Conversion of sampling rate / STFT

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TSIA201

Part I

Conversion of sampling rate

Upsampling (or oversampling)

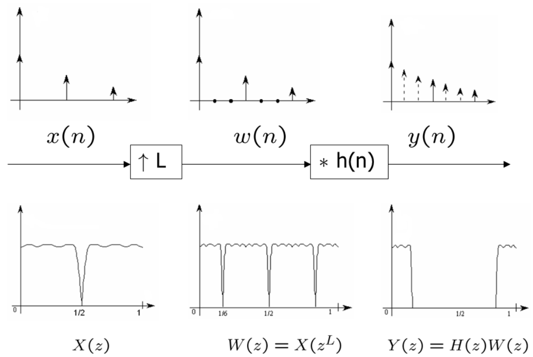
- Let $x(n) = x_a(nT)$, where $\mathrm{supp}(X_a(f)) \subset \left[-\frac{1}{2T}, \frac{1}{2T}\right]$
- Reconstruction formula (Nyquist)

$$x_a(t) = \sum_{m \in \mathbb{Z}} x(m) \,\mathrm{sinc}\left(\pi\left(\frac{t}{T} - m\right)\right)$$

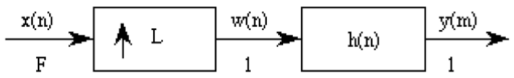
- Let $y(n) = x_a\left(n\frac{T}{L}\right)$ be the interpolated signal $\times L$

$$y(n) = h * w(n), \text{ where } \begin{cases} \forall n \in L\mathbb{Z}, w(n) = x\left(\frac{n}{L}\right) \\ \forall n \notin L\mathbb{Z}, w(n) = 0 \\ \forall n \in \mathbb{Z}, h(n) = \mathrm{sinc}\left(\frac{\pi n}{L}\right) \end{cases}$$

Upsampling (or oversampling)

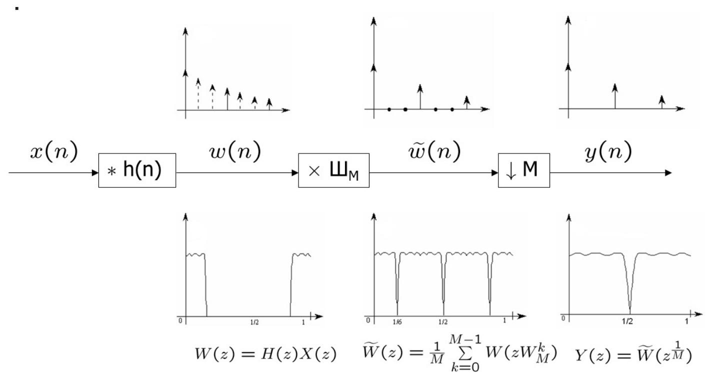


Upsampling (or oversampling)

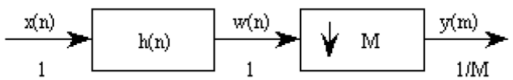


- $W(e^{i2\pi\nu}) = X(e^{i2\pi L\nu})$
- $\begin{cases} H(e^{i2\pi\nu}) = L & \forall|\nu| < \frac{1}{2L} \\ H(e^{i2\pi\nu}) = 0 & \forall \frac{1}{2L} < |\nu| < \frac{1}{2} \end{cases}$
- $\Rightarrow \begin{cases} Y(e^{i2\pi\nu}) = LX(e^{i2\pi L\nu}) & \forall|\nu| < \frac{1}{2L} \\ Y(e^{i2\pi\nu}) = 0 & \forall \frac{1}{2L} < |\nu| < \frac{1}{2} \end{cases}$

Downsampling (or subsampling)

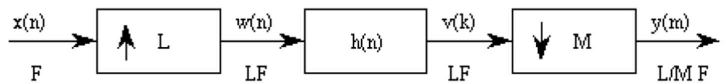


Downsampling (or subsampling)



