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Solution of the tutorial on filter synthesis

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Contexte académique } **sans modifications**
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Représentations des signaux (TSIA201)



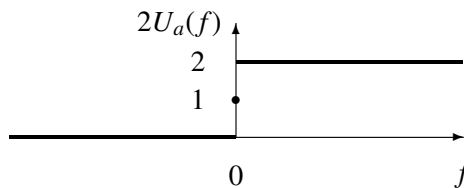
1 Rejactor filter

Let us consider the transfer function $H(z) = \frac{1-2\cos(\theta)z^{-1}+z^{-2}}{1-2\rho\cos(\theta)z^{-1}+\rho^2z^{-2}}$, with $0 < \rho < 1$. Check that $H(z)$ can be factorized in the form $H(z) = \frac{(1-z_0z^{-1})(1-z_0^*z^{-1})}{(1-\rho z_0z^{-1})(1-\rho z_0^*z^{-1})}$, where z_0 is to be expressed as a function of θ . What is the normalized frequency rejected by this filter? What is the domain of convergence of its stable implementation? Is this implementation causal? Write the corresponding input/output relationship.

We get $H(z) = \frac{(1-z_0z^{-1})(1-z_0^*z^{-1})}{(1-\rho z_0z^{-1})(1-\rho z_0^*z^{-1})}$ with $z_0 = e^{i\theta}$. $H(z)$ is zero when $z = z_0$ or $z = z_0^*$, therefore the normalized frequency rejected by this filter is $\frac{\theta}{2\pi}$. The two possible domains of convergence are the disk of radius ρ and its complement. Since $\rho < 1$, the one which contains the unit circle is the complement. Therefore the domain of convergence of its stable implementation is the complement of the disk of radius ρ . This implementation is also causal since its domain is the complement of a disk. Moreover, since $(1-2\rho\cos(\theta)z^{-1}+\rho^2z^{-2})Y(z) = (1-2\cos(\theta)z^{-1}+z^{-2})X(z)$, the corresponding input/output relationship is $y(n) = 2\rho\cos(\theta)y(n-1) - \rho^2y(n-2) + x(n) - 2\cos(\theta)x(n-1) + x(n-2)$.

2 Hilbert filter

Let $x_a(t)$ be a continuous time (analog) real signal. The *analytic* signal associated to $x_a(t)$ is the signal $z_a(t)$ whose the CTFT is expressed as $Z_a(f) = 2U_a(f)X_a(f)$, where $U_a(f)$ is the unit step function, whose value is 1 for $f > 0$, and 0 for $f < 0$. For continuity reasons, we assume that $U_a(0) = \frac{1}{2}$. The filter of frequency response $2U_a(f)$ is referred to as the *analytic filter*.



1. Which property does function $X_a(f)$ satisfy? Deduce the expression of $\frac{1}{2}(Z_a(f) + Z_a^*(-f))$ as a function of $X_a(f)$, and prove that the real part of $z_a(t)$ is equal to $x_a(t)$. We can then write $z_a(t) = x_a(t) + iy_a(t)$, where the real signal $y_a(t)$ is defined as the imaginary part of $z_a(t)$.

Since $x_a(t)$ is a real signal, function $X_a(f)$ is Hermitian symmetric. Therefore $\frac{1}{2}(Z_a(f) + Z_a^*(-f)) = U_a(f)X_a(f) + U_a(-f)X_a^*(-f) = (U_a(f) + U_a(-f))X_a(f) = X_a(f)$. In the time domain, this equality is equivalent to $\frac{1}{2}(z_a(t) + z_a^*(t)) = x_a(t)$, i.e. $\text{Re}(z_a(t)) = x_a(t)$.

2. Prove that $y_a(t)$ can be obtained from $x_a(t)$ by linear filtering of frequency response $H_a(f) = -i\text{sign}(f)$, where $\text{sign}(f) = 1$ for $f > 0$, $\text{sign}(f) = -1$ for $f < 0$, and $\text{sign}(0) = 0$. Filter $H_a(f)$ is referred to as the *Hilbert filter*, and $y_a(t)$ is called the *Hilbert transform* of $x_a(t)$.

By definition, we have $y_a(t) = -i(z_a(t) - x_a(t))$, thus $Y_a(f) = -i(Z_a(f) - X_a(f)) = -i(2U_a(f) - 1)X_a(f) = H_a(f)X_a(f)$, therefore $y_a(t)$ can be obtained from $x_a(t)$ by linear filtering of frequency response $H_a(f)$.

