pi} (1 + 2\rho \cos \lambda)$$

The density is non-negative for all $\lambda$ if and only if $|\rho| leq \frac{1}{2}$, *q.d.e.*.

3. Let us consider a weak white noise $\{\epsilon_t, t \in \mathbb{Z}\}$ and a process $\{X_t = a\epsilon_t + b\epsilon_{t-1}, t \in \mathbb{Z}\}$, with $a, b \in \mathbb{R}$. Then, the new process is real-valued and centered: $\mathbb{E}[X_t] = 0$. Moreover,

$$\begin{aligned} \mathsf{Cov}(X_t, X_{t+h}) &= \mathbb{E}[X_t X_{t+h}] = \mathbb{E}\left[a^2 \epsilon_t \epsilon_{t+h} + b^2 \epsilon_{t-1} \epsilon_{t-1+h} + ab \epsilon_{t+h} \epsilon_{t-1} + ab \epsilon_t \epsilon_{t-1+h}\right] \\ &= \left(a^2 + b^2\right) \delta_h + ab \left(\delta_{h-1} + \delta_{h+1}\right) \end{aligned}$$

Finally, we find $a$ and $b$ by setting:

$$\begin{gathered} \left(a^2 + b^2\right) = 1 \\ ab = \rho \end{gathered}$$

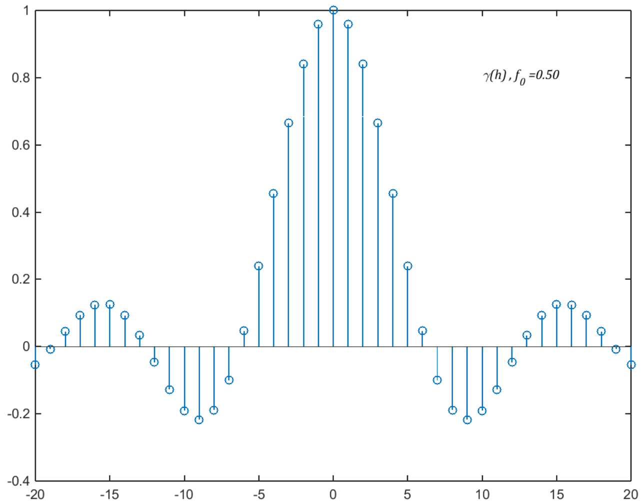


Figure 4: Example of autocovariance function for a band-limited stationary process, Exercise 4.3.

implying $\left(a^2+b^2\right)+2ab = 1+2\rho$ and thus $a+b=\sqrt{1+2\rho}$. Then we have:

$$
\begin{aligned}
b &= \sqrt{1+2\rho}-a \\
b^2 &= a^2+1+2\rho-2a\sqrt{1+2\rho} \\
a^2+b^2 &= 2a^2+1+2\rho-2a\sqrt{1+2\rho} \\
1 &= 2a^2+1+2\rho-2a\sqrt{1+2\rho} \\
2a^2+2\rho-2a\sqrt{1+2\rho} &= 0 \\
a &= \frac{\sqrt{1+2\rho}\pm\sqrt{1-2\rho}}{2} \\
b &= \frac{\sqrt{1+2\rho}\mp\sqrt{1-2\rho}}{2}
\end{aligned}
$$

Note that, since $|\rho|\leq\frac{1}{2}$, $a,b\in\mathbb{R}$.

**Solution of Exercise 4.3**

$$
\begin{aligned}
\gamma(h) &= \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda \\
&= \int_{-f_0}^{f_0} e^{ih\lambda} d\lambda \\
&= \frac{1}{ih}\left(e^{ihf_0}-e^{-ihf_0}\right) \\
&= 2\frac{\sin(hf_0)}{h} = 2f_0\mathsf{Sinc}(f_0 h)
\end{aligned}
$$

An example of this function is given in Fig. 4. It is not $L^1$ since in that case its density would have been continuous.

**Solution of Exercise 4.4** 1. Let $W=\{\ell\in\mathbb{Z}^+|(X_1,\ldots,X_\ell)$ is a set of linearly independent vectors$\}$. If this set is empty, this means that even $(X_1)$ is not a set of linearly independent vectors, thus $\exists a\in\mathbb{R}^+$ such that $\mathsf{Var}\,(aX_1)=0$. Since $a\neq 0$, $\gamma(0)=\mathsf{Var}\,(X_1)=0/a=0$.

If $W$ is not empty, we define $k$ as the maximum value in $W$. Since the elements of $W$ are drawn from $\mathbb{Z}^+$, we have $k \geq 1$. Then, by our choice of $k$, $(X_1, \ldots, X_{k+1})$ is a not set of linearly independent vectors, while $(X_1, \ldots, X_k)$ is such. This imply $X_{k+1} \in \mathsf{Vect}(X_1, \ldots, X_k)$.

2. Since the autocovariance matrix is invertible, its smallest eigenvalue is positive

3. We have to show that, $\forall p \geq 1, X_{k+p} \in \mathsf{Vect}(X_1, \ldots, X_k)$. If $\gamma(0) = 0$ this is trivial. Otherwise, we will prove it by recurrence.

3.1. The basis of the recurrence is already proved: $X_{k+1} \in \mathsf{Vect}(X_1, \ldots, X_k)$

3.2. We have to prove that, if $\forall \ell < p, X_{k+\ell} \in \mathsf{Vect}(X_1, \ldots, X_k)$, then, also $X_{k+p} \in \mathsf{Vect}(X_1, \ldots, X_k)$.

By stationarity, $X_{k+1} \in \mathsf{Vect}(X_1, \ldots, X_k) \Rightarrow X_{k+p} \in \mathsf{Vect}(X_p, \ldots, X_{p+k-1})$.

By recurrence hypothesis, each of $(X_p, \ldots, X_{p+k-1})$ is in $\mathsf{Vect}(X_1, \ldots, X_k)$. Therefore, the same for $X_{k+p}$, *q.e.d.*

4. We rewrite Eq.(1) as $X_{k+p} = \varphi_p^T \mathbf{X} = \mathbf{X}^T \varphi_p$, where $\varphi_p$ is the vector of the scalars $\phi_{p,1}, \ldots, \phi_{p,k}$ and $\mathbf{X}$ is the random vector $[X_1, \ldots, X_k]^T$. We have

$$\gamma(0) = \mathbb{E}\left[|X_{k+p}|^2\right] = \mathbb{E}\left[\varphi_p^H \mathbf{X}\mathbf{X}^H \varphi_p\right] = \varphi_p^H \Gamma_k \varphi_p \geq \lambda_{\mathsf{min}} \|\varphi_p\|^2 \Leftrightarrow \|\varphi_p\|^2 \leq \frac{\gamma(0)}{\lambda_{\mathsf{min}}} < +\infty$$

5.

$$\begin{aligned}\gamma(0) &= \mathsf{Cov}\left(X_{k+p}, X_{k+p}\right) = \mathsf{Cov}\left(X_{k+p}, \sum_{\ell=1}^{k} \phi_{p,\ell} X_\ell\right) = \sum_{\ell=1}^{k} \mathsf{Cov}\left(X_{k+p}, \phi_{p,\ell} X_\ell\right) \\ &= \sum_{\ell=1}^{k} \phi_{p,\ell} \gamma(p+k-\ell) \leq \sum_{\ell=1}^{k} \sqrt{\frac{\gamma(0)}{\lambda_{\mathsf{min}}}} \gamma(p+k-\ell)\end{aligned}$$

By passing to the limit for $p \to +\infty$, we obtain $\gamma(0)$ for the left-hand term and 0 for the right-hand term.

