M2MO Lecture Notes

Derivatives Modeling
Fundamentals

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Foreword

This course is about the fundamentals of financial derivatives modeling. It has three main pillars: the general no arbitrage and hedging theory; the Black-Scholes model, which still covers the majority of the practical uses of the theory; the local volatility model, which still covers the majority of the residual uses of the theory beyond Black-Scholes. However, these basic models are covered in a way, based on the general treatment of the first pillar, that can then be easily extended to any model, including any kind of stochastic volatility or jumps.

We mainly consider vanilla options throughout these notes. But contingent claims are first introduced at a principled level, so as to provide the ground for studying more exotic options in more advanced courses. We also tried to treat European and American options on an equal footing.

A common thread is the notion of profit-and-loss (negative of the tracking error), which is the main focus of a derivative trader or risk manager: see Definition I.8, I.§4.D, Equation I.(33), and Part II.§5.A.

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Contents

Prerequisites

Probabilities at the undergraduate level (including conditional expectation), Brownian motion, continuous Itô calculus, Poisson process.

Notation

Throughout these notes, $(\Omega, \mathcal{A}, \mathbb{Q})$ denotes a probability space, with related expectation, variance and covariance operators denoted by $\mathbb{E}, \mathbb{V}\mathrm{ar}$ and $\mathbb{C}\mathrm{ov}$. The expectation with respect to an arbitrary probability measure, say $\mathbb{P}$, is denoted by $\mathbb{E}^{\mathbb{P}}$ (so $\mathbb{E} = \mathbb{E}^{\mathbb{Q}}$).

From a financial interpretation viewpoint, the probability measure $\mathbb{Q}$ will correspond to a risk-neutral pricing mesure, i.e. a measure representation of financial derivative prices, assuming no arbitrage. We will also resort to the physical[1] probability measure $\mathbb{P}$, in a sense that should be intuitively clear, but is not easy to define in a rigorous and constructive fashion in a non-reproducible and highly non-stationary financial context. At least, it is assumed that the events deemed possible to occur in reality are exactly the ones that have positive $\mathbb{P}$ probability, a property that is then inherited by any probability measure equivalent to $\mathbb{P}$, such as the risk-neutral pricing measure $\mathbb{Q}$.

By default, a random variable is $\mathcal{A}$ measurable; we omit any indication of dependence on $\omega \in \Omega$ in the notation; all inequalities between random variables are meant $\mathbb{Q}$ almost surely (a.s. in shorthand notation); a real function of (possibly multivariate) real arguments is Borel measurable. We write $x^+ = x\mathbb{1}_{x\geq 0}$ and $x^- = (-x)\mathbb{1}_{x\leq 0}$, for any real x, hence $x = x^+ - x^-, |x| = x^+ + x^-$.

Color code Parts in orange should be known and can be *applied* without demonstration in exercises and examinations. Parts in blue are complementary material not required for exercises and examinations. Everything else in black done in class should be understood and assimilated by students, who should therefore be able to redo the corresponding developments, proofs included or variants thereof as requested by exercises and examinations.

We use numbering per chapter. The numbering is local to the current environment, e.g. B means Part B in the current section, §2.B refers to Part B of Section 2 (other than the current one) in the current chapter, IV.§3.B to Part B of Section 3 in Chapter IV (other than the current chapter).

[1] sometimes also dubbed real-world, statistical, historical, or objective.

Chapter I

No Arbitrage and Hedging Principles

§1 Introduction in a One-Period Setup

In a one-period economy, we consider a primary market composed of a zero-coupon bond with maturity $T > 0$ and of a risky asset, which may represent a stock or an equity index, an interest rate, an exchange rate, or the value of a commodity. The bond is worth $B \geq 0$ at time 0 and 1 at time $T > 0$. The risky asset, dubbed stock for concreteness hereafter, is worth $S \geq 0$ at time 0 and a nonnegative random variable S_T at time T. Moreover, the stock earns dividends[1], in the sense that, in order to have one unit of the stock at time T, it is enough to hold $A > 0$ (and "typically" ≤ 1) units of the stock at time 0.

The primary market underlies financial derivatives, which are contracts promising a random cash flow ξ at time T to the holder, where ξ may depend on S_T in ways to be detailed below, in exchange of a time-0 premium π to the issuer.

Long/short positions in a financial asset mean positive/negative positions, where positive refers to the situation of an investor having bought (one or several units of) the asset and negative to the one of an investor having sold the asset. Any linear combination of traded assets with integer coefficients, or portfolio, is assumed to be achievable in the market, and prices are assumed to be linear with respect to trades, so price (or value) of a portfolio means the corresponding linear combination of the prices of its building assets.

A Payoffs

A (European) payoff ξ with maturity T means a random variable representing a cashflow promised at time T under the terms of a financial derivative. A forward with maturity T and strike $K \geq 0$ on the stock corresponds to the payoff $\xi = S_T - K$ at time T. European vanilla call/put options correspond to the payoffs $\xi = (S_T - K)^{\pm}$.

Combining long and short positions in existing assets allows one to generate new payoffs. See Figure 1 and play with the python code by Clint Howard on https://clinthoward.github.io/portfolio/2017/04/16/BlackScholesGreeks for more fancy payoffs. Straddle and butterfly spreads (bottom panels of Figure 1) allow investors to bet on the future riskiness (low or high) of S. Bull and bear spreads (top panels of Figure 1) allow investors to make directional bets on S, at a more affordable price than vanilla calls and puts as we will now see.

In the case of American calls and puts on S, the holder of the option has the choice, which must be done at time 0, between the payoff $(S - K)^+$ at time 0, which is then assumed to be invested into

[1] The exact interpretation of q depends on the nature of the underlying S.

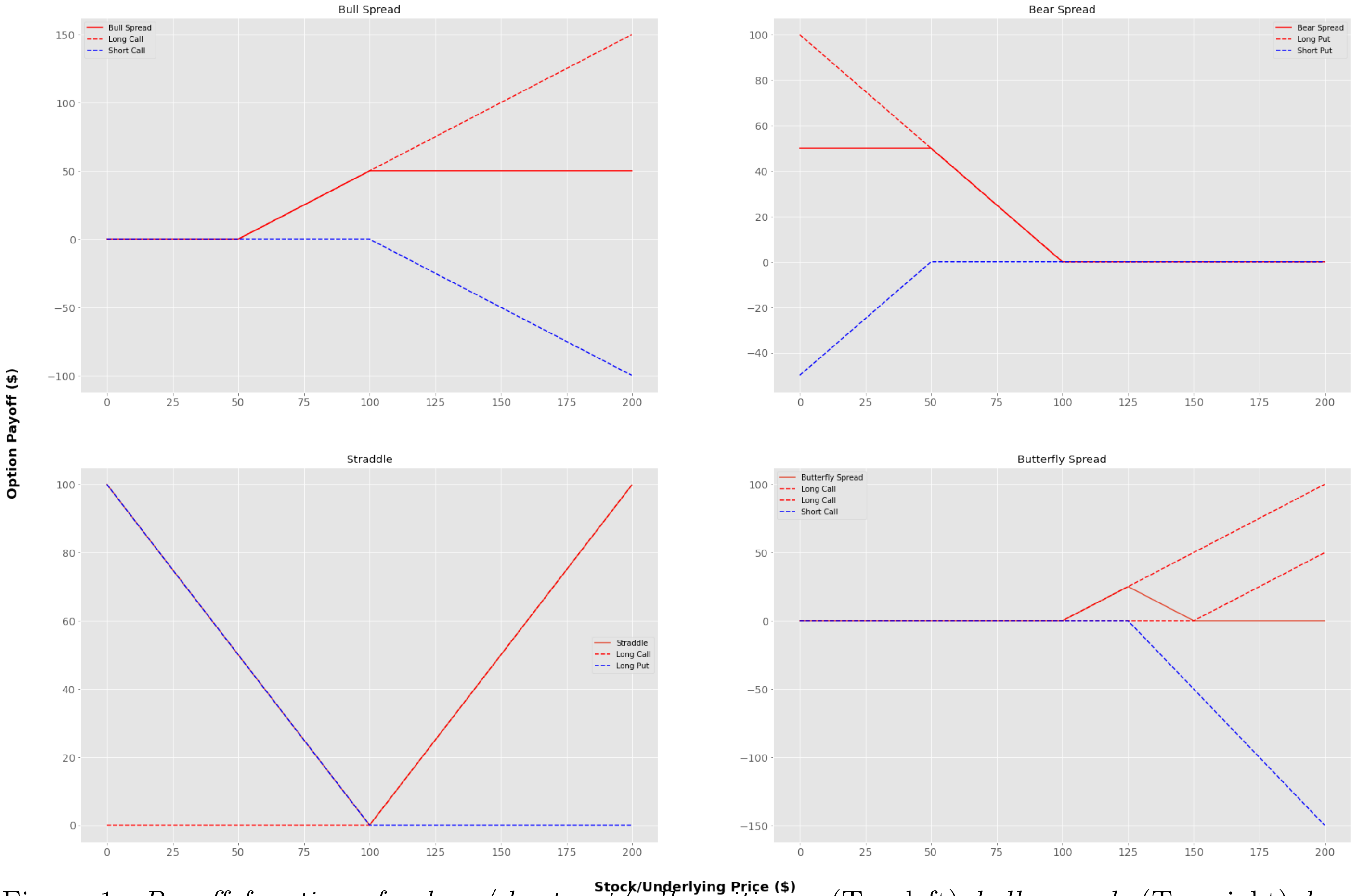

Figure 1: *Payoff functions for long/short put/call positions:* (Top left) *bull spread;* (Top right) *bear spread;* (Bottom left) *straddle;* (Bottom right) *butterfly spread.*

the bond until time T, and the payoff $(S_T - K)^+$ at time T.

B Pricing by Arbitrage

We use "a.s." for almost surely, i.e. "with probability one", and "poss." for possibly, i.e. "with (strictly) positive probability".

Definition 1 *We call arbitrage a trading strategy in the available assets with zero initial cost and terminal payoff* ≥ 0 *a.s. and* > 0 *poss..*

The embedded notion of "a trading strategy in the available assets with zero initial cost" has not been formally defined here. In the case of European-only derivatives that would be available on top of the primary risk-free and risky assets, a trading strategy with zero initial cost just means a zero-valued portfolio. Hence the notion of an arbitrage reduces to a portfolio with time-0 value $\pi = 0$ and time-T value $\xi \geq 0$ a.s. and > 0 poss.. In the case where American options are also available for trading, a definition of a trading strategy with zero initial cost sufficient for our purposes in this section would be the specification of a zero-valued portfolio with only short American positions and, for each exercise decision ("now or later") related to each American claim in the portfolio, of a zero-valued (hedging) portfolio with only long American claims and of an exercise decision ("now or later") for each American claim in the hedging portfolio.

We always assume no arbitrage opportunities in these notes. We always assume no arbitrage opportunities in these notes. Hence, in particular, $B > 0$: otherwise, the strategy consisting in buying the bond and investing the proceeds[2] into the stock would result in a cash flow $\xi \geq 1$ a.s., starting from a zero wealth $\pi = 0$ at time 0, which would be an arbitrage.

Also, as stated above, bull and bear spreads cannot be more expensive than the corresponding vanilla calls and puts, otherwise "buying cheap, selling dear", and investing the proceeds into the bond would result in a cash flow $\xi > 0$ a.s., starting from a zero wealth $\pi = 0$ at time 0, which would be an arbitrage.

C Forwards Contracts and Call-Put Parity

Proposition 1 *A forward contract with payoff $\xi = S_T - K$ has time-0 price*

$$F = SA - KB = C - P, \tag{1}$$

where C and P are the European vanilla call and put prices with payoffs $(S_T - K)^{\pm}$.

Proof. Starting from the initial wealth $F = SA - KB$, borrowing KB and buying A stocks results in the payoff $S_T - K$ at time T. Hence, a time-0 price for the forward that would differ from F would lead to arbitrages obtained by selling the forward, if possible at a price $> F$, or buying it, if possible at a price $< F$, investing the benefit in the risk-free bond, and completing the strategy by the above synthetic replication of the payoff $S_T - K$ at time T or its opposite.

The second identity in (1) follows from the no arbitrage assumption by the the fact that a forward and a long/short call/put position have the same payoff $S_T - K = (S_T - K)^+ - (S_T - K)^-$ at T. ∎

D Calls and Puts: Bounds

We denote by C, P and $\widetilde{C}$, $\widetilde{P}$ the time-0 prices of European and American calls and puts on S, all with maturity T and strike K.

Proposition 2 $\widetilde{C} \geq C\,,\ \widetilde{P} \geq P$.

Proof. Otherwise, supposing say $C > \widetilde{C}$, the strategy consisting in selling the European option, buying the American option, investing the proceeds into the bond and exercising the American option at T results into a constant payoff $\xi = \frac{C - \widetilde{C}}{B} > 0$ at T, starting from a zero wealth $\pi = 0$ at time 0, which would be an arbitrage. ∎

Proposition 3 *We have $(SA - KB)^+ \leq C \leq SA, (KB - SA)^+ \leq P \leq KB$.*

Proof. The no arbitrage assumption implies $C, P \geq 0$ and $C \leq SA$[3] , $P \leq KB$, whereas call-put parity (1) implies $C \geq (SA - KB), P \geq (KB - SA)$. ∎

Proposition 4 **(i)** *If $B \leq 1 \leq A$, then $\widetilde{C} = C \geq (S - K)^+$.*
(ii) *If $B \geq 1 \geq A$, then $\widetilde{P} = P \geq (K - S)^+$.*

[2] assuming $B \leq 0$.

[3] Otherwise selling the call, buying SA stocks and investing the proceeds into the bond would result in a positive payoff ξ, starting from a zero wealth $\pi = 0$ at time 0.

Proof. As both proofs are similar, we only prove (i). By Propositions 2 and 3, we have $(SA-KB)^+ \le C \le \widetilde{C}$. Moreover $B \le 1 \le A$ implies $SA-KB \ge S-K$, hence $(SA-KB)^+ \ge (S-K)^+$. Finally, would $\widetilde{C} \ge C$ be $> C$, an arbitrage would consist in selling the American option, buying the European option and reselling it at the time $\vartheta \le T$ when the American option holder exercises, whence an overall profit

$$(\widetilde{C}-C)B^{-1} + (C_\vartheta - (S_\vartheta - K)^+)B^{-1}\mathbb{1}_{\vartheta=0} \ge (\widetilde{C}-C)B^{-1} > 0. \blacksquare$$

Proposition 5 *Assuming positive S and K:*
(i) *If $B \le 1 \le A$, with at least one of the two inequalities strict, and $S_T > K$ poss., then $C > (S-K)^+$.*
(ii) *If $B \ge 1 \ge A$, with at least one of the two inequalities strict, then $P > (K-S)^+$.*

Proof. As both proofs are similar, we only prove (i). We have already seen in Proposition 4 and its proof that

$$(S-K)^+ \le (SA-KB)^+ \le C.$$

Moreover no arbitrage with $S_T > K$ poss. implies $C > 0$. Hence $(SA-KB)^+ = 0$ implies

$$C > 0 = (SA-KB)^+ = (S-K)^+,$$

whereas $(SA-KB)^+ > 0$ implies

$$C \ge (SA-KB)^+ > (S-K)^+. \blacksquare$$

E Calls and Puts: Shape Constraints

Proposition 6 **(i)** *C and P are respectively nonincreasing and nondecreasing in K;*
(ii) *C and P are B-Lipschitz in K;*
(iii) *C and P are convex in K;*
(iv) *If $B \le 1 \le A$, then C is nondecreasing in T; if $B \ge 1 \ge A$, then P is nondecreasing in T.*

Proof. **(i)** follows from the no arbitrage assumption by nonnegativity of bull and bear spread payoffs (see the top panels in Figure 1).

(ii) follows from the no arbitrage assumption by the bound $|K_1 - K_2|$ on bull and bear spread payoffs, where K_1 and K_2 are the strikes defining the spread options (see the top panels in Figure 1).

(iii) follows from the no arbitrage assumption by nonnegativity of butterly spread payoffs, for any defining strikes $K-k, K, K+k$ (see the bottom right panel in Figure 1).

(iv) for calls follows from the no arbitrage assumption by the fact that the price at T_1 of a call calendar spread (difference between two calls of same strikes and maturities $T_2 \ge T_1$) is nonnegative if $B \le 1 \le A$ (cf. Proposition 4(i) and its proof); similar argument regarding puts. $\blacksquare$

But the argument in the last point of this proof is overstretched (the proof is not rigorous, even if the result is "morally true"), because the one-period setup of this introduction cannot accomodate a situation involving, at least, three dates (the times $0 \le T_1 \le T_2$ in the above). To address this and more general problems, we need a dynamic framework—unless perhaps one could derive formulas for C and P, but explicit formulas for European vanilla options are the prerogative of not only dynamic but even (some) continuous-time models[4].

[4]cf. Proposition II.5.

Hereafter in these notes, we assume a continuous-time filtration $\mathfrak{F} = (\mathfrak{F}_t)_{t\in[0,T]}$ on $\mathcal{A}$ satisfying the usual conditions of right-continuity and completeness, with respect to which all processes that appear are $\mathfrak{F}$-adapted. We assume that $\mathfrak{F}_0$ is the trivial σ-field $\{\emptyset, \mathcal{A}\}$, so that all processes have constant initial conditions at time 0. Unless otherwise stated, all our processes are real valued. All inequalities between processes are meant $dt \times d\mathbb{P}$-almost everywhere, componentwise in case of vector valued processes.

§2 Primary Market Model in Continuous Time

In this probabilistic framework, we consider a primary market model composed of a (locally) risk-free asset, called the savings account, and of q primary risky assets. The savings account P^0 evolves at the short-term risk-free interest rate r_t, which is the rate of a riskless loan[5] between t and $t+dt$. The riskless discount factor β is then defined as the inverse of the savings account. Letting conventionally $\beta_0 = 1$, we thus have, for $t \in [0,T]$,

$$P_t^0 = \beta_t^{-1} = e^{\int_0^t r_s\,ds}, \tag{2}$$

where the short-term interest rate process r is assumed time-integrable and bounded from below.[6]

We denote by P and $\mathcal{D}$ the $\mathbb{R}^q$-valued price process and cumulative dividend value process of the primary risky assets. The price process P may be understood, in a sense to be made more precise by (12) below, as the present value of future cash flows. The finite variation[7] dividend process $\mathcal{D}$ accumulates all the financial cash flows that are granted to the holders of the risky assets during their lifetime. Dividends in this broad sense thus encompass stock dividends in the usual sense in the case of risky assets given as stocks, coupons in the case of coupon-bearing bonds, or recovery of assets upon default of their issuing firm, to state only a few sources of "dividend" incomes.

Given the price process P and the dividend process $\mathcal{D}$, it is convenient mathematically to further introduce the so-called cumulative price $\widehat{P}$ of the primary risky assets, defined as

$$\widehat{P}_t = P_t + \beta_t^{-1} \int_{[0,t]} \beta_s\, d\mathcal{D}_s. \tag{3}$$

In the financial interpretation, the last term in (3) represents the current value at time t of all dividend payments of the asset over the period $[0,t]$, under the assumption that all dividends are immediately reinvested in the savings account.

Also, since the vocation of the primary market is to serve as a pool of hedging instruments for a financial derivative with maturity T, we assume that the primary assets live over $[0,T]$ (or beyond, in which case we simply "ignore" what happens to them beyond T). In particular, we assume that the primary market is "European" in the sense that it doesn't contain assets with early exercise clauses.

We assume that the price P (hence cumulative price process $\widehat{P}$) is a semimartingale[8], so that any càglàd[9] integrand can be stochastically integrated against it[10].

We assume that the primary market model is free of arbitrage opportunities, in a sense to be detailed below. In a broad sense, absence of arbitrage opportunities refers to the impossibility of making a profit

[5] assumed to exist, even though this is irrealistic (as well understood by the market since the 2008-09 global financial crisis).

[6] mostly nonnegative, but not always: in fact, since 2012, a number of central banks introduced negative interest rate policies in response to persistently below-target inflation rates.

[7] see Definition IV.4.

[8] e.g. Itô process of the form IV.(5), see IV.§2.

[9] cf. the introductory paragraph to Chapter IV.

[10] cf. IV.(2).

with positive probability without risking a loss. The idea underlying the no arbitrage assumption is that, every time arbitrage opportunities arise in the market, they are "exercised" and resorbed in quick time by supply and demand, so that it is a reasonable mathematical approximation to consider that no such opportunities remain. Another, more fundamental view on the no arbitrage assumption, seen as prohibition of riskless profits[11], is related to Pareto optimality and the purpose of maximising social welfare.

The no arbitrage condition involves wealth processes of admissible self-financing primary trading strategies.

Definition 2 *A primary trading strategy in the primary market is an $\mathbb{R} \times \mathbb{R}^q$-valued process (ζ^0, ζ), where ζ^0 and the row-vector ζ respectively represent the number of units held in the savings account and in the primary risky assets, with ζ càglàd. The related wealth process V is given, for $t \in [0, T]$, by*

$$V_t = \zeta_t^0 P_t^0 + \zeta_t P_t. \tag{4}$$

The strategy is said to be self-financing if

$$dV_t = \zeta_t^0 dP_t^0 + \zeta_t \left(dP_t + d\mathcal{D}_t\right) \tag{5}$$

or, equivalently,

$$d(\beta_t V_t) = \zeta_t \, d(\beta_t \widehat{P}_t). \tag{6}$$

If, moreover, the discounted wealth process βV is bounded from below, the strategy is said to be admissible.

The equivalence between the intrinsic form (5) and the discounted form (6) of the self-financing condition is easily seen in the present context, so that:

$$\begin{aligned}
& d(\beta_t V_t) = \beta_t (dV_t - r_t V_t dt) \\
& \zeta_t \, d(\beta_t \widehat{P}_t) = \zeta_t \, \left(d(\beta_t P_t) + \beta_t d\mathcal{D}_t\right) \\
& \quad = \beta_t \left(\zeta_t^0 dP_t^0 + \zeta_t \left(dP_t + d\mathcal{D}_t\right) - r_t (\zeta_t^0 P_t^0 + \zeta_t P_t) dt\right) \\
& \quad = \beta_t \left(\zeta_t^0 dP_t^0 + \zeta_t \left(dP_t + d\mathcal{D}_t\right) - r_t V_t dt\right),
\end{aligned}$$

by (4). Hence

$$d(\beta_t V_t) - \zeta_t \, d(\beta_t \widehat{P}_t) = \beta_t \big(dV_t - \zeta_t^0 dP_t^0 - \zeta_t \left(dP_t + d\mathcal{D}_t\right)\big) \tag{7}$$

so that (6) is equivalent to (5).

Remark 1 *One may want to define, instead of (4),*

$$V_t = \zeta_t^0 P_t^0 + \zeta_t (P_t + \Delta \mathcal{D}_t), \tag{8}$$

where $\Delta \mathcal{D} = \mathcal{D} - \mathcal{D}_-$ is the jump process of $\mathcal{D}$. As $\mathcal{D}_t dt = \mathcal{D}_{t-} dt$ (since jump times of the càdlàg process $\mathcal{D}$ are countable), hence $\Delta \mathcal{D}_t dt = 0$. Therefore the identity (7) (by the same proof as above, using $\Delta \mathcal{D}_t dt = 0$) and, in turn, the equivalence between (5) and (6) are still valid with this alternative definition of V. We conclude that, for self-financing strategies, the choice between either definition of V is somehow conventional (only leading to a different ζ^0).

[11] this was de Finetti (1931)'s seminal position.

Remark 2 *The càglàd*[12] *regularity assumption on ζ in Definition 2 implies that the current value of ζ can be "predicted" if we know its past values. In discrete time with a discrete filtration $(\mathfrak{F}_i)_{i\geq 0}$, predictability reduces to adaptedness of the discrete time strategy $(\zeta_i)_{i\geq 1}$ with respect to the "lagged" filtration $(\mathfrak{F}_{i-1})_{i\geq 1}$. From a financial point of view, assuming predictability of a trading strategy is justified by the fact that effectively "implementing" a risky strategy always requires some small amount of time.*

Given the initial wealth π of a self-financing primary trading strategy (ζ^0, ζ), by (6) the wealth process V can also be written as

$$\beta_t V_t = \pi + \int_0^t \zeta_s \, d(\beta_s \widehat{P}_s). \tag{9}$$

The process ζ^0 that gives the number of units held in the savings account is then uniquely determined as (under (4)):

$$\zeta_t^0 = \beta_t \left(V_t - \zeta_t P_t\right). \tag{10}$$

In the sequel we restrict ourselves to self-financing trading strategies. We thus can redefine a (self-financing) primary trading strategy as a pair (π, ζ), formed of an initial wealth $\pi \in \mathbb{R}$, and an $\mathbb{R}^q$-valued càglàd primary strategy in the risky assets ζ, with related wealth process V defined by (9). This redefinition is a convenient trick that allows us to "forget" about the risk-free funding asset and the funding issue (self-financing condition on the strategy), which becomes "absorbed" in the discounting at the risk-free rate.

Recall that $\mathbb{P}$ represents the physical probability measure.

Definition 3 *An arbitrage is a primary trading strategy $(\pi = 0, \zeta)$ such that the process βV is bounded from below and*

$$\mathbb{P}(V_T \geq 0) = 1\,,\ \mathbb{P}(V_T > 0) > 0. \tag{11}$$

The boundedness from below is the admissibility (or feasibility) condition. If violated, then the corresponding strategy would not really be an arbitrage for the bank, since the latter could default on its way.

Definition 4 *A risk-neutral measure on the primary market is a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that $\beta\widehat{P}$ is a $\mathbb{Q}$ local martingale*[13].

We then write $\mathbb{E} = \mathbb{E}^{\mathbb{Q}}$.

Remark 3 *If we assume that $\beta\widehat{P}$ is even a $\mathbb{Q}$ martingale (as opposed to only a local one), then the following "cost-of-carry formula" holds:*

$$P_t = \mathbb{E}_t\Big(\beta_t^{-1}\beta_T P_T + \int_t^T \beta_t^{-1}\beta_s d\mathcal{D}_s\Big), \tag{12}$$

i.e. the time-t price P_t is the discounted $\mathbb{Q}$ expectation of the future earnings related to a long unit position in the risky assets unwound at time T.

Lemma 1 *For any nonnegative, $\mathbb{Q}$ integrable random variable X,*

$$X = 0 \ a.s. \iff \mathbb{E}X = 0. \tag{13}$$

[12] cf. the introductory paragraph to Chapter IV.
[13] cf. Definition IV.3.

Proof of $\Longleftarrow$. We have $\{X=0\}=\cap_{n>0}\{X<\frac{1}{n}\}$, where the Markov inequality yields $\mathbb{Q}(\{X\geq\frac{1}{n}\})\leq n\mathbb{E}X$. Hence $\mathbb{E}X=0$ implies that $\mathbb{Q}(\{X=0\})=1$. ∎

Theorem 1 *If there exists a risk-neutral measure $\mathbb{Q}$ on the primary market, then the latter is free of arbitrage opportunities. It is even free of $\mathbb{Q}$ arbitrage opportunities, which correspond to the relaxed notion of arbitrage opportunities with ζ "admissible, i.e. βV bounded from below" therein replaced by "$\mathbb{Q}$ admissible, i.e. $\beta V\geq$ some $\mathbb{Q}$ martingale"*[14].

Proof. We only show the second (stronger) statement. Assuming that $\beta\widehat{P}$ is a $\mathbb{Q}$ local martingale, the discounted wealth βV of any admissible primary trading strategy $(\pi=0,\zeta)$ is a $\mathbb{Q}$ local martingale, by (6) and Theorem IV.1. If the strategy is $\mathbb{Q}$ admissible, βV is then a $\mathbb{Q}$ supermartingale, by Lemma IV.3. In particular, $\beta_T V_T$ is $\mathbb{Q}$ integrable and $\mathbb{E}(\beta_T V_T)\leq\pi=0$, by assumption. If additionally $V_T\geq 0$ a.s., then $\mathbb{E}(\beta_T V_T)=0$ and therefore $\mathbb{Q}(V_T=0)=1$, by Lemma 1. Hence $\mathbb{P}(V_T=0)=1$, by equivalence between $\mathbb{P}$ and $\mathbb{Q}$. Therefore the second condition in (11) does not hold and the strategy $(\pi=0,\zeta)$ is not an arbitrage. ∎

Remark 4 *$\mathbb{Q}$ admissible strategies include the vector space of the admissible strategies (π,ζ) for which βV is a $\mathbb{Q}$ martingale.*

The converse to the first part in Theorem 1 is also "morally" true (up to technicalities). It is easy to establish in finite Ω and static (de Finetti, 1937) or finitely dynamic (Harrison and Kreps, 1979) setups, where it reduces to a separation hyperplane argument. But it becomes very technical beyond this case (Delbaen and Schachermayer, 2005).

The first part in Theorem 1 and its converse are sometimes jointly referred to as the first fundamental theorem of asset pricing[15]. A companion result to the latter is the characterization of uniqueness of a risk-neutral measure therein. Loosely speaking, the second fundamental theorem of asset pricing states that, assuming the existence of a risk-neutral measure over the primary market, its uniqueness is equivalent to the completeness of the market, which means that every European derivative in the market is replicable[16] Again, this is easily established in the finite case, but beyond this case it becomes demanding technically.

§3 Contingent Claims

We now introduce a contingent claim, or financial derivative, on the primary market.

A Cash Flows

A derivative is a financial claim between an investor or holder of a claim and its counterparty or issuer that involve, as made precise in Definition 5 below, some or all of the following cash flows (or payoffs):

- a cumulative dividend process $D=(D_t)_{t\in[0,T]}$, assumed to be a semimartingale,
- terminal cash flows, consisting of:
  - a payment ξ at maturity T, where ξ denotes a random variable,

[14] which can depend on the strategy ζ that is implicit in V, like the lower bound in the admissibility condition.
[15] see e.g. Shiryaev and Cherny (2002).
[16] at no cost, cf. the last paragraphs of §3.E below.

– in the case of American product with early exercise feature, a put payoff process $L = (L_t)_{t\in[0,T]}$, given as a càdlàg process such that $L_T \le \xi$.

The put payoff L_t corresponds to a payment made by the issuer to the holder of the claim, in case the holder of the claim decides to terminate (or "put") the contract at time t.

The terminology "derivative" comes from the fact that all the above cash flows are typically given as functions of the underlying primary asset price process P.

Example 1 **(i)** *In the respective cases of a forward contract with maturity T on $S = P^1$, the first primary risky asset, and of a European vanilla call/put option with maturity T and strike K on S, one has $D = 0$ and $\xi = (S_T - K)$, respectively $\xi = (S_T - K)^{\pm}$.*
(ii) *In the case of an American vanilla call/put option on S, one has, on top of the above, a put process $L_t = (S_t - K)^{\pm}$.*
(iii) *In the case of a (vanilla) convertible bond on an underlying stock S, the dividend process D consists of a cumulative flow of bond coupons, and we have*

$$\xi = \bar{N} \vee S_T,\ L_t = \bar{P} \vee S_t,$$

for nonnegative constants $\bar{P} \le \bar{N}$.
(iv) *In the case of a credit default swap (CDS for short) on an underlying credit name, there are no terminal cash flows in the above sense. There is only a cumulative dividend process D. From the point of view of the seller of default protection and for a notional conventionally set to one, this process is given by:*

$$D_t = \int_0^t (\Lambda dJ_s + SJ_s ds)\,,$$

where Λ and J are the fractional loss-given-default and the survival indicator process of the underlying credit name and where S is the contractual spread of the CDS.
(v) *In the case of a collateralized debt obligation (CDO tranche) with attachment point a and detachment point b on an underlying portfolio of credit names, there are no terminal cash flows. From the point of view of the seller of default protection and for a notional conventionally set to one, the dividend process D is given by:*

$$D_t = \int_0^t (\Sigma(b - a - L_s)ds - dL_s)\,,$$

where Σ is the tranche contractual spread and the cumulative tranche loss process L is such that

$$L_t = (\mathcal{L}_t - a)^+ - (\mathcal{L}_t - b)^+ = \min\left((\mathcal{L}_t - a)^+, b - a\right),$$

in which $\mathcal{L}_t$ is the aggregated default loss on the portfolio at time t.

