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Solution of the tutorial on filter synthesis

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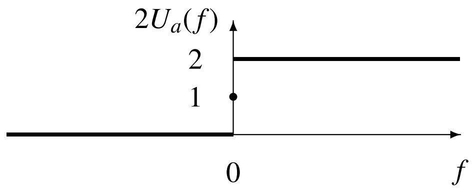
1 Rejector filter

Let us consider the transfer function $H(z) = \frac{1-2\cos(\theta)z^{-1}+z^{-2}}{1-2\rho\cos(\theta)z^{-1}+\rho^2z^{-2}}$, with $0 < \rho < 1$. Check that $H(z)$ can be factorized in the form $H(z) = \frac{(1-z_0z^{-1})(1-z_0^*z^{-1})}{(1-\rho z_0z^{-1})(1-\rho z_0^*z^{-1})}$, where z_0 is to be expressed as a function of θ. What is the normalized frequency rejected by this filter? What is the domain of convergence of its stable implementation? Is this implementation causal? Write the corresponding input/output relationship.

We get $H(z) = \frac{(1-z_0z^{-1})(1-z_0^*z^{-1})}{(1-\rho z_0z^{-1})(1-\rho z_0^*z^{-1})}$ with $z_0 = e^{i\theta}$. $H(z)$ is zero when $z = z_0$ or $z = z_0^*$, therefore the normalized frequency rejected by this filter is $\frac{\theta}{2\pi}$. The two possible domains of convergence are the disk of radius ρ and its complement. Since $\rho < 1$, the one which contains the unit circle is the complement. Therefore the domain of convergence of its stable implementation is the complement of the disk of radius ρ. This implementation is also causal since its domain is the complement of a disk. Moreover, since $(1-2\rho\cos(\theta)z^{-1}+\rho^2z^{-2})Y(z) = (1-2\cos(\theta)z^{-1}+z^{-2})X(z)$, the corresponding input/output relationship is $y(n) = 2\rho\cos(\theta)y(n-1) - \rho^2y(n-2) + x(n) - 2\cos(\theta)x(n-1) + x(n-2)$.

2 Hilbert filter

Let $x_a(t)$ be a continuous time (analog) real signal. The *analytic* signal associated to $x_a(t)$ is the signal $z_a(t)$ whose the CTFT is expressed as $Z_a(f) = 2\,U_a(f)X_a(f)$, where $U_a(f)$ is the unit step function, whose value is 1 for $f > 0$, and 0 for $f < 0$. For continuity reasons, we assume that $U_a(0) = \frac{1}{2}$. The filter of frequency response $2U_a(f)$ is referred to as the *analytic filter*.



1. Which property does function $X_a(f)$ satisfy? Deduce the expression of $\frac{1}{2}(Z_a(f) + Z_a^*(-f))$ as a function of $X_a(f)$, and prove that the real part of $z_a(t)$ is equal to $x_a(t)$. We can then write $z_a(t) = x_a(t) + iy_a(t)$, where the real signal $y_a(t)$ is defined as the imaginary part of $z_a(t)$.

   Since $x_a(t)$ is a real signal, function $X_a(f)$ is Hermitian symmetric. Therefore $\frac{1}{2}(Z_a(f)+Z_a^*(-f)) = U_a(f)X_a(f) + U_a(-f)X_a^*(-f) = (U_a(f)+U_a(-f))X_a(f) = X_a(f)$. In the time domain, this equality is equivalent to $\frac{1}{2}(z_a(t) + z_a^*(t)) = x_a(t)$, i.e. $Re(z_a(t)) = x_a(t)$.

2. Prove that $y_a(t)$ can be obtained from $x_a(t)$ by linear filtering of frequency response $H_a(f) = -i\,\mathrm{sign}(f)$, where $\mathrm{sign}(f) = 1$ for $f > 0$, $\mathrm{sign}(f) = -1$ for $f < 0$, and $\mathrm{sign}(0) = 0$. Filter $H_a(f)$ is referred to as the *Hilbert filter*, and $y_a(t)$ is called the *Hilbert transform* of $x_a(t)$.

   By definition, we have $y_a(t) = -i(z_a(t) - x_a(t))$, thus $Y_a(f) = -i(Z_a(f) - X_a(f)) = -i(2\,U_a(f) - 1)X_a(f) = H_a(f)X_a(f)$, therefore $y_a(t)$ can be obtained from $x_a(t)$ by linear filtering of frequency response $H_a(f)$.