**Solution of Exercise 5.1** We know that, $\forall t, k \in \mathbb{Z}, S_{t+4k} = S_t$

1. $\mathbb{E}[Y_t] = \mathbb{E}[\beta t + S_t + X_t] = \beta t + \mu_S + \mu_X$. Therefore $\{Y_t, t \in \mathbb{Z}\}$ is not w.s. unless $\beta = 0$.

2.1.

$$\forall k \in \mathbb{Z}, \qquad \gamma_S(h) = \mathsf{Cov}\left(S_t, S_{t+h}\right) = \mathsf{Cov}\left(S_t, S_{t+h+4k}\right) = \gamma_S(h+4k)$$

Therefore $\gamma_S$ is periodic with period equal to 4.

2.2. By applying the operator $\left(1 + B + B^2 + B^3\right)$ on $S$, we obtain:

$$\begin{aligned} \forall t \in \mathbb{Z}, &\qquad \bar{S}_t = S_t + S_{t-1} + S_{t-2} + S_{t-3} &\Rightarrow \\ \forall t \in \mathbb{Z}, &\qquad \bar{S}_t - \bar{S}_{t-1} = S_t - S_{t-4} = 0 &\Rightarrow \\ \forall t \in \mathbb{Z}, &\qquad \bar{S}_t = \bar{S}_0 = S_0 + S_1 + S_2 + S_3 & \end{aligned}$$

3. First, we observe that, given a process $\{W_t, t \in \mathbb{Z}\}$, $(1 - B) \circ \left(1 + B + B^2 + B^3\right) \circ W_t = (1 - B^4) \circ W_t$. Therefore,

$$Z_t = (1 - B^4) \circ (\beta t + S_t + X_t) = \beta t + S_t + X_t - \beta(t-4) - S_{t-4} - X_{t-4} = 4\beta + X_t - X_{t-4}$$

Then, $\mathbb{E}[Z_t] = 4\beta$ and:

$$\mathsf{Cov}\left(Z_t, Z_{t+h}\right) = \mathsf{Cov}\left(X_t - X_{t-4}, X_{t+h} - X_{t+h-4}\right) = 2\gamma_X(h) - \gamma_X(h-4) - \gamma_X(h+4)$$

Therefore $\{Z_t, t \in \mathbb{Z}\}$ is w.s. and $\gamma_Z(h) = 2\gamma_X(h) - \gamma_X(h-4) - \gamma_X(h+4)$.

4. As an autocovariance function, $\gamma_S$ is Hermitian, but since $\{S_t, t \in \mathbb{Z}\}$ is real, it is symmetric: $\gamma_S(-h) = \gamma_S(h)$. Moreover, we have shown that $\gamma_S$ is periodic, thus defined by the values of its period. We set:

$$\begin{aligned}\gamma_S(0) &= \gamma_0 \\ \gamma_S(1) &= \gamma_1 \\ \gamma_S(2) &= \gamma_2 \\ \gamma_S(3) &= \gamma_S(-1) = \gamma_S(1) = \gamma_1\end{aligned}$$

Thus $\gamma_S$ has three degrees of freedom. Let us now show that a function

$$\eta(h) = a + b\cos\left(\frac{\pi}{2}h\right) + c\cos(\pi h)$$

satisfies all the constraint of $\gamma_S$. First we observe that $\eta$ is real, periodical of period 4 and symmetric. Moreover,

$$\begin{aligned}\eta(0) &= a + b + c\\ \eta(1) &= a - c\\ \eta(2) &= a - b + c\end{aligned}$$

Finally, the parameters $a, b, c$ are found by solving

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 1\end{bmatrix} \cdot \begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix} \Rightarrow \begin{array}{l} a = \frac{\gamma_0}{4} + \frac{\gamma_1}{2} + \frac{\gamma_2}{4}\\ b = \frac{\gamma_0}{2} - \frac{\gamma_2}{2}\\ c = \frac{\gamma_0}{4} - \frac{\gamma_1}{2} + \frac{\gamma_2}{4}\end{array}$$

As for the spectral measure, from $\gamma_S(h) = a + b\cos\left(\frac{\pi}{2}h\right) + c\cos(\pi h)$, we have that $\nu_S(d\lambda) = a\delta_0(d\lambda) + \frac{b}{2}\delta_{\frac{\pi}{2}}(d\lambda) + \frac{b}{2}\delta_{-\frac{\pi}{2}}(d\lambda) + c\delta_\pi(d\lambda)$.

5.

$$\begin{aligned}&\gamma_Z(h) = 2\gamma_X(h) - \gamma_X(h-4) - \gamma_X(h+4) \Rightarrow\\ &f_Z(\lambda) = 2f_X(\lambda) - f_X(\lambda)e^{-i4\lambda} - f_X(\lambda)e^{i4\lambda}\\ &\qquad = 2f_X(\lambda)\left(1 - \frac{e^{i4\lambda} + e^{-i4\lambda}}{2}\right) = 2f_X(\lambda)\left[1 - \cos(4\lambda)\right] = 4f_X(\lambda)\sin^2(2\lambda)\end{aligned}$$

**Solution of Exercise 5.2** 1. For a centered w.s. process $\{X_t, t \in \mathbb{Z}\}$, the innovation process is defined at each $t$ as the difference between $X_t$ and its projection on the *linear past* of the process. Thus, $\{Z_t, t \in \mathbb{Z}\}$ is the innovation process of $\{X_t, t \in \mathbb{Z}\}$, and as such, it is a white noise (Corollary 2.4.1 in the text book). Let us prove that in this special case.