We will now define general European and American claims. In the following definitions, the put (or maturity) time ϑ represents a stopping time at the holder's convenience.

Definition 5 **(i)** *A European claim is a financial claim with dividend process D and with payment ξ at maturity T.*
(ii) *An American claim is a financial claim with dividend process D and with payment at the terminal (put or maturity) time ϑ given by*

$$\mathbb{1}_{\{\vartheta<T\}} L_\vartheta + \mathbb{1}_{\{\vartheta=T\}} \xi. \tag{14}$$

As visible from Example 1, many real-life financial derivatives have cash flows D, ξ (and also L in the American case) bounded from below.

B $\mathbb{Q}$ Prices

Let Θ^t or simply Θ in the case $t=0$, denote the set of all $[t,T]$-valued stopping times[17].

Definition 6 *For any risk-neutral measure $\mathbb{Q}$[18] with related time-t conditional expectation denoted by $\mathbb{E}_t$:*
(i) *Assuming the random variable $\int_0^T \beta_s\,dD_s + \beta_T\xi$ integrable under $\mathbb{Q}$, the process Π such that, for $t\in[0,T]$,*

$$\beta_t\Pi_t = \mathbb{E}_t\Big\{ \int_t^T \beta_s\,dD_s + \beta_T\xi \Big\}, \tag{15}$$

is called the $\mathbb{Q}$ price of the European claim with cash flows (D,ξ).
(ii) *Assuming the process*

$$\Big(\int_0^t \beta_s\,dD_s + \beta_t\big(\mathbb{1}_{\{t<T\}}L_t + \mathbb{1}_{\{t=T\}}\xi\big)\Big)$$

integrable under $\mathbb{Q}$, the process Π such that, for $t\in[0,T]$,[19]

$$\beta_t\Pi_t = \max_{\vartheta\in\Theta^t}\mathbb{E}_t\Big\{ \int_t^{\vartheta} \beta_s\,dD_s + \beta_\vartheta\big(\mathbb{1}_{\{\vartheta<T\}}L_\vartheta + \mathbb{1}_{\{\vartheta=T\}}\xi\big)\Big\}, \tag{16}$$

is called the $\mathbb{Q}$ price of the American claim with cash flows (D,L,ξ).
(iii) *Given a European or American $\mathbb{Q}$ price Π, the cumulative $\mathbb{Q}$ price of the option is the process $\widehat{\Pi}$ such that, for $t\in[0,T]$,*[20]

$$\beta_t\widehat{\Pi}_t = \beta_t\Pi_t + \int_{[0,t]} \beta_s\,dD_s. \tag{17}$$

Remark 5 *As apparent on (15)-(16), a $\mathbb{Q}$ price Π satisfies $\Pi_T=\xi$,, i.e. we follow the usage*[21] *of including the terminal cash flow ξ in the price of an option at its maturity T. This is also why we formally distinguish ξ from the dividends D in Definition 5, otherwise the price of the option at time T would be 0 but there would be a "dividend" ξ at T. The cumulative price process of the option is the same under both conventions, so this choice is immaterial as far as arbitrage considerations are concerned. It makes life easier in Markovian setups where it avoids an artificial discontinuity of the pricing function at maturity.*

Definition 7 *Given a risk-neutral measure $\mathbb{Q}$ on a primary market encompassing a (risky) stock S such that S_T is $\mathbb{Q}$ integrable:*
(ii) *The futures contract with maturity T on S is the European claim with dividend process $D_t^T = \mathbb{E}_t S_T$, $t\le T$, and no payment at maturity, so $\xi=0$. By application of (15), the $\mathbb{Q}$ price of this futures contract is therefore*

$$\Pi_t = \mathbb{E}_t\int_t^T \beta_t^{-1}\beta_s\,dD_s^T = 0, \tag{18}$$

assuming $\beta_s\,dD_s^T$ a true martingale[22]*. The amount $D_t^T=\mathbb{E}_tS_T$ is known as the T-futures price of S at time t.*
(ii) *The so-called T-forward price of S at time t is the value F_t^T of the strike K of a (T,K)-forward*

[17] cf. Definition IV.2.
[18] cf. Definition 4.
[19] assuming the max achieved in (16).
[20] cf. (3) regarding the primary market.
[21] convenient in Markov setups as it will appear later.
[22] not only a local martingale, which it is by application of Theorem IV.1.

contract[23] *on S such that the $\mathbb{Q}$ price of the contract vanishes at time t, i.e., by application of* (15) *to the data* $(D,\xi)=(0,S_T-K)$*:*

$$F_t^T=\frac{\mathbb{E}_t(\beta_t^{-1}\beta_T S_T)}{\mathbb{E}_t(\beta_t^{-1}\beta_T)}=\frac{\mathbb{E}_t(\beta_t^{-1}\beta_T S_T)}{B_t^T}=\mathbb{E}_t^T S_T, \tag{19}$$

where $B_t^T=\mathbb{E}_t(\beta_t^{-1}\beta_T)$ is the time-t price of the zero-coupon of maturity T, with data $(D,\xi)=(0,1)$, while $\mathbb{E}^T$ is the expectation with respect to the T-forward probability measure $\mathbb{Q}^T$ defined by $\frac{d\mathbb{Q}^T}{d\mathbb{Q}}=\frac{\beta B^T}{B_0^T}$, so that the last equality in (19) *holds by application of Lemma IV.4(i).*

Both futures and forward contracts on S provide some exposure to the evolutions of S in the future, but with counterparty credit risk reduced to daily settlement[24] *in the case of the futures contract. If interest rates are deterministic, then $\mathbb{Q}^T=\mathbb{Q}$, hence $F^T=\mathbb{E}_\cdot^T S_T=\mathbb{E}_\cdot S_T=D^T$.*

In view of (15), the discounted cumulative $\mathbb{Q}$ price process of a European option is a $\mathbb{Q}$ martingale. Hence, the risk neutral measure $\mathbb{Q}$ is still risk-neutral for the augmented market consisting of the primary market and the option trading at that price (assuming the option liquidly traded). Therefore, by application of Theorem 1 to the augmented market, the latter is still arbitrage-free.

In the American case, the discounted cumulative $\mathbb{Q}$ price, defined by (17) with Π from (16), is the Snell envelope of the discounted payoff process given, for $t\in[0,T]$, by

$$\int_{[0,t]}\beta_s\,dD_s+\beta_t\big(\mathbb{1}_{\{t<T\}}L_t+\mathbb{1}_{\{t=T\}}\xi\big)\ ,\ t\in[0,T], \tag{20}$$

i.e. the smallest supermartingale $\geq$ (20)[25]. Let us consider the augmented market consisting of the primary market and the American option trading at that price (assuming the American option liquidly traded). By Kallsen and Kühn (2004), the augmented market is arbitrage-free, under the additional admissibility constraint that the bank cannot short the American claim in scenarios where $\Pi_t<L_t$. The idea is that there may then be no American claims available to short in the market, as all holders may then have exercised their claims (liquidity dry up).

Remark 6 *Risk-neutral prices of liquid European or American claims, i.e. their $\mathbb{Q}$ prices Π relative to a risk-neutral measure $\mathbb{Q}$ on the primary market, are thus no arbitrage prices, in the sense of no arbitrage on the augmented market consisting of the primary market and the derivative. In view of the (partial) converse to the first part of Theorem 1 mentioned after Remark 4, one might even claim that no-arbitrage option prices have to be $\mathbb{Q}$ prices, so assuming $\mathbb{Q}$ prices for options is not a restriction in practice (as soon as no arbitrage is edicted as a minimal requirement). It remains to clarify the meaning of $\mathbb{Q}$ prices from a hedging perspective, which is done hereafter.*

Remark 7 *Consistent with Remark 6, all the no arbitrage results of Propositions 1 through 6, established there for call and put prices in the sense of no arbitrage prices in a static setup, are still valid for call and put $\mathbb{Q}$ prices in a continuous-time setup. In fact, revisiting the proofs of these results shows that they all consist in identifying a payoff $\chi\geq 0$ a.s. (resp. and >0 poss.) to conclude from a static no arbitrage argument that the corresponding static no arbitrage price π is ≥ 0 (resp. >0). In a continuous-time $\mathbb{Q}$ prices setup, the same conclusion follows from the properties of expectation stated as Lemma 1, namely*

$$\begin{aligned}\chi\geq 0 \text{ a.s. } &\Longrightarrow \mathbb{E}\chi\geq 0\\ \chi\geq 0 \text{ a.s. and } >0 \text{ poss.} &\Longrightarrow \mathbb{E}\chi>0.\end{aligned}$$

[23] cf. Example 1(i).

[24] assuming the margin calls dD_t^T implemented on a daily basis.

[25] cf. (El Karoui, 1981).

The only point requiring further verification in these proofs is that continuous-time $\mathbb{Q}$ prices satisfy $\widetilde{C} = C$ (resp. $\widetilde{P} = P$) under the assumptions of Proposition 4(i) (resp. ii), which will be established below as Proposition II.1.

However, in a continuous-time setup, the conditions stated in Propositions 1 through 6 are only no-static-arbitrage conditions on call and put prices. The corresponding no-dynamic-arbitrage conditions, including the treatment of the embedded admissibility issues, are more involved (Cox and Hobson, 2005; Jacquier and Keller-Ressel, 2018).

The following process is the main focus of a derivative trader or risk manager.

Definition 8 *The process p such that*

$$\beta p = \int_0^{\cdot} \left(-d(\beta_s \widehat{\Pi}_s) + \zeta_s d(\beta_s \widehat{P}_s)\right) \tag{21}$$

is the profit-and-loss (or negative of the tracking error) of the issuer using the price-and-hedge strategy (Π, ζ).

In view of the above, assuming a $\mathbb{Q}$ price Π:

Corollary 1 *The profit-and-loss p is a $\mathbb{Q}$ martingale in the European case and a $\mathbb{Q}$ submartingale in the American case.*

Hereafter in this Section[26], we work under a fixed risk-neutral measure $\mathbb{Q}$, with $\mathbb{Q}$ expectation (resp. conditional expectation) denoted by $\mathbb{E}$ (resp. $\mathbb{E}_t \equiv \mathbb{E}(\cdot \,|\, \mathfrak{F}_t)$). All the measure-dependent notions implicitly refer to the probability measure $\mathbb{Q}$.

C American Reflected BSDE

In the American case we introduce the following reflected backward stochastic differential equation (reflected BSDE) with the data β, D, L, ξ :

$$\begin{cases} \beta_t \Pi_t = \beta_T \xi + \int_t^T \beta_s dD_s + \int_t^T \beta_s (dA_s - dM_s), & t \in [0,T] \\ L_t \le \Pi_t, \quad t \in [0,T] \\ \int_0^T (\Pi_s - L_s)\, dA_s = 0 \end{cases} \tag{22}$$

Since $\beta_t = e^{-\int_0^t r_s ds}$, the first line of (22) is equivalent to

$$\Pi_t = \xi + \int_t^T (dD_s - r_s \Pi_s ds) + (A_T - A_t) - (M_T - M_t), \quad t \in [0,T]. \tag{23}$$

Definition 9 *By a solution to (22) we mean a triple of real processes (Π, M, A) such that all conditions in (22) are satisfied, where:*

- *the value process Π is a càdlàg process,*
- *$\int_0^{\cdot} \beta_s dM_s$ is a martingale,*
- *A is a finite variation[27] continuous process.*

[26] except in Theorem 2.
[27] starting from 0 by definition; see Definition IV.4.

By taking the difference between the first line of (22) for times t and 0, we obtain:

$$\beta_t \Pi_t + \int_0^t \beta_s dD_s = \big(\Pi_0 + \int_0^t \beta_s dM_s\big) - \int_0^t \beta_s dA_s, \tag{24}$$

which provides the canonical Doob-Meyer decomposition[28] of the special semimartingale (cf. (17))

$$\beta_t \widehat{\Pi}_t := \beta_t \Pi_t + \int_{[0,t]} \beta_s dDs = \big(\widehat{\Pi}_0 + \int_0^t \beta_s dM_s\big) - \int_0^t \beta_s dA_s\,,\ t \in [0,T]. \tag{25}$$

The "$\mathbb{Q}$ price" notation Π in the solution to (22) and the "cumulative $\mathbb{Q}$ price" notation $\widehat{\Pi}$ in (25) will be justified in Proposition 7.

Assumption 1 *The stochastic pricing equation (22) admits a solution* (Π, M, A).

We have the following *verification principle*:

Proposition 7 *The value process* Π *in the solution to the reflected BSDE* (22) *is the* $\mathbb{Q}$ *price of the American claim.*

Proof. If (Π, M, A) is a solution to (22), then Π satisfies (16), for a maximizer there given by

$$\vartheta^t = \inf\Big\{ s \in [t,T]\,;\, \Pi_s = L_s \Big\} \wedge T$$

In fact, by definition of ϑ^t, we have that A equals 0 on $[t, \vartheta]$. Since A is nondecreasing, taking conditional expectations in the BSDE and also using the facts that $\Pi_{\vartheta^t} = L_{\vartheta^t}$ if $\vartheta^t < T$ and $\Pi_T = \xi$, we obtain for any $\vartheta \in \Theta^t$:

$$\beta_t \Pi_t \geq \mathbb{E}_t\Big(\int_t^{\vartheta} \beta_s dD_s + \Pi_{\vartheta}\Big) \geq \mathbb{E}_t\Big(\int_t^{\vartheta} \beta_s dD_s + \big(\mathbb{1}_{\{\vartheta<T\}} L_{\vartheta} + \mathbb{1}_{\{\vartheta=T\}}\xi\big)\Big),$$

with equality in case $\vartheta = \vartheta^t$. ∎

D Superhedging

When we consider the hedging issue, unless stated otherwise we take the viewpoint of the issuer of the option. In what follows we thus consider an issuer (super)hedge starting at time 0. The corresponding adaptation to a holder hedge, and/or to a hedge starting at an arbitrary time $t \in [0,T]$, is straightforward.

Definition 10 **(i)** *A (resp. $\mathbb{Q}$) superhedge for a European option is a (resp. $\mathbb{Q}$) admissible primary trading strategy* (π, ζ) *with related wealth process* V *such that, almost surely,*

$$\beta_T V_T \geq \int_0^T \beta_s dD_s \,+\, \beta_T \xi. \tag{26}$$

If equality almost surely holds in (30), then we talk of a (resp. $\mathbb{Q}$) replicating strategy for the option.

(ii) *A (resp. $\mathbb{Q}$) superhedge for an American option is a (resp. $\mathbb{Q}$) admissible primary trading strategy* (π, ζ) *with related wealth process* V *such that, for every put time* ϑ *in* Θ*, almost surely,*

$$\beta_{\vartheta} V_{\vartheta} \geq \textstyle\int_0^{\vartheta} \beta_s dD_s \,+\, \beta_{\vartheta}\Big(\mathbb{1}_{\{\vartheta<T\}} L_{\vartheta} + \mathbb{1}_{\{\vartheta=T\}}\xi\Big). \tag{27}$$

[28] see before Lemma IV.7.

In the American case, càdlàg properties of the processes involved in (27) imply that satisfaction of the latter for every put time ϑ in Θ is equivalent to: almost surely,

$$\beta_t V_t \geq \int_0^t \beta_s dD_s + \beta_t \Big(\mathbb{1}_{\{t<T\}} L_t + \mathbb{1}_{\{t=T\}} \xi \Big), \; t \in [0,T]. \tag{28}$$

Remark 8 *In view of Remark 4, European claims with cash flows $(D = 0, \xi)$ that are together $\mathbb{Q}$ integrable*[29] *and $\mathbb{Q}$ replicable*[30] *are a vector space.*

Theorem 2 *In both the European and the American cases,*

$$\begin{aligned} &\sup\{\textit{risk neutral prices } \Pi_0 \textit{ for the option}\} \leq \\ &\inf\{\textit{superhedging prices } \pi \textit{ for the option}\}. \end{aligned} \tag{29}$$

Proof. In the European case we need to show that any time-0 $\mathbb{Q}$ price of the fom $\mathbb{E}\Big\{ \int_0^T \beta_s \, dD_s + \beta_T \xi \Big\}$, where the expectation is taken under some risk-neutral measure $\mathbb{Q}$ on the primary market, is $\leq$ any initial wealth π of a superhedge (π, ζ) for the option. In fact, as a lower bounded $\mathbb{Q}$ local martingale, the discounted wealth βV of (π, ζ)[31] is a $\mathbb{Q}$ supermartingale and, in particular,

$$\pi \geq \mathbb{E}(\beta_T V_T) \geq \mathbb{E}\Big\{ \int_0^T \beta_s \, dD_s + \beta_T \xi \Big\},$$

by (26).

In the American case the proof is similar and much like the one of the second part of Proposition 8, setting $\rho = 0$ there, hence we omit it for brevity. ∎

The inequalities reverse to (29) (whence absence of duality gap) also hold, in both the European and the American cases, but they are much harder to establish, especially in the American case. These reverse inequalities essentially show that the notion of infimal superhedging price is nonarbitrable[32]. But it is also typically too conservative to be usable in practice.

E Hedging at a Cost in Incomplete Markets

We now introduce a very large class of hedges *with semimartingale cost process* ρ[33]. In this perspective the issuer of a financial derivative sets-up a hedge such that the corresponding wealth process V reduces to a cost or hedging error ρ, after accounting for the "dividend cost" $(-D)$ and for the "terminal loss" given by $(-L)$ or $(-\xi)$. The initial wealth π may then be used as a safe issuer price, up to the hedging error ρ, for the derivative at hand.

In the following definition we consider an issuer hedge starting at time 0. The adaptation of this definition to a holder hedge, and/or to a hedge starting at an arbitrary time $t \in [0,T]$, is straightforward.

Definition 11 (i) *A hedge with semimartingale cost process ρ for a European option is a primary trading strategy (π, ζ) with related wealth process V such that the process $\beta V + \int_0^\cdot \beta_s d\rho_s$ is bounded from below and, almost surely,*

$$\beta_T V_T + \int_0^T \beta_s d\rho_s \geq \int_0^T \beta_s dD_s + \beta_T \xi. \tag{30}$$

[29] in the sense of Definition 6(i).

[30] cf. the last sentence of Definition 10(i).

[31] cf. (9).

[32] cf. Remark 6.

[33] see Föllmer and Sondermann (1986) and Schweizer (2001).

If equality almost surely holds in (30), then we talk of a replicating strategy with cost ρ for the option.

(ii) *A hedge with semimartingale cost process ρ for an American option is a primary trading strategy (π, ζ) with related wealth process V such that, for every put time ϑ in Θ, almost surely,*

$$\beta_\vartheta V_\vartheta + \int_0^\vartheta \beta_s d\rho_s \geq \int_0^\vartheta \beta_s dD_s + \beta_\vartheta \Big(\mathbb{1}_{\{\vartheta < T\}} L_\vartheta + \mathbb{1}_{\{\vartheta = T\}} \xi \Big). \tag{31}$$

Process ρ is to be interpreted as the cumulative financing cost, i.e. the amount of cash added to (if $d\rho_t \geq 0$) or withdrawn from (if $d\rho_t \leq 0$) the hedging portfolio in order to get a perfect, but no longer self-financing, hedge. Hedges at no cost, i.e. with $\rho = 0$, are thus in effect superhedges. Hedges with a local martingale cost ρ under a particular risk-neutral measure $\mathbb{Q}$ can also be interpreted as mean-self-financing hedges in the sense of Föllmer and Sondermann (1986) (see also Schweizer (2001)).

In the American case, càdlàg properties of the processes involved in (31) imply that satisfaction of the latter for every put time ϑ in Θ is equivalent to: almost surely,

$$\beta_t V_t + \int_0^t \beta_s d\rho_s \geq \int_0^t \beta_s dD_s + \beta_t \Big(\mathbb{1}_{\{t < T\}} L_t + \mathbb{1}_{\{t = T\}} \xi \Big), \ t \in [0, T]. \tag{32}$$

In relation to admissibility issues (see the end of Definition 2), note that the left-hand side of (32) (discounted wealth process with financing costs included) is bounded from below (but maybe the discounted wealth itself is not), for every hedge (π, ζ).

As Proposition 8 will show below, Definition 11 is consistent with the concept of no arbitrage pricing of European and American options.

E.1 Connection With Risk-Neutral Pricing

We are now ready to interpret the $\mathbb{Q}$ price Π defined via (15) in the European case, or (22) in the American case[34], in terms of the notion of hedging at a cost introduced in Definition 11.

Recalling (21), let the $\mathbb{Q}$ local martingale $\rho(\zeta)$ be such that $\rho_0(\zeta) = 0$ and $(-\int_0^\cdot \beta_t d\rho_t)$ is the canonical local martingale component[35] of the discounted profit-and-loss (βp). That is, in view of I.(21):

$$\int_0^\cdot \beta_t d\rho_t(\zeta) = \int_0^\cdot \big(d(\beta_t \widehat{\Pi}_t) - \zeta_t d(\beta_t \widehat{P}_t) \big) = -\beta p \tag{33}$$

in the European case, respectively (cf. (24))

$$\int_0^\cdot \beta_t d\rho_t(\zeta) = \int_0^\cdot \big(\beta_t dM_t - \zeta_t \, d(\beta_t \widehat{P}_t) \big) \geq \int_0^\cdot \Big(d(\beta_s \widehat{\Pi}_s) - \zeta_s d(\beta_s \widehat{P}_s) \Big) = -\beta p \tag{34}$$

in the American case.

Proposition 8 *In the European case:*

(i) *For any primary strategy ζ, (Π_0, ζ) is a replicating strategy (hence, a hedge) with $\mathbb{Q}$ local martingale cost $\rho(\zeta)$;*

(ii) Π_0 *is the minimal initial wealth of a hedge with $\mathbb{Q}$ local martingale cost.*

In the American case:

(i) *For any primary strategy ζ, (Π_0, ζ) is a hedge with $\mathbb{Q}$ local martingale cost $\rho(\zeta)$;*

(ii) Π_0 *is the minimal initial wealth of a hedge with $\mathbb{Q}$ local martingale cost.*

[34] cf. Assumption 1 and Proposition 7.

[35] see before Lemma IV.7.

Proof. As the European is similar and a bit simpler, we only show the results in the American case.
(i) We must show that for every $\vartheta \in \Theta$, almost surely:

$$\Pi_0 + \int_0^{\vartheta} \zeta_s d(\beta_s \widehat{P}_s) + \int_0^{\vartheta} \beta_s d\rho_s \geq \int_0^{\vartheta} \beta_s dD_s \ + \ \beta_\vartheta \Big(\mathbb{1}_{\{\vartheta < T\}} L_t + \mathbb{1}_{\{\vartheta = T\}} \xi \Big). \tag{35}$$

Using the left identity in (34), this is equivalent to:

$$\Pi_0 + \int_0^{\vartheta} \beta_s dM_s \geq \int_0^{\vartheta} \beta_s dD_s \ + \ \beta_\vartheta \Big(\mathbb{1}_{\{\vartheta < T\}} L_\vartheta + \mathbb{1}_{\{\vartheta = T\}} \xi \Big), \tag{36}$$

where, by (24),

$$\Pi_0 + \int_0^{\vartheta} \beta_s dM_s = \int_0^{\vartheta} \beta_s dD_s + \beta_\vartheta \Pi_\vartheta + \int_0^{\vartheta} \beta_s dA_s.$$

Hence (36) follows from the nondecreasing property of A and from the following relations, which are valid by the terminal and put conditions in (22):

$$\Pi_T = \xi, \ \Pi_\vartheta \geq L_\vartheta.$$

(ii) By part (i), $(\Pi_0, \zeta = 0)$ is a hedge with initial wealth Π_0 and $\mathbb{Q}$ local martingale cost. Moreover, for every hedge (π, ζ) with $\mathbb{Q}$ local martingale cost ρ, for every $t \in [0, T]$ we have by (31) that:

$$\begin{aligned} \pi + \int_0^{t} \zeta_s d(\beta_s \widehat{P}_s) + \int_0^{t} \beta_s d\rho_s \geq \\ \int_0^{t} \beta_s dD_s \ + \ \beta_t \Big(\mathbb{1}_{\{t < T\}} L_t + \mathbb{1}_{\{t = T\}} \xi \Big). \end{aligned} \tag{37}$$

The left-hand side is thus a local martingale that is bounded from below, hence it is a supermartingale. Moreover, (37) also holds with a stopping time $\vartheta \in \Theta$ instead of t there. Taking expectations then yields that

$$\pi \geq \mathbb{E} \left\{ \int_0^{\vartheta} \beta_s dD_s + \beta_\vartheta \Big(\mathbb{1}_{\{\vartheta < T\}} L_\vartheta + \mathbb{1}_{\{\vartheta = T\}} \xi \Big) \right\}.$$

Hence $\pi \geq \Pi_0$ follows by (16). ∎

Proposition 8 thus characterizes the $\mathbb{Q}$ price[36] of a derivative as the smallest initial wealth of a hedge with $\mathbb{Q}$ local martingale cost.

The case where, for each European claim, there exists ζ such that $\rho(\zeta) = 0$, in the previous results, corresponds to market completeness in the sense of replicability (in the usual sense, without a cost) of European options as stated in Proposition 8(i), i.e. profit-and-loss $p = 0$, by (33), in which case the issuer of the option can hedge all the embedded risks.

Unless the market is complete and the usual notion of infimal superhedging price, at no cost, reduces to the notion of replication price[37], the inequalities (29) show that this usual notion of superhedging price is too conservative to be competitive in practice.

E.2 Min-Variance Hedging

In the context of incomplete markets, the choice of a hedging strategy is dependent on an optimality criterion with respect to the hedging cost ρ. For instance, one may wish to minimize the physical, $\mathbb{P}$

[36] given as an assumed solution to the related reflected BSDE in the American case.
[37] in the European case, whereas in the American case the issuer even makes gains if the holder exercises sub-optimally.

variance of $\int_0^T \beta_t d\rho_t$. But the related strategy $\widehat{\zeta}^{va}$ is intractable. Moreover, $\widehat{\zeta}^{va}$ typically depends on the physical drift of the model, but a drift is nearly impossible to estimate statistically from a single realization (the observed past evolution of the market).

As a proxy to this strategy, we can use the strategy, ζ^{va}, which minimizes the risk-neutral variance of the hedging cost. Under mild conditions, $\beta\widehat{\Pi}$ in the European case (resp. $\int_0^\cdot \beta dM$ as per (34) in the American case), as well as $\beta\widehat{P}$, are $\mathbb{Q}$ square integrable martingales (as martingale component of the value processes in the solutions to a related square integrable BSDE[38], regarding $\int_0^\cdot \beta dM$ in (34)). The risk-neutral min-variance hedging strategy ζ^{va} is then given by the following Galtchouk-Kunita-Watanabe decomposition of $\beta\widehat{\Pi}$ (resp. $\int_0^\cdot \beta dM$) with respect to $\beta\widehat{P}$[39]:

$$\beta\widehat{\Pi} \text{ (resp. } \textstyle\int_0^\cdot \beta dM) = \zeta_t^{va} d(\beta_t \widehat{P}_t) + \beta_t d\rho_t^{va}, \tag{38}$$

for some $\mathbb{R}^q$ valued, $\beta\widehat{P}$ integrable process ζ^{va} and a real valued square integrable martingale $\int_0^\cdot \beta_t d\rho_t^{va}$ strongly orthogonal to $\beta\widehat{P}$, i.e. such that their product is again a martingale on $[0,T]$. Setting, in matrix form,

$$< A, B >= (< A^i, B^j >)_i^j, \; < A >=< A, A >,$$

the above strong orthogonality property implies that[40] $< \beta\widehat{P}, \int_0^\cdot \beta_t d\rho_t^{va} >= 0$ and we have, in view of (38) (and (25), in the American case):

$$\zeta_t^{va} = \frac{d<\beta\widehat{\Pi}, \beta\widehat{P}>_t}{dt} \left(\frac{d<\beta\widehat{P}>_t}{dt}\right)^{-1} = \frac{d<\widehat{\Pi}, \widehat{P}>_t}{dt} \left(\frac{d<\widehat{P}>_t}{dt}\right)^{-1}, \tag{39}$$

where invertibility of the matrix $\frac{d<\widehat{P}>_t}{dt}$ is assumed. In the case of finite variation continuous dividend processes, (39) reduces further to

$$\zeta_t^{va} = \frac{d<\Pi, P>_t}{dt} \left(\frac{d<P>_t}{dt}\right)^{-1}, \tag{40}$$

which, in Markov setups, can be computed by means of the formula IV.(10).

§4 From Theory to Practice

A Martingale Modeling

Hereafter we will typically work under a risk-neutral measure $\mathbb{Q}$, rather than under the physical measure $\mathbb{P}$. This doesn't mean that analyzing a financial market under the physical measure $\mathbb{P}$ is not important; such analysis must actually come first in the modeling process. It means simply that this task has already been done. We thus take a pricing model for granted, one that we suppose gives a realistic view of the financial market under consideration, up to an equivalent change of measure.

The above risk-neutral modeling approach can be readily extended to a martingale modeling approach with respect to an arbitrary numéraire, rather than the savings account in the risk-neutral approach. This allows one to apply the previous models to interest rate derivatives (Brigo and Mercurio, 2007) and foreign exchange derivatives (Lipton, 2002).

Let thus be given a numéraire, in the sense of a non-dividend-paying asset with risk-free discounted price process $\beta_t \widetilde{B}_t$, assumed to be a positive $\mathbb{Q}$ martingale. Let $\widetilde{\mathbb{Q}}$ be the pricing measure on $(\Omega, \mathcal{A})$ associated with the numéraire $\widetilde{B}$, defined by

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \nu_T, \tag{41}$$

[38]cf. Part C.

[39]see e.g. Protter (2004, Corollary 1, Chapter IV.3) or Jacod (1979).

[40]see He, Wang, and Yan (1992, Theorem 6.26 page 186).

with $\nu_t = \frac{\beta_t \widetilde{B}_t}{\widetilde{B}_0}$. We denote the $\widetilde{\mathbb{Q}}$ expectation by $\widetilde{\mathbb{E}}$.

Considering a non-dividend-paying European asset with $\mathbb{Q}$ price process Π as per (15), we have the following $\mathbb{Q}$ martingale:

$$\beta\Pi = \nu \frac{\widetilde{B}_0 \Pi}{\widetilde{B}}.$$

Hence, by Lemma IV.4, $\frac{\Pi}{\widetilde{B}}$ is a $\widetilde{\mathbb{Q}}$ martingale, so that

$$\Pi_t = \widetilde{B}_t \, \widetilde{\mathbb{E}}(\widetilde{B}_T^{-1} \xi \,|\, \mathfrak{F}_t) \,, \; t \leq T. \tag{42}$$

Similar developments can be established regarding prices of American options.