Let us assume that the signal $x_a(t)$ satisfies the assumptions of the sampling theorem: there exists a frequency F_s such that the support of $X_a(f)$ is included in $]-\frac{F_s}{2}, \frac{F_s}{2}[$. We then consider the sampled signals $x(n) = x_a(nT_s)$ and $y(n) = y_a(nT_s)$, where $T_s = 1/F_s$. We remind the relationship between the

DTFT $X(e^{2i\pi\nu})$ and the CTFT $X_a(f)$:

$$X(e^{2i\pi\nu}) = \frac{1}{T_s} \sum_{k\in\mathbb{Z}} X_a\left(\frac{\nu+k}{T_s}\right) \tag{1}$$

3. Simplify the expression (1) when $\nu \in]-\frac{1}{2}, \frac{1}{2}[$. Check that $Y(e^{2i\pi\nu})$ satisfies a similar expression. Deduce that the signal $y(n)$ can also be expressed as the output of the discrete filter of frequency response $H(e^{2i\pi\nu}) = -i\,\mathrm{sign}(\nu)$ for $\nu \in]-\frac{1}{2}, \frac{1}{2}[$ (and $H(e^{2i\pi\nu}) = 0$ for $\nu = \pm\frac{1}{2}$), applied to the input signal $x(n)$.

   When $\nu \in]-\frac{1}{2}, \frac{1}{2}[$, we get $X(e^{2i\pi\nu}) = \frac{1}{T_s} X_a\left(\frac{\nu}{T_s}\right)$, and in the same way $Y(e^{2i\pi\nu}) = \frac{1}{T_s} Y_a\left(\frac{\nu}{T_s}\right)$. Therefore $Y(e^{2i\pi\nu}) = \frac{1}{T_s} H_a\left(\frac{\nu}{T_s}\right) X_a\left(\frac{\nu}{T_s}\right) = -i\,\mathrm{sign}(\nu) X(e^{2i\pi\nu}) = H(e^{2i\pi\nu})X(e^{2i\pi\nu})$. Therefore $y(n)$ can also be expressed as the output of the discrete filter of frequency response $H(e^{2i\pi\nu})$ applied to the input signal $x(n)$.

Remark: the discrete filter $H(e^{2i\pi\nu})$ allows us to directly compute the samples $y(n)$ of the Hilbert transform from the samples $x(n)$, without having to perform a digital/analog conversion.

4. By applying the inverse DTFT, prove that the impulse response $h(n)$ satisfies $h(n) = \frac{2}{\pi n}$ if n is odd, and 0 if n is even.

   We have $h(n) = \int_{\nu=-\frac{1}{2}}^{\frac{1}{2}} H(e^{2i\pi\nu}) e^{2i\pi\nu n} d\nu = \int_{\nu=-\frac{1}{2}}^{\frac{1}{2}} -i\,\mathrm{sign}(\nu) e^{2i\pi\nu n} d\nu = -i\left(\int_{\nu=0}^{\frac{1}{2}} e^{2i\pi\nu n} d\nu - \int_{\nu=-\frac{1}{2}}^{0} e^{2i\pi\nu n} d\nu\right) = -i\left(\frac{(-1)^n-1}{2i\pi n} - \frac{1-(-1)^n}{2i\pi n}\right) = \frac{1-(-1)^n}{\pi n}$. Therefore $h(n) = \frac{2}{\pi n}$ if n is odd, and 0 if n is even.

5. Is this filter causal? Stable? Of finite (FIR) or infinite (IIR) impulse response?

   This filter is non-causal, non-stable, and IIR.

6. For a discrete filter of impulse response $g(n)$ and of transfer function $G(z)$, what is the impulse response of the filter of transfer function $G(z^2)$? By using the fact that the even coefficients of $h(n)$ are zero, deduce that there exists a transfer function $G(z)$, such that $H(z) = z^{-1}G(z^2)$. What is the impulse response $g(n)$?

   The impulse response $\tilde{g}(n)$ of the filter of transfer function $\tilde{G}(z) = G(z^2)$ is $\tilde{g}(n) = 0$ if n is odd, and $\tilde{g}(n) = g(\frac{n}{2})$ if n is even. Since the even coefficients of $h(n)$ are zero, we conclude that there exists a transfer function $G(z)$, such that $H(z) = z^{-1}G(z^2)$. Moreover, when n is odd, we have $h(n) = g(\frac{n-1}{2})$, so with $m = \frac{n-1}{2}$, we get $g(m) = h(2m+1) = \frac{1}{\pi(m+\frac{1}{2})}$.

7. We want to approximate the ideal filter $G(z)$ by using the window method, in order to synthesize a linear phase FIR filter, of type 4 (even length N, antisymmetric impulse response $g(n)$). Quickly summarize the principle of the window method, its advantages and its drawbacks.

   The window method is described in the course.

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