It is easy to see that $\mathbb{E}[Z_t] = 0$. We also have that $Z_t \in \mathcal{H}_t$, since both $X_t$ and $\widetilde{X}_t$ are in $\mathcal{H}_t$.

$$\mathsf{Proj}(Z_t|\mathcal{H}_{t-1}) = \mathsf{Proj}(X_t - \widetilde{X}_t|\mathcal{H}_{t-1}) = \widetilde{X}_t - \widetilde{X}_t = 0 \Rightarrow Z_t \perp \mathcal{H}_{t-1} \Rightarrow Z_t \perp \widetilde{X}_t \Rightarrow$$

$$\mathbb{E}\left[|X_t|^2\right] = \mathbb{E}\left[|Z_t|^2 + |\widetilde{X}_t|^2\right] = \mathbb{E}\left[|Z_t|^2\right] + \mathbb{E}\left[|\widetilde{X}_t|^2\right] \Rightarrow \mathbb{E}\left[|Z_t|^2\right] = \mathbb{E}\left[|X_t|^2\right] - \mathbb{E}\left[|\widetilde{X}_t|^2\right] \quad (3)$$

$$\forall s < t, Z_s \in \mathcal{H}_s \subseteq \mathcal{H}_{t-1} \Rightarrow Z_t \perp Z_s \Leftrightarrow \mathsf{Cov}(Z_t, Z_s) = 0 \quad (4)$$

Eq. (3) shows that $\mathsf{Cov}(Z_t, Z_t)$ does not depend on $t$ and Eq. (3) shows that $\mathsf{Cov}(Z_t, Z_{t+h})$ does not depend on $t$ neither, and is null. Therefore, $\{Z_t, t \in \mathbb{Z}\}$ is a weak white noise.

2. $\forall s \le t - q - 1, \mathsf{Cov}(X_t, X_s) = \gamma_X(t - s) = 0$ since $t - s > q$. This means that $\forall s \le t - q - 1, X_t \perp X_s$, q.e.d.

3. We know that $X_t \perp \mathcal{H}_{t-q-1}$ and $X_t \in \mathcal{H}_t$, *i.e.*, $X_t$ is in the orthogonal complement of $\mathcal{H}_{t-q-1}$ in $\mathcal{H}_t$, which is a space with $q + 1$ dimensions. In this space, the set $(Z_s, s \in \{t, t-1, \dots, t-q\})$ is made up of orthogonal vectors, so it is a basis, implying $X_t \in \mathsf{Vect}(Z_s, s \in \{t, t-1, \dots, t-q\})$.

4. From the previous, we can write $X_t = \sum_{p=0}^q \theta_{t,p} Z_{t-p}$. The coefficients of the projection on the orthogonal basis are found as:

$$\begin{aligned}\theta_{t,p} &= \mathsf{Cov}(X_t, Z_{t-p}) = \mathsf{Cov}\left(X_t, X_{t-p} - \widetilde{X}_{t-p}\right)\\ &= \gamma_X(p) - \mathsf{Cov}\left(X_t, \widetilde{X}_{t-p}\right)\end{aligned}$$

By stationarity, $\mathsf{Cov}\left(X_t, \widetilde{X}_{t-p}\right)$ does not depend on $t$, thus $\theta_{t,p}$ also only depends on $p$, and can be referred to as $\theta_p$. In conclusion, we can write:

$$\forall t \in \mathbb{Z}\; X_t = \sum_{p=0}^q \theta_p Z_{t-p},$$

with $\{Z_t, t \in \mathbb{Z}\}$ a white noise: this is the definition of MA($q$) process.

**Solution of Exercise 5.3** 1. Let us compute the average and the covariance for the sum of the MA processes:

$$\begin{aligned}
\mathbb{E}[Z_t] &= \mathbb{E}[X_t] + \mathbb{E}[Y_t] = 0 \\
\mathsf{Cov}(Z_{t+h}, Z_t) &= \mathsf{Cov}(X_{t+h} + Y_{t+h}, X_t + Y_t) = \gamma_X(h) + \gamma_(h)
\end{aligned}$$

Thus, $\{Z_t, t \in \mathbb{Z}\}$ is a w.s. process. Moreover, since $\gamma_Z(h) = \gamma_X(h) + \gamma_(h)$, the support of $\gamma_Z(h)$ is $s = \max\{p, q\}$. As shown in Exercise 5.2, this implies that $\{Z_t, t \in \mathbb{Z}\}$ is an MA($s$) process.

2. Let us use the shortcuts $\theta = \theta_1$ and $\rho = \rho_1$. The process $X$ can be seen as the filtering of the WN $\epsilon$ with an FIR filter with impulse response $a : n \in \mathbb{Z} \to \delta_n + \theta\delta_{n-1}$. This means that $\epsilon$ can be recovered from $X$ by applying the inverse filter with impulse response

$$b : n \in \mathbb{Z} \to \begin{cases} (-\theta)^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we can recover $\eta$ from $Y$. We have

$$\begin{aligned}
\epsilon_t &= \sum_{k=0}^{+\infty} (-\theta)^k X_{t-k} & \eta_t &= \sum_{k=0}^{+\infty} (-\rho)^k Y_{t-k} \\
\mathbb{E}[\epsilon_t, \eta_s] &= \mathbb{E}\left[\sum_{k=0}^{+\infty} (-\theta)^k X_{t-k} \sum_{\ell=0}^{+\infty} (-\rho)^\ell Y_{s-\ell}\right] & &= \sum_{k=0}^{+\infty}\sum_{\ell=0}^{+\infty} (-\theta)^k (-\rho)^\ell \, \mathbb{E}[X_{t-k} Y_{s-\ell}] = 0 \text{ q.e.d.}
\end{aligned}$$

3. In this case, introducing $\xi_t = \epsilon_t + \eta_t$, we have $Z_t = \epsilon_t + \eta_t + \theta(\epsilon_{t-1} + \eta_{t-1}) = \xi_t + \theta\xi_{t-1}$. Now

$$\mathbb{E}[\xi_t] = \mathbb{E}[\epsilon_t] + \mathbb{E}[\eta_t] = 0 \tag{5}$$

$$\mathsf{Cov}(\xi_t, \xi_s) = \mathsf{Cov}(\epsilon_t, \epsilon_s) + \mathsf{Cov}(\epsilon_t, \eta_s) + \mathsf{Cov}(\eta_t, \epsilon_s) + \mathsf{Cov}(\eta_t, \eta_s) = (\sigma_\epsilon^2 + \sigma_\eta^2)\delta_{t-s} \tag{6}$$

$$\xi_t = \sum_{k=0}^{+\infty} (-\theta)^k (X_{t-k} + Y_{t-k}) = \sum_{k=0}^{+\infty} (-\theta)^k Z_{t-k} \in \mathcal{H}_Z^t \tag{7}$$

We observe that by Eqs. (5) and (6), $\xi$ is WN; moreover, by Eq. (7), $\forall s < t, \mathsf{Cov}(\xi_t, Z_s) = \mathsf{Cov}(\xi_t, Z_s) = \mathsf{Cov}\left(\sum_{k=0}^{+\infty} (-\theta)^k Z_{t-k} \in \mathcal{H}_Z^t, Z_s\right) = 0$, thus $\xi_t \perp \mathcal{H}_Z^{t-1}$. Again by Eq. (7), $\theta\xi_{t-1} \in \mathcal{H}_Z^{t-1}$.