B Markovian Setup

In order to be usable in practice, a pricing model needs to be tractable numerically. Concretely this leads to work with a (not too high-dimensional) "Markovian proxy" X for the "true" (if any) market factor process. This is achieved in later chapters by assuming that we are dealing with a Markovian stochastic pricing equation (15) (or (22) in the American case), meaning that the input data of (15) or (22) are given as functions of an underlying Markovian factor process X. The logic here is that this Markovian feature of the pricing equation implies the Markovian property IV.(7) for its solution, the price process Π being then also given as a function u of (t, X_t). Moreover, the pricing function u can be characterized analytically as the unique solution, in some sense, of a related deterministic equation. This deterministic pricing equation can then be solved by various numerical means.

From the point of view of financial interpretation, the components of X are observable factors. Most factors are typically given as primary price processes in P. Additional factors may be required for explaining some path dependence in the payoffs of the derivative at hand, or in the dynamics of the underlying assets. Conversely, some of the primary price processes may not be needed as factors, but are used for hedging purposes.

C Model Calibration

In applications we can think of $\mathbb{Q}$ as "the pricing measure chosen by the market" to price a contingent claim. For hedging purposes or in order to implement bets on specific risk factors, and also for pricing exotic or structured products, traders need to know the market pricing measure $\mathbb{Q}$.

In practice, the measure $\mathbb{Q}$ is typically estimated by calibration of a model to market data. Indeed there are two sets of constraints that the market pricing measure $\mathbb{Q}$ must satisfy. First, $\mathbb{Q}$ must satisfy structural requirements stemming from its equivalence with the physical probability measure $\mathbb{P}$. Any process must thus have the same trajectorial properties (such as continuity or lack of it) under the objective and under an equivalent pricing measure. Second, the cross-section $\Pi_t^{[\pm]}(T, K)$ of the market prices of European vanilla calls and puts quoted at any pricing time t on an underlying S must satisfy

$$\Pi_t^{[\pm]}(T, K) = \beta_t^{-1} \mathbb{E}_t \beta_T (S_T - K)^{\pm}, \; (T, K) \in \mathrm{obs}_t, \tag{43}$$

where obs_t is the set of the most liquid options on S (European vanilla at-the-money or slightly out-of-the-money calls and puts) quoted in the market at time t.

Constraints of type (43) are called calibration constraints. A model is said to fit the market smile[41], at a given time t, if it satisfies the calibration constraints (43). Accounting for synchronization and noise

[41] cf. II.§5.

issues in the market data, one commonly relaxes the calibration equality constraints (43) into inequality constraints within the bid-ask spread. Quite a few classes of models can fit the smile within the bid-ask spread, provided their parameters are suitably calibrated. A further requirement can be to fit the smile dynamics priced by the market. This corresponds to additional calibration constraints associated with market prices of exotic options (Guo and Loeper, 2021).

D Hedging Imperfections

A model calibrated to the market can be used for hedging purposes, and for dealing with more exotic products. When a bank sells a derivative, it immediately sets up a hedge composed of liquid instruments, such as the asset(s) underlying the derivative, and/or further vanilla derivatives. But for feasibility, as well as for transaction cost issues (note that transaction costs are never explicitly considered in our setup), the bank is restricted to piecewise constant hedging strategies ζ^h such that

$$\zeta_t^h = \zeta_{t_i}^h \text{ for } t_i < t \le t_{i+1}, \tag{44}$$

where $(t_i)_{0 \le i \le n}$ is a (possibly random) time-grid over $[0, T]$. In practice n may vary from one (static hedging) to the number of days or weeks between 0 and T . Since derivative payoffs are typically nonlinear, in order to get a good hedge the composition of the hedging portfolio must be updated at a high enough frequency.

In an idealized, complete market model, an appropriate continuously rebalanced hedge ζ provides a perfect hedge to an option's seller (profit-and-loss identically equal to 0 or, in the case of an American option, super-hedge in case of sub-optimal exercise by the option holder). By contrast, in the real-world there are many reasons why a practical strategy ζ^h typically leads to actual profits or losses under some scenarios:

- Hedge slippage as said above, but this is only the tip of the iceberg:
- Transaction and illiquidity costs, which are ignored in our formalism, and, above all perhaps:
- Model misspecification: note that hedging ratios are typically model-dependent, even among models calibrated to the same data set.

Chapter II

Black-Scholes(-Merton)

We consider a primary market composed of the savings account $S^0 = \beta^{-1}$ and of a nonnegative stock S as per I.§1, but in continuous time as in I.§2. The riskless interest rate r in the economy and the dividend yield q on S are assumed to be constant. Hence $\beta_t = e^{-rt}$ and the cumulative stock price $\widehat{S}$ [1] satisfies $\widehat{S}_t = S_t + \int_0^t e^{r(t-s)} q S_s ds$, i.e. $\beta_t \widehat{S}_t = \beta_t S_t + \int_0^t \beta_s q S_s ds$. Hence[2]

$$\begin{aligned} d(\beta_t \widehat{S}_t) &= d(\beta_t S_t) + \beta_t q S_t dt = e^{-qt} d(\beta_t S_t e^{qt}) \\ &= e^{-qt} d(\alpha_t S_t) = e^{-qt} \alpha_t (dS_t - \kappa S_t dt) = \beta_t (dS_t - \kappa S_t dt), \end{aligned} \tag{1}$$

where we set $\alpha_t = e^{-\kappa t}$ with $\kappa = r - q$.

§1 Model-Free Formulas

Hereafter we assume that $dS_t - \kappa S_t dt$ is a $\mathbb{Q}$ local martingale for some $\mathbb{Q} \sim \mathbb{P}$.

Lemma 1 *The above primary market model admits $\mathbb{Q}$ as a risk-neutral measure and it is $\mathbb{Q}$ arbitrage-free.*

Proof. In view of Definition I.4 and I.(3), the identity (1), where $dS_t - \kappa S_t dt$ is assumed to be a $\mathbb{Q}$ local martingale, shows via Theorem IV.1 that $\mathbb{Q}$ is a risk-neutral measure. Theorem I.1 then implies that the model is $\mathbb{Q}$ arbitrage-free. ∎

The processes $\beta \widehat{S}$ or, equivalently in view of Theorem IV.1 and (1), αS, are $\mathbb{Q}$ local martingales. Hereafter we assume that they are even $\mathbb{Q}$ martingales, so that the following cost-of-carry formula holds[3]:

$$S_t = \mathbb{E}_t \Big(e^{-r(T-t)} S_T + \int_t^T e^{-r(s-t)} q S_s ds \Big), \tag{2}$$

as well as

$$S_t = \mathbb{E}_t \big(e^{-\kappa(T-t)} S_T \big), \text{ i.e. } \mathbb{E}_t S_T = S_t e^{\kappa(T-t)}. \tag{3}$$

Following Definition I.6, the $\mathbb{Q}$ price process of a European vanilla option with $\mathbb{Q}$ integrable payoff $\phi(S_T)$ at T, e.g. call/put $(S_T - K)^{\pm}$, is given, for $t \in [0, T]$, by

$$\Pi_t = e^{-r(T-t)} \mathbb{E}_t \phi(S_T). \tag{4}$$

[1] cf. I.(3).
[2] cf. IV.(4).
[3] cf. I.(12).

Hence the discounted price $(e^{-rt}\Pi_t)$ is a $\mathbb{Q}$ martingale. The corresponding American option price is given, for $t \in [0,T]$, by[4]

$$\Pi_t = \max_{\vartheta\in\Theta^t} \mathbb{E}_t e^{-r(\vartheta-t)}\phi(S_\vartheta). \tag{5}$$

The following result, where $C, P, \widetilde{C}$ and $\widetilde{P}$ are the European and American vanilla call and put prices, is the continuous-time $\mathbb{Q}$ prices version of Proposition I.4[5].

Proposition 1 **(i)** *If $q \le 0 \le r$, then $\widetilde{C} = C$.*
(ii) *If $r \le 0 \le q$, then $\widetilde{P} = P$.*

Proof. At time 0 (for notational simplicity), this follows from the martingale property of αS and the conditional Jensen inequality, which yields in the second line, for any $\vartheta \in \Theta^0$:

$$\begin{aligned}
&\mathbb{E}e^{-r\vartheta}(S_\vartheta - K)^\pm = \mathbb{E}(e^{-q\vartheta}e^{-\kappa\vartheta}S_\vartheta - e^{-r\vartheta}K)^\pm = \mathbb{E}\big(e^{-q\vartheta}\mathbb{E}_\vartheta(e^{-\kappa T}S_T) - e^{-r\vartheta}K\big)^\pm \\
&\quad = \mathbb{E}\big(\mathbb{E}_\vartheta(e^{-q\vartheta}e^{-\kappa T}S_T - e^{-r\vartheta}K)\big)^\pm \le \mathbb{E}\mathbb{E}_\vartheta(e^{-q\vartheta}e^{-\kappa T}S_T - e^{-r\vartheta}K)^\pm \\
&\quad \le \mathbb{E}(e^{-rT}S_T - e^{-rT}K)^\pm,
\end{aligned}$$

respectively, under the respective assumptions made in (i) and (ii). ∎

A Call/put parity and the Breeden-Litzenberger Formula

Lemma 2 *In the case of a constant dividend yield q, a strategy with a constant number $\zeta^0 = c_0$ of riskless assets and a number $\zeta = ce^{q\cdot}$ of risky assets, with c constant, is self-financing. The ensuing wealth process βV is a $\mathbb{Q}$ martingale, bounded from below for $c \ge 0$.*

Proof. The self-financing property of the strategy follows from the following identity, where $V = c_0 S^0 + ce^{q\cdot}S$:[6]

$$d(\beta V)_t = c\, d(e^{qt}\beta_t S_t) = ce^{qt}\big(d(\beta S)_t + \beta_t S_t q dt\big).$$

Like αS, the wealth process $\beta V = c_0 + c\alpha S$ is a $\mathbb{Q}$ martingale, bounded from below for $c \ge 0$. ∎

Proposition 2 *The $\mathbb{Q}$ price $(F_t(T,K))$ of a forward contract with payoff $\xi = S_T - K$ is given, for $t \le T$, by*

$$F_t(T,K) = S_t e^{-q\tau} - Ke^{-r\tau} = C_t(T,K) - P_t(T,K), \tag{6}$$

where $\tau = T - t$ and $(C_t(T,K))$ and $(P_t(T,K))$ are the $\mathbb{Q}$ prices of the European vanilla call and put options with payoffs $(S_T - K)^\pm$. The $\mathbb{Q}$ price $F_t(T,K)$ does not depend on the specific risk-neutral measure $\mathbb{Q}$ that is used and it is also a replication price of the payoff ξ. The T forward time-t price of S is $F_t^T = S_t e^{\kappa\tau}$. ∎

Proof. By definition of $\mathbb{Q}$ prices, setting $\tau = T - t$, we have

$$F_t(T,K) = \mathbb{E}_t e^{-r\tau}(S_T - K) = S_t e^{-q\tau} - Ke^{-r\tau},$$

[4] assuming $\mathbb{Q}$ integrability of the process $(\phi(S_t))$ and the existence of a maximiser in (5).
[5] cf. Remark I.7.
[6] cf. I.(6)-(3).

by the assumed $\mathbb{Q}$ martingality of $(e^{-\kappa t}S_t)$, and

$$\mathbb{E}_t e^{-r\tau}(S_T - K) = \mathbb{E}_t e^{-r\tau}(S_T - K)^+ - \mathbb{E}_t e^{-r\tau}(S_T - K)^- = C_t(T,K) - P_t(T,K).$$

Moreover, with $t=0$ for notational simplicity: starting from the initial wealth $\pi = S_0 e^{-qT} - Ke^{-rT}$, borrowing Ke^{-rT}[7], buying e^{-qT} stocks and continuously reinvesting all the dividends in the stock, i.e. $\zeta^0 = -Ke^{-rT}$ and $\zeta^1 = e^{-qT}e^{qt}$, is self-financing and admissible, by Lemma 2, and results in the payoff $S_T - K = \xi$ at time T. The last part of the proposition follows from the first identity in (6) by Definition I.7 of the forward price. ∎

In view of Remark I.8:

Corollary 1 *The $\mathbb{Q}$ price $(-F_t(T,K))$ of the payoff $(K - S_T)$ is a $\mathbb{Q}$ replication price*[8].

Remark 1 *In strict local martingale market models where αS fails to be a true $\mathbb{Q}$ martingale, $\mathbb{Q}$ prices of European vanilla calls and puts may fail to satisfy the call-put parity relationship* (6), *as well as the no-static-arbitrage bounds of Proposition I.3, so that the Black-Scholes implied volatility*[9] *of $\mathbb{Q}$ prices may fail to be well defined, or differ between calls and puts (Jacquier and Keller-Ressel, 2018). It is to avoid these pathologies that we restrict ourselves to models where αS is a true $\mathbb{Q}$ martingale.*

Proposition 3 *Whenever the process (S_t) admits a density of transition probability $\gamma_0(T,K) = \partial_K \mathbb{Q}(S_T \leq K)$ from $(0, S_0)$ to (T,K), $K \geq 0$, the following Breeden and Litzenberger (1978) formula holds:*

$$\partial^2_{K^2} C_0 = e^{-rT}\gamma_0(T,K) = \partial^2_{K^2} P_0. \tag{7}$$

Proof. We have

$$\partial_K C_0 = e^{-rT}\partial_K \mathbb{E}(S_T - K)^+ = -e^{-rT}\mathbb{E}\mathbb{1}_{\{S_T > K\}} = -e^{-rT}\mathbb{Q}(S_T > K). \tag{8}$$

Assuming the existence of $\gamma_0(T,\cdot)$, this implies the first identity in (7). The second one follows by call/put parity. ∎

Proposition 4 *In the present case of constant*[10] *interest and dividend rates, a rolling position of $e^{r\tau}$ at-the-money T forward contracts on S replicates a futures contract on S*[11].

Proof. The dividend process $D^T_\cdot = \mathbb{E}_\cdot S_T$ of a T futures contract on S satisfies

$$\begin{aligned} dD^T_t &= e^{\kappa T} d\mathbb{E}_t(e^{-\kappa T}S_T) = e^{\kappa T} d(e^{-\kappa t}S_t) = e^{\kappa T}e^{-\kappa t}(dS_t - \kappa S_t dt) \\ &= e^{\kappa\tau}(dS_t - \kappa S_t dt) = e^{r\tau}e^{-q\tau}(dS_t - \kappa S_t dt). \end{aligned}$$

Rolling an at-the-money T forward contract on S on a small time step interval $[t-h, t)$ generates dividends[12]

$$F_t(T, F^T_{t-h}) - F_{t-h}(T, F^T_{t-h}) = F_t(T, F^T_{t-h}) = F_t(T, F^T_{t-h}) - F_t(T, F^T_t) = (F^T_t - F^T_{t-h})e^{-r\tau},$$

by (6). In the continuous-time limit, this expression boils down to

$$e^{-r\tau} dF^T_t = e^{-r\tau} d(S_t e^{\kappa\tau}) = e^{-q\tau}(dS_t - \kappa S_t dt). \;∎$$

[7] i.e. selling Ke^{-rT} units of the risk-free asset.
[8] cf. Definition I.10(i), the second part in Theorem I.1, and Remark I.8.
[9] cf. Lemma 7 below.
[10] could be time-deterministic.
[11] cf. Definition I.7.
[12] as $F_t(T, f^T_t) = 0$ holds for all t, by definition of F^T.

§2 Black–Scholes Model

The Black and Scholes (1973); Merton (1973) model postulates the following diffusion for S:

$$dS_t = S_t(\kappa dt + \sigma dW_t), \tag{9}$$

for some standard $(\mathfrak{F}, \mathbb{Q})$-Brownian motion W and a constant volatility parameter σ, where $\mathbb{Q} \sim \mathbb{P}$. Or, explicitly by application of the Itô formula to $\ln(S)$:

$$S_t = S_0 e^{bt+\sigma W_t}, \text{ where } b = \kappa - \tfrac{1}{2}\sigma^2. \tag{10}$$

In terms of the cumulative stock price $\widehat{S}$, (1) yields

$$\begin{aligned} d(\beta_t \widehat{S}_t) &= \beta_t \sigma S_t dW_t \\ d(\alpha_t S_t) &= \sigma(\alpha_t S_t) dW_t. \end{aligned} \tag{11}$$

Remark 2 *By application of Theorems IV.1 and I.1, the Black-Scholes model is $\mathbb{Q}$ arbitrage-free. As one can prove by a direct computation (or an application of the Novikov condition) based on* (10)*, the process αS is even a $\mathbb{Q}$ martingale, as postulated in §1.*

§3 Black-Scholes Formulas

Let $\tau = T - t$, $\mathcal{N}$ and n denote the standard Gaussian cumulative distribution and density functions, and

$$\mathrm{d}_\pm = \mathrm{d}_\pm(t, S, T, K; r, q, \sigma) = \frac{\ln(\frac{S}{K}) + \kappa\tau}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}. \tag{12}$$

The argument $(t, S, T, K; r, q, \sigma)$ will be abbreviated, when no confusion might arise, by (t, S, T, K) or (t, S), or it may sometimes even be omitted (see Table 1).

K/S	$\mathrm{d}_\pm$	$n(\mathrm{d}_\pm)$	$\mathcal{N}(d_\pm)$
0	$+\infty$	0	1
$+\infty$	$-\infty$	0	0

σ	$\mathrm{d}_\pm$	$n(\mathrm{d}_\pm)$	$\mathcal{N}(d_\pm)$
0	$+\infty\mathbb{1}_{\mathrm{ITMF}} - \infty\mathbb{1}_{\mathrm{OTMF}}$	$\frac{1}{\sqrt{2\pi}}\mathbb{1}_{\mathrm{ATMF}}$	$\mathbb{1}_{\mathrm{ITMF}} + \frac{1}{2}\mathbb{1}_{\mathrm{ATMF}}$
$+\infty$	$\pm\infty$	0	1/0

Table 1: *Asymptotics of* $\mathrm{d}_\pm, n(\mathrm{d}_\pm)$, *and* $\mathcal{N}(d_\pm)$ *with respect to K/S and σ. We write* ITMF, ATMF *and* OTMF *for in-the-money forward, at-the-money forward and out-of-the-money forward for a call option, i.e. for the cases $(S/K)e^{\kappa\tau} > 1, = 1$ or < 1.*

Lemma 3 *For every measurable function ϕ such that $\xi = \phi(S_T)$ is $\mathbb{Q}$ integrable, it holds, for any fixed $t \in [0, T]$, that*

$$\mathbb{E}_t \phi(S_T) = \mathbb{E}\big[\phi(S e^{b\tau + \sigma\sqrt{\tau}\varepsilon})\big]|_{S=S_t},$$

with ε standard Gaussian.

Proof. We recall (see e.g. Lamberton and Lapeyre (1996, Proposition A.2.5)) that, if χ is measurable with respect to $\mathcal{B}$ and ξ is independent of a σ-field $\mathcal{B}$, then for every function $\varphi = \varphi(y, z)$ such that $\varphi(\chi, \xi)$ is integrable,

$$\mathbb{E}(\varphi(\chi, \xi)|\mathcal{B}) = \mathbb{E}\varphi(x, \xi)|_{x=\chi}. \tag{13}$$

Hence

$$\begin{aligned}\mathbb{E}_t\phi(S_T) &= \mathbb{E}\big[\phi(S_t e^{b(T-t)+\sigma(W_T-W_t)})\big|\mathfrak{F}_t\big]\\ &= \mathbb{E}\big[\phi(Se^{b\tau+\sigma\sqrt{\tau}\varepsilon})\big]\big|_{S=S_t}\end{aligned}$$

with ε standard Gaussian, by independence of $W_T - W_t$ with respect to $\mathfrak{F}_t$ (followed by the fact that $W_T - W_t$ equals in law $\sqrt{\tau}\varepsilon$). ∎

Lemma 4 *For ε standard Gaussian, we have*

$$\mathbb{E}\big[(Se^{b\tau+\sigma\sqrt{\tau}\varepsilon} - K)^+\big] = Se^{\kappa\tau}\mathcal{N}(\mathrm{d}_+(t,S)) - K\mathcal{N}(\mathrm{d}_-(t,S)).$$

Proof. Decomposing $(Se^{b\tau+\sigma\sqrt{\tau}\varepsilon} - K)^+$ via $(X-K)^+ = (X-K)\mathbb{1}_{X>K}$ yields

$$\mathbb{E}\big[(Se^{b\tau+\sigma\sqrt{\tau}\varepsilon} - K)^+\big] = \mathbb{E}\big(Se^{b\tau+\sigma\sqrt{\tau}\varepsilon}\mathbb{1}_{Se^{b\tau+\sigma\sqrt{\tau}\varepsilon}>K}\big) - K\mathbb{Q}\big(Se^{b\tau+\sigma\sqrt{\tau}\varepsilon} > K\big),$$

where

$$\begin{aligned}\mathbb{Q}\big(Se^{b\tau+\sigma\sqrt{\tau}\varepsilon} > K\big) &= \int_{y=\frac{\ln(K/(Se^{\kappa\tau}))}{\sigma\sqrt{\tau}}+\frac{\sigma\sqrt{\tau}}{2}=-\mathrm{d}_-(t,S)}^{+\infty} n(y)dy\\ &= 1-\mathcal{N}(-\mathrm{d}_-(t,S)) = \mathcal{N}(\mathrm{d}_-(t,S)),\end{aligned}$$

by symmetry of the Gaussian density. Moreover,

$$\begin{aligned}&\mathbb{E}\big(Se^{b\tau+\sigma\sqrt{\tau}\varepsilon}\mathbb{1}_{Se^{b\tau+\sigma\sqrt{\tau}\varepsilon}>K}\big)\\ &= Se^{\kappa\tau}\int_{y=-\mathrm{d}_-(t,S)}^{+\infty} e^{\sigma\sqrt{\tau}y-\frac{1}{2}\sigma^2\tau}e^{-y^2/2}\frac{dy}{\sqrt{2\pi}}\\ &= Se^{\kappa\tau}\int_{y=-\mathrm{d}_-(t,S)}^{+\infty} e^{-(y-\sigma\sqrt{\tau})^2/2}\frac{dy}{\sqrt{2\pi}}\\ &= Se^{\kappa\tau}\int_{z=-\mathrm{d}_-(t,S)-\sigma\sqrt{\tau}=-\mathrm{d}_+(t,S)}^{+\infty} n(z)dz\\ &= Se^{\kappa\tau}\big(1-\mathcal{N}(-\mathrm{d}_+(t,S))\big) = Se^{\kappa\tau}\mathcal{N}(\mathrm{d}_+(t,S)).\end{aligned}$$ ∎

Proposition 5 *At time t with stock worth S, the Black-Scholes call price C^{bs}, delta $\Delta^{bs} = \partial_S C^{bs}$, gamma $\Gamma^{bs} = \partial^2_{S^2}C^{bs}$, theta $\Theta^{bs} = -\partial_\tau C^{bs}$, vega $\mathcal{V}^{bs} = \partial_\sigma C^{bs}$, and rho $\mathrm{P}^{bs} = \partial_r C^{bs}$ are given by:*

$$\begin{aligned}C^{bs} &= Se^{-q\tau}\mathcal{N}(\mathrm{d}_+) - Ke^{-r\tau}\mathcal{N}(\mathrm{d}_-),\\ \Delta^{bs} &= e^{-q\tau}\mathcal{N}(\mathrm{d}_+),\\ \Gamma^{bs} &= e^{-q\tau}\frac{n(\mathrm{d}_+)}{S\sigma\sqrt{\tau}},\\ \Theta^{bs} &= qSe^{-q\tau}\mathcal{N}(\mathrm{d}_+) - rKe^{-r\tau}\mathcal{N}(\mathrm{d}_-) - Se^{-q\tau}n(\mathrm{d}_+)\frac{\sigma}{2\sqrt{\tau}},\\ \mathcal{V}^{bs} &= Se^{-q\tau}\sqrt{\tau}n(\mathrm{d}_+) = S^2\sigma\tau\Gamma^{bs},\\ \mathrm{P}^{bs} &= \tau Ke^{-r\tau}\mathcal{N}(\mathrm{d}_-).\end{aligned}\tag{14}$$

Proof. The formula for the price follows from Lemmas 3 (applied to $\phi(S) = (S-K)^+$) and 4. The formulas for the Greeks follow by differentiation of the price using the identities

$$Se^{-q\tau}n(\mathrm{d}_+) = Ke^{-r\tau}n(\mathrm{d}_-) \text{ and } \mathrm{d}_+ - \mathrm{d}_- = \sigma\sqrt{\tau}.\tag{15}$$

For instance,

$$\begin{aligned}\partial_\tau C^{bs} &= \partial_\tau\big(Se^{-q\tau}\mathcal{N}(\mathrm{d}_+) - Ke^{-r\tau}\mathcal{N}(\mathrm{d}_-)\big)\\ &= Se^{-q\tau}\big(n(\mathrm{d}_+)\partial_\tau \mathrm{d}_+ - q\mathcal{N}(\mathrm{d}_+)\big) - Ke^{-r\tau}\big(n(\mathrm{d}_-)\partial_\tau \mathrm{d}_- - r\mathcal{N}(\mathrm{d}_-)\big)\\ &= Se^{-q\tau}n(\mathrm{d}_+)\frac{\sigma}{2\sqrt{\tau}} - qSe^{-q\tau}\mathcal{N}(\mathrm{d}_+) + rKe^{-r\tau}\mathcal{N}(\mathrm{d}_-),\end{aligned}$$

which yields the formula for Θ^{bs}. ∎

Remark 3 *We have*

$$\begin{aligned}&\Theta^{bs} + \kappa S\Delta^{bs} + \frac{1}{2}\sigma^2 S^2\Gamma^{bs} - rC^{bs} =\\ &\quad qSe^{-q\tau}\mathcal{N}(\mathrm{d}_+) - rKe^{-r\tau}\mathcal{N}(\mathrm{d}_-) - Se^{-q\tau}n(\mathrm{d}_+)\frac{\sigma}{2\sqrt{\tau}}\\ &\quad + \kappa Se^{-q\tau}\mathcal{N}(\mathrm{d}_+) + \frac{1}{2}\sigma^2 S^2 e^{-q\tau}\frac{n(\mathrm{d}_+)}{S\sigma\sqrt{\tau}} - r\big(Se^{-q\tau}\mathcal{N}(\mathrm{d}_+) - Ke^{-r\tau}\mathcal{N}(\mathrm{d}_-)\big) =\\ &\quad - Se^{-q\tau}n(\mathrm{d}_+)\frac{\sigma}{2\sqrt{\tau}} + \frac{1}{2}\sigma^2 S^2 e^{-q\tau}\frac{n(\mathrm{d}_+)}{S\sigma\sqrt{\tau}} = 0.\end{aligned} \tag{16}$$

In the case of a put option, we have the corresponding formulas and results deduced by call-put parity (6), hence the price

$$P_t^{bs}(T,K) = Ke^{-r\tau}\mathcal{N}(-\mathrm{d}_-) - Se^{-q\tau}\mathcal{N}(-\mathrm{d}_+), \tag{17}$$

the delta $(-e^{-q\tau}\mathcal{N}(-\mathrm{d}_+))$, and the same gamma and vega as the call of same characteristics.

Remark 4 **(i)** *P^{bs} and C^{bs} only depend on t and T through their difference τ.*
(ii) *Exchanging S and K and r and q in the Black–Scholes formula for calls yields the Black–Scholes formula for puts.*

Corollary 2 **(i)** *Call and put prices are convex in S.*

(ii) *Call and put prices are differentiable in σ and increase from $(Se^{-q\tau} - Ke^{-r\tau})^+$ to $Se^{-q\tau}$ (for calls) and $(Ke^{-r\tau} - Se^{-q\tau})^+$ to $Ke^{-r\tau}$ (for puts) when σ increases from $0+$ to $+\infty$.*

(iii) *Call prices are increasing in τ if $r \geq 0 \geq q$, put prices are increasing in τ if $r \leq 0 \leq q$.*

Proof. **(i)** By positivity of Γ^{bs}.

(ii) By positivity of $\mathcal{V}^{bs}$ and the variations reported in the second panel of Table 1.

(iii) By positivity of Θ^{bs}, under the stated conditions on r and q. ∎

Remark 5 *Δ^{bs} and Γ^{bs} assess the market sensitivity of the call/put price to small, respectively large, movements of the stock S, whereas $\mathcal{V}^{bs}$ is the sensitivity of the prices to the model parameter σ. Γ^{bs} and $\mathcal{V}^{bs}$, like $n(\mathrm{d}_+)$, are maximum in K when d_+ vanishes, i.e. for $K = Se^{\kappa\tau + \frac{1}{2}\sigma^2\tau}$; Γ^{bs} is mostly significant for close-to-the money options with small time-to-maturity; $\mathcal{V}^{bs}$ is mostly significant for close-to-the money options with large time-to-maturity. In view of the next-to-last identity in (14), a portfolio of European vanilla options of the same maturity is gamma neutral if and only if it is vega neutral.*

Using the identity

$$n'(y) = -yn(y), \tag{18}$$

further formulas can be obtained for other second order sensitivities, e.g. the call/put

$$\text{volga} = \partial^2_{\sigma^2} C^{bs}\,,\ \ \text{vanna} = \partial^2_{S,\sigma} C^{bs} \tag{19}$$

See Figures 1 and 2 and play with the original python code by Clint Howard on https://clinthoward.github.io/portfolio/2017/04/16/BlackScholesGreeks

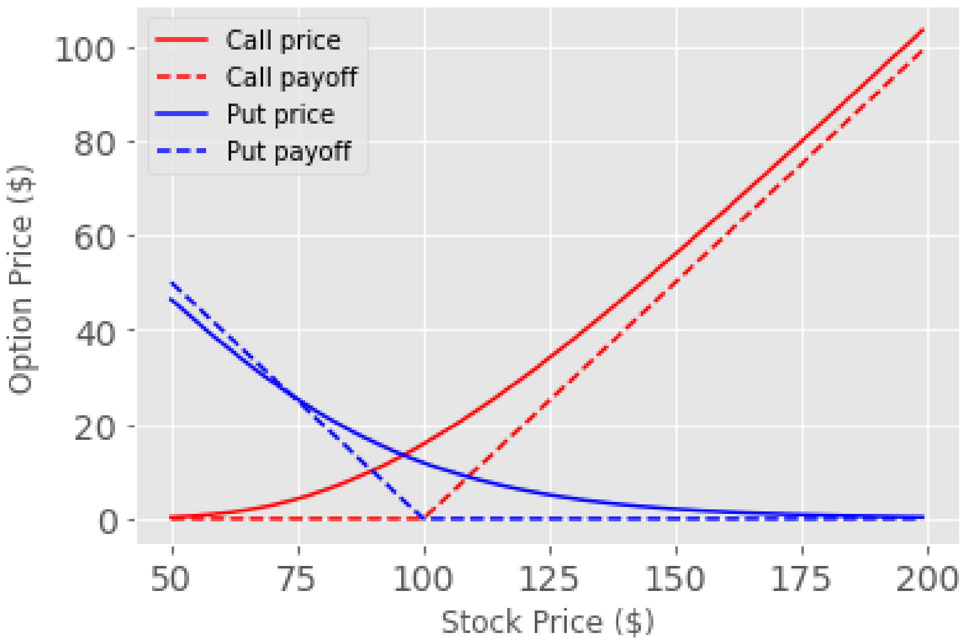

Figure 1: *Option price sensitivity to stock price.*

Remark 6 *Revisiting the above proofs shows that all these formulas admit straightforward extensions to the case where r, q and σ are time-integrable functions: simply replace $r\tau$, $q\tau$ and $\sigma\sqrt{\tau}$ in all the computations and results above by, respectively, $\int_t^T r(s)ds$, $\int_t^T q(s)ds$, and $\big(\int_t^T \sigma^2(s)ds\big)^{\frac{1}{2}}$.*

§4 Black-Scholes Equations

Back to the generic European vanilla pricing formula (4), the Markov property[13] of the risk-neutral Black-Scholes stock S yields

$$\mathbb{E}_t\phi(S_T) = \mathbb{E}\left(\phi(S_T)\,|\,S_t\right)$$

and therefore $\Pi_t = u(t, S_t)$, for a deterministic pricing function u[14].

