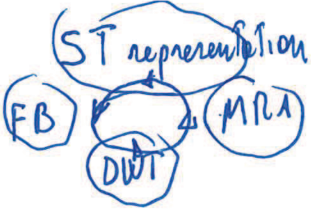
Ondelettes & Filter banks



Basic idea of the Transform-based analysis of signals: project a signal onto a basis where some characteristics are easily highlighted.

Eg. CTFT: $\hat{x}(\nu) = \int x(t)\, e^{-2i\pi\nu t}\, dt = \langle x(t), e^{2i\pi\nu t} \rangle$

$$\mathrm{STFT}(x(t); \nu) = \int x(t)\, w(t)\, e^{-2i\pi\nu t}\, dt = \langle x(t), \overline{w}(t)\, e^{2i\pi\nu t} \rangle$$

Besides frequency analysis, it could be interesting to have other kind of analysis, such as a "Time-scale" one:

$$\Psi_{a,b}(t) = \frac{1}{\sqrt{a}}\, \Psi\left(\frac{t-b}{a}\right)$$

a : scale
b : Time-shift

$$\mathrm{CTWT}\left(x(t); a, b\right) = \langle x(t), \Psi_{a,b}(t) \rangle$$

Ψ: mother wavelet

Continuous function of $a \geq 0$, $b \in \mathbb{R}$

Remember that, if $\hat{\Psi}(\nu) = \mathrm{CTFT}(\Psi(t))$,

Then $\mathrm{CTFT}\left(\Psi\left(\frac{t}{a}\right)\right) = a\, \hat{\Psi}(a\nu)$

Therefore, if $\langle x(t), \Psi(t) \rangle$ Tells about the similarity of x with Ψ at a given scale, changing the scale implies changing both temporal & frequency resolution, i.e. where is concentrated most of the energy of Ψ

If a increases, $\Psi_{a,b}(t)$ is more concentrated in time and more spread in frequency.

[2]

So our "dictionary" $\psi_{a,b}$ makes a time-scale analysis providing good freq resolution at low freq (low scale) and good time resolution at high frequency

This is particularly suitable for image analysis

Having continuous parameters a & b is "too much".

We can discretise: $a = 2^{-j}$, $b = a \cdot k$, $k, j \in \mathbb{Z}$

scale: double the resolution at each level j

time delay: in scale-related units

Therefore, if $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$,

The wavelet coefficients are $c_{j,k} = \langle x(t), \psi_{j,k}(t)\rangle$

The signal can be reconstructed as $x(t) = \sum_j \sum_k c_{j,k} \tilde{\psi}_{j,k}(t)$

where $\tilde{\psi}_{j,k}$ should suitable be found, but we anticipate that it is possible to have $\psi = \tilde{\psi}$ (orthogonal wavelets)

$$\Rightarrow \quad x(t) = \sum_{j,k} \langle x(t), \psi_{j,k}(t)\rangle \, \psi_{j,k}(t)$$

Discretisation of $x(t)$ brings the DWT.

Target: sparse representation: as many zero as possible

Reason: analysis; modeling; compression understanding

(*) page (3) If $f \in V_1$, it exist $g \in V_0$: $f(t) = g(2t)$

but g can be expressed on the basis of V_0: $g(t) = \sum g_k \varphi(t-k)$ $\Rightarrow$

$f(t) = g(2t) = \sum g_k \varphi(2t-k)$ $\Rightarrow$ $\varphi(2t-k)$ is a basis for V_1

Then we can generalise for V_j ; the coeff $2^{j/2}$ is just for normalisation

MRA

Now we need to better formalise those concepts. We introduce the idea of MRA

We consider an MRA over the space $\mathcal{L}^2(\mathbb{R})$.
It is defined as a sequence of vector subspaces V_j such that:

A
1) $\forall j \in \mathbb{Z}, \quad V_j \subset V_{j+1}$ (increasing)
2) $\bigcap V_j = \{0\}$ (constant function returning ϕ) (non redundancy)
3) $\overline{\bigcup V_j} = \mathcal{L}^2$ completeness

B
4) $f(t) \in V_j \iff f(2t) \in V_{j+1}$
5) $f(t) \in V_0 \iff f(t-k) \in V_0 \quad \forall k \in \mathbb{Z}$