Therefore, $Z_t = \xi_t + \theta\xi_{t-1}$ expresses $Z_t$ as a term independent from its linear past and a term in the linear past. In conclusion $\xi$ is the innovation process of $Z$ (and we already proved that it is WN).

4. In this case we have:

$$\begin{aligned}
X_t &= \epsilon_t + \theta\epsilon_{t-1} & Y_t &= \eta_t + \rho\eta_{t-1} & \Rightarrow \\
\gamma_X(h) &= \sigma_\epsilon^2\left[(1+\theta^2)\delta_h + \theta\delta_{h-1} + \theta\delta_{h+1}\right] & \gamma_Y(h) &= \sigma_\eta^2\left[(1+\rho^2)\delta_h + \rho\delta_{h-1} + \rho\delta_{h+1}\right] &
\end{aligned}$$

In Question 1 we have shown that $Z$ must be MA(1). This means that it must exist a WN $\phi$ and a real number $\alpha$ such that $\phi$ is the innovation of $Z$ and

$$\begin{aligned}
Z_t &= \phi_t + \alpha\phi_{t-1} \\
\gamma_Z(h) &= \sigma_\phi^2\left[(1+\alpha^2)\delta_h + \delta_{h-1} + \delta_{h+1}\right]
\end{aligned}$$

The unknown $\alpha$ and $\sigma_\phi^2$ can be found by the identity $\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h)$:

$$\sigma_\phi^2\left[(1+\alpha^2)\delta_h + \delta_{h-1} + \delta_{h+1}\right] = \sigma_\epsilon^2\left[(1+\theta^2)\delta_h + \theta\delta_{h-1} + \theta\delta_{h+1}\right] + \sigma_\eta^2\left[(1+\rho^2)\delta_h + \rho\delta_{h-1} + \rho\delta_{h+1}\right]$$

$$\begin{cases} \sigma_\phi^2(1+\alpha^2) = \sigma_\epsilon^2(1+\theta^2) + \sigma_\eta^2(1+\rho^2) \\ \sigma_\phi^2\alpha = \sigma_\epsilon^2\theta + \sigma_\eta^2\rho \end{cases}$$

Let us first set $a = \sigma_\epsilon^2(1+\theta^2) + \sigma_\eta^2(1+\rho^2)$ and $b = \sigma_\epsilon^2\theta + \sigma_\eta^2\rho$. We find that $\alpha = \frac{b}{\sigma_\phi^2}$ and then:

$$\sigma_\phi^2\left(1+\frac{b^2}{\sigma_\phi^4}\right) = a \qquad\qquad \sigma_\phi^2 + \frac{b^2}{\sigma_\phi^2} - a = 0$$

$$\sigma_\phi^4 - a\sigma_\phi^2 + b^2 = 0 \qquad\qquad \sigma_\phi^2 = \frac{1}{2}\left(a \pm \sqrt{a^2-4b^2}\right)$$

$$\sigma_\phi^2 = \frac{1}{2}\left[\sigma_\epsilon^2(1+\theta^2) + \sigma_\eta^2(1+\rho^2) \pm \sqrt{\sigma_\epsilon^4(1-\theta^2)^2 + \sigma_\eta^4(1-\rho^2)^2 + 2\sigma_\epsilon^2\sigma_\eta^2(1+\theta^2)(1+\rho^2)}\right]$$

**Solution of Exercise 5.4** Let us observe that $\epsilon_t = X_t - aX_{t-1}$ and $\eta_t = Y_t - bY_{t-1}$. We can write the following:

$$\begin{aligned}
Z_t - (a+b)Z_{t-1} + abZ_{t-2} &= X_t + Y_t - aX_{t-1} - aY_{t-1} - bX_{t-1} - bY_{t-1} + abX_{t-2} + abY_{t-2} \\
&= X_t - aX_{t-1} - b(X_{t-1} - bX_{t-2}) + Y_t - bY_{t-1} - a(Y_{t-1} - bY_{t-2}) \\
&= \epsilon_t - b\epsilon_{t-1} + \eta_t - a\eta_{t-1} = W_t + V_t
\end{aligned}$$

Now, both $\{W_t = \epsilon_t - b\epsilon_{t-1}, t \in \mathbb{Z}\}$ and $\{V_t = \eta_t - a\eta_{t-1}, t \in \mathbb{Z}\}$ are MA(1) processes, and thus their sum is also a MA(1) process, meaning that it exists a WN $\xi$ and a real number $\theta \in ]-1,1[$ such that $Z_t - (a+b)Z_{t-1} + abZ_{t-2} = \xi_t - \theta\xi_{t-1}$, *q.e.d.*

2. From the previous point, we can write

$$\xi_t - \theta\xi_{t-1} = \epsilon_t - b\epsilon_{t-1} + \eta_t - a\eta_{t-1} \tag{8}$$

$$(1-\theta B)\circ\xi_t = (1-bB)\circ\epsilon_t + (1-aB)\circ\eta_t \tag{9}$$

where we use the back-shift operator $B$. The left-hand term of this equation can be read as the filtering of $\xi$ with a FIR with impulse response $h_k = \delta_k - \theta\delta_{k-1}$. As shown in Exercise 5.3, this filter can be inversed by applying a filter with impulse response

$$g : n \in \mathbb{Z} \to \begin{cases} (\theta)^n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Apply the inverse filter to both members of Eq. (9), we get:

$$\begin{aligned}
\xi_t &= (1-bB)\sum_{n\geq 0}\theta^n\epsilon_{t-n} + (1-aB)\sum_{n\geq 0}\theta^n\eta_{t-n} \\
&= (1-bB)\left(\epsilon_t + \sum_{n\geq 1}\theta^n\epsilon_{t-n}\right) + (1-aB)\left(\eta_t + \sum_{n\geq 1}\theta^n\eta_{t-n}\right) \\
&= \epsilon_t + \sum_{n\geq 1}\theta^n\epsilon_{t-n} - b\sum_{n\geq 0}\theta^n\epsilon_{t-1-n} + \eta_t + \sum_{n\geq 1}\theta^n\eta_{t-n} - a\sum_{n\geq 0}\theta^n\eta_{t-1-n} \\
&= \epsilon_t + \sum_{m\geq 0}\theta^{m+1}\epsilon_{t-1-m} - b\sum_{n\geq 0}\theta^n\epsilon_{t-1-n} + \eta_t + \sum_{m\geq 0}\theta^{m+1}\eta_{t-1-m} - a\sum_{n\geq 0}\theta^n\eta_{t-1-n} \\
&= \epsilon_t + (\theta-b)\sum_{k\geq 0}\theta^k\epsilon_{t-1-k} + \eta_t + (\theta-a)\sum_{h\geq 0}\theta^h\eta_{t-1-h} \quad q.e.d.
\end{aligned}$$