Lemma 5 *Whenever available*[15]*, a continuous pricing function u on $[0,T]\times(0,+\infty)$ representing the price process Π, i.e. such that $u(t,S_t)=\Pi_t$, $t\le T$, holds almost surely, is uniquely defined.*

Proof. If u and v are two such continuous pricing functions, then, setting $w = u - v$, the process $w(\cdot, S_\cdot)$ is indistinguishable from 0 on $[0,T]$. Assuming by contradiction $w(0,S_0)$[16] nonzero, say >0[17], then by continuity we would have $w>0$ on some open neighborhood $\mathcal{O}$ of $(0,S_0)$, hence $w(\cdot,S_\cdot)>0$ on $[\![0,\eta[\![:= \{(\omega,t); t<\eta(\omega)\}$, where $\eta=\inf\{t>0;(t,S_t)\notin\mathcal{O}\}\wedge T$. So $[\![0,\eta[\![\subseteq \{(\omega,t); t\le T \text{ and } w(t,S_t(\omega))\neq 0\}$,

[13]cf. IV.(7) and the paragraph above it.
[14]measurable in S for each t, by application of Kallenberg (2006, Lemma 1.13 p.7).
[15]which supposes, in particular, a continuous payoff function.
[16]where consideration of the particular point $(0,S_0)$ is for notational simplicity.
[17]the reasoning below works the same for <0.

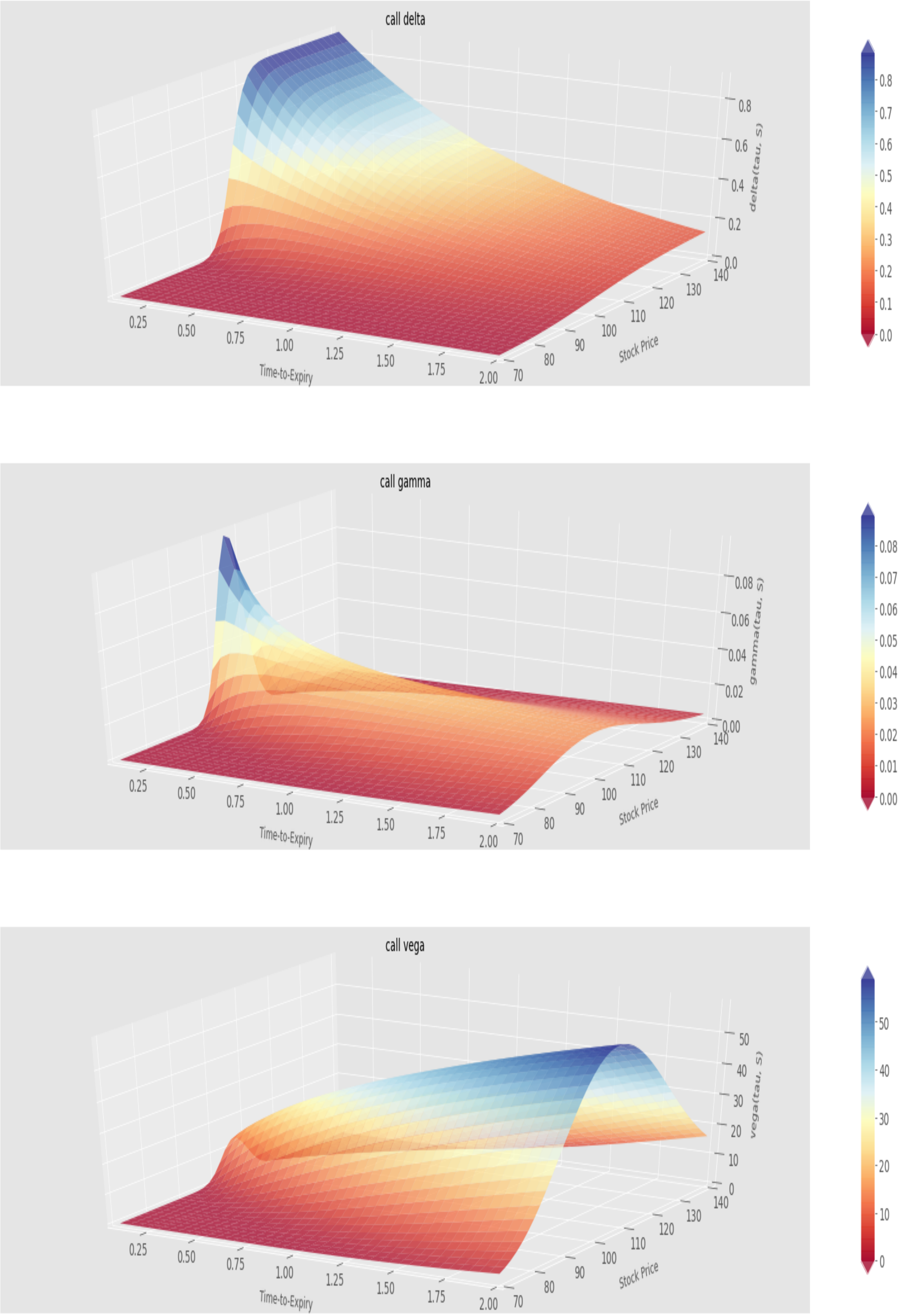

Figure 2: *Greek sensitivities to time-to-expiry and stock price.*

which, by the indistinguishability statement at the beginning of this proof, is of $\mathbb{Q}\times\boldsymbol{\lambda}$ measure 0, where $\boldsymbol{\lambda}$ denotes the Lebesgue measure on $\mathbb{R}_+$. Hence

$$(\mathbb{Q}\times\boldsymbol{\lambda})([\![0,\eta[\![)=0. \tag{20}$$

But, as the process S is continuous, η is a stopping time[18] and we have $\eta>0$ a.s., otherwise by continuity we would have $(0,S_0)\in$ the closed set $\mathcal{O}^c$, which is not the case. Hence

$$(\mathbb{Q}\times\boldsymbol{\lambda})([\![0,\eta[\![)=\int_\Omega\int_{\mathbb{R}_+}\mathbb{1}_{0\le t<\eta(\omega)}dt\mathbb{Q}(d\omega)=\int_\Omega\eta(\omega)\mathbb{Q}(d\omega)=\mathbb{E}\eta>0,$$

by Lemma I.1, in contradiction with (20). ∎

Assuming that u would even be of class $\mathcal{C}^{1,2}$ on $[0,T]\times(0,+\infty)$, an application of the Itô formula would yield:

$$e^{rt}d(e^{-rt}u(t,S_t))=(\partial_t u+\mathcal{A}^{bs}_S u-ru)(t,S_t)dt+\sigma S_t\partial_S u(t,S_t)dW_t, \tag{21}$$

where $\mathcal{A}^{bs}_S=\kappa S\partial_S+\frac{\sigma^2S^2}{2}\partial^2_{S^2}$. Since $e^{-rt}u(t,S_t)=e^{-rt}\Pi_t$ is a martingale, we would then have by Lemma IV.7 that

$$(\partial_t u+\mathcal{A}^{bs}_S u-ru)(t,S_t)=0\,,\ t\le T.$$

Also accounting for the terminal condition $\Pi_T=\phi(S_T)$, this suggests that the following Black-Scholes pricing PDE holds:

$$\begin{cases} u(T,S)=\phi(S),\ S\in(0,+\infty)\\ \partial_t u+\kappa S\partial_S u+\frac{1}{2}\sigma^2S^2\partial^2_{S^2}u-ru=0 \text{ in } [0,T)\times(0,+\infty). \end{cases} \tag{22}$$

In the case of European vanilla call or put options, this was indeed established in (16). Moreover, for any ϕ continuous with polynomial growth, the PDE (22) is known from Friedman (1964) to have a unique classical solution u in $\mathcal{C}^{1,2}\big([0,T)\times(0,+\infty)\big)\cap\mathcal{C}^0\big([0,T]\times(0,+\infty)\big)$ with polynomial growth in S (uniformly in $t\in[0,T]$). An analytic proof[19] of this result relies on related maximum/comparison principles. For instance, the uniqueness of a classical solution to the pricing equation (22) localized on a compact$\mathcal{K}\subseteq(0,+\infty)$ [20] follows from the following maximum principle[21]:

Lemma 6 *Let u,ν in $\mathcal{C}^{1,2}\big([0,T)\times\mathcal{K}\big)\cap\mathcal{C}^0\big([0,T]\times\mathcal{K}\big)$ satisfy*

$$\begin{cases} u(T,S)\le\phi(S),\ S\in\mathcal{K}\\ -\partial_t u-\kappa S\partial_S u-\frac{1}{2}\sigma^2S^2\partial^2_{S^2}u+ru\le 0 \text{ in } [0,T)\times\mathcal{K}\\ \nu(T,S)\ge\phi(S),\ S\in\mathcal{K}\\ -\partial_t \nu-\kappa S\partial_S \nu-\frac{1}{2}\sigma^2S^2\partial^2_{S^2}\nu+r\nu\ge 0 \text{ in } [0,T)\times\mathcal{K}. \end{cases} \tag{23}$$

Then $u\le\nu$.

Proof. Suppose the coefficient r of order 0 of the Black–Scholes equation positive (otherwise apply the reasoning below to the equation satisfied by $e^{-c(T-t)}u$ for $c>-r$). If $\neg(u\le\nu)$, then $(u-\nu)$ admits a positive maximum at some $(t,x)\in[0,T)\times\mathcal{K}$, at which

$$u>\nu\,,\ \partial_t u=\partial_t\nu\,,\ \partial_S u=\partial_S\nu\,,\ \partial^2_{S^2}u\le\partial^2_{S^2}\nu,\ \text{hence}$$
$$-\partial_t u-\kappa S\partial_S u-\frac{1}{2}\sigma^2S^2\partial^2_{S^2}u+ru>-\partial_t\nu-\kappa S\partial_S\nu-\frac{1}{2}\sigma^2S^2\partial^2_{S^2}\nu+r\nu,$$

contradicting (23). ∎

Consistent with the intuition developed from the above:

[18] cf. Lemma IV.1(ii).
[19] vs. a probabilistic proof of Theorem 1 below.
[20] for simplicity, and which is also the version of the equation that can be solved numerically by finite differences.
[21] see also Crépey (2013, Proposition 8.1.1 p.214) and Bouchard and Chassagneux (2016, Theorem 4.5 p.133).

Theorem 1 *Assuming ϕ continuous with polynomial growth in S, a continuous option $\mathbb{Q}$ pricing function u uniquely exists and is also the unique classical solution with polynomial growth in S to (22). The hedging strategy defined, for $t \in [0,T]$, by*

$$\zeta_t^{bs} = \partial_S u(t, S_t) \tag{24}$$

units of stock S and

$$\beta_t \left(u(t, S_t) - S_t \partial_S u(t, S_t)\right) \text{ units of the riskless asset } \beta^{-1}, \tag{25}$$

is self-financing, $\mathbb{Q}$ admissible, and it $\mathbb{Q}$ replicates the option payoff $\phi(S_T)$ at time T, starting from the wealth $u(0, S_0) = \Pi_0$ at time 0.

Proof. *Existence.* An application of Lemma 3 yields that a $\mathbb{Q}$ pricing function u for the option is provided by

$$u(t, S) = \mathbb{E}\big[e^{-r(T-t)}\phi(Se^{\sigma\sqrt{\tau}\varepsilon + b_\tau})\big], \tag{26}$$

where ε is standard Gaussian. Introducing p such that $\phi(S) \leq_c 1 + S^p$, where $\leq_c$ means $\leq$ modulo a multiplicative positive constant, we have

$$\mathbb{E}e^{-r\tau}\phi(xe^{\sigma\sqrt{\tau}\varepsilon + b\tau}) \leq_c 1 + x^p \mathbb{E}e^{p\sigma\sqrt{\tau}\varepsilon + pb\tau} \leq 1 + x^p e^{bpT + \frac{1}{2}p^2\sigma^2 T}.$$

Hence u is of polynomial growth in S, uniformy in $t \in [0,T]$.

By change of variables $xe^{\sigma\sqrt{\tau}y + b\tau} = e^z, \sigma\sqrt{\tau}dy = dz$, for $\tau > 0$, we have

$$\int_y \phi(xe^{\sigma\sqrt{\tau}y + b\tau})n(y)dy = \int_z \phi(e^z) n\Big(\frac{z - \ln(x) - b\tau}{\sigma\sqrt{\tau}}\Big)\frac{dz}{\sigma\sqrt{\tau}}.$$

The Leibnitz rule then ensures that u is in $\mathcal{C}^{1,2}\big([0,T) \times (0,+\infty)\big)$. To establish the continuity at T, letting $(t_n, x_n) \to (t = T, x)$, we have with $\tau_n = T - t_n$:

$$\mathbb{E}e^{-r\tau_n}\phi(x_n e^{\sigma\sqrt{\tau_n}\varepsilon + b\tau_n}) = \int_y e^{-r\tau_n}\phi(x_n e^{\sigma\sqrt{\tau_n}y + b\tau_n})n(y)dy,$$

As $n \to \infty$, $e^{-r\tau_n}\phi(x_n e^{\sigma\sqrt{\tau_n}y + b\tau_n}) \to \phi(x)$ while staying $\leq_c 1 + e^{p\sigma\sqrt{T}|y| + bpT}$, which is $n(y)dy$ integrable. Hence, by the Lebesgue dominated convergence theorem,

$$\mathbb{E}e^{-r\tau_n}\phi(x_n e^{\sigma\sqrt{\tau_n}\varepsilon + b\tau_n}) \to \int_y \phi(x)n(y)dy = \phi(x).$$

We conclude that u sits in the space of $\mathcal{C}^{1,2}\big([0,T) \times (0,+\infty)\big) \cap \mathcal{C}^0\big([0,T] \times (0,+\infty)\big)$ functions and is therefore the unique continuous $\mathbb{Q}$ pricing function of the option, by Lemma 5.

In particular, the Itô formula is applicable to $u(\cdot, S_\cdot)$ on the time interval $[0, T/2]$. Hence (cf. (21))

$$\begin{aligned}\mathbb{1}_{\{t \leq T/2\}} e^{rt} d(e^{-rt}u(t, S_t)) &= \mathbb{1}_{\{t \leq T/2\}}(\partial_t u + \mathcal{A}_S^{bs} u - ru)(t, S_t)dt \\ &+ \mathbb{1}_{\{t \leq T/2\}}\sigma S_t \partial_S u(t, S_t)dW_t\end{aligned} \tag{27}$$

holds on $[0,T]$ and the process $\int_0^{\cdot \wedge T/2}(\partial_t + \mathcal{A}_S - r)u(t, S_t)dt$ is a local martingale, by Lemma IV.5, hence a constant, by Lemma IV.7. Thus[22]

$$(\partial_t + \mathcal{A}_S - r)u(\cdot, S_\cdot) \text{ is indistinguishable from 0 on } [0, T/2]. \tag{28}$$

[22] See e.g. Bouchard and Chassagneux (2016, Proposition 2.1 p.56).

Now, assuming by contradiction $(\partial_t + \mathcal{A}_S - r)u(0, S_0)$[23] nonzero, say > 0[24], then by continuity we would have $(\partial_t + \mathcal{A}_S - r)u > 0$ on some open neighborhood $\mathcal{O}$ of $(0, S_0)$, hence $(\partial_t + \mathcal{A}_S - r)u(\cdot, S_\cdot) > 0$ on $[\![0, \eta[\![:= \{(\omega, t); t < \eta(\omega)\}$, where $\eta = \inf\{t > 0; (t, S_t) \notin \mathcal{O}\} \wedge T/2$. So $[\![0, \eta[\![\subseteq \{(\omega, t); t \leq T/2$ and $(\partial_t + \mathcal{A}_S - r)u(t, S_t(\omega)) \neq 0\}$, which, by (28), is of $\mathbb{Q} \times \boldsymbol{\lambda}$ measure 0[25]. Hence

$$(\mathbb{Q} \times \boldsymbol{\lambda})([\![0, \eta[\![) = 0. \tag{29}$$

But, as the process S is continuous, η is a stopping time[26] and we have $\eta > 0$ a.s., otherwise we would have $(0, S_0) \in$ the closed set $\mathcal{O}^c$, which is not. Hence

$$(\mathbb{Q} \times \boldsymbol{\lambda})([\![0, \eta[\![) = \int_\Omega \int_{\mathbb{R}_+} \mathbb{1}_{0 \leq t < \eta(\omega)} dt \mathbb{Q}(d\omega) = \int_\Omega \eta(\omega) \mathbb{Q}(d\omega) = \mathbb{E}\eta > 0,$$

by Lemma I.1, in contradiction with (29).

In conclusion of this part of the proof, the option admits the continuous $\mathbb{Q}$ pricing function (26), which is a classical solution with polynomial growth in S to the Black–Scholes equation (22).

Uniqueness and replication. Conversely, for any classical solution u with polynomial growth in S to (22), an application of the Itô formula to such a solution u of (22) between times t and T yields

$$\beta_T \phi(S_T) = \beta_t u(t, S_t) + \int_t^T \beta_s \partial_S u(s, S_s) \sigma S_s dW_s, \tag{30}$$

where, by (11),

$$\beta_s \partial_S u(s, S_s) \sigma S_s dW_s = \partial_S u(s, S_s) d(\beta_s \widehat{S}_s).$$

Thus

$$\beta_T \phi(S_T) = \beta_t u(t, S_t) + \textstyle\int_t^T \partial_S u(s, S_s) d(\beta_s \widehat{S}_s) \tag{31}$$

and, in particular,

$$\beta_T \phi(S_T) = u(0, S_0) + \textstyle\int_0^T \partial_S u(s, S_s) d(\beta_s \widehat{S}_s). \tag{32}$$

In view of I.(9), this equals $\beta_T V_T$ for $\pi = u(0, S_0)$ and $\zeta = \partial_S u(\cdot, S_\cdot)$. The discounted wealth

$$\beta V = u(0, S_0) + \textstyle\int_0^\cdot \partial_S u(s, S_s) d(\beta_s \widehat{S}_s) = \beta_t u(t, S_t) \tag{33}$$

(by taking the difference between (31) and (32)) is a $\mathbb{Q}$ local martingale, by Lemma IV.5, sandwiched between two $\mathbb{Q}$ martingales of the form $(\pm c e^{-kt} S_t^p)$ for suitable constants p, k, c, by the polynomial growth condition on u and the Black-Scholes model postulated on S. Hence this process βV is a $\mathbb{Q}$ martingale, by a double application[27] of Lemma IV.3. So $\Pi_t = u(t, S_t)$ follows by taking conditional expectations in (31), i.e. u is a $\mathbb{Q}$ pricing function of the option and, since u is continuous, it is therefore nothing but the[28] continuous $\mathbb{Q}$ pricing function u already identified in the first part of the proof. The identity (32) finally establishes the replicability of the payoff $\phi(S_T)$ by the $\mathbb{Q}$ admissible strategy $(\pi, \zeta = \partial_S u(\cdot, S_\cdot))$[29]. ∎

Note that Theorem 1 covers the special cases of the European vanilla call and put options.

[23] where consideration of the particular point $(0, S_0)$ is for notational simplicity.
[24] the reasoning below works the same for < 0.
[25] $\boldsymbol{\lambda}$ denoting the Lebesgue measure on $\mathbb{R}_+$.
[26] by Lemma IV.1(ii).
[27] to βV and to $(-\beta V)$.
[28] unique, by Lemma 5.
[29] cf. I.(26) and the line following it, as well as I.(10) regarding (25).

Remark 7 *If ϕ is bounded fom below, like with European vanilla put short or long (but call short only) positions, then so are the corresponding $\mathbb{Q}$ pricing function and value of the replicating portfolio, hence replicability holds (not only $\mathbb{Q}$ replicability).*

Remark 8 *In the Black–Scholes model, ($\mathbb{Q}$, at least) replicability of more general, possibly path-dependent, European claims ξ, not necessarily of the form $\phi(S_T)$ but assumed $\mathbb{Q}$ square integrable, follows from the Brownian martingale representation theorem. This replicability of European claims in the Black-Scholes model explains why the Black-Scholes PDE* (22) *(hence, the Black-Scholes prices) doesn't depend on the physical drift of S, even though the latter may be responsible for fat tails or skewness in the physical returns of S.*

Remark 9 *Postulating, instead of* (9)*, a physical model for S, namely the dynamics* (9) *with κ replaced by a constant μ there and W by a Brownian motion $\widehat{W}$ under $\mathbb{P}$, i.e.*

$$dS_t = S_t(\mu dt + \sigma d\widehat{W}_t),$$

then the Girsanov theorem allows showing that the probability measure $\mathbb{Q}$ defined through

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\big(-\lambda W_T - \frac{1}{2}\lambda^2 T\big)\ ,\ \text{where } \lambda = \frac{\mu - r}{\sigma},$$

is the unique risk-neutral measure[30] *on the primary market made of the stock S and the risk-free asset $e^{r\cdot}$.*

Observe the consistency between the results stated in the above remarks and the general comments made in the concluding paragraphs of I.§2.

A American options

Next we consider an American option with payoff process $\phi(S_\cdot)$ in the Black-Scholes model.

For ϕ continuous with polynomial growth, one can show the existence of a solution to the American reflected BSDE I.(23) with data $D = 0, \xi = \phi(S_T)$ and $L = \phi(S_\cdot)$, i.e. Assumption I.1 holds (El Karoui et al., 1997). By Proposition I.7, one can thus consider the no arbitrage price process of the American option given, for $t \in [0, T]$, by:

$$\max_{\vartheta \in \Theta^t} \mathbb{E}_t e^{-r(\vartheta - t)} \phi(S_\vartheta) = v(t, S_t), \tag{34}$$

for some deterministic function v, by the Markov property of the Black-Scholes model. However, as opposed to the European pricing function u, the American pricing function v is not $\mathcal{C}^{1,2}$ before T[31].

Yet, the function v can be shown to be the unique continuous viscosity solution[32] with polynomial growth in S to the following obstacle problem[33]:

$$\begin{cases} v(T, \cdot) = \phi \text{ on } (0, +\infty) \\ \min\left(-\partial_t v - \mathcal{A}_S v + rv, v - \phi\right) = 0 \text{ on } [0, T) \times (0, +\infty). \end{cases} \tag{35}$$

[30] namely, the one that we directly postulated under the formulation (9) of the model.
[31] except for trivial cases such as those of Proposition 1 where $v = u$.
[32] see e.g. Crandall, Ishii, and Lions (1992).
[33] see e.g. Crépey (2013, Definition 13.1.3 page 361).

Proposition 6 *The (admitted continuous[34]) function v is a viscosity solution to* (35)*.*

Proof. In view of (34), v satisfies pointwise the boundary and obstacle conditions $v = \phi$ at T and $v \geq \phi$ before T.

To show that v is viscosity subsolution, suppose by contradiction that, for some test-function φ $\mathcal{C}^{1,2}$ before T,

$$\begin{aligned} &0 = \max_{[0,T)\times(0,+\infty)} (v-\varphi) = (v-\varphi)(0,S_0), \\ &v(0,S_0) > \phi(S_0)\,,\ -(\partial_t + \mathcal{A}_S - r)\varphi(0,S_0) > 0 \end{aligned} \tag{36}$$

(where considering the particular time 0 in $(0,S_0)$ is for notational simplicity). The two inequalities are still valid in an open neighborhood $\mathcal{O}$ of $(0,S_0)$ separated from T. Let $\eta = \inf\{t > 0; S_t \notin \mathcal{O}\} \wedge \inf\{t > 0; |\ln(\frac{S_t}{S_0})| > 1\}$, a stopping time[35] by continuity of S. The American reflected BSDE and the Itô formula respectively yield (as the process A in the BSDE solution is constant on $[0,\eta]$)

$$\begin{aligned} v(0,S_0) &= \beta_\eta v(\eta,S_\eta) - \int_0^\eta \beta_t dM_t \\ \varphi(0,S_0) &= \beta_\eta \varphi(\eta,S_\eta) - \int_0^\eta \beta_t(\partial_t + \mathcal{A}_S)\varphi(t,S_t)dt \\ &- \int_0^\eta \beta_t \sigma S_t \partial_S \varphi(t,S_t) dW_t. \end{aligned}$$

Taking the difference and the expectation yields

$$\begin{aligned} &0 = (v-\varphi)(0,S_0) = \\ &\mathbb{E}\Big[\beta_\eta (v-\varphi)(\eta,S_\eta) + \int_0^\eta \beta_t(\partial_t + \mathcal{A}_S)\varphi(t,S_t)dt\Big] < 0 \end{aligned}$$

(via the second part in the second line of (36)), which contradicts the first line in (36).

To show that v is viscosity supersolution, suppose by contradiction

$$0 = \min_{[0,T)\times(0,+\infty)} (v-\varphi) = (v-\varphi)(0,S_0)\,,\ -(\partial_t + \mathcal{A}_S - r)\varphi(0,S_0) < 0. \tag{37}$$

The inequality is still valid in the open neighborhood $\mathcal{O}$ of $(0,S_0)$ and one can assume $\mathcal{O}$ separated from T wlog. Let $\eta = \inf\{t > 0; |\ln(\frac{S_t}{S_0})| > 1\}$. This time we obtain

$$\begin{aligned} v(0,S_0) &= \beta_\eta v(\eta,S_\eta) - \int_0^\eta \beta_t dM_t + \int_0^\eta \beta_t dA_t \\ \varphi(0,S_0) &= \beta_\eta \varphi(\eta,S_\eta) - \int_0^\eta \beta_t(\partial_t + \mathcal{A}_S)\varphi(t,S_t)dt = \\ &- \int_0^\eta \beta_t \sigma S_t \partial_S \varphi(t,S_t) dW_t. \end{aligned}$$

Hence

$$\begin{aligned} &0 = (v-\varphi)(0,S_0) = \\ &\mathbb{E}\Big[\beta_\eta (v-\varphi)(\eta,S_\eta) + \int_0^\eta \beta_t dA_t + \int_0^\eta \beta_t(\partial_t + \mathcal{A}_S)\varphi(t,S_t)dt\Big] > 0 \end{aligned}$$

(via the second part in (37)), which contradicts the first part in (37). ∎

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[34] see e.g. Crépey (2013, Chapter 12).
[35] cf. Lemma IV.1(ii).

§5 Implied Volatility

The Black-Scholes model is strongly misspecified in practice. In fact, the Black–Scholes pricing formulas are essentially used by traders for conveying information about the relative value of different options in the market. The idea is to express prices in a unit of measurement, implied volatility, that is less sensitive to the strike and maturity of an option than its money-value. Black–Scholes formulas are thus effectively used in the reverse-engineering mode for determining, given a European vanilla price observed on the market, the corresponding value of the Black–Scholes volatility consistent with that option price.

Definition 1 *Given values of r and q inferred at time t, from riskless bonds for r and then from forward prices*[36] *for q, the Black-Scholes implied volatility of a European vanilla (call or put) option at time t is the value Σ_t such that*

$$\Pi^{bs}(t, S_t, T, K; r, q, \Sigma_t) = \Pi^{\star}_t(T, K), \tag{38}$$

where $\Pi^{\star}_t(T, K)$ denotes the market price of the option at time t.

The Black–Scholes formulas are then no more than "wrong formulas into which to put a wrong number [the implied volatility of an option] to get the right result [an option market price]".

Lemma 7 **(i)** *The equation* (38) *has a unique positive solution Σ_t provided the market price lies within the no-static-arbitrage bounds*[37] *$((Se^{-q\tau} - Ke^{-r\tau})^+, Se^{-q\tau})$ for the call price and in $((Ke^{-r\tau} - Se^{-q\tau})^+, Ke^{-r\tau})$ for the put price.*
(ii) *Within the setup of any $\mathbb{Q}$ (true) martingale model for αS, European calls and puts of same characteristics have the same implied volatility.*

Proof. **(i)** is an immediate consequence of Corollary 2(ii).

(ii) follows from the call-put parity (6) that holds in any $\mathbb{Q}$ martingale model for αS[38], which includes the Black–Scholes model as special case. ∎

Remark 10 *In view of the Black–Scholes pricing formulas given by the first line in* (14) *for the call and by* (17) *for the put, where $\mathrm{d}_\pm$ are given by* (12)*, the process Σ_t can be represented as a continuous function of $\Pi^{\star}_t(T, K)$, r, q, $\tau = T - t$, and of the call log-moneyness $\ln(S_t/K)$ (or log-forward moneyness $X_t = \ln(S_t/K) + \kappa\tau$).*

In view of Corollary 2(ii), the equation (38) can be solved numerically by dichotomy. Building on the vega formula in (14), one can also use a Newton–Raphson zero search, i.e. iteratively solve for Σ' the linearized problem $\Pi^{bs} + \mathcal{V}^{bs}(\Sigma' - \Sigma) = \Pi^{\star}$, where Π^{bs} and $\mathcal{V}^{bs}$ are the Black–Scholes price and vega corresponding to the current volatility Σ in the algorithm. This is typically faster than a search by dichotomy, but not for market prices $\Pi^{\star}$ close to the arbitrage bounds of the option, for which the vega sensitivity of the option vanishes.

Proceeding in this way for a range of strikes K and a fixed maturity T, one commonly obtains

- a symmetrical smile on foreign exchange derivative markets,

[36] possibly synthetized from call and put prices by call-put parity.
[37] cf. Proposition I.3 and Remark I.7.
[38] cf. Remark 7.

- a negative skew on equity derivative or markets,
- a smirk on interest rate derivative markets.

See for instance the SPX options smirks on the bottom panel in Figure 3, flattening as the maturity increases.