C
6) It exist an orthonormal basis generated by a suitable function φ:
$\exists \varphi \in V_0 : \{\varphi(t-k)\}$ generates V_0
φ is called father wavelet

Idea of MRA: The projection of $f \in L(\mathbb{R})$ on V_j, for j that increases, gives a better & better approx of f (because of properties A, namely completeness);
moreover, it is a T-S analysis, since 6) implies that

$$\varphi_{j,k}(t) = 2^{j/2}\,\varphi(2^j t - k) \text{ is a basis of } V_j \quad (*)$$

Example
$\varphi(t) = \mathbb{1}_{[0,1]}(t)$

Note that if $\varphi \in C^{(r)}$, the MRA is said to be r-regular

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Here it could be nice to show an example with Haar MRA basis

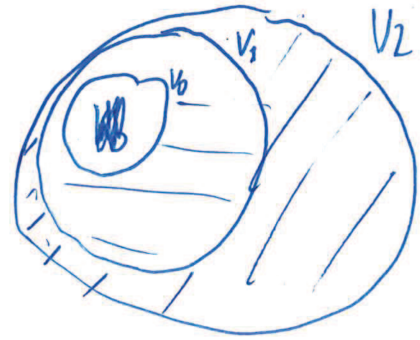
So we consider $f_j = \mathrm{Proj}\{f \mid V_j\}$ for any $f \in L^2(\mathbb{R})$ 4/

Completeness implies $f_j \xrightarrow[j]{} f$

example: if $\varphi(t) = \mathbb{1}_{(0,1)}(t)$, f_j is a piecewise constant version of f, on intervals of size 2^{-j}

We define the details $\Delta f_j = f_{j+1} - f_j$

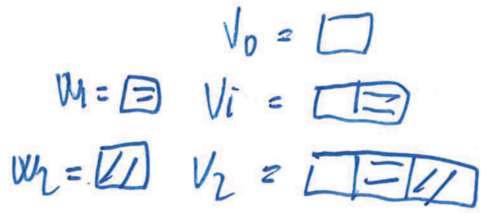
Δf_j belongs to the supplemental space W_j



and $V_j \oplus W_j = V_{j+1}$

Therefore $V_0 \oplus W_0 \oplus \ldots \oplus W_j = V_j$

and $V_j = \bigoplus_{j=-\infty}^{j} W_j$



So we can consider a representation of f as its projection on:

- V_0 and
- $W_0, W_1 \ldots W_j$

(note that the "starting" resolution 0 is arbitrary. It can be any integer

projection on V_0 = similarity with φ

projection on W_0 = details w.r.t. V_1 : or details at a first resolution level

: W_j = details from V_j to V_{j+1}, jth res. level

This representation is equivalent to $f_j = \mathrm{Proj}\{f \mid V_j\}$

Now $f_j(t) = \sum_k a_{j,k}\, \varphi_{j,k}(t)$

where $a_{j,k} = \langle f, \tilde{\varphi}_{j,k} \rangle \overset{\text{if orthogonal}}{=} \langle f, \varphi_{j,k} \rangle$

Now, let us consider the relationship between two consecutive subspaces, V_0 & V_1

Since $\varphi_0 \in V_0$ and $V_0 \subset V_1$, then $\varphi \in V_1$, so it can be expressed using the basis of V_1

$$\boxed{\varphi(t) = \sqrt{2} \sum_k c_k \varphi(2t-k) \quad (1)}$$

This is the dilation equation

example: $\varphi(t) = \mathbb{1}_{(0,1)}(t)$

$\varphi(t) = \varphi(2t) + \varphi(2t-1)$

$c_0 = \frac{1}{\sqrt{2}} \quad c_1 = \frac{1}{\sqrt{2}}$

More in general, the coeffs are computed by projection.

