Examination of the teaching unit
Représentations des signaux **- TSIA201**

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Duration: 1:30

All documents are permitted. However electronic devices (including calculators) are forbidden.

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1 Rejector filter

Let us consider the transfer function $H(z) = \frac{1-2\cos(\theta)z^{-1}+z^{-2}}{1-2\rho\cos(\theta)z^{-1}+\rho^2z^{-2}}$, with $0 < \rho < 1$. Check that $H(z)$ can be factorized in the form $H(z) = \frac{(1-z_0z^{-1})(1-z_0^*z^{-1})}{(1-\rho z_0z^{-1})(1-\rho z_0^*z^{-1})}$, where z_0 is to be expressed as a function of θ. What is the normalized frequency rejected by this filter? What is the domain of convergence of its stable implementation? Is this implementation causal? Write the corresponding input/output relationship.

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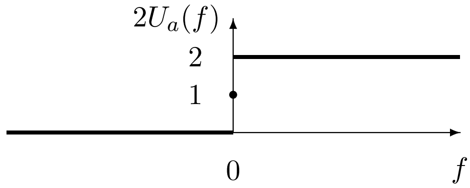
2 Hilbert filter

Let $x_a(t)$ be a continuous time (analog) real signal. The *analytic* signal associated to $x_a(t)$ is the signal $z_a(t)$ whose the CTFT is expressed as $Z_a(f) = 2\,U_a(f)X_a(f)$, where $U_a(f)$ is the unit step function, whose value is 1 for $f > 0$, and 0 for $f < 0$. For continuity reasons, we assume that $U_a(0) = \frac{1}{2}$. The filter of frequency response $2U_a(f)$ is referred to as the *analytic filter*.



1. Which property does function $X_a(f)$ satisfy? Deduce the expression of $\frac{1}{2}(Z_a(f)+Z_a^*(-f))$ as a function of $X_a(f)$, and prove that the real part of $z_a(t)$ is equal to $x_a(t)$. We can then write $z_a(t) = x_a(t) + iy_a(t)$, where the real signal $y_a(t)$ is defined as the imaginary part of $z_a(t)$.

2. Prove that $y_a(t)$ can be obtained from $x_a(t)$ by linear filtering of frequency response $H_a(f) = -i\,\text{sign}(f)$, where $\text{sign}(f) = 1$ for $f > 0$, $\text{sign}(f) = -1$ for $f < 0$, and $\text{sign}(0) = 0$. Filter $H_a(f)$ is referred to as the *Hilbert filter*, and $y_a(t)$ is called the *Hilbert transform* of $x_a(t)$.

Let us assume that the signal $x_a(t)$ satisfies the assumptions of the sampling theorem: there exists a frequency F_s such that the support of $X_a(f)$ is included in $]-\frac{F_s}{2}, \frac{F_s}{2}[$. We then consider the sampled signals $x(n) = x_a(nT_s)$ and $y(n) = y_a(nT_s)$, where $T_s = 1/F_s$. We remind the relationship between the DTFT $X(e^{2i\pi\nu})$ and the CTFT $X_a(f)$:

$$X(e^{2i\pi\nu}) = \frac{1}{T_s} \sum_{k\in\mathbb{Z}} X_a\left(\frac{\nu+k}{T_s}\right) \tag{1}$$

3. Simplify the expression (1) when $\nu \in]-\frac{1}{2}, \frac{1}{2}[$. Check that $Y(e^{2i\pi\nu})$ satisfies a similar expression. Deduce that the signal $y(n)$ can also be expressed as the output of the discrete filter of frequency response $H(e^{2i\pi\nu}) = -i\,\mathrm{sign}(\nu)$ for $\nu \in]-\frac{1}{2}, \frac{1}{2}[$ (and $H(e^{2i\pi\nu}) = 0$ for $\nu = \pm\frac{1}{2}$), applied to the input signal $x(n)$.

Remark: the discrete filter $H(e^{2i\pi\nu})$ allows us to directly compute the samples $y(n)$ of the Hilbert transform from the samples $x(n)$, without having to perform a digital/analog conversion.