3. We write the following:

$$\begin{aligned}
Z_{t+1} &= (a+b)Z_t - abZ_{t-1} + \xi_{t+1} - \theta\xi_t \\
&= (a+b)Z_t - abZ_{t-1} + \epsilon_{t+1} + (\theta-b)\sum_{k\geq 0}\theta^k\epsilon_{t-k} + \eta_{t+1} + (\theta-a)\sum_{h\geq =}\theta^h\eta_{t-h} - \theta\xi_t \\
&= [\epsilon_{t+1} + \eta_{t+1}] + \left[(a+b)Z_t - abZ_{t-1} + (\theta-b)\sum_{k\geq 0}\theta^k\epsilon_{t-k} + (\theta-a)\sum_{h\geq 0}\theta^h\eta_{t-h} - \theta\xi_t\right]
\end{aligned} \tag{10}$$

If we know $(X_s \forall s \leq t)$ and $(Y_s \forall s \leq t)$, we also know $Z_t$, $Z_{t-1}$. Moreover, by applying an inverse filtering, we know also $\epsilon_{t-k} \forall k \geq 0$ and $\eta_{t-h} \forall h \geq 0$. On the contrary, we do not know $\epsilon_{t+1}$ nor $\eta_{t+1}$, and both are uncorrelated with $(X_s \forall s \leq t)$ and $(Y_s \forall s \leq t)$ Therefor the first term in the right-hand part of Eq. (10) is the innovation, while the second term is the prediction.

4. In this case we do not know separately $(X_s \forall s \leq t)$ and $(Y_s \forall s \leq t)$, but only their sum. We write therefore:

$$\begin{aligned} Z_{t+1} &= (a+b)Z_t - abZ_{t-1} + \xi_{t+1} - \theta\xi_t \\ &= \xi_{t+1} + (a+b)Z_t - abZ_{t-1} - \theta\xi_t \\ &= \xi_{t+1} + \widetilde{Z}_t \end{aligned}$$

Thus $\xi_{t+1}$ is the innovation and $\widetilde{Z}_t = (a+b)Z_t - abZ_{t-1} - \theta\xi_t$ is the prediction. Again, $\xi_t$ is obtained by inverse filtering of $Z_t - (a+b)Z_{t-1} + abZ_{t-2}$.

5. In the first case,

$$\mathbb{E}\left[|\eta_{t+1} + \epsilon_{t+1}|^2\right] = \sigma_\eta^2 + \sigma_\epsilon^2.$$

In the second we have:

$$\begin{aligned} \xi_t &= \epsilon_t + (\theta - b)\sum_{k\geq 0}\theta^k\epsilon_{t-1-k} + \eta_t + (\theta - a)\sum_{h\geq 0}\theta^h\eta_{t-1-h} \\ &= \epsilon_t + (\theta - b)\alpha_t + \eta_t + (\theta - a)\beta_t \end{aligned}$$

with:

$$\alpha_t = \sum_{k\geq 0}\theta^k\epsilon_{t-1-k} \qquad \beta_t = \sum_{h\geq 0}\theta^h\eta_{t-1-h}$$

Therefore $\xi$ is expressed as the sum of four uncorrelated processes. We can then compute its variance, referred to as $\sigma^2$, as the sum of the four variances:

$$\sigma^2 = \mathsf{Var}\left(\xi_t\right) = \sigma_\epsilon^2 + (\theta - b)^2\mathsf{Var}\left(\alpha_t\right) + \sigma_\eta^2 + (\theta - a)^2\mathsf{Var}\left(\beta_t\right)$$

We have:

$$\begin{aligned} \mathsf{Var}\left(\alpha_t\right) &= \mathbb{E}\left[\sum_{k\geq 0}\theta^k\epsilon_{t-1-k}\sum_{\ell\geq 0}\theta^\ell\epsilon_{t-1-\ell}\right] = \sum_{k\geq 0}\sum_{\ell\geq 0}\theta^k\theta^\ell\gamma_\epsilon(k-\ell) \\ &= \sum_{k\geq 0}\sum_{\ell\geq 0}\theta^k\theta^\ell\sigma_\epsilon^2\delta_{k-\ell} = \sigma_\epsilon^2\sum_{k\geq 0}\theta^2 k = \frac{\sigma_\epsilon^2}{1-\theta^2} \end{aligned}$$

and, likewise, $\mathsf{Var}\left(\beta_t\right) = \frac{\sigma_\eta^2}{1-\theta^2}$. In conclusion,

$$\begin{aligned} \sigma^2 = \mathsf{Var}\left(\xi_t\right) &= \sigma_\epsilon^2 + (\theta - b)^2\frac{\sigma_\epsilon^2}{1-\theta^2} + \sigma_\eta^2 + (\theta - a)^2\frac{\sigma_\eta^2}{1-\theta^2} \\ &= \sigma_\epsilon^2\left[1 + \frac{(\theta - b)^2}{1-\theta^2}\right] + \sigma_\eta^2\left[1 + \frac{(\theta - a)^2}{1-\theta^2}\right] \end{aligned}$$

Thus we see that the variance of the innovation in the second case is always larger than that of the first case, unless $\theta = a = b$.

**Solution of Exercise 5.5** We observe that $\{X_t, t \in \mathbb{Z}\}$ is a MA(1) process, thus, if $\gamma$ be the autocovariance function of $\{X_t, t \in \mathbb{Z}\}$, its support is $\{-1, 0, +1\}$. In facts, we have:

$$\gamma(h) = \mathbb{E}\left[(Z_t + \theta Z_{t-1})(Z_{t+h} + \theta Z_{t+h-1})\right] = \sigma^2\left[(1+\theta^2)\delta_h + \theta\delta_{h-1} + \theta\delta_{h+1}\right]$$

1. The linear prediction of $X_3$ is written as:

$$\widehat{X}_3 = \alpha X_1 + \beta X_2.$$

Our problem consists in minimizing the mean square error $\mathbb{E}\left[\left(X_3 - \widehat{X}_3\right)^2\right]$. The optimal solution is found the the error $(X_3 - \widehat{X}_3)$ is orthogonal to data $(X_1, X_2)$. Thus we have:

$$\begin{aligned}
\mathsf{Cov}\left(X_3 - \widehat{X}_3, X_1\right) &= 0 \\
\mathsf{Cov}\left(X_3 - \alpha X_1 - \beta X_2, X_1\right) &= 0 \\
\gamma(2) - \alpha\gamma(0) - \beta\gamma(1) &= 0 \\
-\alpha\sigma^2(1+\theta^2) - \beta\sigma^2\theta &= 0
\end{aligned}
\qquad
\begin{aligned}
\mathsf{Cov}\left(X_3 - \widehat{X}_3, X_2\right) &= 0 \\
\mathsf{Cov}\left(X_3 - \alpha X_1 - \beta X_2, X_2\right) &= 0 \\
\gamma(-1) - \alpha\gamma(1) - \beta\gamma(0) &= 0 \\
\sigma^2\theta - \alpha\sigma^2\theta - \beta\sigma^2(1+\theta^2) &= 0
\end{aligned}$$