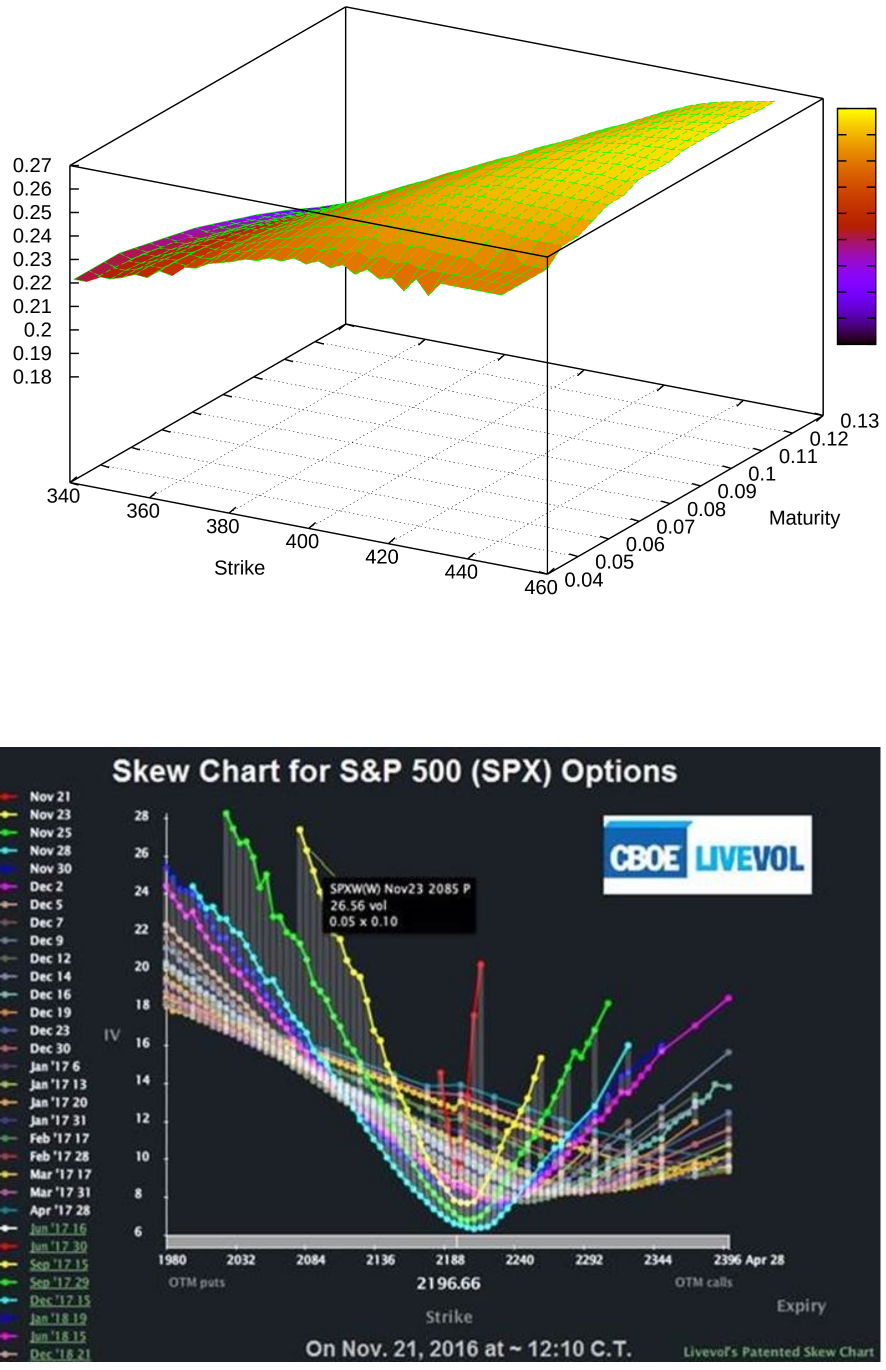


Figure 3: *(Bottom) SPX options smiles and skews; (Top) An implied volatility surface on gold futures [Source: CBOE].*

Implied volatilities tend to be larger than realized volatilities $\widehat{\sigma}$ (statistically estimated standard deviations of stock returns on a time step h divided by $\sqrt{\mathrm{h}}$), because setting up a hedge is more costly in practice than in a theoretical Black–Scholes world with volatility $\widehat{\sigma}$, due to model risk and transaction costs that are all ignored in the Black–Scholes model.

They tend to be negatively skewed at low strikes because a (risk-neutral) Black–Scholes model underestimates the probability of downward jump (or large movements) of a stock or stock index, which traders correct by increasing the price (hence, implied volatility) of far out-of-the-money (OTM) puts. Negative skewness of implied volatilities can be related to a negative skewness, in the sense of a negative third moment, of the risk neutral distribution of stock returns ("$\mathbb{E}(\delta S)^3 < 0$"). Consistent with a jump fear interpretation, the skew is inverted, i.e. a positive implied volatility skew and $\mathbb{E}(\delta S)^3 > 0$ hold in the case of safe haven underliers, such as gold or other negative beta assets in the sense of the capital asset pricing model (CAPM, see the top panel of Figure 3).

Implied volatilities tend to be positively smiled (convex in strike) because of liquidity premia that are ignored in Black–Scholes and larger far from the money. A positive curvature of implied volatilities can be related to an excess kurtosis of the risk neutral distribution of stock returns ("$\mathbb{E}(\delta S)^4 > 3(\mathbb{E}(\delta S)^2)^2$").

The above discussion is summarized by the bottom part of Figure 4, where the upper part refers to further connections guessed in Derman, Kani, and Zou (1996); Derman (1999) and detailed in III.§2 between market implied volatilities and the corresponding local volatilities.

A Implied Black–Scholes Hedging

We now explain how the Black-Scholes model is commonly used in the implied mode for hedging purposes.

We consider the problem of delta-hedging a European option with maturity T on an underlying S. For simplicity of the subsequent analysis, we assume a continuous semimartingale setup for the stock S and the implied volatility Σ of a European vanilla option sold by the bank, zero interest rates and dividends on S, as well as continuous-time hedging by the bank. Recalling (14) and (19), the Itô formula for $\Pi_t^\star = \Pi^{bs}(t, S_t; \Sigma_t)$ yields, with all Greeks valued at the rolling implied volatility Σ_t of the option:

$$\begin{aligned} d\Pi_t^\star &= \Theta_t^{bs} dt + \Delta_t^{bs} dS_t + \frac{1}{2}\Gamma_t^{bs} d\langle S\rangle_t \\ &+ \mathcal{V}_t^{bs} d\Sigma_t + \frac{1}{2}\text{volga}_t^{bs} d\langle \Sigma\rangle_t + \text{vanna}_t^{bs} d\langle S, \Sigma\rangle_t. \end{aligned} \tag{39}$$

Hence, the profit-and-loss of the trader using the hedging strategy Δ^{bs} in S is given, for $t \le T$, by

$$\begin{aligned} dp_t = -d\Pi_t^\star + \Delta_t^{bs} dS_t &= -\Theta_t^{bs} dt \pm \Delta_t^{bs} dS_t \\ - \frac{1}{2}\Gamma_t^{bs} d\langle S\rangle_t - \mathcal{V}_t^{bs} d\Sigma_t &- \frac{1}{2}\text{volga}_t^{bs} d\langle \Sigma\rangle_t - \text{vanna}_t^{bs} d\langle S, \Sigma\rangle_t. \end{aligned}$$

Accounting for the Black–Scholes equation $\Theta^{bs} + \frac{1}{2}\Sigma^2 S^2 \Gamma^{bs} = 0$ (for $r = q = 0$), we obtain

$$\begin{aligned} dp_t &= \frac{1}{2}\Gamma_t^{bs} S_t^2 \big(\Sigma_t^2 - \frac{1}{S_t^2}\frac{d\langle S\rangle_t}{dt}\big) dt \\ &- \mathcal{V}_t^{bs} d\Sigma_t - \frac{1}{2}\text{volga}_t^{bs} d\langle \Sigma\rangle_t - \text{vanna}_t^{bs} d\langle S, \Sigma\rangle_t. \end{aligned} \tag{40}$$

In particular (cf. Figure 2):

- The first term is positive/negative in scenarios where Σ_t^2 is greater/less than the realized variance $\frac{1}{S_t^2}\frac{d\langle S\rangle_t}{dt}$, and significant in the case of a high-gamma, close-to-the money option with small time-to-maturity (cf. Remark 5);

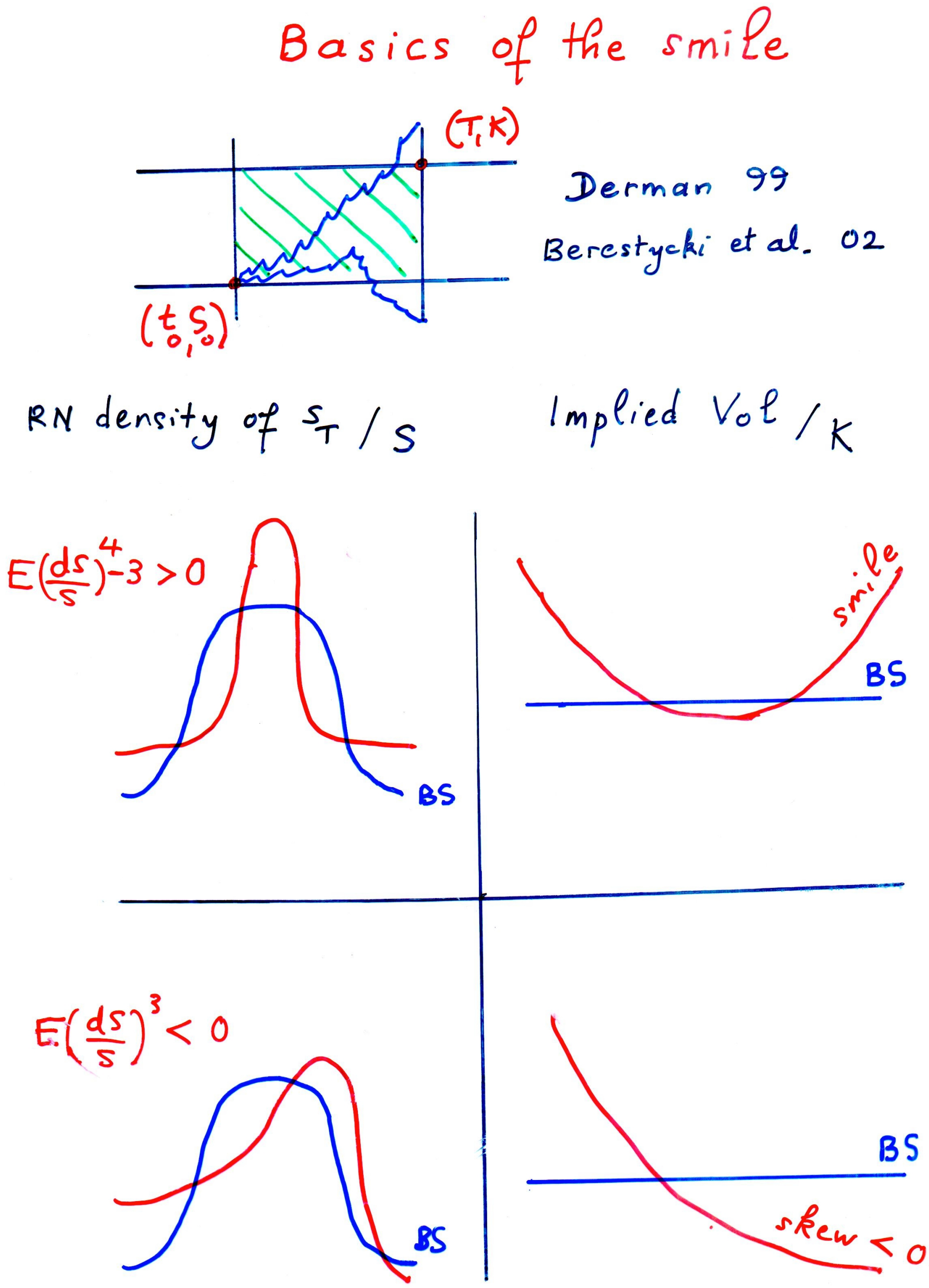


Figure 4: *Risk-neutral density return versus implied volatility patterns.*

- The second term is driven by $d\Sigma_t$ and its coefficient $\mathcal{V}_t^{bs}$ is significant in the case of a high vega, close-to-the money vanilla option with large time-to-maturity (cf. Remark 5).

So the above Black–Scholes implied delta Δ^{bs} only provides a partial hedge, which does no account for volatility risk. To improve on the Black–Scholes implied delta Δ^{bs}, the trader can use a finite difference Black-Scholes implied delta of the form (for some small positive α)

$$(2\alpha S_t)^{-1}\times \\ \Big(\Pi^{bs}(t,(1+\alpha)S_t,T,K;r,q,\widetilde{\Sigma}_t) - \Pi^{bs}(t,(1-\alpha)S_t,T,K;r,q,\Sigma_t)\Big)\,,$$

in which the volatility $\widetilde{\Sigma}_t$ is an update of Σ_t accounting for the correlation between stock returns and implied volatility changes. For instance, the sticky delta rule stipulates that the implied volatility surface evolves deterministically, when parameterized in terms of the time-to-maturity τ and of the put log-forward moneyness $(\ln(\frac{K}{S_t}) - \kappa\tau)$: see Balland (2002) and Figure 5, where "sticky strike" and "sticky moneyness" (aka "sticky delta") speak for themselves, whereas "sticky implied tree"[39] corresponds to the behavior of implied volatilities in a (fixed) local volatility model as per Chapter III.

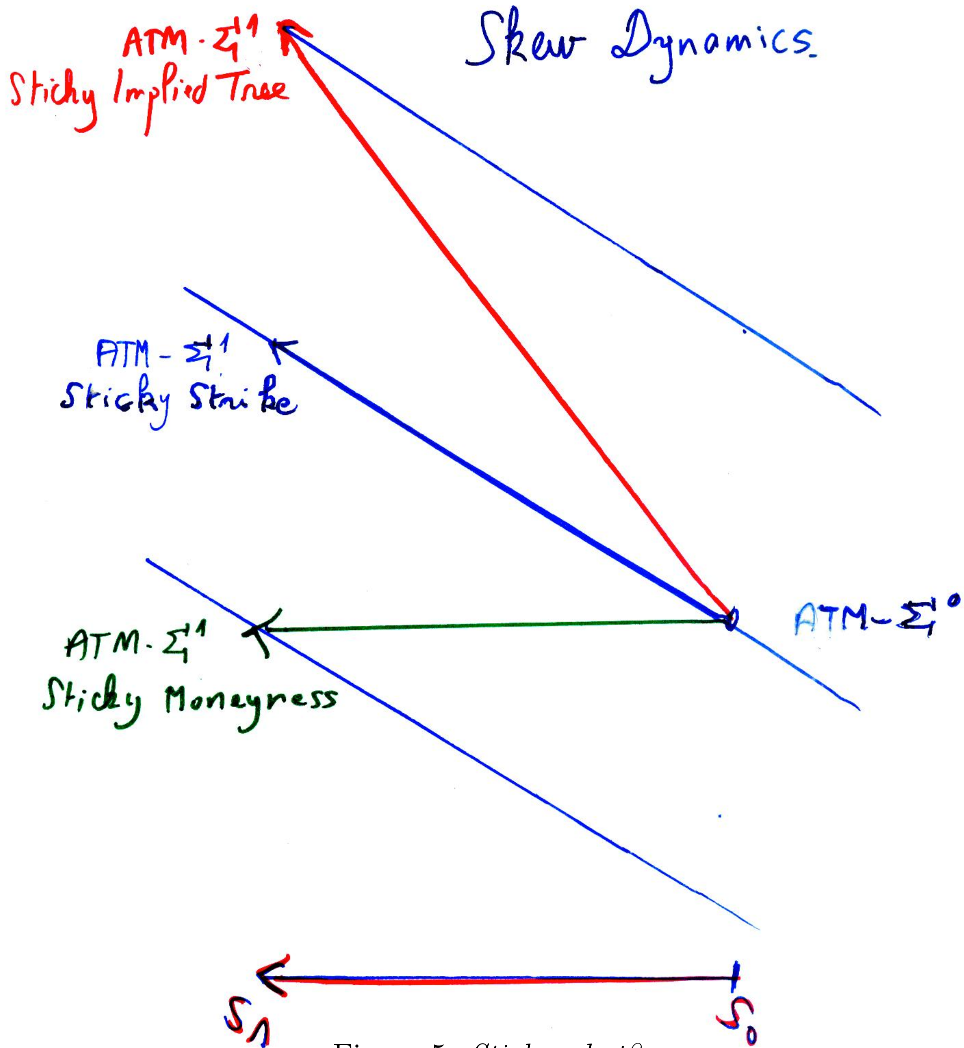


Figure 5: *Sticky what?*

The next step is a multi-(risky) asset hedge by the stock and another option which kills the dS_t and $d\Sigma_t$ terms in the ensuing profit-and-loss. The profit-and-loss then becomes time-differentiable, but it has no reason to vanish in general (unless it is a local martingale). The corresponding "hedge leakage" is a form of model risk depicted in (Albanese et al., 2021).

[39] in the terminology of Derman (1999).

Remark 11 *Hedging model shifts, in the sense of market practice such as vega hedging above, in our opinion does not qualify for "hedging model risk". In fact, the thesis developed in Albanese, Crépey, and Iabichino (2021) is that model risk cannot be hedged, but only compressed by upgrading the "local models" used by traders to a global fair valuation model that would be used consistently for all its purposes by the bank. See also Crépey and Benezet (2022) for an example where vega hedging actually enhances model risk.*

An alternative to the above delta/vega hedge is to be delta/gamma neutral, while staying fully exposed to volatility risk.

The above analysis can also be extended to a barrier and/or American option that would be priced by a market maker bank at its Black–Scholes price, $\widetilde{\Pi}^{\star}$ say, corresponding to the implied volatility Σ of the European vanilla counterpart of the option, with then instead of (39):

$$\begin{aligned} d\widetilde{\Pi}^{\star}_t &= \widetilde{\Theta}^{bs}_t dt + \widetilde{\Delta}^{bs}_t dS_t + \frac{1}{2}\widetilde{\Gamma}^{bs}_t d\langle S\rangle_t + \widetilde{\mathcal{V}}^{bs}_t d\Sigma_t \\ &+ \frac{1}{2}\widetilde{\mathrm{volga}}^{bs}_t d\langle \Sigma\rangle_t + \widetilde{\mathrm{vanna}}^{bs}_t d\langle S, \Sigma\rangle_t, \end{aligned} \tag{41}$$

where

- Σ is meant as the implied volatility of the (European vanilla part of the) options,
- The Greeks denoted by $\widetilde{\cdot}$ are meant as the American and/or barrier option Black–Scholes Greeks valued at Σ,
- the time horizon is restricted to the first time where the barrier is hit or the American option is exercised by the buyer.

Remark 12 *One distinguishes between regular barriers, which are triggered when the option is out-of-the-money, and reverse barriers, which are triggered when the option is in-the-money. Regular barrier options are not much harder to hedge than vanilla options. In contrast, reverse barriers may be very dangerous because of a mixed gamma/vega exposure, i.e. the fact that these Greeks may change of sign in the neighbourhood of the barrier. Namely, in the case of a reverse barrier,* $\widetilde{\Gamma}^{bs}$ *and* $\widetilde{\mathcal{V}}^{bs}$ *can be negative in the vicinity of the barrier as time approaches* T *(Taleb, 1997; Laurence, 2017; Crépey and Benezet, 2022).*

The profit-and-loss $\widetilde{p}$ arising from different hedging strategies (until the barrier is hit or the American option is exercised by the buyer) can then be analyzed much like in the European vanilla case above.

Chapter III

Dupire(-Derman, Gatheral,...)

Figure 1 displays the FTSE 100 index and the corresponding at-the-money (ATM) 3 months rolling implied volatility, as well as the implied volatility of a fixed FTSE 100 option, between October 1 1999 and March 1 2000. In view of this, a natural extension of Black–Scholes is let the volatility σ depend

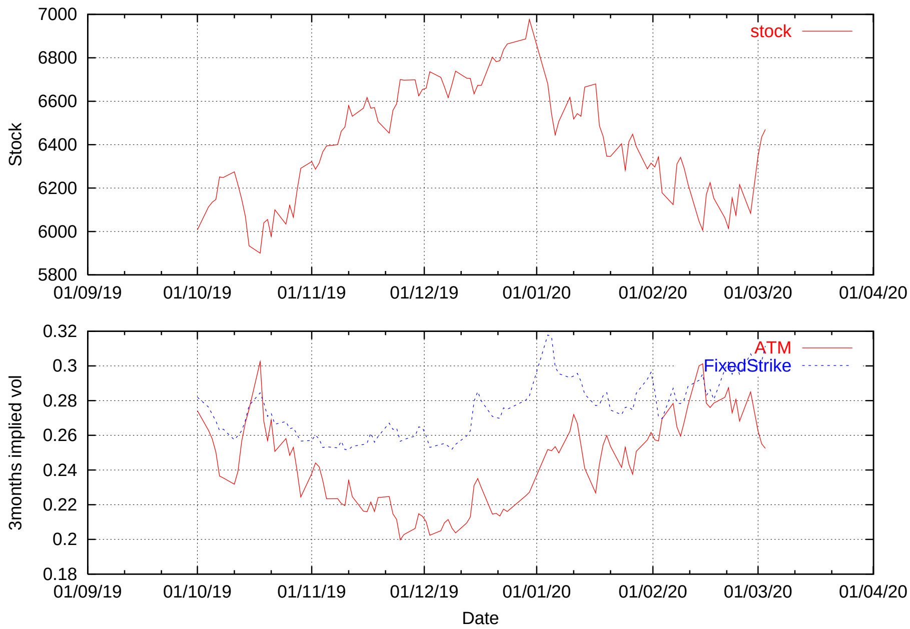


Figure 1: *Inverted mirror picture on the FTSE 100, October 1 1999–March 1 2000.*

on the stock level S[1].

§1 Local Volatility Model

Local volatility models (Dupire, 1994; Derman and Kani, 1994) are the generalization of the Black-Scholes model in which the volatility parameter is no longer constant, but is given by a function $\sigma = \sigma(t, S)$. So

$$dS_t = S_t(\kappa dt + \sigma(t, S_t)dW_t), \tag{1}$$

[1]on top of also possibly time t, but this is already the case in the non-stationary version of the Black–Scholes model with time-deterministic volatility.

where κ denotes, as in Black-Scholes, the difference $(r-q)$ between a risk-free funding rate and a dividend yield q on S (both assumed constant, with immediate extension to time-deterministic), and where W is a Brownian motion under some probability measure $\mathbb{Q} \sim \mathbb{P}$.

Remark 1 *Whenever a model of the stock S satisfies* (1) *for a continuous function $\sigma(\cdot,\cdot)$, it is $\mathbb{Q}$ arbitrage-free, by application of Theorem IV.1 and Lemma II.1.*

An application of the Itô formula yields the following rewriting of (1) in terms of $X = \ln(S)$:

$$dX_t = (\kappa - \frac{\sigma(t,e^{X_t})^2}{2})dt + \sigma(t,e^{X_t})dW_t. \tag{2}$$

Unless explictly stated otherwise, we restrict ourselves hereafter to the "comfort zone" where $(t,x) \mapsto \sigma(t,e^x)$ is Lipschitz continuous—a property we refer to as log-Lipschitz hereafter—and between two positive bounds, so that the stochastic differential equation (SDE) (1) is well posed, by application of the standard SDE result recalled after IV.(6) to (2). Moreover, the corresponding process αS satisfies by application of the elementary integration by parts formula IV.(4):

$$(\alpha S)_t = S_0 e^{\int_0^t \sigma(s,S_s)dW_s - \frac{1}{2}\int_0^t \sigma(s,S_s)^2 ds},$$

a $\mathbb{Q}$ martingale by application of the Novikov criterion (having assumed σ bounded), in line with the general setup postulated in II.§1. One can also show that the corresponding process S has the Markov property IV.(7) and that, for almost every $0 < t < T$, the process S admits a transition probability density $\gamma_t^{lo}(T,\cdot) = \gamma^{lo}(t,S_t,T,\cdot)$ from (t,S_t) to (T,K) (cf. (7)), with $\gamma^{lo}(t,S,T,K)$ jointly continuous in its four variables (t,S,T,K)[2].

However, the above setup is not the only case of interest:

Example 1 *The constant elasticity of variance (CEV) model corresponds to $\sigma(t,S) = \sigma S^\rho$, for positive constants σ and $\rho < 1$. Note that $\frac{\partial(\sigma S^\rho)}{\partial S} / \left(\frac{\sigma S^\rho}{S}\right) =$ the constant ρ (i.e. $\frac{\partial_S(\sigma S^\rho)}{(\sigma S^\rho)} = \frac{\rho}{S}$), whence the name of the model. Even though the corresponding SDE* (1) *is non Lipschitz, one can relate the CEV model to a time-changed squared Bessel process (e.g., for suitable values of ρ, the squared norm of a multivariate Brownian motion)*[3]*. This allows showing that the CEV model is well posed, with αS martingale under $\mathbb{Q}$ as assumed in II.§1, and deriving non-central χ^2 Fourier analytics for European vanilla options in this model.*

Example 2 *One can show that the stochastic differential equation* (1) *admits a unique (weak, i.e. a pair (W,S)) solution, with αS martingale under $\mathbb{Q}$ as assumed in II.§1, for any Borelian function σ of t and S lying between two positive bounds on $[0,T] \times (0,+\infty)$.*[4]

At the level of generality of Example 2, the European vanilla call option $\mathbb{Q}$ price process is defined, for any positive maturity T and strike K and for $t \le T$, by (cf. II.(4)):

$$C_t^{lo}(T,K) = e^{-r(T-t)}\mathbb{E}_t(S_T - K)^+ = u(t,S_t,T,K;\sigma(\cdot,\cdot)). \tag{3}$$

Here u is the pricing function in the state variables t and S, parameterized by the local volatility function $\sigma(\cdot,\cdot)$ and the characteristics (T,K) of the call. Likewise we write $P_t^{lo}(T,K) = e^{-r(T-t)}\mathbb{E}_t(K - S_T)^+$ for puts.

[2]by (Friedman, 1975, Volume 1, Chapter 6, Theorems 5.6 and 4.5).
[3]See Bouchard and Chassagneux (2016, Section 7.4 p.239) and Jeanblanc, Yor, and Chesney (2009, Chapter VI).
[4]See Stroock and Varadhan (1979, exercise 7.3.3) and Karatzas and Shreve (1991, problem 5.6.15 and corollary 3.5.13).

A Dupire Equation

The above pricing function u can be characterized by a related pricing equation in the state variables t and S: this is Theorem 1(i) below. As in Black-Scholes, this result is extendible to any payoff $\phi(S_T)$, where ϕ is a continuous and bounded payoff function. In fact, ***a local volatility model is complete*** in the sense of Theorem II.1 and Remark 8, which still hold in this extended setup, just replacing σ by $\sigma(\cdot,\cdot)$ everywhere, modulo the regularity issues regarding u. Of course, in the general nonparametric $\sigma(t,S)$ case, one loses the analytic formulas for European vanilla calls and puts. But European (as also American) vanilla call and put prices and Greeks can still be computed very efficiently, by finite differences approximation of (5) or (7).

Moreover, regarding the European vanilla calls or puts in a local volatility model[5], the "dual" equation of Theorem 1(ii) also holds for u seen as a function of the option parameters T and K, for any fixed (t,S).

Existence of unique classical solutions, with polynomial growth in their respective spatial variables S and K (locally uniformly in their respective temporal variables t and T), to the PDEs (5) and (7), follows from (Friedman, 1964, Chapter 1, Theorem 12, and Chapter 2, Theorem 10).

Theorem 1 **(i)** *For any (T,K), the function*

$$(t,S) \mapsto u(t,S,T,K;\sigma(\cdot,\cdot)) \tag{4}$$

is the unique classical solution $\leq S$ to the following pricing equation:

$$\begin{cases} u(T,S) = (S-K)^+,\ S>0 \\ (\partial_t u + \kappa S\partial_S u + \frac{1}{2}\sigma(t,S)^2 S^2 \partial^2_{S^2} u - ru)(t,S) = 0, \quad t<T,\ S>0. \end{cases} \tag{5}$$

(ii) *For any (t,S), the function*

$$(T,K) \mapsto u(t,S,T,K;\sigma(\cdot,\cdot)) \tag{6}$$

is the unique classical solution $\leq S$ to the so called Dupire equation:

$$\begin{cases} u(t,K) = (S-K)^+,\ K>0 \\ (\partial_T u + \kappa K\partial_K u - \frac{1}{2}\sigma(T,K)^2 K^2 \partial^2_{K^2} u + qu)(T,K) = 0, \quad T>t,\ K>0. \end{cases} \tag{7}$$

Proof. *that the functions* (4) *and* (6) *satisfy the respective equations* (5) *and* (7) *in the classical sense.*
(i) *(sketched)* Modulo the regularity issue regarding u[6], the pricing equation (5) for the pricing function (4) can be established by similar arguments as the Black–Scholes equation (cf. the proof of Theorem II.1).
(ii) *(assuming $\gamma_t^{lo}(T,K)$ $\mathcal{C}^{1,2}$ in (T,K).)* We have the following Fokker-Planck equation (see Remark 2 below) in the forward variables (T,K) for γ_t^{lo}, given $S_t = S$:

$$\begin{cases} \partial_K \mathbb{Q}_t(S_t \leq K) = \boldsymbol{\delta}_S(dK),\ K>0 \\ \left(\partial_T \gamma_t^{lo} + \partial_K\left(\kappa K \gamma_t^{lo}\right) - \partial^2_{K^2}\left(\frac{1}{2}\sigma(T,K)^2 K^2 \gamma_t^{lo}\right)\right)(T,K) = 0, T>t,\ K>0, \end{cases} \tag{8}$$

where $\boldsymbol{\delta}_S$ in the first line denotes a Dirac measure at S (for $T=t$, the conditional probability $\mathbb{Q}_t(S_T \leq K)$ is only differentiable in the sense of distributions).