- $\begin{cases} H(e^{i2\pi\nu}) = 1 & \forall|\nu| < \frac{1}{2M} \\ H(e^{i2\pi\nu}) = 0 & \forall \frac{1}{2M} < |\nu| < \frac{1}{2} \end{cases}$
- $Y(e^{i2\pi\nu}) = \frac{1}{M} \sum_{k=0}^{M-1} W\left(e^{i2\pi\left(\frac{\nu-k}{M}\right)}\right)$
- $\Rightarrow Y(e^{i2\pi\nu}) = \frac{1}{M} X(e^{i2\pi\frac{\nu}{M}})\ \forall|\nu| < \frac{1}{2}$

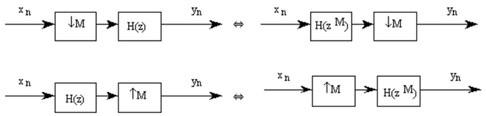
Resampling



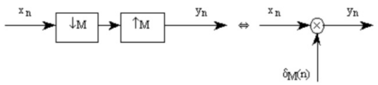
- $\begin{cases} H(e^{i2\pi\nu}) = L & \forall|\nu| < \min\left(\frac{1}{2L}, \frac{1}{2M}\right) \\ H(e^{i2\pi\nu}) = 0 & \forall \min\left(\frac{1}{2L}, \frac{1}{2M}\right) < |\nu| < \frac{1}{2} \end{cases}$
- $Y(e^{i2\pi\nu}) = \frac{L}{M} X(e^{i2\pi\frac{L}{M}\nu})\ \forall|\nu| < \min\left(\frac{1}{2}, \frac{M}{2L}\right)$ (and 0 elsewhere)

Noble identities

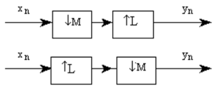
- Permutation of filtering / decimation or insertion



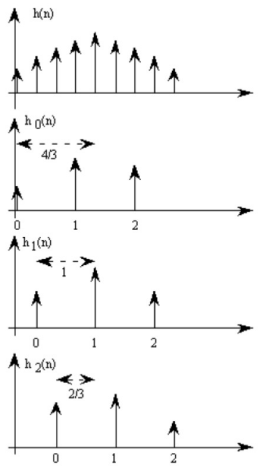
- Simplification insertion / decimation



- Permutation of insertion / decimation ?



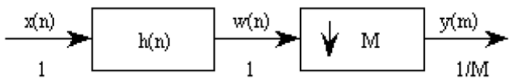
Polyphase components

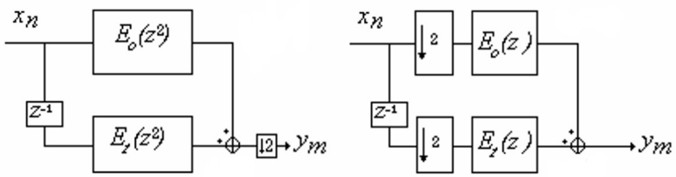


- If h is an ideal low-pass filter
  - h_0, h_1, h_2 are all-pass filters
  - h_0, h_1, h_2 differ by their phase

Type I polyphase components

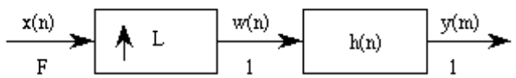
- Definition : $H(z) = \sum_{m=0}^{M-1} E_m(z^M) z^{-m}$
- Efficient structure for downsampling

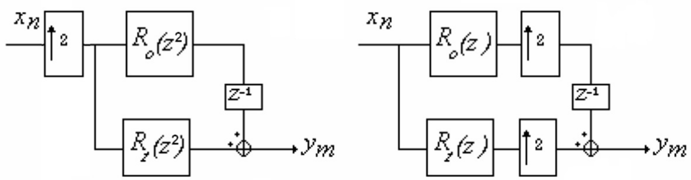




Type II polyphase components

- Definition : $H(z) = \sum_{l=0}^{L-1} R_l(z^L) z^{-(L-1-l)}$
- Efficient structure for upsampling