We consider the case of orthogonal basis:

$$c(k) = \langle \varphi(t), \varphi_{1,k}(t) \rangle = \langle \varphi(t), 2^{1/2} \varphi(2t-k) \rangle$$

$$= \sqrt{2} \int \varphi(t) \varphi(2t-n)\, dt$$

From (1), if we multiply both by $\varphi(t-m)$ & integrate:

$$\int \varphi(t) \varphi(t-m)\, dt = \sqrt{2} \sum_k c_k \int \varphi(2t-k) \varphi(t-m)\, dt$$

$\|$ $\qquad\qquad\qquad\qquad \|$ $\qquad s = t-m$, $t = s+m$

δ_m (by orthog.) $\qquad \sqrt{2} \sum_k c_k \int \varphi(2s+2m-k) \varphi(s)\, ds$

$$\| $$

$$\sum c_k \langle \varphi(s), \varphi_{1,2m-k}(s) \rangle$$

$$\|$$

$$\sum c_k c_{k-2m} = \delta_m \quad \boxed{\text{orthogonality}}$$

It can also be proven that $|\hat{c}(\nu)|^2 + |\hat{c}(\nu+\frac{1}{2})|^2 = 2$

See Mallat page 229

So, The sequence $f_j = \mathrm{Proj}(f, V_j)$ is increasing in resolution, but also redundant, since $V_j \subset V_{j+1}$

Let us consider the decomposition : $V_j = V_0 \oplus W_0 \oplus \dots \oplus W_j$ meaning That f_j can be represented in Terms of f_0 and of projections in W_i, $i \in \{0, \dots j\}$

These projections are the Δf_i

~~Since~~ Let us look for a basis of W_0.

Since $V_0 \oplus W_0 = V_1$, $W_0 \subset V_1$

Mallat proved the following. If $\varphi(\cdot)$ is a father wavelet, i.e., $\varphi(t-k)$ is a basis of V_0 ; and c_k are The coefficient of the dilation equation, i.e.

$$\varphi(t) = \sqrt{2} \sum_k c_k \varphi(t-2k)$$

~~There~~ Moreover, set $C(z) = \sum c_k z^{-1}$, $D(z) = -z^{-(N-1)} C(-z^{-1})$

and $d_k = IZT(D(z))$

defining $\boxed{\psi(t) = \sqrt{2} \sum_k d(k) \phi(2t-k)}$ wavelet equation

Then $\psi_{j,k}(t) = 2^{-j/2} \psi(2^{-j} t - k)$ is a basis for W_j

Proof : see Mallat, Theorem 7.3 page 236.

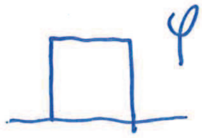
This is very interesting because, since $L^2(\mathbb{R}) = \overline{\bigcup W_j}$,

Then The wavelets $\psi_{j,k}$ are a basis for $L^2(\mathbb{R})$

But The $\psi_{j,k}$ are exactly a Time-Scale analysis

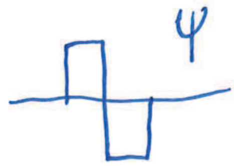
So, a link is established between MRA and ST

Example: if $\varphi = \mathbb{1}_{(0,1)}$, $c_0 = \frac{1}{\sqrt{2}}$, $c_1 = \frac{1}{\sqrt{2}}$



$\Rightarrow$ $d_0 = \frac{1}{\sqrt{2}}$ & $d_1 = -\frac{1}{\sqrt{2}}$

$$\Rightarrow \psi(t) = \mathbb{1}_{(0,1/2)}(t) - \mathbb{1}_{(1/2,1)}(t)$$



$\underline{c}$ and $\underline{d}$ are just the filters of the Haar F.B.

Summary:

$$\varphi(t) = \sqrt{2}\, \Sigma_k\, c_k\, \varphi(2t-k) \qquad \text{dilation eqn.}$$

$$\psi(t) = \sqrt{2}\, \Sigma\, d_k\, \varphi(2t-k) \qquad \text{wavele eqn}$$

with $(2)\ \Sigma\, c_k c_{k-2m} = \delta_m$ ← orthogonality, H.B.

$D(z) = -z^{-(N-1)}\, C(-z^{-1})$ ← $\boxed{\text{CQF filter bank}}$

So, given a filter satisfying (2) and $|\hat{c}(\nu)|^2 + |\hat{c}(\nu+1/2)|^2 = 2$, we deduce d_k: Then (difficult in general) we find φ satisfying the Dilation equation;

Then we find ψ.

