

# Stochastic Dual Coordinate Descent

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## 1 Introduction

In machine learning, the process of fitting a model to the data requires to solve an optimization problem. The difficulty resides in the fact that this optimization quickly becomes very complex when dealing with real problems. The Stochastic Coordinate Descent (SGD) is a very popular algorithm to solve those problems because it has good convergence guaranties. Yet, the SGD does not have a good stopping criteria, and its solutions are often not accurate enough.

The Stochastic Dual Coordinate Ascent tries to solves the optimization problem by solving its dual problem. Instead of optimizing the weights, we optimize a dual variable from which we can compute the weights and thus solve the former. This method can give good results for specific problems : for instance, solving the dual problem of the SVM has proven to be effective and to give interesting results, with a linear convergence in some cases.

In this report, we compile the key theoretical points necessary to have a global understanding of the SDCA.

First we introduce the general forms of the machine learning problems we want to use SDCA on and how we get to the dual form. Then we introduce the duality gap. Then we study computational performances of the method by trying to apply SDCA on concret problems. Finally we conclude on SDCA strengths and weaknesses.

**Note** *We especially added experimentations since the presentation of our poster.*

## 2 Methods

### 2.1 Problems studied

In this report, we study the SDCA algorithm on several machine learning problems. As an introduction, the logistic regression is an interesting problem. The multiclass logistic regression, with the cross entropy loss, is a generalization of the binary logistic regression that requires more work, so for now we stick to the binary problem which is interesting enough to present our work.

We use the following usual notations :

$X \in \mathbf{X} = \mathbb{R}^p$  the random variable for the description space;

$Y \in \mathbf{Y} = \{0, 1\}$  the random variable for the label.

We recall that the model is the following :

$$\frac{\mathbb{P}(y = 1|X = x)}{\mathbb{P}(y = -1|X = x)} = w^T x, \quad w \in \mathbb{R}^p \quad (1)$$

We want to find  $w$  so that it maximizes the likelihood, or log-likelihood, with a term of regularization:

$$\min_w C \sum_i \log(1 + e^{-y_i w^T x_i}) + \frac{1}{2} w^T w \quad (2)$$

In order to get the dual problem, we rewrite it with an artificial constraint  $z_i = e^{-y_i w^T x_i}$ , and we have the following lagrangian :

$$\mathcal{L}(w, z, \alpha) = \sum_i (C \log(1 + z_i) + \alpha_i z_i) - \sum_i \alpha_i e^{-y_i w^T x_i} + \frac{1}{2} w^T w \quad (3)$$

We note  $w^*$  and  $z^*$  the variables solution of the optimization problem

$$\min_{w, z} \mathcal{L}(w, z, \alpha) = \mathcal{L}(w^*, z^*, \alpha) = \psi(\alpha) \quad (4)$$

Where,  $w^* = \sum_i \alpha_i y_i x_i$ .

It leads to the following dual problem :

$$\begin{aligned} \max_{\alpha} \quad & \sum_{i \in I} (-\alpha_i \log(\alpha_i) - (C - \alpha_i) \log(C - \alpha_i)) - \frac{1}{2} \alpha^T Q \alpha \\ \text{s.t.} \quad & I = \{i, 0 < \alpha_i < C\} \\ & 0 \leq \alpha_i \leq C \end{aligned} \quad (5)$$

We used the notation  $Q = (Q_{ij})_{i,j}$  where  $Q_{ij} = y_i x_i^T x_j y_j$

### 2.2 Duality Gap

Let  $x_1, \dots, x_n \in \mathbb{R}^d$ ,  $\phi_1, \dots, \phi_n$  scalar convex functions,  $\lambda > 0$  regularization parameter. Let us focus on the following optimization problem:

$$\min_{w \in \mathbb{R}^d} P(w) \quad \text{where } P(w) = \left[ \frac{1}{n} \sum_{i=1}^n \phi_i(w^\top x_i) + \frac{\lambda}{2} \|w\|^2 \right] \quad (6)$$

with solution  $w^* = \arg \min_{w \in \mathbb{R}^d} P(w)$ .

Moreover, we say that a solution  $w$  is  $\epsilon_P$ -sub-optimal if  $P(w) - P(w^*) \leq \epsilon_P$ . We analyze here the required runtime to find an  $\epsilon_P$ -sub-optimal solution using SDCA.

Let  $\phi_i^* : \mathbb{R} \rightarrow \mathbb{R}$  be the convex conjugate of  $\phi_i$  :  $\phi_i^*(u) = \max_z (zu - \phi_i(z))$ .

The dual problem of (6) is defined as follows:

$$\max_{\alpha \in \mathbb{R}^n} D(\alpha) \quad \text{where } D(\alpha) = \left[ \frac{1}{n} \sum_{i=1}^n -\phi_i^*(-\alpha_i) - \frac{\lambda}{2} \left\| \frac{1}{\lambda n} \sum_{i=1}^n \alpha_i x_i \right\|^2 \right] \quad (7)$$

with solution  $\alpha^* = \arg \max_{\alpha \in \mathbb{R}^n} D(\alpha)$ .

If we define  $w(\alpha) = \frac{1}{\lambda n} \sum_{i=1}^n \alpha_i x_i$ , then we have  $w(\alpha^*) = w^*$ ,  $P(w^*) = D(\alpha^*)$ ,  $\forall (w, \alpha), P(w) = D(\alpha)$  due to classic optimization results, and the duality gap  $P(w(\alpha)) - P(w^*)$ .

## 2.3 The SDCA algorithm

The SDCA algorithm is as follow.