Moreover, by the chain rule, European vanilla put prices in the local volatility model satisfy

$$\partial_T P_t^{lo} = -re^{-r(T-t)}\mathbb{E}_t(K-S_T)^+ + e^{-r(T-t)}\partial_T \mathbb{E}_t(K-S_T)^+,$$

[5] In particular, in a Black–Scholes model, but the Dupire equation is then of a purely anecdotical interest.
[6] About which we can refer to (Friedman, 1964, Chapter 1, Theorem 12, and Chapter 2, Theorem 10).

i.e., developing the second conditional expectation as in integral against $\gamma_t^{lo}(T,k)dk$ and then using the second line in (8),

$$\begin{aligned}\partial_T P_t^{lo} + rP_t^{lo} &= e^{-r(T-t)}\times \\ &\int_0^K (K-k)\Big(-\partial_k\big(\kappa k\gamma_t^{lo}(T,k)\big) + \partial^2_{k^2}\big(\frac{1}{2}\sigma(T,k)^2k^2\gamma_t^{lo}(T,k)\big)\Big)dk,\end{aligned} \tag{9}$$

where

$$\begin{aligned}&\int_0^K (K-k)\Big(-\partial_k\big(\kappa k\gamma_t^{lo}(T,k)\big)\Big)dk = \int_0^K (k-K)\partial_k\big(\kappa k\gamma_t^{lo}(T,k)\big)dk = \\ &\quad[(k-K)\big(\kappa k\gamma_t^{lo}(T,k)\big)]_0^K - \int_0^K \kappa k\gamma_t^{lo}(T,k)dk = -\int_0^K \kappa k\gamma_t^{lo}(T,k)dk\end{aligned} \tag{10}$$

and

$$\begin{aligned}&\int_0^K (K-k)\partial^2_{k^2}\big(\frac{1}{2}\sigma(T,k)^2k^2\gamma_t^{lo}(T,k)\big)dk \\ &= [(K-k)\partial_k\big(\frac{1}{2}\sigma(T,k)^2k^2\gamma_t^{lo}(T,k)\big)]_0^K + \int_0^K \partial_k\big(\frac{1}{2}\sigma(T,k)^2k^2\gamma_t^{lo}(T,k)\big)dk \\ &= \int_0^K \partial_k\big(\frac{1}{2}\sigma(T,k)^2k^2\gamma_t^{lo}(T,k)\big)dk = \frac{1}{2}\sigma(T,K)^2K^2\gamma_t^{lo}(T,K).\end{aligned} \tag{11}$$

Moreover, as $e^{-r(T-t)}\gamma_t^{lo}(T,k) = \partial^2_{k^2}P_t^{lo}(T,k)$ (cf. II.(7)),

$$\begin{aligned}&e^{-r(T-t)}\times\Big(-\int_0^K \kappa k\gamma_t^{lo}(T,k)dk\Big) = -\int_0^K \kappa k\partial^2_{k^2}P_t^{lo}(T,k)dk \\ &= -\kappa\big([k\partial_k P_t^{lo}(T,k)]_0^K - \int_0^K \partial_k P_t^{lo}(T,k)dk\big) \\ &= -\kappa\big(K\partial_K P_t^{lo}(T,K) - P_t^{lo}(T,K)\big)\end{aligned}$$

and

$$\begin{aligned}&e^{-r(T-t)}\times\frac{1}{2}\sigma(T,K)^2K^2\gamma_t^{lo}(T,K) \\ &\quad= \frac{1}{2}\sigma(T,K)^2K^2\partial^2_{K^2}P_t^{lo}(T,K).\end{aligned}$$

Substituting these expressions for $e^{-r(T-t)}\times$ (10) and (11) into (9) yields

$$\partial_T P_t^{lo} + rP_t^{lo} = -\kappa\big(K\partial_K P_t^{lo} - P_t^{lo}\big) + \frac{1}{2}\sigma(T,K)^2K^2\partial^2_{K^2}P_t^{lo},$$

which gives the Dupire equation for puts (with $(K-S)^+$ instead of $(S-K)^+$ in (7)). Moreover, the function $(Se^{-q\tau} - Ke^{-r\tau})$ also satisfies satisfies the second line in (7):

$$\begin{aligned}&(\partial_T + \kappa K\partial_K - \frac{1}{2}\sigma(T,K)^2K^2\partial^2_{K^2} + q)(Se^{-q\tau} - Ke^{-r\tau}) \\ &\quad - qSe^{-q\tau} + rKe^{-r\tau} + \kappa K(-e^{-r\tau}) + q(Se^{-q\tau} - Ke^{-r\tau}) = 0.\end{aligned} \tag{12}$$

Hence so does the function $C^{lo} = P^{lo} + Se^{-q\tau} - Ke^{-r\tau}$ (call/put parity), by linearity of the Dupire equation. ∎

The Dupire equation (7) is thus nothing but the Fokker-Planck equation integrated twice with respect to K.

Remark 2 *The Fokker-Planck equation* (8) *can be regarded as an analytic, weak form of the following integrated Itô formula:*

$$\mathbb{E}\left(\varphi(S_T)\,|\,S_t = S\right) = \varphi(S) + \int_t^T \mathbb{E}\left(\mathcal{A}^{lo}_S\varphi(S_s)\,|\,S_t = S\right) ds\,,\ \forall \text{ test-function } \varphi \tag{13}$$

(function φ of class $\mathcal{C}^2$ and with compact support in $\mathbb{R}_+$), where the generator $\mathcal{A}^{lo}$ of the process S in (1) *acts on φ as*

$$\mathcal{A}^{lo}_S\varphi(S) = \kappa S\partial_S\varphi(S) + \frac{1}{2}\sigma(t,S)^2S^2\partial^2_{S^2}\varphi(S). \tag{14}$$

Proof. that (13) is a weak formulation of (8) (assuming $\gamma^{lo}_t(T,K)$ $\mathcal{C}^{1,2}$ in (T,K).) In terms of γ^{lo}_t, the identity (13) is rewritten as

$$\begin{aligned}&\int_{K>0}\varphi(K)\gamma^{lo}_t(T,K)dK = \\ &\quad\varphi(S) + \int_t^T\int_{K>0}\mathcal{A}^{lo}_K\varphi(K)\gamma^{lo}_t(s,K)dKds,\end{aligned}$$

i.e., by spatial integrations by parts

$$\begin{aligned}&\int_{K>0}\varphi(K)\gamma^{lo}_t(T,K)dK = \\ &\quad\varphi(S) + \int_t^T\int_{K>0}\varphi(K)(\mathcal{A}^{lo,*}_K\gamma^{lo}_t)(s,K)dKds,\end{aligned} \tag{15}$$

where

$$(\mathcal{A}^{lo,*}_K\gamma^{lo}_t)(s,K) := -\partial_K\left(\kappa K\gamma^{lo}_t(s,K)\right) + \frac{1}{2}\partial^2_{K^2}\left(\sigma(s,K)^2K^2\gamma^{lo}_t(s,K)\right). \tag{16}$$

Differentiating (15) with respect to T yields

$$\begin{aligned}&\int_{K>0}\varphi(K)\partial_T\gamma^{lo}_t(T,K)dK = \\ &\int_{K>0}\varphi(K)(\mathcal{A}^{lo,*}_K\gamma^{lo}_t)(T,K)dK.\end{aligned} \tag{17}$$

In view of (16), this equation, to be satisfied for every regular test-function φ, is the weak form of the (second line in the) Fokker-Planck equation (8) for $(T,K)\mapsto\gamma^{lo}_t(T,K)$. ∎

Remark 3 *In terms of the reduced call prices $c_0(T,k) := e^{qT}C_0(T,K)$, where $k = Ke^{-\kappa T}$, the Dupire equation* (7) *at time $t=0$ is rewritten as*

$$\begin{aligned}&c_0(T=0,k) = (S_0-k)^+\,,\ k>0,\ \text{and, for } T>0,\\ &\partial_Tc_0(T,k) - \frac{1}{2}\sigma(T,K)^2k^2\partial^2_{k^2}c_0(T,k) = 0\,,\ k>0.\end{aligned} \tag{18}$$

Proof. Letting $v(T,K) = C_0(T,K)e^{qT}$, note that

$$\begin{aligned}&\partial_KC_0(T,K) = e^{-qT}\partial_Kv(T,K)\,,\ \partial^2_{K^2}C_0(T,K) = e^{-qT}\partial^2_{K^2}v(T,K)\\ &\partial_TC_0(T,K) = e^{-qT}\left(\partial_Tv(T,K) - qv(T,K)\right).\end{aligned}$$

The Dupire equation (second line in (7)) is then rewritten in terms of v as

$$\partial_Tv(T,K) = \frac{1}{2}\sigma^2(T,K)K^2\partial^2_{K^2}v(T,K) - \kappa K\partial_Kv(T,K). \tag{19}$$

Through the additional change of variables $c_0(T,k) = v(T,K)$, where $k = e^{-\kappa T}K$, we obtain

$$\begin{aligned}
\partial_K v(T,K) &= \partial_K c_0(T,k) = \partial_k c_0(T,k)\partial_K k = e^{-\kappa T}\partial_k c_0(T,k)\\
\partial^2_{K^2} v(T,K) &= \partial^2_{K^2} c_0(T,k) = e^{-2\kappa T}\partial^2_{k^2} c_0(T,k)\\
\partial_T v(T,K) &= \partial_T c_0(T,k) + \partial_T k \partial_k c_0(T,k) = \partial_T c_0(T,k) - \kappa k \partial_k c_0(T,k).
\end{aligned}$$

Hence by (19)

$$\partial_T c_0(T,k) - \kappa k \partial_k c_0(T,k) = \frac{1}{2}\sigma(T,K)^2 e^{2\kappa T} k^2 e^{-2\kappa T}\partial^2_{k^2} c_0(T,k) - \kappa k e^{\kappa T} e^{-\kappa T}\partial_k c_0(T,k),$$

which is the reduced Dupire equation (18). ∎

B Dupire Formula and Static Arbitrages

Theorem 1 can be rewritten in the form of the following ***Dupire formula***:

Corollary 1 *(written for time $t = 0$, for notational simplicity). For any continuous surface of call prices $C_0^\star(T,K), (T,K) \in [0,+\infty) \times (0,+\infty)$ in $C^{1,2}((0,+\infty) \times (0,+\infty))$ (for some real $p > 2$) such that $C_0^\star(0,\cdot) = (S_0 - \cdot)^+$ and the ratio*

$$\frac{(\partial_T + \kappa K \partial_K + q)C_0^\star}{K^2 \partial^2_{K^2} C_0^\star}(T,K) \tag{20}$$

defines a log-Lispchitz[7] function of (T,K) sitting between two positive bounds, there exists a unique log-Lispchitz local volatility function $\sigma_0^\star(\cdot,\cdot)$ such that $\frac{\sigma_0^\star(T,K)^2}{2}$ sits between the same bounds and the Dupire model with local volatility function $\sigma_0^\star(\cdot,\cdot)$ is perfectly calibrated to the full surface of call prices $C_0^\star$ at time 0, i.e.

$$C_0^{lo}(T,K; r, q, \sigma_0^\star(\cdot,\cdot)) = C_0^\star(T,K) \text{ holds for all } (T,K).$$

This function $\sigma_0^\star(\cdot,\cdot)$ is the (positive) function with the Dupire ratio (20) as $\frac{\sigma_0^\star(T,K)^2}{2}$.

This is just a reformulation of Theorem 1(ii) and the conditions in Corollary 1 are quite implicit.

In terms of the reduced prices $c_0^*(T,k) := e^{qT}C_0^*(T,K)$, where $k = Ke^{-\kappa T}$, the Dupire formula (20) is rewritten as (cf. (18))

$$\frac{\partial_T c_0^*(T,k)}{k^2 \partial^2_{k^2} c_0^*(T,k)}. \tag{21}$$

In Definition 1 we write $\mathcal{Q}$, instead of $\mathbb{Q}$ as usual, to emphasize that the corresponding "no-static-arbitrage" probability measure $\mathcal{Q}$ may not be a risk-neutral measure. In fact, $\mathcal{Q}$ does not need to be equivalent to the physical measure $\mathbb{P}$.

Definition 1 *The time-0 value s of the underlying stock S being assumed given (observed) in $\mathbb{R}_+$, a time-0 call prices surface $(C_0^\star(T,K))$ is deemed free from static arbitrage if it admits a representation of the form*

$$C_0^\star(T,K) = e^{-rT}\mathbb{E}^{\mathcal{Q}}(S_T - K)^+, \ (T,K) \geq 0, \tag{22}$$
for some probability measure $\mathcal{Q}$ under which αS is a nonnegative martingale starting from $S_0 = s$

[7] see after (2).

or, in a more convenient representation equivalent to (22) via $F_t = S_t e^{-\kappa t}$,

$$c_0^\star(T,k) = \mathbb{E}^{\mathcal{Q}}(F_T - k)^+ \,,\ (T,k) \geq 0, \tag{23}$$
for some probability measure $\mathcal{Q}$ under which F is a nonnegative martingale such that $F_0 = s$.

Corollary 1 provides a set of ***sufficient conditions*** for no static arbitrages in this sense, namely:

(i) *(no calendar spread arbitrage)* c_0^* non-decreasing in T,

(ii) *(no butterfly arbitrage)* c_0^* convex in k ,

(iii) *(initial condition)* $c_0^*(T=0,k) = (s-k)^+$,

and a Dupire ratio (21) well-defined and comprised between two positive bounds. In this case, a suitable, risk-neutral $\mathcal{Q} = \mathbb{Q}$ (also Markov) martingale is provided via the local volatility model (1) determined by the Dupire formula (20). The additional conditions

(iv) *(arbitrage bounds)* $(s-k)^+ \leq c_0^* \leq s$

(v) *(large strikes limit)* $\lim_{k\to\infty} c_0^*(T,k) = 0$

are then also satisfied, as it readily follows from (22).

More generally, it is not hard to verify (left to the reader) that the existence of a representation of the form (23) (or (22)) implies the five above-listed conditions. This and the converse statement are established in Roper (2010), building on Kellerer (1972)'s theorem (see also Hirsch and Roynette (2012)) asserting the existence of a Markovian martingale with given marginals $\mu(T,dF)$, provided the latter increase in the convex order (i.e. $\int_{\mathbb{R}} \phi(F)\mu(T,dF)$ is nondecreasing in T for any convex function ϕ)[8]:

Theorem 2 (Roper (2010)) *The existence of a representation of the form* (23) *(or* (22)*) is equivalent to the above-conditions (i)—(v).*

Proof. (of (22) or, equivalently, (23), assuming the five above-listed conditions, the converse statement having already been established above.) We first show, for each $T \geq 0$, the existence of a probability measure $\mu(T,dF)$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, such that

$$c_0^\star(T,k) = \int_{\mathbb{R}_+} (F-K)^+ \mu(T,dF) \,,\ k \geq 0.$$

Let c' denote the right-hand partial derivative of c_0^* with respect to k. Given the no butterfly arbitrage condition (ii), for each T, c' exists and is right-continuous and nondecreasing[9]. Moreover, the arbitrage bounds condition (iv) then implies that $-1 \leq c' \leq 0$, as well as

$$c_0^*(T,k=0) = s. \tag{24}$$

Hence for all $k \geq 0$

$$c_0^\star(T,k) = s + \int_0^k c'(T,F)dF \tag{25}$$

[8]See also Föllmer and Schied (2016, Theorem 7.25 page 413) for a self-contained proof (not using Kellerer's theorem) in discrete time.

[9](Föllmer and Schied, 2016, Proposition A.7 page 530).

and, via the large strikes limit condition (v),

$$\lim_{k\to\infty}\int_0^k (-c'(T,F))dF = s.$$

As $(-c'(T,F)) \geq 0$ is nonincreasing in F, we conclude that $\lim_{F\to\infty} c'(T,F) = 0$. Summing up,

$$\mu(T,F) := (1 + c'(T,F))\mathbb{1}_{F\geq 0} \tag{26}$$

is, for every fixed T, the cumulative distribution function of some random variable $F_T \geq 0$ (with probability law $\mu(T,dF)$). The expectation of F_T equals[10]

$$\begin{aligned}\int_{\mathbb{R}_+} F\mu(T,dF) &= \int_{F\geq 0}\Big(\int_{f=0}^{F} 1\cdot df\Big)\mu(T,dF) = \int_{f\geq 0}\Big(\int_{F\geq f}\mu(T,dF)\Big)df\\ &= \int_{f\geq 0}(1-\mu(T,f))df = \int_{\mathbb{R}_+}(-c'(T,f))df,\end{aligned} \tag{27}$$

by (26). Besides, for every fixed $k \geq 0$, (25) and (26) yield

$$\begin{aligned}c_0^*(T,k) &= s + \int_0^k c'(T,F)dF = s - \int_0^k (1-\mu(T,F))dF\\ &= s - \int_{F=0}^{k}\int_{f\geq F}\mu(T,df)dF = s - \int_{f\geq 0}\mu(T,df)\int_{F=0}^{k}\mathbb{1}_{f\geq F}dF\\ &= s - \int_{f\geq 0}\mu(T,df)(k\wedge f).\end{aligned} \tag{28}$$

In addition, (24) and the large strikes limit condition (v) yield

$$s = c_0^*(T,0) = c_0^*(T,0) - \lim_{k\to\infty} c_0^*(T,k) = -\int_0^\infty c'(T,F)dF = \int_{\mathbb{R}_+} F\mu(T,dF),$$

by (27). Therefore (28) yields

$$c_0^*(T,k) = \int_{\mathbb{R}_+}(f - k\wedge f)\mu(T,df) = \int_{\mathbb{R}_+}(f-k)^+\mu(T,df). \tag{29}$$

Hence for every $k \geq 0$ and $0 \leq \Theta \leq T$ we have

$$\int_{\mathbb{R}_+}(f-k)^+\mu(\Theta,df) = c_0^*(\Theta,k) \leq c_0^*(T,k) = \int_{\mathbb{R}_+}(f-k)^+\mu(T,df),$$

where the inequality is ensured by the no calendar spread arbitrage condition (i). By Föllmer and Schied (2016, Corollary 2.61 page 98), μ is therefore nondecreasing in the convex order. The passage from (29) to the existence of a representation of the form (23) for the surface $(c_0^*(T,k))$ is then provided by an application of Kellerer (1972)'s theorem. The initial condition (iii) then implies that $F_0 = s$. ∎

C Gyönfi Equation and Formula

Proposition 1 *In a model*[11] $dS_t = \kappa S_t dt + \sigma_t S_t dW_t$ *with stochastic volatility* σ *such that*

$$\mathbb{E}\int_0^T \sigma_t^2 S_t^2 dt < \infty \tag{30}$$

and $\mathbb{E}\int_0^T S_t dt < \infty$ *hold for all* $T > 0$, *if* $\sigma_{gy}^2(t,S) := \mathbb{E}(\sigma_t^2 \,|\, S_t = S)$ *and* $\gamma_0(t,S) = \partial_S \mathbb{Q}(S_t \leq S)$ *are both well-defined and such that the function* $k \mapsto \int_0^T \sigma_{gy}^2(t,k)k^2\gamma_0(t,k)dt$ *is finite and continuous, then the corresponding time-0 European vanilla call and put* $\mathbb{Q}$ *prices,* $C_0(T,K)$ *and* $P_0(T,K)$, *satisfy the Dupire equation for the local volatility function* $\sigma_{gy}(t,S)$.

[10] This is the classical identity $\mathbb{E}X = \int_{\mathbb{R}_+}(1 - cdf_X(x))dx$, for every nonnegative integrable random variable X.

[11] $\mathbb{Q}$ arbitrage-free by application of Lemma II.1.

Proof. We first show the result for puts, resorting to the following twice differentiable approximation of the function $x \mapsto x^-$ (see Figure 2):

$$\varphi_\epsilon(x) = (-x)\mathbb{1}_{(-\infty,-\epsilon/2]}(x) + \frac{(x-\epsilon/2)^2}{2\epsilon}\mathbb{1}_{(-\epsilon/2,\epsilon/2)}(x),$$

worth $(-x)$ before $(-\epsilon/2)$ and 0 after $\epsilon/2$, hence

$$\dot\varphi_\epsilon(x) = (-1)\mathbb{1}_{(-\infty,-\epsilon/2]}(x) + \frac{(x-\epsilon/2)}{\epsilon}\mathbb{1}_{(-\epsilon/2,\epsilon/2)}(x)\,,\ \ddot\varphi_\epsilon(x) = \epsilon^{-1}\mathbb{1}_{(-\epsilon/2,\epsilon/2)}(x).$$

The Itô formula (which, as φ_ϵ is convex, holds even though $\ddot\varphi_\epsilon$ is discontinuous at $(-\epsilon/2)$ and 0[12]) yields

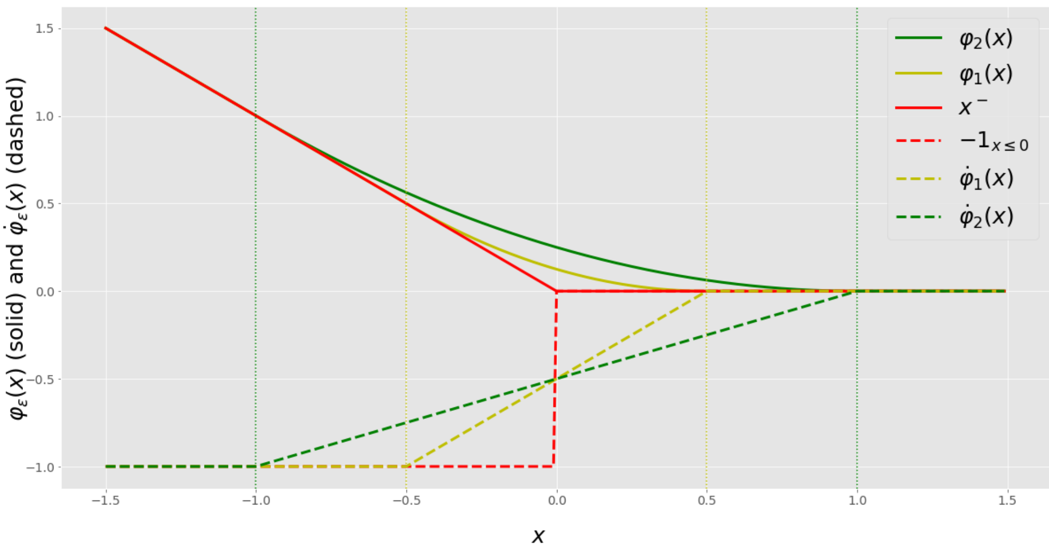


Figure 2: *Twice differentiable approximations of x^- and their derivatives.*

$$\begin{aligned}\varphi_\epsilon(S_T-K) - \varphi_\epsilon(S_0-K) &= \int_0^T \dot\varphi_\epsilon(S_t-K)dS_t \\ &+ \frac{1}{2}\int_0^T \ddot\varphi_\epsilon(S_t-K)\sigma_t^2 S_t^2 dt.\end{aligned} \tag{31}$$

By assumption that $\mathbb{E}\int_0^T \sigma_t^2 S_t^2 dt < \infty$, $\int_0^{\cdot} \dot\varphi_\epsilon(S_t-K)\sigma_t S_t dW_t$ is a true martingale. Hence taking expectation in (31) yields

$$\begin{aligned}\mathbb{E}\varphi_\epsilon(S_T-K) - \varphi_\epsilon(S_0-K) &= \kappa\mathbb{E}\int_0^T [\dot\varphi_\epsilon(S_t-K)S_t]dt \\ &+ \frac{1}{2}\int_0^T \mathbb{E}\ddot\varphi_\epsilon(S_t-K)\sigma_t^2 S_t^2 dt.\end{aligned} \tag{32}$$

As $\lim_{\epsilon\downarrow 0}\varphi_\epsilon(x) = x^-$ with $|\varphi_\epsilon(x)| \le x^- + \varphi_\epsilon(0) = x^- + \frac{\epsilon}{8} \le x^- + 1$ for $\epsilon \le 8$, where $(S_T-K)^- + 1$ is bounded and therefore $\mathbb{Q}$ integrable, we have by the dominated convergence theorem:

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\varphi_\epsilon(S_T-K) = \mathbb{E}(S_T-K)^- = e^{rT}P_0(T,K)$$

(and $\lim_{\epsilon\downarrow 0}\varphi_\epsilon(S_0-K) = (S_0-K)^- = P_0(0,K)$). Likewise, as $|\dot\varphi_\epsilon| \le 1$ and $\lim_{\epsilon\downarrow 0}\dot\varphi_\epsilon(x) = -\mathbb{1}_{x\le 0}$, we have by the dominated convergence theorem (having assumed $\mathbb{E}\int_0^T S_t dt < \infty$) and the Breeden

[12] See e.g. Karatzas and Shreve (1991, Theorem 3.7.3).

Litzenberger relation $\gamma_0(t,k) = e^{rt}\partial^2_{k^2}P_0$[13]:

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\int_0^T [\dot{\varphi}_\epsilon(S_t-K)S_t]dt = -\mathbb{E}\int_0^T [\mathbb{1}_{S_t\le K}S_t]dt = -\int_0^T\int_0^K k\gamma_0(t,k)dkdt$$
$$= -\int_0^T e^{rt}\int_0^K k\partial^2_{k^2}P_0 dkdt = -\int_0^T e^{rt}\big(K\partial_K P_0(t,K) - P_0(t,K)\big)dt,$$

by integrations by parts in the k variable. Moreover,

$$\mathbb{E}\int_0^T \big[\ddot{\varphi}_\epsilon(S_t-K)\sigma_t^2S_t^2\big]dt = \epsilon^{-1}\mathbb{E}\int_0^T \big(\mathbb{1}_{-\epsilon/2<S_t-K<\epsilon/2}S_t^2\sigma_t^2\big)dt =$$
$$\epsilon^{-1}\int_0^T \mathbb{E}\big[\mathbb{1}_{-\epsilon/2<S_t-K<\epsilon/2}S_t^2\mathbb{E}(\sigma_t^2\,|\,S_t)\big]dt =$$
$$\epsilon^{-1}\int_0^T \mathbb{E}\big[\mathbb{1}_{-\epsilon/2<S_t-K<\epsilon/2}S_t^2\sigma_{gy}^2(t,S_t)\big]dt$$
$$= \epsilon^{-1}\int_{K-\epsilon/2}^{K+\epsilon/2}\int_0^T \sigma_{gy}^2(t,k)k^2\gamma_0(t,k)dkdt,$$

which, having assumed $\int_0^T \sigma_{gy}^2(t,k)k^2\gamma_0(t,k)dt$ well-defined and continuous in k, converges to $\int_0^T \sigma_{gy}^2(t,K)K^2\gamma_0(t$ $\int_0^T e^{rt}\sigma_{gy}^2(t,K)K^2\partial^2_{K^2}P_0(t,K)dt$ as $\epsilon\downarrow 0$.

Hence, sending ϵ to 0 in (32) yields

$$e^{rT}P_0(T,K) - P_0(0,K) = -\kappa\int_0^T e^{rt}\big(K\partial_K P_0(t,K) - P_0(t,K)\big)dt$$
$$+\frac{1}{2}\int_0^T e^{rt}\sigma_{gy}^2(t,K)\partial^2_{K^2}P_0(t,K)dt.$$

Differentiating with respect to T then yields

$$e^{rT}(\partial_T P_0 + rP_0) = -\kappa e^{rT}\big(K\partial_K P_0 - P_0\big)$$
$$+\frac{1}{2}e^{rT}\sigma_{gy}^2(T,K)K^2\partial^2_{K^2}P_0(T,K),$$

which is the Dupire equation for puts for the volatility function σ_{gy}. Forward prices also satisfy the Dupire equation (for any volatility function, cf. (12)). Hence, by linearity of the Dupire equation and call/put parity[14], call prices satisfy the Dupire equation for the volatility function σ_{gy}. ∎

In the special case of a local volatility model, we have $\sigma_{gy} = \sigma$. Hence the above provides an alternative proof of the Dupire equation in a local volatility model, which is the proof by Derman and Kani (1994). This probabilistic proof is less demanding in terms of assumptions than the analytic (Fokker-Planck equation based) proof of Theorem 1, essentially requiring a continuous transition probability density γ_0 of (S_t), while the analytic proof needs a $\mathcal{C}^{1,2}$ density.

Proposition 1 generalizes the Dupire equation to stochastic volatility models. Applications include the calibration of local stochastic volatility models[15] or the derivation of robust bounds on exotic prices in calibrated stochastic volatility models. For instance, for a call on σ_T^2 (which, for small T, can be seen as a proxy on a call on the realized variance of S), the conditional Jensen inequality gives in any calibrated stochastic volatility model satisfying the assumptions of Proposition 1:

$$\mathbb{E}\big[(\sigma_T^2-K)^+\big] = \mathbb{E}\mathbb{E}\big[(\sigma_T^2-K)^+\big|S_T\big] \ge \mathbb{E}\Big[\big(\mathbb{E}(\sigma_T^2|S_T)-K)^+\big)\Big]$$
$$= \mathbb{E}\big[(\sigma_{gy}^2(T,S_T)-K)^+\big] = \mathbb{E}\big[(\sigma_{gy}^2(T,S_T^{gy})-K)^+\big],$$

[13] see (7).
[14] having assumed (30).
[15] see e.g. Homescu (2014, Section 4.1.2).

where S^{gy} denotes the model with local volatility σ_{gy} and the last equality is due to the fact that the processes S and S^{gy} have the same marginal distributions. Hence, the calibrated local volatility price of the call is a lower bound on the prices of the call in all the co-calibrated stochastic volatility models.

See Bentata (2012) for further developments around such notions of Markovian projection (in the above case, of a stochastic volatility model into a local volatility model with the same European vanilla call/put prices, i.e. the same marginals, as the original model).

§2 Gatheral Equation and Formulas

Our next goal is to find direct connections between a local volatility function and the corresponding implied volatility surface. By default, the reference model is the local volatility model (1) (with σ log-Lispschitz and between two positive bounds), with generator $\mathcal{A}_S = \kappa S\partial_S + \frac{1}{2}\sigma(t,S)^2S^2\partial^2_{S^2}$ and with density $\gamma_0(t,S)$ of S_t (density known to exist for almost every t).

A Analysis at Fixed (T,K)

We fix the characteristics (T,K) of a European vanilla option. We denote its time-0 implied volatility by Σ_0. An index $\cdot^{bs}$ refers to the Black-Scholes model with constant volatility Σ_0, with generator $\mathcal{A}^{bs}_S = \kappa S\partial_S + \frac{1}{2}\Sigma_0^2 S^2\partial^2_{S^2}$.

Proposition 2 *We have*

$$\Sigma_0^2 = \iint_{[0,T]\times(0,+\infty)} w_0(t,S)\sigma^2(t,S)dtdS,$$

for weights $w_0(t,S)$ proportional to $e^{-rt}S^2\Gamma^{bs}(t,S)\gamma_0(t,S)$ on the integration domain $[0,T]\times(0,+\infty)$, i.e.

$$w_0(t,S) = \frac{e^{-rt}S^2\Gamma^{bs}(t,S)\gamma_0(t,S)}{\iint_{[0,T]\times(0,+\infty)} e^{-rs}S^2\Gamma^{bs}(s,k)\gamma_0(s,k)dsdk}, \ \forall (t,S)\in[0,T]\times(0,+\infty).$$

Proof. The equations for the option pricing functions $u = u(t,S)$ and $u^{bs} = u^{bs}(t,S)$ in the respective local volatility σ and constant volatility Σ_0 models yield (cf. Theorem 1(i))

$$(\partial_t + \mathcal{A}_S - r)u = (\partial_t + \mathcal{A}^{bs}_S - r)u^{bs} = 0.$$

Hence the function $\delta u = u - u^{bs}$ satisfies

$$(\partial_t + \mathcal{A}_S - r)(\delta u) + \frac{1}{2}(\sigma^2 - \Sigma_0^2)\Gamma^{bs}S^2 = 0.$$

The Itô formula then yields

$$\begin{aligned} d\big(e^{-rt}(\delta u)(t,S_t)\big) &= e^{-rt}(\partial_t + \mathcal{A}_S - r)(\delta u)(t,S_t)dt \\ &\quad + e^{-rt}\partial_S(\delta u)(t,S_t)\sigma(t,S_t)S_t dW_t. \end{aligned} \tag{33}$$

As $\delta u(T,\cdot) = 0$, this implies the following Feynman-Kac representation for the time-0 price difference (assuming the stochastic integral in the above is a true martingale):

$$(\delta u)(0,S_0) = \frac{1}{2}\mathbb{E}\int_0^T e^{-rt}(\sigma^2(t,S_t) - \Sigma_0^2)\Gamma^{bs}(t,S_t)S_t^2 dt.$$

But $u^{bs}(0, S_0; \Sigma_0) = u(0, S_0)$, i.e. $\delta u(0, S_0) = 0$, hence we deduce from (33) that

$$\begin{aligned}\Sigma_0^2 &= \frac{\mathbb{E}\int_0^T e^{-rt}\sigma^2(t, S_t)\Gamma^{bs}(t, S_t)S_t^2 dt}{\mathbb{E}\int_0^T e^{-rt}\Gamma^{bs}(t, S_t)S_t^2 dt} \\ &= \frac{\iint_{[0,T]\times(0,+\infty)} e^{-rt}\sigma^2(t, S)\Gamma^{bs}(t, S)S^2\gamma_0(t, S)dSdt}{\iint_{[0,T]\times(0,+\infty)} e^{-rt}\Gamma^{bs}(t, S)S^2\gamma_0(t, S)dSdt}. \blacksquare\end{aligned}$$

For more details see (Gatheral, 2011, p.25–31).