In conclusion we represent $f_j = \mathrm{Proj}(f|V_j) =$

$$= \underbrace{\mathrm{Proj}(f|V_0)}_{\text{"approximation"}} + \underbrace{\sum_{i=0}^{j} \mathrm{proj}(f|W_i)}_{\text{"details"}}$$

basis of W_j

$$= \sum_k a_{0,k}\, \varphi(t-k) + \sum_j \sum_k b_{j,k}\, \psi_{j,k}(t)$$

Now we have to compute $a_{j,k}$, $b_{j,k}$

in terms of f, c, d

The coefficients $a_{j,k}$, $b_{j,k}$ are the MRA representation

of f in V_j : for $j \to +\infty$, they fully represent f

It is a recursive relationship. Let us start at level 0.

If $f_1 \in V_1$, also $f_1 \in V_0 \oplus W_0$

So we can use a basis of V_0 ; or the 2 basis, V_0's an W_0's :

$$f_1(t) = \sum_k a_{1k}\varphi_{1k} = \sum_k a_{0k}\varphi_{0k} + \sum_k b_{0k}\psi_{0,k} \quad (3)$$

From DE & WE,

$$\boxed{\begin{array}{l} l = 2m+k \\ k = l-2m \end{array}}$$

$$\varphi_{0,m}(t) = \varphi(t-m) = \sqrt{2}\sum_k c_k\,\varphi(2t-2m-k)$$

$$= \sum_l c(l-2m)\sqrt{2}\,\varphi(2t-l)$$

$$= \sum_l c(l-2m)\,\varphi_{1,l}(t) \quad (4)$$

likewise $\quad \psi_{0,m}(t) = \sum_l d(l-2m)\,\varphi_{1,l}(t) \quad (5)$

Now, $\quad a_{0,k} = \langle f(t), \varphi_{0k}(t)\rangle = \sum_l c(l-2m)\langle f(t), \varphi_{1,l}(t)\rangle$

$$= \sum_l c(l-2k)\,a_{1,l}$$

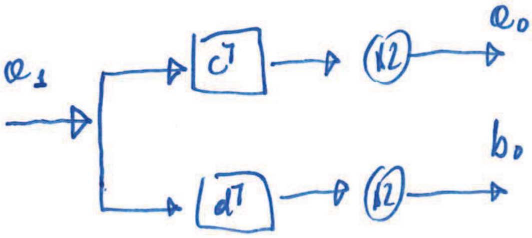
likewise $\quad b_{0,k} = \langle f(t), \psi_{0,k}(t)\rangle = \sum_l d(l-2m)\,a_{1,l}$

Defining $\quad c^T(n) = c(-n)$

$$d^T(n) = d(-n)$$

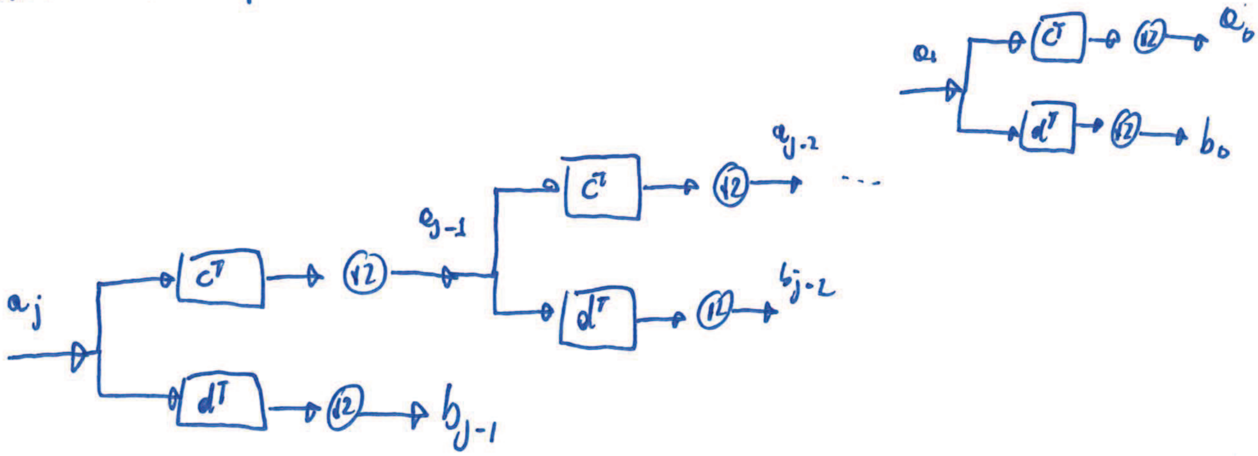
$$a_{0,k} = \sum_l c^T(2k-l)\, a_{1,l} = (c^T * a_1) \downarrow 2$$