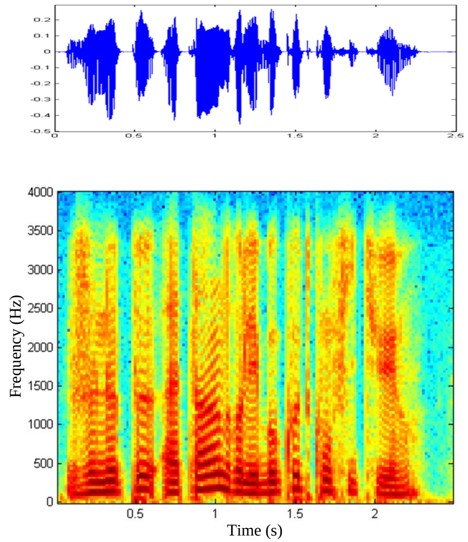




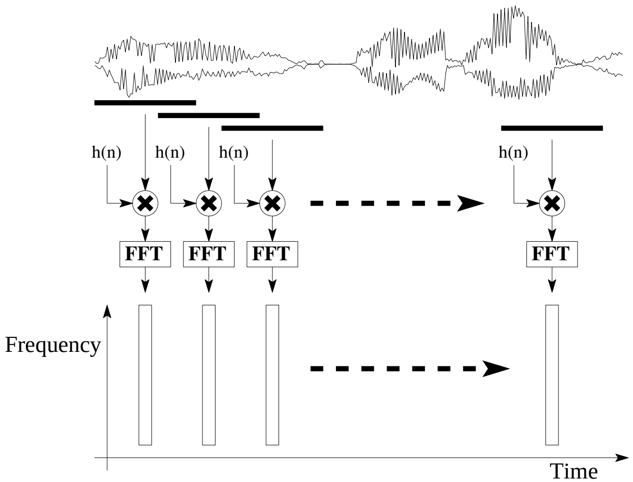
Part II

Short Time Fourier Transform

Spectrogram



Diagram

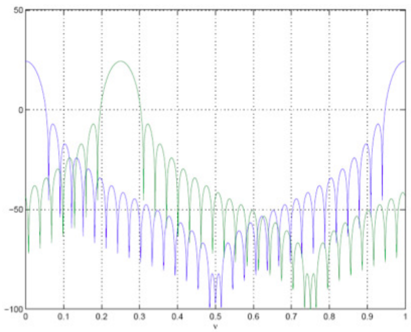


Short Time Fourier Transform

- Definition : $\widetilde{X}(t_a, \nu) = \sum_{n\in\mathbb{Z}} x(n+t_a)\, w_a(n)\, e^{-i2\pi\nu n}$
  - $w_a(n)$ is finite, real and symmetric
  - the analysis times t_a are indexed by an integer m
- Discrete version of the STFT :
  - let $\nu_k = k/N$:
    $\widetilde{X}(t_a, \nu_k) = \sum_{n\in\mathbb{Z}} x(n+t_a)\, w_a(n)\, e^{-i2\pi\nu_k n}$
  - the length of $w_a(n)$ must be $\leq N$ ($\mathrm{supp}(w_a) \subset [0, N-1]$)

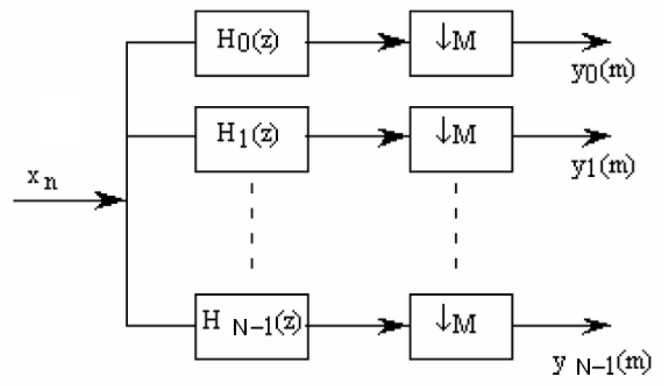
Equivalent band-pass filter

- $\widetilde{X}(t_a(m), \nu_k) = [h_k \star x](t_a(m))$ where $h_k(n) = w_a(-n)\, e^{i2\pi\nu_k n}$
- the FT of $h_k(n)$ is $H_k(e^{i2\pi\nu}) = W_a\left(e^{i2\pi(\nu_k - \nu)}\right)$
  $H_k(z) = H_0\left(zW_N^k\right)$ where $W_N^k = e^{-i2\pi \frac{k}{N}}$

