Point estimation

Exercise 1 (Statistical model)**:**
An agency measures the concentration of $CO2$ in Earth's atmosphere θ through n independent observations. Measurement errors are i.i.d. with known mean μ and known variance σ^2.

1. Describe the statistical model.
2. Is the model parametric? Same question if the measurement errors are gaussian.
3. Is the parameter θ identifiable?

Exercise 2 (Identifiability)**:**
You can go to Telecom Paris with a shared bike only if there is one available at home and there is room for leaving it at Telecom Paris, two events that occur independently at random with respective probabilities p and q. You observe the number of times you've used a shared bike over n days. Is the parameter $\theta = (p, q)$ identifiable?

Exercise 3 (Bernoulli model)**:**
You would like to estimate the probability θ of winning a lottery game through n independent observations. Describe the statistical model and give the maximum likelihood estimator (MLE) of θ.

Exercise 4 (Geometric model)**:**
In the previous exercise, the observations are now the numbers of attempts between two wins, that are i.i.d. random variables with a geometric distribution with parameter θ. Describe the statistical model for n observations and give the MLE of θ.

Exercise 5 (Gaussian model – mean)**:**
The height of people in a given country is supposed to have a gaussian distribution. You want to estimate the mean θ through n independent observations; the variance σ^2 is known. Give the MLE of θ.

Exercise 6 (Gaussian model – variance)**:**
In the previous exercise, the mean μ is now supposed to be known while the variance θ is unknown and to be estimated. Give the MLE of θ.

Exercise 7 (Gaussian model – mean and variance)**:**
In the previous exercise, both the mean μ and the variance σ^2 are unknown. We want to estimate the parameter $\theta = (\mu, \sigma^2)$. Give the MLE of θ.

Exercise 8 (Method of moments):
Propose an estimator by the method of moments for each of the models of exercices 3–7. Compare with the MLE.

Exercise 9 (Uniform model):
The lifetime of a smartphone is supposed to be uniformly distributed over $[0, \theta]$, where θ is unknown. You have n independent observations. Compare the MLE of θ to another given by the method of moments.

Exercise 10 (Mixture model):
A proportion θ of smartphones are defective: their lifetime has an exponential distribution with parameter μ while the lifetime of a regular smartphone has an exponential distribution with parameter $\lambda < \mu$. Both parameters λ and μ are known. You observe the lifetime of n smartphones. Propose an estimator of θ by the method of moments.

Exercise 11 (Gamma distribution):
The Gamma distribution with parameter $\theta = (a, \lambda)$ has density:

$$p_\theta(x) = \frac{1}{\Gamma(a)} \lambda^a x^{a-1} e^{-\lambda x}, \quad x > 0.$$

The mean and variance are respectively given by $\frac{a}{\lambda}$ and $\frac{a}{\lambda^2}$. You get n i.i.d. observations. Propose an estimator of θ by the method of moments.

Exercise 12 (Linear regression):
The cost of an apartment of surface x is equal to $y = \theta x + \epsilon$ where θ is some unknown parameter (in Euros / m^2) and ϵ is a Gaussian variable with zero mean and known variance σ^2. You observe n independent samples of the pair (x, y), say $(x_1, y_1), \ldots, (x_n, y_n)$. Give the MLE of θ.