$$b_{0,k} = \sum_l d^T(2k-l)\, a_{1,l} = (d^T * a_1) \downarrow 2$$



c & d are CQF!

With multiple levels:



so f_j is represented by the coefficients

a_0, b_0, b_1, ... b_{j-2}, b_{j-1}

lowest resolution approx.

increasing resolution details

$a_j \longrightarrow \{a_0, b_0 \ldots b_{j-2}, b_{j-1}\}$ DWT over j levels

$a_1 \longrightarrow \{a_0, b_0\}$ DWT over 1 level $\Leftrightarrow$ Two ways FB

Synthesis F.B.

$$f_1(t) = \sum a_{0,k}\, \varphi_{0,k}(t) + \sum b_{0,k}\, \psi_{0,k}(t) \qquad (\text{it es } (3))$$

$$\Rightarrow \quad \langle f_1(t), \varphi_{1l} \rangle = \sum a_{0,k} \langle \varphi_{0,k}(t), \varphi_{1l}(t) \rangle + \sum b_{0k} \langle \psi_{0,k}(t), \varphi_{1,l}(t) \rangle$$

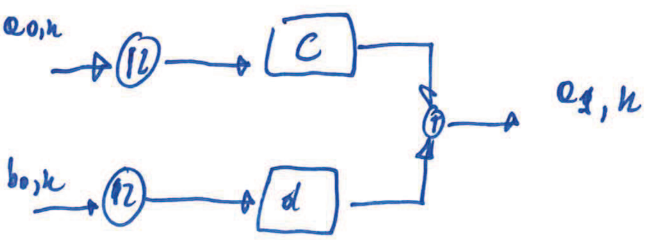
$$a_{1,l} = \sum a_{0,k} \sqrt{2} \int \varphi(t-k)\varphi(2t-l)\,dt + \dots$$

$$\overset{\substack{s=t-k \\ t=s+k}}{=} \sum a_{0k} \sqrt{2} \int \varphi(s)\, \varphi(2s+2k-l)\,dt + \dots$$

$$= \sum a_{0k}\, c_{l-2k} + \sum b_{0k}\, d_{l-2k} \quad \text{likewise}$$

Now, $\sum a_{0k}\, c_{l-2k} = a_{00} c_l + a_{01} c_{l-2} + a_{02} c_{l-4} + \dots$

$$= a_{00} c_l + 0 \cdot c_{l-1} + a_{01} c_{l-2} + 0 \cdot c_{l-3} + a_{0,2} c_{l-4} + \dots$$