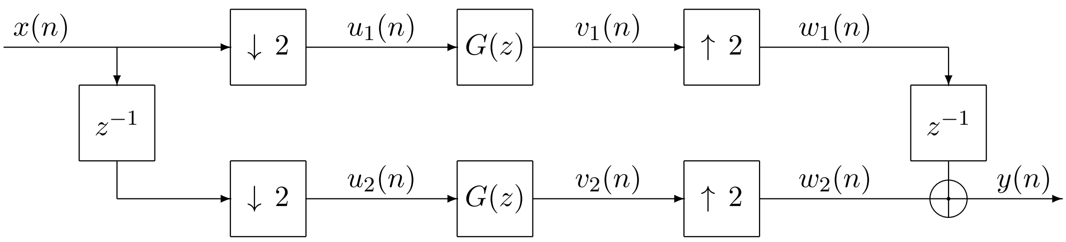
4. By applying the inverse DTFT, prove that the impulse response $h(n)$ satisfies $h(n) = \frac{2}{\pi n}$ if n is odd, and 0 if n is even.

5. Is this filter causal? Stable? Of finite (FIR) or infinite (IIR) impulse response?

6. For a discrete filter of impulse response $g(n)$ and of transfer function $G(z)$, what is the impulse response of the filter of transfer function $G(z^2)$? By using the fact that the even coefficients of $h(n)$ are zero, deduce that there exists a transfer function $G(z)$, such that $H(z) = z^{-1}G(z^2)$. What is the impulse response $g(n)$?

7. We want to approximate the ideal filter $G(z)$ by using the window method, in order to synthesize a linear phase FIR filter, of type 4 (even length N, antisymmetric impulse response $g(n)$). Quickly summarize the principle of the window method, its advantages and its drawbacks.

8. Now, we want to prove that the following diagram provides an efficient implementation of the discrete Hilbert filter $H(z)$:



We remind that $U_1(z) = \frac{1}{2}(X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}}))$. Express $U_2(z)$ as a function of $X(z)$, then $V_1(z)$ and $V_2(z)$ as a function of $U_1(z)$ and $U_2(z)$, then $W_1(z)$ and $W_2(z)$ as a function of $V_1(z)$ and $V_2(z)$, and finally $Y(z)$ as a function of $W_1(z)$ and $W_2(z)$. By substitution, retrieve the relationship $Y(z) = H(z)X(z)$.

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3 MRA and Wavelet transform

1. Show that, for the Haar filter ($h_0[0] = h_0[1] = \frac{\sqrt{2}}{2}$, the other coefficients are zero), a suitably scaled indicator function of the interval $(0, 1)$ is the father wavelet for an MRA (hint: use the dilation equation). Propose a mother wavelet function and show that with such a choice of ψ the wavelet equation holds.

2. Let $\phi(t)$ be following function, see also Fig.3:

$$\phi(t) = \begin{cases} t & \text{if } t \in (0,1) \\ 2-t & \text{if } t \in (12) \\ 0 & \text{otherwise} \end{cases}$$

   It can be shown that this ϕ is the father wavelet of an MRA. We say that $\phi \in V_0$ and that the integer-shifts of ϕ produce a basis of V_0. By using the dilation equation, show (by sketching the graphs of $c_k\phi(2t-k)$) that , with suitable normalization, the representation of ϕ on V_1 is given with coefficients c_k as follows:

$$c_k = \begin{cases} 1/2 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ 1/2 & \text{if } k = 2 \\ 0 & \text{otherwise} \end{cases}$$

   Find the coefficients d_k from the coefficients c_k (normalization can be ignored for simplicity); finally, find the corresponding mother wavelet ψ by using the wavelet equation.

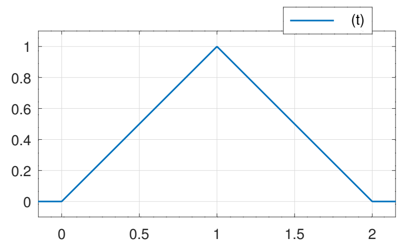

Figure 1: Father wavelet $\phi(t)$

3. We want to compute the projection of a signal $x : t \in \mathbb{R} \to \mathbb{R}$ onto a MRA, but we only have access to the samples $x[n] = x(t)|_{t=n\in\mathbb{Z}}$. Is it correct to use these samples as input of the analysis filterbank? Is there any implicit assumption in doing this, in particular on the father wavelet function ϕ?