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**Algorithm 1** Procedure SCDA with averaging option

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procedure SCDA( $\alpha^{(0)}, \phi, T_0$ )
   $w^{(0)} \leftarrow w(\alpha^{(0)})$ 
  for  $t = 1, \dots, T$  do
    Randomly pick  $i$ 
     $\Delta \alpha_i \leftarrow \arg \max -\phi_i^*(-(\alpha_i^{(t-1)} + \Delta \alpha_i)) - \frac{\lambda n}{2} \|w^{(t-1)} + (\lambda n)^{-1} \Delta \alpha_i x_i\|^2$ 
     $\alpha^{(t)} \leftarrow \alpha^{(t-1)} + \Delta \alpha_i e_i$ 
     $w^{(t)} \leftarrow w^{(t-1)} + (\lambda n)^{-1} \Delta \alpha_i x_i$ 
  return  $\bar{w} = \frac{1}{T-T_0} \sum_{i=T_0+1}^T w^{(i)}$ 

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$T_0$  can be chosen between 1 to  $T$ , and is generally chosen equal to  $T/2$ . However, in practice, these parameters are not required as the duality gap is used to terminate the algorithm.

## 2.4 SDCA vs SGD

The SDCA consists in taking a coordinate, optimize the problem on that coordinate by solving a subproblem, and then update the weights.

In classic stochastic gradient descent (SGD), we optimize the primal problem by looking at its gradient only for one  $x$ .

In this approach, we

### 3 Experiments

In this study, SDCA is computed either for  $L$ -Lipschitz loss functions or for  $(1/\gamma)$ -smooth loss functions. We recall that a function  $\phi_i : \mathbb{R} \rightarrow \mathbb{R}$  is  $L$ -Lipschitz if  $\forall a, b \in \mathbb{R}, |\phi_i(a) - \phi_i(b)| \leq L|a - b|$ , and that a function  $\phi_i : \mathbb{R} \rightarrow \mathbb{R}$  is  $(1/\gamma)$ -smooth if it is differentiable and its derivative is  $(1/\gamma)$ -Lipschitz. Moreover, if  $\phi_i$  is  $(1/\gamma)$ -smooth, then  $\phi_i^*$  is  $\gamma$ -strongly convex. The different loss functions used are described in the table below.

<p><b>Squared loss:</b>  <math>\phi_i(a) = (a - y_i)^2</math>  <math>\phi_i^*(-a) = -ay_i + a^2/4</math>  <math>\Delta\alpha_i = \frac{y_i - x_i^\top w^{(t-1)} - 0.5\alpha_i^{(t-1)}}{0.5 + \ x_i\ ^2/(\lambda n)}</math></p>	<p><b>Absolute deviation loss:</b>  <math>\phi_i(a) =  a - y_i </math>  <math>\phi_i^*(-a) = -ay_i, a \in [-1, 1]</math>  <math>\Delta\alpha_i = \max\left(1, \min\left(1, \frac{y_i - x_i^\top w^{(t-1)}}{\ x_i\ ^2/(\lambda n)} + \alpha_i^{(t-1)}\right)\right) - \alpha_i^{(t-1)}</math></p>
<p><b>Log loss:</b>  <math>\phi_i(a) = \log(1 + \exp(-y_i a))</math>  <math>\phi_i^*(-a) = -ay_i \log(ay_i) + (1 - ay_i) \log(1 - ay_i)</math>  <math>\Delta\alpha_i = \frac{(1 + \exp(x_i^\top w^{(t-1)} y_i))^{-1} y_i - \alpha_i^{(t-1)}}{\max(1, 0.25 + \ x_i\ ^2/(\lambda n))}</math></p>	<p><b><math>(\gamma</math>-smoothed) Hinge loss:</b>  <math>\phi_i(a) = \max\{0, 1 - y_i a\}</math>  <math>\phi_i^*(-a) = -ay_i + \gamma a^2/2, ay_i \in [0, 1]</math>  <math>\Delta\alpha_i = y_i \max\left(0, \min\left(1, \frac{1 - x_i^\top w^{(t-1)} y_i - \gamma \alpha_i^{(t-1)} y_i}{\ x_i\ ^2/(\lambda n) + \gamma} + \alpha_i^{(t-1)} y_i\right)\right) - \alpha_i^{(t-1)}</math></p>

Table 1: Used loss functions, convex conjugates and closed form of solutions of problem (\*).

For the sake of simplicity, we consider the following assumptions:  $\forall i, \|x_i\| \leq 1, \forall (i, a), \phi_i(a) \geq 0$  and  $\forall i, \phi_i(0) \leq 1$ . Under these assumptions, we have the following theorem:

**Theorem** Consider Procedure SDCA with  $\alpha^{(0)} = 0$ . Assume that  $\forall i, \phi_i$  is  $L$ -Lipschitz (resp.  $(1/\gamma)$ -smooth). To obtain an expected duality gap of  $\mathbb{E}[P(\bar{w}) - D(\bar{\alpha})] \leq \epsilon_P$ , it suffices to have a total number of iterations of

$$T \geq n + \max\left(0, \left\lceil n \log\left(\frac{\lambda n}{2L^2}\right) \right\rceil\right) + \frac{20L^2}{\lambda \epsilon_P} \quad \left(\text{resp. } T > \left(n + \frac{1}{\lambda \gamma}\right) \log\left[\frac{1}{(T - T_0)\epsilon_P} \left(n + \frac{1}{\lambda \gamma}\right)\right]\right) \quad (8)$$

## 4 Conclusion