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Quadrature methods (I)

Exercice 1 (Error of simple quadrature rules).
Let f be a smooth function over $[a,b]$. The purpose of this exercise is to evaluate the approximation error of different simple quadrature rules defined by :

$$e(t) = \int_a^b (f(u) - Lf_n(u))\mathrm{d}u \ .$$

Where Lf_n is the Lagrange polynomial of f defined with the interpolation nodes $(x_0, \ldots, x_n)$. We are going to assume the following result :

$$(\forall t \in [a,b])(\exists c_t \in]a,b[) \quad f(t) = Lf_n(t) + \frac{f^{(n+1)}(c_t)}{n+1!} w_{n+1}(t)$$

where $w_n(t) = \prod_{i=0}^{n-1}(t - x_i)$.

1. Let f, g be two continuous functions over $[a,b]$. Assume that g has constant sign over $[a,b]$. Show that there exists $c \in [a,b]$ such that :

$$\int_a^b f(u)g(u) = f(c)\int_a^b g(u)du$$

2. Consider the left rectangle and trapezoidal rules :

$$I_r(f) = (b-a)f(a), \qquad I_T(t) = \frac{(b-a)}{2}(f(a) + f(b))$$

Show that their respective errors are given by :

$$e_r(t) = \frac{f'(c_1)}{2}(b-a)^2, \qquad e_T(t) = -\frac{f"(c_2)}{12}(b-a)^3$$

3. Now we turn to the more delicate cases where w does not have a constant sign. Assume that $\int w(t)dt = 0$. Show that one can add an additional interpolation point without changing the error.
4. Deduce the error for the midpoint rule and Simpson's rule :

$$I_m(f) = (b-a)f(\frac{a+b}{2}), \qquad I_s(t) = \frac{(b-a)}{6}\left(2f(a) + 4f(\frac{a+b}{2}) + 2f(b)\right)$$

Exercice 2 (Cavalieri-Simpson).
Let f be a smooth (C^∞) function over $[a,b]$. We consider the problem of evaluating the integral $I(f) = \int_a^b f(x)\,\mathrm{d}x$. We approximate $I(f)$ using a simple quadrature method of Newton-Cotes of order l with the nodes : $x_i = a + i\frac{b-a}{l}$, $i = 0, \ldots, l$, and weights $\lambda_0, \ldots, \lambda_l$, such that :

$$\hat{I}(f) = (b-a)\sum_{i=0}^{l} \lambda_i f(x_i)\,.$$

1. Show that the weights λ_i are independent of (a,b). Without loss of generality, we can assume that $(a,b) = (-1,1)$.
2. Find λ_i for $l = 2$.
3. Show that this quadrature method is of order 3.

Exercice 3 (Legendre-Gauss quadrature).
Let $I(f) = \int_a^b f$, where f is a smooth function over $[a,b]$.

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1. Show that the corresponding weights λ_0, λ_1 are independent of $[a, b]$ and that the points corresponding to the interval $[a', b'] = [-1, 1]$ are related to (x_0, x_1) via

$$x_i = \frac{b+a}{2} + \frac{b-a}{2} u_i \,.$$

2. Find $(u_0, u_1, \lambda_0, \lambda_1)$ for $[a, b] = [-1, 1]$.

Exercice 4 (Homework)**.**

Consider the problem of approximating $I_{a,b}(f) = \int_a^b f(x)\,\mathrm{d}x$, where f is in $\mathcal{C}^\infty$ via

$$\hat{I}_{a,b}(f) = (b-a)(\lambda_0 f(a) + \lambda_1 f(b) + \lambda_2 f'(a)),$$

with $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3$.

1. For $a = 0$, $b = 1$, find $(\lambda_0, \lambda_1, \lambda_2)$ such that the order of the method is exact for polynomials of degree ≤ 2.
2. Deduce an expression of $(\lambda_0, \lambda_1, \lambda_2)$ as function of a and b such that the method is of order 2 for any interval $[a, b]$.