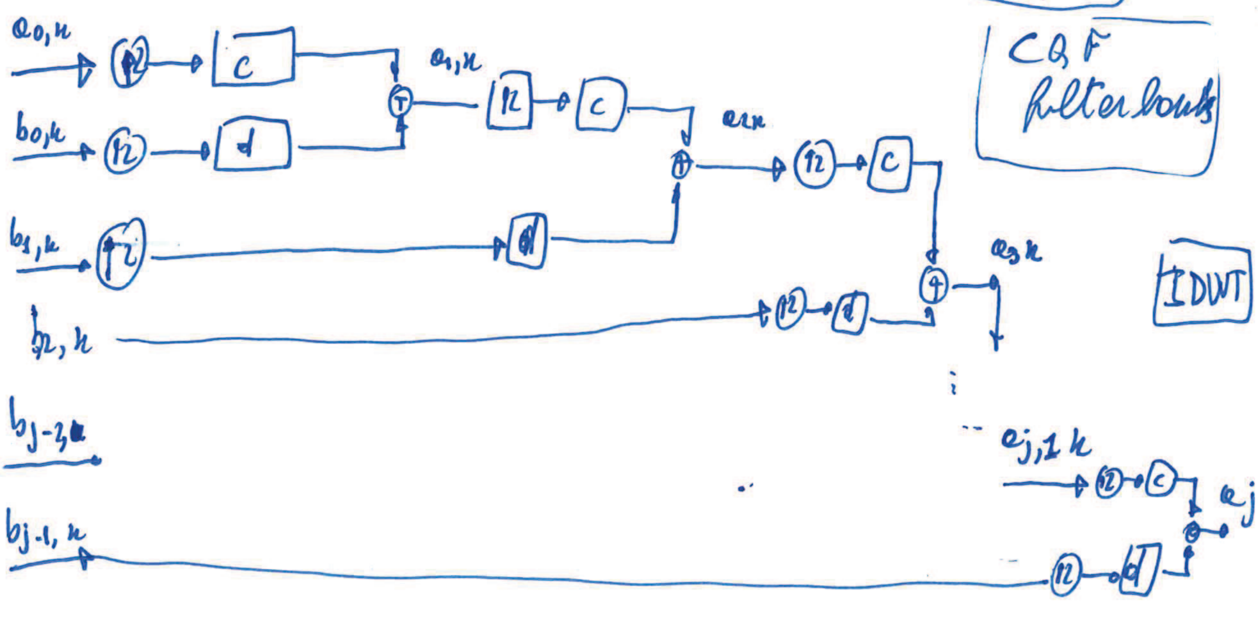
one level of wavelet analysis

Synthesis FB

c^T, d^T

c, d

CQF filterbanks



IDWT

Other properties of FB for wavelet analysis

Vanishing moments

$$f_j(t) = \sum a_k \varphi_{0k} + \sum_l \sum_k b_{l,k} \psi_{l,k}$$

It can be proven that the approx error is

$$\| f(t) - f_j(t) \| \approx C (2^{-j})^p \, \| f^{(p)}(t) \| \qquad (6)$$

$f \in C^p$

where p is the multiplicity of the zero of $\hat{h}_0(\nu)$ in $1/2$:

$$\hat{h}_0(\nu) = \left(\frac{1 + e^{-i2\pi\nu}}{2} \right)^p \cdot Q(\nu) \iff H(z) = \left(\frac{1 + z^{-1}}{2} \right)^p Q(z)$$

This is equivalent to:

$$\forall j \in \{0, \dots p-1\} \ , \ \sum_n (-1)^n n^j h_0(n) = 0$$

(6): polynomials of order up to p are perfectly reconstructed

Also, it can be proved that $h_1 * s$ where s is polynomial is null

So, having a high number of V.M. allows to represent s only with a, since the b will be zero.

If s is piecewise polynomial, b is non zero only "near discontinuities"

All this is important ~~so~~ since we achieve a sparse representation of the signal

Linear phase and borders

What happens when using DWT on finite signals?

- coeff. expansion

card $\{a_0\} = \frac{M+N-1}{2}$ $\xrightarrow{\text{Total}}$ $M+N-1$

card $\{b_0\} = \frac{M+N-1}{2}$ expansion

→ at each level!

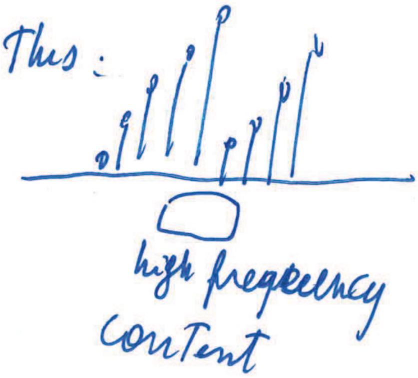
- how to avoid this?