B Analysis at Fixed (t, S)

What follows (most of the notation included, e.g. f_x for the partial derivative of a function f with respect to x) is drawn from Berestycki, Busca, and Florent (2002) (see also Gatheral (2011)). Given (t, S, T, K), let

$$\tau = T - t\,,\ x = \ln(S/K) + \kappa(T - t) \tag{34}$$

denote the time-to-maturity and the log-forward moneyness of the (T, K) call at time t when $S_t = S$.

Given $S_t = S$, we denote by

$$\varsigma(\tau, x) = \sigma(T, K)\,,\ v(\tau, x) = e^{r(T-\tau)}C_t^{lo}(T, K)/K \tag{35}$$

the local volatility in the (τ, x) variables and the “reduced price”, in the (τ, x) variables, of the European vanilla call of maturity T and strike K.

Lemma 1 *The function v satisfies the following “reduced Dupire equation”:*

$$v_{|\tau=0} = (e^x - 1)^+\,,\ v_\tau = \frac{1}{2}\varsigma^2(\tau, x)(v_{xx} - v_x). \tag{36}$$

Proof. Writing $C(T, K)$ for $C_t^{lo}(T, K)$, we have through (34)

$$v(\tau = 0, x = \ln(S/K)) = C(T = t, K)/K = (S/K - 1)^+ = (e^x - 1)^+$$

and

$$\begin{aligned}&e^{r(T-t)}C(T, K) = Kv(\tau, x)\\ &e^{r(T-t)}(rC + \partial_T C) = K(\partial_\tau v + \partial_x v\, \partial_T x) = K(\partial_\tau v + \kappa\partial_x v)\\ &e^{r(T-t)}\partial_K C = v + K\partial_x v\partial_K x = v + K\partial_x v(\frac{-1}{K}) = v - \partial_x v\\ &e^{r(T-t)}\partial^2_{K^2} C = (\frac{-1}{K})\partial_x v + (\frac{1}{K})\partial^2_{x^2} v = \frac{1}{K}(\partial^2_{x^2} - \partial_x)v.\end{aligned}$$

Hence by the Dupire equation (7)

$$\begin{aligned}0 =& e^{r(T-t)}\big(\partial_T C + \kappa K\partial_K C - \frac{1}{2}\sigma(T, K)^2K^2\partial^2_{K^2}C + qC\big)(T, K)\\ &= e^{r(T-t)}\big((\partial_T C + rC) + \kappa K\partial_K C - \frac{1}{2}\sigma(T, K)^2K^2\partial^2_{K^2}C - \kappa C\big)(T, K)\\ &= \big(K(\partial_\tau v + \kappa\partial_x v) + \kappa K(v - \partial_x v) - \frac{1}{2}\varsigma^2 K(\partial^2_{x^2}v - \partial_x v) - \kappa Kv\big)(\tau, x)\\ &= K\big(\partial_\tau v - \frac{1}{2}\varsigma^2(\partial^2_{x^2}v - \partial_x v)\big)(\tau, x). \blacksquare\end{aligned}$$

Let

$$u(\theta, x) := e^x\mathcal{N}(\delta_+(\theta, x)) - \mathcal{N}(\delta_-(\theta, x)),\ \text{where } \delta_\pm(\theta, x) = \frac{x}{\sqrt{\theta}} \pm \frac{\sqrt{\theta}}{2}. \tag{37}$$

Lemma 2 **(i)** *u coincides with the function v corresponding to $\varsigma \equiv 1$.*
(ii) *Denoting by $\varphi(\tau, x)$ the time-t implied volatility of the European vanilla option with time-to-maturity τ and call log-forward moneyness x (in the reference local volatility model (1)), we have*

$$v(\tau, x) = u(\varphi^2(\tau, x)\tau, x). \tag{38}$$

Proof. (i) We recognize in (37) the reduced price (cf. (35)) of a call option for a volatility set to one, i.e. u coincides with the function v for $\varsigma \equiv 1$.
(ii) By (37),

$$\begin{aligned}
&u(\varphi^2(\tau, x)\tau, x) \\
&= e^x \mathcal{N}\Big(\frac{x}{\varphi(\tau, x)\sqrt{\tau}} + \frac{\varphi(\tau, x)\sqrt{\tau}}{2}\Big) - \mathcal{N}\Big(\frac{x}{\varphi(\tau, x)\sqrt{\tau}} - \frac{\varphi(\tau, x)\sqrt{\tau}}{2}\Big) \\
&= e^{r\tau} C^{bs}\big(t, S, T, K; r, q, \varphi(\tau, x)\big)/K = e^{r(T-\tau)} C_t^{lo}(T, K)/K = v(\tau, x)
\end{aligned}$$

where (34) and the Black-Scholes call pricing formula (first line in (14)) were used for passing to the last line, while the last two identities hold by definition of the implied volatility and by definition (35) of v. ∎

Lemma 3 *We have*

$$\begin{aligned}
\frac{u_{\theta x}}{u_\theta} &= \frac{1}{2} - \frac{x}{\theta} \\
\frac{u_{\theta\theta}}{u_\theta} &= \frac{x^2}{2\theta^2} - \frac{1}{8} - \frac{1}{2\theta}
\end{aligned} \tag{39}$$

Proof. Exploiting the Θ^{bs} formula II.(14) and the identity $n'(y) = -yn(y)$, we get

$$\begin{aligned}
u_\theta &= e^x n(\delta_+(\theta, x))\frac{1}{2\sqrt{\theta}} \\
\frac{u_{\theta x}}{u_\theta} &= 1 - \delta_+(\theta, x)\partial_x(\delta_+(\theta, x)) = 1 - \Big(\frac{x}{\sqrt{\theta}} + \frac{\sqrt{\theta}}{2}\Big)\frac{1}{\sqrt{\theta}} = \frac{1}{2} - \frac{x}{\theta} \\
\frac{u_{\theta\theta}}{u_\theta} &= -\frac{1}{2\theta} - \delta_+(\theta, x)\partial_\theta(\delta_+(\theta, x)) = -\frac{1}{2\theta} - \frac{1}{2}\partial_\theta(\delta_+^2(\theta, x)) = \\
&\quad -\frac{1}{2\theta} - \frac{1}{2}\Big(\frac{-x^2}{\theta^2} + \frac{1}{4}\Big) = \frac{x^2}{2\theta^2} - \frac{1}{8} - \frac{1}{2\theta}. \ \blacksquare
\end{aligned}$$

Lemma 4 *For any function $\psi = \psi(\tau, x)$, denoting $w(\tau, x) = u\big(\psi^2(\tau, x)\tau, x\big)$, we have*

$$\begin{aligned}
&w_\tau - \frac{1}{2}\varsigma^2(\tau, x)(w_{xx} - w_x) = u_\theta(\psi^2\tau, x)\times \\
&\quad \Big(2\tau\psi\psi_\tau + \psi^2 - \varsigma^2 \times \big((1 - \frac{x\psi_x}{\psi})^2 + \tau\psi\psi_{xx} - \frac{1}{4}\tau^2\psi^2\psi_x^2\big)\Big).
\end{aligned} \tag{40}$$

Proof. We abbreviate $\psi^2(\tau, x)\tau = \vartheta$. The (ordinary) chain differentiation rule yields

$$\begin{aligned}
w_\tau &= u_\theta(\vartheta, x)(\psi^2 + 2\tau\psi\psi_\tau) \\
w_x &= u_\theta(\vartheta, x)2\tau\psi\psi_x + u_x(\vartheta, x) \\
w_{xx} &= \big(u_{\theta\theta}(\vartheta, x)2\tau\psi\psi_x + u_{\theta x}(\vartheta, x)\big)2\tau\psi\psi_x + u_\theta(\vartheta, x)2\tau(\psi_x^2 + \psi\psi_{xx}) \\
&\quad + u_{xx}(\vartheta, x) + u_{\theta x}(\vartheta, x)2\tau\psi\psi_x.
\end{aligned}$$

Hence, it comes by Lemma 3:

$$
\begin{aligned}
&\big(w_\tau - \frac{1}{2}\varsigma^2(\tau,x)(w_{xx}-w_x)\big)/u_\theta(\vartheta,x) = \psi^2 + 2\tau\psi\psi_\tau - \varsigma^2(\tau,x)\times \\
&\frac{1}{2}\Big\{\big[(\frac{x^2}{2\vartheta^2} - \frac{1}{8} - \frac{1}{2\vartheta})2\tau\psi\psi_x + (\frac{1}{2} - \frac{x}{\vartheta})\big]2\tau\psi\psi_x + 2\tau(\psi_x^2 + \psi\psi_{xx}) \\
&\quad + \frac{u_{xx}}{u_\theta} + (\frac{1}{2} - \frac{x}{\vartheta})2\tau\psi\psi_x - (2\tau\psi\psi_x + \frac{u_x}{u_\theta})\Big\},
\end{aligned}
$$

where $\frac{u_{xx}}{u_\theta} - \frac{u_x}{u_\theta} = 2$, by (36) with ς set equal to one. Hence, the $\frac{1}{2}\{\ldots\}$ term in the above equates

$$
\begin{aligned}
&\big[(\frac{x^2}{2\vartheta^2} - \frac{1}{8} - \frac{1}{2\vartheta})2\tau\psi\psi_x + (\frac{1}{2} - \frac{x}{\vartheta})\big]\tau\psi\psi_x + \tau(\psi_x^2 + \psi\psi_{xx}) \\
&\quad + 1 + (\frac{1}{2} - \frac{x}{\vartheta})\tau\psi\psi_x - \tau\psi\psi_x \\
&= (\frac{x^2}{2\vartheta^2} - \frac{1}{8} - \frac{1}{2\vartheta})2\tau^2\psi^2\psi_x^2 - \frac{x}{\vartheta}\tau\psi\psi_x + \tau(\psi_x^2 + \psi\psi_{xx}) + 1 - \frac{x}{\vartheta}\tau\psi\psi_x.
\end{aligned}
$$

Substituting $\psi^2\tau$ for ϑ, we obtain for this $\frac{1}{2}\{\ldots\}$ term the value

$$
\begin{aligned}
&(\frac{x^2}{2\vartheta^2} - \frac{1}{8} - \frac{1}{2\vartheta})2\tau^2\psi^2\psi_x^2 - \frac{x}{\vartheta}\tau\psi\psi_x + \tau(\psi_x^2 + \psi\psi_{xx}) + 1 - \frac{x}{\vartheta}\tau\psi\psi_x \\
&= (\frac{x^2}{2\psi^4\tau^2} - \frac{1}{8} - \frac{1}{2\psi^2\tau})2\tau^2\psi^2\psi_x^2 \\
&\qquad - \frac{x}{\psi^2\tau}\tau\psi\psi_x + \tau(\psi_x^2 + \psi\psi_{xx}) + 1 - \frac{x}{\psi^2\tau}\tau\psi\psi_x \\
&= \frac{x^2}{\psi^2}\psi_x^2 - \frac{1}{4}\tau^2\psi^2\psi_x^2 - \tau\psi_x^2 - \frac{x}{\psi}\psi_x + \tau(\psi_x^2 + \psi\psi_{xx}) + 1 - \frac{x}{\psi}\psi_x \\
&= (1 - \frac{x\psi_x}{\psi})^2 + \tau\psi\psi_{xx} - \frac{1}{4}\tau^2\psi^2\psi_x^2,
\end{aligned}
$$

which proves (40). ∎

Theorem 3 *The implied volatilities $\varphi(\tau,x)$ satisfy*
(i) *the following Gatheral equation: For $\tau > 0$,*

$$2\tau\varphi\varphi_\tau + \varphi^2 - \varsigma^2\big((1 - \frac{x\varphi_x}{\varphi})^2 + \tau\varphi\varphi_{xx} - \frac{1}{4}\tau^2\varphi^2\varphi_x^2\big) = 0, \tag{41}$$

i.e. the following Gatheral formula holds:

$$\varsigma^2 = \frac{2\tau\varphi\varphi_\tau + \varphi^2}{(1 - \frac{x\varphi_x}{\varphi})^2 + \tau\varphi\varphi_{xx} - \frac{1}{4}\tau^2\varphi^2\varphi_x^2}, \tag{42}$$

(ii) *along with the following initial condition:*

$$\lim_{\tau\to 0}\varphi^{-1}(\tau,x) = \frac{1}{x}\int_0^x \varsigma^{-1}(0,z)dz = \int_0^1 \varsigma^{-1}(0,yx)dy, \tag{43}$$

i.e. the short-term implied volatility is the harmonic mean of the local volatility of zero time-to-maturity.

Proof. **(i)** For $\psi = \varphi$, one has

$$w(\tau,x) = u(\varphi(\tau,x)\tau,x) = v(t,x),$$

by (38), where v satisfies (36). Hence (40) then implies (41).

(ii) *(idea).* The function $\phi(x) := \big(x^{-1}\int_0^x \varsigma^{-1}(0,z)dz\big)^{-1} = \big(\int_0^1 \varsigma^{-1}(0,yx)dy\big)^{-1}$ satisfies

$$-\partial_x\phi = \phi^2\partial_x\int_0^1 \varsigma^{-1}(0,yx)dy = -\phi^2\int_0^1 \frac{\partial_2\varsigma}{\varsigma^2}(0,xy)ydy.$$

Hence

$$-x\partial_x\phi = \phi^2\int_0^1 y\partial_y\varsigma^{-1}(0,xy)dy.$$

An integration by parts yields

$$\int_0^1 y\partial_y\varsigma^{-1}(0,xy)dy = [y\varsigma^{-1}(0,xy)]_{y=0}^1 - \int_0^1 \varsigma^{-1}(0,xy)dy = \varsigma^{-1}(0,x) - \phi^{-1}(x).$$

Therefore ϕ satisfies the differential equation $-\frac{x}{\phi}\partial_x\phi = \frac{\phi}{\varsigma} - 1$, i.e. $1 - \frac{x}{\phi}\partial_x\phi = \frac{\phi}{\varsigma}$, i.e. $\varsigma^2 = \frac{\phi^2}{(1-\frac{x}{\phi}\partial_x\phi)^2}$, which is the limit of the Gatheral formula (41) as $\tau \to 0$. ∎

Remark 4 *As a sanity check for Theorem 3(i), note that it is consistent with the usual formula $\varphi^2(\tau) = \frac{1}{\tau}\int_0^\tau \varsigma^2(s)ds$ in a time-dependent Black-Scholes model, where the Gatheral formula (42) degenerates to*

$$\varsigma^2 = 2\tau\varphi\varphi_\tau + \varphi^2 = \partial_\tau(\varphi^2\tau),$$

i.e., for any fixed $\bar{T} \geq t$,

$$\varphi^2(\bar{T}-t)(\bar{T}-t) = \int_{\tau=0}^{\bar{T}-t} \varsigma^2(\tau)d\tau = \int_{T=t}^{\bar{T}} \varsigma^2(T-t)dT,$$

i.e. via (34)-(35) and the definition of φ[16]:

$$\Sigma^2(\bar{T})(\bar{T}-t) = \int_{T=t}^{\bar{T}} \sigma^2(T)dT,$$

consistent with the last expression in Remark II.6.

See Berestycki, Busca, and Florent (2002) for the rigorous proof of part (ii), and the fact that the implied volatility φ is the *unique* solution to (41) and (43).

Corollary 2 *The short-term at-the-money implied volatility and local volatility coincide, i.e.*

$$\lim_{\tau\to 0}\varphi(\tau,0) = \varsigma(0,0), \tag{44}$$

and the following half-slope rule holds:

$$\lim_{\tau\to 0}\varphi_x(\tau,x) = \frac{1}{2}\varsigma_x(0,x). \tag{45}$$

Proof. By application of (43) at $x = 0$, for (44), and of (43) and Lemma 5 to $\zeta(z) = \varsigma(0,z)$ and $\psi(x) = \lim_{\tau\to 0}\varphi(\tau,x)$, regarding (45). ∎

Lemma 5 *A harmonic mean function $\psi(x) = \big(\frac{1}{x}\int_0^x \zeta^{-1}(z)dz\big)^{-1}$ satisfies $\psi'(0) = \frac{1}{2}\zeta'(0)$ (provided $\zeta'(0)$ exists).*

[16] see Lemma 2(ii).

Proof. We have:

$$\frac{1}{x}\big(\psi(x)-\psi(0)\big)=\frac{1}{x}\Big(\big(\frac{1}{x}\int_0^x\zeta^{-1}(z)dz\big)^{-1}-\big(\frac{1}{x}\int_0^x\zeta^{-1}(0)dz\big)^{-1}\Big)$$
$$=\frac{1}{x}\frac{\frac{1}{x}\int_0^x\zeta^{-1}(0)dz-\frac{1}{x}\int_0^x\zeta^{-1}(z)dz}{\zeta^{-1}(0)\frac{1}{x}\int_0^x\zeta^{-1}(z)dz},$$

which for $x\sim 0$ is equivalent to

$$\frac{1}{x^2\zeta^{-2}(0)}\int_0^x z(-\zeta^{-1})'(0)dz=\frac{1}{x^2\zeta^{-2}(0)}\big(\frac{\zeta'}{\zeta^2}\big)(0)\int_0^x zdz=\frac{1}{2}\zeta'(0).$$

Hence $\psi'(0)=\frac{1}{2}\zeta'(0)$. ∎

In view of Corollary 2, *local volatility models are dynamically inconsistent with persistently smiled (e.g. FX) derivative markets.* In fact, as Figure 3 illustrates, Corollary 2 implies that an implied volatility smile at time 0 tends to become a positive, respectively negative, skew around the new stock position in a bullish, respectively bearish scenario (stock increase, respectively decrease). By contrast, see the top panel in Figure 5, labelled as "sticky implied tree", for the dynamics of implied volatilities embedded in a local volatility model. Here there is no apparent dynamic inconsistency: local volatility models may be dynamically consistent with persistently skewed (e.g. equity index) derivative markets. In conclusion, local volatility models may be fine for dealing with predominantly skewed (such as equity index) option markets, but they are dynamically wrong, leading to flawed sensitivities and hedging strategies, regarding predominantly smiled (such as FX) option markets.

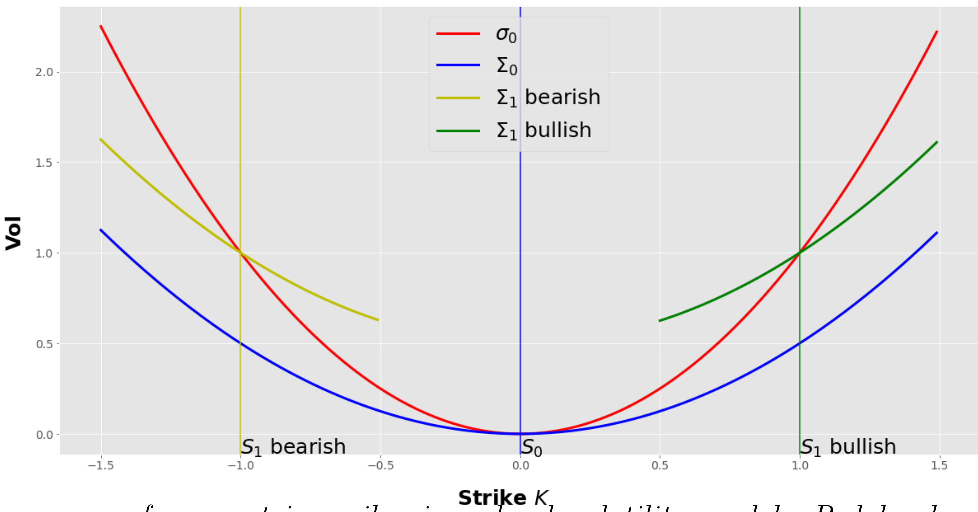


Figure 3: *Evanescence of symmetric smiles in a local volatility model: Red local volatility function calibrated (geometrically by the half-slope rule at time 0) to the initial implied volatility smile in blue, and future negative, respectively positive, implied volatility skews predicted by the local volatility model in bearish (yellow), respectively bullish (green), scenarios.*

Remark 5 *For a tentative restatement in terms of time scaled implied volatilities $\varphi\sqrt{\tau}$, based on* (41)-(42), *of the necessary and sufficient conditions listed in Section §1.B for a call price surface to be free from static arbitrage, see Roper (2010, Section 2.2). In many market models, various asymptotics regarding the behavior of the implied volatility at small and large maturity or strike are also available, such as the implied variance (implied volatility squared) being asymptotically linear with respect to the log-strike in the limits where the latter goes to $\pm\infty$ (Lee, 2004).*

§3 Extracting the Local Volatility

The Dupire formula (20) is only constructive in appearance. In fact, the local volatility calibration problem is an ill-posed inverse problem and this formula is unstable numerically.

To appreciate this, we consider a DAX index options real data set consisting of about 300 European vanilla option prices distributed throughout 6 maturities with moneyness $K/S_0 \in [0.8, 1.2]$, corresponding to the implied volatility surface displayed in the bottom panel of Figure 4. A naive numerical

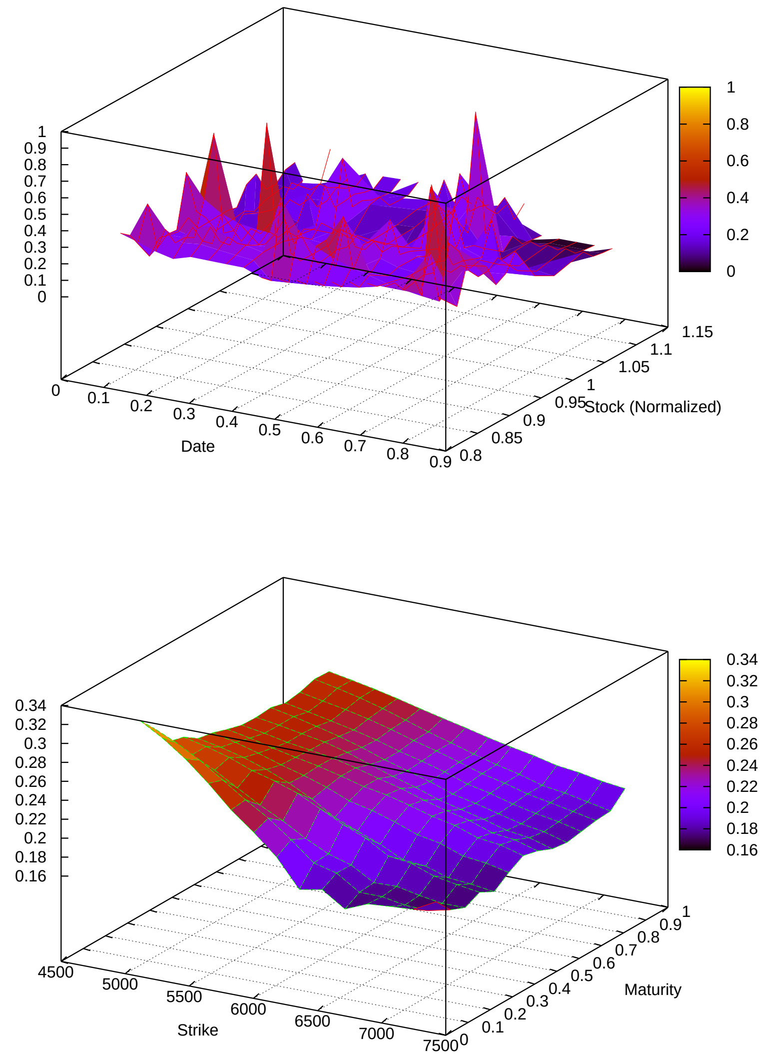


Figure 4: (Bottom) *Implied volatility surface corresponding to the DAX index options real data set that is used in the calibrations of Parts A and B;* (Top) *Local volatility surface obtained by the Dupire formula run on the prices corresponding to these implied volatilities.*

differentiation approach based on the Dupire formula gives a local volatility surface that is both very irregular at a fixed calibration time t (top panel in Figure 4) and unstable in t. Since market prices are only available for a finite set of strikes and maturities, the problem is also under-determined.

Differentiation of well chosen parametric approximations of the market implied volatilities through the Gatheral formula (42) work better in practice, as could be expected from homogeneity considerations. An industry standard in this regard is the SSVI approach of Gatheral and Jacquier (2014).

In what follows, we discuss various non-parametric approaches with regularization.

A Tikhonov Regularization

To stabilize the local volatility calibration problem, we can reformulate it in the form of the following nonlinear minimization problem:

$$\min_{\{\sigma\equiv\sigma(\cdot,\cdot);\underline{\sigma}\leq\sigma\leq\overline{\sigma}\}} J(\sigma) = \|\Pi(\sigma) - \pi\|^2 + \alpha\|\sigma - \sigma_*\|^2_{\mathcal{H}^1}, \tag{46}$$

where:

- α is a positive regularization parameter,
- the bounds $\underline{\sigma}$ and $\overline{\sigma}$ are positive constants,
- π is a vector of vanilla option prices observed in the market at the calibration time $t = 0$,
- $\Pi(\sigma)$ is the corresponding vector of prices in the local volatility model with volatility function σ,
- σ_* is a prior on σ and
- $\|u\|^2_{\mathcal{H}^1} \equiv \int_0^\infty \int_0^\infty (u(t,S)^2 + (\partial_t u(t,S))^2 + (\partial_S u(t,S))^2)\, dtdS.$

Stability, convergence and convergence rates for this formulation of the local volatility calibration problem are established in Crépey (2003a). A trinomial tree implementation developed in Crépey (2003b) draws its efficiency from an exact computation of the gradient of the (discretized) cost criterion J in (46). Figure 5 displays the local volatility surface thus calibrated, along with the accuracy of the calibration, using the prices corresponding to the implied volatilities of Figure 4 (bottom) as calibration input data.

This approach can be extended to the calibration of a local volatility function based on American option prices (Crépey, 2003b).

B Entropic Regularization

An alternative is to use entropic regularization, rewriting the calibration problem as the following nonlinear minimization problem (Avellaneda et al., 1997; Samperi, 2002):

$$\min_{\{\sigma\equiv\sigma(\cdot,\cdot);\underline{\sigma}\leq\sigma\leq\overline{\sigma}\}} J(\sigma) = \|\Pi(\sigma) - \pi\|^2 + \alpha\|\sigma - \sigma_*\|^2_{\mathcal{L}^2}, \tag{47}$$

with

$$\|\sigma - \sigma_*\|^2_{\mathcal{L}^2} := \mathbb{E}\int_0^\infty (\sigma(t,S_t) - \sigma_*(t,S_t))^2\, dt.$$

Using a dual formulation, the minimization problem (47) can be solved in time $\mathrm{O}(d)$, where d is the number of options in the calibration data set, versus $\mathrm{O}(n^2)$ in the case of Tikhonov regularization implemented on a trinomial tree with n time steps. The numerical solution is thus typically faster than by Tikhonov regularization. However, it is also less stable, since the regularization term doesn't involve the gradient, but only the values, of $(\sigma - \sigma_*)$. See Figure 6, which displays the results obtained by this method for the data set of Figure 4 (bottom).

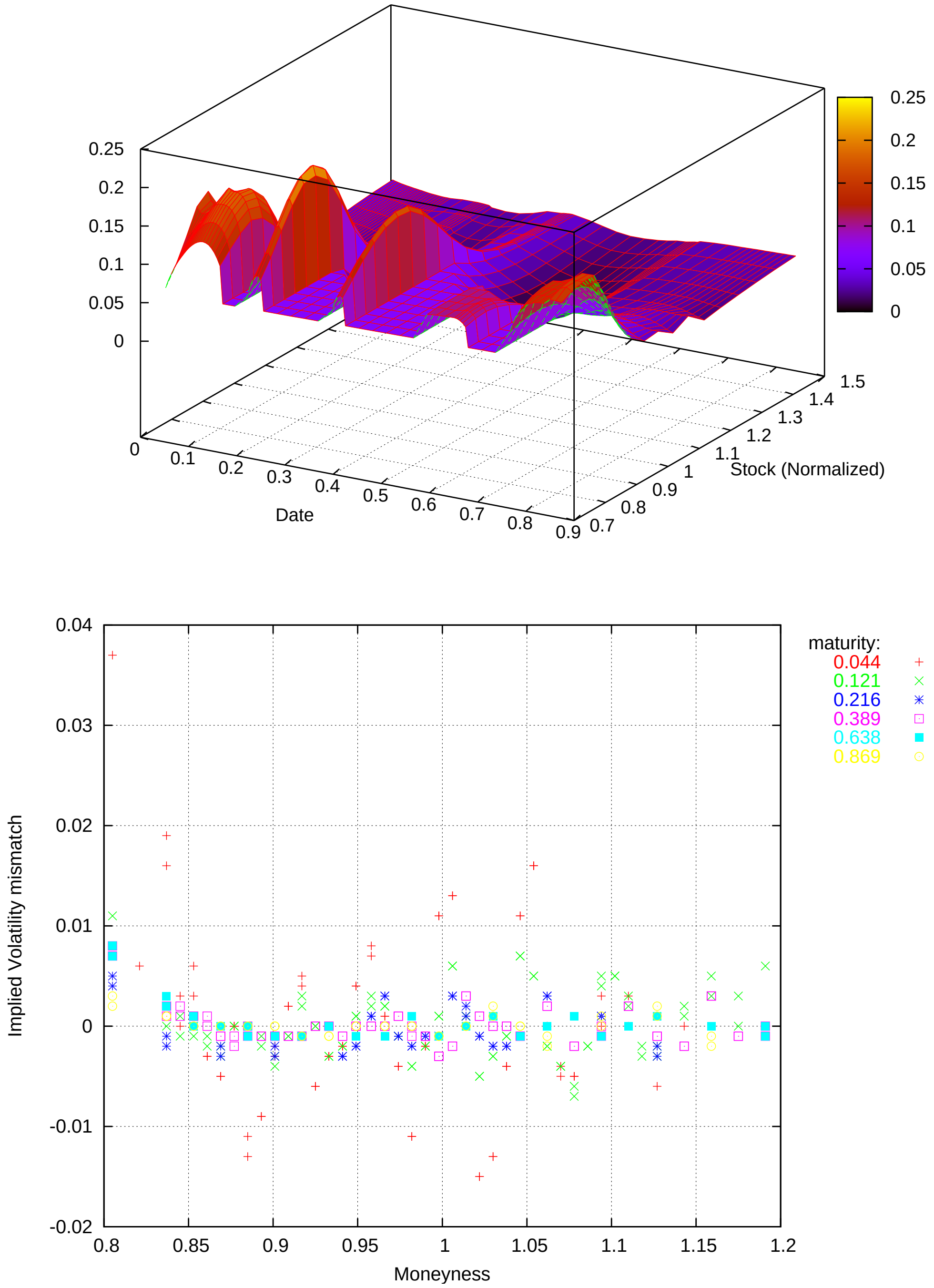
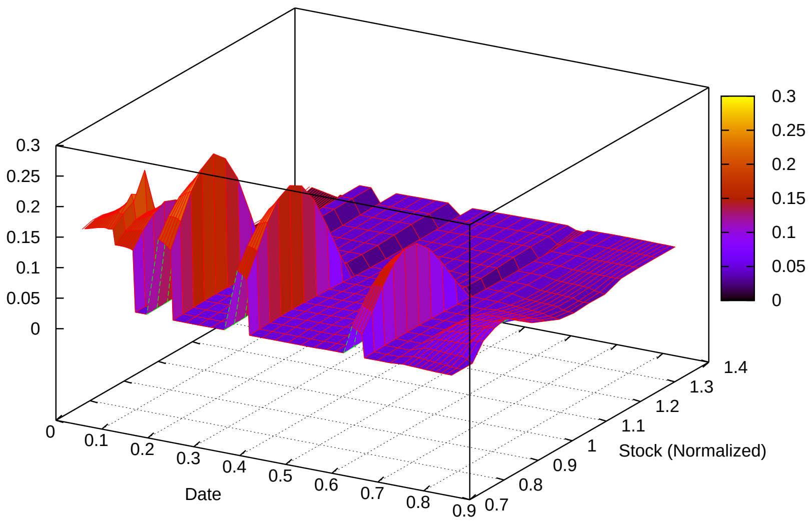


Figure 5: *Calibration by Tikhonov regularization to the DAX options data of Figure 4 (bottom): local volatility surface and calibration accuracy.*



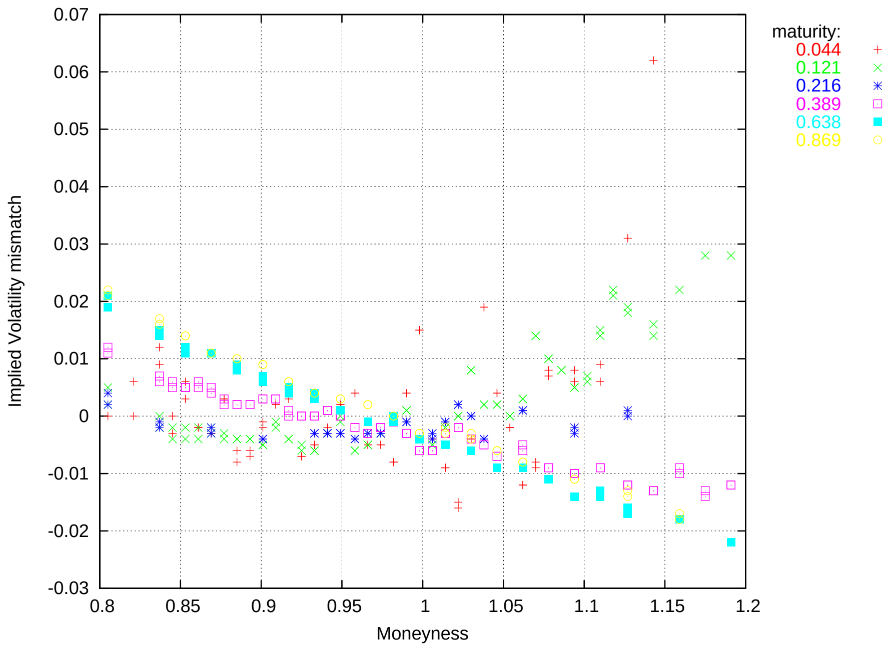


Figure 6: *Calibration by entropic regularization to the DAX options data of Figure 4 (bottom): local volatility surface and calibration accuracy.*

C Deep Local Volatility

Figure 7 (top) was obtained by applying the Gatheral formula (42) to a suitably penalized neural net interpolation of the market implied volatilities displayed in Figure 7 (middle) (Chataigner et al., 2021).