Let us assume that the signal $x_a(t)$ satisfies the assumptions of the sampling theorem: there exists a frequency F_s such that the support of $X_a(f)$ is included in $]-\frac{F_s}{2}, \frac{F_s}{2}[$. We then consider the sampled signals $x(n) = x_a(nT_s)$ and $y(n) = y_a(nT_s)$, where $T_s = 1/F_s$. We remind the relationship between the

DTFT $X(e^{2i\pi\nu})$ and the CTFT $X_a(f)$:

$$X(e^{2i\pi\nu}) = \frac{1}{T_s} \sum_{k \in \mathbb{Z}} X_a\left(\frac{\nu + k}{T_s}\right) \quad (1)$$

3. Simplify the expression (1) when $\nu \in]-\frac{1}{2}, \frac{1}{2}[$. Check that $Y(e^{2i\pi\nu})$ satisfies a similar expression. Deduce that the signal $y(n)$ can also be expressed as the output of the discrete filter of frequency response $H(e^{2i\pi\nu}) = -i \operatorname{sign}(\nu)$ for $\nu \in]-\frac{1}{2}, \frac{1}{2}[$ (and $H(e^{2i\pi\nu}) = 0$ for $\nu = \pm \frac{1}{2}$), applied to the input signal $x(n)$.

When $\nu \in]-\frac{1}{2}, \frac{1}{2}[$, we get $X(e^{2i\pi\nu}) = \frac{1}{T_s} X_a\left(\frac{\nu}{T_s}\right)$, and in the same way $Y(e^{2i\pi\nu}) = \frac{1}{T_s} Y_a\left(\frac{\nu}{T_s}\right)$. Therefore $Y(e^{2i\pi\nu}) = \frac{1}{T_s} H_a\left(\frac{\nu}{T_s}\right) X_a\left(\frac{\nu}{T_s}\right) = -i \operatorname{sign}(\nu) X(e^{2i\pi\nu}) = H(e^{2i\pi\nu}) X(e^{2i\pi\nu})$. Therefore $y(n)$ can also be expressed as the output of the discrete filter of frequency response $H(e^{2i\pi\nu})$ applied to the input signal $x(n)$.

Remark: the discrete filter $H(e^{2i\pi\nu})$ allows us to directly compute the samples $y(n)$ of the Hilbert transform from the samples $x(n)$, without having to perform a digital/analog conversion.

4. By applying the inverse DTFT, prove that the impulse response $h(n)$ satisfies $h(n) = \frac{2}{\pi n}$ if n is odd, and 0 if n is even.

We have $h(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} H(e^{2i\pi\nu}) e^{2i\pi\nu n} d\nu = \int_{-\frac{1}{2}}^{\frac{1}{2}} -i \operatorname{sign}(\nu) e^{2i\pi\nu n} d\nu = -i \left(\int_{\nu=0}^{\frac{1}{2}} e^{2i\pi\nu n} d\nu - \int_{\nu=-\frac{1}{2}}^0 e^{2i\pi\nu n} d\nu \right) = -i \left(\frac{(-1)^n - 1}{2i\pi n} - \frac{1 - (-1)^n}{2i\pi n} \right) = \frac{1 - (-1)^n}{\pi n}$. Therefore $h(n) = \frac{2}{\pi n}$ if n is odd, and 0 if n is even.

5. Is this filter causal? Stable? Of finite (FIR) or infinite (IIR) impulse response?

This filter is non-causal, non-stable, and IIR.

6. For a discrete filter of impulse response $g(n)$ and of transfer function $G(z)$, what is the impulse response of the filter of transfer function $G(z^2)$? By using the fact that the even coefficients of $h(n)$ are zero, deduce that there exists a transfer function $\tilde{G}(z)$, such that $H(z) = z^{-1} \tilde{G}(z^2)$. What is the impulse response $\tilde{g}(n)$?

The impulse response $\tilde{g}(n)$ of the filter of transfer function $\tilde{G}(z) = G(z^2)$ is $\tilde{g}(n) = 0$ if n is odd, and $\tilde{g}(n) = g(\frac{n}{2})$ if n is even. Since the even coefficients of $h(n)$ are zero, we conclude that there exists a transfer function $\tilde{G}(z)$, such that $H(z) = z^{-1} \tilde{G}(z^2)$. Moreover, when n is odd, we have $h(n) = g(\frac{n-1}{2})$, so with $m = \frac{n-1}{2}$, we get $g(m) = h(2m+1) = \frac{1}{\pi(m+\frac{1}{2})}$.

7. We want to approximate the ideal filter $G(z)$ by using the window method, in order to synthesize a linear phase FIR filter, of type 4 (even length N , antisymmetric impulse response $g(n)$). Quickly summarize the principle of the window method, its advantages and its drawbacks.

The window method is described in the course.





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