$x \to \tilde{x}$, periodised version of x

$$\begin{cases} a = \tilde{x} * h_0 \\ b = \tilde{x} * h_1 \end{cases}$$ a, b are also periodical with the same period as $\tilde{x}$

Then, M coeff of a represent perfectly $\tilde{x}$, and so x

But this means that we are filtering this:



Periodisation generates high frequency content, i.e. more non-null coeffs in b

Solution: symmetric periodization

$$\tilde{x} = \begin{cases} x(n) & n: 0 \dots M-1 \\ x(M-1-n) & n: M \dots 2M-1 \end{cases}$$

Then

$$a = (h_0 * \tilde{x}) \downarrow 2$$

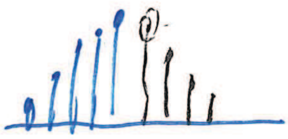
$$b = (h_1 * \tilde{x}) \downarrow 2$$

you need 2M coeffs,

__unless__ h_0, h_1 are __linear phase__

In that case, a & b are also symmetrical, and M samples are enough.

The advantage is that we filter: less high freq. content



smoother signal

__But__ There is no linear phase CQF except Haar, which has only 1 VM!

So, for having more VM & linear phase, we must go beyond CQF, and use "biorthogonal" filters

Why orthogonality is important!

because it can be proven that, with orthog. filters,

⚠ Core of finite signals

$$\| a_1 \|^2 = \| a_0 \|^2 + \| b_0 \|^2 = \| w_0 \|^2 \qquad w_0 = \begin{bmatrix} a_0 \\ \hline b_0 \end{bmatrix}$$

Now, if we perform compression, we reduce the info in w_0

: $\tilde{w}_0$ is a degradation of w_0 :

Now, by orthogonality, $a_1 = \underline{\underline{W}}^T \underline{w_0}$, $\underline{\underline{W}}$ orthogonal matrix.

and $\underline{w_0} = \underline{\underline{W}} \, \underline{a_0}$

and $\tilde{a}_1 = \underline{\underline{W}}^T \tilde{w}_0$

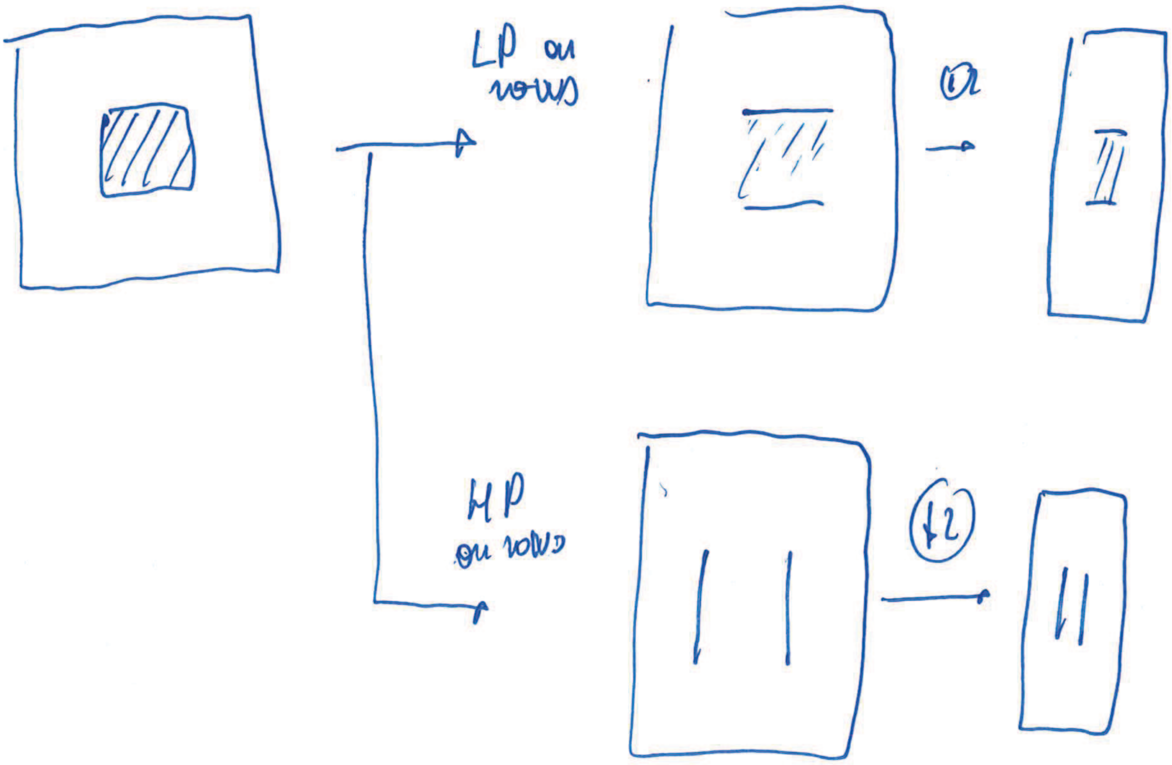
$$\| a_1 - \tilde{a}_1 \|^2 = (a_1 - \tilde{a}_1)^T (a_1 - \tilde{a}_1) = (w_0 - \tilde{w}_0)^T \underline{\underline{W}}^T \underline{\underline{W}} (w_0 - \tilde{w}_0)$$