Once robustly extracted from the market prices or implied volatilities, a local volatility function can be used for various purposes, such as pricing exotic options and/or Greeking consistent with the market, or calibrating more general stochastic volatility models (see Gatheral (2011)).

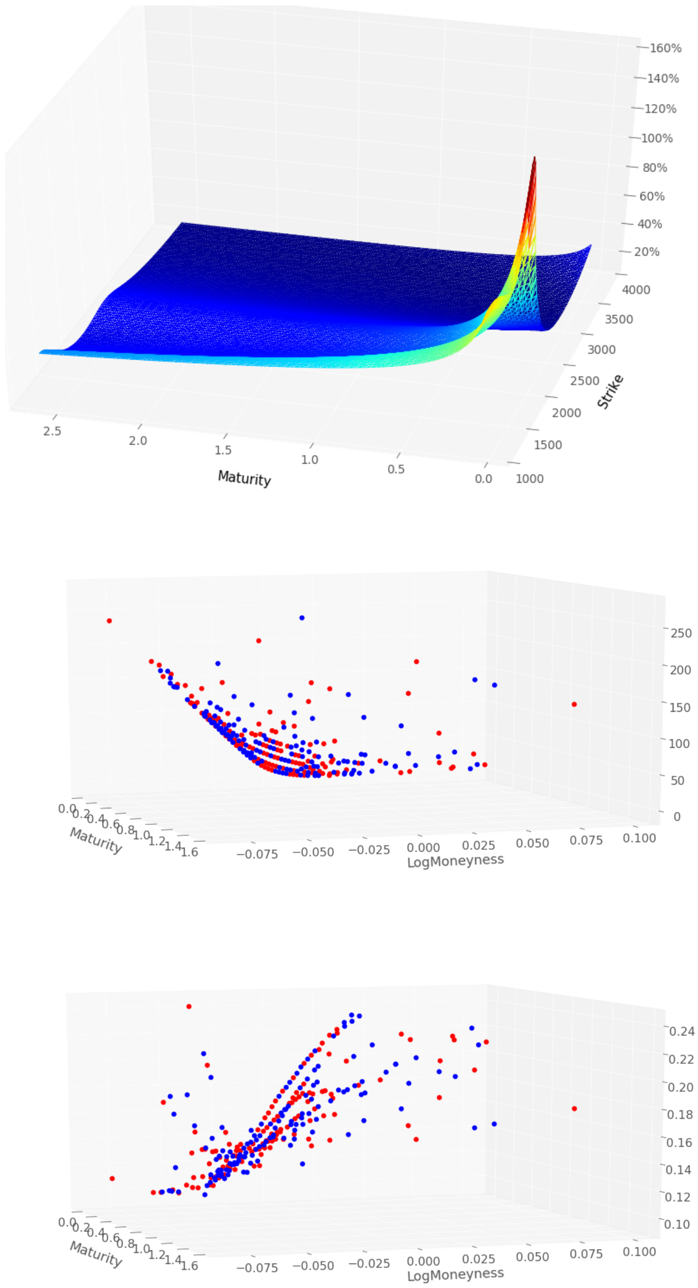


Figure 7: *Bottom to top: May 18, 2019 SPX option prices, implied volatilities, and local volatility computed by an application of the Gatheral formula to a suitably penalized neural net interpolation of implied volatilities. Blue and red dots identify the training and testing sets used in the learning process, namely 135 option market prices used for training and 144 used for testing. Local volatility on top predicted on 10,000 nodes.*

Chapter IV

Stochastic Analysis Toolbox

We provide a toolbox of results in stochastic analysis (some of them without proofs) that are used in the main body of the notes.

We use the French acronym càdlàg for "left limited and right continuous" (a.s., with finite left and right limits everywhere). In these notes we also use càglàd for the left-limit processes of càdlàg processes[1].

§1 Martingales and Stopping Times

Definition 1 *The process Y is said to be a martingale (resp. supermartingale) with respect to the filtration $(\mathfrak{F}_t\,,\ t \in [0,T])$, if (shortening $\mathbb{E}[\cdot|\mathfrak{F}_t]$ into $\mathbb{E}_t$):*
(i) *Y is $\mathfrak{F}$ adapted and $\mathbb{E}|Y_t| < +\infty$, $\forall t \leq T$;*
(ii) *$\forall s \leq t$, we have $Y_s = \mathbb{E}_s Y_t$ (resp. $Y_s \geq \mathbb{E}_s Y_t$).*

We assume that the model filtration $\mathfrak{F}$ satisfies the so-called usual conditions of completeness[2] and right-continuity[3]. Then every martingale (or more general local martingale below) admits a càdlàg modification[4]. Modifying a process does not modify the family of its finite-dimensional distributions, hence preserves its law. In these notes all local martingales are assumed càdlàg.

Definition 2 *A $[0,T]$ valued random variable ϑ is said to be a stopping time with respect to the continuous-time filtration $\mathfrak{F} = (\mathfrak{F}_t)_{t\in[0,T]}$ if, for each t, the indicator function $\mathbb{1}_{\{\vartheta \leq t\}}$ of the event $\{\vartheta \leq t\}$ is $\mathfrak{F}_t$ measurable.*

Lemma 1 [5] **(i)** *The hitting time of an open set $\mathcal{O}$ by an adapted càdlàg process X, i.e. $\vartheta = \inf\{t > 0;\ X_t \in \mathcal{O}\} \wedge T$ (restricting attention to $[0,T]$), is a stopping time.*
(ii) *The hitting time of a closed set $\mathcal{C}$ by an adapted continuous process X, i.e. $\vartheta = \inf\{t > 0;\ X_t \in \mathcal{C}\} \wedge T$ (likewise), is a stopping time.*
(iii) *For any $a \in \mathbb{R}$ and adapted nondecreasing process X, $\vartheta = \inf\{t > 0;\ X_t \geq a\} \wedge T$ is a stopping time.*

---

[1] more restrictive than the common use of càglàd for "left limited and right continuous" processes.
[2] i.e. $\mathfrak{F}_0$ contains all the null sets of $(\Omega, \mathbb{Q})$.
[3] i.e. $\mathfrak{F}_{t+} = \mathfrak{F}_t$, where $\mathfrak{F}_{t+} = \cap_{s>t}\mathfrak{F}_s$ represents the model information "right after time t".
[4] see He, Wang, and Yan (1992, Corollary 2.48 page 56).
[5] see Karatzas and Shreve (1991, Corrected problems 2.6 and 2.7 p.39), He, Wang, and Yan (1992, Problems and Complements 3.1 through 3.3 pages 106-107), and Revuz and Yor (1999, 1.§4).

Proof (sketched). Given a denumerable dense subset $\mathbb{D}$of $\mathbb{R}$, e.g. the rational numbers: **(i)** Show that $\{\tau < s\} = \cup_{\mathbb{D}\ni r<s}\{X_r \in \mathcal{O}\} \in \mathfrak{F}_s$. Deduce that $\{\tau \le t\} \in \mathfrak{F}_{t+} = \mathfrak{F}_t$.
(ii) Noting that $X_s \in \mathcal{C} \iff Y_s := d(X_s, \mathcal{C}) = 0$, show using the continuity of X that $\{\tau \le t\} = \{(\inf_{s\in(\mathbb{D}\cap[0,t])\cup\{t\}} Y_s) = 0\}$. Deduce that $\{\tau \le t\} \in \mathfrak{F}_t$.
(iii) Show that $\{\tau \ge s\} = \cap_{\mathbb{D}\ni r<s}\{X_r < a\}$. Deduce that $\{\tau < s\} \in \mathfrak{F}_s$, hence $\{\tau \le t\} \in \mathfrak{F}_{t+} = \mathfrak{F}_t$. ∎

Lemma 2 [6] *Let M be a martingale on $[0,T]$ and ϑ be a $[0,T]$ valued stopping time, then*

$$\mathbb{E}M_\vartheta = \mathbb{E}M_0.$$

The following definition is the variant on $[0,T]$, obtained "by stopping at T" of the usual definition on $\mathbb{R}_+$.

Definition 3 *The process Y is said to be a local martingale on $[0,T]$ with respect to the filtration $\mathfrak{F}$ if it admits a nondecreasing (dubbed* localizing*) sequence of $[0,T]$ valued stopping times τ_n such that $\mathbb{D}(\cup_n\{\tau_n = T\}) = 1$ and every stopped process $Y_{\cdot\wedge\tau_n}$ is a martingale on $[0,T]$.*

Lemma 3 *Any local martingale Y dominating a martingale M is a supermartingale.*

Proof. The general case is readily deduced by application to $Y - M$ of the special case when $M = 0$, to which we therefore restrict ourselves hereafter. Let there be given a local martingale $Y \ge 0$ with a localizing sequence of stopping times (τ_n). For every n,

$$Y_{s\wedge\tau_n} = \mathbb{E}[Y_{t\wedge\tau_n}|\mathfrak{F}_s]\,,\ 0 \le s \le t \le T.$$

Sending n to ∞, we obtain by the (conditional) Fatou lemma that

$$\begin{aligned} Y_s = \liminf_n Y_{s\wedge\tau_n} &= \liminf_n \mathbb{E}[Y_{t\wedge\tau_n}|\mathfrak{F}_s] \\ &\ge \mathbb{E}[\liminf_n Y_{t\wedge\tau_n}|\mathfrak{F}_s] = \mathbb{E}[Y_t|\mathfrak{F}_s]\,,\ 0 \le s \le t \le T. \end{aligned}$$

In addition, $Y_0 = Y_{0\wedge\tau_0}$ is integrable, hence so is Y_t for any fixed t, as $\mathbb{E}Y_t \le \mathbb{E}Y_0$ holds by the already established inequality $\mathbb{E}_0 Y_t \le Y_0$. ∎

Lemma 4 *Given a positive $\mathbb{D}$ martingale ν with $\nu_0 = 1$, let $\widetilde{\mathbb{D}}$ be the probability measure on $(\Omega, \mathfrak{F}_T)$ such that $\frac{d\widetilde{\mathbb{D}}}{d\mathbb{D}} = \nu_T$, with related expectation denoted by $\widetilde{\mathbb{E}}$. For any given nonnegative process $\widetilde{X}$:*
(i) *For $s \le t$, $\widetilde{\mathbb{E}}_s(\widetilde{X}_t) = \frac{1}{\nu_s}\mathbb{E}_s(\nu_t\widetilde{X}_t)$.*
(ii) *$\widetilde{X}$ is a $\widetilde{\mathbb{D}}$ (resp. local) martingale if and only if $(\nu\widetilde{X})$ is a $\mathbb{D}$ (resp. local) martingale.*

Proof. For each time s, we have

$$\widetilde{\mathbb{E}}\widetilde{X}_s = \mathbb{E}(\widetilde{X}_s\nu_T) = \mathbb{E}\mathbb{E}_s(\widetilde{X}_s\nu_T) = \mathbb{E}(\widetilde{X}_s\nu_s),$$

by the martingale property of ν. Hence an adapted process $\widetilde{X}$ is $\widetilde{\mathbb{D}}$ integrable if and only if $(\nu\widetilde{X})$ is $\mathbb{D}$ integrable. Then the formula in (i) holds because, for any $A \in \mathfrak{F}_s$,

$$\begin{aligned} \widetilde{\mathbb{E}}\Big(\frac{1}{\nu_s}\mathbb{E}_s\big(\nu_t\widetilde{X}_t\big)\mathbb{1}_A\Big) &= \mathbb{E}\Big(\frac{\nu_T}{\nu_s}\mathbb{E}_s\big(\nu_t\widetilde{X}_t\big)\mathbb{1}_A\Big) = \mathbb{E}\mathbb{E}_s\Big(\frac{\nu_T}{\nu_s}\mathbb{E}_s\big(\nu_t\widetilde{X}_t\big)\mathbb{1}_A\Big) \\ &= \mathbb{E}\Big(\mathbb{E}_s\big(\nu_t\widetilde{X}_t\big)\mathbb{1}_A\Big) = \mathbb{E}\big(\nu_t\widetilde{X}_t\mathbb{1}_A\big) = \mathbb{E}\big((\mathbb{E}_t\nu_T)\widetilde{X}_t\mathbb{1}_A\big) = \mathbb{E}(\widetilde{X}_t\mathbb{1}_A\nu_T) = \widetilde{\mathbb{E}}(\widetilde{X}_t\mathbb{1}_A). \end{aligned}$$

Part (i) implies the statement in (ii) regarding martingales, from which the one regarding local martingales follows by localization. ∎

[6] cf. He, Wang, and Yan (1992, Theorem 2.58 page 60).

Lemma 5 *A progressive process Z[7] such that $\int_0^T \|Z_s\|^2 ds < \infty$ is integrable against a standard Brownian motion W (possibly multivariate, with then Z of the same dimension as W) in the sense of local martingales on $[0,T]$.*

Proof. Assuming $\int_0^T \|Z_s\|^2 ds < \infty$, the

$$\tau_n = \inf\{t; \int_0^t \|Z_s\|^2 ds \geq n\} \wedge T$$

are a nondecreasing sequence of stopping times[8] such that $\mathbb{D}(\cup_n\{\tau_n = T\}) = 1$[9]. For each n we have

$$\int_0^T \mathbb{1}_{\{s\leq\tau_n\}} \|Z_s\|^2 ds = \int_0^{\tau_n} \|Z_s\|^2 ds \leq n,$$

hence

$$\mathbb{E}\Big(\int_0^T \mathbb{1}_{\{s\leq\tau_n\}} Z_s dW_s\Big)^2 = \mathbb{E}\Big(\int_0^T \mathbb{1}_{\{s\leq\tau_n\}} \|Z_s\|^2 ds\Big) \leq n$$

is finite, so that $\int_0^\cdot \mathbb{1}_{\{s\leq\tau_n\}} Z_s dW_s = \int_0^{\cdot\wedge\tau_n} Z_s dW_s$ is a (square integrable) martingale on $[0,T]$, by the standard Itô stochastic integration theory[10]. In conclusion $\int_0^\cdot Z_s dW_s$ is a local martingale on $[0,T]$, with τ_n as a localizing sequence of stopping times. ∎

Lemma 6 *Progressive processes Z such that $\int_0^T \|Z_s\|^2 ds < \infty$ include all the càglàd or càdlàg (adapted) processes on $[0,T]$.*

Proof. Any (adapted) càdlàg process $\widetilde{Z}$ or càglàd process $Z = \widetilde{Z}_-$ is progressive[11]. The

$$\tau_n = \inf\{t; \sup_{s\leq t} \|\widetilde{Z}_s\| \geq n\} \wedge T$$

are a nondecreasing sequence of stopping times[12] such that $\mathbb{D}(\cup_n\{\tau_n = T\}) = 1$. In fact, setting $\chi = \sup_{t\in[0,T]} \|\widetilde{Z}_t\|$ (which is finite for $\widetilde{Z}$ càdlàg), we have

$$\Big\{\chi < \infty\Big\} = \cup_{\nearrow n\infty}\Big\{\chi < n\Big\},$$

hence $\mathbb{D}(\{\chi < n\}) \nearrow_{n\infty} 1$ and

$$\mathbb{D}(\{\tau_n < T\}) \leq \mathbb{D}(\chi \geq n) \searrow_{n\infty} 0.$$

Therefore $\mathbb{D}(\cap_n\{\tau_n < T\}) = 0$ and $\mathbb{D}(\cup_n\{\tau_n = T\}) = 1$, as claimed. Moreover, on $\{\tau_n = T\}$, we have $\|Z\| < n$ on $[0,T]$, hence $\int_0^T \|Z_s\|^2 ds \leq n^2 T < \infty$. In conclusion, $\int_0^T \|Z_s\|^2 ds = \int_0^T \|\widetilde{Z}_s\|^2 ds$ is almost surely finite. ∎

[7] i.e. such that $Z|_{\Omega\times[0,t]}$ is $\mathfrak{F}_t \times \mathcal{B}([0,t])$ measurable for each $t \geq 0$ (He, Wang, and Yan, 1992, Definition 3.10 page 86).
[8] by Lemma 1(i).
[9] as $\cup_n\{\tau_n = T\}$ contains $\{\int_0^T \|Z_s\|^2 ds < +\infty\}$, which is of $\mathbb{D}$ probability measure 1.
[10] for square integrable progressive integrands.
[11] see He, Wang, and Yan (1992, Theorem 3.11 page 86).
[12] by Lemma 1(iii).

§2 Semimartingales

Semimartingales are a class of integrators giving rise to a flexible theory of stochastic integration encompassing the Itô integral with respect to the Brownian motion as presented in, for instance, Karatzas and Shreve (1991, Section 3.1), as a special case. For detailed treatments, see e.g. Meyer (1976) and He, Wang, and Yan (1992) or, for a renewed pedagogical presentation building on Dellacherie (1980), Protter (2004). In one of these two equivalent characterizations (see Meyer (1976, 1 DEFINITION page 32) and Dellacherie (1980, Définition 1 page 119, Théorème 6 page 125)):

Definition 4 *A semimartingale X on $[0,T]$ corresponds to the sum of a finite variation process D and a local martingale M, where a finite variation process $D = D^{[+]} - D^{[-]}$ is a difference between two nondecreasing càdlàg (adapted) processes $D^{[\pm]}$ starting from 0, while local martingales on $[0,T]$ were introduced in Definition 3.*

A financial motivation for modeling prices of traded assets as semimartingales is that price processes outside this class give rise to arbitrages unless rather stringent conditions are imposed on the trading strategies[13].

Any representation $X = D + M$ as above is called a Doob-Meyer decomposition of the semimartingale X. A Doob-Meyer decomposition is not generally unique. However, there is at most one such representation of a process X with D continuous[14]. One then speaks of "the canonical Doob-Meyer decomposition of a special semimartingale X". Equivalently to this uniqueness result:

Lemma 7 *The only finite variation continuous local martingale M is the null process.*

Proof. [15] As M is continuous and the $M^{[\pm]}$ are càdlàg, we can in fact assume the $M^{[\pm]}$ continuous, by subtracting from them their (necessarily at most countable and common) jumps in the first place. Since M has finite variation, we have

$$|M_t| \le M_t^{[+]} + M_t^{[-]} < +\infty, \qquad (1)$$

for all $t \in [0,T]$[16]. The $\tau_n := \inf\{t > 0;\, M_t^{[+]} + M_t^{[-]} \ge n\} \wedge T$ are a nondecreasing sequence of stopping times[17] such that $\mathbb{D}(\cup_n\{\tau_n = T\}) = 1$[18]. In view of (1), for any fixed k the process $K := M_{\cdot\wedge\tau_k}$ is a bounded martingale, by a double application[19] of Lemma 3. Then, denoting $t_i^n = i\frac{T}{n}$, we have $\mathbb{E}_{t_{i-1}^n} K_{t_i^n} = K_{t_{i-1}^n}$, hence

$$\begin{aligned}
\mathbb{E}(K_{t_i^n} - K_{t_{i-1}^n})^2 &= \mathbb{E}\mathbb{E}_{t_{i-1}^n}(K_{t_i^n}^2 - 2K_{t_i^n}K_{t_{i-1}^n} + K_{t_{i-1}^n}^2) \\
&= \mathbb{E}(K_{t_{i-1}^n}^2 - 2K_{t_{i-1}^n}\mathbb{E}_{t_{i-1}^n}K_{t_i^n} + \mathbb{E}_{t_{i-1}^n}K_{t_i^n}^2) \\
&= \mathbb{E}K_{t_i^n}^2 - \mathbb{E}K_{t_{i-1}^n}^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}K_t^2 = \sum_{i=1}^n \mathbb{E}(K_{t_i^n} - K_{t_{i-1}^n})^2 &\le \mathbb{E}\big(\max_{1\le i\le n}|K_{t_i^n} - K_{t_{i-1}^n}| \cdot \sum_{i=1}^n |K_{t_i^n} - K_{t_{i-1}^n}|\big) \\
&\le k\mathbb{E}\max_{1\le i\le n}|K_{t_i^n} - K_{t_{i-1}^n}| \to_{n\infty} 0,
\end{aligned}$$

[13] see Delbaen and Schachermayer (2005).
[14] see e.g. He, Wang, and Yan (1992, Theorem 8.5 page 210).
[15] adapted from https://faculty.math.illinois.edu/~psdey/MATH562FA21/lec11.pdf accessed on 26 Nov 2021.
[16] see e.g. He, Wang, and Yan (1992, Theorem 3.44 page 101).
[17] by Lemma 1(ii).
[18] as $\cup_n\{\tau_n = T\}$ contains the set $\{M_T^{[+]} + M_T^{[-]} < +\infty\}$, which is of $\mathbb{D}$ probability measure 1.
[19] to K and to $(-K)$.

by the dominated convergence theorem and the fact that (K_t) is continuous a.s.. Hence $K_t = M_{t\wedge\tau_k} = 0$, for all k. Since $\mathbb{D}(\cup_n\{\tau_n = T\}) = 1$ we conclude that $M_t = 0$ almost surely holds. Then $M = 0$ almost surely holds on $[0, T]$, by continuity of M. ∎

Let $X = D + M$ denote a Doob-Meyer decomposition of a semimartingale X. The stochastic integral of a càglàd process Z[20] with respect to X is defined as

$$Y_t = \int_0^t Z_s dX_s := \int_0^t Z_s dD_s + \int_0^t Z_s dM_s. \tag{2}$$

Here $\int_0^t Z_s dD_s$ is the corresponding pathwise Lebesgue-Stieltjes integral, e.g.[21], for any sequence of stopping times T_l increasing to infinity, random variables c_l, and $D_t = \sum_l c_l \mathbb{1}_{\{T_l\leq t\}}$,

$$\int_0^t Z_s dD_s = \sum_{T_l\leq t} Z_{T_l} c_l. \tag{3}$$

Along with the case where $dD_t = c_t dt$ for some progressive Lebesgue-integrable process c, (3) covers all our purposes in these notes. We will not detail further the definition of $\int_0^t Z_s dM_s$ when M has infinite variation (and the integral is truly stochastic, like an Itô integral). It will be enough for us to know that the corresponding notion of stochastic integral is independent of the Doob-Meyer decomposition of X that is used in (2)[22], and:

Theorem 1 [23] *In the case where X is a local martingale, the integral process Y is again a local martingale.*

Elementary integration by parts formula, for any semimartingale X and $\beta = e^{-\int_0^\cdot r_s ds}$ for a time-integrable process r:

$$d(\beta_t X_t) = \beta_t(dX_t - r_t X_t dt) \tag{4}$$

(without bracket term, by time differentiability of $\beta = e^{-\int_0^\cdot r_s ds}$).

§3 Itô and Markov Processes

Let there be given an $\mathbb{R}^d$ valued drift coefficient b, an $\mathbb{R}^{d\times d}$ valued diffusion coefficient σ, and an $\mathbb{R}^d$ valued jump size process δ, given as càdlàg processes such that $\int_0^T(|b_t| + |\delta_t| + |\sigma|_t^2)dt < +\infty$ a.s..

We consider an Itô process in the sense of a d-variate process X obeying the following dynamics: $X_0 = x$ and, for $t \geq 0$,

$$dX_t = b_t\, dt + \sigma_t\, dW_t + \delta_{t-} dN_t, \tag{5}$$

for some d-variate standard Brownian motion W and a Poisson process N[24] with prescribed intensity λ, i.e. compensated martingale $M = N - \lambda t$.

The corresponding Markov (jump-diffusion) setup is when

$$b_t = b(t, X_t)\,,\ \ \sigma_t = \sigma(t, X_t)\,,\ \ \delta_t = \delta(t, X_t), \tag{6}$$

[20] cf. Lemma 6 and He, Wang, and Yan (1992, Theorem 7.7 1) page 192).
[21] see https://en.wikipedia.org/wiki/Lebesgue-Stieltjes_integration#Definition.
[22] see e.g. He, Wang, and Yan (1992, Lemma 9.12 and Definition 9.13 page 234).
[23] see e.g. Protter (2004, Theorem IV.29 page 173).
[24] hence $\int_0^\cdot \delta_{t-} dN_t = \sum_{T_l\in\cdot} \delta_{T_l-}$, by (3).

for continuous *functions* $b(\cdot,\cdot), \sigma(\cdot,\cdot), \delta(\cdot,\cdot)$, so that (5) is in effect a stochastic differential equation (forward SDE). This SDE has a unique (strong, i.e. given the driving processes W and N) solution X provided the coefficients are Lipschitz with linear growth in x, uniformly in t[25]. A notable feature of the solution is the so-called ***Markov property***, meaning that[26]

$$\mathbb{E}_t(\Phi(X_s, s \in [t,T]) = \mathbb{E}(\Phi(X_s, s \in [t,T]) \mid X_t) \tag{7}$$

holds for every functional Φ of X that makes sense on both sides of the equality. Thus the past of X doesn't influence its future; the present of X provides all the relevant information.

Given a real valued and $\mathcal{C}^{1,2}$ function $u = u(t,x)$, an application of the diffusive Itô formula between jumps combined with a direct inspection of the impact of the jumps yield the following ***Itô formula with Poisson driven jumps***: for any $t \in [0,T]$,

$$\begin{aligned} du(t,X_t) = \\ &(\partial_t + \mathcal{A}_x)\, u(t,X_t)dt + \partial_x u(t,X_t)\sigma_t\, dW_t + \delta u(t,X_{t-})dM_t, \end{aligned} \tag{8}$$

where $\partial_x u$ is the row gradient of u with respect to x, where in the compensated jump local martingale of the last term

$$\delta u(t,x) = u(t, x+\delta(t,x)) - u(t,x),$$

and where the infinitesimal generator $\mathcal{A}_x$ of X acts on u as follows:

$$(\mathcal{A}_x u)(t,x) = \partial_x u(t,x)b(t,x) + \frac{1}{2}\mathrm{tr}\left(\partial^2_{x^2} u(t,x)a(t,x)\right) + \lambda\delta u(t,x), \tag{9}$$

with $a = \sigma\sigma^\top$. In addition, for all $\mathcal{C}^{1,2}$ functions u and v of (t,x) such that

$$\mathbb{E}\int_0^T \left(\partial u(t,X_t)a_t(\partial v(t,X_t))^\top\right)^2 dt + \lambda\mathbb{E}\int_0^T \left(\delta u(t,X_t)\delta v(t,X_t)\right)^2 dt < \infty,$$

we have with $Y_t = u(t,X_t)$ and $Z_t = v(t,X_t)$:

$$\frac{d\langle Y,Z\rangle_t}{dt} = \partial u(t,X_t)a_t(\partial v(t,X_t))^\top + \lambda\delta u(t,X_t)\delta v(t,X_t), \tag{10}$$

where $\frac{d\langle Y,Z\rangle_t}{dt} = \lim_{\mathrm{h}\to 0} \mathrm{h}^{-1}\mathbb{C}\mathrm{ov}(Y_{t+\mathrm{h}} - Y_t, Z_{t+\mathrm{h}} - Z_t | \mathfrak{F}_t)$.

By Girsanov transform based measure change, one can extend the above to models with a random intensity (λ_t) of jumps or, in a Markovian specification, $\lambda(t,X_t)$. This allows designing models with dependent driving noises W and N, where the latter is a point process (increasing by one at increasing random times) with intensity *process* λ[27].

Further extension outside the scope of these notes would be to "unpredictable" jump size (like with compound Poisson processes) and/or "infinite intensity" jumps[28].

[25] this can be shown iteratively on larger and larger time intervals $[0,T_l]$, where the T_l are the increasing jump times of N, by using Karatzas and Shreve (1991, Theorems 2.5 page 287 and 2.9 page 289) between jumps; see also Élie (2006, Section II.1.5 p.125) for direct computations with jumps.

[26] see Protter (2004, theorems I.45 page 35 and V.32 page 300).

[27] By contrast, independence always holds between a standard Brownian motion and Poisson process on a common filtered probability space (He, Wang, and Yan, 1992, Theorem 11.43 page 316).

[28] see e.g. Crépey, Bielecki, and Brigo (2014, Sections 13.2-13.3) or Cont and Tankov (2003).

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