This is a linear system, and we can actually get rid of $\sigma^2$:

$$\begin{bmatrix} (1+\theta^2) & \theta \\ \theta & (1+\theta^2) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ \theta \end{bmatrix}$$

We find:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\theta^4+\theta^2+1} \begin{bmatrix} (1+\theta^2) & -\theta \\ -\theta & (1+\theta^2) \end{bmatrix} \begin{bmatrix} 0 \\ \theta \end{bmatrix} = \begin{bmatrix} \frac{-\theta^2}{\theta^4+\theta^2+1} \\ \frac{\theta+\theta^3}{\theta^4+\theta^2+1} \end{bmatrix}$$

2. If we now set

$$\widehat{X}_3 = \alpha X_5 + \beta X_4.$$

and we look for $\alpha, \beta$ minimizing the MSE, we end up exactly with the same equation as before, since for real processes, $\gamma(h) = \gamma(-h)$. Therefore, the same optimal values of the coefficients are found:

$$\alpha = \frac{-\theta^2}{\theta^4+\theta^2+1} \qquad \beta = \frac{\theta+\theta^3}{\theta^4+\theta^2+1}$$

3. Let us define the spaces $V_1 = \mathsf{Vect}(X_1, X_2)$ and $V_2 = \mathsf{Vect}(X_4, X_5)$. Any element of $V_1$ is uncorrelated to any element of $V_2$ (*i.e.*, they are orthogonal):

$$\begin{aligned}
&\mathsf{Cov}\left(aX_1 + bX_2, cX_4 + dX_5\right) \\
=&ac\mathsf{Cov}\left(X_1, X_4\right) + ad\mathsf{Cov}\left(X_1, X_5\right) + bc\mathsf{Cov}\left(X_2, X_4\right) + bd\mathsf{Cov}\left(X_2, X_5\right) \\
=&ac\gamma(-3) + ad\gamma(-4) + bc\gamma(-2) + bd\gamma(-3) = 0
\end{aligned}$$

Thus, $\mathsf{Vect}(X_1, X_2, X_4, X_5) = V_1 \oplus V_2$, which implies that

$$\widehat{X}_3 = \mathsf{Proj}(X_3|V_1 \oplus V_2) = \mathsf{Proj}(X_3|V_1) + \mathsf{Proj}(X_3|V_2) = \widehat{X}_{3,1} + \widehat{X}_{3,2}$$

Since $\widehat{X}_{3,1}$ and $\widehat{X}_{3,2}$ are orthogonal, when we impose $\mathsf{Cov}\left(X_3 - \widehat{X}_3, X_i\right) = 0$, with $i \in \{1,2,4,5\}$, only one between $\widehat{X}_{3,1}$ and $\widehat{X}_{3,2}$ gives a non-zero covariance (depending on $i$). Therefore, we end up with $\mathsf{Cov}\left(X_3 - \widehat{X}_{3,1}, X_i\right) = 0$ or $\mathsf{Cov}\left(X_3 - \widehat{X}_{3,2}, X_i\right) = 0$, *i.e.*, the same equations as in Questions 1 and 2. Therefore we find the same partial solutions. In conclusion:

$$\begin{aligned}
\widehat{X}_3 &= \frac{-\theta^2}{\theta^4+\theta^2+1}X_1 + \frac{\theta+\theta^3}{\theta^4+\theta^2+1}X_2 + \frac{\theta+\theta^3}{\theta^4+\theta^2+1}X_4 + \frac{-\theta^2}{\theta^4+\theta^2+1}X_5 \\
&= \frac{\theta+\theta^3}{\theta^4+\theta^2+1}(X_2 + X_4) - \frac{\theta^2}{\theta^4+\theta^2+1}(X_1 + X_5)
\end{aligned}$$

**Solution of Exercise 1** 1. Let us first rewrite the equation defining $X$ as an ARMA$(p,q)$ equation:

$$X_t - \sum_{k=1}^{p} \phi_k X_{t-k} = \epsilon_t + \sum_{k=1}^{p} \theta_k \epsilon_{t-k} \tag{11}$$

Let us introduce the polynomials $\Phi(z)$, $\Theta(z)$:

$$\Phi(z) = 1 - \sum_{k=1}^{p} \phi_k z^k \qquad \Theta(z) = 1 + \sum_{k=1}^{p} \theta_k z^k$$

Introducing the backshift operator $B$, the ARMA equation (Eq. (11)) can be written as:

$$\Phi(B)X = \Theta(B)\epsilon \tag{12}$$

Now we have just to check that a) $\Phi(z)$ and $\Theta(z)$ do not have common roots and that b) $\Phi(z)$ does not vanish on the unit circle of $\mathbb{C}$. This is straighforward since the only root of $\Phi$ is $1/2$ while the only root of $\Theta$ is $-1/4$. We can then apply theorem 3.3.2: $X$ is the unique w.s. solution of Eq. (11), and it admits a spectral density function given by:

$$f_Z(\lambda) = \frac{\sigma^2}{2\pi} \frac{\left|\Theta(e^{-i\lambda})\right|^2}{\left|\Phi(e^{-i\lambda})\right|^2}$$

In our case we have the following function, shown in Fig. 1:

$$f_Z(\lambda) = \frac{\sigma^2}{2\pi} \frac{\left|1 + 4e^{-i\lambda}\right|^2}{\left|1 - 2e^{-i\lambda}\right|^2} = \frac{\sigma^2}{2\pi} \frac{8\cos\lambda + 17}{5 - 4\cos\lambda}.$$

2. We remind that a canonical representation of an ARMA process is characterized by the fact that $X$ is a causal and invertible filtering of weak noise. This is equivalent to say that neither $\Phi$ nor $\Theta$ vanish on the closed unit disk $\Delta_1 = \{z \in \mathbb{C} : |z| \leq 1\}$.

A given representation of an ARMA process is not necessarily canonical but it is possible to get a canonical representation by using an *all-pass filter*. We recall that, given $\psi \in \ell^1$, the filter $F_\psi$ is an all-pass filter if and only if:

$$\forall z \in \Gamma_1, \left|\sum_{k\in\mathbb{Z}} \psi_k z^k\right| = c,$$

where $\Gamma_1 = \{\{z \in \mathbb{C} : |z| = 1\}$ is the complex unit circle and $c > 0$ is a constant.