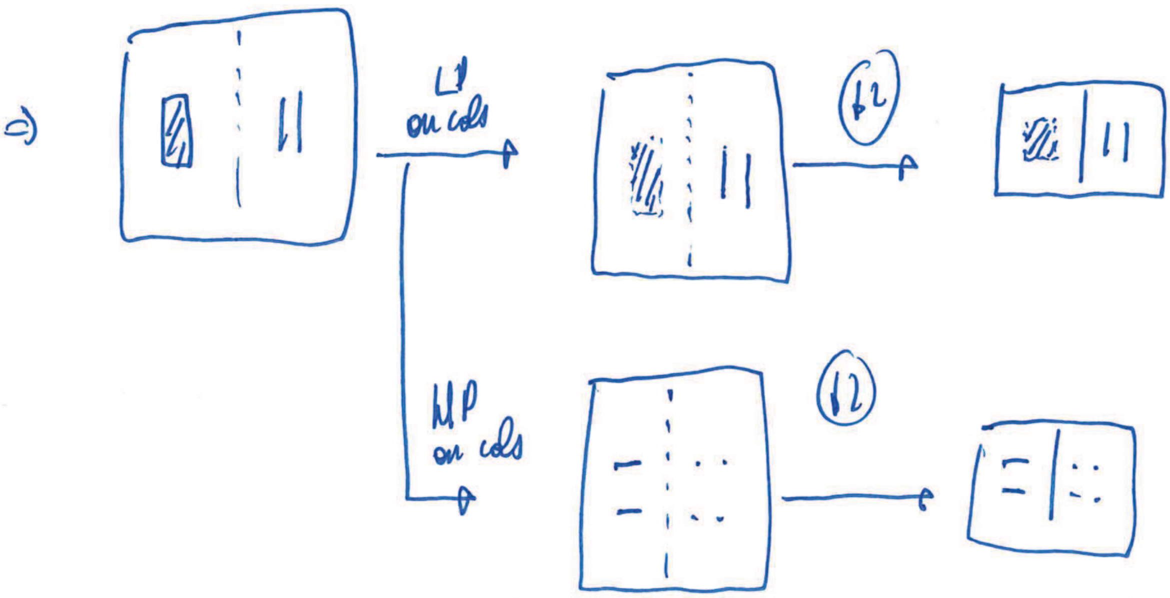
$$= \| w_0 - \tilde{w}_0 \|^2$$

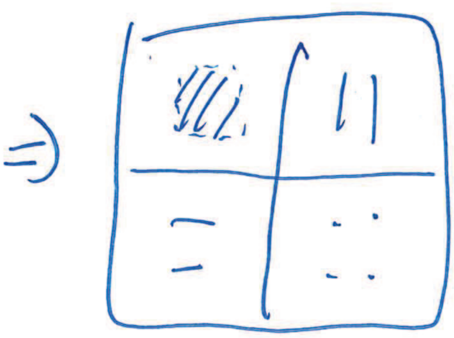
So, an orthogonal DWT is an isometry.

Therefore, we can control the error on a_1 by controlling the error on w_0 : useful in compression and denoising, since we can optimise our processing by minimizing $\| w - \tilde{w} \|$, which is simpler than minimizing $\| W^T w - W^T \tilde{w} \| = \| a_1 - \tilde{a}_1 \|$

Examples on images

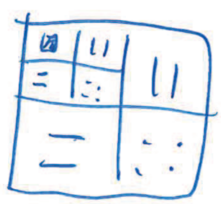






1 level of DWT

2 levels:



⇒ ST analysis