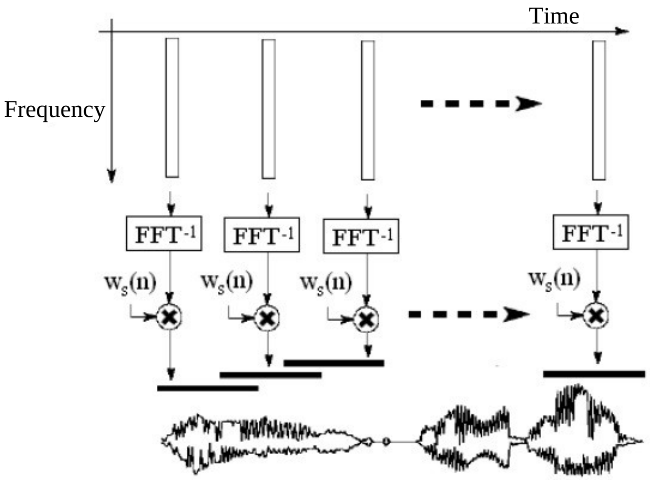


Equivalent filter banks

- Let $t_a(m) = Mm$ and $y_k(m) = \widetilde{X}(Mm, \nu_k)$
- Equivalent diagram :



Reconstruction diagram

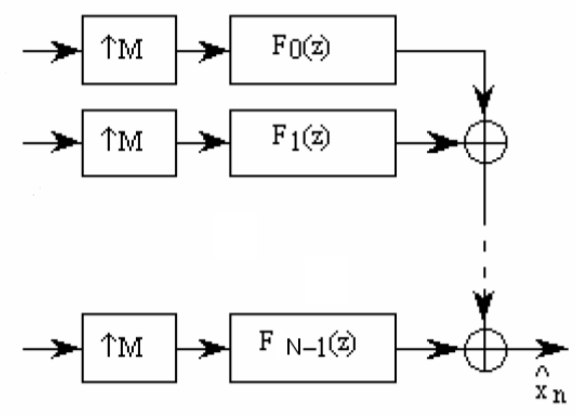


Signal reconstruction

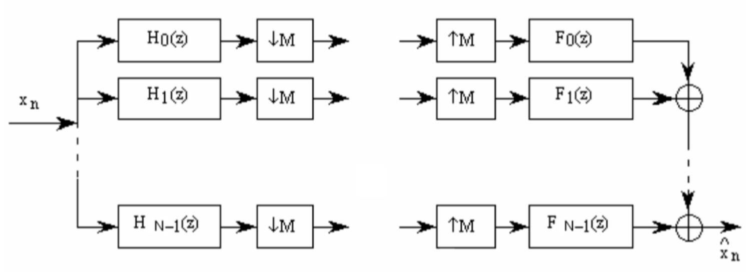
- Overlap-add synthesis
  - $\widehat{x}(n) = \sum_m w_s(n - t_a(m))\, y_w(n - t_a(m), t_a(m))$ where $\mathrm{supp}(w_s) \subset [0, N-1]$ and $y_w(n, t_a(m)) = \frac{1}{N} \sum_{k=0}^{N-1} \widetilde{X}(t_a(m), \nu_k)\, e^{i2\pi\nu_k n}$
  - Perfect reconstruction condition : $\sum_m w_a(n - t_a(m))\, w_s(n - t_a(m)) \equiv 1$
- Equivalent band-pass filter
  Let $\widetilde{y}_k(mM) = y_k(m)$, and $\widetilde{y}_k(n) = 0$ everywhere else
  Then $\widehat{x}(n) = \sum_{k=0}^{N-1} [f_k \star \widetilde{y}_k](n)$ where $f_k(n) = \frac{1}{N} w_s(n)\, e^{i2\pi\nu_k n}$
  $F_k(z) = \frac{1}{N} W_s(zW_N^k) = F_0(zW_N^k)$

Equivalent filter bank

- Equivalent diagram :



Analysis / synthesis diagram



- Particular case of perfect reconstruction filter bank, with $H_k(z) = H_0(zW_N^k)$