A key property of all-pass filters is that they transform a WN process $A_t$ into another WN process $B_t$. To prove this, let us first recall that, since $\psi \in \ell^1$, then theorem 3.1.2 and corollary 3.1.3 apply. Thus $B = F_\psi(A)$ is a w.s. centered process, with spectral density function

$$f_B(\lambda) = \frac{\sigma_A^2}{2\pi} \left|\sum_{k\in\mathbb{Z}} \psi_k e^{-ik\lambda}\right|^2 = \frac{\sigma_A^2}{2\pi} c^2,$$

where we applied the definition of all-pass filter for $z = e^{-i\lambda} \in \Gamma_1$. We also have that:

$$f_B(\lambda) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \gamma_B(h) e^{-ik\lambda}$$

Comparing the two last equations and remembering that the Discrete-Time Fourier Transform is injective for $\ell^1$ sequences, we get $\gamma_B(h) = c^2\sigma_A^2\delta_h$, $\square$.

A second, crucial property of all-pass filters is that they can be used to invert the moduli of the roots of a polynomial (example 3.2.2): let $Q$ be a polynomial defined by $Q(z) = \prod_{k=1}^{q}(1 - \nu_k z)$, such that none of the $\nu_k$ have neither unitary nor zero modulus. We observe that $Q(0) = 1$ and that the $q$ roots of $Q$ are $\nu_k^{-1}$ for $k = 1, \ldots, q$.

Now we define the polinomial $\widetilde{Q}(z) = \prod_{k=1}^{n}\left(1 - \overline{\nu_k^{-1}}z\right)$ and the function $\Xi : z = \frac{Q}{\widetilde{Q}}(z)$. $\Xi$ is a rational function with poles $\overline{\nu_k} \neq \Gamma_1$. Then we know that it exists a unique $\ell^1$ sequence $\xi_k$ such that $\Xi(z) = \sum_{k\in\mathbb{Z}} \xi_k z^k$. Let us now prove that the filter $F_\xi$ is then an all-pass. First, we have:

$$\left|\sum_{k\in\mathbb{Z}} \xi_k z^k\right| = \prod_{k=1}^{n} \frac{|1-\nu_k z|}{\left|1-\overline{\nu_k^{-1}}z\right|}. \tag{13}$$

Now, since $z \in \Gamma_1 \Rightarrow \overline{z} = z^{-1}$, for any $k = 1, \dots, n$ and for $z \in \Gamma_1$ we have:

$$\begin{aligned}\left|1-\overline{\nu_k^{-1}}z\right| &= \left|-\overline{\nu_k^{-1}}z\right|\left|-\overline{\nu_k}z^{-1}+1\right| = \left|\overline{\nu_k^{-1}}\right||z|\left|1-\overline{\nu_k}z^{-1}\right| = \left|\overline{\nu_k^{-1}}\right||z|\left|1-\overline{\nu_k z}\right| \\ &= \left|\overline{\nu_k^{-1}}\right|\left|\overline{1-\nu_k z}\right| = \left|\overline{\nu_k^{-1}}\right|\left|1-\nu_k z\right|.\end{aligned}$$

Replacing in Eq. (13), we get:

$$\left|\sum_{k\in\mathbb{Z}} \xi_k z^k\right| = \prod_{k=1}^{n} \frac{|1-\nu_k z|}{\left|\overline{\nu_k^{-1}}\right||1-\nu_k z|} = \prod_{k=1}^{n} |\nu_k| = c > 0 \qquad \square$$

Equipped with the all-pass filter properties, we can rewrite an ARMA filter in canonical form. Let us consider an all-pass filter in the form $\Xi = a\frac{Q}{\widetilde{Q}}$. The roots $\nu_k^{-1}$ and the constant $a$ will be defined later on. If $\Phi$ has no roots on $\Gamma_1$ we know that $F_\phi$ is invertible. Likewise, by construction $\Xi$ is invertible. Using a fraction notation to refer to inverse filters, we can formally rewrite Eq. (12) as:

$$X = \frac{\Theta}{\Phi} \circ \epsilon = \frac{\Theta}{\Phi} \circ \frac{\Xi}{\Xi} \circ \epsilon = \left(\frac{\Theta}{\Phi\Xi}\right) \circ (\Xi \circ \epsilon) = \frac{\widetilde{\Theta}}{\widetilde{\Phi}} \circ \eta$$

with:

$$\frac{\widetilde{\Theta}}{\widetilde{\Phi}} = \frac{\Theta}{\Phi\Xi} \qquad\qquad \eta = \Xi \circ \epsilon.$$

We already know that $\eta$ is a WN process, since it is an all-pass filtering of a WN. We have to show that we can build such a $\Xi(B) = a\frac{Q}{\widetilde{Q}}(B)$ that $\widetilde{\Theta}$ and $\widetilde{\Phi}$ do not have roots in the closed unit disk $\Delta_1$. This is always possible since we can write:

$$\frac{\widetilde{\Theta}(z)}{\widetilde{\Phi}(z)} = \frac{\Theta(z)}{\Phi(z)}\frac{1}{\Xi(z)} = \frac{\prod_{k=1}^{q}(1-\nu_k^{(\theta)}z)}{\prod_{k=1}^{p}(1-\nu_k^{(\phi)}z)}\frac{1}{a}\prod_{k=1}^{n}\frac{1-\overline{\nu_k^{-1}}z}{1-\nu_k z}$$

where $\nu_k^{(\phi)}$ (resp. $\nu_k^{(\theta)}$) are the inverse of the roots of $\Phi$ (resp. of $\Theta$). Now we build $\Xi$ such that we cancel out the roots of $\Phi$ and of $\Theta$ in $\Delta_1$. More precisely, to cancel out a given $\nu_k^{(\theta)}$ we introduce as a root of $Q$ the number $\nu_k = \nu_k^{(\theta)}$ and to cancel out a given $\nu_k^{(\phi)}$ we introduce as a root of $Q$ the number $\nu_k = \left(\overline{\nu_k^{(\theta)}}\right)^{-1}$.

In our case, we have: $\frac{\Theta(z)}{\Phi(z)} = \frac{1+4z}{1-2z}$ with roots $-\frac{1}{4}$ and $\frac{1}{2}$. To cancel out these roots, we set:

$$\begin{aligned}\frac{\widetilde{\Theta}(z)}{\widetilde{\Phi}(z)} &= \frac{\Theta(z)}{\Phi(z)}\frac{1}{\Xi(z)} = \frac{1+4z}{1-2z} \cdot \frac{1}{a}\frac{1+\frac{1}{4}z}{1+4z}\frac{1-2z}{1-\frac{1}{2}z} = \frac{1}{a}\frac{1+\frac{1}{4}z}{1-\frac{1}{2}z} \\ \Xi(z) &= a\,\frac{1+4z}{1+\frac{1}{4}z}\,\frac{1-\frac{1}{2}z}{1-2z}\end{aligned}$$

Since $\forall z \in \Gamma_1, |\Xi(z)| = c$, given that $\Xi(1) = a\frac{5}{5/4}\frac{1/2}{-1} = -2a$, choosing $a = -1/2$ we get $\forall z \in \Gamma_1\, |\Xi(z)| =$

$|\Xi(1)| = 1$. This also implies $f_\eta(\lambda) = f_\epsilon(\lambda)$ and thus $\mathsf{Var}(\eta) = \mathsf{Var}(\epsilon)$. In conclusion,

$$\boxed{\begin{aligned} \frac{\widetilde{\Theta}(z)}{\widetilde{\Phi}(z)} &= \frac{-2-\frac{1}{2}z}{1-\frac{1}{2}z} \\ \eta &= -\frac{1}{2}\,\frac{1+4z}{1+\frac{1}{4}z}\,\frac{1-\frac{1}{2}z}{1-2z}\,\epsilon \\ X_t - \frac{1}{2}X_{t-1} &= -2\eta_t - \frac{1}{2}\eta_{t-1} \end{aligned}}$$

3. Let us recall here the results of theorem 3.5.1. The canonical representation of an ARMA process is desirable since it express the former as an *causal* and *inversible* filtering of WN:

$$X_t = \widetilde{\phi}_1 X_{t-1} + \ldots + \widetilde{\phi}_p X_{t-p} + \widetilde{\theta}_0\eta_t + \widetilde{\theta}_1\eta_{t-1} + \ldots + \widetilde{\theta}_q\eta_{t-q}$$

This means that there exist two causal $\ell^1$ sequences, $\xi$ and $\widetilde{\xi}$, such that:

$$X = F_\xi(\eta) \tag{14}$$
$$\eta = F_{\widetilde{\xi}}(X) \tag{15}$$

From Eq. (14), since $\xi$ is causal, we deduce that $\mathcal{H}_X^t \subseteq \mathcal{H}_Z^t$. From Eq. (15), since $\widetilde{\xi}$ is causal, we deduce that $\mathcal{H}_Z^t \subseteq \mathcal{H}_X^t$. In conclusion, $\mathcal{H}_X^t = \mathcal{H}_Z^t$. If we set:

$$\widehat{X}_t = \widetilde{\phi}_1 X_{t-1} + \ldots + \widetilde{\phi}_p X_{t-p} + +\widetilde{\theta}_1\eta_{t-1} + \ldots + \widetilde{\theta}_q\eta_{t-q}$$

we see that $X_t - \widehat{X}_t = \widetilde{\theta}_0\eta_t$. Since $\eta$ is WN, $X_t - \widehat{X}_t \perp \mathcal{H}_\eta^{t-1}$ but then $X_t - \widehat{X}_t \perp \mathcal{H}_X^{t-1}$. This means that $\widehat{X}_t$ is the projection of $X_t$ onto its linear past, and therefore $\widetilde{\theta}_0\eta_t$ is the innovation process of $X$.

The canonical form gives therefore a direct access to the innovation of an ARMA process.

Now we can answer immediately to the question. The variance of the innovation is:

$$\mathsf{Var}(-2\eta_t) = 4\mathsf{Var}(\eta_t) = 4\mathsf{Var}(\epsilon_t)\,.$$

4. From the definition of $X$ we can write: $(1-2B)X_t = (1+4B)\epsilon_t$. Setting the AR process $W_t$ such that $(1-2B)W_t = \epsilon_t$, we have $X_t = (1+4B)W_t$.

Now,

$$\begin{aligned} W_t &= \frac{1}{1-2B}\epsilon_t = -\frac{1}{2B}\,\frac{1}{1-\frac{1}{2}B^{-1}}\epsilon_t = -\left(\frac{1}{2}B^{-1}\right)\sum_{k\geq 0}\left(\frac{1}{2}B^{-1}\right)^k \epsilon_t \\ &= -\sum_{k\geq 1}\left(\frac{1}{2}B^{-1}\right)^k \epsilon_t = -\sum_{k\geq 1}\left(\frac{1}{2}\right)^k \epsilon_{t+k} \\ X_t &= W_t + 4W_{t-1} = -\left[\sum_{k\geq 1}\left(\frac{1}{2}\right)^k \epsilon_{t+k}\right] - 4\left[\sum_{n\geq 1}\left(\frac{1}{2}\right)^n \epsilon_{t+n-1}\right] \quad \text{set } \ell = n-1 \\ &= -\left[\sum_{k\geq 1}\left(\frac{1}{2}\right)^k \epsilon_{t+k}\right] - 4\left[\sum_{\ell\geq 0}\left(\frac{1}{2}\right)^\ell \frac{1}{2}\epsilon_{t+\ell}\right] \\ &= -\left[\sum_{k\geq 1}\left(\frac{1}{2}\right)^k \epsilon_{t+k}\right] - 4\left[\frac{1}{2}\epsilon_t + \sum_{\ell\geq 1}\left(\frac{1}{2}\right)^\ell \frac{1}{2}\epsilon_{t+\ell}\right] \\ &= -2\epsilon_t - \left[\sum_{k\geq 1}\left(\frac{1}{2}\right)^k \epsilon_{t+k}\right] - 2\left[\sum_{\ell\geq 1}\left(\frac{1}{2}\right)^\ell \epsilon_{t+\ell}\right] \\ &= -2\epsilon_t - \sum_{k\geq 1}\frac{3}{2^k}\epsilon_{t+k} \end{aligned}$$

**Solution of Exercise 5.7** We have to compute the impulse response of a recursive filter. Since $|\phi| < 1$, a stable, causal solution exists. The weights $\psi_k$ are such that:

$$\sum_{k\in\mathbb{Z}} \psi_k z^k = \frac{1}{1-\phi z} = \sum_{k\geq} \phi^k z^k \Rightarrow \psi_k = \begin{cases} \phi^k & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

Therefore, $X_t = \sum_{k\geq 0} \phi^k \epsilon_{t-k}$

2. We can apply Corollary 3.1.3 on the linear filtering of WN. Therefore, observing that $\psi_k$ is real,

$$\begin{aligned}\gamma_X(h) &= \sigma_\epsilon^2 \sum_{k\in\mathbb{Z}} \psi_{k+h}\psi_k = \\ &= \begin{cases} \sigma_\epsilon^2 \phi^h \sum_{k\geq 0} \phi^{2k} = \frac{\sigma_\epsilon^2 \phi^h}{1-\phi^2} & \text{if } h \geq 0 \\ \sigma_\epsilon^2 \sum_{k\geq -h} \phi^{k+h}\phi^k = \sigma_\epsilon^2 \sum_{n\geq 0} \phi^n \phi^{n-h} = \sigma_\epsilon^2 \phi^{-h} \sum_{k\geq 0} \phi^{2k} = \frac{\sigma_\epsilon^2 \phi^{-h}}{1-\phi^2} & \text{if } h < 0 \end{cases} \\ &= \frac{\sigma_\epsilon^2 \phi^{|h|}}{1-\phi^2}\end{aligned}$$